Math 4504: Student Packet

Jenny Sheldon and Bart Snapp

Spring 2018

Contents

1 Earliest Mathematics

We begin our study of the history of mathematics as far back in history as we can. The earliest form of mathematics that we know is counting, as our ancestors worked to keep track of how many of various things they had. and the earliest evidence we have is a prehistoric bone on which have been marked some tallies, which sometimes appear to be in groups of five. You can see a picture of these marks on what is now called the "Ishago bone" at Prehistoric Mathematics[1] . The earliest civilization we know to then develop methods of adding, subtracting, multiplying, and dividing are the ancient Egyptians. In the readings below, we will see some history of the time period, as well as the methods the Egyptians used for counting and basic mathematical operations. The third reading is a timeline, which you might find helpful during this first part of our course.

Readings

First Reading: (Video) The Language of the Universe: Mathematics in Ancient Times[2]

- Watch at least Section 1: Emergence of a New Universe

Second Reading: Egyptian Mathematics[3]

- Section 1: Basic Facts About Ancient Egypt
- Section 2: Counting and Arithmetic: Basics

Third Reading: Egyptian Fractions: Ahmes to Fibonacci to Today[4]

Fourth Reading: Chronology for 30000BC to 500BC[5]

Questions

Question 1 *When are the first symbols for numbers used?* 3400*BC*

Question 2 *What kind of fractions did the Ancient Egyptians use?*

Multiple Choice:

(a) *They did not use fractions.*

(b) *Unit fractions* $\frac{1}{n}$

(c) *Only the fractions* $\frac{1}{2}$, $\frac{2}{3}$, *and* $\frac{3}{4}$

[1] See Prehistoric Mathematics at http://www.storyofmathematics.com/prehistoric.html

[2] See The Language of the Universe: Mathematics in Ancient Times at http://fod.infobase.com.proxy.lib.ohio-state.edu/p_ViewVideo.aspx?xtid=40029

[3] See Egyptian Mathematics at http://www.math.tamu.edu/~don.allen/history/egypt/egypt.html

[4] See Egyptian Fractions: Ahmes to Fibonacci to Today at http://www.jstor.org.proxy.lib.ohio-state.edu/stable/27967280

[5] See Chronology for 30000BC to 500BC at http://www-history.mcs.st-and.ac.uk/Chronology/30000BC_500BC.html

(d) *All fractions that we use today.*

Question 3 *What are the most important points of this reading?*

2 Begin to Count

Activity: Bunt Chapter 1, Part 1

Problem 4 *Write the numbers from 1 to 100 (counting by fives) in hieroglyphic.*

Problem 5 *Compare and contrast the hieroglyphic counting system with our (Hindu-Arabic) system.*

Problem 6 *Perform the following operations in hieroglyphic, using the method that ancient Egyptians would have:*

(a) $2371 + 185$

(b) $3914 - 1609$

Problem 7 *Write* $\frac{4}{17}$ *as a sum of unique unit fractions in both Hindu-Arabic symbols and in hieroglyphic.*

Problem 8 *Solve the following using the "doubling and adding" method of multiplication. You may use Hindu-Arabic numerals if you'd like!*

(a) 13×33

(b) $36 \div 5$

(c) $6 \div 17$

Problem 9 *What is the value of the tables that ancient Egyptians often used for their calculations?*

3 Solve Like an Egyptian

We have seen how the ancient Egyptians worked through basic arithmetic problems. Our next question should then be: what kinds of problems did they solve? Generally, the ancient Egyptians are known for solving practical, every-day problems that had to do with administering their large empire. Scribes would solve geometry and arithmetic problems as part of their jobs, and other people in the empire would generally not know or use such mathematics. Scribes were trained in scribal schools, where they learned mathematics. The best examples we have come from papyrus rolls which have been preserved and translated. Our goal for the readings below is to understand some examples of how the ancient Egyptians solved arithmetic and geometric problems.

Readings

First Reading: Mathematics Problems from Ancient Egyptian Papyri[6]

Second Reading: Mathematics in Egyptian Papyri[7]

Third Reading: The Moscow Papyrus[8]

Fourth Reading: Summary of Egyptian Mathematics[9]

Questions

Question 10 *What was the Egyptian value for π?* 3.1605

Question 11 *What is the first step in the Method of False Position?*

Multiple Choice:

(a) *Moving everything to one side of the equals sign.*

(b) *Labeling a variable.*

(c) *Guessing.*

(d) *Doubling numbers using a chart.*

Question 12 *What are the most important points of this reading?*

[6]See Mathematics Problems from Ancient Egyptian Papyri at `http://www.jstor.org.proxy.lib.ohio-state.edu/stable/20876630`

[7]See Mathematics in Egyptian Papyri at `http://www-history.mcs.st-and.ac.uk/HistTopics/Egyptian_papyri.html`

[8]See The Moscow Papyrus at `http://www.math.tamu.edu/~don.allen/history/egypt/node4.html`

[9]See Summary of Egyptian Mathematics at `http://www.math.tamu.edu/~don.allen/history/egypt/node5.html`

4 The method of false position

In this activity we will seek to understand the method of false position.

Exercise 13 *Solve the following algebra problem:*

$$x^2 + y^2 = 52$$
$$2x = 3y$$

Question 14 *While I am not sure which method you used to solve this problem, ancient Egyptians used the method of false position to solve problems like this. Moreover, such a method was taught in American schools until the mid 1800's. Here is the solution using false position—without any explanation!*

(a) *Set* $x = 3$ *and* $y = 2$.

(b) $3^2 + 2^2 = 13$.

(c) $52/13 = 4$.

(d) $\sqrt{4} = 2$.

(e) $x = 2 \cdot 3$ *and* $y = 2 \cdot 2$.

Explain the algorithm used and give another example to show you know how it is done.

Exploration 15 *Can you explain* ***why*** *the method of false position works?*

Exercise 16 *Solve the following problem: One hundred dollars is to be split among four siblings: Ali, Brad, Cara, and Denise where Brad gets four more dollars than Ali, Cara gets eight more dollars than Brad, and Denise gets twice as much as Cara. How much does each sibling get?*

Question 17 *Here is the solution by double false position:*

(a) *Suppose Ali gets 6 dollars.*

(b) *The total now is not 100, but 70. We are too low by 30.*

(c) *Now suppose Ali gets 8 dollars.*

(d) *The total now is not 100, but 80. We are too low by 20.*

(e) *Compute*

$$\frac{8 \cdot 30 - 6 \cdot 20}{30 - 20} = 12.$$

This is the correct answer.

Explain the algorithm used and give another example to show you know how it is done.

Exploration 18 *Can you explain **why** the method of double false position works?*

5 Babylonian Basics

The next ancient civilization we will study are the ancient Babylonians. As with the Egyptians, we will begin with their number system and ways of doing arithmetic, then move on in the next section to problem-solving methods. The two readings below have some overlap, which will hopefully help to clarify the material. Especially when it comes to these ancient civilizations, it's also good to have several perspectives, as historians' opinions can differ quite wildly. For instance, historians have long thought that the tablet called "Plimpton 322" was a list of Pythagorean Triples, and used this evidence to cite that the ancient Babylonians knew this theorem long before Pythagoras. However, historians are still looking at this tablet to determine whether this conclusion is accurate. An article whose author takes a different interpretation is Words and Pictures: New Light on Plimpton 322[10] . An even more recent article (August 2017) takes a different approach still: Mathematical secrets of ancient tablet unlocked after nearly a century of study[11] .

If after you read these selections and still have some questions, the MacTutor articles on this topic may also be helpful. In particular, the Babylonian method of division is often very confusing! The optional third reading about reciprocals of numbers addresses this topic.

MacTutor articles (optional):

- An overview of Babylonian mathematics[12]
- Babylonian numerals[13]

Readings

First Reading: Counting in Cuneiform[14]

Second Reading: Babylonian Mathematics[15]

Optional Third Reading: Babylonian Mathematical Texts I: Reciprocals of Regular Sexagesimal Numbers[16]

Questions

Question 19 *What base did the Babylonians use for their number system?* 60

Question 20 *Why did the Babylonians make tables? Choose the best answer.*

Multiple Choice:

(a) *For fun.*

[10]See Words and Pictures: New Light on Plimpton 322 at `http://www.jstor.org/stable/2695324`

[11]See Mathematical secrets of ancient tablet unlocked after nearly a century of study at `https://www.theguardian.com/science/2017/aug/24/mathematical-secrets-of-ancient-tablet-unlocked-after-nearly-a-century-of-study?CMP=fb_gu`

[12]See An overview of Babylonian mathematics at `http://www-history.mcs.st-and.ac.uk/HistTopics/Babylonian_mathematics.html`

[13]See Babylonian numerals at `http://www-history.mcs.st-and.ac.uk/HistTopics/Babylonian_numerals.html`

[14]See Counting in Cuneiform at `http://www.jstor.org.proxy.lib.ohio-state.edu/stable/30211866`

[15]See Babylonian Mathematics at `http://www.math.tamu.edu/~dallen/masters/egypt_babylon/babylon.pdf`

[16]See Babylonian Mathematical Texts I: Reciprocals of Regular Sexagesimal Numbers at `http://www.jstor.org.proxy.lib.ohio-state.edu/stable/1359434`

(b) *For use by non-mathematicians.*

(c) *To demonstrate solutions to problems.*

(d) *To simplify calculations.*

Question 21 *What are the most important points of this reading?*

6 Babylonian numbers

In this activity we explore the number system of the ancient Babylonians.

The ancient Babylonians used cuneiform characters to write their numbers.

Exercise 22 *What are the 2 basic ancient Babylonian numerical symbols and what do they mean?*

Exploration 23 *Discuss the limitations of the Babylonian system. Then debate whether these so-called limitations were actually limitations at all.*

Exploration 24 *Is the Babylonian system more of a place-value system or a concatenation system?*

Problem 25 *Fill out the following table, simplifying any calculations.*

$Hindu-Arabic$	$Cuneiform$	$Hindu-Arabic$	$Cuneiform$	$Hindu-Arabic$	$Cuneiform$
5×1		5×2		5×3	
5×4		5×5		5×6	
5×7		5×8		5×9	
5×10		5×20		5×30	
5×40		5×50		$\frac{1}{5}$	
$\frac{1}{4}$		$\frac{1}{9}$		$\frac{1}{10}$	
$\frac{5}{6}$		$\frac{1}{20}$		$\frac{1}{100}$	

Problem 26 *Use your table to make the following calculations. You should work in base sixty, though you may use Hindu-Arabic numerals.*

(a) 34×5

(b) $1, 47 \div 5$

(c) $150 \div 4$

(d) $8, 6, 15 \div 6, 40$

7 Babylonian Geometry

As we began to see in the previous section's readings, the Babylonians could solve many different kinds of problems. In some ways, their methods were more advanced than the Egyptian methods. We would like to study (or look again at) the ways the Babylonians approximated square roots, how they solved quadratic equations, and several different kinds of geometry problems. Again, with material that is repeated in more than one place, it's good to pay attention to the different opinions of different authors!

Readings

First Reading: A Babylonian Geometrical Algebra[17]

Second Reading: Ancient Babylonian Astronomers Calculated Jupiter's Position from the Area Under a Time-Velocity Graph[18]

Third Reading: Pythagoras's Theorem in Babylonian Mathematics[19]

Questions

Question 27 *The time interval over which the ancient Babylonians made observations about the position of Jupiter was how many days long?* 60 *days.*

Question 28 *Which of the following is not a challenge when translating ancient Babylonian tablets?*

Multiple Choice:

(a) *We do not know the ancient Babylonian language.*

(b) *Some parts of the documents are broken off or missing.*

(c) *Scribes occasionally made errors when copying documents.*

(d) *Tables of numbers are not usually labeled with their purpose.*

Question 29 *What are the most important points of this reading?*

---

[17]See A Babylonian Geometrical Algebra at `http://www.jstor.org.proxy.lib.ohio-state.edu/stable/2686867`

[18]See Ancient Babylonian Astronomers Calculated Jupiter's Position from the Area Under a Time-Velocity Graph at `http://science.sciencemag.org.proxy.lib.ohio-state.edu/content/351/6272/482`

[19]See Pythagoras's Theorem in Babylonian Mathematics at `http://www-history.mcs.st-and.ac.uk/HistTopics/Babylonian_Pythagoras.html`

8 The Beginning of the Greek Era

Now that we have looked at the mathematics done by the ancient Egyptians and Babylonians, we turn our attention to the Greeks. Here, we find the first evidence of what we might recognize today as "mathematics": statements about objects and their properties, followed by proofs of these statements. We will begin by considering two of the most well-known Greek mathematicians: Thales and Pythagoras.

Readings

First Reading: Thales: Our Founder?[20]

Second Reading: Pythagoras biography[21]

Third Reading: A Brief History of Numbers[22]

- Read Section 2.2, pages 28 - 30. (The earlier parts of Section 2.2 are a review of Egypt and Babylon if you'd like to read them.)
- Read Chapter 3, pages 31 - 42

Questions

Question 30 *How many theorems was Thales said to have proven?* 5

Question 31 *What is Pythagoras primarily considered?*

Multiple Choice:

(a) *A philosopher.*

(b) *A mathematician.*

(c) *A traveler.*

(d) *An astronomer.*

Question 32 *What are the most important points of this reading?*

[20] See Thales: Our Founder? at `http://www.jstor.org/stable/3615512`

[21] See Pythagoras biography at `http://www-history.mcs.st-and.ac.uk/Biographies/Pythagoras.html`

[22] See A Brief History of Numbers at `http://ebookcentral.proquest.com.proxy.lib.ohio-state.edu/lib/ohiostate-ebooks/detail.action?docID=2084891`

9 Rational numbers and similarity

In this activity we play a game of "what if" and see a reason that the ancient Greeks might have wanted every number to be rational.

Exploration 33 *Think about plain old plane geometry. What are some theorems that you would want to be true?*

Question 34 *What are the basic theorems involving similar triangles?*

OK—now we are going to do something very strange. Let's suppose that every number is rational. In essence, let's put ourselves into the mindset of the ancient Greeks, **before** they knew that irrational numbers existed.

Exploration 35 *Suppose that you have two triangles whose angles are congruent. Can you make a fairly simple argument, using the fact that the sides are rational numbers, that shows that the sides are proportional? Hint: You may need to use ASA.*

Exploration 36 *Suppose that you have two triangles whose sides are proportional. Can you make a fairly simple argument, using the fact that the sides are rational numbers, that shows that the angles are congruent? Hint: You may need to use SSS.*

10 Pythagorean Mathematics

Pythagoras is certainly an interesting historical character, and many people have researched his life to try to divine fact from fiction. For instance, perhaps the most famous mathematical theorem is called "The Pythagorean Theorem", but scholars debate how early the theorem was known. Perhaps it was known to the Babylonians, with scholars citing Plimpton 322 as evidence. Perhaps some cases were known to the Egyptians, where legend says they used a 3-4-5 triangle to ensure their buildings had square corners. Some people argue that the first known proof of the theorem comes from the school Pythagoras founded, while other scholars find the evidence for this claim to be thin.

In this section, we will focus on the school Pythagoras founded, sometimes called the Pythagorean Society, the School of Pythagoras, or the Pythagorean Cult. We will examine their beliefs, particularly concerning numbers and geometry, and see what came out of what was perhaps the first school of mathematics in the ancient world.

Readings

First Reading: The Cult of Pythagoras[23]

Second Reading: Greek Mathematics - Pythagoras[24]

Third Reading: The End of a Perfect Number[25]

Fourth Reading: Means Appearing in Geometric Figures[26]

Questions

Question 37 *What is the harmonic mean of* $\frac{1}{2}$ *and* $\frac{3}{4}$*?* 0.6

Question 38 *Which of the following was not a supposed belief or practice of the Pythagoreans?*

Multiple Choice:

(a) *Not eating beans.*

(b) *Never passing an ass lying in the street.*

(c) *Only wearing the color red.*

(d) *Vegetarianism.*

Question 39 *What are the most important points of this reading?*

[23]See The Cult of Pythagoras at http://classicalwisdom.com/cult-of-pythagoras/

[24]See Greek Mathematics - Pythagoras at http://www.storyofmathematics.com/greek_pythagoras.html

[25]See The End of a Perfect Number at http://www.jstor.org.proxy.lib.ohio-state.edu/stable/27957201

[26]See Means Appearing in Geometric Figures at http://www.jstor.org.proxy.lib.ohio-state.edu/stable/3219084

11 Pythagorean means

In this activity we explore the three different means of the ancient Greeks.

The arithmetic mean

The *arithmetic mean* is the good-old mean that we are all familiar with.

Question 40 *What is the mean that we are all familiar with? Explain how to compute the mean of $a_1, a_2, \ldots, a_n$. Give some examples.*

The geometric mean

The *geometric mean* is a bit different. The geometric mean of $a_1, a_2, \ldots, a_n$ is given by:

$$\left(\prod_{i=1}^{n} a_i\right)^{1/n}$$

Question 41 *Explain an analogy between the arithmetic mean and the geometric mean.*

Question 42 *Can you explain the geometric mean in terms of geometry? First do it for 2 numbers. Next do it for three.*

The harmonic mean

The *harmonic mean* might be the most mysterious of all. The harmonic mean of $a_1, a_2, \ldots, a_n$ is given by:

$$\frac{n}{\sum_{i=1}^{n} \frac{1}{a_i}}$$

Exploration 43 *Can you find a connection between the harmonic mean and music?*

Question 44 *In the United States, the fuel efficiency of a car is usually given in the units:*

$$\frac{\textit{miles}}{\textit{gallon}}$$

However, in Europe, the fuel efficiency of a car is usually given in the units:

$$\frac{\textit{liters}}{100\text{km}}$$

Give some examples of fuel efficiency (both efficient and inefficient) with each set of units.

Question 45 *Now suppose that a car gets* $60\frac{\text{miles}}{\text{gallon}}$ *and another car gets* $20\frac{\text{miles}}{\text{gallon}}$. *What is the average fuel efficiency?*

Question 46 *Now suppose that a car gets* $4\frac{\text{liters}}{100\text{km}}$ *and another car gets* $20\frac{\text{liters}}{100\text{km}}$. *What is the average fuel efficiency?*

Exploration 47 *Compare your answers to the last two questions. Something fishy is going on, what is it?*

12 Circles and Lunes

In this section, we begin to read in our textbook for the course. Dunham will begin from the beginning of mathematics as we know it, so much of the first reading will be a brief review of things we have studied so far. Dunham's book is generally organized so that there is one "Great Theorem" in each chapter of the book. The first of these is the earliest mathematical proof we have in its original form, by a Greek mathematician named Hippocrates.

This result was not Hippocrates' only result, and he will play a role in the solutions to the other problems we will discuss in the next two sections. The second reading looks at some of Hippocrates other results from a more modern perspective. I encourage you to try some of the dynamic geometry exercises; as the article mentions, GeoGebra is free to use online. You do not need to be able to recite the proofs of all of the lunes considered, but you should be able to give at least the proof of the Great Theorem in our textbook.

Readings

First Reading: Dunham, Chapter 1, pages 1 - 20

Second Reading: Exploring the lunes of Hippocrates in a dynamic geometry environment[27]

Questions

Question 48 *How many types of lunes was Hippocrates able to square?* 3

Question 49 *What does it mean to "square" a particular figure?*

Multiple Choice:

(a) *To cut up and rearrange the figure until it looks like a square.*

(b) *To use compass and straight edge to produce a triangle with the same area.*

(c) *To use compass and straight edge to produce any other figure with the same area which is known to be quadrable.*

(d) *To use compass and straight edge to construct a square whose corners intersect with the sides of the figure.*

Question 50 *What are the most important points of this reading?*

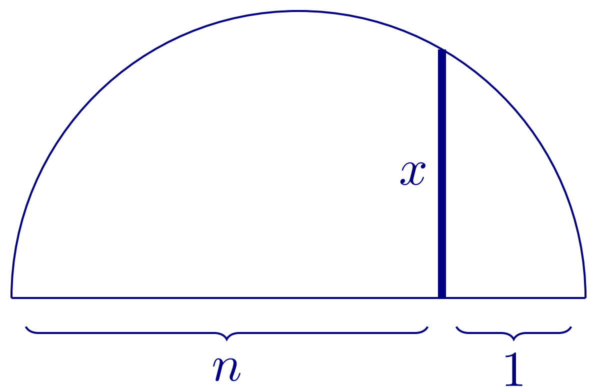
[27] See Exploring the lunes of Hippocrates in a dynamic geometry environment at `http://dx.doi.org.proxy.lib.ohio-state.edu/10.1080/17498430.2015.1122301`

13 Computing quadratures

In this activity we will compute some basic quadratures.

When computing a quadrature of a shape in the method of the ancient Greeks, one needs to produce a line segment whose length gives the side of a square of equal area to the original shape.

Question 51 *Consider the figure below. Explain how one could construct it and what segment x represents.*



Question 52 *Construct a rectangle whose side lengths are 8 units and 5 units. Then construct its quadrature. Explain your construction step-by-step, and tell why it works!*

Question 53 *Construct a triangle whose base has length 8 units and whose height has length 5 units. Then construct its quadrature. Explain your construction step-by-step, and tell why it works!*

Question 54 *Suppose you have a square whose side length is 8 units and another square whose side length is 15 units. How would you construct the quadrature of the two areas together? Explain how you know.*

Question 55 *How do you compute the quadrature of a polygon?*

14 The Three Problems of Antiquity

Hippocrates' squaring of a particular lune in the Great Theorem of our first chapter of Dunham's book is just the beginning of a discussion about three famous problems on which ancient Greek mathematicians worked. These three problems are:

(a) Can every circle be squared using only compass and straightedge?

(b) Can every cube be doubled using only compass and straightedge?

(c) Can every angle be trisected using only compass and straightedge?

We will find that the answer to each of these questions is actually "no", and that the negative answer is potentially the more interesting answer for this question. As the Greek mathematicians explored these problems, they also began to change the questions slightly as they asked them. Changing the question slightly in order to get a related answer is a very common mathematical method of today, and one of the best ways to begin to chip away at a difficult mathematical problem.

In this section, we will explore briefly the solution to the circle squaring problem, and we will also look at the cube doubling problem. In the next section, we will tackle the angle trisection problem.

Readings

First Reading: Dunham, Chapter 1, pages 20 - 26

Second Reading: Doubling the Cube[28]

Third Reading: A Method of Duplicating the Cube[29]

Questions

Question 56 *How many stories are told in the readings about how the cube doubling problem originated?*
2

Question 57 *What geometric subject did Menaechmus discover while trying to duplicate the cube?*

Multiple Choice:

(a) *Conic sections.*

(b) *Calculus.*

(c) *Formulas for volumes of pyramids.*

(d) *Formulas for slopes of lines.*

Question 58 *What are the most important points of this reading?*

[28] See Doubling the Cube at `http://www-history.mcs.st-and.ac.uk/HistTopics/Doubling_the_cube.html`

[29] See A Method of Duplicating the Cube at `https://www.maa.org/sites/default/files/Graef32917.pdf`

15 It's All Greek To Me!

We investigate solutions to the Problems of Antiquity.

Question 59 *Use your compass and straightedge to double a square with side length s. That is, construct a square whose area is twice that of your original square. Why can't we do the same for the cube?*

Hippocrates used a "continued mean proportional" to double the cube. Let's let a be the side length of the original cube, and x be the side length of the new, larger cube.

Question 60 *Just to check: write an equation relating a and x.*

Here is an outline for how Hippocrates might have gotten his mean proportional:

(a) Start with two cubes of side length a next to each other.

(b) Rearrange the volume of these two cubes into a rectangular prism so that the height is unchanged, but the base rectangle has one side of length x. Call the other side length y.

(c) Rearrange the volume again, but this time into a cube. Use a different side as the base, and leave the "height" x unchanged.

Question 61 *Draw pictures representing the geometry in the method described above.*

Question 62 *Write a proportion corresponding to the first rearrangement using the variables x, y, a, and $2a$. Hint: we know the height of the box stays the same. What does this mean about the area of the base?*

Question 63 *Write a proportion corresponding to the second rearrangement. One of the fractions should be equal to one of the fractions from the previous question! Then write Hippocrates' continued mean proportional by setting three fractions equal to each other.*

Question 64 *How is this continued mean proportional related to the equation you found in Question 2? How might you use this information to duplicate the cube?*

About 420BC, Hippias invented a curve called the "quadratrix". Here is its construction:

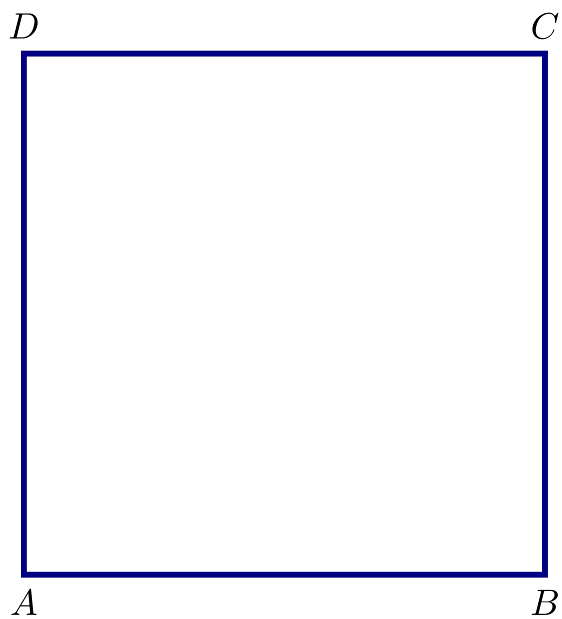
(a) Start with a square $ABCD$.

(b) A line segment congruent with AB and coinciding with AB rotates with center A a quarter turn.

(c) At the same time, and at the same speed, another segment congruent with AB and coinciding with AB moves using straight-line motion through the square until it coincides with CD.

(d) Points on the quadratrix are where the two moving segments intersect.

Question 65 *On the square below, use a ruler and a protractor to construct at least four points on the quadratrix, and then sketch the entire curve.*



Question 66 *Let X be any point on the quadratrix, and X' be the point directly below X on segment AB. If l is the length of segment AB and n is the length of segment XX', explain why it's always true that*

$$\frac{m\angle XAB}{m\angle DAB} = \frac{n}{l}.$$

Question 67 *Use the quadratrix to trisect an angle of 45°, then an angle of 60°. Then, explain how the quadratrix can be used to trisect any angle.*

Question 68 *Can you use the quadratrix to square the circle? Explain how!*

16 Trisecting

We arrive now at our third of three problems of antiquity, the angle trisection problem. Again, we will look at the origin of this problem in history, some ways the ancient Greeks solved it, some modern ways of solving it, and the eventual solution to the problem.

If you are interested in one proof that an arbitrary angle cannot be trisected, you can see Proof of the Impossibility of Trisecting an Angle with Euclidean Tools[30] . The proof we'll look at in class is essentially that of the third reading.

Readings

First Reading: Trisecting an Angle[31]

Second Reading: The Quadratrix, a Simple but Remarkable Curve[32]

Third Reading: Proof that one can not Trisect an angle of 60 degrees with Straight Edge and Compass[33]

Fourth Reading: An Iterative Angle Trisection[34]

Questions

Question 69 *In what year were both the cube duplication and angle trisection problems solved by Wantzel?* 1837

Question 70 *Which of the following is not possible with only straightedge and compass?*

Multiple Choice:

(a) *Trisecting an angle of* 90°.

(b) *Trisecting any line segment.*

(c) *Trisecting an angle of* 45°.

(d) *Trisecting an angle of* 60°.

Question 71 *What are the most important points of this reading?*

[30] See Proof of the Impossibility of Trisecting an Angle with Euclidean Tools at `http://www.jstor.org.proxy.lib.ohio-state.edu/stable/2688093`

[31] See Trisecting an Angle at `http://www-history.mcs.st-and.ac.uk/HistTopics/Trisecting_an_angle.html`

[32] See The Quadratrix, a Simple but Remarkable Curve at `http://onlinelibrary.wiley.com.proxy.lib.ohio-state.edu/doi/10.1111/j.1949-8594.1952.tb06937.x/abstract`

[33] See Proof that one can not Trisect an angle of 60 degrees with Straight Edge and Compass at `http://www.math.toronto.edu/rosent/Mat246Y/OLDPDF/week19.pdf`

[34] See An Iterative Angle Trisection at `http://www.jstor.org.proxy.lib.ohio-state.edu/stable/27646441`

17 Proofs of the Pythagorean Theorem

We will study Euclid for two chapters - the first focused on geometry and the second focused on number theory. Euclid's name is worth knowing because of his work called the "Elements", where he attempts to construct all of the mathematics known at the time from basic assumptions he calls "common notions" and "postulates". By the time we are finished with this chapter, you should be able to state Euclid's Fifth Postulate and say something about why it was controversial.

Two important people who influenced Euclid's thinking are Eudoxus[35] , most famous for his "method of exhaustion", and Aristotle[36] who wrote on what proof in mathematics should be, and may have been the first to use the phrase "common notion".

Readings

First reading: Dunham Chapter 2

Second Reading: Proofs of the Pythagorean Theorem[37]

In the second reading, you should read the introduction, and then pick a few of these proofs to study. You do not need to know all of the proofs on this site! You should be able to give, in full detail, the proof from our textbook (which is also Proof #1 on the site) as well as two other proofs of your choice.

Questions

Question 72 *How many proofs are listed on this site?* 118

Question 73 *Which of the following is NOT a category of proofs of the theorem mentioned in the remarks?*

Multiple Choice:

(a) *Proofs by contradiction.*

(b) *Algebraic proofs.*

(c) *Geometric proofs.*

(d) *Trigonometric proofs.*

Question 74 *What are the most important points from this reading?*

[35] See Eudoxus at `http://www-groups.dcs.st-and.ac.uk/history/Biographies/Eudoxus.html`
[36] See Aristotle at `http://www-groups.dcs.st-and.ac.uk/history/Biographies/Aristotle.html`
[37] See Proofs of the Pythagorean Theorem at `http://www.cut-the-knot.org/pythagoras/`

18 Euclid's Elements

We prove some propositions from Euclid's Elements.

A full list of the definitions, common notions, postulates, and propositions in Book I is posted on Carmen, downloaded from `http://aleph0.clarku.edu/~djoyce/java/elements/bookI/bookI.html`

Proposition 1 (I.5)**.** *In isosceles triangles the angles at the base equal one another, and, if the equal straight lines are produced further, then the angles under the base equal one another.*

Question 75 *Prove Proposition I.5.*

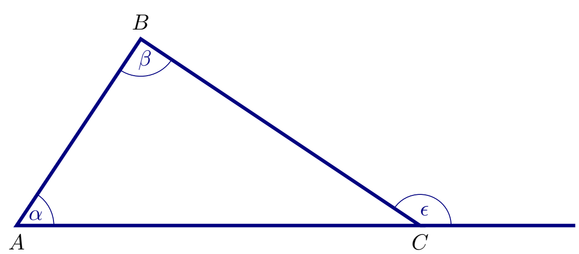
Proposition 2 (I.6)**.** *If in a triangle two angles equal one another, then the sides opposite the equal angles also equal one another.*

Question 76 *Prove Proposition I.6.*

Proposition 3 (I.15)**.** *If two straight-lines cut one another then they make the vertically opposite angles equal to one another.*

Question 77 *Prove Proposition I.15.*

Proposition 4 (I.16)**.** *For any triangle, when one of the sides is produced, the external angle is greater than each of the internal and opposite angles.*



Question 78 *Prove Proposition I.16.*

Proposition 5 (I.27)**.** *If a straight-line falling across two straight-lines makes the alternate angles equal to one another then the (two) straight-lines will be parallel to one another.*

Question 79 *Prove Proposition I.27.*

Proposition 6 (I.29)**.** *A straight-line falling across parallel straight-lines makes the alternate angles equal to one another, the external (angle) equal to the internal and opposite (angle), and the (sum of the) internal (angles) on the same side equal to two right-angles.*

Question 80 *Prove Proposition I.29.*

Proposition 7 (I.32)**.** *In any triangle, (if) one of the sides (is) produced (then) the external angle is equal to the (sum of the) two internal and opposite (angles), and the (sum of the) three internal angles of the triangle is equal to two right-angles.*

Question 81 *Prove Proposition I.32.*

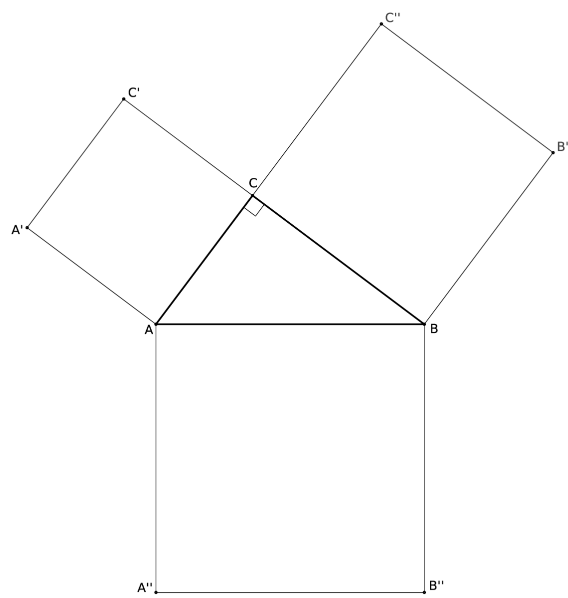
19 The Pythagorean Theorem

In this activity we will prove the most famous theorem of all.

Question 82 *Remind us, what is the most famous theorem of all and what exactly does it assert?*

Euclid's proof

Question 83 *What would one need to prove about the following diagram to prove the Pythagorean Theorem?*



Let's see if we can do this!

Question 84 *Draw a line perpendicular to $\overline{AB}$ that passes though both C and $\overline{A''B''}$. Call the intersection between this line and $\overline{AB}$, point E; call the intersection point between this line and $\overline{A''B''}$, point E'. Explain why $\triangle ACA''$ has half the area of rectangle $AEE'A''$.*

Question 85 *Explain why $\triangle ABA'$ has half the area of square $ACC'A'$.*

Question 86 *Explain why $\triangle ACA''$ is congruent to $\triangle ABA'$.*

Question 87 *Explain why area of square $ACC'A'$ is equal to the area of rectangle $AEE'A''$.*

Question 88 *Use similar ideas to complete a proof the Pythagorean Theorem.*

The converse

Question 89 *What is the converse to the Pythagorean Theorem? Is it true? How do you prove it?*

20 Unsolved Problems

Our next chapter is the first where we'll discuss number theory, a branch of mathematics wherein we study properties of whole numbers and relationships between them. Number theory is sometimes used as an introduction to "higher mathematics", because the definitions are usually easy to grasp. Examples are often easy to come by, and conjectures seem to follow naturally.

Once we begin to ask questions about the relationships between numbers, we quickly realize that many of these questions are incredibly easy to state, but incredibly difficult to prove! In fact, some of the most well-known unsolved problems in mathematics come from the branch of number theory.

In the second reading, we'll be introduced to some famous unsolved problems in mathematics. Some of these problems are mentioned in the first reading, and some of them are not. As you look at this article, make sure to click on some of the links to get more information about topics that look interesting to you. The Collatz problem, for instance, is very easy to try out for yourself. The notion that 10 is a solitary number is related to some things we read about the Pythagoreans, and of course we've already discussed the case of odd perfect numbers.

Readings

First reading: Dunham, Chapter 3, pages 61-73

Second reading: Difficult Problems[38] .

In the second reading, please click on the links on the page to get more information about these problems. You should read at least the following.

(a) The Goldbach Conjecture[39]

(b) Twin Primes[40]

(c) The Twin Prime Conjecture[41]

Questions

Question 90 *How many versions or types of Goldbach's conjecture are listed in the article on that topic?*
7

Question 91 *Which of the following are twin primes?*

Multiple Choice:

(a) 1 *and* 3

(b) 11 *and* 23

[38] See Difficult Problems at http://mathworld.wolfram.com/UnsolvedProblems.html
[39] See The Goldbach Conjecture at http://mathworld.wolfram.com/GoldbachConjecture.html
[40] See Twin Primes at http://mathworld.wolfram.com/TwinPrimes.html
[41] See The Twin Prime Conjecture at http://mathworld.wolfram.com/TwinPrimeConjecture.html

(c) 23 *and* 46

(d) 29 *and* 31

Question 92 *What are the most important points from this reading?*

21 The unique factorization theorem

In this activity we investigate unique factorization theorems.

Consider this proposition from Euclid's *Elements*:

Proposition 8 (IX.14)**.** *If a number is the least that is measured by prime numbers, then it is not measured by any other prime number except those originally measuring it.*

Question 93 *Explain what the proposition above is saying.*

Question 94 *Now consider Euclid's proof:*

> *Let the number A be the least that is measured by the prime numbers B, C, and D. I say that A is not measured by any other prime number except B, C, or D. If possible, let it be measured by the prime number E, and let E not be the same as any one of the numbers B, C, or D.*
>
> *Now, since E measures A, let it measure it according to F, therefore E multiplied by F makes A. And A is measured by the prime numbers B, C, and D. But, if two numbers multiplied by one another make some number, and any prime number measures the product, then it also measures one of the original numbers, therefore each of B, C, and D measures one of the numbers E or F. Now they do not measure E, for E is prime and not the same with any one of the numbers B, C, or D. Therefore they measure F, which is less than A, which is impossible, for A is by hypothesis the least number measured by B, C, and D. Therefore no prime number measures A except B, C, and D. Therefore, if a number is the least that is measured by prime numbers, then it is not measured by any other prime number except those originally measuring it.*

Can you explain what this proof is saying?

Now let's consider a crazy set of numbers—all multiples of 3. Let's use the symbol $3\mathbb{Z}$ to denote the set consisting of all multiples of 3. As a gesture of friendship, I have written down the first 100 nonnegative integers in $3\mathbb{Z}$:

0	3	6	9	12	15	18	21	24	27
30	33	36	39	42	45	48	51	54	57
60	63	66	69	72	75	78	81	84	87
90	93	96	99	102	105	108	111	114	117
120	123	126	129	132	135	138	141	144	147
150	153	156	159	162	165	168	171	174	177
180	183	186	189	192	195	198	201	204	207
210	213	216	219	222	225	228	231	234	237
240	243	246	249	252	255	258	261	264	267
270	273	276	279	282	285	288	291	294	297

Question 95 *Given any two integers in $3\mathbb{Z}$, will their sum be in $3\mathbb{Z}$? Explain your reasoning.*

Question 96 *Given any two integers in $3\mathbb{Z}$, will their difference be in $3\mathbb{Z}$? Explain your reasoning.*

Question 97 *Given any two integers in $3\mathbb{Z}$, will their product be in $3\mathbb{Z}$? Explain your reasoning.*

Question 98 *Given any two integers in $3\mathbb{Z}$, will their quotient be in $3\mathbb{Z}$? Explain your reasoning.*

Definition 1. *Call a positive integer* ***prome*** *in $3\mathbb{Z}$ if it cannot be expressed as the product of two integers both in $3\mathbb{Z}$.*

As an example, I tell you that 6 is prome number in $3\mathbb{Z}$. You may object because $6 = 2 \cdot 3$, but remember—2 is not in $3\mathbb{Z}$!

Question 99 *List all the prome numbers less than 297.*

Question 100 *Can you give some sort of algebraic characterization of prome numbers in $3\mathbb{Z}$?*

Question 101 *Can you find numbers that factor completely into prome numbers in two different ways? How many can you find?*

22 On the Infinitude of Primes

The great theorem of this chapter is, essentially, that there are infinitely many primes. In our readings, we'll see Euclid's proof of this fact as well as another proof by a mathematician named Hillel Furstenberg. Furstenberg is probably most famous for his contributions to an area of mathematics called "ergodic theory", in which we study moving systems.

Readings

First reading: Dunham, Chapter 3, pages 73-83

Second reading: On Furstenburg's Proof of the Infinitude of Primes [42]

Questions

Question 102 *What is the example given of an arithmetic progression?* $2 + 7\mathbb{Z}$

Question 103 *To which branch of mathematics is Furstenberg's proof method most related? In other words, what makes his approach different from Euclid's?*

Multiple Choice:

(a) *Number Theory*

(b) *Calculus*

(c) *Topology*

(d) *Ergodic Theory*

Question 104 *What are the most important points from this reading?*

[42] See On Furstenburg's Proof of the Infinitude of Primes at http://www.jstor.org.proxy.lib.ohio-state.edu/stable/40391095

23 Pi

We are now introduced to perhaps the greatest mathematician of antiquity, Archimedes. There are many things for which he is famous, but for our Great Theorem, Dunham has chosen his proof that the area formula for circles is $A = \pi r^2$. Of course, Archimedes won't write the formula quite this way, and we will see what he does to get around the notation. Another circle-related result for which Archimedes is quite famous is his approximation for π, which we still sometimes use today. Our second reading will be a history of this constant throughout history. People are still studying π today! For more information about modern estimations of π, see the third reading. The optional fourth reading has more detail than the second one.

In case you are curious: the current (as of July 2017) record for digits of π computed is 22.4 trillion. See "Chronology of computation of π" on Wikipedia (and its linked sources) for an overview.

Readings

First reading: Dunham, Chapter 4, pages 84 - 99

Second reading: A History of Pi[43]

Third reading: 12.1 Trillion Digits of Pi[44]

Fourth reading: π: A Brief History[45]

Questions

Question 105 *How long did the computation of 12.1 trillion digits of π take? 94 days*

Question 106 *Which of the following had the most accurate estimation for π?*

Multiple Choice:

(a) *Archimedes*

(b) *Ptolemy*

(c) *Zu Chongzhi*

(d) *al-Khwarizmi*

Question 107 *What are the most important points from this reading?*

[43] See A History of Pi at `http://www-groups.dcs.st-and.ac.uk/~history/HistTopics/Pi_through_the_ages.html`
[44] See 12.1 Trillion Digits of Pi at `http://www.numberworld.org/misc_runs/pi-12t/`
[45] See π: A Brief History at `http://www.math.tamu.edu/~dallen/masters/alg_numtheory/pi.pdf`

24 Estimating Pi

Question 108 *List as many ways as you can think of for estimating the value of π.*

Draw a (fairly large) circle on a blank sheet of paper. We'll think of this as a unit circle.

Problem 109 *Divide the unit circle into $2^2 = 4$ equal wedges each with its vertex at the center of the circle O. On each wedge, call the two corners of the wedge that lie on the circle A and B_2. Let $\mathcal{A}_2$ denote the area of the triangle $\triangle OAB_2$ and let θ_2 denote the measure of the angle at O. Explain how to estimate the area of the circle with triangle $\triangle OAB_2$. What is your estimate?*

Problem 110 *Divide the unit circle into $2^3 = 8$ equal wedges each with its vertex at the center of the circle O. On each wedge, call the two corners of the wedge that lie on the circle A and B_3. Let $\mathcal{A}_3$ denote the area of the triangle $\triangle OAB_3$ and let θ_3 denote the measure of the angle at O. Explain how to estimate the area of the circle with triangle $\triangle OAB_3$. What information do you need to know to actually do this computation?*

Problem 111 *Given an angle θ, explain the relation of $\sin(\theta)$ and $\cos(\theta)$ to the unit circle. How could these values help with the calculation described above?*

Problem 112 *Divide the unit circle into 2^n equal wedges each with its vertex at the center of the circle O. On each wedge, call the two corners of the wedge that lie on the circle A and B_n. Let $\mathcal{A}_n$ denote the area of the triangle $\triangle OAB_n$ and let θ_n denote the measure of the angle at O. Explain why someone would be interested in the value of:*

$$\sin\left(\frac{\theta_n}{2}\right)$$

Problem 113 *Recalling that:*

$$\sin\left(\frac{\theta}{2}\right) = \sqrt{\frac{1-\cos(\theta)}{2}} \qquad \textit{and} \qquad \cos(\theta)^2 + \sin(\theta)^2 = 1$$

Explain why:

$$2\mathcal{A}_{n+1} = \sqrt{\frac{1-\sqrt{1-(2\mathcal{A}_n)^2}}{2}}$$

Problem 114 *Let's fill out the following table (a calculator will help!):*

n	$\mathcal{A}_n$	*Approx. Area*	$\sqrt{1-(2\mathcal{A}_n)^2}$	$\dfrac{1-\sqrt{1-(2\mathcal{A}_n)^2}}{2}$	$2\mathcal{A}_{n+1} = \sqrt{\dfrac{1-\sqrt{1-(2\mathcal{A}_n)^2}}{2}}$
2					
3					
4					
5					
6					
7					
8					

What do you notice?

25 Archimedes and Estimation

As we look at the Great Theorem of this chapter, one question many people have when they look at Archimedes' work is how he came to his conclusions. In particular, Archimedes spent a lot of time estimating values that today we would usually find using a calculator. There are a number of different ways to estimate things like π, which we talked about in the previous readings, and square roots. In the second reading, we consider some ways that Archimedes might have arrived at his conclusions. For the second reading, you should read at least the introduction and Section 4.

Readings

First reading: Dunham, Chapter 4, pages 99 - 112

Second reading: Archimedes' calculations of square roots[46]

Questions

Question 115 *The fourth step in the interpolation method places $\sqrt{3}$ between 5/3 and 7/4*

Question 116 *The author believes that the interpolation method shows that Archimedes understood what topic?*

Multiple Choice:

(a) *Astronomy*

(b) *Derivatives*

(c) *Geometry*

(d) *Limits*

Question 117 *What are the most important points from this reading?*

[46]See Archimedes' calculations of square roots at https://arxiv.org/abs/1101.0492

26 Mathematicians after Archimedes

Archimedes was certainly a mathematical genius, and following in his footsteps would certainly have been challenging. Our subject now is these mathematicians, living in changing times. It is relatively fitting to the story of mathematics that Archimedes was killed by the Romans, as the Roman takeover of much of the Greek world changed the way that people thought about and did mathematics for many years.

Here are some names of mathematicians whose contributions we will discuss at least briefly in class. By the end of our time on this chapter, you should be able to say a few words about what each person did.

- Eratosthenes (the subject of our second reading)
- Apollonius
- Heron (the subject of our first reading)
- Ptolemy
- Diophantus

The second reading is a little more history focused than mathematically focused. You should make sure to first read Dunham's version of Eratosthenes' proof in the first reading. This reading helps us to see some of the challenges that historical scholars face with mathematical works.

Readings

First reading: Dunham, Chapter 5, pages 113 - 121

Second reading: The Origin and Value of the Stadion Unit used by Eratosthenes in the Third Century B.C.[47]

Questions

Question 118 *How Greek feet are in one stadion?* 600

Question 119 *What does the author suggest was the usual precision for measuring long distances?*

Multiple Choice:

(a) *Round to the nearest inch.*

(b) *Round to the nearest foot.*

(c) *Round to the nearest stadion.*

(d) *Round to the nearest 10 stadia.*

[47] See The Origin and Value of the Stadion Unit used by Eratosthenes in the Third Century B.C. at http://www.jstor.org/stable/41133819

Question 120 *What are the most important points from this reading?*

27 Heron's formula

In this activity we will give two proofs of Heron's formula.

We'll start by giving a proof using synthetic geometry.

Part I

Proposition 9. *The bisectors of the angles of a triangle meet at a point that is the center of the triangle's inscribed circle.*

Question 121 *How can we prove this?*

Question 122 *Now draw a triangle with vertices A, B, and C. Draw the incircle. Explain why the radii of the incircle touch the sides of the triangle at right angles.*

Question 123 *Label the intersection of the radii with D between A and B, E between B and C, and F between C and A. Compute the areas of the following triangles:*

$$\triangle AOB, \qquad \triangle BOC, \qquad \triangle COA.$$

Use this to express the area of $\triangle ABC$.

Part II

Question 124 *Explain why*

$$\triangle AOD \cong \triangle AOF, \qquad \triangle BOD \cong \triangle BOE, \qquad \triangle COF \cong \triangle EOF.$$

Question 125 *If $AG \cong CE$, explain why $|BG|$ is the semiperimeter.*

Question 126 *Find segments in your drawing equal to the length of*

$$s - a, \qquad s - b, \qquad s - c.$$

Part III

Proposition 10. *If quadrilateral $AHBO$ has diagonals AB and OH with $\angle HAB$ and $\angle HOB$ being right angles, then $AHOB$ can be inscribed in a circle.*

Question 127 *Can you prove this proposition?*

Proposition 11. *The opposite angles of a cyclic quadrilateral sum to two right angles.*

Question 128 *Can you prove this proposition?*

Question 129 *Now we need to decorate our triangle even more:*

(a) *Draw OL perpendicular to OB cutting AB at K.*

(b) *Draw AM perpendicular to OB.*

(c) *Call the intersection of OL and OM, H.*

(d) *Draw BH.*

Consider quadrilateral $AHBO$, explain why opposite angles sum to two right angles.

Question 130 *Explain why $\triangle COF$ is similar to $\triangle BHA$. Use this to explain why*

$$\frac{|AB|}{|AG|} = \frac{|AH|}{r}.$$

Question 131 *Explain why $\triangle KAH$ is similar to $\triangle KDO$. Use this to explain why*

$$\frac{|AK|}{|KD|} = \frac{|AH|}{r}.$$

Question 132 *Now we see*

$$\frac{|AB|}{|AG|} = \frac{|AK|}{|KD|}.$$

Add 1 to both sides to obtain

$$\frac{|BG|}{|AG|} = \frac{|AD|}{|KD|}.$$

Question 133 *Explain why $\triangle KDO$ is similar to $\triangle ODB$. Use this to explain why*

$$|KD| \cdot |BD| = r^2.$$

Question 134 *Multiply both sides of*

$$\frac{|BG|}{|AG|} = \frac{|AD|}{|KD|}$$

by $\dfrac{|BD|}{|BD|}$ *to obtain*

$$r^2|BG|^2 = |AG| \cdot |BG| \cdot |AD| \cdot |BD|.$$

Question 135 *Explain how to deduce Heron's formula.*

A modern proof

Question 136 *Now give a modern proof that a high school student might give.*

Question 137 *Which proof was harder? Why didn't the ancient Greeks use our modern proof?*

28 Arabic Mathematics

Dunham leaves a pretty big gap after Chapter 5, which he covers just a little bit in the epilogue to the chapter. During the years between Heron (we'll say after 150 AD) and Cardano (in the 1500s), people were still doing mathematics, of course! During this period, mathematics was being studied mostly in the Middle East and India. Our second reading helps us get a better sense of the contributions of Arabic mathematicians during this time period.

Readings

First reading: Dunham, Chapter 5, pages 121 - 132

Second reading: Arabic Mathematics: forgotten brilliance?[48]

Questions

Question 138 *How many of Archimedes' works were translated into Arabic?* 2

Question 139 *Why did Arabic mathematicians use trigonometric tables?*

Multiple Choice:

(a) *For examples of algebra problems.*

(b) *To teach school children.*

(c) *In astronomy.*

(d) *In building projects.*

Question 140 *What are the most important points from this reading?*

[48] See Arabic Mathematics: forgotten brilliance? at http://www-history.mcs.st-and.ac.uk/HistTopics/Arabic_mathematics.html

29 The Beginning of the Modern Era

Our final reading associated with Chapter 5 helped bridge the gap in years between Heron and Cardano by looking more closely at Arabic mathematics and mathematicians. Europeans were also doing some mathematics during this period, and one of the most famous such is Fibonacci. Much of the foundation for the mathematics that was developed in Europe starting in the 1500s and 1600s was laid by merchants and other such professionals in the 1200s through the 1400s. These people, including Fibonacci, helped to bring knowledge from the intellectual center of the world in the Middle East to Europe.

Readings

First reading: Dunham, Chapter 6, pages 133 - 142

Second reading: Fibonacci Biography[49]

Questions

Question 141 *Fibonacci defines the concept of a congruum, and then proves that such a number must be divisible by 24.*

Question 142 *In what context were the Fibonacci numbers introduced?*

Multiple Choice:

(a) *In a problem about a spider climbing the wall.*

(b) *In a problem about a ship's voyages.*

(c) *In a problem about rabbit breeding.*

(d) *Without any context; simply a list of numbers.*

Question 143 *What are the most important points from this reading?*

[49] See Fibonacci Biography at `http://www-history.mcs.st-andrews.ac.uk/Biographies/Fibonacci.html`

30 Solving equations

In this activity we will solve second and third degree equations.

Finding roots of quadratic polynomials is somewhat complex. We want to find x such that

$$ax^2 + bx + c = 0.$$

I know you already know how to do this. However, pretend for a moment that you don't. This would be a really hard problem. We have evidence that it took humans around 1000 years to solve this problem in generality, with the first general solutions appearing in Babylon and China around 2500 years ago. Let's begin with an easier problem: make $a = 1$ and try to solve $x^2 + bx = c$.

Problem 144 *Geometrically, you could visualize $x^2+bx = c$ as an $x\times x$ square along with a $b\times x$ rectangle. Make a blob for c on the other side. Draw a picture of this!*

Question 145 *What is the total area of the shapes in your picture?*

Problem 146 *Now draw a new picture: take your $b\times x$ rectangle and divide it into two $(b/2)\times x$ rectangles.*

Question 147 *What is the total area of the shapes in your picture?*

Problem 148 *Draw a next picture in your sequence: take both of your $(b/2) \times x$ rectangles and snuggie them next to your $x \times x$ square on adjacent sides. You should now have what looks like an $(x+\frac{b}{2})\times(x+\frac{b}{2})$ square with a corner cut out of it.*

Question 149 *What is the total area of the shapes in your picture?*

Finally, your big $(x + \frac{b}{2}) \times (x + \frac{b}{2})$ has a piece missing, a $(b/2) \times (b/2)$ square, right? So if you add that piece in on both sides, the area of both sides of your picture had better be $c + (b/2)^2$. From your picture you will find that:

$$\left(x+\frac{b}{2}\right)^2 = c + \left(\frac{b}{2}\right)^2$$

Question 150 *Can you find x at this point?*

Question 151 *Explain how to solve $ax^2 + bx + c = 0$.*

Cubic Equations

While the quadratic formula was discovered around 2500 years ago, cubic equations proved to be a tougher nut to crack. A general solution to a cubic equation was not found until the 1500's - and under some pretty strange circumstances! See your text for the low-down on all of the drama.

We'll show you the Ferro-Tartaglia method for finding at least one root of a cubic of the form

$$x^3 + px + q.$$

All I can tell you are these three steps:

(a) Replace x with $u + v$.

(b) Set uv so that all of the terms are eliminated except for u^3, v^3, and constant terms.

(c) Clear denominators and use the quadratic formula.

Question 152 *How many solutions are we supposed to have in total?*

Question 153 *Use the Ferro-Tartaglia method to solve $x^3 + 9x - 26 = 0$.*

Question 154 *How many solutions should our equation above have? Where/what are they? Hint: Make use of an old forgotten foe...*

Question 155 *Is the method described here the same as the one in our text as the proof of the "great theorem"? Explain why or why not.*

Question 156 *Use the Ferro-Tartaglia method to solve $x^3 = 15x + 4$. What do you notice?*

Question 157 *How do we do this procedure for other equations of the form*

$$x^3 + px + q = 0?$$

Give an algebraic formula as your solution.

Question 158 *Would Ferro, Tartaglia, Cardano, or Ferrari have answered the previous question differently than you might?*

31 Mathematical Notation

As we consider Cardano's proof of our Great Theorem, one important concept to keep in mind is that much of our modern algebraic notation has not yet been developed. You can see an example in the text on page 146. An important following question is then: when was our modern algebraic notation developed, and by who? The second reading sheds some light on this question. An optional third reading gives a concise list of sources for the earliest uses of some common symbols.

Readings

First reading: Dunham, Chapter 6, pages 142 - 154

Second reading: A Brief History of Algebraic Notation[50]

Third reading: Earliest Uses of Various Mathematical Symbols[51]

Questions

Question 159 *In the Greek system, what was the value of the symbol π?* 80

Question 160 *What kind of algebra was Cardano using in his proof?*

Multiple Choice:

(a) *Rhetorical algebra.*

(b) *Syncopated algebra.*

(c) *Modern algebra.*

(d) *An early form of symbolic algebra.*

Question 161 *What are the most important points from this reading?*

[50] See A Brief History of Algebraic Notation at http://onlinelibrary.wiley.com.proxy.lib.ohio-state.edu/doi/10.1111/j.1949-8594.2000.tb17262.x/epdf

[51] See Earliest Uses of Various Mathematical Symbols at http://jeff560.tripod.com/mathsym.html

32 Logarithms and Multiplication

In the late 1500s and early 1600s, astronomy was really taking off. (Kepler, for instance, lived from 1571 - 1630.) The more accurate astronomy became, the more mathematicians needed to do computations with ever-larger numbers. The invention of the logarithm (published in 1614 and 1619) by John Napier, with assistance from Henry Briggs, was designed to shorten these calculations. In Napier's own words:

Seeing there is nothing (right well-beloved Students of the Mathematics) that is so troublesome to mathematical practice, nor that doth more molest and hinder calculators, than the multiplications, divisions, square and cubical extractions of great numbers, which besides the tedious expense of time are for the most part subject to many slippery errors, I began therefore to consider in my mind by what certain and ready art I might remove those hindrances. And having thought upon many things to this purpose, I found at length some excellent brief rules to be treated of (perhaps) hereafter. But amongst all, none more profitable than this which together with the hard and tedious multiplications, divisions, and extractions of roots, doth also cast away from the work itself even the very numbers themselves that are to be multiplied, divided and resolved into roots, and putteth other numbers in their place which perform as much as they can do, only by addition and subtraction, division by two or division by three.

Our second reading is a video explaining some ways we know to speed up multiplication, including the use of a logarithm. These methods are an important backdrop to our first reading, which is about the life and mathematics of Isaac Newton.

Sources:

- http://www-history.mcs.st-and.ac.uk/history/Biographies/Napier.html
- http://ualr.edu/lasmoller/napier.html
- https://en.wikipedia.org/wiki/Johannes_Kepler

Readings

First reading: Dunham, Chapter 7, pages 155 - 174

Second reading: How Can We Multiply Quickly?[52]

Questions

Question 162 *The video considers the multiplication problem* 17×37*.*

Question 163 *Which of the following is not mentioned in the video as a way to speed up multiplication?*

Multiple Choice:

(a) *Using Roman Numerals.*

(b) *Using quarter squares.*

(c) *Using logarithms.*

[52] See How Can We Multiply Quickly? at https://www.youtube.com/watch?v=LcJPGsWErdQ

(d) *Using a slide rule.*

Question 164 *What are the most important points from this reading?*

33 The binomial theorem and π

In this activity we investigate a generalization of the binomial theorem and its connection to an approximation of π.

Question 165 *The binomial theorem states*

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k.$$

Why would we be interested in this?

Question 166 *Newton says*

$$(1+x)^r = \sum_{k=0}^{\infty} \left(\frac{x^k}{k!} \prod_{\ell=0}^{k-1} (r-\ell) \right).$$

How did Newton come up with this? Hint: Calculus!

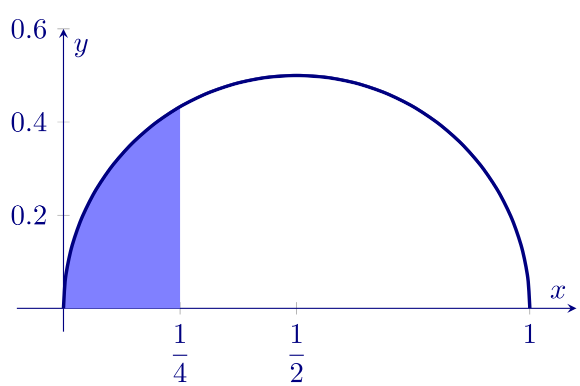
Now we're going to use this to approximate π.

Question 167 *Come up with a function of x for the semicircle of radius $1/2$ centered at $(1/2, 0)$.*

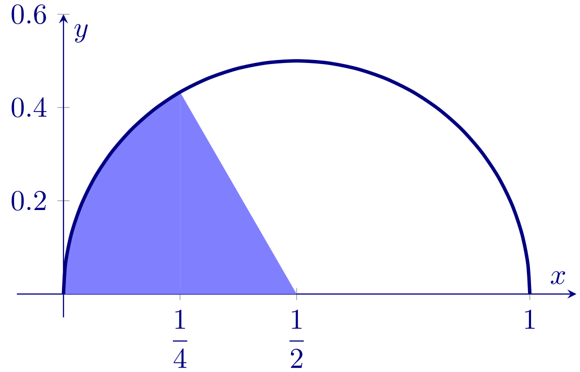
Question 168 *Use Newton's binomial theorem to show your function above is equal to:*

$$x^{1/2} - \frac{x^{3/2}}{2} - \frac{x^{5/2}}{8} - \frac{x^{7/2}}{16} - \frac{5x^{9/2}}{128} - \frac{7x^{11/2}}{256} - \cdots$$

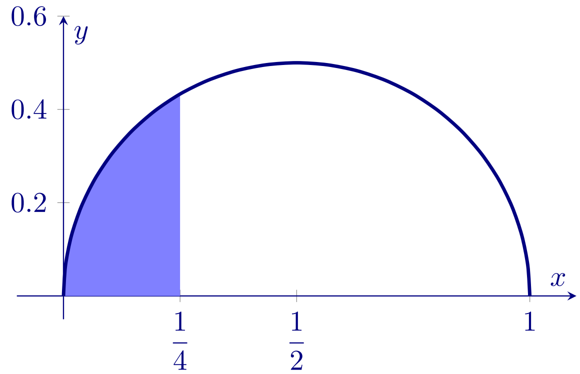
Question 169 *Use calculus to compute the area of the shaded region:*



Question 170 *Use proportional reasoning to compute the area of the sector below:*



Question 171 *Use the previous problem, along with the area of a certain 30-60-90 right triangle to give a different computation of the area below.*



Question 172 *Use your work from above to give an approximation of π.*