# Rectangles and Cartesian coordinates

*In this activity, we explore some consequences of Euclid's fifth axiom.*

## Euclid's fifth axiom, the parallel postulate

After answering the following questions, students should be able to:

- Define Euclidean geometry,
- Prove that in Euclidean geometry, the sum of the interior angles of a triangle is $180°$.
- Show that Euclidean geometry has a Cartesian coordinate system.

We saw in the last chapter that neutral geometry does not assume that two parallel lines must intersect a third line in such a way that the sum of the interior angles (ie, the two angles between the parallel lines) is the same as the sum of two right angles (ie, $180°$). A (very not obviously) equivalent statement is:

**Axiom 1** (E5)**.** *Through a point not on a line there passes a unique parallel line.*

Neutral geometry together with E5 is called *Euclidean* or *flat geometry* (Euclidean geometry). Euclidean geometry is probably what you think about when you think about geometry. We will see later that there is another geometry called *hyperbolic geometry* (**HG**) that satisfies all the postulates of neutral geometry but not E5. In it, the sum of the interior angles of a triangle will *always* be less than $180°$!

In Problem 16, we will show that in Euclidean geometry, two parallel lines must intersect a third line in such a way that the sum of the interior angles is $180°$.

**Problem 1** *Show that if two parallel lines are cut by a transversal line, then alternate interior angles are equal.*

***Hint:*** *Draw a picture and seek a contradiction.*

Learning outcomes:
Author(s):

***Hint:*** *Watch the solution at* `https://youtu.be/tMBADmI6LnY` *or read the file at* `https://osu.instructure.com/courses/84670/files/23572065/`.

*Your solution should make sense to someone who has not watched the video or read the notes file. In particular, you should explain how to draw the diagram and put the proof in your own words.*

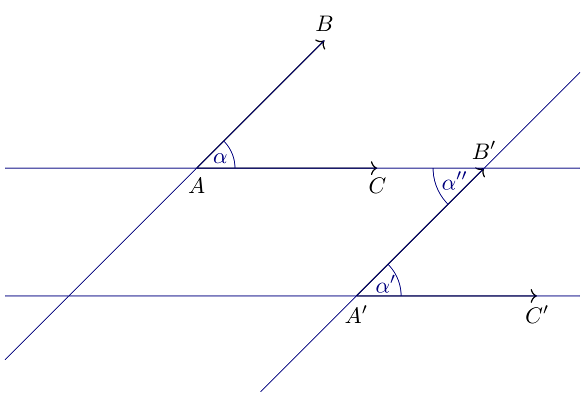
**Problem 2** *Show that if two parallel lines are cut by a transverse line, then the sum of the interior angles is* $180°$.

***Hint:*** *Watch hint video at* `https://youtu.be/7ttySwGJpZg` *or read the notes at* `https://osu.instructure.com/courses/84670/files/23572065/`.

**Problem 3** *Show that angles* $\angle BAC$ *and* $\angle B'A'C'$ *in the Euclidean plane are equal (have the same measure) if corresponding rays are parallel. Hence if* $\angle B'A'C'$ *can be rotated around* $A'$ *to obtain an angle* $\angle B''A'C''$ *with corresponding rays parallel to those of* $\angle BAC$*, then* $\angle BAC$ *and* $\angle B'A'C'$ *are equal.*

***Hint:*** *Draw your angles and extend their legs so that you have two sets of parallel sides.*

***Proof*** *Start with the given angles and extend their legs so that we now have two sets of parallel lines:*




*By our previous work with alternate interior angles,*

$$\alpha = \alpha'' \qquad \text{and} \qquad \alpha'' = \alpha',$$

*hence* $\alpha = \alpha'$. ■

**Problem 4** *Use the 'uniqueness' assertion in E5 together with what we have established about neutral geometry to show that in Euclidean geometry the sum of the interior angles of any triangle is* $180°$.

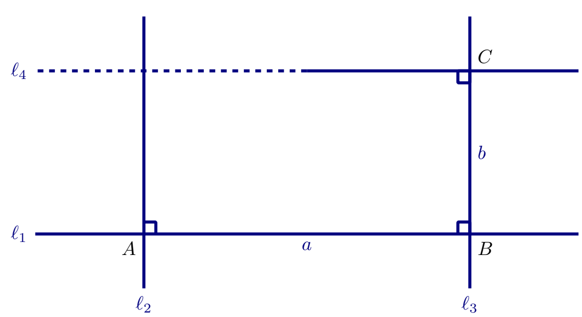
***Hint:*** *Make two parallel lines, the first being the base of a given triangle and the second being the unique parallel line that passes through the vertex opposite to the base.*

***Hint:*** *Watch hint video at* `https://youtu.be/7ttySwGJpZg` *or read the notes at* `https://osu.instructure.com/courses/84670/files/23572065/`.

**Problem 5** *Show that in Euclidean geometry the sum of the interior angles of a quadrilateral is* $360°$.

**Problem 6** *Now we will show in Euclidean geometry that given any positive real numbers $a$ and $b$, there exist rectangles with adjacent side of lengths $a$ and $b$. Your task is to fill-in the details of the proof below. The notes from when we talked about this problem in class are at* `https://osu.instructure.com/files/23572263/` *or you can rewatch the class recording* `https://osu.zoom.us/rec/share/UH_Zr42lQKe2VYVa9t_AVoiPiKuvXj5rcA5M9h9y30-7oMjUSiSoxnFCwXIo8vs.U26znUPozxyOrGaT` *(Access Passcode: Gq7%m.jc)*

*Start by constructing lines $\ell_1$ and $\ell_2$ with $\ell_1$ perpendicular to $\ell_2$ at point $A$. On $\ell_1$ add point $B$ so that $|AB| = a$. Next construct $\ell_3$ perpendicular to $\ell_1$ through $B$. On $\ell_3$ add point $C$ such that $|BC| = b$. Finally add $\ell_4$ through $C$ so that $\ell_4$ is perpendicular to $\ell_3$.*




*Explain why $\ell_3$ is parallel to $\ell_2$.*

*Explain why $\ell_4$ intersects $\ell_2$.*

*Call the intersection of $\ell_2$ and $\ell_4$ point $D$ and label the two remaining sides $a'$ and $b'$. Explain why $\angle ADC = 90°$.*

*Explain why $\ell_1$ is parallel to $\ell_4$.*

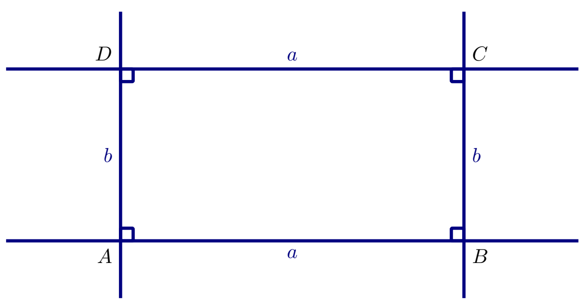
*Finally, add a segment to our figure and use a triangle congruence theorem to explain why $a = a'$ and $b = b'$.*

**Problem 7** *Consider the ordered pair $(x, y) \in \mathbb{R}^2$ to be the corner of rectangle with one vertex at the origin, side lengths $|x|$ and $|y|$, and the sign of $x$ and $y$ moving in either the positive or negative direction along the $x$ and $y$ axes. This is a* Cartesian coordinate system.

*Show that there is a Cartesian coordinate system on Euclidean geometry. This means you must show that there is a bijection between the points in Euclidean geometry and elements of $\mathbb{R}^2$, the set of pairs of real numbers.*

***Hint:*** *For a longer explanation of what the question is asking and where the map comes from, watch the video at* `https://youtu.be/ffal4JkqrEw`.

***Proof*** *Pick any point of Euclidean geometry, call it $A$. Our map will take this point to $(0, 0)$. By our previous problem, we can construct an $a \times b$ rectangle $ABCD$ in Euclidean geometry.*




*Our map will take the the point $C$ to $(a, b)$. On the other hand, if $B$ was placed to the left of $A$, our map takes $C$ to the point $(-a, b)$. If $D$ was placed below $A$, then our map takes $C$ to $(a, -b)$. Finally if $B$ was placed to the left of $A$ and $D$ was placed below $A$, then our map takes $C$ to $(-a, -b)$. By construction, this map is one-to-one and onto and hence shows that we have a Cartesian coordinate system in Euclidean geometry.* ■

## The distance formula in Euclidean geometry

After answering the following questions, students should be able to:

- Use the axioms of Euclidean geometry to prove the Pythagorean theorem,
- Prove the Euclidean distance formula.

It is the existence of a Cartesian coordinate system in Euclidean geometry that allows us to define distance between points

$$d\left((a_1, b_1), (a_2, b_2)\right) = \sqrt{(a_2 - a_1)^2 + (b_2 - b_1)^2}$$

and so gives rigorous mathematical meaning to a concept that the ancient Greeks were never able to describe precisely, namely the similarity of figures in Euclidean geometry. For that we will require the notion of a *dilation* or *magnification* in Euclidean geometry. We need a Cartesian coordinate system to describe dilation precisely, a reality backed up by the fact that similarities do not exist in hyperbolic or spherical. (Try drawing two triangles that are similar but not congruent on a perfectly spherical balloon!)

**Problem 8** *State and prove the Pythagorean theorem in Euclidean geometry. Do not use the distance formula.*

***Hint:*** *In the Cartesian plane, construct a square with vertices*

$$(0,0)\,, \qquad (a+b,0)\,, \qquad (0,a+b)\,, \qquad (a+b,a+b)\,.$$

*Inside that square, construct the square with vertices*

$$(a,0)\,, \qquad (a+b,a)\,, \qquad (b,a+b)\,, \qquad (0,b)\,.$$

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**Problem 9** *Use the Pythagorean theorem to justify the Euclidean distance formula,*

$$d\left((a_1, b_1), (a_2, b_2)\right) = \sqrt{(a_2 - a_1)^2 + (b_2 - b_1)^2}.$$

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**Problem 10** *Summarize the results from this section. In particular, indicate which results follow from the others.*

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