Lines in hyperbolic geometry

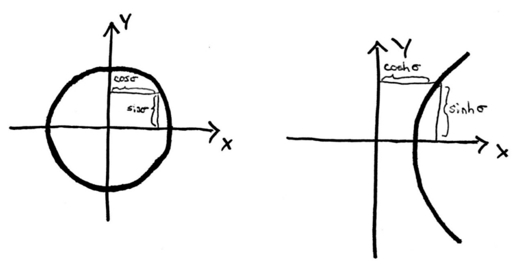
Here we examine "lines" in hyperbolic geometry and prove a hyperbolic version of the Pythagorean Theorem.

Hyperbolic coordinates, a shortest path from the North Pole

After answering the following questions, students should be able to:

- Convert K-warped space to hyperbolic coordinates
- Derive the formula for the hyperbolic (σ, τ)-coordinate dot product
- Show that the shortest distance from the north pole of the K-surface to a point on the surface in the $y = 0$ plane is on the path that stays in the $y = 0$ plane.

We next will figure out what is the shortest path you can take between two points in hyperbolic geometry. Since K is negative, we must do our calculation using only (x, y, z)-coordinates. However, this will allow us to see the full power of working in K-warped space, since our work will be essentially the same as when K was positive—though our parametrization will be different.



Just as $(\cos\sigma, \sin\sigma)$ parametrize the unit circle, hyperbolic functions

$$\left(\cosh\sigma = \frac{e^{\sigma}+e^{-\sigma}}{2}, \sinh\sigma = \frac{e^{\sigma}-e^{-\sigma}}{2}\right)$$

Learning outcomes:
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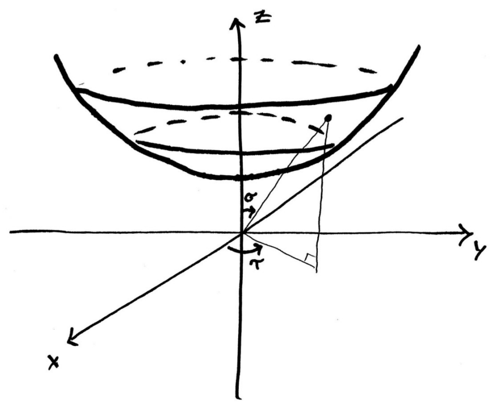
parametrize the 'unit' hyperbola. Hence we define

$$\begin{aligned} x(\sigma,\tau) &= |K|^{-1/2}\cdot\sinh\sigma\cdot\cos\tau,\\ y(\sigma,\tau) &= |K|^{-1/2}\cdot\sinh\sigma\cdot\sin\tau,\\ z(\sigma,\tau) &= \cosh\sigma, \end{aligned}$$

where $0 \le \sigma < \infty$ and $0 \le \tau < 2\pi$.



Problem 1 *Show that these hyperbolic coordinates do actually parametrize K-geometry.*

Hint: *Remember, K is negative.*

Hint: *Remember,*

$$-\sinh^2\sigma + \cosh^2\sigma = 1.$$

Hint: *This is an exercise in "double-containment." To show the one direction, show*

$$K\left(x(\sigma,\tau)^2 + y(\sigma,\tau)^2\right) + z(\sigma,\tau)^2 = 1$$

for all (σ,τ). To show the other direction, appeal to the diagram above.

Just as we did on the R-sphere, we can write a path on the K-surface by giving a path in the (σ, τ)-plane. Again, we will need to figure out the K-dot product in (σ, τ)-coordinates so that we can compute the lengths of paths in these coordinates.

Problem 2 *Suppose we have a curve γ in K-warped space which we can decompose as*

$$\begin{tikzcd} t \arrow[r, mapsto, "\gamma_{\text{hyp}}"] \arrow[rr, bend right, "\gamma"'] & (\sigma(t), \tau(t)) \arrow[r, mapsto, "\begin{bmatrix} x(\sigma,\tau) \\ y(\sigma,\tau) \\ z(\sigma,\tau) \end{bmatrix}"] & \gamma(t) = (x(t), y(t), z(t)) \end{tikzcd}$$

Use the chain rule to compute

$$\frac{dx}{dt}, \qquad \frac{dy}{dt}, \qquad \frac{dz}{dt},$$

in terms of $\dfrac{d\sigma}{dt}, \dfrac{d\tau}{dt}, \dfrac{\partial x}{\partial \sigma}, \dfrac{\partial y}{\partial \sigma}, \dfrac{\partial z}{\partial \sigma}, \dfrac{\partial x}{\partial \tau}, \dfrac{\partial y}{\partial \tau}$, and $\dfrac{\partial z}{\partial \tau}$.

Hint: *Simply write down the answer from a previous problem.*

Problem 3 *With the same setting as in the previous problem, rewrite the result of your computation in matrix notation to find* D_{hyp} *such that*

$$\begin{bmatrix} dx/dt \\ dy/dt \\ dz/dt \end{bmatrix} = D_{\text{hyp}} \cdot \begin{bmatrix} d\sigma/dt \\ d\tau/dt \end{bmatrix}$$

in terms of $\dfrac{\partial x}{\partial \sigma}, \dfrac{\partial y}{\partial \sigma}, \dfrac{\partial z}{\partial \sigma}, \dfrac{\partial x}{\partial \tau}, \dfrac{\partial y}{\partial \tau}$, and $\dfrac{\partial z}{\partial \tau}$.

Hint: *Simply write down the answer from a previous problem.*

Problem 4 *Now find P_{hyp} in terms of K, $\frac{\partial x}{\partial \sigma}, \frac{\partial y}{\partial \sigma}, \frac{\partial z}{\partial \sigma}, \frac{\partial x}{\partial \tau}, \frac{\partial y}{\partial \tau}$, and $\frac{\partial z}{\partial \tau}$ such that*

$$\left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right) \bullet_K \left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right) = \begin{bmatrix} \frac{d\sigma}{dt} & \frac{d\tau}{dt} \end{bmatrix} \cdot P_{\text{hyp}} \cdot \begin{bmatrix} \frac{d\sigma}{dt} \\ \frac{d\tau}{dt} \end{bmatrix}.$$

Hint: *Simply write down the answer from a previous problem.*

Problem 5 *Set*

$$x(\sigma,\tau) = |K|^{-1/2} \cdot \sinh\sigma \cdot \cos\tau,$$
$$y(\sigma,\tau) = |K|^{-1/2} \cdot \sinh\sigma \cdot \sin\tau,$$
$$z(\sigma,\tau) = \cosh\sigma,$$

and show that P_{hyp} *from the problem above is*

$$P_{\mathrm{hyp}} = \begin{bmatrix} |K|^{-1} & 0 \\ 0 & |K|^{-1} \cdot \sinh^2\sigma \end{bmatrix}.$$

Definition 1. *Let* $\mathbf{v}_{\text{hyp}} = \begin{bmatrix} a & b \end{bmatrix}$ *and* $\mathbf{w}_{\text{hyp}} = \begin{bmatrix} c & d \end{bmatrix}$ *be a vectors in* (σ, τ)*-coordinates originating at the same* (σ, τ)*-coordinate. Define*

$$\mathbf{v}_{\text{hyp}} \bullet_{\text{hyp}} \mathbf{w}_{\text{hyp}} = \frac{ac}{|K|^{-1}} + \frac{bd \sinh^2 \sigma}{|K|^{-1}}$$

or in matrix notation,

$$\mathbf{v}_{\text{hyp}} \bullet_{\text{hyp}} \mathbf{w}_{\text{hyp}} = \mathbf{v}_{\text{hyp}} \cdot P_{\text{hyp}} \cdot \mathbf{w}_{\text{hyp}}^{\intercal}$$

where

$$P_{\text{hyp}} = \begin{bmatrix} |K|^{-1} & 0 \\ 0 & |K|^{-1} \cdot \sinh^2 \sigma \end{bmatrix}$$

and σ *is determined by the coordinate that the vectors originate from.*

Problem 6 *How does this definition of the hyperbolic dot product compare with the definition of the K-dot product from Chapter 12 and the spherical dot product from Chapter 15?*

Now notice that you can write a path on the K-surface by giving a path $(\sigma(t), \tau(t))$ in the (σ, τ)-plane. To write a path that starts at the North Pole, just write

$$(\sigma(t), \tau(t)), \qquad 0 \leq t \leq e$$

and demand that $\sigma(0) = \tau(0) = 0$. If you want the path to end on the plane $y = \widehat{y} = 0$, demand additionally that $\tau(e) = 0$.

Now given a path on the K-surface

$$(\sigma(t), \tau(t)), \qquad 0 \leq t \leq e$$

satisfying $\sigma(0) = \tau(0) = 0$ and $\tau(e) = 0$, its length is given by the formula

$$L = \int_0^e \sqrt{\left(\frac{d\sigma}{dt}, \frac{d\tau}{dt}\right) \bullet_{\mathrm{hyp}} \left(\frac{d\sigma}{dt}, \frac{d\tau}{dt}\right)}\, dt. \tag{$*$}$$

Problem 7 *Prove that the shortest path on the K-surface from the North Pole*

$$N = \left(|K|^{-1/2} \cdot \sinh 0 \cdot \cos 0, |K|^{-1/2} \cdot \sinh 0 \cdot \sin 0, \cosh 0\right)$$

to a point

$$(x, y, z) = \left(|K|^{-1/2} \sinh e, 0, \cosh e\right)$$

is the path lying in the plane $y = 0$.

Hint: *Use the same steps you did in the sphere case.*

Shortest path between any two points

After answering the following questions, students should be able to:

- Show that the shortest distance between two points on the hyperbolic plane lies on the plane containing the points and the origin

Just as we proved in spherical geometry that the shortest path is the path cut out by

$$K(x^2 + y^2) + z^2 = 1$$

and the plane containing the origin and the two points in question, we will see that a completely analogous result is true in hyperbolic geometry. Before we start, we will need one more class of rigid motions to add to our collection.

Problem 8 *Assuming K is negative, show*

$$N_\psi = \begin{bmatrix} \cosh\psi & 0 & |K|^{-1/2} \cdot \sinh\psi \\ 0 & 1 & 0 \\ |K|^{1/2} \cdot \sinh\psi & 0 & \cosh\psi \end{bmatrix}$$

is a K-rigid motion.

Problem 9 *Assuming K is negative, consider*

$$N_\psi = \begin{bmatrix} \cosh\psi & 0 & |K|^{-1/2}\cdot\sinh\psi \\ 0 & 1 & 0 \\ |K|^{1/2}\cdot\sinh\psi & 0 & \cosh\psi \end{bmatrix}.$$

Can you describe geometrically what this mapping is doing to the points in K-warped space?

Hint: *First look at the image of the point $(0,0,1)$.*

Theorem 1. *Given any two points* $X_1 = (x_1, y_1, z_1)$ *and* $X_2 = (x_2, y_2, z_2)$ *in* K*-geometry, the shortest path between the two points is the path cut out by the set*

$$K\left(x^2 + y^2\right) + z^2 = 1$$

and the plane containing $(0,0,0)$*,* X_1*, and* X_2*.*

Problem 10 *Explain in words, with pictures as needed, how to prove this theorem by using the* K*-rigid motions*

$$M_\theta = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \text{and} \qquad N_\psi = \begin{bmatrix} \cosh\psi & 0 & |K|^{-1/2}\cdot\sinh\psi \\ 0 & 1 & 0 \\ |K|^{1/2}\cdot\sinh\psi & 0 & \cosh\psi \end{bmatrix}.$$

Hint: *M_θ is a K-rigid motion that rotates around the z-axis and N_ψ is a K-rigid motion that "slides" the K-surface past the y-axis.*

Hint: *You should apply two K-rigid motions of the form M_θ (for different angles) and one K-rigid motion of the form N_ψ—though not necessarily in that order!*

Definition 2. *A* ***line*** *in hyperbolic geometry will be a curve that extends infinitely in each direction and has the property that, given any two points X_1 and X_2 on the path, the shortest path between X_1 and X_2 lies along that curve. Lines in hyperbolic geometry are the intersections of the K-geometry with planes through* $(0,0,0)$. *The length of the shortest path between two points in K-geometry will be called the K-distance.*

The hyperbolic Pythagorean Theorem

After answering the following questions, students should be able to:

- Prove the hyperbolic version of the Pythagorean Theorem
- Show that for very small triangles, the hyperbolic version of the Pythagorean Theorem is approximately the same as the Euclidean version.

To start we need some basic facts about lengths of lines in hyperbolic geometry.

Problem 11 *Given a line in hyperbolic geometry lying entire in the plane* $y = 0$,

$$\begin{aligned} x(t) &= |K|^{-1/2}\sinh t, \\ y(t) &= 0, \\ z(t) &= \cosh(t), \end{aligned}$$

show that the length of the segment on the interval $0 \le t \le e$ *is exactly* $|K|^{-1/2}e$.

Hint: *Use a previous problem.*

Problem 12 *Explain in words how to prove that given two points on the surface*

$$K(x^2+y^2)+z^2=1,$$

say X_A and X_B, the length of the hyperbolic line connecting them is given by

$$|K|^{-1/2}\cdot\varepsilon = |K|^{-1/2}\cdot \operatorname{arcosh}\left(\frac{X_A \bullet_K X_B}{|X_A|_K\cdot|X_B|_K}\right).$$

by using the K-rigid motions

$$M_\theta=\begin{bmatrix}\cos\theta & -\sin\theta & 0\\ \sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{bmatrix} \qquad \text{and} \qquad N_\psi=\begin{bmatrix}\cosh\psi & 0 & |K|^{-1/2}\cdot\sinh\psi\\ 0 & 1 & 0\\ |K|^{1/2}\cdot\sinh\psi & 0 & \cos\psi\end{bmatrix}.$$

We will now give the hyperbolic analogue of the Pythagorean Theorem.

Theorem 2 (Hyperbolic Pythagorean Theorem). *If $\triangle X_A X_B X_C$ is a right triangle on the surface*

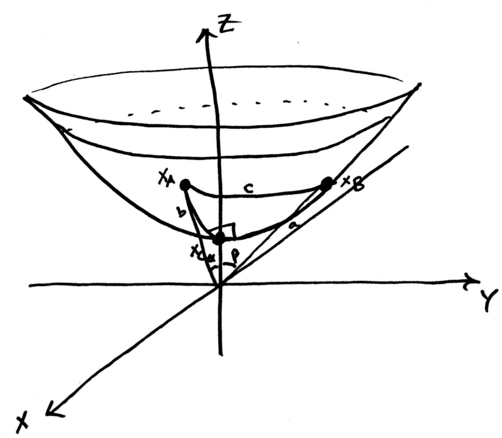
$$K(x^2+y^2)+z^2 \qquad \textit{where} \qquad K<0$$

with right angle $\angle X_A X_C X_B$, and side a opposite X_A, b opposite X_B, and c opposite X_C, then

$$\cosh\left(|K|^{1/2}\cdot c\right)=\cosh\left(|K|^{1/2}\cdot a\right)\cosh\left(|K|^{1/2}\cdot b\right).$$

Let's see why this theorem is true. We may via K-rigid motions place the triangle so that X_C is at the North Pole, X_A is in the plane $y=0$, and X_B is in the plane $x=0$ (note X_A and X_B may be switched—if this is the case, simply rename them). In this case,

$$X_A=(|K|^{-1/2}\cdot\sinh\alpha, 0, \cosh\alpha),$$
$$X_B=(0, |K|^{-1/2}\cdot\sinh\beta, \cosh\beta).$$



Hence the length of side b is $|K|^{-1/2}\cdot\alpha$. Using a rigid motion of the form

$$M_\theta=\begin{bmatrix}\cos\theta & -\sin\theta & 0\\ \sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{bmatrix}$$

when $\theta=\pi/2$ we see that the length of side a is $|K|^{-1/2}\cdot\beta$. Set

$$\gamma=\operatorname{arcosh}\left(\frac{X_A\bullet_K X_B}{|X_A|_K\cdot|X_B|_K}\right).$$

Since we are working on the K-surface,

$$\begin{aligned} K^{-1} \cdot \cosh\gamma &= X_A \bullet_K X_B \\ &= \begin{bmatrix} |K|^{-1/2} \cdot \sinh\alpha & 0 & \cosh\alpha \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & K^{-1} \end{bmatrix} \begin{bmatrix} 0 \\ |K|^{-1/2} \cdot \sinh\beta \\ \cosh\beta \end{bmatrix} \\ &= K^{-1} \cdot \cosh\alpha \cdot \cosh\beta. \end{aligned}$$

Problem 13 *Explain how to progress from the fact that*

$$K^{-1} \cdot \cosh\gamma = K^{-1} \cdot \cosh\alpha \cdot \cosh\beta.$$

to the conclusion of the theorem

$$\cosh(|K|^{1/2} \cdot c) = \cosh(|K|^{1/2} \cdot a)\cosh(|K|^{1/2} \cdot b).$$

Problem 14 *Use the Taylor series expansion of* $\cosh(x)$ *centered around* $x = 0$,

$$\cosh(x) = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots$$

to show that for "small" triangles, the hyperbolic Pythagorean Theorem reduces to the euclidean Pythagorean Theorem, meaning

$$c^2 \approx a^2 + b^2.$$

Problem 15 *Summarize the results from this section. In particular, indicate which results follow from the others.*