

Rigid motions in (x, y, z) -coordinates

Here we dig deeper to understand our different coordinates.

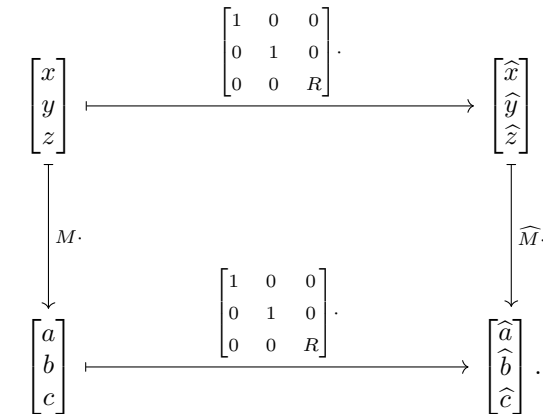
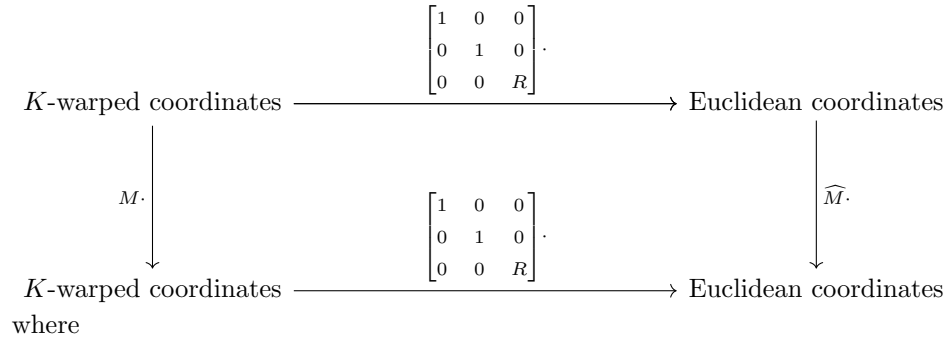
We now wish to figure out how to convert a transformation

$$\widehat{T}(\widehat{x}, \widehat{y}, \widehat{z}) = \begin{bmatrix} \widehat{a} \\ \widehat{b} \\ \widehat{c} \end{bmatrix} = \widehat{M} \cdot \begin{bmatrix} \widehat{x} \\ \widehat{y} \\ \widehat{z} \end{bmatrix}$$

from Euclidean space to (x, y, z) -coordinates. Recall to convert a point from (x, y, z) -coordinates to Euclidean coordinates, we write

$$\begin{bmatrix} \widehat{x} \\ \widehat{y} \\ \widehat{z} \end{bmatrix} = \begin{bmatrix} x \\ y \\ Rz \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & R \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

We summarize this in the diagrams



Learning outcomes:
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Thus, we start at $[x \ y \ z]^\top$ and end at $[\hat{a} \ \hat{b} \ \hat{c}]^\top$ whether we convert to Euclidean coordinates then apply \widehat{M} , or we apply M then convert to Euclidean coordinates.

Problem 1 *Using the diagram above, explain why*

$$M = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & R^{-1} \end{bmatrix} \cdot \widehat{M} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & R \end{bmatrix}.$$

Hint: We can also go from Euclidean coordinates to K -warped coordinates using the map defined by $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & R \end{bmatrix}^{-1}$

Hint: We start at $[x \ y \ z]^\top$ and end at $[\hat{a} \ \hat{b} \ \hat{c}]^\top$ whether we convert to Euclidean coordinates then apply \widehat{M} , or we apply M then convert to Euclidean coordinates.

Problem 2 Show that a transformation M in (x, y, z) -coordinates preserves distances in K -warped space if and only if

$$M^T \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & K^{-1} \end{bmatrix} \cdot M = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & K^{-1} \end{bmatrix}. \quad (*)$$

Hint: Explain how the previous problem gives a correspondence between transformations defined by a matrix in K -warped space and transformations defined by a matrix in Euclidean space. Then show that $(*)$ is equivalent to the corresponding Euclidean matrix \widehat{M} being orthogonal.

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The equation (??) is the condition (in (x, y, z) -coordinates) which affirms that the transformation which takes the path $\gamma(t) = (x(t), y(t), z(t))$ to the path $\gamma_M(t) = M \cdot (x(t), y(t), z(t))^T$ preserves lengths of tangent vectors at corresponding points. Therefore, by integrating, the (total) length of the curve γ_M is the same as the total length of the curve γ .

Problem 3 Verify that this is the correct condition by showing that any 3×3 matrix M satisfying (??) also satisfies

$$(M \cdot \mathbf{v}) \bullet_K (M \cdot \mathbf{v}) = \mathbf{v} \bullet_K \mathbf{v},$$

where

$$\mathbf{v} = X_2 - X_1.$$

That is, the transformation given in (x, y, z) -coordinates by a matrix M that satisfies your condition preserves the K -dot product.

Definition 1. A K -distance-preserving transformation of K -geometry is called a **K -rigid motion** or a **K -congruence**.

With this definition, and our work above, we make a new definition:

Definition 2. A 3×3 matrix M is called **K -orthogonal** if

$$M^T \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & K^{-1} \end{bmatrix} \cdot M = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & K^{-1} \end{bmatrix}.$$

Problem 4 For $K \neq 0$, show that if M is K -orthogonal, then the transformation

$$T(x, y, z) = \begin{bmatrix} a \\ b \\ c \end{bmatrix} = M \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

takes the set of points (x, y, z) such that

$$1 = K(x^2 + y^2) + z^2$$

to the set of points (a, b, c) such that

$$1 = K(a^2 + b^2) + c^2.$$

That is, M gives a one-to-one and onto mapping of K -geometry to itself.

Hint: Explain how you can write the equation

$$1 = K(a^2 + b^2) + c^2.$$

as

$$\begin{bmatrix} a & b & c \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & K^{-1} \end{bmatrix} \cdot \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \frac{1}{K}.$$

Problem 5 For $K \neq 0$, show that the set of K -orthogonal matrices M forms a group. That is, show that

- (a) multiplication of K -orthogonal matrices is associative,

Hint: Recall that function composition is always associative.

- (b) the product of two K -orthogonal matrices is K -orthogonal,

- (c) the identity matrix is K -orthogonal,

- (d) the inverse matrix M^{-1} of a K -orthogonal matrix M is K -orthogonal.
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Problem 6 *Convert the orthogonal matrix*

$$\widehat{M}_\theta = \begin{bmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

into its K -orthogonal counterpart M_θ . Are you surprised? Why or why not?

Problem 7 *Convert the orthogonal matrix*

$$\widehat{M}_\psi = \begin{bmatrix} \cos \psi & 0 & -\sin \psi \\ 0 & 1 & 0 \\ \sin \psi & 0 & \cos \psi \end{bmatrix}$$

into its K -orthogonal counterpart M_ψ . Are you surprised? Why or why not?

Why use K -coordinates?

We have seen that we could measure the usual Euclidean lengths of curves $\hat{\gamma}$ in terms of the formulas of curves γ in K -warped space using the K -dot product. The short reason for this is that

$$\frac{d\hat{\gamma}}{dt} \bullet \frac{d\hat{\gamma}}{dt} = \frac{d\gamma}{dt} \bullet_K \frac{d\gamma}{dt}$$

where

$$\frac{d\gamma}{dt} \bullet_K \frac{d\gamma}{dt} = \left[\frac{d\gamma}{dt} \right] \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & K^{-1} \end{bmatrix} \cdot \left[\frac{d\gamma}{dt} \right]^\top.$$

In other words, the usual geometry of the sphere of radius R is simply the geometry of the set

$$\{(x, y, z) \in \mathbb{R}^3 : 1 = K(x^2 + y^2) + z^2\}$$

with $K = 1/R^2$ and with lengths (and areas) given by the K -dot product. Said another way, we can do all of spherical geometry in (x, y, z) -coordinates. All we need is the set defined by the relation

$$1 = K(x^2 + y^2) + z^2$$

and the K -dot product. But the set defined by the equation above continues to exist even if $K = 0$ or $K < 0$, and the K -dot product formula continues to make sense even if $K < 0$. In short we have the following table:

	Spherical ($K > 0$)	Euclidean ($K = 0$)	Hyperbolic ($K < 0$)
Surface in Euclidean space	$\hat{x}^2 + \hat{y}^2 + \hat{z}^2 = R^2$	DNE	DNE
Euclidean dot product	$\hat{\mathbf{v}}^\top \cdot \hat{\mathbf{w}}$	DNE	DNE
Surface in K -warped space	$1 = K(x^2 + y^2) + z^2$	$1 = K(x^2 + y^2) + z^2$	$1 = K(x^2 + y^2) + z^2$
K -dot product	$\mathbf{v}^\top \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & K^{-1} \end{bmatrix} \mathbf{w}$	DNE	$\mathbf{v}^\top \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & K^{-1} \end{bmatrix} \mathbf{w}$

This table tells us that ‘there is something else out there,’ that is, some other type of two-dimensional geometry beyond plane geometry and spherical geometry. But the gap in the bottom row of the table is a bit disturbing. If we can’t express the usual dot-product in plane geometry as the K -dot product for

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$K = 0$, we can't pass smoothly from spherical through plane geometry to hyperbolic geometry using (x, y, z) -coordinates. Later, we will examine two ways to produce coordinates uniformly for spherical, plane and hyperbolic geometry that overcome this difficulty.

Problem 8 *Summarize the results from this section. In particular, indicate which results follow from the others.*
