Dilations and similarity in Euclidean geometry

Now we explore some properties that are somewhat unique to Euclidean geometry.

Dilations in Euclidean Geometry

After answering the following questions, students should be able to:

- Show how dilations affect vectors.
- Show how dilations affect lines.
- Show how dilations affect angles.

We start this section off by asking you to prove a general result.

Problem 1 *Let X be a set and $f : X \to X$. Prove that a function f^{-1} exists if and only if f is one-to-one and onto.*

Hint: *Remember the definition of a function means that f^{-1} is a function if and only if for every input x in the domain X, there exists a unique output y in the range (in this case, also X).*

Definition 1. *A* transformation *or* mapping *is a function f that takes a set X to itself. That is $f : X \to X$. Mathematicians often use function, map, and transformation interchangeably.*

Definition 2. *A **dilation** is a transformation of the Cartesian plane to itself that:*

(a) *Is one-to-one and onto.*

(b) *Fixes one point called the **center** of the dilation.*

(c) *Takes each line through the center of the dilation to itself.*

Learning outcomes:
Author(s):

(d) *Multiplies all distances by a fixed positive real number called the **magnification factor** of the dilation.*

Dilations are like inflation and deflation or zooming in or out on an image. Inflation or zooming in does not introduce gaps (onto) or folds (one-to-one). One point stays the same when zooming in and lines through the center are stretched.

Definition 3. *Given a point (x_0, y_0) in the plane and a positive real number r, we define a mapping D with center (x_0, y_0) and magnification factor r by the formula*

$$D(x, y) = (x_0, y_0) + r(x - x_0, y - y_0).$$

We will also denote the output $D(x, y)$ of the dilation as $\left(\underline{x}, \underline{y}\right)$.

Problem 2 *Using Cartesian coordinates for the plane, show that the mapping D defined above is a dilation with magnification factor r and center (x_0, y_0).*

Hint: *Use the parametric formula for a line:*

$$\ell(t) = \text{point} + t \cdot \text{vector}$$

Hint: *See video solution at `https://youtu.be/ZkNg-dp2ijQ` or notes `https://osu.instructure.com/courses/84670/files/23562177/`. The video and notes give a proof that D is one-to-one and onto without using the result of the previous problem.*

Proof *We need to verify that D satisfies the four parts of the definition of a dilation.*

(a) *By the previous problem, we can prove that D is one-to-one and onto by finding its inverse. To do this, let $(x, y), (z, w) \in \mathbb{R}^2$ such that*

$$(z, w) := D(x, y) = (x_0, y_0) + r(x - x_0, y - y_0). \tag{1}$$

Since $x, y, x_0, y_0, r \in \mathbb{R}$, we know such a (z, w) exists (This is the proof of onto!). Now we solve for (x, y).

$$(z, w) = (x_0, y_0) + r(x - x_0, y - y_0) \tag{2}$$
$$(z - x_0, w - y_0) = r(x - x_0, y - y_0) \tag{3}$$
$$\left(\frac{z - x_0}{r}, \frac{w - y_0}{r}\right) = (x - x_0, y - y_0) \tag{4}$$
$$\left(\frac{z - x_0}{r} + x_0, \frac{w - y_0}{r} + y_0\right) = (x, y). \tag{5}$$

Thus, $D^{-1}(z, w) := \left(\frac{z - x_0}{r} + x_0, \frac{w - y_0}{r} + y_0\right)$. Since we found an inverse function, D must be one-to-one and onto.

(b) *We must show that the center of the dilation is fixed, that is we must show that*

$$D(x_0, y_0) = (x_0, y_0).$$

Write

$$\begin{aligned} D(x_0, y_0) &= (x_0, y_0) + r(x_0 - x_0, y_0 - y_0) \\ &= (x_0, y_0) + r(0, 0) \\ &= (x_0, y_0). \end{aligned}$$

(c) *Next we must show that every line through the fixed point goes to itself. To do this, consider the line*

$$\ell(t) = (x_0, y_0) + t(u, v)$$

that goes through the center of the dilation in the direction of the vector (u, v)*. Write*

$$\begin{aligned} D(\ell(t)) &= D(x_0 + tu, y_0 + tv) \\ &= (x_0, y_0) + r(x_0 + tu - x_0, y_0 + tv - y_0) \\ &= (x_0, y_0) + r(tu, tv) \\ &= (x_0, y_0) + t(ru, rv). \end{aligned}$$

This is the line that passes through the center of the dilation in the direction of (ru, rv)*, but since* r *is a positive real number, the vector* (u, v) *goes in the same direction as the vector* (ru, rv)*. Hence a dilation maps any line that passes through its center to itself.*

(d) *Finally, we must show that all distances are multiplied by the magnification factor of the dilation. Consider two points* (a_1, b_1) *and* (a_2, b_2)*. Write*

$$\begin{aligned} D(a_1, b_1) &= (x_0, y_0) + r(a_1 - x_0, b_1 - y_0) \\ &= (x_0 + ra_1 - rx_0, y_0 + rb_1 - ry_0) \\ &= (\underline{a_1}, \underline{b_1}). \end{aligned}$$

and

$$\begin{aligned} D(a_2, b_2) &= (x_0, y_0) + r(a_2 - x_0, b_2 - y_0) \\ &= (x_0 + ra_2 - rx_0, y_0 + rb_2 - ry_0) \\ &= (\underline{a_2}, \underline{b_2}). \end{aligned}$$

Now

$$\begin{aligned}
d((\underline{a_1},\underline{b_1}),(\underline{a_2},\underline{b_2})) &= \sqrt{(\underline{a_2}-\underline{a_1})^2+(\underline{b_2}-\underline{b_1})^2}\\
&= \sqrt{(x_0+ra_2-rx_0-x_0-ra_1+rx_0)^2+(y_0+rb_2-ry_0-y_0-rb_1+ry_0)^2}\\
&= \sqrt{(ra_2-ra_1)^2+(rb_2-rb_1)^2}\\
&= \sqrt{r^2(a_2-a_1)^2+r^2(b_2-b_1)^2}\\
&= r\sqrt{(a_2-a_1)^2+(b_2-b_1)^2}\\
&= r\cdot d((a_1,b_1),(a_2,b_2)).
\end{aligned}$$

Hence we see that the distance between any two points is multiplied by the magnification factor.

■

Problem 3 *Show that the inverse mapping of a dilation is again a dilation with the same center but with magnification factor r^{-1}.*

Hint: *Solve for (x,y) in terms of $(\underline{x},\underline{y})$. Look for where this is done in the previous problem.*

Problem 4 *Show that a dilation by a factor of r takes any vector to r times itself.*

Hint: *Write a vector $\vec{u}$ as (a,b), where the head of $\vec{u}$ is on (a,b) when the tail is on $(0,0)$. That is, $\vec{u}$ has slope $\frac{b}{a}$ and length $\sqrt{a^2+b^2}$.*

Look for where this was done in a previous problem.

Problem 5 *Show that a dilation takes a line to a line parallel (or equal) to itself.*

Hint: *Use the parametric formula for a line. For a longer review of parameterizing lines using vectors, see* https://youtu.be/ZwgeCXuIROA *and* https://youtu.be/ZkNg-dp2ijQ?t=532 *at 8:52.*

Problem 6 *Show that a dilation of the plane preserves angles.*

Problem 7 *Show using several-variable calculus that a dilation with magnification factor r multiplies all areas by a factor of r^2.*

Hint: *Recall that if R is a region in the plane,*

$$\iint_{D(R)} f(x,y)\,dxdy = \iint_R f(D(x,y))\,|\det J_D(x,y)|\;dxdy,$$

where

$$J_D(x,y) = \begin{bmatrix} \dfrac{\partial D_x}{\partial x} & \dfrac{\partial D_y}{\partial x} \\ \dfrac{\partial D_x}{\partial y} & \dfrac{\partial D_y}{\partial y} \end{bmatrix}$$

and D_x and D_y are the components of $D(x,y)$.

Proof *Let R be the region in the plane. Write*

$$\int_{D(R)} 1\,dxdy = \int_R |\det J_D(x,y)|\;dxdy,$$

where

$$J_D(x,y) = \begin{bmatrix} \dfrac{\partial D_x}{\partial x} & \dfrac{\partial D_x}{\partial y} \\ \dfrac{\partial D_y}{\partial x} & \dfrac{\partial D_y}{\partial y} \end{bmatrix} = \begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix}.$$

Hence $|\det J_D(x,y)| = r^2$ and so

$$\int_{D(R)} 1\,dxdy = r^2 \cdot \int_R 1\,dxdy.$$

This shows that a dilation with magnification factor r multiplies all areas by a factor of r^2. ■

Problem 8 *Give an explanation that a middle grades student would understand that a dilation with magnification factor r multiplies all areas by a factor of r^2.*

Hint: *Break the region into rectangles. Focus on explaining why this fact is true for rectangles, then why it makes sense to be true in general.*

Similarity in Euclidean Geometry

After answering the following questions, students should be able to:

- Prove two triangles are similar if and only if the corresponding sides are proportionally with the same constant of proportionality.
- Prove two triangles are similar if and only if the corresponding angles are congruent.
- Prove two triangles are similar if the corresponding sides are parallel or perpendicular.

Definition 4. *Two triangles are* ***similar*** *if there is a dilation of the plane that takes one to a triangle which is congruent to the other. We write*

$$\triangle ABC \sim \triangle A'B'C'$$

to denote that these two triangles are similar (where the order of the vertices tells us which vertices correspond).

Problem 9

(a) *Show that, if two triangles are similar, then corresponding sides are proportional with the same constant of proportionality.*

(b) *Show that, if corresponding sides of two triangles are proportional with the same constant of proportionality, then the two triangles are similar.*

Hint: *For the first part, you have to start from the hypothesis that the two triangles satisfy our definition of similar triangles.*

Hint: *For the second part, you have to start from the assumption that corresponding sides of the two triangles are proportional and use SSS to show that there is a dilation of $\triangle ABC$ is congruent to $\triangle A'B'C'$.*

Proof *For the first part, suppose that we have two similar triangles. This means that there is a dilation that takes one triangle to a triangle congruent to the other. Since a dilation with magnification factor r takes a vector to r times that vector, and congruence preserves lengths, we have that*

$$|AB| = r \cdot |A'B'|$$
$$|AC| = r \cdot |A'C'|$$
$$|BC| = r \cdot |B'C'|.$$

■

Problem 10

(a) *Show that, if two triangles are similar, then corresponding angles are equal.*

(b) *Show that, if corresponding angles of two triangles are equal, then the two triangles are similar.*

Hint: *For the first part, you have to start from the assumption that the two triangles satisfy our definition of similar triangles.*

Hint: *For the second part, you have to start from the assumption that corresponding angles of the two triangles are equal, then use a dilation with $r = |A'B'|/|AB|$ and ASA to show that the dilation of one triangle is congruent to the other.*

Proof *For the second part, suppose that corresponding angles of two triangles are equal. Set the magnification factor of the dilation D to be*

$$r = \frac{|A'B'|}{|AB|}.$$

In this case, D maps $\overline{AB}$ to a segment congruent to $\overline{A'B'}$. Moreover, dilations preserve angles. Hence by ASA, D maps $\triangle ABC$ to a triangle congruent to $\triangle A'B'C'$, and so $\triangle ABC \sim \triangle A'B'C'$. ■

Problem 11 *Show that two triangles are similar if corresponding sides are parallel.*

Hint: *Use the fact that angles are equal if corresponding rays are parallel.*

Problem 12 *Show that two triangles are similar if corresponding sides are perpendicular.*

Hint: *Just think about one angle at a time.*

Proof *It is sufficient to show that given two angles with perpendicular legs, then the angles are equal.*

Extend the legs of the angles until they cross at right angles. There could be several configurations, but all proofs will be similar. Right triangles will be formed containing the angle in question. Using the fact that right triangles are similar if and only if two nonright angles are equal one can show the result for any configuration. ■

Problem 13 *Summarize the results from this section. In particular, indicate which results follow from the others.*