Precalculus with Review 2: Unit 10

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Preparing for Calculus

11.1 Linear Systems of Equations

Learning Objectives

- From Systems to Solutions
  - What is a system of equations?
  - What is a solution to a system?
  - Solving systems via graphs
- Solving Systems Algebraically
  - Eliminating Variables
  - Substitution Method
- Applications of Systems of Equations
  - Word problems
  - Mixture Problems

11.1.1 From Systems to Solutions

Motivating Questions

- What is a system of equations?
- What is a solution to a system?
- How can we solve systems of equations using graphs?

Introduction

We have already seen many techniques for solving equations. Until now, however, we have only solved equations of the form $f(x) = 0$ for the variable x. In this section, we will consider equations with more than one variable and discuss how to solve them.

Consider a peculiar grocery store where the prices of all the items for sale are not listed, and you only find out the total cost of your purchase. Say you buy 6 mangos and 3 bananas, and your total cost is 9 dollars. Assume all the mangos cost the same amount and all the bananas cost the same amount. Without making any more purchases, is it possible find out how much a mango and a banana cost on their own?

Let's create an equation to describe this situation. Let x be a variable representing the cost of a mango, and let y be a variable representing the cost of a banana. Then, the equation

$$6x + 3y = 9$$

represents that buying 6 mangos at a cost of x dollars and 3 bananas at a cost of y dollars yields a total cost of 9 dollars.

You might have noticed that plugging $x = 1$ and $y = 1$ into the equation gives us a true statement, so you might conclude that mangos and bananas both cost 1 dollar. However, notice that plugging $x = 1.20$ and $y = 0.60$ into the equation also gives us a true statement, so it's also possible that mangos cost \$1.20 and bananas cost \$0.60. Even more worrying is that $x = 0$ and $y = 3$ also gives us a solution to the equation: is this store peculiar enough to be giving away mangos for free and charging \$3 per banana?

Examining the equation we set up can give us more insight. Let's rearrange the equation to solve for y in terms of x:

$$\begin{aligned} 6x + 3y &= 9 \\ 3y &= 9 - 6x \\ y &= 3 - 2x. \end{aligned}$$

Now it becomes clearer what's going on. Whatever x is, we can find a value of y that satisfies our original equation. No matter the cost of a single mango, there's a way to price the bananas so that our equation is true! This means it's impossible to find the price of a single mango or a single banana with the information you've been given. We need more data!

Systems of linear equations

In order to collect more information, you go back to the store and buy 2 mangos and 2 banana for a total cost of \$3.20. This can be modeled by the equation

$$2x + 2y = 3.2.$$

Keep in mind that this x and y are the same x and y from before, so in order to find the cost of a mango and a banana, we must find x and y that satisfy both equations

$$\begin{cases} 6x + 3y = 9 \\ 2x + 2y = 3.2 \end{cases}$$

at the same time. This coupling of two (or more) linear equations is called a *system of linear equations*.

Definition A **linear equation of two variables** is an equation of the form

$$a_1x + a_2y = c,$$

where a_1, a_2, and c are real numbers and at least one of a_1 and a_2 is nonzero.
A **system of linear equations of two variables** is a collection of two or more linear equations of two variables.
We say a **solution** to a system of linear equations of two variables is a point (x, y) satisfying all equations in the system.

It is clear that some systems of equations have solutions, and some do not. Those which have solutions are called *consistent*, those with no solution are called *inconsistent*.

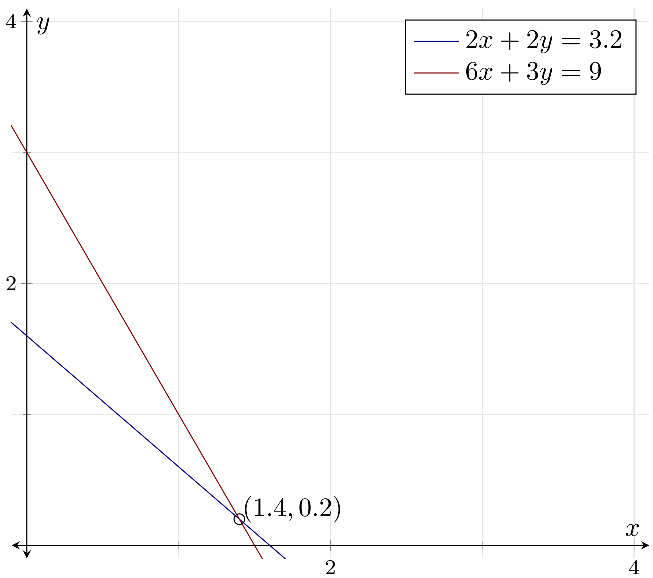
The key to identifying linear equations is to note that the variables involved are to the first power and that the coefficients of the variables are numbers. Some examples of equations which are non-linear are $x^2 + y = 1$, $xy = 5$ and $e^{2x} + \ln(y) = 1$. Note that we can still have systems of non-linear equations, but they can be much more difficult to solve.

Finding solutions graphically

Let's return to our example from earlier and try to find a solution to the system of linear equation:

$$\begin{cases} 6x + 3y = 9 \\ 2x + 2y = 3.2 \end{cases}.$$

We want to find x and y satisfying both equations in the system. If x and y satisfy $6x + 3y = 9$, then the point (x, y) lies on the graph of $6x + 3y = 9$. Similarly, if x and y satisfy $2x+2y = 3.2$, then the point (x, y) lies on the graph of $2x + 2y = 3.2$. Therefore, to find any solutions, we can look at the graphs of $6x + 3y = 9$ and $2x + 2y = 3.2$, and see if there are any points that lie at the intersection of the two graphs:



By inspecting the graph, we see that these two lines intersect only at $(1.4, 0.2)$, so the only solution to the system is $x = 1.4$ and $y = 0.2$.

In context, this means that mangos cost \$1.40 each and bananas cost \$0.20 each. Note that in order to have exactly one solution to our system of linear equations in two variables, we needed the system to have two equations.

Note that not every system of linear equations will have one solution. If the graphs of the two equations are parallel, they will never intersect, so there won't be any solutions. Additionally, if the two equations are represented by the same graph, there will be infinitely many intersection points, and therefore, infinitely many solutions.

Next, we will see some methods for solving systems of equations algebraically.

11.1.2 Solving Systems of Equations Algebraically

Substitution

In the previous section, we focused on solving systems of equations by graphing. In addition to being time consuming, graphing can be an awkward method to determine the exact solution when the solution has large numbers, fractions, or decimals. There are two symbolic methods for solving systems of linear equations, and in this section we will use one of them: substitution.

Example 1. *In 2014, the New York Times posted the following about the movie, "The Interview":*

> *"The Interview" generated roughly* \$15 *million in online sales and rentals during its first four days of availability, Sony Pictures said on Sunday. Sony did not say how much of that total represented* \$6 *digital rentals versus* \$15 *sales. The studio said there were about two million transactions overall.*

A few days later, Joey Devilla cleverly pointed out in his blog, that there is enough information given to find the amount of sales versus rentals.

Explanation Using algebra, we can write a system of equations and solve it to find the two quantities. Although since the given information uses approximate values, the solutions we will find will only be approximations too.

First, we will define variables. We need two variables, because there are two unknown quantities: how many sales there were and how many rentals there were. Let r be the number of rental transactions and let s be the number of sales transactions.

If you are unsure how to write an equation from the background information, use the units to help you. The units of each term in an equation must match because we can only add like quantities. Both r and s are in transactions. The article says that the total number of transactions is 2 million. So our first equation will add the total number of rental and sales transactions and set that equal to 2 million. Our equation is:

$$(r\,\text{transactions}) + (s\,\text{transactions}) = 2{,}000{,}000\,\text{transactions}$$

Without the units:

$$r + s = 2{,}000{,}000$$

The price of each rental was \$6. That means the problem has given us a *rate* of $6\,\dfrac{\text{dollars}}{\text{transaction}}$ to work with. The rate unit suggests this should be multiplied by something measured in transactions. It makes sense to multiply by r, and then the number of dollars generated from rentals was $6r$. Similarly, the price of each sale was \$15, so the revenue from sales was $15s$. The total revenue was \$15 million, which we can represent with this equation:

$$\left(6\,\frac{\text{dollars}}{\text{transaction}}\right)(r\text{ transactions}) + \left(15\,\frac{\text{dollars}}{\text{transaction}}\right)(s\text{ transactions}) = \$15{,}000{,}000$$

Without the units:

$$6r + 15s = 15{,}000{,}000$$

Here is our system of equations:

$$\begin{array}{rcrcr} r & + & s & = & 2{,}000{,}000 \\ 6r & + & 15s & = & 15{,}000{,}000 \end{array}$$

To solve the system, we will use the **substitution** method. The idea is to use *one* equation to find an expression that is equal to r but, cleverly, does not use the variable "r." Then, substitute this for r into the *other* equation. This leaves you with *one* equation that only has *one* variable.

The first equation from the system is an easy one to solve for r:

$$\begin{aligned} r + s &= 2{,}000{,}000 \\ r &= 2{,}000{,}000 - s \end{aligned}$$

This tells us that the expression $2{,}000{,}000 - s$ is equal to r, so we can *substitute* it for r in the second equation:

$$\begin{aligned} 6r + 15s &= 15{,}000{,}000 \\ 6(\color{blue}{2{,}000{,}000 - s}) + 15s &= 15{,}000{,}000 \end{aligned}$$

Now we have an equation with only one variable, s, which we will solve for:

$$\begin{aligned} 6(2{,}000{,}000 - s) + 15s &= 15{,}000{,}000 \\ 12{,}000{,}000 - 6s + 15s &= 15{,}000{,}000 \\ 12{,}000{,}000 + 9s &= 15{,}000{,}000 \\ 9s &= 3{,}000{,}000 \\ \frac{9s}{\color{blue}{9}} &= \frac{3{,}000{,}000}{\color{blue}{9}} \\ s &= 333{,}333.\overline{3} \end{aligned}$$

At this point, we know that $s = 333{,}333.\overline{3}$. This tells us that out of the 2 million transactions, roughly 333,333 were from online sales. Recall that we solved the first equation for r, and found $r = 2{,}000{,}000 - s$.

$$\begin{aligned} r &= 2{,}000{,}000 - s \\ r &= 2{,}000{,}000 - \color{blue}{333{,}333.\overline{3}} \\ r &= 1{,}666{,}666.\overline{6} \end{aligned}$$

To check our answer, we will see if $s = 333{,}333.\overline{3}$ and $r = 1{,}666{,}666.\overline{6}$ make the original equations true:

$$\begin{aligned}
r + s &= 2{,}000{,}000 \\
1{,}666{,}666.\overline{6} + 333{,}333.\overline{3} &= 2{,}000{,}000 \\
2{,}000{,}000 &= 2{,}000{,}000 \\
6r + 15s &= 15{,}000{,}000 \\
6\left(1{,}666{,}666.\overline{6}\right) + 15\left(333{,}333.\overline{3}\right) &= 15{,}000{,}000 \\
10{,}000{,}000 + 5{,}000{,}000 &= 15{,}000{,}000
\end{aligned}$$

In summary, there were roughly 333,333 copies sold and roughly 1,666,667 copies rented.

Elimination

We just learned how to solve a system of linear equations using substitution above. Now, we will learn a second symbolic method for solving systems of linear equations.

Example 2. *Alicia has* \$1000 *to give to her two grandchildren for New Year's. She would like to give the older grandchild* \$120 *more than the younger grandchild, because that is the cost of the older grandchild's college textbooks this term. How much money should she give to each grandchild?*

Explanation To answer this question, we will demonstrate a new technique. You may have a very good way for finding how much money Alicia should give to each grandchild, but right now we will try to see this new method.

Let A be the dollar amount she gives to her older grandchild, and B be the dollar amount she gives to her younger grandchild. (As always, we start solving a word problem like this by defining the variables, including their units.) Since the total she has to give is \$1000, we can say that $A + B = 1000$. And since she wants to give \$120 more to the older grandchild, we can say that $A - B = 120$. So we have the system of equations:

$$\begin{aligned}
A + B &= 1000 \\
A - B &= 120
\end{aligned}$$

We could solve this system by substitution as we learned previously but there is an easier method. If we add together the *left* sides from the two equations, it should equal the sum of the *right* sides:

$$\begin{array}{rcl}
A + B & = & 1000 \\
+A - B & & +120
\end{array}$$

So we have:

$$2A = 1120$$

Note that the variable B is eliminated. This happened because the $+B$ and the $-B$ perfectly cancel each other out when they are added. With only one variable left, it doesn't take much to finish:

$$\begin{aligned} 2A &= 1120 \\ A &= 560 \end{aligned}$$

To finish solving this system of equations, we need the value of B. For now, an easy way to find B is to substitute in our value of A into one of the original equations:

$$\begin{aligned} A + B &= 1000 \\ \color{blue}{560} + B &= 1000 \\ B &= 440 \end{aligned}$$

To check our work, substitute $A = 560$ and $B = 440$ into the original equations:

$$\begin{aligned} A + B &= 1000 \\ \color{blue}{560} + \color{blue}{440} &= 1000 \\ 1000 &= 1000 \\ A - B &= 120 \\ \color{blue}{560} - \color{blue}{440} &= 120 \\ 120 &= 120 \end{aligned}$$

This confirms that our solution is correct. In summary, Alicia should give \$560 to her older grandchild, and \$440 to her younger grandchild.

This method for solving the system of equations in the example above worked because B and $-B$ add to zero. Once the B-terms were eliminated we were able to solve for A. This method is called the **elimination method**. Some people call it the **addition method**, because we added the corresponding sides from the two equations to eliminate a variable.

If neither variable can be immediately eliminated, we can still use this method but it will require that we first adjust one or both of the equations. Let's look at an example where we need to adjust one of the equations.

Example 3. *Solve the system of equations using the elimination method.*

$$\begin{array}{rcrcr} 3x & - & 4y & = & 2 \\ 5x & + & 8y & = & 18 \end{array}$$

Explanation To start, we want to see whether it will be easier to eliminate x or y. We see that the coefficients of x in each equation are 3 and 5, and the coefficients of y are -4 and 8. Because 8 is a multiple of 4 and the coefficients already have opposite signs, the y variable will be easier to eliminate.

To eliminate the y terms, we will multiply each side of the first equation by 2 so that we will have $-8y$. We can call this process scaling the first equation by 2.

$$\begin{aligned} {\color{blue}2} \cdot (3x &- 4y) = {\color{blue}2} \cdot (2) \\ 5x &+ 8y = 18 \end{aligned}$$

$$\begin{aligned} 6x - 8y &= 4 \\ 5x + 8y &= 18 \end{aligned}$$

We now have an equivalent system of equations where the y-terms can be eliminated:

$$\begin{aligned} 6x - 8y &= 4 \\ +5x + 8y & \quad +18 \end{aligned}$$

So we have:

$$\begin{aligned} 11x &= 22 \\ x &= 2 \end{aligned}$$

To solve for y, we can substitute 2 for x into either of the original equations or the new one. We use the first original equation, $3x - 4y = 2$:

$$\begin{aligned} 3x - 4y &= 2 \\ 3({\color{blue}2}) - 4y &= 2 \\ 6 - 4y &= 2 \\ -4y &= -4 \\ y &= 1 \end{aligned}$$

Our solution is $x = 2$ and $y = 1$. We will check this in both of the original equations:

$$\begin{aligned} 5x + 8y &= 18 \\ 5({\color{blue}2}) + 8({\color{blue}1}) &= 18 \\ 10 + 8 &= 18 \\ 3x - 4y &= 2 \\ 3({\color{blue}2}) - 4({\color{blue}1}) &= 2 \\ 6 - 4 &= 2 \end{aligned}$$

The solution to this system is $(2, 1)$ and the solution set is $\{(2, 1)\}$.

11.1.3 Applications of Systems of Equations

Example 4. *Two Different Interest Rates Notah made some large purchases with his two credit cards one month and took on a total of* \$8,400 *in debt from the two cards. He didn't make any payments the first month, so the two credit card debts each started to accrue interest. That month, his Visa card charged* 2% *interest and his Mastercard charged* 2.5% *interest. Because of this, Notah's total debt grew by* \$178*. How much money did Notah charge to each card?*

Explanation To start, we will define two variables based on our two unknowns. Let v be the amount charged to the Visa card (in dollars) and let m be the amount charged to the Mastercard (in dollars).

To determine our equations, notice that we are given two different totals. We will use these to form our two equations. The total amount charged is \$8,400 so we have:

$$(v\,\text{dollars}) + (m\,\text{dollars}) = \$8400$$

Or without units:

$$v + m = 8400$$

The other total we were given is the total amount of interest, \$178, which is also in dollars. The Visa had v dollars charged to it and accrues 2% interest. So $0.02v$ is the dollar amount of interest that comes from using this card. Similarly, $0.025m$ is the dollar amount of interest from using the Mastercard. Together:

$$0.02(v\,\text{dollars}) + 0.025(m\,\text{dollars}) = \$178$$

Or without units:

$$0.02v + 0.025m = 178$$

As a system, we write:

$$\begin{aligned} v \quad &+ \quad m &= 8400 \\ 0.02v \quad &+ \quad 0.025m &= 178 \end{aligned}$$

To solve this system by substitution, notice that it will be easier to solve for one of the variables in the first equation. We'll solve that equation for v:

$$\begin{aligned} v + m &= 8400 \\ v &= 8400 - m \end{aligned}$$

Now we will substitute $8400 - m$ for v in the second equation:

$$
\begin{aligned}
0.02v + 0.025m &= 178 \\
0.02({\color{blue}8400 - m}) + 0.025m &= 178 \\
168 - 0.02m + 0.025m &= 178 \\
168 + 0.005m &= 178 \\
0.005m &= 10 \\
\frac{0.005m}{\color{blue}0.005} &= \frac{10}{\color{blue}0.005} \\
m &= 2000
\end{aligned}
$$

Lastly, we can determine the value of v by using the earlier equation where we isolated v:

$$
\begin{aligned}
v &= 8400 - m \\
v &= 8400 - {\color{blue}2000} \\
v &= 6400
\end{aligned}
$$

In summary, Notah charged \$6400 to the Visa and \$2000 to the Mastercard. We should check that these numbers work as solutions to our original system ¡em¿and¡/em¿ that they make sense in context. (For instance, if one of these numbers were negative, or was something small like \$0.50, they wouldn't make sense as credit card debt.)

Mixture Problems

The next two examples are called **mixture problems**, because they involve mixing two quantities together to form a combination and we want to find out how much of each quantity to mix.

Example 5. *Mixing Solutions with Two Different Concentrations LaVonda is a meticulous bartender and she needs to serve* 600 *milliliters of Rob Roy, an alcoholic cocktail that is* 34% *alcohol by volume. The main ingredients are scotch that is* 42% *alcohol and vermouth that is* 18% *alcohol. How many milliliters of each ingredient should she mix together to make the concentration she needs?*
Explanation The two unknowns are the quantities of each ingredient. Let s be the amount of scotch (in mL) and let v be the amount of vermouth (in mL).

One quantity given to us in the problem is 600 mL. Since this is the total volume of the mixed drink, we must have:

$$(s\,\mathrm{mL}) + (v\,\mathrm{mL}) = 600\,\mathrm{mL}$$

Or without units:

$$s + v = 600$$

To build the second equation, we have to think about the alcohol concentrations for the scotch, vermouth, and Rob Roy. It can be tricky to think about percentages like these correctly. One strategy is to focus on the *amount* (in mL) of *alcohol* being mixed. If we have s milliliters of scotch that is 42% alcohol, then $0.42s$ is the actual *amount* (in mL) of alcohol in that scotch. Similarly, $0.18v$ is the amount of alcohol in the vermouth. And the final cocktail is 600 mL of liquid that is 34% alcohol, so it has $0.34(600) = 204$ milliliters of alcohol. All this means:

$$0.42(s\,\text{mL}) + 0.18(v\,\text{mL}) = 204\,\text{mL}$$

Or without units:

$$0.42s + 0.18v = 204$$

So our system is:

$$\begin{aligned} s \quad + \quad v \quad &= 600 \\ 0.42s \quad + \quad 0.18v \quad &= 204 \end{aligned}$$

To solve this system, we'll solve for s in the first equation: $\begin{aligned} s + v &= 600 \\ s &= 600 - v \end{aligned}$

And then substitute s in the second equation with $600 - v$:

$$\begin{aligned} 0.42s + 0.18v &= 204 \\ 0.42(600 - v) + 0.18v &= 204 \\ 252 - 0.42v + 0.18v &= 204 \\ 252 - 0.24v &= 204 \\ -0.24v &= -48 \\ \frac{-0.24v}{-0.24} &= \frac{-48}{-0.24} \\ v &= 200 \end{aligned}$$

As a last step, we will determine s using the equation where we had isolated s:

$$\begin{aligned} s &= 600 - v \\ s &= 600 - 200 \\ s &= 400 \end{aligned}$$

In summary, LaVonda needs to combine 400 mL of scotch with 200 mL of vermouth to create 600 mL of Rob Roy that is 34% alcohol by volume.

As a check for the previous example, we can use estimation to see that our solution is reasonable. Since LaVonda is making a 34% solution, she would need to use more of the 42% concentration than the 18% concentration, because 34% is closer to 42% than to 18%. This agrees with our answer because we found that she needed 400 mL of the 42% solution and

200 mL of the 18% solution. This is an added check that we have found reasonable answers.

Example 6. *Mixing a Coffee Blend Desi owns a coffee shop and they want to mix two different types of coffee beans to make a blend that sells for* \$12.50 *per pound. They have some coffee beans from Columbia that sell for* \$9.00 *per pound and some coffee beans from Honduras that sell for* \$14.00 *per pound. How many pounds of each should they mix to make* 30 *pounds of the blend?*
Explanation Before we begin, it may be helpful to try to estimate the solution. Let's compare the three prices. Since \$12.50 is between the prices of \$9.00 and \$14.00, this mixture is possible. Now we need to estimate the amount of each type needed. The price of the blend (\$12.50 per pound) is closer to the higher priced beans (\$14.00 per pound) than the lower priced beans (\$9.00 per pound). So we will need to use more of that type. Keeping in mind that we need a total of 30 pounds, we roughly estimate 20 pounds of the \$14.00 Honduran beans and 10 pounds of the \$9.00 Columbian beans. How good is our estimate? Next we will solve this exercise exactly.

To set up our system of equations we define variables, letting C be the amount of Columbian coffee beans (in pounds) and H be the amount of Honduran coffee beans (in pounds).

The equations in our system will come from the total amount of beans and the total cost. The equation for the total amount of beans can be written as:

$$(C\,\mathrm{lb}) + (H\,\mathrm{lb}) = 30\,\mathrm{lb}$$

Or without units:

$$C + H = 30$$

To build the second equation, we have to think about the cost of all these beans. If we have C pounds of Columbian beans that cost \$9.00 per pound, then $9C$ is the cost of those beans in dollars. Similarly, $14H$ is the cost of the Honduran beans. And the total cost is for 30 pounds of beans priced at \$12.50 per pound, totaling $12.5(30) = 37.5$ dollars. All this means:

$$\left(9\,\tfrac{\mathrm{dollars}}{\mathrm{lb}}\right)(C\,\mathrm{lb}) + \left(14\,\tfrac{\mathrm{dollars}}{\mathrm{lb}}\right)(H\,\mathrm{lb}) = \left(12.50\,\tfrac{\mathrm{dollars}}{\mathrm{lb}}\right)(30\,\mathrm{lb})$$

Or without units and carrying out the multiplication on the right:

$$9C + 14H = 37.5$$

Now our system is:

$$\begin{array}{rcrl} C & + & H & = 30 \\ 9C & + & 14H & = 37.50 \end{array}$$

To solve the system, we'll solve the first equation for C:

$$\begin{aligned} C + H &= 30 \\ C &= 30 - H \end{aligned}$$

Next, we'll substitute C in the second equation with $30 - H$:

$$\begin{aligned} 9C + 14H &= 375 \\ 9(30 - H) + 14H &= 375 \\ 270 - 9H + 14H &= 375 \\ 270 + 5H &= 375 \\ 5H &= 105 \\ H &= 21 \end{aligned}$$

Since $H = 21$, we can conclude that $C = 9$.

In summary, Desi needs to mix 21 pounds of the Honduran coffee beans with 9 pounds of the Columbian coffee beans to create this blend. Our estimate at the beginning was pretty close, so we feel this answer is reasonable.

11.2 Non-linear Systems

Learning Objectives

- Famous Formulas
  - Reviewing famous functions
  - Introducing conic sections and their formulas
- Solving Non-linear Systems Graphically
  - What is a non-linear system?
  - Finding solutions graphically
  - What can we say about when solutions exist?
- Eliminating Variables
  - Algebra of reducing multivariable systems to a single equation
  - Systems created from functions
  - Reviewing some algebra and misconceptions

11.2.1 Famous Formulas

Introduction

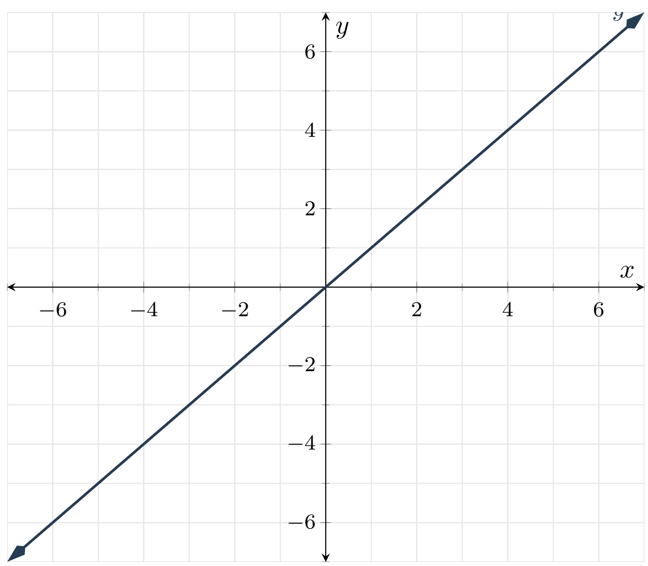
As a review, we go over the list of famous functions from earlier. Then, we move to a discussion of conic sections.

Linear Functions

Recall that the graph of a linear function is a line.

Example 7. *A prototypical example of a linear function is*

$$y = x.$$



Important Values of $y = x$	
x	y
-2	-2
-1	-1
0	0
1	1
2	2

In general, linear functions can be written as $y = mx + b$ where m and b can be any numbers. We learned that m represents the slope, and b is the y-coordinate of the y-intercept. You can play with changing the values of m and b on the graph using Desmos and see how that changes the line.

Desmos link: `https://www.desmos.com/calculator/japnhapzvn`

Table 1: Properties of Linear Functions $y = mx + b$

Periodic?	If $m = 0$
Odd?	If $b = 0$
Even?	If $m = 0$
One-to-one/invertible	If $b = 0$

Note that any real number can be plugged into $f(x) = mx + b$, so the domain of linear functions is $(-\infty, \infty)$. Unless $m = 0$, we can find a y such that $y = mx+b$, so the range of linear functions with $m \neq 0$ is $(-\infty, \infty)$. If $m = 0$, then the only output of the linear function is b, so its range is $\{b\}$.

Table 2: Domain and Range of Linear Functions $y = mx + b$

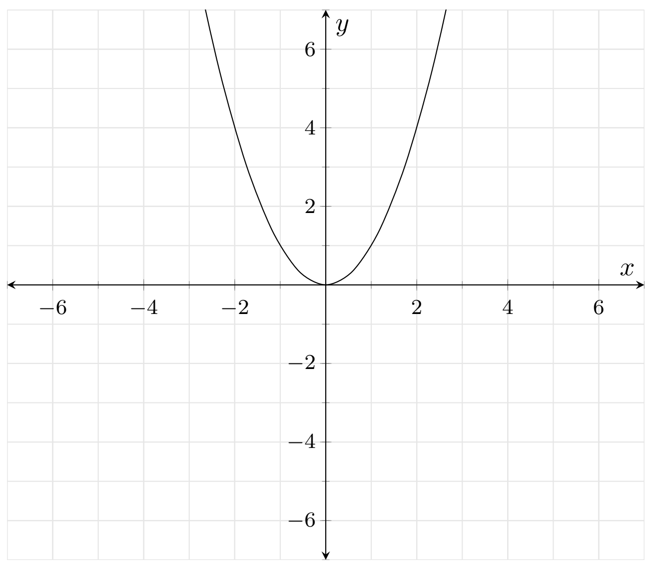
Domain	$(-\infty, \infty)$
Range	If $m \neq 0$, $(-\infty, \infty)$; if $m = 0$, $\{b\}$

Quadratic Functions

Recall that the graph of a quadratic function is a parabola.

Example 8. *A prototypical example of a quadratic function is*

$$y = x^2.$$



Important Values of $y = x^2$	
x	y
-2	4
-1	1
0	0
1	1
2	4

In general, quadratic functions can be written as $y = ax^2 + bx + c$ where a, b, and c can be any numbers. You can play with changing the values of a, b, and c on the graph using Desmos and see how that changes the parabola.

Desmos link: `https://www.desmos.com/calculator/nmlghfrws9`

Note that any real number can be plugged into $f(x) = ax^2+bx+c$, so the domain of quadratic functions is $(-\infty, \infty)$. In Chapter 4, we saw that all quadratic

Table 3: Properties of Quadratic Functions $y = ax^2 + bx + c$

Periodic?	If $a = 0$ and $b = 0$
Odd?	If $a = 0$, $b = 0$, and $c = 0$
Even?	If $b = 0$
One-to-one/invertible	If $a = 0$ and $c = 0$

functions have a vertex form $f(x) = d(x-h)^2 + k$, where the vertex is at (h, k). If $d > 0$, all points above the vertex, that is $[k, \infty)$ are in the range of the quadratic, and if $d < 0$, all points below the vertex, that is $(\infty, k]$ are in the range of the quadratic.

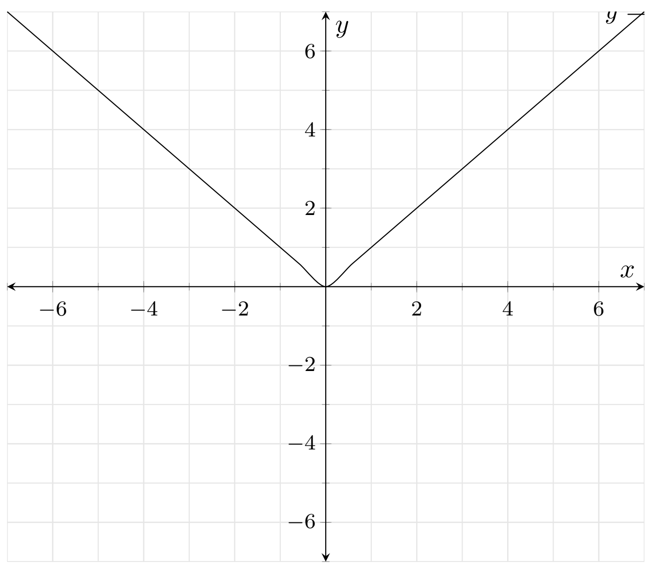
Table 4: Domain and Range of Quadratic Functions $y = d(x-h)^2 + k$

Domain	$(-\infty, \infty)$
Range	If $d > 0$, $[k, \infty)$; if $d < 0$, $(\infty, k]$

Absolute Value

Another important type of function is the absolute value function. This is the function that takes all y-values and makes them positive. The absolute value function is written as

$$y = |x|.$$



Important Values of $y = \|x\|$	
x	y
-2	2
-1	1
0	0
1	1
2	2

Table 5: Properties of The Absolute Value Function $y = |x|$

Periodic?	No
Odd?	No
Even?	Yes
One-to-one/invertible	No

Note that any real number has an absolute value, so the domain of the absolute value function is $(-\infty, \infty)$. Furthermore, by looking at the graph, we can see that all non-negative numbers are in the range of the absolute value function.

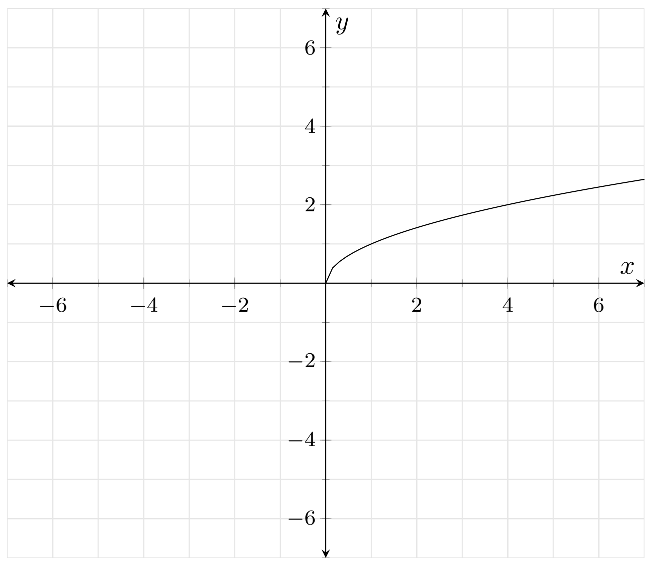
Table 6: Domain and Range of The Absolute Value Function $y = |x|$

Domain	$(-\infty, \infty)$
Range	$[0, \infty)$

Square Root

Another famous function is the square root function,

$$y = \sqrt{x}.$$



Important Values of $y = \sqrt{x}$	
x	y
0	0
1	1
4	2
9	3
25	5

Table 7: Properties of The Square Root Function $y = \sqrt{x}$

Periodic?	No
Odd?	No
Even?	No
One-to-one/invertible	Yes

Note that only non-negative numbers have square roots, so the domain of the square root function is $[0, \infty)$. Furthermore, by looking at the graph, we can see that all non-negative numbers are in the range of the square root function. Algebraically, we can say that for any non-negative y, $\sqrt{(y^2)} = y$, so y is in the range of the square root function.

Table 8: Domain and Range of The Square Root Function $y = \sqrt{x}$

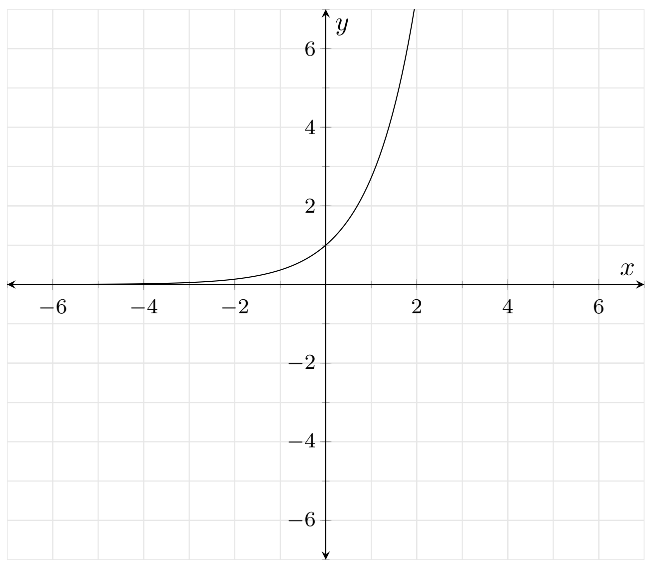
Domain	$[0, \infty)$
Range	$[0, \infty)$

Exponential

Another famous function is the exponential growth function,

$$y = e^x.$$

Here e is the mathematical constant known as Euler's number. $e \approx 2.71828..$



Important Values of $y = e^x$	
x	y
0	1
1	e
−1	$\frac{1}{e}$

In general, we can talk about exponential functions of the form $y = b^x$ where b is a positive number not equal to 1. You can play with changing the values of b on the graph using Desmos and see how that changes the graph. Pay particular attention to the difference between $b > 1$ and $0 < b < 1$.

Desmos link: `https://www.desmos.com/calculator/qsmvb7tiex`

Note that the domain of the exponential functions is $(-\infty, \infty)$. Furthermore, by looking at the graph, we can see that all non-negative numbers are in the range of the exponential functions.

Table 9: Properties of The Exponential Functions $y = b^x$

Periodic?	No
Odd?	No
Even?	No
One-to-one/invertible	Yes

Table 10: Domain and Range of The Exponential Functions $y = b^x$

Domain	$(-\infty, \infty)$
Range	$[0, \infty)$

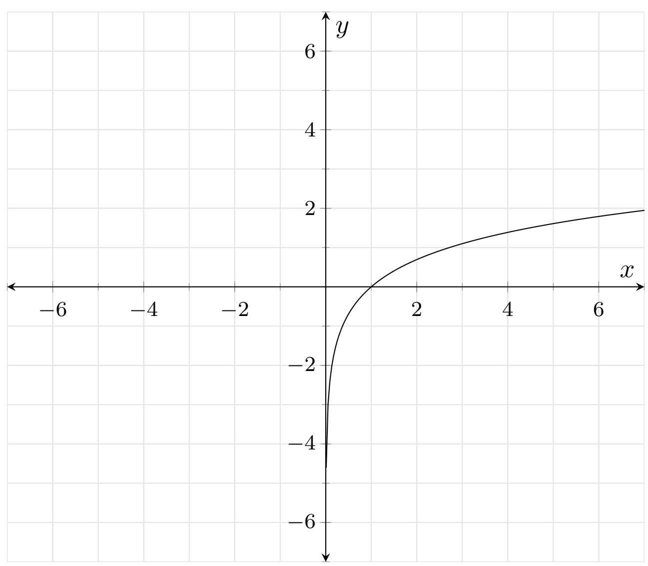
Logarithm

Another group of famous functions are logarithms.

Example 9. *The most famous logarithm function is*

$$y = \ln(x) = \log_e(x).$$

Here e is the mathematical constant known as Euler's number. $e \approx 2.71828$.



Important Values of $y = ln(x)$	
x	y
0	undefined
$\frac{1}{e}$	-1
1	0
e	1

You may notice that the table of values for $y = \ln(x)$ and $y = e^x$ are similiar. This is becase these two functions are interconnected. We will explore this more later in the course.

In general, we can talk about logarithmic functions of the form $y = \log_b(x)$ where b is a positive number not equal to 1. You can play with changing the values of b on the graph using Desmos and see how that changes the graph. Pay particular attention to the difference between $b > 1$ and $0 < b < 1$.

Desmos link: `https://www.desmos.com/calculator/lxllnpdi6w`

Table 11: Properties of The Logarithm Functions $y = \log_b(x)$

Periodic?	No
Odd?	No
Even?	No
One-to-one/invertible	Yes

Note that since the logarithm is the inverse of the exponential, the domain of the logarithms is the range of the exponentials: $[0, \infty)$. Furthermore, the range of the logarithms is the range of the exponentials: $(-\infty, \infty)$.

Table 12: Domain and Range of The Logarithms $y = \log_b(x)$

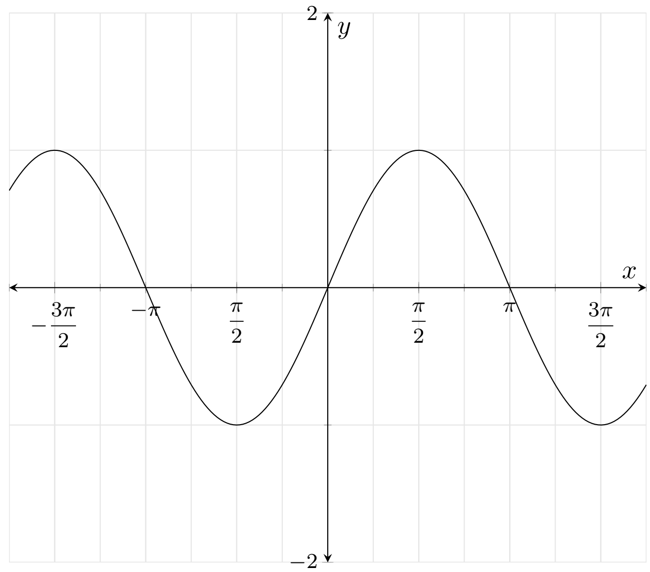
Domain	$[0, \infty)$
Range	$(-\infty, \infty)$

Sine

Another important function is the sine function,

$$y = \sin(x).$$

This function comes from trigonometry. In the table below we will use another mathematical constant, π ("pi" pronounced pie). $\pi \approx 3.14159$.



Important Values of $y = \sin(x)$	
x	y
$-\pi$	0
$-\frac{\pi}{2}$	-1
0	0
$\frac{\pi}{2}$	1
π	0
$\frac{3\pi}{2}$	-1
2π	0

Note that the domain of the sine function is $(-\infty, \infty)$. Furthermore, by looking at the graph, we can see that its range is $[-1, 1]$

Table 13: Properties of The Sine Function $y = \sin(x)$

Periodic?	Yes, with period 2π
Odd?	Yes
Even?	No
One-to-one/invertible	No

Table 14: Domain and Range of The Sine Function $y = \sin(x)$

Domain	$(-\infty, \infty)$
Range	$[-1, 1]$

In general, we can consider $y = a\sin(bx)$. You can play with changing the values of a and b on the graph using Desmos and see how that changes the graph.

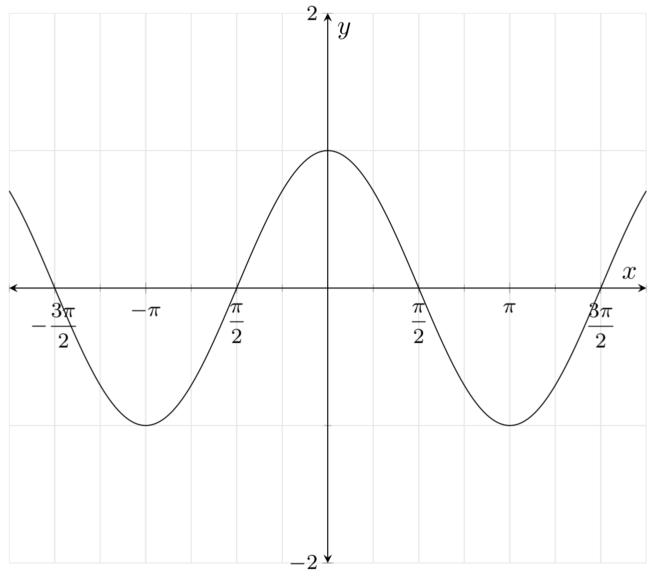
Desmos link: `https://www.desmos.com/calculator/vkxzcfv2aq`

Cosine

A function introduced in Section 3-2 is the cosine function,

$$y = \cos(x).$$

As with sine, the cosine function comes from trigonometry. In the table below we will again use π.



Important Values of $y = \cos(x)$	
x	y
$-\pi$	-1
$-\frac{\pi}{2}$	0
0	1
$\frac{\pi}{2}$	0
π	-1
$\frac{3\pi}{2}$	0
2π	1

As mentioned earlier, the cosine function is even and periodic with period 2π. Since it is periodic, however, it cannot be one-to-one, since its values repeat. We summarize some information in Table 15.

Table 15: Properties of The Cosine Function $y = \cos(x)$

Periodic?	Yes, with period 2π
Odd?	No
Even?	Yes
One-to-one/invertible	No

Note that the domain of the cosine function is $(-\infty, \infty)$. Furthermore, by looking at the graph, we can see that its range is $[-1, 1]$

Table 16: Domain and Range of The Cosine Function $y = \cos(x)$

Domain	$(-\infty, \infty)$
Range	$[-1, 1]$

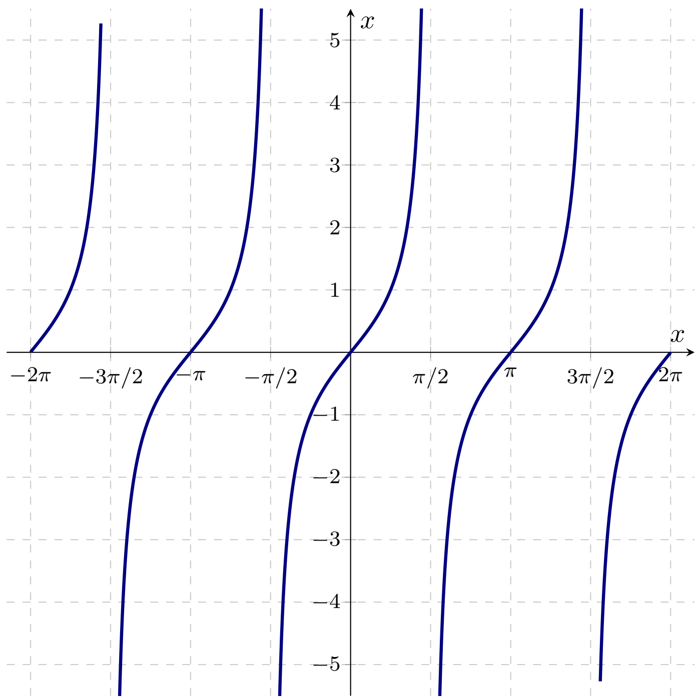
In general, we can consider $y = a\cos(bx)$. You can play with changing the values of a and b on the graph using Desmos and see how that changes the graph.

Desmos link: https://www.desmos.com/calculator/kvmz1kt19n

Tangent

A function introduced in Section 4-1 is the tangent function,

$$y = \tan(x).$$



Important Values of $y = \tan(x)$	
x	y
$-\pi$	0
$-\dfrac{\pi}{2}$	undefined
0	0
$\dfrac{\pi}{2}$	undefined
π	0
$\dfrac{3\pi}{2}$	undefined
2π	0

As mentioned earlier, the tangent function is odd and periodic with period π.

Since it is periodic, however, it cannot be one-to-one, since its values repeat. We summarize some information in Table 17.

Table 17: Properties of The Tangent Function $y = \tan(x)$

Periodic?	Yes, with period π
Odd?	Yes
Even?	No
One-to-one/invertible	No

Note that the domain of the tangent function is all real numbers except for odd multiples of $\frac{\pi}{2}$, since tangent is undefined at those places. Furthermore, by looking at the graph, we can see that its range is $(-\infty, \infty)$.

Table 18: Domain and Range of The Tangent Function $y = \tan(x)$

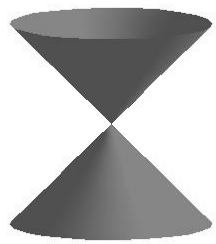
Domain	$\cdots \cup \left(-\frac{5\pi}{2}, -\frac{3\pi}{2}\right) \cup \left(-\frac{3\pi}{2}, -\frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \frac{3\pi}{2}\right) \cup \left(\frac{3\pi}{2}, \frac{5\pi}{2}\right) \cup \cdots$
Range	$(-\infty, \infty)$

In general, we can consider $y = a\tan(bx)$. You can play with changing the values of a and b on the graph using Desmos and see how that changes the graph.

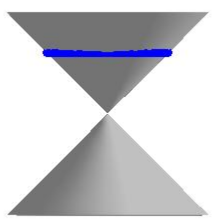
Desmos link: `https://www.desmos.com/calculator/1je3xt6hag`

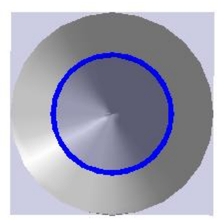
Conic Sections

In this section, we study the **Conic Sections** - literally 'sections of a cone'. Imagine a double-napped cone as seen below being 'sliced' by a plane.

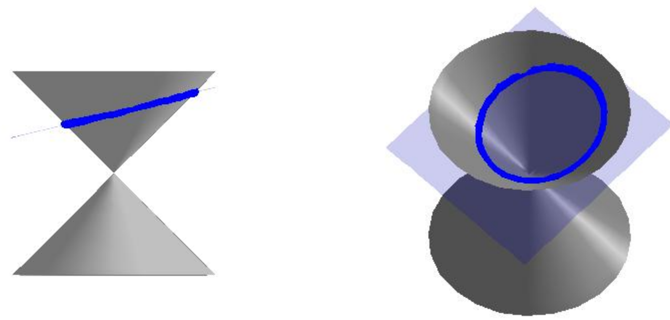


If we slice the cone with a horizontal plane the resulting curve is a **circle**.

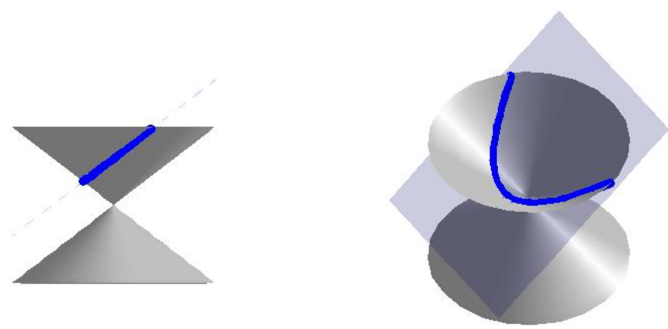




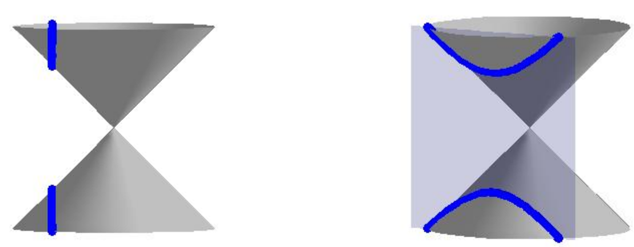
Tilting the plane ever so slightly produces an **ellipse**.



If the plane cuts parallel to the cone, we get a **parabola**.

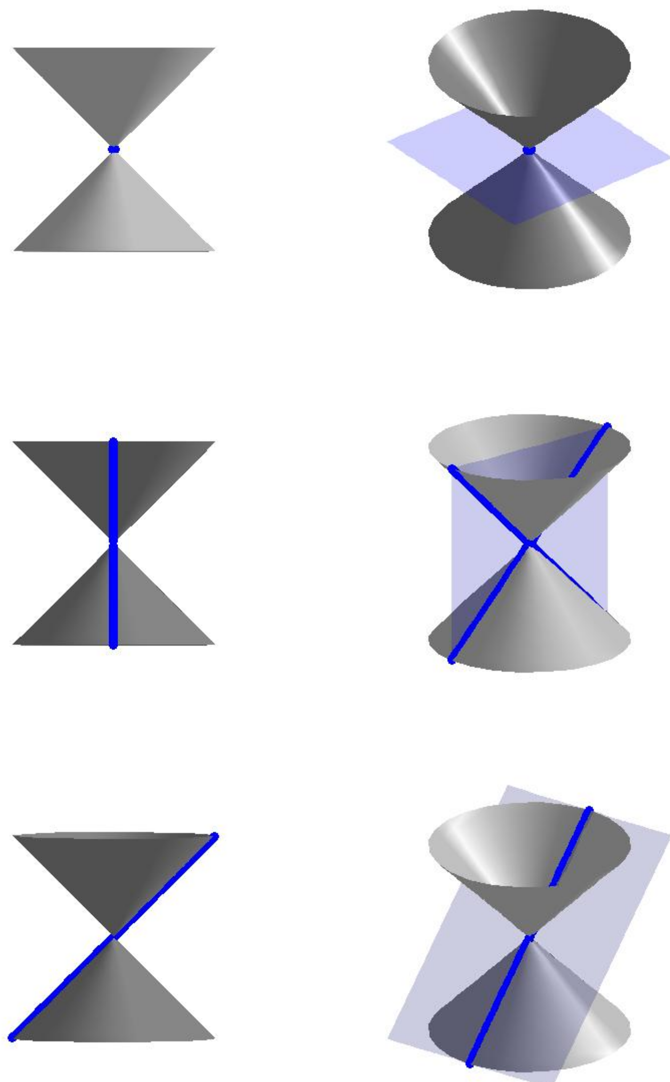


If we slice the cone with a vertical plane, we get a **hyperbola**.



For a wonderful animation describing the conics as intersections of planes and cones, see Dr. Louis Talman's Mathematics Animated Website.

If the slicing plane contains the vertex of the cone, we get the so-called 'degenerate' conics: a point, a line, or two intersecting lines.

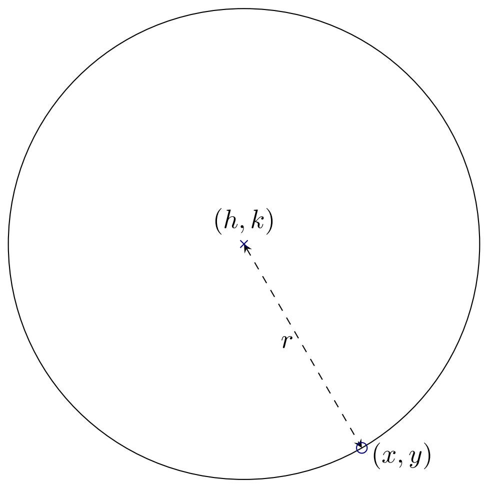


We will focus the discussion on the non-degenerate cases: circles, parabolas, ellipses, and hyperbolas, in that order. It's not necessary to memorize the description of each conic section. We'd just like you to understand that each conic section is the graph of an equation which can be rearranged into a certain *standard equation*. This standard equation is useful, because it allows us to say something about various geometric properties of the graph. In addition, we will only discuss conic sections centered at the origin.

Circles

Recall from Geometry that a circle can be determined by fixing a point (called the *center*) and a positive number (called the *radius*) as follows.

> **Definition** A **circle** with center (h, k) and radius $r > 0$ is the set of all points (x, y) in the plane whose distance to (h, k) is r.



We express this relationship algebraically using the Distance Formula as

$$r = \sqrt{(x-h)^2 + (y-k)^2}$$

By squaring both sides of this equation, we get an equivalent equation (since $r > 0$) which gives us the standard equation of a circle.

> **Definition** **The standard equation of a circle** with center (h, k) and radius $r > 0$ is $(x-h)^2 + (y-k)^2 = r^2$.

This is the first example of a standard equation. If we are given a standard equation of a circle, we can easily find its center and its radius, which is all that we need to be able to draw the circle in the xy-plane. In other courses, we would spend a lot of time taking an equation, converting it into a standard equation, recognizing it as the standard equation of a conic section, and then using the information provided by the standard equation to graph the relation. For our purposes, we only need to know that this process can be done.

We close this section with the most important circle in all of mathematics: the *unit circle*.

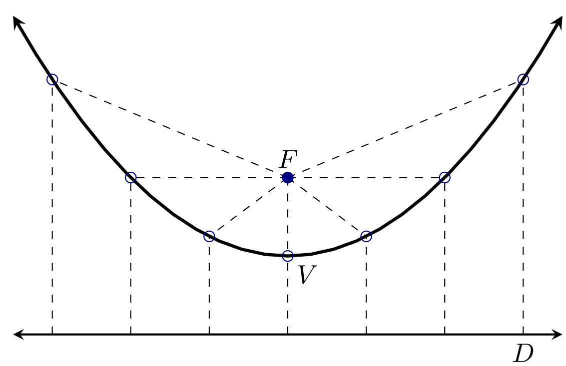
> **Definition** The **unit circle** is the circle centered at $(0, 0)$ with a radius of 1. The standard equation of the unit circle is $x^2 + y^2 = 1$.

As you will soon see, the unit circle is central to the study of trigonometry.

Parabolas

We know that parabolas are the graphs of quadratic functions. To our surprise and delight, we may also define parabolas in terms of distance.

> **Definition** Let F be a point in the plane and D be a line not containing F. A **parabola** is the set of all points equidistant from F and D. The point F is called the **focus** of the parabola and the line D is called the **directrix** of the parabola. The **vertex** is the point on the parabola closest to the focus.



Each dashed line from the point F to a point on the curve has the same length as the dashed line from the point on the curve to the line D. The point suggestively labeled V is, as you should expect, the vertex. Notice that the focus F is not actually a point on the parabola, but only serves to help in its construction.

As with circles, there is a standard equation for parabolas.

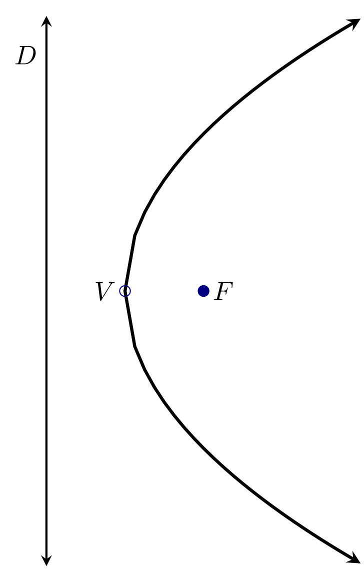
Definition The **standard equation of a parabola** which opens up or down with vertex (h, k) and focal length $|p|$ is

$$(x - h)^2 = 4p(y - k)$$

If $p > 0$, the parabola opens upwards; if $p < 0$, it opens downwards. The **focal length** of the parabola is the distance from the focus to the vertex.

Notice that in the standard equation of the parabola above, only one of the variables, x, is squared. This is a quick way to distinguish an equation of a parabola from that of a circle because in the equation of a circle, both variables are squared.

Recall from our earlier discussion of inverse functions that interchanging the roles of x and y results in reflecting the graph across the line $y = x$. Therefore, if we interchange the roles of x and y, we can produce 'horizontal' parabolas: parabolas which open to the left or to the right. The directrices (plural of 'directrix') of such animals would be vertical lines and the focus would either lie to the left or to the right of the vertex, as seen below.



Definition The standard equation of a parabola that opens to the left or right with vertex (h, k) and focal length $|p|$ is

$$(y - k)^2 = 4p(x - h)$$

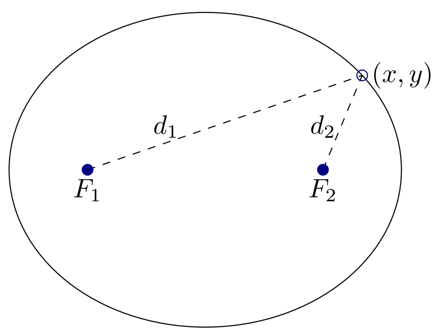
If $p > 0$, the parabola opens to the right; if $p < 0$, it opens to the left.

Ellipses

In the definition of a circle, we fixed a point called the *center* and considered all of the points which were a fixed distance r from that one point. For our next conic section, the ellipse, we fix two distinct points and a distance d to use in our definition.

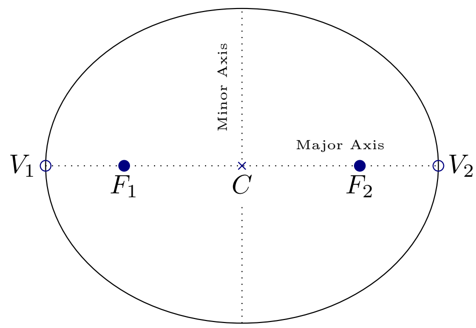
Definition Given two distinct points F_1 and F_2 in the plane and a fixed distance d, an **ellipse** is the set of all points (x, y) in the plane such that the sum of each of the distances from F_1 and F_2 to (x, y) is d. The points F_1 and F_2 are called the **foci** (the plural of 'focus') of the ellipse.

In the figure below, d_1 is the distance from (x, y) to F_1, and d_2 is the distance from (x, y) to F_2. Since (x, y) is on the ellipse, $d_1 + d_2 = d$ for some fixed d.



We may imagine taking a length of string and anchoring it to two points on a piece of paper. The curve traced out by taking a pencil and moving it so the string is always taut is an ellipse. Notice again that the foci are not actually points on the ellipse, but only serve to help in its construction.

The *center* of the ellipse is the midpoint of the line segment connecting the two foci. The *major axis* of the ellipse is the line segment connecting two opposite ends of the ellipse which also contains the center and foci. The *minor axis* of the ellipse is the line segment connecting two opposite ends of the ellipse which contains the center but is perpendicular to the major axis. The major axis is always the longer of the two segments. The *vertices* of an ellipse are the points of the ellipse which lie on the major axis. Notice that the center is also the midpoint of the major axis, hence it is the midpoint of the vertices. In pictures we have,



There is also a standard equation for ellipses.

> **Definition** For positive unequal numbers a and b, **the standard equation of an ellipse** with center (h, k) is
>
> $$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

First note that the values a and b determine how far in the x and y directions, respectively, one counts from the center to arrive at points on the ellipse. Also take note that if $a > b$, then we have an ellipse whose major axis is horizontal, and hence, the foci lie to the left and right of the center. In this case, the distance from the center to the focus, c, can be found by $c = \sqrt{a^2 - b^2}$. If $b > a$, the roles of the major and minor axes are reversed, and the foci lie above and below the center. In this case, $c = \sqrt{b^2 - a^2}$. In either case, c is the distance from the center to each focus, and $c = \sqrt{\text{bigger denominator} - \text{smaller denominator}}$. Finally, it is worth mentioning that if we take the standard equation of a circle and divide both sides by r^2, we get

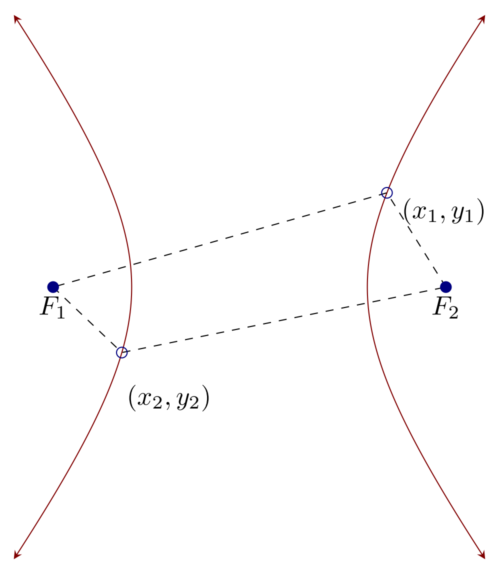
> **Definition** **The alternate standard equation of a circle** with center (h, k) and radius $r > 0$ is
>
> $$\frac{(x-h)^2}{r^2} + \frac{(y-k)^2}{r^2} = 1$$

Notice the similarity between the two equations. Both involve a sum of squares equal to 1; the difference is that with a circle, the denominators are the same, and with an ellipse, they are different. If we take a transformational approach, we can consider both equations as shifts and stretches of the unit circle $x^2+y^2=1$. Replacing x with $(x-h)$ and y with $(y-k)$ causes the usual horizontal and vertical shifts. Replacing x with $\frac{x}{a}$ and y with $\frac{y}{b}$ causes the usual vertical and horizontal stretches. In other words, it is perfectly fine to think of an ellipse as the deformation of a circle in which the circle is stretched farther in one direction than the other.

Hyperbolas

In the definition of an ellipse, we fixed two points called foci and looked at points whose distances to the foci always *added* to a constant distance d. Those prone to syntactical tinkering may wonder what, if any, curve we'd generate if we replaced *added* with *subtracted*. The answer is a hyperbola.

> **Definition** Given two distinct points F_1 and F_2 in the plane and a fixed distance d, a **hyperbola** is the set of all points (x, y) in the plane such that the difference of each of the distances from F_1 and F_2 to (x, y) is d. The points F_1 and F_2 are called the **foci** of the hyperbola.



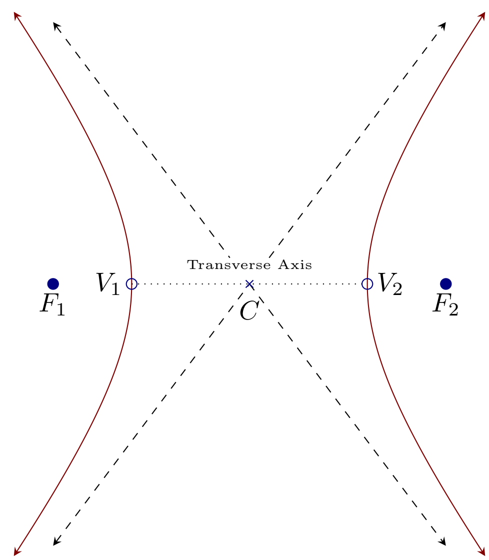
In the figure above:

$$\text{the distance from } F_1 \text{ to } (x_1, y_1) - \text{the distance from } F_2 \text{ to } (x_1, y_1) \quad = \quad d$$

and

$$\text{the distance from } F_2 \text{ to } (x_2, y_2) - \text{the distance from } F_1 \text{ to } (x_2, y_2) \quad = \quad d$$

Note that the hyperbola has two parts, called *branches*. The *center* of the hyperbola is the midpoint of the line segment connecting the two foci. The *transverse axis* of the hyperbola is the line segment connecting two opposite ends of the hyperbola which also contains the center and foci. The *vertices* of a hyperbola are the points of the hyperbola which lie on the transverse axis. In addition, there are lines called *asymptotes* which the branches of the hyperbola approach for large x and y values. They serve as guides to the graph. In pictures,



The above hyperbola has center C, foci F_1 and F_2, and vertices V_1 and V_2. The asymptotes are represented by dashed lines.

The *conjugate axis* of a hyperbola is the line segment through the center which is perpendicular to the transverse axis and has the same length as the line segment through a vertex which connects the asymptotes.

As with all the other conic sections, we have a standard equation for hyperbolas.

Definition For positive numbers a and b, **the equation of a hyperbola** opening left and right with center (h, k) is

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$

If the roles of x and y were interchanged, then the hyperbola's branches would open upwards and downwards and we would get a 'vertical' hyperbola.

Definition For positive numbers a and b, **the equation of a hyperbola** opening upwards and downwards with center (h, k) is

$$\frac{(y-k)^2}{b^2} - \frac{(x-h)^2}{a^2} = 1$$

The values of a and b determine how far in the x and y directions, respectively, one counts from the center to determine the rectangle through which the asymptotes pass. In both cases, the distance from the center to the foci, c, can be found by the formula $c = \sqrt{a^2 + b^2}$. Lastly, note that we can quickly distinguish the equation of a hyperbola from that of a circle or ellipse because the hyperbola formula involves a *difference* of squares where the circle and ellipse formulas both involve the *sum* of squares.

11.2.2 Solving Non-linear Systems Graphically

Motivating Questions

- What is a non-linear system of equations?
- How can we find solutions graphically?
- What can we say about when solutions exist?

Introduction

In this section, we study systems of non-linear equations. Unlike the systems of linear equations for which we have developed several algorithmic solution techniques, there is no general algorithm to solve systems of non-linear equations. Moreover, all of the usual hazards of non-linear equations like extraneous solutions and unusual function domains are once again present. Along with the tried and true techniques of substitution and elimination, we shall often need equal parts tenacity and ingenuity to see a problem through to the end. You may find it necessary to review topics throughout the text which pertain to solving equations involving the various functions we have studied thus far.

What are non-linear systems of equations?

The key to identifying non-linear equations is to note that the variables involved are not necessarily to the first power, and the coefficients of the variables may not just be real numbers. Some examples of equations which are non-linear are $x^2 + y = 1$, $xy = 5$ and $e^{2x} + \ln(y) = 1$. An example of a non-linear system of equations is given by

$$\begin{cases} x^2 + y^2 &= 4 \\ 4x^2 - 9y^2 &= 36 \end{cases}.$$

Note that this system is non-linear because the variables x and y are raised to the second power.

Another example of a non-linear system of equations is given by

$$\begin{cases} x^2 + y^2 &= 4 \\ y - 2x &= 0 \end{cases}.$$

Even though y and x are both raised to the first power in the second equation above, the first equation still contains second powers of variables, so this is a non-linear system.

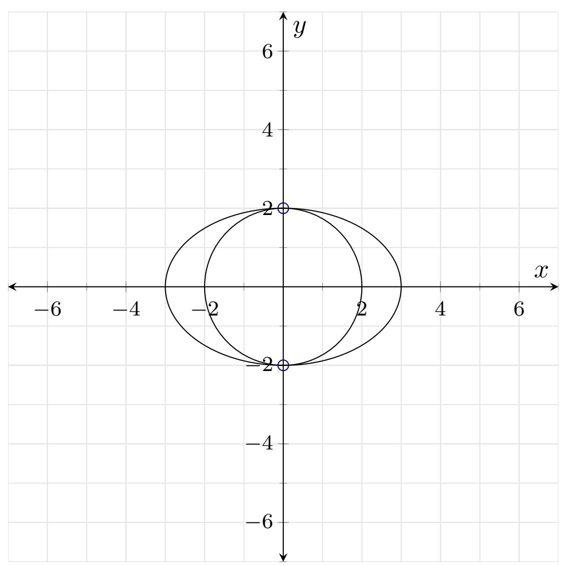
Solving systems graphically

Finding solutions to non-linear systems is the same concept as finding solutions to linear systems. This means that we can also think about finding solutions as finding intersections points of the graphs of the equations in our system.

Example 10. *Find all solutions to the following system of equations:*

$$\begin{cases} x^2 + y^2 & = 4 \\ 4x^2 + 9y^2 & = 36 \end{cases}.$$

Explanation We sketch the graphs of both equations and look for their points of intersection. The graph of $x^2 + y^2 = 4$ is a circle centered at $(0, 0)$ with a radius of 2, whereas the graph of $4x^2 + 9y^2 = 36$, when written in the standard form $\frac{x^2}{9} + \frac{y^2}{4} = 1$ can be recognized as an ellipse centered at $(0, 0)$ with a major axis along the x-axis of length 6 and a minor axis along the y-axis of length 4. This is illustrated in the figure below.



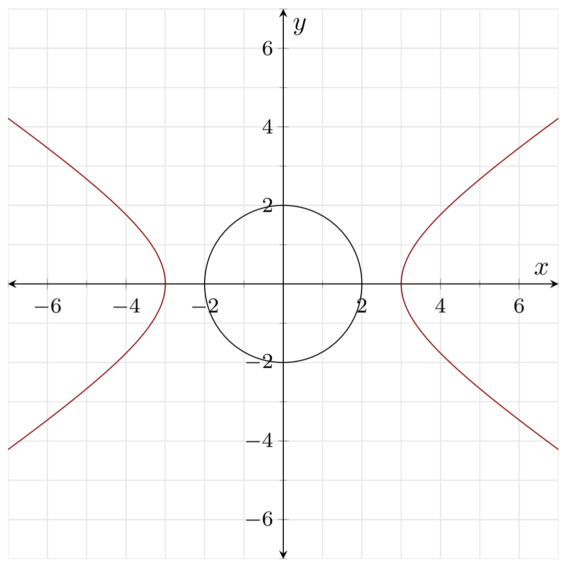
We see from the figure that the two graphs intersect at their y-intercepts only, $(0, 2)$ and $(0, -2)$. Recalling that points of intersection correspond to solutions to the system of equations, $(0, 2)$ and $(0, -2)$ are the only solutions to the system.

Example 11. *Find all solutions to the following system of equations:*

$$\begin{cases} x^2 + y^2 & = 4 \\ 4x^2 - 9y^2 & = 36 \end{cases}.$$

Explanation First, notice that this system only differs from the previous example in that it has a minus sign in front of the $9y^2$ in the bottom equation.

We again sketch the graphs of both equations and look for their points of intersection. The graph of $x^2 + y^2 = 4$ is a circle centered at $(0, 0)$ with a radius of 2, as in the previous example. However, the graph of $4x^2 - 9y^2 = 36$, when written in the standard form $\dfrac{x^2}{9} - \dfrac{y^2}{4} = 1$ can be recognized as a hyperbola centered at $(0, 0)$ opening to the left and right with a transverse axis of length 6 and a conjugate axis of length 4. This is illustrated in the figure below.



We see that the circle and the hyperbola have no points in common. Recalling that points of intersection correspond to solutions to the system of equations, we say that the system has no solutions.

Note that we can characterize systems of nonlinear equations as being consistent or inconsistent, just like their linear counterparts. Unlike systems of linear equations, however, it is possible for a system of non-linear equations to have more than one solution without having infinitely many solutions. Secondly, as we have seen above, sometimes making a quick sketch of the problem situation can save a lot of time and effort. While in general the graphs of equations in a non-linear system may not be easily visualized, it sometimes pays to take advantage of visualization when you are able.

11.2.3 Eliminating Variables

Solving Non-linear Systems Algebraically

Algebraically, we can use the methods of substitution and elimination outlined in Section 8.1 to solve non-linear systems of equations. However, we need to exercise care when solving non-linear systems, especially since the operations involved may not always result in valid solutions!

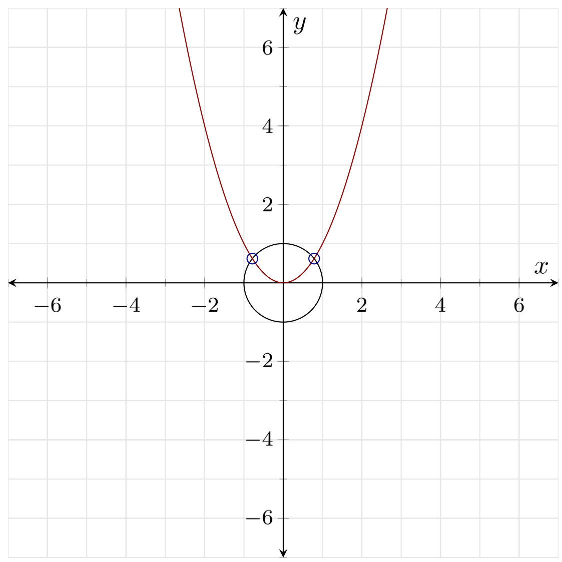
For example, consider the system given by

$$\begin{cases} y - x^2 &= 0 \\ x^2 + y^2 &= 1 \end{cases}.$$

Let's try to use substitution here. From the top equation, we can see that $y = x^2$. Substituting this into the bottom equation results in $x^2 + (x^2)^2 = 1$, or $x^4 + x^2 = 1$, which we can rewrite as $x^4 + x^2 - 1 = 0$. We can now use the quadratic formula on $x^4 + x^2 - 1$ to find that $x^2 = \dfrac{-1 \pm \sqrt{5}}{2}$. Taking a square root, we find that $x = \pm\sqrt{\dfrac{-1 \pm \sqrt{5}}{2}}$ are possible values of x. Note that there are actually *four* separate possible values of x, one for each choice of plus or minus in the expression above: $\sqrt{\dfrac{-1+\sqrt{5}}{2}}$, $\sqrt{\dfrac{-1-\sqrt{5}}{2}}$, $-\sqrt{\dfrac{-1+\sqrt{5}}{2}}$, and $-\sqrt{\dfrac{-1-\sqrt{5}}{2}}$.

However, $\dfrac{-1-\sqrt{5}}{2}$ is actually negative! Since the square root of a negative number is not a real number, $\sqrt{\dfrac{-1-\sqrt{5}}{2}}$ and $-\sqrt{\dfrac{-1-\sqrt{5}}{2}}$ are not valid x-values of a solution to this system. The other two solutions are fine. Therefore, keeping in mind that $y = x^2$, the solutions to our system are given by $\left(\sqrt{\dfrac{-1+\sqrt{5}}{2}}, \dfrac{-1+\sqrt{5}}{2}\right)$ and $\left(-\sqrt{\dfrac{-1+\sqrt{5}}{2}}, \dfrac{-1+\sqrt{5}}{2}\right)$. We can plug these back into the original equations to make sure that they satisfy both.

Taking a look at the graphs of these equations should shed some light on what's happening here.



The only intersection points of the two graphs have a positive y-coordinate. This could have tipped us off earlier that some of the x-values we got wouldn't be valid. Indeed, from the first equation, we have that $x = \sqrt{y}$, and this ensures that y must be positive.

The above example illustrates the importance of always checking that the solutions you find are real numbers, and also checking that the solutions you find are actually solutions to the system.

Eliminating Variables

Now we illustrate the method of elimination, which can be used when you notice that the equations in the system have like terms. The difference from before is that we now may have non-linear terms that we can eliminate.

Let's apply this technique to a system we saw previously.

Example 12. *Find all solutions to the following system of equations:*

$$\begin{cases} x^2 + y^2 &= 4 \\ 4x^2 + 9y^2 &= 36 \end{cases}.$$

Explanation We can multiply the top equation by -4, so we get the equivalent system of equations

$$\begin{cases} -4x^2 - 4y^2 &= -16 \\ 4x^2 + 9y^2 &= 36 \end{cases}.$$

Now we can eliminate the x^2 terms to obtain $5y^2 = 20$. From here, we see that $y^2 = 4$, so $y = \pm 2$. To find the associated x values, we substitute each value of y into one of the equations to find the resulting value of x. Choosing $x^2 + y^2 = 4$, we find that for both $y = -2$ and $y = 2$, we get $x = 0$. Our solution set is thus $\{(0, 2), (0, -2)\}$.

Example 13. *Find all solutions to the following system of equations:*

$$\begin{cases} x^2 + y^2 & = 4 \\ 4x^2 - 9y^2 & = 36 \end{cases}.$$

Explanation We proceed as before to eliminate one of the variables. We can multiply the top equation by -4, so we get the equivalent system of equations

$$\begin{cases} -4x^2 - 4y^2 & = -16 \\ 4x^2 - 9y^2 & = 36 \end{cases}.$$

Now we can eliminate the x^2 terms to obtain $-13y^2 = 20$. From here, we see that $y^2 = -\dfrac{20}{13}$. Since the square root of a negative number is not a real number, we see that there are no real values of y that solve this equation. Therefore, we conclude that this system has no solution. Recall that a system that has no solution is called inconsistent.

Example 14. *Find all solutions to the following system of equations:*

$$\begin{cases} x^2 + 2xy - 16 & = 0 \\ y^2 + 2xy - 16 & = 0 \end{cases}.$$

Explanation At first glance, it doesn't appear as though elimination will do us any good since it's clear that we cannot completely eliminate one of the variables. The alternative, solving one of the equations for one variable and substituting it into the other, is full of unpleasantness. Returning to elimination, we note that it is possible to eliminate the troublesome xy term, and the constant term as well, by elimination and doing so we get a more tractable relationship between x and y. We can multiply the top equation by -1, so we get the equivalent system of equations

$$\begin{cases} -x^2 - 2xy + 16 & = 0 \\ y^2 + 2xy - 16 & = 0 \end{cases}.$$

Eliminating, we find that $y^2 - x^2 = 0$, so $y^2 = x^2$, and $y = \pm x$. Substituting $y = x$ into the top equation, we get $x^2 + 2x^2 - 16 = 0$, so that $x^2 = \dfrac{16}{3}$ or $x = \pm\dfrac{4\sqrt{3}}{3}$. On the other hand, when we substitute $y = -x$ into the top equation, we get $x^2 - 2x^2 - 16 = 0$ or $x^2 = -16$, which gives no real solutions. Substituting each of $x = \pm\frac{4\sqrt{3}}{3}$ into the substitution equation $y = x$ yields the solution set $\left\{\left(\frac{4\sqrt{3}}{3}, \frac{4\sqrt{3}}{3}\right), \left(-\frac{4\sqrt{3}}{3}, -\frac{4\sqrt{3}}{3}\right)\right\}$. Try plugging these into the original system to see that they are actually solutions. Verifying this graphically would be a fun exercise, but we leave that up to you.

Some Common Issues and Techniques

Example 15. *Find all solutions to the following system of equations:*

$$\begin{cases} x^2 + y &= 12 \\ 3xy &= 0 \end{cases}.$$

Explanation Notice that this is, in fact, a non-linear system, since the second equation contains an xy term. Since we can't see any like terms in the two equations, it makes sense to try to use substitution. We might be tempted to divide both sides of the bottom equation by $3x$, so as to isolate y, but as always with division, we need to be careful! Indeed, $x = 0$ is still a possibility, so we cannot divide through by $3x$, since we'd then be dividing by 0.

Instead, it helps to think about what it means for the product of two numbers to equal 0. In fact, the product of two nonzero numbers can never be 0. In our situation, we know that $3xy = 0$, so either $3x = 0$ or $y = 0$.

If $3x = 0$, then dividing by 3 (since $3 \neq 0$) gives us $x = 0$. We can plug that into the top equation and find that $0^2 + y = 12$, so $y = 12$. We can then check that $(0, 12)$ is a solution to our original system.

If $y = 0$, we can plug that into the top equation to find that $x^2 + 0 = 12$. Solving for x yields $x = \pm\sqrt{12} = \pm 2\sqrt{3}$. We can then check that $(2\sqrt{3}, 0)$ and $(-2\sqrt{3}, 0)$ are solutions to the system.

Our final solution set is $\{(2\sqrt{3}, 0), (-2\sqrt{3}, 0), (0, 12)\}$.

Example 16. *Find all solutions to the following system of equations:*

$$\begin{cases} \dfrac{4}{x} + \dfrac{3}{y} &= 1 \\ \\ \dfrac{3}{x} + \dfrac{2}{y} &= -1 \end{cases}.$$

Explanation Notice that this is, in fact, a non-linear system, since both equations divide by the variables we're using.

If we define new (but related) variables by letting $u = \dfrac{1}{x}$ and $v = \dfrac{1}{y}$ then the system becomes

$$\begin{cases} 4u + 3v &= 1 \\ 3u + 2v &= -1 \end{cases}.$$

This associated system of linear equations can then be solved using any of the techniques you've learned earlier to find that $u = -5$ and $v = 7$. Therefore, $x = \dfrac{1}{u} = -\dfrac{1}{5}$ and $y = \dfrac{1}{v} = \dfrac{1}{7}$, and our solution set is $\left\{\left(-\dfrac{1}{5}, \dfrac{1}{7}\right)\right\}$.

We say that the original system is linear in form because its equations are not linear, but a few substitutions reveal a structure that we can treat like a system of linear equations. However, the substitutions may introduce some complexity, as seen in the following example.

Example 17. *Find all solutions to the following system of equations:*

$$\begin{cases} 5e^x + 3e^{2y} &= 1 \\ \\ 3e^x + e^{2y} &= 2 \end{cases}.$$

Explanation Notice that this is, in fact, a non-linear system, since both equations divide by the variables we're using.

If we define new (but related) variables by letting $u = e^x$ and $v = e^{2y}$ then the system becomes

$$\begin{cases} 5u + 3v &= 1 \\ 3u + v &= 2 \end{cases}.$$

This associated system of linear equations can then be solved using any of the techniques you've learned earlier to find that $u = \dfrac{3}{4}$ and $v = -\dfrac{1}{4}$. Therefore, $x = \ln(u) = \ln\left(\dfrac{3}{4}\right)$ and $y = \dfrac{\ln(v)}{2} = \dfrac{\ln\left(-\frac{1}{4}\right)}{7}$. However, the nonlinearity of the system throws us a wrench! Logarithms are not defined on negative numbers, so $\dfrac{\ln\left(-\frac{1}{4}\right)}{7}$ does not exist, and there is actually no value of y that satisfies both equations. Therefore, this system does not have a solution.

Exploration Consider the following system.

$$\begin{cases} 4\ln(x) + 3y^2 &= 1 \\ 3\ln(x) + 2y^2 &= -1 \end{cases}.$$

(a) Is the system linear in form?

(b) If so, make substitutions by defining variables u and v so that the system in terms of u and v is linear. What is u? What is v? What is our new associated linear system?

(c) What is the solution set to our associated linear system?

(d) What is the solution set to our original system?

11.3 Applications of Systems

Learning Objectives

- Applications of Systems
  - Word problems with extraneous variables
  - Word problems similar to related rates or optimization

11.3.1 Applications of Systems

Introduction

Suppose a rectangle has width w and length l, with area 24 and perimeter 20. The area of the rectangle is wl and the perimeter is given by $2w+2l$, giving the following system of equations

$$\begin{cases} wl &= 24 \\ 2w + 2l &= 20. \end{cases}$$

Since the first equation here is not a linear equation, this is a nonlinear system of equations. If we want to find the dimensions of the corresponding rectangle, we must solve this system. Since calculations of areas and volumes are nonlinear in general, situations involving geometric shapes often result in nonlinear systems of equations.

To solve this system, we will start by dividing both sides of the second equation by 2, to obtain the following equivalent system.

$$\begin{cases} wl &= 24 \\ w + l &= 10. \end{cases}$$

If this second equation is satisfied, that means $l = 10 - w$, which can be substituted into the top equation to eliminate the variable l.

$$\begin{aligned} wl &= 24 \\ w(10-w) &= 24 \\ 10w - w^2 &= 24 \\ w^2 - 10w + 24 &= 0 \\ (w-6)(w-4) &= 0. \end{aligned}$$

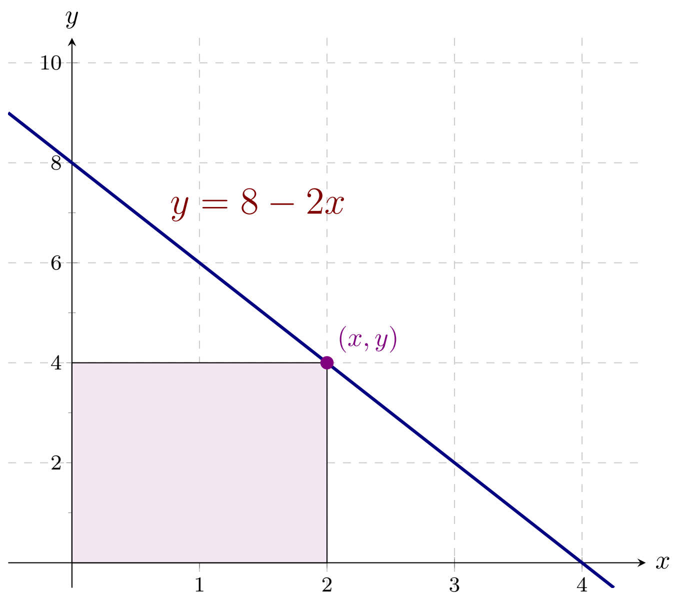
The $w-6$ factor gives a solution of $w = 6$, and the $w-4$ factor gives a solution of $w = 4$.

Looking back at $l = 10 - w$ we see that if $w = 6$, then $l = 10 - 6 = 4$ and if $w = 4$ then $l = 10 - 4 = 6$.

There are two possible rectangles: One with width 6 and length 4, and the other with width 4 and length 6.

Applications of Systems

Exercise 1 *A rectangle is drawn in the first quadrant with one side along the x-axis, one side along the y-axis, the lower left corner at the origin, and* *upper right corner on the graph of the equation $y = 8 - 2x$. Denote this* upper right *vertex as (x, y). Find the coordinates of the point (x, y) if the area of the rectangle is $\frac{15}{2}$.*



Explanation Since (x, y) are the coordinates of the upper right vertex, this tells us that x and y are both positive. It also tells us that the distance from the x-axis is y, and the distance from the y-axis is x. In other words, the height of the rectangle is just y, and the width of the rectangle is x. In terms of x and y, the area is given by xy, giving us one equation $xy = \frac{15}{2}$.

Since the upper right corner (x, y) is on the graph of the line, we also know that $y = 8 - 2x$. This leaves us with the following system: This gives a system of nonlinear equations

$$\begin{cases} xy &= \frac{15}{2} \\ y &= 8 - 2x. \end{cases}$$

This bottom equation is already solved for y, so the easiest way to eliminate a variable would be to substitute it into the y in the top equation.

$$\begin{aligned} xy &= \frac{15}{2} \\ x(8-2x) &= \frac{15}{2} \\ 8x - 2x^2 &= \frac{15}{2} \\ 2x^2 - 8x &= -\frac{15}{2} \\ x^2 - 4x &= -\frac{15}{4}. \end{aligned}$$

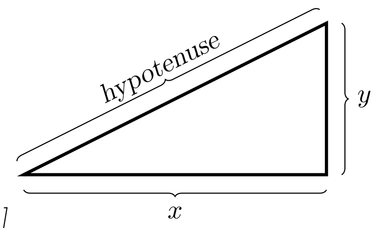
We'll solve this equation by completing the square.

$$\begin{aligned} x^2 - 4x &= -\frac{15}{4} \\ x^2 - 4x + 4 &= -\frac{15}{4} + 4 \\ x^2 - 4x + 4 &= \frac{1}{4} \\ (x-2)^2 &= \frac{1}{4} \\ x - 2 &= \pm\sqrt{\frac{1}{4}} \\ x - 2 &= \pm\frac{1}{2} \\ x &= 2 \pm \frac{1}{2} \\ x &= \frac{3}{2}, \frac{5}{2} \end{aligned}$$

If $x = \frac{3}{2}$ then $y = 8 - 2\left(\frac{3}{2}\right) = 5$, and if $x = \frac{5}{2}$ then $y = 8 - 2\left(\frac{5}{2}\right) = 3$.

There are two possibilities. One has coordinates $\left(\frac{3}{2}, 5\right)$ and the other has coordinates $\left(\frac{5}{2}, 3\right)$.

Exercise 2 *A right triangle has hypotenuse of length* $13m$ *and area* $30m^2$. *Find the lengths of the two legs of the triangle.*



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Explanation

Since this is a right triangle, the Pythagorean Theorem tells us that $x^2 + y^2 = 13^2 = 169$. The area of a triangle is given by $\frac{1}{2} \times \text{base} \times \text{height}$ which means $\frac{1}{2}xy = 30$, or equivalently $xy = 60$.

This gives a system of nonlinear equations

$$\begin{cases} x^2 + y^2 & = 169 \\ xy & = 60. \end{cases}$$

A direct way to solve this system of equations would be to solve the bottom equation for y, giving $y = \frac{60}{x}$, and substitute this into the top equation eliminating the y variable. After simplification that will yield a degree 4 polynomial equation to solve for x. Instead of following that method, we will make use of a different algebraic trick.

We know that there is a difference between $x^2 + y^2$ and $(x + y)^2$. If we multiply out $(x + y)^2$ we get $x^2 + 2xy + y^2$. That means if we add $2xy$ to $x^2 + y^2$, it becomes $(x + y)^2$. To make use of that, we need to know the value of $2xy$ so we can add it to the other side of our top equation. Notice that the bottom equation of our system $xy = 60$ means that $2xy = 2(60) = 120$. If the bottom equation is satisfied, the top equation can be rewritten as:

$$\begin{aligned} x^2 + y^2 &= 169 \\ x^2 + y^2 + 2xy &= 169 + 2xy \\ x^2 + 2xy + y^2 &= 169 + 120 \\ (x + y)^2 &= 289. \end{aligned}$$

Taking square roots of both sides gives $|x + y| = \sqrt{289} = 17$. That is, $x + y = \pm 17$. Neither x nor y can be negative (since they denote lengths of the sides of this triangle), this results in $x + y = 17$. Our system of equations is equivalent to:

$$\begin{cases} x + y & = 17 \\ xy & = 60. \end{cases}$$

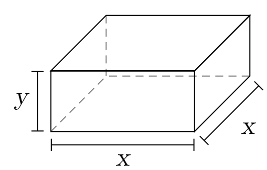
Now that we've been able to simplify the first equation, we will proceed with substitution as mentioned above. If $y = \dfrac{60}{x}$ is satisfied, then the top equation gives:

$$\begin{aligned} x + y &= 17 \\ x + \frac{60}{x} &= 17 \\ x\left(x + \frac{60}{x}\right) &= x(17) \\ x^2 + 60 &= 17x \\ x^2 - 17x + 60 &= 0 \\ (x-12)(x-5) &= 0. \end{aligned}$$

The $x-12$ factor yields a solution of $x = 12$, and the $x-5$ factor gives a solution of $x = 5$. Using $y = \dfrac{60}{x}$ we find that if $x = 12$ then $y = \dfrac{60}{12} = 5$, and if $x = 5$ then $y = \dfrac{60}{5} = 12$.

The two legs of the triangle have lengths $5m$ and $12m$.

Exercise 3 *Suppose we have a box with square base, as illustrated below, constructed to have volume $8cm^3$ and surface area $24cm^2$. Call the side-lengths of the base as x, and the height of the box as y.*



Find the dimensions of the box.

Explanation We know that the volume of the box is given by x^2y (length $\times$ width $\times$ height), giving the nonlinear equation $x^2y = 8$. The surface of the box consists of six rectangles. The top and bottom each have area x^2, and the four sides each have area xy. The full surface area of the box is given by $2x^2 + 4xy$. This setup gives us a system of two nonlinear equations with two unknowns:

$$\begin{cases} x^2y & = 8 \\ 2x^2 + 4xy & = 24. \end{cases}$$

If we want to find the dimensions of the box, we will have to solve this system of equations.

As you have seen in the previous section, solving systems of nonlinear equations involves finding a way to eliminate one of the variables by performing operations on the two equations and/or using substutition. In the case of these equations, notice that the variable y only occurs in a single term in each equation. In the top equation there is an x^2y term, while in the bottom equation there is an xy term. These are not like terms, so we will need to deal with that. Let us begin by multiplying both sides of the second equation by x. This gives the system

$$\begin{cases} x^2y &= 8 \\ 2x^3 + 4x^2y &= 24x. \end{cases}$$

Since no solution has x-coordinate equal to 0, this system is equivalent to the original one. This modification has given that the y variable appears in like terms in boh equations. Substituting $x^2y = 8$ into the new bottom equation gives:

$$\begin{aligned} 2x^3 + 4x^2y &= 24x \\ 2x^3 + 4\left(x^2y\right) &= 24x \\ 2x^3 + 4\left(8\right) &= 24x \\ 2x^3 + 32 &= 24x \\ 2x^3 - 24x + 32 &= 0 \\ x^3 - 12x + 16 &= 0. \end{aligned}$$

That is, if the $x^2y = 8$ equation is satisfied, then the bottom equation of the system is equivalent to $x^3 - 12x + 16 = 0$. This is a polynomial equation in the single variable, x. (Notice that if we had taken the original top equation $x^2y = 8$, solved it for y to obtain $y = \dfrac{8}{x^2}$, and substituted that into the original bottom equation, we would have arrived at this exact same result.)

Notice that $2^3 - 12(2) + 16 = 8 - 24 + 16 = 0$. That means $x = 2$ is a solution to this cubic equation, and that $x - 2$ is a factor of the polynomial $x^3 - 12x + 16$. By long-division we can find that $x^3 - 12x + 16 = (x - 2)(x^2 + 2x - 8)$. Since $x^2 + 2x - 8 = (x + 4)(x - 2)$ we see that $x^3 - 12x + 16 = (x - 2)^2(x + 4)$. The zeroes of this polymial are $x = 2$ and $x = -4$. Since x represents a length of the side of the box, the $x = -4$ solution is extraneous and should be dropped.

The only solution to the system has $x = 2$. Looking back at the first equation of the system:

$$\begin{aligned} x^2y &= 8 \\ (2)^2y &= 8 \\ 4y &= 8 \\ y &= 2. \end{aligned}$$

The solution is for $(x, y) = (2, 2)$. Since the question asks us to find the dimensions of the box, we say that the box is $2cm \times 2cm \times 2cm$. That is, it's a cube with side length $2cm$.

11.4 Average Rate of Change: Difference Quotients

Learning Objectives

- Secant Lines
  - Definition
  - Finding secant lines
  - Applications
- Difference Quotients
  - Average rate of change when one or both points are given as letters
  - Simplify with algebra (early examples)
  - Finding slopes of secant lines

11.4.1 Average Rate of Change and Secant Lines

Motivating Questions

- What does a line passing through two points of a function represent?
- How does this inform our understanding of the function?

Introduction

We begin by recalling the definitions of *average rate of change* of a function and *secant line* to the graph of a function.

Definition For a function f defined on an interval $[a, b]$,

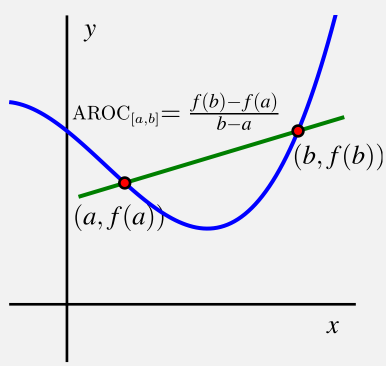
- the **average rate of change of f on** $[a, b]$ is the quantity

$$\text{AROC}_{[a,b]} = \frac{f(b) - f(a)}{b - a}.$$

- a **secant line** to the graph of f is a line passing through two points $(a, f(a))$ and $(b, f(b))$, with $a \neq b$.

  Recall: The slope of a secant line is the average rate of change of the function on the interval $[a, b]$.

This is illustrated in the figure below, where the green line (between the red points on the graph) is the secant line of f from $(a, f(a))$ to $(b, f(b))$.



Recall that given two points (x_0, y_0) and (x_1, y_1) in the plane, with $x_0 \neq x_1$, we can find the equation of the line passing through them by using the slope ("rise-over-run"):

$$m = \frac{y_1 - y_0}{x_1 - x_0}.$$

Then the line equation is given by $y - y_0 = m(x - x_0)$, simply because any given point (x, y) in such line must realize the *same* slope:

$$m = \frac{y - y_0}{x - x_0}.$$

Of course, one may also use the point (x_1, y_1) instead of (x_0, y_0) and consider the equation $y - y_1 = m(x - x_1)$, as it describes the same line.

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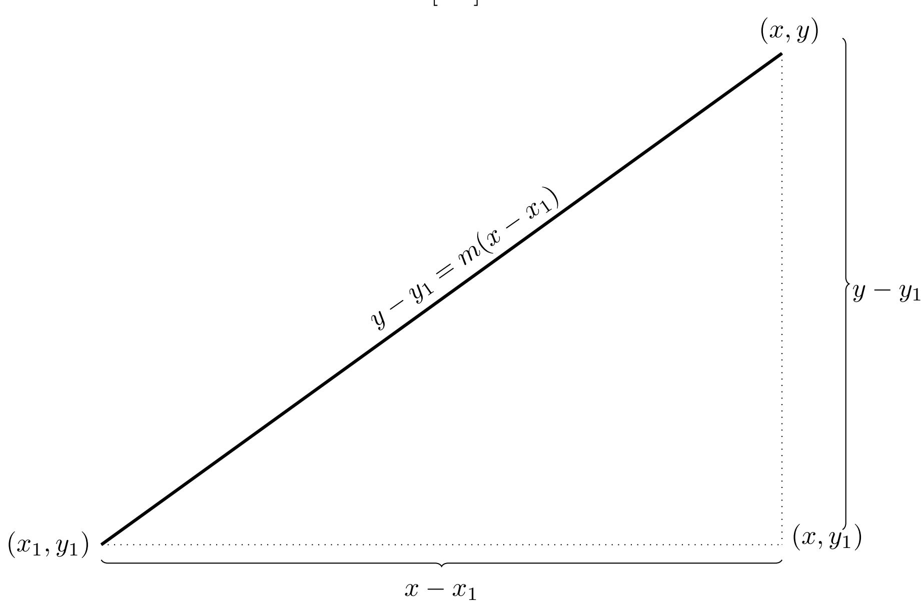
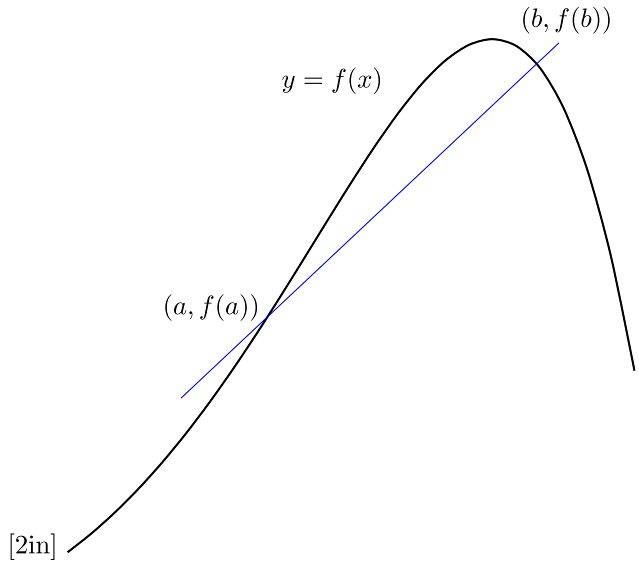


With this in place, we'll focus on the situation where two such points lie in the graph of some function $y = f(x)$.



Definitions and examples

Definition: Consider a function $y = f(x)$. A line passing through two points $(a, f(a))$ and $(b, f(b))$, with $a \neq b$, in the graph of $y = f(x)$, is called a **secant line** to the graph.
Recall: The slope of a secant line is the average rate of change of the function on the interval $[a, b]$.

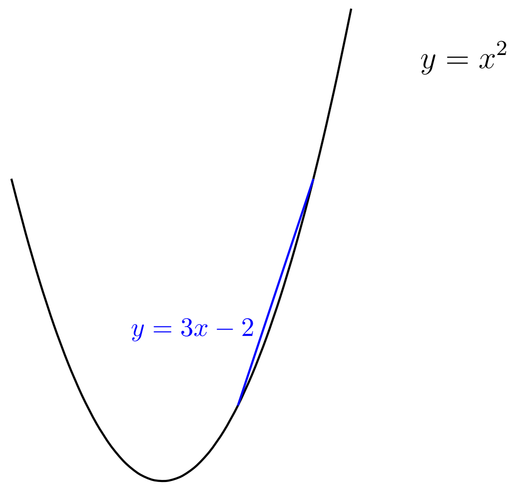
Example 18. *On the following situations, given a function $y = f(x)$ and two points in the graph, find the equation of the secant line they determine.*

a. $f(x) = x^2$, *points* $(1, f(1))$ *and* $(2, f(2))$.

Explanation First, we have that $f(1) = 1^2 = 1$ and $f(2) = 2^2 = 4$, so the points given are actually $(1, 1)$ and $(2, 4)$. So

$$m = \frac{4-1}{2-1} = 3$$

means that the line equation we're looking for is $y - 1 = 3(x - 1)$, which may be rewritten simply as $y = 3x - 2$.



[2in]

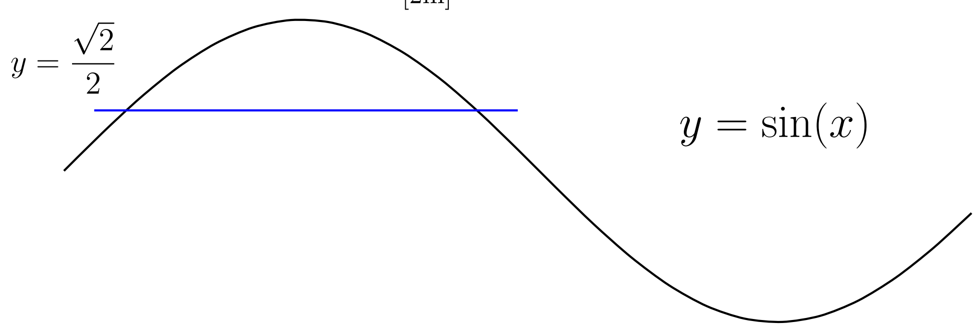
b. $f(x) = \sin x$, *points* $(\pi/4, f(\pi/4))$ *and* $(3\pi/4, f(3\pi/4))$.

Explanation This time, we have that $f(\pi/4) = \sin(\pi/4) = \sqrt{2}/2$ and also that $f(3\pi/4) = \sin(3\pi/4) = \sqrt{2}/2$, so the points given were in fact $(\pi/4, \sqrt{2}/2)$ and $(3\pi/4, \sqrt{2}/2)$. Hence the slope of the secant line is

$$m = \frac{\sqrt{2}/2 - \sqrt{2}/2}{3\pi/4 - \pi/4} = \frac{0}{\pi/2} = 0,$$

so the line equation $y - \sqrt{2}/2 = 0(x - \pi/4)$ boils down to $y = \sqrt{2}/2$. Again, you should see $y = \sqrt{2}/2$ not as a single value of y, but as a line equation for which it just happens that x does not appear — thus describing a horizontal line.

[2in]



c. $f(x) = 2x + 3$, *points* $(-1, f(-1))$ *and* $(3, f(3))$.

Explanation Now, we have $f(-1) = 2(-1) + 3 = 1$ and $f(3) = 2 \cdot 3 + 3 = 9$, so the points given were $(-1, 1)$ and $(3, 9)$. The slope these points determine is

$$m = \frac{9-1}{3-(-1)} = \frac{8}{4} = 2,$$

so we obtain $y - 1 = 2(x - (-1))$, which can be rewritten as $y = 2x + 3$. This is not a coincidence! The secant line to the graph of a line must be the line itself. This is because a line is determined by two points, and since both the original line and the secant line must share the two given points, they must be, in fact, equal.

11.4.2 Slopes of Secant Lines as a Function of h

Motivating Questions

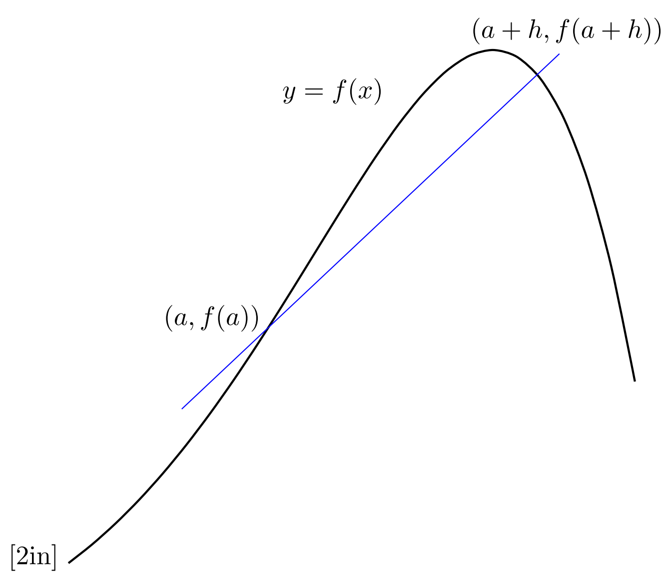
- How can we rewrite the average rate of change of a function in terms of the horizontal distance, h, between the points? Why would we want to do that?
- What are some algebra techniques that allow us to simplify the average rate of change for an arbitrary h?
- What does it tell us when we put in small values for h?

Introduction

We have discussed a secant line to the graph of a function $y = f(x)$ from $(a, f(a))$ to $(b, f(b))$, and the fact that the slope, m, of this line is the average rate of change of the function f on the interval $[a, b]$, $\text{AROC}_{[a,b]}$. Furthermore, recall that we can let $h = b - a$, so that the slope expression becomes

$$m = \frac{f(a+h) - f(a)}{h},$$

where h is the horizontal distance between the points.

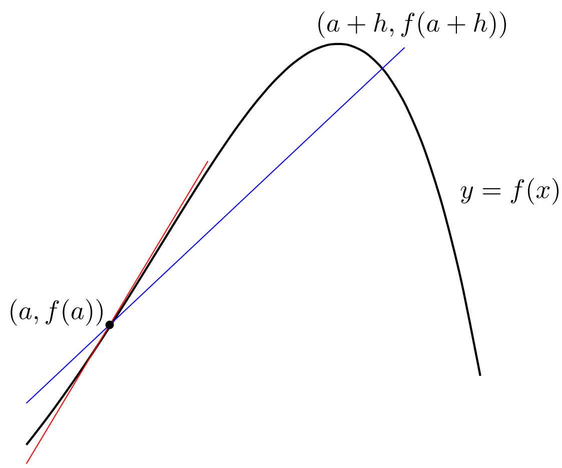


One of the main objectives of Calculus is to understand instantaneous rates of change, as opposed to average rates of change. Namely, what is the behavior of the expression

$$\frac{f(a+h)-f(a)}{h}$$

when h gets very small? Geometrically, making h become very small is making the secant line through $(a, f(a))$ and $(a+h, f(a+h))$ approach a certain line — the tangent line to the graph of $y = f(x)$ at the point a. This is demonstrated in the figure below, where the secant line is blue, and the line tangent to the graph at a is in red.

[2.5in]



Follow the link below to an example Desmos graph where you can see the effect of changing the value of h on the secant line in real-time. Desmos link: https://www.desmos.com/calculator/f6fh2wkrrn

The slope of such a tangent line, when it exists, is called the derivative of f at a. Here, we'll discuss difference quotients and several examples, to prepare you to learn those things in more detail in a future Calculus class.

Definitions and examples

Definition: The difference quotient of a function $y = f(x)$ at a point a of its domain is the quantity

$$\frac{f(a+h)-f(a)}{h},$$

i.e., the average rate of change of f on the interval $[a, a+h]$.

Example 19. *Find the difference quotients of the following functions, at the given point.*

a. $f(x) = x^2$, $a = 2$.

Explanation Let's evaluate it directly:

$$\begin{aligned}\frac{f(2+h)-f(2)}{h} &= \frac{(2+h)^2 - 2^2}{h} = \frac{2^2 + 4h + h^2 - 2^2}{h} \\ &= \frac{4h + h^2}{h} = h + 4.\end{aligned}$$

We thus have an equation for the slope of the secant line from $(2, f(2))$ to $(2+h, f(2+h))$:

$$\mathrm{AROC}_{[2,2+h]} = m = h + 4.$$

Recall from Example 1(a) of Section 12-3-1, that we calculated the slope of the secant line from $(1, f(1))$ to $(2, f(2))$. Letting $h = -1$, we see that this gives the same answer for the slope of that secant line.

Furthermore, if we now let $h \to 0$, this expression, $\dfrac{f(2+h)-f(2)}{h} \to 4$.

b. $f(x) = \sin(x)$, $a = \dfrac{\pi}{3}$.

Explanation Again, we evaluate directly:

$$\frac{f(\frac{\pi}{3}+h) - f(\frac{\pi}{3})}{h} = \frac{\sin(\frac{\pi}{3}+h) - \sin(\frac{\pi}{3})}{h}$$

Recognizing that we cannot further simplify this expression in its current form, we replace $\sin((\pi/3)+h)$ using the sine sum expression:

$$\begin{aligned}\frac{\sin(\frac{\pi}{3}+h) - \sin(\frac{\pi}{3})}{h} &= \frac{\sin(\frac{\pi}{3})\cos(h) + \cos(\frac{\pi}{3})\sin(h) - \frac{\sqrt{3}}{2}}{h} \\ &= \frac{1}{2h}\big(\sqrt{3}(\cos(h) - 1) + \sin(h)\big)\end{aligned}$$

This gives a less easy to visualize equation for the slope of the secant line from $(a, f(a))$ to $(a+h, f(a+h))$, for $a = \dfrac{\pi}{3}$:

$$\mathrm{AROC}_{[\frac{\pi}{3}, \frac{\pi}{3}+h]} = m = \frac{1}{2h}\big(\sqrt{3}(\cos(h) - 1) + \sin(h)\big).$$

However, consider $h = \dfrac{\pi}{2}$, so that we are looking at the secant line from $(\pi/3, f(\pi/3))$ to $(5\pi/6, f(5\pi/6))$. We then see that

$$\frac{f(\frac{\pi}{3}+h) - f(\frac{\pi}{3})}{h} = \frac{1}{2} \cdot \frac{\pi}{2}\big(\sqrt{3}(\cos\left(\frac{\pi}{2}\right) - 1) + \sin\left(\frac{\pi}{2}\right)\big) = 0,$$

as before.

Furthermore, what happens as we let $h \to 0$. Consider $h = \dfrac{\pi}{6}$, then we have

$$\begin{aligned}\frac{f(\frac{\pi}{3}+h)-f(\frac{\pi}{3})}{h} &= \frac{1}{2}\cdot\frac{6}{\pi}(\sqrt{3}(\cos\left(\frac{\pi}{6}\right)-1)+\sin\left(\frac{\pi}{6}\right)) \\ &= \frac{3}{\pi}\Big(\sqrt{3}\Big(\frac{\sqrt{3}}{2}-1\Big)+\frac{1}{2}\Big) \\ &= \frac{6-3\sqrt{3}}{\pi},\end{aligned}$$

Note that this is greater than 0. Think about the graph of $y = sin(x)$. It is increasing on the interval $[\dfrac{\pi}{3},\dfrac{\pi}{2}]$.

Follow the Desmos link to explore more initial values of x and see what happens as you adjust h smaller and smaller to zero. Desmos link: `https://www.desmos.com/calculator/xbi081bx3w`

c. $f(x) = 2x+3,\ a = -1.$
Explanation Evaluate directly:

$$\frac{f(-1+h)-f(-1)}{h} = \frac{2(h-1)+3-(2(-1)+3)}{h} = \frac{2h-2+2}{h} = 2$$

Observe that the equation for the slope of the secant line from $(-1, f(-1))$ to $(-1+h, f(-1+h))$ is simply

$$\text{AROC}_{[-1,-1+h]} = m = 2.$$

Recall from Example 1(c) of Section 12-3-1, that the equation of the secant line from $(-1, f(-1))$ to $(3, f(3))$ was simply $f(x)$. Why was this?

Now, this tells us that regardless of the points we choose, the secant line between them will have the same slope and equation as the line itself.

11.4.3 Algebra of Secant Lines

Motivating Questions

- What are some algebra techniques that allow us to simplify the equation of a secant line?
- Why is this important?

Introduction

Given the graph of a function $y = f(x)$, we have discussed methods to determine the slope of the secant line between two points, $(a, f(a))$ and $(b, f(b))$, on the graph. We know that this slope represents the average rate of change of the function f on the interval $[a, b]$, denoted by $\mathrm{AROC}_{[a,b]}$. Both of these can be rewritten by letting $b = h - a$, so that we have the value h representing the horizontal distance between the points. This means that as $h \to 0$, the secant line, or the average rate of change of the function, approaches a value known as the slope of the tangent line of f at a. This will be discussed extensively in future calculus courses, but in this section we will focus on tools to simplify the expression $\mathrm{AROC}_{[a,a+h]}$, as they are essential to calculating this limit.

Definitions and examples

Recall the special formula for difference of squares, $a^2 - b^2 = (a - b)(a + b)$. For non-square values of a and b we can use the same idea to rationalize differences (or sums) of square roots through multiplication by the corresponding sum (or difference), which we call the *conjugate*. Given any expression $\sqrt{a} \pm \sqrt{b}$, a, b real numbers, the conjugate of this expression is $\sqrt{a} \mp \sqrt{b}$. Multiplying such an expression by its conjugate rationalizes it through the distributive property: $(\sqrt{a} + \sqrt{b}) \cdot (\sqrt{a} - \sqrt{b}) = (\sqrt{a})^2 + \sqrt{a}\sqrt{b} - \sqrt{a}\sqrt{b} - (\sqrt{b})^2 = a - b$

Definition: Given any difference of positive values $a - b$, we know from the difference of squares, that $a - b = (\sqrt{a} - \sqrt{b})(\sqrt{a} + \sqrt{b})$.
The sum $\sqrt{a} + \sqrt{b}$ is the *conjugate* of the difference $\sqrt{a} - \sqrt{b}$. Likewise, the difference $\sqrt{a} - \sqrt{b}$ is the *conjugate* of the sum $\sqrt{a} + \sqrt{b}$.
Multiplying such an expression by its conjugate will rationalize the expression.

Note that this is one of the most important tools in your simplification toolbox. Other tools include simplifying polynomials and fractions (finding the common denominator), moving coefficients inside or outside the square root, and the trigonometric identities introduced in Section 10-2.

Example 20. *For the following, find the difference quotient. Simplify as much as possible*

(a) $f(x) = \sqrt{x}$, $x \geq 0$

Explanation We consider $h > 0$ to avoid any potential undefined values plugged into our function f since its domain is $[0, \infty)$.

$$\frac{f(x+h) - f(x)}{h} = \frac{\sqrt{x+h} - \sqrt{x}}{h} \tag{1}$$

Observe that we cannot combine any terms in (1), but the numerator is of the form $\sqrt{a} - \sqrt{b}$. Hence, we will multiply by the conjugate to rationalize the numerator:

$$\begin{aligned} \frac{\sqrt{x+h} - \sqrt{x}}{h} &= \frac{(\sqrt{x+h} - \sqrt{x})}{h} \cdot \frac{(\sqrt{x+h} + \sqrt{x})}{(\sqrt{x+h} + \sqrt{x})} \qquad (2) \\ &= \frac{(\sqrt{x+h})^2 - (\sqrt{x})^2}{h(\sqrt{x+h} + \sqrt{x})} \qquad (3) \end{aligned}$$

Remember, in (2), that in order to avoid changing the value of the expression, we must multiply by the conjugate over itself, i.e., multiply by 1. Then (3) has a difference of squares in the numerator and is equal to

$$\begin{aligned} \frac{(x+h) - x}{h(\sqrt{x+h} + \sqrt{x})} &= \frac{h}{h(\sqrt{x+h} + \sqrt{x})} \\ &= \frac{1}{\sqrt{x+h} + \sqrt{x}}, \end{aligned}$$

by cancelling out the h in the numerator and the denominator.

This expression that we have found is now in a form that allows us to consider what happens when $h \to 0$ by removing the h from the denominator.

(b) $g(x) = \sqrt{8 - 2x}$, $x \geq 4$.
Explanation We consider $h > 0$ to avoid any potential undefined values plugged into our function g since its domain is $[4, \infty)$.

$$\frac{g(x+h) - g(x)}{h} = \frac{\sqrt{8 - 4(x+h)} - \sqrt{8 - 4x}}{h} \tag{4}$$

Once again, we cannot combine any terms in the numerator of (4), so we will multiply by the conjugate to rationalize the numerator, hoping we will be able to simplify the equation. (4) is equal to

$$\begin{aligned}
\frac{(\sqrt{8-4(x+h)}-\sqrt{8-4x})}{h}\cdot\frac{(\sqrt{8-4(x+h)}+\sqrt{8-4x})}{(\sqrt{8-4(x+h)}+\sqrt{8-4x})}
&= \frac{(\sqrt{8-4(x+h)})^2-(\sqrt{8-4x})^2}{h(\sqrt{8-4(x+h)}+\sqrt{8-4x})}\\
&= \frac{(8-4x-4h)-(8-4x)}{h(\sqrt{4(2-(x+h))}+\sqrt{4(2-x)})}\\
&= \frac{-4h}{2h(\sqrt{2-(x+h)}+\sqrt{2-x})}
\end{aligned}$$

Now, we simply cancel the $2h$ in the numerator and the denominator, giving

$$\frac{g(x+h)-g(x)}{h}=\frac{-2}{\sqrt{2-(x+h)}+\sqrt{2-x}}.$$

(c) $f(x)=\cos(2x)$

Explanation Note that $\cos(z)$ is defined for all real numbers z, so we need not worry about the values of x and h plugged into the difference quotient formula.

$$\frac{f(x+h)-f(x)}{h}=\frac{\cos(2(x+h))-\cos(2x)}{h}$$

Now, we can expand out using the summation formula for cosine:

$$\begin{aligned}
\frac{\cos(2x)\cos(2h)-\sin(2x)\sin(2h)-\cos(2x)}{h} &= \frac{\cos(2x)(\cos(2h)-1)-\sin(2x)\sin(2h)}{h}\\
&= \cos(2x)\frac{\cos(2h)-1}{h}-\sin(2x)\frac{\sin(2h)}{h}
\end{aligned}$$

Here we can plug in decreasing values of h for cosine and sine and start to notice a pattern..

Explore this further by changing the x and h values in the following Desmos graph: Desmos link: https://www.desmos.com/calculator/lf9wkgurwz

(d) $f(x)=|x-1|$

Explanation We will consider two regions and ranges of h: (1) $x\in(-\infty,1)$ with $h<0$ and (2) $x\in(1,\infty)$ with $h>0$.

Let's start with region (1), where $x<1$ and $h<0$. From this, we know that $x+h-1<x-1<0$, so $|x-1|=-(x-1)$. Hence we have

$$\begin{aligned}
\frac{f(x+h)-f(x)}{h} &= \frac{|x+h-1|-|x-1|}{h}\\
&= \frac{-x-h+1+(x-1)}{h}\\
&= \frac{-h}{h}=-1
\end{aligned}$$

Alternatively, if we consider region (2), where $x > 1$ and $h > 0$, then we have $x + h - 1 > x - 1 > 0$, so that

$$\begin{aligned}\frac{|x+h+1| - |x+1|}{h} &= \frac{x+h-1-(x-1)}{h} \\ &= \frac{h}{h} = 1\end{aligned}$$

Notice that this is not as clear-cut if we consider say $x < 1$ and $h > 0$. Then we would need to consider if h is large enough that $x + h > 1$. Let's explore this some more.

Let $x < 1$ and $h > 0$. Further, assume $h > 1 - x$, then

$$\begin{aligned}\frac{f(x+h) - f(x)}{h} &= \frac{|x+h-1| - |x-1|}{h} \\ &= \frac{x+h+1+(x-1)}{h} \\ &= \frac{2x+h}{h}\end{aligned}$$

Alternatively, if $h < 1 - x$, then

$$\begin{aligned}\frac{f(x+h) - f(x)}{h} &= \frac{|x+h-1| - |x-1|}{h} \\ &= \frac{-x-h+1+(x-1)}{h} \\ &= \frac{-h}{h} = -1\end{aligned}$$

As when $x < 1$ and $h < 0$.

(e) $g(x) = \dfrac{2x}{x^2+3}$

Explanation First note that the denominator of our function g is greater than zero for all real values of x, so the function is defined for all real numbers. Thus, we may calculate the difference quotient without concern for input values of x and h.

$$\frac{g(x+h) - g(x)}{h} = \frac{\frac{2(x+h)}{(x+h)^2+3} - \frac{2x}{x^2+3}}{h}$$

This expression for the difference quotient looks rather messy, so let's find the common denominator and see if we can cancel out some terms in the numerator by combining the fractions. We will leave the terms in the denominator in their current format, but multiply out the $(x+h)^2$ in the numerator for ease of simplification.

Note that the common denominator is $((x+h)^2+3)(x^2+3)$. Then we have

$$\begin{aligned}\frac{g(x+h)-g(x)}{h} &= \frac{\frac{(2x+2h)(x^2+3)-2x(x^2+2xh+h^2+3)}{((x+h)^2+3)(x^2+3)}}{h}\\ &= \frac{2x^3+2x^2h+6x+6h-(2x^3+4x^2h+2xh^2+6x)}{h\big((x+h)^2+3\big)(x^2+3)}\end{aligned}$$

Observe that we have combined the denominators now and have many common terms in the numerator that can be subtracted from each other, so that

$$\frac{g(x+h)-g(x)}{h} = \frac{-2x^2h+6h-2xh^2}{h\big((x+h)^2+3\big)(x^2+3)}.$$

Now, all the terms in the numerator have a factor of h, so we can cancel the h in the numerator and denominator for a final, simplified difference quotient of

$$\frac{-2x^2-2xh+6}{\big((x+h)^2+3\big)(x^2+3)}.$$

Summary Useful tools for simplification:

- Simplifying polynomials.
- Simplifying fractions by finding common denominators.
- Multiplying by the conjugate to rationalize the numerator.
- Considering regions for absolute value functions.

11.5 Functions: The Big Picture

Learning Objectives

- Functions: A Summary
  -
  -
  -
- What is Calculus?
  -
  -
  -

11.5.1 Functions: A Summary

> **Motivating Questions**
>
> - What have we learned about functions in this course?

Introduction

Over the past two semesters, you've learned quite a bit about functions. When we started, we didn't even say what a function was, but we've now talked about many functions and discussed their properties, as well as how to work with them.

Here are some questions you should be able to answer.

- What is a function?
- What are some properties that all functions share?
- What are the domain and range of a function, and how can they be calculated?
- What are zeros of functions, and how can they be found?
- How can functions be built out of other functions?
- Which functions have inverses, and how can they be found?
- What kinds of symmetries can the graphs of functions show?
- What are some famous kinds of functions? What do their graphs look like? Why are they important? What do they model?
- How can we describe the average rate of change of a function?
- How can we go back and forth between different representations of a function?

Using Functions to Solve Problems

Analyzing a Function

Once we have a function that models some phenomenon, we can ask all sorts of questions about our function. In this section, we'll take a particular function, and see what kinds of interesting things we can discover. Our hope is to demonstrate how you can use the tools we have developed in this course to gain information about complicated functions.

A model used in many fields is the *logistic function*. The standard logistic function is a function f defined by $f(x) = \dfrac{1}{1+e^{-x}}$. Looking at this function can be intimidating, but we have all the tools at our disposal to be able to analyze this function.

Domain and Range Let's start by finding the domain and range of f. To find the domain, notice that the only possible obstruction to $f(x)$ being defined is if the denominator were to equal zero. This tells us that to find the domain, we need to solve the equation $1 + e^{-x} = 0$. To do this, we take the following steps:

$$\begin{aligned} 1 + e^{-x} &= 0 \\ 1 &= -e^{-x} \\ -1 &= e^{-x} \\ \ln(-1) &= \ln(e^{-x}) \\ \ln(-1) &= -x \\ -\ln(-1) &= x \end{aligned}$$

However, as we learned, the domain of the natural logarithm is $(0, \infty)$, so -1 is not in the domain of the natural logarithm, and therefore, this equation does not have a solution. Since the equation does not have a solution, there are no obstructions to $f(x)$ being defined, and the domain of f is $(-\infty, \infty)$.

To find the range of f, we first must recall that the range of e^{-x} is $(0, \infty)$. Let's consider what happens as e^{-x} gets arbitrarily large. In this case, the denominator of $\dfrac{1}{1+e^{-x}}$ becomes arbitrarily large, and therefore, $f(x)$ becomes arbitrarily close to 0, but always remains positive.

As e^{-x} gets arbitrarily close to 0, the denominator of $\dfrac{1}{1+e^{-x}}$ becomes arbitrarily close to 1, but is always greater than 1, therefore, $f(x)$ can be arbitrarily close to 1, but never greater than or equal to 1.

Combining these two statements tells us that the range of f is $(0, 1)$. The above arguments use reasoning that you will develop further in calculus: the idea of getting arbitrarily close to a point is a major topic in that subject. Another way to get an idea of the range is to graph the function on a graphing utility such as Desmos.

This makes it easier to see what's going on, but being able to understand how to find the range using reasoning about the range of e^{-x} is an important skill to develop.

Average Rate of Change Notice that the graph of f has an S-shape. It flattens out as the absolute value of x becomes large. We will make this more concrete in the following exploration.

> **Exploration** In this exploration, we will learn about the average rate of change of f over various intervals. Use a calculator to get a sense of how large or small the rates of change are.
>
> a. Find the average rate of change of f over the interval $[0, 1]$.
>
> b. Find the average rate of change of f over the interval $[2, 3]$.
>
> c. Find the average rate of change of f over the interval $[5, 6]$.
>
> d. Find the average rate of change of f over the interval $[-7, -6]$.
>
> e. Of the intervals above, which had the highest rate of change? The lowest?

Adapting Models One application of the logistic function is to model population growth. At time x, the logistic growth model says that the population is $f(x)$. The rationale behind this is that various factors (space, resources, etc.) put a limit, or "carrying capacity" on how many individuals can survive in a population. Therefore, population growth should slow down based on how close the population is to the carrying capacity. That is, the closer $f(x)$ is to the carrying capacity, the slower the rate of change of f should be. Population should also be slower when $f(x)$ is close to 0, since there are fewer individuals to reproduce.

A very reasonable question to ask would be "How can this be used to model populations if its range is $(0, 1)$?" The answer is that function transformations allow us to fit the function to our specific need. For example, if the carrying capacity is 5000, instead of using $f(x)$ to model the population, we would use $5000f(x)$. We can use horizontal stretches and compressions to adjust how steep the growth is and use horizontal shifts to adjust the starting population.

A more general form of the logistic function would then be something of the form

$$Kf(r(x-h)) = \frac{K}{1+e^{-r(x-h)}}.$$

The value K adjusts the vertical stretch and therefore the carrying capacity. The value r corresponds to a horizontal compression or stretch. For the logistic function, this affects the steepness of the graph. As usual, h represents a horizontal shift.

Exploration Here is a table containing Columbus population data from Wikipedia.

Year	Population (in thousands)
1812	0.300
1820	1.450
1830	2.435
1840	6.048
1850	17.882
1860	18.554
1870	31.274
1880	51.647
1890	88.150
1900	125.560
1910	181.511
1920	237.031
1930	290.564
1940	306.087
1950	375.901
1960	471.316
1970	539.677
1980	564.871
1990	632.910
2000	711.470
2010	787.033

Use the following Desmos link to answer the following questions.

Desmos link: `https://www.desmos.com/calculator/ymiuslnbqw`

a. Experiment with the sliders to find values of K, r, and h that make $\dfrac{K}{1+e^{-r(x-h)}}$ a model for the data above that is as suitable as possible. Answers may vary.

b. Based on your values of K, r, and h, what is the carrying capacity of the population of Columbus?

c. Use your model to estimate the population of Columbus in 2020.

d. The actual population of Columbus in 2020 was 905,748. Does this agree with the model you found? Why do you think this is the case?

Inverse Function Another fun fact about the function f is that it is one-to-one, and therefore, invertible. What's more, we can use the tools we developed during the sections on inverse functions to be able to find an inverse for the logistic function. Since the logistic function takes a time as an input and returns the population at that time, its inverse takes a number and returns the time when the population has reached that number.

Exploration Use your model from the previous section. Call it f.

a. Find a formula for $f^{-1}(x)$.

b. Use your formula to estimate when the population of Columbus will be 1,000,000.

c. What is the domain of f^{-1}?

11.5.2 What is Calculus?

Motivating Questions

- What are the main ideas of calculus?

Introduction

This course aims to provide you with a background in all the tools you'll need to be successful in calculus. In this section, we'll provide a brief overview of some of the types of problems you'll be able to solve with calculus.

Speed

Say you're running, and the number of miles you run in t minutes can be given by $f(t) = \sqrt{\frac{t}{6}}$. What is your speed at the start of mile 2?

Calculus allows us to talk about instantaneous speeds and rates of change, rather than just average rates of change.

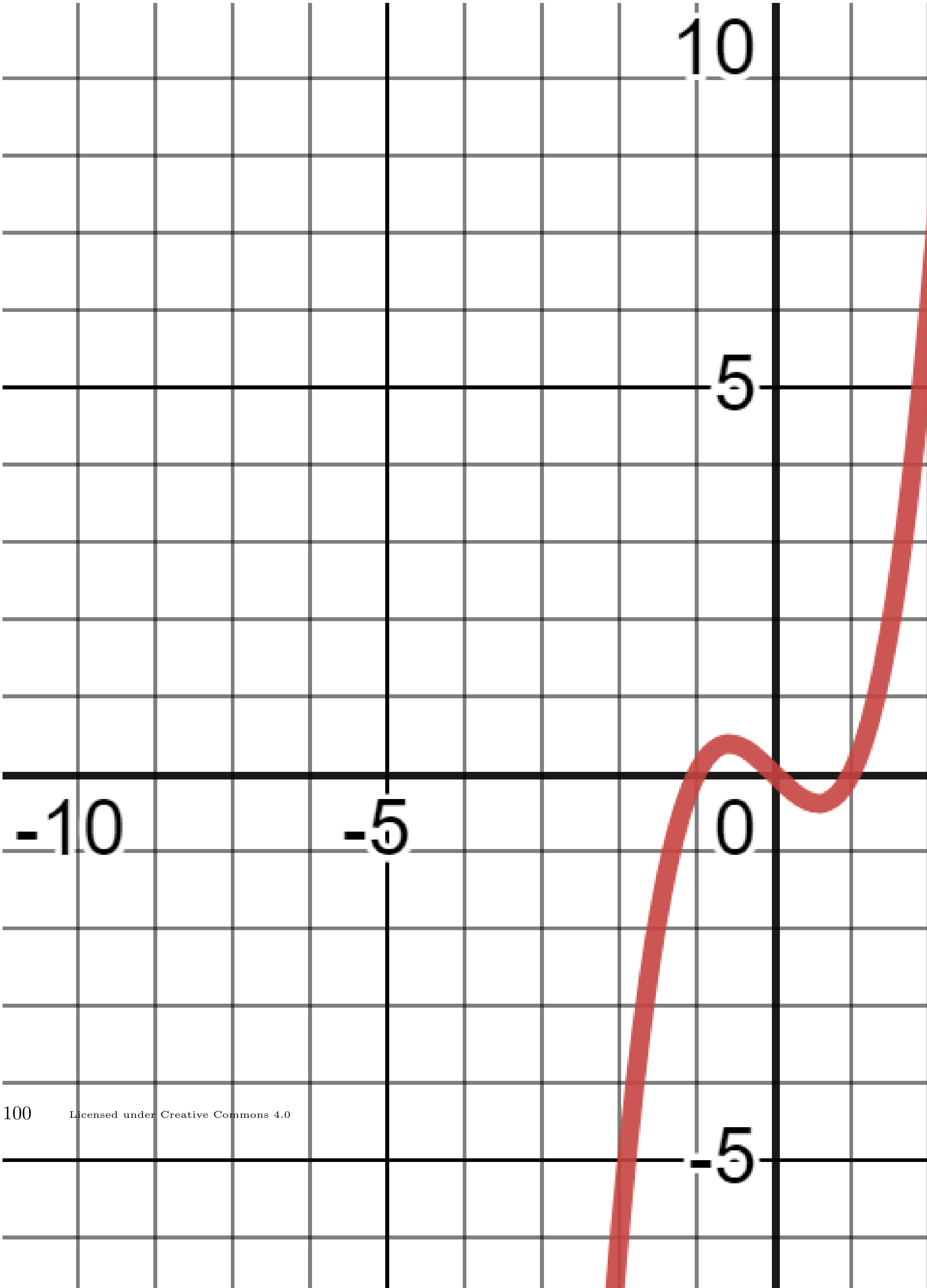
Optimization

Say you want to build a fence, and one side of the fenced-in area is already bounded by a building. If your budget only allows for the purchase of 1000 feet of fencing material, what is the largest area which can be enclosed by the fence?

Calculus can answer many questions about maximizing or minimizing certain values.

Finding Maxima and Minima

When we talked about rational functions and polynomials, there was certain information about their graphs that we couldn't provide. For example, the graph of the function defined by $f(x) = x^3 - x$ is shown below.


10
5
-10
-5
0
-5

We can answer questions about its end behavior with the concepts learned in this course. However, we notice that there are a peak and a valley on the graph of f. Calculus will give us the tools to be able to find the exact location of those points.

Calculus helps us find relative maxima and minima.

Volume of Solids

Say you have a bowl whose silhouette viewed from the side can be described by a parabola. Assume its cross-sections are perfect circles. What is the volume of the bowl?

Calculus can help us find the volume of many different kinds of 3D solids.

Related Rates

Say you're filling a cone with base radius 100 centimeters and height 100 centimeters with water at a rate of 4 centimeters cubed per minute. How quickly is the depth of water in the cone is changing when the water is at a height of 50 centimeters?

Calculus can allow us to describe rates of change in relation to other rates of change.

Newton's Method

Say you're looking at a really complicated function, like $x^6-2x^5-x^4+x^2-\pi x+e$. Finding the roots of this function algebraically is impossible, but if you start with a good estimate of the root, a tool from calculus called Newton's Method can refine your solution into a better estimate. What's more, you can then use Newton's method on that better estimate to obtain an even better estimate!

Calculus can help us estimate roots of functions.

Approximations

We know that the sine function is extremely complicated. We saw in the section on inverse trigonometric functions, that sometimes, we can rewrite functions involving trig functions in purely algebraic terms. For example, you can show that $\sin(\arctan(x)) = \dfrac{x}{\sqrt{x^2+1}}$. However, we haven't seen a way to rewrite sin by itself. However, in certain situations, we can come up with an approximation to functions, and this approximation is linear, which is often much easier to work with! There is a famous linear approximation of $\sin(x)$, but it only works when x is very close to zero. Part of calculus will be learning when certain approximations are good substitutes for the actual function.

Also, there's no reason to stop at linear functions! We can use higher and higher degree polynomials to get better and better approximations of trig functions, as well as other functions.

Calculus can allow us to approximate complicated functions with polynomials, but these approximations are only good for certain values of x.

Average Rates of Change

Say you're driving in an area where the speed limit is 60 miles per hour. If you drive 6 miles in 5 minutes, your average speed is greater than 60 miles per hour, but your speed at any given point on your trip need not be greater than 60 miles per hour: you may slow down and speed up. Is it possible to go 6 miles in 5 minutes without speeding? In other words, if your average rate of change is greater than 60 miles per hour, is it possible for your instantaneous rate of change to be below 60 miles per hour for the entire trip?

Calculus can give us important information on the relationship between average rates of change and instantaneous rates of change.

Finding the Area

In the project this semester, you wrote equations and inequalities that defined the shapes of letters of the alphabet in the plane. However, if you were using these as a font, a reasonable question would be, "How much ink does each letter use?" To answer this question, we need to find the area of each letter. With the tools we have now, this isn't possible. However, in calculus, you will learn a way to calculate the exact area of each of your letters.

Calculus provides a method for finding the area bounded by curves in the plane.

Part 12

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