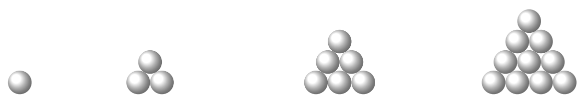
Leibniz and series

In this activity we investigate some of the series that Leibniz investigated.

Series pop-up at an early age. I distinctly remember being in fourth grade, sitting at my desk, starring at my ruler, wondering how 1/3 of a foot could simultaneously be 4 inches (clearly a finite number) and $0.333333\ldots$ of a foot (a number that somehow seemed finite and infinite at the same time). I was struggling with the implicit concept that

$$\frac{1}{3} = \frac{3}{10} + \frac{3}{100} + \frac{3}{1000} + \frac{3}{10000} + \cdots$$

Leibniz (and other mathematicians of the era) had similar feelings regarding series. Leibniz's mentor, Christian Huygens, suggested that Leibniz work on computing the sum of the reciprocal of the triangular numbers. Recall that the triangular numbers are the number of dots in discrete equilateral triangles:



Question 1 *Consider Leibniz's "proof."*

$$\begin{aligned}
S &= 1 + \frac{1}{3} + \frac{1}{6} + \frac{1}{10} + \frac{1}{15} + \frac{1}{21} + \cdots \\
\frac{S}{2} &= \frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \frac{1}{30} + \frac{1}{42} + \cdots \\
\frac{S}{2} &= \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \left(\frac{1}{4} - \frac{1}{5}\right) + \cdots \\
\frac{S}{2} &= 1 + \left(-\frac{1}{2} + \frac{1}{2}\right) + \left(-\frac{1}{3} + \frac{1}{3}\right) + \left(-\frac{1}{4} + \frac{1}{4}\right) + \cdots \\
\frac{S}{2} &= 1 \\
S &= 2.
\end{aligned}$$

Can you explain what is going on here? Where might Leibniz want to be a bit more rigorous?

Question 2 *What does*

$$1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + \cdots$$

sum to?

Question 3 *Now consider this summation of*

$$1 + \frac{1}{3} + \frac{1}{6} + \frac{1}{10} + \frac{1}{15} + \frac{1}{21} + \cdots$$

Learning outcomes:
Author(s):

Write

$$\begin{aligned}
\sum_{k=1}^{\infty} \frac{2}{k(k+1)} &= 1 + \frac{1}{3} + \frac{1}{6} + \frac{1}{10} + \frac{1}{15} + \frac{1}{21} + \cdots \\
\sum_{k=1}^{n} \frac{2}{k(k+1)} &= \sum_{k=1}^{n} 2\left(\frac{1}{n} - \frac{1}{n+1}\right) \\
\sum_{k=1}^{n} \frac{2}{k(k+1)} &= 2\left(1 - \frac{1}{n+1}\right) \\
\lim_{n\to\infty} \sum_{k=1}^{n} \frac{2}{k(k+1)} &= \lim_{n\to\infty} 2\left(1 - \frac{1}{n+1}\right) \\
\sum_{k=1}^{\infty} \frac{2}{k(k+1)} &= 2.
\end{aligned}$$

What's going on here? How does this compare to Leibniz's proof?

In addition to (co)inventing calculus, Leibniz, dreamed up this beast while trying to solve problems like the one posed to him by Huygens:

$$\begin{array}{ccccccccc}
 & & & & \frac{1}{1} & & & & \\
 & & & \frac{1}{2} & & \frac{1}{2} & & & \\
 & & \frac{1}{3} & & \frac{1}{6} & & \frac{1}{3} & & \\
 & \frac{1}{4} & & \frac{1}{12} & & \frac{1}{12} & & \frac{1}{4} & \\
\frac{1}{5} & & \frac{1}{20} & & \frac{1}{30} & & \frac{1}{20} & & \frac{1}{5}
\end{array}$$

Question 4 *What relationships can you find between the entries of the triangle as we move from row to row?*

Question 5 *What are the next two rows? Clearly articulate how to produce more rows of the harmonic triangle.*

Question 6 *Explain how the following expression*

$$\frac{1}{r \cdot \binom{r-1}{c-1}}$$

corresponds to entries of the harmonic triangle. Feel free to draw diagrams and give examples.

Question 7 *Explain how the harmonic triangle is formed. In your explanation, use the notation*

$$\frac{1}{r \cdot \binom{r-1}{c-1}}$$

If you were so inclined to do so, could you state a single equation that basically encapsulates your explanation above?

Question 8 *Can you explain why the numerators of the fractions in the harmonic triangle must always be 1?*

Question 9 *Explain how to use the harmonic triangle to go from:*

$$\frac{1}{2}+\frac{1}{6}+\frac{1}{12}+\frac{1}{20}+\cdots$$

to

$$\left(1-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\left(\frac{1}{4}-\frac{1}{5}\right)+\cdots$$

Conclude by explaining why Leibniz said:

$$\frac{1}{2}+\frac{1}{6}+\frac{1}{12}+\frac{1}{20}+\cdots=1$$

Question 10 *Can you generalize the results above? Can you give a list of infinite sums and conjecture what they will converge to?*