Rigid motions in central projection

Here we study rigid motions in central projection coordinates.

Rigid motions in central projection coordinates

Suppose now we have a K-rigid motion

$$\begin{bmatrix} \underline{x} \\ \underline{y} \\ \underline{z} \end{bmatrix} = M \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

of K-geometry, given by a K-orthogonal matrix

$$M = \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{bmatrix}.$$

Let's convert this K-rigid motion to a rigid motion in central projection coordinates. This new rigid motion will not necessarily be a mapping defined by a matrix, so we'll have to use some new notation.

$$\begin{tikzcd}
(x,y,z) \in \mathbb{R}^3 \arrow[d, mapsto, "M\cdot"] & (x_c,y_c,1) \in \mathbb{R}^2 \times \{1\} \arrow[l, mapsto, "\cdot\lambda"'] \arrow[d, mapsto, "\mu_{c'} =?"] & (x_c,y_c) \in \mathbb{R}^2 \arrow[l, mapsto] \arrow[d, mapsto, "\mu_c =?"] \\
(\underline{x},\underline{y},\underline{z}) & (\underline{x_c},\underline{y_c},1) \arrow[l, mapsto, "\cdot\underline{\lambda}"'] & (\underline{x_c},\underline{y_c}) \arrow[l, mapsto]
\end{tikzcd}$$

Problem 1 *Using the diagram above, explain why the formula for* $(\underline{x_c},\underline{y_c}) = \mu_c(x_c,y_c)$ *is*

$$\mu_c(x_c,y_c) = \left(\frac{m_{11}x_c + m_{12}y_c + m_{13}}{m_{31}x_c + m_{32}y_c + m_{33}}, \frac{m_{21}x_c + m_{22}y_c + m_{23}}{m_{31}x_c + m_{32}y_c + m_{33}} \right).$$

Learning outcomes:
Author(s):

From K-rigid motions to rigid motions in central projection

Now let's use our new tool to convert K-rigid motions to rigid motions in central projection.

Problem 2 *For any K, consider the K-rigid motion of K-geometry*

$$M_\theta = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Can you describe geometrically what this mapping is doing to the points in central projection?

Problem 3 *For any K, consider the K-rigid motion of K-geometry*

$$M_\theta = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Convert this to a rigid motion in central projection.

Problem 4 *Assuming $K > 0$, consider the K-rigid motion of the R-sphere*

$$N_\psi = \begin{bmatrix} \cos\psi & 0 & -R \cdot \sin\psi \\ 0 & 1 & 0 \\ R^{-1} \cdot \sin\psi & 0 & \cos\psi \end{bmatrix}.$$

Can you describe geometrically what this mapping is doing to the points in central projection?

Problem 5 *Assuming $K > 0$, consider the K-rigid motion of the R-sphere*

$$N_\psi = \begin{bmatrix} \cos\psi & 0 & -R\cdot\sin\psi \\ 0 & 1 & 0 \\ R^{-1}\cdot\sin\psi & 0 & \cos\psi \end{bmatrix}.$$

Convert this to a rigid motion in central projection.

Problem 6 *Assuming $K < 0$, consider the K-rigid motion of the K-surface*

$$N_\psi = \begin{bmatrix} \cosh\psi & 0 & |K|^{-1/2} \cdot \sinh\psi \\ 0 & 1 & 0 \\ |K|^{1/2} \cdot \sinh\psi & 0 & \cosh\psi \end{bmatrix}$$

Can you describe geometrically what this mapping is doing to the points in central projection?

Problem 7 *Assuming $K < 0$, consider the K-rigid motion of the K-surface*

$$N_\psi = \begin{bmatrix} \cosh\psi & 0 & |K|^{-1/2}\cdot\sinh\psi \\ 0 & 1 & 0 \\ |K|^{1/2}\cdot\sinh\psi & 0 & \cosh\psi \end{bmatrix}$$

Convert this to a rigid motion in central projection.

Problem 8 *Summarize the results from this section. In particular, indicate which results follow from the others.*