Heron's formula

In this activity we will give two proofs of Heron's formula.

We'll start by giving a proof using synthetic geometry.

Part I

Proposition 1. *The bisectors of the angles of a triangle meet at a point that is the center of the triangle's inscribed circle.*

Question 1 *How can we prove this?*

Question 2 *Now draw a triangle with vertices A, B, and C. Draw the incircle. Explain why the radii of the incircle touch the sides of the triangle at right angles.*

Question 3 *Label the intersection of the radii with D between A and B, E between B and C, and F between C and A. Compute the areas of the following triangles:*

$$\triangle AOB, \qquad \triangle BOC, \qquad \triangle COA.$$

Use this to express the area of $\triangle ABC$.

Part II

Question 4 *Explain why*

$$\triangle AOD \cong \triangle AOF, \qquad \triangle BOD \cong \triangle BOE, \qquad \triangle COF \cong \triangle EOF.$$

Question 5 *If $AG \cong CE$, explain why $|BG|$ is the semiperimeter.*

Question 6 *Find segments in your drawing equal to the length of*

$$s-a, \qquad s-b, \qquad s-c.$$

Learning outcomes:
Author(s):

Part III

Proposition 2. *If quadrilateral $AHBO$ has diagonals AB and OH with $\angle HAB$ and $\angle HOB$ being right angles, then $AHOB$ can be inscribed in a circle.*

Question 7 *Can you prove this proposition?*

Proposition 3. *The opposite angles of a cyclic quadrilateral sum to two right angles.*

Question 8 *Can you prove this proposition?*

Question 9 *Now we need to decorate our triangle even more:*

(a) *Draw OL perpendicular to OB cutting AB at K.*

(b) *Draw AM perpendicular to OB.*

(c) *Call the intersection of OL and OM, H.*

(d) *Draw BH.*

Consider quadrilateral $AHBO$, explain why opposite angles sum to two right angles.

Question 10 *Explain why $\triangle COF$ is similar to $\triangle BHA$. Use this to explain why*

$$\frac{|AB|}{|AG|} = \frac{|AH|}{r}.$$

Question 11 *Explain why $\triangle KAH$ is similar to $\triangle KDO$. Use this to explain why*

$$\frac{|AK|}{|KD|} = \frac{|AH|}{r}.$$

Question 12 *Now we see*

$$\frac{|AB|}{|AG|} = \frac{|AK|}{|KD|}.$$

Add 1 to both sides to obtain

$$\frac{|BG|}{|AG|} = \frac{|AD|}{|KD|}.$$

Question 13 *Explain why* $\triangle KDO$ *is similar to* $\triangle ODB$. *Use this to explain why*

$$|KD| \cdot |BD| = r^2.$$

Question 14 *Multiply both sides of*

$$\frac{|BG|}{|AG|} = \frac{|AD|}{|KD|}$$

by $\dfrac{|BD|}{|BD|}$ *to obtain*

$$r^2 |BG|^2 = |AG| \cdot |BG| \cdot |AD| \cdot |BD|.$$

Question 15 *Explain how to deduce Heron's formula.*

A modern proof

Question 16 *Now give a modern proof that a high school student might give.*

Question 17 *Which proof was harder? Why didn't the ancient Greeks use our modern proof?*