Solving equations

In this activity we will solve second and third degree equations.

Finding roots of quadratic polynomials is somewhat complex. We want to find x such that

$$ax^2 + bx + c = 0.$$

I know you already know how to do this. However, pretend for a moment that you don't. This would be a really hard problem. We have evidence that it took humans around 1000 years to solve this problem in generality, the first general solution appearing in Babylon and China around 2500 years ago. With this in mind, I think this topic warrants some attention. If you want to solve $ax^2 + bx + c = 0$, a good place to start would be with an easier problem. Let's make $a = 1$ and try to solve

$$x^2 + bx = c$$

Geometrically, you could visualize this as an $x \times x$ square along with a $b \times x$ rectangle. Make a blob for c on the other side.

Question 1 *What would a picture of this look like?*

Question 2 *What is the total area of the shapes in your picture?*

Take your $b \times x$ rectangle and divide it into two $(b/2) \times x$ rectangles.

Question 3 *What would a picture of this look like?*

Question 4 *What is the total area of the shapes in your picture?*

Now take both of your $(b/2) \times x$ rectangles and snuggie them next to your $x \times x$ square on adjacent sides. You should now have what looks like an $(x + \frac{b}{2}) \times (x + \frac{b}{2})$ square with a corner cut out of it.

Question 5 *What would a picture of this look like?*

Question 6 *What is the total area of the shapes in your picture?*

Finally, your big $(x + \frac{b}{2}) \times (x + \frac{b}{2})$ has a piece missing, a $(b/2) \times (b/2)$ square, right? So if you add that piece in on both sides, the area of both sides of your picture had better be $c + (b/2)^2$. From your picture you will find that:

$$\left(x + \frac{b}{2}\right)^2 = c + \left(\frac{b}{2}\right)^2$$

Learning outcomes:
Author(s):

Question 7 *Can you find x at this point?*

Question 8 *Explain how to solve $ax^2 + bx + c = 0$.*

Cubic Equations

While the quadratic formula was discovered around 2500 years ago, cubic equations proved to be a tougher nut to crack. A general solution to a cubic equation was not found until the 1500's. At the time mathematicians were a secretive and competitive bunch. Someone would solve a particular cubic equation, then challenge another mathematician to a sort of "mathematical duel." Each mathematician would give the other a list of problems to solve by a given date. The one who solved the most problems was the winner and glory everlasting was theirs. One of the greatest duelists was Niccolò Fontana Tartaglia (pronounced *Tar-tah-lee-ya*). Why was he so great? He developed a general method for solving cubic equations! However, neither was he alone in this discovery nor was he the first. As sometimes happens, the method was discovered some years earlier by another mathematician, Scipione del Ferro. However, due to the secrecy and competitiveness, very few people knew of Ferro's method. Since these discoveries were independent, we'll call the method the *Ferro-Tartaglia method*.

We'll show you the Ferro-Tartaglia method for finding at least one root of a cubic of the form:

$$x^3 + px + q$$

We'll illustrate with a specific example—you'll have to generalize yourself! Take

$$x^3 + 3x - 2 = 0$$

All I can tell you are these three steps:

(a) Replace x with $u + v$.

(b) Set uv so that all of the terms are eliminated except for u^3, v^3, and constant terms.

(c) Clear denominators and use the quadratic formula.

Question 9 *How many solutions are we supposed to have in total?*

Question 10 *Use the Ferro-Tartaglia method to solve $x^3 + 9x - 26 = 0$.*

Question 11 *How many solutions should our equation above have? Where/what are they? Hint: Make use of an old forgotten foe...*

Question 12 *How do we do this procedure for other equations of the form*

$$x^3 + px + q = 0?$$

This might be a slight exaggeration.