Changing coordinates

Now we will change coordinates.

After answering the following questions, students should be able to:

- Define K-warped space
- Change from Euclidean coordinates to K-warped space coordinates
- Prove facts about the angle between vectors and the area of a parallelogram in K-warped space

Bringing the North Pole of the R-sphere to $(0,0,1)$

We are now ready to change coordinates on Euclidean 3-space so that we can fill up that space with plane geometry and all the spherical and hyperbolic geometries. We have reserved the notation (x,y,z) for these new coordinates that we will put on the 'same' objects we have been studying in Euclidean $(\widehat{x},\widehat{y},\widehat{z})$-coordinates. These new coordinates will be chosen to keep the north and south poles from going to infinity as the radius R of a sphere increases without bound. In these now (x,y,z)-coordinates the sphere of radius R will be given by the equation

$$K(x^2+y^2)+z^2=1$$

where $K=1/R^2$. Notice that the above equation has solutions even when K is negative. It is on those solution sets that hyperbolic geometries will live. So this change of viewpoint will eventually let us go hyperbolic or, in the language of Buzz Lightyear, will let R go 'to infinity and beyond.' The idea will be like the change from rectangular to polar coordinates for the plane that you encountered in calculus, only easier.

We are now ready to introduce this slightly different set of coordinates for $\mathbb{R}^3$, three-dimensional Euclidean space. To understand a bit better why we are doing this, suppose we are standing at the North Pole

$$N=(0,0,R)$$

of the sphere

$$\widehat{x}^2+\widehat{y}^2+\widehat{z}^2=R^2$$

Learning outcomes:
Author(s):

of radius R. As R increases, but we stay our same size, the sphere around us becomes more and more like a flat, plane surface. However it can never get completely flat because we are zooming out the positive $\widehat{z}$-axis and we would have to be 'at infinity' for our surface to become exactly flat. We remedy that unfortunate situation by considering another copy of $\mathbb{R}^3$, that we will call ***K*-warped space**, whose coordinates we denote as (x, y, z). We make the following rule in order to pass between the two $\mathbb{R}^3$'s:

$$\begin{aligned}\widehat{x} &= x\\ \widehat{y} &= y\\ \widehat{z} &= Rz.\end{aligned}$$

We think of the (x, y, z)-coordinates as simply being a different set of addresses for the points in Euclidean space. For example,

$$(x, y, z) = (0, 0, 1)$$

tells me that the point in Euclidean space that I'm referring to is

$$(\widehat{x}, \widehat{y}, \widehat{z}) = (0, 0, R) = N.$$

Continuing with this "change of addresses" the sphere of radius R in Euclidean space is given by

$$\begin{aligned}R^2 &= \widehat{x}^2 + \widehat{y}^2 + \widehat{z}^2\\ &= x^2 + y^2 + R^2 z^2\end{aligned}$$

that is, by the equation

$$1 = \frac{1}{R^2}\left(x^2 + y^2\right) + z^2.$$

Definition 1. *For the surface defined by*

$$1 = \frac{1}{R^2}\left(x^2 + y^2\right) + z^2.$$

The quantity $K = \dfrac{1}{R^2}$ *is called the* ***curvature*** *of the R-sphere.*

Problem 1 *What happens to the surface when K goes to 0? How does this relate to the colloquial sense of "curvature"?*

Problem 2

(a) *Sketch the solution set in* (x, y, z)*-coordinates representing the sphere*

$$R^2 = \widehat{x}^2 + \widehat{y}^2 + \widehat{z}^2 = 1.$$

(b) *Sketch the solution set in* (x, y, z)*-coordinates representing the sphere*

$$R^2 = \widehat{x}^2 + \widehat{y}^2 + \widehat{z}^2 = 10^2.$$

(c) *Sketch the solution set in* (x, y, z)*-coordinates representing the sphere*

$$R^2 = \widehat{x}^2 + \widehat{y}^2 + \widehat{z}^2 = 10^{-2}.$$

Formulas for Euclidean lengths, angles, and areas in terms of (x, y, z)-coordinates

To prepare ourselves to do hyperbolic geometry, which (in some sense) has no satisfactory model in Euclidean space, we will 'practice' by doing spherical geometry (which *does* have a completely satisfactory model in Euclidean space) using these 'slightly strange' (x, y, z)-coordinates. Gradually throughout this course we will discover that the same rules that govern spherical geometry, expressed in (x, y, z)-coordinates, also govern flat and hyperbolic geometry! In all three cases, the surface in (x, y, z)-coordinates that we will study is

$$1 = K\left(x^2 + y^2\right) + z^2.$$

If $K > 0$, the geometry we will be studying is the geometry of the the Euclidean sphere of radius

$$R = \frac{1}{\sqrt{K}}.$$

If $K = 0$ we will be studying flat (plane) geometry. If $K < 0$, we will be studying hyperbolic geometry.

In short, we want to use (x, y, z)-coordinates to compute with, but we want lengths and angles to be the usual Euclidean ones in $(\widehat{x}, \widehat{y}, \widehat{z})$-coordinates.

Problem 3 *Suppose we have a curve $\widehat{\gamma}$ in Euclidean space and we think of it as a composition of a curve γ in K-warped space with a transformation. In other words, we're looking at a diagram*

$$\begin{tikzcd} t \arrow[r, mapsto, "\gamma"] \arrow[rr, mapsto, bend right, "\widehat{\gamma}"'] & (x(t), y(t), z(t)) \arrow[r, mapsto, "{\begin{bmatrix} \widehat{x}(x,y,z) \\ \widehat{y}(x,y,z) \\ \widehat{z}(x,y,z) \end{bmatrix}}"] & \widehat{\gamma}(t) = (\widehat{x}(t), \widehat{y}(t), \widehat{z}(t)) \end{tikzcd}$$

$$\begin{tikzcd} \mathbb{R} \arrow[r, "\gamma"] \arrow[rr, bend right, "\widehat{\gamma}"'] & K - warped\ space \arrow[r, "{\begin{bmatrix} \widehat{x}(x,y,z) \\ \widehat{y}(x,y,z) \\ \widehat{z}(x,y,z) \end{bmatrix}}"] & Euclidean\ geometry \end{tikzcd}$$

Use the chain rule to compute

$$\frac{d\widehat{x}}{dt}, \qquad \frac{d\widehat{y}}{dt}, \qquad \frac{d\widehat{z}}{dt},$$

in terms of $\dfrac{dx}{dt}, \dfrac{dy}{dt}, \dfrac{dz}{dt}, \dfrac{\partial\widehat{x}}{\partial x}, \dfrac{\partial\widehat{y}}{\partial x}, \dfrac{\partial\widehat{z}}{\partial x}, \dfrac{\partial\widehat{x}}{\partial y}, \dfrac{\partial\widehat{y}}{\partial y}, \dfrac{\partial\widehat{z}}{\partial y}, \dfrac{\partial\widehat{x}}{\partial z}, \dfrac{\partial\widehat{y}}{\partial z}$, and $\dfrac{\partial\widehat{z}}{\partial z}$.

Hint: *Recall that if F is a differentiable function of x, y, and z; and if x, y, and z are all differentiable functions of t, then the chain rule states*

$$\frac{dF}{dt} = \nabla F \cdot \begin{bmatrix} \frac{dx}{dt} & \frac{dy}{dt} & \frac{dz}{dt} \end{bmatrix}^{\intercal}.$$

Problem 4 *With the same setting as in the previous problem, rewrite the result of your computation in matrix notation to find D_K such that*

$$\begin{bmatrix} d\widehat{x}/dt \\ d\widehat{y}/dt \\ d\widehat{z}/dt \end{bmatrix} = D_K \cdot \begin{bmatrix} dx/dt \\ dy/dt \\ dz/dt \end{bmatrix}$$

in terms of $\frac{\partial \widehat{x}}{\partial x}, \frac{\partial \widehat{y}}{\partial x}, \frac{\partial \widehat{z}}{\partial x}, \frac{\partial \widehat{x}}{\partial y}, \frac{\partial \widehat{y}}{\partial y}, \frac{\partial \widehat{z}}{\partial y}, \frac{\partial \widehat{x}}{\partial z}, \frac{\partial \widehat{y}}{\partial z}$, and $\frac{\partial \widehat{z}}{\partial z}$.

Problem 5 *Use the previous problems and the relationship between Euclidean and (x, y, z)-coordinates to show that*

$$\left[\frac{d\widehat{\gamma}}{dt}\right] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & R \end{bmatrix} \cdot \left[\frac{d\gamma}{dt}\right].$$

This last computation shows that if

$$\widehat{\mathbf{v}}_1 = \begin{pmatrix} \widehat{a}_1 \\ \widehat{b}_1 \\ \widehat{c}_1 \end{pmatrix} \qquad \text{and} \qquad \widehat{\mathbf{v}}_2 = \begin{pmatrix} \widehat{a}_2 \\ \widehat{b}_2 \\ \widehat{c}_2 \end{pmatrix}$$

are vectors tangent to a curve in $(\widehat{x}, \widehat{y}, \widehat{z})$-coordinates and

$$\mathbf{v}_1 = \begin{pmatrix} a_1 \\ b_1 \\ c_1 \end{pmatrix} \qquad \text{and} \qquad \mathbf{v}_2 = \begin{pmatrix} a_2 \\ b_2 \\ c_2 \end{pmatrix}$$

are their transformations into (x, y, z)-coordinates, then

$$\widehat{\mathbf{v}}_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & R \end{bmatrix} \mathbf{v}_1 \qquad \text{and} \qquad \widehat{\mathbf{v}}_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & R \end{bmatrix} \mathbf{v}_2$$

and

$$\begin{aligned}
\widehat{\mathbf{v}}_1 \bullet \widehat{\mathbf{v}}_2 = \widehat{\mathbf{v}}_1^{\mathsf{T}} \cdot \widehat{\mathbf{v}}_2 &= \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & R \end{bmatrix} \mathbf{v}_1 \right)^{\mathsf{T}} \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & R \end{bmatrix} \mathbf{v}_2 \right) \\
&= \mathbf{v}_1^{\mathsf{T}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & R \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & R \end{bmatrix} \mathbf{v}_2 \\
&= \mathbf{v}_1^{\mathsf{T}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & K^{-1} \end{bmatrix} \mathbf{v}_2.
\end{aligned}$$

Hence:

We can compute the Euclidean dot product without ever referring to Euclidean coordinates!

We incorporate that fact into the following definition.

Definition 2. *The* ***K-dot-product*** *of vectors:*

$$\begin{aligned}
\mathbf{v}_1 \bullet_K \mathbf{v}_2 &= \mathbf{v}_1^{\mathsf{T}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & K^{-1} \end{bmatrix} \mathbf{v}_2 \\
&= \begin{bmatrix} a_1 & b_1 & c_1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & K^{-1} \end{bmatrix} \begin{bmatrix} a_2 \\ b_2 \\ c_2 \end{bmatrix}.
\end{aligned}$$

Computing length

Problem 6 *Show that if a vector is given to us in (x, y, z)-coordinates as*

$$\mathbf{v} = (a, b, c)^{\mathsf{T}},$$

then the length of its image in Euclidean space is given by

$$|\mathbf{v}|_K = \sqrt{\mathbf{v} \bullet_K \mathbf{v}}.$$

Problem 7 *Consider all vectors in K-warped space with their tips on the surface*

$$1 = K(x^2 + y^2) + z^2$$

and their tails at the origin. What can you say about the length of these vectors? What does this tell you about the surface for all values of $K > 0$?

Computing angles

Problem 8 *Show that when $K > 0$ if two vectors are given to us in (x, y, z)-coordinates as*

$$\mathbf{v}_1 = \begin{pmatrix} a_1 \\ b_1 \\ c_1 \end{pmatrix} \qquad \textit{and} \qquad \mathbf{v}_2 = \begin{pmatrix} a_2 \\ b_2 \\ c_2 \end{pmatrix}$$

then the angle between their image in Euclidean space is given by

$$\theta = \arccos\left(\frac{\mathbf{v}_1 \bullet_K \mathbf{v}_2}{|\mathbf{v}_1|_K \cdot |\mathbf{v}_2|_K}\right).$$

Computing area

Problem 9 *Show that if two vectors are given to us in (x, y, z)-coordinates as*

$$\mathbf{v}_1 = \begin{pmatrix} a_1 \\ b_1 \\ c_1 \end{pmatrix} \qquad \text{and} \qquad \mathbf{v}_2 = \begin{pmatrix} a_2 \\ b_2 \\ c_2 \end{pmatrix}$$

then the area of the parallelogram spanned by the image of those two vectors in Euclidean space is

$$\sqrt{\det \begin{bmatrix} \mathbf{v}_1 \bullet_K \mathbf{v}_1 & \mathbf{v}_2 \bullet_K \mathbf{v}_1 \\ \mathbf{v}_1 \bullet_K \mathbf{v}_2 & \mathbf{v}_2 \bullet_K \mathbf{v}_2 \end{bmatrix}} = \sqrt{\det \left(\begin{bmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & K^{-1} \end{bmatrix} \begin{bmatrix} \mathbf{v}_1^\intercal & \mathbf{v}_2^\intercal \end{bmatrix} \right)}.$$

Moral of the story: The dot-product rules! That is, if you know the dot-product you know everything there is to know about a geometry, lengths, areas, angles, everything. And the set

$$1 = K\left(x^2 + y^2\right) + z^2$$

continues to make sense even when K is negative. And as we will see later on, the definition of the K-dot product also makes sense for tangent vectors to that set when K is negative. The geometry we get when the constant K is chosen to be negative is called a hyperbolic geometry. The geometry we get, when the constant K is just chosen to be non-zero is called a non-Euclidean geometry. In fact all the non-Euclidean 2-dimensional geometries are either spherical or hyperbolic.

Problem 10 *Summarize the results from this section. In particular, summarize the results in your own words and indicate which results follow from the others.*