Stereographic projection

November 2, 2020

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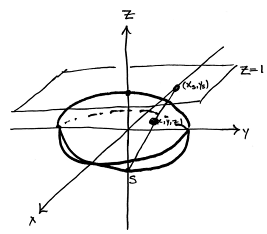
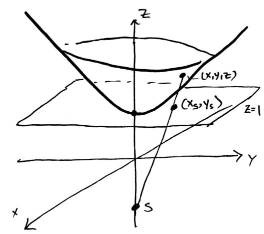
Stereographic projection coordinates

Now we will examine the second way to produce coordinates uniformly for spherical, plane, and hyperbolic geometry that will allow us to pass smoothly through the three geometries. Now let's project K-geometry,

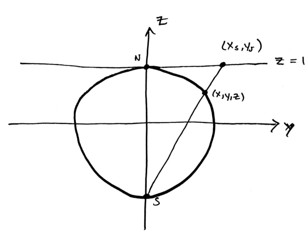
$$1 = K\left(x^2 + y^2\right) + z^2$$

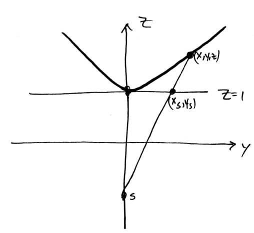
onto the plane $z = 1$ using the 'south pole' $S = (0, 0, -1)$ as the center of projection:





Let's look at this from a different vantage point, say with our eye along the edge of the plane $z = 1$:





Problem 1 *If $K = 1$ where does the 'equator' map to? What about the 'Northern hemisphere?' How about the 'Southern hemisphere?'*

Problem 2 *Use similar triangles to explain why for any given (x, y, z), there is a number ρ such that*

$$\rho \cdot (x_s, y_s, 2) = (x, y, z + 1).$$

Problem 3 *For the projection of the set* $1 = K\left(x^2 + y^2\right) + z^2$ *onto the* $z = 1$ *plane with center of projection* S*, write* (x_s, y_s) *as a function of* (x, y, z)*.*

Problem 4 *For the projection of the set* $1 = K\left(x^2 + y^2\right) + z^2$ *onto the* $z = 1$ *plane with center of projection* S*, write* (x, y, z) *as a function of* (x_s, y_s)*.*

Hint: *Note that*

$$\rho \cdot (x_s, y_s, 2) = (x, y, z + 1) \qquad \text{where} \qquad z = 2\rho - 1.$$

Hence if

$$1 = K\left(x^2 + y^2\right) + z^2$$

we may write

$$1 = K\left((\rho \cdot x_s)^2 + (\rho \cdot y_s)^2\right) + (2\rho - 1)^2$$

and solve for ρ*.*

Stereographic projection dot product

Once more, we want to be able to find a dot product that will allow us to compute lengths in stereographic projection coordinates that will agree with the K-dot product, and hence the euclidean dot product.

Problem 5 *Suppose we have a curve γ in K-warped space which we can decompose as*

$$\begin{tikzcd} t \arrow[r, mapsto, "\gamma_s"] \arrow[rr, bend right, "\gamma"'] & (x_s(t), y_s(t)) \arrow[r, mapsto, "{\begin{bmatrix} x(x_s,y_s) \\ y(x_s,y_s) \\ z(x_s,y_s) \end{bmatrix}}"] & \gamma(t) = (x(t), y(t), z(t)) \end{tikzcd}$$

Use the chain rule to compute

$$\frac{dx}{dt}, \qquad \frac{dy}{dt}, \qquad \frac{dz}{dt},$$

in terms of $\dfrac{dx_s}{dt}$, $\dfrac{dy_s}{dt}$, $\dfrac{\partial x}{\partial x_s}$, $\dfrac{\partial y}{\partial x_s}$, $\dfrac{\partial z}{\partial x_s}$, $\dfrac{\partial x}{\partial y_s}$, $\dfrac{\partial y}{\partial y_s}$, and $\dfrac{\partial z}{\partial y_s}$.

Hint: *Simply write down the answer from a previous problem with some minor changes.*

Problem 6 *With the same setting as in the previous problem, rewrite the result of your computation in matrix notation to find D_s such that*

$$\begin{bmatrix} dx/dt \\ dy/dt \\ dz/dt \end{bmatrix} = D_s \cdot \begin{bmatrix} dx_s/dt \\ dy_s/dt \end{bmatrix}$$

in terms of $\dfrac{\partial x}{\partial x_s}, \dfrac{\partial y}{\partial x_s}, \dfrac{\partial z}{\partial x_s}, \dfrac{\partial x}{\partial y_s}, \dfrac{\partial y}{\partial y_s}$, and $\dfrac{\partial z}{\partial y_s}$.

Hint: *Simply write down the answer from a previous problem with some minor changes.*

Problem 7 *Now find P_s in terms of K, $\dfrac{\partial x}{\partial x_s}$, $\dfrac{\partial y}{\partial x_s}$, $\dfrac{\partial z}{\partial x_s}$, $\dfrac{\partial x}{\partial y_s}$, $\dfrac{\partial y}{\partial y_s}$, and $\dfrac{\partial z}{\partial y_s}$ such that*

$$\left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right) \bullet_K \left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right) = \begin{bmatrix} \frac{dx_s}{dt} & \frac{dy_s}{dt} \end{bmatrix} \cdot P_s \cdot \begin{bmatrix} \frac{dx_s}{dt} \\ \frac{dy_s}{dt} \end{bmatrix}.$$

Hint: *Simply write down the answer from a previous problem with some minor changes.*

Before we actually compute P_s, it will help to compute the partial derivatives.

Problem 8 *Set*

$$\begin{aligned} x(x_s, y_s) &= \rho \cdot x_s, \\ y(x_s, y_s) &= \rho \cdot y_s, \\ z(x_s, y_s) &= 2\rho - 1, \end{aligned}$$

and show that

$$\begin{aligned} \frac{\partial x}{\partial x_s} &= \rho - \left(\frac{K}{2}\right)\rho^2 x_s^2, & \frac{\partial x}{\partial y_s} &= -\left(\frac{K}{2}\right)\rho^2 x_s y_s, \\ \frac{\partial y}{\partial x_s} &= -\left(\frac{K}{2}\right)\rho^2 x_s y_s, & \frac{\partial y}{\partial y_s} &= \rho - \left(\frac{K}{2}\right)\rho^2 y_s^2, \\ \frac{\partial z}{\partial x_s} &= -K\rho^2 x_s, & \frac{\partial z}{\partial y_s} &= -K\rho^2 y_s. \end{aligned}$$

Hint: *Work in the following way:*

(a) *Recall* $x = \rho \cdot x_s$.

(b) *Note that* $\dfrac{\partial x}{\partial x_s} = \rho + x_s \cdot \dfrac{\partial \rho}{\partial x_s}$.

(c) *Express the partial derivative in terms of* ρ, K, x_s, *and* y_s.

Problem 9 *With the same setting as above, show that P_s is*

$$P_s = \begin{bmatrix} \rho^2 & 0 \\ 0 & \rho^2 \end{bmatrix}.$$

Hint: *When simplifying, combine the terms with the highest degree of ρ and note that*

$$\rho^{-1} = \frac{K\left(x_s^2 + y_s^2\right) + 4}{4}.$$

Definition 1. *Let $\mathbf{v}_s$ and $\mathbf{w}_s$ be vectors in (x_s, y_s)-coordinates originating at the same (x_s, y_s)-coordinate. Define*

$$\mathbf{v}_s \bullet_s \mathbf{w}_s = \mathbf{v}_s^\intercal P_s \mathbf{w}_s$$

where

$$P_s = \begin{bmatrix} \rho^2 & 0 \\ 0 & \rho^2 \end{bmatrix}$$

is determined by the coordinate that the vectors originate from.

Again, when the finite is infinite and the infinite is finite

Just like central projection, stereographic projection allows us to think about both "finite" areas and "infinite" areas in new ways.

When K is positive, the finite is infinite When $K > 0$, we are working with a sphere. As we know, a sphere has finite surface area.

Problem 10 *Where does the Northern hemisphere of the R-sphere map to under stereographic projection?*

Hint: *A picture is worth a thousand words!*

Problem 11 *Where does the Southern hemisphere of the R-sphere map to under stereographic projection?*

Hint: *A picture is worth a thousand words!*

Problem 12 *As a point approaches the south pole of the R-sphere where does it move to under stereographic projection?*

Hint: *Simply transform the relevant coordinate.*

Problem 13 *Explain why one might say that stereographic projection makes the "finite" seem "infinite."*

When K is negative, the infinite is finite Recall that, if $K < 0$, the equation of K-geometry becomes

$$z^2 - |K|(x^2 + y^2) = 1.$$

This describes a 2-sheeted hyperboloid with the z-axis as major axis. We will only consider the sheet on which z is positive as forming the K-geometry. Recall:

- This hyperboloid has infinite surface area.
- It is obtained by rotating the hyperbola

  $$z^2 - |K|\,x^2 = 1$$

  in the (x, z)-plane around the z-axis.

Problem 14 *Consider points of the form $(x, 0, z)$ in K-geometry, $K < 0$. As x and z shoot off to infinity, what happens to the corresponding stereographic projection coordinate x_s?*

Hint: *As you did with central projection, use similar triangles and the fact that*

$$z = \sqrt{1 + |K|x^2}$$

and compute a limit.

Problem 15 *When $K < 0$, what is the set of points (x_s, y_s) in stereographic projection which correspond to points on the hyperboloid?*

Hint: *Again, use the previous question and rotational symmetry.*

Problem 16 *Explain why one might say that stereographic projection makes the "infinite" seem "finite."*

The (x_s, y_s)-coordinates are called **Poincaré coordinates** for hyperbolic geometry and the disk of radius $2/\sqrt{|K|}$ called the **Poincaré model** for hyperbolic geometry in honor of the famous French geometer, Henri Poincaré.

Problem 17 *Summarize the results from this section. In particular, indicate which results follow from the others.*

Lines, angles, and areas in stereographic projection

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Stereographic projection sends lines to circles (or lines)

As we know, we cannot make perfect flat maps of 3D surfaces. In stereographic projection, shortest paths are sent to either lines or circles, and (perhaps surprisingly!) circles are sent to circles.

Problem 18 *Show that "lines" in K-geometry correspond to either lines or circles in (x_s, y_s)-coordinates under stereographic projection.*

Hint: (a) *Recall that intersecting the K-surface*

$$1 = K\left(x^2 + y^2\right) + z^2$$

with the plane

$$ax + by + cz = 0$$

produces a K-geometry line.

(b) *Use the projection formulas*

$$x = \frac{4x_s}{K(x_s^2 + y_s^2) + 4},$$
$$y = \frac{4y_s}{K(x_s^2 + y_s^2) + 4},$$
$$z = \frac{4 - K(x_s^2 + y_s^2)}{4 + K(x_s^2 + y_s^2)},$$

and note that the points generated this way automatically satisfy the condition

$$1 = K(x^2 + y^2) + z^2.$$

(c) *If $c = 0$, then you will find the line*

$$ax_s + by_s = 0.$$

(d) *If $c \neq 0$, then you will find the circle*

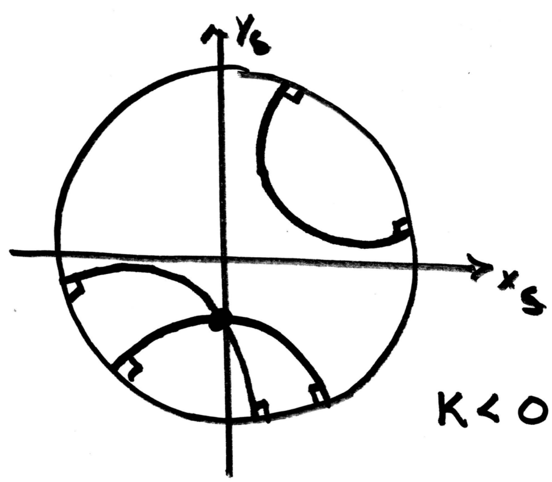
$$\left(x_s - \frac{2a}{cK}\right)^2 + \left(y_s - \frac{2b}{cK}\right)^2 = \frac{4a^2 + 4b^2 + 4c^2K}{(cK)^2}.$$

You will need to complete the square to convert it into this form.

Remark 1. *With entirely similar reasoning, you can show that circles in stereographic projection are also sent to either circles (or lines).*

When we project shortest paths, they map to circles (unless $c = 0$ as above).

When $K < 0$, the circles meet the "circle at infinity" at right angles. **We won't make you prove this, but try to convince yourself that it's true!**



Problem 19 *If $K < 0$ explain why Euclid's fifth axiom:*

Through a point not on a line there passes a unique parallel line.

fails.

Stereographic projection preserves angles

A remarkable fact about stereographic projection is that it "preserves" angles. This means that the angles we see in stereographic projection are true representations of the angle in euclidean coordinates. Now that we are masters of dot products, we will prove this fact with ease!

Problem 20 *Show that*

$$\frac{\mathbf{v}_s \bullet_s \mathbf{w}_s}{|\mathbf{v}_s|_s \cdot |\mathbf{w}_s|_s} = \frac{\mathbf{v}_s \bullet \mathbf{w}_s}{|\mathbf{v}_s| \cdot |\mathbf{w}_s|}.$$

Hint: *On the right-hand side we are using the euclidean dot product and euclidean length formula.*

Problem 21 *Explain how the last problem shows that stereographic projection "preserves" angles.*

Areas in stereographic projection coordinates

Let's use the power of stereographic coordinates to compute the areas of spherical, hyperbolic, and euclidean circles.

We know that the area of a region in stereographic coordinates is given by

$$\iint_{C_s} \sqrt{\det \begin{bmatrix} \frac{\partial X}{\partial x_s} \bullet_K \frac{\partial X}{\partial x_s} & \frac{\partial X}{\partial y_s} \bullet_K \frac{\partial X}{\partial x_s} \\ \frac{\partial X}{\partial x_s} \bullet_K \frac{\partial X}{\partial y_s} & \frac{\partial X}{\partial y_s} \bullet_K \frac{\partial X}{\partial y_s} \end{bmatrix}}\, dx_s\, dy_s$$

Problem 22 *Give a heuristic explanation of why this integral computes what we say it computes.*

You have already shown that

$$\iint_{C_s} \sqrt{\det \begin{bmatrix} \frac{\partial X}{\partial x_s} \bullet_K \frac{\partial X}{\partial x_s} & \frac{\partial X}{\partial y_s} \bullet_K \frac{\partial X}{\partial x_s} \\ \frac{\partial X}{\partial x_s} \bullet_K \frac{\partial X}{\partial y_s} & \frac{\partial X}{\partial y_s} \bullet_K \frac{\partial X}{\partial y_s} \end{bmatrix}}\, dx_s\, dy_s$$

$$= \iint_{C_s} \sqrt{\det \left(\begin{bmatrix} \leftarrow & \frac{\partial X}{\partial x_s} & \rightarrow \\ \leftarrow & \frac{\partial X}{\partial y_s} & \rightarrow \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & K^{-1} \end{bmatrix} \begin{bmatrix} \uparrow & \uparrow \\ \frac{\partial X}{\partial x_s} & \frac{\partial X}{\partial y_s} \\ \downarrow & \downarrow \end{bmatrix} \right)}\, dx_s\, dy_s.$$

Problem 23 *Explain why*

$$\begin{bmatrix} \leftarrow & \frac{\partial X}{\partial x_s} & \rightarrow \\ \leftarrow & \frac{\partial X}{\partial y_s} & \rightarrow \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & K^{-1} \end{bmatrix} \begin{bmatrix} \uparrow & \uparrow \\ \frac{\partial X}{\partial x_s} & \frac{\partial X}{\partial y_s} \\ \downarrow & \downarrow \end{bmatrix} = P_s.$$

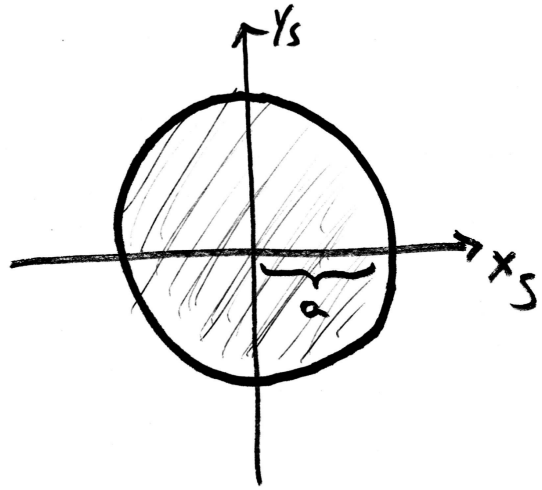
Hint: *No new computations need to be done, just look at how P_s was derived.*

Hence now we see

$$\iint_{C_s} \sqrt{\det \begin{bmatrix} \frac{\partial X}{\partial x_s} \bullet_K \frac{\partial X}{\partial x_s} & \frac{\partial X}{\partial y_s} \bullet_K \frac{\partial X}{\partial x_s} \\ \frac{\partial X}{\partial x_s} \bullet_K \frac{\partial X}{\partial y_s} & \frac{\partial X}{\partial y_s} \bullet_K \frac{\partial X}{\partial y_s} \end{bmatrix}}\, dx_s\, dy_s = \iint_{C_s} \sqrt{\det P_s}\, dx_s\, dy_s.$$

Problem 24 *Compute* $\sqrt{\det P_s}$ *in terms of* K, x_s, *and* y_s.

For the next few problems, consider the following diagram:



Problem 25 *Compute the area of C_s in K-geometry by taking the integral*

$$\iint_{C_s} \sqrt{\det P_s} dx_s dy_s.$$

Hint: *In the previous problem, you should have obtained*

$$\sqrt{\det P_s} = \frac{16}{(K(x_s^2 + y_s^2) + 4)^2}.$$

Hint: *This integral is easiest to compute in polar coordinates. Recall that to convert to polar coordinates, you must set*

$$r = \sqrt{x_s^2 + y_s^2},$$
$$\theta = \arctan(y_s/x_s),$$

and replace $dx_s\, dy_s$ with $r\, dr\, d\theta$.

Problem 26 *Explain why the radius of C_s in K-geometry is given by*

$$r = \int_0^a \sqrt{\left(\frac{dx_s}{dt}, \frac{dy_s}{dt}\right) \bullet_s \left(\frac{dx_s}{dt}, \frac{dy_s}{dt}\right)} dt$$

where

$$(x_s(t), y_s(t)) = (t, 0).$$

Problem 27 *When $K > 0$, find a in terms of the radius r from the previous problem.*

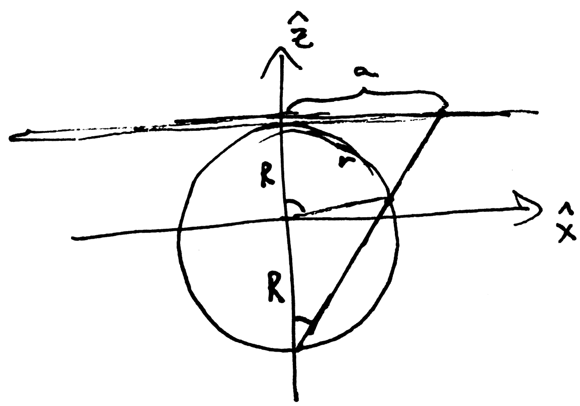
Hint: *There are two ways of going about this:*

- *Compute the integral from the previous problem, using the fact that*

$$\frac{d}{dx}\arctan(Ax) = \frac{A}{1+(Ax)^2}.$$

  Then solve for a.

- ***Or,*** *consider the following diagram in Euclidean geometry and use facts about circles and triangles:*



Problem 28 *When $K > 0$, plug in and simplify to get a formula for the area of a circle in K-geometry in terms of its radius r.*

Problem 29 *What happens as r approaches $\pi R = \pi K^{-1/2}$?*

When r gets very small, what happens to $\dfrac{\text{area}}{r^2}$?

Explain how these computations square with our understanding of spheres.

Problem 30 *When $K < 0$, find a in terms of the radius r.*

Hint: *There are two ways of going about this:*

- *Compute the integral from a previous problem, using the fact that*

$$\frac{d}{dx}\operatorname{artanh}(Ax) = \frac{A}{1-(Ax)^2}.$$

  Then solve for a.

- ***Or,*** *use the fact that the K-length of the line*

$$\textit{from} \quad (1,0,0) \quad \textit{to} \quad (|K|^{-1/2}\sinh(\sigma), 0, \cosh(\sigma))$$

  is $|K|^{-1/2}\sigma$, and compare that to the length of its stereographic projection onto the plane $z = 1$. Use the identities

$$\cosh(2x) = 2\cosh^2(x) - 1$$
$$\sinh(2x) = 2\sinh(x)\cosh(x).$$

  to simplify.

Hint: *This problem is potentially more involved than the spherical version, but in the end you should get the same answer with an extra "h".*

Problem 31 *When $K < 0$, plug in to get a formula for the area of a circle in K-geometry in terms of its radius r.*

Notice that for big enough x, $\sinh(x) \approx e^x/2$, since the e^{-x} term is pretty much zero. So the area of a circle in hyperbolic geometry grows exponentially with radius, unlike in Euclidean geometry where it only grows quadratically. There's a **lot** of space within radius r of a point in the hyperbolic plane.

Problem 32 *When r gets very small, what happens to $\frac{\text{area}}{r^2}$? Explain why this makes sense.*

Stereographic projection unifies euclidean, spherical, and hyperbolic geometry

Under stereographic projection, our new dot product defined by

$$\mathbf{v}_s \bullet_s \mathbf{w}_s = \mathbf{v}_s^{\intercal} \cdot P_s \cdot \mathbf{w}_s$$

make sense when K is zero, positive, and negative. Hence this dot product makes sense for euclidean, spherical, and hyperbolic geometry. However, in stereographic projection, shortest paths on the K-surface

$$K(x^2 + y^2) + z^2 = 1$$

map to circles (or lines) in the plane $z = 1$. The advantage of stereographic projection over central projection is that angles are preserved in stereographic projection. This means that when angles are projected into the plane via stereographic projection, the angle we see in the (x_s, y_s)-plane is the actual angle between two vectors. Summarizing, we have:

	Spherical $(K > 0)$	Euclidean $(K = 0)$	Hyperbolic $(K < 0)$
Surface in euclidean space	$\widehat{x}^2 + \widehat{y}^2 + \widehat{z}^2 = R^2$	DNE	DNE
Euclidean dot product	$\widehat{\mathbf{v}}^{\intercal} \cdot \widehat{\mathbf{w}}$	DNE	DNE
Surface in K-warped space	$1 = K(x^2 + y^2) + z^2$	$1 = K(x^2 + y^2) + z^2$	$1 = K(x^2 + y^2) + z^2$
K-dot product	$\mathbf{v}^{\intercal} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & K^{-1} \end{bmatrix} \mathbf{w}$	DNE	$\mathbf{v}^{\intercal} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & K^{-1} \end{bmatrix} \mathbf{w}$
Central dot product	$\mathbf{v}_c^{\intercal} \cdot P_c \cdot \mathbf{w}_c = \mathbf{v}_c^{\intercal} \begin{bmatrix} (Ky_c^2+1)\lambda^4 & -Kx_cy_c\lambda^4 \\ -Kx_cy_c\lambda^4 & (Kx_c^2+1)\lambda^4 \end{bmatrix} \mathbf{w}_c$		
Stereographic dot product	$\mathbf{v}_s^{\intercal} \cdot P_s \cdot \mathbf{w}_s = \mathbf{v}_s^{\intercal} \begin{bmatrix} \rho^2 & 0 \\ 0 & \rho^2 \end{bmatrix} \mathbf{w}_s$		

Problem 33 *Summarize the results from this section. In particular, indicate which results follow from the others.*