Limits of axioms

In this activity, we discuss how statements can be independent of axioms.

We will motivate our discussion with questions about cardinality.

Question 1 *Given any finite set S, can you prove that the power set of S has a larger cardinality?*

Consider $S = \mathbb{N}$. We wish to show that $|\mathbb{N}| < |\mathcal{P}(\mathbb{N})|$. Proceed as follows: Seeking a contradiction, suppose that they are equinumerous and imagine a bijection between every element of $\mathbb{N}$ and $\mathcal{P}(\mathbb{N})$. Call a natural number **selfish** if by your bijection it is pared with a set containing itself. Call a natural number **nonselfish** if it is paired with a set not containing itself.

Question 2 *Give part of an example map between $\mathbb{N}$ and $\mathcal{P}(\mathbb{N})$ and clearly identify selfish numbers and nonselfish numbers based on your map.*

Question 3 *Let B be the set of all nonselfish numbers. Explain why $B \in \mathcal{P}(\mathbb{N})$. Arrive at a contradiction. Hint: Which integers map to B?*

Question 4 *So far we have only shown $|\mathbb{N}| \neq |\mathcal{P}(\mathbb{N})|$. How do you conclude that $|\mathbb{N}| < |\mathcal{P}(\mathbb{N})|$?*

Question 5 *Given any set S, can you prove that the power set of S has a larger cardinality? Hint: repeat the argument above.*

We know that $\mathbb{N}$ is countable and that $\mathcal{P}(\mathbb{N})$ is uncountable. Define:

$$\begin{aligned}
\beth_0 &:= |\mathbb{N}| \\
\beth_1 &:= |\mathcal{P}(\mathbb{N})| \\
\beth_2 &:= |\mathcal{P}(\mathcal{P}(\mathbb{N}))| \\
\beth_3 &:= |\mathcal{P}(\mathcal{P}(\mathcal{P}(\mathbb{N})))| \\
&\vdots
\end{aligned}$$

and so on. These are *beth* numbers. From our work above we see that

$$\beth_0 \leq \beth_1 \leq \beth_2 \leq \beth_3 \leq \cdots$$

Question 6 *Let*

$$\varphi : \mathcal{P}(\mathbb{N}) \to [0,1]$$

Author(s):

via the map

$$A \mapsto \sum_{x \in A} \frac{1}{2^x}$$

where $A \subset \mathbb{N}$. *Explain how every element in the image of* φ *can be associated to exactly one number* $0.a_1a_2a_3\ldots$ *where* $a_i = 0$ *or* $a_i = 1$.

Question 7 *Argue that* $\beth_1 = \mathcal{P}(\mathbb{N}) = |[0,1]| = |\mathbb{R}|$.

On the other hand there are also *aleph* numbers. Here

$$\aleph_0 := |\mathbb{N}|$$

but $\aleph_1$ is defined to be the *smallest* infinite cardinal number larger than $\aleph_0$. In general, $\aleph_n$ is the smallest infinite cardinal number larger than $\aleph_{n-1}$. So from this definition we find:

$$\aleph_0 \leq \aleph_1 \leq \aleph_2 \leq \aleph_3 \leq \cdots$$

Question 8 *Say everything you can about the relationship between the aleph numbers and the beth numbers.*

Hilbert's first problem

In 1900 Hilbert made a list of problems to guide the mathematicians of the 20th Century. Here is the first problem on the list:

> Prove the continuum hypothesis.

What is this so-called continuum hypothesis? It states

$$\aleph_1 = \beth_1 = |\mathbb{R}|.$$

Hilbert's second problem

I speculate that finding the "holes" in Euclid's arguments led Hilbert to question the validity of our own proofs. This speculation is supported by the fact that in 1900 the second problem in Hilbert's list of problems was:

> Prove that the axioms of arithmetic are consistent.

Question 9 *What would an counterexample to this claim imply? What would an affirmative proof imply?*

With Hilbert's second problem in mind, in 1901 Bertrand Russell showed that the naive set theory of Cantor cannot be used to answer Hilbert's second problem. Russell proposed that one consider the set of all sets that do not contain themselves.

Question 10 *How does one express this set in "set-builder" notation?*

Question 11 *What is the problem with this set?*

Question 12 *Does this set remind you of anything you've seen before?*

The reader should rest assured that the foundations of mathematics will not come collapsing upon our heads. Russell himself has a resolution based on something called *type theory*, though we cannot discuss this at the moment.

(In)completeness

Now we will turn our attention to Kurt Gödel. In 1931, Gödel proved his (first) incompleteness theorem. To paraphrase:

> Any set of axioms powerful enough to describe "elementary number theory" will have statements that are *true* but *unprovable*, and hence this set of axioms is incomplete.

In 1940, Gödel proved that the continuum hypothesis cannot be disproved using the standard axioms of set theory. Around 1964, Paul Cohen showed that the continuum hypothesis cannot be proved using the standard axioms of set theory. To use the language of vector-spaces,

> The continuum hypothesis is outside the "span" of our standard axioms!

Hence the continuum hypothesis is one of these unprovable statements. There are in fact, many others. Once upon a time, mathematical statements were either true or false. Now we have a third option, the statement could be undecidable. We have matured much since the birth of numbers.