# What is a Function?

## Motivating Questions

- How can functions be represented?
- When is a relation not a function?

### Respresentations of Functions

While the formal definition of a function is a set of ordered pairs, there are many ways that we represent functions when studying them. Each representation has advantages and disadvantages and being able to change between different representations of the same function is an important skill. Let's look at some different types of representations.

**Tables** Functions on a finite set of points are often represented by tables. One advantage of a table is that you can easily see all the information for a function. One disadvantage is if you have too many input values, it can be difficult to analyze them all in a table format. The exploration below shows and example of a function given by a table.

**Exploration** The function $f$ is defined by the table below. Note: this means the table gives all the values of the function.
Answer the questions about the function below.

| $x$ | $f(x)$ |
|---|---|
| banana | .5 |
| apple | 1.59 |
| pear | 2.50 |
| orange | 1 |

a. Is $f$ a function?

b. What is $f(\text{apple})$?

c. What are the inputs of $f$?

d. What are the outputs of $f$?

e. Rewrite $f$ as a set of ordered pairs.



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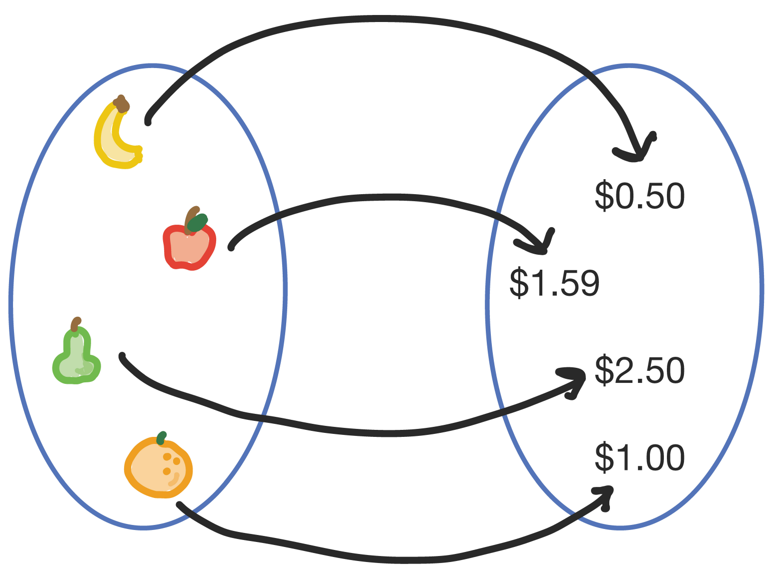
f. Assume that $f$ gives the price of a single item of a given fruit at a grocery store. Interpret $f(\text{orange}) = 1$ in this context.

If the set of ordered pairs that makes up your function is an infinite set, you cannot represent your entire function as a table because it would go on forever! Instead, you might see a table with a sampling of points from a function. In that case, unless you have additional information about your function, you cannot know what the outputs are for inputs not listed in the table. In fact, without additional information, you cannot even say which values are allowed to be inputs for the function!

**Arrow Diagrams** Arrow diagrams are another tool used to represent functions. Here is an arrow diagram that corresponds to the function $f$ in the exploration above.




Arrow diagrams can help make it easier to see when multiple inputs go to the same output. Arrow diagrams can also be used to represent relations, and can make it easier to see if the relation is a function. The three arrow diagrams below show relations. The first two relations are functions, but the last relation is not a function because the same input goes to two different outputs.



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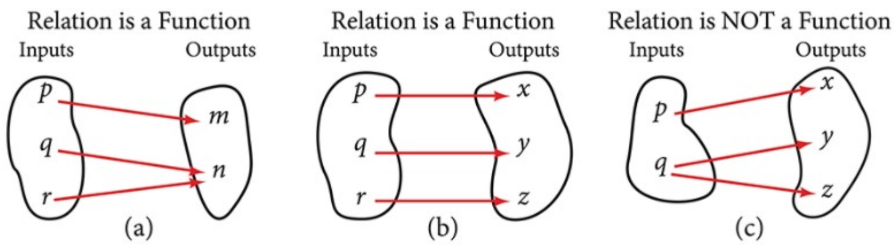


> **Remark** Remember: A function is always a relation, but not every relation is a function!

**Graphs** Graphing points on a coordinate plane is a great way to represent a function and one which is used often. Graphs are most often used for functions where both the inputs and outputs are numbers. When graphing functions, typically the horizontal axis represents the input values and the vertical axis represents the output values.

**Example 1.** *Here are some graphs of relations. Can you idenfity which of the relations represented here are functions?*




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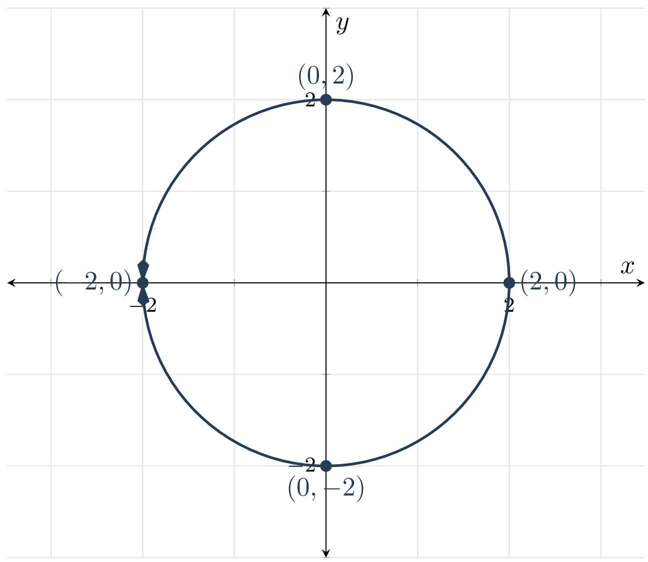


**Explanation**

a. A graph of a (nonvertical) line is the graph of a function. This is because, for each $x$-value, there is only one $y$-value that corresponds to it on the line.


The line in the graph above is the set of all points of the form $(x, 3)$ for any real number value of $x$.



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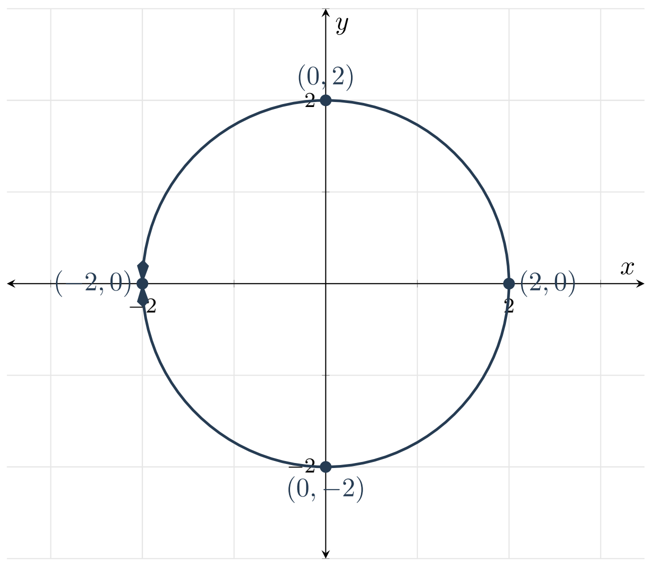
b. This graph of a circle is not the graph of a function. There exists an input value (many of them, actually) for which there are multiple different outputs.




For example, one the graph above, the points $(0, 2)$ and $(0, -2)$ are both on the graph. This tells us that for the input of $x = 0$, we have two outputs, $y = 2$ and $y = -2$. This circle represents a relation but not a function.

**Example 2.** *Here are some more graphs of relations. Can you idenfity which of the relations represented here are functions?*



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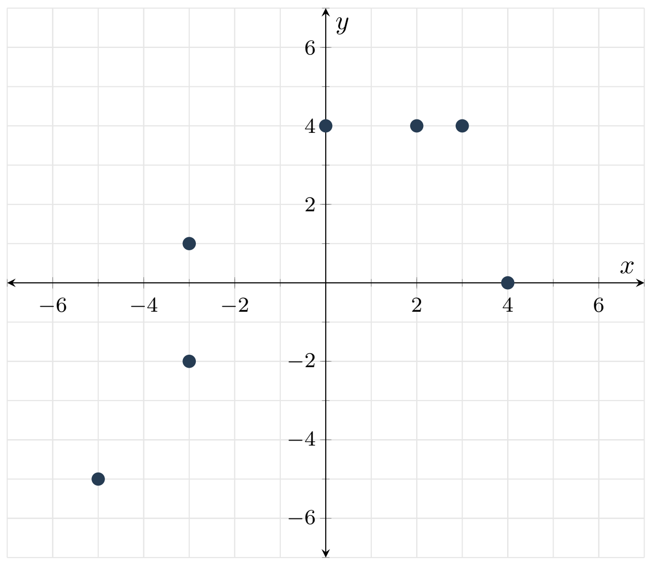


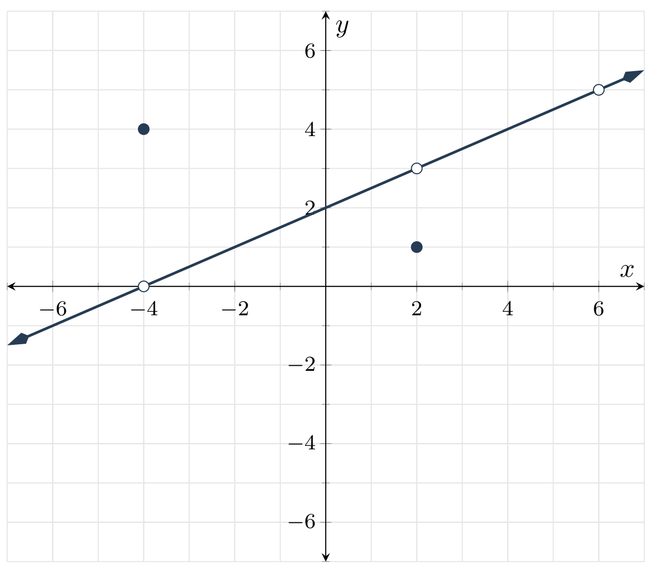




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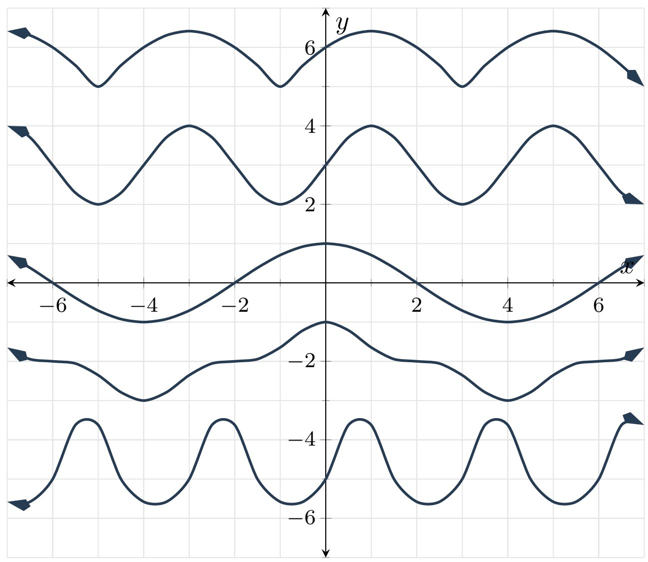


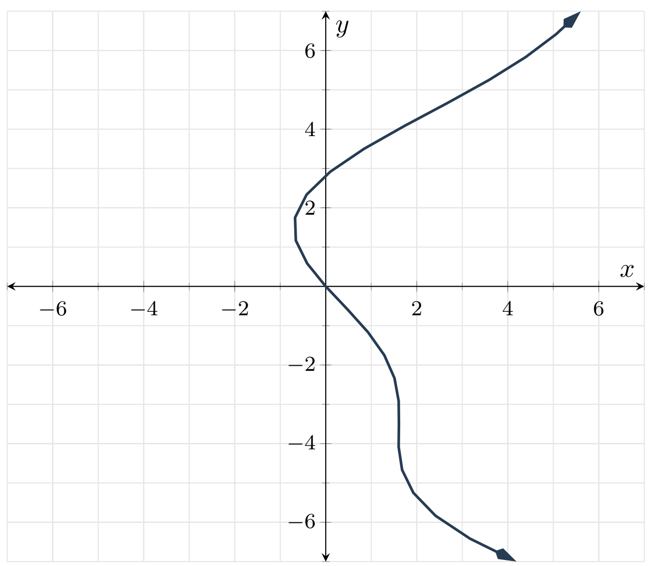


**Explanation** As in the previous example, we are looking to determine if there are any values of $x$ for which there are multiple outputs. Visually, what that means is there are places on the graph that are directly above/below each other. Thinking about this leads to a quick visual "test" to determine if a graph gives $y$ as a function of $x$.


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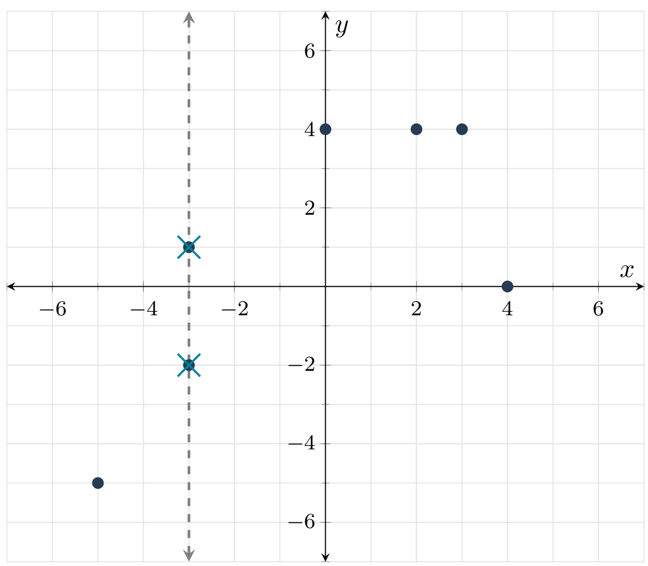
**Vertical Line "Test"** Given a graph in the $xy$-plane, if there exists a vertical line that touchs the graph in more than one place, the graph does not represent a function. If no such vertical line exists, then the relation represented by the graph is also a function.

Let's use this test to analyze each of the graphs in this example.

a. We notice that it is possible to draw a vertical line that touches this graph in two places.




Therefore, by the vertical line test, this graph is not the graph of a function. Really, the vertical line we found at $x = -3$ is helping us to quickly identify that both $(-3, -2)$ and $(-3, 1)$ are points on this graph so that for the same input, 3, there are two different outputs, $-2$ and 1.

b. This graph is the graph of a function. Here are some examples of some vertical lines we might consider.



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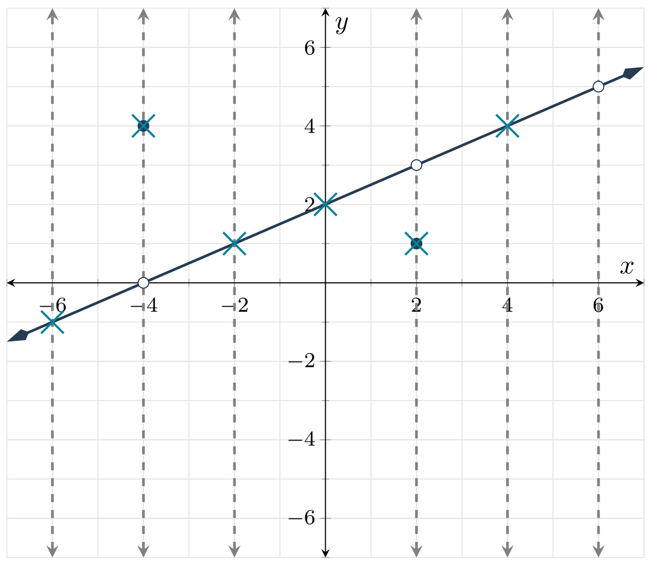


Note that we cannot draw all possible vertical lines. Really, we only need to look at the points on this graph which are not a line. We already said the graphs of lines (without any other points or curves above or below that line) are functions. So on this graph, the interesting points we want to look at more closely are $x = -4$, $x = 2$, and $x = 6$. In each of these cases, there is either one or no outputs. In particular, for $x = -4$ the corresponding output is $y = 4$. There is an open dot at $(-4, 0)$ so this point is not in the relation. Similarly, $(2, 1)$ is a point on this graph but $(2, 3)$ is not. At $x = 6$, there are no corresponds outputs. We say are function is not defined at $x = 6$.

c. This graph is not a function. Any vertical line we draw will cross the graph multiple times. Here is an example of a vertical line at $x = -4$.



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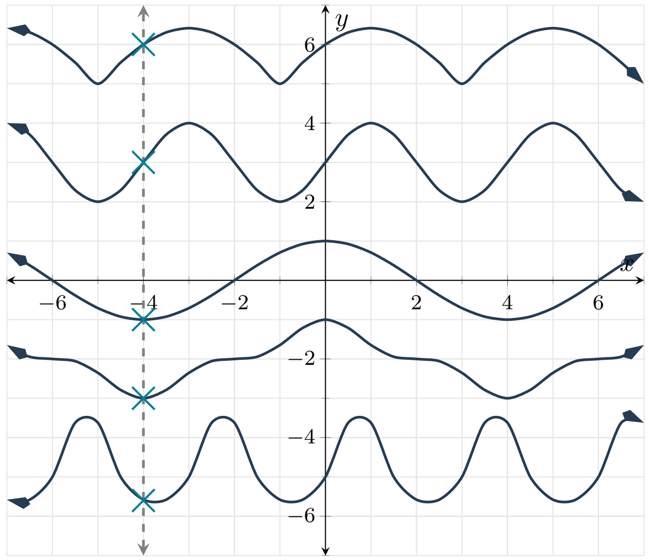
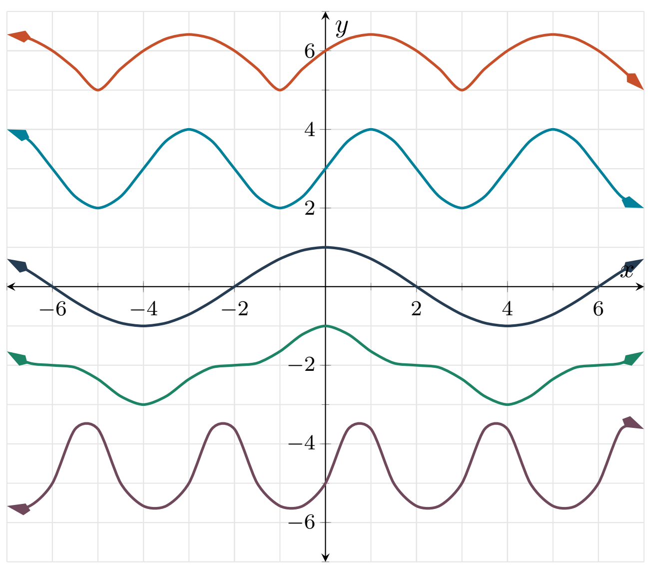



You could consider this a graph of five separate functions all graphed on the same coordinate axes, but that is a different question from the one being asked.




d. This graph is also not a function. Even though it is a single curve, it has input values, such as $x = 2$ with multiple corresponding output values.



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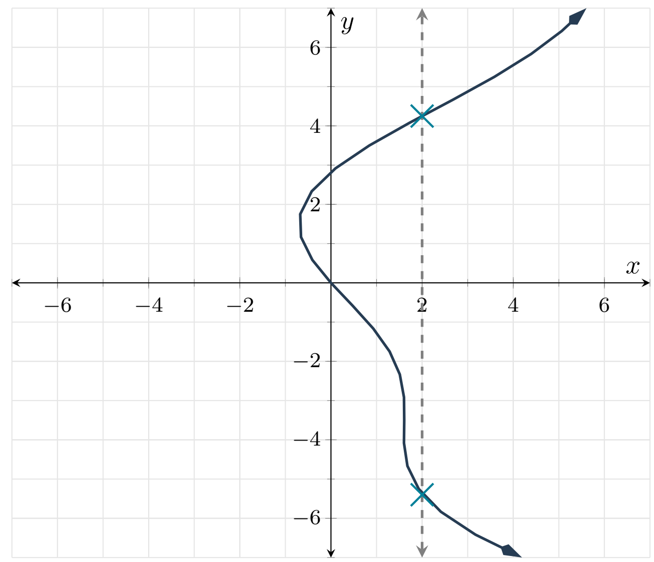


**Formula Representation** Another common way to represent functions is using a formula. In the example of Dolbear's Law, the function which models this law is given as a formula by

$$D(N) = 40 + 0.25N \text{ for } 50 \leq x \leq 85$$

When we give functions as a formula, we also need to say which input values are allowed. In this case, the allowed inputs are $50 \leq x \leq 85$. If allowed inputs are not given, the inputs are assumed to be all the values for which the formula used to defined the function makes sense.

Here are some examples of functions represented by a formula.

- $f(x) = x^2$
- $g(x) = 5x - 7$
- $h(x) = \sin(x)$

  Recall that this is the famous function named sine.
- $z(x) = \dfrac{x^2 \sin(x) - 2}{72 - x}$

Here is an example of working with the formula representation of a function.

**Example 3.** *Let $f(x) = 5x - 2$. Find $f(1)$.*

**Explanation** Recall that the notation $f(1)$ means to evaluate the function $f$ at the input $x = 1$. That is, $f(1) = 5(1) - 2 = 3$.


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### Intercepts

Zero is a very important number, and as we will see later, knowing where a function's $x$- or $y$-value equals zero can be powerful information.

**Definition** [Intercepts] Say $f$ is a function.
An $x$**-intercept** of $f$ is a point $(x, 0)$ such that $f(x) = 0$. That is, a point in which the graph of the function touches the $x$-axis.
The $y$**-intercept** of $f$ is a point $(0, y)$ such that $f(0) = y$. That is, a point in which the graph of the function touches the $y$-axis. Unlike $x$-intercepts, there can be only one $y$-intercept for each function.

## Summary

- Functions can be represented in many ways including as:
  - sets of ordered pairs
  - tables
  - arrow diagrams
  - graphs
  - formula
  - domain and range



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