Bernoulli, Euler, and series

Here we see some topics that both Bernoulli and Euler were interested in.

Finding the sum of the following series is called "The Basel Problem" as it interested several mathematicians with connections to the city of Basel, Switzerland. (Who were they?)

$$1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \frac{1}{36} + \frac{1}{49} \cdots$$

Notice: we are asking for the sum of the reciprocals of the square numbers.

Question 1 *Consider:*

$$f(x) = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \frac{x^8}{9!} - \frac{x^{10}}{11!} + \dots$$

Can you explain why

$$f(x) = \frac{\sin(x)}{x} \qquad x \neq 0?$$

Question 2 *Let $g(x)$ be a polynomial with roots $a_1, \dots, a_n$. Suppose also that $g(0) = 0$. What are the factors of $g(x)$?*

Question 3 *Let $g(x)$ be a polynomial with roots $a_1, \dots, a_n$. Suppose also that $g(0) = 1$. What are the factors of $g(x)$?*

Question 4 *What exactly are the roots of $f(x)$? What is $f(0)$? Explain why:*

$$f(x) = \left(1 - \frac{x}{\pi}\right)\left(1 - \frac{x}{-\pi}\right)\left(1 - \frac{x}{2\pi}\right)\left(1 - \frac{x}{-2\pi}\right)\left(1 - \frac{x}{3\pi}\right)\left(1 - \frac{x}{-3\pi}\right)\cdots$$

Question 5 *Explain why:*

$$f(x) = \prod_{n=1}^{\infty}\left(1 - \frac{x^2}{n^2\pi^2}\right)$$

Question 6 *Explain why:*

$$f(x) = 1 - x^2\sum_{n=1}^{\infty}\frac{1}{n^2\pi^2} + x^4\left(\cdots\right) - x^6\left(\cdots\right) + \cdots$$

Learning outcomes:
Author(s):

Question 7 *Explain why:*

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

Exploration 8 *Compute*

$$\sum_{n=1}^{\infty} \frac{1}{n^3}.$$