Lines in spherical geometry

Here we examine "lines" in spherical geometry and prove a spherical version of the Pythagorean Theorem.

Spherical coordinates, a shortest path from the North Pole

After answering the following questions, students should be able to:

- Convert K-warped space to spherical coordinates
- Derive the formula for the spherical or (σ, τ)-coordinate dot product
- Show that the shortest distance from the north pole of the R-sphere to a point on the sphere in the $y = 0$ plane is on the path that stays in the $y = 0$ plane.

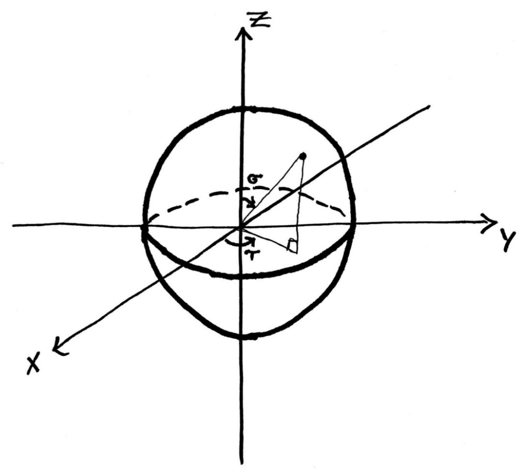
We next will figure out what is the shortest path you can take between two points on the Euclidean R-sphere. However, we will do our calculation using (x, y, z)-coordinates, as we won't have $(\widehat{x}, \widehat{y}, \widehat{z})$-coordinates when we get to hyperbolic geometry.

For our purposes, it will be convenient to parameterize the sphere in K-geometry:

$$\begin{aligned} x(\sigma, \tau) &= R \cdot \sin\sigma \cdot \cos\tau, \\ y(\sigma, \tau) &= R \cdot \sin\sigma \cdot \sin\tau, \\ z(\sigma, \tau) &= \cos\sigma, \end{aligned}$$

where $0 \leq \sigma \leq \pi$ and $0 \leq \tau < 2\pi$.

Learning outcomes:
Author(s):



Problem 1 *Show that these functions actually parameterize the R-sphere.*

Hint: *This is an exercise in "double-containment": you must show that every point specified in this way is a point on the sphere and vice versa. To show every point is on the sphere, show*

$$K\left(x(\sigma,\tau)^2 + y(\sigma,\tau)^2\right) + z(\sigma,\tau)^2 = 1$$

for all (σ, τ). To show the every point on the sphere can be written this way, appeal to the diagram above.

If we are going to describe paths on the R-sphere by paths in the (σ, τ)-plane we are going to need to figure out the K-dot product in (σ, τ)-coordinates so that we can compute the lengths of paths in these coordinates.

Problem 2 *Suppose we have a curve γ in K-warped space which we can decompose as*

$$\begin{tikzcd} t \arrow[r, mapsto, "\gamma_{\mathrm{sph}}"] \arrow[rr, bend right, "\gamma"'] & (\sigma(t), \tau(t)) \arrow[r, mapsto, "\begin{bmatrix} x(\sigma,\tau) \\ y(\sigma,\tau) \\ z(\sigma,\tau) \end{bmatrix}"] & \gamma(t) = (x(t), y(t), z(t)) \end{tikzcd}$$

Use the chain rule to compute

$$\frac{dx}{dt}, \qquad \frac{dy}{dt}, \qquad \frac{dz}{dt},$$

in terms of $\dfrac{d\sigma}{dt}, \dfrac{d\tau}{dt}, \dfrac{\partial x}{\partial \sigma}, \dfrac{\partial y}{\partial \sigma}, \dfrac{\partial z}{\partial \sigma}, \dfrac{\partial x}{\partial \tau}, \dfrac{\partial y}{\partial \tau}$, and $\dfrac{\partial z}{\partial \tau}$.

Hint: *Recall that if F is a differentiable function of a and b; and if a and b are all differentiable functions of t, then the chain rule states*

$$\frac{dF}{dt} = \nabla F \cdot \begin{bmatrix} da/dt \\ db/dt \end{bmatrix}.$$

Problem 3 *With the same setting as in the previous problem, rewrite the result of your computation in matrix notation to find D_{sph} such that*

$$\begin{bmatrix} dx/dt \\ dy/dt \\ dz/dt \end{bmatrix} = D_{\text{sph}} \cdot \begin{bmatrix} d\sigma/dt \\ d\tau/dt \end{bmatrix}$$

in terms of $\dfrac{\partial x}{\partial \sigma}, \dfrac{\partial y}{\partial \sigma}, \dfrac{\partial z}{\partial \sigma}, \dfrac{\partial x}{\partial \tau}, \dfrac{\partial y}{\partial \tau}$, and $\dfrac{\partial z}{\partial \tau}$.

Problem 4 *Now find P_{sph} in terms of K, $\frac{\partial x}{\partial \sigma}$, $\frac{\partial y}{\partial \sigma}$, $\frac{\partial z}{\partial \sigma}$, $\frac{\partial x}{\partial \tau}$, $\frac{\partial y}{\partial \tau}$, and $\frac{\partial z}{\partial \tau}$ such that*

$$\left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right) \bullet_K \left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right) = \begin{bmatrix} \frac{d\sigma}{dt} & \frac{d\tau}{dt} \end{bmatrix} \cdot P_{\text{sph}} \cdot \begin{bmatrix} \frac{d\sigma}{dt} \\ \frac{d\tau}{dt} \end{bmatrix}.$$

Problem 5 *Set*

$$x(\sigma,\tau) = R\cdot\sin\sigma\cdot\cos\tau,$$
$$y(\sigma,\tau) = R\cdot\sin\sigma\cdot\sin\tau,$$
$$z(\sigma,\tau) = \cos\sigma,$$

and show that P_{sph} from the problem above is

$$P_{\mathrm{sph}} = \begin{bmatrix} R^2 & 0 \\ 0 & R^2\cdot\sin^2\sigma \end{bmatrix}.$$

Definition 1. *Let* $\mathbf{v}_{\mathrm{sph}} = \begin{bmatrix} a & b \end{bmatrix}$ *and* $\mathbf{w}_{\mathrm{sph}} = \begin{bmatrix} c & d \end{bmatrix}$ *be a vectors in* (σ, τ)*-coordinates based at the same point* (σ, τ)*-coordinate. Define*

$$\mathbf{v}_{\mathrm{sph}} \bullet_{\mathrm{sph}} \mathbf{w}_{\mathrm{sph}} = R^2 ac + R^2 bd \sin^2 \sigma$$

or in matrix notation,

$$\mathbf{v}_{\mathrm{sph}} \bullet_{\mathrm{sph}} \mathbf{w}_{\mathrm{sph}} = \mathbf{v}_{\mathrm{sph}} \cdot P_{\mathrm{sph}} \cdot \mathbf{w}_{\mathrm{sph}}^{\mathsf{T}}$$

where

$$P_{\mathrm{sph}} = \begin{bmatrix} R^2 & 0 \\ 0 & R^2 \cdot \sin^2 \sigma \end{bmatrix}$$

and σ *is determined by the coordinate that the vectors originate from.*

Now notice that you can write a path on the R-sphere by giving a path $(\sigma(t), \tau(t))$ in the (σ, τ)-plane. To write a path that starts at the North Pole, just write

$$(\sigma(t), \tau(t)), \qquad 0 \le t \le e$$

and demand that $\sigma(0) = 0$. If you want the path to end on the plane $y = \widehat{y} = 0$, demand additionally that $\tau(e) = 0$.

Now given a path on the R-sphere

$$(\sigma(t), \tau(t)), \qquad 0 \le t \le e$$

satisfying $\sigma(0) = \tau(0) = 0$ and $\tau(e) = 0$, its length is given by the formula

$$L = \int_0^e \sqrt{\left(\frac{d\sigma}{dt}, \frac{d\tau}{dt}\right) \bullet_{\mathrm{sph}} \left(\frac{d\sigma}{dt}, \frac{d\tau}{dt}\right)}\, dt. \tag{$*$}$$

(Why?)

Problem 6 *Prove that the shortest path on the* R*-sphere from the North Pole* $(0, 0, 1)$ *(in* (σ, τ)*-coordinates,* $(0, 0)$*) to a point*

$$(x, y, z) = (R \sin e, 0, \cos e)$$

(in $\sigma, \tau)$*-coordinates,* $(e, 0)$*) is the downwards path lying in the plane* $y = 0$*.*

Hint: *Start with some arbitrary path* $(\sigma(t), \tau(t))$ *which begins and ends at this point and show that it's longer than this path, in two steps:*

- *First use the equation* $(*)$ *to show that it gets shorter if you keep the same* σ *but make* τ *always zero (squishing the curve onto the plane* $y = 0$.*)*
- *Finish the argument by eliminating any backtracking.*

Shortest path between any two points

After answering the following questions, students should be able to:

- Show that the shortest distance between two points on a sphere lies on a great circle

We next prove the theorem that shows that shortest path on the surface of the Earth from Rio de Janeiro to Los Angeles is the one cut on the surface of the Earth by the plane that passes through the center of the Earth and through Rio and through Los Angeles. That is usually the route an airplane would take when flying between the two cities.

Theorem 1. *Given any two points $X_1 = (x_1, y_1, z_1)$ and $X_2 = (x_2, y_2, z_2)$ in K-geometry, the shortest path between the two points is the path cut out by the set*

$$K\left(x^2 + y^2\right) + z^2 = 1$$

and the plane containing $(0,0,0)$, X_1, and X_2.

Problem 7 *(**Tricky!**) Explain in words how to prove this theorem by using K-rigid motions of the form*

$$M_\theta = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \text{and} \qquad N_\psi = \begin{bmatrix} \cos\psi & 0 & -R\cdot\sin\psi \\ 0 & 1 & 0 \\ R^{-1}\cdot\sin\psi & 0 & \cos\psi \end{bmatrix}.$$

Hint: *M_θ is a K-rigid motion that rotates around the z-axis and N_ψ is a K-rigid motion that rotates around the y-axis.*

Hint: *You should apply two K-rigid motions of the form M_θ (for different angles) and one K-rigid motion of the form N_ψ, though not necessarily in that order!*

Definition 2. *A **line** in spherical geometry will be a curve that extends infinitely in each direction and has the property that, given any two points X_1 and X_2 on the path, the shortest path between X_1 and X_2 lies along that curve.*

Lines in spherical geometry are usually called great circles on the R-sphere. They are the intersections of the R-sphere with planes through $(0, 0, 0)$.

The spherical Pythagorean Theorem

After answering the following questions, students should be able to:

- Prove the spherical version of the Pythagorean Theorem
- Show that for very small triangles, the spherical version of the Pythagorean Theorem is approximately the same as the Euclidean version.

To start we need some basic facts about lengths of lines in spherical geometry.

Problem 8 *Given a line in spherical geometry lying entirely in the plane* $y = 0$,

$$\begin{aligned} x(t) &= R\sin t, \\ y(t) &= 0, \\ z(t) &= \cos t, \end{aligned}$$

show that the length of the segment at $0 \leq t \leq e$ is exactly Re.

Hint: *Use a previous problem.*

Problem 9 *Explain in words how to prove that given two points on R-sphere, say X_A and X_B, the length of the spherical line connecting them is given by*

$$R \cdot e = R \cdot \arccos\left(\frac{X_A \bullet_K X_B}{|X_A|_K \cdot |X_B|_K}\right).$$

by using K-rigid motions of the form

$$M_\theta = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad N_\psi = \begin{bmatrix} \cos\psi & 0 & -R \cdot \sin\psi \\ 0 & 1 & 0 \\ R^{-1} \cdot \sin\psi & 0 & \cos\psi \end{bmatrix}.$$

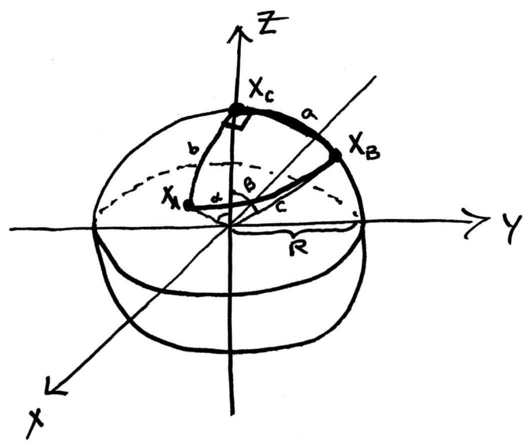
We will now give the spherical analogue of the Pythagorean Theorem.

Theorem 2 (Spherical Pythagorean Theorem)**.** *If $\triangle X_A X_B X_C$ is a right triangle on the R-sphere with right angle $\angle X_A X_C X_B$, and side a opposite X_A, b opposite X_B, and c opposite X_C, then*

$$\cos\left(\frac{c}{R}\right) = \cos\left(\frac{a}{R}\right)\cos\left(\frac{b}{R}\right).$$

Let's see why this theorem is true. We may via K-rigid motions place the triangle so that X_C is at the North Pole, X_A is in the plane $y = 0$, and X_B is in the plane $x = 0$ (note X_A and X_B may be switched—if this is the case, simply rename them). In this case,

$$X_A = (R \cdot \sin\alpha, 0, \cos\alpha),$$
$$X_B = (0, R \cdot \sin\beta, \cos\beta).$$



Hence the length of side b is $R \cdot \alpha$. Using a rigid motion of the form

$$M_\theta = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

when $\theta = \pi/2$ we see that the length of side a is $R \cdot \beta$. Set

$$\gamma = \arccos\left(\frac{X_A \bullet_K X_B}{|X_A|_K \cdot |X_B|_K}\right).$$

Since we are working on the R-sphere,

$$\begin{aligned} R^2 \cdot \cos\gamma &= X_A \bullet_K X_B \\ &= \begin{bmatrix} R\cdot\sin\alpha & 0 & \cos\alpha \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & R^2 \end{bmatrix} \begin{bmatrix} 0 \\ R\cdot\sin\beta \\ \cos\beta \end{bmatrix} \\ &= R^2 \cdot \cos\alpha \cdot \cos\beta. \end{aligned}$$

Problem 10 *Explain how to progress from the fact that*

$$R^2 \cdot \cos\gamma = R^2 \cdot \cos\alpha \cdot \cos\beta.$$

to the conclusion of the theorem

$$\cos\left(\frac{c}{R}\right) = \cos\left(\frac{a}{R}\right)\cos\left(\frac{b}{R}\right).$$

Problem 11 *Use the Taylor series expansion of* $\cos(x)$ *centered around* $x = 0$,

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

to show that for "small" triangles, the spherical Pythagorean Theorem reduces to the Euclidean Pythagorean Theorem, meaning

$$c^2 \approx a^2 + b^2.$$

Problem 12 *Summarize the results from this section. In particular, indicate which results follow from the others.*