Precalculus with Review 2: Unit 10

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Inverse Functions In Depth

10.1 Review of Inverse Functions

Learning Objectives

- Inverses Review
  - Reviewing the definitions and properties of inverse functions.
- Finding Inverses Algebraically
- Graphs of Inverse Functions
  - Understanding the relationship between graphs of functions and their inverses.
  - Finding the inverse of compositions of functions.

10.1.1 Review of Inverse Functions

In Section 3-2-2, we briefly introduced the concept of *inverse functions*. Recall that for a one-to-one function f, we can define the inverse function f^{-1}. If we think of f as a process that takes some input x and produces some output $f(x)$, then providing $f(x)$ as an input to f^{-1} produces the original input x, and vice versa. Symbolically, we wrote that $f^{-1}(f(x)) = x$ and $f(f^{-1}(x)) = x$.

We learned several important principles, which we summarize below.

- A function f has an inverse function if and only if there exists a function g that undoes the work of f: that is, there is some function g for which $g(f(x)) = x$ for each x in the domain of f, and $f(g(y)) = y$ for each y in the range of f. We call g the inverse of f, and write $g = f^{-1}$.
- A function f has an inverse function if and only if the graph of f passes the *Horizontal Line Test*.
- A function f has an inverse function if and only if f is a *one-to-one* function.
- When f has an inverse, we know that writing "$y = f(t)$" and "$t = f^{-1}(y)$" are two different perspectives on the same statement.
- If (x, y) is a point on the graph of f, then (y, x) is a point on the graph of f^{-1}.
- The graph of f^{-1} is the graph of f reflected across the line $y = x$.
- The domain of f is the range of f^{-1} and the range of f is the domain of f^{-1}.
- If f^{-1} is the inverse of f, then f is the inverse of f^{-1}.

In this section, we'll explore inverse functions more in-depth.

10.1.2 Finding Inverses Algebraically

In this section, we will practice how to actually algebraically determine whether the inverse for a given function f exists, and how to find it.

Example 1. *Let f be the function given by $f(x) = 4x^3+1$, for every real number x. Is f one-to-one? If so, what is the inverse f^{-1}? What is its domain?*

Explanation Recall here that f is one-to-one if $f(x_1) = f(x_2)$ always implies that $x_1 = x_2$. What happens in this case? We start with the equality

$$4x_1^3 + 1 = 4x_2^3 + 1,$$

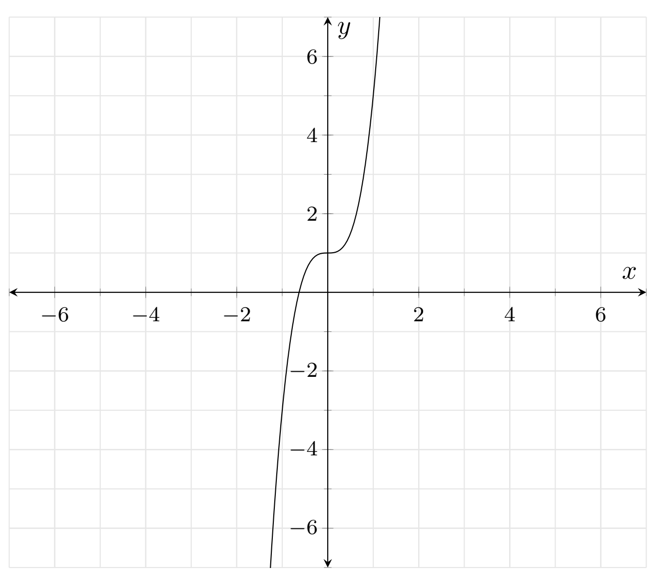
and if we can arrive at $x_1 = x_2$, then our function f is one-to-one. Subtracting 1 from both sides gives $4x_1^3 = 4x_2^3$. Dividing both sides by 4, we have that $x_1^3 = x_2^3$. As the function "taking the third power" is one-to-one (namely, its inverse is "taking cube roots"), it follows that $x_1 = x_2$. This means that one can start with $y = 4x^3 + 1$ and solve for x as a function of y. When this is finished, one replaces x and y with $f^{-1}(x)$ and x, respectively, for the sake of conventions. We have that

$$y = 4x^3 + 1 \implies y - 1 = 4x^3 \implies \frac{y-1}{4} = x^3 \implies \sqrt[3]{\frac{y-1}{4}} = x.$$

Therefore, we conclude that

$$f^{-1}(x) = \sqrt[3]{\frac{x-1}{4}}.$$

Observe that the domain of f^{-1} is precisely the set of all real numbers, as there is no positivity restriction for taking "odd roots". This turns out to be exactly the range of the original function f, suitably read in the y variable: namely, the range of f consists of all real numbers y, as the graph shows:



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Observe that the graph also shows that f passes the Horizontal Line Test, a useful sanity check.

What happened with the domain of f^{-1} on the previous example is a very general phenomenon: if a given function f is one-to-one, so that f^{-1} exists, the domain of f^{-1} is exactly the range of the original function f.

Exploration Study the function $f(x) = -x^5 + 10$. Is it one-to-one? If yes, find a formula for f^{-1}.

Example 2. *Let g be the rational function given by $g(x) = 2x/(1-x)$ for every real number x not equal to 1. Is g one-to-one? If so, what is the inverse g^{-1}? What is its domain?*

Explanation The formula defining g is different from the formula defining the function from the previous example: there, we had a polynomial, while here we have a rational function. The process to study it, however, is the same. To decide whether g is one-to-one or not, we will start with $g(x_1) = g(x_2)$, and try to conclude that $x_1 = x_2$. So, we start with

$$\frac{2x_1}{1-x_1} = \frac{2x_2}{1-x_2}.$$

By cross multiplying, we have that

$$2x_1(1-x_2) = 2x_2(1-x_1).$$

Distributing the products on both sides, we obtain

$$2x_1 - 2x_1x_2 = 2x_2 - 2x_2x_1.$$

Since $2x_2x_1 = 2x_1x_2$, adding this quantity to both sides gives us that $2x_1 = 2x_2$. Finally, dividing everything through by 2, it follows that $x_1 = x_2$. Therefore, g is one-to-one. With this in place, we can attempt to find g^{-1}. Starting with

$$y = \frac{2x}{1-x},$$

the only thing it seems we might be able to try is to cross multiply terms, and distribute the products. So

$$(1-x)y = 2x \implies x - xy = 2x.$$

As our goal is to solve for x in terms of y, let's move everything that contains x in it to one side, and leave the rest which does not contain it on the other side. We obtain $y = 2x + xy$. Factoring x, it follows that $y = (2+y)x$. Finally, we conclude that

$$x = \frac{y}{y+2} \implies g^{-1}(x) = \frac{x}{x+2},$$

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by replacing x and y with $g^{-1}(x)$ and x, respectively. We observe that the domain of g^{-1} consists of all real numbers x different from -2, in the same way that the range of g consists of all real numbers y different from -2 (to wit, if we had $-2 = 2x/(1-x)$, then cross multiplying and simplifying yields $-x = 1-x$, leading to a terribly nonsensical $0 = 1$).

Exploration Study the function $g(x) = 3x/(4-5x)$. Is it one-to-one? If yes, find a formula for g^{-1}.

Example 3. *Let h be the function given by $h(x) = e^{\sqrt{x-3}}$ for every real number x greater or equal to 3. Is h one-to-one? If so, what is the inverse h^{-1}? What is its domain?*

Explanation Let's start verifying whether h is one-to-one or not. So, as before, we start with $h(x_1) = h(x_2)$ and try to obtain that $x_1 = x_2$. In other words, we start with $e^{\sqrt{x_1-3}} = e^{\sqrt{x_2-3}}$. From arguing either that the exponential function itself is one-to-one, or applying ln on both sides of this equality, it follows that $\sqrt{x_1-3} = \sqrt{x_2-3}$. Raising both sides to the square, we have that $x_1 - 3 = x_2 - 3$, and adding 3 to everything yields $x_1 = x_2$, as desired. Therefore, h is one-to-one. To find a formula for h^{-1}, we start with $y = e^{\sqrt{x-3}}$ and solve for x in terms of y. First, we apply ln on both sides of the equality, as to obtain $\ln y = \sqrt{x-3}$. Next, we take squares on both sides, so $(\ln y)^2 = x - 3$. Finally, we add 3 to both sides:

$$x = (\ln y)^2 + 3 \implies h^{-1}(x) = (\ln x)^2 + 3.$$

Note that $(\ln x)^2$ is not the same thing as $\ln(x^2)$ (to wit, the latter equals $2\ln x$, by properties of the logarithm function we shall soon see). The inverse h^{-1} is defined for all $x > 0$, due to the presence of $\ln x$ in its formula.

Exploration Study the function $h(x) = \ln\sqrt{x+10}$. Is it one-to-one? If yes, find a formula for h^{-1}.

10.1.3 Graphs of Inverse Functions

Motivating Questions

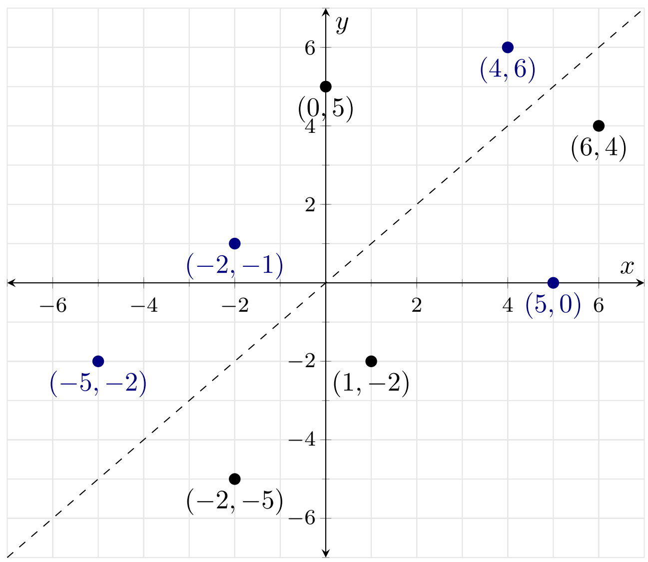
- How to obtain, from the graph of a given one-to-one function f, the graph of inverse function f^{-1}?
- Knowing formulas for f^{-1} and g^{-1}, is there a nice formula for the inverse $(g \circ f)^{-1}$ of the composition $g \circ f$, whenever it makes sense?

Introduction

Recall that given a function f from a set A to a set B, the graph of f is the set of all points $(x, f(x))$, where x belongs to A. When A and B are intervals of real numbers, we are able to draw the graph of f in the xy-plane.

When f is one-to-one, we may talk about the inverse function f^{-1} (from the range of f inside B, to A), which is characterized by the equivalence of the relations $y = f(x)$ and $f^{-1}(y) = x$. However, as we have seen in the previous section, when solving algebraically for the inverse f^{-1}, we switch the letters x and y after all calculations are done. This step, while a priori seemingly arbitrary, allows us to draw the graphs of f and f^{-1} together to compare them.

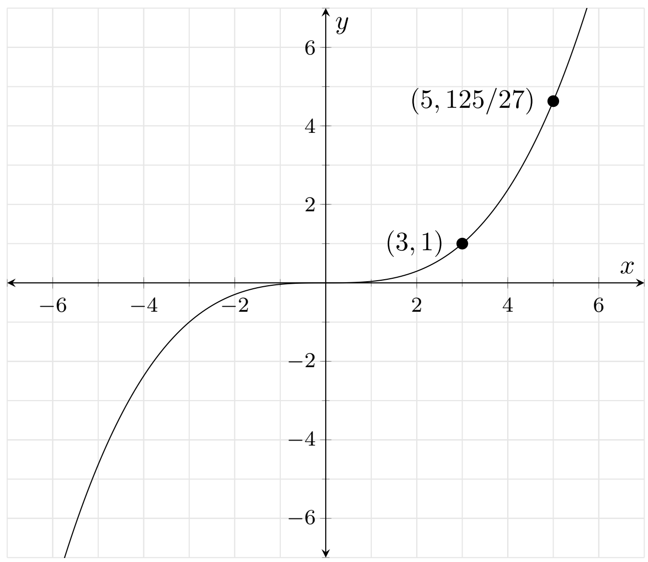
With this in mind, we have that a point (b, a) is in the graph of f^{-1} precisely when (a, b) is in the graph of f. But what is the geometric relation between the points (a, b) and (b, a)? Or, in other words, what does it mean to switch the coordinates of a point? Let's see what happens with a few points, say, $(-2, -5)$, $(1, -2)$, $(0, 5)$, and $(6, 4)$. They will be indicated in black, while the corresponding points with the switched coordinates will be indicated in blue.



As the picture indicates, switching the coordinates of a point amounts to reflecting it about the principal diagonal line, whose line equation is $y = x$.

> **Conclusion:** to obtain the graph of f^{-1}, just reflect the graph of f about the line $y = x$.

To elaborate more on this conclusion, we note that it allows us to read values of f^{-1} even without actually graphing it! Consider the following graph, of a function $y = f(x)$:



Even though we do not know the actual formula defining f, just from knowing that $(3, 1)$ and $(5, 125/27)$ are in the graph of f, we may safely conclude that $(1, 3)$ and $(125/7, 5)$ are in the graph of f^{-1}, which is to say that

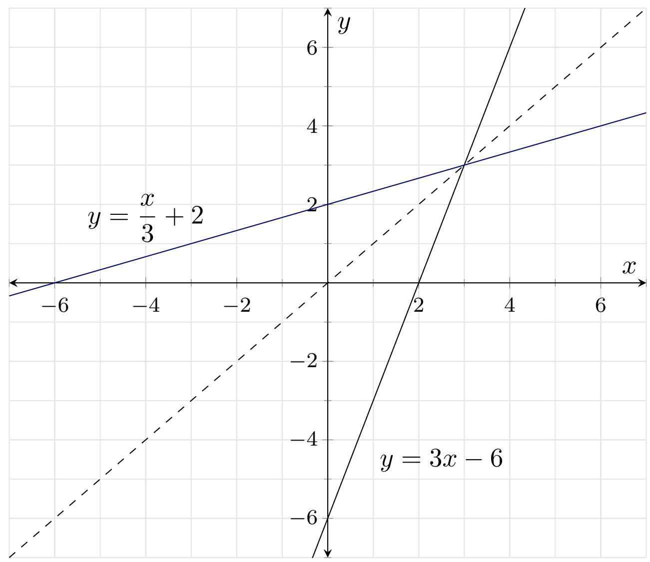
$$f^{-1}(1) = 3 \quad \text{and} \quad f^{-1}(125/7) = 5.$$

Practicing

Let's explore several situations on what follows. To draw the graph of f to begin with, we rely on what was discussed on previous chapters.

Example 4. *Consider the function $f(x) = 3x - 6$. Draw the graph of f^{-1} from the graph of f. Then, algebraically find f^{-1} to confirm your work.*

Explanation Since f is a linear function, all we need to know to draw its graph are its slope and y-intercept. The formula immediately tells us that the slope is 3 and the y-intercept is -6. The inverse of a one-to-one linear function is also a linear function, and a line is determined by two points. So in this case, all we have to do is to reflect two points in the graph of f about the line $y = x$, and draw the line passing through the reflected points: this is the graph of f^{-1}.



Let's make a few observations, based on the picture.

- The x-intercept of f became the y-intercept of f^{-1}.
- The y-intercept of f became the x-intercept of f^{-1}.
- The graphs of f and f^{-1} meet precisely at the line $y = x$.
- Both f and f^{-1} are increasing functions.

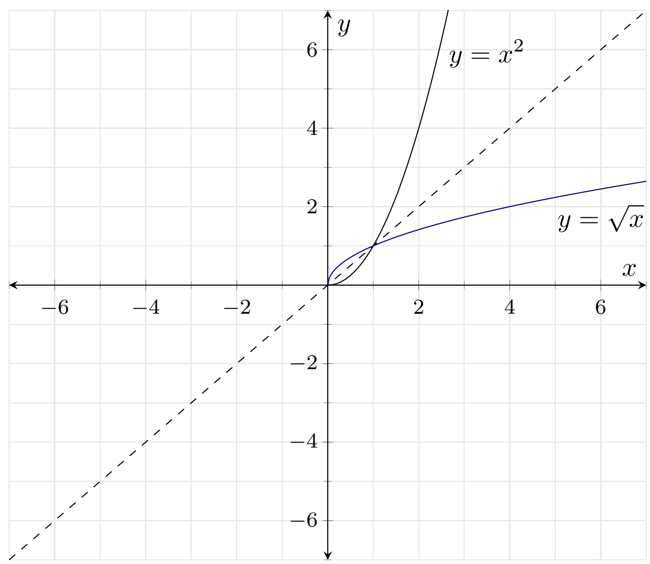
The first and second points are actually true for *any* (non-horizontal, and hence one-to-one) linear function; the third point is true for any one-to-one function (with the caveat that the graphs may meet more than once, but still always an points lying on the line $y = x$); the fourth point illustrates a very general phenomenon, also true for any one-to-one function – namely, that a one-to-one function must necessarily be (strictly) increasing or decreasing, and f^{-1} will have the same behavior as f (explictly, f^{-1} is increasing if f is increasing, and f^{-1} is decreasing if f is decreasing).

Finally, let's double-check our work algebraically. If $y = 3x - 6$, then $y + 6 = x$. Dividing everything by 3 gives $x = (y/3) + 2$. Replacing x and y with $f^{-1}(x)$ and x, respectively, yields $f^{-1}(x) = (x/3) + 2$.

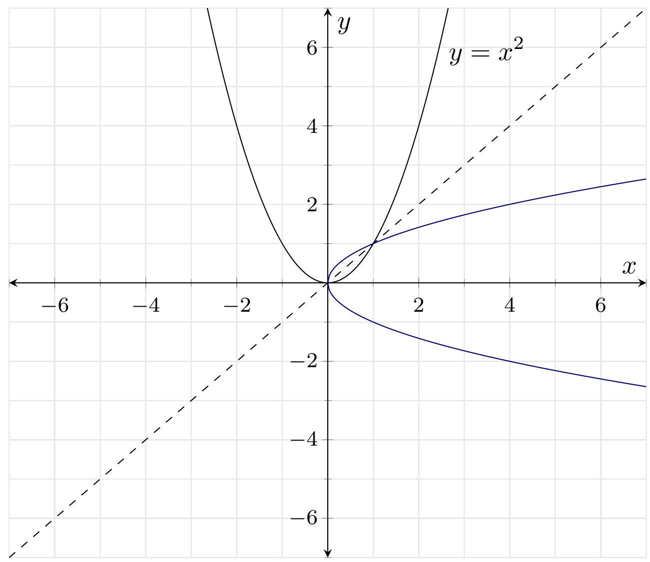
Exploration Consider the function $f(x) = -2x + 5$. Draw the graph of f^{-1} from the graph of f, deduce the formula for $f^{-1}(x)$ from the picture, and then double-check your work algebraically.

Example 5. *Consider the function $f(x) = x^2$, defined for all $x \geq 0$, and draw the graph of f^{-1} from the graph of f. What happens if you take f to be defined for every real number x instead?*

Explanation We know that the graph of $y = x^2$ is the classical parabola we are already familiar with. As we're only considering the domain to be the set of real numbers with $x \geq 0$, we must look only at the right half of the parabola (which passes the Horizontal Line Test), and then reflect it about the line $y = x$.



If we were to consider the graph of $y = x^2$, this time defined for all real numbers, the full parabola enters the picture again. Reflecting it, we obtain the following curve indicated in blue:



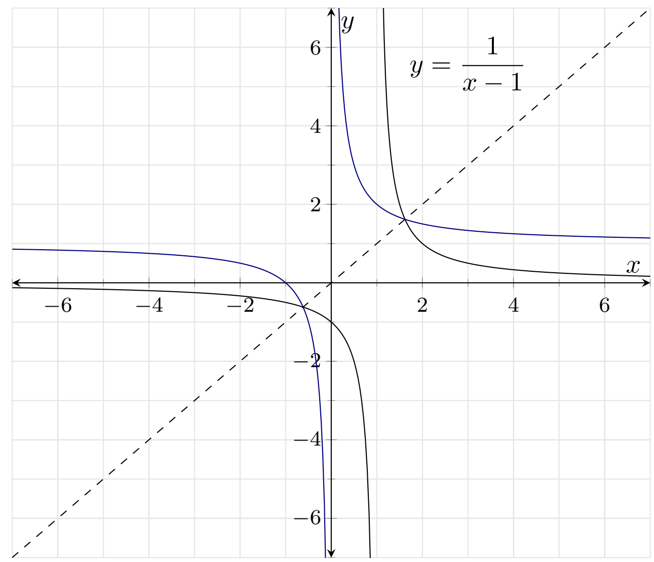
The problem here is that the reflected curve does not pass the Vertical Line Test, which means that it is not the graph of any function. To understand why this should be expected, it suffices to observe that the function $f(x) = x^2$ defining the original parabola, defined for all real numbers x, is not one-to-one. This time, we have the freedom to switch signs, leading us to things like $f(1) = f(-1)$, $f(2) = f(-2)$ and, more generally, $f(x) = f(-x)$ for all real numbers x (you might recall this last property as saying that f is an "even function"; in fact, every even function is not one-to-one).

Graphically, the original parabola does not pass the Horizontal Line Test. Algebraically, while $(\sqrt{x})^2 = x$ for every $x \geq 0$ (with $\sqrt{x}$ undefined for negative x), we have that $\sqrt{x^2} = |x|$ for every real x, with $|x| = x$ only for $x \geq 0$.

Exploration Consider the function $f(x) = -(x-1)^2+3$, defined for all $x \geq 1$, and draw the graph of f^{-1} from the graph of f. What happens if you take f to be defined for every real number x instead?

Example 6. *Consider the function $f(x) = 1/(x-1)$, defined for all numbers $x \neq 1$. Draw the graph of f^{-1} from the graph of f. Does it look similar to the graph of f? Algebraically find f^{-1} to double-check your possible suspicions.*

Explanation We start from the graph of the famous function $1/x$, which we have already encountered before, and shift it one unit to the right.

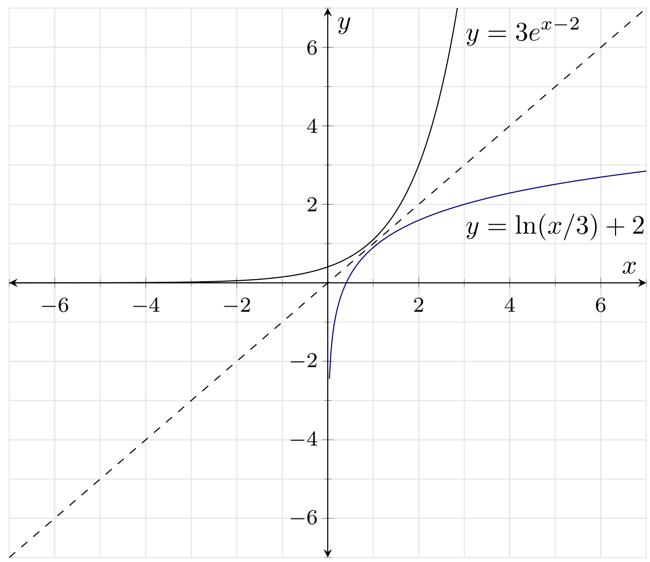


The graph of f^{-1}, also formed by branches of hyperbolas and having a vertical asymptote at the line $x = 0$, suggests that the actual formula for $f^{-1}(x)$ should contain some term like $1/x$ itself. To verify this, we proceed by starting with $y = 1/(x-1)$, which is equivalent to $x - 1 = 1/y$, and adding 1 to both sides as to obtain $x = 1 + 1/y$. Replacing x and y with $f^{-1}(x)$ and x, respectively, gives us that $f^{-1}(x) = 1 + 1/x$, as expected.

Exploration Consider the function $f(x) = 3/(x+2)$, defined for all numbers $x \neq -2$. Draw the graph of f^{-1} from the graph of f, and then algebraically find a formula for f^{-1}.

Example 7. *Consider the function $f(x) = 3e^{x-2}$, defined for all real numbers x. Draw the graph of f^{-1} from the graph of f. Then, algebraically find f^{-1} to confirm your work.*

Explanation To draw the graph of f, we start from the graph of the exponential $y = e^x$, shift it 2 units to the right, and then apply a vertical stretch by a factor of 3.



To verify that the inverse function is indeed given by $f^{-1}(x) = \ln(x/3) + 2$, we start with $y = 3\mathrm{e}^{x-2}$ and solve for x in terms of y. First, we divide both sides by 3, to obtain $y/3 = \mathrm{e}^{x-2}$. Then, we apply ln to both sides, to obtain $\ln(y/3) = x-2$. Finally, we add 2 to both sides, getting to $x = \ln(y/3)+2$. With this in place, we replace x and y with $f^{-1}(x)$ and x, respectively, concluding that $f^{-1}(x) = \ln(x/3) + 2$, as required.

Exploration Consider the function $f(x) = \ln(9x - 3)$, defined for all $x \leq 1/3$. Draw the graph of f^{-1} from the graph of f. Then, algebraically find f^{-1} to confirm your work.

We also observe that, as suggested by the last exploration exercise above, the strategy illustrated here also allows us to find the graph of the original one-to-one function f, if we're given the graph of the inverse function f^{-1} instead. The reason for this is that the inverse of f^{-1} is $(f^{-1})^{-1} = f$, by design.

Inverses of Compositions

Lastly, there's a general phenomenon worth mentioning, which sometimes makes things simpler. To understand it, let's go back to the underpinning idea of what is the inverse function f^{-1}'s job: to undo what f does. So, if we have two one-to-one functions f and g for which the composition $g \circ f$ makes sense, then $g \circ f$ should also be one-to-one. In other words, the composition of two "reversible" processes must also be "reversible". The next question, then, is what should be the inverse of the full composition $g \circ f$. Since $g \circ f$ first applies f, and then g, reversing it should be done on the opposite order of things, first applying g^{-1}, and then f^{-1}.

> **Theorem:** The composition of two one-to-one functions f and g, whenever it makes sense, is also one-to-one, and the relation
>
> $$(g \circ f)^{-1} = f^{-1} \circ g^{-1}$$
>
> holds. More generally, the composition of any number of one-to-one functions is one-to-one, and the inverse of the composition is the composition of the inverses, *but in the reverse order.*

In practice, this result can be used to quickly find formulas for inverse function, provided one is able to express (or reverse engineer) the given function as a composition of simpler functions whose inverses are known. Properly applied, this also allows us to bypass the verification that the given function is one-to-one (again, if the simpler functions are more well-known to be one-to-one).

Let's revisit the last example above as an application of this:

Example 8. *Consider again the function given by $h(x) = e^{\sqrt{x-3}}$, for all $x \geq 3$, on Example 3. What does h do to a number x? First, it subtracts 3. Secondly, it takes the square root. Lastly, it exponentiates it. Writing it mathematically, we have that $h = h_1 \circ h_2 \circ h_3$, where*

$$h_1(x) = e^x, \quad h_2(x) = \sqrt{x}, \quad \textit{and} \quad h_3(x) = x - 3.$$

All of those functions are one-to-one, with inverses given by

$$h_1^{-1}(x) = \ln x, \quad h_2^{-1}(x) = x^2, \quad \textit{and} \quad h_3^{-1}(x) = x + 3.$$

Hence

$$\begin{aligned}
h^{-1}(x) &= (h_1 \circ h_2 \circ h_3)^{-1}(x) \\
&= h_3^{-1} \circ h_2^{-1} \circ h_1^{-1}(x) \\
&= h_3^{-1} \circ h_2^{-1}(\ln x) \\
&= h_3^{-1}((\ln x)^2) \\
&= (\ln x)^2 + 3,
\end{aligned}$$

which agrees with what was obtained before.

10.2 Logarithms

Learning Objectives

- Reviewing the definition of the logarithm and its basic properties.
- Learning new properties of the logarithm and their relationship to the exponential function.
- Solving equations involving logarithms and exponential functions.

10.2.1 Definition of Logarithms

Motivating Questions

- How is the base-10 logarithm defined?
- What is the "natural logarithm" and how is it different from the base-10 logarithm?
- How can we solve an equation that involves e to some unknown quantity?

In previous sections, we introduced the idea of an inverse function. The fundamental idea is that f has an inverse function if and only if there exists another function g such that f and g "undo" one another's respective processes. In other words, the process of the function f is reversible to generate a related function g.

More formally, recall that a function $y = f(x)$ (where f goes from a set A to a set B) has an inverse function if and only if there exists another function g going from B to A such that $g(f(x)) = x$ for every x in the domain of f and $f(g(y)) = y$ for every y in the range of f. We know that given a function f, we can use the Horizontal Line Test to determine whether or not f has an inverse function. Finally, whenever a function f has an inverse function, we call its inverse function f^{-1} and know that the two equations $y = f(x)$ and $x = f^{-1}(y)$ say the same thing from different perspectives.

Exploration
Let $P(t)$ be the "powers of 10" function, which is given by $P(t) = 10^t$.

a. Complete the following table to generate certain values of P.

t	-3	-2	-1	0	1	2	3
$y = P(t) = 10^t$							

b. Why does P have an inverse function?

c. Since P has an inverse function, we know there exists some other function, say L, such that writing "$y = P(t)$" says the exact same thing as writing "$t = L(y)$". In words, where P produces the result of raising 10 to a given power, the function L reverses this process and instead tells us the power to which we need to raise 10, given a desired result. Complete the table to generate a collection of values of L.

y	10^{-3}	10^{-2}	10^{-1}	10^0	10^1	10^2	10^3
$L(y)$							

d. What are the domain and range of the function P? What are the domain and range of the function L?

The base-10 logarithm

The powers-of-10 function $P(t) = 10^t$ is an exponential function with base $b = 10 > 1$. As such, P is always increasing, and thus its graph passes the Horizontal Line Test, so P has an inverse function. We therefore know there exists some other function, L, such that writing $y = P(t)$ is equivalent to writing $t = L(y)$. For instance, we know that $P(2) = 100$ and $P(-3) = \frac{1}{1000}$, so it's equivalent to say that $L(100) = 2$ and $L\left(\frac{1}{1000}\right) = -3$. This new function L we call the *base* 10 *logarithm*, which is formally defined as follows.

Given a positive real number y, the *base*-10 *logarithm of* y is the power to which we raise 10 to get y. We use the notation "$\log_{10}(y)$" to denote the base-10 logarithm of y.

The base-10 logarithm is therefore the inverse of the powers of 10 function. Whereas $P(t) = 10^t$ takes an input whose value is an exponent and produces the result of taking 10 to that power, the base-10 logarithm takes an input number we view as a power of 10 and produces the corresponding exponent such that 10 to that exponent is the input number.

In the notation of logarithms, we can now update our earlier observations with the functions P and L and see how exponential equations can be written in two equivalent ways. For instance,

$$10^2 = 100 \text{ and } \log_{10}(100) = 2 \tag{1}$$

each say the same thing from two different perspectives. The first says 100 is 10 to the power 2 , while the second says 2 is the power to which we raise 10 to get 100. Similarly,

$$10^{-3} = \frac{1}{1000} \text{ and } \log_{10}\left(\frac{1}{1000}\right) = -3. \tag{2}$$

If we rearrange the statements of the facts, we can see yet another important relationship between the powers of 10 and base-10 logarithm function. Noting that $\log_{10}(100) = 2$ and $100 = 10^2$ are equivalent statements, and substituting the former equation into the latter shows, we see that

$$\log_{10}(10^2) = 2. \tag{3}$$

In words, the equation says that "the power to which we raise 10 to get 10^2, is 2". That is, the base-10 logarithm function undoes the work of the powers of 10 function.

In a similar way, we can observe that by replacing -3 with $\log_{10}\left(\frac{1}{1000}\right)$ we have

$$10^{\log_{10}\left(\frac{1}{1000}\right)} = \frac{1}{1000}. \tag{4}$$

In words, this says that "when 10 is raised to the power to which we raise 10 in order to get $\frac{1}{1000}$, we get $\frac{1}{1000}$".

We summarize the key relationships between the powers-of-10 function and its inverse, the base-10 logarithm function, more generally as follows. Let $P(t) = 10^t$ and $L(y) = \log_{10}(y)$.

- The domain of P is the set of all real numbers and the range of P is the set of all positive real numbers.
- The domain of L is the set of all positive real numbers and the range of L is the set of all real numbers.
- For any real number t, $\log_{10}(10^t) = t$. That is, $L(P(t)) = t$.
- For any positive real number y, $10^{\log_{10}(y)} = y$. That is, $P(L(y)) = y$.
- $10^0 = 1$ and $\log_{10}(1) = 0$.

The base-10 logarithm function is like the sine or cosine function in this way: for certain special values, it's easy to know by heart the value of the logarithm function. While for sine and cosine the familiar points come from specially placed points on the unit circle, for the base-10 logarithm function, the familiar points come from powers of 10. In addition, like sine and cosine, for all other input values, (a) calculus ultimately determines the value of the base-10 logarithm function at other values, and (b) we use computational technology in order to compute these values. For most computational devices, the command $\log(y)$ produces the result of the base-10 logarithm of y.

It's important to note that the logarithm function produces exact values. For instance, if we want to solve the equation $10^t = 5$, then it follows that $t = \log_{10}(5)$ is the exact solution to the equation. Like $\sqrt{2}$ or $\cos(1)$, $\log_{10}(5)$ is a number that is an exact value. A computational device can give us a decimal approximation, and we normally want to distinguish between the exact value and the approximate one. For the three different numbers here, $\sqrt{2} \approx 1.414$, $\cos(1) \approx 0.540$, and $\log_{10}(5) \approx 0.699$.

Exploration

For each of the following equations, determine the exact value of the unknown variable. If the exact value involves a logarithm, use a computational device to also report an approximate value. For instance, if the exact value is $y = \log_{10}(2)$, you can also note that $y \approx 0.301$.

a. $10^t = 0.00001$

b. $\log_{10}(1000000) = t$

c. $10^t = 37$

d. $\log_{10}(y) = 1.375$

e. $10^t = 0.04$

f. $3 \cdot 10^t + 11 = 147$

g. $2\log_{10}(t) + 5 = 1$

The natural logarithm

The base-10 logarithm is a good starting point for understanding how logarithmic functions work because powers of 10 are easy to mentally compute. We could similarly consider the powers of 2 or powers of 3 function and develop a corresponding logarithm of base 2 or 3. But rather than have a whole collection of different logarithm functions, in the same way that we now use the function e^t and appropriate scaling to represent any exponential function, we develop a single logarithm function that we can use to represent any other logarithmic function through scaling. In correspondence with the natural exponential function, e^t, we now develop its inverse function, and call this inverse function the *natural logarithm*.

Given a positive real number y, the *natural logarithm of* y is the power to which we raise e to get y. We use the notation "$\ln(y)$" to denote the natural logarithm of y.

We can think of the natural logarithm, $\ln(y)$, as the "base-e logarithm". For instance,

$$\ln(e^2) = 2$$

and

$$e^{\ln(-1)} = -1.$$

The former equation is true because "the power to which we raise e to get e^2 is 2"; the latter equation is true since "when we raise e to the power to which we raise e to get -1, we get -1".

Exploration Let $E(t) = e^t$ and $N(y) = \ln(y)$ be the natural exponential function and the natural logarithm function, respectively.

a. What are the domain and range of E?

b. What are the domain and range of N?

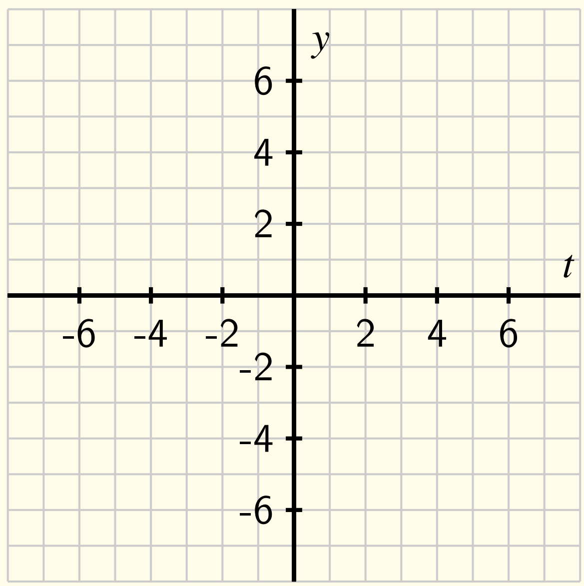
c. What can you say about $\ln(e^t)$ for every real number t?

d. What can you say about $e^{\ln(y)}$ for every positive real number y?

e. Complete the following tables with both exact and approximate of E and N. Then, plot the corresponding ordered pairs from each table on the axes below and connect the points in an intuitive way. When you plot the ordered pairs on the axes, in both cases view the first line of the table as generating values on the horizontal axis and the second line of the table as producing values on the vertical axis label each ordered pair you plot appropriately.

t	-2	-1	0	1	2
$E(t) = e^t$	$e^{-2} \approx 0.135$				

y	e^{-2}	e^{-1}	1	e^1	e^2
$N(y) = \ln(y)$	-2				



$\log_b$ or logarithms in general

In the previous sections, we looked at two specific (and the most common) types of logarithms, base-10 and natural log. In order to fully discuss logarithms, we need to talk about logarithms in general with any base. Let $b > 1$. Because the function $y = f(t) = b^t$ has an inverse function, it makes sense to define its inverse like we did when $b = 10$ or $b = e$. The base-b logarithm, denoted $\log_b(y)$ is defined to be the power to which we raise b to get y.

$$\log_b(y) = t$$

$$y = f(t) = b^t$$

Example 9. *Evaluate the following base-b logarithms.*

(a) $\log_2(8)$

(b) $\log_5(25)$

Explanation

(a)

$$\begin{aligned}\log_2(8) &= \log_2(2^3)\\ \log_2(8) &= 3\end{aligned}$$

(b)

$$\begin{aligned}\log_5(25) &= \log_5(5^2)\\ \log_5(25) &= 2\end{aligned}$$

Revisiting $f(t) = b^t$

In earlier sections, we saw that that function $f(t) = b^t$ plays a key role in modeling exponential growth and decay, and that the value of b not only determines whether the function models growth ($b > 1$) or decay ($0 < b < 1$), but also how fast the growth or decay occurs. Furthermore, once we introduced the natural base e, we realized that we could write every exponential function of form $f(t) = b^t$ as a horizontal scaling of the function $E(t) = e^t$ by writing

$$f(t) = b^t = e^{kt} = E(kt)$$

for some value k. Our development of the natural logarithm function in the current section enables us to now determine k exactly.

Example 10. *Determine the exact value of k for which $f(t) = 3^t = e^{kt}$.*

Explanation Since we want $3^t = e^{kt}$ to hold for every value of t and $e^{kt} = (e^k)^t$, we need to have $3^t = (e^k)^t$, and thus $3 = e^k$. Therefore, k is the power to which we raise e to get 3, which by definition means that $k = \ln(3)$.

In modeling important phenomena using exponential functions, we will frequently encounter equations where the variable is in the exponent, like in the example where we had to solve $e^k = 3$. It is in this context where logarithms find one of their most powerful applications.

Example 11. *Solve each of the following equations for the exact value of the unknown variable. If there is no solution to the equation, explain why not.*

a. $e^t = \frac{1}{10}$

b. $5e^t = 7$

c. $\ln(t) = -\frac{1}{3}$

d. $e^{1-3t} = 4$

e. $2\ln(t) + 1 = 4$

f. $4 - 3e^{2t} = 2$

g. $4 + 3e^{2t} = 2$

h. $\ln(5 - 6t) = -2$

Explanation

a.

$$\begin{aligned} e^t &= \frac{1}{10} \\ \ln(e^t) &= \ln\left(\frac{1}{10}\right) \\ t &= \ln\left(\frac{1}{10}\right) \end{aligned}$$

b.

$$\begin{aligned} 5e^t &= 7 \\ e^t &= \frac{7}{5} \\ \ln\left(e^t\right) &= \ln\left(\frac{7}{5}\right) \\ t &= \ln\left(\frac{7}{5}\right) \end{aligned}$$

c.

$$\begin{aligned} \ln(t) &= -\frac{1}{3} \\ e^{\ln(t)} &= e^{-\frac{1}{3}} \\ t &= e^{-\frac{1}{3}} \end{aligned}$$

d.

$$\begin{aligned}
e^{1-3t} &= 4 \\
\ln(e^{1-3t}) &= \ln(4) \\
1-3t &= \ln(4) \\
-3t &= \ln(4)-1 \\
t &= \frac{\ln(4)-1}{-3}
\end{aligned}$$

e.

$$\begin{aligned}
2\ln(t)+1 &= 4 \\
2\ln(t) &= 3 \\
\ln(t) &= \frac{3}{2} \\
e^{ln(t)} &= e^{\frac{3}{2}} \\
t &= e^{\frac{3}{2}}
\end{aligned}$$

f.

$$\begin{aligned}
4-3e^{2t} &= 2 \\
-3e^{2t} &= -2 \\
e^{2t} &= \frac{2}{3} \\
\ln(e^{2t}) &= \ln\left(\frac{2}{3}\right) \\
2t &= \ln\left(\frac{2}{3}\right) \\
t &= \frac{\ln\left(\frac{2}{3}\right)}{2}
\end{aligned}$$

g.

$$\begin{aligned}
4+3e^{2t} &= 2 \\
3e^{2t} &= -2 \\
e^{2t} &= \frac{-2}{3}
\end{aligned}$$

No solution, because $\frac{-2}{3}$ is outside of the range of e^{2t}

h.

$$\begin{aligned} \ln(5-6t) &= -2 \\ \ln(5-6t) &= -2 \\ e^{\ln(5-6t)} &= e^{-2} \\ 5-6t &= e^{-2} \\ t &= \frac{e^{-2}-5}{-6} \end{aligned}$$

Summary

(a) The base-10 logarithm of y, denoted $\log_{10}(y)$ is defined to be the power to which we raise 10 to get y. For instance, $\log_{10}(1000) = 3$, since $10^3 = 1000$. The function $L(y) = \log_{10}(y)$ is thus the inverse of the powers-of-10 function, $P(t) = 10^t$.

(b) The natural logarithm $N(y) = \ln(y)$ differs from the base-10 logarithm in that it is the logarithm with base e instead of 10, and thus $\ln(y)$ is the power to which we raise e to get y. The function $N(y) = \ln(y)$ is the inverse of the natural exponential function $E(t) = e^t$.

(c) The natural logarithm often enables us solve an equation that involves e to some unknown quantity. For instance, to solve $2e^{3t-4} + 5 = 13$, we can first solve for e^{3t-4} by subtracting 5 from each side and dividing by 2 to get

$$e^{3t-4} = 4.$$

This last equation says "e to some power is 4". We know that it is equivalent to say

$$\ln(4) = 3t - 4.$$

Since $\ln(4)$ is a number, we can solve this most recent linear equation for t. In particular, $3t = 4 + \ln(4)$, so

$$t = \frac{1}{3}(4 + \ln(4)).$$

10.2.2 Properties of Logarithms

The key to understanding logarithms is through their relationship with exponential functions. Since $f(x) = \log_b(x)$ is the inverse function to $g(x) = b^x$, many of the properties of exponential functions can be translated into properties of logarithms. In this section, we'll try to discover these and find several other interesting properties of logarithms along the way.

We highlight several important principles from our previous discussion of inverse functions:

- A function f has an inverse function if and only if there exists a function g that undoes the work of f: that is, there is some function g for which $g(f(x)) = x$ for each x in the domain of f, and $f(g(y)) = y$ for each y in the range of f. We call g the inverse of f, and write $g = f^{-1}$.
- When f has an inverse, we know that writing "$y = f(t)$" and "$t = f^{-1}(y)$" are two different perspectives on the same statement.

Inverse Property of Logarithms

An important fact to recall is that the range of the function $g(x) = b^x$ is $(0, \infty)$, the set of all positive real numbers. This means that any positive real number can be written as the output of the exponential function with base b. Let's fix $b = 10$ and try to write the number 17 as an output of the function $g(x) = 10^x$. If 17 is an output of g, then $17 = 10^x$ for some real number x. Taking log of both sides of this equation, we find that $\log(17) = \log(10^x)$.

Now we use the most important property of logarithms: the logarithms and exponential of the same base are inverses. With our base being set to 10, this tells us that $\log(10^x) = x$. It is important to remember that even though our notation for the exponential function writes its input as an exponent, and not by wrapping it in parenthesis, x is the input to the exponential function in 10^x.

Returning to our original quest to write 17 as an output of the exponential with base 10, we use the inverse property of logarithms to say that $\log(17) = x$, and therefore,

$$17 = 10^{\log(17)}.$$

Another way to see this is by using the fact that the function $g(x) = 10^x$ is the inverse of $f(x) = \log(x)$.

There was nothing special about 10 and 17 in what we just showed, so this allows us to arrive at a very general way to write positive real numbers as exponentials.

> [Inverse Property of Logarithms] If x and b are positive real numbers and $b \neq 1$, we can write $x = b^{\log_b(x)}$.

Another way to understand this is to remember the definition of the logarithm. $\log_b(x)$ is precisely the power to which you have to raise b in order to obtain x.

Finally, this can also be viewed as a statement about inverse functions. If $f(x) = \log_b(x)$, then $f^{-1}(x) = b^x$. In this setup, the statement $f^{-1}(f(x)) = x$ becomes $b^{\log_b(x)} = x$.

Product Property of Logarithms

You might think that the method in the previous section of writing positive real numbers as exponentials unnecessarily complicates things, but we can use it to adapt properties of exponents into properties of logarithms.

Recall that multiplying exponential expressions of the same base results in another exponential expression whose exponent is the sum of the two original exponents: in symbols,

$$b^u \cdot b^v = b^{u+v}$$

for any real numbers u and v.

Let's see if we can use this fact, again restricting our attention to $b = 10$. Since 2 and 3 are positive real numbers, we can write $2 = 10^{\log(2)}$ and $3 = 10^{\log(3)}$. Then,

$$\log(6) = \log(2\cdot 3) = \log(10^{\log(2)}\cdot 10^{\log(3)}) = \log(10^{\log(2)+\log(3)}) = \log(2)+\log(3).$$

Notice again how we used the fact that the logarithm and exponential with base 10 are inverses! There's nothing special about 2 and 3, so for any positive real numbers x and y, $\log(xy) = \log(x) + \log(y)$. Even more, there's nothing special about base 10, allowing us to come up with a general rule.

> [Product Property of Logarithms] If x, y, and b are positive real numbers with $b \neq 1$, then $\log_b(xy) = \log_b(x) + \log_b(y)$.

Quotient Property of Logarithms

Now that we've dealt with multiplication, it makes sense to deal with division. If x and y are positive real numbers, we can think about the quotient x/y as a product: $x \cdot (1/y)$. What's more, we can write $1/y$ as a power of y: $1/y = y^{-1}$. Using the product property of logarithms from the previous section, we can conclude that $\log_b(x/y) = \log_b(x) + \log_b(y^{-1})$.

It would be really nice if there was a nice relationship between $\log_b(y^{-1})$ and $\log_b(y)$. Indeed, there is! Using the definition of the logarithm, $\log_b(y)$ is the power to which you have to raise b to obtain y, but to obtain y^{-1}, we can use the negative power. As an example, note that $\log(1000) = \log(10^3) = 3$, but $\log\left(\frac{1}{1000}\right) = \log(10^{-3}) = -3$. In general,

$$\log_b(y^{-1}) = -\log_b(y).$$

Combining this with our previous work, we obtain the following quotient property of logarithms.

[Quotient Property of Logarithms] If x, y, and b are positive real numbers with $b \neq 1$, then $\log_b\left(\frac{x}{y}\right) = \log_b(x) - \log_b(y)$.

Power Property of Logarithms

Something else you might remember about exponents is that repeated exponentiation is the same thing as multiplying exponents. For example, $(7^3)^2 = 7^{(3\cdot 2)} = 7^6$ (check this yourself!). In words, this says that raising 7 to the 3rd power, then raising that result to the 2nd power is the same as raising 7 to the $3 \cdot 2 = 6$th power. Since $7^3 = 343$, $\log_7(343) = 3$. So in the language of logarithms, the above says that $\log_7(343^2) = 2 \cdot \log_7(343)$.

In general,

$$(b^u)^v = b^{u\cdot v}$$

for all real numbers b, u, and v.

Let's see if this fact has any consequences for logarithms! Recall that for positive b and x, $\log_b(x^u)$ is the power to which we need to raise b in order to obtain x^u. However, another way to obtain x^u is to raise b to the power $\log_b(x)$ (yielding x) and then raise that result to the power u. Since repeated exponentiation is the same thing as multiplying exponents, this amounts to raising b to the power $u \log_b(x)$. In symbols, we've shown that

[Power Property of Logarithms] If x and b are positive real numbers, and u is a real number, then $\log_b(x^u) = u \log_b(x)$.

In essence, taking the logarithm of a power of x is the same thing as multiplying the logarithm of x by the power. An intuitive way to think about this property is in the context of the product property from above. Since logarithms "turn multiplication into addition" and exponentiation is repeated multiplication, logarithms should "turn exponentiation into repeated addition", that is, multiplication. As an example, notice that

$$\begin{aligned}\log_2(3^4) &= \log_2(3^2 \cdot 3^2)\\ &= \log_2(3^2) + \log_2(3^2)\\ &= \log_2(3 \cdot 3) + \log_2(3 \cdot 3)\\ &= \log_2(3) + \log_2(3) + \log_2(3) + \log_2(3)\\ &= 4\log_2(3).\end{aligned}$$

The above calculation uses the product property to arrive at the same conclusion as the power property.

Change-of-Base Formula

One important thing to recognize is that logarithms can have any positive number (except 1) as their base. Sometimes, when doing calculations, it may be preferable to use one base over another. The good news is that any logarithm can be computed using this preferred base.

As an example, consider the quantity $\log_3(7)$. Many calculators are unable to directly calculate logarithms with a base other than e or 10, so let's convert this into a natural logarithm (logarithm with base e). Rewriting 7 as $3^{\log_3(7)}$ using the inverse property of logarithms, we see that $\ln(7) = \ln(3^{\log_3(7)})$. Now, using the power property of logarithms, we see that $\ln(3^{\log_3(7)}) = \log_3(7) \cdot \ln(3)$. This gives us the equality $\ln(7) = \log_3(7) \cdot \ln(3)$, so dividing both sides by $\ln(3)$, $\log_3(7) = \dfrac{\ln(7)}{\ln(3)}$. If you have an aversion to $\log_3$ and a fondness for ln, then this allows you to calculate $\ln(7)/\ln(3)$ instead of $\log_3(7)$.

Of course, there's nothing special about 3, 7, and the natural logarithm. In general, we have the following formula.

> [Change-of-Base Formula] If a, b, and x are positive real numbers with $a \neq 1$ and $b \neq 1$, then $\log_b(x) = \dfrac{\log_a(x)}{\log_a(b)}$.

In words, this formula says that instead of using $\log_b$ to calculate $\log_b(x)$, we can make two calculations with $\log_a$ and divide, which will yield the same result.

Logarithm Properties in Action

Example 12. *Say* $\log_b(3)$ *is approximately* 0.388 *and* $\log_b(2)$ *is approximately* 0.245. *Using the properties of logarithms, approximate* $\log_b(108)$.

Explanation To use the properties of logarithms, we can make use of the factorization of 108: $108 = 4 \cdot 27 = 2^2 \cdot 3^3$. Using the product property of logarithms, $\log_b(108) = \log_b(2^2 \cdot 3^3) = \log_b(2^2) + \log_b(3^3)$. Now we can apply the product property of logarithms to simplify each term. By substituting in our approximations, we conclude that $\log_b(2^2) + \log_b(3^3) = 2\log_b(2) + 3\log_b(3) \approx 2(0.388) + 3(0.245) = 1.511$.

Therefore, $\log_b(108)$ is approximately 1.511.

Example 13. *Use the properties of logarithms to write* $5\log_5(u) - \dfrac{1}{3}\log_5(v) + \log_5(v)$ *as a single logarithm with coefficient 1. Simplify as much as possible.*

Explanation We can first use the power property to rewrite $5\log_5(u) = \log_5(u^5)$ and $\dfrac{1}{3}\log_5(v) = \log_5(v^{1/3})$. Then we can use the product and quotient properties to combine the terms of the expression.

$$\begin{aligned} \log_5(u^5) - \log_5(v^{1/3}) + \log_5(v) &= \log_5\left(\frac{u^5}{v^{1/3}}\right) + \log_5(v) \\ &= \log_5\left(\frac{u^5 v}{v^{1/3}}\right) \\ &= \log_5(u^5 v^{2/3}) \end{aligned}$$

There are other ways to approach this problem as well. See if you can find another way to do this problem!

Example 14. *Use the properties of logarithms to rewrite*

$$\log\left(\frac{(x^2-1)\sqrt{yz}}{w-7}\right)$$

as a sum and difference of logarithms. Expand as much as possible.

Explanation In the previous example, we wanted to combine logarithms, while in this one, we want to pull them apart. The first way we can do this is by recognizing that the argument of the logarithm is a fraction. This provides us with an opportunity to use the quotient property:

$$\log\left(\frac{(x^2-1)\sqrt{yz}}{w-7}\right) = \log((x^2-1)\sqrt{yz}) - \log(w-7).$$

Now our strategy is to look at each log term and see if we can further expand it into more logarithms. In the first term, note that we have a product, so it can expand using the product property:

$$\log((x^2-1)\sqrt{yz}) - \log(w-7) = \log(x^2-1) + \log(\sqrt{yz}) - \log(w-7).$$

Note that $\log(w-7)$ does not contain a power, product, or quotient, so it cannot be expanded. However, now we have $\log(x^2-1)$ as a term in our expression. Since x^2-1 factors as $(x-1)(x+1)$, we technically have a product in the argument of that term. We can therefore expand further using the product property:

$$\log(x^2-1)+\log(\sqrt{yz})-\log(w-7) = \log(x-1)+\log(x+1)+\log(\sqrt{yz})-\log(w-7).$$

Note that now, $\log(x-1)$ and $\log(x+1)$ cannot be expanded further. That leaves $\log(\sqrt{yz})$. Recall that taking the square root of a quantity is the same as raising it to the $1/2$ power. Therefore, $\log(\sqrt{yz}) = \log((yz)^{1/2})$, and we can use the power property to expand, bringing down the exponent:

$$\log(x-1) + \log(x+1) + \frac{1}{2}\log(yz) - \log(w-7).$$

However, this now creates a log term with a product in it. We can therefore expand our expression by replacing $\log(yz)$ with $\log(y) + \log(z)$, being careful to use parentheses so that the whole quantity is multiplied by $\frac{1}{2}$:

$$\log(x-1) + \log(x+1) + \frac{1}{2}(\log(y) + \log(z)) - \log(w-7).$$

By distributing the $\frac{1}{2}$, we've arrived at a state where nothing can be further expanded:

$$\log(x-1) + \log(x+1) + \frac{1}{2}\log(y) + \frac{1}{2}\log(z) - \log(w-7).$$

In the previous two examples, we illustrated a trade-off that occurs. When we combine logarithms into one, we often find that their arguments become messy. However, in order to simplify the arguments of logarithms, we need to separate them out into sums and differences of logarithms. We can have simple arguments, but multiple logs, or we can have one log, but complex arguments.

Summary

If x, y, and b are positive real numbers with $b \neq 1$,

- $\log_b(xy) = \log_b(x) + \log_b(y)$
- $\log_b\left(\frac{x}{y}\right) = \log_b(x) - \log_b(y)$
- $\log_b(x^u) = u\log_b(x)$ for all real numbers u.
- $\log_b(x) = \frac{\log_a(x)}{\log_a(b)}$ for all positive real numbers $a \neq 1$.

10.2.3 Solving Logarithmic Equations

Using Inverses to Solve Equations

Now that we have an understanding of the properties of logarithms, we're prepared to solve equations involving logarithms and exponential functions. Before we do that, however, let's discuss a method of solving equations that you're already familiar with.

Consider the equation

$$x + 2 = 7.$$

You may have already found that the solution is $x = 5$, but let's think about the process of finding the solution.

Our general plan when solving equations is to isolate the variable we're solving for. In this case, we'd like to isolate x by itself on one side of the equation. However, x is not by itself: it's contained in a sum! Naturally, to undo the addition of 2, we subtract 2 from both sides and obtain $x = 5$. The key here is that it was stuck in some operation, and in order to "access" the x, we had to undo that operation.

We can also view this process in the context of functions. Let f be a function defined by $f(x) = x+2$. Then, our equation becomes $f(x) = 7$. In the language of functions, "undoing" f corresponds to applying the inverse function f^{-1}. In this case, $f^{-1}(x) = x - 2$. By applying f^{-1} to both sides of our original equation, we find that

$$\begin{aligned} f(x) &= 7 \\ f^{-1}(f(x)) &= f^{-1}(7) \\ x &= 7 - 2 \\ x &= 5. \end{aligned}$$

This may seem like an awfully strange way to subtract 2, but it has the benefit of being usable for any invertible function.

For example, say we want to solve the equation $\dfrac{x+1}{x} = 4$. If we define a function g by $g(x) = \dfrac{x+1}{x}$, our equation becomes $g(x) = 4$. We can find that the inverse is defined by $g^{-1}(x) = \dfrac{1}{x-1}$. Therefore,

$$\begin{aligned} g(x) &= 4 \\ g^{-1}(g(x)) &= g^{-1}(4) \\ x &= \frac{1}{4-1} \\ x &= \frac{1}{3} \end{aligned}$$

yields the solution to the equation.

Since we had to do quite a bit of work to find the equation for g^{-1} in the above scenario, this method may not be useful in that context. However, there are many functions for which we already know the inverse! For example, the inverse function of $h(x) = x^3$ is $h^{-1}(x) = \sqrt[3]{x}$. Therefore, if we want to solve $h(x) = 343$, we can apply h^{-1} on both sides to find that

$$\begin{aligned} h^{-1}(h(x)) &= h^{-1}(343) \\ x &= 7. \end{aligned}$$

Another important example of inverse functions that we know instantly comes from logarithms! If $f(x) = b^x$, then we know from our previous discussion that $f^{-1}(x) = \log_b(x)$. This is the definition of the logarithm, and looking at solving equations from the point of view of applying inverses is key to solving logarithmic and exponential equations.

For example, if we want to solve the equation $\log(2t-5) = 7$, we can define $f(x) = \log(x)$, so $f^{-1}(x) = 10^x$. This means our equation is $f(2t-5) = 7$. Therefore,

$$\begin{aligned} f(2t-5) &= 7 \\ f^{-1}(f(2t-5)) &= f^{-1}(7) \\ 2t-5 &= 10^7 \\ 2t &= 1000000 + 5 \\ t &= \frac{1000005}{2} \end{aligned}$$

yields the solution to the equation.

Exponential Equations

Example 15. *Solve the equation* $-4^{x-1} + 6 = 3$.

Explanation Notice that the variable we're solving for in this equation is located in the exponent of the exponential expression 4^{x-1}. Whenever this occurs, we call the equation an *exponential equation*.

If we define a function by $f(x) = 4^x$, then our equation becomes $-f(x-1)+6 = 2$. In order to solve this equation, we must use the inverse function: $f^{-1}(x) = \log_4(x)$. However, before we can apply this to both sides of the equation, we need to isolate $f(x-1)$ like so:

$$\begin{aligned} -f(x-1)+6 &= 3 \\ -f(x-1) &= -3 \\ f(x-1) &= 3. \end{aligned}$$

Now we can take f^{-1} of both sides of the equation and obtain

$$\begin{aligned} f^{-1}(f(x-1)) &= f^{-1}(3) \\ x-1 &= \log_4(3) \\ x &= \log_4(3)+1. \end{aligned}$$

Therefore, the solution to the equation $-4^x + 6 = 3$ is $\log_4(3) + 1$. Your first instinct might be that this doesn't seem like a solution, since there's still a logarithm in our expression! However, there is no nicer way to write the number $\log_4(3)$. If you were to plug this into a calculator, you would get a decimal approximation to the value of $\log_4(3)$, but the decimal approximation loses some information, so the exact value of the solution is $\log_4(3) + 1$.

The process of writing out a function $f(x) = 4^x$ and then taking inverses may seem unnecessary, and indeed, there's no need to actually be so explicit when doing your own calculations. For example, the work

$$\begin{aligned} -4^{x-1}+6 &= 3 \\ -4^{x-1} &= -3 \\ 4^{x-1} &= 3 \\ \log_4(4^{x-1}) &= \log_4(3) \\ x-1 &= \log_4(3) \\ x &= \log_4(3)+1 \end{aligned}$$

would be perfectly sufficient, and is usually how work for this kind of problem would be written. However, it must be emphasized that solving exponential equations involves more than just the basic operations of addition, subtraction, multiplication, and division. We now need to involve the process of taking logarithms of both sides of the equation.

Example 16. *Solve the equation* $3^x = 5^{2-x}$.

Explanation At first glance, this problem seems fundamentally different from the previous example. Instead of dealing with an exponential function with one base, we're dealing with two different bases: 3 and 5.

However, recall from the previous section that any positive real number can be written as a power of any number we want. In this case, 3 can be written as $3 = 5^{\log_5(3)}$. Therefore, our equation becomes

$$5^{x\log_5(3)} = 5^{2-x}.$$

If $f(x) = 5^x$, then our equation has become $f(x\log_5(3)) = f(2-x)$. To solve this, we can take $f^{-1} = \log_5$ of both sides of the equation and do some more algebra to isolate x.

$$\begin{aligned}\log_5(5^{x\log_5(3)}) &= \log_5(5^{2-x})\\ x\log_5(3) &= 2-x\\ x\log_5(3)+x &= 2\\ x(\log_5(3)+1) &= 2\\ x &= \frac{2}{\log_5(3)+1},\end{aligned}$$

which is our solution.

A more conventional way to solve this equation is to take $\log_5$ of both sides at the very beginning:

$$\begin{aligned}\log_5(3^x) &= \log_5(5^{2-x})\\ \log_5(3^x) &= 2-x,\end{aligned}$$

using the fact that logs and exponentials are inverses. Now we can apply the power property of logarithms to simplify the left-hand side:

$$x\log_5(3) = 2-x.$$

From here, we can isolate all the x terms on one side and factor:

$$\begin{aligned}x\log_5(3) &= 2-x\\ x\log_5(3)+x &= 2\\ x(\log_5(3)+1) &= 2\\ x &= \frac{2}{\log_5(3)+1}.\end{aligned}$$

In fact, we can take $\log_b$ from the beginning for any positive $b \neq 1$ and obtain the same answer. However, it may look different. The key is understanding that the change-of-base formula ensures that the answers are in fact equal. Try doing this problem by taking ln on both sides to start, and check that your answer is the same as ours.

Logarithmic Equations

We now turn our attention to equations involving logarithms. These are similar to equations involving exponentials, but now our function we want to isolate is a logarithm. There's also a bit of a catch. Solutions that we find must be in the domain of all logarithm functions that were in the original equation, so there's some checking to be done at the end. Let's see what we mean.

Example 17. *Solve the equation* $5\log_2(x+3) = -2$.

Explanation To start off, divide both sides by 5 to isolate the $\log_2(x+3)$ on the left-hand side.

$$\log_2(x+3) = -\frac{2}{5}$$

Next, we apply the inverse of $\log_2$ to both sides of the equation, obtaining

$$\begin{aligned} 2^{\log_2(x+3)} &= 2^{-2/5} \\ x+3 &= \frac{1}{\sqrt[5]{4}} \\ x &= \frac{1}{\sqrt[5]{4}} - 3. \end{aligned}$$

Since logarithms are not always defined (their domain is only positive real numbers), we should check that plugging in our solution for x does not result in any part of our original equation being undefined. In our case, this amounts to checking that $\log_2(x+3)$ is defined, that is, that $x+3$ is positive. Since $x+3 = \frac{1}{\sqrt[5]{4}} > 0$, our solution is $\frac{1}{\sqrt[5]{4}} - 3$.

Example 18. *Solve the equation* $\log_6(x) = 1 - \log_6(x-1)$.

Explanation At first glance, there appear to be too many functions going on here. However, if we add $\log_6(x-1)$ to both sides of the equation and use the product property of logarithms, we obtain:

$$\begin{aligned} \log_6(x) + \log_6(x-1) &= 1 \\ \log_6(x(x-1)) &= 1. \end{aligned}$$

Next, we can apply the inverse function of $\log_6$, which is given by $f(x) = 6^x$. Doing so, we see that

$$\begin{aligned} 6^{\log_6(x(x-1))} &= 6^1 \\ x(x-1) &= 6 \\ x^2 - x - 6 &= 0. \end{aligned}$$

This results in a quadratic equation! This is something we know how to solve. By using our preferred method, we find that $x = 3$ or $x = -2$.

We're not done yet, however! We need to check that these x-values don't cause any logarithms in our original equation to be undefined. Note that $\log_6(-2)$ is undefined, since -2 is negative, so $x = -2$ is not a solution to our equation. Since $\log_6(3)$ and $\log_6(3-1)$ are both defined, $x = 3$ is our only solution.

Summary

- When solving exponential equations, our strategy is to isolate a single exponential on one side of the equation, then apply a logarithm to both sides to undo the exponential.
- When solving logarithmic equations, our strategy is to isolate a single logarithm on one side of the equation, then apply an exponential function to both sides to undo the logarithm.
- Since the domain of logarithms is only $(0, \infty)$, we need to check that our solutions do not make our original logarithmic equations undefined.

10.3 Inverse Trigonometric Functions

Learning Objectives

- Inverse Cosine
  - Defining the arccosine function as the inverse of a restriction of the cosine function.
  - Exploring properties of the arccosine function.
  - Evaluating compositions of the arccosine function with the cosine function.
- Other Inverse Trig Functions
  - Defining the other inverse trig functions.
  - Exploring the properties of the other inverse trig functions.
- Applications of Inverse Trig
  - Exploring how inverse trig functions can be used in the context of word problems.
  - Using inverse trig functions to find missing angles.
  - Using inverse trig to solve equations.

10.3.1 Inverse Cosine

Motivating Questions

- Is it possible for a periodic function that fails the Horizontal Line Test to have an inverse?
- For the restricted cosine function, how do we define the corresponding arccosine function?
- What are the key properties of arccosine?

Introduction

In our prior work with inverse functions, we learned several important principles, including

- A function f has an inverse function if and only if there exists a function g that undoes the work of f: that is, there is some function g for which $g(f(x)) = x$ for each x in the domain of f, and $f(g(y)) = y$ for each y in the range of f. We call g the inverse of f, and write $g = f^{-1}$.
- A function f has an inverse function if and only if the graph of f passes the *Horizontal Line Test*.
- When f has an inverse, we know that writing "$y = f(t)$" and "$t = f^{-1}(y)$" are two different perspectives on the same statement.

The trigonometric function $g(t) = \cos(t)$ is periodic, so it fails the horizontal line test. Hence, considering this function on its full domain, it does not have an inverse function. At the same time, it is reasonable to think about changing perspective and viewing angles as outputs in certain restricted settings. For instance, we may want to say both

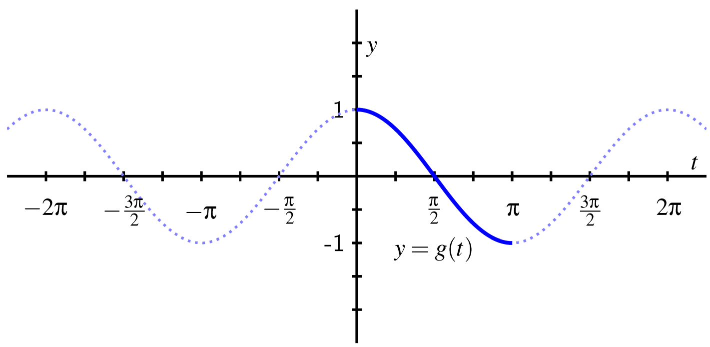
$$\frac{\sqrt{3}}{2} = \cos\left(\frac{\pi}{6}\right) \quad \text{and} \quad \frac{\pi}{6} = \cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$$

depending on the context in which we are considering the relationship between the angle and side length.

It is also helpful to contextualize the importance of finding an angle in terms of a known value of a trigonometric function. Suppose we know the following information about a right triangle: one leg has length 2.5, and the hypotenuse has length 4. If we let θ be the angle adjacent to the side of length 2.5, it follows that $\cos(\theta) = \frac{2.5}{4}$. We naturally want to use the inverse of the cosine function to solve the most recent equation for θ. But the cosine function does not have an inverse function, so how can we address this situation?

While the original trigonometric function $g(t) = \cos(t)$ does not have an inverse function, we can instead consider a restricted version of the function that does. We thus investigate how we can think differently about the trigonometric functions so that we can discuss inverses in a meaningful way.

Consider the plot of the standard cosine function on $\left[-\frac{5\pi}{2}, \frac{5\pi}{2}\right]$ with the portion on $[0, \pi]$ emphasized below.



Exploration Let g be the function whose domain is $0 \leq t \leq \pi$ and whose outputs are determined by the rule $g(t) = \cos(t)$.
The key observation here is that g is defined in terms of the cosine function, but because it has a different domain, it is *not* the cosine function.

a. What is the domain of g?

b. What is the range of g?

c. Does g pass the horizontal line test? Why or why not?

d. Explain why g has an inverse function, g^{-1}, and state the domain and range of g^{-1}.

e. We know that $g\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$. What is the exact value of $g^{-1}\left(\frac{\sqrt{2}}{2}\right)$?

   How about the exact value of $g^{-1}\left(-\frac{\sqrt{2}}{2}\right)$?

f. Determine the exact values of $g^{-1}\left(-\frac{1}{2}\right)$, $g^{-1}\left(\frac{\sqrt{3}}{2}\right)$, $g^{-1}(0)$, and $g^{-1}(-1)$. Use proper notation to label your results.

The Arccosine Function

For the cosine function restricted to the domain $[0, \pi]$ that we considered above, the function is strictly decreasing on its domain and thus passes the Horizontal Line Test. Therefore, this restricted version of the cosine function has an inverse function; we will call this inverse function the *arccosine* function.

Definition Let $y = g(t) = \cos(t)$ be defined on the domain $[0, \pi]$, and observe that g has the range $[-1, 1]$. For any real number y that satisfies $-1 \leq y \leq 1$, the **arccosine of y**, denoted

$$\arccos(y)$$

is the angle t satisfying $0 \leq t \leq \pi$ such that $\cos(t) = y$. Note that we use $t = \cos^{-1}(y)$ interchangeably with with $t = \arccos(y)$.

In particular, we note that the output of the arccosine function is an angle. Recall that in the context of the unit circle, an angle measured in radians and the corresponding arc length along the unit circle are numerically equal. This is the origin of the "arc" in "arccosine": given a value $-1 \leq y \leq 1$, the arccosine function produces the corresponding *arc* (measured counterclockwise from $(1, 0)$) such that the cosine of that arc is y.

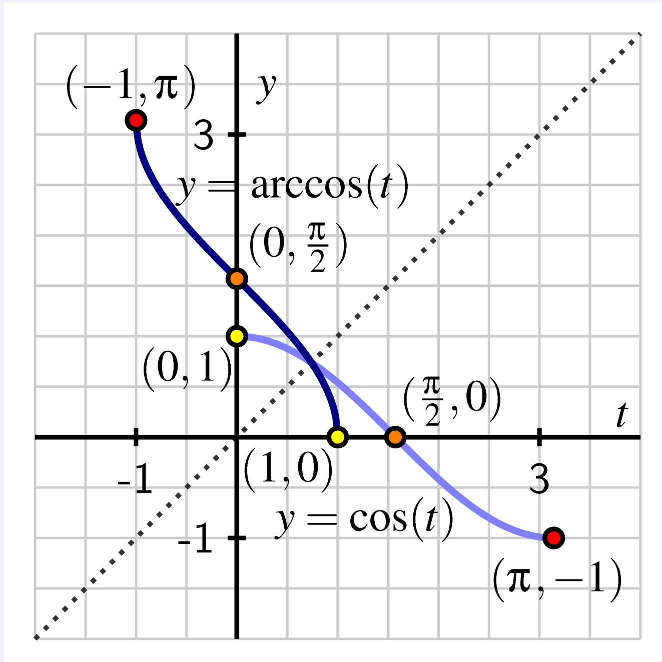
For any function with an inverse function, the inverse function reverses the process of the original function. Thus, given $y = \cos(t)$, we can read this statement as saying "y is the cosine of the angle t". Changing perspective and writing the equivalent statement, $t = \arccos(y)$, we read this statement as "t is the angle whose cosine is y". Just as $y = f(t)$ and $t = f^{-1}(y)$ mean the same thing for a function and its inverse in general. To summarize, both expressions

$$y = \cos(t) \text{ and } t = \arccos(y)$$

mean the same thing for any angle t that satisfies $0 \leq t \leq \pi$. We read $t = \cos^{-1}(y)$ as "t is the angle whose cosine is y" or "t is the inverse cosine of y". Key properties of the arccosine function can be summarized as follows.

Properties of the arccosine function.

- The restricted cosine function, $y = g(t) = \cos(t)$, is defined on the domain $[0, \pi]$ with range $[-1, 1]$. This function has an inverse function that we call the arccosine function, denoted $t = g^{-1}(y) = \arccos(y)$.
- The domain of $y = g^{-1}(t) = \arccos(t)$ is $[-1, 1]$ with range $[0, \pi]$.
- The arccosine function is always decreasing on its domain.
- Below we have a plot of the restricted cosine function (in light blue) and its corresponding inverse, the arccosine function (in dark blue).



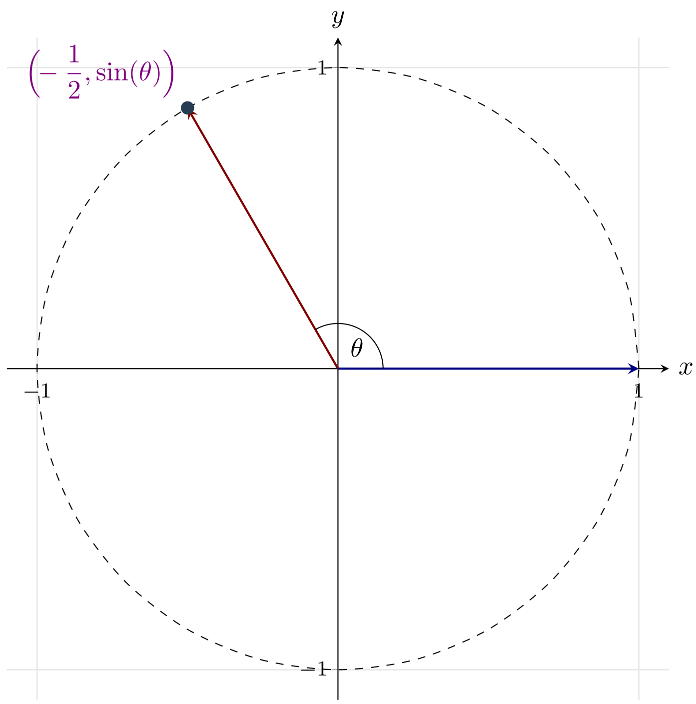
Just as the natural logarithm function allowed us to rewrite exponential equations in an equivalent way (for instance, $y = e^t$ and $t = \ln(y)$ give the same information), the arccosine function allows us to do likewise for certain angles and cosine outputs. For instance, saying $\cos\left(\frac{\pi}{2}\right) = 0$ is the same as writing $\frac{\pi}{2} = \arccos(0)$, which reads "$\frac{\pi}{2}$ is the angle whose cosine is 0". Indeed, these relationships are reflected in the plot above, where we see that any point (a, b) that lies on the graph of $y = \cos(t)$ corresponds to the point (b, a) that lies on the graph of $y = \arccos(t)$.

Exploring Arccosine

Example 19. *Use the special points on the unit circle to determine the exact values of each of the following numerical expressions. Do so without using a computational device.*

(a) $\cos\left(\arccos\left(-\frac{1}{2}\right)\right)$

Explanation We start by finding $\arccos\left(-\frac{1}{2}\right)$. Remember that for x in $[-1, 1]$, $\arccos(x)$ is the angle θ in $[0, \pi]$ such that $\cos(\theta) = x$. Hence we are looking for the value of θ corresponding to the point on the upper hemisphere of the unit circle with x-value $-\dfrac{1}{2}$.



Hence, θ is $\dfrac{2\pi}{3}$, and we now see that

$$\cos\left(\arccos\left(-\frac{1}{2}\right)\right) = \cos\left(\frac{2\pi}{3}\right) = -\frac{1}{2}.$$

Now, if you're thinking, "Hey, we didn't need that extra step!" Then you would be correct. But ***why*** didn't we need that final step?

Let's recall how we defined arccosine. Since cosine is a periodic function, it fails the horizontal line test. However, if we *restrict* cosine to a portion of its domain on which it is only decreasing, $[0, \pi]$, then we may define a function g on this domain such that $g(x) = \cos(x)$ for x in $[0, \pi]$. Arccosine is defined as the inverse of this function g. Therefore, g is the inverse of arccosine. Thus, in practice, cosine is the inverse of arccosine.

A word of caution: arccosine is only the inverse of restricted cosine, as we will demonstrate with the next example.

(b) $\arccos\left(\cos\left(\dfrac{7\pi}{6}\right)\right)$

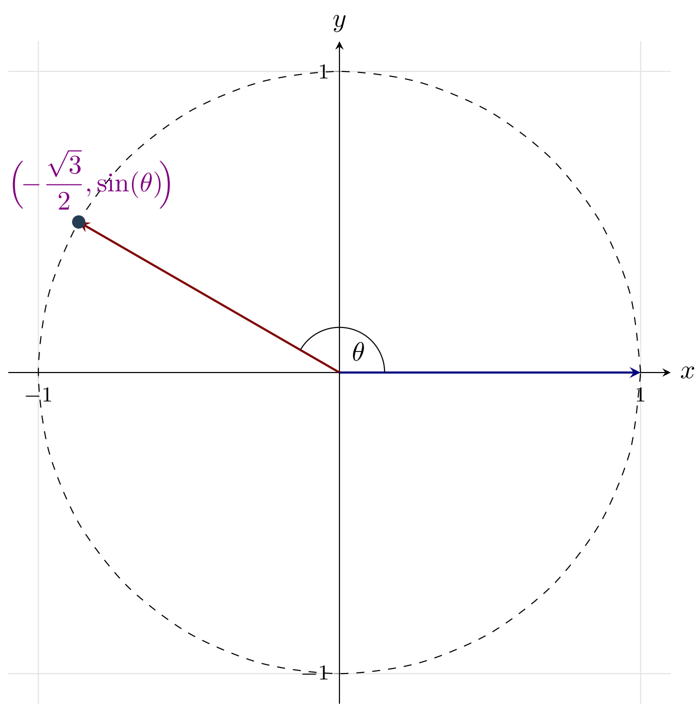
Explanation It may be tempting to take a look at this expression and conclude that the solution is $\dfrac{7\pi}{6}$ since arccosine is the inverse of cosine.

But wait!

Remember, we had to restrict the domain of cosine in order to define an inverse function, which we called arccosine. Arccosine is the inverse of the *restricted* cosine function, whose domain is $[0, \pi]$. $\frac{7\pi}{6}$ is larger than π, so it is not within the domain of this restricted cosine.

Thus, we begin by simplifying $\cos\left(\frac{7\pi}{6}\right) = -\frac{\sqrt{3}}{2}$.

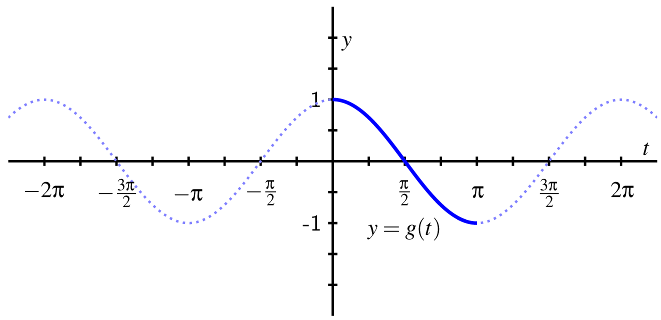
Now, when we consider $\arccos\left(-\frac{\sqrt{3}}{2}\right)$, we will once again recall the unit circle. We are looking at the upper hemisphere, but this time we want to find the angle θ in $[0, \pi]$ that corresponds to the point with x-value $-\frac{\sqrt{3}}{2}$.



Hence, θ is $\frac{5\pi}{6}$, and we now see that

$$\arccos\left(\cos\left(\frac{7\pi}{6}\right)\right) = \arccos\left(-\frac{\sqrt{3}}{2}\right) = \frac{5\pi}{6}.$$

Now, let's look again at the graph of cosine. Here we highlight $g : [0, \pi] \to [0, \pi]$ defined by $y = g(x) = \cos(x)$, the restricted cosine function. We may use the symmetry of the graph of cosine to help find the appropriate values for arccosine.



(c) *We can also solve trig equations as in the section Some Applications of Trig Functions. Solve the equation* $4\arccos(x) - 3\pi = 0$.

Explanation We start by isolating the arccosine term so that our equation is now

$$\arccos(x) = \frac{3\pi}{4}.$$

We observe that $\frac{3\pi}{4}$ is in the range of arccosine, so we may use the fact that cosine is the inverse of arccosine. Thus, $\arccos(x) = \frac{3\pi}{4}$ is equivalent to

$$\cos(\arccos(x)) = \cos\left(\frac{3\pi}{4}\right).$$

This is further equivalent to $x = -\frac{\sqrt{2}}{2}$.

Summary

- Any function that fails the Horizontal Line Test cannot have an inverse function. However, for a periodic function that fails the horizontal line test, if we restrict the domain of the function to an interval that is the length of a single period of the function, we then determine a related function that does, in fact, have an inverse function. This makes it possible for us to develop the inverse function of the restricted cosine function.
- We choose to define the restricted cosine function on the domain $[0, \pi]$. On this interval, the restricted function is strictly decreasing, and thus has an inverse function. The restricted cosine function has range $[-1, 1]$.

10.3.2 Other Inverse Trig Functions

Motivating Questions

- For the restricted sine, tangent, and secant functions, how do we define the corresponding arcsine, arctangent, and arcsecant functions?
- What are the key properties of arcsine, arctangent, and arcsecant?

Introduction

In the last section we defined *arccosine*, the inverse for cosine restricted to a single period. In this section we will explore the definition of similar inverse functions on restricted domains of sine, tangent, and secant.

As we recalled last time,

- A function f has an inverse function if and only if there exists a function g that undoes the work of f: that is, there is some function g, the inverse of f, for which $g(f(x)) = x$ for each x in the domain of f, and $f(g(y)) = y$ for each y in the range of f.
- A function f has an inverse function if and only if the graph of f passes the *Horizontal Line Test*.
- When f has an inverse, we know that "$y = f(t)$" and "$t = f^{-1}(y)$" are two different perspectives on the same statement.

As with the cosine function, the trigonometric functions $f(t) = \sin(t)$, $h(t) = \tan(t)$, and $k(t) = \sec(t)$ are periodic, so they fail the horizontal line test. Hence, considering these functions on their full domains, neither has an inverse function. At the same time, it is reasonable to think about changing perspective and viewing angles as outputs in certain restricted settings, as we did with cosine.

The Arcsine Function

We can develop an inverse function for a restricted version of the sine function in a similar way. As with the cosine function, we need to choose an interval on which the sine function is always increasing or always decreasing in order to have the function pass the horizontal line test. The standard choice is the domain $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ on which $f(t) = \sin(t)$ is increasing and attains all of the values in the range of the sine function. Thus, we consider $f(t) = \sin(t)$ so that f has

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domain $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and range $\left[-1, 1\right]$ and use this restricted function to define the corresponding arcsine function.

Definition Let $y = f(t) = \sin(t)$ be defined on the domain $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, and observe f has the range $\left[-1, 1\right]$. For any real number y that satisfies $-1 \leq y \leq 1$, the **arcsine of y**, denoted

$$\arcsin(y)$$

is the angle t satisfying $-\frac{\pi}{2} \leq t \leq \frac{\pi}{2}$ such that $\sin(t) = y$. Note that we use $t = \sin^{-1}(y)$ interchangeably with with $t = \arcsin(y)$.

Problem 1 *The goal of this activity is to understand key properties of the arcsine function in a way similar to our discussion of the arccosine function in the previous section. We will use our deductive reasoning skills a la Sherlock Holmes to build off our discussion from the last section.*

(a) *Using the definition of arcsine given above, what are the domain and range of the arcsine function?*

- *The domain of arcsine is* $[\boxed{?}, \boxed{?}]$.
- *The range of arcsine is* $\left[\boxed{?}, \boxed{?}\right]$.

(b) *Determine the following values exactly:*

- $\arcsin(-1) = \boxed{?}$
- $\arcsin\left(-\frac{\sqrt{2}}{2}\right) = \boxed{?}$
- $\arcsin(0) = \boxed{?}$
- $\arcsin\left(\frac{1}{2}\right) = \boxed{?}$
- $\arcsin\left(\frac{\sqrt{3}}{2}\right) = \boxed{?}$

(c) *On the axes provided below, sketch a careful plot of the restricted sine function on the interval* $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ *along with its corresponding inverse, the arcsine function. Label at least three points on each curve so that each point on the sine graph corresponds to a point on the arcsine graph. In addition, sketch the line* $y = t$ *to demonstrate how the graphs are reflections of one another across this line.*

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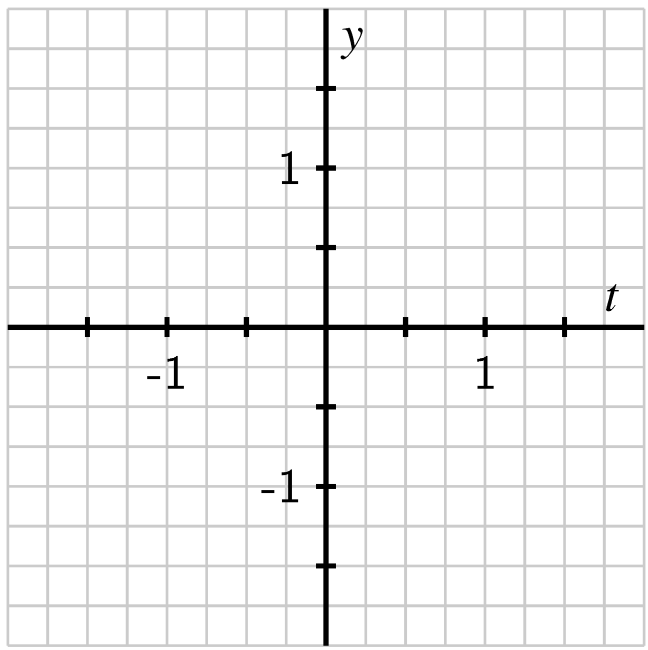
(d) *True or false:* $\arcsin(\sin(5\pi)) = 5\pi$? *(true/ false)*
Write a complete sentence to explain your reasoning.



Properties of the arcsine function.

- The restricted sine function, $y = f(t) = \sin(t)$, is defined on the domain $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ with range $[-1, 1]$. This function has an inverse function that we call the arcsine function, denoted $t = f^{-1}(y) = \arcsin(y)$.
- The domain of $y = f^{-1}(t) = \arcsin(t)$ is $[-1, 1]$ with range $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.
- The arcsine function is always increasing on its domain.
- Below we have a plot of the restricted sine function (in light blue) and its corresponding inverse, the arcsine function (in dark blue).

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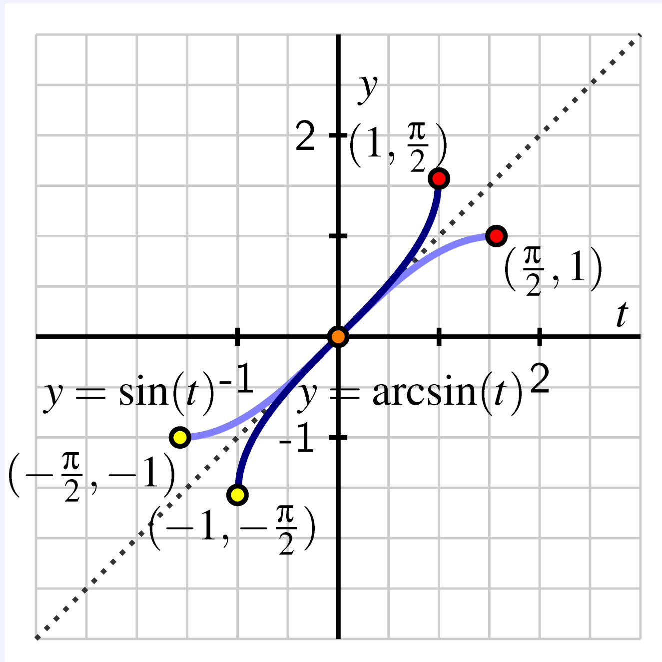

Exploring Arcsine

Example 20. *Let's solve the following equations analytically, then we can consider the graph of arcsine.*

(a) $\sin\Big(\arcsin\Big(-\dfrac{\sqrt{2}}{2}\Big)\Big)$

Explanation We start by finding $\arcsin\Big(-\dfrac{\sqrt{2}}{2}\Big)$. Remember that for x in $[-1,1]$, $\arcsin(x)$ is the value y in $\Big[-\dfrac{\pi}{2},\dfrac{\pi}{2}\Big]$ such that $\sin(y)=x$. Hence, y is $-\dfrac{\pi}{4}$, and we now see that

$$\sin\Big(\arcsin\Big(-\frac{\sqrt{2}}{2}\Big)\Big)=\sin\Big(-\frac{\pi}{4}\Big)=-\frac{\sqrt{2}}{2}.$$

Now, if you're thinking, "Hey, we didn't need that extra step!" Then you would be correct. But ***why*** didn't we need that final step?

Let's recall how we defined arcsine. Since sine is a periodic function, it fails the horizontal line test. However, if we *restrict* sine to a portion of its domain on which it is only increasing, $\Big[-\dfrac{\pi}{2},\dfrac{\pi}{2}\Big]$, then we may define

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a function f on this domain such that $f(x) = \sin(x)$ for x in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. Arcsine then is defined as the inverse of this function f. Therefore, f is the inverse of arcsine. Thus, in practice, sine is the inverse of arcsine.

A word of caution: As was the case with arccosine and cosine, arcsine is only the inverse of restricted sine. We will illustrate this with the next example.

(b) $\arcsin\left(\sin\left(\frac{5\pi}{4}\right)\right)$

Explanation It may be tempting to take a look at this expression and conclude that the solution is $\frac{5\pi}{4}$ since arcsine is the inverse of sine.

Hold those horses!

Remember, we had to restrict the domain of sine in order to define an inverse function, which we called arcsine. Arcsine is the inverse of the *restricted* sine function, whose domain is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. $\frac{5\pi}{4}$ is larger than $\frac{\pi}{2}$, so it is not within the domain of this restricted sine function.

Thus, we begin by simplifying $\sin\left(\frac{5\pi}{4}\right) = -\frac{\sqrt{2}}{2}$.

Now, let's consider $\arcsin\left(-\frac{\sqrt{2}}{2}\right)$, recalling again the *range* of arcsine. We are looking for the value of y in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ such that $\sin(y) = -\frac{\sqrt{2}}{2}$.

Hence, y is $-\frac{\pi}{4}$, and we now see that

$$\arcsin\left(\sin\left(\frac{5\pi}{4}\right)\right) = \arcsin\left(-\frac{\sqrt{2}}{2}\right) = -\frac{\pi}{4}.$$

(c) $\arcsin(2x) = \frac{\pi}{3}$

Explanation First, we observe that $\frac{\pi}{3}$ is in the range of arcsine, so there should be a solution. We will now use the fact that sine is the inverse of arcsine to reduce this to a linear equation.

$$\begin{aligned}\arcsin(2x) &= \frac{\pi}{3}\\ \sin(\arcsin(2x)) &= \sin\left(\frac{\pi}{3}\right)\end{aligned}$$

Thus, we have

$$2x = \frac{\sqrt{3}}{2},$$

which is equivalent to $x = \frac{\sqrt{3}}{4}$.

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The Arctangent Function

Next, we develop an inverse function for a restricted version of the tangent function. We choose the domain $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ on which $h(t) = \tan(t)$ is increasing and attains all of the values in the range of the tangent function.

> **Definition** Let $y = h(t) = \tan(t)$ be defined on the domain $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, and observe h has the range $\big(-\infty, \infty\big)$. For any real number y, the **arctangent of y**, denoted
>
> $$\arctan(y)$$
>
> is the angle t satisfying $-\frac{\pi}{2} < t < \frac{\pi}{2}$ such that $\tan(t) = y$. Note that we use $t = \tan^{-1}(y)$ interchangeably with with $t = \arctan(y)$.

Problem 2 *Let us once again channel our inner Sherlock Holmes to understand key properties of the arctangent function.*

(a) *Using the definition given above, what are the domain and range of the arctangent function?*

- *The domain of arctangent is* $(\boxed{?}, \boxed{?})$.
- *The range of arctangent is* $\left(\boxed{?}, \boxed{?}\right)$.

(b) *Determine the following values exactly:*

- $\arctan(-\sqrt{3}) = \boxed{?}$
- $\arctan(-1) = \boxed{?}$
- $\arctan(0) = \boxed{?}$
- $\arctan\left(\frac{1}{\sqrt{3}}\right) = \boxed{?}$.

(c) *The restricted tangent function on the interval* $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ *is plotted below. On the same axes, sketch its corresponding inverse function (arctangent). Label at least three points on each curve so that each point on the tangent graph corresponds to a point on the arctangent graph. In addition, sketch the line* $y = t$ *to demonstrate how the graphs are reflections of one another across this line.*

(d) *Complete the following sentence: "as t increases without bound,* $\arctan(t)$ *. . ." (increases without bound/ decreases without bound/ increases toward* $\frac{\pi}{2}$*/ decreases toward* $-\frac{\pi}{2}$*)*

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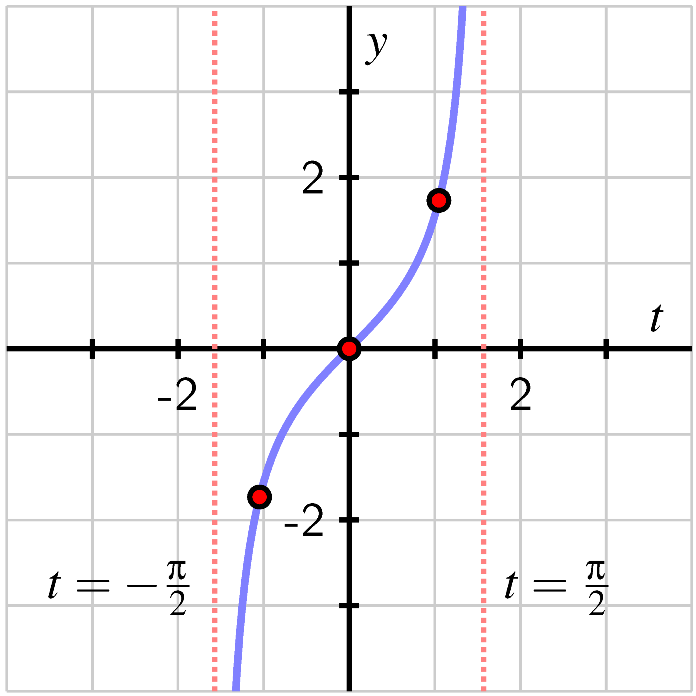

Properties of the arctangent function.

- The restricted tangent function, $y = h(t) = \tan(t)$, is defined on the domain $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ with range $(-\infty, \infty)$. This function has an inverse function that we call the arctangent function, denoted $t = h^{-1}(y) = \arctan(y)$.
- The domain of $y = h^{-1}(t) = \arctan(t)$ is $(-\infty, \infty)$ with range $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.
- The arctangent function is always increasing on its domain.
- Below we have a plot of the restricted tangent function (in light blue) and its corresponding inverse, the arctangent function (in dark blue).

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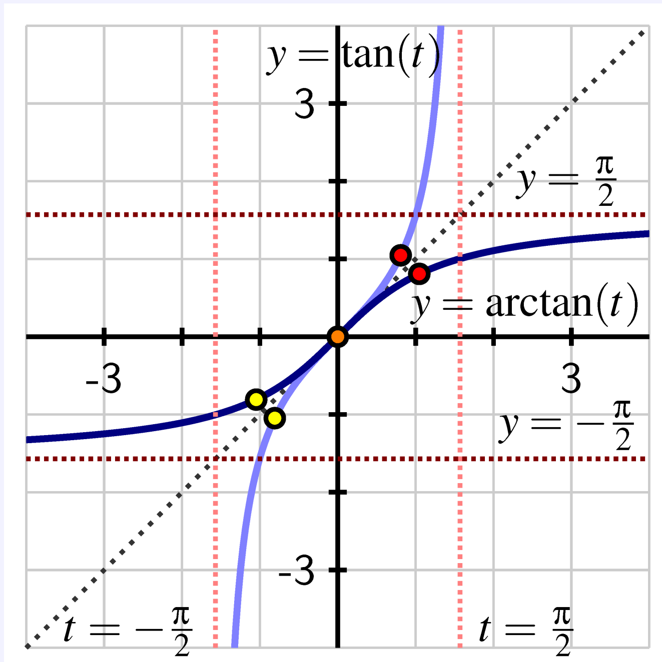

Exploring Arctangent

Example 21. *Let's solve the following equations analytically, then we can consider the graph of arctangent.*

(a) $\tan(\arctan(-\sqrt{3}))$
Explanation We start by finding $\arctan(-\sqrt{3})$. Remember that for x in $(-\infty, \infty)$, $\arctan(x)$ is the value y in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ such that $\tan(y) = x$.

Hence, y is $-\frac{\pi}{3}$, and we now see that

$$\tan(\arctan(-\sqrt{3})) = \tan\left(-\frac{\pi}{3}\right) = -\sqrt{3}.$$

Now, I know you're thinking, "Hey, why do you keep making us do an extra step?" It's because it is imperative that you ***consider the range*** of the arc trig functions. These are considerations you will also need to make when we start combining different trig functions with the inverses of others (say sine of arctangent of a value).

Let's recall how we defined arctangent. Since tangent is a periodic function, it fails the horizontal line test. However, if we *restrict* tangent to a

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single period (note tangent only increasing), $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, then we may define a function h on this domain such that $h(x) = \tan(x)$ for x in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Arctangent then is defined as the inverse of this function h. Therefore, h is the inverse of arctangent. Thus, in practice, tangent is the inverse of arctangent.

A word of caution: As was the case with the previous two trig functions and their respective inverses, arctangent is only the inverse of restricted tangent. We will illustrate this with the next example.

(b) $\arctan\left(\tan\left(\frac{5\pi}{3}\right)\right)$

Explanation It may be tempting to take a look at this expression and conclude that the solution is $\frac{5\pi}{3}$ since arctangent is the inverse of tangent.

But, wait!

Remember, we had to restrict the domain of tangent in order to define an inverse function, which we called arctangent. Arctangent is the inverse of the *restricted* tangent function, whose domain is $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. $\frac{5\pi}{3}$ is larger than $\frac{\pi}{2}$, so it is not within the domain of this restricted tangent function.

Thus, we begin by simplifying $\tan\left(\frac{5\pi}{3}\right) = -\sqrt{3}$.

Now, let's consider $\arctan(-\sqrt{3})$, recalling again the *range* of arctangent. We are looking for the value of y in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ such that $\tan(y) = -\sqrt{3}$.

Hence, y is $-\frac{\pi}{3}$, and we now see that

$$\arctan\left(\tan\left(\frac{5\pi}{3}\right)\right) = \arctan(-\sqrt{3}) = -\frac{\pi}{3}.$$

(c) $4\arctan^2(x) - 3\pi\arctan(x) - \pi^2 = 0$

Explanation We will treat this like a quadratic equation to begin, as we did in the section Some Applications of Trig Functions.

Let $y = \arctan(x)$, then we have a standard quadratic equation: $4y^2 - 3\pi y - \pi^2 = 0$. Factoring, we see that this is equivalent to

$$(4y + \pi)(y - \pi) = 0.$$

This has two solutions: $y = -\frac{\pi}{4}$ and $y = \pi$. In other words, we now simply solve (a) $\arctan(x) = -\frac{\pi}{4}$ and (b) $\arctan(x) = \pi$. π is not in the range of arctangent, so (b) does not have a solution. Hence, this cannot be a solution to our equation, and we must look at (a). $-\frac{\pi}{4}$ is in the range

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of arctangent, so the solution to (a) will be a solution to our original equation.

Since tangent is the inverse to arctangent, the equation (a) is equivalent to

$$\tan(\arctan(x)) = \tan\Big(-\frac{\pi}{4}\Big),$$

which is further equivalent to $x = -1$

The Arcsecant Function

We will also consider the inverse function for a restricted version of the secant function. As with the cosine and sine functions, we need to choose an interval on which the secant function is always increasing or always decreasing in order to have the function pass the horizontal line test. In the case of secant, this means choosing two distinct intervals. A word of caution in working with the restricted secant function and its associated inverse, there is not a "standard" choice for the domain of restricted secant. *However*, we will establish a convention in this course.

We restrict the domain of the function $k(t) = \sec(t)$ to $[0, \pi/2) \cup (\pi/2, \pi]$, where secant is increasing on each interval and attains all the values within the range of the secant function. By reflecting across the line $y = t$ and switching the t and y coordinates we are able to define the function $k^{-1}(t) = \operatorname{arcsec}(t)$ as follows.

Definition Let $y = k(t) = \sec(t)$ be defined on the domain $[0, \pi/2) \cup (\pi/2, \pi]$, and observe that k has the range $(-\infty, -1] \cup [1, \infty)$. For any real number y, the **arcsecant of y**, denoted

$$\operatorname{arcsec}(y)$$

is the angle t satisfying $0 \leq t < \dfrac{\pi}{2}$ or $\dfrac{\pi}{2} < t \leq \pi$. Note that we use $t = \sec^{-1}(y)$ interchangeably with $t = \operatorname{arcsec}(y)$.

Problem 3 *Take the lead Watson, and we will deduce the key properties of the arcsecant function as we did above for arcsine and arctangent.*

(a) *Using the definition of arcsecant given above, what are the domain and range of the arcsecant function?*

- *The domain of arcsecant is* $(\boxed{?}, \boxed{?}] \cup [\boxed{?}, \boxed{?})$.
- *The range of arcsecant is* $\Big[\boxed{?}, \boxed{?}\Big) \cup \Big(\boxed{?}, \boxed{?}\Big]$.

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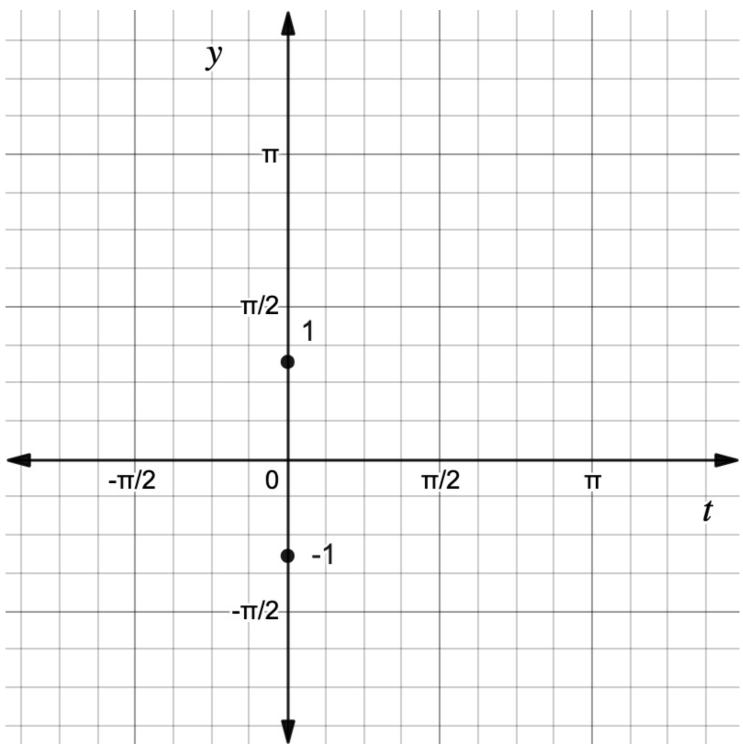
(b) *Determine the following values exactly:*

- $\text{arcsec}(1) = \boxed{?}$
- $\text{arcsec}(-1) = \boxed{?}$
- $\text{arcsec}(2) = \boxed{?}$

(c) *On the axes provided below, sketch a careful plot of the restricted secant function on the intervals* $\left[0, \frac{\pi}{2}\right)$ *and* $\left(\frac{\pi}{2}, \pi\right]$ *along with its corresponding inverse, the arcsecant function. Label at least three points on each curve so that each point on the secant graph corresponds to a point on the arcsecant graph.*

(d) *True or false:* $\text{arcsec}(\sec(2\pi)) = 2\pi$*? (true/ false)*
Write a complete sentence to explain your reasoning.



Properties of the arcsecant function.

- The restricted secant function, $y = k(t) = \sec(t)$, is defined on

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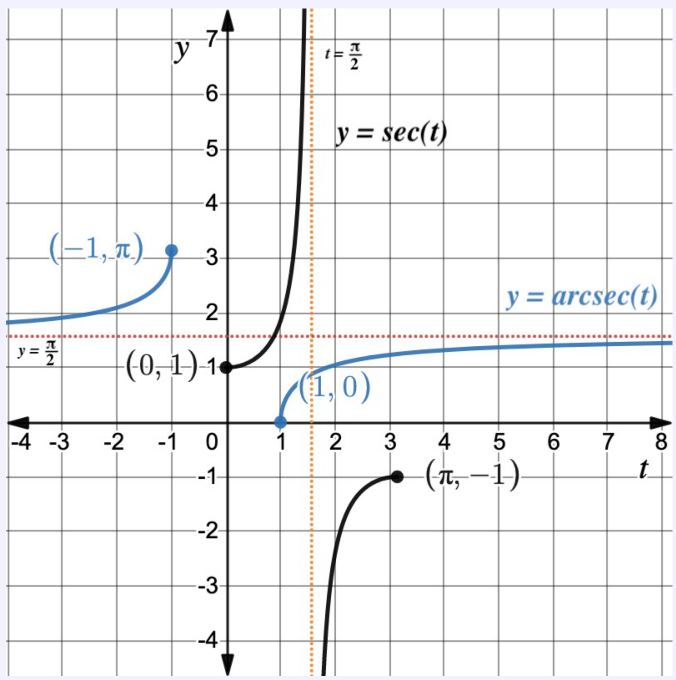
the domain $\left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$ with range $(-\infty, -1] \cup [1, \infty)$. This function has an inverse function that we call the arcsecant function, denoted $t = k^{-1}(y) = \text{arcsec}(y)$.

- The domain of $y = k^{-1}(t) = \text{arcsec}(t)$ is $(-\infty, -1] \cup [1, \infty)$ with range $\left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$.
- The arcsecant function is always increasing on each interval in its domain.
- Recall that $\sec(\theta) = \dfrac{1}{\cos(\theta)}$. Arcsecant and arccosine maintain a relationship as well, though they are *not* reciprocals:
  For t in the domain of arcsecant, $\text{arcsec}(t) = \arccos\left(\dfrac{1}{t}\right)$.



Exploring Arcsecant

Example 22. *Sometimes we must rely on other properties of these functions and their relations to more familiar functions to find solutions. In the following examples, we wish to find x in the range of arcsecant such that*

(a) $x = \text{arcsec}(-\sqrt{2})$

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Explanation We may use the relationship between arcsecant and arccosine to rewrite this equation in terms of arccosine. In other words, since $\operatorname{arcsec}(y) = \arccos\left(\frac{1}{y}\right)$, for y in the domain of arcsecant,

$$\operatorname{arcsec}(-\sqrt{2}) = \arccos\left(-\frac{\sqrt{2}}{2}\right) = \frac{3\pi}{4}.$$

(b) $x = \operatorname{arcsec}\left(-\frac{2\sqrt{3}}{3}\right)$

Explanation Again, we use the relationship $\operatorname{arcsec}(y) = \arccos\left(\frac{1}{y}\right)$, for y in the domain of arcsecant:

$$\operatorname{arcsec}\left(-\frac{2\sqrt{3}}{3}\right) = \arccos\left(-\frac{\sqrt{3}}{2}\right),$$

since the reciprocal of $\frac{2\sqrt{3}}{3} = \frac{3}{2\sqrt{3}} = \frac{\sqrt{3}}{2}$. Thus, we have

$$\operatorname{arcsec}\left(-\frac{2\sqrt{3}}{3}\right) = \frac{5\pi}{6}.$$

Let's consider a couple more traditional problems combining secant and arcsecant. Remember that we must be cautious of their respective domains and ranges as with combinations of sine and arcsine and tangent and cotangent explored above.

(c) $\sec(\operatorname{arcsec}(-\sqrt{2}))$

Explanation Recall from part (a), that we already solved the equation $x = \operatorname{arcsec}(-\sqrt{2})$, and found that $x = \frac{3\pi}{4}$. Hence, we can now plug that in to solve our current equation:

$$\sec(\operatorname{arcsec}(-\sqrt{2})) = \sec\left(\frac{3\pi}{4}\right) = -\sqrt{2}.$$

As we have observed previously with other trig inverses, we have $\sec(\operatorname{arcsec}(x)) = x$ for x in the domain of arcsecant. However, we must be careful in our application of this, as exemplified by the next example.

(d) $\operatorname{arcsec}\left(\sec\left(\frac{5\pi}{4}\right)\right)$

Explanation Remember that we had to restrict the domain of secant in order to define the inverse function, arcsecant. Arcsecant is thus the inverse of the *restricted* secant function, which has domain $[0, \pi/2) \cup (\pi/2, \pi]$. Observing that $\frac{5\pi}{4} > \pi$, and is therefore not in the domain of the restricted secant function, we cannot simply treat arcsecant as the inverse.

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Instead, we begin by finding $\sec\left(\frac{5\pi}{4}\right)$, which is equal to $-\sqrt{2}$.

We now have $\operatorname{arcsec}\left(\sec\left(\frac{5\pi}{4}\right)\right) = \operatorname{arcsec}(-\sqrt{2})$, which we found to be equal to $\frac{3\pi}{4}$ in part (a). Thus, we may conclude that

$$\operatorname{arcsec}\left(\sec\left(\frac{5\pi}{4}\right)\right) = \frac{3\pi}{4}.$$

Summary

- We choose to define the restricted cosine, sine, tangent, and secant functions on the respective domains $[0, \pi]$, $[-\pi/2, \pi/2]$, $(-\pi/2, \pi/2)$, and $[0, \pi/2) \cup (\pi/2, \pi]$. On each such interval, the restricted function is strictly decreasing (cosine) or strictly increasing (sine, tangent, and secant), and thus has an inverse function. The restricted sine and cosine functions each have range $[-1, 1]$, while the restricted tangent's range is the set of all real numbers, and the restricted secant's range is $(-\infty, 1] \cup [1, \infty)$. We thus define the inverse function of each as follows:

    i. For any y such that $-1 \leq y \leq 1$, the arccosine of y (denoted $\arccos(y)$) is the angle t in the interval $[0, \pi]$ such that $\cos(t) = y$. That is, t is the angle whose cosine is y.

    ii. For any y such that $-1 \leq y \leq 1$, the arcsine of y (denoted $\arcsin(y)$) is the angle t in the interval $[-\pi/2, \pi/2]$ such that $\sin(t) = y$. That is, t is the angle whose sine is y.

    iii. For any real number y, the arctangent of y (denoted $\arctan(y)$) is the angle t in the interval $(-\pi/2, \pi/2)$ such that $\tan(t) = y$. That is, t is the angle whose tangent is y.

    iv. For any real number y, the arcsecant of y (denoted $\operatorname{arcsec}(y)$) is the angle t in the interval $[0, \pi/2) \cup (\pi/2, \pi]$ such that $\sec(t) = y$. That is, t is the angle whose secant is y.

- The domain of $y = g^{-1}(t) = \arccos(t)$ is $[-1, 1]$ with corresponding range $[0, \pi]$, and the arccosine function is always decreasing. These facts correspond to the domain and range of the restricted cosine function and the fact that the restricted cosine function is decreasing on $[0, \pi]$.

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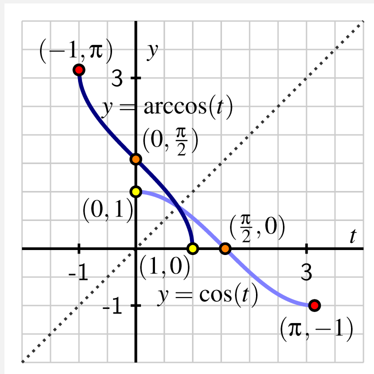
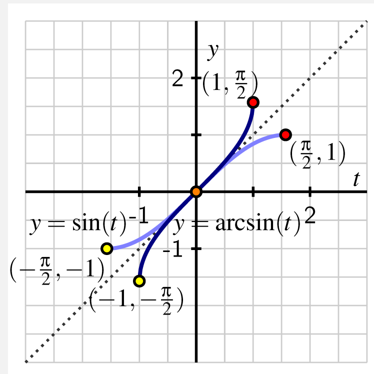
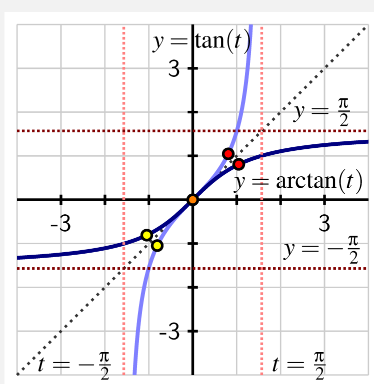
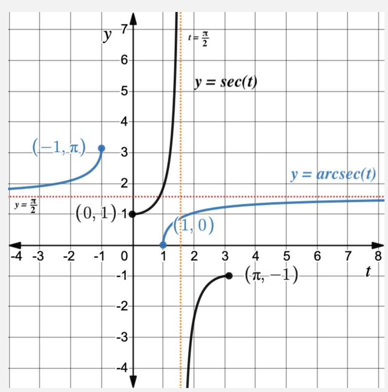




The domain of $y = f^{-1}(t) = \arcsin(t)$ is $[-1, 1]$ with corresponding range $[-\pi/2, \pi/2]$, and the arcsine function is always increasing. These facts correspond to the domain and range of the restricted sine function and the fact that the restricted sine function is increasing on $[-\pi/2, \pi/2]$.

The domain of $y = h^{-1}(t) = \arctan(t)$ is the set of all real numbers with corresponding range $(-\pi/2, \pi/2)$, and the arctangent function is always increasing. These facts correspond to the domain and range of the restricted tangent function and the fact that the restricted tangent function is increasing on $(-\pi/2, \pi/2)$.





The domain of $y = k^{-1}(t) = \text{arcsec}(t)$ is $(-\infty, -1] \cup [1, \infty)$,[a] with corresponding range $[0, \pi/2) \cup (\pi/2, \pi]$. These facts correspond to the domain and range of the restricted secant function.

[a]We note that this may also be written as $\{x : |x| \geq 1\}$.

10.3.3 Applications of Inverse Trigonometry

Motivating Questions

- How can we use inverse trigonometric functions to determine missing angles in right triangles?
- What other situations may require us to use inverse trigonometric functions?

Introduction

When we learned about trig functions, we observed that in any right triangle, if we know the measure of one additional angle and the length of one additional side, we can determine all of the other parts of the triangle. With the inverse trigonometric functions that we developed in the last two sections, we are now also able to determine the missing angles in any right triangle where we know the lengths of two sides.

While the original trigonometric functions take a particular angle as input and provide an output that can be viewed as the ratio of two sides of a right triangle, the inverse trigonometric functions take an input that can be viewed as a ratio of two sides of a right triangle and produce the corresponding angle as output. Indeed, it's imperative to remember that statements such as

$$\arccos(x) = \theta \text{ and } \cos(\theta) = x$$

say the exact same thing from two different perspectives, and that we read "$\arccos(x)$" as "the angle whose cosine is x".

Exploration Consider a right triangle that has one leg of length 3 and another leg of length $\sqrt{3}$. Let θ be the angle that lies opposite the shorter leg. Sketch a labeled picture of the triangle.

(a) What is the exact length of the triangle's hypotenuse?

(b) What is the exact value of $\sin(\theta)$?

(c) Rewrite your equation from (b) using the arcsine function in the form $\arcsin(\Box) = \Delta$, where $\Box$ and Δ are numerical values.

(d) What special angle from the unit circle is θ?

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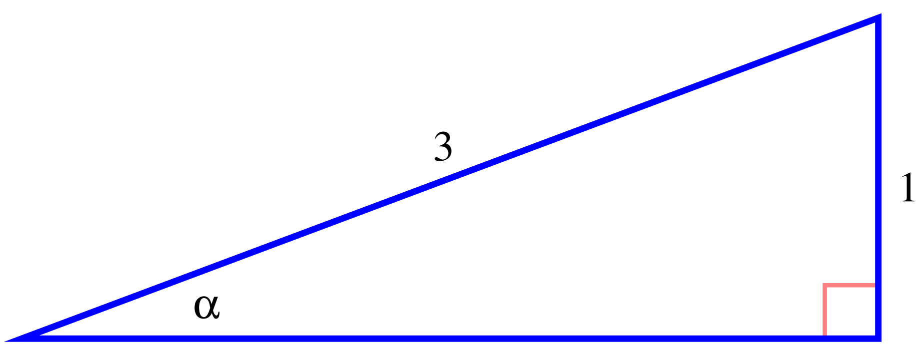
Evaluating Inverse Trigonometric Functions

Like the trigonometric functions themselves, there are a handful of important values of the inverse trigonometric functions that we can determine exactly without the aid of a computer. For instance, we know from the unit circle that $\arcsin\left(-\frac{\sqrt{3}}{2}\right) = -\frac{\pi}{3}$, $\arccos\left(-\frac{\sqrt{3}}{2}\right) = \frac{5\pi}{6}$, and $\arctan\left(-\frac{1}{\sqrt{3}}\right) = -\frac{\pi}{6}$. In these evaluations, we have to be careful to remember that the range of the arccosine function is $[0, \pi]$, while the range of the arcsine function is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and the range of the arctangent function is $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, in order to ensure that we choose the appropriate angle that results from the inverse trigonometric function. This is why our emphasis is now turning to the *graphs* of these functions.

In addition, there are many other values at which we may wish to know the angle that results from an inverse trigonometric function. To determine such values, one can use a computational device (such as *Desmos*) in order to find an approximation; however, in this class we leave it in the form $\arccos(a)$, as this is the exact value.

Example 23. *Consider the right triangle pictured below and assume we know that the vertical leg has length* 1 *and the hypotenuse has length* 3*. Let* α *be the angle opposite the known leg. Determine exact values for all of the remaining parts of the triangle.*



Explanation Because we know the hypotenuse and the side opposite α, we observe that $\sin(\alpha) = \frac{1}{3}$. Rewriting this statement using inverse function notation, we have equivalently that $\alpha = \arcsin\left(\frac{1}{3}\right)$, which is the exact value of α. Since this is not one of the known special angles on the unit circle, we leave it in this form.

We can now find the remaining leg's length and the remaining angle's measure. If we let x represent the length of the horizontal leg, by the Pythagorean

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Theorem we know that

$$x^2 + 1^2 = 3^2,$$

and thus $x^2 = 8$ so $x = \sqrt{8}$. Calling the remaining angle β, since $\alpha + \beta = \frac{\pi}{2}$, it follows that

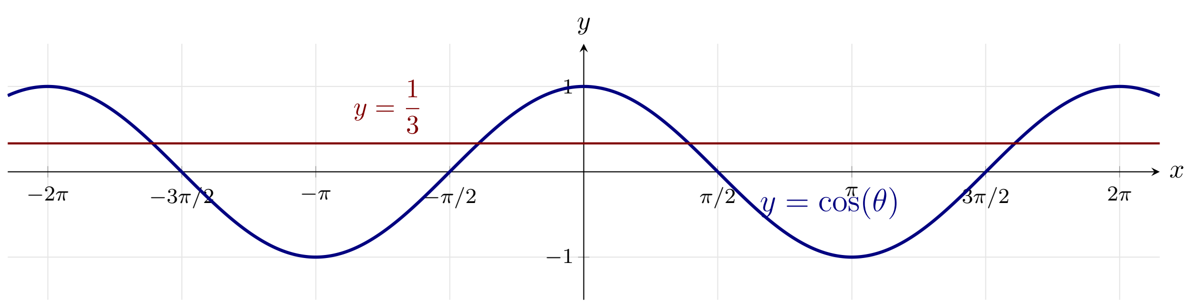
$$\beta = \frac{\pi}{2} - \arcsin\left(\frac{1}{3}\right).$$

We can also use inverse trigonometric functions to solve equations that up until now, have been unsolvable for us.

Example 24. *Solve the equation* $\cos(t) = \frac{1}{3}$.

Explanation Before learning about inverse trig, since $\frac{1}{3}$ was not a famous value of sine or cosine, we couldn't solve this equation. But now, we know a value of t that gives us $\cos(t) = \frac{1}{3}$, namely, $t = \arccos\left(\frac{1}{3}\right)$. Even though this feels weird, since we don't have a decimal expression for $\arccos\left(\frac{1}{3}\right)$, this does give us a solution to our equation.

However, we need to be careful. This isn't the only solution to our equation. The picture below reminds us that there are infinitely many places where the graphs of $y = \cos(t)$ and $y = \frac{1}{3}$ intersect.



Our strategy to find all solutions to $\cos(t) = \frac{1}{3}$ is to first find all solutions within a single period, then add all multiples of 2π. This should be familiar from an earlier section. Note that the interval $[-\pi, \pi]$ contains a full period of cos, and from the graph above, there are two solutions on that interval: one on the interval $\left[0, \frac{\pi}{2}\right]$ and one on the interval $\left[-\frac{\pi}{2}, 0\right]$. The solution on the interval $\left[0, \frac{\pi}{2}\right]$ is the solution we've already found, $\arccos\left(\frac{1}{3}\right)$, since we know the range of arccos is $[0, \pi]$.

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As in similar problems we've done before, the other solution has the same reference angle as $\arccos\left(\frac{1}{3}\right)$. We know that $\arccos\left(\frac{1}{3}\right)$ lives in Quadrant I, since it corresponds to a positive cosine value, so it is its own reference angle. The other solution therefore, has reference angle $\arccos\left(\frac{1}{3}\right)$ and because cosine is positive, lies in Quadrant IV. A Quadrant IV angle that has this reference angle is given by $-\arccos\left(\frac{1}{3}\right)$. Since $\arccos\left(\frac{1}{3}\right)$ and $-\arccos\left(\frac{1}{3}\right)$ give a complete list of all solutions on one period of cosine, a complete list of all solutions to our original equation is

$$\arccos\left(\frac{1}{3}\right)+2\pi k \quad \text{and} \quad -\arccos\left(\frac{1}{3}\right)+2\pi k$$

for all integers k.

Example 25. *Solve the equation* $\sin(t)=-\frac{2}{3}$.

Explanation As in the previous example, applying the inverse trig function $\arcsin\left(-\frac{2}{3}\right)$ gives us *a* solution to our equation. First let's find out where our solution lives. Since the range of arcsin is $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$ and the associated sine value is negative, we know that $\arcsin\left(-\frac{2}{3}\right)$ must be a negative angle in Quadrant IV, which corresponds to $\left[-\frac{\pi}{2},0\right]$. The only other quadrant in which sine is negative is Quadrant III, so the second angle in the period of sine that we're looking for is in Quadrant III with reference angle $-\arcsin\left(-\frac{2}{3}\right)$. The angle $\pi-\arcsin\left(-\frac{2}{3}\right)$ fits the bill. Therefore, a list of all solutions is

$$\arcsin\left(-\frac{2}{3}\right)+2\pi k \quad \text{and} \quad \pi-\arcsin\left(-\frac{2}{3}\right)+2\pi k$$

for all integers k.

Example 26. *Let's consider the composite function* $h(x)=\cos(\arcsin(x))$.
Does it makes sense to consider this function? Let's think ...
This function makes sense to consider since the arcsine function has range $[-1,1]$*, on which we may evaluate the cosine function. In the questions that follow, we investigate how to express* h *without using trigonometric functions at all.*

(a) *What is the domain of* h*? The range of* h*?*
Explanation The domain of h is the domain of the inner function,

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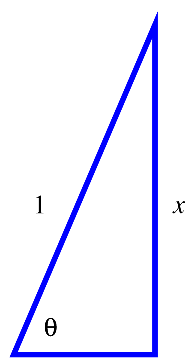
$\arcsin(x)$, which produces values within the domain of the outer function, $\cos(z)$. As noted at the beginning, since the range of $\arcsin(x)$ is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ contained in $(-\infty, \infty)$, the domain of $\cos(z)$, the domain of h is simply the domain of $\arcsin(x)$.
The domain of h is therefore $[-1, 1]$.

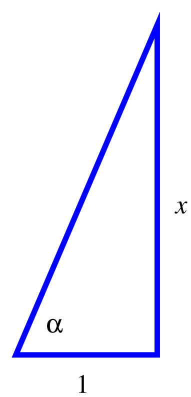
Now, the range of h will be the output of the outer function when the input is the range of the inner function. In other words, we are looking for the values that $\cos(z)$ attains on the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. Since cosine is symmetric about the y-axis, this is the same as the values attained by $\cos(z)$ on the interval $\left[0, \frac{\pi}{2}\right]$. Thus, we have a range of $[0, 1]$.

(b) *Since the arcsine function produces a value we can consider as an angle, let's say that* $\theta = \arcsin(x)$*, so that* θ *is the angle whose sine is* x*. By definition, we can picture* θ *as an angle in a right triangle with hypotenuse* 1 *and a vertical leg of length* x*, as shown in the image on the left below. Use the Pythagorean Theorem to determine the length of the horizontal leg as a function of* x*.*
Explanation First we recall the Pythagorean Theorem, $a^2 + b^2 = c^2$, where c is the hypotenuse of a right triangle with legs of lengths a, b. Hence, in this instance, let's denote the length of the horizontal leg by y, so we have $y^2 + x^2 = 1^2$. In other words

$$y = \sqrt{1 - x^2},$$

since a triangle leg will have positive length.





The right triangle on the left corresponds to the angle $\theta = \arcsin(x)$*. The right triangle on the right corresponds to the angle* $\alpha = \arctan(x)$*.*

(c) *What is the value of* $\cos(\theta)$ *as a function of* x*? What have we shown about* $h(x) = \cos(\arcsin(x))$*?*

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Explanation Here, we use the results of part (b). Since we know that $\cos(\theta)$ is $\dfrac{adj}{hyp}$ and $\theta = \arcsin(x)$,

$$\cos(\theta) = \frac{y}{1} = \sqrt{1-x^2}.$$

From this we see that

$$h(x) = \cos(\arcsin(x)) = \sqrt{1-x^2}.$$

(d) *How about the function $p(x) = \cos(\arctan(x))$? How can you reason similarly to write p in a way that doesn't involve any trigonometric functions at all? (Hint: let $\alpha = \arctan(x)$ and consider the right triangle on the right above.)*
Explanation We can now use a similar approach to determine p as an algebraic function of x. Let $\alpha = \arctan(x)$, so that $p(x) = \cos(\arctan(x)) = \cos(\alpha)$.

In the second triangle we must find the value of the hypotenuse, call it y. Then

$$y^2 = 1^2 + x^2 \text{ which implies } y = \sqrt{1+x^2}.$$

Now, $\cos(\alpha) = \dfrac{1}{y} = \dfrac{1}{\sqrt{1+x^2}}$. Therefore,

$$p(x) = \cos(\arctan(x)) = \frac{1}{\sqrt{1+x^2}}.$$

Using Inverse Trig in Applied Contexts

Now that we have developed the (restricted) sine, cosine, tangent, and secant functions and their respective inverses, in any setting in which we have a right triangle together with one side length and any one additional piece of information (another side length or a non-right angle measurement), we can determine all of the remaining pieces of the triangle. In the example that follows and the homework, we explore these possibilities in a variety of different applied contexts.

Example 27. *A roof is being built with a "7-12 pitch." This means that the roof rises 7 inches vertically for every 12 inches of horizontal span; in other words, the slope of the roof is $\dfrac{7}{12}$.*

(a) *What is the exact measure of the angle the roof makes with the horizontal?*
Explanation Looking at a side view of the house, we may divide the triangle of the roof in half to get a right triangle with legs 7 and 12 feet long. We want to find the angle of inclination (θ in the diagram below),

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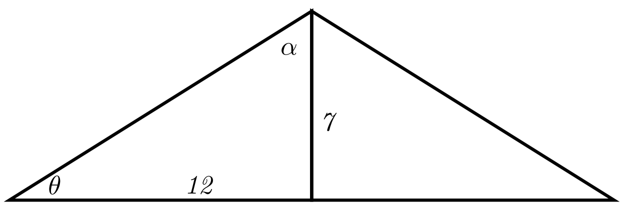
which satisfies the equation $\tan(\theta) = \frac{7}{12}$. In other words, we wish to find $\theta = \arctan\left(\frac{7}{12}\right)$. As in Example 1, this does not match one of our known values, so we leave it in this form since this is the *exact* value.



The image above is a side-view of the roof.

(b) *What is the exact measure of the angle at the peak of the roof (made by the front and back portions of the roof that meet to form the ridge)?*
Explanation This will be double the angle at the top of the right triangle we used for part (a), since we had bisected this angle to form the right triangle. We now wish to find the angle α satisfying $\tan(\alpha) = \frac{12}{7}$. In other words, we are looking for $\alpha = \arctan\left(\frac{12}{7}\right)$. Once again, this is not a common angle, so it is the *exact* value. We need double this angle, so $2\alpha = 2\arctan\left(\frac{12}{7}\right)$ is our solution.

Exploration On a baseball diamond (which is a square with 90-foot sides), the third baseman fields the ball right on the line from third base to home plate and 10 feet away from third base (towards home plate). Give exact solutions without using a computational device.

(a) When he throws the ball to first base, what angle does the line the ball travels make with the first base line?

(b) What angle does it make with the third base line? Draw a well-labeled diagram.

(c) What angles arise if he throws the ball to second base instead?

Exploration Give exact solutions without using a computational device. A camera is tracking the launch of a SpaceX rocket. The camera is located 4000' from the rocket's launching pad, and the camera elevates in order to keep the rocket in focus.

(a) At what angle is the camera tilted when the rocket is 3000' off the ground?

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Now, rather than considering the rocket at a fixed height of 3000 ft, let its height vary and call the rocket's height h.

(b) Determine the camera's angle, θ as a function of h, and compute the average rate of change of θ on the intervals $[3000, 3500]$, $[5000, 5500]$, and $[7000, 7500]$.

(c) What do you observe about how the camera angle is changing?

Further Exploration

When composing trigonometric functions with inverse trigonometric functions, the expressions can often be rewritten as algebraic expressions of x. We will see two examples of this below.

Example 28. *Rewrite the following values as algebraic expressions of x and give the domain on which these equivalences are valid.*

(a) $\cos(\arctan(x))$.
Explanation Recall that we found this expression in Example 2, part (d), to be

$$\cos(\arctan(x)) = \frac{1}{\sqrt{1+x^2}}.$$

Now, we must find the domain for which this is true.

We start by checking the domain of the outer function, $\cos(y)$. Since the domain of the cosine function is all real numbers, we do not have any restrictions to consider here. Thus, our only concern is the domain of the arctangent function, which is also the real line. Thus, we see that

$$\cos(\arctan(x)) = \frac{1}{\sqrt{1+x^2}} \text{ for all } x \text{ in } (-\infty, \infty).$$

(b) $\sin(\arccos(2x))$.
Explanation $\sin(\arccos(2x)) = \sqrt{1-4x^2}$ given x in $\left[-\frac{1}{2}, \frac{1}{2}\right]$
Again, to see this, we begin by setting $t = \arccos(2x)$, so that $\cos(t) = 2x$ for t in the domain of restricted cosine, $[0, \pi]$. In other words, we have $\cos(t) = 2x$ for t in $[0, \pi]$, and must find a formula for $\sin(t)$. Now, we must relate sine and cosine, for which we use the well-known trigonometric identity $\sin^2(t) + \cos^2(t) = 1$. Re-writing this to solve our equation, we see that we have $\sin^2(t) + (2x)^2 = 1$, which is equivalent to

$$\sin(t) = \pm\sqrt{1-4x^2}.$$

Since sine is positive on the interval $[0, \pi]$, where we defined t, we choose the positive square root, and observe that $\sin(\arccos(2x)) = \sqrt{1-4x^2}$, as desired.

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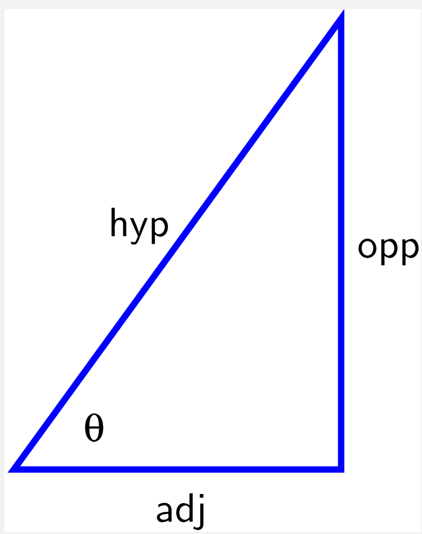
Finally, to establish the domain on which this equivalence holds, we recall that the domain of arccosine is $[-1,1]$. Since we consider $\arccos(2x)$, we want $2x$ in $[-1,1]$. This is equivalent to x in $\left[-\frac{1}{2},\frac{1}{2}\right]$, and so our work is done.

A Note on Triangles We can now use trigonometry to find angles of right triangles if we know the side lengths and side lengths of right triangles if we know the angles. You might be wondering, "What about triangles that are not right triangles? Can we use trig to learn anything about those?" It turns out that the Law of Sines and the Law of Cosines gives use a way to analyze other triangles beyond just right triangles using trig functions. For more information about this topic, see Laws of Sines and Cosines by Katherine Yoshiwara .

Summary Anytime we know two side lengths in a right triangle, we can use one of the inverse trigonometric functions to determine the measure of one of the non-right angles. For instance, if we know the values of opp and adj in the triangle pictured below, then since

$$\tan(\theta) = \frac{\text{opp}}{\text{adj}},$$

it follows that $\theta = \arctan\left(\frac{\text{opp}}{\text{adj}}\right)$.



If we instead know the hypotenuse and one of the two legs, we can use either the arcsine or arccosine function accordingly.
Similarly, we may use this relationship along with the Pythagorean Theorem to find algebraic expressions for compositions of trig functions with

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trig inverses (see Example 2). The trig identities we learned in Section 10 are also useful to rewrite the compositions of functions as algebraic expressions (see Example 4).

Part 11

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