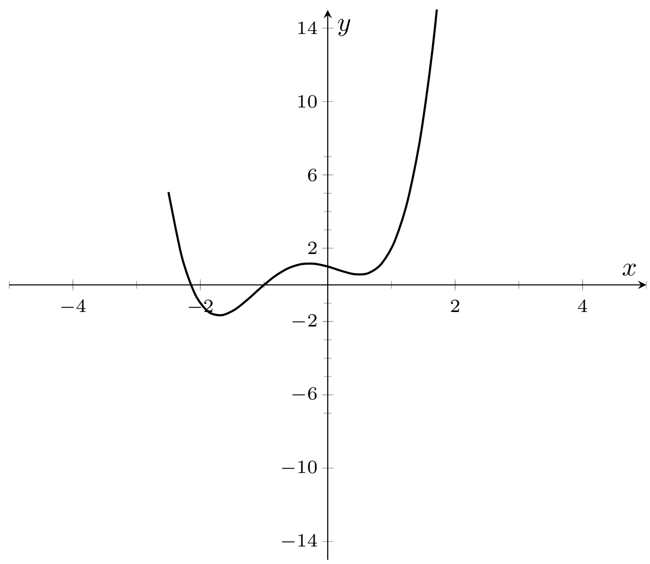
Roots

Remark 1. ***Motivating Questions***

- *Can we find the inverse of a polynomial?*
- *What does it mean to take the* n^{th} *root of a value?*

Consider the polynomial $p(x) = x^4 + 2x^3 - x^2 - x + 1$. A good question to ask would be whether the function p is invertible. To help us decide, here is the graph of $p(x)$.



Notice that this graph does not pass the Horizontal Line Test, so the function p is not one-to-one, and therefore not invertible.

Our polynomial $p(x)$ has many terms, so to simplify the situation, we'll look only at polynomials of the form x^n, where n is a positive integer.

Odd Roots

Recall that every polynomial $p(x) = x^n$, where n is odd, has the same basic shape. This is demonstrated in the figure below by the graphs of $y = x^n$ for $n = 3, 5, 7, 9$.

Learning outcomes:

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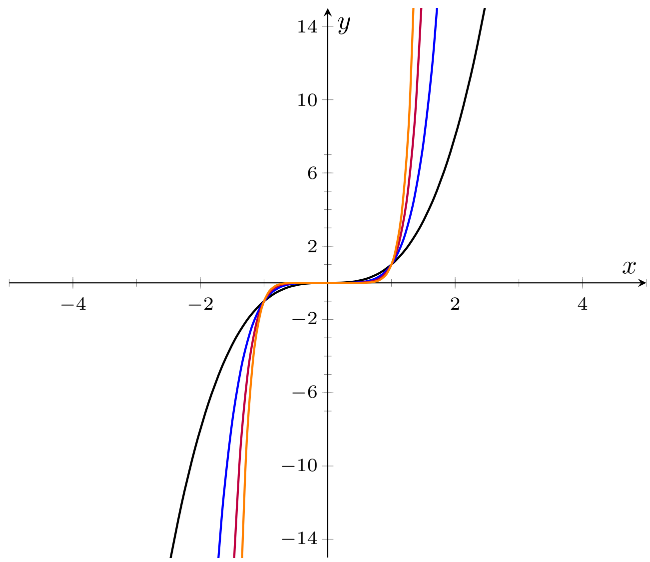


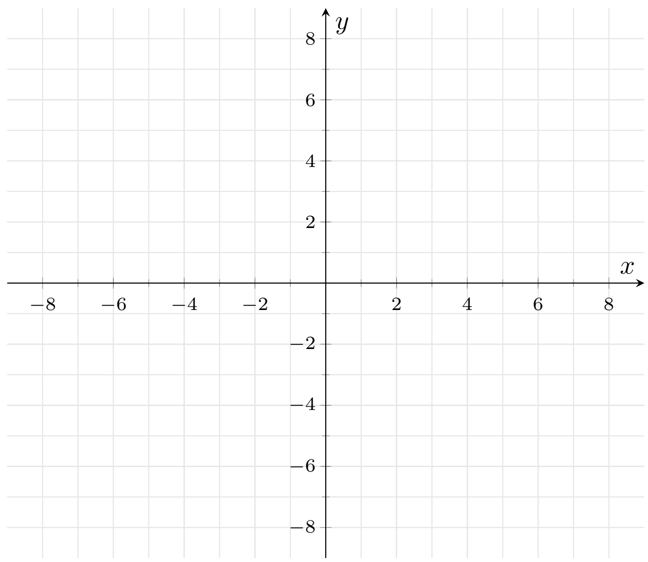
To see more of how these graphs change with n, follow the Desmos link: `https://www.desmos.com/calculator/viuyei80a0`.

Now, are these functions invertible? Looking at the graphs, we see that these functions pass the Horizontal Line Test. Thus, the functions are one-to-one, and therefore invertible.

Definition 1. *When n is an odd positive integer, we define* ***the nth root function*** $\sqrt[n]{x}$ *to be the inverse of the function defined by x^n. The number n is the* ***index*** *of the root, and x is the* ***radicand****. We call the symbol $\sqrt{\ }$ the* ***radical****.*

Let's delve more deeply into the $p(x) = x^3$ example. We have now established that it is invertible, and it's inverse is $r(x) = \sqrt[3]{x}$.

Exploration 1 *Draw both functions on the axes provided, then answer the following questions about the function $r(x)$.*



(a) *What is the x-intercept of $r(x)$?* $(\boxed{0}, \boxed{0})$

(b) *What is the y-intercept of $r(x)$?* $(\boxed{0}, \boxed{0})$

(c) *What is the domain of $r(x)$?* $(\boxed{-\infty}, \boxed{\infty})$

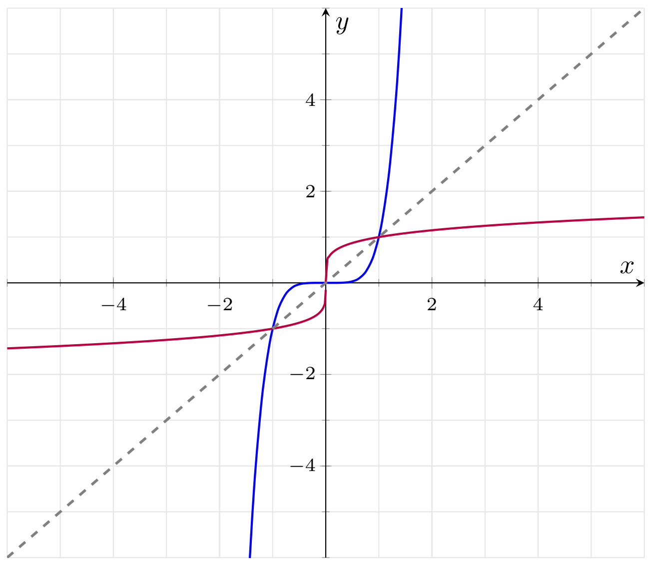
(d) *What is the range of $r(x)$?* $(\boxed{-\infty}, \boxed{\infty})$

(e) *As x goes to ∞, y goes to* $\boxed{\infty}$.

(f) *As x goes to $-\infty$, y goes to* $\boxed{-\infty}$.

(g) *Does this function have any vertical asymptotes? (yes/ no ✓)*

Example 1. *Below, we have an example of the graph of $y = x^5$ and the two functions*

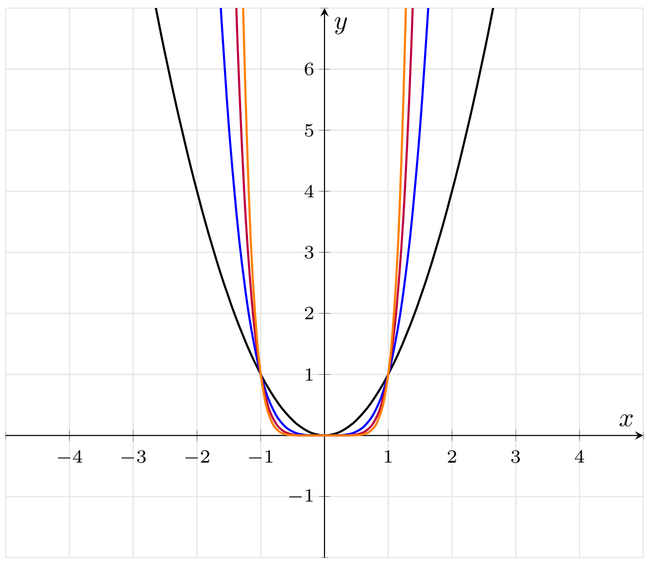


Recall that a function f is *even* if $f(-x) = f(x)$ for all x in its domain, and f is *odd* if $f(-x) = -f(x)$ for all x. Otherwise, the function is neither. Let's consider an example. Given $x = 8$, $r(x) = \sqrt[3]{8} = 2$ and $r(-x) = \sqrt[3]{-8} = -2$, since $(-2) \cdot (-2) \cdot (-2) = (4) \cdot (-2) = -8$. Based on this example, do you think $r(x)$ is even, odd, or neither?

If you guessed odd, then you are correct! All odd-index root functions are odd functions.

Even Roots

We again begin by recalling the general shape of $p(x) = x^n$, but this time for n *even*. These functions also have the same basic shape for all even n. This is demonstrated by the graphs of $y = x^n$ for $n = 2, 4, 6, 8$ given below.

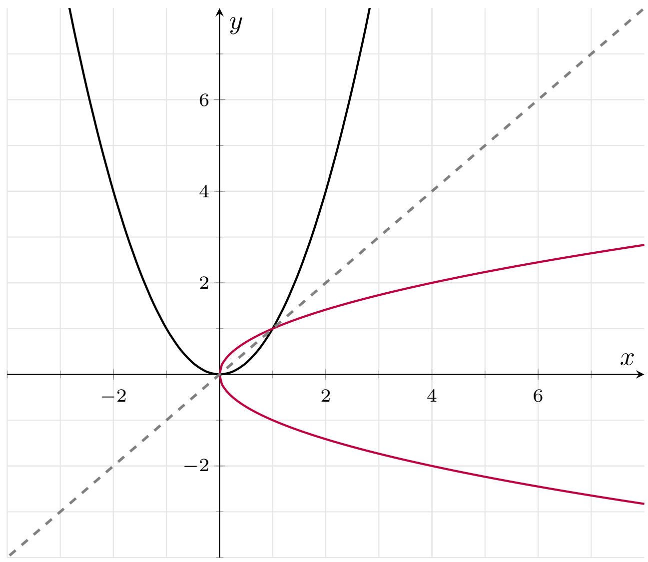


To see more of how these graphs change with n, follow the Desmos link: `https://www.desmos.com/calculator/qpqwtrppqt`.

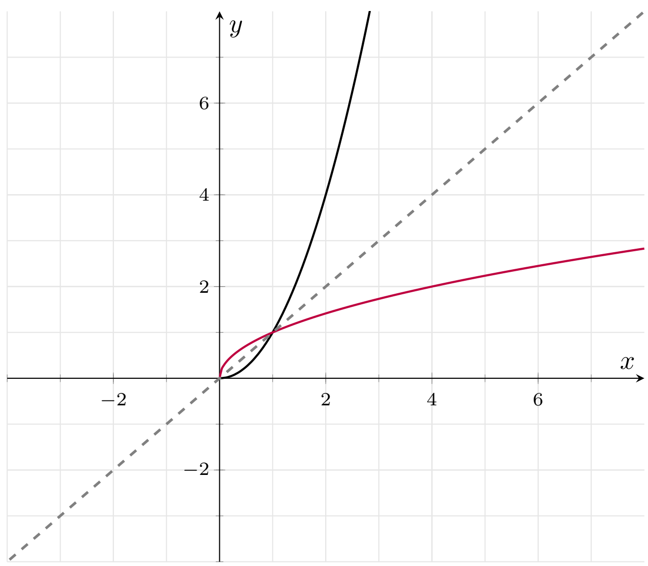
Now, are these functions invertible? All of the graphs in the figure above are symmetric about the y-axis (example $2^2 = 4 = (-2)^2$), so they do *not* pass the horizontal line test. Thus, these functions are *not* one-to-one, and therefore *not* invertible.

So, how can we define an even root function? For example, what does $\sqrt{x}$ really mean, and how is it related to x^2?

Consider the polynomial $p(x) = x^2$, graphed below with its inverse relation $\{(x^2, x) : x \text{ is a real number}\}$.



Observe that by restricting the domain of p to $x \geq 0$, we now have a function which passes the horizontal line test, and can thus be inverted. The following picture illustrates the situation.

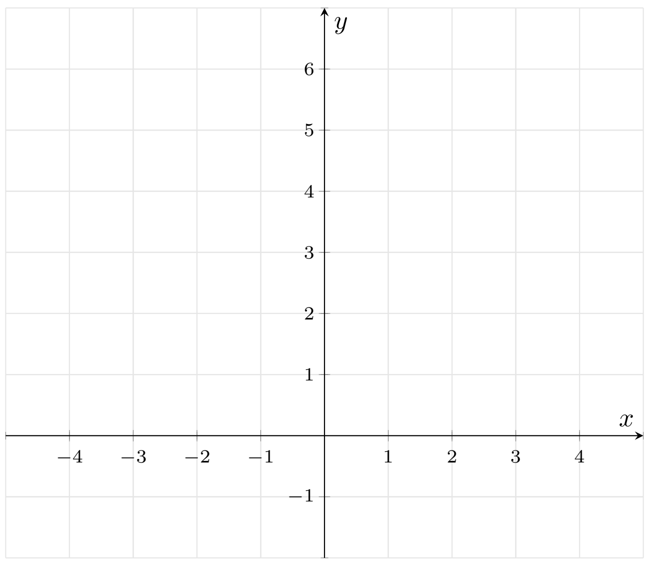


Now, our inverse relation is actually a function, since it passes the Horizontal Line Test. Therefore, if we let $p(x) = x^2$ for $x \geq 0$, we can then define $r(x) = \sqrt[2]{x} = \sqrt{x}$ as the inverse function of $p(x)$ on this restricted domain.

Definition 2. *When n is an even positive integer, we define* ***the*** $\boldsymbol{n}$***th root***

function *$\sqrt[n]{x}$ to be the inverse of the function defined by x^n restricted to the domain $x \geq 0$.*

Exploration 2 *We now repeat Exploration 1 for $r(x) = \sqrt[4]{x}$. Draw both functions on the axes provided, then answer the following questions about the function $r(x)$.*



(a) *What is the x-intercept of $r(x)$?* $(\boxed{0}, \boxed{0})$

(b) *What is the y-intercept of $r(x)$?* $(\boxed{0}, \boxed{0})$

(c) *What is the domain of $r(x)$?* $[\boxed{0}, \boxed{\infty})$

(d) *What is the range of $r(x)$?* $[\boxed{0}, \boxed{\infty})$

(e) *As x goes to ∞, y goes to* $\boxed{\infty}$.

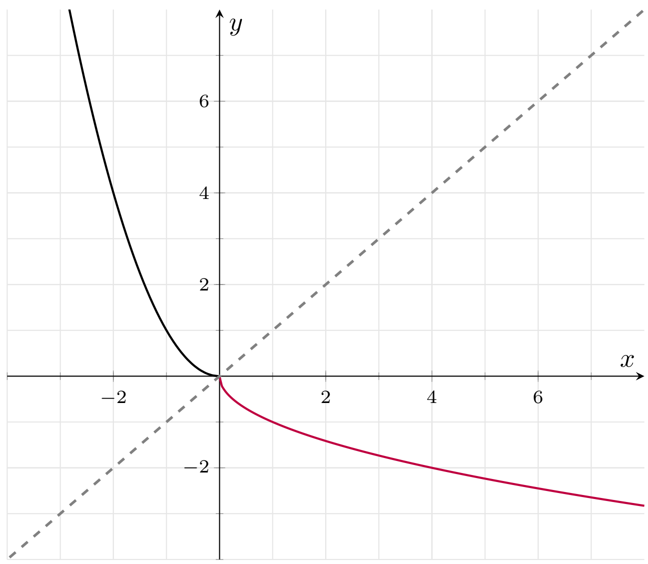
(f) *Does this function have any vertical asymptotes? (yes/ no ✓)*

One question we might ask is whether $r(x) = \sqrt{x}$ and $p(x) = x^2$ are truly inverses. The answer may seem like an obvious "yes!", but since we restricted the domain of p in order to define r, we need to check. To check whether r and p are inverses, we need to confirm that $r(p(x)) = \sqrt{x^2} = x$ and $p(r(x)) = (\sqrt{x})^2 = x$. That is, when we plug in a number to $\sqrt{x^2}$ and $(\sqrt{x})^2$, we should get the same number as the output. Let's try plugging in -1 to $r(p(x))$. This gives us

$$r(p(-1)) = r((-1)^2) = \sqrt{(-1)^2} = \sqrt{1} = 1,$$

which is not the same as -1. If we repeat this process with a few more numbers, we find that $\sqrt{(-2)^2} = 2$, $\sqrt{1^2} = 1$, $\sqrt{(-45)^2} = 45$, and $\sqrt{98^2} = 98$. We can conclude that $\sqrt{x^2}$ is a function that takes its input and returns its absolute value. That is, $\sqrt{x^2} = |x|$. Since $\sqrt{x^2} = r(p(x))$, we conclude that $r(p(x))$ does not output its input, and therefore, r and p are not inverses. This is something that will be extremely important when solving equations using even roots.

Now, what if we instead restricted our domain to $x \leq 0$? Consider $q(x) = x^2$ defined for $x \leq 0$. The graph of this function is below.



By the Horizontal Line Test, this restriction is one-to-one, and therefore invertible. The inverse of this function as shown above is $s(x) = -\sqrt{x} = -r(x)$.

Example 2. *We demonstrate a few common even and odd n^{th} roots to highlight this distinction.*

(a) $\sqrt[3]{8} = 2$, *since* $2 \cdot 2 \cdot 2 = 8$.

(b) $\sqrt[3]{-8} = -2$, *since* $-2 \cdot (-2) \cdot (-2) = 4 \cdot (-2) = -8$.

(c) $\sqrt[4]{16} = 2$, *since* $2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 4 \cdot 4 = 16$. *However, the* 4^{th} *root of -16 is not defined.*

(d) $\sqrt[4]{0} = 0$, *since zero times any number is always zero. This is* the *example of an even* n^{th} *root that has only* one *solution.*

(e) *Likewise,* $\sqrt[125]{0} = 0$.

Using Roots to Solve Equations

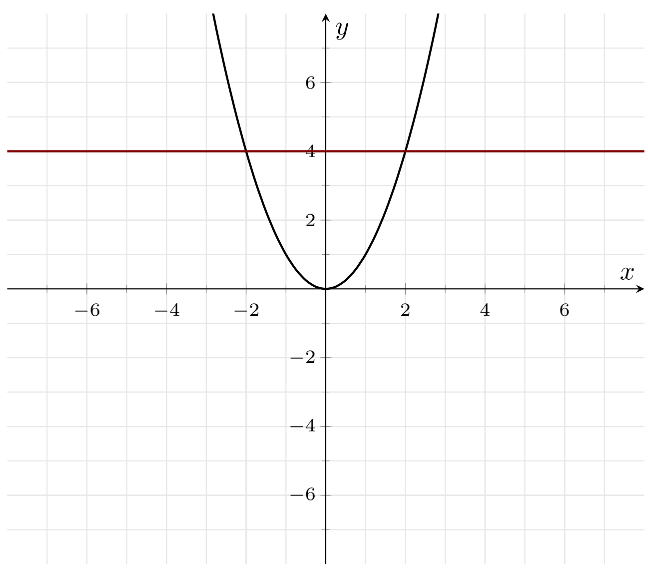
If we are asked to find all values x such that $x^2 = 4$, then the question is asking which values of x multiplied by themselves give 4. In other words, find x such that $x \cdot x$ is equal to 4. It is simple to see that there are two values which make this true:

$$2 \cdot 2 = 4 \text{ and } (-2) \cdot (-2) = 4.$$

In solving an equation, it is common to express this as follows.

$$\begin{aligned} x^2 &= 4 \\ \sqrt{x^2} &= \sqrt{4} \\ |x| &= 2 \\ x &= \pm 2. \end{aligned}$$

Since $f(x) = x^2$ is **not** one-to-one, there are two values of x which make it equal to any positive number, as demonstrated in the following graph.



Example 3. (a) *Solve the equation $x^3 = 8$.*

Taking the cube root on both sides, we find that

$$\begin{aligned} \sqrt[3]{x^3} &= \sqrt[3]{8} \\ x &= 2. \end{aligned}$$

Note that $\sqrt[3]{x^3} = x$, since 3 is odd, and odd roots are really inverses to their corresponding power functions.

(b) *Solve the equation* $x^3 = -8$.

Taking the cube root on both sides, we find that

$$\begin{aligned}\sqrt[3]{x^3} &= \sqrt[3]{-8}\\ x &= -2.\end{aligned}$$

(c) *Solve the equation* $x^2 = 16$.

Taking the square root on both sides, we find that

$$\begin{aligned}\sqrt{x^2} &= \sqrt{16}\\ |x| &= 4\\ x &= \pm 4.\end{aligned}$$

Therefore, there are two solutions to this equation: -4 *and* 4.

(d) *Solve the equation* $2x^4 - 4 = 28$.

First, rearrange the equation. Add 4 to both sides to find $2x^4 = 32$. *Divide both sides by 2 to find* $x^4 = 16$. *Taking the 4th root on both sides, we find that*

$$\begin{aligned}\sqrt[4]{x^4} &= \sqrt[4]{16}\\ |x| &= 2\\ x &= \pm 2.\end{aligned}$$

Therefore, there are two solutions to this equation: -2 *and* 2.

Recall that for any even integer n, $\sqrt[n]{x^n} = |x|$.

(e) *Solve the equation* $-3x^4 = 32$.

First, rearrange the equation by dividing both sides by -3. This yields $x^4 = -\dfrac{32}{3}$. *Taking the 4th root on both sides, we find that*

$$\begin{aligned}\sqrt[4]{x^4} &= \sqrt[4]{-\frac{32}{3}}\\ |x| &= \sqrt[4]{-\frac{32}{3}}.\end{aligned}$$

Since any even-index root of a negative number is not defined, there are no solutions to this equation.

These examples illustrate a general principle that is good to have in your toolbox for solving equations. If $a^3 = b$, then we know that $a = \sqrt[3]{b}$. This is true for any odd powers. However, if $a^2 = b$, then **either** $a = \sqrt{b}$ **or** $a = -\sqrt{b}$. This is true for any even powers.

Finding x-intercepts of a Quadratic in Vertex Form

Now, we can use our understanding of the squareroot function to find the x-intercepts of a quadratic given in vertex form.

Example 4. *Find the x-intercepts of the quadratic $2(x-3)^2 - 5 = 0$ which is written in vertex form.*

Explanation. *First, rearrange the equation by adding 5 to both sides. This yields*

$$2(x-3)^2 = 5.$$

Then divide each side by 2, resulting in

$$(x-3)^2 = \frac{5}{2}.$$

Taking the square root on both sides, we find that

$$\begin{aligned} \sqrt{(x-3)^2} &= \sqrt{\frac{5}{2}} \\ |x-3| &= \sqrt{\frac{5}{2}} \\ x-3 &= \pm\sqrt{\frac{5}{2}} \\ x &= 3 \pm \sqrt{\frac{5}{2}} \end{aligned}$$

Notice that this gives us a third method for finding the roots (x-intercepts) of a quadratic in general. We can use any of these methods to solve a quadratic.

(a) Factor the quadratic and write it in Root Form

(b) Use the quadratic formula to find the roots

(c) Write the quadratic in vertex form and then solve using a squareroot

Mathematically, these last two methods are actually related. The quadratic formula is just what happens when you rewrite the general quadratic $f(x) = ax^2 + bx + c$ in vertex form and then solve for x!

Composing $f(x) = x^2$ and $g(x) = \sqrt{x}$

This final example is going to be a very important one that comes up often so we we will give it its own section.

Example 5. *Let $f(x) = x^2$ and let $g(x) = \sqrt{x}$.*

a. *Find the domain and range of $f \circ g$ and compare this function to $id(x) = x$ and $abs(x) = |x|$.*

b. *Find the domain and range of $g \circ f$ and compare this function to $id(x) = x$ and $abs(x) = |x|$.*

Explanation. *You probably have the idea that the squaring and squarerooting actions undo one another. This is true for nonnegative values of x, but can get tricky when x is allowed to be negative. Let's look at each of these situations closely.*

a. *First we consider $f \circ g$ and compare this function to $id(x) = x$ and $abs(x) = |x|$. We have $f(x) = x^2$ and $g(x) = \sqrt{x}$ so*

$$(f \circ g)(x) = f(g(x)) = (g(x))^2 = (\sqrt{x})^2.$$

Let's consider the domain of this function. Recall that the domain of a composite function $f \circ g$ is the set of those inputs x in the domain of g for which $g(x)$ is in the domain of f. In this case, this means that the domain of $f(g(x)) = (\sqrt{x})^2$ is the set of those inputs x in the domain of $g(x) = \sqrt{x}$ for which $\sqrt{x}$ is in the domain of $f(x) = x^2$. The implied domain of $g(x) = \sqrt{x}$ is $[0, \infty)$ since we cannot take the square root of a negative number. Therefore, since the domain of the composition has to be only values in the domain of $g(x)$, this means the largest our domain can be is $[0, \infty)$. Now, the only additional limiting factor is that the values $\sqrt{x}$ must be in the domain of f but since the domain of f is all real numbers, that will not limit the domain of the composition. Therefore, the domain of $f \circ g$ is $[0, \infty)$.

Now that we know the domain and we know that squaring and square-rooting undo each other for nonnegative values of x, we can conclude that $f \circ g$ is the identity function, $id(x) = x$ but restricted to the domain $[0, \infty)$. That is,

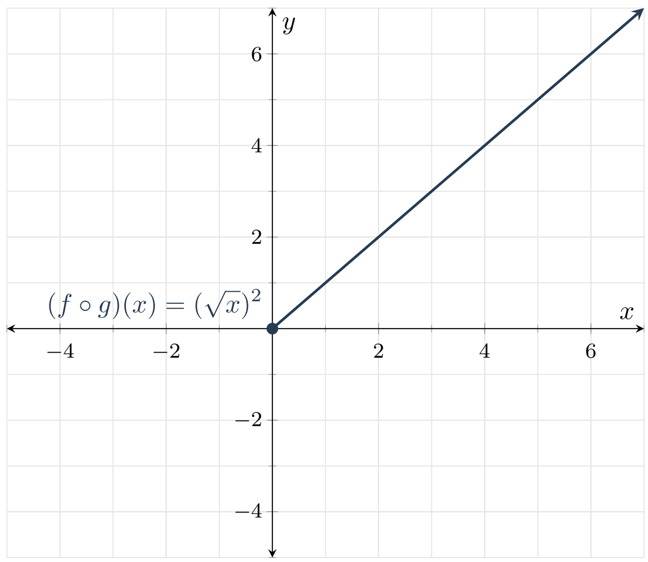
$$(f \circ g)(x) = (\sqrt{x})^2 = x, x \geq 0$$

Since the absolute value function is the same as the identity function when $x \geq 0$. Therefore, we could also say that

$$(f \circ g)(x) = (\sqrt{x})^2 = |x|, x \geq 0$$

From this information, we also know that the range of $f \circ g$ will also be $[0, \infty)$, since $(f \circ g)(x) = (\sqrt{x})^2$.

Here is a graph of $(f \circ g)(x) = (\sqrt{x})^2$.



b. *Now we consider $g \circ f$ and compare this function to $id(x) = x$ and $abs(x) = |x|$. The domain of $g(f(x)) = \sqrt{x^2}$ is the set of those inputs x in the domain of $f(x) = x^2$ for which x^2 is in the domain of $g(x) = \sqrt{x}$. The domain of $f(x) = x^2$ is all real numbers, so this does not reduce the domain of the composite function. The range of $f(x) = x^2$ is $[0, \infty)$ since the square of every number will be greater than or equal to zero. The implied domain of $g(x) = \sqrt{x}$ is $[0, \infty)$. Therefore, every output from $f(x) = x^2$ is in the domain of $f(x) = \sqrt{x}$. Therefore, the domain of $g \circ f$ is $(-\infty, \infty)$.*

Now, let's consider the range of $g \circ f$. We know that the range of $g \circ f$ must be contained in the range of $g(x) = \sqrt{x}$. The range of $g(x) = \sqrt{x}$ is $[0, \infty)$, so that is the largest range possible for $g \circ f$. We know that for values of $x \geq 0$, squaring and squarerooting undo one another so we know that all the values of $[0, \infty)$ are contained in the range of $g \circ f$. More precisely, for any value x_0 in $[0, \infty)$, $g(f(x_0)) = x_0$ so x_0 will be in the range of $g \circ f$. Thus, the range of $g \circ f$ is $[0, \infty)$.

Now, since we know that this function $g \circ f$ only outputs positive numbers, we know it cannot equal the identity function for inputs of $x < 0$. Let's explore what this function does for values of $x < 0$ by considering $x = -2$.

$$g(f(-2)) = \sqrt{(-2)^2} = \sqrt{4} = 2$$

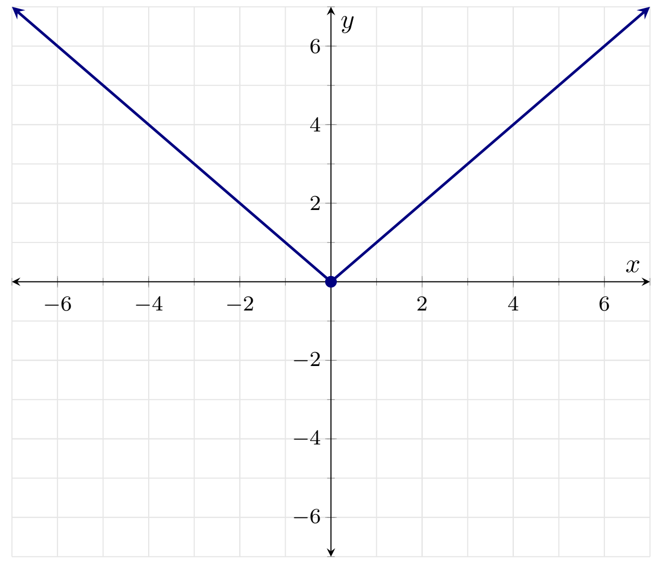
Notice, that we input $x = -2$ but the output was positive 2. In fact, for all values of $x < 0$, $g(f(x)) = \sqrt{x^2} = -x = |x|$.

Since the absolute value function is the same as the identity function when

$x \geq 0$ but the negative of the idenity function for $x < 0$, we have that

$$(g \circ f)(x) = \sqrt{x^2} = |x|$$

Here is a graph of $(g \circ f)(x) = \sqrt{x^2} = |x|$.



Summary 1. *In general, we are not able to simply find the inverse of polynomials.*

However, when the polynomial is $p(x) = x^n$ for a positive odd *integer n, the polynomial is invertible as the n^{th} root function $r(x) = \sqrt[n]{x}$.*

When n is even*, it is possible to define an inverse function $r(x) = \sqrt[n]{x}$ on a restricted domain of $[0, \infty)$. The n^{th} root is defined as the inverse of $p(x)$ on the restricted domain $[0, \infty)$.*

