Precalculus with Review 1: Unit 5

November 9, 2022

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5.1 Introduction to Function Transformations

Learning Objectives

- Vertical and Horizontal Shifts
  - How to shift a function vertically
  - How to shift a function horizontally
  - Combining shifts and properties of quadratics (vertex, completing the square)
- Stretching Functions
  - Vertical stretch
  - Horizontal stretch

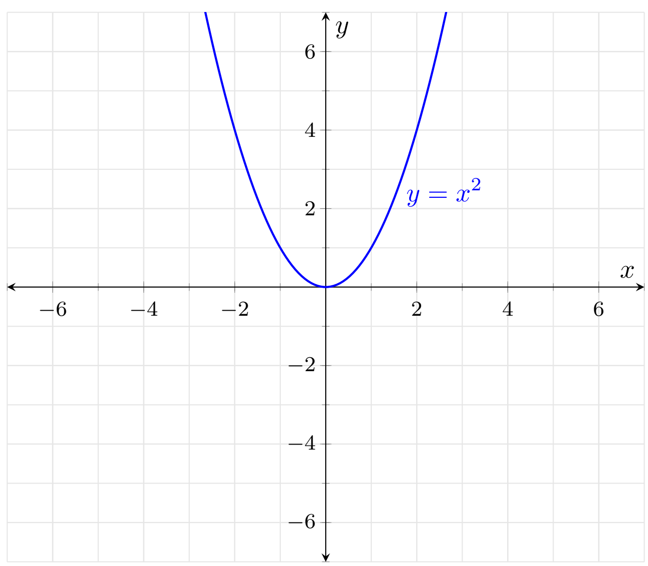
5.1.1 Vertical and Horizontal Shifts

Motivating Questions

- By adding a constant a to a function $f(x)$, what is the relation between the graphs of $y = f(x)$ and $y = f(x) + a$?
- By performing a "change of variable" $x \mapsto x - a$, what is the relation between the graphs of $y = f(x)$ and $y = f(x - a)$?
- How to use this new understanding to gain a deeper understanding of graphs of quadratic functions (i.e., parabolas)?

Introduction

Let's consider the two quadratic functions $f(x) = x^2$ and $g(x) = x^2 - 2x + 1$, defined for all real values of x. We know what their graphs look like:



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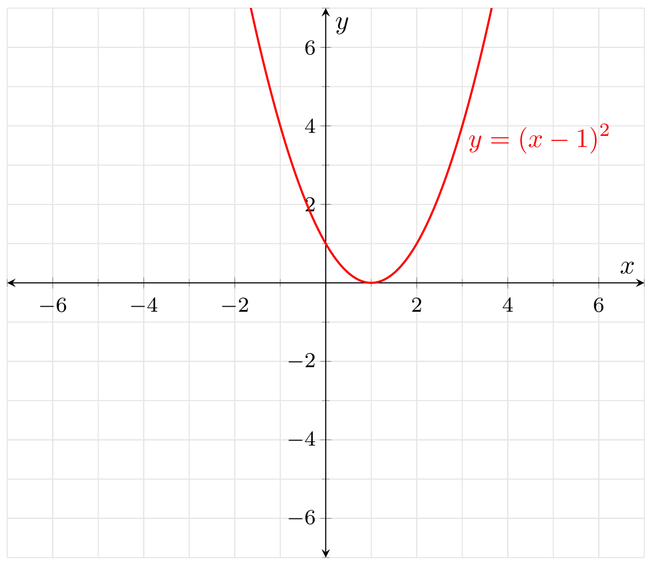
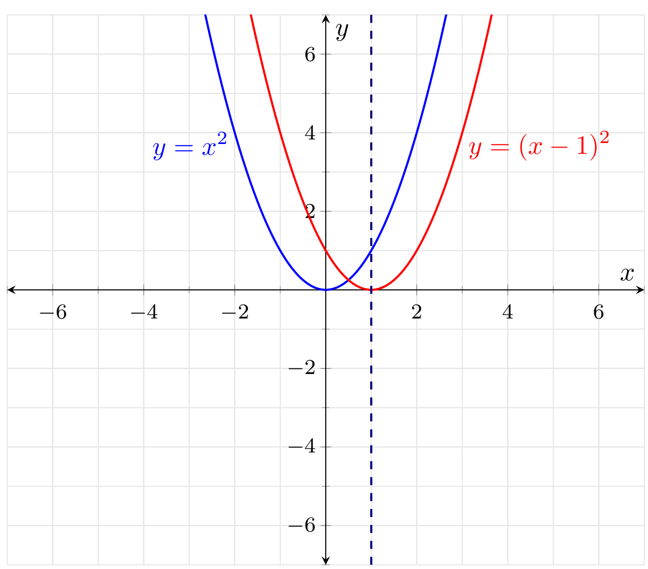


The graphs are very similar, down to the horizontal "width". In fact, drawing them together, we may see that they only differ by a horizontal translation:



Algebraically, one can see that this happens because

$$g(x) = x^2 - 2x + 1 = (x-1)^2 = f(x-1).$$

This hints at the following general fact: doing horizontal shifts to the graph of a function amounts to replacing x with "$x \pm \text{shift}$" inside $f(\cdot)$. In this unit, we'll

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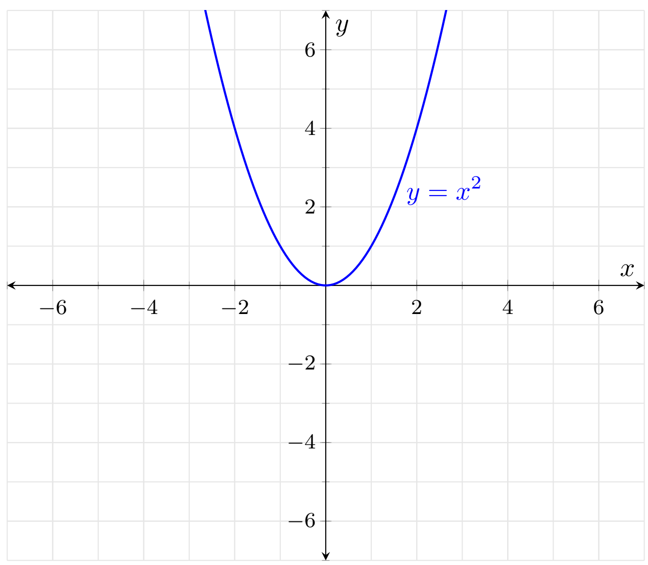
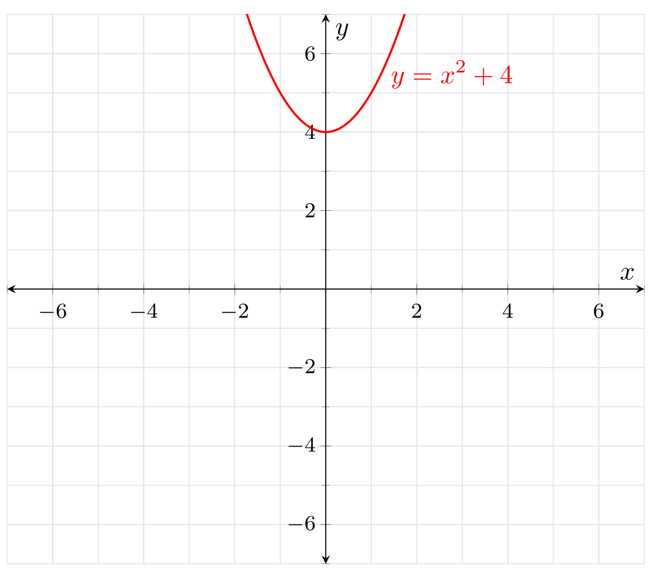
understand in more detail how to work with this, and also how to deal with vertical shifts, as opposed to horizontal shifts. Since vertical shifts are much easier to understand, that's where we'll begin.

Shifting a function vertically

Let's consider a very simple situation, where we have two functions $f(x) = x^2$ and $g(x) = x^2 + 4$. Graphing them, in order, we have that





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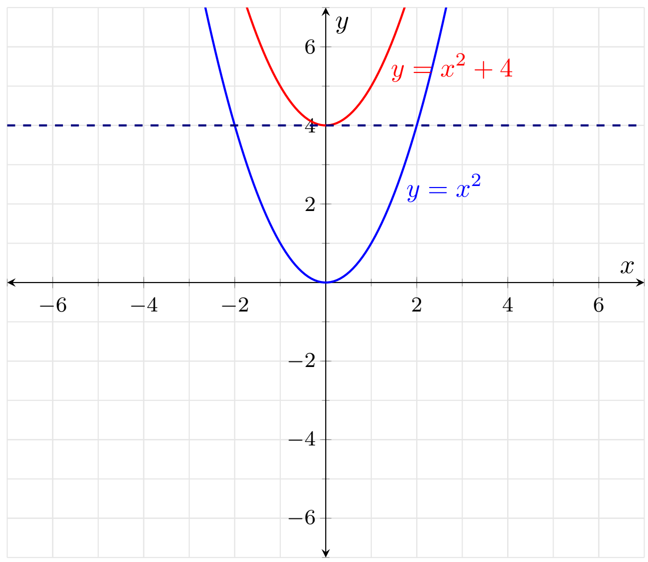
Clearly, $f(x)$ and $g(x)$ are directly related via $g(x) = f(x) + 4$, and seeing their graph together, we have that:



In other words, the graph of $y = g(x)$ was obtained from the graph of $y = f(x)$ by shifting it up exactly by 4 units. This is a very general phenomenon, that happens for any functions who differ by a constant.

> **Theorem (vertical shifts):** Suppose f is a function and a is a positive number.
>
> - To graph $y = f(x) + a$, shift the graph of $y = f(x)$ up a units, by adding a to the y-coordinates of the points on the graph of f.
> - To graph $y = f(x) - a$, shift the graph of $y = f(x)$ down a units, by subtracting a from the y-coordinates of the points on the graph of f.

In the above setting, it is useful to call f the **parent function**.

> **Convention:** Here, we'll always draw graphs of parent functions in blue, and graphs of the "child" functions in red. We'll indicate with a dashed line where the shift has happened.

Example 1. *For each of the following functions, find the parent function. How would the graph look like, in terms of the graph of the original function?*

a. $g(x) = 2e^{-x} - 1$.

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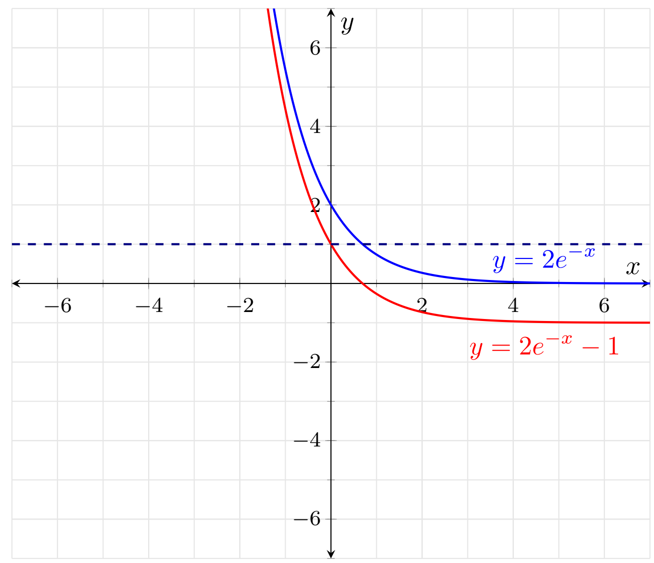
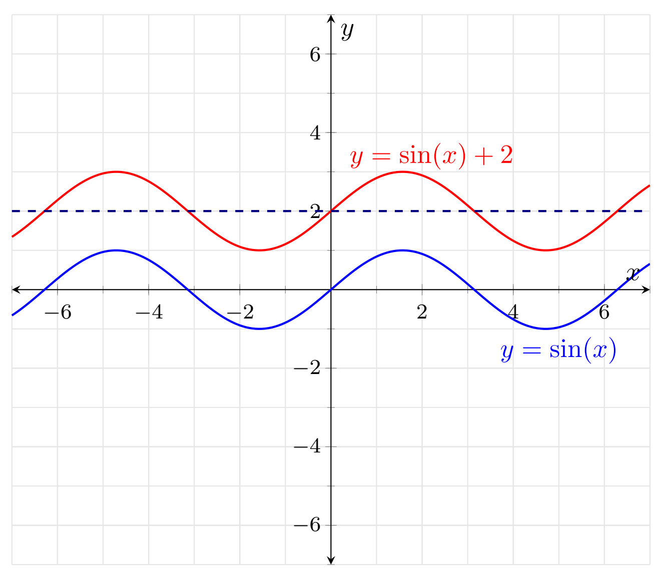
Explanation Setting $f(x) = 2e^{-x}$, we have that $g(x) = f(x) - 1$. So, to graph $y = g(x)$, we just need to consider the graph of $y = f(x)$ and shift it one unit down.



b. $g(x) = \sin(x) + 2$.

Explanation Here, we can just recognize that for $f(x) = \sin(x)$, we have $g(x) = f(x) + 2$. Thus, we just need to shift the graph of $y = \sin(x)$ up by 2 units.



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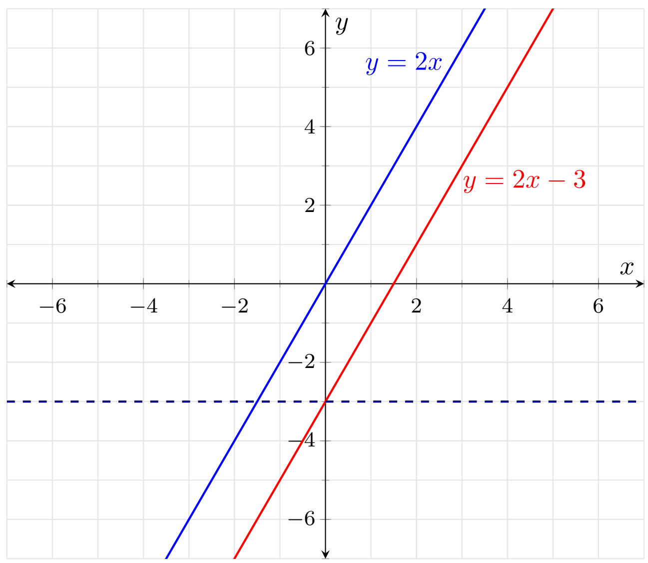
c. $g(x) = 2x - 3$

Explanation Granted, graphing a linear function poses little to no challenge, but understanding how things work in this setting might offer us some general insight on y-intercepts. If $f(x) = 2x$, then $g(x) = f(x) - 3$. Graphing $g(x) = 2x$ is easier than easy: just a line with slope 2 passing through the origin. And the shift down by 3 units comes last, as you would expect:



Shifting a function horizontally

Consider again the example given in the introduction, where we have $f(x) = x^2$ and $f(x-1) = x^2 - 2x + 1$. The first thing we would like to address is a source of frequent confusion when first learning this topic. Namely, we have replaced x with $x - 1$ in the formula for $f(x)$, but the graph of the modified function ended up shifted to the *right*, even though one might expect the shift to have happened to the *left*, due to the negative sign in the $x - 1$ factor!

Here is one safe way to think about it: imagine that you are standing on the x-axis and, say, at the origin of the cartesian plane, but that the graph of $y = f(x)$ is already drawn. Replacing x with $x - 1$ *does move* the x-axis to the left. But *you, the observer*, standing on the x-axis, sees the graph move to the right!

Alternatively, compare this with what happened with vertical shifts, but switching the roles of the x-axis and y-axis. More precisely, start with the graph of $y = f(x)$, then rotate it by 90° clockwise (this switches the axes). Replacing x with $x-1$ now brings the graph down by 1 unit. Finally, rotate everything back by 90° counterclockwise (this undoes the switching of the axes). The resulting

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graph is obtained from the original one by shifting it to the *right*, not left.

Theorem (horizontal shifts): Suppose f is a function and a is a positive number.

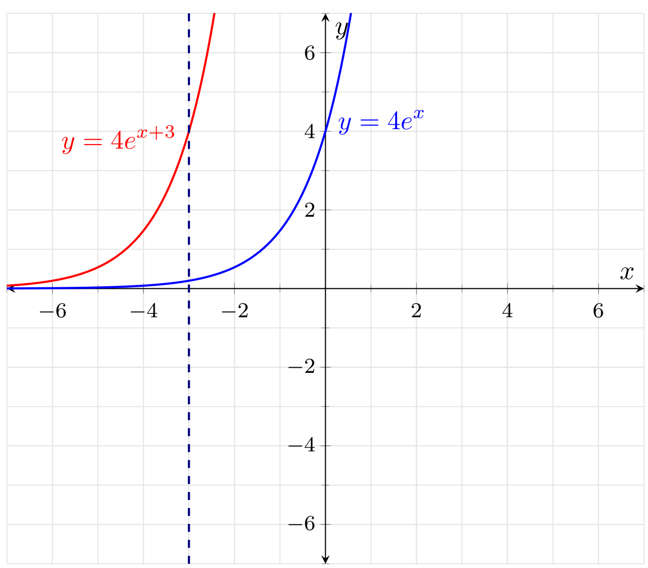
- To graph $y = f(x-a)$, shift the graph of $y = f(x)$ right a units, by adding a to the x-coordinates of the points on the graph of f.
- To graph $y = f(x+a)$, shift the graph of $y = f(x)$ left a units, by subtracting a from the x-coordinates of the points on the graph of f.

As before we'll continue to call f the **parent function**, whose graph will be drawn in blue, while the graphs of the "child" functions will be indicated in red.

Example 2. *For each of the following functions, find the parent function. How would the graph look like, in terms of the graph of the original function?*

a. $g(x) = 4e^{x+3}$.

Explanation Here, if we look at $f(x) = 4e^x$, then we have that $g(x) = 4e^{x+3} = f(x+3)$. So, to graph $y = g(x)$, we may just graph $y = f(x)$, and then shift it 3 units to the left.



b. $g(x) = \sin(x - (\pi/2))$.

Explanation Consider this time $f(x) = \sin x$. Then we have that $g(x) = f(x-(\pi/2))$, which means that to graph $y = g(x)$, we may take the graph of $y = \sin x$ and shift it to the right by $\pi/2$ units. We obtain:

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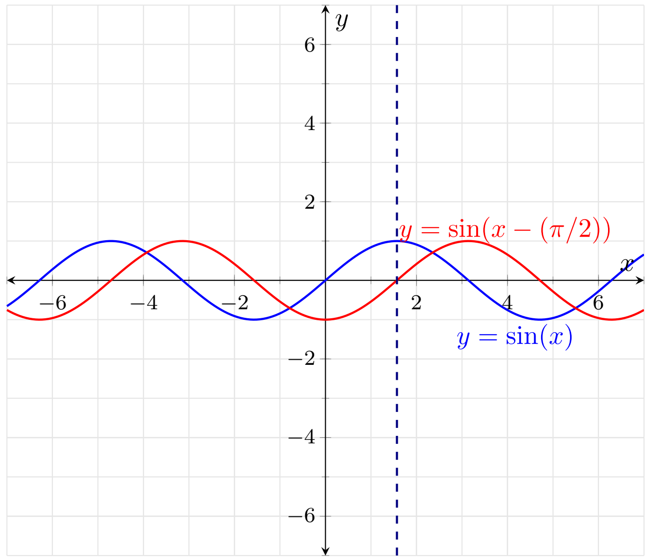

You might be thinking now that the graph of $y = \sin(x - (\pi/2))$ looks an awful lot like the graph of $y = -\cos x$. This is not a coincidence and indeed $\sin(x - (\pi/2)) = -\cos x$ is true for all values of x. We will explore such relations (and more) between trigonometric functions, in future units.

c. $g(x) = (x - 2)^2 - 5(x - 2) + 6$.

Explanation As you might be guessing by now, the parent function can be found by just seeking the shifted variable (in this case, $x - 2$), and replacing it with x. Meaning that if $f(x) = x^2 - 5x + 6$, then $g(x) = f(x - 2)$. Thus, to graph $y = g(x)$, we can just graph $y = f(x)$ and shift it 2 units to the right. Since we can write $f(x) = (x - 2)(x - 3)$, we know that f is a parabola which crosses the x-axis at $x = 2$ and $x = 3$, and it is concave up (we'll understand how to graph parabolas in a bit more of detail, finding also its vertex, on the next section). Hence:

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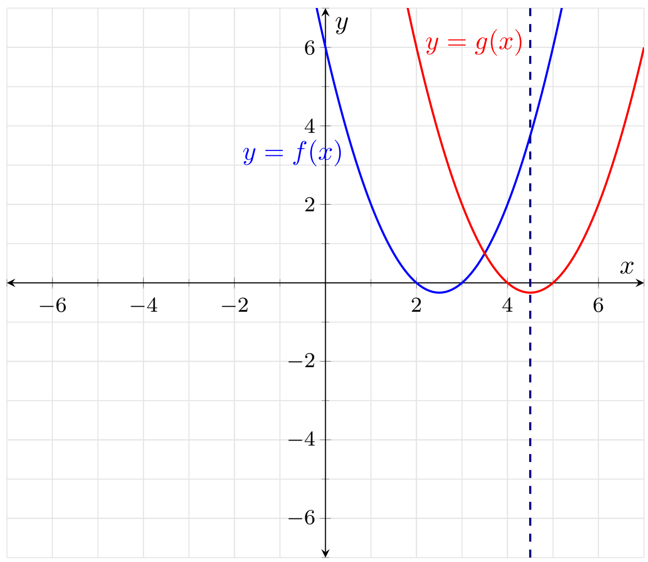

More on parabolas

Let's discuss what happens with parabolas and quadratic functions more precisely. Consider a generic $f(x) = ax^2 + bx + c$, with a, b and c real numbers, and assume that $a \neq 0$. We assume this because if a were zero, we would have a linear function instead of a quadratic one, which is not the focus here. The quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a},$$

used to find the values of x for which $f(x) = 0$ or, in other words, the possible x-intercepts, is usually a source of grief for people studying quadratic functions for the first time. Let's try to remedy this by understanding where such formula comes from. The main strategy here is a little algebraic device called "completing the squares", which is also useful for finding the coordinates (h, k) of the vertex of the parabola given by the graph of $f(x)$.

Very briefly, the procedure consists in noting that $(x - h)^2 = x^2 - 2hx + h^2$ and paying close attention to the $-2hx$ term. Comparing this with the linear term you were given will tell you what h should be. Add and subtract whatever you need to in the quadratic function you were given, to produce $(x - h)^2$ (or, more generally, the multiple $a(x - h)^2$), and whatever constant term is left outside the factored $(\cdots)^2$ will be the desired k. We'll see several examples below.

Example 3. *For each of the following quadratic functions, rewrite it in the form $f(x) = a(x - h)^2 + k$, for suitable numbers h and k. Such point (h, k) is automatically the vertex of the parabola in question.*

a. $f(x) = x^2 - 4x + 7$.

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Explanation Comparing x^2-4x with x^2-2hx suggests that $h=2$ does the trick. Since $h^2=4$, we add and subtract 4 in the expression for the given $f(x)$, to get

$$f(x)=x^2-4x+7=(x^2-4x+4)-4+7=(x-2)^2+3.$$

Hence, the vertex of the parabola $y=x^2-4x+7$ is the point $(2,3)$.

b. $f(x)=2x^2+6x+12$.

Explanation This time, look at $2x^2+6x=2(x^2+3x)$, and compare x^2+3x with x^2-2hx. It seems like $h=-3/2$ is what we need. Note that $h^2=9/4$. Because of the 2 we had to factor out, we'll add and subtract $2\cdot(9/4)=9/2$ to complete the square. So

$$\begin{aligned}f(x)=2x^2+6x+12&=2x^2+6x+\frac{9}{2}-\frac{9}{2}+12\\&=2\left(x^2+3x+\frac{9}{4}\right)-\frac{9}{2}+12=2\left(x+\frac{3}{2}\right)^2+\frac{15}{2}\\&=2\left(x-\left(-\frac{3}{2}\right)\right)^2+\frac{15}{2}.\end{aligned}$$

Thus $(h,k)=(-3/2,15/2)$ is the vertex of this parabola.

c. $f(x)=3x^2+12x+14$.

Explanation As before, we start focusing on $3x^2+12x=3(x^2+4x)$. Compare x^2+4x with x^2-2hx to see that we need $h=-2$ here. Since $h^2=4$, let's add and subtract $3\cdot 4=12$ from the original expression, to obtain

$$\begin{aligned}f(x)=3x^2+12x+14&=(3x^2+12x+12)-12+14\\&=3(x^2+4x+4)+2=3(x+2)^2+2\\&=3(x-(-2))^2+2.\end{aligned}$$

So, the vertex of this parabola has coordinates $(h,k)=(-2,2)$.

Very generally, consider $f(x)=ax^2+bx+c$, with a, b and c real numbers, with $a\neq 0$. Let's repeat everything we have done in the previous example, with a, b and c instead of concrete numbers. Here are the steps we can follow:

- First, look only at $ax^2+bx=a(x^2+bx/a)$.
- Compare x^2+bx/a with x^2-2hx. The h we need here is $h=-b/2a$. Note that $h^2=b^2/4a^2$.
- Because of the a we had to factor out in the beginning, let's add and subtract $a\cdot(b^2/4a^2)=b^2/4a$ from the original expression.

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- Compute

$$\begin{aligned}
ax^2+bx+c &= \left(ax^2+bx+\frac{b^2}{4a}\right)-\frac{b^2}{4a}+c \\
&= a\left(x^2+\frac{bx}{a}+\frac{b^2}{4a^2}\right)+\frac{-b^2+4ac}{4a} \\
&= a\left(x+\frac{b}{2a}\right)^2+\frac{-b^2+4ac}{4a} \\
&= a\left(x-\left(-\frac{b}{2a}\right)\right)^2+\frac{-(b^2-4ac)}{4a}.
\end{aligned}$$

- Hence, the vertex of the parabola described by $y=ax^2+bx+c$ is given by

$$(h,k)=\left(-\frac{b}{2a},-\frac{b^2-4ac}{4a}\right).$$

And with those computations in place, we are in fact very close to understanding where the quadratic formula came from. Solving $ax^2+bx+c=0$ is, by the above, the same as solving

$$a\left(x+\frac{b}{2a}\right)^2-\frac{b^2-4ac}{4a}=0.$$

Reorganize as

$$a\left(x+\frac{b}{2a}\right)^2=\frac{b^2-4ac}{4a},$$

and divide by a to get

$$\left(x+\frac{b}{2a}\right)^2=\frac{b^2-4ac}{4a^2}.$$

Assuming that $b^2-4ac\geq 0$, we may take square roots on both sides:

$$\left|x+\frac{b}{2a}\right|=\frac{\sqrt{b^2-4ac}}{2|a|}.$$

Eliminating the absolute values, we have

$$x+\frac{b}{2a}=\frac{\pm\sqrt{b^2-4ac}}{2a}.$$

Now solve for x, by putting everything on the right side under a common denominator:

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}.$$

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Mystery solved. For each choice of sign $\pm$, we get one x-intercept. Now, we also observe that the average of such solutions does give the x-coordinate of the vertex, as you might expect:

$$\frac{1}{2}\left(\frac{-b-\sqrt{b^2-4ac}}{2a}+\frac{-b+\sqrt{b^2-4ac}}{2a}\right)=\frac{1}{2}\left(\frac{-2b}{2a}\right)=-\frac{b}{2a}.$$

The y-coordinate of the vertex will be, naturally, $k = f(-b/2a)$. This can also be used as a shortcut to write quadratic functions in vertex-form.

Example 4. *Write* $f(x) = x^2 - 8x + 15$ *in vertex-form* $f(x) = a(x-h)^2 + k$ *without completing the squares explicitly.*

Explanation We can factor the quadratic as $f(x) = (x-3)(x-5)$. This means that the x-coordinate of the vertex is $h = (3+5)/2 = 4$, and so

$$k = f(4) = 4^2 - 8 \cdot 4 + 15 = -1.$$

Hence $x^2 - 8x + 15 = (x-4)^2 - 1$. As a quick sanity-check (particular to *this* example), note that factoring this last result as a difference of squares (because $1^2 = 1$) does give $(x-3)(x-5)$.

Summary

- Vertical shifts: given the graph of $y = f(x)$, we can draw the graph of $y = f(x) + a$, with $a > 0$, by shifting the graph of $y = f(x)$ up by a units. Similarly, the graph of $y = f(x) - a$ is obtained by shifting the original graph down by a units.
- Horizontal shitfs: given the graph of $y = f(x)$, we can draw the graph of $y = f(x-a)$, with $a > 0$, by shifting the graph of $y = f(x)$ by a units to the left. Similarly, the graph of $y = f(x+a)$ is obtained by shifting the original graph by a units to the right.
- For an arbitrary quadratic function $f(x) = ax^2 + bx + c$, we found formulas for the coordinates (h, k) of its vertex by completing the squares. We have also concluded that the x-coordinate h of the vertex is, in fact, the average of the x-intercepts of the parabola described as the graph of $y = f(x)$ and, in particular, we have seen how to deduce the famous quadratic formula with this general strategy.

5.1.2 Stretching Functions

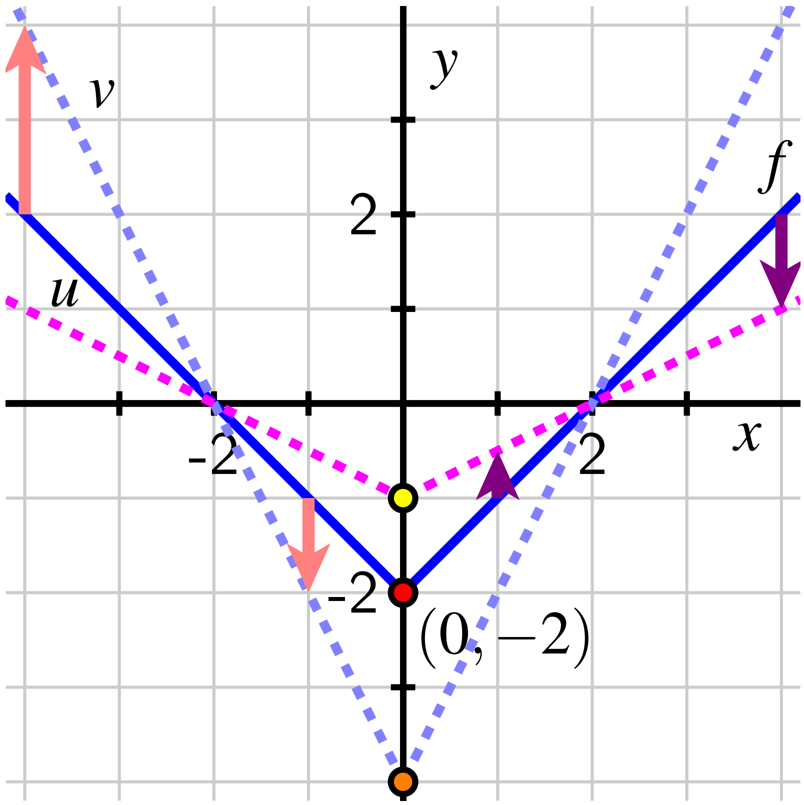
Vertical stretches and reflections

So far, we have seen the possible effects of adding a constant value to function output $f(x)+a$ and adding a constant value to function input $f(x+b)$. Each of these actions results in a translation of the function's graph (either vertically or horizontally), but otherwise leaving the graph the same. Next, we investigate the effects of multiplication the function's output by a constant.

Example 5. *Given the parent function $y = f(x)$ pictured in below , what are the effects of the transformation $y = v(x) = cf(x)$ for various values of c?*

Explanation We first investigate the effects of $c = 2$ and $c = \frac{1}{2}$. For $v(x) = 2f(x)$, the algebraic impact of this transformation is that every output of f is multiplied by 2. This means that the only output that is unchanged is when $f(x) = 0$, while any other point on the graph of the original function f will be stretched vertically away from the x-axis by a factor of 2. We can see this in **image** where each point on the original dark blue graph is transformed to a corresponding point whose y-coordinate is twice as large, as partially indicated by the red arrows.

 Acknowledgements: Active Prelude to Calculus (https://activecalculus.org/prelude)


y
v
f
2
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-2
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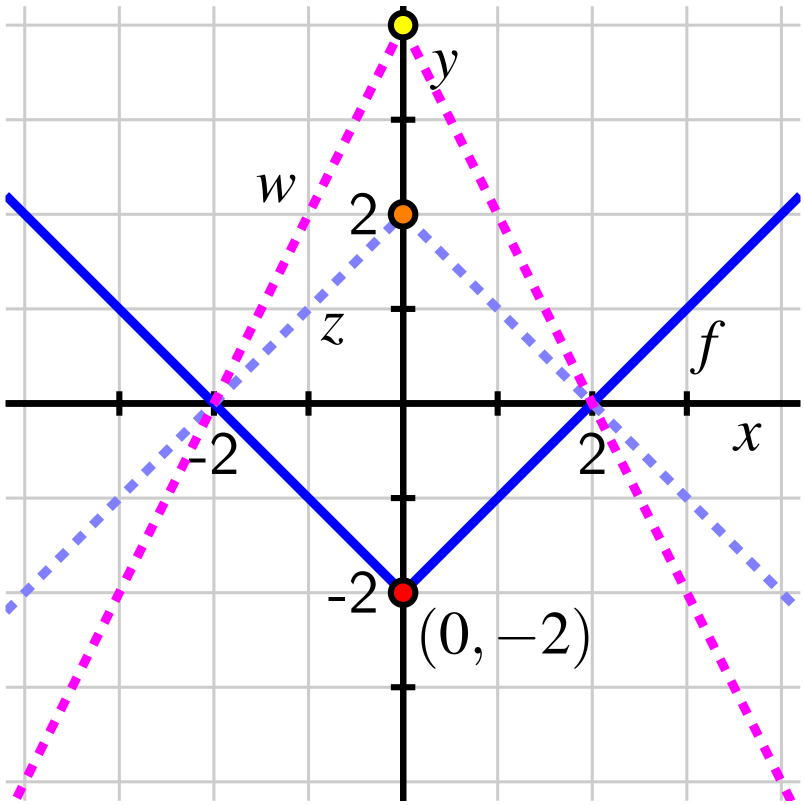
-2
$(0,-2)$



In contrast, the transformation $u(x) = \frac{1}{2}f(x)$ is stretched vertically by a factor of $\frac{1}{2}$, which has the effect of compressing the graph of f towards the x-axis, as all function outputs of f are multiplied by $\frac{1}{2}$. For instance, the point $(0, -2)$ on the graph of f is transformed to the graph of $(0, -1)$ on the graph of u, and others are transformed as indicated by the purple arrows.

To consider the situation where $c < 0$, we first consider the simplest case where $c = -1$ in the transformation $z(x) = -f(x)$. Here the impact of the transformation is to multiply every output of the parent function f by -1; this takes any point of form (x, y) and transforms it to $(x, -y)$, which means we are reflecting each point on the original function's graph across the x-axis to generate the resulting function's graph. This is demonstrated in second graph where $y = z(x)$ is the reflection of $y = f(x)$ across the x-axis and will be discussed more in the next section.

 Acknowledgements: Active Prelude to Calculus (https://activecalculus.org/prelude)

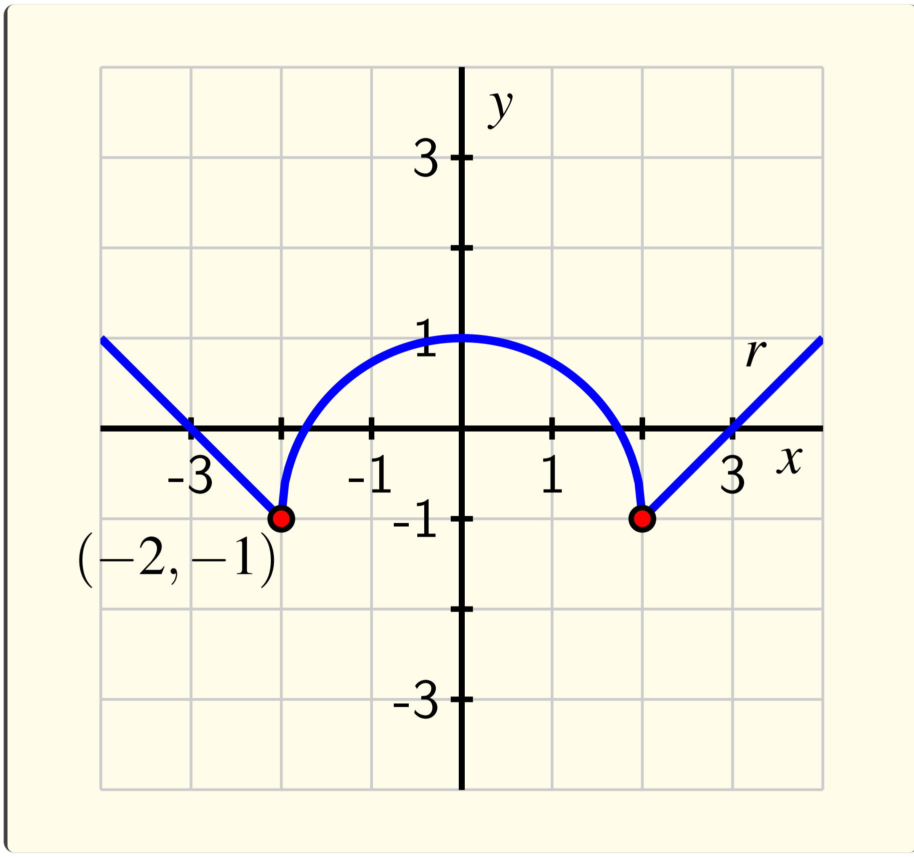
Finally, we also investigate the case where $c = -2$, which generates $y = w(x) = -2f(x)$. Here we can think of -2 as $-2 = 2(-1)$: the effect of multiplying by -1 first reflects the graph of f across the x-axis (resulting in w), and then multiplying by 2 stretches the graph of z vertically to result in w, as shown in second graph .

We summarize and generalize our observations from the graphs above as follows.

Given a function $y = f(x)$ and a real number $c > 0$, the transformed function $y = v(x) = cf(x)$ is a *vertical stretch* of the graph of f. Every point $(x, f(x))$ on the graph of f gets stretched vertically to the corresponding point $(x, cf(x))$ on the graph of v. If $0 < c < 1$, the graph of v is a compression of f toward the x-axis; if $c > 1$, the graph of v is a stretch of f away from the x-axis. Points where $f(x) = 0$ are unchanged by the transformation.
Given a function $y = f(x)$ and a real number $c < 0$, the transformed function $y = v(x) = cf(x)$ is a reflection of the graph of f across the x-axis followed by a vertical stretch by a factor of $|c|$.

Exploration
Consider the functions r and s given in the following graphs.

Acknowledgements: Active Prelude to Calculus (https://activecalculus.org/prelude)



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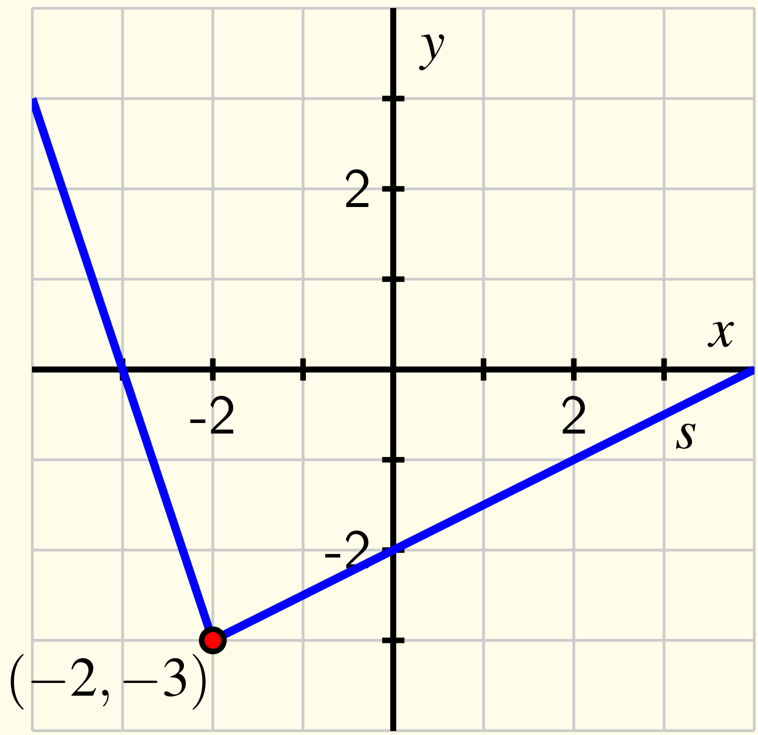
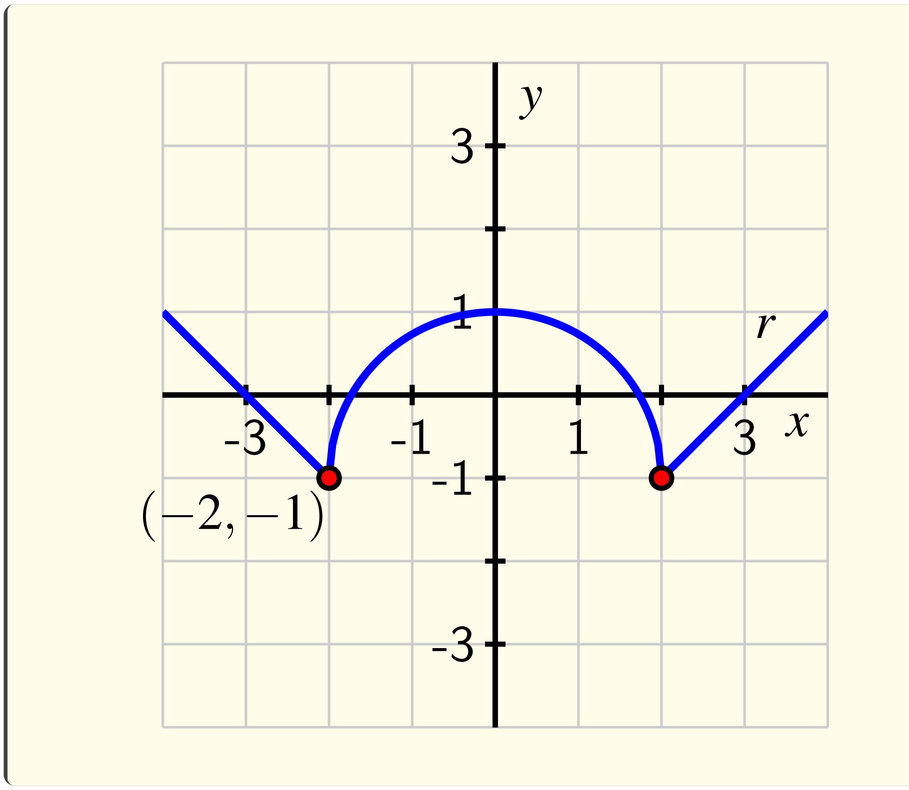


a. On the same axes as the plot of $y = r(x)$, sketch the following graphs: $y = g(x) = 3r(x)$ and $y = h(x) = \frac{1}{3}r(x)$. Be sure to label several points on each of r, g, and h with arrows to indicate their correspondence. In addition, write one sentence to explain the overall transformations that have resulted in g and h from r.

b. On the same axes as the plot of $y = s(x)$, sketch the following graphs: $y = k(x) = -s(x)$ and $y = j(x) = -\frac{1}{2}s(x)$. Be sure to label several points on each of r, g, and h with arrows to indicate their correspondence. In addition, write one sentence to explain the overall transformations that have resulted in g and h from r.

c. On the additional copies of the two figures below, sketch the graphs of the following transformed functions: $y = m(x) = 2r(x+1) - 1$ (at left) and $y = n(x) = \frac{1}{2}s(x-2) + 2$. As above, be sure to label several points on each graph and indicate their correspondence to points on the original parent function.

Acknowledgements: Active Prelude to Calculus (https://activecalculus.org/prelude)



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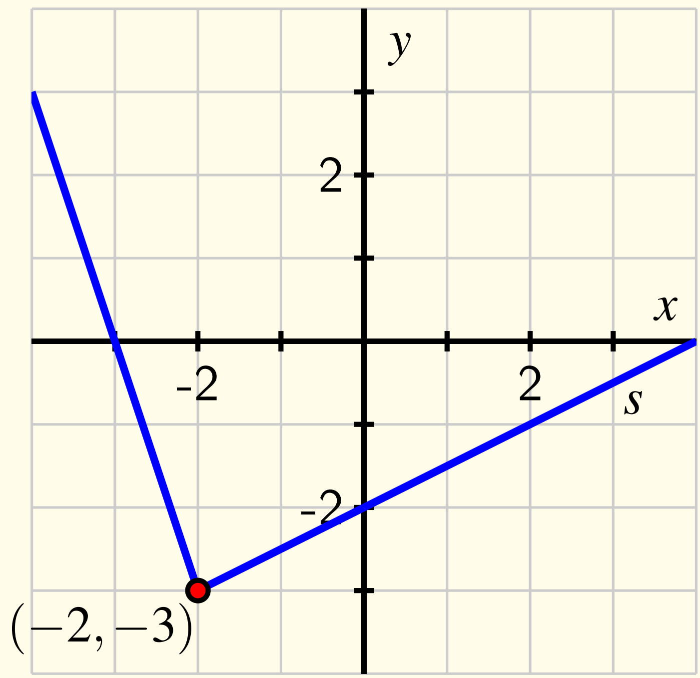

d. Describe in words how the function $y = m(x) = 2r(x+1) - 1$ is the result of three elementary transformations of $y = r(x)$. Does the order in which these transformations occur matter? Why or why not?

Horizontal Stretches

Exploration Follow the Link to Desmos `https://www.desmos.com/calculator/xjem27frqi`.

(a) Make sure that the following graphs are enabled.

- $f(x) = \sqrt{4 - x^2}$
- $1.5f(x)$ or $1.5\sqrt{4 - x^2}$
- $2f(x)$ or $2\sqrt{4 - x^2}$
- $0.5f(x)$ or $0.5\sqrt{4 - x^2}$
- $0.25f(x)$ or $0.25\sqrt{4 - x^2}$

Acknowledgements: Active Prelude to Calculus (https://activecalculus.org/prelude)

What effect do the 1.5, 2, 0.5 and 0.25 seem to have?

(b) Now disable the previous graphs and make sure that the following graphs are enabled.

- $f(x) = \sqrt{4 - x^2}$
- $f(1.5x)$ or $\sqrt{4 - (1.5x)^2}$
- $f(2x)$ or $r\sqrt{4 - (2x)^2}$
- $f(0.5x)$ or $\sqrt{4 - (0.5x)^2}$
- $f(0.25x)$ or $\sqrt{4 - (0.25x)^2}$

What effect do the 1.5, 2, 0.5 and 0.25 seem to have?

Let c be a positive real number then the following transformations result in stretches or shrinks of the graph $y = f(x)$

Horizontal Stretches or Shrinks

$$y = f\left(\frac{x}{c}\right)$$

The transformation is a stretch by factor c if $c > 1$.
The transformation is a shrink by factor c if $c < 1$.

Vertical Stretches or Shrinks

$$y = c \cdot f(x)$$

The transformation is a stretch by factor c if $c > 1$
The transformation is a shrink by factor c if $c < 1$

Example 6. *Let $f(x) = x^3 - 16x$. Find equations for the following transformations of $f(x)$.*

(a) *$g(x)$ is a vertical stretch of $f(x)$ by a factor of 3.*

(b) *$h(x)$ is a horizontal shrink of $f(x)$ by a factor of $\frac{1}{2}$.*

Explanation

(a) Transformation from $f(x)$ to $g(x)$

$$\begin{aligned} g(x) &= 3 \cdot f(x) \\ &= 3(x^3 - 16x) \\ &= 3x^3 - 48x \end{aligned}$$

(b) Transformation from $f(x)$ to $h(x)$

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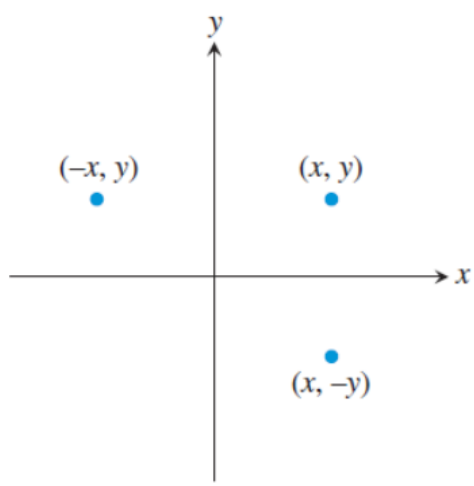
$$
\begin{aligned}
h(x) &= f\left(\frac{x}{\frac{1}{2}}\right) \\
&= f(2x) \\
&= (2x)^3 - 16(2x) \\
&= 8x^3 - 32x
\end{aligned}
$$

Reflections Across Axes

Points (x, y) and $(x, -y)$ are reflections of each other across the x-axis. Points (x, y) and $(-x, y)$ are reflections of each other across the y-axis. In general, two points that are symmetric with respect to a line are reflections of each other across that line.



The following transformations result in reflections of the graph of $y = f(x)$

- Reflection across the x-axis

$$y = -f(x)$$

- Reflection across the y-axis

$$y = f(-x)$$

- Reflection through the origin

$$y = -f(-x)$$

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Notice that this is just a special case of horizontal or vertical stretching where the factor we are multiplying by is -1!

Example 7. *Find an equation for the reflection of* $f(x) = \dfrac{5x-9}{x^2+3}$ *across each axis.*

Explanation Across the x-axis: $y = -f(x) = -\dfrac{5x-9}{x^2+3} = \dfrac{9-5x}{x^2+3}$

Across the y-axis: $y = f(-x) = \dfrac{5(-x)-9}{(-x)^2+3} = \dfrac{-5x-9}{x^2+3}$

5.2 Exponential Functions Revisited

Learning Objectives

- Exponential Functions
  - Function output changes at a proportional rate
  - Definition as functions of the form $f(x) = ab^x$
  - Finding the formula of an exponential function from a table
  - Increasing (at an increasing rate) vs decreasing (at an increasing rate) exponentials
- Modeling with Exponential Functions Revisited
  - Modeling cooling coffee with $f(x) = ab^x + c$
  - Identifying vertical shifts from $+c$ terms
  - Defining long term (end) behavior, introducing arrows to infinity
  - Discussing the concavity of exponentials
- The Special Number e
  - Varying b in b^t is a horizontal scaling
  - Definition of e
  - Average rate of change of e^t approaches the function value
  - $f(t) = e^t$ is invertible. Name its inverse $f(t) = \ln(t)$

5.2.1 Exponential Functions

Motivating Questions

- What does it mean to say that a function is "exponential"?
- How much data do we need to know in order to determine the formula for an exponential function?
- Are there important trends that all exponential functions exhibit?

Introduction

Linear functions have constant average rate of change and model many important phenomena. In other settings, it is natural for a quantity to change at a rate that is proportional to the amount of the quantity present. For instance, whether you put \$100 or \$100000 or any other amount in a mutual fund, the investment's value changes at a rate proportional the amount present. We often measure that rate in terms of the annual percentage rate of return.

Suppose that a certain mutual fund has a 10% annual return. If we invest \$100, after 1 year we still have the original \$100, plus we gain 10% of \$100, so

$$100 \xrightarrow{\text{year 1}} 1 \cdot 100 + 0.1 \cdot 100 = (1 + 0.1) \cdot 100 = 100 + 0.1(100) = 1.1(100).$$

If we instead invested \$100000, after 1 year we again have the original \$100000, but now we gain 10% of \$100000, and thus

$$100000 \xrightarrow{\text{year 1}} 100000 + 0.1(100000) = 1.1(100000).$$

We therefore see that regardless of the amount of money originally invested, say P, the amount of money we have after 1 year is $1.1P$.

If we repeat our computations for the second year, we observe that

$$1.1(100) \xrightarrow{\text{year 2}} 1.1(100) + 0.1(1.1(100) = 1.1(1.1(100)) = 1.1^2(100).$$

The ideas are identical with the larger dollar value, so

$$1.1(100000) \xrightarrow{\text{year 2}} 1.1(100000){+}0.1(1.1(100000) = 1.1(1.1(100000)) = 1.1^2(100000),$$

and we see that if we invest P dollars, in 2 years our investment will grow to 1.1^2P.

Of course, in 3 years at 10%, the original investment P will have grown to 1.1^3P. Here we see a new kind of pattern developing: annual growth of 10% is leading to *powers* of the base 1.1, where the power to which we raise 1.1 corresponds to the number of years the investment has grown. We often call this phenomenon *exponential growth.*

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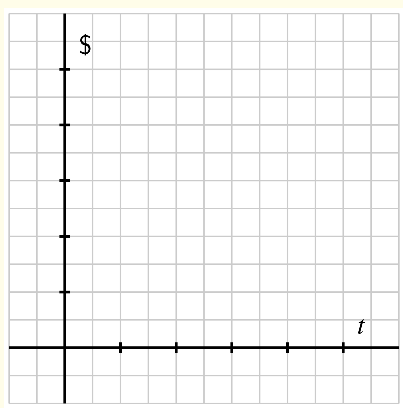
Exploration Suppose that at age 20 you have \$20000 and you can choose between one of two ways to use the money: you can invest it in a mutual fund that will, on average, earn 8% interest annually, or you can purchase a new automobile that will, on average, depreciate 12% annually. Let's explore how the 20000 changes over time.
Let $I(t)$ denote the value of the \$20000 after t years if it is invested in the mutual fund, and let $V(t)$ denote the value of the automobile t years after it is purchased.

a. Determine $I(0)$, $I(1)$, $I(2)$, and $I(3)$.

b. Note that if a quantity depreciates 12% annually, after a given year, 88% of the quantity remains. Compute $V(0)$, $V(1)$, $V(2)$, and $V(3)$.

c. Based on the patterns in your computations in (a) and (b), determine formulas for $I(t)$ and $V(t)$.

d. Use *Desmos* to define $I(t)$ and $V(t)$. Plot each function on the interval $0 \le t \le 20$ and record your results on the axes below, being sure to label the scale on the axes. What trends do you observe in the graphs? How do $I(20)$ and $V(20)$ compare?



Exponential functions of form $f(t) = ab^t$

In the exploration above, we encountered the functions $I(t)$ and $V(t)$ that had the same basic structure. Each can be written in the form $g(t) = ab^t$ where a and b are positive constants and $b \neq 1$. Based on our earlier work with transformations, we know that the constant a is a vertical scaling factor, and thus the main behavior of the function comes from b^t, which we call an "exponential function".

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Definition Let b be a real number such that $b > 0$ and $b \neq 1$. We call the function defined by

$$f(t) = b^t$$

an **exponential function** with **base** b.

For an exponential function $f(t) = b^t$, we note that $f(0) = b^0 = 1$, so an exponential function of this form always passes through $(0, 1)$. In addition, because a positive number raised to any power is always positive (for instance, $2^{10} = 1096$ and $2^{-10} = \dfrac{1}{2^{10}} = \dfrac{1}{2096}$), the output of an exponential function is also always positive.

Remark In particular, $f(t) = b^t$ is never zero and thus has no x-intercepts.

Because we will be frequently interested in functions such as $I(t)$ and $V(t)$ with the form ab^t, we will also refer to functions of this form as "exponential", understanding that technically these are vertical stretches of exponential functions according to our definition of exponential function. In the exploration above, we found that $I(t) = 20000(1.08)^t$ and $V(t) = 20000(0.88)^t$. It is natural to call 1.08 the "growth factor" of I and similarly 0.88 the growth factor of V. In addition, we note that these values stem from the actual growth rates: 0.08 for I and -0.12 for V, the latter being negative because value is depreciating.

Definition In general, for a function of form $f(t) = ab^t$, we call b the **growth factor**. Moreover, if $b = 1 + r$, we call r the **growth rate**. Whenever $b > 1$, we often say that the function f is exhibiting **exponential growth**, whereas if $0 < b < 1$, we say f exhibits **exponential decay**.

Exploration Suppose that at age 20 you have \$20000 and you can choose between one of two ways to use the money: you can invest it in a mutual fund that will, on average, earn 8% interest annually, or you can purchase a new automobile that will, on average, depreciate 12% annually. Let's explore how the 20000 changes over time.
Let $I(t)$ denote the value of the \$20000 after t years if it is invested in the mutual fund, and let $V(t)$ denote the value of the automobile t years after it is purchased.

a. What is the domain of $g(t) = ab^t$?

b. What is the range of $g(t) = ab^t$?

c. What is the y-intercept of $g(t) = ab^t$?

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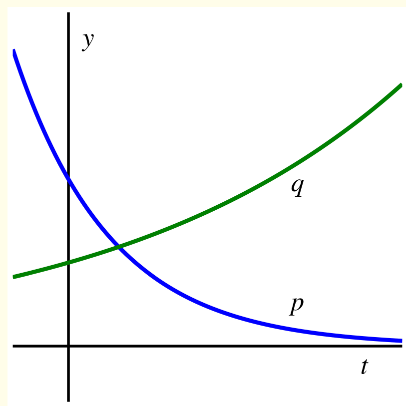
d. How does changing the value of b affect the shape and behavior of the graph of $g(t) = ab^t$? Write several sentences to explain.

e. For what values of the growth factor b is the corresponding growth rate positive? For which b-values is the growth rate negative?

f. Consider the graphs of the exponential functions p and q provided in the figure below. If $p(t) = ab^t$ and $q(t) = cd^t$, what can you say about the values a, b, c, and d (beyond the fact that all are positive and $b \neq 1$ and $d \neq 1$)? For instance, can you say a certain value is larger than another? Or that one of the values is less than 1?



Determining formulas for exponential functions

To better understand the roles that a and b play in an exponential function, let's compare exponential and linear functions. In the tables below, we see output for two different functions r and s that correspond to equally spaced input.

t	$r(t)$
0	12
3	10
6	8
9	6

t	$s(t)$
0	12
3	9
6	6.75
9	5.0625

In the leftside table for $r(t)$, we see a function that exhibits constant average rate of change since the change in output is always $\triangle r = -2$ for any change in input of $\triangle t = 3$. Said differently, r is a linear function with slope $m = -\frac{2}{3}$. Since its y-intercept is $(0, 12)$, the function's formula is $y = r(t) = 12 - \frac{2}{3}t$.

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In contrast, the function s given by rightside table for $s(t)$ does not exhibit constant average rate of change. Instead, another pattern is present. Observe that if we consider the ratios of consecutive outputs in the table, we see that

$$\frac{9}{12} = \frac{3}{4}, \frac{6.75}{9} = 0.75 = \frac{3}{4}, \text{ and } \frac{5.0625}{6.75} = 0.75 = \frac{3}{4}.$$

So, where the *differences* in the outputs in the table for $r(t)$ are constant, the *ratios* in the outputs in the table for $s(t)$ are constant. The latter is a hallmark of exponential functions and may be used to help us determine the formula of a function for which we have certain information.

> **Remark** A function growing exponentially doesn't just mean that it grows faster and faster, but that the ratio between outputs corresponding to equally-spaced inputs is constant.

If we know that a certain function is linear, it suffices to know two points that lie on the line to determine the function's formula. It turns out that exponential functions are similar: knowing two points on the graph of a function known to be exponential is enough information to determine the function's formula. In the following example, we show how knowing two values of an exponential function enables us to find both a and b exactly.

Example 8. *Suppose that p is an exponential function and we know that $p(2) = 11$ and $p(5) = 18$. Determine the exact values of a and b for which $p(t) = ab^t$.*

Explanation Since we know that $p(t) = ab^t$, the two data points give us two equations in the unknowns a and b. First, using $t = 2$,

$$ab^2 = 11,$$

and using $t = 5$ we also have

$$ab^5 = 18.$$

Because we know that the quotient of outputs of an exponential function corresponding to equally-spaced inputs must be constant, we thus naturally consider the quotient $\frac{18}{11}$. Using $ab^2 = 11$ and $ab^5 = 18$, it follows that

$$\frac{18}{11} = \frac{ab^5}{ab^2}.$$

Simplifying the fraction on the right, we see that $\frac{18}{11} = b^3$. Solving for b, we find that $b = \sqrt[3]{\frac{18}{11}}$ is the exact value of b. Substituting this value for b in $ab^2 = 11$, it then follows that $a\left(\sqrt[3]{\frac{18}{11}}\right)^2 = 11$, so $a = \frac{11}{\left(\frac{18}{11}\right)^{2/3}}$. Therefore,

$$p(t) = \frac{11}{\left(\frac{18}{11}\right)^{2/3}}\left(\sqrt[3]{\frac{18}{11}}\right)^t \approx 7.9215 \cdot 1.1784^t,$$

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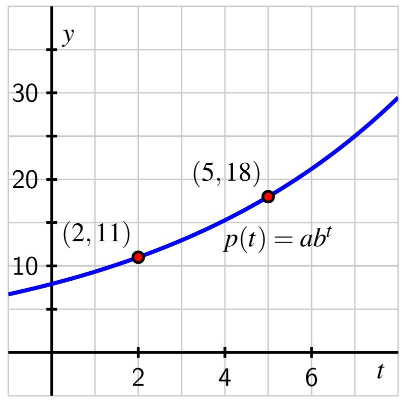
and a plot of $y = p(t)$ confirms that the function indeed passes through $(2, 11)$ and $(5, 18)$ as shown in the figure below.



Exploration The value of an automobile is depreciating. When the car is 3 years old, its value is \$12500; when the car is 7 years old, its value is \$6500.

a. Suppose the car's value t years after its purchase is given by the function $V(t)$ and that V is exponential with form $V(t) = ab^t$. What are the exact values of a and b?

b. Use the exponential model determined in (a), determine the purchase value of the car and estimate when the car will be worth less than \$1000.

c. Suppose instead that the car's value is modeled by a linear function L and satisfies the values stated at the outset of this activity. Find a formula for $L(t)$ and determine both the purchase value of the car and when the car will be worth \$1000.

d. Which model do you think is more realistic? Why?

Recall that a function is increasing on an interval if its value always increasing as we move from left to right. Similarly, a function is decreasing on an interval provided that its value always decreases as we move from left to right.

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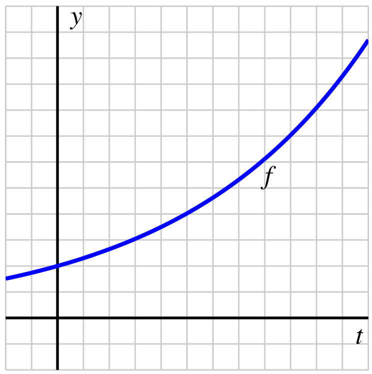

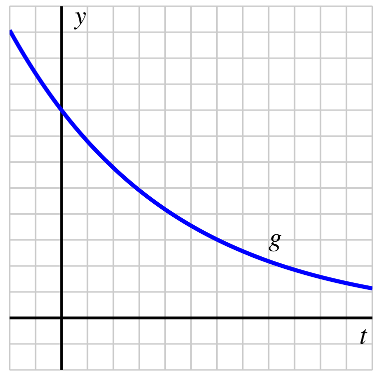

If we consider an exponential function f with a growth factor $b > 1$, such as the function pictured in the left-hand graph above, then the function is always increasing because higher powers of b are greater than lesser powers (for example, $(1.2)^3 > (1.2)^2$). On the other hand, if $0 < b < 1$, then the exponential function will be decreasing because higher powers of positive numbers less than 1 get smaller (e.g., $(0.9)^3 < (0.9)^2$), as seen for the exponential function in the right-hand graph above.

An additional trend is apparent in the graphs in above. Each graph bends upward and is therefore concave up. We can better understand why this is so by considering the average rate of change of both f and g on consecutive intervals of the same width. We choose adjacent intervals of length 1 and note particularly that as we compute the average rate of change of each function on such intervals,

$$\text{AROC}_{[t,t+1]} = \frac{f(t+1) - f(t)}{t+1-1} = f(t+1) - f(t).$$

Thus, these average rates of change are also measuring the total change in the function across an interval that is 1-unit wide. We now assume that $f(t) = 2(1.25)^t$ and $g(t) = 8(0.75)^t$ and compute the rate of change of each function on several consecutive intervals.

The average rate of change of $f(t) = 2(1.25)^t$

t	$f(t)$	$\text{AROC}_{[t,t+1]}$
0	2	0.5
1	2.5	0.625
2	3.125	0.78215
3	3.90625	0.97656

 Acknowledgements: Active Prelude to Calculus (https://activecalculus.org/prelude)

The average rate of change of $g(t) = 8(0.75)^t$

t	$g(t)$	$\text{AROC}_{[t,t+1]}$
0	8	-2
1	6	-1.5
2	4.5	-1.125
3	3.375	-0.84375

From the data in the first table about $f(t)$ we see that the average rate of change is increasing as we increase the value of t. We naturally say that f appears to be "increasing at an increasing rate". For the function g, we first notice that its average rate of change is always negative, but also that the average rate of change gets *less negative* as we increase the value of t. Said differently, the average rate of change of g is also increasing as we increase the value of t. Since g is always decreasing but its average rate of change is increasing, we say that g appears to be "decreasing at an increasing rate". These trends hold for exponential functions generally according to the conditions given below. It takes calculus to justify this claim fully and rigorously.

Trends in exponential function behavior.

For an exponential function of the form $f(t) = ab^t$ where a and b are both positive with $b \neq 1$,

- if $b > 1$, then f is always increasing and always increases at an increasing rate;
- if $0 < b < 1$, then f is always decreasing and always decreases at an increasing rate.

If a function f is always increasing and always increases at an increasing rate, it is concave up, and vice-versa. If a function f is always decreasing and always decreases at an increasing rate, it is concave down, and vice-versa.

Observe how a function's average rate of change helps us classify the function's behavior on an interval: whether the average rate of change is always positive or always negative on the interval enables us to say if the function is always increasing or always decreasing, and then how the average rate of change itself changes enables us to potentially say *how* the function is increasing or decreasing through phrases such as "decreasing at an increasing rate".

Exploration
For each of the following prompts, give an example of a function that satisfies the stated characteristics by both providing a formula and sketching

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a graph.

a. A function p that is always decreasing and decreases at a constant rate.

b. A function q that is always increasing and increases at an increasing rate.

c. A function r that is always increasing for $t < 2$, always decreasing for $t > 2$, and is always changing at a decreasing rate.

d. A function s that is always increasing and increases at a decreasing rate. (Hint: to find a formula, think about how you might use a transformation of a familiar function.)

e. A function u that is always decreasing and decreases at a decreasing rate.

Summary

- We say that a function is exponential whenever its algebraic form is $f(t) = ab^t$ for some positive constants a and b where $b \neq 1$. (Technically, the formal definition of an exponential function is one of form $f(t) = b^t$, but in our everyday usage of the term "exponential" we include vertical stretches of these functions and thus allow a to be any positive constant, not just $a = 1$.)
- To determine the formula for an exponential function of form $f(t) = ab^t$, we need to know two pieces of information. Typically this information is presented in one of two ways.
  - If we know the amount, a, of a quantity at time $t = 0$ and the rate, r, at which the quantity grows or decays per unit time, then it follows $f(t) = a(1 + r)^t$. In this setting, r is often given as a percentage that we convert to a decimal (e.g., if the quantity grows at a rate of 7% per year, we set $r = 0.07$, so $b = 1.07$).
  - If we know any two points on the exponential function's graph, then we can set up a system of two equations in two unknowns and solve for both a and b exactly. In this situation, it is useful to consider the quotient of the two known outputs, as demonstrated in Example 8.
- Exponential functions of the form $f(t) = ab^t$ (where a and b are both positive and $b \neq 1$) exhibit the following important characteristics:

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- The domain of any exponential function is the set of all real numbers and the range of any exponential function is the set of all positive real numbers.
- The y-intercept of the exponential function $f(t) = ab^t$ is $(0, a)$ and the function has no x-intercepts.
- If $b > 1$, then the exponential function is always increasing and always increases at an increasing rate. If $0 < b < 1$, then the exponential function is always decreasing and always decreases at an increasing rate.

5.2.2 Modeling with Exponential Functions Revisited

Motivating Questions

- What can we say about the behavior of an exponential function as the input gets larger and larger?
- How do vertical stretches and shifts of an exponontial function affect its behavior?
- Why is the temperature of a cooling or warming object modeled by a function of the form $F(t) = ab^t + c$?

Introduction

If a quantity changes so that its growth or decay occurs at a constant percentage rate with respect to time, the function is exponential. This is because if the growth or decay rate is r, the total amount of the quantity at time t is given by $A(t) = a(1 + r)^t$, where a is the amount present at time $t = 0$. Many different natural quantities change according to exponential models: money growth through compounding interest, the growth of a population of cells, and the decay of radioactive elements

A related situation arises when an object's temperature changes in response to its surroundings. For instance, if we have a cup of coffee at an initial temperature of 186° Fahrenheit and the cup is placed in a room where the surrounding temperature is 71°, our intuition and experience tell us that over time the coffee will cool and eventually tend to the 71° temperature of the surroundings. From an experiment with an actual temperature probe, we have the data in the table below.

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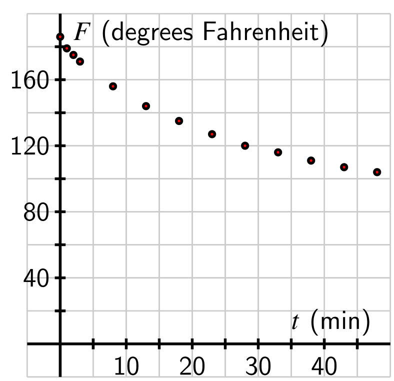
Data for cooling coffee

(measured in degrees Fahrenheit at the time t in minutes)

t	$F(t)$
0	186
1	179
2	175
3	171
8	156
13	144
18	135
23	127
28	120
33	116
38	111
43	107
48	104

Here is a graph of these points.



In one sense, the data looks exponential: the points appear to lie on a curve that is always decreasing and decreasing at an increasing rate. However, we know that the function can't have the form $f(t) = ab^t$ because such a function's range is the set of all positive real numbers, and it's impossible for the coffee's temperature to fall below room temperature (71°). It is natural to wonder if a function of the form $g(t) = ab^t + c$ will work. Thus, in order to find a function that fits the data in a situation such as this, we begin by investigating and understanding the roles of a, b, and c in the behavior of $g(t) = ab^t + c$.

 Acknowledgements: Active Prelude to Calculus (https://activecalculus.org/prelude)

Exploration In *Desmos* , define $g(t) = ab^t + c$ and accept the prompt for sliders for a, b, and c. Edit the sliders so that a has values from $a = 5$ to $a = 50$, b has values from $b = 0.7$ to $b = 1.3$, and c has values from $c = -5$ to $b = 5$ (also with a step-size of 0.01). In addition, in *Desmos* let $P = (0, g(0))$ and check the box to show the label. Finally, zoom out so that the window shows an interval of t-values from $-30 \le t \le 30$

a. Set $b = 1.1$ and explore the effects of changing the values of a and c. Write several sentences to summarize your observations.

b. Follow the directions for (a) again, this time with $b = 0.9$

c. Set $a = 5$ and $c = 4$. Explore the effects of changing the value of b; be sure to include values of b both less than and greater than 1. Write several sentences to summarize your observations.

d. When $0 < b < 1$, what happens to the graph of g when we consider positive t-values that get larger and larger?

End behavior of exponential functions

We have already established that any exponential function of the form $f(t) = ab^t$ where a and b are positive real numbers with $b \neq 1$ is always concave up and is either always increasing or always decreasing. We next introduce precise language to describe the behavior of an exponential function's value as t gets bigger and bigger. To start, let's consider the two basic exponential functions $p(t) = 2^t$ and $q(t) = \left(\frac{1}{2}\right)^t$ and their respective values at $t = 10$, $t = 20$, and $t = 30$, as displayed below.

t	$p(t)$
10	$2^{10} = 1024$
20	$2^{20} = 1048576$
30	$2^{30} = 1073741824$

t	$q(t)$
10	$\left(\frac{1}{2}\right)^{10} = \frac{1}{1024} \approx 0.0009765625$
20	$\left(\frac{1}{2}\right)^{20} = \frac{1}{1048576} \approx 0.00000095367$
30	$\left(\frac{1}{2}\right)^{30} = \frac{1}{1073741824} \approx 0.00000000093192$

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For the increasing function $p(t) = 2^t$, we see that the output of the function gets very large very quickly. In addition, there is no upper bound to how large the function can be. Indeed, we can make the value of $p(t)$ as large as we'd like by taking t sufficiently big. We thus say that as t increases, $p(t)$ **increases without bound**.

For the decreasing function $q(t) = \left(\frac{1}{2}\right)^t$, we see that the output $q(t)$ is always positive but getting closer and closer to 0. Indeed, becasue we can make 2^t as large as we like, it follows that we can make its reciprocal $\frac{1}{2^t} = \left(\frac{1}{2}\right)^t$ as small as we'd like. We thus say that as t increases, $q(t)$ **approaches** 0.

To represent these two common phenomena with exponential functions the value increasing without bound or the value approaching 0, we will use shorthand notation. First, it is natural to write "$q(t) \to 0$" as t increases without bound. Moreover, since we have the notion of the infinite to represent quantities without bound, we use the symbol for infinity (∞) and write "$p(t) \to \infty$" as t increases without bound in order to indicate that $p(t)$ increases without bound.

In the exploration above, we saw how the value of b affects the steepness of the graph of $f(t) = ab^t$, as well as how all graphs with $b > 1$ have the similar increasing behavior, and all graphs with $0 < b < 1$ have similar decreasing behavior. For instance, by taking t sufficiently large, we can make $(1.01)^t$ as large as we want; it just takes much larger t to make $(1.01)^t$ big in comparison to 2^t. In the same way, we can make $(0.99)^t$ as close to 0 as we wish by taking t sufficiently big, even though it takes longer for $(0.99)^t$ to get close to 0 in comparison to $\left(\frac{1}{2}\right)^t$. For an arbitrary choice of b, we can say the following.

End behavior of exponential functions. Let $f(t) = b^t$ with $b > 0$ and $b \neq 1$.

- If $0 < b < 1$, then $b^t \to 0$ as $t \to \infty$. We read this notation as "b^t tends to 0 as t increases without bound." Also, $b^t \to \infty$ as $t \to -\infty$. We read this notation as "b^t increases without bound as t decreases without bound."
- If $b > 1$, then $b^t \to \infty$ as $t \to \infty$. We read this notation as "b^t increases without bound as t increases without bound." Also, $b^t \to 0$ as $t \to -\infty$. We read this notation as "b^t tends to 0 as t decreases without bound."

In addition, we make a key observation about the use of exponents. For the function $q(t) = \left(\frac{1}{2}\right)^t$, there are three equivalent ways we may write the func-

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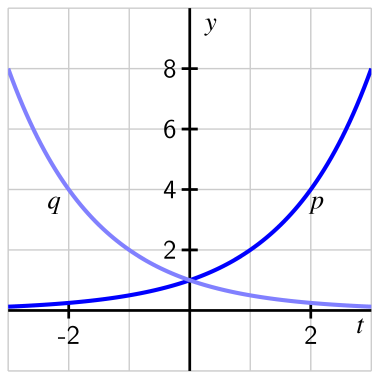
tion:

$$\left(\frac{1}{2}\right)^t = \frac{1}{2^t} = 2^{-t}.$$

In our work with transformations involving horizontal scaling in Exercise 2.4.5.3, we saw that the graph of $y = h(-t)$ is the reflection of the graph of $y = h(t)$ across the y-axis. Therefore, we can say that the graphs of $p(t) = 2^t$ and $q(t) = \left(\frac{1}{2}\right)^t = 2^{-t}$ are reflections of one another across the y-axis since $p(-t) = 2^{-t} = q(t)$. We see this fact verified in the graph below.



Similar observations hold for the relationship between the graphs of b^t and $\frac{1}{b^t} = b^{-t}$ for any positive $b \neq 1$.

Summary

- If the base b of an exponential function b^t satisfies $b > 1$, $b^t \to \infty$ as $t \to \infty$ and $b^t \to 0$ as $t \to -\infty$.
- If the base b of an exponential function b^t satisfies $0 < b < 1$, $b^t \to 0$ as $t \to \infty$ and $b^t \to \infty$ as $t \to -\infty$.

The role of c in $g(t) = ab^t + c$

The function $g(t) = ab^t + c$ is a vertical translation of the function $f(t) = ab^t$. We now have extensive understanding of the behavior of $f(t)$ and how that behavior depends on a and b. Since a vertical translation by c does not change the shape of any graph, we expect that g will exhibit very similar behavior to f. Indeed, we can compare the two functions' graphs as shown in the graphs belowA and then make the following general observations.

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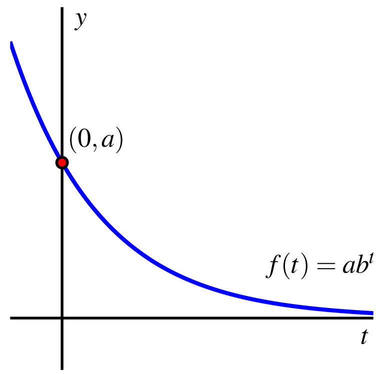

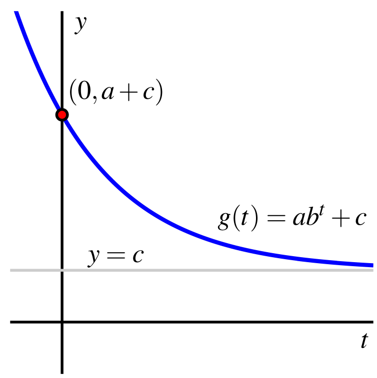

Behavior of vertically shifted exponential functions. Let $g(t) = ab^t + c$ with $a > 0$, $b > 0$ and $b \neq 1$, and c any real number.

- If $0 < b < 1$, then $g(t) = ab^t + c \to c$ as $t \to \infty$. The function g is always decreasing, always concave up, and has y-intercept $(0, a+c)$. The range of the function is all real numbers greater than c.
- If $b > 1$, then $g(t) = ab^t + c \to \infty$ as $t \to \infty$. The function g is always increasing, always concave up, and has y-intercept $(0, a+c)$. The range of the function is all real numbers greater than c.

It is also possible to have $a < 0$. In this situation, because $g(t) = ab^t$ is both a reflection of $f(t) = b^t$ across the x-axis and a vertical stretch by $|a|$, the function g is always concave down. If $0 < b < 1$ so that f is always decreasing, then g is always increasing; if instead $b > 1$ so f is increasing, then g is decreasing. Moreover, instead of the range of the function g having a lower bound as when

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$a > 0$, in this setting the range of g has an upper bound. These ideas are explored further below.

It's an important skill to be able to look at an exponential function of the form $g(t) = ab^t + c$ and form an accurate mental picture of the graph's main features in light of the values of a, b, and c.

Exploration For each of the following functions, *without* using graphing technology, determine whether the function is

i. always increasing or always decreasing;

ii. always concave up or always concave down; and

iii. increasing without bound, decreasing without bound, or increasing/decreasing toward a finite value.

In addition, state the y-intercept and the range of the function. For each function, write a sentence that explains your thinking and sketch a rough graph of how the function appears.

a. $p(t) = 4372(1.000235)^t + 92856$

b. $q(t) = 27931(0.97231)^t + 549786$

c. $r(t) = -17398(0.85234)^t$

d. $s(t) = -17398(0.85234)^t + 19411$

e. $u(t) = -7522(1.03817)^t$

f. $v(t) = -7522(1.03817)^t + 6731$

Modeling temperature data

Newton's Law of Cooling states that the rate that an object warms or cools occurs in direct proportion to the difference between its own temperature and the temperature of its surroundings. If we return to the coffee temperature data and recall that the room temperature in that experiment was 71°, we can see how to use a transformed exponential function to model the data. In the table below, we add a row of information to the table where we compute $F(t) - 71$ to subtract the room temperature from each reading.

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Data for cooling coffee

(measured in degrees Fahrenheit at the time t in minutes)

t	$F(t)$	$f(t) = F(t) - 71$
0	186	115
1	179	108
2	175	104
3	171	100
8	156	85
13	144	73
18	135	64
23	127	56
28	120	49
33	116	45
38	111	40
43	107	36
48	104	33

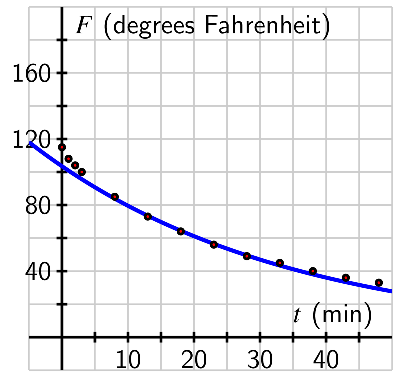
The data in the last row of the table appears exponential, and if we test the data by computing the quotients of output values that correspond to equally-spaced input, we see a nearly constant ratio. In particular,

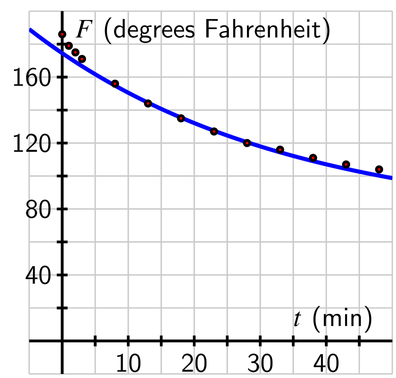
$$\frac{73}{85} \approx 0.86,\ \frac{64}{73} \approx 0.88,\ \frac{56}{64} \approx 0.88,\ \frac{49}{56} \approx 0.88,\ \frac{45}{49} \approx 0.92, \text{and} \frac{40}{45} \approx 0.89.$$

Of course there is some measurement error in the data (plus it is only recorded to accuracy of whole degrees), so these computations provide convincing evidence that the underlying function is exponential. In addition, we expect that if the data continued in the last of the table, the values would approach 0 because $F(t)$ will approach 71.

If we choose two of the data points, say $(18, 64)$ and $(23, 56)$, and assume that $f(t) = ab^t$, we can determine the values of a and b. Since the above points are on the graph, we know that $f(18) = 64$ and $f(23) = 56$, that is, $ab^{18} = 64$ and $ab^{23} = 56$. Solving for a in both equations, we find that $a = 64/b^{18}$ and $a = 56/b^{23}$. Equating these expressions tells us that $64/b^{18} = 56/b^{23}$, and cross-multiplying gives us $64b^{23} = 56b^{18}$. Cancelling gives us $b^5 = 56/64$, or $b = \sqrt[5]{56/64}$. Now we can plug this into our earlier expression for a to find that $a = 64/(\sqrt[5]{56/64})^{18}$.

Using a calculator, we can see that $a \approx 103.503$ and $b \approx 0.974$, so $f(t) \approx 103.503(0.974)^t$. We'll use the name $g(t)$ to refer to the approximate function $103.503(0.974)^t$. Since $f(t) = F(t) - 71$, we see that $F(t) = f(t) + 71$, so $F(t) \approx 103.503(0.974)^t + 71$. We'll use the name $G(t)$ to refer to the approximate function $103.503(0.974)^t + 71$. Plotting g against the shifted data and G along with the original data in the graphs above, we see that the curves go exactly through the points where $t = 18$ and $t = 23$ as expected, but also that the function provides a reasonable model for the observed behavior at any time t. If our data was even more accurate, we would expect that the curve's fit would be even better.





Our preceding work with the coffee data can be done similarly with data for any cooling or warming object whose temperature initially differs from its surroundings. Indeed, it is possible to show that Newton's Law of Cooling implies that the object's temperature is given by a function of the form $F(t) = ab^t + c$.

Exploration A can of soda (at room temperature) is placed in a refrigerator at time $t = 0$ (in minutes) and its temperature, $F(t)$, in degrees Fahrenheit, is computed at regular intervals. Based on the data, a model is formulated for the object's temperature, given by

$$F(t) = 42 + 30(0.95)^t.$$

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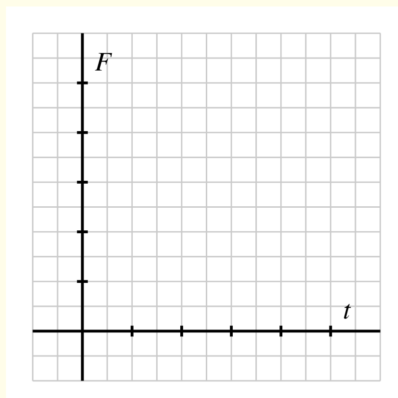
a. Consider the simpler (parent) function $p(t) = (0.95)^t$. How do you expect the graph of this function to appear? How will it behave as time increases? Without using graphing technology, sketch a rough graph of p and write a sentence of explanation.

b. For the slightly more complicated function $r(t) = 30(0.95)^t$, how do you expect this function to look in comparison to p? What is the long-range behavior of this function as t increases? Without using graphing technology, sketch a rough graph of r and write a sentence of explanation.

c. Finally, how do you expect the graph of $F(t) = 42 + 30(0.95)^t$ to appear? Why? First sketch a rough graph without graphing technology, and then use technology to check your thinking and report an accurate, labeled graph on the axes provided.



d. What is the temperature of the refrigerator? What is the room temperature of the surroundings outside the refrigerator? Why?

e. Determine the average rate of change of F on the intervals $[10, 20]$, $[20, 30]$, and $[30, 40]$. Write at least two careful sentences that explain the meaning of the values you found, including units, and discuss any overall trend in how the average rate of change is changing.

Exploration A potato initially at room temperature (68°) is placed in an oven (at 350°) at time $t = 0$. It is known that the potato's temperature at time t is given by the function $F(t) = a - b(0.98)^t$ for some positive constants a and b, where F is measured in degrees Fahrenheit and t is time in minutes.

a. What is the numerical value of $F(0)$? What does this tell you

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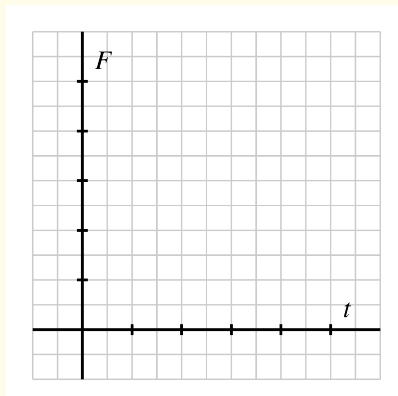
about the value of $a - b$?

b. Based on the context of the problem, what should be the long-range behavior of the function $F(t)$? Use this fact along with the behavior of $(0.98)^t$ to determine the value of a. Write a sentence to explain your thinking.

c. What is the value of b? Why?

d. Check your work above by plotting the function F using graphing technology in an appropriate window. Record your results on the axes provided, labeling the scale on the axes. Then, use the graph to estimate the time at which the potato's temperature reaches 325 degrees.



e. How can we view the function $F(t) = a - b(0.98)^t$ as a transformation of the parent function $f(t) = (0.98)^t$? Explain.

Summary

- For an exponential function of the form $f(t) = b^t$, the function either approaches zero or grows without bound as the input gets larger and larger. In particular, if $0 < b < 1$, then $f(t) = b^t \to 0$ as $t \to \infty$, while if $b > 1$, then $f(t) = b^t \to \infty$ as $t \to \infty$. Scaling f by a positive value a (that is, the transformed function ab^t) does not affect the long-range behavior: whether the function tends to 0 or increases without bound depends solely on whether b is less than or greater than 1.

- The function $f(t) = b^t$ passes through $(0, 1)$, is always concave up, is either always increasing or always decreasing, and its range is the set of all positive real numbers. Among these properties, a

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vertical stretch by a positive value a only affects the y-intercept, which is instead $(0, a)$. If we include a vertical shift and write $g(t) = ab^t + c$, the biggest changes is that the range of g is the set of all real numbers greater than c. In addition, the y-intercept of g is $(0, a + c)$.

- In the situation where $a < 0$, several other changes are induced. Here, because $g(t) = ab^t$ is both a reflection of $f(t) = b^t$ across the x-axis and a vertical stretch by $|a|$, the function g is now always concave down. If $0 < b < 1$ so that f is always decreasing, then g (the reflected function) is now always increasing; if instead $b > 1$ so f is increasing, then g is decreasing. Finally, if $a < 0$, then the range of $g(t) = ab^t + c$ is the set of all real numbers c.

- An exponential function can be thought of as a function that changes at a rate proportional to itself, like how money grows with compound interest or the amount of a radioactive quantity decays. Newton's Law of Cooling says that the rate of change of an object's temperature is proportional to the *difference* between its own temperature and the temperature of its surroundings. This leads to the function that measures the difference between the object's temperature and room temperature being exponential, and hence the object's temperature itself is a vertically-shifted exponential function of the form $F(t) = ab^t + c$.

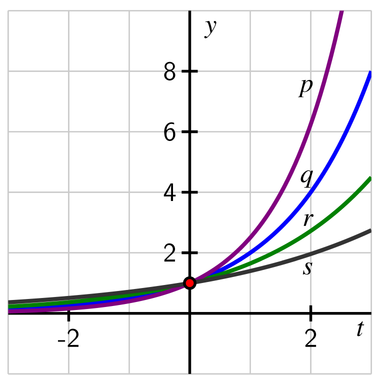
5.2.3 The Special Number e

Motivating Questions

- Why can every exponential function of form $f(t) = b^t$ (where $b > 0$ and $b \neq 1$) be thought of as a horizontal scaling of a single special exponential function?
- What is the natural base e and what makes this number special?

Introduction

We have observed that the behavior of functions of the form $f(t) = b^t$ is very consistent, where the only major differences depend on whether $0 < b < 1$ or $b > 1$. Indeed, if we stipulate that $b > 1$, the graphs of functions with different bases b look nearly identical, as seen in the plots of p, q, r, and s below.



Because the point $(0, 1)$ lies on the graph of each of the four functions in the figure above, the functions cannot be vertical scalings of one another. However, it is possible that the functions are *horizontal* scalings of one another. This leads us to a natural question: might it be possible to find a single exponential function with a special base, say e, for which every other exponential function $f(t) = b^t$ can be expressed as a horizontal scaling of $E(t) = e^t$?

Exploration Open a new *Desmos* worksheet and define the following functions: $f(t) = 2^t$, $g(t) = 3^t$, $h(t) = \left(\frac{1}{3}\right)^t$, and $p(t) = f(kt)$. After you define p, accept the slider for k, and set the range of the slider to be $-2 \leq k \leq 2$.

a. By experimenting with the value of k, find a value of k so that

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the graph of $p(t) = f(kt) = 2^{kt}$ appears to align with the graph of $g(t) = 3^t$. What is the value of k?

b. Similarly, experiment to find a value of k so that the graph of $p(t) = f(kt) = 2^{kt}$ appears to align with the graph of $h(t) = (\frac{1}{3})^t$. What is the value of k?

c. For the value of k you determined in (a), compute 2^k. What do you observe?

d. For the value of k you determined in (b), compute 2^k. What do you observe?

e. Given any exponential function of the form b^t, do you think it's possible to find a value of k to that $p(t) = f(kt) = 2^{kt}$ is the same function as b^t? Why or why not?

The natural base e

In the exploration above, we found that it appears possible to find a value of k so that given any base b, we can write the function b^t as the horizontal scaling of 2^t given by

$$b^t = 2^{kt}.$$

It's also apparent that there's nothing particularly special about "2": we could similarly write any function b^t as a horizontal scaling of 3^t or 4^t, albeit with a different scaling factor k for each. Thus, we might also ask: is there a *best* possible single base to use?

Through the central topic of the *rate of change* of a function, calculus helps us decide which base is best to use to represent all exponential functions. While we study *average* rate of change extensively in this course, in calculus there is more emphasis on the *instantaneous* rate of change. In that context, a natural question arises: is there a nonzero function that grows in such a way that its *height* is exactly how *fast* its height is increasing?

Amazingly, it turns out that the answer to this questions is "yes," and the function with this property is **the exponential function with the natural base**, denoted e^t. The number e (named in homage to the great Swiss mathematician Leonard Euler (1707-1783)) is complicated to define. Like π, e is an irrational number that cannot be represented exactly by a ratio of integers and whose decimal expansion never repeats. Advanced mathematics is needed in order to make the following formal definition of e.

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Definition [The natural base, e] The number e is the infinite sum[a]

$$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots$$

where $n! = 1 \cdot 2 \cdot 3 \cdots (n-1) \cdot n$. From this, $e \approx 2.718281828$.

[a] Infinite sums are usually studied in second semester calculus.

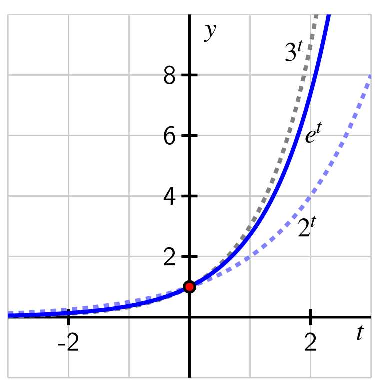
For instance, $1 + \frac{1}{1} + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120} = \frac{163}{60} \approx 2.7167$ is an approximation of e generated by taking the first 6 terms in the infinite sum that defines it. Most calculators know the number e and we will normally work with this number by using technology appropriately.

Initially, it's important to note that $2 < e < 3$, and thus we expect the function e^t to lie between 2^t and 3^t. The following table gives decimal approximations to the values of 2^t, e^t, and 3^t.

t	-2	-1	0	1	2
2^t	0.25	0.5	1	2	4

t	-2	-1	0	1	2
e^t	0.135	0.368	1	2.718	7.389

t	-2	-1	0	1	2
3^t	0.111	0.333	1	3	9



If we compare the graphs and some selected outputs of each function, as in the table and figure above, we see that the function e^t satisfies the inequality

$$2^t < e^t < 3^t$$

for all positive values of t. When t is negative, we can view the values of each function as being reciprocals of powers of 2, e, and 3. For instance, since $2^2 < e^2 < 3^2$, it follows $\frac{1}{3^2} < \frac{1}{e^2} < \frac{1}{2^2}$, or

$$3^{-2} < e^{-2} < 2^{-2}.$$

Thus, for any $t < 0$,

$$3^t < e^t < 2^t$$

Like 2^t and 3^t, the function e^t passes through $(0, 1)$ is always increasing and always concave up, and its range is the set of all positive real numbers.

Exploration Recall that the average rate of change of a function f on an interval $[a, b]$ is

$$\text{AROC}_{[a,b]} = \frac{f(b) - f(a)}{b - a}.$$

In this activity we explore the average rate of change of $f(t) = e^t$ near the points where $t = 1$ and $t = 2$.

In a new *Desmos* worksheet, let $f(t) = e^t$ and define the function A by the rule

$$A(t) = \frac{f(t) - f(1)}{t - 1}.$$

a. What is the meaning of $A(1.5)$ in terms of the function f and its graph?

b. Compute the value of $A(t)$ for at least 6 difference values of t that are close to 1, both above and below 1. For instance, one value to try might be $h = 1.0001$. Record a table of your results.

c. What do you notice about the values you found in (b)? How do they compare to an important number?

d. Explain why the following sentence makes sense: "The function e^t is increasing at an average rate that is about the same as its value on small intervals near $t = 1$."

e. Adjust your definition of A in *Desmos* by changing 1 to 2 so that

$$A(t) = \frac{f(t) - f(2)}{t - 2}.$$

How does the value of $A(t)$ for values of t near 2 compare to $f(2)$?

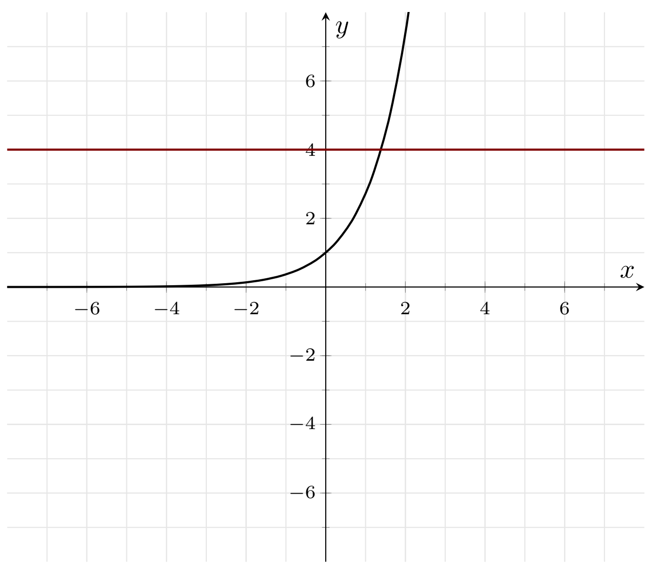
Earlier, we saw graphical evidence that any exponential function $f(t) = b^t$ can be written as a horizontal scaling of the function $g(t) = 2^t$, plus we observed that there wasn't anything particularly special about 2^t. Because of the importance of e^t in calculus, we will choose instead to use the natural exponential function, $E(t) = e^t$ as the function we scale to generate any other exponential function $f(t) = b^t$. We claim that for any choice of $b > 0$ (with $b \neq 1$), there exists a horizontal scaling factor k such that $b^t = f(t) = E(kt) = e^{kt}$.

By the rules of exponents, we can rewrite this last equation equivalently as

$$b^t = (e^k)^t.$$

Since this equation has to hold for every value of t, it follows that $b = e^k$. Thus, our claim that we can scale $E(t)$ to get $f(t)$ requires us to show that regardless of the choice of the positive number b, there exists a single corresponding value of k such that $b = e^k$.

Given $b > 0$, we can always find a corresponding value of k such that $e^k = b$ because the function $f(t) = e^t$ passes the Horizontal Line Test, as seen in the figure below.



In this figure, we can think of b as a point on the positive vertical axis. From there, we draw a horizontal line over to the graph of $f(t) = e^t$, and then from the (unique) point of intersection we drop a vertical line to the x-axis. At that corresponding point on the x-axis we have found the input value k that corresponds to b. We see that there is always exactly one such k value that corresponds to each chosen b because $f(t) = e^t$ is always increasing, and any always increasing function passes the Horizontal Line Test.

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It follows that the function $f(t) = e^t$ has an inverse function, and hence there must be some other function g such that writing $y = f(t)$ is the same as writing $t = g(y)$. This important function g will be be developed more later and will enable us to find the value of k exactly for a given b. For now, we are content to work with these observations graphically and to hence find estimates for the value of k.

Exploration By graphing $f(t) = e^t$ and appropriate horizontal lines, estimate the solution to each of the following equations. Note that in some parts, you may need to do some algebraic work in addition to using the graph.

a. $e^t = 2$

b. $e^{3t} = 5$

c. $2e^t - 4 = 7$

d. $3e^{0.25t} + 2 = 6$

e. $4 - 2e^{-0.7t} = 3$

f. $2e^{1.2t} = 1.5e^{1.6t}$

Summary

- Any exponential function $f(t) = b^t$ can be viewed as a horizontal scaling of $E(t) = e^t$ because there exists a unique constant k such that $E(kt) = e^{kt} = b^t = f(t)$ is true for every value of t. This holds since the exponential function e^t is always increasing, so given an output b there exists a unique input k such that $e^k = b$, from which it follows that $e^{kt} = b^t$.

- The natural base e is the special number that defines an increasing exponential function whose rate of change at any point is the same as its height at that point, a fact that is established using calculus. The number e turns out to be given exactly by an infinite sum and approximately by $e \approx 2.7182818$.

5.3 Introduction to Logarithms

Learning Objectives

- Introduction to Logarithms
  - What is the inverse function for an exponential function? What are it's function properties?

5.3.1 Definition of Logarithms

Motivating Questions

- How is the base-10 logarithm defined?
- What is the "natural logarithm" and how is it different from the base-10 logarithm?
- How can we solve an equation that involves e to some unknown quantity?

In previous sections, we introduced the idea of an inverse function. The fundamental idea is that f has an inverse function if and only if there exists another function g such that f and g "undo" one another's respective processes. In other words, the process of the function f is reversible to generate a related function g.

More formally, recall that a function $y = f(x)$ (where $f : A \to B$) has an inverse function if and only if there exists another function $g : B \to A$ such that $g(f(x)) = x$ for every x in the domain of f and $f(g(y)) = y$ for every y in the domain of g. We know that given a function f, we can use the Horizontal Line Test to determine whether or not f has an inverse function. Finally, whenever a function f has an inverse function, we call its inverse function f^{-1} and know that the two equations $y = f(x)$ and $x = f^{-1}(y)$ say the same thing from different perspectives.

Exploration
Let $P(t)$ be the "powers of 10" function, which is given by $P(t) = 10^t$.

a. Completethe following table to generate certain values of P.

t	-3	-2	-1	0	1	2	3
$y = P(t) = 10^t$							

b. Why does P have an inverse function?

c. Since P has an inverse function, we know there exists some other function, say L, such that writing "$y = P(t)$" says the exact same thing as writing "$t = L(y)$". In words, where P produces the result of raising 10 to a given power, the function L reverses this process and instead tells us the power to which we need to raise 10, given a desired result. Complete the table to generate a collection of values of L.

y	10^{-3}	10^{-2}	10^{-1}	10^0	10^1	10^2	10^3
$L(y)$							

d. What are the domain and range of the function P? What are the domain and range of the function L?

The base-10 logarithm

The powers-of-10 function $P(t) = 10^t$ is an exponential function with base $b > 1$. As such, P is always increasing, and thus its graph passes the Horizontal Line Test, so P has an inverse function. We therefore know there exists some other function, L, such that writing $y = P(t)$ is equivalent to writing $t = L(y)$. For instance, we know that $P(2) = 100$ and $P(-3) = \frac{1}{1000}$, so it's equivalent to say that $L(100) = 2$ and $L\left(\frac{1}{1000}\right) = -3$. This new function L we call the *base* 10 *logarithm*, which is formally defined as follows.

Given a positive real number y, the *base*-10 *logarithm of* y is the power to which we raise 10 to get y. We use the notation "$\log_{10}(y)$" to denote the base-10 logarithm of y.

The base-10 logarithm is therefore the inverse of the powers of 10 function. Whereas $P(t) = 10^t$ takes an input whose value is an exponent and produces the result of taking 10 to that power, the base-10 logarithm takes an input number we view as a power of 10 and produces the corresponding exponent such that 10 to that exponent is the input number.

In the notation of logarithms, we can now update our earlier observations with the functions P and L and see how exponential equations can be written in two equivalent ways. For instance,

$$10^2 = 100 \text{ and } \log_{10}(100) = 2$$

each say the same thing from two different perspectives. The first says 100 is 10 to the power 2 , while the second says 2 is the power to which we raise 10 to get 100. Similarly,

$$10^{-3} = \frac{1}{1000} \text{ and } \log_{10}\left(\frac{1}{1000}\right) = -3.$$

If we rearrange the statements of the facts, we can see yet another important relationship between the powers of 10 and base-10 logarithm function. Noting that $\log_{10}(100) = 2$ and $100 = 10^2$ are equivalent statements, and substituting the former equation into the latter shows, we see that

$$\log_{10}(10^2) = 2.$$

In words, the equation says that "the power to which we raise 10 to get 10^2, is 2". That is, the base-10 logarithm function undoes the work of the powers of 10 function.

In a similar way, we can observe that by replacing -3 with $\log_{10}\left(\frac{1}{1000}\right)$ we have

$$10^{\log_{10}\left(\frac{1}{1000}\right)} = \frac{1}{1000}.$$

In words, this says that "when 10 is raised to the power to which we raise 10 in order to get $\frac{1}{1000}$, we get $\frac{1}{1000}$".

We summarize the key relationships between the powers-of-10 function and its inverse, the base-10 logarithm function, more generally as follows.$P(t) = 10^t$ and $L(y) = \log_{10}(y)$.

- The domain of P is the set of all real numbers and the range of P is the set of all positive real numbers.
- The domain of L is the set of all positive real numbers and the range of L is the set of all real numbers.
- For any real number t, $\log_{10}(10^t) = t$. That is, $L(P(t)) = t$.
- For any positive real number y, $10^{\log_{10}(y)} = y$. That is, $P(L(y)) = y$.
- $10^0 = 1$ and $\log_{10}(1) = 0$.

The base-10 logarithm function is like the sine or cosine function in this way: for certain special values, it's easy to know by heart the value of the logarithm function. For the base-10 logarithm function, the familiar points come from powers of 10. In addition, like sine and cosine, for all other input values, (a) calculus ultimately determines the value of the base-10 logarithm function at other values, and (b) we use computational technology in order to compute these values. For most computational devices, the command $\log(y)$ produces the result of the base-10 logarithm of y.

Remark It's important to note that the logarithm function produces exact values. For instance, if we want to solve the equation $10^t = 5$, then it follows that $t = \log_{10}(5)$ is the exact solution to the equation. Like $\sqrt{2}$ or $\cos(1)$, $\log_{10}(5)$ is a number that is an exact value. A computational device can give us a decimal approximation, and we normally want to distinguish between the exact value and the approximate one. For the three different numbers here, $\sqrt{2} \approx 1.414$, $\cos(1) \approx 0.540$, and $\log_{10}(5) \approx 0.699$.

Exploration
For each of the following equations, determine the exact value of the unknown variable. If the exact value involves a logarithm, use a computational device to also report an approximate value. For instance, if the exact value is $y = \log_{10}(2)$, you can also note that $y \approx 0.301$.

a. $10^t = 0.00001$

b. $\log_{10}(1000000) = t$

c. $10^t = 37$

d. $\log_{10}(y) = 1.375$

e. $10^t = 0.04$

f. $3 \cdot 10^t + 11 = 147$

g. $2\log_{10}(y) + 5 = 1$

The natural logarithm

The base-10 logarithm is a good starting point for understanding how logarithmic functions work because powers of 10 are easy to mentally compute. We could similarly consider the powers of 2 or powers of 3 function and develop a corresponding logarithm of base 2 or 3. But rather than have a whole collection of different logarithm functions, in the same way that we now use the function e^t and appropriate scaling to represent any exponential function, we develop a single logarithm function that we can use to represent any other logarithmic function through scaling. In correspondence with the natural exponential function, e^t, we now develop its inverse function, and call this inverse function the *natural logarithm*.

Definition Given a positive real number y, the **natural logarithm of** y is the power to which we raise e to get y. We use the notation "$\ln(y)$" to denote the natural logarithm of y. The domain of ln is all positive numbers, and the range of ln is all real numbers.

We can think of the natural logarithm, $\ln(y)$, as the "base-e logarithm". For instance,

$$\ln(e^2) = 2$$

and

$$e^{\ln(\pi)} = \pi.$$

The former equation is true because "the power to which we raise e to get e^2 is 2"; the latter equation is true since "when we raise e to the power to which we raise e to get π, we get π".

Exploration Let $E(t) = e^t$ and $N(y) = \ln(y)$ be the natural exponential function and the natural logarithm function, respectively.

a. What are the domain and range of E?

b. What are the domain and range of N?

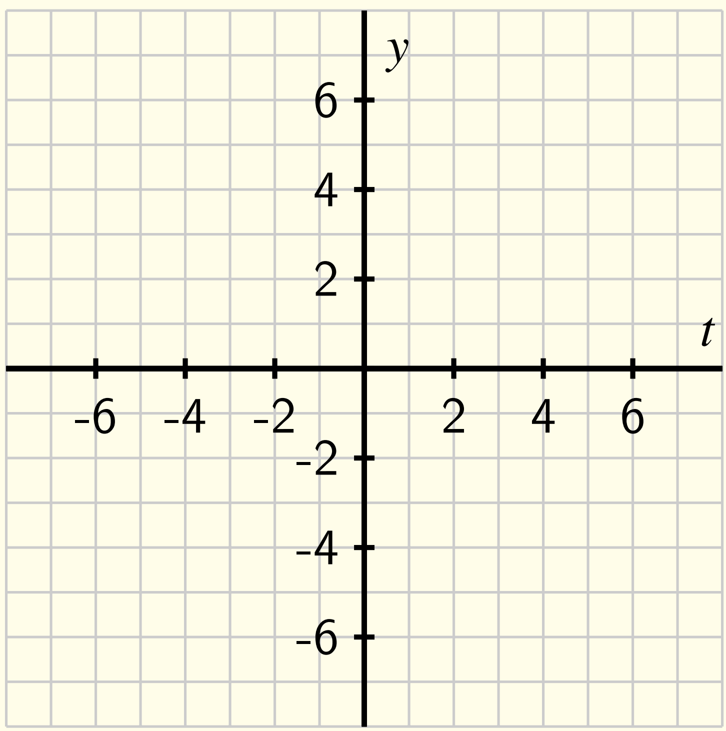
c. What can you say about $\ln(e^t)$ for every real number t?

d. What can you say about $e^{\ln(y)}$ for every positive real number y?

e. Complete the following tables with both exact and approximate of E and N. Then, plot the corresponding ordered pairs from each table on the axes below and connect the points in an intuitive way. When you plot the ordered pairs on the axes, in both cases view the first line of the table as generating values on the horizontal axis and the second line of the table as producing values on the vertical axis label each ordered pair you plot appropriately.

t	-2	-1	0	1	2
$E(t) = e^t$	$e^{-2} \approx 0.135$				

y	e^{-2}	e^{-1}	1	e^1	e^2
$N(y) = \ln(y)$	-2				



Logarithms in general

In the previous sections, we looked at two specific (and the most common) types of logarithms, base-10 and natural log. In order to fully discuss logarithms, we need to talk about logarithms in general with any base. Let $b > 1$. Because the function $y = f(t) = b^t$ has an inverse function, it makes sense to define its inverse like we did when $b = 10$ or $b = e$. The base-b logarithm, denoted $\log_b(y)$ is defined to be the power to which we raise b to get y.

$$\log_b(y) = t$$
$$y = f(t) = b^t$$

Example 9. *Evaluate the following base-b logarithms.*

(a) $\log_2(8)$

(b) $\log_5(25)$

Explanation

(a)

$$\log_2(8) = \log_2(2^3)$$
$$\log_2(8) = 3$$

(b)

$$\log_5(25) = \log_5(5^2)$$
$$\log_5(25) = 2$$

Revisiting $\mathbf{f(t) = b^t}$

In earlier sections, we saw that that function $f(t) = b^t$ plays a key role in modeling exponential growth and decay, and that the value of b not only determines whether the function models growth ($b > 1$) or decay ($0 < b < 1$), but also how fast the growth or decay occurs. Furthermore, once we introduced the natural base e, we realized that we could write every exponential function of form $f(t) = b^t$ as a horizontal scaling of the function $E(t) = e^t$ by writing

$$b^t = f(t) = E(kt) = e^{kt}$$

for some value k. Our development of the natural logarithm function in the current section enables us to now determine k exactly.

Example 10. *Determine the exact value of k for which $f(t) = 3^t = e^{kt}$.*

Explanation Since we want $3^t = e^{kt}$ to hold for every value of t and $e^{kt} = (e^k)^t$, we need to have $3^t = (e^k)^t$, and thus $3 = e^k$. Therefore, k is the power to which we raise e to get 3, which by definition means that $k = \ln(3)$.

In modeling important phenomena using exponential functions, we will frequently encounter equations where the variable is in the exponent, like in the example where we had to solve $e^k = 3$. It is in this context where logarithms find one of their most powerful applications.

Example 11. *Solve each of the following equations for the exact value of the unknown variable. If there is no solution to the equation, explain why not.*

(a) $e^t = \frac{1}{10}$

(b) $5e^t = 7$

(c) $\ln(t) = -\frac{1}{3}$

(d) $e^{1-3t} = 4$

(e) $2\ln(t) + 1 = 4$

(f) $4 - 3e^{2t} = 2$

(g) $4 + 3e^{2t} = 2$

(h) $\ln(5 - 6t) = -2$

Explanation

a.

$$\begin{aligned} e^t &= \frac{1}{10} \\ \ln(e^t) &= \ln\left(\frac{1}{10}\right) \\ t &= \ln\left(\frac{1}{10}\right) \end{aligned}$$

b.

$$\begin{aligned} 5e^t &= 7 \\ e^t &= \frac{7}{5} \\ \ln\left(e^t\right) &= \ln\left(\frac{7}{5}\right) \\ t &= \ln\left(\frac{7}{5}\right) \end{aligned}$$

c.

$$\begin{aligned}\ln(t) &= -\frac{1}{3}\\ e^{\ln(t)} &= e^{-\frac{1}{3}}\\ t &= e^{-\frac{1}{3}}\end{aligned}$$

d.

$$\begin{aligned}e^{1-3t} &= 4\\ \ln(e^{1-3t}) &= \ln(4)\\ 1-3t &= \ln(4)\\ -3t &= \ln(4)-1\\ t &= \frac{\ln(4)-1}{-3}\end{aligned}$$

e.

$$\begin{aligned}2\ln(t)+1 &= 4\\ 2\ln(t) &= 3\\ \ln(t) &= \frac{3}{2}\\ e^{ln(t)} &= e^{\frac{3}{2}}\\ t &= e^{\frac{3}{2}}\end{aligned}$$

f.

$$\begin{aligned}4-3e^{2t} &= 2\\ -3e^{2t} &= -2\\ e^{2t} &= \frac{2}{3}\\ \ln(e^{2t}) &= \ln\left(\frac{2}{3}\right)\\ 2t &= \ln\left(\frac{2}{3}\right)\\ t &= \frac{\ln\left(\frac{2}{3}\right)}{2}\end{aligned}$$

g.

$$\begin{aligned}4+3e^{2t} &= 2\\ 3e^{2t} &= -2\\ e^{2t} &= \frac{-2}{3}\end{aligned}$$

No solution, because $\dfrac{-2}{3}$ is outside of the range of e^{2t}

h.

$$
\begin{aligned}
\ln(5-6t) &= -2 \\
\ln(5-6t) &= -2 \\
e^{\ln(5-6t)} &= e^{-2} \\
5-6t &= e^{-2} \\
t &= \frac{e^{-2}-5}{-6}
\end{aligned}
$$

Summary

(a) The base-10 logarithm of y, denoted $\log_{10}(y)$ is defined to be the power to which we raise 10 to get y. For instance, $\log_{10}(1000) = 3$, since $10^3 = 1000$. The function $L(y) = \log_{10}(y)$ is thus the inverse of the powers-of-10 function, $P(t) = 10^t$.

(b) The natural logarithm $N(y) = \ln(y)$ differs from the base-10 logarithm in that it is the logarithm with base e instead of 10, and thus $\ln(y)$ is the power to which we raise e to get y. The function $N(y) = \ln(y)$ is the inverse of the natural exponential function $E(t) = e^t$.

(c) The base-b logarithm $\log_b(y)$ is the logarithm with base b for $b > 0$ and thus $\log_b(y)$ is the power to which we raise b to get y. The function $\log_b(y)$ is the inverse of the powers-of-b function $B(t) = b^t$.

(d) The natural logarithm often enables us solve an equation that involves e to some unknown quantity. For instance, to solve $2e^{3t-4} + 5 = 13$, we can first solve for e^{3t-4} by subtracting 5 from each side and dividing by 2 to get

$$e^{3t-4} = 4.$$

This last equation says "e to some power is 4". We know that it is equivalent to say

$$\ln(4) = 3t - 4.$$

Since $\ln(4)$ is a number, we can solve this most recent linear equation for t. In particular, $3t = 4 + \ln(4)$, so

$$t = \frac{1}{3}(4 + \ln(4)).$$

Part 6

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