# Lines in hyperbolic geometry

*Here we examine "lines" in hyperbolic geometry.*

## Hyperbolic coordinates, a shortest path from the North Pole

We next will figure out what is the shortest path you can take between two points in **HG**. Again we will do our calculation using only $(x, y, z)$-coordinates (since, as we have seen we don't have $(\hat{x}, \hat{y}, \hat{z})$-coordinates). The $(x, y, z)$-coordinates for **SG**, namely

$$\begin{aligned} x(\sigma, \tau) &= R \cdot \sin\sigma \cdot \cos\tau \\ y(\sigma, \tau) &= R \cdot \sin\sigma \cdot \sin\tau \\ z(\sigma, \tau) &= \cos\sigma \end{aligned}$$

won't work this time because they involve $R$ which has gone off to infinity. Fortunately there are hyperbolic coordinates

$$\left(\cosh\sigma = \frac{e^{\sigma} + e^{-\sigma}}{2}, \sinh\sigma = \frac{e^{\sigma} - e^{-\sigma}}{2}\right)$$

that parametrize the 'unit' hyperbola just like $(\cos\sigma, \sin\sigma)$ parametrize the unit circle. So we define

$$\begin{aligned} x(\sigma, \tau) &= |K|^{-1/2} \cdot \sinh\sigma \cdot \cos\tau \\ y(\sigma, \tau) &= |K|^{-1/2} \cdot \sinh\sigma \cdot \sin\tau \\ z(\sigma, \tau) &= \cosh\sigma \end{aligned}$$

**Exercise 1** *Show that these hyperbolic coordinates do actually parametrize the $K$-geometry, that is, that*

$$K\left(x(\sigma, \tau)^2 + y(\sigma, \tau)^2\right) + z(\sigma, \tau)^2 \equiv 1$$

*for all $(\sigma, \tau)$.*

Again notice that you can write a path on the $R$-sphere by giving a path $(\sigma(t), \tau(t))$ in the $(\sigma, \tau)$-plane. In fact, you can use $\sigma$ as the parameter $t$ and just write

$$(\sigma, \tau(\sigma))$$

Learning outcomes:
Author(s):

where $\tau$ is a function of $\sigma$. To write a path that starts at the North Pole, just write

$$(\sigma, \tau(\sigma)), \ 0 \leq \sigma \leq \varepsilon$$

and demand that

$$\tau(0) = 0.$$

If you want the path to end on the plane $y = 0$, demand additionally that

$$\tau(\varepsilon) = 0.$$

But if we are going to describe paths on **HG** by paths in the $(\sigma, \tau)$-plane we are going to need to figure out the $K$-dot product in $(\sigma, \tau)$-coordinates so that we can compute the lengths of paths in these coordinates.

### Exercise 2

(a) *Compute the* $2 \times 3$ *matrix* $D_{hyp}$ *such that*

$$\left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right) = \left(\frac{d\sigma}{dt}, \frac{d\tau}{dt}\right) \cdot D_{hyp}$$

*when a path in* $K$*-geometry is given by a path in the* $(\sigma, \tau)$*-plane.*

*Hint: By the Chain Rule from several variable calculus*

$$D_{hyp} = \left(\begin{array}{ccc} \frac{dx}{d\sigma} & \frac{dy}{d\sigma} & \frac{dz}{d\sigma} \\ \frac{dx}{d\tau} & \frac{dy}{d\tau} & \frac{dz}{d\tau} \end{array}\right).$$

(b) *Use a) to compute the* $K$*-dot product in* $(\sigma, \tau)$*-coordinates, namely compute the matrix* $P_{hyp}$ *in the equation*

$$\begin{aligned}
\left(\frac{d\sigma_1}{dt}, \frac{d\tau_1}{dt}\right) \bullet_{hyp} \left(\frac{d\sigma_2}{dt}, \frac{d\tau_2}{dt}\right) &= \left(\frac{dx_1}{dt}, \frac{dy_1}{dt}, \frac{dz_1}{dt}\right) \bullet_K \left(\frac{dx_2}{dt}, \frac{dy_2}{dt}, \frac{dz_2}{dt}\right) \\
&= \left(\frac{dx_1}{dt}, \frac{dy_1}{dt}, \frac{dz_1}{dt}\right) \cdot \left(\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & K^{-1} \end{array}\right) \cdot \left(\frac{dx_2}{dt}, \frac{dy_2}{dt}, \frac{dz_2}{dt}\right)^t \\
&= \left(\frac{d\sigma_1}{dt}, \frac{d\tau_1}{dt}\right) \cdot D_{hyp} \cdot \left(\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & K^{-1} \end{array}\right) \cdot D_{hyp}^t \cdot \left(\frac{d\sigma_1}{dt}, \frac{d\tau_1}{dt}\right)^t \\
&= \left(\frac{d\sigma_1}{dt}, \frac{d\tau_1}{dt}\right) \cdot P_{hyp} \cdot \left(\frac{d\sigma_1}{dt}, \frac{d\tau_1}{dt}\right)^t.
\end{aligned}$$

**Exercise 3** *Show that the length $L$ of any path in our $K$-geometry is given by*

$$(\sigma, \tau(\sigma)), \ 0 \leq \sigma \leq \varepsilon$$

*with*

$$\tau(0) = 0.$$

*and*

$$\tau(\varepsilon) = 0$$

*is given by the formula*

$$L = |K|^{-1/2} \int_0^{\varepsilon} \sqrt{\left(1, \frac{d\tau}{d\sigma}\right) \cdot \begin{pmatrix} 1 & 0 \\ 0 & \sinh^2 \sigma \end{pmatrix} \cdot \left(1, \frac{d\tau}{d\sigma}\right)^t} d\sigma.$$

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This last formula for $L$ lets us figure out the shortest path from

$$N = (\sinh 0 \cdot \cos 0, R \cdot \sinh 0 \cdot \sin 0, \cosh 0)$$

to

$$\left(|K|^{-1/2} \cdot \sinh \varepsilon, 0, \cosh \varepsilon\right) = \left(|K|^{-1/2} \cdot \sinh \varepsilon \cdot \cos 0, |K|^{-1/2} \cdot \sinh 0 \cdot \sin 0, \cosh \varepsilon\right).$$

Since

$$L = |K|^{-1/2} \cdot \int_0^{\varepsilon} \sqrt{1 + \sinh^2 \sigma \cdot \left(\frac{d\tau}{d\sigma}\right)^2} d\sigma$$

and $\sinh^2 \sigma$ is is positive for almost all $\sigma \in [0, \varepsilon]$, $L$ is minimal only when $\frac{d\tau}{d\sigma}$ is identically 0. But this means that $\tau(\sigma)$ is a constant function. Since $\tau(0) = 0$, this means that $\tau(\sigma)$ is identically 0. So we have the shown the following result.

**Theorem 1.** *The shortest path in $K$-geometry from the North Pole to a point $(x, y, z) = \left(|K|^{-1/2} \cdot \sinh \varepsilon, 0, \cosh \varepsilon\right)$ is the path lying in the plane $y = 0$. The $K$-length of that shortest path is*

$$|K|^{-1/2} \cdot \varepsilon.$$

## Shortest path between any two points

**Theorem 2.** *Given any two points $X_1 = (x_1, y_1, z_1)$ and $X_2 = (x_2, y_2, z_2)$ in $K$-geometry, the shortest path between the two points is the path cut out by the two equations*

$$K\left(x^2 + y^2\right) + z^2 = 1$$

*and the plane*

$$\left|\begin{pmatrix} x & y & z \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{pmatrix}\right| = 0, \tag{1}$$

*that is, the plane containing* $(0,0,0)$ *and* $X_1$ *and* $X_2$.

***Proof*** Let $V_1 = (a_1, b_1, c_1)$ be the tangent vector at $X_1$ of $K$-length 1 that is tangent to the path cut out by the plane given by equation (**??**). Then $(x, y, z) = (a_1, b_1, c_1)$ also satisfies equation (**??**) and so the equation for that plane can also be written

$$\left|\begin{pmatrix} x & y & z \\ x_1 & y_1 & z_1 \\ a_1 & b_1 & c_1 \end{pmatrix}\right| = 0. \tag{2}$$

By a previous exercise, there is a $K$-rigid motion $M$ that takes $X_1$ to the North Pole $N$ and $V_1$ to $(1,0,0)$. So $M$ takes the plane (**??**) to the plane given by the equation

$$\left|\begin{pmatrix} x & y & z \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}\right| = 0,$$

namely the plane.

$$y = 0.$$

So $X_2 \cdot M$ must also line in this plane since $X_2$ lies in the plane (**??**). So

$$X_2 \cdot M = \left(|K|^{-1/2} \cdot \sinh\varepsilon, 0, \cosh\varepsilon\right)$$

for some $\varepsilon$ since all points in $K$-geometry with $y = 0$ can be written as $\left(|K|^{-1/2} \cdot \sinh\varepsilon, 0, \cosh\varepsilon\right)$ for some $\varepsilon$. Since $M$ is a $K$-rigid motion it must take the shortest path from $X_1$ to $X_2$ to the shortest path from $X_1 \cdot M = N$ to $X_2 \cdot M = \left(|K|^{-1/2} \cdot \sinh\varepsilon, 0, \cosh\varepsilon\right)$. But we already know that the shortest path from $X_1 \cdot M$ to $X_2 \cdot M$ is the one cut out by the plane $y = 0$. But that path comes from the path cut out by the plane given by equation (**??**), or, what is the same thing, the plane given by the equation (**??**). This path is called the *great hyperbolic arc* between $X_1$ and $X_2$. ■

**Definition 1.** *A* ***line*** *in* ***HG*** *will be a curve that extends infinitely in each direction and has the property that, given any two points* $X_1$ *and* $X_2$ *on the path, the shortest path between* $X_1$ *and* $X_2$ *lies along that curve. Lines in* ***HG*** *are the intersections of the* $K$*-geometry with planes through* $(0,0,0)$. *The length of the shortest path between two points in* $K$*-geometry will be called the* $K$*-distance.*