Euclidean three-space as a metric space

In this activity we will work in three dimensional space and see the importance of the dot product to geometry.

Points and vectors in Euclidean 3-space

Up to this point in this book, we have studied two and three-dimensional shapes, the *objects* found in two (and three) dimensional geometry. Now we will study geometries as objects unto themselves. In particular, we will study plane, spherical, and hyperbolic geometry in two-dimensions. What does it mean to study a "whole geometry?" Well, geometry is the study of

- length,
- angle, and
- area.

Hence, if we can explain how to compute lengths, angles, and areas, we understand something about the geometry as a whole. Moreover, the congruences found in geometry are simply the transformations that preserve length, angle, and area: the *rigid motions*. Thus we now seek to understand how to compute length, angle, and area, along with a description of the congruences, for Euclidean, spherical, and hyperbolic geometry. Our plan of attack is as follows: we will visualize each of the geometries as different surfaces lying in a common three-dimensional space. This will allow us to use the techniques from calculus and linear algebra to describe Euclidean, spherical, and hyperbolic geometry in a unified way. We start by introducing the concepts we need in the more familiar setting in ordinary 3-dimensional Euclidean space:

$$\mathbb{R}^3 = \{(\widehat{x}, \widehat{y}, \widehat{z}) : \widehat{x}, \widehat{y}, \widehat{z} \in \mathbb{R}\}.$$

We reserve the notation (x, y, z) for some new coordinates that we will put on the 'same' objects later in this course. In Euclidean space, there is a standard way to measure distance between two points

$$\begin{aligned} \widehat{X}_1 &= (\widehat{x}_1, \widehat{y}_1, \widehat{z}_1) \\ \widehat{X}_2 &= (\widehat{x}_2, \widehat{y}_2, \widehat{z}_2), \end{aligned}$$

Learning outcomes:
Author(s):

namely

$$d(\widehat{X}_1, \widehat{X}_2) = \sqrt{(\widehat{x}_2 - \widehat{x}_1)^2 + (\widehat{y}_2 - \widehat{y}_1)^2 + (\widehat{z}_2 - \widehat{z}_1)^2}.$$

When you see two point $\widehat{X}_1$ and $\widehat{X}_2$ in what follows, the 'hats' mean that distance between points is measured by the formula above. One more thing, in Euclidean three-space it will be important throughout to make the distinction between **points** and **vectors**: Although each will be represented by a triple of real numbers we will use

$$\widehat{X} = (\widehat{x}, \widehat{y}, \widehat{z})$$

to denote **points**, that is, **position** in Euclidean 3-space, and

$$\widehat{\mathbf{v}} = (\widehat{a}, \widehat{b}, \widehat{c})$$

to denote **vectors**, that is, **displacement** by which we mean the amount and direction a given point is being moved. So vectors always indicate *motion* from an explicit (or implicit) *point* of reference.

The dot product determines length, angle, and area

There are various operations we can perform on one or more vectors when we think of them as based at the same point in Euclidean 3-space. The first is the dot product of two vectors.

Definition 1. *The dot product of two vectors*

$$\widehat{\mathbf{v}}_1 = (\widehat{a}_1, \widehat{b}_1, \widehat{c}_1),$$
$$\widehat{\mathbf{v}}_2 = (\widehat{a}_2, \widehat{b}_2, \widehat{c}_2),$$

based at the same point in 3-dimensional Euclidean space is defined as the real number given by the formula

$$\widehat{a}_1\widehat{a}_2 + \widehat{b}_1\widehat{b}_2 + \widehat{c}_1\widehat{c}_2$$

or in matrix notation as

$$\begin{bmatrix} \widehat{a}_1 & \widehat{b}_1 & \widehat{c}_1 \end{bmatrix} \begin{bmatrix} \widehat{a}_2 \\ \widehat{b}_2 \\ \widehat{c}_2 \end{bmatrix}.$$

It is also denoted as

$$\widehat{\mathbf{v}}_1 \bullet \widehat{\mathbf{v}}_2$$

or in matrix notation as

$$\widehat{\mathbf{v}}_1 \cdot \widehat{\mathbf{v}}_2^{\intercal}.$$

Problem 1 *Give the formula for the length* $|\widehat{\mathbf{v}}|$ *of a vector* $\widehat{\mathbf{v}} = (\widehat{a}, \widehat{b}, \widehat{c})$ *in 3-dimensional Euclidean space in terms of dot product.*

Problem 2 *Prove that*

$$\widehat{x}^2 + \widehat{y}^2 + \widehat{z}^2 = R^2$$

is a sphere in Euclidean three-space.

Hint: *Remember, a sphere in Euclidean three-space is the set of points equidistant from the origin.*

Lemma 1 (Law of Cosines). *The (smaller) angle θ between two vectors $\widehat{\mathbf{v}}_1$ and $\widehat{\mathbf{v}}_2$ based at $O = (0,0,0)$ satisfies the relation*

$$|\widehat{\mathbf{v}}_2 - \widehat{\mathbf{v}}_1|^2 = |\widehat{\mathbf{v}}_1|^2 + |\widehat{\mathbf{v}}_2|^2 - 2\,|\widehat{\mathbf{v}}_1| \cdot |\widehat{\mathbf{v}}_2| \cdot \cos\theta.$$

Now we will prove this lemma. Your task is to fill in the details of the proof below.

Problem 3 *Start by noting that without loss of generality we can assume that $|\widehat{\mathbf{v}}_1| \leq |\widehat{\mathbf{v}}_2|$. Consider the triangle OP_1P_2, where P_1 and P_2 are the endpoints of $\mathbf{v}_1$ and $\mathbf{v}_2$, respectively. Let P be the point on OP_2 so that the segment between P_1 and P is perpendicular to OP_2.*

(a) *Illustrate the diagram described above.*

(b) *Now explain how the Pythagorean theorem gives*

$$\begin{aligned} |P_1P_2|^2 - |P_2P|^2 &= |PP_1|^2 \\ &= |OP_1|^2 - |OP|^2. \end{aligned}$$

(c) *Explain each of the following lines:*

$$\begin{aligned} |P_1P_2|^2 &= |OP_1|^2 + \left(|P_2P|^2 - |OP|^2\right) && (1) \\ &= |OP_1|^2 + (|P_2P| + |OP|)\,(|P_2P| - |OP|) && (2) \\ &= |OP_1|^2 + |OP_2|\,(|P_2P| - |OP|) && (3) \\ &= |OP_1|^2 + |OP_2|\,(|OP_2| - 2|OP|)\,. && (4) \end{aligned}$$

(d) *Now note that*

$$|OP| = |OP_1| \cdot \cos\theta.$$

Explain how this completes the proof.

Theorem 1. *The angle θ between two vectors $\widehat{\mathbf{v}}_1$ and $\widehat{\mathbf{v}}_2$ based at the same point in Euclidean 3-space satisfies the relation*

$$\widehat{\mathbf{v}}_1 \bullet \widehat{\mathbf{v}}_2 = |\widehat{\mathbf{v}}_1| \cdot |\widehat{\mathbf{v}}_2| \cdot \cos\theta. \tag{5}$$

Problem 4 *Prove this theorem.*

Hint: *Use the law of cosines together with algebraic properties of the dot product.*

The significance of the theorem above is that the measure of angles between vectors depends only on the definition of the dot product.

Corollary 1. *The formula for the angle θ between two vectors $\widehat{\mathbf{v}}_1 = (\widehat{a}_1, \widehat{b}_1, \widehat{c}_1)$ and $\widehat{\mathbf{v}}_2 = (\widehat{a}_2, \widehat{b}_2, \widehat{c}_2)$ in 3-dimensional Euclidean space depends only on the dot products of the two vectors with themselves and with each other. Namely,*

$$\theta = \arccos\left(\frac{\widehat{\mathbf{v}}_1 \bullet \widehat{\mathbf{v}}_2}{|\widehat{\mathbf{v}}_1| \cdot |\widehat{\mathbf{v}}_2|}\right).$$

Similarly, the formula for the area of the parallelogram determined by two vectors $\widehat{\mathbf{v}}_1$ and $\widehat{\mathbf{v}}_2$ depends only on the dot products of the two vectors with themselves and with each other.

For a slower look at the next two propositions, see `https://www.khanacademy.org/math/linear-algebra/matrix-transformations/determinant-depth/v/linear-algebra-determinant-and-area-of-a-parallelogram`

Proposition 1. *The area of the parallelogram determined by $\widehat{\mathbf{v}}_1$ and $\widehat{\mathbf{v}}_2$ based at the same point in Euclidean 3-space is given by*

$$|\widehat{\mathbf{v}}_1| \cdot |\widehat{\mathbf{v}}_2| \cdot \sin\theta.$$

Proof Let $|\widehat{\mathbf{v}}_1|$ be the base and h be the height of the parallelogram determined by $\widehat{\mathbf{v}}_1$ and $\widehat{\mathbf{v}}_2$. If θ is the angle between $\widehat{\mathbf{v}}_1$ and $\widehat{\mathbf{v}}_2$ then $|\widehat{\mathbf{v}}_2| \cdot \sin\theta = h$. Therefore, $A = b \cdot h = |\widehat{\mathbf{v}}_1| \cdot |\widehat{\mathbf{v}}_2| \cdot \sin\theta$.

$\blacksquare$

Proposition 2. *The area of the parallelogram determined by $\widehat{\mathbf{v}}_1$ and $\widehat{\mathbf{v}}_2$ based at the same point in Euclidean 3-space is also given by*

$$\sqrt{\det\begin{bmatrix} \widehat{\mathbf{v}}_1 \bullet \widehat{\mathbf{v}}_1 & \widehat{\mathbf{v}}_2 \bullet \widehat{\mathbf{v}}_1 \\ \widehat{\mathbf{v}}_1 \bullet \widehat{\mathbf{v}}_2 & \widehat{\mathbf{v}}_2 \bullet \widehat{\mathbf{v}}_2 \end{bmatrix}}.$$

Proof By the theorem:

$$\begin{aligned} A^2 = |\widehat{\mathbf{v}}_1|^2 \cdot |\widehat{\mathbf{v}}_2|^2 \cdot \sin^2\theta &= |\widehat{\mathbf{v}}_1|^2 \cdot |\widehat{\mathbf{v}}_2|^2 \cdot \left(1 - \cos^2\theta\right) \\ &= |\widehat{\mathbf{v}}_1|^2 \cdot |\widehat{\mathbf{v}}_2|^2 - |\widehat{\mathbf{v}}_1|^2 \cdot |\widehat{\mathbf{v}}_2|^2 \cdot \cos^2\theta \\ &= |\widehat{\mathbf{v}}_1|^2 \cdot |\widehat{\mathbf{v}}_2|^2 - |\widehat{\mathbf{v}}_1|^2 \cdot |\widehat{\mathbf{v}}_2|^2 \left(\frac{\widehat{\mathbf{v}}_1 \bullet \widehat{\mathbf{v}}_2}{|\widehat{\mathbf{v}}_1| \cdot |\widehat{\mathbf{v}}_2|}\right)^2 \\ &= |\widehat{\mathbf{v}}_1|^2 \cdot |\widehat{\mathbf{v}}_2|^2 - \left(\widehat{\mathbf{v}}_1 \bullet \widehat{\mathbf{v}}_2\right)^2 \\ &= \left(\widehat{\mathbf{v}}_1 \bullet \widehat{\mathbf{v}}_1\right)\left(\widehat{\mathbf{v}}_2 \bullet \widehat{\mathbf{v}}_2\right) - \left(\widehat{\mathbf{v}}_1 \bullet \widehat{\mathbf{v}}_2\right)^2 \\ &= \det\begin{bmatrix} \widehat{\mathbf{v}}_1 \bullet \widehat{\mathbf{v}}_1 & \widehat{\mathbf{v}}_2 \bullet \widehat{\mathbf{v}}_1 \\ \widehat{\mathbf{v}}_1 \bullet \widehat{\mathbf{v}}_2 & \widehat{\mathbf{v}}_2 \bullet \widehat{\mathbf{v}}_2 \end{bmatrix} \end{aligned}$$

Therefore, the area of the parallelogram is also given by $\sqrt{\det\begin{bmatrix}\widehat{\mathbf{v}}_1 \bullet \widehat{\mathbf{v}}_1 & \widehat{\mathbf{v}}_2 \bullet \widehat{\mathbf{v}}_1 \\ \widehat{\mathbf{v}}_1 \bullet \widehat{\mathbf{v}}_2 & \widehat{\mathbf{v}}_2 \bullet \widehat{\mathbf{v}}_2\end{bmatrix}}$.

Alternatively, given that $A = |\widehat{\mathbf{v}}_1| \cdot |\widehat{\mathbf{v}}_2| \cdot \sin\theta$, we know

$$\theta = \arccos\left(\frac{\widehat{\mathbf{v}}_1 \bullet \widehat{\mathbf{v}}_2}{|\widehat{\mathbf{v}}_1| \cdot |\widehat{\mathbf{v}}_2|}.\right)$$

Then, using a right triangle whose "adjacent" side is $\widehat{\mathbf{v}}_1 \bullet \widehat{\mathbf{v}}_2$ and hypotenuse is $|\widehat{\mathbf{v}}_1| \cdot |\widehat{\mathbf{v}}_2|$,

$$\begin{aligned}
\sin\theta = \sin\left(\arccos\left(\frac{\widehat{\mathbf{v}}_1 \bullet \widehat{\mathbf{v}}_2}{|\widehat{\mathbf{v}}_1| \cdot |\widehat{\mathbf{v}}_2|}\right)\right) &= \frac{\sqrt{\left(|\widehat{\mathbf{v}}_1| \cdot |\widehat{\mathbf{v}}_2|\right)^2 - \left(\widehat{\mathbf{v}}_1 \bullet \widehat{\mathbf{v}}_2\right)^2}}{|\widehat{\mathbf{v}}_1| \cdot |\widehat{\mathbf{v}}_2|} \\
&= \frac{\sqrt{\left(\widehat{\mathbf{v}}_1 \bullet \widehat{\mathbf{v}}_1\right) \cdot \left(\widehat{\mathbf{v}}_2 \bullet \widehat{\mathbf{v}}_2\right) - \left(\widehat{\mathbf{v}}_1 \bullet \widehat{\mathbf{v}}_2\right)^2}}{|\widehat{\mathbf{v}}_1| \cdot |\widehat{\mathbf{v}}_2|} \\
&= \frac{\sqrt{\det\begin{bmatrix}\widehat{\mathbf{v}}_1 \bullet \widehat{\mathbf{v}}_1 & \widehat{\mathbf{v}}_2 \bullet \widehat{\mathbf{v}}_1 \\ \widehat{\mathbf{v}}_1 \bullet \widehat{\mathbf{v}}_2 & \widehat{\mathbf{v}}_2 \bullet \widehat{\mathbf{v}}_2\end{bmatrix}}}{|\widehat{\mathbf{v}}_1| \cdot |\widehat{\mathbf{v}}_2|}
\end{aligned}$$

Therefore, the area of the parallelogram is also given by $\sqrt{\det\begin{bmatrix}\widehat{\mathbf{v}}_1 \bullet \widehat{\mathbf{v}}_1 & \widehat{\mathbf{v}}_2 \bullet \widehat{\mathbf{v}}_1 \\ \widehat{\mathbf{v}}_1 \bullet \widehat{\mathbf{v}}_2 & \widehat{\mathbf{v}}_2 \bullet \widehat{\mathbf{v}}_2\end{bmatrix}}$.

■

Problem 5 *Using the preceding two propositions, show that we have the following equality:*

$$\sqrt{\det\begin{bmatrix}\widehat{\mathbf{v}}_1 \bullet \widehat{\mathbf{v}}_1 & \widehat{\mathbf{v}}_2 \bullet \widehat{\mathbf{v}}_1 \\ \widehat{\mathbf{v}}_1 \bullet \widehat{\mathbf{v}}_2 & \widehat{\mathbf{v}}_2 \bullet \widehat{\mathbf{v}}_2\end{bmatrix}} = \sqrt{\det\left(\begin{bmatrix}\widehat{\mathbf{v}}_1 \\ \widehat{\mathbf{v}}_2\end{bmatrix} \cdot \begin{bmatrix}\widehat{\mathbf{v}}_1^{\intercal} & \widehat{\mathbf{v}}_2^{\intercal}\end{bmatrix}\right)}.$$

Again, the significance of the previous problems is that, to compute areas, we only need to know how to compute dot products. Hence, it is the definition of the dot product of the vectors completely determines the calculation of the area of the parallelogram they generate.

Curves in Euclidean 3-space and vectors tangent to them

Definition 2. *A* ***smooth curve in 3-dimensional Euclidean space*** *is given by a differentiable mapping*

$$\begin{aligned} \widehat{\gamma} : [b, e] &\to \mathbb{R}^3 \\ t &\mapsto (\widehat{x}(t), \widehat{y}(t), \widehat{z}(t)) \end{aligned}$$

from an interval $[a, b]$ on the real line whose tangent vector

$$\frac{d\widehat{\gamma}}{dt} = \left(\frac{d\widehat{x}}{dt}, \frac{d\widehat{y}}{dt}, \frac{d\widehat{z}}{dt} \right)$$

is not the zero vector for any t in $[a, b]$. We will sometimes use column vector notation if it looks better:

$$\widehat{\gamma}(t) = \begin{pmatrix} \widehat{x}(t) \\ \widehat{y}(t) \\ \widehat{z}(t). \end{pmatrix}$$

Problem 6

(a) *Give a second example of a smooth curves,*

$$\begin{aligned} \widehat{\gamma}_1(s) &= (s, s, \cos(s)) \\ \widehat{\gamma}_2(t) &= (\widehat{x}_2(t), \widehat{y}_2(t), \widehat{z}_2(t)) \end{aligned}$$

where

- *neither $\widehat{\gamma}_1$ nor $\widehat{\gamma}_2$ are a straight line,*
- *the two curves $\widehat{\gamma}_1$ and $\widehat{\gamma}_2$ pass through a common point* **and** *go in different tangent directions at that point.*
- *None of the components of $\widehat{\gamma}_1$ and $\widehat{\gamma}_2$ are constant functions.*

The first applet of `https://www.geogebra.org/m/cvn97uzz` *allows you to graph 3d parametric equations to see if they intersect. Notice that the equations in the second applet are not the same as the ones in this problem, but do illustrate the same concepts.*

(b) *Compute the tangent vectors of each of the two curves at each of their points.*

(c) *For the two curves you defined in a), what are the coordinates of the point in Euclidean 3-space at which the two curves intersect?*

(d) *Use the dot product formula to compute the angle θ between (the tangent vectors to) your two example curves in a) at the point at which the curves intersect.*

There are two kinds of vectors that naturally arise from curves: the displacement between times t_1 and t_2, given by

$$\widehat{\mathbf{v}} = \widehat{\gamma}(t_2) - \widehat{\gamma}(t_1) = (\widehat{x}(t_2) - \widehat{x}(t_1), \widehat{y}(t_2) - \widehat{y}(t_1), \widehat{z}(t_2) - \widehat{z}(t_1)),$$

and the instantaneous velocity of a point moving along a curve, given by

$$\widehat{\mathbf{v}} = \frac{d\widehat{\gamma}(t)}{dt} = \left(\frac{d\widehat{x}(t)}{dt}, \frac{d\widehat{y}(t)}{dt}, \frac{d\widehat{z}(t)}{dt}\right).$$

In matrix notation we can think of the difference between two points

$$\widehat{X}_2 - \widehat{X}_1 = (\widehat{x}_2 - \widehat{x}_1, \widehat{y}_2 - \widehat{y}_1, \widehat{z}_2 - \widehat{z}_1)$$

as a 1×3 matrix $\left[\widehat{X}_2 - \widehat{X}_1\right]$. Then we can write the formula for the distance between $\widehat{X}_1$ and $\widehat{X}_2$ in Euclidean 3-space in terms of the dot-product:

$$d(\widehat{X}_1, \widehat{X}_2) = \sqrt{(\widehat{X}_2 - \widehat{X}_1) \bullet (\widehat{X}_2 - \widehat{X}_1)} \tag{6}$$

or in terms of the matrix product

$$d(\widehat{X}_1, \widehat{X}_2) = \sqrt{\left[\widehat{X}_2 - \widehat{X}_1\right] \cdot \left[\widehat{X}_2 - \widehat{X}_1\right]^{\mathsf{T}}}.$$

Length of a smooth curve in Euclidean 3-space

Problem 7 *Compute the length of the tangent vector*

$$\ell(t) = \sqrt{\frac{d\widehat{\gamma}}{dt} \bullet \frac{d\widehat{\gamma}}{dt}}$$

to each of your two example curves in the previous problem at each of their points.

Definition 3. *The length L of the curve $\widehat{\gamma}(t)$, $t \in [a, b]$, in Euclidean 3-space is obtained by integrating the length of the tangent vector to the curve, that is,*

$$L = \int_a^b \ell(t)\, dt.$$

Notice that the length of any curve only depends on the definition of the dot product. That is, if we know the formula for the dot product, we know (the formula for) the length of any curve.

Our first example is the path

$$(\widehat{x}(t), \widehat{y}(t), \widehat{z}(t)) = (R \cdot \sin(t), 0, R \cdot \cos(t)) \qquad 0 \leq t \leq \pi.$$

Notice that this path lies on the sphere of radius R.

Problem 8 *Write the formula for the tangent vector to the path above. Show that the length of this path is $R\pi$.*

Problem 9 *(Try to) compute the length of each of your two example curves in the previous problem.*

Remark 1. *In this last problem, you may easily be confronted with an integral that you cannot compute. For example, if your curve $\widehat{\gamma}_1(t)$ happens to describe an ellipse that is not circular, it was proved in the 19th century that no formula involving only the standard functions from calculus will give you the length of your path from a fixed beginning point to a variable ending point on the ellipse. If that kind of thing occurs, go back and change the definitions of your curves in the previous problem until you get two curves for which you can compute length of your path from a fixed beginning point to a fixed ending point.*

To study plane geometry, spherical geometry, and hyperbolic geometry in a uniform way we will have to *change* the coordinate system we use. In essence, this means that we will have to change the distance formula slightly for each geometry. These new coordinates will be chosen to keep the north and south poles from going to infinity as the radius R of a sphere increases without bound. This change of viewpoint will eventually let us go non-Euclidean or, in the language of Buzz Lightyear "to infinity and beyond." The idea will be like the change from rectangular to polar coordinates for the plane that you encountered in calculus, only easier.

Problem 10 *Summarize the results from this section. In particular, indicate which results follow from the others.*