Concurrence theorems in Euclidean geometry

Now we explore when three lines meet at a point.

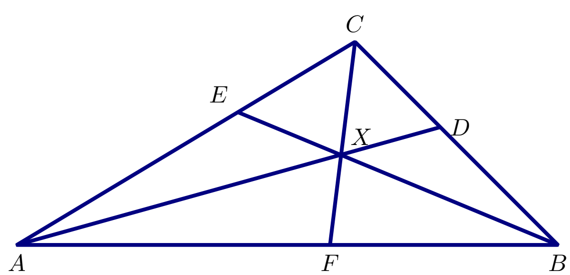
After answering the following questions, students should be able to:

- State and prove facts about the ratio of sides and areas of triangles
- State and prove Ceva's theorem

Let's look at a some *concurrence* theorems. Concurrence theorems deal with situations when three or more lines (or curves) pass through the same point.

Problem 1 *Denote the measure or area of a triangle* $\triangle ABC$ *as* $|\triangle ABC|$. *Show that, in the diagram below,*

$$\frac{|AF|}{|FB|} = \frac{|\triangle AFC|}{|\triangle CFB|} = \frac{|\triangle AFX|}{|\triangle XFB|}.$$



Hint: *Mark the height of the relevant triangles.*

Hint: *Video for the first equality is at* `https://youtu.be/nALZ_REZV74` *with notes at* `https://osu.instructure.com/courses/84670/files/23758741`.

Learning outcomes:
Author(s):

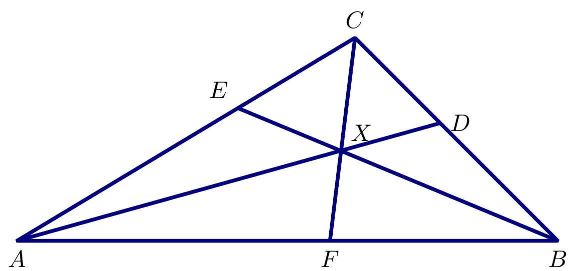
Problem 2 *Use the previous problem to show using only algebra that*

$$\frac{|AF|}{|FB|} = \frac{|\triangle AXC|}{|\triangle CXB|}.$$

Now we will present, and you will prove, *Ceva's Theorem.*

Historical note: The following fact was proved in the 1000s by Yusuf al-Mu'taman ibn-Hūd, the Muslim king of Zaragoza in Spain. But we call it Ceva's theorem. Giovanni Ceva (*CHEH-vah*) was an Italian mathematician who proved this theorem in the 1600s. In fact, it

Theorem 1 (Ceva's Theorem)**.** *Three segments* $\overline{AD}$*,* $\overline{BE}$*, and* $\overline{CF}$

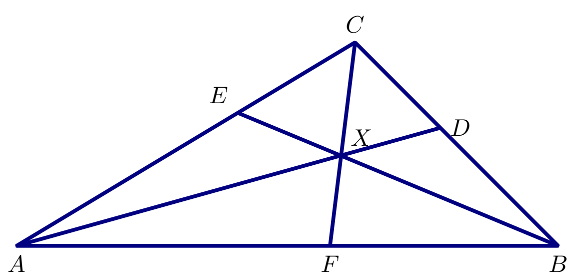


are concurrent if and only if

$$\frac{|AF|}{|FB|} \cdot \frac{|BD|}{|DC|} \cdot \frac{|CE|}{|EA|} = 1.$$

Problem 3 *Prove the Ceva's theorem:*

(a) *For three concurrent segments* $\overline{AD}$, $\overline{BE}$ and $\overline{CF}$



show that

$$\frac{|AF|}{|FB|} \cdot \frac{|BD|}{|DC|} \cdot \frac{|CE|}{|EA|} = 1.$$

Hint: *Use the previous problem repeatedly. Video for finding* $\frac{|BD|}{|CD|}$ *is at* `https://youtu.be/nALZ_REZV74?t=249` *with notes at* `https://osu.instructure.com/courses/84670/files/23758741`*.*

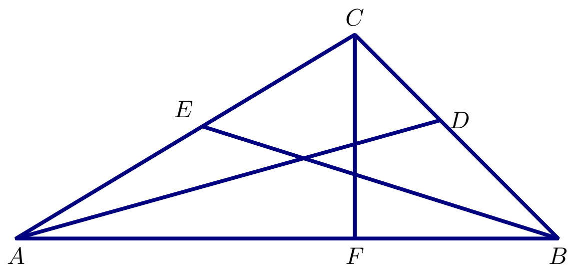
Interactive demonstration is at `https://www.geogebra.org/m/s7m8xVDu`*.*

(b) *Prove the reverse direction of Ceva's Theorem: If*

$$\frac{|AF|}{|FB|} \cdot \frac{|BD|}{|DC|} \cdot \frac{|CE|}{|EA|} = 1$$

then the lines AD, BE, and CF pass through a common point.

Hint: *Suppose that they do not pass through a common point.*



Hint: *Notice that if, for example, F moves along the segment $\overline{AB}$ from A to B, then $\dfrac{|AF|}{|FB|}$ is a strictly increasing function of $|AF|$. Now use a previous problem to determine a position F' for F along the segment $\overline{AB}$ at which*

$$\frac{|AF'|}{|F'B|} \cdot \frac{|BD|}{|DC|} \cdot \frac{|CE|}{|EA|} = 1.$$

Hint: *Video showing F moving from A to B is at* `https://youtu.be/nALZ_REZV74?t=496` *with notes at* `https://osu.instructure.com/courses/84670/files/23758741`.

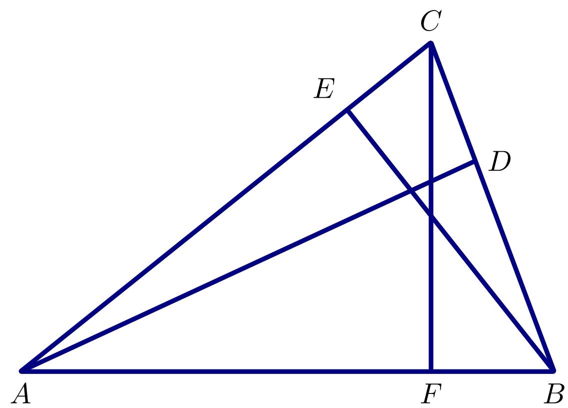
Interactive demonstration is at `https://www.geogebra.org/m/s7m8xVDu`.

Problem 4 *A* ***median*** *of a triangle is a line segment from a vertex to the midpoint of the opposite side. Show that the medians of any triangle meet in a common point.*

Hint: *Use Ceva's Theorem.*

Definition 1. *An* ***altitude*** *of a triangle is a line segment originating at a vertex of the triangle that meets the line containing the opposite side at a right angle.*

Problem 5 *Use Ceva's theorem to show that the three lines containing altitudes of a triangle are concurrent.*



Hint: *Use all three similarities of the form* $\triangle CEB \sim \triangle CDA$ *and then apply Ceva's theorem.*

Problem 6 *Summarize the results from this section. In particular, indicate which results follow from the others.*