Workbook of two-dimensional geometries

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1 Introduction

Seeing different geometries as a variation on a theme.

As a young mathematician I was introduced to the classic *Leçons sur la Géométrie des Espaces de Riemann*, written by the great French geometer Élie Cartan. Early in his treatise on geometries in all dimensions, the author presents the case of two-dimensional geometries, in particular, those two-dimensional geometries that look the same at all points and in all directions. (For example, a cylinder looks the same at each of its points but not in all directions emanating from any one of its points, whereas a sphere looks the same at all points and in all directions.) It turns out that there is one and only one such geometry for each real number K, called the *curvature* of the geometry. The case $K = 0$ is the (flat) Euclidean geometry that you learned in high school.

These twenty-five pages of Cartan's book (Chapter VI, §i-v) so captivated me that I have returned to them regularly throughout my career and have adapted and taught them many times at the advanced undergraduate level. They form the basis for this little book. To me they tell one of the most beautiful and satisying stories in all of geometry, one which exemplifies a fundamental principle of all great mathematics, namely that, using the tools at hand but in a slightly novel way, the clouds part and one sees that objects and relationships that seemed so different are in fact parts of a single elegant story!

When $K > 0$ it turns out that the K-geometry is the geometry of the sphere of radius $R = 1/K^{1/2}$ that we can see as a subset of euclidean three-space $\mathbb{R}^3$. But the geometries with $K < 0$ are not so easy to visualize. They are the so-called 'hyperbolic' geometries. In fact it took mathematicians a couple thousand years to realize that the existed at all! It turns out that the secret to understanding all the two-dimensional geometries, including the ones with $K < 0$, in a unified way is to simply rescale the third coordinate in $\mathbb{R}^3$ and use these 'unusual' coordinates to look at each two-dimensional geometry as the solution set to the equation

$$K\left(x^2 + y^2\right) + z^2 = 1.$$

However the idea of changing coordinates without changing the underlying geometry described by those coordinates is a challenging one that did not come into mathematics until a couple of centuries ago. It will require that, before we get into the beautiful uniform study of all two-dimensional geometries, we practice the coordinate change we are going to use, namely the rescaling of the third coordinate in euclidean 3-space. That practice, together with a review of some concepts from several variable calculus and linear algebra, will comprise much of this book.

It has often been said that "mathematics is not a spectator sport." This truism is very much in evidence in the writing of this book. It is written so as to guide you through the entire story, yet permit you, when possible, to construct the mathematical story for yourself, that is, to do some mathematics yourself rather than just observe it done by others. This 'doing mathematics oneself' takes the form of Exercises with enough help (Hints) provided so that the 'doing' is not so onerous as to get in the way of the story itself.

Strong evidence has been provided by students of mathematics over many centuries that such guided 'doing' is indispensible for understanding and retention. In fact the very form of this book, as a loose-leaf or electronic notebook, is intended to encourage you to write out (in correctable form) solutions to the problems that can be inserted at the appropriate places into the text.

The second half of this book supposes familiarity with several variable calculus and the linear algebra of matrices. In particular, it will often be useful to consider a vector, for example $V = (a, b, c)$, as a 1×3 matrix

$$V = \begin{bmatrix} a & b & c \end{bmatrix}$$

with

$$V^{\mathsf{T}} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}.$$

This will allow us to write the scalar product of two vectors

$$\begin{aligned} V \bullet W &= (a, b, c) \bullet (d, e, f) \\ &= ad + be + cf \end{aligned}$$

as a product of matrices

$$V \cdot W^{\mathsf{T}} = \begin{bmatrix} a & b & c \end{bmatrix} \cdot \begin{bmatrix} d \\ e \\ f \end{bmatrix}.$$

Furthermore we will use the notations $\det(A)$ and $|A|$ interchangeably for the determinant of a square matrix A.

You will also need to remember and apply the Chain Rule for differentiable functions of several variables, written in matrix notation. Here's the gradual build-up to this general form of the Chain Rule using matrix notation and matrix multiplication.

Theorem 1 (Chain Rule)**.** *We will present three different versions:*

(a) *Given differentiable functions $y(x)$ and $z(y)$ and substituting we have*

$$z\left(y\left(x\right)\right)$$

making z a function of x and

$$\frac{dz}{dx} = \frac{dz}{dy} \cdot \frac{dy}{dx}.$$

(b) *Given differentiable functions*

$$\begin{aligned} &(y_1(x), \ldots, y_n(x)) \\ &z(y_1, \ldots, y_n) \end{aligned}$$

then substituting we have

$$z(y_1(x), \ldots, y_n(x))$$

making z a function of x and

$$\frac{dz}{dx} = \begin{bmatrix} \dfrac{\partial z}{\partial y_1} & \cdots & \dfrac{\partial z}{\partial y_n} \end{bmatrix} \cdot \begin{bmatrix} \dfrac{dy_1}{dx} \\ \vdots \\ \dfrac{dy_n}{dx} \end{bmatrix} = \nabla z \cdot \begin{bmatrix} \dfrac{dy_1}{dx} \\ \vdots \\ \dfrac{dy_n}{dx} \end{bmatrix}$$

(c) *Given differentiable mappings*

$$\begin{aligned} &(y_1(x_1, \ldots, x_m), \ldots, y_n(x_1, \ldots, x_m)) \\ &(z_1(y_1, \ldots, y_n), \ldots, z_p(y_1, \ldots, y_n)) \end{aligned}$$

then substituting

$$z_k(y_1(x_1, \ldots, x_m), \ldots, y_n(x_1, \ldots, x_m))$$

and fixing $x_1, \ldots, x_{i-1}$ and $x_{i+1}, \ldots, x_m$ we make z_k a function of x_i and

$$\frac{\partial z_k}{\partial x_i} = \begin{bmatrix} \dfrac{\partial z_k}{\partial y_1} & \cdots & \dfrac{\partial z_k}{\partial y_n} \end{bmatrix} \cdot \begin{bmatrix} \dfrac{\partial y_1}{\partial x_i} \\ \vdots \\ \dfrac{\partial y_n}{\partial x_i} \end{bmatrix} = \nabla z_k \cdot \begin{bmatrix} \dfrac{\partial y_1}{\partial x_i} \\ \vdots \\ \dfrac{\partial y_n}{\partial x_i} \end{bmatrix}.$$

(d) *Putting the previous work together for all indices k and i, we have the matrix equation*

$$\left[\frac{\partial z_k}{\partial x_i}\right] = \left[\frac{\partial z_k}{\partial y_j}\right] \cdot \left[\frac{\partial y_j}{\partial x_i}\right]$$

where $\left[\dfrac{\partial z_k}{\partial x_i}\right]$ is the $p \times m$ matrix whose (k,i)-th entry is $\dfrac{\partial z_k}{\partial x_i}$, etc.

One of the Chain Rule's important applications is the Substitution Rule for integrals of functions of several variables.

Theorem 2 (Substitution Rule)**.** *We will present two different versions:*

(a) *If $f(y)$ is a function of y and $y(x)$ is a function of x,*

$$\int_{[y(a),y(b)]} f(y) \cdot dy = \int_{[a,b]} f((y(x))) \cdot \frac{dy}{dx}\, dx.$$

(b) *If $f(\mathbf{y})$ is a function of $\mathbf{y} = (y_1, \ldots, y_m)$ and the function*

$$\begin{bmatrix} y_1(x_1, \ldots, x_m) \\ \vdots \\ y_m(x_1, \ldots, x_m) \end{bmatrix}$$

takes the region R_x to the region R_y, then

$$\int_{R_y} f(\mathbf{y})\, d\mathbf{y} = \int_{R_x} f((y_1(\mathbf{x}), \ldots, y_m(\mathbf{x}))) \cdot \det \left[\frac{\partial y_j}{\partial x_i}\right] d\mathbf{x}$$

where $\mathbf{x} = (x_1, \ldots, x_m)$.

As a help, at some points in the text and in some of the exercises, a more complete treatment of a particular topic can be found in one of the following two texts:

[MJG]: Greenberg, Marvin Jay. *Euclidean and Non-Euclidean Geometry: Development and History.* W.H. Freeman & Co. 3rd Ed., 1994.

[DS]: Davis, H. and Snider, A.D. *Introduction to Vector Analysis.* Wm. C. Brown Publishers, 7th Ed. 1994.

The corresponding topics in these texts are referenced. For example, [MJG,311] refers to page 311 in the Greenberg book and [DS,59ff] refers to page 59 and those pages just following page 59 in the Davis-Snider book.

Some final remarks about notation in this book. The letters '**EG**' will always mean euclidean (usually plane but occasionally 3-dimensional) Geometry, the letters '**SG**' will always mean Spherical Geometry, and the letters '**HG**' will always mean Hyperbolic Geometry. One further kind of geometry, which we call Neutral Geometry, will be explained in the book and denoted by '**NG**.'

It is my hope and intention in writing this little book that you engage with and enjoy this uniform way of understanding all two-dimensional geometries as much as I did!

Remark 1. *Special message to current or future teachers of high school geometry: Many parts of this book are especially relevant to your teaching of the subject. Look especially closely at the treatment of congruence (rigid motion), similarity (dilation), circles, expressing geometric properties with equations, and geometric measurement and dimension, and compare them with the high school geometry sections of the Common Core State Standards in Mathematics. The latter can be found at:*

http://www.corestandards.org/Math/Content/HSG/introduction.

A useful companion course to one based on this book, one that might be called *Geometry for Teaching*, would explicitly make the connections between the material covered as in this book and what you do (or will do) in your high school geometry classroom. The idea is *not* that the material we will cover will tell you how to teach that material but rather that the treatment given here will give you the depth and breadth of geometric understanding that will allow you to design what you teach and bring it into your classroom in ways that those who lack that understanding cannot.

Remark 2. *This book can also be used as a bridge to a first course in Riemannian geometry. It treats the case of two-dimensional geometries that are homogeneous, that is, that look the same at all their points. But to treat these geometries efficiently, we introduce the notion of changing coordinates for the geometry without changing the geometry itself. It is that notion that allowed geometers to treat surfaces and higher-dimensional smooth spaces that look different at different points, ones that can often not be treated at all their points using a single set of coordinates.*

2 Euclid's postulates for plane geometry

We begin with Euclid's first four assumptions and explore neutral geometries.

Neutral geometry

After answering the following questions, students should be able to:

- State the axioms of neutral geoemtry.
- State the side-angle-side congruence axiom.
- Apply side-angle-side.
- Prove that the sum of the interior angles of a triangle in neutral geomtry is always less than or equal to 180°.

In Western civilization, the primary source of our understanding of plane geometry comes from Euclid's *Elements*. The treatise is of transcendent importance well beyond geometry itself. It is among the first examples of formal logical deductive reasoning. Certain fundamentals, that are called *axioms*, are postulated or 'given,' providing the platform on which a 'geometry' is built. This creates a mathematical entity modeling a physical 'reality.' Its properties are arrived at by applying the laws of logic to the given fundamentals. Euclid gives five axioms for plane geometry, the first four of which seem to be 'obvious' reflections of physical reality. In paraphrased form, they are:

Axiom 1 (E1)**.** *Through any point P and any other point Q, there lies a unique line.*

Axiom 2 (E2)**.** *Given any two segments $\overline{AB}$ and $\overline{CD}$, there is a segment $\overline{AE}$ such that B lies on $\overline{AE}$ and $|CD| = |BE|$.*

Note: We will use both $|AB|$ and $d(A, B)$ to denote the distance between two points A and B.

Axiom 3 (E3)**.** *Given a point P and any positive real number r, there exists a (unique) circle of radius r and center P.*

Said another way, if you move away from point P along a line in any direction, you will encounter a unique point at distance r from P.

Axiom 4 (E4)**.** *All right angles are congruent.*

Note: A right angle is defined as follows. Let C be the midpoint on the segment $\overline{AB}$. Let E be any point not equal to C. The angle $\angle ACE$ is called a right angle if $\angle ACE$ is congruent to $\angle ECB$. [MJG,17-18]

Definition 1. *If we are only given axioms E1–E4, we will call our geometry* ***neutral geometry (NG)****.*

Definition 2. *In* ***neutral geometry****, two distinct lines are called* ***parallel*** *if and only if they don't intersect.*

Definition 3. *We will call the set of points on a line which lie on one side of a given point a* ***ray****. We call the given point the* ***origin*** *of the ray.*

Remark 3. *Spheres do not have neutral geometry. There are infinitely many lines between the north pole and south pole; you can see some of them in this image from Wikipedia:*

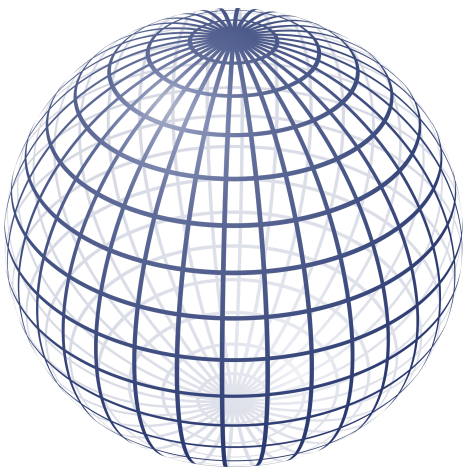


Figure 1: Image of a wireframe sphere from https://en.wikipedia.org/wiki/Sphere. By Geek3 - Own work, CC BY 3.0,

Definition 4. *We call two rays in the plane* ***parallel*** *if they lie on parallel lines and they both lie on the same side of the transversal line passing through their origins.*

Definition 5. *An* ***angle*** *in the plane is the union of two ordered rays with common origin and choice of one of the two connected regions into which the union of the rays divides the plane. We often denote angles by $\angle BAC$ where A is the common origin and B a point along one of the rays, called the initial ray, and C is a point along the other ray, called the final ray.*

The choice of the region is either clear from the context or explicitly given.

Problem 1 *Given an angle $\angle BAC$ show by drawings the two regions into which it divides the plane. Show how the (signed) measure of the angle depends on:*

(a) *Which region you pick as the interior/exterior.*

(b) *Which ray is the initial ray and which is the final ray of the angle.*

One implicit assumption of two-dimensional neutral geometry is the existence of (a group of) rigid motions or congruences. That is, it is assumed that given any point A and any vector $\vec{v}$ emanating from A and given any second point B in the geometry and any vector $\vec{w}$ emanating from B, then there is a transformation (thought of as a matrix that we will multiply on the left) $\mathbf{M}$ such that:

(a) $\mathbf{M} \cdot A = B$.

(b) $\mathbf{M} \cdot \vec{v}$ is a positive scalar multiple of $\vec{w}$.

(c) For all points P and Q in the geometry, $\mathbf{M}$ leaves the distance between them unchanged:
$$d\left(\mathbf{M} \cdot P, \mathbf{M} \cdot Q\right) = d\left(P, Q\right).$$

(d) For any two vectors $\vec{u}$ and $\vec{v}$ emanating from A, the angle between $\mathbf{M} \cdot \vec{u}$ and $\mathbf{M} \cdot \vec{v}$ is the same as the angle between $\vec{u}$ and $\vec{v}$.

Problem 2 *Think back to high school days and write the congruence rules side-side-side (SSS), side-angle-side (SAS), and angle-side-angle (ASA). Translate them into the language of* ***rigid motions****.*

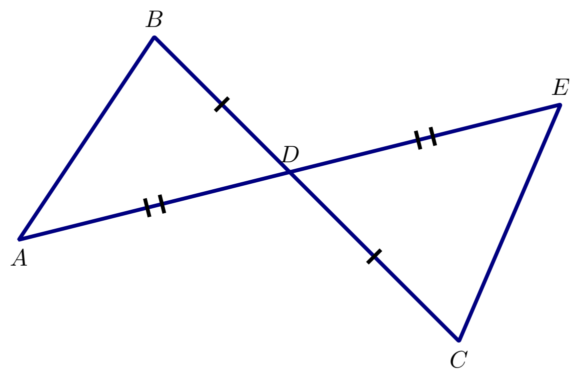
You may look up these congruence rules without citing the source.

Problem 3 *Give an example to show that there is no universal side-side-angle (SSA) law.*

Can you find a restriction that will allow for an "side-side-angle-type" law?

Although it is a bit tedious to show (and we will not ask you to do it here), using only E1–E4 you can derive the usual rules for congruent triangles (SSS, SAS, ASA). Thus these laws hold in any neutral geometry.

Problem 4 *In the diagram below, we see the intersection of* $\overline{BC}$ *and* $\overline{AE}$*. Suppose that* $|BD| = |CD|$ *and* $|AD| = |ED|$*.*



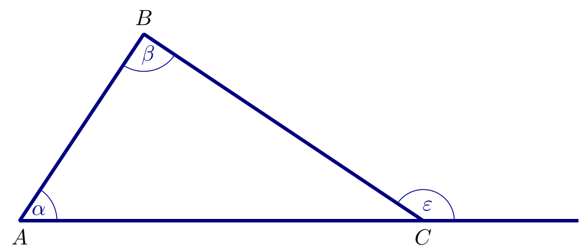
Show that triangle $\triangle BDA$ *and triangle* $\triangle CDE$ *are congruent. [MJG,138]*

Hint: *First you should explain why* $\angle BDA = \angle CDE$*.*

Hint: *Next you should use one of the congruence properties above.*

Problem 5

(a) *Show in neutral geometry that, for $\triangle ABC$*



the exterior angle ε of the triangle at C is greater than either remote interior angle α or β. [MJG,119]

(b) *Use the previous part to show that the sum of any two angles of a triangle is less than $180°$.*

Hint: *Add line segments to the diagram above, and attempt to use the previous problem.*

Problem 6 *Show in* ***neutral geometry*** *that, if two lines cut by a transversal line have a pair of congruent alternate interior angles, then they are parallel. [MJG,117]*

Hint: *Suppose the assertion is false for some pair of lines. Find a triangle that violates the conclusion of the previous problem.*

Sum of angles in a triangle in neutral geometry

After answering the following questions, students should be able to:

- Construct a new triangle $\triangle ABC$ where the sum of the angles of $\triangle ABC$ has certain properties relating to a given triangle.

- Show that there are no triangles in neutral geometry whose angles sum to more than $180°$.

You may have learned these five postulates of two-dimensional geometry:

(a) You can draw a straight line through any two points

(b) You can extend any straight line segment to an infinite line

(c) Given any straight light segment, you can draw a circle with that segment as the radius and one endpoint as the center

(d) All right angles are equivalent

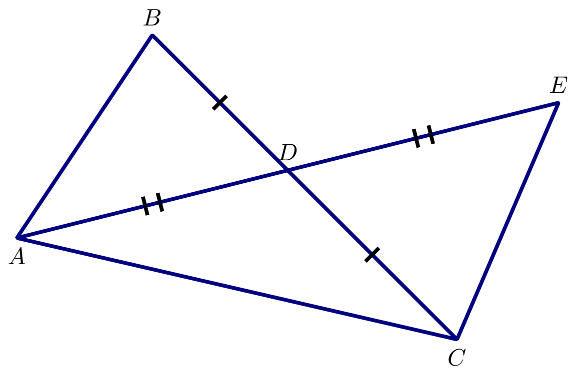
(e) Given any two lines intersect with a third line such that the sum of the interior angles on one side is less that two right angles, then the two lines must intersect if the lines are extended far enough.

How do these five postulates relate to (E1)-(E4) in the previous section?

You might also remember that the sum of the interior angles of a triangle is $180°$. While it might seem like this fact could follow from (E1)-(E4), we will eventually see that is false. It turns out the fifth postulate about parallel lines is true **if and only if** the sum of the interior angles of a triangle is always $180°$. In fact, there is an entire list of geometry facts that are true if and only if the parallel postulate is true. You can find some at: `https://orion.math.iastate.edu/lhogben/classes/parallelpostulate.pdf`.

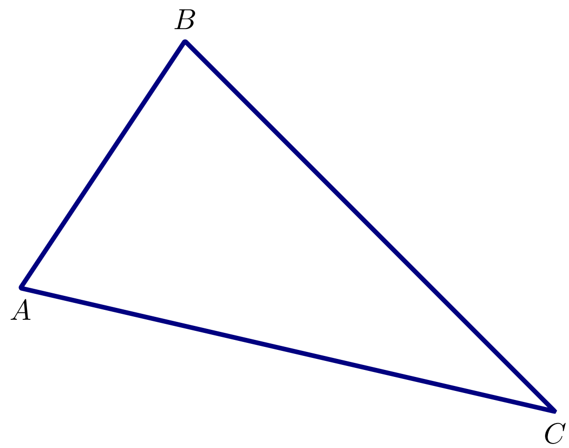
Remember: The parallel postulate is not part of neutral geometry. In fact, that is what make it neutral, although I'd argue a *truly* neutral geometry would also apply to spherical geometry.

Problem 7 *Show that the sum of the angles in $\triangle ACE$ is the same as the sum of the angles in $\triangle ACB$*

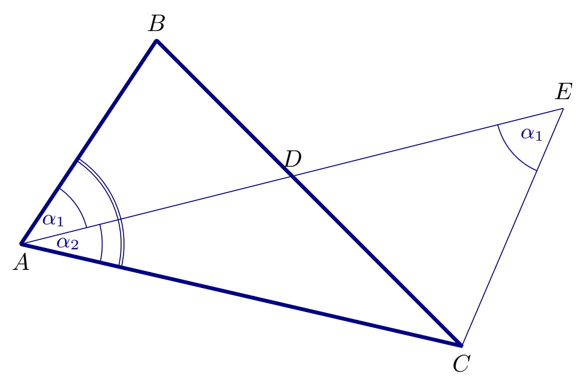


Hint: *Use the previous problems.*

Problem 8 *Suppose that there is a triangle $\triangle ABC$ in neutral geometry for which the sum of the angles in a triangle $\triangle ABC$ is $\Delta°$. Construct a new triangle, $\triangle ACE$ such that the sum of the angles in a triangle is still $\Delta°$, but one of the angles of $\triangle ACE$ is no more than half the size of $\angle CAB$.*



Hint: *Consider the following additions to our diagram above:*



Be sure to carefully explain how points D and E are constructed.

Problem 9 *Suppose that there is a triangle $\triangle ABC$ in neutral geometry for which the sum of the angles in a triangle $\triangle ABC$ is $(180+x)^\circ$ with $x>0$. Show that there is a triangle with sum of its angles still equal to $(180+x)^\circ$ but with one of its angles having measure less than x. [MJG,125-127]*

Hint: *Use the previous problem repeatedly to construct a triangle where the sum of the interior angles is $(180+x)^\circ$ but one of this triangles measures is less than*

$$\frac{\angle CAB}{2^n},$$

for any value of n.

Hint: *Next, use the fact that for any positive real number M,*

$$\lim_{n\to\infty}\frac{M}{2^n}=0.$$

On the other hand, by our previous work you cannot have a triangle with two angles summing to more than $180°$. Hence, from our work above, we can prove the following theorem:

Theorem 3. *In neutral geometry, the sum of the interior angles in any triangle is no greater than* $180°$.

Proof Seeking a contradiction, suppose there exists a triangle $\triangle ABC$ where the sum of the interior angles is

$$(180 + x)^\circ$$

with $x > 0$. We have seen that we can construct triangle $\triangle A'B'C'$ such that the sum of the interior angles is $(180 + x)^\circ$, and if α', β', γ' are the measures of the angles at A', B', and C' respectively, we can ensure

$$\alpha' < x.$$

We now have two relations:

$$\begin{aligned} \alpha' + \beta' + \gamma' &= 180 + x, \\ \alpha' &< x. \end{aligned}$$

So

$$\begin{aligned} \beta' + \gamma' &= 180 + (x - \alpha') \\ &> 180. \end{aligned}$$

As we have seen, this is impossible as two interior angles of a triangle cannot sum to be greater than $180°$. ■

Problem 10 *Considering the proof above, give a sketch of the proof. This means you should state major steps, but exclude the details.*

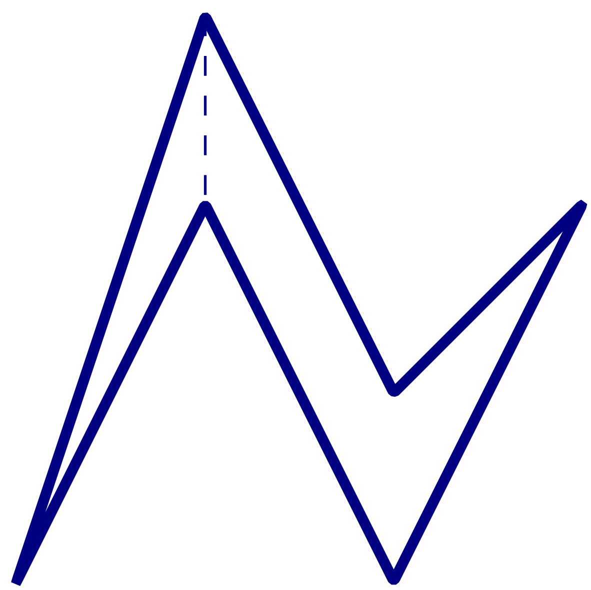
Problem 11 *Show the following:*

(a) *The sum of the interior angles in any quadrilateral is no greater than* $360°$.

(b) *The sum of the interior angles of an* n*-gon is no greater than* $(n-2)\cdot 180°$.

Hint: *Given any quadrilateral, we can break it into two triangles.*

Hint: *You may use the* ***two ears theorem****: For any polygon, even a non-convex one, there are always two vertices where you can "clip off an ear", as in the illustration.*



This fact wasn't written down and proved until 1899!

Problem 12 *Let $ABCD$ be a quadrilateral with $\angle ABC = \angle BCD$ right angles. (We denote polygons by naming their vertices in counterclockwise order.) Show in neutral geometry that:*

(a)
- $|AB| = |CD|$ *implies that* $\angle BAD = \angle ADC$,
- $|AB| > |CD|$ *implies that* $\angle BAD < \angle ADC$,
- $|AB| < |CD|$ *implies that* $\angle BAD > \angle ADC$.

(b) *Use the previous part and pure logic to show that*
- $\angle BAD < \angle ADC$ *implies that* $|AB| > |CD|$,
- $\angle BAD = \angle ADC$ *implies that* $|AB| = |CD|$,
- $\angle BAD > \angle ADC$ *implies that* $|AB| < |CD|$.

Hint: *For the first implication in a) show that quadrilateral $ABCD$ is (self-) congruent to the quadrilateral $DCBA$. For the second implication in the first part suppose that $|AB| > |CD|$. Construct A' on $\overline{AB}$ so that $|A'B| = |CD|$. By a previous problem*

$$\angle BAD < \angle BA'D.$$

By the (already proved) first implication

$$\angle BA'D = \angle CDA'.$$

Finally

$$\angle A'DC < \angle ADC$$

since the segment DA' lies between the segment DA and the segment DC. Stringing these last three relations together, we get the second implication.

The proof of the third implication in is the same as the proof of the second implication– just interchange A and D and interchange B and C.

For the second part, write out the contrapositive of each part and then prove that.

Problem 13 *Explain why a) in the last Problem does not prove the existence of rectangles in Neutral Geometry.*

Problem 14 *Summarize the results from this chapter. In particular, indicate which results follow from the others.*

3 Rectangles and Cartesian coordinates

In this activity, we explore some consequences of Euclid's fifth axiom.

Euclid's fifth axiom, the parallel postulate

After answering the following questions, students should be able to:

- Define Euclidean geometry,
- Prove that in Euclidean geometry, the sum of the interior angles of a triangle is $180°$.
- Show that Euclidean geometry has a Cartesian coordinate system.

We saw in the last chapter that neutral geometry does not assume that two parallel lines must intersect a third line in such a way that the sum of the interior angles (ie, the two angles between the parallel lines) is the same as the sum of two right angles (ie, $180°$). A (very not obviously) equivalent statement is:

Axiom 5 (E5)**.** *Through a point not on a line there passes a unique parallel line.*

Neutral geometry together with E5 is called *Euclidean* or *flat geometry* (Euclidean geometry). Euclidean geometry is probably what you think about when you think about geometry. We will see later that there is another geometry called *hyperbolic geometry* (**HG**) that satisfies all the postulates of neutral geometry but not E5. In it, the sum of the interior angles of a triangle will *always* be less than $180°$!

In Problem 16, we will show that in Euclidean geometry, two parallel lines must intersect a third line in such a way that the sum of the interior angles is $180°$.

Problem 15 *Show that if two parallel lines are cut by a transversal line, then alternate interior angles are equal.*

Hint: *Draw a picture and seek a contradiction.*

Hint: *Watch the solution at* `https://youtu.be/tMBADmI6LnY` *or read the file at* `https://osu.instructure.com/courses/84670/files/23572065/`.

Your solution should make sense to someone who has not watched the video or read the notes file. In particular, you should explain how to draw the diagram and put the proof in your own words.

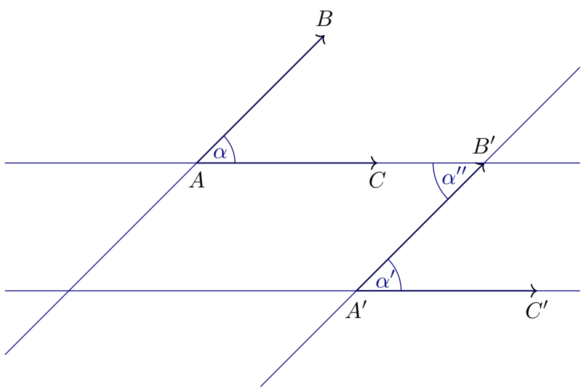
Problem 16 *Show that if two parallel lines are cut by a transverse line, then the sum of the interior angles is 180°.*

Hint: *Watch hint video at* `https://youtu.be/7ttySwGJpZg` *or read the notes at* `https://osu.instructure.com/courses/84670/files/23572065/`.

Problem 17 *Show that angles $\angle BAC$ and $\angle B'A'C'$ in the Euclidean plane are equal (have the same measure) if corresponding rays are parallel. Hence if $\angle B'A'C'$ can be rotated around A' to obtain an angle $\angle B''A'C''$ with corresponding rays parallel to those of $\angle BAC$, then $\angle BAC$ and $\angle B'A'C'$ are equal.*

Hint: *Draw your angles and extend their legs so that you have two sets of parallel sides.*

Proof *Start with the given angles and extend their legs so that we now have two sets of parallel lines:*



By our previous work with alternate interior angles,

$$\alpha = \alpha'' \qquad \text{and} \qquad \alpha'' = \alpha',$$

hence $\alpha = \alpha'$. ■

Problem 18 *Use the 'uniqueness' assertion in E5 together with what we have established about neutral geometry to show that in Euclidean geometry the sum of the interior angles of any triangle is* 180°.

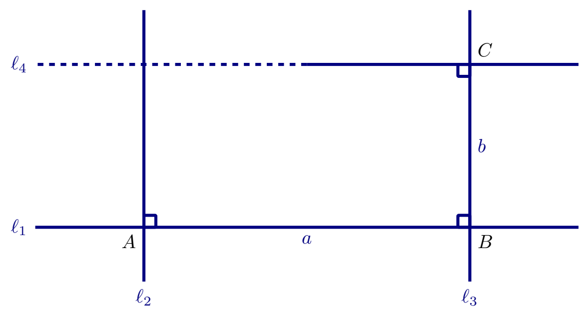
Hint: *Make two parallel lines, the first being the base of a given triangle and the second being the unique parallel line that passes through the vertex opposite to the base.*

Hint: *Watch hint video at* `https://youtu.be/7ttySwGJpZg` *or read the notes at* `https://osu.instructure.com/courses/84670/files/23572065/`.

Problem 19 *Show that in Euclidean geometry the sum of the interior angles of a quadrilateral is* 360°.

Problem 20 *Now we will show in Euclidean geometry that given any positive real numbers a and b, there exist rectangles with adjacent side of lengths a and b. Your task is to fill-in the details of the proof below. The notes from when we talked about this problem in class are at* `https://osu.instructure.com/files/23572263/` *or you can rewatch the class recording* `https://osu.zoom.us/rec/share/UH_Zr42lQKe2VYVa9t_AVoiPiKuvXj5rcA5M9h9y30-7oMjUSiSoxnFCwXIo8vs.U26znUPozxyOrGaT` *(Access Passcode: Gq7%m.jc)*

Start by constructing lines ℓ_1 and ℓ_2 with ℓ_1 perpendicular to ℓ_2 at point A. On ℓ_1 add point B so that $|AB| = a$. Next construct ℓ_3 perpendicular to ℓ_1 through B. On ℓ_3 add point C such that $|BC| = b$. Finally add ℓ_4 through C so that ℓ_4 is perpendicular to ℓ_3.



Explain why ℓ_3 is parallel to ℓ_2.

Explain why ℓ_4 intersects ℓ_2.

Call the intersection of ℓ_2 and ℓ_4 point D and label the two remaining sides a' and b'. Explain why $\angle ADC = 90°$.

Explain why ℓ_1 is parallel to ℓ_4.

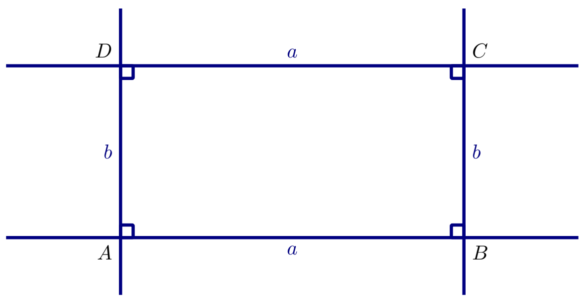
Finally, add a segment to our figure and use a triangle congruence theorem to explain why $a = a'$ and $b = b'$.

Problem 21 *Consider the ordered pair $(x, y) \in \mathbb{R}^2$ to be the corner of rectangle with one vertex at the origin, side lengths $|x|$ and $|y|$, and the sign of x and y moving in either the positive or negative direction along the x and y axes. This is* a Cartesian coordinate system.

Show that there is a Cartesian coordinate system on Euclidean geometry. This means you must show that there is a bijection between the points in Euclidean geometry and elements of $\mathbb{R}^2$, the set of pairs of real numbers.

Hint: *For a longer explanation of what the question is asking and where the map comes from, watch the video at* `https://youtu.be/ffal4JkqrEw`.

Proof *Pick any point of Euclidean geometry, call it A. Our map will take this point to $(0, 0)$. By our previous problem, we can construct an $a \times b$ rectangle $ABCD$ in Euclidean geometry.*



Our map will take the the point C to (a, b). On the other hand, if B was placed to the left of A, our map takes C to the point $(-a, b)$. If D was placed below A, then our map takes C to $(a, -b)$. Finally if B was placed to the left of A and D was placed below A, then our map takes C to $(-a, -b)$. By construction, this map is one-to-one and onto and hence shows that we have a Cartesian coordinate system in Euclidean geometry. ■

The distance formula in Euclidean geometry

After answering the following questions, students should be able to:

- Use the axioms of Euclidean geometry to prove the Pythagorean theorem,
- Prove the Euclidean distance formula.

It is the existence of a Cartesian coordinate system in Euclidean geometry that allows us to define distance between points

$$d\left((a_1, b_1), (a_2, b_2)\right) = \sqrt{(a_2 - a_1)^2 + (b_2 - b_1)^2}$$

and so gives rigorous mathematical meaning to a concept that the ancient Greeks were never able to describe precisely, namely the similarity of figures in Euclidean geometry. For that we will require the notion of a *dilation* or *magnification* in Euclidean geometry. We need a Cartesian coordinate system to describe dilation precisely, a reality backed up by the fact that similarities do not exist in hyperbolic or spherical. (Try drawing two triangles that are similar but not congruent on a perfectly spherical balloon!)

Problem 22 *State and prove the Pythagorean theorem in Euclidean geometry. Do not use the distance formula.*

Hint: *In the Cartesian plane, construct a square with vertices*

$$(0,0)\,, \qquad (a+b,0)\,, \qquad (0,a+b)\,, \qquad (a+b,a+b)\,.$$

Inside that square, construct the square with vertices

$$(a,0)\,, \qquad (a+b,a)\,, \qquad (b,a+b)\,, \qquad (0,b)\,.$$

Problem 23 *Use the Pythagorean theorem to justify the Euclidean distance formula,*

$$d\left((a_1, b_1), (a_2, b_2)\right) = \sqrt{(a_2 - a_1)^2 + (b_2 - b_1)^2}.$$

Problem 24 *Summarize the results from this section. In particular, indicate which results follow from the others.*

4 Dilations and similarity in Euclidean geometry

Now we explore some properties that are somewhat unique to Euclidean geometry.

Dilations in Euclidean Geometry

After answering the following questions, students should be able to:

- Show how dilations affect vectors.
- Show how dilations affect lines.
- Show how dilations affect angles.

We start this section off by asking you to prove a general result.

Problem 25 *Let X be a set and $f : X \to X$. Prove that a function f^{-1} exists if and only if f is one-to-one and onto.*

Hint: *Remember the definition of a function means that f^{-1} is a function if and only if for every input x in the domain X, there exists a unique output y in the range (in this case, also X).*

Definition 6. *A* transformation *or* mapping *is a function f that takes a set X to itself. That is $f : X \to X$. Mathematicians often use function, map, and transformation interchangeably.*

Definition 7. *A **dilation** is a transformation of the Cartesian plane to itself that:*

(a) *Is one-to-one and onto.*

(b) *Fixes one point called the **center** of the dilation.*

(c) *Takes each line through the center of the dilation to itself.*

(d) *Multiplies all distances by a fixed positive real number called the **magnification factor** of the dilation.*

Dilations are like inflation and deflation or zooming in or out on an image. Inflation or zooming in does not introduce gaps (onto) or folds (one-to-one). One point stays the same when zooming in and lines through the center are stretched.

Definition 8. *Given a point (x_0, y_0) in the plane and a positive real number r, we define a mapping D with center (x_0, y_0) and magnification factor r by the formula*

$$D(x, y) = (x_0, y_0) + r(x - x_0, y - y_0).$$

We will also denote the output $D(x, y)$ of the dilation as $(\underline{x}, \underline{y})$.

Problem 26 *Using Cartesian coordinates for the plane, show that the mapping D defined above is a dilation with magnification factor r and center (x_0, y_0).*

Hint: Use the parametric formula for a line:

$$\ell(t) = \text{point} + t \cdot \text{vector}$$

Hint: See video solution at `https://youtu.be/ZkNg-dp2ijQ` or notes `https://osu.instructure.com/courses/84670/files/23562177/`. The video and notes give a proof that D is one-to-one and onto without using the result of the previous problem.

Proof We need to verify that D satisfies the four parts of the definition of a dilation.

(a) By the previous problem, we can prove that D is one-to-one and onto by finding its inverse. To do this, let $(x, y), (z, w) \in \mathbb{R}^2$ such that

$$(z, w) := D(x, y) = (x_0, y_0) + r(x - x_0, y - y_0). \tag{1}$$

Since $x, y, x_0, y_0, r \in \mathbb{R}$, we know such a (z, w) exists (This is the proof of onto!). Now we solve for (x, y).

$$
\begin{aligned}
(z,w) &= (x_0,y_0) + r\,(x-x_0, y-y_0) && (2)\\
(z-x_0, w-y_0) &= r(x-x_0, y-y_0) && (3)\\
\left(\frac{z-x_0}{r}, \frac{w-y_0}{r}\right) &= (x-x_0, y-y_0) && (4)\\
\left(\frac{z-x_0}{r}+x_0, \frac{w-y_0}{r}+y_0\right) &= (x,y). && (5)
\end{aligned}
$$

Thus, $D^{-1}(z, w) := \left(\frac{z-x_0}{r}+x_0, \frac{w-y_0}{r}+y_0\right)$. Since we found an inverse function, D must be one-to-one and onto.

(b) *We must show that the center of the dilation is fixed, that is we must show that*

$$D(x_0, y_0) = (x_0, y_0).$$

Write

$$
\begin{aligned}
D(x_0,y_0) &= (x_0,y_0) + r(x_0-x_0, y_0-y_0)\\
&= (x_0,y_0) + r(0,0)\\
&= (x_0,y_0).
\end{aligned}
$$

(c) *Next we must show that every line through the fixed point goes to itself. To do this, consider the line*

$$\ell(t) = (x_0, y_0) + t(u, v)$$

that goes through the center of the dilation in the direction of the vector (u, v). Write

$$
\begin{aligned}
D(\ell(t)) &= D(x_0+tu, y_0+tv)\\
&= (x_0,y_0) + r(x_0+tu-x_0, y_0+tv-y_0)\\
&= (x_0,y_0) + r(tu, tv)\\
&= (x_0,y_0) + t(ru, rv).
\end{aligned}
$$

This is the line that passes through the center of the dilation in the direction of (ru, rv), but since r is a positive real number, the vector (u, v) goes in the same direction as the vector (ru, rv). Hence a dilation maps any line that passes through its center to itself.

(d) *Finally, we must show that all distances are multiplied by the magnification factor of the dilation. Consider two points (a_1, b_1) and (a_2, b_2). Write*

$$
\begin{aligned}
D(a_1,b_1) &= (x_0,y_0) + r(a_1-x_0, b_1-y_0)\\
&= (x_0+ra_1-rx_0, y_0+rb_1-ry_0)\\
&= (\underline{a_1}, \underline{b_1}).
\end{aligned}
$$

and

$$\begin{aligned} D(a_2, b_2) &= (x_0, y_0) + r(a_2 - x_0, b_2 - y_0) \\ &= (x_0 + ra_2 - rx_0, y_0 + rb_2 - ry_0) \\ &= (\underline{a_2}, \underline{b_2}). \end{aligned}$$

Now

$$\begin{aligned} d((\underline{a_1}, \underline{b_1}), (\underline{a_2}, \underline{b_2})) &= \sqrt{(\underline{a_2} - \underline{a_1})^2 + (\underline{b_2} - \underline{b_1})^2} \\ &= \sqrt{(x_0 + ra_2 - rx_0 - x_0 - ra_1 + rx_0)^2 + (y_0 + rb_2 - ry_0 - y_0 - rb_1 + ry_0)^2} \\ &= \sqrt{(ra_2 - ra_1)^2 + (rb_2 - rb_1)^2} \\ &= \sqrt{r^2(a_2 - a_1)^2 + r^2(b_2 - b_1)^2} \\ &= r\sqrt{(a_2 - a_1)^2 + (b_2 - b_1)^2} \\ &= r \cdot d((a_1, b_1), (a_2, b_2)). \end{aligned}$$

Hence we see that the distance between any two points is multiplied by the magnification factor.

■

Problem 27 *Show that the inverse mapping of a dilation is again a dilation with the same center but with magnification factor r^{-1}.*

Hint: *Solve for (x, y) in terms of $(\underline{x}, \underline{y})$. Look for where this is done in the previous problem.*

Problem 28 *Show that a dilation by a factor of r takes any vector to r times itself.*

Hint: *Write a vector $\vec{u}$ as (a,b), where the head of $\vec{u}$ is on (a,b) when the tail is on $(0,0)$. That is, $\vec{u}$ has slope $\frac{b}{a}$ and length $\sqrt{a^2+b^2}$.*

Look for where this was done in a previous problem.

Problem 29 *Show that a dilation takes a line to a line parallel (or equal) to itself.*

Hint: *Use the parametric formula for a line. For a longer review of parameterizing lines using vectors, see* `https://youtu.be/ZwgeCXuIROA` *and* `https://youtu.be/ZkNg-dp2ijQ?t=532` *at 8:52.*

Problem 30 *Show that a dilation of the plane preserves angles.*

Problem 31 *Show using several-variable calculus that a dilation with magnification factor r multiplies all areas by a factor of r^2.*

Hint: *Recall that if R is a region in the plane,*

$$\iint_{D(R)} f(x,y)\,dxdy = \iint_R f(D(x,y))\,|\det J_D(x,y)|\;dxdy,$$

where

$$J_D(x,y) = \begin{bmatrix} \dfrac{\partial D_x}{\partial x} & \dfrac{\partial D_y}{\partial x} \\ \dfrac{\partial D_x}{\partial y} & \dfrac{\partial D_y}{\partial y} \end{bmatrix}$$

and D_x and D_y are the components of $D(x,y)$.

Proof *Let R be the region in the plane. Write*

$$\int_{D(R)} 1\,dxdy = \int_R |\det J_D(x,y)|\;dxdy,$$

where

$$J_D(x,y) = \begin{bmatrix} \dfrac{\partial D_x}{\partial x} & \dfrac{\partial D_x}{\partial y} \\ \dfrac{\partial D_y}{\partial x} & \dfrac{\partial D_y}{\partial y} \end{bmatrix} = \begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix}.$$

Hence $|\det J_D(x,y)| = r^2$ and so

$$\int_{D(R)} 1\,dxdy = r^2 \cdot \int_R 1\,dxdy.$$

This shows that a dilation with magnification factor r multiplies all areas by a factor of r^2. ■

Problem 32 *Give an explanation that a middle grades student would understand that a dilation with magnification factor r multiplies all areas by a factor of r^2.*

Hint: *Break the region into rectangles. Focus on explaining why this fact is true for rectangles, then why it makes sense to be true in general.*

Similarity in Euclidean Geometry

After answering the following questions, students should be able to:

- Prove two triangles are similar if and only if the corresponding sides are proportionally with the same constant of proportionality.
- Prove two triangles are similar if and only if the corresponding angles are congruent.
- Prove two triangles are similar if the corresponding sides are parallel or perpendicular.

Definition 9. *Two triangles are **similar** if there is a dilation of the plane that takes one to a triangle which is congruent to the other. We write*

$$\triangle ABC \sim \triangle A'B'C'$$

to denote that these two triangles are similar (where the order of the vertices tells us which vertices correspond).

Problem 33

(a) *Show that, if two triangles are similar, then corresponding sides are proportional with the same constant of proportionality.*

(b) *Show that, if corresponding sides of two triangles are proportional with the same constant of proportionality, then the two triangles are similar.*

Hint: *For the first part, you have to start from the hypothesis that the two triangles satisfy our definition of similar triangles.*

Hint: *For the second part, you have to start from the assumption that corresponding sides of the two triangles are proportional and use SSS to show that there is a dilation of $\triangle ABC$ is congruent to $\triangle A'B'C'$.*

Proof *For the first part, suppose that we have two similar triangles. This means that there is a dilation that takes one triangle to a triangle congruent to the other. Since a dilation with magnification factor r takes a vector to r times that vector, and congruence preserves lengths, we have that*

$$\begin{aligned}|AB| &= r \cdot |A'B'|\\ |AC| &= r \cdot |A'C'|\\ |BC| &= r \cdot |B'C'|.\end{aligned}$$

■

Problem 34

(a) *Show that, if two triangles are similar, then corresponding angles are equal.*

(b) *Show that, if corresponding angles of two triangles are equal, then the two triangles are similar.*

Hint: *For the first part, you have to start from the assumption that the two triangles satisfy our definition of similar triangles.*

Hint: *For the second part, you have to start from the assumption that corresponding angles of the two triangles are equal, then use a dilation with $r = |A'B'|/|AB|$ and ASA to show that the dilation of one triangle is congruent to the other.*

Proof *For the second part, suppose that corresponding angles of two triangles are equal. Set the magnification factor of the dilation D to be*

$$r = \frac{|A'B'|}{|AB|}.$$

In this case, D maps $\overline{AB}$ to a segment congruent to $\overline{A'B'}$. Moreover, dilations preserve angles. Hence by ASA, D maps $\triangle ABC$ to a triangle congruent to $\triangle A'B'C'$, and so $\triangle ABC \sim \triangle A'B'C'$. ■

Problem 35 *Show that two triangles are similar if corresponding sides are parallel.*

Hint: *Use the fact that angles are equal if corresponding rays are parallel.*

Problem 36 *Show that two triangles are similar if corresponding sides are perpendicular.*

Hint: *Just think about one angle at a time.*

Proof *It is sufficient to show that given two angles with perpendicular legs, then the angles are equal.*

Extend the legs of the angles until they cross at right angles. There could be several configurations, but all proofs will be similar. Right triangles will be formed containing the angle in question. Using the fact that right triangles are similar if and only if two nonright angles are equal one can show the result for any configuration. ■

Problem 37 *Summarize the results from this section. In particular, indicate which results follow from the others.*

5 Concurrence theorems in Euclidean geometry

Now we explore when three lines meet at a point.

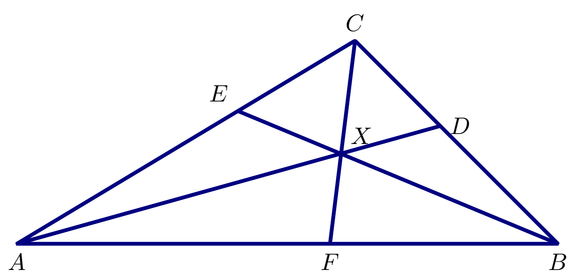
After answering the following questions, students should be able to:

- State and prove facts about the ratio of sides and areas of triangles
- State and prove Ceva's theorem

Let's look at a some *concurrence* theorems. Concurrence theorems deal with situations when three or more lines (or curves) pass through the same point.

Problem 38 *Denote the measure or area of a triangle $\triangle ABC$ as $|\triangle ABC|$. Show that, in the diagram below,*

$$\frac{|AF|}{|FB|} = \frac{|\triangle AFC|}{|\triangle CFB|} = \frac{|\triangle AFX|}{|\triangle XFB|}.$$



Hint: *Mark the height of the relevant triangles.*

Hint: *Video for the first equality is at* `https://youtu.be/nALZ_REZV74` *with notes at* `https://osu.instructure.com/courses/84670/files/23758741`.

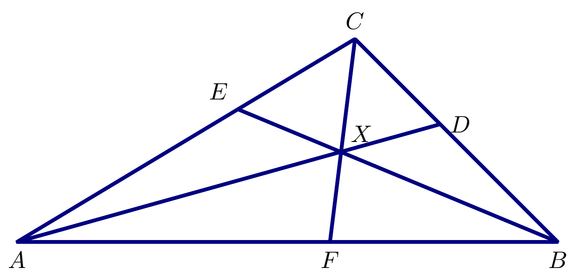
Problem 39 *Use the previous problem to show using only algebra that*

$$\frac{|AF|}{|FB|} = \frac{|\triangle AXC|}{|\triangle CXB|}.$$

Now we will present, and you will prove, *Ceva's Theorem.*

Historical note: The following fact was proved in the 1000s by Yusuf al-Mu'taman ibn-Hūd, the Muslim king of Zaragoza in Spain. But we call it Ceva's theorem. Giovanni Ceva (*CHEH-vah*) was an Italian mathematician who proved this theorem in the 1600s. In fact, it

Theorem 4 (Ceva's Theorem)**.** *Three segments $\overline{AD}$, $\overline{BE}$, and $\overline{CF}$*

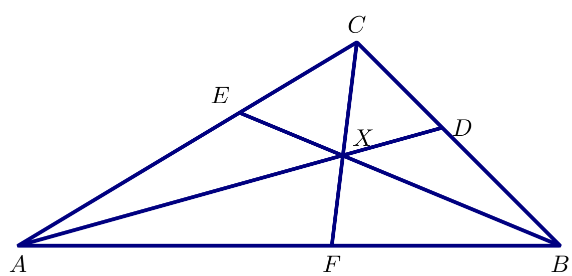


are concurrent if and only if

$$\frac{|AF|}{|FB|} \cdot \frac{|BD|}{|DC|} \cdot \frac{|CE|}{|EA|} = 1.$$

Problem 40 *Prove the Ceva's theorem:*

(a) *For three concurrent segments $\overline{AD}$, $\overline{BE}$ and $\overline{CF}$*



show that

$$\frac{|AF|}{|FB|} \cdot \frac{|BD|}{|DC|} \cdot \frac{|CE|}{|EA|} = 1.$$

Hint: *Use the previous problem repeatedly. Video for finding* $\frac{|BD|}{|CD|}$ *is at* https://youtu.be/nALZ_REZV74?t=249 *with notes at* https://osu.instructure.com/courses/84670/files/23758741*.*

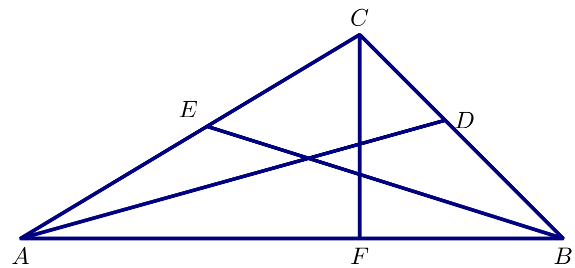
Interactive demonstration is at https://www.geogebra.org/m/s7m8xVDu*.*

(b) *Prove the reverse direction of Ceva's Theorem: If*

$$\frac{|AF|}{|FB|} \cdot \frac{|BD|}{|DC|} \cdot \frac{|CE|}{|EA|} = 1$$

then the lines AD, BE, and CF pass through a common point.

Hint: *Suppose that they do not pass through a common point.*



Hint: *Notice that if, for example, F moves along the segment $\overline{AB}$ from A to B, then $\frac{|AF|}{|FB|}$ is a strictly increasing function of $|AF|$. Now use a previous problem to determine a position F' for F along the segment $\overline{AB}$ at which*

$$\frac{|AF'|}{|F'B|} \cdot \frac{|BD|}{|DC|} \cdot \frac{|CE|}{|EA|} = 1.$$

Hint: *Video showing F moving from A to B is at* `https://youtu.be/nALZ_REZV74?t=496` *with notes at* `https://osu.instructure.com/courses/84670/files/23758741`.

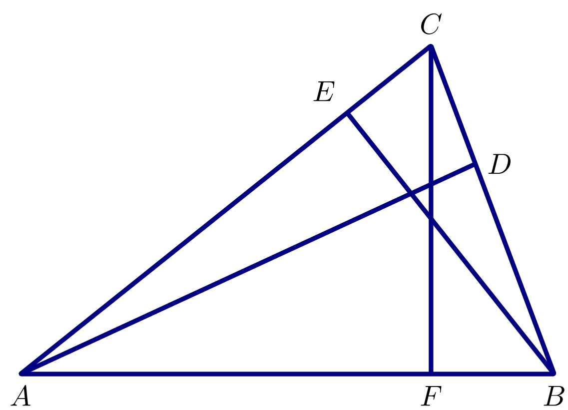
Interactive demonstration is at `https://www.geogebra.org/m/s7m8xVDu`.

Problem 41 *A **median** of a triangle is a line segment from a vertex to the midpoint of the opposite side. Show that the medians of any triangle meet in a common point.*

Hint: *Use Ceva's Theorem.*

Definition 10. *An* ***altitude*** *of a triangle is a line segment originating at a vertex of the triangle that meets the line containing the opposite side at a right angle.*

Problem 42 *Use Ceva's theorem to show that the three lines containing altitudes of a triangle are concurrent.*



Hint: *Use all three similarities of the form $\triangle CEB \sim \triangle CDA$ and then apply Ceva's theorem.*

Problem 43 *Summarize the results from this section. In particular, indicate which results follow from the others.*

6 Central and inscribed angles in Euclidean geometry

In this activity, we study central and inscribed angles in Euclidean geometry.

After answering the following questions, students should be able to:

- Prove that the measure of any angle inscribed in a circle is one-half of the measure of the corresponding central angle.
- State and prove the extended Law of Sines for Euclidean geometry.

A basic fact

Now we prove a basic fact about isosceles triangles.

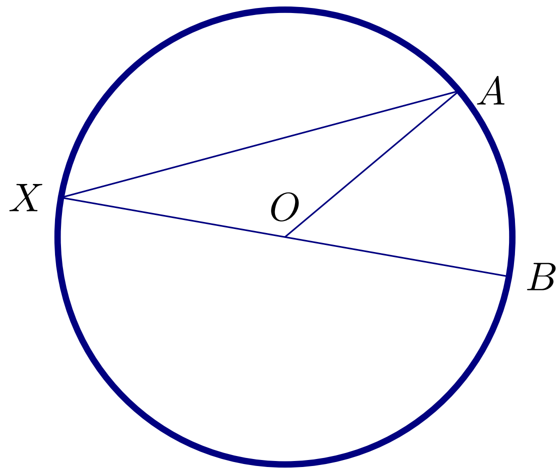
Problem 44 *Prove that given an isosceles triangle, angles opposite the congruent sides are also congruent.*

Hint: *Use a congruence theorem.*

Central and inscribed angles

The next topic in Euclidean geometry concerns central and inscribed angles in circles. We include this partly for its own interest, and partly because the properties we visit here will be useful later on.

Problem 45 *On the circle with center O below,*

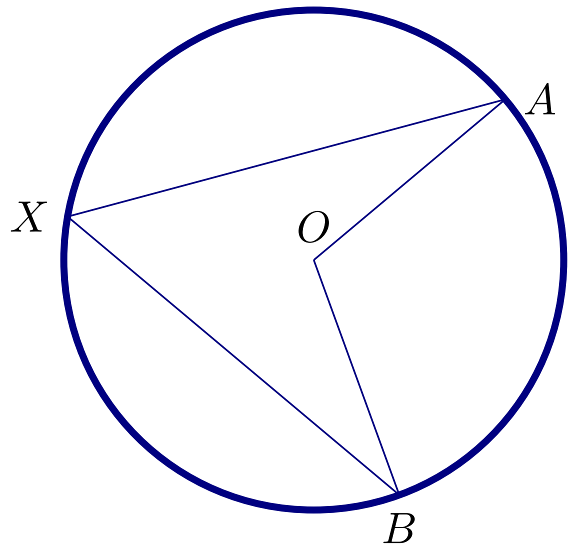


show that

$$\angle BXA = (1/2)(\angle BOA).$$

Hint: $\triangle OAX$ *is isosceles.*

Problem 46 *On the circle with center O below,*

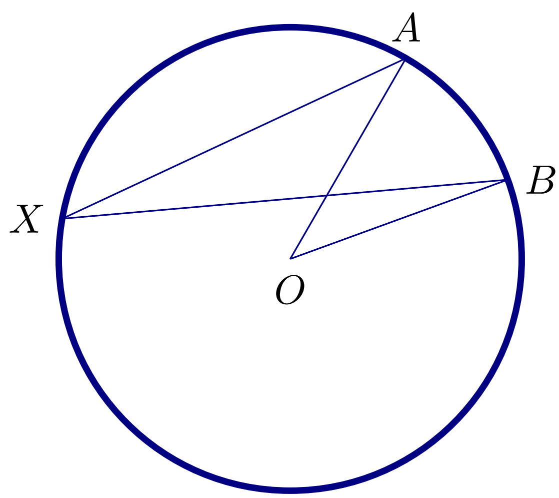


show that

$$\angle BXA = (1/2)(\angle BOA).$$

Hint: *Draw the diameter through O and X and add.*

Problem 47 *On the circle with center O below,*



show that

$$\angle BXA = (1/2)(\angle BOA).$$

Hint: *Draw the diameter through O and X and subtract.*

We can summarize the results of the last three problems into the following theorem.

Theorem 5. *The measure of any angle inscribed in a circle is one-half of the measure of the corresponding central angle.*

Definition 11. *An* inscribed angle *is formed by two chords of a circle, while a* central angle *is formed by two radii.*

The Law of Sines

Recall the (extended) Law of Sines:

Theorem 6 (Law of Sines)**.** *Given a triangle with angles α, β, and γ, with side a opposite α, side b opposite β, and side c opposite γ, we have*

$$\frac{a}{\sin(\alpha)} = \frac{b}{\sin(\beta)} = \frac{c}{\sin(\gamma)} = 2R$$

where R is the radius of a circle circumscribing the triangle.

To prove this theorem, we must check several things.

Definition 12. *The **midset** of two given points is the set of points equidistant from the given points.*

Problem 48 *Show that in Euclidean geometry, the midset of two points is given by the perpendicular bisector of the segment connecting the two given points.*

Hint: *Find the midpoint of the segment connecting the two given points. Draw two congruent triangles, each with one side on the segment connecting the two points and one vertex on their midpoint.*

Problem 49 *Explain why every triangle can be circumscribed by a circle.*

Hint: *Consider the intersection of the perpendicular bisectors of the sides.*

Problem 50 *Prove the Law of Sines.*

Hint: *Start by circumscribing the triangle.*

Hint: *Next, construct two (congruent) right triangles using the center of circle and two vertices of the triangle.*

Hint: *Finally, use the definition of sine.*

Problem 51 *Summarize the results from this section. In particular, indicate which results follow from the others.*

7 The cross-ratio and Ptolemy's Theorem

In this activity, we do some basic projective geometry and learn Ptolemy's Theorem.

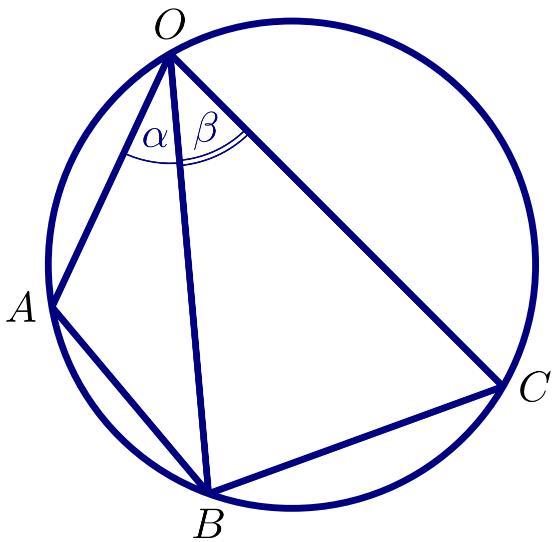
In this section, we will study some basic ideas of *projective geometry*. The main topic, one closely related to the notion of perspective in painting, is called the *cross-ratio*. In particular, we will use the cross-ratio to prove a famous mathematical relationship, *Ptolemy's Theorem*. We will then explore several corollaries of Ptolemy's Theorem.

The cross-ratio

After answering the following questions, students should be able to:

- Prove results about the ratio of the length of sides and the ratio of the sine of the corresponding angle
- Draw diagrams to illustrate the results results
- Define the *cross-ratio of the ordered sequence of four points* on a line and on a circle
- Prove that the two cross-ratios are the same if the four points have a certain relationship.

Problem 52 *In the diagram*



show that

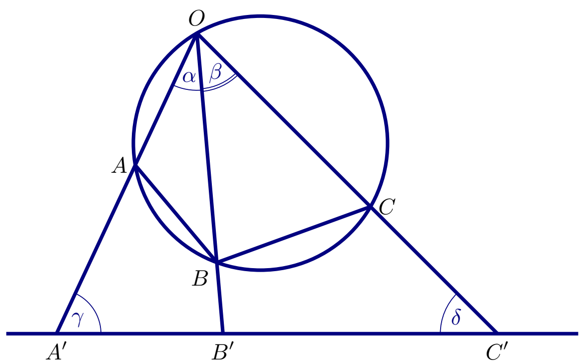
$$\frac{|AB|}{|CB|} = \frac{\sin\alpha}{\sin\beta} = \frac{\sin(\angle AOB)}{\sin(\angle COB)}.$$

Problem 53 *Note that if, in the above figure, B moves along the circle to the other side of C, it is still true that*

$$\frac{|AB|}{|CB|} = \frac{\sin(\angle AOB)}{\sin(\angle COB)}.$$

Carefully draw the diagram for this situation. You do not need to give the proof again.

Problem 54 *In the diagram*



show that

$$\frac{|A'B'|}{|C'B'|} = \frac{\sin\alpha}{\sin\beta}\cdot\frac{\sin\delta}{\sin\gamma} = \frac{\sin(\angle A'OB')}{\sin(\angle C'OB')}\cdot\frac{\sin(\angle B'C'O)}{\sin(\angle B'A'O)}.$$

Problem 55 *Note that if, in the above figure, B moves along the circle to the other side of C, it is still true that*

$$\frac{|A'B'|}{|C'B'|} = \frac{\sin\alpha}{\sin\beta} \cdot \frac{\sin\delta}{\sin\gamma} = \frac{\sin(\angle A'OB')}{\sin(\angle C'OB')} \cdot \frac{\sin(\angle B'C'O)}{\sin(\angle B'A'O)}.$$

Carefully draw the diagram for this situation.

These problems allow us to define the cross-ratio of four points on a circle.

Definition 13. *For a sequence of four (ordered) points A, B, C, and D on a circle, we define*

$$(A : B : C : D) = \frac{|AB|}{|CB|} \cdot \frac{|CD|}{|DA|}$$

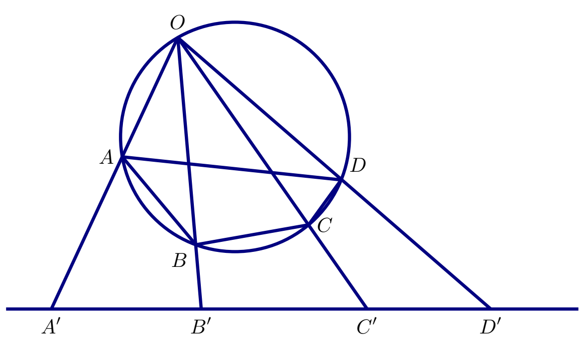
which we call the ***cross-ratio of the ordered sequence of four points on a circle****. Similarly for a sequence of four (ordered) points A', B', C', and D' on a line, we define*

$$(A' : B' : C' : D') = \frac{|A'B'|}{|C'B'|} \cdot \frac{|C'D'|}{|D'A'|}$$

which we call the ***cross-ratio of the ordered sequence of the four points on a line****.*

Based on the following result, we say that "Cross-ratio is invariant under stereographic projection." This statement will have more meaning

Problem 56 *Show that, in the figure*



we have the equality

$$(A : B : C : D) = (A' : B' : C' : D') .$$

Hint: *Use the previous problems.*

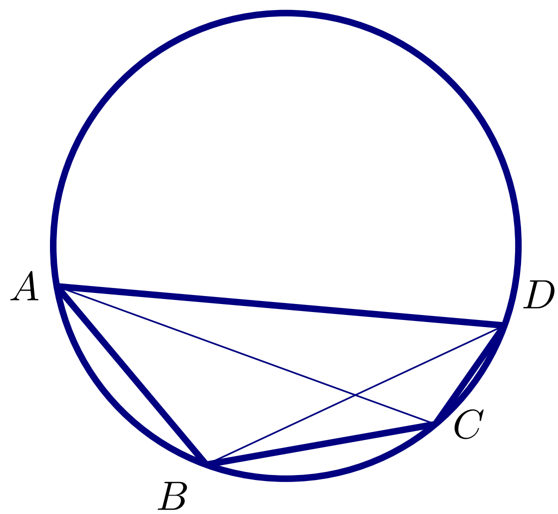
Ptolemy's Theorem

After answering the following questions, students should be able to:

- Finish the proof off Ptolemy's Thoerem
- Give an alternate proof of the Pythagorean Theorem
- Prove the angle addition and subtraction formulas for sine.

You can convince yourself with a few examples that, given four non-collinear points A, B, C and D in the plane, it is not always true that there is a circle that passes through all four. A famous theorem of classical Euclidean geometry gives a condition when there is a circle that passes through all four.

Theorem 7 (Ptolemy)**.** *If the ordered sequence of points A, B, C and D lies on a circle,*



then

$$|AC| \cdot |BD| = |AD| \cdot |BC| + |AB| \cdot |CD| \, .$$

That is, the product of the diagonals of the quadrilateral $ABCD$ is the sum of the products of pairs of opposite sides.

Proof We need to check that

$$|AC| \cdot |BD| = |AB| \cdot |CD| + |BC| \cdot |DA|$$

or, what is the same, we need to check that

$$\frac{|AC| \cdot |BD|}{|DA| \cdot |BC|} = \frac{|AB| \cdot |CD|}{|DA| \cdot |BC|} + 1.$$

That is, we need to check that

$$(A : C : B : D) = (A : B : C : D) + 1.$$

But by a previous problem this is the same as checking that

$$(A' : C' : B' : D') = (A' : B' : C' : D') + 1$$

for the projection of the four points onto a line from a point O on the circle. But that is the same thing as showing that

$$\frac{|A'C'| \cdot |B'D'|}{|D'A'| \cdot |B'C'|} = \frac{|A'B'| \cdot |C'D'|}{|D'A'| \cdot |B'C'|} + 1$$

which is the same thing as showing that

$$|A'C'| \cdot |B'D'| = |D'A'| \cdot |B'C'| + |A'B'| \cdot |C'D'| . \tag{$*$}$$

In order to finish the proof, we need to show ($*$). ■

Problem 57 *Check the equality ($*$) using high-school algebra.*

Hint: *Use that* $|A'C'| = |A'B'| + |B'C|, |B'D'| = |B'C'| + |C'D'|$, *and* $|D'A'| = |A'B'| + |B'C + |C'D'|$.

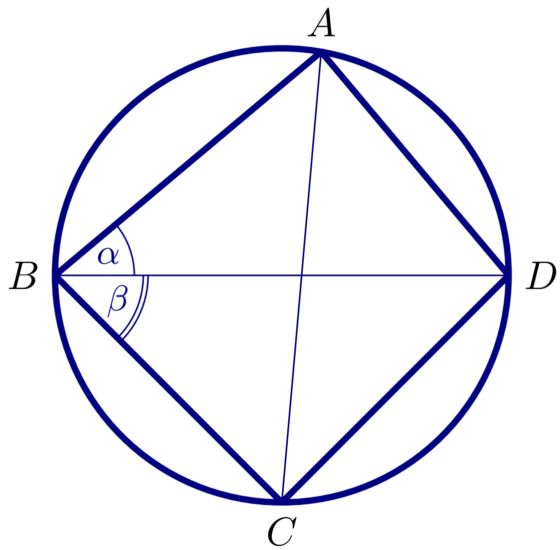
Problem 58 *Use Ptolemy's Theorem to give another proof of the Pythagorean Theorem.*

Hint: *Let the four points in Ptolemy's Theorem form a rectangle.*

Problem 59 *Prove the addition formula for sine:*

$$\sin(\alpha+\beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta)$$

Hint: *Consider this setup,*

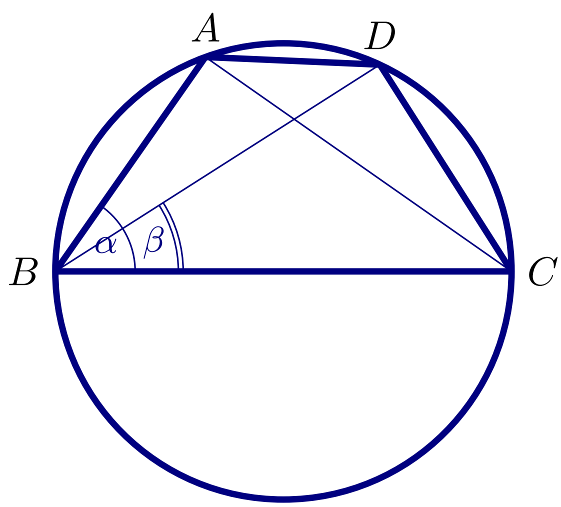


where $\overline{BD}$ is a unit diameter for the circle.

Problem 60 *Prove the subtraction formula for sine:*

$$\sin(\alpha - \beta) = \sin(\alpha)\cos(\beta) - \cos(\alpha)\sin(\beta)$$

Hint: *Consider this setup,*



where $\overline{BC}$ is a unit diameter for the circle.

Both of the above formulas were crucial to Ptolemy. Ptolemy's magnum opus was the *Almagest*, a book that contained information about the stars and their motion, including information on how a potential reader could reproduce the observations and conclusions. A key technical hurdle was the computation of sines and cosines of lots of different angles. The two formulas above were the key to this endeavor.

Problem 61 *Summarize the results from this section. In particular, indicate which results follow from the others.*

8 Surface area and volume of the R-sphere

In this activity we begin to explore spherical geometry.

We are now going to study the geometry of the R-sphere in Euclidean 3-space. This is the sphere of radius R in normal 3-space. We will show why there is a factor of $1/3$ in many formulas for volumes in 3-dimensional Euclidean geometry, just like there is a factor of $1/2$ in many formulas for areas in 2-dimensional Euclidean geometry.

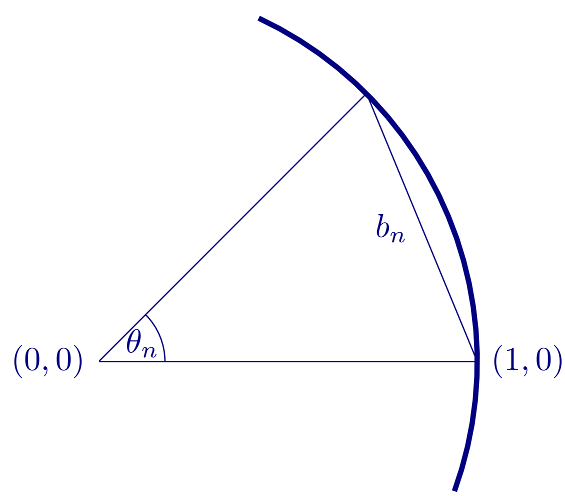
Circumference and area of circles

After answering the following questions, students should be able to:

- Prove that the circumference of a circle with radius 1 is 2π.
- Give two different proofs that the area of a circle with radius 1 is pi.

We begin with perhaps the most basic fact of all about circles in Euclidean geometry. We will use calculus to compute the perimeter of a unit circle.

Problem Set Up Consider a unit circle with n equal inscribed triangles meeting at the center. Here we have shown the nth triangle:



Find θ_n and b_n in terms of n.

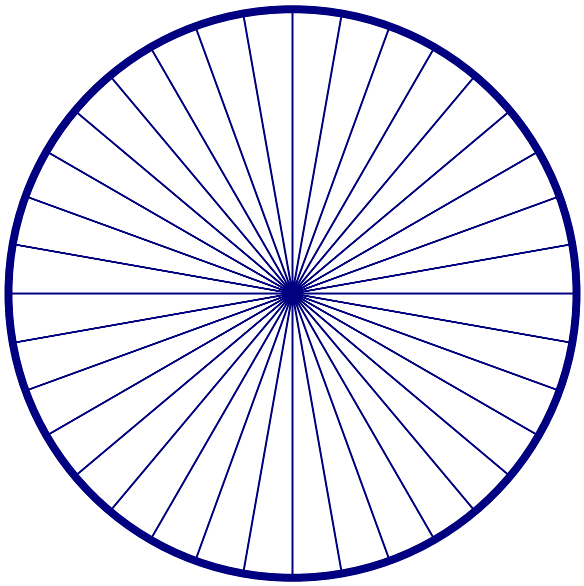
Problem 62 *Write θ_n and b_n in terms of n.*

Explain what the limit

$$\lim_{n\to\infty} (n \cdot b_n)$$

computes and compute the limit.

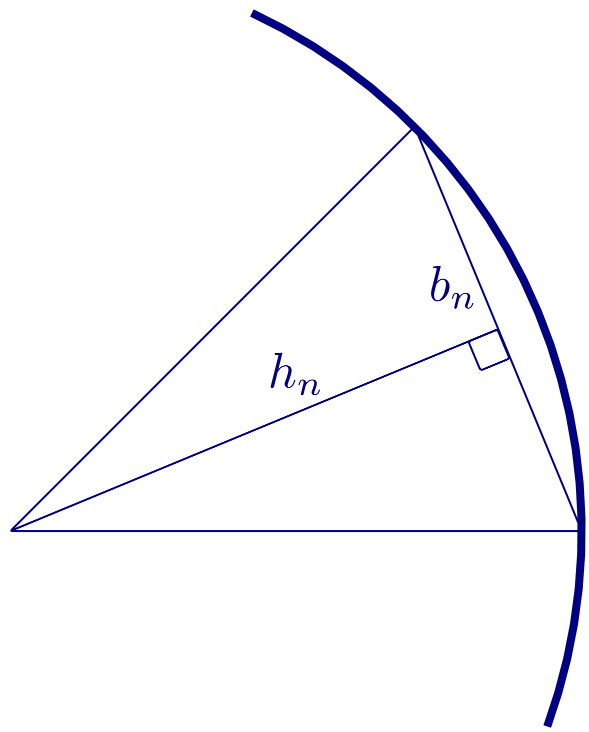
Problem 63 *The circle of radius 1 has circumference 2π. Use this to show that the circle of radius 1 has area π.*



Hint: *Approximate a rectangle by rearranging the slices in the picture. Compute the area of the "rectangle."*

Not content to give just one explanation, let's give another. The techniques we use in this explanation will be used in later problems.

Problem 64 *Again, consider the circle of radius* 1 *with circumference* 2π*. Suppose we inscribe* n *triangles (with one vertex at the center of the circle), here we have shown the* n*th triangle:*



In each triangle, label b_n *and* h_n*, where* b_n *is conceptualized as a "base" and* h_n *is a "height." Find*

$$\lim_{n\to\infty} h_n$$

and explain your reasoning.

Problem 65 *Consider the limit*

$$\lim_{n\to\infty} \frac{n \cdot b_n \cdot h_n}{2}.$$

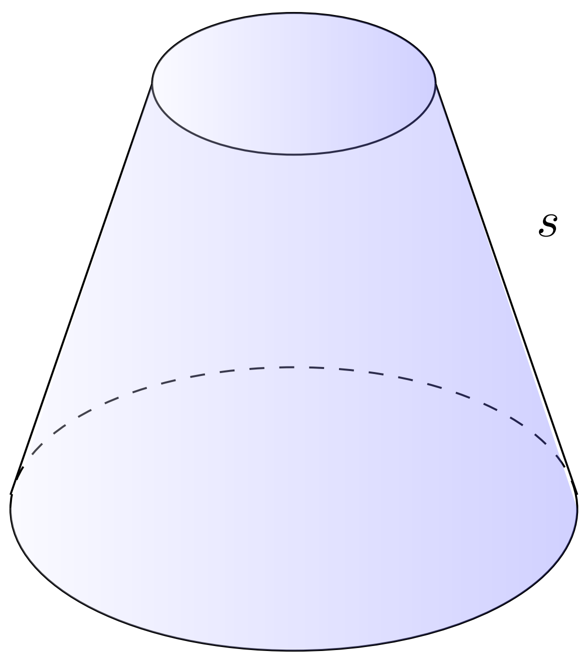
Explain what this limit is computing, and compute this limit.

Surface area of spheres

After answering the following questions, students should be able to:

- Prove the formula for the surface area of a sphere.

To compute the surface area of the sphere of radius R in 3-dimensional Euclidean space, we will show that its surface area is equal to the surface area of something we can lay out flat. The argument for this goes way back to the great physicist and mathematician, Archimedes of Alexandria, in the 2nd century B.C. To follow his argument, we have to begin by computing the area of a 'lamp shade' or 'collar.' We think of a circular collar as in the figure below



as approximated by an arrangement of trapezoids. To achieve this, we approximate the bottom circle of the collar by an inscribed regular n-gon whose vertices are the points of intersection with the slant lines in the figure. Similarly approximate the top circle by an inscribed regular n-gon positioned directly above the bottom one, again with vertices given by the points of intersection with the slant lines. Complete a side of the bottom n-gon and the side of the top n-gon directly above it to a trapezoid by adjoining the two slant lines in the figure that connect endpoints. Let

- b_n denote the length of a side of the bottom regular n-gon and let
- t_n denote the length of a side of the top n-gon.

Then the trapezoid has area

$$\left(\frac{b_n + t_n}{2}\right) \cdot h_n$$

where h_n is the vertical height of the trapezoid. The collar is approximated by the union of these n trapezoids, so the area of the collar is approximated by the sum of the areas of the n congruent trapezoids, namely

$$n \cdot \left(\frac{b_n + t_n}{2}\right) \cdot h_n = \left(\frac{n \cdot b_n + n \cdot t_n}{2}\right) \cdot h_n.$$

As n goes to infinity, the area of the approximation approaches the area of the collar. But if

- c_b is the circumference of the bottom circle and
- c_t is the circumference of the top circle and
- s is the slant height of the collar as shown in the above figure,

then

$$\lim_{n\to\infty} n \cdot b_n = c_b$$
$$\lim_{n\to\infty} n \cdot t_n = c_t$$
$$\lim_{n\to\infty} h_n = s.$$

Problem 66 *Let*

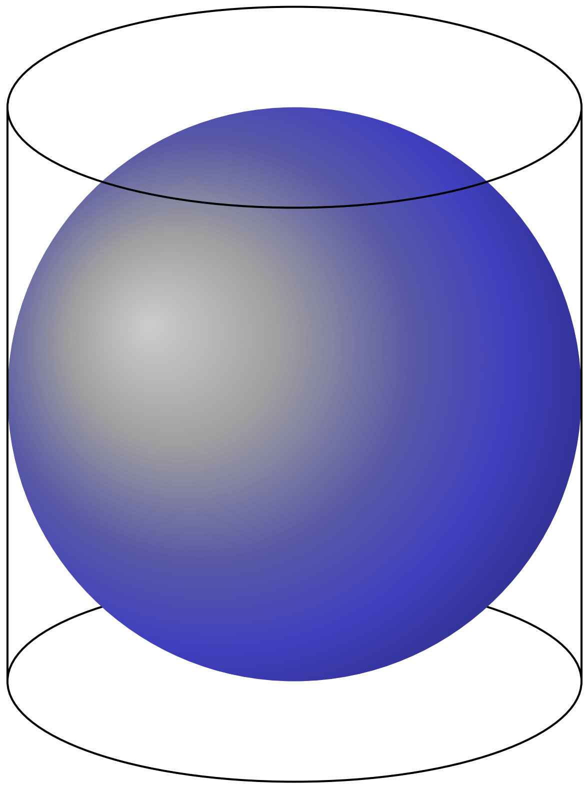
- r_b *be the radius of the circle defining the base of the collar,*
- r_t *be the radius of the circle defining the top of the collar,*
- r_a *be the average of* r_b *and* r_t*,*
- s *be the slant height of the collar.*

Explain why the area of the collar is

$$\pi \cdot (r_b + r_t) \cdot s.$$

(This can also be written as $2\pi \cdot r_a \cdot s$*.)*

Theorem 8. *The surface area of the sphere of radius R is the same as the surface area of the label of the smallest can into which the sphere will fit.*



Namely the surface area of the sphere of radius R is

$$2\pi R \cdot 2R = 4\pi R^2.$$

Problem 67 *Show why the above theorem is true.*

Hint: *Slice the picture above into n horizontal slices. Approximate the piece of the surface of the sphere between the ith pair of successive slices by a collar C_i. Let $a(C_i)$ denote the area of C_i, let r_i denote its average radius and Explain why the surface area of the sphere is*

$$\lim_{n\to\infty} \sum_{i=1}^{n} a(C_i).$$

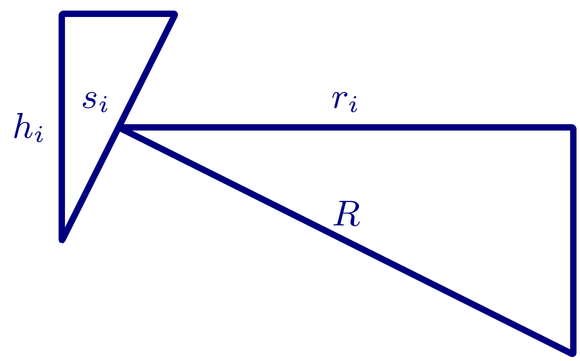
Explain further how we can conclude that the area of the sphere is given by

$$\lim_{n\to\infty} \sum_{i=1}^{n} 2\pi \cdot r_i \cdot s_i.$$

Hint: *Let h_i denote the vertical height of the label on the can between the ith pair of successive slices. Explain why the area of the label is exactly*

$$\sum_{i=1}^{n} 2\pi R \cdot h_i.$$

Hint: *Explain why the relationship between each r_i, s_i and h_i is given by the picture below.*



Now use facts about similar triangles to explain why

$$r_i \cdot s_i = h_i \cdot R.$$

Volumes of pyramids and spheres

After answering the following questions, students should be able to:

- State the magnification principle.
- Prove the formula for the volume of a rectangular base pyramid.

We will start by looking at the volumes of pyramids.

Problem 68 *Show that an $r \times r \times r$ cube can be constructed from three equal pyramids with an $r \times r$ square base. Conclude that the volume of each pyramid is $1/3$ the volume of the cube, namely*

$$\frac{r^3}{3}.$$

Hint: *Suppose the cube had a hollow interior and infinitely thin faces. Put your (infinitely tiny) eye at one vertex of the cube and look inside. How many faces of the cube can you see?*

We next want to show why all pyramids with an $r \times r$ square base and vertical altitude r have the same volume. That is, if we put the vertex of the pyramid anywhere in a plane parallel to the base and at distance r, the volume is unchanged.

This fact is an example of *Cavalieri's Principle*: Shearing a figure parallel to a fixed direction does not change the n-dimensional measure of an object in Euclidean n-space. Think of a stack of (very thin) books. We'll give a proof in Euclidean 3-space, and we will use the coordinates $(\widehat{x}, \widehat{y}, \widehat{z})$ to denote this space.

Problem 69 *Show that Cavalieri's Principle is true for the pyramid using multivariable calculus.*

Hint: *Put the base of the pyramid P so that its vertices are $(0,0)$, $(r,0)$, $(0,r)$ and (r,r) in 3-dimensional Euclidean space. Consider the transformation*

$$T(x,y,z) = \begin{bmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

and use the 3-dimensional change of variables formula,

$$\iiint_{T(P)} f(x,y,z)dxdydz = \iiint_{P} f(T(x,y,z))|\det DT(x,y,z)|dxdydz,$$

where

$$DT(x,y,z) = \begin{bmatrix} \frac{\partial T}{\partial x} & \frac{\partial T}{\partial y} & \frac{\partial T}{\partial z} \end{bmatrix}.$$

(Each partial derivative is itself a vector since $T(x,y,z)$ has three components.)

The magnification principle: If an object in Euclidean n-space is magnified by factors of $r_1, \dots, r_n$, its n-dimensional measure is multiplied by $r_1 \cdots r_n$.

Remark 4. *When $n = 2$ and $r_1 = r_2$, this is a dilation, as in problem 31.*

Theorem 9 (The magnification principle)**.** *We now prove the magnification principle using multivariable calculus.*

Proof In Chapter 4, we used the 2-dimensional change of variables formula. In Cavalieri's Principle, we used the three dimensional change of variables formula. Now, we extend that idea to the n-dimensional change of variables formula:

$$\begin{aligned} &\int\int\cdots\int_{J(R)} f(x_1, x_2, \dots, x_n) dx_1 dx_2 \dots dx_n \\ &= \int\int\cdots\int_{R} f(J(x_1, x_2, \dots, x_n)) |\det J(x_1, x_2, \dots, x_n)| dx_1 dx_2 \dots dx_n. \end{aligned} \tag{6}$$

As we saw in the two dimensional version, when computing area, $f(x, y) = 1$. Here, we are computing the n-dimensional version of volume, so $f(x_1, x_2, \dots, x_n)$.

Now we translate the magnification principle into a function and into a matrix. We want to magnify x_1 by r_1, x_2 by r_2, all the way through the list to multiplying x_n by r_n. As a function, the magnification is

$$J(x_1, x_2, \dots, x_n) = (r_1 x_1, r_2 x_2, \dots, r_n x_n).$$

As a matrix, $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$, and

$$J(\vec{x}) = \begin{bmatrix} r_1 & 0 & \dots & 0 \\ 0 & r_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & r_n \end{bmatrix} \vec{x}.$$

Since $J(\vec{x})$ is a diagonal matrix, the absolute value of the determinant is $|r_1 r_2 r_3 \cdots r_n|$. Now we can plug all of our formulas into the change of variables formula (6):

$$\begin{aligned} \int\int\cdots\int_{J(R)} 1 dx_1 dx_2 \dots dx_n &= \int\int\cdots\int_{R} 1 |r_1 r_2 r_3 \cdots r_n| dx_1 dx_2 \dots dx_n \\ &= |r_1 r_2 r_3 \cdots r_n| \int\int\cdots\int_{R} dx_1 dx_2 \dots dx_n. \end{aligned} \tag{7}$$

■

Problem 70 *Explain how the statement and proof of the magnification principle is related to the statement and proof of Problem 31.*

Problem 71 *Use this magnification principle to justify the volume formula*

$$(1/3)B \cdot h$$

for any pyramid with rectangular base of area B and vertical altitude h.

Hint: *Use a previous problem that found the formula for the volume of a pyramid with square base of area r^2 and vertical altitude r.*

Relation between volume and surface area of a sphere

After answering the following questions, students should be able to:

- Use pyramids to approximate the volume and surface area of a sphere.
- Use pyramids to find a relationship between the volume and surface area of a sphere.

Think of a disco-ball. Its surface is approximately a sphere, but that surface is made up of tiny flat mirrors.

Problem 72

(a) *Explain why you can think of the disco-ball as being made up of pyramids, with each pyramid having base one of the tiny mirrors and vertex at the interior point O at the center of the disco-ball.*

(b) *Argue that the volume of the disco-ball is* $(1/3)$ *times the distance* h *from a mirror to* O *times the sum of the areas of all the mirrors.*

Problem 73 *Argue that, as the mirrors are made to be smaller and smaller,*

(a) *the sum of the areas of the mirrors approaches the surface area of a sphere,*

(b) *the distance h approaches the radius R of that sphere,*

(c) *the volume of the disco-ball approaches the volume of the sphere.*

Conclude that, for a sphere of radius R in Euclidean 3-space, the relation between the volume V of the sphere and the surface area S of the sphere is given by the formula

$$V = \frac{R \cdot S}{3}.$$

Problem 74 *Use the work above to deduce a formula for the volume of a sphere.*

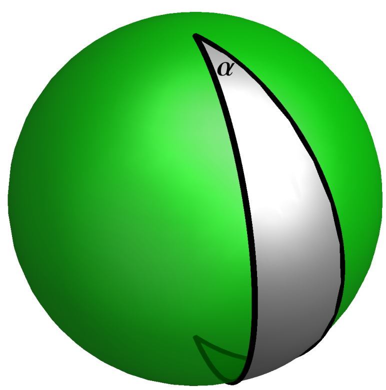
Problem 75 *Summarize the results from this section. In particular, state the conclusion of each problem in your own words and indicate which results follow from the others.*

9 Spherical lunes and triangles

In this activity we explore the areas of lunes and triangles on the sphere. For now we will assume that the equivalent of "lines" on the sphere are great circles*: circles that cut the sphere exactly in half. Later we will see an explanation of why this is the right notion.*

Spherical lunes

In the picture we have shaded in an 'α-lune' on the R-sphere in euclidean 3-space.

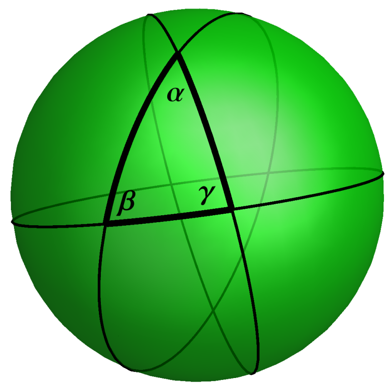


The lune has two vertices. They are at opposite (antipodal) points on the R-sphere, that is, the line in euclidean 3-space that joins the two vertices runs through the center of the sphere. The angle at a vertex of the lune is α radians.

Problem 76 *Explain why the area of the α-lune is $2\alpha \cdot R^2$.*

Spherical triangles

If a triangle on the sphere of radius R has interior angles with radian measures α, β, and γ, it can be covered three times by lunes as shown in the figure below.



Notice that each lune has one vertex at a vertex of the triangle and angle equal to that interior angle of the triangle. The other vertices of each lune are vertices of an 'opposite' triangle that has the same area as the given one since it is just the image of the given one under the rigid motion

$$\begin{bmatrix} -x \\ -y \\ -z \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

The three lunes cover the triangle three times. The three opposite lunes cover the opposite triangle three times. If you take all six lunes together, they cover each of the two triangles three times and everything else exactly once.

Problem 77 *Show that the area of the spherical triangle is given by the formula*

$$R^2\left((\alpha+\beta+\gamma)-\pi\right).$$

Hint: *Use the previous problem.*

Problem 78 *Use induction to derive a formula for the area of any spherical n-gon.*

Hint: *Divide the spherical n-gon into spherical triangles.*

Problem 79 *Summarize the results from this section. In particular, indicate which results follow from the others.*

10 Euclidean three-space as a metric space

In this activity we will work in three dimensional space and see the importance of the dot product to geometry.

Points and vectors in Euclidean 3-space

Up to this point in this book, we have studied two and three-dimensional shapes, the *objects* found in two (and three) dimensional geometry. Now we will study geometries as objects unto themselves. In particular, we will study plane, spherical, and hyperbolic geometry in two-dimensions. What does it mean to study a "whole geometry?" Well, geometry is the study of

- length,
- angle, and
- area.

Hence, if we can explain how to compute lengths, angles, and areas, we understand something about the geometry as a whole. Moreover, the congruences found in geometry are simply the transformations that preserve length, angle, and area: the *rigid motions*. Thus we now seek to understand how to compute length, angle, and area, along with a description of the congruences, for Euclidean, spherical, and hyperbolic geometry. Our plan of attack is as follows: we will visualize each of the geometries as different surfaces lying in a common three-dimensional space. This will allow us to use the techniques from calculus and linear algebra to describe Euclidean, spherical, and hyperbolic geometry in a unified way. We start by introducing the concepts we need in the more familiar setting in ordinary 3-dimensional Euclidean space:

$$\mathbb{R}^3 = \left\{ (\widehat{x}, \widehat{y}, \widehat{z}) : \widehat{x}, \widehat{y}, \widehat{z} \in \mathbb{R} \right\}.$$

We reserve the notation (x, y, z) for some new coordinates that we will put on the 'same' objects later in this course. In Euclidean space, there is a standard way to measure distance between two points

$$\begin{aligned} \widehat{X}_1 &= (\widehat{x}_1, \widehat{y}_1, \widehat{z}_1) \\ \widehat{X}_2 &= (\widehat{x}_2, \widehat{y}_2, \widehat{z}_2), \end{aligned}$$

namely

$$d(\widehat{X}_1, \widehat{X}_2) = \sqrt{(\widehat{x}_2 - \widehat{x}_1)^2 + (\widehat{y}_2 - \widehat{y}_1)^2 + (\widehat{z}_2 - \widehat{z}_1)^2}.$$

When you see two point $\widehat{X}_1$ and $\widehat{X}_2$ in what follows, the 'hats' mean that distance between points is measured by the formula above. One more thing, in Euclidean three-space it will be important throughout to make the distinction between **points** and **vectors**: Although each will be represented by a triple of real numbers we will use

$$\widehat{X} = (\widehat{x}, \widehat{y}, \widehat{z})$$

to denote **points**, that is, **position** in Euclidean 3-space, and

$$\widehat{\mathbf{v}} = (\widehat{a}, \widehat{b}, \widehat{c})$$

to denote **vectors**, that is, **displacement** by which we mean the amount and direction a given point is being moved. So vectors always indicate *motion* from an explicit (or implicit) *point* of reference.

The dot product determines length, angle, and area

There are various operations we can perform on one or more vectors when we think of them as based at the same point in Euclidean 3-space. The first is the dot product of two vectors.

Definition 14. *The dot product of two vectors*

$$\widehat{\mathbf{v}}_1 = (\widehat{a}_1, \widehat{b}_1, \widehat{c}_1),$$
$$\widehat{\mathbf{v}}_2 = (\widehat{a}_2, \widehat{b}_2, \widehat{c}_2),$$

based at the same point in 3-dimensional Euclidean space is defined as the real number given by the formula

$$\widehat{a}_1\widehat{a}_2 + \widehat{b}_1\widehat{b}_2 + \widehat{c}_1\widehat{c}_2$$

or in matrix notation as

$$\begin{bmatrix} \widehat{a}_1 & \widehat{b}_1 & \widehat{c}_1 \end{bmatrix} \begin{bmatrix} \widehat{a}_2 \\ \widehat{b}_2 \\ \widehat{c}_2 \end{bmatrix}.$$

It is also denoted as

$$\widehat{\mathbf{v}}_1 \bullet \widehat{\mathbf{v}}_2$$

or in matrix notation as

$$\widehat{\mathbf{v}}_1 \cdot \widehat{\mathbf{v}}_2^{\mathsf{T}}.$$

Problem 80 *Give the formula for the length $|\widehat{\mathbf{v}}|$ of a vector $\widehat{\mathbf{v}} = (\widehat{a}, \widehat{b}, \widehat{c})$ in 3-dimensional Euclidean space in terms of dot product.*

Problem 81 *Prove that*

$$\widehat{x}^2 + \widehat{y}^2 + \widehat{z}^2 = R^2$$

is a sphere in Euclidean three-space.

Hint: *Remember, a sphere in Euclidean three-space is the set of points equidistant from the origin.*

Lemma 1 (Law of Cosines)**.** *The (smaller) angle θ between two vectors $\widehat{\mathbf{v}}_1$ and $\widehat{\mathbf{v}}_2$ based at $O = (0,0,0)$ satisfies the relation*

$$|\widehat{\mathbf{v}}_2 - \widehat{\mathbf{v}}_1|^2 = |\widehat{\mathbf{v}}_1|^2 + |\widehat{\mathbf{v}}_2|^2 - 2\,|\widehat{\mathbf{v}}_1| \cdot |\widehat{\mathbf{v}}_2| \cdot \cos\theta.$$

Now we will prove this lemma. Your task is to fill in the details of the proof below.

Problem 82 *Start by noting that without loss of generality we can assume that $|\widehat{\mathbf{v}}_1| \leq |\widehat{\mathbf{v}}_2|$. Consider the triangle OP_1P_2, where P_1 and P_2 are the end-points of $\mathbf{v}_1$ and $\mathbf{v}_2$, respectively. Let P be the point on OP_2 so that the segment between P_1 and P is perpendicular to OP_2.*

(a) *Illustrate the diagram described above.*

(b) *Now explain how the Pythagorean theorem gives*

$$\begin{aligned} |P_1P_2|^2 - |P_2P|^2 &= |PP_1|^2 \\ &= |OP_1|^2 - |OP|^2. \end{aligned}$$

(c) *Explain each of the following lines:*

$$\begin{aligned} |P_1P_2|^2 &= |OP_1|^2 + \left(|P_2P|^2 - |OP|^2\right) && (8)\\ &= |OP_1|^2 + (|P_2P| + |OP|)\,(|P_2P| - |OP|) && (9)\\ &= |OP_1|^2 + |OP_2|\,(|P_2P| - |OP|) && (10)\\ &= |OP_1|^2 + |OP_2|\,(|OP_2| - 2|OP|)\,. && (11) \end{aligned}$$

(d) *Now note that*

$$|OP| = |OP_1| \cdot \cos\theta.$$

Explain how this completes the proof.

Theorem 10. *The angle θ between two vectors $\widehat{\mathbf{v}}_1$ and $\widehat{\mathbf{v}}_2$ based at the same point in Euclidean 3-space satisfies the relation*

$$\widehat{\mathbf{v}}_1 \bullet \widehat{\mathbf{v}}_2 = |\widehat{\mathbf{v}}_1| \cdot |\widehat{\mathbf{v}}_2| \cdot \cos\theta. \tag{12}$$

Problem 83 *Prove this theorem.*

Hint: *Use the law of cosines together with algebraic properties of the dot product.*

The significance of the theorem above is that the measure of angles between vectors depends only on the definition of the dot product.

Corollary 1. *The formula for the angle θ between two vectors $\widehat{\mathbf{v}}_1 = (\widehat{a}_1, \widehat{b}_1, \widehat{c}_1)$ and $\widehat{\mathbf{v}}_2 = (\widehat{a}_2, \widehat{b}_2, \widehat{c}_2)$ in 3-dimensional Euclidean space depends only on the dot products of the two vectors with themselves and with each other. Namely,*

$$\theta = \arccos\left(\frac{\widehat{\mathbf{v}}_1 \bullet \widehat{\mathbf{v}}_2}{|\widehat{\mathbf{v}}_1| \cdot |\widehat{\mathbf{v}}_2|}\right).$$

Similarly, the formula for the area of the parallelogram determined by two vectors $\widehat{\mathbf{v}}_1$ and $\widehat{\mathbf{v}}_2$ depends only on the dot products of the two vectors with themselves and with each other.

For a slower look at the next two propositions, see `https://www.khanacademy.org/math/linear-algebra/matrix-transformations/determinant-depth/v/linear-algebra-determinant-and-area-of-a-parallelogram`

Proposition 1. *The area of the parallelogram determined by $\widehat{\mathbf{v}}_1$ and $\widehat{\mathbf{v}}_2$ based at the same point in Euclidean 3-space is given by*

$$|\widehat{\mathbf{v}}_1| \cdot |\widehat{\mathbf{v}}_2| \cdot \sin\theta.$$

Proof Let $|\widehat{\mathbf{v}}_1|$ be the base and h be the height of the parallelogram determined by $\widehat{\mathbf{v}}_1$ and $\widehat{\mathbf{v}}_2$. If θ is the angle between $\widehat{\mathbf{v}}_1$ and $\widehat{\mathbf{v}}_2$ then $|\widehat{\mathbf{v}}_2| \cdot \sin\theta = h$. Therefore, $A = b \cdot h = |\widehat{\mathbf{v}}_1| \cdot |\widehat{\mathbf{v}}_2| \cdot \sin\theta$.

■

Proposition 2. *The area of the parallelogram determined by $\widehat{\mathbf{v}}_1$ and $\widehat{\mathbf{v}}_2$ based at the same point in Euclidean 3-space is also given by*

$$\sqrt{\det\begin{bmatrix}\widehat{\mathbf{v}}_1 \bullet \widehat{\mathbf{v}}_1 & \widehat{\mathbf{v}}_2 \bullet \widehat{\mathbf{v}}_1 \\ \widehat{\mathbf{v}}_1 \bullet \widehat{\mathbf{v}}_2 & \widehat{\mathbf{v}}_2 \bullet \widehat{\mathbf{v}}_2\end{bmatrix}}.$$

Proof By the theorem:

$$\begin{aligned}
A^2 = |\widehat{\mathbf{v}}_1|^2 \cdot |\widehat{\mathbf{v}}_2|^2 \cdot \sin^2\theta &= |\widehat{\mathbf{v}}_1|^2 \cdot |\widehat{\mathbf{v}}_2|^2 \cdot \left(1 - \cos^2\theta\right) \\
&= |\widehat{\mathbf{v}}_1|^2 \cdot |\widehat{\mathbf{v}}_2|^2 - |\widehat{\mathbf{v}}_1|^2 \cdot |\widehat{\mathbf{v}}_2|^2 \cdot \cos^2\theta \\
&= |\widehat{\mathbf{v}}_1|^2 \cdot |\widehat{\mathbf{v}}_2|^2 - |\widehat{\mathbf{v}}_1|^2 \cdot |\widehat{\mathbf{v}}_2|^2 \left(\frac{\widehat{\mathbf{v}}_1 \bullet \widehat{\mathbf{v}}_2}{|\widehat{\mathbf{v}}_1| \cdot |\widehat{\mathbf{v}}_2|}\right)^2 \\
&= |\widehat{\mathbf{v}}_1|^2 \cdot |\widehat{\mathbf{v}}_2|^2 - \left(\widehat{\mathbf{v}}_1 \bullet \widehat{\mathbf{v}}_2\right)^2 \\
&= \left(\widehat{\mathbf{v}}_1 \bullet \widehat{\mathbf{v}}_1\right)\left(\widehat{\mathbf{v}}_2 \bullet \widehat{\mathbf{v}}_2\right) - \left(\widehat{\mathbf{v}}_1 \bullet \widehat{\mathbf{v}}_2\right)^2 \\
&= \det\begin{bmatrix}\widehat{\mathbf{v}}_1 \bullet \widehat{\mathbf{v}}_1 & \widehat{\mathbf{v}}_2 \bullet \widehat{\mathbf{v}}_1 \\ \widehat{\mathbf{v}}_1 \bullet \widehat{\mathbf{v}}_2 & \widehat{\mathbf{v}}_2 \bullet \widehat{\mathbf{v}}_2\end{bmatrix}
\end{aligned}$$

Therefore, the area of the parallelogram is also given by $\sqrt{\det\begin{bmatrix}\widehat{\mathbf{v}}_1 \bullet \widehat{\mathbf{v}}_1 & \widehat{\mathbf{v}}_2 \bullet \widehat{\mathbf{v}}_1 \\ \widehat{\mathbf{v}}_1 \bullet \widehat{\mathbf{v}}_2 & \widehat{\mathbf{v}}_2 \bullet \widehat{\mathbf{v}}_2\end{bmatrix}}$.

Alternatively, given that $A = |\widehat{\mathbf{v}}_1| \cdot |\widehat{\mathbf{v}}_2| \cdot \sin\theta$, we know

$$\theta = \arccos\left(\frac{\widehat{\mathbf{v}}_1 \bullet \widehat{\mathbf{v}}_2}{|\widehat{\mathbf{v}}_1| \cdot |\widehat{\mathbf{v}}_2|}.\right)$$

Then, using a right triangle whose "adjacent" side is $\widehat{\mathbf{v}}_1 \bullet \widehat{\mathbf{v}}_2$ and hypotenuse is $|\widehat{\mathbf{v}}_1| \cdot |\widehat{\mathbf{v}}_2|$,

$$\begin{aligned}
\sin\theta = \sin\left(\arccos\left(\frac{\widehat{\mathbf{v}}_1 \bullet \widehat{\mathbf{v}}_2}{|\widehat{\mathbf{v}}_1| \cdot |\widehat{\mathbf{v}}_2|}\right)\right) &= \frac{\sqrt{\left(|\widehat{\mathbf{v}}_1| \cdot |\widehat{\mathbf{v}}_2|\right)^2 - \left(\widehat{\mathbf{v}}_1 \bullet \widehat{\mathbf{v}}_2\right)^2}}{|\widehat{\mathbf{v}}_1| \cdot |\widehat{\mathbf{v}}_2|} \\
&= \frac{\sqrt{\left(\widehat{\mathbf{v}}_1 \bullet \widehat{\mathbf{v}}_1\right) \cdot \left(\widehat{\mathbf{v}}_2 \bullet \widehat{\mathbf{v}}_2\right) - \left(\widehat{\mathbf{v}}_1 \bullet \widehat{\mathbf{v}}_2\right)^2}}{|\widehat{\mathbf{v}}_1| \cdot |\widehat{\mathbf{v}}_2|} \\
&= \frac{\sqrt{\det\begin{bmatrix}\widehat{\mathbf{v}}_1 \bullet \widehat{\mathbf{v}}_1 & \widehat{\mathbf{v}}_2 \bullet \widehat{\mathbf{v}}_1 \\ \widehat{\mathbf{v}}_1 \bullet \widehat{\mathbf{v}}_2 & \widehat{\mathbf{v}}_2 \bullet \widehat{\mathbf{v}}_2\end{bmatrix}}}{|\widehat{\mathbf{v}}_1| \cdot |\widehat{\mathbf{v}}_2|}
\end{aligned}$$

Therefore, the area of the parallelogram is also given by $\sqrt{\det\begin{bmatrix}\widehat{\mathbf{v}}_1 \bullet \widehat{\mathbf{v}}_1 & \widehat{\mathbf{v}}_2 \bullet \widehat{\mathbf{v}}_1 \\ \widehat{\mathbf{v}}_1 \bullet \widehat{\mathbf{v}}_2 & \widehat{\mathbf{v}}_2 \bullet \widehat{\mathbf{v}}_2\end{bmatrix}}$.

■

Problem 84 *Using the preceding two propositions, show that we have the following equality:*

$$\sqrt{\det\begin{bmatrix}\widehat{\mathbf{v}}_1 \bullet \widehat{\mathbf{v}}_1 & \widehat{\mathbf{v}}_2 \bullet \widehat{\mathbf{v}}_1 \\ \widehat{\mathbf{v}}_1 \bullet \widehat{\mathbf{v}}_2 & \widehat{\mathbf{v}}_2 \bullet \widehat{\mathbf{v}}_2\end{bmatrix}} = \sqrt{\det\left(\begin{bmatrix}\widehat{\mathbf{v}}_1 \\ \widehat{\mathbf{v}}_2\end{bmatrix} \cdot \begin{bmatrix}\widehat{\mathbf{v}}_1^{\mathsf{T}} & \widehat{\mathbf{v}}_2^{\mathsf{T}}\end{bmatrix}\right)}.$$

Again, the significance of the previous problems is that, to compute areas, we only need to know how to compute dot products. Hence, it is the definition of the dot product of the vectors completely determines the calculation of the area of the parallelogram they generate.

Curves in Euclidean 3-space and vectors tangent to them

Definition 15. *A* ***smooth curve in 3-dimensional Euclidean space*** *is given by a differentiable mapping*

$$\begin{aligned} \widehat{\gamma} : [b, e] &\to \mathbb{R}^3 \\ t &\mapsto (\widehat{x}(t), \widehat{y}(t), \widehat{z}(t)) \end{aligned}$$

from an interval $[a, b]$ *on the real line whose tangent vector*

$$\frac{d\widehat{\gamma}}{dt} = \left(\frac{d\widehat{x}}{dt}, \frac{d\widehat{y}}{dt}, \frac{d\widehat{z}}{dt} \right)$$

is not the zero vector for any t *in* $[a, b]$. *We will sometimes use column vector notation if it looks better:*

$$\widehat{\gamma}(t) = \begin{pmatrix} \widehat{x}(t) \\ \widehat{y}(t) \\ \widehat{z}(t). \end{pmatrix}$$

Problem 85

(a) *Give a second example of a smooth curves,*

$$\begin{aligned} \widehat{\gamma}_1(s) &= (s, s, \cos(s)) \\ \widehat{\gamma}_2(t) &= (\widehat{x}_2(t), \widehat{y}_2(t), \widehat{z}_2(t)) \end{aligned}$$

where

- *neither* $\widehat{\gamma}_1$ *nor* $\widehat{\gamma}_2$ *are a straight line,*
- *the two curves* $\widehat{\gamma}_1$ *and* $\widehat{\gamma}_2$ *pass through a common point* ***and*** *go in different tangent directions at that point.*
- *None of the components of* $\widehat{\gamma}_1$ *and* $\widehat{\gamma}_2$ *are constant functions.*

The first applet of `https://www.geogebra.org/m/cvn97uzz` *allows you to graph 3d parametric equations to see if they intersect. Notice that the equations in the second applet are not the same as the ones in this problem, but do illustrate the same concepts.*

(b) *Compute the tangent vectors of each of the two curves at each of their points.*

(c) *For the two curves you defined in a), what are the coordinates of the point in Euclidean 3-space at which the two curves intersect?*

(d) *Use the dot product formula to compute the angle θ between (the tangent vectors to) your two example curves in a) at the point at which the curves intersect.*

There are two kinds of vectors that naturally arise from curves: the displacement between times t_1 and t_2, given by

$$\widehat{\mathbf{v}} = \widehat{\gamma}(t_2) - \widehat{\gamma}(t_1) = (\widehat{x}(t_2) - \widehat{x}(t_1), \widehat{y}(t_2) - \widehat{y}(t_1), \widehat{z}(t_2) - \widehat{z}(t_1)),$$

and the instantaneous velocity of a point moving along a curve, given by

$$\widehat{\mathbf{v}} = \frac{d\widehat{\gamma}(t)}{dt} = \left(\frac{d\widehat{x}(t)}{dt}, \frac{d\widehat{y}(t)}{dt}, \frac{d\widehat{z}(t)}{dt}\right).$$

In matrix notation we can think of the difference between two points

$$\widehat{X}_2 - \widehat{X}_1 = (\widehat{x}_2 - \widehat{x}_1, \widehat{y}_2 - \widehat{y}_1, \widehat{z}_2 - \widehat{z}_1)$$

as a 1×3 matrix $\left[\widehat{X}_2 - \widehat{X}_1\right]$. Then we can write the formula for the distance between $\widehat{X}_1$ and $\widehat{X}_2$ in Euclidean 3-space in terms of the dot-product:

$$d(\widehat{X}_1, \widehat{X}_2) = \sqrt{(\widehat{X}_2 - \widehat{X}_1) \bullet (\widehat{X}_2 - \widehat{X}_1)} \tag{13}$$

or in terms of the matrix product

$$d(\widehat{X}_1, \widehat{X}_2) = \sqrt{\left[\widehat{X}_2 - \widehat{X}_1\right] \cdot \left[\widehat{X}_2 - \widehat{X}_1\right]^{\mathsf{T}}}.$$

Length of a smooth curve in Euclidean 3-space

Problem 86 *Compute the length of the tangent vector*

$$\ell(t) = \sqrt{\frac{d\widehat{\gamma}}{dt} \bullet \frac{d\widehat{\gamma}}{dt}}$$

to each of your two example curves in the previous problem at each of their points.

Definition 16. *The length L of the curve $\widehat{\gamma}(t)$, $t \in [a,b]$, in Euclidean 3-space is obtained by integrating the length of the tangent vector to the curve, that is,*

$$L = \int_a^b \ell(t)\, dt.$$

Notice that the length of any curve only depends on the definition of the dot product. That is, if we know the formula for the dot product, we know (the formula for) the length of any curve.

Our first example is the path

$$(\widehat{x}(t), \widehat{y}(t), \widehat{z}(t)) = (R \cdot \sin(t), 0, R \cdot \cos(t)) \qquad 0 \leq t \leq \pi.$$

Notice that this path lies on the sphere of radius R.

Problem 87 *Write the formula for the tangent vector to the path above. Show that the length of this path is $R\pi$.*

Problem 88 *(Try to) compute the length of each of your two example curves in the previous problem.*

Remark 5. *In this last problem, you may easily be confronted with an integral that you cannot compute. For example, if your curve* $\widehat{\gamma}_1(t)$ *happens to describe an ellipse that is not circular, it was proved in the 19th century that no formula involving only the standard functions from calculus will give you the length of your path from a fixed beginning point to a variable ending point on the ellipse. If that kind of thing occurs, go back and change the definitions of your curves in the previous problem until you get two curves for which you can compute length of your path from a fixed beginning point to a fixed ending point.*

To study plane geometry, spherical geometry, and hyperbolic geometry in a uniform way we will have to *change* the coordinate system we use. In essence, this means that we will have to change the distance formula slightly for each geometry. These new coordinates will be chosen to keep the north and south poles from going to infinity as the radius R of a sphere increases without bound. This change of viewpoint will eventually let us go non-Euclidean or, in the language of Buzz Lightyear "to infinity and beyond." The idea will be like the change from rectangular to polar coordinates for the plane that you encountered in calculus, only easier.

Problem 89 *Summarize the results from this section. In particular, rephrase the results in your own words and indicate which results follow from the others.*

11 Congruences, that is, rigid motions

We explore rigid motions as matrix multiplication.

After answering the following questions, students should be able to:

- Define an orthogonal matrix
- Prove that orthogonal matrices define rigid motions of 3-d Euclidean space
- Prove that orthogonal mappings are bijective mappings of the sphere
- Describe certain orthogonal maps geometrically
- Prove that orthogonal mappings preserve the lengths of curves

Transformations of Euclidean space

Consider the following mapping of Euclidean space to itself:

$$\begin{bmatrix}\underline{\widehat{x}} & \underline{\widehat{y}} & \underline{\widehat{z}}\end{bmatrix}^{\intercal} = \widehat{M}\begin{bmatrix}\widehat{x} & \widehat{y} & \widehat{z}\end{bmatrix}^{\intercal},$$

where $\widehat{M}$ is an invertible 3×3 matrix. Since the matrix is invertible, this mapping is one-to-one and onto.

Notation: In this section, points and vectors will be arranged vertically.

Definition 17. *Such a mapping is called a* ***rigid motion*** *if the distance between any two points in Euclidean space is left unchanged by the mapping, that is, for any two points $\widehat{X}_1$ and $\widehat{X}_2$ in Euclidean space,*

$$d(\widehat{M}\widehat{X}_1, \widehat{M}\widehat{X}_2) = d(\widehat{X}_1, \widehat{X}_2).$$

It turns out that there is a special class of matrices that give rise to rigid motions.

Definition 18. *A matrix $\widehat{M}$ satisfying*

$$\widehat{M}^{\intercal} \cdot \widehat{M} = I = \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}.$$

is called an ***orthogonal*** *matrix.*

Problem 90 *Prove that a matrix $\widehat{M}$ defines a rigid motion (a congruence) via*

$$\begin{bmatrix}\underline{\widehat{x}} & \underline{\widehat{y}} & \underline{\widehat{z}}\end{bmatrix}^{\intercal} = \widehat{M}\begin{bmatrix}\widehat{x} & \widehat{y} & \widehat{z}\end{bmatrix}^{\intercal}$$

if and only if it is orthogonal.

Hint: *Note that the square of the distance between* $\widehat{X}_1$ *and* $\widehat{X}_2$ *is the dot product of the vector*

$$\widehat{\mathbf{v}} = \widehat{X}_2 - \widehat{X}_1$$

with itself. Also recall the identity $(AB)^\intercal = B^\intercal A^\intercal$.

Hint: *If* $\widehat{M}$ *is orthogonal, write*

$$(\widehat{M}\widehat{\mathbf{v}}) \bullet (\widehat{M}\widehat{\mathbf{v}})$$

and deduce that this equals $\widehat{\mathbf{v}} \bullet \widehat{\mathbf{v}}$.

Hint: *Now suppose that* $\widehat{M}$ *defines a rigid motion. Explain why this means that*

$$(\widehat{M}\widehat{\mathbf{v}}) \bullet (\widehat{M}\widehat{\mathbf{v}}) = \widehat{\mathbf{v}} \bullet \widehat{\mathbf{v}}$$

for every $\widehat{\mathbf{v}}$. *Now rewrite as:*

$$\widehat{\mathbf{v}}^\intercal \cdot \widehat{\mathbf{v}} = (\widehat{M}\widehat{\mathbf{v}})^\intercal \cdot (\widehat{M}\widehat{\mathbf{v}}).$$

Write

$$\widehat{\mathbf{v}} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \qquad \text{and} \qquad \widehat{M} = \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{bmatrix}$$

and view the equation

$$\widehat{\mathbf{v}}^\intercal \cdot \widehat{\mathbf{v}} = (\widehat{M}\widehat{\mathbf{v}})^\intercal \bullet (\widehat{M}\widehat{\mathbf{v}})$$

as a polynomial equation in the variables a, b, *and* c.

Hint: *Polynomials are equal if and only if their coefficients are equal.*

Problem 91 *Show that, if $\widehat{M}$ is orthogonal, then the transformation*

$$\begin{bmatrix} \underline{\widehat{x}} & \underline{\widehat{y}} & \underline{\widehat{z}} \end{bmatrix}^{\mathsf{T}} = \widehat{M} \cdot \begin{bmatrix} \widehat{x} & \widehat{y} & \widehat{z} \end{bmatrix}^{\mathsf{T}}.$$

takes the set of points $(\widehat{x}, \widehat{y}, \widehat{z})$ such that

$$\widehat{x}^2 + \widehat{y}^2 + \widehat{z}^2 = R^2$$

to the set of points $\left(\widehat{a}, \widehat{b}, \widehat{c}\right)$ such that

$$\widehat{a}^2 + \widehat{b}^2 + \widehat{c}^2 = R^2.$$

That is, $\widehat{M}$ gives a one-to-one and onto mapping of the R-sphere to itself.

Hint: *Write the equation*

$$\widehat{a}^2 + \widehat{b}^2 + \widehat{c}^2 = R^2$$

as

$$\begin{bmatrix} \widehat{a} \\ \widehat{b} \\ \widehat{c} \end{bmatrix} \bullet \begin{bmatrix} \widehat{a} \\ \widehat{b} \\ \widehat{c} \end{bmatrix} = R^2.$$

Problem 92 *Show that the set of orthogonal matrices $\widehat{M}$ form a group. That is, show that*

(a) *multiplication of orthogonal matrices is associative,*

Hint: *Use the fact that function composition os associatvie.*

(b) *the product of two orthogonal matrices is orthogonal,*

(c) *the identity matrix is orthogonal,*

(d) *the inverse matrix $\widehat{M}^{-1}$ of a orthogonal matrix $\widehat{M}$ is orthogonal.*

Problem 93 *Consider*

$$\widehat{M}_\theta = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(a) *Describe what this mapping does geometrically? ie, Does it translate in a certain direction, scale/dilate by a certain factor, rotate about a certain line, or some other geometric action?*

(b) *Show that the matrix $\widehat{M}_\theta$ is orthogonal.*

Problem 94 *Consider*

$$\widehat{N}_\psi = \begin{bmatrix} \cos\psi & 0 & -\sin\psi \\ 0 & 1 & 0 \\ \sin\psi & 0 & \cos\psi \end{bmatrix}$$

(a) *Describe what this mapping does geometrically? ie, Does it translate in a certain direction, scale/dilate by a certain factor, rotate about a certain line, or some other geometric action?*

(b) *Show that the matrix $\widehat{N}_\psi$ is orthogonal.*

Problem 95 *If a curve*

$$\widehat{\gamma}(t) = (\widehat{x}(t), \widehat{y}(t), \widehat{z}(t)), \qquad b \le t \le e$$

is moved by a transformation given by an orthogonal matrix $\widehat{M}$, show that its length is unchanged.

Hint: *Recall that the length of a curve is given by*

$$\int_b^e \sqrt{\frac{d\widehat{\gamma}}{dt} \bullet \frac{d\widehat{\gamma}}{dt}}\, dt.$$

Problem 96 *Summarize the results from this section. In particular, indicate which results follow from the others.*

12 Changing coordinates

Now we will change coordinates.

After answering the following questions, students should be able to:

- Define K-warped space
- Change from Euclidean coordinates to K-warped space coordinates
- Prove facts about the angle between vectors and the area of a parallelogram in K-warped space

Bringing the North Pole of the R-sphere to $(0, 0, 1)$

We are now ready to change coordinates on Euclidean 3-space so that we can fill up that space with plane geometry and all the spherical and hyperbolic geometries. We have reserved the notation (x, y, z) for these new coordinates that we will put on the 'same' objects we have been studying in Euclidean $(\widehat{x}, \widehat{y}, \widehat{z})$-coordinates. These new coordinates will be chosen to keep the north and south poles from going to infinity as the radius R of a sphere increases without bound. In these now (x, y, z)-coordinates the sphere of radius R will be given by the equation

$$K(x^2 + y^2) + z^2 = 1$$

where $K = 1/R^2$. Notice that the above equation has solutions even when K is negative. It is on those solution sets that hyperbolic geometries will live. So this change of viewpoint will eventually let us go hyperbolic or, in the language of Buzz Lightyear, will let R go 'to infinity and beyond.' The idea will be like the change from rectangular to polar coordinates for the plane that you encountered in calculus, only easier.

We are now ready to introduce this slightly different set of coordinates for $\mathbb{R}^3$, three-dimensional Euclidean space. To understand a bit better why we are doing this, suppose we are standing at the North Pole

$$N = (0, 0, R)$$

of the sphere

$$\widehat{x}^2 + \widehat{y}^2 + \widehat{z}^2 = R^2$$

of radius R. As R increases, but we stay our same size, the sphere around us becomes more and more like a flat, plane surface. However it can never get completely flat because we are zooming out the positive $\widehat{z}$-axis and we would have to be 'at infinity' for our surface to become exactly flat. We remedy that unfortunate situation by considering another copy of $\mathbb{R}^3$, that we will call ***K*-warped space**, whose coordinates we denote as (x, y, z). We make the following rule in order to pass between the two $\mathbb{R}^3$'s:

$$\begin{aligned}\widehat{x} &= x\\ \widehat{y} &= y\\ \widehat{z} &= Rz.\end{aligned}$$

We think of the (x, y, z)-coordinates as simply being a different set of addresses for the points in Euclidean space. For example,

$$(x, y, z) = (0, 0, 1)$$

tells me that the point in Euclidean space that I'm referring to is

$$(\widehat{x}, \widehat{y}, \widehat{z}) = (0, 0, R) = N.$$

Continuing with this "change of addresses" the sphere of radius R in Euclidean space is given by

$$\begin{aligned}R^2 &= \widehat{x}^2 + \widehat{y}^2 + \widehat{z}^2\\ &= x^2 + y^2 + R^2 z^2\end{aligned}$$

that is, by the equation

$$1 = \frac{1}{R^2}\left(x^2 + y^2\right) + z^2.$$

Definition 19. *For the surface defined by*

$$1 = \frac{1}{R^2}\left(x^2 + y^2\right) + z^2.$$

The quantity $K = \dfrac{1}{R^2}$ *is called the* ***curvature*** *of the R-sphere.*

Problem 97 *What happens to the surface when K goes to 0? How does this relate to the colloquial sense of "curvature"?*

Problem 98

(a) *Sketch the solution set in (x, y, z)-coordinates representing the sphere*

$$R^2 = \widehat{x}^2 + \widehat{y}^2 + \widehat{z}^2 = 1.$$

(b) *Sketch the solution set in (x, y, z)-coordinates representing the sphere*

$$R^2 = \widehat{x}^2 + \widehat{y}^2 + \widehat{z}^2 = 10^2.$$

(c) *Sketch the solution set in (x, y, z)-coordinates representing the sphere*

$$R^2 = \widehat{x}^2 + \widehat{y}^2 + \widehat{z}^2 = 10^{-2}.$$

Formulas for Euclidean lengths, angles, and areas in terms of (x, y, z)-coordinates

To prepare ourselves to do hyperbolic geometry, which (in some sense) has no satisfactory model in Euclidean space, we will 'practice' by doing spherical geometry (which *does* have a completely satisfactory model in Euclidean space) using these 'slightly strange' (x, y, z)-coordinates. Gradually throughout this course we will discover that the same rules that govern spherical geometry, expressed in (x, y, z)-coordinates, also govern flat and hyperbolic geometry! In all three cases, the surface in (x, y, z)-coordinates that we will study is

$$1 = K\left(x^2 + y^2\right) + z^2.$$

If $K > 0$, the geometry we will be studying is the geometry of the the Euclidean sphere of radius

$$R = \frac{1}{\sqrt{K}}.$$

If $K = 0$ we will be studying flat (plane) geometry. If $K < 0$, we will be studying hyperbolic geometry.

In short, we want to use (x, y, z)-coordinates to compute with, but we want lengths and angles to be the usual Euclidean ones in $(\widehat{x}, \widehat{y}, \widehat{z})$-coordinates.

Problem 99 *Suppose we have a curve $\widehat{\gamma}$ in Euclidean space and we think of it as a composition of a curve γ in K-warped space with a transformation. In other words, we're looking at a diagram*

$$\begin{tikzcd}
t \arrow[r, mapsto, "\gamma"] \arrow[rr, bend right, "\widehat{\gamma}"'] & (x(t), y(t), z(t)) \arrow[r, mapsto, "\begin{bmatrix}\widehat{x}(x,y,z)\\ \widehat{y}(x,y,z)\\ \widehat{z}(x,y,z)\end{bmatrix}"] & \widehat{\gamma}(t) = (\widehat{x}(t), \widehat{y}(t), \widehat{z}(t))
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbb{R} \arrow[r, "\gamma"] \arrow[rr, bend right, "\widehat{\gamma}"'] & K - \text{\textit{warped space}} \arrow[r, "\begin{bmatrix}\widehat{x}(x,y,z)\\ \widehat{y}(x,y,z)\\ \widehat{z}(x,y,z)\end{bmatrix}"] & \text{\textit{Euclidean geometry}}
\end{tikzcd}$$

Use the chain rule to compute

$$\frac{d\widehat{x}}{dt}, \qquad \frac{d\widehat{y}}{dt}, \qquad \frac{d\widehat{z}}{dt},$$

in terms of $\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}, \frac{\partial\widehat{x}}{\partial x}, \frac{\partial\widehat{y}}{\partial x}, \frac{\partial\widehat{z}}{\partial x}, \frac{\partial\widehat{x}}{\partial y}, \frac{\partial\widehat{y}}{\partial y}, \frac{\partial\widehat{z}}{\partial y}, \frac{\partial\widehat{x}}{\partial z}, \frac{\partial\widehat{y}}{\partial z}$, and $\frac{\partial\widehat{z}}{\partial z}$.

Hint: *Recall that if F is a differentiable function of x, y, and z; and if x, y, and z are all differentiable functions of t, then the chain rule states*

$$\frac{dF}{dt} = \nabla F \cdot \begin{bmatrix} \frac{dx}{dt} & \frac{dy}{dt} & \frac{dz}{dt} \end{bmatrix}^{\mathsf{T}}.$$

Problem 100 *With the same setting as in the previous problem, rewrite the result of your computation in matrix notation to find D_K such that*

$$\begin{bmatrix} d\widehat{x}/dt \\ d\widehat{y}/dt \\ d\widehat{z}/dt \end{bmatrix} = D_K \cdot \begin{bmatrix} dx/dt \\ dy/dt \\ dz/dt \end{bmatrix}$$

in terms of $\dfrac{\partial \widehat{x}}{\partial x}, \dfrac{\partial \widehat{y}}{\partial x}, \dfrac{\partial \widehat{z}}{\partial x}, \dfrac{\partial \widehat{x}}{\partial y}, \dfrac{\partial \widehat{y}}{\partial y}, \dfrac{\partial \widehat{z}}{\partial y}, \dfrac{\partial \widehat{x}}{\partial z}, \dfrac{\partial \widehat{y}}{\partial z}$, and $\dfrac{\partial \widehat{z}}{\partial z}$.

Problem 101 *Use the previous problems and the relationship between Euclidean and (x, y, z)-coordinates to show that*

$$\left[\frac{d\widehat{\gamma}}{dt}\right] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & R \end{bmatrix} \cdot \left[\frac{d\gamma}{dt}\right].$$

This last computation shows that if

$$\widehat{\mathbf{v}}_1 = \begin{pmatrix} \widehat{a}_1 \\ \widehat{b}_1 \\ \widehat{c}_1 \end{pmatrix} \qquad \text{and} \qquad \widehat{\mathbf{v}}_2 = \begin{pmatrix} \widehat{a}_2 \\ \widehat{b}_2 \\ \widehat{c}_2 \end{pmatrix}$$

are vectors tangent to a curve in $(\widehat{x}, \widehat{y}, \widehat{z})$-coordinates and

$$\mathbf{v}_1 = \begin{pmatrix} a_1 \\ b_1 \\ c_1 \end{pmatrix} \qquad \text{and} \qquad \mathbf{v}_2 = \begin{pmatrix} a_2 \\ b_2 \\ c_2 \end{pmatrix}$$

are their transformations into (x, y, z)-coordinates, then

$$\widehat{\mathbf{v}}_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & R \end{bmatrix} \mathbf{v}_1 \qquad \text{and} \qquad \widehat{\mathbf{v}}_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & R \end{bmatrix} \mathbf{v}_2$$

and

$$\begin{aligned} \widehat{\mathbf{v}}_1 \bullet \widehat{\mathbf{v}}_2 = \widehat{\mathbf{v}}_1^{\mathsf{T}} \cdot \widehat{\mathbf{v}}_2 &= \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & R \end{bmatrix} \mathbf{v}_1 \right)^{\mathsf{T}} \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & R \end{bmatrix} \mathbf{v}_2 \right) \\ &= \mathbf{v}_1^{\mathsf{T}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & R \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & R \end{bmatrix} \mathbf{v}_2 \\ &= \mathbf{v}_1^{\mathsf{T}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & K^{-1} \end{bmatrix} \mathbf{v}_2. \end{aligned}$$

Hence:

We can compute the Euclidean dot product without ever referring to Euclidean coordinates!

We incorporate that fact into the following definition.

Definition 20. *The* ***K-dot-product*** *of vectors:*

$$\begin{aligned} \mathbf{v}_1 \bullet_K \mathbf{v}_2 &= \mathbf{v}_1^{\mathsf{T}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & K^{-1} \end{bmatrix} \mathbf{v}_2 \\ &= \begin{bmatrix} a_1 & b_1 & c_1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & K^{-1} \end{bmatrix} \begin{bmatrix} a_2 \\ b_2 \\ c_2 \end{bmatrix}. \end{aligned}$$

Computing length

Problem 102 *Show that if a vector is given to us in (x, y, z)-coordinates as*

$$\mathbf{v} = (a, b, c)^\intercal,$$

then the length of its image in Euclidean space is given by

$$|\mathbf{v}|_K = \sqrt{\mathbf{v} \bullet_K \mathbf{v}}.$$

Problem 103 *Consider all vectors in K-warped space with their tips on the surface*

$$1 = K(x^2 + y^2) + z^2$$

and their tails at the origin. What can you say about the length of these vectors? What does this tell you about the surface for all values of $K > 0$?

Computing angles

Problem 104 *Show that when $K > 0$ if two vectors are given to us in (x, y, z)-coordinates as*

$$\mathbf{v}_1 = \begin{pmatrix} a_1 \\ b_1 \\ c_1 \end{pmatrix} \qquad \textit{and} \qquad \mathbf{v}_2 = \begin{pmatrix} a_2 \\ b_2 \\ c_2 \end{pmatrix}$$

then the angle between their image in Euclidean space is given by

$$\theta = \arccos\left(\frac{\mathbf{v}_1 \bullet_K \mathbf{v}_2}{|\mathbf{v}_1|_K \cdot |\mathbf{v}_2|_K}\right).$$

Computing area

Problem 105 *Show that if two vectors are given to us in (x, y, z)-coordinates as*

$$\mathbf{v}_1 = \begin{pmatrix} a_1 \\ b_1 \\ c_1 \end{pmatrix} \qquad \text{and} \qquad \mathbf{v}_2 = \begin{pmatrix} a_2 \\ b_2 \\ c_2 \end{pmatrix}$$

then the area of the parallelogram spanned by the image of those two vectors in Euclidean space is

$$\sqrt{\det \begin{bmatrix} \mathbf{v}_1 \bullet_K \mathbf{v}_1 & \mathbf{v}_2 \bullet_K \mathbf{v}_1 \\ \mathbf{v}_1 \bullet_K \mathbf{v}_2 & \mathbf{v}_2 \bullet_K \mathbf{v}_2 \end{bmatrix}} = \sqrt{\det \left(\begin{bmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & K^{-1} \end{bmatrix} \begin{bmatrix} \mathbf{v}_1^\intercal & \mathbf{v}_2^\intercal \end{bmatrix} \right)}.$$

Moral of the story: The dot-product rules! That is, if you know the dot-product you know everything there is to know about a geometry, lengths, areas, angles, everything. And the set

$$1 = K\left(x^2 + y^2\right) + z^2$$

continues to make sense even when K is negative. And as we will see later on, the definition of the K-dot product also makes sense for tangent vectors to that set when K is negative. The geometry we get when the constant K is chosen to be negative is called a hyperbolic geometry. The geometry we get, when the constant K is just chosen to be non-zero is called a non-Euclidean geometry. In fact all the non-Euclidean 2-dimensional geometries are either spherical or hyperbolic.

Problem 106 *Summarize the results from this section. In particular, summarize the results in your own words and indicate which results follow from the others.*

13 Tessellations of the Euclidean plane and Platonic solids

In this activity we begin to see the number of equilateral triangles around a point affects geometry.

This material was adapted from the Illinois Geometry Labs workshops on tessellations and Platonic solids, the Summer Illinois Math Camp course "When Straight Lines Curve," and the Buckeye Aha! Math Moments summer 2020 Beyond the Classroom.

We will start by returning to Euclidean geometry.

Tessellations of the Euclidean plane

After answering the following questions, students should be able to:

- Prove which regular polygons tile the Euclidean plane

Definition 21. *A* regular polygon *is a polygon where all the edges are the same length and all of the angles are congruent.*

Definition 22. *A* tessellation *or* tiling *of a surface is the covering of the surface using one or more geometric shapes, called tiles, with no overlaps and no gaps.An* edge-to-edge tiling *is tessellation with polygonal tiles where adjacent tiles only share one full side, i.e., no tile shares a partial side or more than one side with any other tile.*

A regular tiling *is an edgge-to-edge tiling where every tile is the same regular polygon and the vertex.*

We will be looking at tilings of the Euclidean plane using regular polygons.

Problem 107 (a) *In neutral and spherical geometry, we saw that the formula for the sum of the interior angles of an n-gon is*

$$(n-2) * (sum\ of\ the\ interior\ angles\ of\ a\ triangle).$$

This is also true in Euclidean geometry, even though we didn't prove it. Use this fact to write down formula for the the sum of the interior angles of a Euclidean n-gon. Your answer should be a formula and nothing else.

(b) *What is the formula for the measure of one interior angle of a regular n-gon? Your answer should be a formula and nothing else.*

Problem 108 *Now we want to see which regular polygons can be used as tiles for regular tilings. Draw or cut out at least 3 copies of the each regular polygon from the end of this chapter. For some of the smaller shapes, you will need more. You can draw them on paper or a tablet, or you can use the website* https://mathigon.org/polypad

Fill out the following chart:

Shape	*Number of vertices*	*Measure of one interior angle*	*Tessellate?*	*Number that meet at a point*
Triangle				
Square				
Pentagon				
Hexagon				
Heptagon				
Octagon				
⋮				
n-*gon*				

Problem 109 *Which regular polygons can be used a tile for a regular tiling? How do you know that you found all of them?*

Hint: *Argue why number of regular polygons that meet at a vertex must be a positive integer, then explain why you know you have found all possible polygons that can be used as tiles in a regular tessellation.*

Platonic solids

After answering the following questions, students should be able to:

- Find all Platonic solids.
- Find a relationship between the faces, vertices, and edges of Platonic solids.

If we have less than 360° around a point, like if we glued together 5 equilateral triangles, we would get a cone point.

Definition 23. *A* Platonic solid *is a polyhedra where every face is a regular polygon, the adjacent faces share a full side, and the same number of faces meet at each vertex.*

Problem 110 (a) *How many Platonic solids can you make with equilateral triangles?*

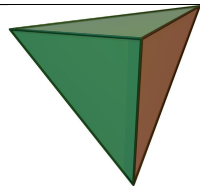
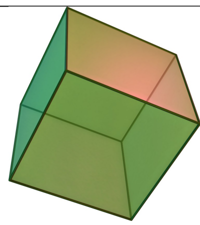
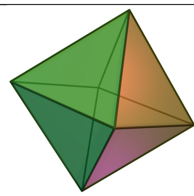
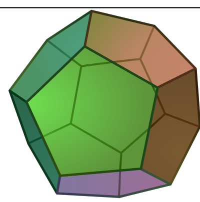
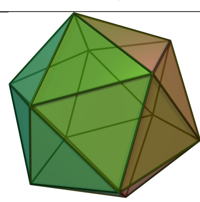
__Hint:__ Use Platonic solids have the same number of faces at each vertex, and the Platonic solid with 5 faces at each vertex is different from the solid with four faces at each vertex.

(b) *How many Platonic solids can you make with squares?*

(c) *How many Platonic solids can you make with pentagons?*

(d) *Are there any other Platonic solids?*

Problem 111 *Now fill out this other table*

Polyhedra	# *Faces*	# *Vertices*	#*Edges*
			
			
			
			
			

Images from `https://en.wikipedia.org/wiki/Platonic_solid`

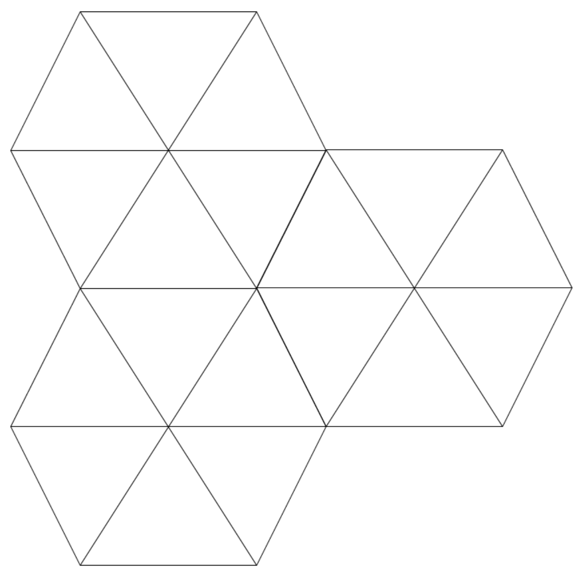
Problem 112 *What patterns are in the table above? Find at least 3 patterns in the table, at least one which describes a relationship between the faces, vertices, and edges.*

Problem 113 *Print, trace or draw two copies of the following diagram. You will also need extra equilateral triangle congruent to the ones in the diagram.*

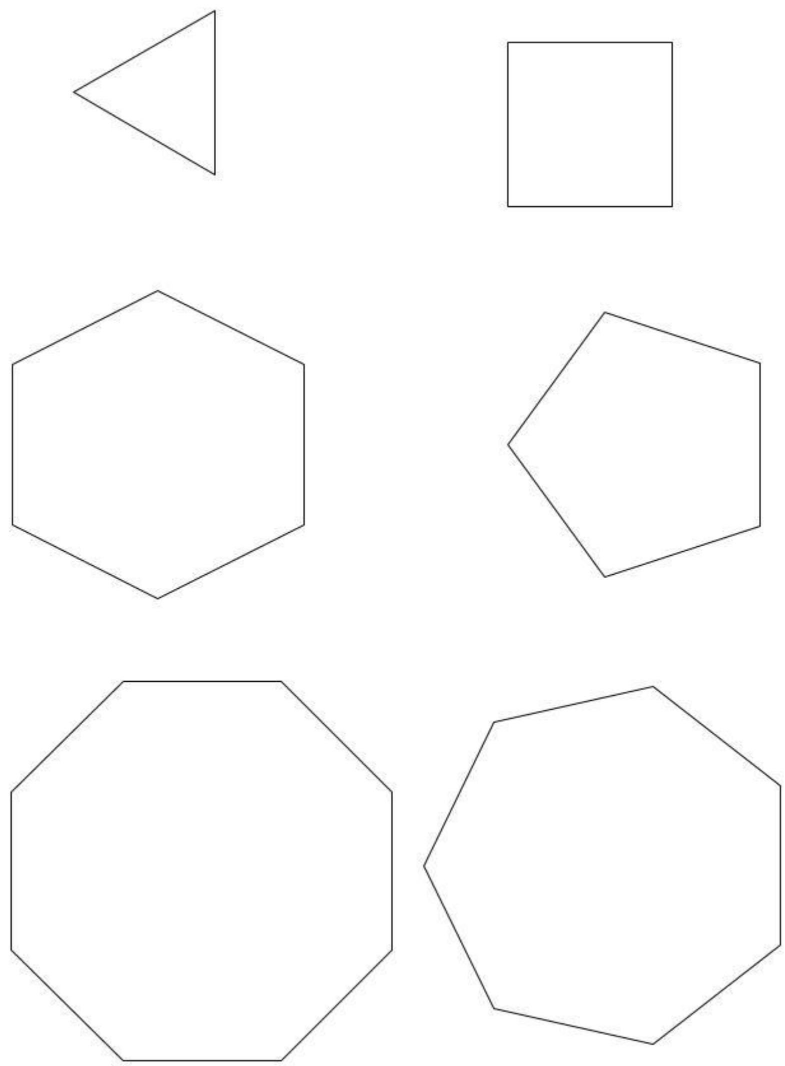
(a) *Cut out one copy and remove triangles so that every vertex is either surrounded by 5 equilateral triangle or is on an edge of the shape. What happens to the paper when you do this?*

(b) *Cut out one copy and add triangles so that every vertex is either surrounded by 7 equilateral triangle or is on an edge of the shape. What happens to the paper when you do this?*

(c) *We can think of the shape with 5 triangles at a vertex as having positive curvature. Why? Why would it make sense to also say that the shape with 7 triangles at a vertex has negative curature?*



Problem 114 *Summarize the results from this section. In particular, rephrase the results in your own words and indicate which results follow from the others.*



14 Rigid motions in (x, y, z)-coordinates

Here we dig deeper to understand our different coordinates.

We now wish to figure out how to convert a transformation

$$\widehat{T}(\widehat{x}, \widehat{y}, \widehat{z}) = \begin{bmatrix} \widehat{a} \\ \widehat{b} \\ \widehat{c} \end{bmatrix} = \widehat{M} \cdot \begin{bmatrix} \widehat{x} \\ \widehat{y} \\ \widehat{z} \end{bmatrix}$$

from Euclidean space to (x, y, z)-coordinates. Recall to convert a point from (x, y, z)-coordinates to Euclidean coordinates, we write

$$\begin{bmatrix} \widehat{x} \\ \widehat{y} \\ \widehat{z} \end{bmatrix} = \begin{bmatrix} x \\ y \\ Rz \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & R \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

We summarize this in the diagrams

$$\begin{tikzcd} K\text{-warped coordinates} \arrow[r, "\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & R \end{bmatrix}\cdot"] \arrow[d, "M\cdot"'] & \text{Euclidean coordinates} \arrow[d, "\widehat{M}\cdot"] \\ K\text{-warped coordinates} \arrow[r, "\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & R \end{bmatrix}\cdot"] & \text{Euclidean coordinates} \end{tikzcd}$$

where

$$\begin{tikzcd} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \arrow[r, mapsto, "\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & R \end{bmatrix}\cdot"] \arrow[d, mapsto, "M\cdot"] & \begin{bmatrix} \widehat{x} \\ \widehat{y} \\ \widehat{z} \end{bmatrix} \arrow[d, mapsto, "\widehat{M}\cdot"] \\ \begin{bmatrix} a \\ b \\ c \end{bmatrix} \arrow[r, mapsto, "\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & R \end{bmatrix}\cdot"] & \begin{bmatrix} \widehat{a} \\ \widehat{b} \\ \widehat{c} \end{bmatrix} \end{tikzcd}.$$

Thus, we start at $[x \quad y \quad z]^\intercal$ and end at $[\widehat{a} \quad \widehat{b} \quad \widehat{c}]^\intercal$ whether we convert to Euclidean coordinates then apply $\widehat{M}$, or we apply M then convert to Euclidean coordinates.

Problem 115 *Using the diagram above, explain why*

$$M = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & R^{-1} \end{bmatrix} \cdot \widehat{M} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & R \end{bmatrix}.$$

Hint: *We can also go from Euclidean coordinates to K-warped coordinates using the map defined by* $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & R \end{bmatrix}^{-1}$

Hint: *We start at $[x\ y\ z]^{\intercal}$ and end at $[\widehat{a}\ \widehat{b}\ \widehat{c}]^{\intercal}$ whether we convert to Euclidean coordinates then apply $\widehat{M}$, or we apply M then convert to Euclidean coordinates.*

Problem 116 *Show that a transformation M in (x, y, z)-coordinates preserves distances in K-warped space if and only if*

$$M^{\mathsf{T}} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & K^{-1} \end{bmatrix} \cdot M = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & K^{-1} \end{bmatrix}. \tag{$*$}$$

Hint: *Prove that $(*)$ is true if and only if $\widehat{M}$ is orthogonal.*

Hint: *Use a previous problem to show $|\mathbf{v}|_K = |\hat{\mathbf{v}}|$ and conclude $|M\mathbf{v}|_K = |\widehat{M}\hat{\mathbf{v}}|$ if and only if $\widehat{M}$ is orthogonal.*

The equation $(*)$ is the condition (in (x, y, z)-coordinates) which affirms that the transformation which takes the path $\gamma(t) = (x(t), y(t), z(t))$ to the path $\gamma_M(t) = M \cdot (x(t), y(t), z(t))^{\mathsf{T}}$ preserves lengths of tangent vectors at corresponding points. Therefore, by integrating, the (total) length of the curve γ_M is the same as the total length of the curve γ.

Problem 117 *Verify that this is the correct condition by showing that any 3×3 matrix M satisfying $(*)$ also satisfies*

$$(M \cdot \mathbf{v}) \bullet_K (M \cdot \mathbf{v}) = \mathbf{v} \bullet_K \mathbf{v},$$

where

$$\mathbf{v} = X_2 - X_1.$$

That is, the transformation given in (x, y, z)-coordinates by a matrix M that satisfies your condition preserves the K-dot product.

Definition 24. *A K-distance-preserving transformation of K-geometry is called a K-**rigid motion** or a K-**congruence**.*

With this definition, and our work above, we make a new definition:

Definition 25. *A 3×3 matrix M is called **K-orthogonal** if*

$$M^{\intercal} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & K^{-1} \end{bmatrix} \cdot M = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & K^{-1} \end{bmatrix}.$$

Problem 118 *For $K \neq 0$, show that if M is K-orthogonal, then the transformation*

$$T(x, y, z) = \begin{bmatrix} a \\ b \\ c \end{bmatrix} = M \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

takes the set of points (x, y, z) such that

$$1 = K\left(x^2 + y^2\right) + z^2$$

to the set of points (a, b, c) such that

$$1 = K\left(a^2 + b^2\right) + c^2.$$

That is, M gives a one-to-one and onto mapping of K-geometry to itself.

Hint: *Explain how you can write the equation*

$$1 = K\left(a^2 + b^2\right) + c^2.$$

as

$$\begin{bmatrix} a & b & c \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & K^{-1} \end{bmatrix} \cdot \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \frac{1}{K}.$$

Problem 119 *For $K \neq 0$, show that the set of K-orthogonal matrices M forms a group. That is, show that*

(a) *multiplication of K-orthogonal matrices is associative,*

Hint: *Recall that function composition is always associative.*

(b) *the product of two K-orthogonal matrices is K-orthogonal,*

(c) *the identity matrix is K-orthogonal,*

(d) *the inverse matrix M^{-1} of a K-orthogonal matrix M is K-orthogonal.*

Problem 120 *Convert the orthogonal matrix*

$$\widehat{M}_\theta = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

into its K-orthogonal counterpart M_θ. Are you surprised? Why or why not?

Problem 121 *Convert the orthogonal matrix*

$$\widehat{M}_{\psi} = \begin{bmatrix} \cos\psi & 0 & -\sin\psi \\ 0 & 1 & 0 \\ \sin\psi & 0 & \cos\psi \end{bmatrix}$$

into its K-orthogonal counterpart M_{ψ}. Are you surprised? Why or why not?

Why use K-coordinates?

We have seen that we could measure the usual Euclidean lengths of curves $\widehat{\gamma}$ in terms of the formulas of curves γ in K-warped space using the K-dot product. The short reason for this is that

$$\frac{d\widehat{\gamma}}{dt} \bullet \frac{d\widehat{\gamma}}{dt} = \frac{d\gamma}{dt} \bullet_K \frac{d\gamma}{dt}$$

where

$$\frac{d\gamma}{dt} \bullet_K \frac{d\gamma}{dt} = \left[\frac{d\gamma}{dt}\right] \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & K^{-1} \end{bmatrix} \cdot \left[\frac{d\gamma}{dt}\right]^{\mathsf{T}}.$$

In other words, the usual geometry of the sphere of radius R is simply the geometry of the set

$$\left\{(x, y, z) \in \mathbb{R}^3 : 1 = K\left(x^2 + y^2\right) + z^2\right\}$$

with $K = 1/R^2$ and with lengths (and areas) given by the K-dot product. Said another way, we can do all of spherical geometry in (x, y, z)-coordinates. All we need is the set defined by the relation

$$1 = K\left(x^2 + y^2\right) + z^2$$

and the K-dot product. But the set defined by the equation above continues to exist even if $K = 0$ or $K < 0$, and the K-dot product formula continues to make sense even if $K < 0$. In short we have the following table:

	Spherical $(K > 0)$	Euclidean $(K = 0)$	Hyperbolic $(K < 0)$
Surface in Euclidean space	$\widehat{x}^2 + \widehat{y}^2 + \widehat{z}^2 = R^2$	DNE	DNE
Euclidean dot product	$\widehat{\mathbf{v}}^{\mathsf{T}} \cdot \widehat{\mathbf{w}}$	DNE	DNE
Surface in K-warped space	$1 = K(x^2 + y^2) + z^2$	$1 = K(x^2 + y^2) + z^2$	$1 = K(x^2 + y^2) + z^2$
K-dot product	$\mathbf{v}^{\mathsf{T}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & K^{-1} \end{bmatrix} \mathbf{w}$	DNE	$\mathbf{v}^{\mathsf{T}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & K^{-1} \end{bmatrix} \mathbf{w}$

This table tells us that 'there is something else out there,' that is, some other type of two-dimensional geometry beyond plane geometry and spherical geometry. But the gap in the bottom row of the table is a bit disturbing. If we can't express the usual dot-product in plane geometry as the K-dot product for

$K = 0$, we can't pass smoothly from spherical through plane geometry to hyperbolic geometry using (x, y, z)-coordinates. Later, we will examine two ways to produce coordinates uniformly for spherical, plane and hyperbolic geometry that overcome this difficulty.

Problem 122 *Summarize the results from this section. In particular, indicate which results follow from the others.*

15 Lines in spherical geometry

Here we examine "lines" in spherical geometry and prove a spherical version of the Pythagorean Theorem.

Spherical coordinates, a shortest path from the North Pole

After answering the following questions, students should be able to:

- Convert K-warped space to spherical coordinates

- Derive the formula for the spherical or (σ, τ)-coordinate dot product

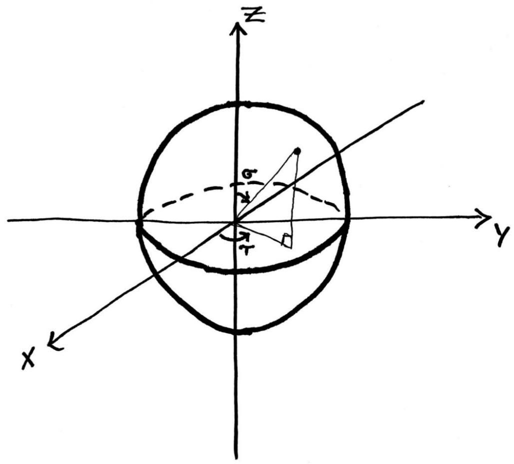
- Show that the shortest distance from the north pole of the R-sphere to a point on the sphere in the $y = 0$ plane is on the path that stays in the $y = 0$ plane.

We next will figure out what is the shortest path you can take between two points on the Euclidean R-sphere. However, we will do our calculation using (x, y, z)-coordinates, as we won't have $(\widehat{x}, \widehat{y}, \widehat{z})$-coordinates when we get to hyperbolic geometry.

For our purposes, it will be convenient to parameterize the sphere in K-geometry:

$$\begin{aligned}
x(\sigma, \tau) &= R \cdot \sin\sigma \cdot \cos\tau, \\
y(\sigma, \tau) &= R \cdot \sin\sigma \cdot \sin\tau, \\
z(\sigma, \tau) &= \cos\sigma,
\end{aligned}$$

where $0 \leq \sigma \leq \pi$ and $0 \leq \tau < 2\pi$.



Problem 123 *Show that these functions actually parameterize the R-sphere.*

Hint: *This is an exercise in "double-containment": you must show that every point specified in this way is a point on the sphere and vice versa. To show every point is on the sphere, show*

$$K\left(x(\sigma,\tau)^2 + y(\sigma,\tau)^2\right) + z(\sigma,\tau)^2 = 1$$

for all (σ,τ)*. To show the every point on the sphere can be written this way, appeal to the diagram above.*

If we are going to describe paths on the R-sphere by paths in the (σ, τ)-plane we are going to need to figure out the K-dot product in (σ, τ)-coordinates so that we can compute the lengths of paths in these coordinates.

Problem 124 *Suppose we have a curve γ in K-warped space which we can decompose as*

$$\begin{tikzcd} t \arrow[r, mapsto, "\gamma_{\text{sph}}"] \arrow[rr, bend right, "\gamma"'] & (\sigma(t), \tau(t)) \arrow[r, mapsto, "{\begin{bmatrix} x(\sigma,\tau) \\ y(\sigma,\tau) \\ z(\sigma,\tau) \end{bmatrix}}"] & \gamma(t) = (x(t), y(t), z(t)) \end{tikzcd}$$

Use the chain rule to compute

$$\frac{dx}{dt}, \qquad \frac{dy}{dt}, \qquad \frac{dz}{dt},$$

in terms of $\frac{d\sigma}{dt}, \frac{d\tau}{dt}, \frac{\partial x}{\partial \sigma}, \frac{\partial y}{\partial \sigma}, \frac{\partial z}{\partial \sigma}, \frac{\partial x}{\partial \tau}, \frac{\partial y}{\partial \tau}$, and $\frac{\partial z}{\partial \tau}$.

Hint: *Recall that if F is a differentiable function of a and b; and if a and b are all differentiable functions of t, then the chain rule states*

$$\frac{dF}{dt} = \nabla F \cdot \begin{bmatrix} da/dt \\ db/dt \end{bmatrix}.$$

Problem 125 *With the same setting as in the previous problem, rewrite the result of your computation in matrix notation to find D_{sph} such that*

$$\begin{bmatrix} dx/dt \\ dy/dt \\ dz/dt \end{bmatrix} = D_{\text{sph}} \cdot \begin{bmatrix} d\sigma/dt \\ d\tau/dt \end{bmatrix}$$

in terms of $\dfrac{\partial x}{\partial \sigma}, \dfrac{\partial y}{\partial \sigma}, \dfrac{\partial z}{\partial \sigma}, \dfrac{\partial x}{\partial \tau}, \dfrac{\partial y}{\partial \tau}$, and $\dfrac{\partial z}{\partial \tau}$.

Problem 126 *Now find P_{sph} in terms of K, $\frac{\partial x}{\partial \sigma}$, $\frac{\partial y}{\partial \sigma}$, $\frac{\partial z}{\partial \sigma}$, $\frac{\partial x}{\partial \tau}$, $\frac{\partial y}{\partial \tau}$, and $\frac{\partial z}{\partial \tau}$ such that*

$$\left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right) \bullet_K \left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right) = \begin{bmatrix} \frac{d\sigma}{dt} & \frac{d\tau}{dt} \end{bmatrix} \cdot P_{\text{sph}} \cdot \begin{bmatrix} \frac{d\sigma}{dt} \\ \frac{d\tau}{dt} \end{bmatrix}.$$

Problem 127 *Set*

$$
\begin{aligned}
x(\sigma,\tau) &= R\cdot\sin\sigma\cdot\cos\tau,\\
y(\sigma,\tau) &= R\cdot\sin\sigma\cdot\sin\tau,\\
z(\sigma,\tau) &= \cos\sigma,
\end{aligned}
$$

and show that P_{sph} *from the problem above is*

$$
P_{\mathrm{sph}} = \begin{bmatrix} R^2 & 0 \\ 0 & R^2\cdot\sin^2\sigma \end{bmatrix}.
$$

Definition 26. *Let* $\mathbf{v}_{\text{sph}} = \begin{bmatrix} a & b \end{bmatrix}$ *and* $\mathbf{w}_{\text{sph}} = \begin{bmatrix} c & d \end{bmatrix}$ *be a vectors in* (σ, τ)*-coordinates based at the same point* (σ, τ)*-coordinate. Define*

$$\mathbf{v}_{\text{sph}} \bullet_{\text{sph}} \mathbf{w}_{\text{sph}} = R^2 ac + R^2 bd \sin^2 \sigma$$

or in matrix notation,

$$\mathbf{v}_{\text{sph}} \bullet_{\text{sph}} \mathbf{w}_{\text{sph}} = \mathbf{v}_{\text{sph}} \cdot P_{\text{sph}} \cdot \mathbf{w}_{\text{sph}}^{\mathsf{T}}$$

where

$$P_{\text{sph}} = \begin{bmatrix} R^2 & 0 \\ 0 & R^2 \cdot \sin^2 \sigma \end{bmatrix}$$

and σ *is determined by the coordinate that the vectors originate from.*

Now notice that you can write a path on the R-sphere by giving a path $(\sigma(t), \tau(t))$ in the (σ, τ)-plane. To write a path that starts at the North Pole, just write

$$(\sigma(t), \tau(t)), \qquad 0 \leq t \leq e$$

and demand that $\sigma(0) = 0$. If you want the path to end on the plane $y = \widehat{y} = 0$, demand additionally that $\tau(e) = 0$.

Now given a path on the R-sphere

$$(\sigma(t), \tau(t)), \qquad 0 \leq t \leq e$$

satisfying $\sigma(0) = \tau(0) = 0$ and $\tau(e) = 0$, its length is given by the formula

$$L = \int_0^e \sqrt{\left(\frac{d\sigma}{dt}, \frac{d\tau}{dt}\right) \bullet_{\text{sph}} \left(\frac{d\sigma}{dt}, \frac{d\tau}{dt}\right)} dt. \tag{$*$}$$

(Why?)

Problem 128 *Prove that the shortest path on the* R*-sphere from the North Pole* $(0, 0, 1)$ *(in* (σ, τ)*-coordinates,* $(0, 0)$*) to a point*

$$(x, y, z) = (R \sin e, 0, \cos e)$$

(in $\sigma, \tau)$*-coordinates,* $(e, 0)$*) is the downwards path lying in the plane* $y = 0$*.*

Hint: *Start with some arbitrary path* $(\sigma(t), \tau(t))$ *which begins and ends at this point and show that it's longer than this path, in two steps:*

- *First use the equation* $(*)$ *to show that it gets shorter if you keep the same* σ *but make* τ *always zero (squishing the curve onto the plane* $y = 0$*.)*
- *Finish the argument by eliminating any backtracking.*

Shortest path between any two points

After answering the following questions, students should be able to:

- Show that the shortest distance between two points on a sphere lies on a great circle

We next prove the theorem that shows that shortest path on the surface of the Earth from Rio de Janeiro to Los Angeles is the one cut on the surface of the Earth by the plane that passes through the center of the Earth and through Rio and through Los Angeles. That is usually the route an airplane would take when flying between the two cities.

Theorem 11. *Given any two points $X_1 = (x_1, y_1, z_1)$ and $X_2 = (x_2, y_2, z_2)$ in K-geometry, the shortest path between the two points is the path cut out by the set*

$$K\left(x^2 + y^2\right) + z^2 = 1$$

and the plane containing $(0, 0, 0)$, X_1, and X_2.

Problem 129 *(**Tricky!**) Explain in words how to prove this theorem by using K-rigid motions of the form*

$$M_\theta = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \text{and} \qquad N_\psi = \begin{bmatrix} \cos\psi & 0 & -R \cdot \sin\psi \\ 0 & 1 & 0 \\ R^{-1} \cdot \sin\psi & 0 & \cos\psi \end{bmatrix}.$$

Hint: *M_θ is a K-rigid motion that rotates around the z-axis and N_ψ is a K-rigid motion that rotates around the y-axis.*

Hint: *You should apply two K-rigid motions of the form M_θ (for different angles) and one K-rigid motion of the form N_ψ, though not necessarily in that order!*

Definition 27. *A **line** in spherical geometry will be a curve that extends infinitely in each direction and has the property that, given any two points X_1 and X_2 on the path, the shortest path between X_1 and X_2 lies along that curve.*

Lines in spherical geometry are usually called great circles on the R-sphere. They are the intersections of the R-sphere with planes through $(0, 0, 0)$.

The spherical Pythagorean Theorem

After answering the following questions, students should be able to:

- Prove the spherical version of the Pythagorean Theorem
- Show that for very small triangles, the spherical version of the Pythagorean Theorem is approximately the same as the Euclidean version.

To start we need some basic facts about lengths of lines in spherical geometry.

Problem 130 *Given a line in spherical geometry lying entirely in the plane $y = 0$,*

$$\begin{aligned} x(t) &= R\sin t, \\ y(t) &= 0, \\ z(t) &= \cos t, \end{aligned}$$

show that the length of the segment at $0 \leq t \leq e$ is exactly Re.

Hint: *Use a previous problem.*

Problem 131 *Explain in words how to prove that given two points on R-sphere, say X_A and X_B, the length of the spherical line connecting them is given by*

$$R \cdot e = R \cdot \arccos\left(\frac{X_A \bullet_K X_B}{|X_A|_K \cdot |X_B|_K}\right).$$

by using K-rigid motions of the form

$$M_\theta = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad N_\psi = \begin{bmatrix} \cos\psi & 0 & -R\cdot\sin\psi \\ 0 & 1 & 0 \\ R^{-1}\cdot\sin\psi & 0 & \cos\psi \end{bmatrix}.$$

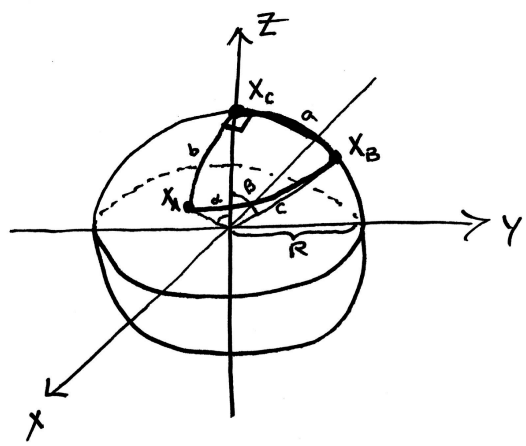
We will now give the spherical analogue of the Pythagorean Theorem.

Theorem 12 (Spherical Pythagorean Theorem)**.** *If $\triangle X_A X_B X_C$ is a right triangle on the R-sphere with right angle $\angle X_A X_C X_B$, and side a opposite X_A, b opposite X_B, and c opposite X_C, then*

$$\cos\left(\frac{c}{R}\right) = \cos\left(\frac{a}{R}\right)\cos\left(\frac{b}{R}\right).$$

Let's see why this theorem is true. We may via K-rigid motions place the triangle so that X_C is at the North Pole, X_A is in the plane $y = 0$, and X_B is in the plane $x = 0$ (note X_A and X_B may be switched—if this is the case, simply rename them). In this case,

$$\begin{aligned} X_A &= (R \cdot \sin\alpha, 0, \cos\alpha), \\ X_B &= (0, R \cdot \sin\beta, \cos\beta). \end{aligned}$$



Hence the length of side b is $R \cdot \alpha$. Using a rigid motion of the form

$$M_\theta = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

when $\theta = \pi/2$ we see that the length of side a is $R \cdot \beta$. Set

$$\gamma = \arccos\left(\frac{X_A \bullet_K X_B}{|X_A|_K \cdot |X_B|_K}\right).$$

Since we are working on the R-sphere,

$$\begin{aligned} R^2 \cdot \cos\gamma &= X_A \bullet_K X_B \\ &= \begin{bmatrix} R\cdot\sin\alpha & 0 & \cos\alpha \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & R^2 \end{bmatrix} \begin{bmatrix} 0 \\ R\cdot\sin\beta \\ \cos\beta \end{bmatrix} \\ &= R^2 \cdot \cos\alpha \cdot \cos\beta. \end{aligned}$$

Problem 132 *Explain how to progress from the fact that*

$$R^2 \cdot \cos\gamma = R^2 \cdot \cos\alpha \cdot \cos\beta.$$

to the conclusion of the theorem

$$\cos\left(\frac{c}{R}\right) = \cos\left(\frac{a}{R}\right)\cos\left(\frac{b}{R}\right).$$

Problem 133 *Use the Taylor series expansion of* $\cos(x)$ *centered around* $x = 0$,

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

to show that for "small" triangles, the spherical Pythagorean Theorem reduces to the Euclidean Pythagorean Theorem, meaning

$$c^2 \approx a^2 + b^2.$$

Problem 134 *Summarize the results from this section. In particular, indicate which results follow from the others.*

16 The upper half plane and hyperbolic triangles

In this activity, we explore rigid motions of the upper half plane model, revisit the axioms of neutral geometry, and find the areas of polygons in hyperbolic geometry.

Upper half plane model

To work with hyperbolic space, we want a model that we can draw and look at. We cannot create a model that preserves lengths of geodesics and angles between curves. This is similar to how we cannot draw spherical geometry on a flat plane without distorting lengths or angles, but spherical geometry has the nice feature of working on spheres.

There are several models for hyperbolic geometry, but one of the most common is *the upper half plane* $\mathbb{H} = \{x + iy | y > 0\}$

You can get a sense of the upper half plane model of hyperbolic space by drawing in the GeoGebra applet https://www.geogebra.org/m/dqc3uKhY#material/DBXvbruR

Problem 135 *Using the GeoGebra applet, describe what the following look like in hyperbolic geometry:*

- *Lines between two points*
- *Circle centered at some point*
- *Triangles*

Definition 28. *A* geodesic *is in* $\mathbb{H}$ *is either a semicircle meeting the real axis at right angles (ie, a circle centered on the x-axis), or a vertical ray emanating from a point on the real axis.*

Definition 29. *The* angle *between two geodesics is defined to be the angle between the tangents to the geodesics at their intersection point.*

Definition 30. *Let* p *and* q *be two points in* $\mathbb{H}$. *Let* o *and* r *denote the points where the geodesic meets the real axis, as in Figure . We define*

$$distance(p,q) = \left| \ln \frac{|o-p||p-r|}{|o-q||q-r|} \right|$$

Here $|o-p|$ *computes the Euclidean distance from* o *to* p. *Likewise for the other terms. When* p *and* q *lie on the same vertical segment, we interpret* $r = \infty$. *In this case, the second two factors cancel out, and the distance is:*

$$distance(p,q) = \left| \ln \frac{|o-p|}{|o-q|} \right|.$$

To save space, we will write this function as d_{hyp}.

Definition 31. *Let* $f : \mathbb{H} \to \mathbb{H}$. *Such a mapping is called a* rigid motion of hyperbolic space *if the distance between any two points in hyperbolic space is left unchanged by the mapping, that is, for any two points* X_1 *and* X_2 *in hyperbolic space,*

$$d_{hyp}(X_1, X_2) = d_{hyp}(f(X_1), f(X_2)).$$

Problem 136 *Let* $z = x + iy, y > 0$. *Show that the following maps are rigid motions of hyperbolic space:*

- *Let* t *be any real number and* $f(z) = z + t$.
- *Let* t *be a positive real number and* $f(z) = tz$.
- $f(x + iy) = -(x - iy)$.

Problem 137 *Show that* $f(z) = \frac{-1}{z}$ *is a one-to-one and onto map from the hyperbolic plane to itself.*

Note: Since we are dealing with the entire hyperbolic plane and not a subset, any preimages will be in our set.

Problem 138 *Show that the map* $f(z) = \dfrac{-1}{z}$ *maps hyperbolic geodesics to hyperbolic geodesics.*

Problem 139 *Show that the map* $f(z) = \dfrac{-1}{z}$ *is a rigid motion of hyperbolic space.*

We can use $z \mapsto z + t$ and $z \mapsto sz$ to map any point P onto any other point Q. The map $z = (x + iy) \mapsto -(x - iy) = -\overline{z}$ is a reflection across the imaginary axis. Combining these three maps and $z \mapsto \dfrac{-1}{z}$ lets us prove:

Theorem 13. *Let γ be any geodesic in $\mathbb{H}$. There is a rigid motion r such that $r(p) = p$ for all $p \in \gamma$ and $r \circ r$ is the identity. That is, there is a mirror reflection rigid motion which fixes γ.*

Proof Consider first the case when γ is a vertical ray. There is a rigid motion of the form $f(z) = z + t$ so that $f(\gamma)$ is the vertical ray through the origin, and $g(z) = -\overline{z}$. Then $f \circ g \circ f^{-1}$ is the desired rigid motion.

Now consider the case when γ is a semicircle. There is a symmetry of the form $f(z) = az + b$ so that $f(\gamma)$ is the semicircle contained in the unit circle $|z| = d_{hyp}(z, 0) = 1$, and $h(z) = 1/z$. Then$f \circ h \circ f^{-1}$ is the desired rigid motion. ■

Theorem 14. *Let p be any point in the hyperbolic plane, and let θ be any angle. Then there is a rigid motion of hyperbolic space which fixes p and rotates by p about p.*

Proof We can find geodesics C_1 and C_2 which intersect at p. Let R_1 and R_2 be the reflections through C_1 and C_2. Both R_1 and R_2 fix the point p, since p lies on both C_1 and C_2. The composition $R = R1R2$ rotates around p by 2α, where α is the smaller of the two angles between C_1 and C_2. So. if we take $\alpha = \frac{\theta}{2}$, we get the desired result. ■

Problem 140 *Why do the rigid motions we have considered ($f(z) = z + t, f(z) = sz, f(z) = -\overline{z}$ and $f(z) = \dfrac{-1}{z}$) preserve the angle between geodesics?*

Euclids postulates

Remember back to Chapter 2, in neutral geometry:

Axiom 6 (E1)**.** *Through any point P and any other point Q, there lies a unique line.*

Axiom 7 (E2)**.** *Given any two segments $\overline{AB}$ and $\overline{CD}$, there is a segment $\overline{AE}$ such that B lies on $\overline{AE}$ and $|CD| = |BE|$.*

Note: We will use both $|AB|$ and $d(A, B)$ to denote the distance between two points A and B.

Axiom 8 (E3)**.** *Given a point P and any positive real number r, there exists a (unique) circle of radius r and center P.*

Said another way, if you move away from point P along a line in any direction, you will encounter a unique point at distance r from P.

Axiom 9 (E4)**.** *All right angles are congruent.*

Note: A right angle is defined as follows. Let C be the midpoint on the segment $\overline{AB}$. Let E be any point not equal to C. The angle $\angle ACE$ is called a right angle if $\angle ACE$ is congruent to $\angle ECB$. [MJG,17-18]

Definition 32. *If we are only given axioms E1–E4, we will call our geometry **neutral geometry (NG)**.*

Definition 33. *We call two rays in the plane **parallel** if they lie on parallel lines and they both lie on the same side of the transversal line passing through their origins.*

and Chapter 3 on Euclidean geometry:

Axiom 10 (E5)**.** *Through a point not on a line there passes a unique parallel line.*

We want to show that in hyperbolic geometry, axioms E1–E4 are true, but E5 is not.

Historically, showing that there is a geometry where E1–E4 are true, but E5 is not was very difficult. But now that we know about hyperbolic geometry, it is not too difficult to show that it has the properties we desire.

For each of the following 4 problems, your answer should be at most 2 paragraphs.

Problem 141 *Use the upper half plane model $\mathbb{H}$ and the fact that the shortest distance between two points lies on a geodesic. Given any two points P and Q, explain how to construct the unique line (here: geodesic) between them.*

Problem 142 *Use the upper half plane model $\mathbb{H}$ and the fact that the shortest distance between two points lies on a geodesic. Given any two segments (here: geodesic arcs) $\overline{AB}$ and $\overline{CD}$, explain how to construct the geodesic arc $\overline{AE}$ such that B lies on $\overline{AE}$ and $d_{hyp}(C, D) = d_{hyp}(B, E)$.*

Problem 143 *Use the upper half plane model and the fact the a circle of radius r is the set of points distance r from the center. Explain how to construct a circle of radius r around a point P for any real number r.*

Problem 144 *Use the upper half plane model, the fact that there is a rigid motion of hyperbolic space which fixes p and rotates by p about p, and that these rigid motions preserve the angles between geodesics. Explain why all right angles are congruent.*

Problem 145 *Using the upper half plane model $\mathbb{H}$ and the fact that the shortest distance between two points lies on a geodesic, draw a hyperbolic geodesic ℓ, a point P not on that geodesic, and two geodesics that are parallel to ℓ and pass through P.*

Hyperbolic triangles

We know that the area of a triangle on the R-sphere with angles α, β, and γ is given by

$$R^2(\alpha + \beta + \gamma - \pi).$$

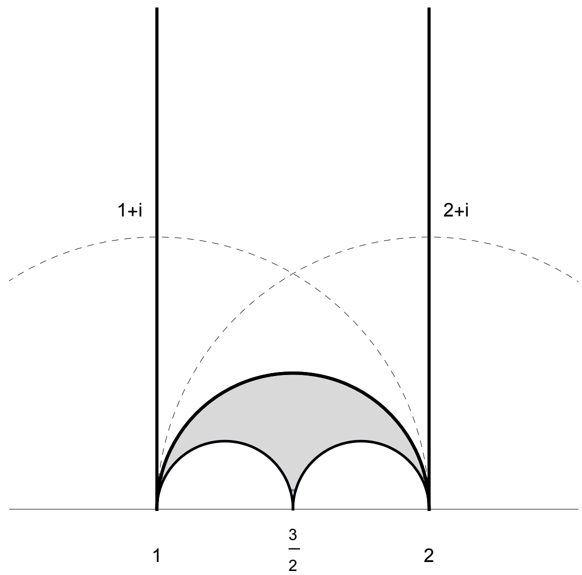
Problem 146 *Briefly (one paragraph or less) sketch the line of reasoning used to deduce the formula for the area of a triangle on the R-sphere.*

Problem 147 *In one paragraph or less, explain why the formula for the area of a triangle on the R sphere is the same in Euclidean coordinates and in K-warped coordinates.*

Problem 148 *Use K-warped coordinates to give the formula for the area of a triangle on the hyperbolic plane, where $K < 0$.*

Definition 34. *An **ideal triangle** is a triangle in hyperbolic geometry with all of its vertices at infinity.*

Problem 149 *Calculate the area of the shaded hyperbolic triangle*



Definition 35. *An **ideal n-gon** is an n-gon in hyperbolic geometry with its vertices at infinity.*

Problem 150 *Use induction to derive a formula for the area of any ideal n-gon hyperbolic geometry.*

Problem 151 *Summarize the results from this section. In particular, rephrase the results in your own words and indicate which results follow from the others.*

17 Lines in hyperbolic geometry

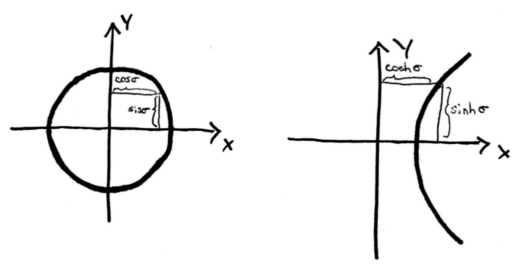
Here we examine "lines" in hyperbolic geometry and prove a hyperbolic version of the Pythagorean Theorem.

Hyperbolic coordinates, a shortest path from the North Pole

After answering the following questions, students should be able to:

- Convert K-warped space to hyperbolic coordinates
- Derive the formula for the hyperbolic (σ, τ)-coordinate dot product
- Show that the shortest distance from the north pole of the K-surface to a point on the surface in the $y = 0$ plane is on the path that stays in the $y = 0$ plane.

We next will figure out what is the shortest path you can take between two points in hyperbolic geometry. Since K is negative, we must do our calculation using only (x, y, z)-coordinates. However, this will allow us to see the full power of working in K-warped space, since our work will be essentially the same as when K was positive—though our parametrization will be different.



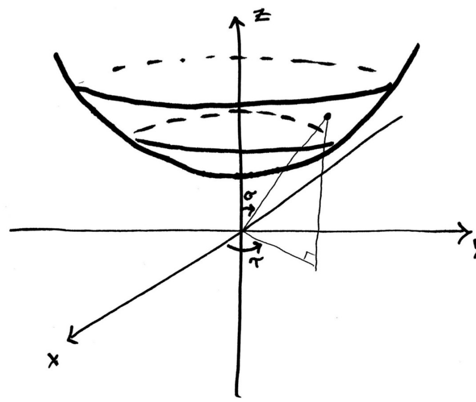
Just as $(\cos\sigma, \sin\sigma)$ parametrize the unit circle, hyperbolic functions

$$\left(\cosh\sigma = \frac{e^{\sigma} + e^{-\sigma}}{2}, \sinh\sigma = \frac{e^{\sigma} - e^{-\sigma}}{2}\right)$$

parametrize the 'unit' hyperbola. Hence we define

$$x(\sigma, \tau) = |K|^{-1/2} \cdot \sinh\sigma \cdot \cos\tau,$$
$$y(\sigma, \tau) = |K|^{-1/2} \cdot \sinh\sigma \cdot \sin\tau,$$
$$z(\sigma, \tau) = \cosh\sigma,$$

where $0 \leq \sigma < \infty$ and $0 \leq \tau < 2\pi$.



Problem 152 *Show that these hyperbolic coordinates do actually parametrize K-geometry.*

Hint: *Remember, K is negative.*

Hint: *Remember,*

$$-\sinh^2\sigma + \cosh^2\sigma = 1.$$

Hint: *This is an exercise in "double-containment." To show the one direction, show*

$$K\left(x(\sigma,\tau)^2 + y(\sigma,\tau)^2\right) + z(\sigma,\tau)^2 = 1$$

for all (σ,τ). To show the other direction, appeal to the diagram above.

Just as we did on the R-sphere, we can write a path on the K-surface by giving a path in the (σ, τ)-plane. Again, we will need to figure out the K-dot product in (σ, τ)-coordinates so that we can compute the lengths of paths in these coordinates.

Problem 153 *Suppose we have a curve γ in K-warped space which we can decompose as*

$$\begin{tikzcd} t \arrow[r, mapsto, "\gamma_{\text{hyp}}"] \arrow[rr, bend right, "\gamma"'] & (\sigma(t),\tau(t)) \arrow[r, mapsto, "\begin{bmatrix} x(\sigma,\tau)\\ y(\sigma,\tau)\\ z(\sigma,\tau)\end{bmatrix}"] & \gamma(t) = (x(t),y(t),z(t)) \end{tikzcd}$$

Use the chain rule to compute

$$\frac{dx}{dt}, \qquad \frac{dy}{dt}, \qquad \frac{dz}{dt},$$

in terms of $\dfrac{d\sigma}{dt}, \dfrac{d\tau}{dt}, \dfrac{\partial x}{\partial \sigma}, \dfrac{\partial y}{\partial \sigma}, \dfrac{\partial z}{\partial \sigma}, \dfrac{\partial x}{\partial \tau}, \dfrac{\partial y}{\partial \tau}$, and $\dfrac{\partial z}{\partial \tau}$.

Hint: *Simply write down the answer from a previous problem.*

Problem 154 *With the same setting as in the previous problem, rewrite the result of your computation in matrix notation to find D_{hyp} such that*

$$\begin{bmatrix} dx/dt \\ dy/dt \\ dz/dt \end{bmatrix} = D_{\text{hyp}} \cdot \begin{bmatrix} d\sigma/dt \\ d\tau/dt \end{bmatrix}$$

in terms of $\dfrac{\partial x}{\partial \sigma}$, $\dfrac{\partial y}{\partial \sigma}$, $\dfrac{\partial z}{\partial \sigma}$, $\dfrac{\partial x}{\partial \tau}$, $\dfrac{\partial y}{\partial \tau}$, and $\dfrac{\partial z}{\partial \tau}$.

Hint: *Simply write down the answer from a previous problem.*

Problem 155 *Now find P_{hyp} in terms of K, $\frac{\partial x}{\partial \sigma}$, $\frac{\partial y}{\partial \sigma}$, $\frac{\partial z}{\partial \sigma}$, $\frac{\partial x}{\partial \tau}$, $\frac{\partial y}{\partial \tau}$, and $\frac{\partial z}{\partial \tau}$ such that*

$$\left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right) \bullet_K \left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right) = \begin{bmatrix} \frac{d\sigma}{dt} & \frac{d\tau}{dt} \end{bmatrix} \cdot P_{\text{hyp}} \cdot \begin{bmatrix} \frac{d\sigma}{dt} \\ \frac{d\tau}{dt} \end{bmatrix}.$$

Hint: *Simply write down the answer from a previous problem.*

Problems 154 and 125 have the same set up as each other. Problems 126 and 155 are the same set up, so have the same answer.

Problem 156 *Set*

$$\begin{aligned} x(\sigma, \tau) &= |K|^{-1/2} \cdot \sinh \sigma \cdot \cos \tau, \\ y(\sigma, \tau) &= |K|^{-1/2} \cdot \sinh \sigma \cdot \sin \tau, \\ z(\sigma, \tau) &= \cosh \sigma, \end{aligned}$$

and show that P_{hyp} *from the problem above is*

$$P_{\text{hyp}} = \begin{bmatrix} |K|^{-1} & 0 \\ 0 & |K|^{-1} \cdot \sinh^2 \sigma \end{bmatrix}.$$

Definition 36. *Let* $\mathbf{v}_{\text{hyp}} = \begin{bmatrix} a & b \end{bmatrix}$ *and* $\mathbf{w}_{\text{hyp}} = \begin{bmatrix} c & d \end{bmatrix}$ *be a vectors in* (σ, τ)*-coordinates originating at the same* (σ, τ)*-coordinate. Define*

$$\mathbf{v}_{\text{hyp}} \bullet_{\text{hyp}} \mathbf{w}_{\text{hyp}} = \frac{ac}{|K|^{-1}} + \frac{bd \sinh^2 \sigma}{|K|^{-1}}$$

or in matrix notation,

$$\mathbf{v}_{\text{hyp}} \bullet_{\text{hyp}} \mathbf{w}_{\text{hyp}} = \mathbf{v}_{\text{hyp}} \cdot P_{\text{hyp}} \cdot \mathbf{w}_{\text{hyp}}^{\intercal}$$

where

$$P_{\text{hyp}} = \begin{bmatrix} |K|^{-1} & 0 \\ 0 & |K|^{-1} \cdot \sinh^2 \sigma \end{bmatrix}$$

and σ *is determined by the coordinate that the vectors originate from.*

Problem 157 *How does this definition of the hyperbolic dot product compare with the definition of the K-dot product from Chapter 12 and the spherical dot product from Chapter 15?*

Now notice that you can write a path on the K-surface by giving a path $(\sigma(t), \tau(t))$ in the (σ, τ)-plane. To write a path that starts at the North Pole, just write

$$(\sigma(t), \tau(t)), \qquad 0 \leq t \leq e$$

and demand that $\sigma(0) = \tau(0) = 0$. If you want the path to end on the plane $y = \widehat{y} = 0$, demand additionally that $\tau(e) = 0$.

Now given a path on the K-surface

$$(\sigma(t), \tau(t)), \qquad 0 \leq t \leq e$$

satisfying $\sigma(0) = \tau(0) = 0$ and $\tau(e) = 0$, its length is given by the formula

$$L = \int_0^e \sqrt{\left(\frac{d\sigma}{dt}, \frac{d\tau}{dt}\right) \bullet_{\text{hyp}} \left(\frac{d\sigma}{dt}, \frac{d\tau}{dt}\right)} dt. \tag{$*$}$$

Problem 158 *Prove that the shortest path on the K-surface from the North Pole*

$$N = \left(|K|^{-1/2} \cdot \sinh 0 \cdot \cos 0, |K|^{-1/2} \cdot \sinh 0 \cdot \sin 0, \cosh 0\right)$$

to a point

$$(x, y, z) = \left(|K|^{-1/2} \sinh e, 0, \cosh e\right)$$

is the path lying in the plane $y = 0$.

Hint: *Use the same steps you did in the sphere case.*

Shortest path between any two points

After answering the following questions, students should be able to:

- Show that the shortest distance between two points on the hyperbolic plane lies on the plane containing the points and the origin

Just as we proved in spherical geometry that the shortest path is the path cut out by

$$K(x^2 + y^2) + z^2 = 1$$

and the plane containing the origin and the two points in question, we will see that a completely analogous result is true in hyperbolic geometry. Before we start, we will need one more class of rigid motions to add to our collection.

Problem 159 *Assuming K is negative, show*

$$N_\psi = \begin{bmatrix} \cosh\psi & 0 & |K|^{-1/2}\cdot\sinh\psi \\ 0 & 1 & 0 \\ |K|^{1/2}\cdot\sinh\psi & 0 & \cosh\psi \end{bmatrix}$$

is a K-rigid motion.

Problem 160 *Assuming K is negative, consider*

$$N_\psi = \begin{bmatrix} \cosh\psi & 0 & |K|^{-1/2}\cdot\sinh\psi \\ 0 & 1 & 0 \\ |K|^{1/2}\cdot\sinh\psi & 0 & \cosh\psi \end{bmatrix}.$$

Can you describe geometrically what this mapping is doing to the points in K-warped space?

Hint: *First look at the image of the point $(0,0,1)$.*

Theorem 15. *Given any two points $X_1 = (x_1, y_1, z_1)$ and $X_2 = (x_2, y_2, z_2)$ in K-geometry, the shortest path between the two points is the path cut out by the set*

$$K\left(x^2 + y^2\right) + z^2 = 1$$

and the plane containing $(0, 0, 0)$, X_1, and X_2.

Problem 161 *Explain in words, with pictures as needed, how to prove this theorem by using the K-rigid motions*

$$M_\theta = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \text{and} \qquad N_\psi = \begin{bmatrix} \cosh\psi & 0 & |K|^{-1/2}\cdot\sinh\psi \\ 0 & 1 & 0 \\ |K|^{1/2}\cdot\sinh\psi & 0 & \cosh\psi \end{bmatrix}.$$

Hint: *M_θ is a K-rigid motion that rotates around the z-axis and N_ψ is a K-rigid motion that "slides" the K-surface past the y-axis.*

Hint: *You should apply two K-rigid motions of the form M_θ (for different angles) and one K-rigid motion of the form N_ψ—though not necessarily in that order!*

Definition 37. *A **line** in hyperbolic geometry will be a curve that extends infinitely in each direction and has the property that, given any two points X_1 and X_2 on the path, the shortest path between X_1 and X_2 lies along that curve. Lines in hyperbolic geometry are the intersections of the K-geometry with planes through $(0,0,0)$. The length of the shortest path between two points in K-geometry will be called the K-distance.*

The hyperbolic Pythagorean Theorem

After answering the following questions, students should be able to:

- Prove the hyperbolic version of the Pythagorean Theorem
- Show that for very small triangles, the hyperbolic version of the Pythagorean Theorem is approximately the same as the Euclidean version.

To start we need some basic facts about lengths of lines in hyperbolic geometry.

Problem 162 *Given a line in hyperbolic geometry lying entire in the plane $y = 0$,*

$$\begin{aligned} x(t) &= |K|^{-1/2} \sinh t, \\ y(t) &= 0, \\ z(t) &= \cosh(t), \end{aligned}$$

show that the length of the segment on the interval $0 \le t \le e$ is exactly $|K|^{-1/2} e$.

Hint: Use a previous problem.

Problem 163 *Explain in words how to prove that given two points on the surface*

$$K(x^2 + y^2) + z^2 = 1,$$

say X_A and X_B, the length of the hyperbolic line connecting them is given by

$$|K|^{-1/2} \cdot \varepsilon = |K|^{-1/2} \cdot \operatorname{arcosh}\left(\frac{X_A \bullet_K X_B}{|X_A|_K \cdot |X_B|_K}\right).$$

by using the K-rigid motions

$$M_\theta = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad N_\psi = \begin{bmatrix} \cosh\psi & 0 & |K|^{-1/2} \cdot \sinh\psi \\ 0 & 1 & 0 \\ |K|^{1/2} \cdot \sinh\psi & 0 & \cos\psi \end{bmatrix}.$$

We will now give the hyperbolic analogue of the Pythagorean Theorem.

Theorem 16 (Hyperbolic Pythagorean Theorem). *If $\triangle X_A X_B X_C$ is a right triangle on the surface*

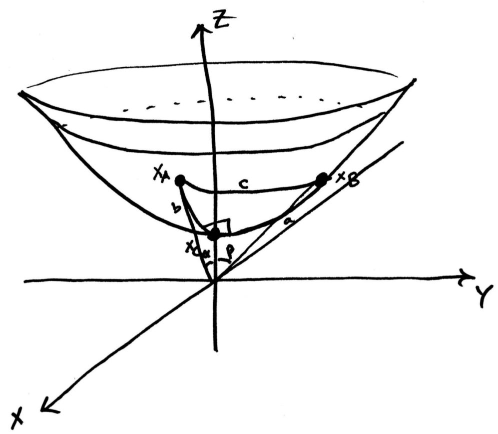
$$K(x^2+y^2)+z^2 \qquad \textit{where} \qquad K<0$$

with right angle $\angle X_A X_C X_B$, and side a opposite X_A, b opposite X_B, and c opposite X_C, then

$$\cosh\left(|K|^{1/2}\cdot c\right) = \cosh\left(|K|^{1/2}\cdot a\right)\cosh\left(|K|^{1/2}\cdot b\right).$$

Let's see why this theorem is true. We may via K-rigid motions place the triangle so that X_C is at the North Pole, X_A is in the plane $y=0$, and X_B is in the plane $x=0$ (note X_A and X_B may be switched—if this is the case, simply rename them). In this case,

$$\begin{aligned} X_A &= (|K|^{-1/2}\cdot\sinh\alpha, 0, \cosh\alpha),\\ X_B &= (0, |K|^{-1/2}\cdot\sinh\beta, \cosh\beta). \end{aligned}$$



Hence the length of side b is $|K|^{-1/2}\cdot\alpha$. Using a rigid motion of the form

$$M_\theta = \begin{bmatrix} \cos\theta & -\sin\theta & 0\\ \sin\theta & \cos\theta & 0\\ 0 & 0 & 1 \end{bmatrix}$$

when $\theta=\pi/2$ we see that the length of side a is $|K|^{-1/2}\cdot\beta$. Set

$$\gamma = \operatorname{arcosh}\left(\frac{X_A \bullet_K X_B}{|X_A|_K\cdot|X_B|_K}\right).$$

Since we are working on the K-surface,

$$\begin{aligned}
K^{-1} \cdot \cosh\gamma &= X_A \bullet_K X_B \\
&= \begin{bmatrix} |K|^{-1/2} \cdot \sinh\alpha & 0 & \cosh\alpha \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & K^{-1} \end{bmatrix} \begin{bmatrix} 0 \\ |K|^{-1/2} \cdot \sinh\beta \\ \cosh\beta \end{bmatrix} \\
&= K^{-1} \cdot \cosh\alpha \cdot \cosh\beta.
\end{aligned}$$

Problem 164 *Explain how to progress from the fact that*

$$K^{-1} \cdot \cosh\gamma = K^{-1} \cdot \cosh\alpha \cdot \cosh\beta.$$

to the conclusion of the theorem

$$\cosh(|K|^{1/2} \cdot c) = \cosh(|K|^{1/2} \cdot a) \cosh(|K|^{1/2} \cdot b).$$

Problem 165 *Use the Taylor series expansion of* $\cosh(x)$ *centered around* $x = 0$,

$$\cosh(x) = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots$$

to show that for "small" triangles, the hyperbolic Pythagorean Theorem reduces to the euclidean Pythagorean Theorem, meaning

$$c^2 \approx a^2 + b^2.$$

Problem 166 *Summarize the results from this section. In particular, indicate which results follow from the others.*