Precalculus with Review 1: Unit 4

September 13, 2022

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Part 1

Building New Functions

1.1 Building New Functions

Learning Objectives

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- Creating a new Famous Function: Tangent
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1.1.1 Algebra of Functions

Motivating Questions

- We know that we can add, subtract, multiply, and divide numbers. What kinds of operations can we perform on functions?

Introduction

In arithmetic, we execute processes where we take two numbers to generate a new number. For example, $2 + 3 = 5$. The number 5 results from adding the numbers 2 and 3. Similarly, we can multiply two numbers to generate a new one: $2 \cdot 3 = 6$.

We can work similarly with functions. Just as we can add, subtract, multiply, and divide numbers, we can also add, subtract, multiply, and divide functions to create a new function from two or more given functions.

Algebra of Functions

In most mathematics up until calculus, the main object we study is *numbers*. We ask questions such as

- "What number(s) form solutions to the equation $x^2 - 4x - 5 = 0$?"
- "What number is the slope of the line $3x - 4y = 7$?"
- "What number is generated as output by the function $f(x) = \sqrt{x^2+1}$ by the input $x = -2$?"

Certainly we also study overall patterns as seen in functions and equations, but this usually occurs through an examination of numbers themselves, and we think of numbers as the main objects being acted upon.

This changes in calculus. In calculus, the fundamental objects being studied are functions themselves. A function is a much more sophisticated mathematical object than a number, in part because a function can be thought of in terms of its graph, which is an infinite collection of ordered pairs of the form $(x, f(x))$.

It is often helpful to look at a function's formula and observe algebraic structure. For instance, given the quadratic function

$$q(x) = -3x^2 + 5x - 7$$

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we might benefit from thinking of this as the sum of three simpler functions: the constant function $c(x) = -7$, the linear function $s(x) = 5x$ that passes through the point $(0, 0)$ with slope $m = 5$, and the concave down basic quadratic function $w(x) = -3x^2$. Indeed, each of the simpler functions c, s, and w contribute to making q be the function that it is. Likewise, if we were interested in the function $p(x) = (3x^2 + 4)(9 - 2x^2)$, it might be natural to think about the two simpler functions $f(x) = 3x^2 + 4$ and $g(x) = 9 - 2x^2$ that are being multiplied to produce p.

We thus naturally arrive at the ideas of adding, subtracting, multiplying, or dividing two or more functions, and hence introduce the following definitions and notation.

Definition Let f and g be functions.

- The **sum of** f **and** g is the function $f + g$ defined by $(f + g)(x) = f(x) + g(x)$.

- The **difference of** f **and** g is the function $f - g$ defined by $(f - g)(x) = f(x) - g(x)$.

- The **product of** f **and** g is the function $f \cdot g$ defined by $(f \cdot g)(x) = f(x) \cdot g(x)$.

- The **quotient of** f **and** g is the function $\dfrac{f}{g}$ defined by $\left(\dfrac{f}{g}\right)(x) = \dfrac{f(x)}{g(x)}$ for all x such that $g(x) \neq 0$.

We are thinking here about f and g being functions with real numbers as outputs. Performing these operations on the functions means applying the corresponding operation to the output values of the functions.

Example 1. *Consider the functions f and g defined by the table of values below.*

x	$f(x)$
2	0
4	3
6	7
8	-2

x	$g(x)$
1	5
2	9
3	-1
4	4

(a) *Determine the value of* $(f + g)(2)$.

(b) *Determine the value of* $(f - g)(4)$.

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(c) *Determine the value of* $(f \cdot g)(2)$.

(d) *Determine the value of* $\left(\frac{f}{g}\right)(4)$.

(e) *What can we say about the value of* $(f+g)(3)$*?*

Explanation

(a) We know that $(f+g)(2) = f(2) + g(2)$. From the tables above $f(2) = 0$ and $g(2) = 9$, so $(f+g)(2) = 0 + 9 = 9$.

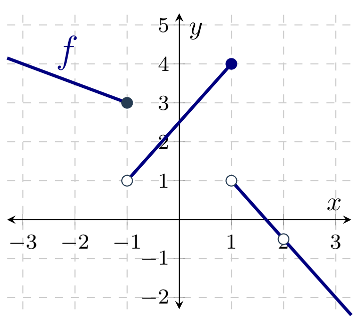
(b) Since $f(4) = 3$ and $g(4) = 4$, we know $(f-g)(4) = 3 - 4 = -1$.

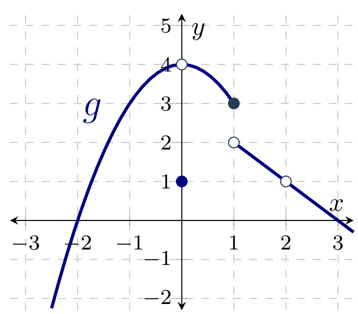
(c) $(f \cdot g)(2) = f(2) \cdot g(2) = 0 \cdot 9 = 0$.

(d) $\left(\frac{f}{g}\right)(4) = \frac{f(4)}{g(4)} = \frac{3}{4}$.

(e) The value of $(f+g)(3)$ would be given by $f(3) + g(3)$. we are given the value of $g(3)$ in the table above, but there is no listed value for $f(3)$. That means $f(3)$ is undefined, since 3 is not a valid input. $(f+g)(3)$ is undefined.

Example 2. *Consider the functions f and g defined by*





(a) *Determine the exact value of* $(f+g)(0)$.

(b) *Determine the exact value of* $(g-f)(1)$.

(c) *Determine the exact value of* $(f \cdot g)(-1)$.

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(d) *Are there any values of x for which* $\left(\dfrac{f}{g}\right)(x)$ *is undefined? If not, explain why. If so, determine the values and justify your answer.*

(e) *For what values of x is $(f \cdot g)(x) = 0$? Why?*

Explanation

(a) The notation $(f + g)(0)$ means we are plugging the input 0 into both functions f and g, then *adding* the results. That is, $(f+g)(0) = f(0)+g(0)$. From the graphs above we see $f(0) = \dfrac{5}{2}$ and $g(0) = 1$. That means $(f + g)(0) = \dfrac{5}{2} + 1 = \dfrac{7}{2}$.

(b) The notation $(g - f)(1)$ means we are plugging the input 1 into both functions g and f, then *subtracting* the results.That is, $(g-f)(1) = g(1)-f(1)$. From the graphs above we see $f(1) = 4$ and $g(1) = 3$. That means $(g - f)(1) = 3 - 4 = -1$.

(c) The notation $(f \cdot g)(-1)$ means we are plugging the input -1 into both functions f and g, then *multiplying* the results. That is, $(f \cdot g)(-1) = f(-1) \cdot g(-1)$. From the graphs above we see $f(-1) = 3$ and $g(-1) = 3$, which tells us $(f \cdot g)(-1) = 3 \cdot 3 = 9$.

(d) For any valid value of the input x, $\left(\dfrac{f}{g}\right)(x) = \dfrac{f(x)}{g(x)}$. In order for that fraction to be defined $f(x)$ has to exist, $g(x)$ has to exist, and $g(x) \neq 0$ since division by zero is undefined. From the graphs above, f is defined for all x-values except $x = 2$, and g is defined for all x-values except $x = 2$. That tells us that $\left(\dfrac{f}{g}\right)(2)$ is undefined. Notice that $g(-2) = 0$ and $g(3) = 0$? That means $\left(\dfrac{f}{g}\right)(x)$ is undefined at $x = -2$ and $x = 3$ as well

(e) Since $(f \cdot g)(x) = f(x) \cdot g(x)$, if an x-value makes $(f \cdot g)(x) = 0$, then $f(x) \cdot g(x) = 0$. The only way a product of two real numbers can be zero is if at least one of the factors is itself zero. That means we are looking for all of the x-values satisfying either $f(x) = 0$ or $g(x) = 0$. (In other words, we're looking for the x-intercepts of these graphs.)

From the graph of g we see that $g(-2) = 0$ and $g(3) = 0$. The graph of f crosses the x-axis somewhere between the points $(1, 0)$ and $(2, 0)$, but we will have to be more careful to find the exact value we are looking for.

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Notice that the graph of f looks to be a straight line if we only look at those x-values with $x > 1$. The straight line that f follows travels through the point $(1, 1)$ and $(3, -2)$. Its slope is given by $m = \dfrac{-2-1}{3-1} = -\dfrac{3}{2}$. Since the line contains the point $(1, 1)$, the point-slope form of the equation of the line can be written as $y - 1 = -\dfrac{3}{2}(x-1)$. This line crosses the x-axis when its y-coordinate is zero. Solving for the corresponding x-value gives us:

$$\begin{aligned}
0 - 1 &= -\frac{3}{2}(x-1) \\
-1 &= -\frac{3}{2}(x-1) \\
-\frac{2}{3}\cdot(-1) &= -\frac{2}{3}\cdot\left(-\frac{3}{2}(x-1)\right) \\
\frac{2}{3} &= x - 1 \\
\frac{2}{3} + 1 &= x \\
\frac{5}{3} &= x.
\end{aligned}$$

That means the point $\left(\dfrac{5}{3}, 0\right)$ is on the graph of f, so $f\left(\dfrac{5}{3}\right) = 0$.

The x-values with $(f \cdot g)(x) = 0$ are $x = -2$, $x = \dfrac{5}{3}$, and $x = 3$.

Consider the functions $f(x) = 2x$ and $g(x) = x$. These are functions whose graphs are straight lines with slopes 2 and 1 respectively.

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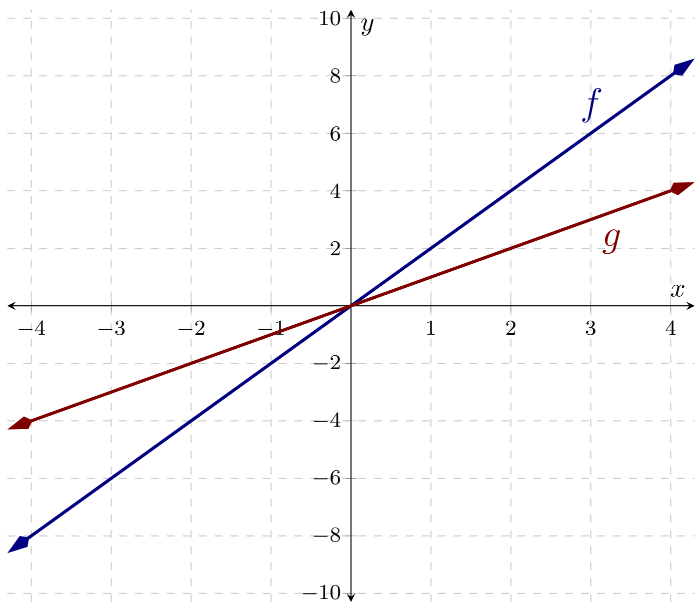
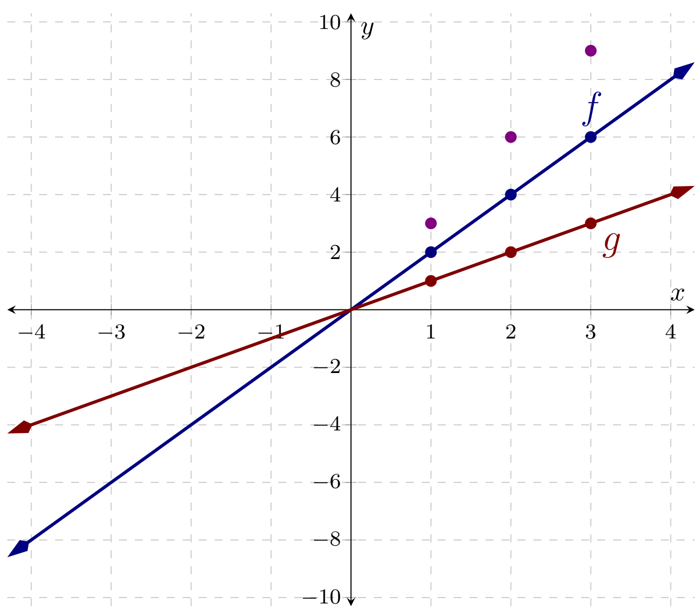


Let's look at a few points on these graphs. Since $f(1) = 2$ and $g(1) = 1$, the sum of those output values is 3, so we'll mark the point $(1, 3)$. Similarly $f(2) = 4$ and $g(2) = 2$, with $4 + 2 = 6$ so we'll mark the point $(2, 6)$. As $f(3) = 6$ and $g(3) = 3$, we'll also mark $(3, 9)$.



Notice that the points we've marked $(1, 3)$, $(2, 6)$, and $(3, 9)$ are starting to form

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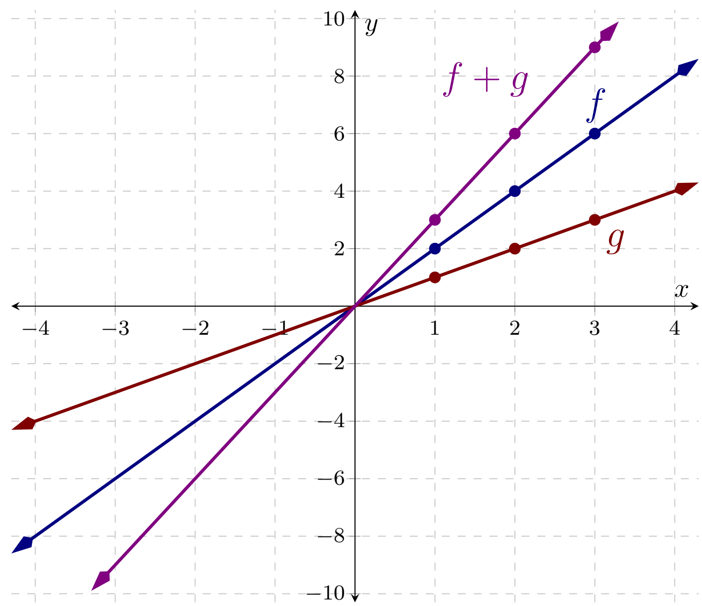
a straight line. Let's connect those dots to examine the line constructed this way.



The graph obtained this way is the graph of $f+g$. This graph is a straight line passing through the points $(0,0)$ and $(1,3)$, so the line has equation $y=3x$.

For a given value of x, we know $(f+g)(x)=f(x)+g(x)$. This means $(f+g)(x)=2x+x=3x$ by combining the like terms. Notice that this aligns with the graph we found above. This example shows that we can work with these operations through formulas for our functions as well.

Example 3. *Let f and g be the functions given by the formulas $f(x)=2\sin(x)$ and $g(x)=5x-4$.*

(a) *Find the value of $(f-g)\left(\frac{\pi}{3}\right)$.*

(b) *Find a formula for $(f+g)(x)$.*

(c) *Find a formula for $(f\cdot g)(x)$.*

(d) *Find a formula for $\left(\frac{f}{g}\right)(x)$.*

Explanation

 Acknowledgements: Active Prelude to Calculus (https://activecalculus.org/prelude)

(a) $(f-g)\left(\frac{\pi}{3}\right) = f\left(\frac{\pi}{3}\right) - g\left(\frac{\pi}{3}\right)$. Sine is one of our famous functions, which has $\sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$, so $f\left(\frac{\pi}{3}\right) = 2\sin\left(\frac{\pi}{3}\right) = 2\frac{\sqrt{3}}{2} = \sqrt{3}$. $g\left(\frac{\pi}{3}\right) = 5\left(\frac{\pi}{3}\right) - 4 = \frac{5\pi}{3} - 4$.

Together this gives $(f-g)\left(\frac{\pi}{3}\right) = \sqrt{3} - \frac{5\pi}{3} + 4$

(b) $(f+g)(x) = f(x) + g(x) = 2\sin(x) + 5x - 4$.

(c) $(f \cdot g)(x) = 2\sin(x)\,(5x-4)$.

(d) $\left(\frac{f}{g}\right)(x) = \frac{2\sin(x)}{5x-4}$.

When we work in applied settings with functions that model phenomena in the world around us, it is often useful to think carefully about the units of various quantities. Analyzing units can help us both understand the algebraic structure of functions and the variables involved, as well as assist us in assigning meaning to quantities we compute. We have already seen this with the notion of average rate of change: if a function $P(t)$ measures the population in a city in year t and we compute $\text{AROC}_{[5,11]}$, then the units on $\text{AROC}_{[5,11]}$ are "people per year," and the value of $\text{AROC}_{[5,11]}$ is telling us the average rate at which the population changes in people per year on the time interval from year 5 to year 11.

Example 4. *Say that an investor is regularly purchasing stock in a particular company. Let $N(t)$ represent the number of shares owned on day t, where $t = 0$ represents the first day on which shares were purchased. Let $S(t)$ give the value of one share of the stock on day t; note that the units on $S(t)$ are dollars per share. How is the total value, $V(t)$, of the held stock on day t determined?*
Explanation Observe that the units on $N(t)$ are "shares" and the units on $S(t)$ are "dollars per share". Thus when we compute the product

$$N(t)\,\text{shares} \cdot S(t)\,\text{dollars per share},$$

it follows that the resulting units are "dollars", which is the total value of held stock. Hence,

$$V(t) = N(t) \cdot S(t).$$

Exploration Let f be a function that measures a car's fuel economy in the following way. Given an input velocity v in miles per hour, $f(v)$ is the number of gallons of fuel that the car consumes per mile (i.e., "gallons per mile"). We know that $f(60) = 0.04$.

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(a) What is the meaning of the statement "$f(60) = 0.04$" in the context of the problem? That is, what does this say about the car's fuel economy? Write a complete sentence.

(b) Consider the function $g(v) = \dfrac{1}{f(v)}$. What is the value of $g(60)$? What are the units on g? What does g measure?

(c) Consider the function $h(v) = v \cdot f(v)$. What is the value of $h(60)$? What are the units on h? What does h measure?

(d) Do $f(60)$, $g(60)$, and $h(60)$ tell us fundamentally different information, or are they all essentially saying the same thing? Explain.

(e) Suppose we also know that $f(70) = 0.045$. Find the average rate of change of f on the interval $[60, 70]$. What are the units on the average rate of change of f? What does this quantity measure? Write a complete sentence to explain.

Taking a complicated function and determining how it is constructed out of simpler ones is an important skill to develop. At the beginning of this section we split the function $q(x) = -3x^2 + 5x - 7$ into the sum/difference of three simple functions, $-3x^2$, $5x$, and 7. Let us experiment with splitting a few more complicated functions.

Example 5. (a) *Find functions h and k so that $f(x) = 4x^2 \sin(x)$ can be written as $(h \cdot k)(x)$.*

(b) *Find functions f and g so that $h(x) = \dfrac{2x+3}{x-1}$ can be written as $\left(\dfrac{f}{g}\right)(x)$.*

(c) *Find functions r and s so that $t(x) = \dfrac{x+1}{3x-1} + xe^x$ can be written as $(r+s)(x)$.*

(d) *Find functions f, g, and h so that $k(x) = \dfrac{\sin(x)\sqrt{2x+3}}{1+\ln(x)}$ can be written as $\left(\dfrac{f \cdot g}{h}\right)(x)$.*

Explanation

Before working through these questions, we want to remind you that the answers we give are not unique. There are many different, equally valid choices for the simpler functions requested. For the first question, we will mention multiple possibilities. We leave it to you to find other answers for the remaining questions.

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(a) Notice that if $h(x) = 4x^2$ and $k(x) = \sin(x)$, then $(h \cdot k)(x) = 4x^2 \sin(x)$, so this choice of h and k is one valid answer. What if we had chosen $h(x) = 4x$ and $k(x) = x\sin(x)$ instead? Then $(h \cdot k)(x) = (4x)(x\sin(x)) = 4x^2 \sin(x)$, so this choice would also be a valid answer. Another possibility would have been to choose $h(x) = 4$ and $k(x) = x^2 \sin(x)$.

(b) If we set $f(x) = 2x + 3$ and $g(x) = x - 1$, then $\left(\frac{f}{g}\right)(x) = \frac{2x+3}{x-1}$.

(c) Since $t(x) = \frac{x+1}{3x-1} + xe^x$ is written as a sum of two terms, take $r(x) = \frac{x+1}{3x-1}$ to be the first term and $s(x) = xe^x$ to be the seond term. Then $(r + s)(x) = t(x)$.

(d) We are trying to identify $\frac{\sin(x)\sqrt{2x+3}}{1+\ln(x)}$ as a fraction with the numerator a product. The denominator is $1 + \ln(x)$ and the numerator is a product of $\sin(x)$ and $\sqrt{2x+3}$. We will choose $f(x) = \sin(x)$, $g(x) = \sqrt{2x+3}$, and $h(x) = 1 + \ln(x)$.

Exploration

(a) Find functions f and g so that $h(x) = \frac{5e^x}{1+\sin(x)}$ can be written as $(f \cdot g)(x)$.

(b) Find functions h and k so that $f(x) = 3x^2 - \sqrt{x+1}$ can be written as $(h + k)(x)$. Find two other choices for h and k.

(c) If f and g are functions, we know that $f + g$ is the function given by $(f + g)(x) = f(x) + g(x)$. What function do you think the notation $f + 3g$ means? Find functions f and g so that $h(x) = 4x^2 - 5\sqrt{x} + 7 \cdot 2^x$ can be written as $(f + 3g)(x)$.

1.1.2 Composition of Functions

Motivating Questions

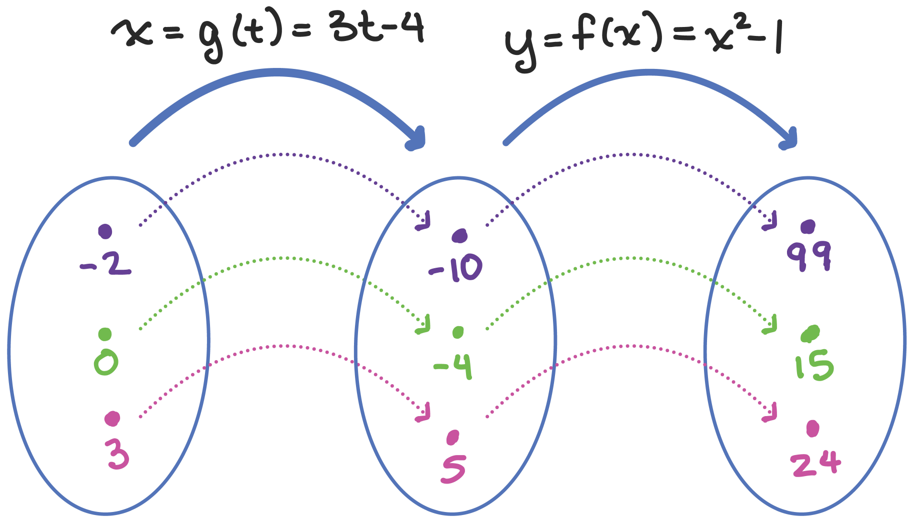
- How does the process of function composition produce a new function from two other functions?
- In the composite function $h(x) = f(g(x))$, what do we mean by the "inner" and "outer" function?
- How does the expression for $\text{AROC}_{[a,a+h]}$ involve a composite function?

Introduction

Recall that a function, by definition, is a process that takes a collection of inputs and produces a corresponding collection of outputs in such a way that the process produces one and only one output value for any single input value. Because every function is a process, it makes sense to think that it may be possible to take two function processes and do one of the processes first, and then apply the second process to the result.

Example 6. *Suppose we know that y is a function of x according to the process defined by $y = f(x) = x^2 - 1$ and, in turn, x is a function of t via $x = g(t) = 3t - 4$. Is it possible to combine these processes to generate a new function so that y is a function of t?*

Explanation Since y depends on x and x depends on t, it follows that we can also think of y depending directly on t. Let's looks at this as an arrow diagram with a few sample points.



Notice that if we take a point such as $t = 2$, we can put that value in for t in the function $x = g(t) = 3t - 4$. This will give

$$x = g(-2) = 3(-2) - 4 = -6 - 4 - 10.$$

Now we have an x-value of -10. $g(x)$ take in x-values so we can put -10 into $f(x) = x^2 - 1$. This will give

$$f(-10) = (-10)^2 - 1 = 100 - 1 = 99.$$

You should verify that the arrow diagram above gives the correct values of y that corresponds to $t = 0$ and $t = 3$.

Now, we would like to create a new function that will directly take in any t value and give us the corresponding y value. We can use substitution and the notation of functions to determine this function.

First, it's important to realize what the rule for f tells us. In words, f says "to generate the output that corresponds to an input, take the input and square it, and then subtract 1." In symbols, we might express f more generally by writing "$f(\square) = \square^2 - 1$."

Now, observing that $y = f(x) = x^2 - 1$ and that $x = g(t) = 3t - 4$, we can substitute the expression $g(t)$ for x in f. Doing so,

$$\begin{aligned} y &= f(x) \\ &= f(g(t)) \\ &= f(3t - 4). \end{aligned}$$

Applying the process defined by the function f to the input $3t - 4$, we see that

$$y = (3t - 4)^2 - 1,$$

which defines y as a function of t.

One way to think about the substitution above is that we are putting the entire expression $3t-4$ inside the input box in "$f(\square) = \square^2-1$." That is, $f\left(\boxed{3t-4}\right) = \left(\boxed{3t-4}\right)^2 - 1$. For the substitution, we are thinking of $3t-4$ as a single object!

When we have a situation such as in the example above where we use the output of one function as the input of another, we often say that we have **composed** two functions. In addition, we use the notation $h(t) = f(g(t))$ to denote that a new function, h, results from composing the two functions f and g.

Exploration

a. Let $y = p(x) = 3x - 4$ and $x = q(t) = t^2$. Determine a formula for r that depends only on t and not on p or q. What is the biggest difference between your work in this problem compared to the example above?

b. Let $t = s(z) = \dfrac{1}{t+4}$ and recall that $x = q(t) = t^2$. Determine a formula for $x = q(s(z))$ that depends only on z.

c. Suppose that $h(t) = \sqrt{2t^2 + 5}$. Determine formulas for two related functions, $y = f(x)$ and $x = g(t)$, so that $h(t) = f(g(t))$.

Composing Two Functions

Whenever we have two functions, g and f, where the outputs of g match inputs of f, it is possible to link the two processes together to create a new process that we call the *composition* of f and g.

Definition If f and g are functions, we define the **composition of** f **and** g to be the new function h given by

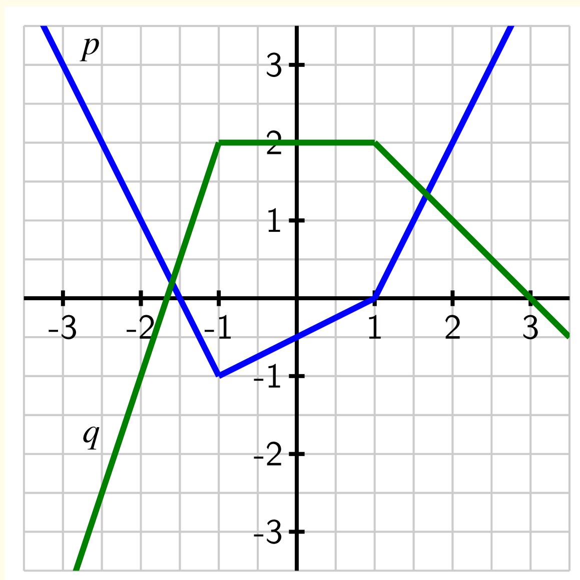
$$h(t) = f(g(t)).$$

This composition is denoted by $h = f \circ g$, where $f \circ g$ means the single function defined by $(f \circ g)(t) = f(g(t))$.

We sometimes call g the "inner function" and f the "outer function". It is important to note that the inner function is actually the first function that gets applied to a given input, and then the outer function is applied to the output of the inner function. In addition, in order for a composite function to make sense, we need to ensure that the outputs of the inner function are values that it makes sense to put into the outer function so that the resulting composite function is defined.

In addition to the possibility that functions are given by formulas, functions can be given by tables or graphs. We can think about composite functions in these settings as well, and the following activities prompt us to consider functions given in this way.

Exploration Let functions p and q be given by the graphs below (which are each piecewise linear - that is, parts that look like straight lines are straight lines) and let f and g be given by the table below.



x	$f(x)$	$g(x)$
0	6	1
1	4	3
2	3	0
3	4	4
4	6	2

Compute each of the following quantities or explain why they are not defined.

a. $p(q(0))$

b. $q(p(0))$

c. $(p \circ p)(-1)$

d. $(f \circ g)(2)$

e. $(g \circ f)(3)$

f. $g(f(0))$

g. For what value(s) of x is $f(g(x)) = 4$?

h. For what value(s) of x is $q(p(x)) = 1$?

Composing functions in content

In the late 1800s, the physicist Amos Dolbear was listening to crickets chirp and noticed a pattern: how frequently the crickets chirped seemed to be connected to the outside temperature. If we let T represent the temperature in degrees Fahrenheit and N the number of chirps per minute, we can summarize Dolbear's observations with the following function, $T = D(N) = 40 + 0.25T$. Scientists who made many additional cricket chirp observations following Dolbear's initial counts found that this formauls holds with remarkable accuracy for the snowy tree cricket in temperatures ranging from 50° to 85°. This function is called Dolbear's Law.

In what follows, we replace T with F to emphasize that temperature is measured in Fahrenheit degrees.

The Celcius and Fahrenheit temperature scales are connected by a linear function. Indeed, the function that converts Fahrenheit to Celcius is

$$C = G(F) = \frac{5}{9}(F - 32).$$

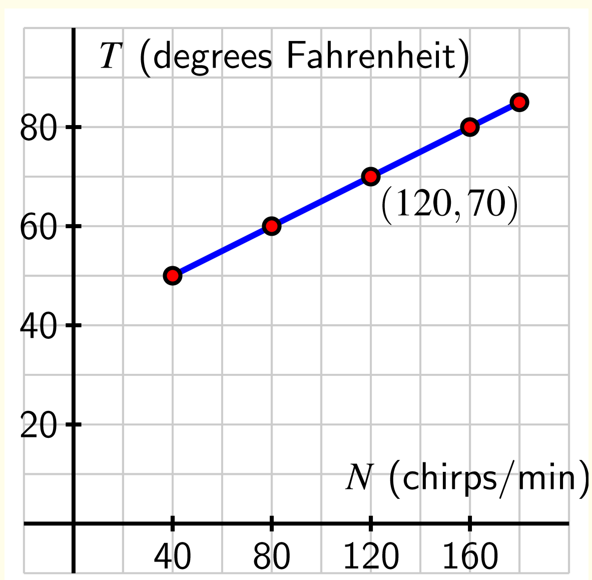
For instance, a Fahrenheit temperature of 32 degrees corresponds to $C = G(32) = \frac{5}{9}(32 - 32) = 0$ degrees Celcius.

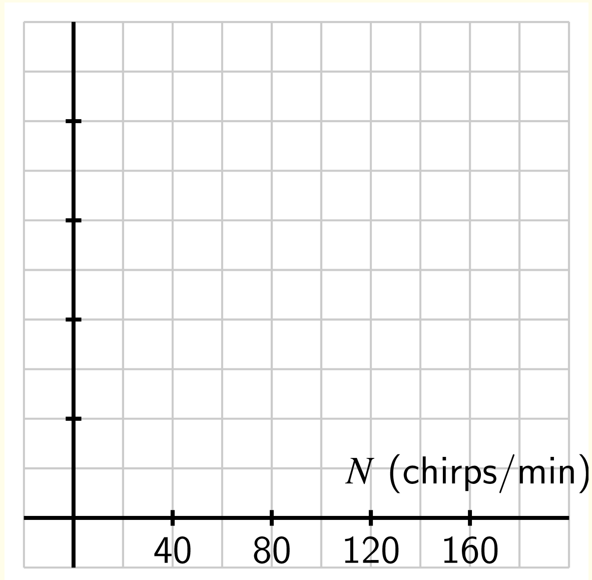
Exploration Let $D(N) = 40 + 0.25N$ be Dolbear's function that converts an input of number of chirps per minute to degrees Fahrenheit, and let $G(F) = \frac{5}{9}(F - 32)$ be the function that converts an input of degrees Fahrenheit to an output of degrees Celcius.

a. Determine a formula for the new function $(G \circ D)(N)$ that depends only on the variable N.

b. What is the meaning of the function you found in (a)?

c. Let $H = G \circ D$. How does a plot of the function H compare to the that of Dolbear's function? Sketch a plot of H on the blank axes to the right of the plot of Dolbear's function, and discuss the similarities and differences between them. Be sure to label the vertical scale on your axes.





Summary

- When defined, the composition of two functions f and g produces a single new function $f \circ g$ according to the rule $(f \circ g)(x) = f(g(x))$. We note that g is applied first to the input x, and then f is applied to the output $g(x)$ that results from g.
- In the composite function $h(x) = f(g(x))$, the "inner" function is g and the *outer* function is f. Note that the inner function gets applied to x first, even though the outer function appears first when we read from left to right.

1.2 Polynomials

Learning Objectives

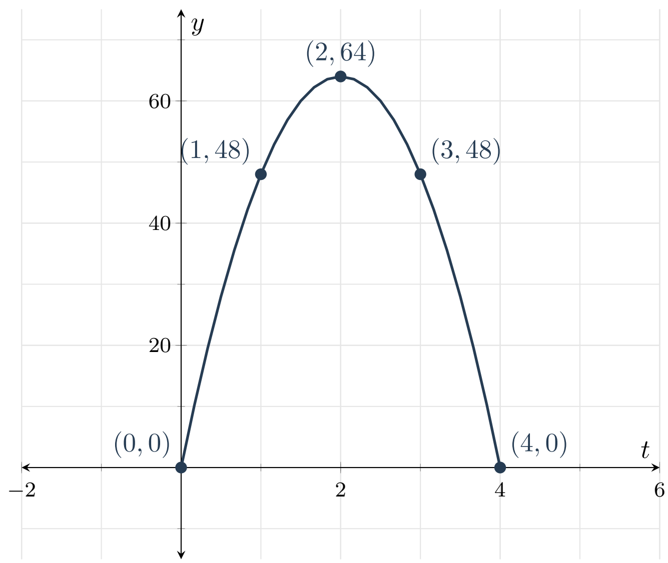
- Quadratics
  - What does a parabola look like? When do they open up versus down?
  - What is the vertex of a parabola and how can one find it?
  - Forms of parabolas, including general polynomial form

1.2.1 Definition of Quadratics

Quadratic Graphs

Example 7. *Hannah fired a toy rocket from the ground, which launched into the air with an initial speed of* 64 *feet per second. The height of the rocket can be modeled by the equation* $y = -16t^2 + 64t$*, where* t *is how many seconds had passed since the launch. To see the shape of the graph made by this equation, we make a table of values and plot the points.*

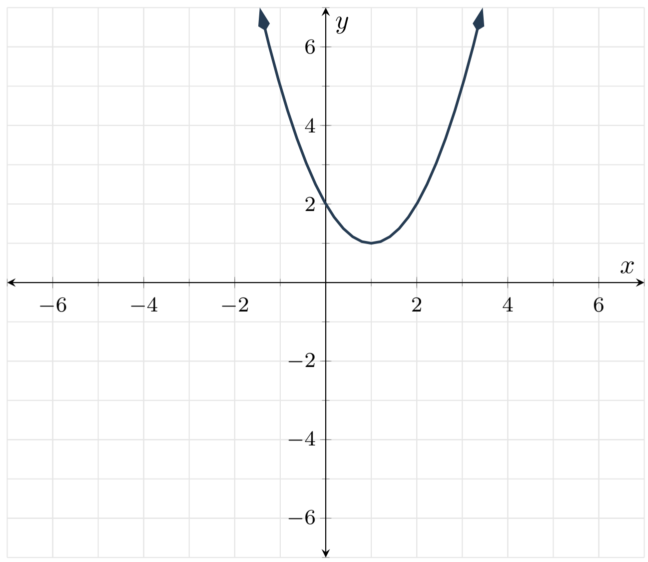
t	$-16t^2 + 64t$	*Point*
0	$-16(0)^2 + 64(0) = 0$	$(0, 0)$
1	$-16(1)^2 + 64(1) = 48$	$(1, 48)$
2	$-16(2)^2 + 64(2) = 64$	$(2, 64)$
3	$-16(3)^2 + 64(3) = 48$	$(3, 48)$
4	$-16(4)^2 + 64(4) = 0$	$(4, 0)$

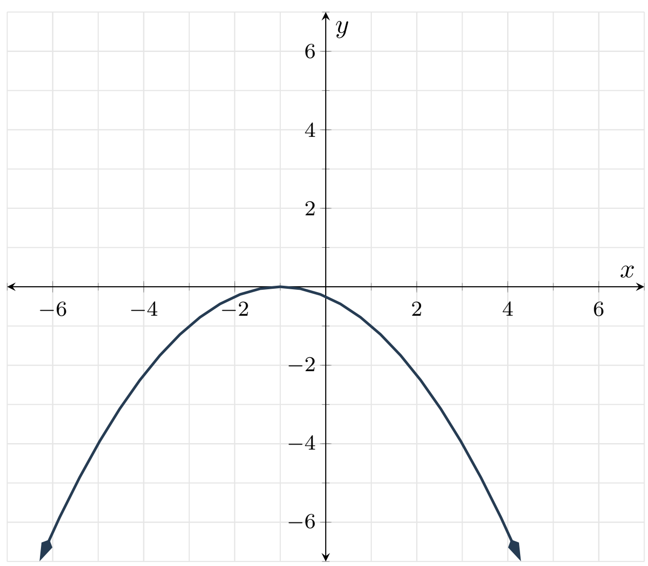


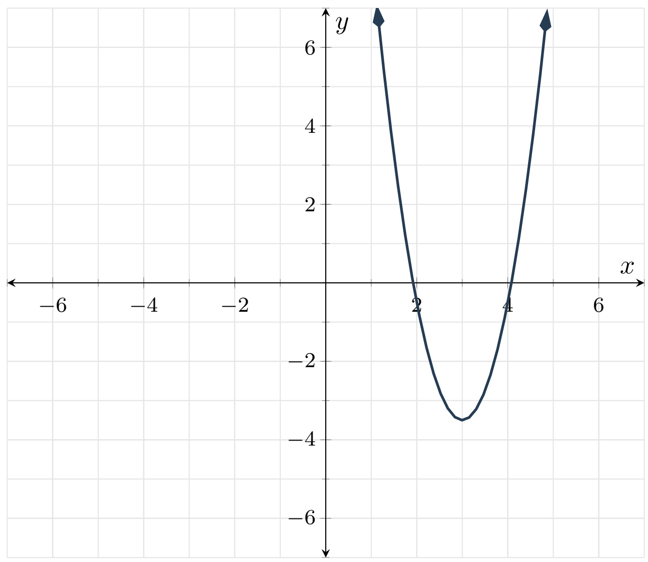
A curve with the shape that we see in the above figure is called a **parabola**. Notice the symmetry in figure, how the y-values in rows above the middle row match those below the middle row. Also notice the symmetry in the shape of the graph, how its left side is a mirror image of its right side.

The first feature that we will talk about is the direction that a parabola opens. All parabolas open either upward or downward. This parabola in the rocket example opens downward because a is negative. That means that for large values of t, the at^2 term will be large and negative, and the resulting y-value will be low on the y-axis. So the negative leading coefficient causes the arms of the parabola to point downward.

Here are some more quadratic graphs so we can see which way they open.







The graph of a quadratic equation $y = ax^2 + bx + c$ is a parabola which opens upward or downward according to the sign of the leading coefficient a. If the leading coefficient is positive, the parabola opens upward. If the leading coefficient is negative, the parabola opens downward.

The **vertex** of a parabola is the highest or lowest point on the graph, depending uponon whether the graph opens downward or upward. In Example 1, the vertex is $(2, 64)$. This tells us that Hannah's rocket reached its maximum height of 64 feet after 2 seconds. If the parabola opens downward, as in the rocket example, then the y-value of the vertex is the **maximum** y-value. If the parabola opens upward then the y-value of the vertex is the **minimum** y-value. The **axis of symmetry** is a vertical line that passes through the vertex, cutting the parabola into two symmetric halves. We write the axis of symmetry as an equation of a vertical line so it always starts with "$x =$". In Example 1, the equation for the axis of symmetry is $x = 2$.

The **vertical intercept** is the point where the parabola crosses the vertical axis. The vertical intercept is the y-intercept if the vertical axis is labeled y. In Example 1, the point $(0, 0)$ is the starting point of the rocket, and it is where the graph crosses the y-axis, so it is the vertical intercept. The y-value of 0 means the rocket was on the ground when the t-value was 0, which was when the rocket launched.

The **horizontal intercept(s)** are the points where the parabola crosses the horizontal axis. They are the x-intercepts if the horizontal axis is labeled x. The point $(0, 0)$ on the path of the rocket is also a horizontal intercept. The t-value of 0 indicates the time when the rocket was launched from the ground. There is another horizontal intercept at the point $(4, 0)$, which means the rocket came back to hit the ground after 4 seconds.

It is possible for a quadratic graph to have zero, one, or two horizontal intercepts. The figures below show an example of each.

Example 8. *Use technology to graph and make a table of the quadratic function f defined by $f(x) = 2x^2 + 4x - 3$ and find each of the key points or features.*

(a) *Find the vertex.*

(b) *Find the vertical intercept (i.e. the y-intercept).*

(c) *Find the horizontal or (i.e. the x-intercept(s)).*

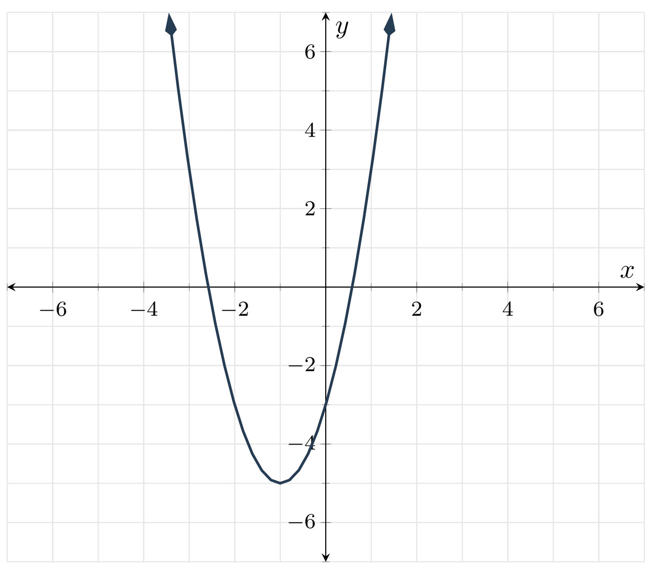
(d) *Find $f(-2)$.*

(e) *Solve $f(x) = 3$ using the graph.*

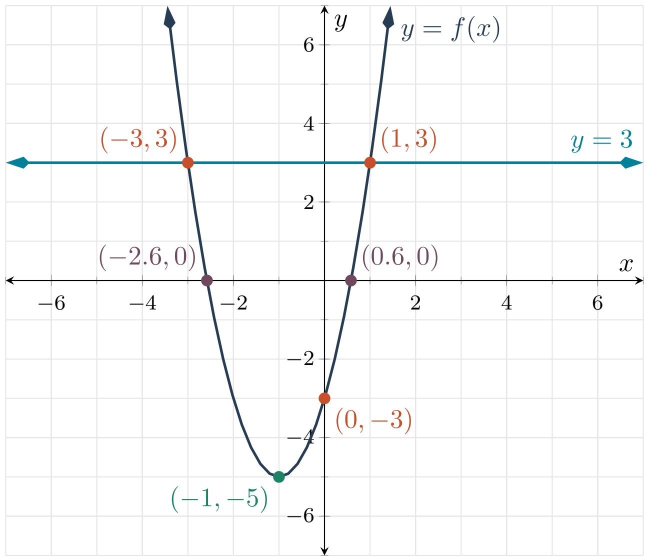
(f) *Solve $f(x) \leq 3$ using the graph.*

Explanation The specifics of how to use any one particular technology tool vary. Whether you use an app, a physical calculator, or something else, a table and graph should look like:

x	$f(x)$
-2	-3
-1	-5
0	-3
1	3
2	13



Additional features of your technology tool can enhance the graph to help answer these questions. You may be able to make the graph appear like:



(a) The vertex is $(-1, -5)$.

(b) The vertical intercept is $(0, -3)$.

(c) The horizontal intercepts are approximately $(-2.6, 0)$ and $(0.6, 0)$.

(d) When $x = -2$, $y = -3$, so $f(-2) = -3$.

(e) The solutions to $f(x) = 3$ are the x-values where $y = 3$. We graph the horizontal line $y = 3$ and find the x-values where the graphs intersect. The solution set is $\{-3, 1\}$.

(f) The solutions are all of the x-values where the function's graph is below (or touching) the line $y = 3$. The interval is $[-3, 1]$.

A polynomal is a particular type of algebraic expression

(a) A company's sales, s (in millions of dollars), can be modeled by $2.2t + 5.8$, where t stands for the number of years since 2010.

(b) The height of an object from the ground, h (in feet), launched upward from the top of a building can be modeled by $-16t^2 + 32t + 300$, where t represents the amount of time (in seconds) since the launch.

(c) The volume of an open-top box with a square base, V (in cubic inches), can be calculated by $30s^2 - \frac{1}{2}s^2$, where s stands for the length of the square base, and the box sides have to be cut from a certain square piece of metal.

Polynomial Vocabulary

A polynomial is an expression with one or more terms summed together. A term of a polynomial must either be a plain number or the product of a number and one or more variables raised to natural number powers. The expression 0 is also considered a polynomial, with zero terms.

Example 9. *Here are some examples of polynomials*

(a) *Here are three polynomials:* $x^2 - 5x + 2$, $t^3 - 1$, $7y$.

(b) *The expression* $3x^4y^3 + 7xy^2 - 12xy$ *is an example of a polynomial in more than one variable.*

(c) *The polynomial* $x^2 - 5x + 3$ *has three terms:* x^2, $-5x$, *and* 3.

(d) *The polynomial* $3x^4 + 7xy^2 - 12xy$ *also has three terms.*

(e) *The polynomial* $t^3 - 1$ *has two terms.*

> **Definition** The coefficient (or numerical coefficient) of a term in a polynomial is the numerical factor in the term.

Example 10. (a) *The coefficient of the term* $\frac{4}{3}x^6$ *is* $\frac{4}{3}$.

(b) *The coefficient of the second term of the polynomial* $x^2 - 5x + 3$ *is* -5.

(c) *The coefficient of the term* $\frac{y^7}{4}$ *is* $\frac{1}{4}$, *because we can rewrite* $\frac{y^7}{4}$ *as* $\frac{1}{4}y^7$.

A term in a polynomial with no variable factor is called a constant term.

Example 11. *The constant term of the polynomial* $x^2 - 5x + 3$ *is* 3.

> **Definition** The degree of a term is one way to measure how large it is. When a term only has one variable, its degree is the exponent on that variable. When a term has more than one variable, its degree is the sum of the exponents on the variables. A nonzero constant term has degree 0.

Example 12. (a) *The degree of* $5x^2$ *is* 2.

(b) *The degree of* $-\frac{4}{7}y^5$ *is* 5.

(c) *The degree of* $-4x^2y^3$ *is* 5.

(d) *The degree of* 17 *is* 0. *Constant terms always have* 0 *degree.*

Definition The **degree** of a nonzero polynomial is the greatest degree that appears amongst its terms

Remark To help us recognize a polynomial's degree, the standard convention at this level is to write a polynomial's terms in order from highest degree to lowest degree. When a polynomial is written in this order, it is written in standard form. For example, it is standard practice to write $7-4x-x^2$ as $-x^2-4x+7$ since $-x^2$ is the leading term. By writing the polynomial in standard form, we can look at the first term to determine both the polynomial's degree and leading term.

Adding and Subtracting Polynomials

Bayani started a company that makes one product: one-gallon ketchup jugs for industrial kitchens. The company's production expenses only come from two things: supplies and labor. The cost of supplies, S (in thousands of dollars), can be modeled by $S = 0.05x^2 + 2x + 30$, where x is number of thousands of jugs of ketchup produced. The labor cost for his employees, L (in thousands of dollars), can be modeled by $0.1x^2 + 4x$, where x again represents the number of jugs they produce (in thousands of jugs). Find a model for the company's total production costs.

Evaluating Polynomial Expressions

Recall that evaluating expressions involves replacing the variable(s) in an expression with specific numbers and calculating the result. Here, we will look at evaluating polynomial expressions.

Example 13. *Evaluate the expression*

$$-12y^3 + 4y^2 - 9y + 2 \textit{ for } y = -5$$

Explanation We will replace y with -5 and simplify the result:

$$\begin{aligned} 12y^3 + 4y^2 - 9y + 2 &= -12(-5)^3 + 4(-5)^2 - 9(-5) + 2 \\ &= -12(-125) + 4(25) + 45 + 2 \\ &= 1647 \end{aligned}$$

1.2.2 Vertex Form

The Vertex Form of a Quadratic

We have learned the standard form of a quadratic function's formula, which is $f(x) = ax^2 + bx + c$. we will learn another form called the vertex form.

> **Vertex Form of a Quadratic Function** A quadratic function whose graph has vertex at the point (h, k) is given by
>
> $$f(x) = a(x-h)^2 + k$$

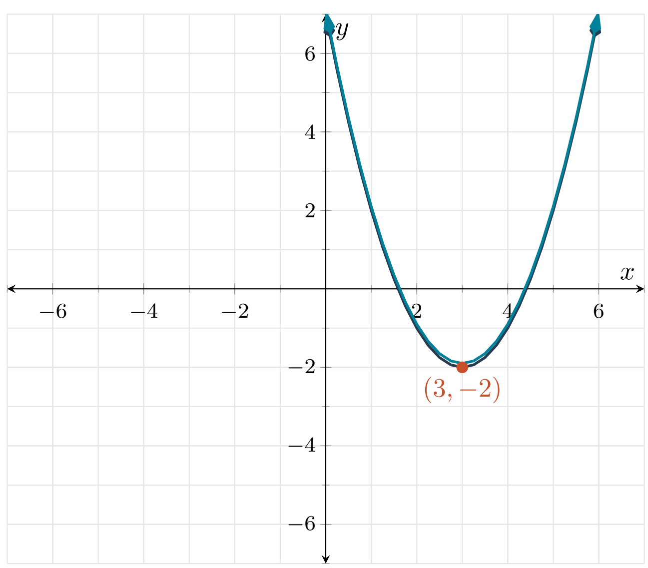
Using graphing technology, consider the graphs of $f(x) = x^2 - 6x + 7$ and $g(x) = (x-3)^2 - 2$ on the same axes.

We see only one parabola because these are two different forms of the same function. Indeed, if we convert $g(x)$ into standard form:

$$\begin{aligned} g(x) &= (x-3)^2 - 2 \\ &= (x^2 - 6x + 9) - 2 \\ &= x^2 - 6x + 7 \end{aligned}$$

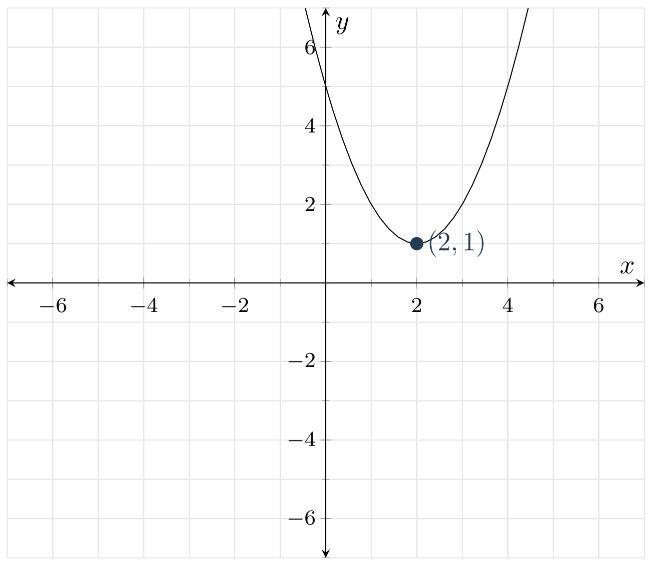
it is clear that f and g are the same function.

Graph of $f(x) = x^2 - 6x + 7$ and $g(x) = (x-3)^2 - 2$ the graphs of the two parabolas overlap each other completely

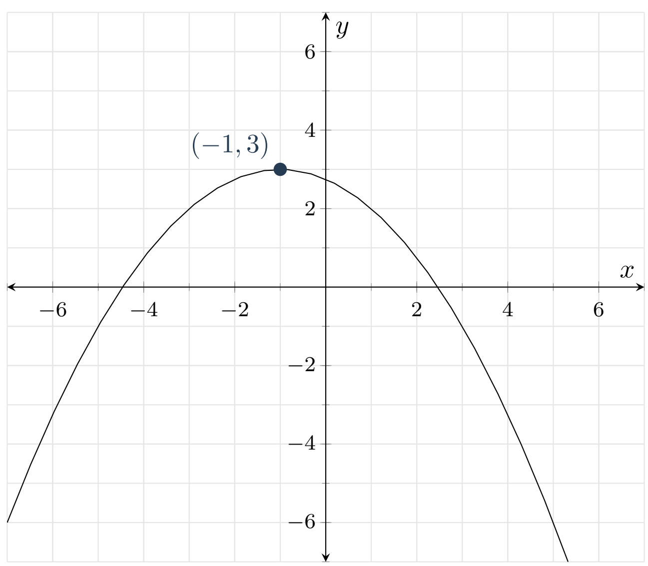


The formula given for g is said to be in vertex form because it allows us to read the vertex without doing any calculations. The vertex of the parabola is $(3, -2)$. We can see those numbers in $g(x) = (x-3)^2 - 2$. The x-value is the solution to $(x-3) = 0$, and the y-value is the constant added at the end.

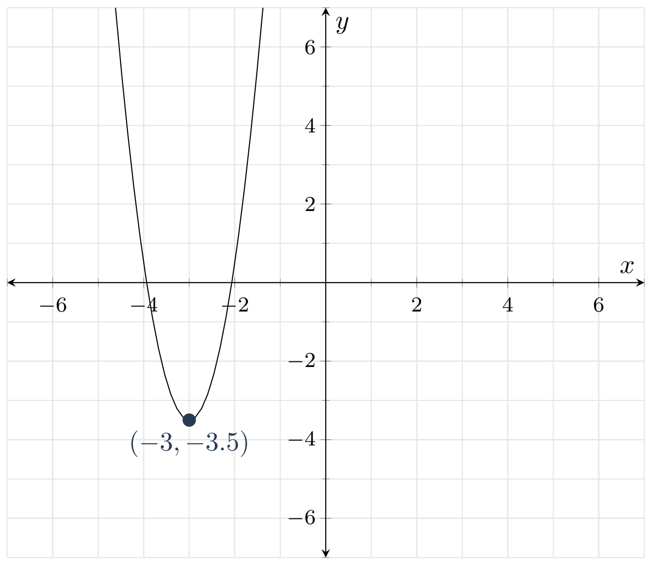
Example 14. *Here are the graphs of three more functions with formulas in vertex form. Compare each function with the vertex of its graph.*



$r(x) = (x-2)^2 + 1$



$$s(x) = -\frac{1}{4}(x+1)^2 + 3$$



$$t(x) = 4(x+3)^2 - 3.5$$

Notice that the x-coordinate of the vertex has the opposite sign as the value in the function formula. On the other hand, the y-coordinate of the vertex has the same sign as the value in the function formula. Let's look at an example to understand why. We will evaluate $r(2)$.

$r(2) = (2-2)^2 + 1 = 1$

The x-value is the solution to $(x-2) = 0$, which is positive 2. When we substitute 2 for x we get the value $y = 1$. Note that these coordinates create the vertex at $(2,1)$. Now we can define the vertex form of a quadratic function.

1.2.3 Root Form

We have previously looked at different forms of quadratic functions. We've looked at standard form and vertex form, where characteristics like y-intercept and vertex can be found easily by looking at the function. Another useful way to look at quadratic functions is to have them written out as a product of linear factors. This can help us to quickly determine the x-intercepts of a quadratic function and to get a good idea of the position and shape of the graph. Not all quadratics can be written in factored form, so we will begin by addressing those.

Remark **Irreducible quadratic factors** are quadratic factors that when set equal to zero only have complex roots. As a result they cannot be reduced into factors containing only real numbers, hence the name irreducible.

As seen in the graphs below, the graphs of the functions do not cross the x-axis, so they do not have x-intercepts. The first graph, $y = x^2 + x + 1$ is entirely above the x-axis and the second graph, $y = -x^2 + x - 1$ is entirely below the x-axis. Since neither of them cross the x-axis, they have no x-intercepts and are irreducible.

Root Form

Root Form of a Quadratic Function

A quadratic function whose graph has x-intercepts (called roots) at the points $(r, 0)$ and $(s, 0)$ can be written as:

$$f(x) = a(x - r)(x - s)$$

This form is called **Root Form** because the roots of the quadratic can be easily read off from this form.

Factoring from Standard Form when $a = 1$

When $a = 1$, putting a quadratic in Root Form is the same as factoring a quadratic. In general, **factoring** refers to writing as a product of linear terms, but does not necessarily imply that the a term is pulled out front like it is in Root Form.

Example 15. *Factor the following quadratic into a product of linear factors:*

$$x^2 + 3x + 2$$

Explanation For us to begin factoring this quadratic, we have to look at the b and c terms. We are looking for 2 numbers that muliply to 2 (or c) and add up to 3 (or b). By going through the factors of 2 we can see that the only numbers that satisfy these conditions are 2 and 1.

$$2 + 1 = 3$$

$$2 \cdot 1 = 2$$

This means that we can factor the quadratic the following way:

$$x^2 + 3x + 2 = (x + 2)(x + 1)$$

The quadratic is now written as a product of linear factors and because $a = 1$, these are also our x-intercepts (or roots) for our function.

Factoring from Standard Form when $a > 1$

Example 16. *Rewrite the following quadratic in Root Form.*

$$f(x) = 3x^2 - 4x - 7$$

Explanation First we will start by pulling out a 3 from every term.

$$3\left(x^2 - \frac{4}{3}x - \frac{7}{3}\right)$$

We now have to find factors that add up to $-\frac{4}{3}$ and multiply to $-\frac{7}{3}$. In this particular case, we can see that the difference between the numerators (7&4) is 3, and since $\frac{3}{3} = 1$ our job will be a little easier. This leads us to the following factors:

$$\frac{3}{3} + \frac{-7}{3} = -\frac{4}{3}$$
$$\frac{3}{3} \cdot \frac{-7}{3} = -\frac{7}{3}$$

So, our factored form is as follows:

$$f(x) = 3\left(x + \frac{3}{3}\right)\left(x - \frac{7}{3}\right)$$

Note that in the previous example, it is not necessary to pull at the 3 *as the first step. Instead, we could pull out the* 3 *as the last step and still have the root form.*

Explanation This way, we start with

$$f(x) = 3x^2 - 4x - 7$$

We now have to find numbers $m_1, m_2, b_1,$ and b_2 such that:

$$\begin{aligned}(m_1x+b_1)(m_2x+b_2) &= m_1m_2x^2 + m_1b_2x + m_2b_1x + b_1b_2\\ &= m_1m_2x^2 + (m_1b_2+m_2b_1)x + b_1b_2\\ &= 3x^2 - 4x - 7\end{aligned}$$

This means that we need

$$\begin{aligned}m_1m_2 &= 3\\ m_1b_2 + m_2b_1 &= -4\\ b_1b_2 &= -7\end{aligned}$$

because the only way two quadratics in standard form can be equal is if they have the same coefficients for each term.

Through a little trial and error, we find that:

$$\begin{aligned}m_1 &= 1\\ b_1 &= 1\\ m_2 &= 3\\ b_2 &= 7\end{aligned}$$

will work.

We now have the equation written as a product of linear components

$$f(x) = 3x^2 - 4x - 7 = (x+1)(3x-7)$$

Now, to write our answer in Root Form, we just need to factor out both m_1 and m_2. Since in this example, $m_1 = 1$, we don't actually have to do any thing for that one.

$$f(x) = 3x^2-4x-7 = (x+1)(3x-7) = (x+1)\left[3\left(x-\frac{7}{3}\right)\right] = 3(x+1)\left(x-\frac{7}{3}\right)$$

Now, we have our quadratic in Root Form and can read off our roots as $x = 1$ *and* $x = \dfrac{-7}{3}$.

Quadratic Formula

When there appears to be no easy way to factor a quadratic, our best option is to use the Quadratic Formula. Let's try the previous example with the Quadratic Formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

when $ax^2 + bx + c = 0$

Example 17. *Find the solutions to the following quadratic equation:*

$$3x^2 - 4x - 7 = 0$$

Explanation

$$\begin{aligned} x &= \frac{-(-4) \pm \sqrt{(-4)^2 - 4(3)(-7)}}{2(3)} \\ &= \frac{4 \pm \sqrt{16 - (-84)}}{6} \\ &= \frac{4 \pm \sqrt{100}}{6} \\ &= \frac{4 \pm 10}{6} \end{aligned}$$

$$x = \frac{4+10}{6} \qquad x = \frac{4-10}{6}$$

$$x = \frac{14}{6} \qquad x = \frac{-6}{6}$$

$$x = -\frac{7}{3} \qquad x = -1$$

We still get $x = -1$ and $x = \frac{7}{3}$ as our roots. This can be a very useful tool especially with more complicated quadratic equations.

Now that we know the roots, we can use the a-value we see in standard form, $a = 3$ and the two roots $r = 1$ and $s = \frac{-7}{3}$ to plug into our Root Form formula $a(x - r)(x - s)$, so again we get as a final answer that

$$f(x) = 3(x + 1)\left(x - \frac{7}{3}\right)$$

Factoring when missing a term

We've talked about factoring $ax^2 + bx + c$ when all terms are present, but what do we do when one of the terms is missing? Since there are three terms we have three different cases to address.

The first case also happens to be the easiest to solve. How do you a solve a quadratic that is missing the ax^2 term? This is a bit of a trick question, because without an x^2 term, we are no longer dealing with a quadratic.

Example 18. *Solve the following equation*

$$2x - 9 = 0$$

Explanation We can see here that we are only dealing with a linear terms and there are no quadratic (x^2) terms. This means we do not have to factor and we can solve for x directly.

$$\begin{aligned} 2x - 9 &= 0 \\ 2x &= 9 \\ x &= \frac{9}{2} \end{aligned}$$

The second case is when the middle bx term is missing.

Example 19. *Factor the quadratic* $f(x) = x^2 - 9$ *into linear components.*

Explanation This quadratic is a special case called "difference of squares." There is no "middle" term and the remaining two terms are both perfect squares, so we can use a shortcut when factoring.

> **Difference of Squares**
> When a,b are non zero.
>
> $$a^2 - b^2 = (a + b)(a - b)$$

In our case, we can see that x^2 is a perfect square and 9 is also a perfect square because $9 = 3^2$. This means that our original quadratic will be factored like this:

$$x^2 - 9 = (x + 3)(x - 3)$$

We can also think of it in the same way as factoring other quadratics. Since there is no middle term, we can look at factors of -9 that add up to 0. 3 and -3 add up to 0 and multiply out to -9. The difference of squares is just a useful pattern that helps to speed up our factoring process.

The last case is when there is no constant or c term.

Example 20. *Factor the quadratic* $f(x) = x^2 + 2x$ *into linear components.*

Explanation Since there is a common x factor in both terms we can pull out that factor and we are left with a product of linear components.

$$\begin{aligned} f(x) &= x^2 + 2x \\ &= x \cdot x + 2x \\ &= x(x + 2) \end{aligned}$$

Again, our factoring is already simplified. We do not have go through the whole process of factoring. If we have a quadratic function with only the ax^2 and bx term, then we will always be able to pull out at least an x term when factoring.

1.3 Polynomials

Learning Objectives

- Definition of Polynomials
  - Understand polynomials as functions
  - Understand lines and parabolas as graphs of polynomials
  - Evaluate polynomials
  - Put polynomial expressions into standard form
- Shape of Polynomials
  - Even and Odd polynomials
  - Importance of highest term and sign of highest term
  - End behavior of polynomials

1.3.1 Definition of Polynomials

A polynomal is a particular type of algebraic expression

(a) A company's sales, s (in millions of dollars), can be modeled by $2.2t + 5.8$, where t stands for the number of years since 2010.

(b) The height of an object from the ground, h (in feet), launched upward from the top of a building can be modeled by $-16t^2 + 32t + 300$, where t represents the amount of time (in seconds) since the launch.

(c) The volume of an open-top box with a square base, V (in cubic inches), can be calculated by $30s^2 - \frac{1}{2}s^2$, where s stands for the length of the square base, and the box sides have to be cut from a certain square piece of metal.

Polynomial Vocabulary

A polynomial is an expression with one or more terms summed together. A term of a polynomial must either be a plain number or the product of a number and one or more variables raised to natural number powers. The expression 0 is also considered a polynomial, with zero terms.

Example 21. *Here are some examples of polynomials*

(a) *Here are three polynomials:* $x^2 - 5x + 2$, $t^3 - 1$, $7y$.

(b) *The expression* $3x^4y^3 + 7xy^2 - 12xy$ *is an example of a polynomial in more than one variable.*

(c) *The polynomial* $x^2 - 5x + 3$ *has three terms:* x^2, $-5x$, *and* 3.

(d) *The polynomial* $3x^4 + 7xy^2 - 12xy$ *also has three terms.*

(e) *The polynomial* $t^3 - 1$ *has two terms.*

> **Definition** The coefficient (or numerical coefficient) of a term in a polynomial is the numerical factor in the term.

Example 22. (a) *The coefficient of the term* $\frac{4}{3}x^6$ *is* $\frac{4}{3}$.

(b) *The coefficient of the second term of the polynomial* $x^2 - 5x + 3$ *is* -5.

(c) *The coefficient of the term* $\frac{y^7}{4}$ *is* $\frac{1}{4}$, *because we can rewrite* $\frac{y^7}{4}$ *as* $\frac{1}{4}y^7$.

A term in a polynomial with no variable factor is called a constant term.

Example 23. *The constant term of the polynomial* $x^2 - 5x + 3$ *is* 3.

Definition The degree of a term is one way to measure how large it is. When a term only has one variable, its degree is the exponent on that variable. When a term has more than one variable, its degree is the sum of the exponents on the variables. A nonzero constant term has degree 0.

Example 24. (a) *The degree of* $5x^2$ *is* 2.

(b) *The degree of* $-\frac{4}{7}y^5$ *is* 5.

(c) *The degree of* $-4x^2y^3$ *is* 5.

(d) *The degree of* 17 *is* 0. *Constant terms always have* 0 *degree.*

Definition The **degree** of a nonzero polynomial is the greatest degree that appears amongst its terms

Remark To help us recognize a polynomial's degree, the standard convention at this level is to write a polynomial's terms in order from highest degree to lowest degree. When a polynomial is written in this order, it is written in standard form. For example, it is standard practice to write $7-4x-x^2$ as $-x^2-4x+7$ since $-x^2$ is the leading term. By writing the polynomial in standard form, we can look at the first term to determine both the polynomial's degree and leading term.

Adding and Subtracting Polynomials

Bayani started a company that makes one product: one-gallon ketchup jugs for industrial kitchens. The company's production expenses only come from two things: supplies and labor. The cost of supplies, S (in thousands of dollars), can be modeled by $S = 0.05x^2 + 2x + 30$, where x is number of thousands of jugs of ketchup produced. The labor cost for his employees, L (in thousands of dollars), can be modeled by $0.1x^2 + 4x$, where x again represents the number of jugs they produce (in thousands of jugs). Find a model for the company's total production costs.

Evaluating Polynomial Expressions

Recall that evaluating expressions involves replacing the variable(s) in an expression with specific numbers and calculating the result. Here, we will look at evaluating polynomial expressions.

Example 25. *Evaluate the expression*

$$-12y^3 + 4y^2 - 9y + 2 \textit{ for } y = -5$$

Explanation We will replace y with -5 and simplify the result:

$$\begin{aligned} 12y^3 + 4y^2 - 9y + 2 \quad &= -12(-5)^3 + 4(-5)^2 - 9(-5) + 2 \\ &= -12(-125) + 4(25) + 45 + 2 \\ &= 1647 \end{aligned}$$

1.3.2 Shape of Polynomials

We know that linear functions are the simplest of all funcitons we can consider: their graphs have the simplest shape, their average rate of change is always constant (regardless of the interval chosen), and their formula is elementary. Moreover, computing the value of a linear function only requires multiplication and addition.

If we think of a linear function as having formula $L(x) = b + mx$, and the next-simplest functions, quadratic functions, as having form $Q(x) = c + bx + ax^2$, we can see immediate parallels between their respective forms and realize that it's natural to consider slightly more complicated functions by adding additional power functions.

Indeed, if we instead view linear functions as having form

$$L(x) = a_0 + a_1 x$$

(for some constants a_0 and a_1) and quadratic functions as having form

$$Q(x) = a_0 + a_1 x + a_2 x^2,$$

then it's natural to think about more general functions of this same form, but with additional power functions included.

Definition Given real numbers $a_0, a_1, \ldots, a_n$ where $a_n \neq 0$, we say that the function a function P is a **polynomial** of degree n if it can be written in the form

$$P(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_{n-1} x^{n-1} + a_n x^n,$$

We say that the a_i are the *coefficients* of the polynomial and the individual power functions $a_i x^i$ are the *terms* of the polynomial. Any value of x for which $P(x) = 0$ is called a *zero* of the polynomial.

Example 26. *The polyomial function* $P(x) = 3 - 7x + 4x^2 - 2x^3 + 9x^5$ *has degree* 5*, its constant term is* 3*, and its linear term is* $-7x$.

Since a polynomial is simply a sum of constant multiples of various power functions with positive integer powers, we often refer to those individual terms by referring to their individual degrees: the linear term, the quadratic term, and so on. In addition, since the domain of any power function of the form $p(x) = x^n$ where n is a positive whole number is the set of all real numbers, it's also true the the domain of any polynomial function is the set of all real numbers.

Exploration Point your browser to the *Desmos* worksheet at `http://gvsu.edu/s/0zy`. There you'll find a degree 4 polynomial of the form $p(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4$, where $a_0, \ldots, a_4$ are set up as sliders. In the questions that follow, you'll experiment with different values of $a_0, \ldots, a_4$ to investigate different possible behaviors in a degree 4 polynomial.

(a) What is the largest number of distinct points at which $p(x)$ can cross the x-axis?For a polynomial p, we call any value r such that $p(r) = 0$ a zero of the polynomial. Report the values of $a_0, \ldots, a_4$ that lead to that largest number of zeros for $p(x)$.

(b) What other numbers of zeros are possible for $p(x)$? Said differently, can you get each possible number of fewer zeros than the largest number that you found in (a)? Why or why not?

(c) We say that a function has a turning pointif the function changes from decreasing to increasing or increasing to decreasing at the point. For example, any quadratic function has a turning point at its vertex.What is the largest number of turning points that $p(x)$ (the function in the Desmos worksheet) can have? Experiment with the sliders, and report values of $a_0, \ldots, a_4$ that lead to that largest number of turning points for $p(x)$.

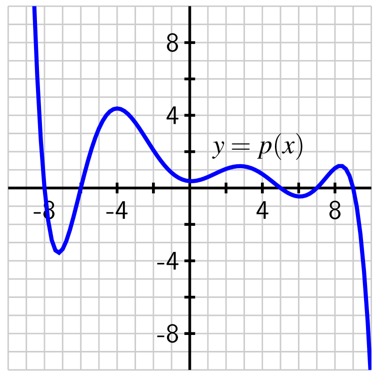
(d) What other numbers of turning points are possible for $p(x)$? Can it have no turning points? Just one? Exactly two? Experiment and explain.

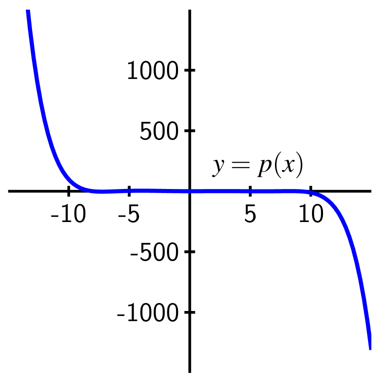
(e) What long-range behavior is possible for $p(x)$? Said differently, what are the possible results for $\lim_{x\to-\infty} p(x)$ and $\lim_{x\to\infty} p(x)$?

(f) What happens when we plot $y = a_4x^4$ in and compare $p(x)$ and a_4x^4? How do they look when we zoom out? (Experiment with different values of each of the sliders, too.)

We know that each of the power functions x, x^2, ..., x^n grow without bound as $x \to \infty$. Intuitively, we sense that x^5 grows faster than x^4 (and likewise for any comparison of a higher power to a lower one). This means that for large values of x, the most important term in any polynomial is its highest order term. when we compared $p(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4$ and $y = a_4x^4$.

For any degree n polynomial $p(x) = a_0 + a_1x + \cdots + a_{n-1}x^{n-1} + a_nx^n$, its long-range behavior is the same as its highest-order term $q(x) = a_nx^n$. Thus, any polynomial of even degree appears "U-shaped" ($\cup$ or $\cap$, like x^2 or $-x^2$) when we zoom way out, and any polynomial of odd degree appears "chair-shaped" (like x^3 or $-x^3$) when we zoom way out.





In the second graph we see how the degree 7 polynomial pictured there (and in the first graph as well) appears to look like $q(x) = -x^7$ as we zoom out.

Summary

- Polynomial

1.4 Roots

Learning Objectives

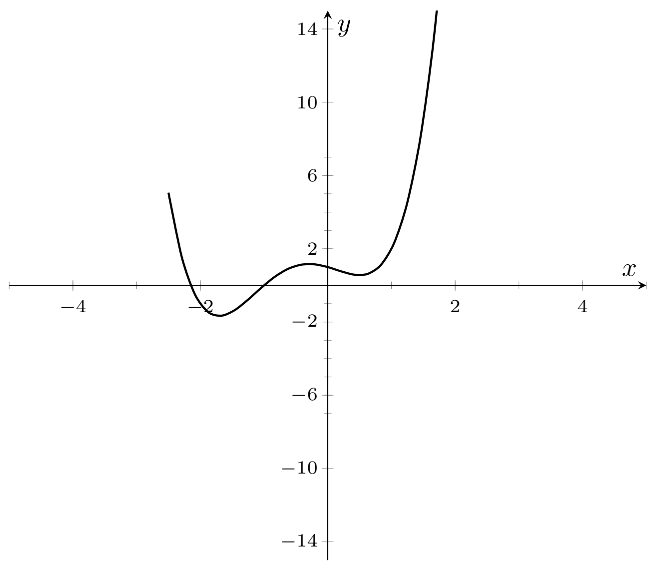
- Roots
  - When in the inverse function for $f(x) = x^n$ defined? What are it's function properties?

1.4.1 Roots

Motivating Questions

- Can we find the inverse of a polynomial?
- What does it mean to take the n^{th} root of a value?

Consider the polynomial $p(x) = x^4 + 2x^3 - x^2 - x + 1$. A good question to ask would be whether the function p is invertible. To help us decide, here is the graph of $p(x)$.

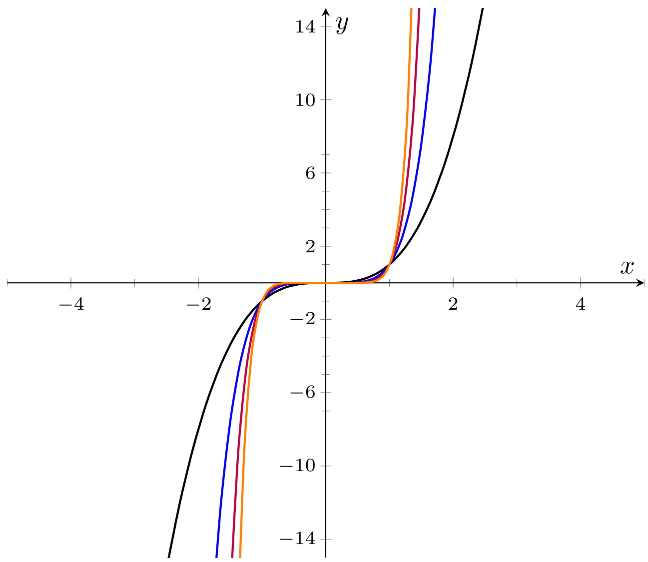


Notice that this graph does not pass the Horizontal Line Test, so the function p is not one-to-one, and therefore not invertible.

Our polynomial $p(x)$ has many terms, so to simplify the situation, we'll look only at polynomials of the form x^n, where n is a positive integer.

Odd Roots

Recall that every polynomial $p(x) = x^n$, where n is odd, has the same basic shape. This is demonstrated in the figure below by the graphs of $y = x^n$ for $n = 3, 5, 7, 9$.



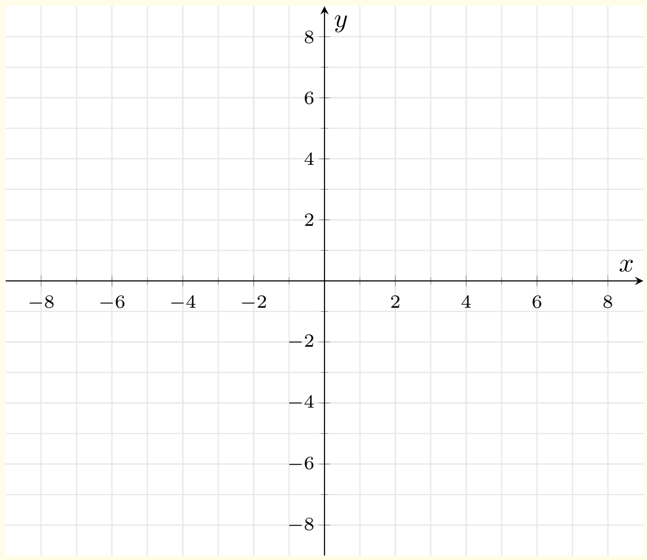
To see more of how these graphs change with n, follow the Desmos link: `https://www.desmos.com/calculator/viuyei80a0`.

Now, are these functions invertible? Looking at the graphs, we see that these functions pass the Horizontal Line Test. Thus, the functions are one-to-one, and therefore invertible.

Definition When n is an odd positive integer, we define **the nth root function** $\sqrt[n]{x}$ to be the inverse of the function defined by x^n. The number n is the **index** of the root, and x is the **radicand**. We call the symbol $\sqrt{\ }$ the **radical**.

Let's delve more deeply into the $p(x) = x^3$ example. We have now established that it is invertible, and it's inverse is $r(x) = \sqrt[3]{x}$.

Exploration Draw both functions on the axes provided, then answer the following questions about the function $r(x)$.



(a) What is the x-intercept of $r(x)$? ($\boxed{?}$, $\boxed{?}$)

(b) What is the y-intercept of $r(x)$? ($\boxed{?}$, $\boxed{?}$)

(c) What is the domain of $r(x)$? ($\boxed{?}$, $\boxed{?}$)

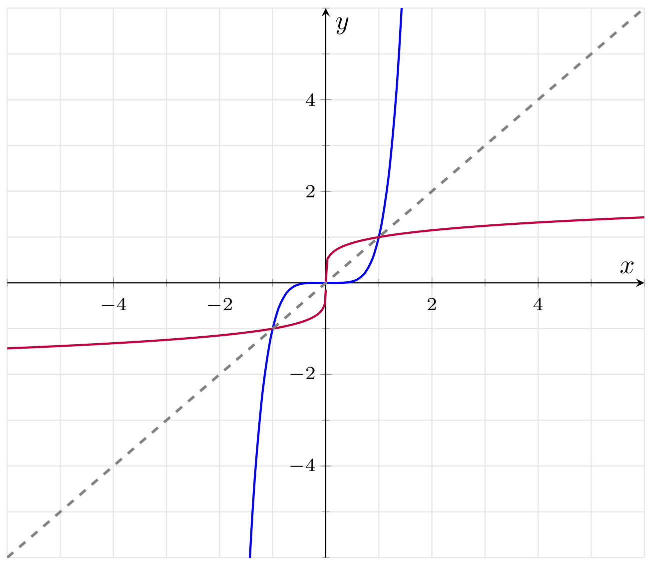
(d) What is the range of $r(x)$? ($\boxed{?}$, $\boxed{?}$)

(e) As x goes to ∞, y goes to $\boxed{?}$.

(f) As x goes to $-\infty$, y goes to $\boxed{?}$.

(g) Does this function have any vertical asymptotes? (yes/ no)

Example 27. *Below, we have an example of the graph of* $y = x^5$ *and the two functions*

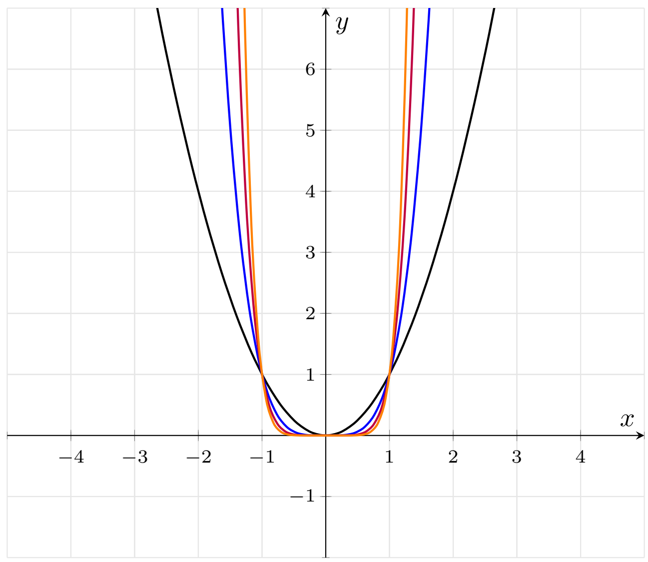


Recall that a function f is *even* if $f(-x) = f(x)$ for all x in its domain, and f is *odd* if $f(-x) = -f(x)$ for all x. Otherwise, the function is neither. Let's consider an example. Given $x = 8$, $r(x) = \sqrt[3]{8} = 2$ and $r(-x) = \sqrt[3]{-8} = -2$, since $(-2) \cdot (-2) \cdot (-2) = (4) \cdot (-2) = -8$. Based on this example, do you think $r(x)$ is even, odd, or neither?

If you guessed odd, then you are correct! All odd-index root functions are odd functions.

Even Roots

We again begin by recalling the general shape of $p(x) = x^n$, but this time for n *even*. These functions also have the same basic shape for all even n. This is demonstrated by the graphs of $y = x^n$ for $n = 2, 4, 6, 8$ given below.

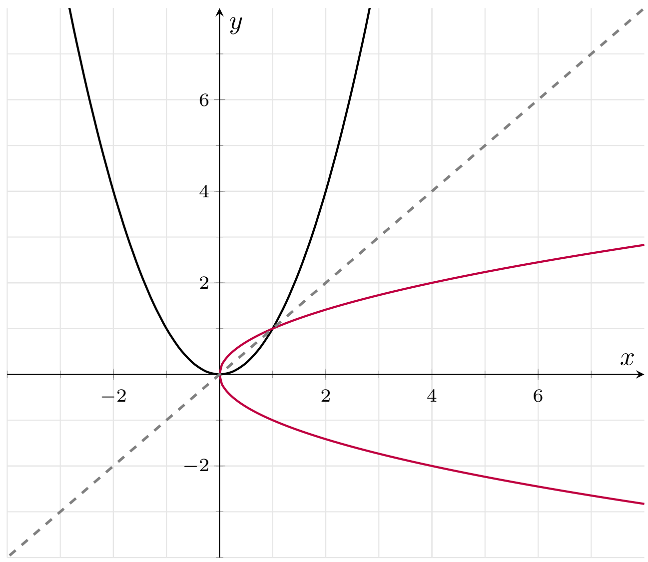


To see more of how these graphs change with n, follow the Desmos link: https://www.desmos.com/calculator/qpqwtrppqt.

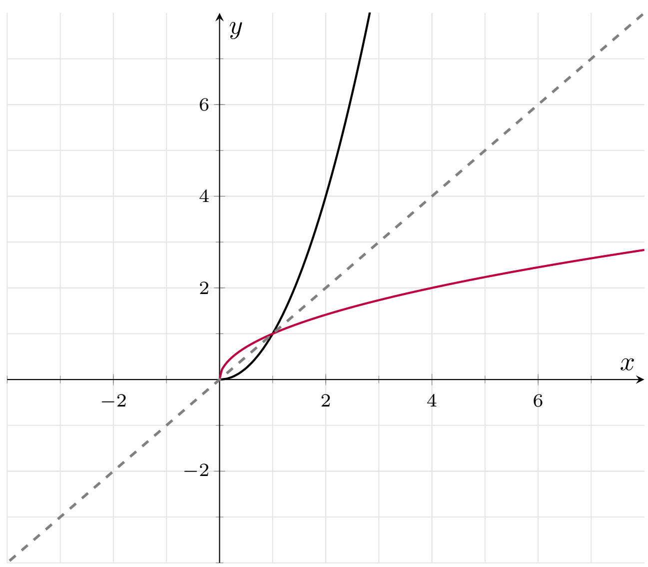
Now, are these functions invertible? All of the graphs in the figure above are symmetric about the y-axis (example $2^2 = 4 = (-2)^2$), so they do *not* pass the horizontal line test. Thus, these functions are *not* one-to-one, and therefore *not* invertible.

So, how can we define an even root function? For example, what does $\sqrt{x}$ really mean, and how is it related to x^2?

Consider the polynomial $p(x) = x^2$, graphed below with its inverse relation $\{(x^2, x) : x \text{ is a real number}\}$.



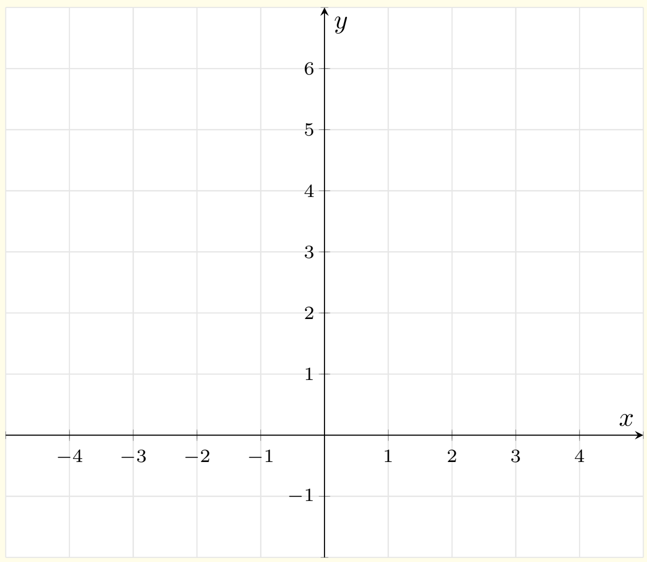
Observe that by restricting the domain of p to $x \geq 0$, we now have a function which passes the horizontal line test, and can thus be inverted. The following picture illustrates the situation.



Now, our inverse relation is actually a function, since it passes the Horizontal Line Test. Therefore, if we let $p(x) = x^2$ for $x \geq 0$, we can then define $r(x) = \sqrt[2]{x} = \sqrt{x}$ as the inverse function of $p(x)$ on this restricted domain.

Definition When n is an even positive integer, we define **the nth root function** $\sqrt[n]{x}$ to be the inverse of the function defined by x^n restricted to the domain $x \geq 0$.

Exploration We now repeat Exploration 1 for $r(x) = \sqrt[4]{x}$. Draw both functions on the axes provided, then answer the following questions about the function $r(x)$.



(a) What is the x-intercept of $r(x)$? $(\boxed{?}, \boxed{?})$

(b) What is the y-intercept of $r(x)$? $(\boxed{?}, \boxed{?})$

(c) What is the domain of $r(x)$? $[\boxed{?}, \boxed{?})$

(d) What is the range of $r(x)$? $[\boxed{?}, \boxed{?})$

(e) As x goes to ∞, y goes to $\boxed{?}$.

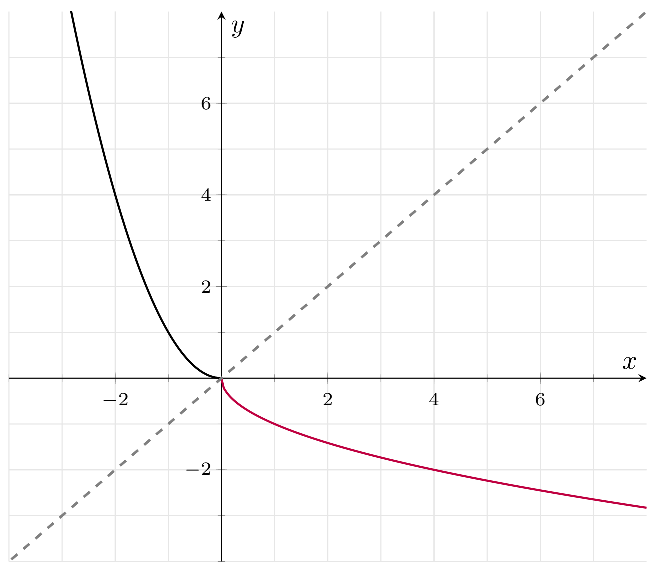
(f) Does this function have any vertical asymptotes? (yes/ no)

One question we might ask is whether $r(x) = \sqrt{x}$ and $p(x) = x^2$ are truly inverses. The answer may seem like an obvious "yes!", but since we restricted the domain of p in order to define r, we need to check. To check whether r and p are inverses, we need to confirm that $r(p(x)) = \sqrt{x^2} = x$ and $p(r(x)) = (\sqrt{x})^2 = x$. That is, when we plug in a number to $\sqrt{x^2}$ and $(\sqrt{x})^2$, we should get the same number as the output. Let's try plugging in -1 to $r(p(x))$. This gives us

$$r(p(-1)) = r((-1)^2) = \sqrt{(-1)^2} = \sqrt{1} = 1,$$

which is not the same as -1. If we repeat this process with a few more numbers, we find that $\sqrt{(-2)^2} = 2$, $\sqrt{1^2} = 1$, $\sqrt{(-45)^2} = 45$, and $\sqrt{98^2} = 98$. We can conclude that $\sqrt{x^2}$ is a function that takes its input and returns its absolute value. That is, $\sqrt{x^2} = |x|$. Since $\sqrt{x^2} = r(p(x))$, we conclude that $r(p(x))$ does not output its input, and therefore, r and p are not inverses. This is something that will be extremely important when solving equations using even roots.

Now, what if we instead restricted our domain to $x \leq 0$? Consider $q(x) = x^2$ defined for $x \leq 0$. The graph of this function is below.



By the Horizontal Line Test, this restriction is one-to-one, and therefore invertible. The inverse of this function as shown above is $s(x) = -\sqrt{x} = -r(x)$.

Example 28. *We demonstrate a few common even and odd n^{th} roots to highlight this distinction.*

(a) $\sqrt[3]{8} = 2$, *since* $2 \cdot 2 \cdot 2 = 8$.

(b) $\sqrt[3]{-8} = -2$, *since* $-2 \cdot (-2) \cdot (-2) = 4 \cdot (-2) = -8$.

(c) $\sqrt[4]{16} = 2$, *since* $2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 4 \cdot 4 = 16$. *However, the* 4^{th} *root of -16 is not defined.*

(d) $\sqrt[4]{0} = 0$, *since zero times any number is always zero. This is* the *example of an even n^{th} root that has only* one *solution.*

(e) *Likewise,* $\sqrt[125]{0} = 0$.

Using Roots to Solve Equations

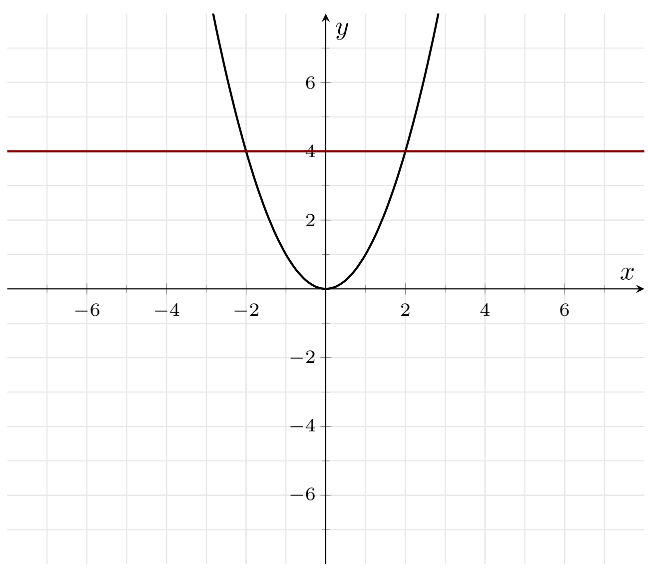
If we are asked to find all values x such that $x^2 = 4$, then the question is asking which values of x multiplied by themselves give 4. In other words, find x such that $x \cdot x$ is equal to 4. It is simple to see that there are two values which make this true:

$$2 \cdot 2 = 4 \text{ and } (-2) \cdot (-2) = 4.$$

In solving an equation, it is common to express this as follows.

$$\begin{aligned} x^2 &= 4 \\ \sqrt{x^2} &= \sqrt{4} \\ |x| &= 2 \\ x &= \pm 2. \end{aligned}$$

Since $f(x) = x^2$ is **not** one-to-one, there are two values of x which make it equal to any positive number, as demonstrated in the following graph.



Example 29. (a) *Solve the equation* $x^3 = 8$.

Taking the cube root on both sides, we find that

$$\begin{aligned} \sqrt[3]{x^3} &= \sqrt[3]{8} \\ x &= 2. \end{aligned}$$

Note that $\sqrt[3]{x^3} = x$, *since 3 is odd, and odd roots are really inverses to their corresponding power functions.*

(b) *Solve the equation* $x^3 = -8$.

Taking the cube root on both sides, we find that

$$\begin{aligned}\sqrt[3]{x^3} &= \sqrt[3]{-8}\\ x &= -2.\end{aligned}$$

(c) *Solve the equation* $x^2 = 16$.

Taking the square root on both sides, we find that

$$\begin{aligned}\sqrt{x^2} &= \sqrt{16}\\ |x| &= 4\\ x &= \pm 4.\end{aligned}$$

Therefore, there are two solutions to this equation: -4 *and* 4.

(d) *Solve the equation* $2x^4 - 4 = 28$.

First, rearrange the equation. Add 4 to both sides to find $2x^4 = 32$. *Divide both sides by 2 to find* $x^4 = 16$. *Taking the 4th root on both sides, we find that*

$$\begin{aligned}\sqrt[4]{x^4} &= \sqrt[4]{16}\\ |x| &= 2\\ x &= \pm 2.\end{aligned}$$

Therefore, there are two solutions to this equation: -2 *and* 2.

Recall that for any even integer n, $\sqrt[n]{x^n} = |x|$.

(e) *Solve the equation* $-3x^4 = 32$.

First, rearrange the equation by dividing both sides by -3. This yields $x^4 = -\frac{32}{3}$. *Taking the 4th root on both sides, we find that*

$$\begin{aligned}\sqrt[4]{x^4} &= \sqrt[4]{-\frac{32}{3}}\\ |x| &= \sqrt[4]{-\frac{32}{3}}.\end{aligned}$$

Since any even-index root of a negative number is not defined, there are no solutions to this equation.

These examples illustrate a general principle that is good to have in your toolbox for solving equations. If $a^3 = b$, then we know that $a = \sqrt[3]{b}$. This is true for any odd powers. However, if $a^2 = b$, then **either** $a = \sqrt{b}$ **or** $a = -\sqrt{b}$. This is true for any even powers.

Summary In general, we are not able to simply find the inverse of polynomials.
However, when the polynomial is $p(x) = x^n$ for a positive *odd* integer n, the polynomial is invertible as the n^{th} root function $r(x) = \sqrt[n]{x}$.
When n is *even*, it is possible to define an inverse function $r(x) = \sqrt[n]{x}$ on a restricted domain ($[0, \infty)$ or $(-\infty, 0]$). The n^{th} root is defined as the inverse of $p(x)$ on the restricted domain $[0, \infty)$.

Part 2

Back Matter

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