Rigid motions in (x, y, z)-coordinates

Here we dig deeper to understand our different coordinates.

We now wish to figure out how to convert a transformation

$$\widehat{T}(\widehat{x}, \widehat{y}, \widehat{z}) = \begin{bmatrix} \widehat{a} \\ \widehat{b} \\ \widehat{c} \end{bmatrix} = \widehat{M} \cdot \begin{bmatrix} \widehat{x} \\ \widehat{y} \\ \widehat{z} \end{bmatrix}$$

from Euclidean space to (x, y, z)-coordinates. Recall to convert a point from (x, y, z)-coordinates to Euclidean coordinates, we write

$$\begin{bmatrix} \widehat{x} \\ \widehat{y} \\ \widehat{z} \end{bmatrix} = \begin{bmatrix} x \\ y \\ Rz \end{bmatrix} = \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & R \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

We summarize this in the diagrams

$$\begin{tikzcd}
K\text{-warped coordinates} \arrow[r, "\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & R \end{bmatrix}\cdot"] \arrow[d, "M\cdot"'] & \text{Euclidean coordinates} \arrow[d, "\widehat{M}\cdot"] \\
K\text{-warped coordinates} \arrow[r, "\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & R \end{bmatrix}\cdot"] & \text{Euclidean coordinates}
\end{tikzcd}$$

where

$$\begin{tikzcd}
\begin{bmatrix} x \\ y \\ z \end{bmatrix} \arrow[r, mapsto, "\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & R \end{bmatrix}\cdot"] \arrow[d, mapsto, "M\cdot"] & \begin{bmatrix} \widehat{x} \\ \widehat{y} \\ \widehat{z} \end{bmatrix} \arrow[d, mapsto, "\widehat{M}\cdot"] \\
\begin{bmatrix} a \\ b \\ c \end{bmatrix} \arrow[r, mapsto, "\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & R \end{bmatrix}\cdot"] & \begin{bmatrix} \widehat{a} \\ \widehat{b} \\ \widehat{c} \end{bmatrix}
\end{tikzcd}.$$

Learning outcomes:
Author(s):

Thus, we start at $[x \quad y \quad z]^\intercal$ and end at $[\widehat{a} \quad \widehat{b} \quad \widehat{c}]^\intercal$ whether we convert to Euclidean coordinates then apply $\widehat{M}$, or we apply M then convert to Euclidean coordinates.

Problem 1 *Using the diagram above, explain why*

$$M = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & R^{-1} \end{bmatrix} \cdot \widehat{M} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & R \end{bmatrix}.$$

Hint: *We can also go from Euclidean coordinates to K-warped coordinates using the map defined by* $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & R \end{bmatrix}^{-1}$

Hint: *We start at $[x \; y \; z]^\intercal$ and end at $[\widehat{a} \; \widehat{b} \; \widehat{c}]^\intercal$ whether we convert to Euclidean coordinates then apply $\widehat{M}$, or we apply M then convert to Euclidean coordinates.*

Problem 2 *Show that a transformation M in (x, y, z)-coordinates preserves distances in K-warped space if and only if*

$$M^{\mathsf{T}} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & K^{-1} \end{bmatrix} \cdot M = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & K^{-1} \end{bmatrix}. \tag{$*$}$$

Hint: *Explain how the previous problem gives a correspondence between transformations defined by a matrix in K-warped space and transformations defined by a matrix in Euclidean space. Then show that (**??**) is equivalent to the corresponding Euclidean matrix $\widehat{M}$ being orthogonal.*

The equation (**??**) is the condition (in (x, y, z)-coordinates) which affirms that the transformation which takes the path $\gamma(t) = (x(t), y(t), z(t))$ to the path $\gamma_M(t) = M \cdot (x(t), y(t), z(t))^{\intercal}$ preserves lengths of tangent vectors at corresponding points. Therefore, by integrating, the (total) length of the curve γ_M is the same as the total length of the curve γ.

Problem 3 *Verify that this is the correct condition by showing that any 3×3 matrix M satisfying* (**??**) *also satisfies*

$$(M \cdot \mathbf{v}) \bullet_K (M \cdot \mathbf{v}) = \mathbf{v} \bullet_K \mathbf{v},$$

where

$$\mathbf{v} = X_2 - X_1.$$

That is, the transformation given in (x, y, z)-coordinates by a matrix M that satisfies your condition preserves the K-dot product.

Definition 1. *A K-distance-preserving transformation of K-geometry is called a K-**rigid motion** or a K-**congruence**.*

With this definition, and our work above, we make a new definition:

Definition 2. *A 3×3 matrix M is called **K-orthogonal** if*

$$M^{\intercal} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & K^{-1} \end{bmatrix} \cdot M = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & K^{-1} \end{bmatrix}.$$

Problem 4 *For $K \neq 0$, show that if M is K-orthogonal, then the transformation*

$$T(x, y, z) = \begin{bmatrix} a \\ b \\ c \end{bmatrix} = M \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

takes the set of points (x, y, z) such that

$$1 = K\left(x^2 + y^2\right) + z^2$$

to the set of points (a, b, c) such that

$$1 = K\left(a^2 + b^2\right) + c^2.$$

That is, M gives a one-to-one and onto mapping of K-geometry to itself.

Hint: *Explain how you can write the equation*

$$1 = K\left(a^2 + b^2\right) + c^2.$$

as

$$\begin{bmatrix} a & b & c \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & K^{-1} \end{bmatrix} \cdot \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \frac{1}{K}.$$

Problem 5 *For $K \neq 0$, show that the set of K-orthogonal matrices M forms a group. That is, show that*

(a) *multiplication of K-orthogonal matrices is associative,*

Hint: *Recall that function composition is always associative.*

(b) *the product of two K-orthogonal matrices is K-orthogonal,*

(c) *the identity matrix is K-orthogonal,*

(d) *the inverse matrix M^{-1} of a K-orthogonal matrix M is K-orthogonal.*

Problem 6 *Convert the orthogonal matrix*

$$\widehat{M}_\theta = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

into its K-orthogonal counterpart M_θ. Are you surprised? Why or why not?

Problem 7 *Convert the orthogonal matrix*

$$\widehat{M_\psi} = \begin{bmatrix} \cos\psi & 0 & -\sin\psi \\ 0 & 1 & 0 \\ \sin\psi & 0 & \cos\psi \end{bmatrix}$$

into its K-orthogonal counterpart M_ψ. Are you surprised? Why or why not?

Why use K-coordinates?

We have seen that we could measure the usual Euclidean lengths of curves $\widehat{\gamma}$ in terms of the formulas of curves γ in K-warped space using the K-dot product. The short reason for this is that

$$\frac{d\widehat{\gamma}}{dt} \bullet \frac{d\widehat{\gamma}}{dt} = \frac{d\gamma}{dt} \bullet_K \frac{d\gamma}{dt}$$

where

$$\frac{d\gamma}{dt} \bullet_K \frac{d\gamma}{dt} = \left[\frac{d\gamma}{dt}\right] \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & K^{-1} \end{bmatrix} \cdot \left[\frac{d\gamma}{dt}\right]^{\mathsf{T}}.$$

In other words, the usual geometry of the sphere of radius R is simply the geometry of the set

$$\{(x, y, z) \in \mathbb{R}^3 : 1 = K\left(x^2 + y^2\right) + z^2\}$$

with $K = 1/R^2$ and with lengths (and areas) given by the K-dot product. Said another way, we can do all of spherical geometry in (x, y, z)-coordinates. All we need is the set defined by the relation

$$1 = K\left(x^2 + y^2\right) + z^2$$

and the K-dot product. But the set defined by the equation above continues to exist even if $K = 0$ or $K < 0$, and the K-dot product formula continues to make sense even if $K < 0$. In short we have the following table:

	Spherical ($K > 0$)	Euclidean ($K = 0$)	Hyperbolic ($K < 0$)
Surface in Euclidean space	$\widehat{x}^2 + \widehat{y}^2 + \widehat{z}^2 = R^2$	DNE	DNE
Euclidean dot product	$\widehat{\mathbf{v}}^{\mathsf{T}} \cdot \widehat{\mathbf{w}}$	DNE	DNE
Surface in K-warped space	$1 = K(x^2 + y^2) + z^2$	$1 = K(x^2 + y^2) + z^2$	$1 = K(x^2 + y^2) + z^2$
K-dot product	$\mathbf{v}^{\mathsf{T}} \left[\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & K^{-1} \end{smallmatrix}\right] \mathbf{w}$	DNE	$\mathbf{v}^{\mathsf{T}} \left[\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & K^{-1} \end{smallmatrix}\right] \mathbf{w}$

This table tells us that ‘there is something else out there,’ that is, some other type of two-dimensional geometry beyond plane geometry and spherical geometry. But the gap in the bottom row of the table is a bit disturbing. If we can’t express the usual dot-product in plane geometry as the K-dot product for

$K = 0$, we can't pass smoothly from spherical through plane geometry to hyperbolic geometry using (x, y, z)-coordinates. Later, we will examine two ways to produce coordinates uniformly for spherical, plane and hyperbolic geometry that overcome this difficulty.

Problem 8 *Summarize the results from this section. In particular, indicate which results follow from the others.*