Central projection

November 2, 2020

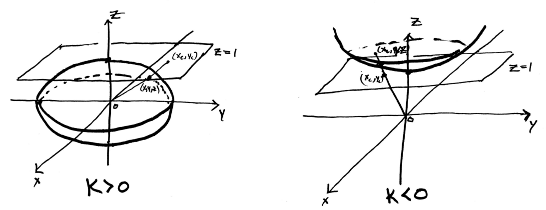
Contents

Central projection coordinates

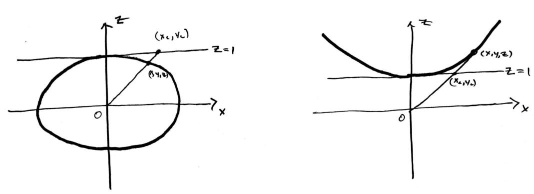
Now we will examine one way to produce coordinates uniformly for spherical, plane, and hyperbolic geometry that will allow us to pass smoothly through the three geometries. Let's project K-geometry,

$$1 = K\left(x^2 + y^2\right) + z^2$$

onto the plane $z = 1$ using the origin $O = (0, 0, 0)$ as the center of projection:



Let's look at this from a different vantage point, say with our eye along the edge of the plane $z = 1$:



Note, we are only projecting the top of the surface

$$1 = K\left(x^2 + y^2\right) + z^2$$

onto the plane $z = 1$. So when $K > 0$, this is the "Northern hemisphere" of the sphere, when $K < 0$, this is the upper hyperboloid, and when $K = 0$, this is the plane $z = 1$.

Problem 1 *Use similar triangles to explain why for any given (x, y, z), there is a number λ such that*

$$\lambda \cdot (x_c, y_c, 1) = (x, y, z).$$

Problem 2 *What range of values could λ take when $K > 0$?*

Problem 3 *What range of values could λ take when $K < 0$?*

Problem 4 *What range of values could λ take when $K = 0$?*

Problem 5 *For the projection of the set* $1 = K\left(x^2+y^2\right)+z^2$ *onto the* $z=1$ *plane with center of projection* O*, write* (x_c, y_c) *as a function of* (x, y, z).

Problem 6 *For the projection of the set* $1 = K\left(x^2 + y^2\right) + z^2$ *onto the* $z = 1$ *plane with center of projection* O*, write* (x, y, z) *as a function of* (x_c, y_c)*.*

Central projection dot product

Like we have done before, now we want to be able to find a dot product that will allow us to compute lengths in central projection coordinates that will agree with the K-dot product, and hence the euclidean dot product.

Problem 7 *Suppose we have a curve γ in K-warped space which we can decompose as*

$$\begin{tikzcd} t \arrow[r, mapsto, "\gamma_c"] \arrow[rr, bend right, "\gamma"'] & (x_c(t), y_c(t)) \arrow[r, mapsto, "\begin{bmatrix} x(x_c,y_c) \\ y(x_c,y_c) \\ z(x_c,y_c) \end{bmatrix}"] & \gamma(t) = (x(t), y(t), z(t)) \end{tikzcd}$$

Use the chain rule to compute

$$\frac{dx}{dt}, \qquad \frac{dy}{dt}, \qquad \frac{dz}{dt},$$

in terms of $\dfrac{dx_c}{dt}$, $\dfrac{dy_c}{dt}$, $\dfrac{\partial x}{\partial x_c}$, $\dfrac{\partial y}{\partial x_c}$, $\dfrac{\partial z}{\partial x_c}$, $\dfrac{\partial x}{\partial y_c}$, $\dfrac{\partial y}{\partial y_c}$, and $\dfrac{\partial z}{\partial y_c}$.

Hint: *Simply write down the answer from a previous problem with some minor changes.*

Problem 8 *With the same setting as in the previous problem, rewrite the result of your computation in matrix notation to find D_c such that*

$$\begin{bmatrix} dx/dt \\ dy/dt \\ dz/dt \end{bmatrix} = D_c \cdot \begin{bmatrix} dx_c/dt \\ dy_c/dt \end{bmatrix}$$

in terms of $\dfrac{\partial x}{\partial x_c}, \dfrac{\partial y}{\partial x_c}, \dfrac{\partial z}{\partial x_c}, \dfrac{\partial x}{\partial y_c}, \dfrac{\partial y}{\partial y_c}$, and $\dfrac{\partial z}{\partial y_c}$.

Hint: *Simply write down the answer from a previous problem with some minor changes.*

Problem 9 *Now find P_c in terms of K, $\dfrac{\partial x}{\partial x_c}$, $\dfrac{\partial y}{\partial x_c}$, $\dfrac{\partial z}{\partial x_c}$, $\dfrac{\partial x}{\partial y_c}$, $\dfrac{\partial y}{\partial y_c}$, and $\dfrac{\partial z}{\partial y_c}$ such that*

$$\left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right) \bullet_K \left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right) = \begin{bmatrix} \dfrac{dx_c}{dt} & \dfrac{dy_c}{dt} \end{bmatrix} \cdot P_c \cdot \begin{bmatrix} \dfrac{dx_c}{dt} \\ \dfrac{dy_c}{dt} \end{bmatrix}.$$

Hint: *Simply write down the answer from a previous problem with some minor changes.*

Before we actually compute P_c, it will help to compute the partial derivatives.

Problem 10 *Set*

$$x(x_c, y_c) = \lambda \cdot x_c,$$
$$y(x_c, y_c) = \lambda \cdot y_c,$$
$$z(x_c, y_c) = \lambda,$$

and show that

$$\frac{\partial x}{\partial x_c} = \lambda - K x_c^2 \lambda^3, \qquad \frac{\partial x}{\partial y_c} = -K x_c y_c \lambda^3,$$
$$\frac{\partial y}{\partial x_c} = -K x_c y_c \lambda^3, \qquad \frac{\partial y}{\partial y_c} = \lambda - K y_c^2 \lambda^3,$$
$$\frac{\partial z}{\partial x_c} = -K x_c \lambda^3, \qquad \frac{\partial z}{\partial y_c} = -K y_c \lambda^3.$$

Hint: *Work in the following way:*

(a) *Remember that λ is itself a function of x_c and y_c, so its partial derivatives also matter. Use the product rule:*

$$\frac{\partial x}{\partial x_c} = \lambda + x_c \cdot \frac{\partial \lambda}{\partial x_c}.$$

(b) *Express the partial derivative in terms of λ, K, x_c, and y_c.*

Problem 11 *With the same setting as above, show that P_c is*

$$P_c = \begin{bmatrix} \left(Ky_c^2 + 1\right)\lambda^4 & -Kx_cy_c\lambda^4 \\ -Kx_cy_c\lambda^4 & \left(Kx_c^2 + 1\right)\lambda^4 \end{bmatrix}.$$

Hint: *When simplifying, combine the terms with the highest degree of λ and note that*

$$\lambda^{-2} = K\left(x_c^2 + y_c^2\right) + 1.$$

Definition 1. *Let $\mathbf{v}_c$ and $\mathbf{w}_c$ be vectors in (x_c, y_c)-coordinates originating at the same (x_c, y_c)-coordinate. Define*

$$\mathbf{v}_c \bullet_c \mathbf{w}_c = \mathbf{v}_c^{\intercal} P_c \mathbf{w}_c$$

where

$$P_c = \begin{bmatrix} \left(K y_c^2 + 1\right) \lambda^4 & -K x_c y_c \lambda^4 \\ -K x_c y_c \lambda^4 & \left(K x_c^2 + 1\right) \lambda^4 \end{bmatrix}$$

and λ, x_c, and y_c are determined by the coordinate that the vectors originate from.

When the finite is infinite and the infinite is finite

Central projection allows us to think about both "finite" areas and "infinite" areas in new ways.

When K is positive, the finite is infinite When $K > 0$, we are working with a sphere. As we know, a sphere has finite surface area.

Problem 12 *As a point approaches the equator of the R-sphere from the North Pole, where does it move to under central projection?*

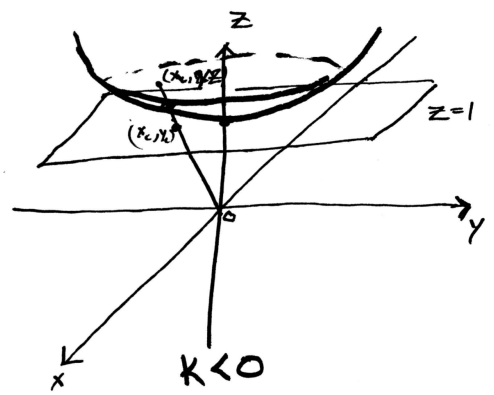
Hint: *A picture is worth a thousand words!*

Problem 13 *Explain why one might say that central projection makes the "finite" seem "infinite."*

When K is negative, the infinite is finite Notice that, if $K < 0$, the equation for the K-plane becomes

$$z^2 - |K|(x^2 + y^2) = 1.$$

This describes a 2-sheeted hyperboloid with the z-axis as major axis. We will only consider the sheet on which z is positive as forming the K-geometry. In plain English, this means we are doing geometry on this hyperboloid:



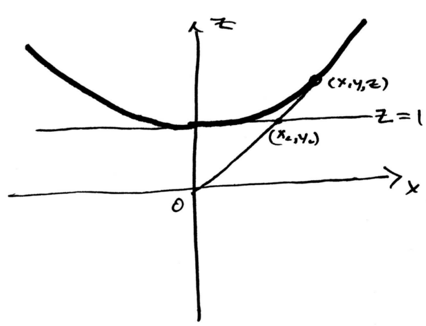
Of critical importance, note

this hyperboloid has infinite surface area.

The hyperboloid above is obtained by rotating the hyperbola

$$z^2 - |K|\,x^2 = 1$$

in the (x, z)-plane around the z-axis.



Problem 14 *Consider points of the form $(x, 0, z)$ in K-geometry. As x and z shoot off to infinity, what happens to the corresponding central projection coordinate x_c? Relate this to the asymptote of the hyperbola as $x \to \infty$.*

Hint: *First rewrite $z^2 - |K|x^2 = 1$ as*

$$z = \sqrt{1 + |K|x^2}$$

and explain why this is acceptable.

Hint: *Explain why computing this limit*

$$\lim_{x \to \infty} \frac{\sqrt{1 + |K|x^2}}{x}$$

helps us in this context.

Problem 15 *When $K < 0$, what is the set of points (x_c, y_c) in central projection which correspond to points on the hyperboloid?*

Hint: *Use the answer to the previous question and the fact that both the hyperboloid and the process of central projection are symmetric with respect to rotation around the z-axis.*

Problem 16 *Explain why one might say that central projection makes the "infinite" seem "finite."*

The (x_c, y_c)-coordinates are called **Klein coordinates** for hyperbolic geometry and the disk of radius $1/\sqrt{|K|}$ called the **Klein model** for hyperbolic geometry in honor of the famous German geometer, Felix Klein.

Problem 17 *Summarize the results from this section. In particular, indicate which results follow from the others.*

Rigid motions in central projection

November 2, 2020

Contents

Rigid motions in central projection coordinates

Suppose now we have a K-rigid motion

$$\begin{bmatrix} \underline{x} \\ \underline{y} \\ \underline{z} \end{bmatrix} = M \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

of K-geometry, given by a K-orthogonal matrix

$$M = \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{bmatrix}.$$

Let's convert this K-rigid motion to a rigid motion in central projection coordinates. This new rigid motion will not necessarily be a mapping defined by a matrix, so we'll have to use some new notation.

$$\begin{tikzcd}
(x,y,z) \in \mathbb{R}^3 & (x_c, y_c, 1) \in \mathbb{R}^2 \times \{1\} \arrow[l, mapsto, "\cdot\lambda"'] \arrow[d, mapsto, "\mu_{c'} =?"] & (x_c, y_c) \in \mathbb{R}^2 \arrow[l, mapsto] \arrow[d, mapsto, "\mu_c =?"] \\
(\underline{x}, \underline{y}, \underline{z}) \arrow[from=u, mapsto, "M\cdot"] & (\underline{x_c}, \underline{y_c}, 1) \arrow[l, mapsto, "\cdot\underline{\lambda}"'] & (\underline{x_c}, \underline{y_c}) \arrow[l, mapsto]
\end{tikzcd}$$

Problem 18 *Using the diagram above, explain why the formula for* $(\underline{x_c}, \underline{y_c}) = \mu_c(x_c, y_c)$ *is*

$$\mu_c(x_c, y_c) = \left(\frac{m_{11}x_c + m_{12}y_c + m_{13}}{m_{31}x_c + m_{32}y_c + m_{33}}, \frac{m_{21}x_c + m_{22}y_c + m_{23}}{m_{31}x_c + m_{32}y_c + m_{33}} \right).$$

From K-rigid motions to rigid motions in central projection

Now let's use our new tool to convert K-rigid motions to rigid motions in central projection.

Problem 19 *For any K, consider the K-rigid motion of K-geometry*

$$M_\theta = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Can you describe geometrically what this mapping is doing to the points in central projection?

Problem 20 *For any K, consider the K-rigid motion of K-geometry*

$$M_\theta = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Convert this to a rigid motion in central projection.

Problem 21 *Assuming $K > 0$, consider the K-rigid motion of the R-sphere*

$$N_{\psi} = \begin{bmatrix} \cos\psi & 0 & -R \cdot \sin\psi \\ 0 & 1 & 0 \\ R^{-1} \cdot \sin\psi & 0 & \cos\psi \end{bmatrix}.$$

Can you describe geometrically what this mapping is doing to the points in central projection?

Problem 22 *Assuming $K > 0$, consider the K-rigid motion of the R-sphere*

$$N_\psi = \begin{bmatrix} \cos\psi & 0 & -R\cdot\sin\psi \\ 0 & 1 & 0 \\ R^{-1}\cdot\sin\psi & 0 & \cos\psi \end{bmatrix}.$$

Convert this to a rigid motion in central projection.

Problem 23 *Assuming $K < 0$, consider the K-rigid motion of the K-surface*

$$N_\psi = \begin{bmatrix} \cosh\psi & 0 & |K|^{-1/2} \cdot \sinh\psi \\ 0 & 1 & 0 \\ |K|^{1/2} \cdot \sinh\psi & 0 & \cosh\psi \end{bmatrix}$$

Can you describe geometrically what this mapping is doing to the points in central projection?

Problem 24 *Assuming $K < 0$, consider the K-rigid motion of the K-surface*

$$N_\psi = \begin{bmatrix} \cosh\psi & 0 & |K|^{-1/2} \cdot \sinh\psi \\ 0 & 1 & 0 \\ |K|^{1/2} \cdot \sinh\psi & 0 & \cosh\psi \end{bmatrix}$$

Convert this to a rigid motion in central projection.

Problem 25 *Summarize the results from this section. In particular, indicate which results follow from the others.*

Lines, angles, and areas in central projection

November 2, 2020

Contents

Central projection preserves lines

We all probably realize that you can't make a perfect flat map of our spherical world.

What this means is that either great circular arcs on the sphere won't correspond to straight lines on the map, or angles on the sphere be equal to the corresponding angles on the map.

We do the next best thing—we make two maps of the sphere! One map has the property that straight lines on the map correspond to shortest paths, and the other map has the property that angles are faithfully represented. We will now show that in central projection, straight lines correspond to shortest paths, but angles are not faithfully represented.

Problem 26 *Show that "lines" in K-geometry correspond to actual lines in (x_c, y_c)-coordinates under central projection.*

Hint: *Referring to previous handouts, explain why intersecting the K-surface*

$$1 = K(x^2 + y^2) + z^2$$

with the plane

$$ax + by + cz = 0$$

produces a K geometry line.

Hint: *Argue that the central projection to the plane $z = 1$ of a point on the K-surface which lies in the plane*

$$ax + by + cz = 0$$

also lies in that plane. What is the intersection of two planes?

Problem 27 *Explain why the answer from the previous question makes perfect sense if $K = 0$.*

Problem 28 *If $K < 0$ explain why Euclid's fifth axiom:*

> *Through a point not on a line there passes a unique parallel line.*

fails.

Problem 29 *If $K > 0$ reconcile the fact*

> *in spherical geometry, there are no parallel lines,*

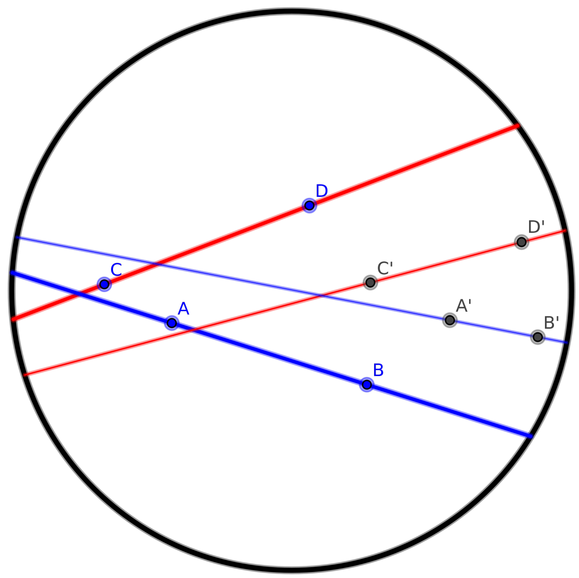
with the fact that

> *The lines $y_c = 1$ and $y_c = -1$ seem to be parallel.*

Angles in central projection coordinates

While central projection coordinates "preserve" lines, meaning that shortest paths in euclidean geometry, spherical geometry, and hyperbolic geometry are all mapped to lines, it does not preserve angles. This means that the angle that we see in central projection coordinates may or may not be the same as the angle in euclidean geometry. To see this, we will give an example where the appearance of the angle is changed by a rigid motion.

Problem 30 *Consider the following lines in the Klein disk,*

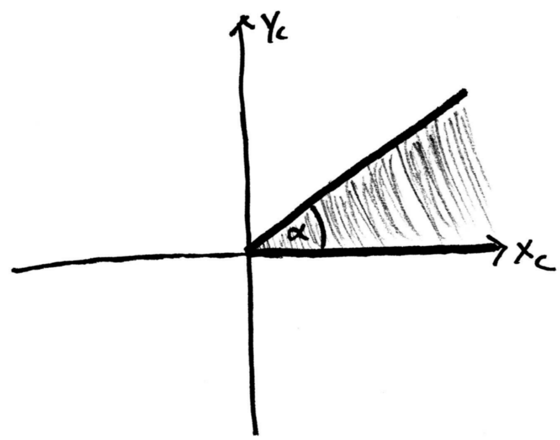


where $\overline{A'B'}$ and $\overline{C'D'}$ are the images of $\overline{AB}$ and $\overline{CD}$ under a K-rigid motion. Explain how this image shows that central projection does not preserve angles.

Areas in central projection coordinates

We have already seen that on the R-sphere, the area of an α-lune is $2\alpha \cdot R^2$. Let's compute this a different way.

Problem 31 *Let $K > 0$ and consider the following region in central projection*



Use the fact that the area of an α-lune is $2\alpha \cdot R^2$ to compute the area of the region.

On the other hand, we know that the area in spherical geometry represented by the region in central projection is given by

$$\iint_{L_c} \sqrt{\det \begin{bmatrix} \dfrac{\partial X}{\partial x_c} \bullet_K \dfrac{\partial X}{\partial x_c} & \dfrac{\partial X}{\partial y_c} \bullet_K \dfrac{\partial X}{\partial x_c} \\ \dfrac{\partial X}{\partial x_c} \bullet_K \dfrac{\partial X}{\partial y_c} & \dfrac{\partial X}{\partial y_c} \bullet_K \dfrac{\partial X}{\partial y_c} \end{bmatrix}} \, dx_c \, dy_c$$

Problem 32 *Give a heuristic explanation of why this integral computes what we say it computes.*

Hint: *Think about a tiny square in (x_c, y_c)-coordinates.*

You have already shown that

$$\iint_{L_c} \sqrt{\det \begin{bmatrix} \frac{\partial X}{\partial x_c} \bullet_K \frac{\partial X}{\partial x_c} & \frac{\partial X}{\partial y_c} \bullet_K \frac{\partial X}{\partial x_c} \\ \frac{\partial X}{\partial x_c} \bullet_K \frac{\partial X}{\partial y_c} & \frac{\partial X}{\partial y_c} \bullet_K \frac{\partial X}{\partial y_c} \end{bmatrix}}\, dx_c\, dy_c$$

$$= \iint_{L_c} \sqrt{\det \left(\begin{bmatrix} \leftarrow & \frac{\partial X}{\partial x_c} & \rightarrow \\ \leftarrow & \frac{\partial X}{\partial y_c} & \rightarrow \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & K^{-1} \end{bmatrix} \begin{bmatrix} \uparrow & \uparrow \\ \frac{\partial X}{\partial x_c} & \frac{\partial X}{\partial y_c} \\ \downarrow & \downarrow \end{bmatrix} \right)}\, dx_c\, dy_c$$

Problem 33 *Explain why*

$$\begin{bmatrix} \leftarrow & \frac{\partial X}{\partial x_c} & \rightarrow \\ \leftarrow & \frac{\partial X}{\partial y_c} & \rightarrow \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & K^{-1} \end{bmatrix} \begin{bmatrix} \uparrow & \uparrow \\ \frac{\partial X}{\partial x_c} & \frac{\partial X}{\partial y_c} \\ \downarrow & \downarrow \end{bmatrix} = P_c.$$

Hint: *No new computations need to be done, just look at how P_c was derived.*

Hence now we see

$$\iint_{L_c} \sqrt{\det \begin{bmatrix} \dfrac{\partial X}{\partial x_c} \bullet_K \dfrac{\partial X}{\partial x_c} & \dfrac{\partial X}{\partial y_c} \bullet_K \dfrac{\partial X}{\partial x_c} \\ \dfrac{\partial X}{\partial x_c} \bullet_K \dfrac{\partial X}{\partial y_c} & \dfrac{\partial X}{\partial y_c} \bullet_K \dfrac{\partial X}{\partial y_c} \end{bmatrix}}\, dx_c\, dy_c = \iint_{L_c} \sqrt{\det P_c}\, dx_c\, dy_c$$

Problem 34 *Compute* $\sqrt{\det P_c}$.

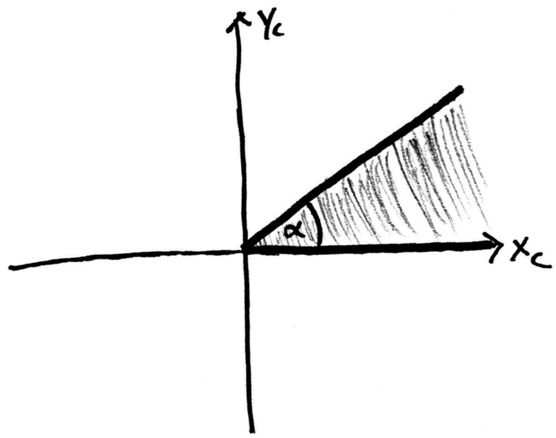
Hint: *As a gesture of friendship, we'll remind you that*

$$P_c = \begin{bmatrix} \left(K y_c^2 + 1\right) \lambda^4 & -K x_c y_c \lambda^4 \\ -K x_c y_c \lambda^4 & \left(K x_c^2 + 1\right) \lambda^4 \end{bmatrix}.$$

At this point we see we are interested in the following integral

$$\iint_{L_c} (K(x_c^2 + y_c^2) + 1)^{-3/2}\, dx_c\, dy_c.$$

Problem 35 *Examining the following diagram*



convert

$$\iint_{L_c} (K(x_c^2 + y_c^2) + 1)^{-3/2}\, dx_c\, dy_c.$$

to polar coordinates and compute the integral.

Hint: *Recall that to convert to polar coordinates, set*

$$r = \sqrt{x_c^2 + y_c^2},$$
$$\theta = \arctan(y_c/x_c),$$

and replace $dx_c\, dy_c$ with $r\, dr\, d\theta$.

Hopefully you got the same area that you predicted in the previous problem!

Central projection unifies euclidean, spherical, and hyperbolic geometry

Under central projection, our new dot product defined by

$$\mathbf{v}_c \bullet_c \mathbf{w}_c = \mathbf{v}_c^{\mathsf{T}} \cdot P_c \cdot \mathbf{w}_c$$

make sense when K is zero, positive, and negative. Hence this dot product makes sense for euclidean, spherical, and hyperbolic geometry. Moreover, in central projection, shortest paths on the K-surface

$$K(x^2 + y^2) + z^2 = 1$$

map to lines in the plane $z = 1$. However, there is one trade-off: Angles are not preserved in central projection. This means that when angles are projected into the plane via central projection, the angle we see in the (x_c, y_c)-plane may or may not be the actual angle between two vectors. Summarizing, we have:

	Spherical $(K > 0)$	Euclidean $(K = 0)$	Hyperbolic $(K < 0)$
Surface in euclidean space	$\widehat{x}^2 + \widehat{y}^2 + \widehat{z}^2 = R^2$	DNE	DNE
Euclidean dot product	$\widehat{\mathbf{v}}^{\mathsf{T}} \cdot \widehat{\mathbf{w}}$	DNE	DNE
Surface in K-warped space	$1 = K(x^2 + y^2) + z^2$	$1 = K(x^2 + y^2) + z^2$	$1 = K(x^2 + y^2) + z^2$
K-dot product	$\mathbf{v}^{\mathsf{T}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & K^{-1} \end{bmatrix} \mathbf{w}$	DNE	$\mathbf{v}^{\mathsf{T}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & K^{-1} \end{bmatrix} \mathbf{w}$
Central dot product	$\mathbf{v}_c^{\mathsf{T}} \cdot P_c \cdot \mathbf{w}_c = \mathbf{v}_c^{\mathsf{T}} \begin{bmatrix} (Ky_c^2+1)\lambda^4 & -Kx_cy_c\lambda^4 \\ -Kx_cy_c\lambda^4 & (Kx_c^2+1)\lambda^4 \end{bmatrix} \mathbf{w}_c$		

Problem 36 *Summarize the results from this section. In particular, indicate which results follow from the others.*