# Bernoulli, Euler, and series

*Here we see some topics that both Bernoulli and Euler were interested in.*

While Leibniz was investigating the sum of the reciprocals of the triangular numbers, Johann Bernoulli was investigating the sum of the reciprocals of the integers:

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\cdots$$

This is called the *harmonic series*. Johann Bernoulli's proof started with the following definitions:

$$\begin{aligned} A &= \frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\cdots \\ B &= \frac{1}{2}+\frac{2}{6}+\frac{3}{12}+\frac{4}{20}+\frac{5}{30}+\frac{6}{42}+\cdots \end{aligned}$$

**Question 1** *Next he defined*

$$\begin{aligned} C &= \frac{1}{2}+\frac{1}{6}+\frac{1}{12}+\frac{1}{20}+\frac{1}{30}+\frac{1}{42}+\cdots \\ D &= \phantom{\frac{1}{2}+{}} \frac{1}{6}+\frac{1}{12}+\frac{1}{20}+\frac{1}{30}+\frac{1}{42}+\cdots \\ E &= \phantom{\frac{1}{2}+\frac{1}{6}+{}} \frac{1}{12}+\frac{1}{20}+\frac{1}{30}+\frac{1}{42}+\cdots \\ F &= \phantom{\frac{1}{2}+\frac{1}{6}+\frac{1}{12}+{}} \frac{1}{20}+\frac{1}{30}+\frac{1}{42}+\cdots \end{aligned}$$

*Compute $C$, $D$, $E$, and $F$. Can you convince yourself that the pattern will continue?*

**Question 2** *Explain why*

$$C+D+E+F+\cdots = A.$$

**Question 3** *Explain why*

$$C+D+E+F+\cdots = A+1.$$

**Question 4** *Explain why Johann concluded that the series*

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\cdots$$

*diverges.*

Learning outcomes:
Author(s):

**Question 5** *Explicitly explain how a calculus student could see that the series*

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\cdots$$

*diverges.*

**Question 6** *Do you know any other proofs that the harmonic series diverges?*

On the other hand Euler investigated this sum:

$$1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\frac{1}{25}+\frac{1}{36}+\frac{1}{49}\cdots$$

or the sum of the reciprocals of the square numbers.

**Question 7** *Consider:*

$$f(x)=1-\frac{x^2}{3!}+\frac{x^4}{5!}-\frac{x^6}{7!}+\frac{x^8}{9!}-\frac{x^{10}}{11!}+\dots$$

*Can you explain why*

$$f(x)=\frac{\sin(x)}{x} \qquad x\neq 0?$$

**Question 8** *Let $g(x)$ be a polynomial with roots $a_1,\dots,a_n$. Suppose also that $g(0)=0$. What are the factors of $g(x)$?*

**Question 9** *Let $g(x)$ be a polynomial with roots $a_1,\dots,a_n$. Suppose also that $g(0)=1$. What are the factors of $g(x)$?*

**Question 10** *What exactly are the roots of $f(x)$? What is $f(0)$? Explain why:*

$$f(x)=\left(1-\frac{x}{\pi}\right)\left(1-\frac{x}{-\pi}\right)\left(1-\frac{x}{2\pi}\right)\left(1-\frac{x}{-2\pi}\right)\left(1-\frac{x}{3\pi}\right)\left(1-\frac{x}{-3\pi}\right)\cdots$$

**Question 11** *Explain why:*

$$f(x)=\prod_{n=1}^{\infty}\left(1-\frac{x^2}{n^2\pi^2}\right)$$

**Question 12** *Explain why:*

$$f(x)=1-x^2\sum_{n=1}^{\infty}\frac{1}{n^2\pi^2}+x^4\left(\cdots\right)-x^6\left(\cdots\right)+\cdots$$

**Question 13** *Explain why:*

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

**Exploration 14** *Compute*

$$\sum_{n=1}^{\infty} \frac{1}{n^3}.$$