

Bernoulli, Euler, and series

Here we see some topics that both Bernoulli and Euler were interested in.

While Leibniz was investigating the sum of the reciprocals of the triangular numbers, Johann Bernoulli was investigating the sum of the reciprocals of the integers:

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \cdots$$

This is called the *harmonic series*. Johann Bernoulli's proof started with the following definitions:

$$A = \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \cdots$$

$$B = \frac{1}{2} + \frac{2}{6} + \frac{3}{12} + \frac{4}{20} + \frac{5}{30} + \frac{6}{42} + \cdots$$

Question 1 Next he defined

$$C = \frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \frac{1}{30} + \frac{1}{42} + \cdots$$

$$D = \frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \frac{1}{30} + \frac{1}{42} + \cdots$$

$$E = \frac{1}{12} + \frac{1}{20} + \frac{1}{30} + \frac{1}{42} + \cdots$$

$$F = \frac{1}{20} + \frac{1}{30} + \frac{1}{42} + \cdots$$

Compute C , D , E , and F . Can you convince yourself that the pattern will continue?

Question 2 Explain why

$$C + D + E + F + \cdots = A.$$

Question 3 Explain why

$$C + D + E + F + \cdots = A + 1.$$

Question 4 Explain why Johann concluded that the series

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \cdots$$

diverges.

Question 5 Explicitly explain how a calculus student could see that the series

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \cdots$$

diverges.

Question 6 Do you know any other proofs that the harmonic series diverges?

On the other hand Euler investigated this sum:

$$1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \frac{1}{36} + \frac{1}{49} \cdots$$

or the sum of the reciprocals of the square numbers.

Question 7 Consider:

$$f(x) = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \frac{x^8}{9!} - \frac{x^{10}}{11!} + \cdots$$

Can you explain why

$$f(x) = \frac{\sin(x)}{x} \quad x \neq 0?$$

Question 8 Let $g(x)$ be a polynomial with roots a_1, \dots, a_n . Suppose also that $g(0) = 0$. What are the factors of $g(x)$?

Question 9 Let $g(x)$ be a polynomial with roots a_1, \dots, a_n . Suppose also that $g(0) = 1$. What are the factors of $g(x)$?

Question 10 What exactly are the roots of $f(x)$? What is $f(0)$? Explain why:

$$f(x) = \left(1 - \frac{x}{\pi}\right) \left(1 - \frac{x}{-\pi}\right) \left(1 - \frac{x}{2\pi}\right) \left(1 - \frac{x}{-2\pi}\right) \left(1 - \frac{x}{3\pi}\right) \left(1 - \frac{x}{-3\pi}\right) \cdots$$

Question 11 Explain why:

$$f(x) = \prod_{n=1}^{\infty} \left(1 - \frac{x^2}{n^2\pi^2}\right)$$

Question 12 Explain why:

$$f(x) = 1 - x^2 \sum_{n=1}^{\infty} \frac{1}{n^2\pi^2} + x^4 \left(\cdots\right) - x^6 \left(\cdots\right) + \cdots$$

Question 13 *Explain why:*

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

Exploration 14 *Compute*

$$\sum_{n=1}^{\infty} \frac{1}{n^3}.$$