Shape of Polynomials

We explore polynomial functions.

We know that linear functions are the simplest of all funcitons we can consider: their graphs have the simplest shape, their average rate of change is always constant (regardless of the interval chosen), and their formula is elementary. Moreover, computing the value of a linear function only requires multiplication and addition.

If we think of a linear function as having formula $L(x) = b + mx$, and the next-simplest functions, quadratic functions, as having form $Q(x) = c + bx + ax^2$, we can see immediate parallels between their respective forms and realize that it's natural to consider slightly more complicated functions by adding additional power functions.

Indeed, if we instead view linear functions as having form

$$L(x) = a_0 + a_1 x$$

(for some constants a_0 and a_1) and quadratic functions as having form

$$Q(x) = a_0 + a_1 x + a_2 x^2,$$

then it's natural to think about more general functions of this same form, but with additional power functions included.

Definition Given real numbers $a_0, a_1, \ldots, a_n$ where $a_n \neq 0$, we say that the function a function P is a **polynomial** of degree n if it can be written in the form

$$P(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_{n-1} x^{n-1} + a_n x^n,$$

We say that the a_i are the *coefficients* of the polynomial and the individual power functions $a_i x^i$ are the *terms* of the polynomial. Any value of x for which $P(x) = 0$ is called a *zero* of the polynomial.

Example 1. *The polyomial function $P(x) = 3 - 7x + 4x^2 - 2x^3 + 9x^5$ has degree 5, its constant term is 3, and its linear term is $-7x$.*

Since a polynomial is simply a sum of constant multiples of various power functions with positive integer powers, we often refer to those individual terms by referring to their individual degrees: the linear term, the quadratic term, and so

Learning outcomes:
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on. In addition, since the domain of any power function of the form $p(x) = x^n$ where n is a positive whole number is the set of all real numbers, it's also true the the domain of any polynomial function is the set of all real numbers.

Exploration Point your browser to the *Desmos* worksheet at http://gvsu.edu/s/0zy. There you'll find a degree 4 polynomial of the form $p(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4$, where $a_0, \ldots, a_4$ are set up as sliders. In the questions that follow, you'll experiment with different values of $a_0, \ldots, a_4$ to investigate different possible behaviors in a degree 4 polynomial.

(a) What is the largest number of distinct points at which $p(x)$ can cross the x-axis?For a polynomial p, we call any value r such that $p(r) = 0$ a zero of the polynomial. Report the values of $a_0, \ldots, a_4$ that lead to that largest number of zeros for $p(x)$.

(b) What other numbers of zeros are possible for $p(x)$? Said differently, can you get each possible number of fewer zeros than the largest number that you found in (a)? Why or why not?

(c) We say that a function has a turning pointif the function changes from decreasing to increasing or increasing to decreasing at the point. For example, any quadratic function has a turning point at its vertex.What is the largest number of turning points that $p(x)$ (the function in the Desmos worksheet) can have? Experiment with the sliders, and report values of $a_0, \ldots, a_4$ that lead to that largest number of turning points for $p(x)$.

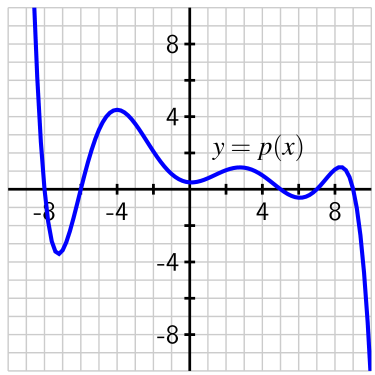
(d) What other numbers of turning points are possible for $p(x)$? Can it have no turning points? Just one? Exactly two? Experiment and explain.

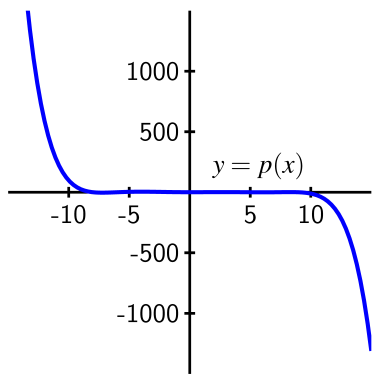
(e) What long-range behavior is possible for $p(x)$? Said differently, what are the possible results for $\lim_{x\to-\infty} p(x)$ and $\lim_{x\to\infty} p(x)$?

(f) What happens when we plot $y = a_4x^4$ in and compare $p(x)$ and a_4x^4? How do they look when we zoom out? (Experiment with different values of each of the sliders, too.)

We know that each of the power functions x, x^2, …, x^n grow without bound as $x \to \infty$. Intuitively, we sense that x^5 grows faster than x^4 (and likewise for any comparison of a higher power to a lower one). This means that for large values of x, the most important term in any polynomial is its highest order term. when we compared $p(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4$ and $y = a_4x^4$.

For any degree n polynomial $p(x) = a_0 + a_1x + \cdots + a_{n-1}x^{n-1} + a_nx^n$, its long-range behavior is the same as its highest-order term $q(x) = a_nx^n$. Thus, any polynomial of even degree appears "U-shaped" ($\cup$ or $\cap$, like x^2 or $-x^2$) when we zoom way out, and any polynomial of odd degree appears "chair-shaped" (like x^3 or $-x^3$) when we zoom way out.





In the second graph we see how the degree 7 polynomial pictured there (and in the first graph as well) appears to look like $q(x) = -x^7$ as we zoom out.

Summary

- Polynomial

