Precalculus with Review 2: Unit 9

February 7, 2023

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Trigonometric Functions

9.1 The Unit Circle to the Function Graph

Learning Objectives

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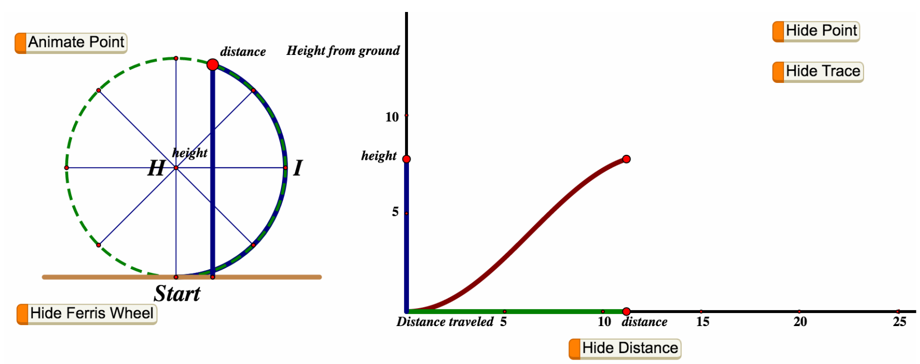
9.1.1 Traversing A Circle

Motivating Questions

- How does a point traversing a circle naturally generate a function?
- What are some important properties that characterize a function generated by a point traversing a circle?

Certain naturally occurring phenomena eventually repeat themselves, especially when the phenomenon is somehow connected to a circle. You may recall from when we first studied periodic functions that we considered the case of taking a ride on a ferris wheel. We considered your height, h, above the ground and how your height changed in tandem with the distance, d, that you have traveled around the wheel. We saw a snapshot of this situation, which is available as a full animation at http://gvsu.edu/s/0Dt .

A snapshot of the motion of a cab moving around a ferris wheel. Reprinted with permission from Illuminations by the National Council of Teachers of Mathematics. All rights reserved.



Because we have two quantities changing in tandem, it is natural to wonder if it is possible to represent one as a function of the other.

Exploration In the context of the ferris wheel pictured above, assume that the height, h, of the moving point (the cab in which you are riding), and the distance, d, that the point has traveled around the circumference of the ferris wheel are both measured in meters. Further, assume that the circumference of the ferris wheel is 150 meters. In addition, suppose that after getting in your cab at the lowest point on the wheel, you traverse the full circle several times.

a. Recall that the circumference, C, of a circle is connected to the

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circle's radius, r, by the formula $C = 2\pi r$. What is the radius of the ferris wheel? How high is the highest point on the ferris wheel?

b. How high is the cab after it has traveled $1/4$ of the circumference of the circle?

c. How much distance along the circle has the cab traversed at the moment it first reaches a height of $\frac{150}{\pi} \approx 47.75$ meters?

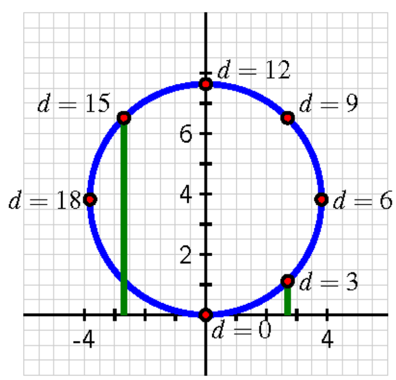
d. Can h be thought of as a function of d? Why or why not?

e. Can d be thought of as a function of h? Why or why not?

f. Why do you think the curve shown above has the shape that it does? Write several sentences to explain.

Circular Functions

The natural phenomenon of a point moving around a circle leads to interesting relationships. Let's consider a point traversing a circle of circumference 24 and examine how the point's height, h, changes as the distance traversed, d, changes. Note particularly that each time the point traverses $\frac{1}{8}$ of the circumference of the circle, it travels a distance of $24 \cdot \frac{1}{8} = 3$ units, as seen below where each noted point lies 3 additional units along the circle beyond the preceding one.



Note that we know the exact heights of certain points. Since the circle has circumference $C = 24$, we know that $24 = 2\pi r$ and therefore $r = \frac{12}{\pi} \approx 3.82$. Hence, the point where $d = 6$ (located $1/4$ of the way along the circle) is at a height of $h = \frac{12}{\pi} \approx 3.82$. Doubling this value, the point where $d = 12$ has

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height $h = \frac{24}{\pi} \approx 7.64$. Other heights, such as those that correspond to $d = 3$ and $d = 15$ (identified on the figure by the green line segments) are not obvious from the circle's radius, but can be estimated from the grid in the graph above as $h \approx 1.1$ (for $d = 3$) and $h \approx 6.5$ (for $d = 15$). Using all of these observations along with the symmetry of the circle, we can determine the other entries in the table below.

Data for height, h, as a function of distance traversed, d.

d	0	3	6	9	12	15	18	21	24
h	0	1.1	3.82	6.5	7.64	6.5	3.82	1.1	0

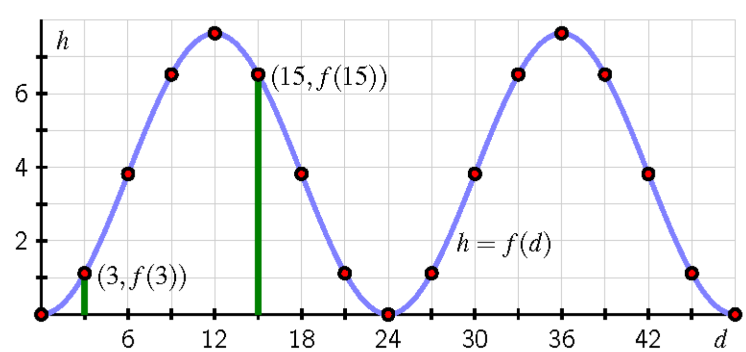
Moreover, if we now let the point continue traversing the circle, we observe that the d-values will increase accordingly, but the h-values will repeat according to the already-established pattern, resulting in the data in the table below.

Additional data for height, h, as a function of distance traversed, d.

d	24	27	30	33	36	39	42	45	48
h	0	1.1	3.82	6.5	7.64	6.5	3.82	1.1	0

It is apparent that each point on the circle corresponds to one and only one height, and thus we can view the height of a point as a function of the distance the point has traversed around the circle, say $h = f(d)$. Using the data from the two tables and connecting the points in an intuitive way, we get the graph shown below.

The height, h, of a point traversing a circle of radius 24 as a function of distance, d, traversed around the circle.



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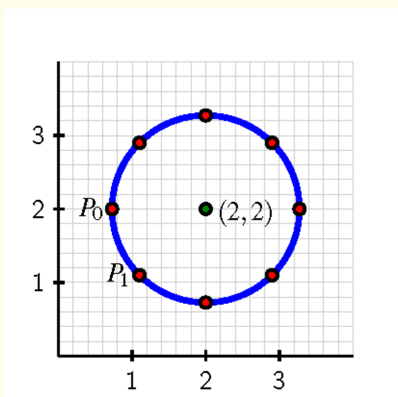
The function $h = f(d)$ we have been discussing is an example of what we will call a **circular function**. Indeed, it is apparent that if we:

- take any circle in the plane,
- choose a starting location for a point on the circle,
- let the point traverse the circle continuously,
- and track the height of the point as it traverses the circle,

the height of the point is a function of distance traversed and the resulting graph will have the same basic shape as the curve shown in the graph above. It also turns out that if we track the location of the x-coordinate of the point on the circle, the x-coordinate is also a function of distance traversed and its curve has a similar shape to the graph of the height of the point (the y-coordinate). Both of these functions are circular functions because they are generated by motion around a circle.

Exploration Consider the circle pictured below that is centered at the point $(2, 2)$ and that has circumference 8. Assume that we track the y-coordinate (that is, the height, h) of a point that is traversing the circle counterclockwise and that it starts at P_0 as pictured.



a. How far along the circle is the point P_1 from P_0? Why?

b. Label the subsequent points in the figure P_2, P_3, ... as we move counterclockwise around the circle. What are the exact coordinates of P_2? of P_4? Why?

c. Determine the coordinates of the remaining points on the circle (exactly where possible, otherwise approximately) and hence complete the entries in the table below that track the height, h, of the

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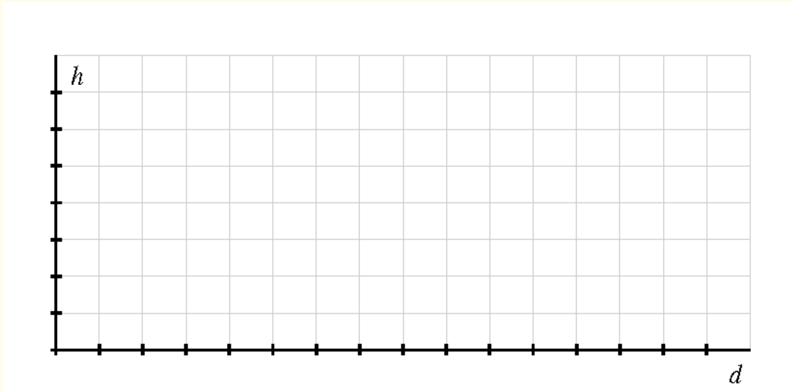
point traversing the circle as a function of distance traveled, d.

d	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
h	2																

d. By plotting the points in the table and connecting them in an intuitive way, sketch a graph of h as a function of d on the axes provided over the interval $0 \leq d \leq 16$. Clearly label the scale of your axes and the coordinates of several important points on the curve.



e. What is similar about your graph in comparison to the one in we created at the beginning of this section? What is different?

f. What will be the value of h when $d = 51$? How about when $d = 102$?

Properties of Circular Functions

Every circular function has several important features that are connected to the circle that defines the function. For the discussion that follows, we focus on circular functions that result from tracking the y-coordinate of a point traversing counterclockwise a circle of radius a centered at the point (k, m). Further, we will denote the circumference of the circle by the letter p.

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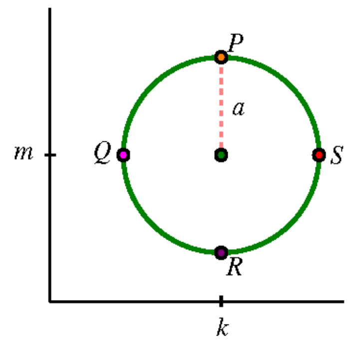
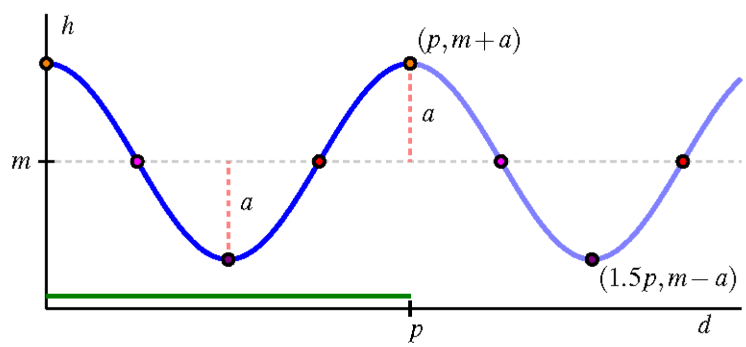


We assume that the point traversing the circle starts at P in the graph of the circle above. Its height is initially $y = m + a$, and then its height decreases to $y = m$ as we traverse to Q. Continuing, the point's height falls to $y = m - a$ at R, and then rises back to $y = m$ at S, and eventually back up to $y = m + a$ at the top of the circle. If we plot these heights continuously as a function of distance, d, traversed around the circle, we get the curve shown below.



This curve has several important features for which we introduce important terminology.

Definition The **midline** of a circular function is the horizontal line $y = m$ for which half the curve lies above the line and half the curve lies below. If the circular function results from tracking the y-coordinate of a point traversing a circle, $y = m$ corresponds to the y-coordinate of the center of the circle. In addition, the **amplitude** of a circular function is the maximum deviation of the curve from the midline. Note particularly that the value of the amplitude, a, corresponds to the radius of the circle that generates the curve.

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Because we can traverse the circle in either direction and for as far as we wish, the domain of any circular function is the set of all real numbers. From our observations about the midline and amplitude, it follows that the range of a circular function with midline $y = m$ and amplitude a is the interval $[m - a, m + a]$.

This graph is an example of a periodic function. Recall the definition of a periodic function.

Definition Let f be a function whose domain and codomain are each the set of all real numbers. We say that f is **periodic** provided that there exists a real number k such that $f(x+k) = f(x)$ for every possible choice of x. The smallest value p for which $f(x + p) = f(x)$ for every choice of x is called the **period** of f.

For a circular function, the period is always the circumference of the circle that generates the curve. In the graph of the function above, we see how the curve has completed one full cycle of behavior every p units, regardless of where we start on the curve.

Circular functions arise as models for important phenomena in the world around us, such as in a *harmonic oscillator*. Consider a mass attached to a spring where the mass sits on a frictionless surface. After setting the mass in motion by stretching or compressing the spring, the mass will oscillate indefinitely back and forth, and its distance from a fixed point on the surface turns out to be given by a circular function.

The Average Rate of Change of a Circular Function

Just as there are important trends in the values of a circular function, there are also interesting patterns in the average rate of change of the function. These patterns are closely tied to the geometry of the circle.

For the next part of our discussion, we consider a circle of radius 1 centered at $(0, 0)$, and consider a point that travels a distance d counterclockwise around the circle with its starting point viewed as $(1, 0)$. We use this circle to generate the circular function $h = f(d)$ that tracks the height of the point at the moment the point has traversed d units around the circle from $(1, 0)$. Let's consider the average rate of change of f on several intervals that are connected to certain fractions of the circumference.

Remembering that h is a function of distance traversed along the circle, it follows that the average rate of change of h on any interval of distance between two points P and Q on the circle is given by

$$\text{AROC}_{[P,Q]} = \frac{\text{change in height}}{\text{distance along the circle}},$$

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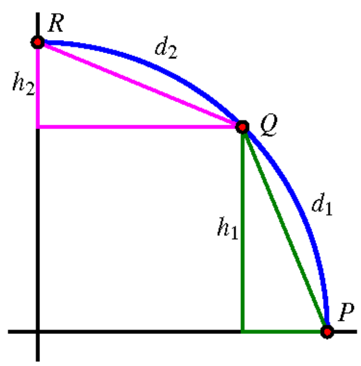
where both quantities are measured from point P to point Q.

First, we consider points P, Q, and R where Q results from traversing 1/8 of the circumference from P, and R 1/8 of the circumference from Q. In particular, we note that the distance d_1 along the circle from P to Q is the same as the distance d_2 along the circle from Q to R, and thus $d_1 = d_2$. At the same time, it is apparent from the geometry of the circle that the change in height h_1 from P to Q is greater than the change in height h_2 from Q to R, so $h_1 > h_2$. Thus, we can say that

$$\text{AROC}_{[P,Q]} = \frac{h_1}{d_1} > \frac{h_2}{d_2} = \text{AROC}_{[Q,R]}.$$



The differences in certain average rates of change appear to become more extreme if we consider shorter arcs along the circle. Next we consider traveling 1/20 of the circumference along the circle. In the graph below, points P and Q lie 1/20 of the circumference apart, as do R and S, so here $d_1 = d_5$. In this situation, it is the case that $h_1 > h_5$ for the same reasons as above, but we can say even more. From the green triangle, we see that $h_1 \approx d_1$ (while $h_1 < d_1$), so that $\text{AROC}_{[P,Q]} = \dfrac{h_1}{d_1} \approx 1$. At the same time, in the magenta triangle in the figure we see that h_5 is very small, especially in comparison to d_5, and thus $\text{AROC}_{[R,S]} = \dfrac{h_5}{d_5} \approx 0$. Hence, in this graph,

$$\text{AROC}_{[P,Q]} \approx 1 \text{ and } \text{AROC}_{[R,S]} \approx 0.$$

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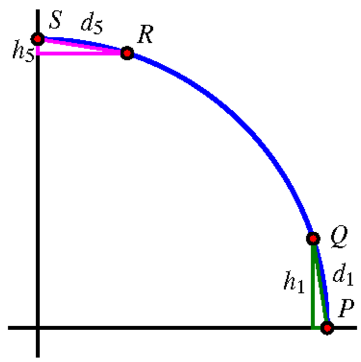
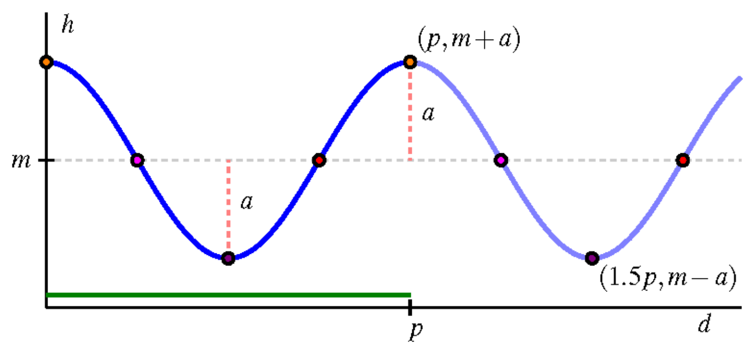


This information tells us that a circular function appears to change most rapidly for points near its midline and to change least rapidly for points near its highest and lowest values.

We can study the average rate of change not only on the circle itself, but also on a generic circular function graph, and thus make conclusions about where the function is increasing, decreasing, concave up, and concave down.



Summary

- When a point traverses a circle, a corresponding function can be generated by tracking the height of the point as it moves around the circle, where height is viewed as a function of distance traveled around the circle. We call such a function a *circular function.*
- Circular functions have several standard features. The function has a *midline* that is the line for which half the points on the curve lie above the line and half the points on the curve lie below. A circular function's *amplitude* is the maximum deviation of the

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function value from the midline; the amplitude corresponds to the radius of the circle that generates the function. Circular functions also repeat themselves, and we call the smallest value of p for which $f(x+p) = f(x)$ for all x the period of the function. The period of a circular function corresponds to the circumference of the circle that generates the function.

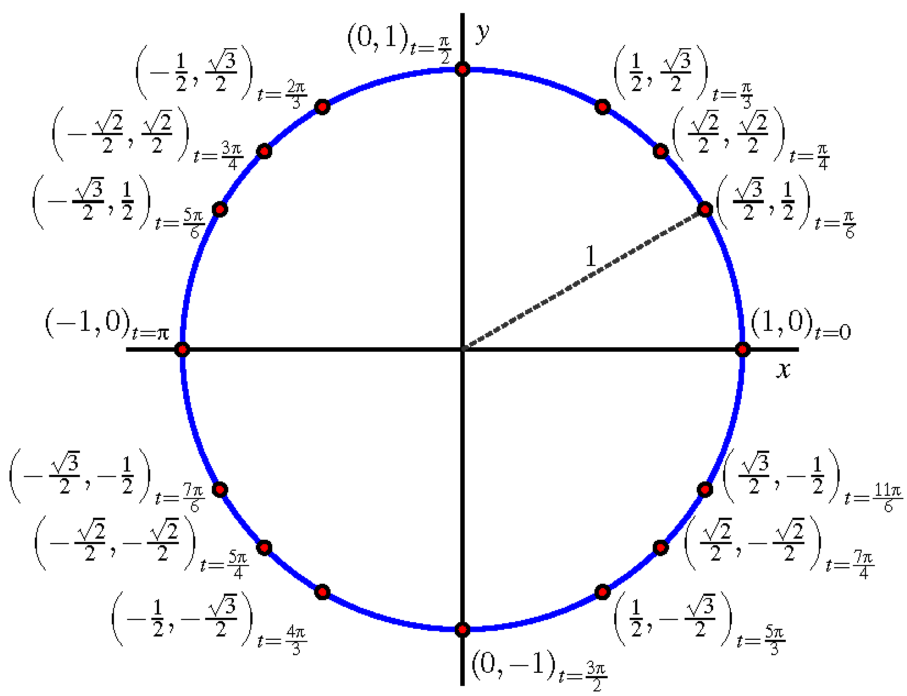
- Non-constant linear functions are either always increasing or always decreasing; quadratic functions are either always concave up or always concave down. Circular functions are sometimes increasing and sometimes decreasing, plus sometimes concave up and sometimes concave down. These behaviors are closely tied to the geometry of the circle.

9.1.2 The Sine and Cosine Functions

Motivating Questions

- What are the sine and cosine functions and how do they arise from a point traversing the unit circle?
- What important properties do the sine and cosine functions share?

In the last section, we saw how tracking the height of a point that is traversing a circle generates a periodic function. Previously, we also identified a collection of special points on the unit circle.



You can also use the *Desmos* file:

Desmos link: https://www.desmos.com/calculator/jgddn7tzxg

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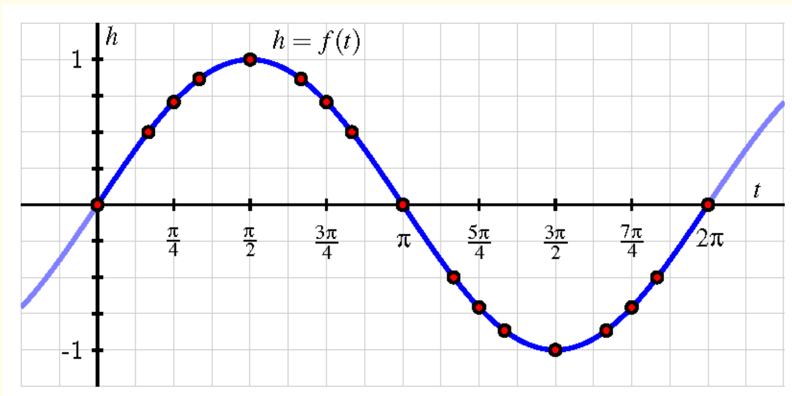
Exploration

If we consider the unit circle, start at $t = 0$, and traverse the circle counterclockwise, we may view the height, h, of the traversing point as a function of the angle, t, in radians. From there, we can plot the resulting (t, h) ordered pairs and connect them to generate the circular function pictured below.



a. What is the exact value of $h\left(\frac{\pi}{4}\right)$? of $h\left(\frac{\pi}{3}\right)$?

b. Complete the following table with the exact values of h that correspond to the stated inputs.

t	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
h									

t	π	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{7\pi}{4}$	$\frac{11\pi}{6}$	2π
h									

c. What is the exact value of $h\left(\frac{11\pi}{4}\right)$? of $h\left(\frac{14\pi}{3}\right)$?

d. Give four different values of t for which $h(t) = -\frac{\sqrt{3}}{2}$.

The Definition of the Sine Function

The circular function that tracks the height of a point on the unit circle traversing counterclockwise from $(1, 0)$ as a function of the corresponding central angle

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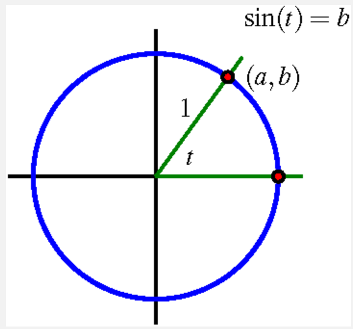
(in radians) is one of the most important functions in mathematics. As such, we give the function a name: the **sine** function.

Definition



Given a central angle in the unit circle that measures t radians and that intersects the circle at both $(1,0)$ and (a,b), we define the **sine of** t, denoted $\sin(t)$, by the rule

$$\sin(t) = b.$$

Because of the correspondence between radian angle measure and distance traversed on the unit circle, we can also think of $\sin(t)$ as identifying the y-coordinate of the point after it has traveled t units counterclockwise along the circle from $(1,0)$. Note particularly that we can consider the sine of negative inputs: for instance, $\sin\left(-\frac{\pi}{2}\right) = -1$.

Based on our earlier work with the unit circle, we know many different exact values of the sine function, and summarize these in in the table below:

t	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
$\sin(t)$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0

t	π	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{7\pi}{4}$	$\frac{11\pi}{6}$	2π
$\sin(t)$	0	$-\frac{1}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{3}}{2}$	-1	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{1}{2}$	0

Moreover, if we now plot these points in the usual way, we get the familiar circular wave function that comes from tracking the height of a point traversing a circle. We often call this graph the **sine wave**.

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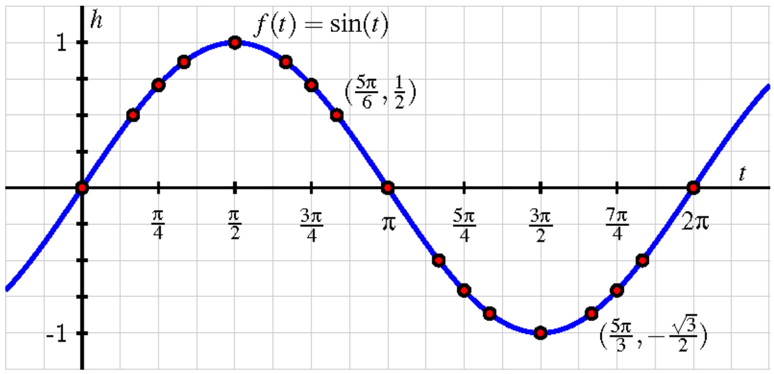


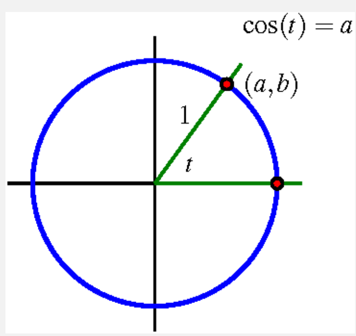
At https://www.desmos.com/calculator/f9foqx24ct you can explore and investigate a helpful Desmos animation that shows how this motion around the circle generates the sine graph.

Desmos link: https://www.desmos.com/calculator/f9foqx24ct

The Definition of the Cosine Function

Given any central angle of radian measure t in the unit circle with one side passing through the point $(1, 0)$, the angle generates a unique point (a, b) that lies on the circle. Just as we can view the y-coordinate as a function of t, the x-coordinate is likewise a function of t. We therefore make the following definition.

Definition



Given a central angle in the unit circle that measures t radians and that intersects the circle at both $(1, 0)$ and (a, b), we define the **cosine of** t, denoted $\cos(t)$, by the rule

$$\cos(t) = a.$$

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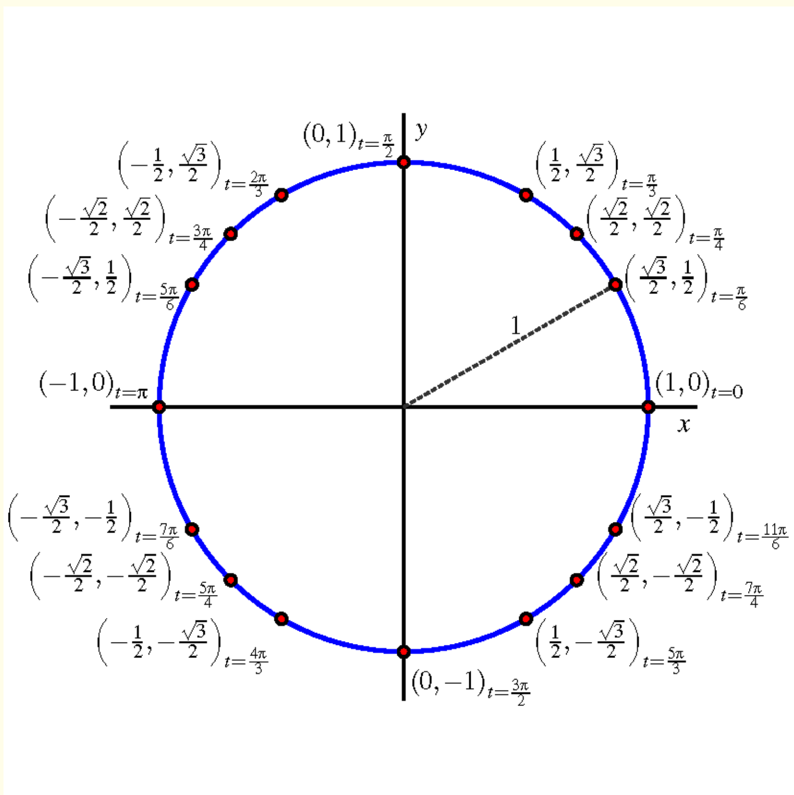
Again because of the correspondence between the radian measure of an angle and arc length along the unit circle, we can view the value of $\cos(t)$ as tracking the x-coordinate of a point traversing the unit circle clockwise a distance of t units along the circle from $(1, 0)$. We now use the data and information we have developed about the unit circle to build a table of values of $\cos(t)$ as well as a graph of the curve it generates.

Exploration Let $k = g(t)$ be the function that tracks the x-coordinate of a point traversing the unit circle counterclockwise from $(1, 0)$. That is, $g(t) = \cos(t)$. Use the information we know about the unit circle to respond to the following questions.



a. What is the exact value of $\cos\left(\frac{\pi}{6}\right)$? of $\cos\left(\frac{5\pi}{6}\right)$? $\cos\left(-\frac{\pi}{3}\right)$?

b. Complete the following table with the exact values of k that cor-

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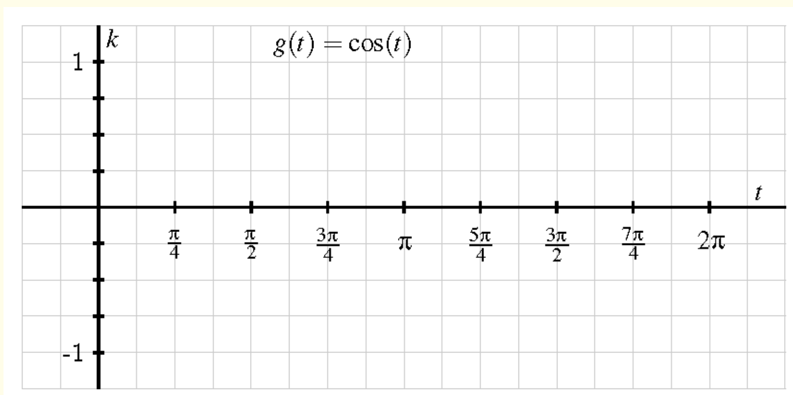
respond to the stated inputs.

t	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
k									

t	π	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{7\pi}{4}$	$\frac{11\pi}{6}$	2π
k									

c. On the axes provided, sketch an accurate graph of $k = \cos(t)$. Label the exact location of several key points on the curve.



d. What is the exact value of $\cos\left(\frac{11\pi}{4}\right)$? of $\cos\left(\frac{14\pi}{3}\right)$?

e. Give four different values of t for which $\cos(t) = -\frac{\sqrt{3}}{2}$.

f. How is the graph of $k = \cos(t)$ different from the graph of $h = \sin(t)$? How are the graphs similar?

As we work with the sine and cosine functions, it's always helpful to remember their definitions in terms of the unit circle and the motion of a point traversing the circle. At `https://www.desmos.com/calculator/9s1ms0nlyf` you can explore and investigate a helpful Desmos animation that shows how this motion around the circle generates the cosine graph.

Desmos link: `https://www.desmos.com/calculator/9s1ms0nlyf`

 Acknowledgements: Active Prelude to Calculus (https://activecalculus.org/prelude)

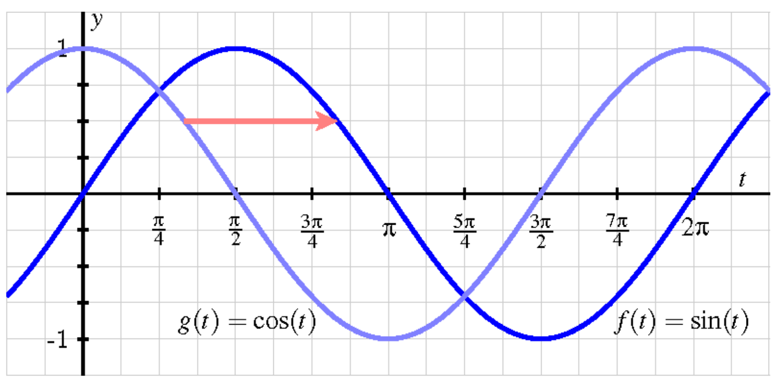
Properties of Sine and Cosine

Because the sine function results from tracking the y-coordinate of a point traversing the unit circle and the cosine function from the x-coordinate, the two functions have several shared properties of circular functions.

For both $f(t) = \sin(t)$ and $g(t) = \cos(t)$,

- the domain of the function is all real numbers;
- the range of the function is $[-1, 1]$;
- the midline of the function is $y = 0$;
- the amplitude of the function is $a = 1$;
- the period of the function is $P = 2\pi$.

It is also insightful to juxtapose the sine and cosine functions' graphs on the same coordinate axes. When we do, as seen in the figure below, we see that the curves can be viewed as horizontal translations of one another.



In particular, since the sine graph can be viewed as the cosine graph shifted $\frac{\pi}{2}$ units to the right, it follows that for any value of t,

$$\sin(t) = \cos\left(t - \frac{\pi}{2}\right).$$

Similarly, since the cosine graph can be viewed as the sine graph shifted left,

$$\cos(t) = \sin\left(t + \frac{\pi}{2}\right).$$

Because each of the two preceding equations hold for every value of t, they are often referred to as *identities*.

 Acknowledgements: Active Prelude to Calculus (https://activecalculus.org/prelude)

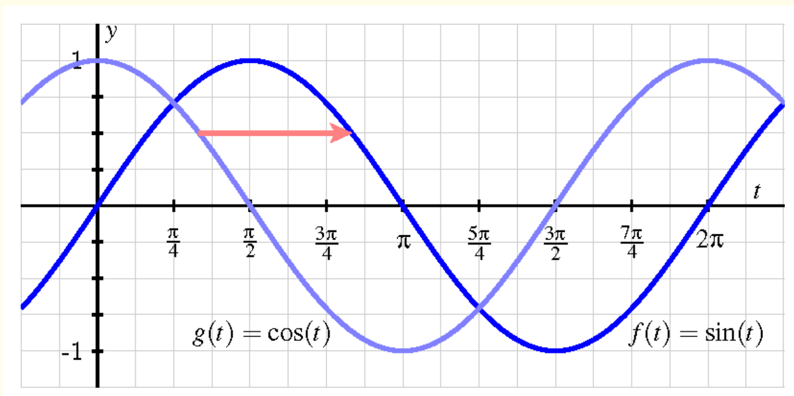
In light of the definitions of the sine and cosine functions, we can now view any point (x, y) on the unit circle as being of the form $(\cos(t), \sin(t))$, where t is the measure of the angle whose vertices are $(1, 0)$, $(0, 0)$, and (x, y). Note particularly that since $x^2 + y^2 = 1$, it is also true that $\cos^2(t) + \sin^2(t) = 1$. We call this fact the Fundamental Trigonometric Identity.

For any real number t,

$$\cos^2(t) + \sin^2(t) = 1.$$

There are additional trends and patterns in the two functions' graphs that we explore further in the following activity.

Exploration Use the figure below to assist in answering the following questions.



a. Give an example of the largest interval you can find on which $f(t) = \sin(t)$ is decreasing.

b. Give an example of the largest interval you can find on which $f(t) = \sin(t)$ is decreasing and concave down.

c. Give an example of the largest interval you can find on which $g(t) = \cos(t)$ is increasing.

d. Give an example of the largest interval you can find on which $g(t) = \sin(t)$ is increasing and concave up.

e. Without doing any computation, on which interval is the average rate of change of $g(t) = \cos(t)$ greater: $[\pi, \pi + 0.1]$ or $\left[\frac{3\pi}{2}, \frac{3\pi}{2} + 0.1\right]$? Why?

 Acknowledgements: Active Prelude to Calculus (https://activecalculus.org/prelude)

f. In general, how would you characterize the locations on the sine and cosine graphs where the functions are increasing or decreasingly most rapidly?

g. For which quadrants of the x-y plane is $\cos(t)$ negative for an angle in that quadrant?

Using Computing Technology

We have established that we know the exact value of $\sin(t)$ and $\cos(t)$ for any of the t-values labeled on the unit circle, as well as for any such $t \pm 2j\pi$, where j is a whole number, due to the periodicity of the functions. But what if we want to know $\sin(1.35)$ or $\cos\left(\frac{\pi}{5}\right)$ or values for other inputs not in the table?

Any standard computing device a scientific calculator, *Desmos*, *Geogebra*, *WolframAlpha*, etc. has the ability to evaluate the sine and cosine functions at any input we desire. Because the input is viewed as an angle, each computing device has the option to consider the angle in radians or degrees. *It is always essential that you are sure which type of input your device is expecting.* Our computational device of choice is *Desmos*. In *Desmos*, you can change the input type between radians and degrees by clicking the wrench icon in the upper right and choosing the desired units. Radians is the default, and radians is what we will primarily use in both this class and calculus.

It take substantial and sophisticated mathematics to enable a computational device to evaluate the sine and cosine functions at any value we want; the algorithms involve an idea from calculus known as an infinite series. While your computational device is powerful, it's both helpful and important to understand the meaning of these values on the unit circle and to remember the special points for which we know the outputs of the sine and cosine functions exactly.

Exploration
Answer the following questions exactly wherever possible. If you estimate a value, do so to at least 5 decimal places of accuracy.

a. The x coordinate of the point on the unit circle that lies in the third quadrant and whose y-coordinate is $y = -\frac{3}{4}$.

b. The y-coordinate of the point on the unit circle generated by a central angle in standard position that measures $t = 2$ radians.

c. The x-coordinate of the point on the unit circle generated by a central angle in standard position that measures $t = -3.05$ radians.

d. The value of $\cos(t)$ where t is an angle in Quadrant II that satisfies

Acknowledgements: Active Prelude to Calculus (https://activecalculus.org/prelude)

$\sin(t) = \frac{1}{2}$.

e. The value of $\sin(t)$ where t is an angle in Quadrant III for which $\cos(t) = -0.7$.

f. The average rate of change of $f(t) = \sin(t)$ on the intervals $[0.1, 0.2]$ and $[0.8, 0.9]$.

g. The average rate of change of $g(t) = \cos(t)$ on the intervals $[0.1, 0.2]$ and $[0.8, 0.9]$.

Summary

- The sine and cosine functions result from tracking the y- and x-coordinates of a point traversing the unit circle counterclockwise from $(1, 0)$. The value of $\sin(t)$ is the y-coordinate of a point that has traversed t units along the circle from $(1, 0)$ (or equivalently that corresponds to an angle of t radians), while the value of $\cos(t)$ is the x-coordinate of the same point.

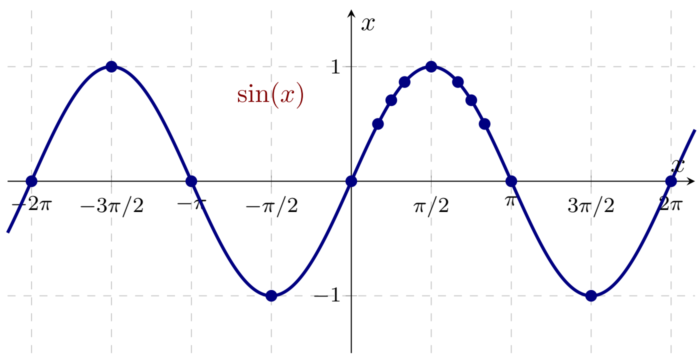
- The sine and cosine functions are both periodic functions that share the same domain (the set of all real numbers), range (the interval $[-1, 1]$), midline ($y = 0$), amplitude ($a = 1$), and period ($P = 2\pi$). In addition, the sine function is horizontal shift of the cosine function by $\frac{\pi}{2}$ units to the right, so $\sin(t) = \cos\left(t - \frac{\pi}{2}\right)$ for any value of t.

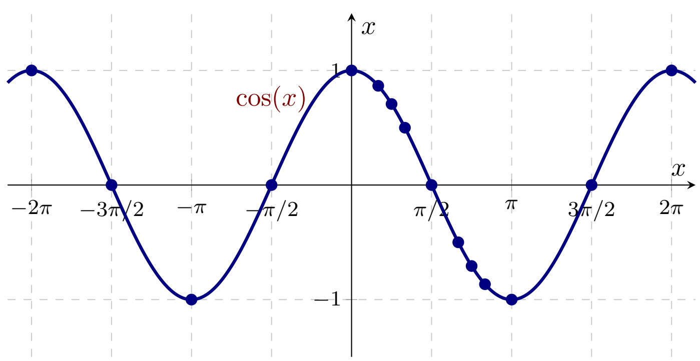
- If t corresponds to one of the special angles that we know on the unit circle, we can compute the values of $\sin(t)$ and $\cos(t)$ exactly. For other values of t, we can use a computational device to estimate the value of either function at a given input; when we do so, we must take care to know whether we are computing in terms of radians or degrees.

9.1.3 Creating a New Function: Tangent

Introduction

We are now going to determine the graph of the tangent function by analyzing what we now know about the sine and cosine functions. As a reminder, here is a graph of those functions with some important points marked. Specifically the points at all multiples of $\frac{\pi}{2}$ have been marked, as well as at the standard points $x = \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{2\pi}{3}, \frac{3\pi}{4}$, and $\frac{5\pi}{6}$.



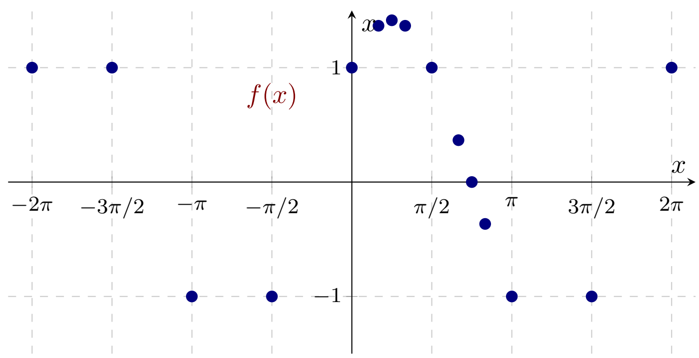


Graph of $f(x) = \sin(x) + \cos(x)$

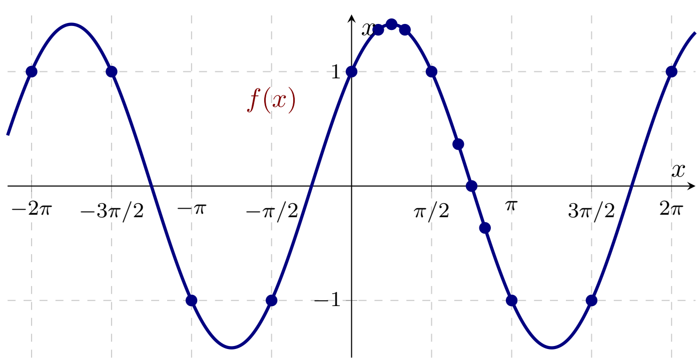
Before considering the function $\tan(x) = \dfrac{\sin(x)}{\cos(x)}$, lets consider the (possibly) more straightforward $f(x) = \sin(x) + \cos(x)$. Let's practice with this function creating a graph of a sum of two known functions.

What is the value of $f(0)$? We know $f(0) = \sin(0) + \cos(0) = 0 + 1 = 1$.

We can easily calculate the values of f at all of the important points marked in the graphs above. Let us plot the points of f corresponding to them.



Those extra points plotted between $x = 0$ and $x = \pi$ show us the behavior of this function f. Notice that between $x = 0$ and $x = \frac{\pi}{2}$, the graph increases to a peak, then decreases in a very sinusoidal manner. If we continue this with standard values and "connect the dots", we end up with the following graph.



We've ended up with another periodic function that looks like a stretched and shifted version of sine or cosine.

Now, let's try again using division instead of addition.

Determining the Graph of Tangent

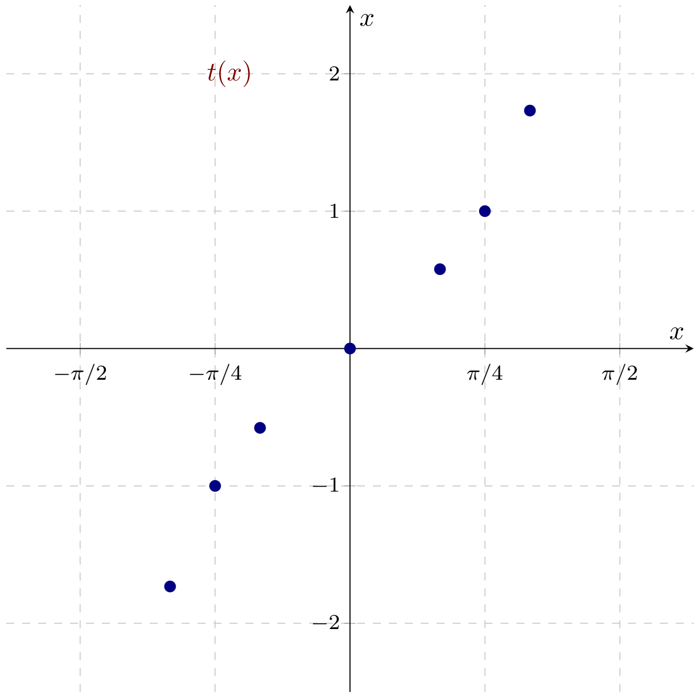
Recall that $\tan(x) = \dfrac{\sin(x)}{\cos(x)}$.

Notice that this function is undefined at all x-values with $\cos(x) = 0$. That means the function $\tan(x)$ is not defined for $x = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \ldots$.

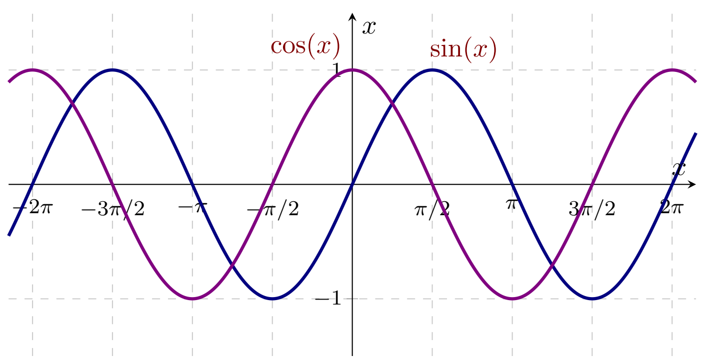
We can calculate values of $\tan(x)$ for other inputs. $\tan(0) = \dfrac{\sin(0)}{\cos(0)} = \dfrac{0}{1} = 0.$ The following table lists some of the other values arising from this division. Notice that in this table we have chosen to rationalize the denominators of the fractions that have appeared. That is, we have written $\dfrac{1}{\sqrt{2}}$ as $\dfrac{\sqrt{2}}{2}$, by multiplying the fraction by 1 written in the form $\dfrac{\sqrt{2}}{\sqrt{2}}$. Similarly $\dfrac{1}{\sqrt{3}}$ is written as $\dfrac{\sqrt{3}}{3}$.

x	$\sin(x)$	$\cos(x)$	$t(x) = \dfrac{\sin(x)}{\cos(x)}$
$-\dfrac{\pi}{3}$	$-\dfrac{\sqrt{3}}{2}$	$\dfrac{1}{2}$	$-\sqrt{3}$
$-\dfrac{\pi}{4}$	$-\dfrac{\sqrt{2}}{2}$	$\dfrac{\sqrt{2}}{2}$	-1
$-\dfrac{\pi}{6}$	$-\dfrac{1}{2}$	$\dfrac{\sqrt{3}}{2}$	$-\dfrac{\sqrt{3}}{3}$
0	0	1	0
$\dfrac{\pi}{6}$	$\dfrac{1}{2}$	$\dfrac{\sqrt{3}}{2}$	$\dfrac{\sqrt{3}}{3}$
$\dfrac{\pi}{4}$	$\dfrac{\sqrt{2}}{2}$	$\dfrac{\sqrt{2}}{2}$	1
$\dfrac{\pi}{3}$	$\dfrac{\sqrt{3}}{2}$	$\dfrac{1}{2}$	$\sqrt{3}$

If we plot these points, we find the following graph.



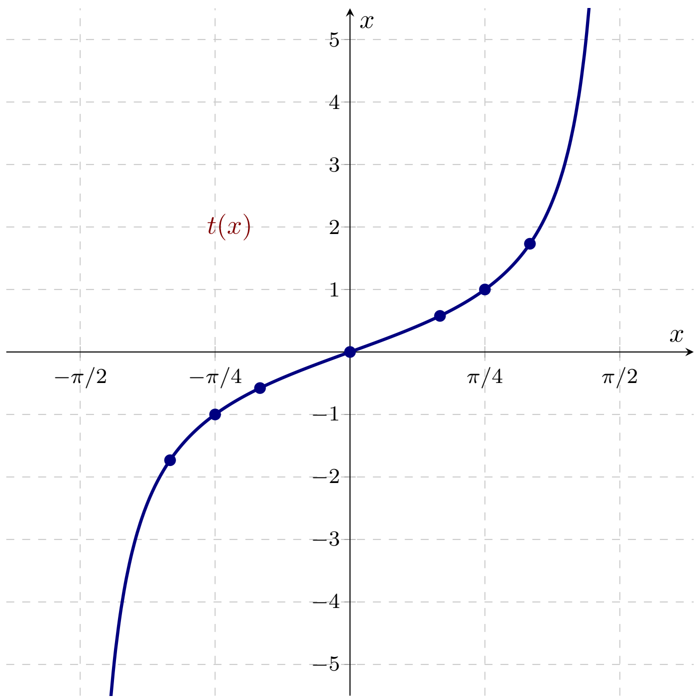
Let's think about what happens if x is a number really close to $\frac{\pi}{2}$, but just a little bit smaller than $\frac{\pi}{2}$. Notice from the graphs that the value of $\sin(x)$ will be a positive number that is really close to 1 and the value of $\cos(x)$ will be really close to 0 but still positive.



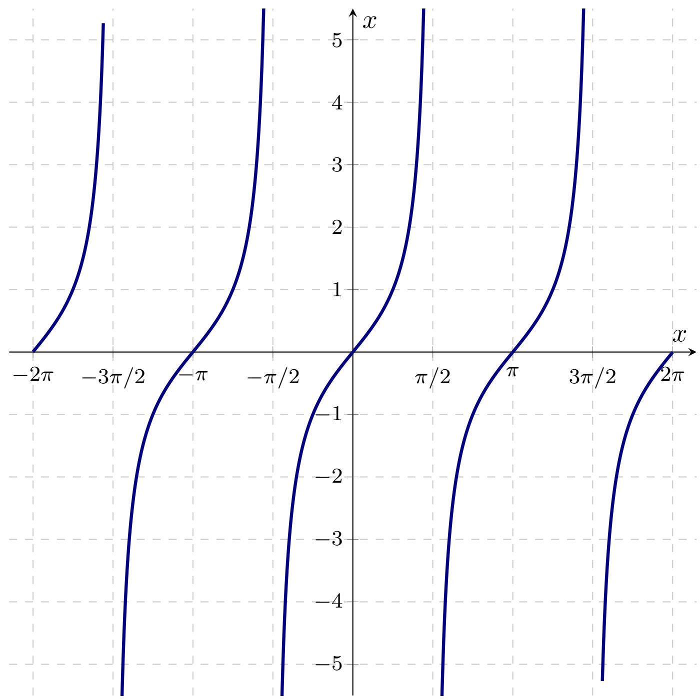
What happens if we take 1 and divide it by a small positive number? Let's look at a table of values to see.

z	$\frac{1}{z}$
1	1
$\frac{1}{2}$	2
$\frac{1}{3}$	3
$\frac{1}{10}$	10
$\frac{1}{100}$	100
$\frac{1}{1000}$	1000

Notice that as the numbers $z = 1,\ 1/2,\ 1/3,\ \ldots$ got smaller and smaller, the values of $\dfrac{1}{z} = 1,\ 2,\ 3,\ \ldots$ got larger and larger? That is the same thing we are noticing in the graph of the function t we are building above. For values of x really close to $\pi/2$, but still less than $\pi/2$, the value of $t(x)$ is basically 1 divided by a very small positive number. This table of values tells us that the smaller that denominator gets, the larger the fraction becomes. Adding this behavior to the graph of t gives the following.



By repeating similar calculations for other standard inputs, we arrive at the following graph.



As you can see from the graph, $\tan(x)$ is an odd, periodic function with period π (not 2π like sine and cosine).

9.1.4 Graphs of Secant, Cosecant, and Cotangent

Motivating Questions

- What do the graphs of secant, cosecant, and cotangent look like?
- What are some important properties of these graphs? Where do they have asymptotes?

Like the tangent function, the secant, cosecant, and cotangent functions are defined in terms of the sine and cosine functions, so we can determine the exact values of these functions at each of the special points on the unit circle. In addition, we can use our understanding of the unit circle and the properties of the sine and cosine functions to determine key properties of these other trigonometric functions.

The Secant Function

We begin by investigating the secant function. Using the fact that $\sec(t) = \frac{1}{\cos(t)}$, we note that anywhere $\cos(t) = 0$, the value of $\sec(t)$ is undefined. We record such instances in the following table by writing "u". At all other points, the value of the secant function is simply the reciprocal of the cosine function's value. Since $|\cos(t)| \le 1$ for all t, it follows that $|\sec(t)| \ge 1$ for all t (for which the secant's value is defined).

Values of Secant in Quadrant I

t	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$\cos(t)$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
$\sec(t)$	1	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2	u

 Acknowledgements: Active Prelude to Calculus (https://activecalculus.org/prelude)

Values of Secant in Quadrant II

t	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
$\cos(t)$	0	$-\frac{1}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{3}}{2}$	-1
$\sec(t)$	u	-2	$-\sqrt{2}$	$-\frac{2}{\sqrt{3}}$	-1

Values of Secant in Quadrant III

t	π	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$
$\cos(t)$	-1	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{1}{2}$	0
$\sec(t)$	-1	$-\frac{2}{\sqrt{3}}$	$-\sqrt{2}$	-2	u

Values of Secant in Quadrant IV

t	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{7\pi}{4}$	$\frac{11\pi}{6}$	2π
$\cos(t)$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
$\sec(t)$	u	2	$\sqrt{2}$	$\frac{2}{\sqrt{3}}$	1

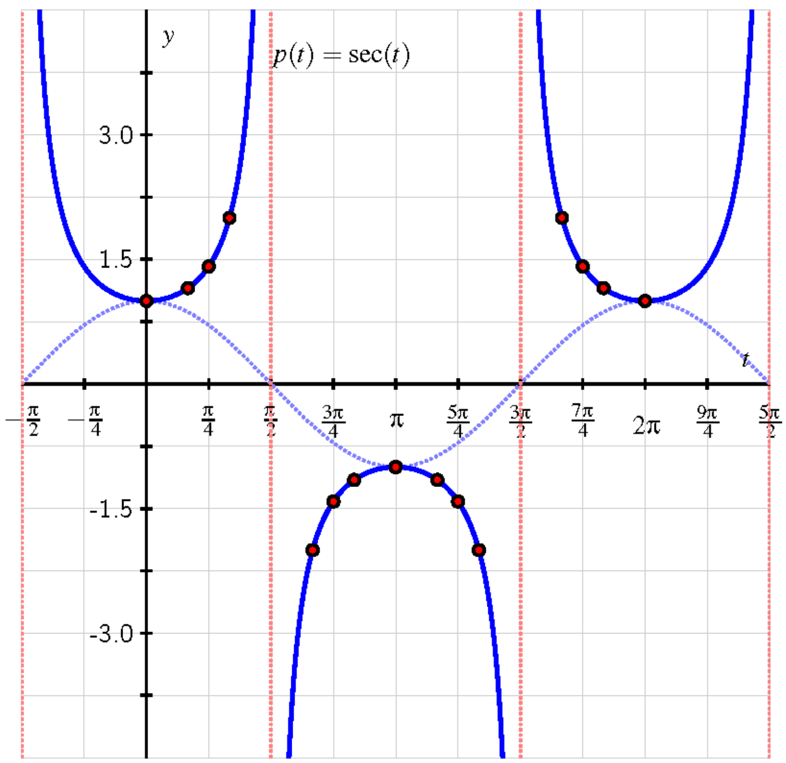
These tables help us identify trends in the secant function. The sign of $\sec(t)$ matches the sign of $\cos(t)$ and thus is positive in Quadrant I, negative in Quadrant II, negative in Quadrant III, and positive in Quadrant IV.

In addition, we observe that as t-values in the first quadrant get closer to $\frac{\pi}{2}$, $\cos(t)$ gets closer to 0 (while being always positive). Since the numerator of the secant function is always 1, having its denominator approach 0 means that $\sec(t)$ increases without bound as t approaches $\frac{\pi}{2}$ from the left side. Once t

 Acknowledgements: Active Prelude to Calculus (https://activecalculus.org/prelude)

is slightly greater than $\frac{\pi}{2}$ in Quadrant II, the value of $\cos(t)$ is negative (and close to zero). This makes the value of $\sec(t)$ decrease without bound (negative and getting further away from 0) for t approaching $\frac{\pi}{2}$ from the right side, and results in $p(t) = \sec(t)$ having a vertical asymptote at $t = \frac{\pi}{2}$. The periodicity and sign behavior of $\cos(t)$ mean this asymptotic behavior of the secant function will repeat.

Plotting the data in the table along with the expected asymptotes and connecting the points intuitively, we see the graph of the secant function.



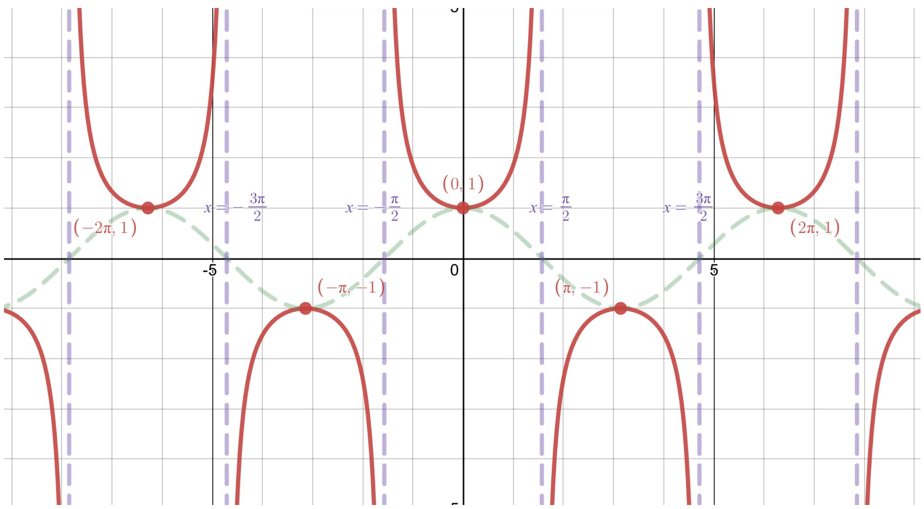
We see from both the table and the graph that the secant function has period $P = 2\pi$. We summarize our recent work as follows.

Properties of the secant function.
For the function $p(t) = \sec(t)$,

- its domain is the set of all real numbers except $t = \frac{\pi}{2} \pm k\pi$ where k is any whole number;

- its range is the set of all real numbers y such that $|y| \geq 1$;
- its period is $P = 2\pi$.

We can see the secant function in Desmos as well.



Try playing with the secant graph yourself.

Desmos link: `https://www.desmos.com/calculator/gmsanlrjza`

The Cosecant Function

Graphing the cosecant function is extremely similar to graphing the secant function, except we use $\csc(t) = \dfrac{1}{\sin(t)}$ so we are flipping over the values of sine instead of the values of cosine. Since the sine and cosine graphs look very similar except they are are shifted by $\dfrac{\pi}{2}$, the secant and cosecant graphs will also look very similar but be shifted from one another in the exact same way.

Let's create the table of famous values for cosecant.

Values of Cosecant in Quadrant I

t	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$\sin(t)$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
$\csc(t)$	u	2	$\sqrt{2}$	$\frac{2}{\sqrt{3}}$	1

Values of Cosecant in Quadrant II

t	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
$\sin(t)$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
$\csc(t)$	1	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2	u

Values of Cosecant in Quadrant III

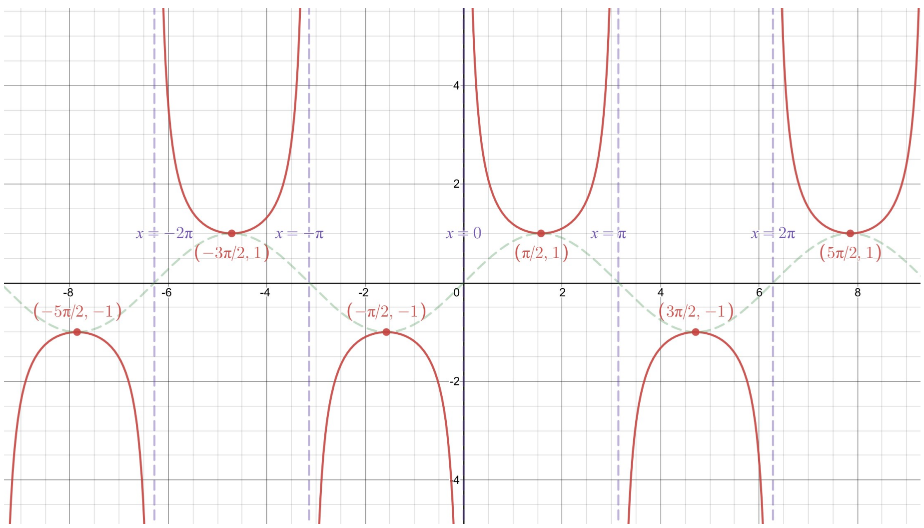
t	π	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$
$\sin(t)$	0	$-\frac{1}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{3}}{2}$	-1
$\csc(t)$	u	-2	$-\sqrt{2}$	$-\frac{2}{\sqrt{3}}$	-1

 Acknowledgements: Active Prelude to Calculus (https://activecalculus.org/prelude)

Values of Cosecant in Quadrant IV

t	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{7\pi}{4}$	$\frac{11\pi}{6}$	2π
$\sin(t)$	-1	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{1}{2}$	0
$\csc(t)$	-1	$-\frac{2}{\sqrt{3}}$	$-\sqrt{2}$	-2	u

Plotting the data in the table along with the expected asymptotes and connecting the points intuitively, we see the graph of the cosecant function.



We see from both the table and the graph that the cosecant function has period $P = 2\pi$. We summarize our recent work as follows.

Properties of the cosecant function.
For the function $p(t) = \csc(t)$,

- its domain is the set of all real numbers except $t = \pm k\pi$ where k is any whole number;
- its range is the set of all real numbers y such that $|y| \geq 1$;
- its period is $P = 2\pi$.

Try playing with the cosecant graph yourself.

 Acknowledgements: Active Prelude to Calculus (https://activecalculus.org/prelude)

Desmos link: https://www.desmos.com/calculator/norjxi7z4r

The Cotangent Function

Graphing the cotangent function is similar to graphing the secant and cosecant functions, except we use $\cot(t) = \dfrac{1}{\tan(t)}$ so we are flipping over the values of tangent. Since the tangent graph has a period of π, the graph of cotangent will also have a period of π. Therefore, we only need to calculate tables of values for the first two quadrants.

Let's create the table of famous values for cotangent.

Values of Cotangent in Quadrant I

t	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$\tan(t)$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	u
$\cot(t)$	u	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$	0

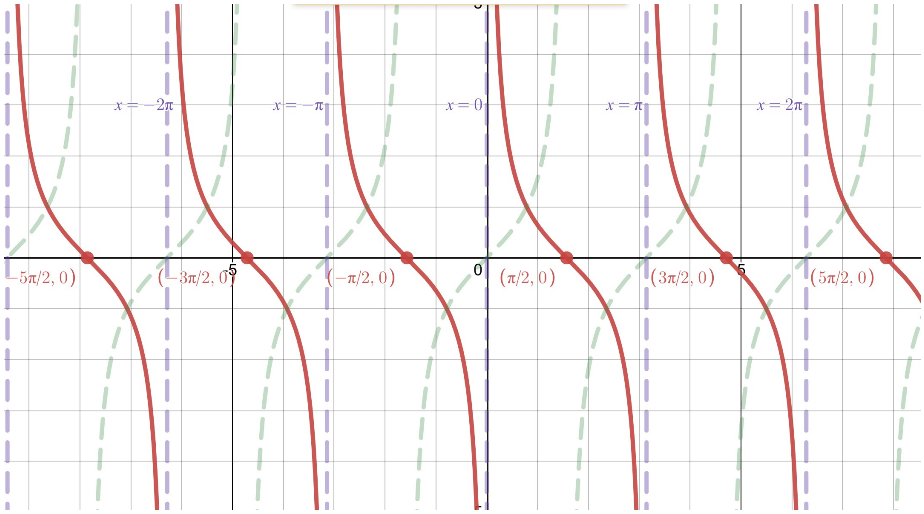
Notice that something unusual happened here. Even though tangent is undefined at $t = \dfrac{\pi}{2}$, we have that $\cot\left(\dfrac{\pi}{2}\right) = 0$. This is because $\cot(t) = \dfrac{\cos(t)}{\sin(t)}$ and so we can define $\cot\left(\dfrac{\pi}{2}\right) = \dfrac{\cos\left(\frac{\pi}{2}\right)}{\sin\left(\frac{\pi}{2}\right)} = \dfrac{0}{1} = 0$.

Values of Cotangent in Quadrant II

t	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
$\tan(t)$	u	$-\sqrt{3}$	-1	$-\frac{1}{\sqrt{3}}$	0
$\cot(t)$	0	$-\frac{1}{\sqrt{3}}$	-1	$-\sqrt{3}$	u

Plotting the data in the table along with the expected asymptotes and connecting the points intuitively, we see the graph of the cotangent function.

 Acknowledgements: Active Prelude to Calculus (https://activecalculus.org/prelude)



We summarize our recent work as follows.

Properties of the cotangent function.
For the function $p(t) = \csc(t)$,

- its domain is the set of all real numbers except $t = \pm k\dfrac{\pi}{2}$ where k is any whole number;
- its range is the set of all real numbers;
- its period is $P = \pi$.

Try playing with the cotangent graph yourself.

Desmos link: `https://www.desmos.com/calculator/yhmzxaq7ep`

9.2 Trig Functions As Functions

Learning Objectives

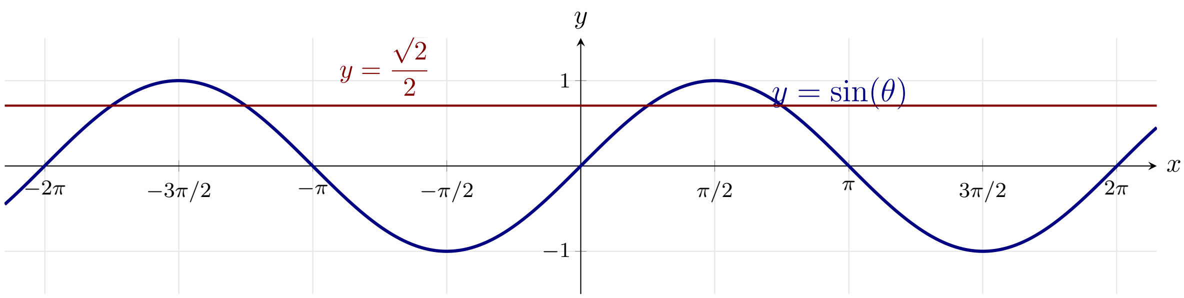
- Solving Trigonometric Equations
  - Solving elementary trigonometric equations.
  - Solving trigonometric equations using identities.
  - FInding solutions on restricted domains.

9.2.1 Finding All Solutions to Trig Equations

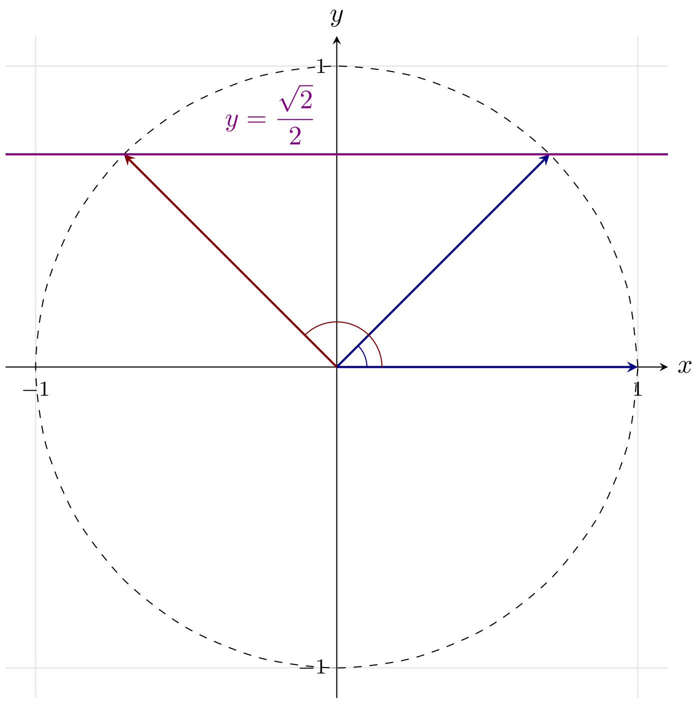
Introduction

Frequently we are in the situation of having to determine precisely which angles satisfy a particular equation. Something like $\sin(\theta) = \frac{\sqrt{2}}{2}$. We know that $\sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$, meaning that $\theta = \frac{\pi}{4}$ is a solution of this equation, but is that the only solution or *are there more*?

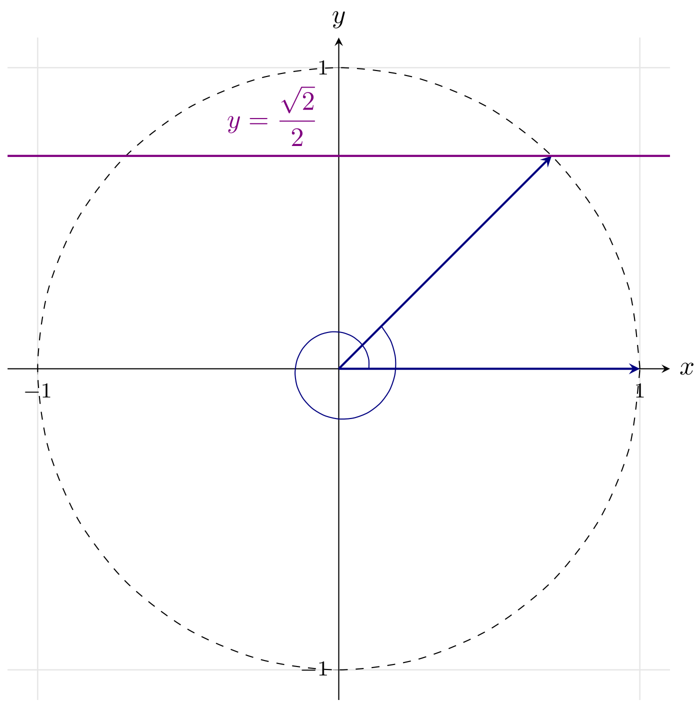
Let's look at the graph of the sine function.



Notice that the graph of $\sin(\theta)$ and the graph of the constant function $y = \frac{\sqrt{2}}{2}$ intersect many times, not just once. In fact, since sine is a periodic function, these graphs intersect infinitely many times. Each of these intersections represents a single solution of the equation $\sin(\theta) = \frac{\sqrt{2}}{2}$. We need a process to identify and write down each of these solutions. Let's start by looking at the unit circle. Remember that sine values correspond to the y-coordinate of points on the unit circle. This equation is asking us to find all the points on the unit circle with a y-coordinate of $\frac{\sqrt{2}}{2}$.



You see that there are two locations on the unit circle with y-coordinate equal to $\frac{\sqrt{2}}{2}$, one in the first quadrant and another in the second. As we mentioned earlier, the first quadrant angle is $\theta = \frac{\pi}{4}$. The angle in the second quadrant has reference angle $\frac{\pi}{4}$, which means the angle is $\frac{3\pi}{4}$. Those are the only two points on the circle with that y-coordinate, but remember that there are many other angles which are coterminal with those. For instance:



The only solutions are the angles $\frac{\pi}{4}$, $\frac{3\pi}{4}$, and *all the angles coterminal with them*. Since the sine function has period 2π, that means any other solution has to be an integer multiple of 2π away from one of these first two solutions. Putting that together, our solutions are:

$$\theta = \frac{\pi}{4} + 2\pi k,\ \frac{3\pi}{4} + 2\pi k,\ k \text{ any integer.}$$

The steps we've followed are summarized in the following.

To solve a triginometric equation:

(a) Find the reference angle of the solutions. Typically the standard values will help identify this.

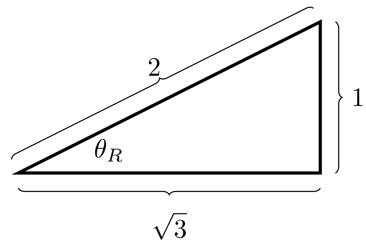
(b) Find all solutions on a single period of the function. Use the graph, the unit circle, and the reference angle to identify these.

(c) Find all solutions. Use the period of the function to find all requested solutions.

Example 1. *Solve the equation:*

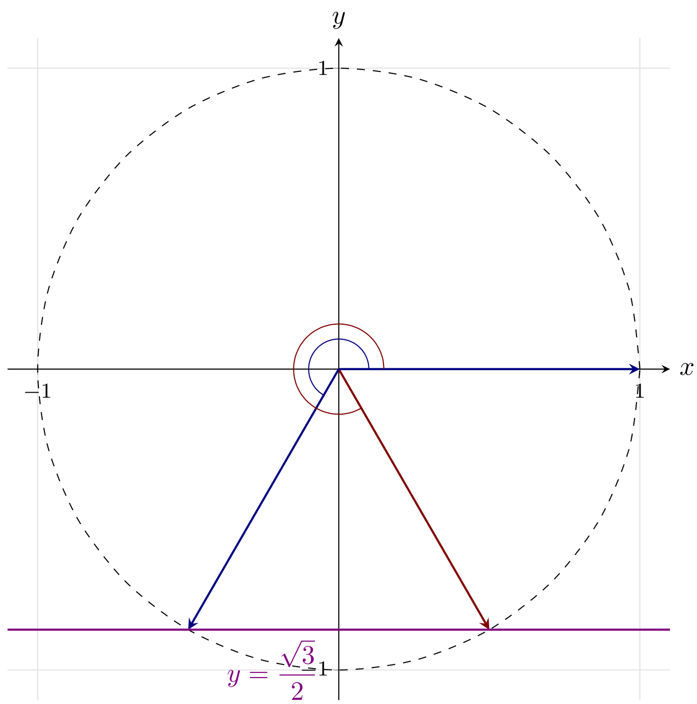
$$\sin(\theta) = -\frac{1}{2}.$$

Explanation We'll start by finding the reference angle, θ_R, the acute angle between the terminal side of θ and the x-axis. The reference angle satisfies $\sin(\theta_R) = \frac{1}{2}$ and the negative sign will be used to indicate the quadrant of the angle.



[2in]

From the picture we see $\theta_R = \frac{\pi}{6}$. Let's look at the unit circle.



In one period $[0, 2\pi)$, there are two angles that have reference angle $\frac{\pi}{6}$ and have negative sine value. One is in quadrant 3, and one in quadrant 4. That means the solutions in the interval $[0, 2\pi)$ are $\frac{7\pi}{6}$ and $\frac{11\pi}{6}$.

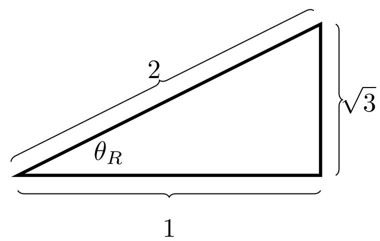
To find all solutions, we have to add all multiples of 2π to these. The solutions are then

$$\theta = \frac{5\pi}{6} + 2\pi k,\ \frac{11\pi}{6} + 2\pi k,\ k \text{ any integer.}$$

Example 2. *Solve the equation:*

$$\tan(\theta) = -\sqrt{3}.$$

Explanation We'll start by finding the reference angle, θ_R, the acute angle between the terminal side of θ and the x-axis. The reference angle satisfies $\tan(\theta_R) = \sqrt{3}$ and the negative sign will be used to indicate the quadrant of the angle. Since tangent is opposite over adjacent, we have the following triangle.



[2in]

From the picture we see $\theta_R = \frac{\pi}{3}$.

The tangent function goes through one period on the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. In the interval $\left(-\frac{\pi}{2}, 0\right)$ (which is Quadrant IV), tangent is negative while in $\left(0, \frac{\pi}{2}\right)$ (which is Quadrant I), tangent is positive. For $\tan(\theta)$ to be negative in this interval, we need θ to be in $\left(-\frac{\pi}{2}, 0\right)$. The only angle in that interval with reference angle $\frac{\pi}{3}$ is $\theta = -\frac{\pi}{3}$. This is the only solution on this period.

> Remember that the tangent function has period π, unlike sine and cosine which have period 2π. On the period $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, tangent is one-to-one, so there is exactly one angle which gives the desired output value. Sine and cosine are not one-to-one across a full period.

To find all solutions, we have to add all multiples of π to this. We use π instead of 2π because the period of tangent is only π. The solutions are then

$$\theta = -\frac{\pi}{3} + \pi k,\ k \text{ any integer.}$$

Let's try one a bit more complicated.

Example 3. *Solve the equation:*

$$\cos(\theta)\left(\cos(\theta) + 1\right) = \sin^2(\theta).$$

Explanation We'll start by simplifying a bit. Note that we use a rearranged form of the Pythagorean identity, $\sin^2(\theta) = 1 - \cos^2(\theta)$.

$$\begin{aligned}
\cos(\theta)\left(\cos(\theta)+1\right) &= \sin^2(\theta)\\
\cos^2(\theta)+\cos(\theta) &= \sin^2(\theta)\\
\cos^2(\theta)-\sin^2(\theta)+\cos(\theta) &= 0\\
\cos^2(\theta)-\left(1-\cos^2\theta\right)+\cos(\theta) &= 0\\
2\cos^2(\theta)+\cos(\theta)-1 &= 0.
\end{aligned}$$

Notice that this equation is quadratic in $\cos(\theta)$. We can factor it like we try to do to solve any other quadratic equation:

$$\left(\cos(\theta)+1\right)\left(2\cos(\theta)-1\right) = 0.$$

Now, we can set each factor equal to zero and solve the two resulting equations:

$$\cos(\theta)+1 = 0 \text{ and } 2\cos(\theta)-1 = 0$$

yield the equations $\cos(\theta) = -1$ and $\cos(\theta) = \frac{1}{2}$. On the interval $[0, 2\pi)$, $\cos(\theta) = -1$ has only one solution, $\theta = \pi$. For $\cos(\theta) = \frac{1}{2}$, we see that the reference angle $\theta_R = \frac{\pi}{3}$. Since cosine is positive in Quadrants I and IV, we find solutions $\theta = \frac{\pi}{3}$ and $\frac{5\pi}{3}$.

All solutions are:

$$\theta = \pi + 2\pi k,\ \frac{\pi}{3} + 2\pi k,\ \frac{5\pi}{3} + 2\pi k,\ k \text{ any integer.}$$

9.2.2 Trig Functions as Functions

Motivating Questions

- How do trigonometric functions interact with other functions?
- How do we find zeros of trigonometric functions?
- What does average rate of change look like with trigonometric functions?

Trig Function Compositions

Trigonometric functions can be composed with any of the types of functions that we have already seen. Just as with other function compostions, we need to be mindful of the domains and ranges of our functions.

Example 4. *Let's consider the following functions:* $f(x) = \sin(x)$ *and* $g(x) = 3x^2$.
Find the function below and state its domain and range.

(a) $(f \circ g)(x)$

(b) $(g \circ f)(x)$

(c) $(f \circ f)(x)$

Explanation First let's find the domain and range of $f(x)$ and $g(x)$.
The domain for both $f(x)$ and $g(x)$ is $(-\infty, \infty)$. The range for $f(x)$ is $[-1, 1]$ and the range for $g(x)$ is $[0, \infty)$. Now we can look at the compositions.

(a) $f(g(x)) = \sin(3x^2)$
The domain of $f \circ g$ consists of all inputs to g whose corresponding outputs are in the domain of f. Since the domain of f is $(-\infty, \infty)$, all outputs of g are in the domain of f, so the domain of $f \circ g$ is the entire domain of g, namely $(-\infty, \infty)$.

Finding the range of $f \circ g$ is a bit trickier. Since the range of g is only non-negative numbers, only non-negative numbers will be plugged into f when evaluating the composition. The question therefore becomes: what outputs of f correspond to non-negative inputs? The answer is: all of them! By looking at the graph of the sine function, we can see that all y-values have a corresponding non-negative x-value. Therefore, the range of f, namely $[-1, 1]$, is the range of $f \circ g$.

(b) $g(f(x)) = 3\sin^2(x)$. Remember that $\sin^2(x)$ means $(\sin(x))^2$.
All real numbers can be plugged into $\sin(x)$, and the results can all be

squared. The results of *that* process can all be multiplied by 3, so the domain of $g \circ f$ is $(-\infty, \infty)$.

Let's start by finding the range of $\sin^2(x)$. Since we are squaring $\sin(x)$, its outputs are all non-negative. However, the maximum absolute value of $\sin(x)$ is 1, so the maximum value of $\sin^2(x)$ is 1. Since $\sin(x)$ ranges through all values in $[-1, 1]$, $\sin^2(x)$ ranges through all values in $[0, 1]$. Multiplying $\sin^2(x)$ by 3 results in a vertical stretch by a factor of 3, so the range of $3\sin^2(x)$ is $[0, 3]$.

(c) $f(f(x)) = \sin(\sin(x))$
The domain of $\sin(x)$ is all real numbers, and therefore, all outputs of $\sin(x)$ can be plugged into $\sin(x)$, meaning that the domain of $f \circ f$ is all real numbers, $(-\infty, \infty)$.

Since $\sin(x)$ increases from -1 to 1 as x goes from $-\pi/2$ to $\pi/2$, $\sin(\sin(x))$ increases from $\sin(-1)$ to $\sin(1)$ as x goes from $-\pi/2$ to $\pi/2$. Since going from $-\pi/2$ to $\pi/2$ takes us through the entire range of $\sin(x)$, the range of $f \circ f$ is $[\sin(-1), \sin(1)]$. Recall that sin is an odd function, meaning $\sin(-x) = -\sin(x)$. Therefore, another way to write the range of $f \circ f$ is $[-\sin(1), \sin(1)]$.

Finding Zeros of Trigonometric Funtions.

Example 5. *Let f be a function defined by $f(x) = \sin^2(x) - 1$. Find the zeros of f in the interval $[0, 2\pi)$.*

Explanation First we need to set $f(x)$ equal to 0:

$$\sin^2(x) - 1 = 0$$

Now we can recognize that we have a difference of squares, so we have the following:

$$(\sin(x) + 1)(\sin(x) - 1) = 0.$$

Now we can set each factor on the left equal to zero. So we have

$$\sin(x) + 1 = 0 \text{ and } \sin(x) - 1 = 0.$$

After simplifying them a bit, we have

$$\sin(x) = -1 \text{ and } \sin(x) = 1.$$

Because these are famous values of $\sin(x)$, we can find values for x without using inverse trigonometry. The solution to $\sin(x) = -1$ is $x = \dfrac{3\pi}{2}$ and the solution to $\sin(x) = 1$ is $x = \dfrac{\pi}{2}$. Note that there are other x-values for which $\sin(x) = \pm 1$, but these are the only ones for which $0 \leq x < 2\pi$.

If we wanted to find those other x-values, we could add or subtract multiples of 2π to our solutions in the interval $[0, 2\pi)$. This is because our function is periodic with period 2π. A complete list of all the zeros of f would then be $\frac{\pi}{2} + 2\pi k$ and $\frac{3\pi}{2} + 2\pi k$ for all integers k. The notation with the k means that for any integer value of k $(\ldots, -2, -1, 0, 1, 2, \ldots)$, $\frac{\pi}{2} + 2\pi k$ gives us another zero of the function f, and likewise for $\frac{3\pi}{2} + 2\pi k$.

Example 6. *Let f be a function defined by $f(x) = \sin^2(x) + \frac{5}{2}\sin(x) - \frac{3}{2}$. Find the zeros of f in the interval $[0, 2\pi)$.*

Explanation First we need to set f equal to 0:

$$\sin^2(x) + \frac{5}{2}\sin(x) - \frac{3}{2} = 0$$

Thankfully the left-hand side can factor nicely, so we have

$$\sin^2(x) + \frac{5}{2}\sin(x) - \frac{3}{2} = \left(\sin(x) - \frac{1}{2}\right)(\sin(x) + 3).$$

Now we set each factor equal to zero:

$$\sin(x) - \frac{1}{2} = 0 \text{ and } \sin(x) + 3 = 0.$$

After simplifying them a bit, we have

$$\sin(x) = \frac{1}{2} \text{ and } \sin(x) = -3.$$

-3 is outside of the range of $\sin(x)$ so we get no solutions to our equation from that factor. Our only concern is the famous value where $\sin(x) = \frac{1}{2}$. This happens to occur in two places for $0 \le x < 2\pi$, so our answer is $x = \frac{\pi}{6}, \frac{5\pi}{6}$.

A complete list of all the zeros of f would be $\frac{\pi}{6} + 2\pi k$ and $\frac{5\pi}{6} + 2\pi k$ for all integers k.

Example 7. *Let f be a function defined by $f(x) = \cos(x) + \cos(-x) - \sqrt{2}$. Find the zeros of f in the interval $[0, 2\pi)$.*

Explanation First we need to set f equal to 0:

$$\cos(x) + \cos(-x) - \sqrt{2} = 0$$

Recall that cos is an even function, so $\cos(-x) = \cos(x)$. Making this substitution, we have that

$$\begin{aligned}
\cos(x) + \cos(-x) - \sqrt{2} &= 0 \\
\cos(x) + \cos(x) - \sqrt{2} &= 0 \\
2\cos(x) - \sqrt{2} &= 0 \\
2\cos(x) &= \sqrt{2} \\
\cos(x) &= \frac{\sqrt{2}}{2}.
\end{aligned}$$

But we know values of x that satisfy this equation from our earlier study: $\frac{\pi}{4}$ and $\frac{7\pi}{4}$.

A complete list of all the zeros of f would be $\frac{\pi}{4} + 2\pi k$ and $\frac{7\pi}{4} + 2\pi k$ for all integers k.

Example 8. *Let f be a function defined by $f(x) = \csc(x) + \sqrt{2}$. Find the zeros of f in the interval $[0, 2\pi)$.* **Explanation** First we need to set f equal to 0:

$$\begin{aligned}
\csc(x) + \sqrt{2} &= 0 \\
\csc(x) &= -\sqrt{2}.
\end{aligned}$$

Now we use the fact that cosecant is the reciprocal of sine:

$$\begin{aligned}
\frac{1}{\sin(x)} &= -\sqrt{2} \\
\sin(x) &= -\frac{1}{\sqrt{2}} \\
\sin(x) &= -\frac{\sqrt{2}}{2}.
\end{aligned}$$

We now know values of x that satisfy this equation from our earlier study: $\frac{5\pi}{4}$ and $\frac{7\pi}{4}$.

A complete list of all the zeros of f would be $\frac{5\pi}{4} + 2\pi k$ and $\frac{7\pi}{4} + 2\pi k$ for all integers k.

Example 9. *Let f be a function defined by $f(x) = \sin(2x) - \frac{1}{2}$. Find the zeros of f in the interval $[0, 2\pi)$.*

Explanation First we need to set f equal to 0:

$$\sin(2x) - \frac{1}{2} = 0 \implies \sin(2x) = \frac{1}{2}$$

Now comes a tricky part. We know values that we can plug into $\sin(x)$ to make it equal $\frac{1}{2}$, but we don't have the same comfort level with $\sin(2x)$. What we'll do is make a substitution $u = 2x$. Now our equation has become

$$\sin(u) = \frac{1}{2}.$$

But we know all the solutions to this equation! If $u = \frac{\pi}{6}$ or $u = \frac{5\pi}{6}$, then $\sin(u) = \frac{1}{2}$. In fact, we know that all solutions are $u = \frac{\pi}{6} + 2\pi k$ and $u = \frac{5\pi}{6} + 2\pi k$ for all integers k.

However, to solve our original equation of $\sin(2x) = \frac{1}{2}$ in terms of x, we need to undo our substitution: if $u = 2x$, then $x = \frac{u}{2}$. Dividing our solutions in terms of u by 2, we obtain the solutions $x = \frac{\pi}{12} + \pi k$ and $x = \frac{5\pi}{12} + \pi k$ for all integers k. This is a complete list of zeros of our function f.

It now remains to see which of these zeros are in the interval $[0, 2\pi)$. A surefire way to do this is to plug in different values for k. Plugging in $k = -1$ for $x = \frac{\pi}{12} + \pi k$ gives us a negative, which isn't in the interval $[0, 2\pi)$, so we plug in 0 and 1 for k to obtain $\frac{\pi}{12}$ and $\frac{\pi}{12} + \pi = \frac{13\pi}{12}$. If we plug in 2 for k, we obtain $\frac{\pi}{12} + 2\pi = \frac{25\pi}{12}$, which is larger than 2π, and so is not in the correct interval. All other values of k will similarly produce numbers outside $[0, 2\pi)$. Therefore, the only zeros in $[0, 2\pi)$ of the form $x = \frac{\pi}{12} + \pi k$ are $\frac{\pi}{12}$ and $\frac{13\pi}{12}$. Likewise, the only solutions in the interval $[0, 2\pi)$ of the form $x = \frac{5\pi}{12} + \pi k$ are $\frac{5\pi}{12}$ and $\frac{17\pi}{12}$. Therefore, a list of the zeros of f in the interval $[0, 2\pi)$ is given by $\frac{\pi}{12}$, $\frac{5\pi}{12}$, $\frac{13\pi}{12}$, and $\frac{17\pi}{12}$.

The previous example should seem quite complicated, so let's try and summarize what we did. First, we isolated the trigonometric function on one side of the equation. Then, we made a substitution to simplify the equation in x into an equation in u we knew how to solve. Upon solving, we needed to undo the substitution. The twist here is that we needed to undo the substitution for *all* the solutions to the u-equation, then see which of them were in $[0, 2\pi)$.

Example 10. *Let f be a function defined by $f(x) = \cos(3x + \pi) - \frac{1}{2}$. Find the zeros of f in the interval $[0, 2\pi)$.*

Explanation First we need to set f equal to 0:

$$\cos(3x + \pi) - \frac{1}{2} = 0 \implies \cos(3x + \pi) = \frac{1}{2}$$

Now comes the substitution part. We'll make a substitution $u = 3x + \pi$. Now our equation has become

$$\cos(u) = \frac{1}{2}.$$

But we know all the solutions to this equation! If $u = \frac{\pi}{3}$ or $u = \frac{5\pi}{3}$, then $\cos(u) = \frac{1}{2}$. In fact, we know that all solutions are $u = \frac{\pi}{3} + 2\pi k$ and $u = \frac{5\pi}{3} + 2\pi j$ for all integers k.

However, to solve our original equation of $\cos(3x + \pi) = \frac{1}{2}$ in terms of x, we need to undo our substitution: if $u = 3x + \pi$, then $x = \frac{u}{3} - \frac{\pi}{3}$. Undoing our substitution by subtracting π and dividing by 3, we obtain the solutions $x = -\frac{2\pi}{9} + \frac{2\pi}{3}k$ and $x = \frac{2\pi}{9} + \frac{2\pi}{3}k$ for all integers k. This is a complete list of zeros of our function f.

It now remains to see which of these zeros are in the interval $[0, 2\pi)$. Going through all the relevant values of k, we find that $\frac{2\pi}{9}$, $\frac{4\pi}{9}$, $\frac{8\pi}{9}$, $\frac{10\pi}{9}$, $\frac{14\pi}{9}$, and $\frac{16\pi}{9}$ are the only zeros in the correct interval.

Average Rate of Change with Trigonometry

We can still find average rate of change with trigonometric functions, but because they are periodic there can be some interesting results.

Example 11. *Let* $f(x) = \sin(x)$.

(a) *Find* $\text{AROC}_{\left[\frac{\pi}{6}, \frac{3\pi}{4}\right]}$.

(b) *Find* $\text{AROC}_{\left[\frac{\pi}{3}, \frac{2\pi}{3}\right]}$.

Explanation

(a)

$$\text{AROC}_{\left[\frac{\pi}{6}, \frac{3\pi}{4}\right]} = \frac{\sin(\frac{3\pi}{4}) - \sin(\frac{\pi}{6})}{\frac{3\pi}{4} - \frac{\pi}{6}}$$

First we substitute in our trig values:

$$\frac{\sin(\frac{3\pi}{4}) - \sin(\frac{\pi}{6})}{\frac{3\pi}{4} - \frac{\pi}{6}} = \frac{\frac{\sqrt{2}}{2} - \frac{1}{2}}{\frac{3\pi}{4} - \frac{\pi}{6}}$$

Next, we simplify our fractions:

$$\frac{\frac{\sqrt{2}}{2}-\frac{1}{2}}{\frac{3\pi}{4}-\frac{\pi}{6}} = \frac{\frac{\sqrt{2}-1}{2}}{\frac{9\pi}{12}-\frac{2\pi}{12}} = \frac{\frac{\sqrt{2}-1}{2}}{\frac{7\pi}{12}}$$

$$= \frac{\frac{\sqrt{2}-1}{2}}{\frac{7\pi}{12}}$$

$$= \frac{6(\sqrt{2}-1)}{7\pi}.$$

Therefore,

$$\text{AROC}_{\left[\frac{\pi}{6},\frac{3\pi}{4}\right]} = \frac{6\sqrt{2}-6}{7\pi}.$$

(b)

$$\text{AROC}_{\left[\frac{\pi}{3},\frac{2\pi}{3}\right]} = \frac{\sin(\frac{2\pi}{3})-\sin(\frac{\pi}{3})}{\frac{2\pi}{3}-\frac{\pi}{3}}$$

First we substitute in our trig values:

$$\frac{\sin(\frac{2\pi}{3})-\sin(\frac{\pi}{3})}{\frac{2\pi}{3}-\frac{\pi}{3}} = \frac{\frac{\sqrt{3}}{2}-\frac{\sqrt{3}}{2}}{\frac{2\pi}{3}-\frac{\pi}{3}}.$$

Next, we simplify:

$$\frac{\frac{\sqrt{3}}{2}-\frac{\sqrt{3}}{2}}{\frac{2\pi}{3}-\frac{\pi}{3}} = \frac{0}{\frac{\pi}{3}}.$$

We can see that we have 0 in the numerator, which means that our average rate of change will be 0. This is because we picked x-values with the same y-value. Even though we had positive and negative rates of change at some point between $\dfrac{\pi}{3}$ and $\dfrac{2\pi}{3}$, both endpoints of the interval have the same y-value, so *on average* there was no change.

9.3 Transformations of Trig Functions

Learning Objectives

- –

9.3.1 Trig Function Transformations

Motivating Questions

- How do the three standard transformations (vertical translation, horizontal translation, and vertical scaling) affect the midline, amplitude, range, and period of sine and cosine curves?
- What algebraic transformation results in horizontal stretching or scaling of a function?
- How can we determine a formula involving sine or cosine that models any circular periodic function for which the midline, amplitude, period, and an anchor point are known?

Recall our previous work with transformations, where we studied how the graph of the function g defined by $g(x) = af(bx - h) + k$ is related to the graph of f, where a, b, h, and k are real numbers with $a \neq 0$. Because such transformations can shift and stretch a function, we are interested in understanding how we can use transformations of the sine and cosine functions to fit formulas to circular functions.

Example 12. *Let $f(t) = \cos(t)$. First, answer all of the questions below* without *using* Desmos*; then use* Desmos *to confirm your conjectures. For each prompt, describe the graphs of g and h as transformations of f and, in addition, state the amplitude, midline, and period of both g and h.*

a. $g(t) = 3\cos(t)$ *and* $h(t) = -\dfrac{1}{4}\cos(t)$

b. $g(t) = \cos(t - \pi)$ *and* $h(t) = \cos\left(t + \dfrac{\pi}{2}\right)$

c. $g(t) = \cos(t) + 4$ *and* $h(t) = \cos(t) - 2$

d. $g(t) = 3\cos(t - \pi) + 4$ *and* $h(t) = -\dfrac{1}{4}\cos\left(t + \dfrac{\pi}{2}\right) - 2$

Explanation

a. The graph of g has a vertical stretch by 3 and an amplitude of 3. The graph of h has a reflection across the x-axis with a vertical shrink of $\dfrac{1}{4}$ and an amplitude of $\dfrac{1}{4}$. $g(t)$ and $h(t)$ both have a midline at $y = 0$ and a period of 2π.

b. The graph of g has a horizontal shift to the right of π. The graph of h has a horizontal shift to the left of $\dfrac{\pi}{2}$. $g(t)$ and $h(t)$ both have a midline at $y = 0$, a period of 2π, and an amplitude of 1.

c. The graph of g has a vertical shift up of 4 and a midline at $y = 4$. The graph of h has a vertical shift down of 2 and a midline of $y = -2$. $g(t)$ and $h(t)$ both have a period of 2π and an amplitude of 1.

d. The graph of g has a vertical stretch by 3, a horizontal shift to the right of π,and a vertical shift up of 4. $g(t)$ has an amplitude of 3 and a midline at $y = 4$. The graph of h has a reflection across the x-axis with a vertical shrink of $\frac{1}{4}$, a horizontal shift to the left of $\frac{\pi}{2}$, and a vertical shift down of 2. $h(t)$ has an amplitude of $\frac{1}{4}$ and a midline of $y = -2$. $g(t)$ and $h(t)$ both have a period of 2π.

Note in the exercise above several patterns. Given a function $a\cos(t - h) + k$, the amplitude of the function was $|a|$, and the midline of the function was k. In addition, each function above had period 2π, unchanged from the period of cos. We will see later that this is related to the fact that there were no horizontal stretches or compressions of cos.

Shifts and vertical stretches of the sine and cosine functions

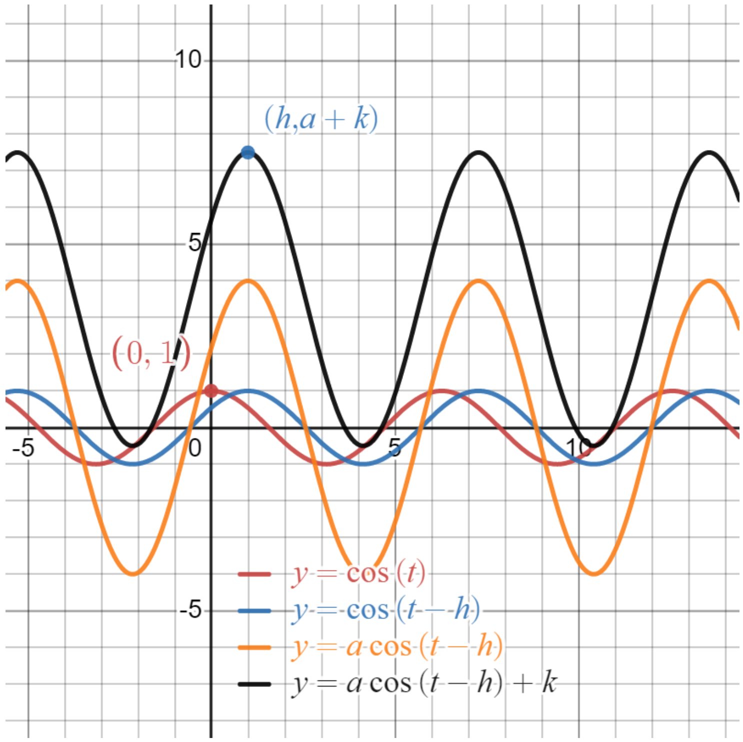
We know that the standard functions $f(t) = \sin(t)$ and $g(t) = \cos(t)$ are circular functions that each have midline $y = 0$, amplitude $a = 1$, period $P = 2\pi$, and range $[-1, 1]$. This suggests the following general principles.

Transformations of sine and cosine with shifts and vertical stretches.
Given real numbers a, h, and k with $a \neq 0$, the functions

$$f(t) = a\cos(t - h) + k \text{ and } g(t) = a\sin(t - h) + k$$

each represent a horizontal shift by h units to the right, followed by a vertical stretch by a units, followed by a vertical shift of k units, applied to the parent function ($\cos(t)$ or $\sin(t)$, respectively). The resulting circular functions have midline $y = k$, amplitude a, range $[k - a, k + a]$, and period $P = 2\pi$. In addition, the point $(h, a + k)$ lies on the graph of f and the point (h, k) lies on the graph of g.

In the figure below, we see how the overall transformation $f(t) = a\cos(t-h)+k$ comes from executing a sequence of simpler ones. The original parent function $y = \cos(t)$ (in red) is first shifted h units right to generate the blue graph of $y = \cos(t - h)$. In turn, that graph is then scaled vertically by a to generate the orange graph of $y = a\cos(t - h)$. Finally, the orange graph is shifted k units vertically to result in the final graph of $y = a\cos(t - h) + k$ in black.



It is often useful to follow one particular point through a sequence of transformations. In the above figure, we see the red point that is located at $(0, 1)$ on the original function $y = \cos(t)$, as well as the point $(h, a+k)$ that is the corresponding point on $f(t) = a\cos(t-h)+k$ under the overall transformation. Note that the point $(h, a+k)$ results from the input, $t = h$, that makes the argument of the cosine function zero: $f(h) = a\cos(h - h) + k = a\cos(0) + k = a + k$.

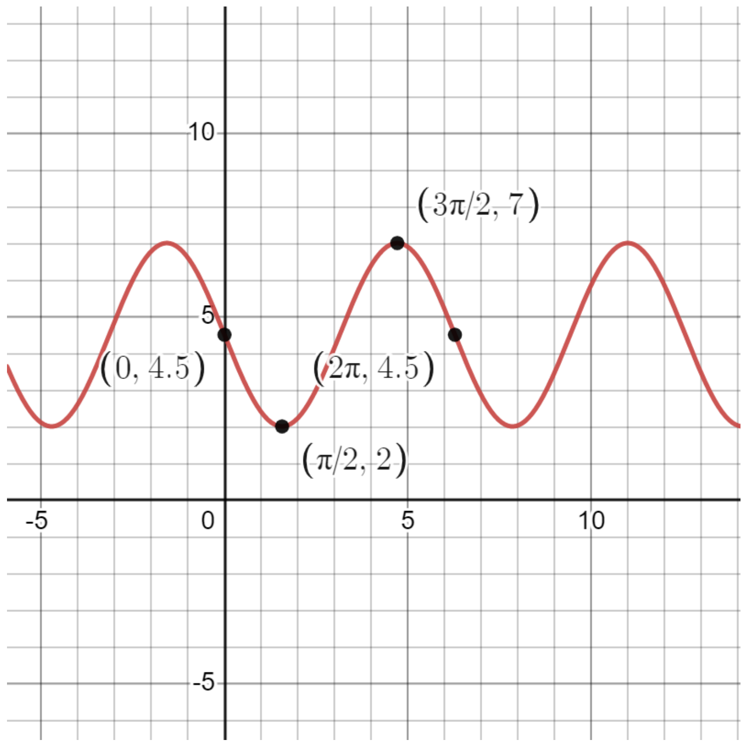
While the sine and cosine functions extend infinitely in either direction, it's natural to think of the point $(0, 1)$ as the "starting point" of the cosine function, and similarly the point $(0, 0)$ as the starting point of the sine function. We will refer to the corresponding points $(h, a+k)$ and (h, k) on $f(t) = a\cos(t-h)+k$ and $g(t) = a\sin(t - h) + k$ as anchor points. Anchor points, along with other information about a circular function's amplitude, midline, and period help us to determine a formula for a function that fits a given situation.

Example 13. *Consider a spring-mass system where the weight resting on a frictionless table. We let $s(t)$ denote the distance from the wall (where the spring is attached) to the weight at time t in seconds and know that the weight oscillates periodically with a minimum value of $s(t) = 2$ feet and a maximum value of $s(t) = 7$ feet with a period of 2π. We also know that $s(0) = 4.5$ and $s\left(\frac{\pi}{2}\right) = 2$.*

Determine a formula for $s(t)$ in the form $s(t) = a\cos(t - b) + c$ or $s(t) =$

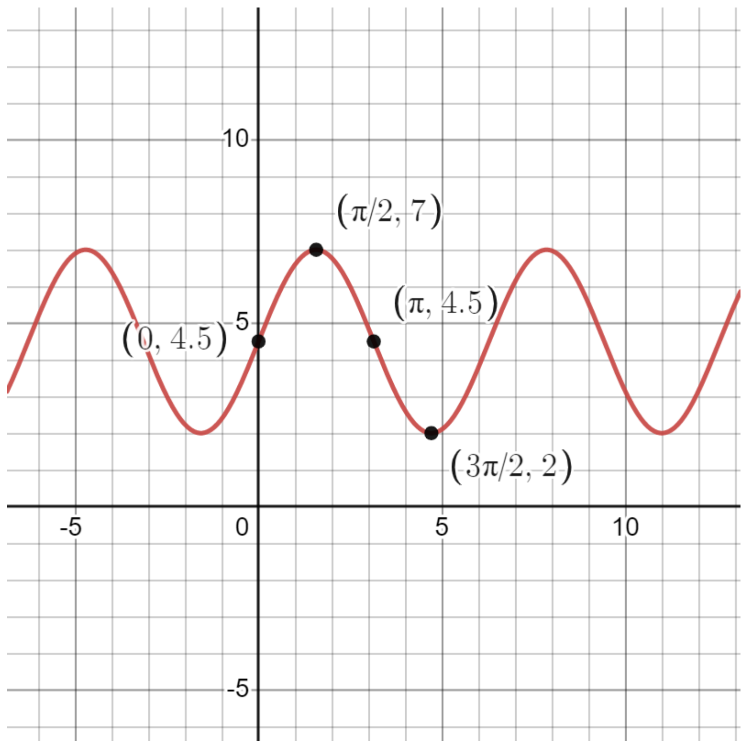
$a\sin(t-b)+c$. Is it possible to find two different formulas that work?

Explanation Let's begin by drawing an accurate picture:



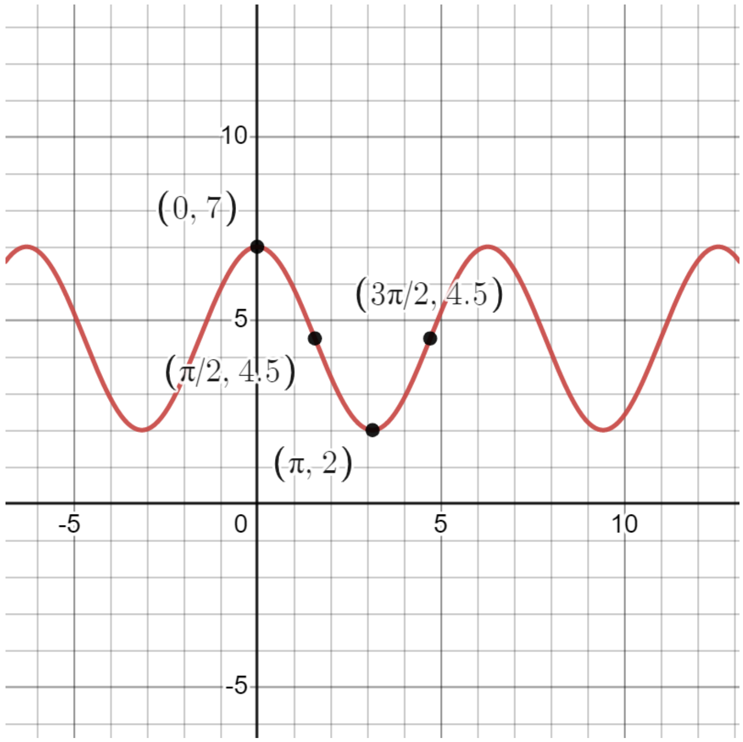
Since the minimum value of s is 2 and the maximum value is 7, the midline of s must be their midpoint, or $\dfrac{7+2}{2}=\dfrac{9}{2}=4.5$. Now we can find the amplitude by finding the distance from the midline to the minimum (or maximum), which would be $|4.5-2|=2.5$.

Now we can try to find a formula that uses the sine function. Because the amplitude of $\sin(x)$ is 1, we must stretch by a factor of 2.5 to obtain the correct amplitude of s. This results in a transformed function of $s_1(x)=2.5\sin(x)$. Because the midline of $\sin(x)$ is 0, we must shift up by 4.5 to get the midline that we want, resulting in the transformed function $s_2(x)=s_1(x)+4.5=2.5\sin(x)+4.5$. Now s_2 has the correct amplitude, midline, and period (unchanged from 2π). The graph of s_2 looks like this:



After some examination, we can see that the only change we need to make to obtain the graph of s is to shift π units to the left, so that the points $(0, 4.5)$ and $\left(\frac{\pi}{2}, 2\right)$ are on the graph of s. Therefore, $s_3(x) = s_2(x+\pi) = 2.5\sin(x+\pi)+4.5$ is a formula for s in terms of the sine function.

Note that we said that $2.5\sin(x+\pi)+4.5$ was only *a* formula for s. Let's try to find a formula using cos. Since the amplitude of $\cos(x)$ is also 1, we still need to stretch by a factor of 2.5, yielding $s_1(x) = 2.5\cos(x)$. Again, since the midline of s_1 is 0, we need to shift up by 4.5 units, yielding $s_2(x) = s_1(x)+4.5 = 2.5\cos(x)+4.5$. The graph of s_2 this time looks as follows:



After examining the picture, we see that to obtain the graph of s, we need to shift left by $\frac{\pi}{2}$ units, meaning that $s_3(x) = s_2\left(x + \frac{\pi}{2}\right) = 2.5\cos\left(x + \frac{\pi}{2}\right) + 4.5$ is another formula for s, this time in terms of the cosine function.

The previous example illustrates a general technique for giving the formula of the graph of a circular function of period 2π with a given midline, amplitude, and y-intercept. Say f is a circular function with period 2π. If the midline is m, the amplitude is a, and we know that $f(0) = z$, then we have a procedure to follow to produce a formula for f.

(a) Stretch the graph vertically by a factor of a to obtain $a\sin(x)$

(b) Shift the graph vertically up by m units to obtain $a\sin(x) + m$

(c) Find x such that $a\sin(x) + m = z$

(d) Shift the function horizontally so that the point (x, z) transforms into $(0, z)$.

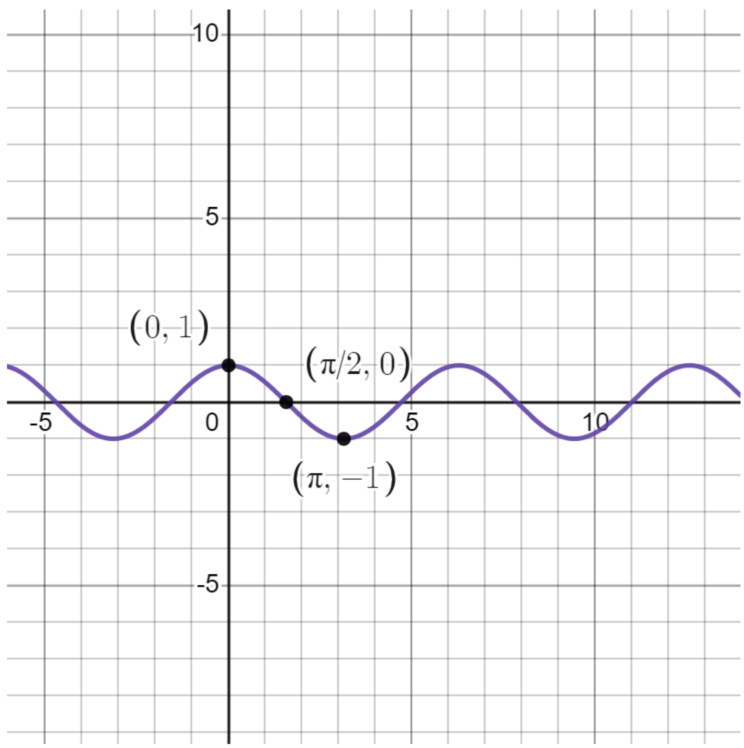
Note that this procedure can work with cos as well as sin. Let's see an example.

Example 14. *Give the formula for a circular function f with period 2π whose midline is -2, whose amplitude is 5, and whose y-intercept is $(0, -7)$.*

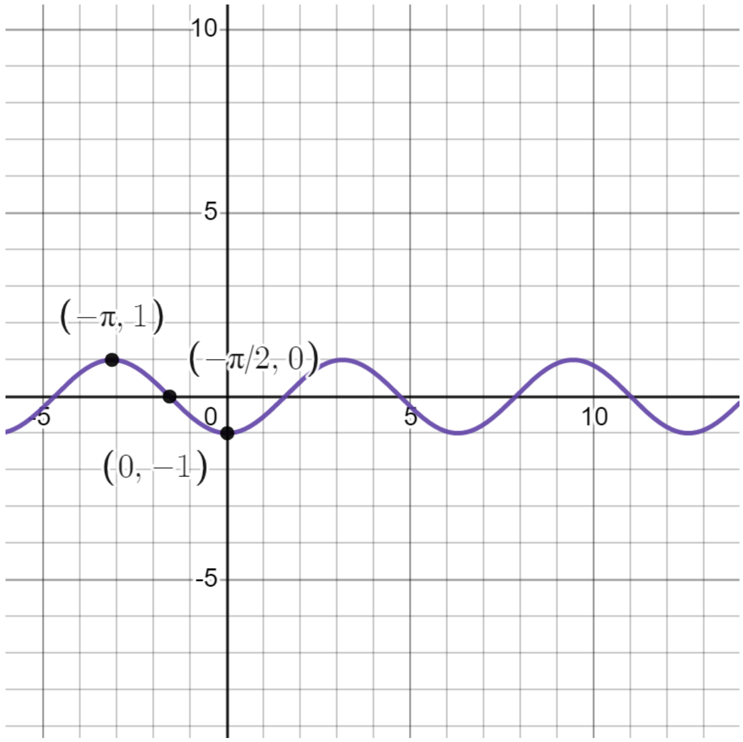
Explanation Since the amplitude is 5, the first transformation we apply to $\cos(x)$ is a vertical stretch by a factor of 5 to obtain $f_1(x) = 5\cos(x)$. Then, we need to shift down by 2 to obtain the correct midline: $f_2(x) = f_1(x) - 2 = 5\cos(x) - 2$. Finally, by looking at a graph, we can tell that $f_2(\pi) = 5\cos(\pi) - 2 = -5 - 2 = -7$, so we need to shift π units to the left. Doing this gives us $f(x) = f_2(x + \pi) = 5\cos(x + \pi) - 2$. To check our work, we can graph the function and calculate its midline, amplitude, and y-intercept.

Example 15. *Graph the function f given by $f(x) = 7\cos(x + \pi) - 1$. Keep track of the points $(0, 1)$, $\left(\frac{\pi}{2}, 0\right)$, and $(\pi, -1)$ in each step. Give the midline, amplitude, and period of f.*

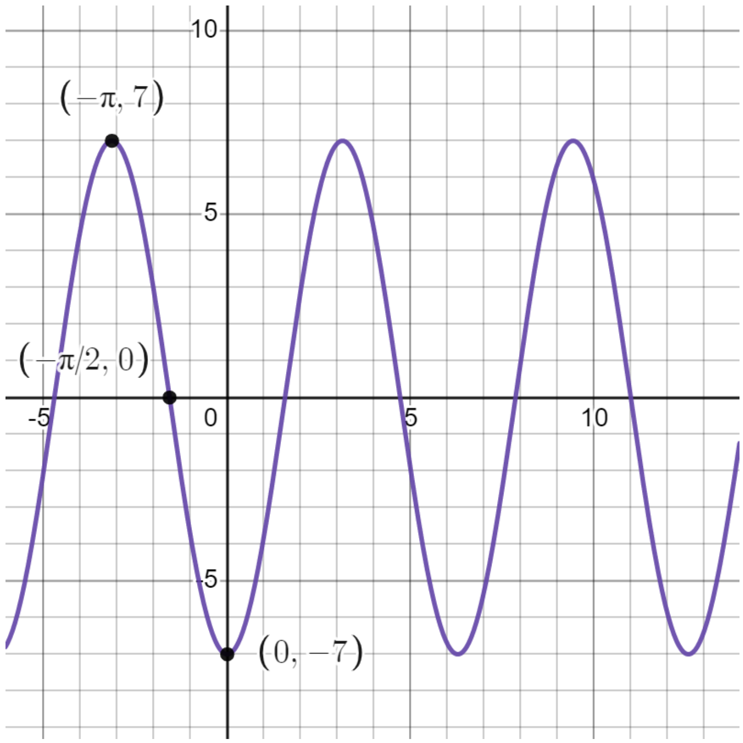
Explanation We first graph the parent function, which is defined in this case by $f_0(x) = \cos(x)$. This graph has the points $(0, 1)$, $\left(\frac{\pi}{2}, 0\right)$, and $(\pi, -1)$, and is shown below.



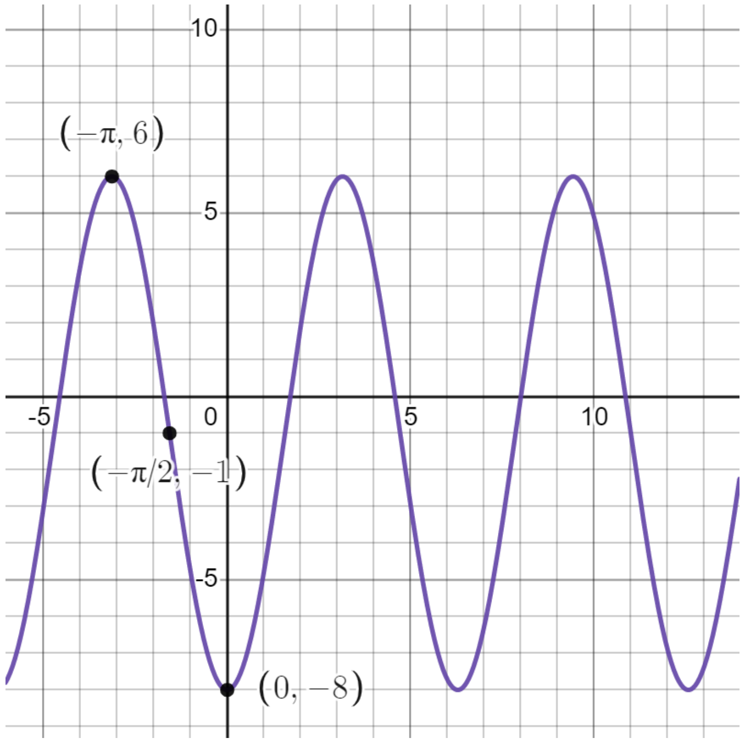
We then proceed using the order of transformations outlined in a previous section. The first transformation to take place is the horizontal shift left by π units. That results in the function $f_1(x) = f_0(x + \pi) = \cos(x + \pi)$ and moves our points to $(-\pi, 1)$, $\left(-\frac{\pi}{2}, 0\right)$, and $(0, -1)$.



Next, we have a vertical stretch by a factor of 7. This results in the function $f_2(x) = 7f_1(x) = 7\cos(x + \pi)$ and moves our points to $(-\pi, 7)$, $\left(-\dfrac{\pi}{2}, 0\right)$, and $(0, -7)$.



Finally, we shift down by 1 unit. This results in the function $f(x) = f_2(x) - 1 = 7\cos(x + \pi) - 1$ and moves our points to $(-\pi, 6)$, $\left(-\dfrac{\pi}{2}, -1\right)$, and $(0, -8)$.



We can see from the graph that the midline of our function is -1, and the amplitude is 7. The period can also be read off as 2π. Note that we could have read this information off from the formula, since $a = 7$, $k = -1$, and there is no horizontal stretch or compression.

Horizontal scaling

There is one more very important transformation of a function that we've not yet explored in the trigonometric context. Given a function $y = f(x)$, we want to understand the related function $g(x) = f(bx)$, where b is a positive real number. The sine and cosine functions are ideal functions with which to explore these effects; moreover, this transformation is crucial for being able to use the sine and cosine functions to model phenomena that oscillate at different frequencies.

By using a graphing utility such as Desmos, we can explore the effect of the transformation $h(t) = f(bt)$, where $f(t) = \sin(t)$.

Desmos link: `https://www.desmos.com/calculator/zftuh2cfzr`

By experimenting with the slider, we gain an intuitive sense for how the value of b affects the graph of $h(t) = f(bt)$ in comparision to the graph of $f(t)$. When

$b = 2$, we see that the graph of h is oscillating twice as fast as the graph of f since $h(t) = f(2t)$ completes two full cycles over an interval in which f completes one full cycle. In contrast, when $b = \frac{1}{2}$, the graph of h oscillates half as fast as the graph of f, as $h(t) = f\left(\frac{1}{2}t\right)$ completes only half of one cycle over an interval where $f(t)$ completes a full one.

We can also understand this from the perspective of function composition. To evaluate $h(t) = f(2t)$, at a given value of t, we first multiply the input t by a factor of 2, and then evaluate the function f at the result. An important observation is that

$$h\left(\frac{1}{2}t\right) = f\left(2 \cdot \frac{1}{2}t\right) = f(t).$$

This tells us that the point $\left(\frac{1}{2}t, f(t)\right)$ lies on the graph of h since an input of $\frac{1}{2}t$ in h results in the value $f(t)$. At the same time, the point $(t, f(t))$ lies on the graph of f. Thus we see that the correlation between points on the graphs of f and h (where $h(t) = f(2t)$) is

$$(t, f(t)) \to \left(\frac{1}{2}t, f(t)\right).$$

We can therefore think of the transformation $h(t) = f(2t)$ as achieving the output values of f twice as fast as the original function $f(t)$ does. Analogously, the transformation $h(t) = f\left(\frac{1}{2}t\right)$ will achieve the output values of f only half as quickly as the original function.

Recall that given a function $y = f(t)$ and a real number $b > 0$, the transformed function $y = h(t) = f(bt)$ is a *horizontal stretch* of the graph of f. Every point $(t, f(t))$ on the graph of f gets stretched horizontally to the corresponding point $\left(\frac{1}{b}t, f(t)\right)$ on the graph of h. If $0 < b < 1$, the graph of v is a stretch of f away from the y-axis by a factor of b; if $b > 1$, the graph of h is a compression of f toward the y-axis by a factor of b. The only point on the graph of f that is unchanged by the transformation is $(0, f(0))$.

Circular functions with different periods

Because the circumference of the unit circle is 2π, the sine and cosine functions each have period 2π. Of course, as we think about using transformations of the sine and cosine functions to model different phenomena, it is apparent that we will need to generate functions with different periods than 2π. For instance, if a ferris wheel makes one revolution every 5 minutes, we'd want the period of the function that models the height of one car as a function of time to be $P = 5$.

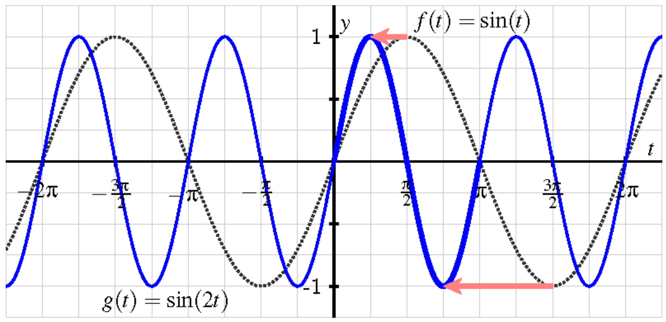


Figure 1: A plot of the parent function, $f(t) = \sin(t)$ (dashed, in gray), and the transformed function $g(t) = f(2t) = \sin(2t)$ (in blue).

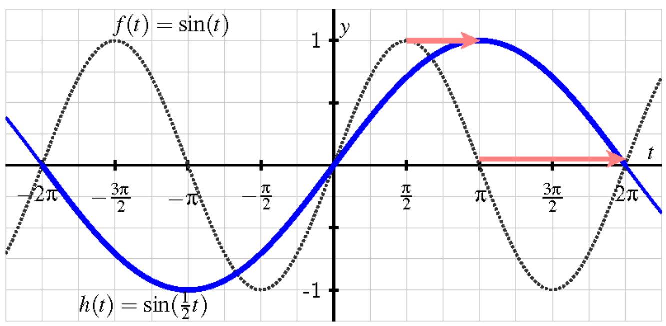


Figure 2: A plot of the parent function, $f(t) = \sin(t)$ (dashed, in gray), and the transformed function $h(t) = f\left(\frac{1}{2}t\right) = \sin\left(\frac{1}{2}t\right)$ (in blue).

Horizontal scaling of functions enables us to generate circular functions with any period we desire.

We begin by considering two basic examples. First, let $f(t) = \sin(t)$ and $g(t) = f(2t) = \sin(2t)$. We know from our most recent work that this transformation results in a horizontal compression of the graph of $\sin(t)$ by a factor of $\frac{1}{2}$ toward the y-axis. If we plot the two functions on the same axes, it becomes apparent how this transformation affects the period of f.

From the graph, we see that $g(t) = \sin(2t)$ oscillates twice as frequently as $f(t) = \sin(t)$, and that g completes a full cycle on the interval $[0, \pi]$, which is half the length of the period of f. Thus, the "2" in $f(2t)$ causes the period of f to be $\frac{1}{2}$ as long; specifically, the period of g is $P = \frac{1}{2}(2\pi) = \pi$.

On the other hand, if we let $h(t) = f\left(\frac{1}{2}t\right) = \sin\left(\frac{1}{2}t\right)$, the transformed graph h is stretched away from the y-axis by a factor of 2. This has the effect of doubling the period of f, so that the period of h is $P = 2 \cdot 2\pi = 4\pi$, as seen in the previous figure.

Our observations generalize for any positive constant $b > 0$. In the case where $b = 2$, we saw that the period of $g(t) = \sin(2t)$ is $P = \frac{1}{2} \cdot 2\pi$, whereas in the case where $b = \frac{1}{2}$, the period of $h(t) = \sin(\frac{1}{2}t)$ is $P = 2 \cdot 2\pi = \frac{1}{\frac{1}{2}} \cdot 2\pi$. Identical reasoning holds if we are instead working with the cosine function. In general, we can say the following.

For any constant $b > 0$, the period of the functions $\sin(bt)$ and $\cos(bt)$ is

$$P = \frac{2\pi}{b}.$$

Thus, if we know the b-value from the given function, we can deduce the period. If instead we know the desired period, we can determine b by the rule $b = \frac{2\pi}{P}$.

Transformations of sine and cosine with any shift or stretch. Given real numbers a, b, h, and k with $a > 0$ and $b > 0$, the functions

$$f(t) = a\cos(b(t-h)) + k \text{ or } g(t) = a\sin(b(t-h)) + k.$$

each represent a horizontal stretch or compression by a factor of b, followed by a vertical stretch by a units, followed by a vertical shift of k units, applied to the parent function ($\cos(t)$ or $\sin(t)$, respectively). They then contain a horizontal shift to the right by h units. The resulting circular functions have midline $y = k$, amplitude a, range $[k-a, k+a]$, and period $P = 2\pi/b$. In addition, the point $(h, a+k)$ lies on the graph of f and the point (h, k) lies on the graph of g.

Be careful! These calculations only apply to sin and cos. For example, $\tan(bx)$ has a period of $\frac{\pi}{b}$, not $\frac{2\pi}{b}$, since the original period of tan is just π. When you're faced with a problem involving a transformation of a trig function, it's best to start from what you already know about the trig function, then see how each transformation affects its properties.

Example 16. *Determine the exact period, amplitude, and midline of each of the following functions. In addition, state the range of each function and any horizontal shift that has been introduced to the graph. Make your conclusions without consulting* Desmos*, and then use the program to check your work.*

a. $p(x) = \sin(10x) + 2$

b. $q(x) = -3\cos(0.25x) - 4$

c. $r(x) = 2\sin\left(\frac{\pi}{4}x\right) + 5$

d. $w(x) = 2\cos\left(\frac{\pi}{2}(x-3)\right) + 5$

e. $u(x) = -0.25\sin(3x-6) + 5$

Explanation

a. $p(x)$ has a period of $\frac{\pi}{5}$, an amplitude of 1 and a midline of $y = 2$.
The range of $p(x)$ is $[1, 3]$.

b. $q(x)$ has a period of 8π, an amplitude of 3 and a midline of $y = -4$.
The range of $q(x)$ is $[-7, -1]$.

c. $r(x)$ has a period of 8, an amplitude of 2 and a midline of $y = 5$.
The range of $r(x)$ is $[3, 7]$.

d. $w(x)$ has a period of 4, an amplitude of 2 and a midline of $y = 5$.
The range of $w(x)$ is $[3, 7]$.
There is a horizontal shift of 3 to the right.

e. $u(x)$ has a period of $\frac{2\pi}{3}$, an amplitude of 0.25 and a midline of $y = 5$.
The range of $u(x)$ is $[4.75, 5.25]$.
There is a horizontal shift of 2 to the right. This is because we have to factor out the 3 inside the parentheses: $-0.25\sin(3(x-2)) + 5$.

You might wonder why we've chosen to use the formula $a\sin(b(t-h)) + k$ with the parentheses instead of $a\sin(bt-h)+k$ without the parentheses. In the former formula, we save the horizontal shift until the end, but in the latter, we do the horizontal shift before anything else. The reason we prefer the former is that the midline, amplitude, and period of a function are easy to plug into the formula, and plugging these in results in a formula of the form $f_1(t) = a\sin(bt) + k$. We can then find the horizontal shift we need to apply to the graph of f_1 to obtain $f(t) = f_1(t-h) = a\sin(b(t-h)) + k$.

Example 17. *Consider a spring-mass system where the weight is hanging from the ceiling in such a way that the following is known: we let $d(t)$ denote the distance from the ceiling to the weight at time t in seconds and know that the weight oscillates periodically with a minimum value of $d(t) = 1.5$ and a maximum value of $d(t) = 4$, with a period of 3, and you know $d(0.5) = 2.75$ and $d(1.25) = 4$.*

State the midline, amplitude, range, and an anchor point for the function, and hence determine a formula for $d(t)$ in the form $a\cos(b(t-h)) + k$ or $a\sin(b(t-$

h)) + k. Show your work and thinking, and use Desmos *appropriately to check that your formula generates the desired behavior.*

Explanation Since the period of our function is 3, we know that $b = \frac{2\pi}{3}$. The midline is the midpoint of the minimum and maximum values: $\frac{4+1.5}{2} = \frac{5.5}{2} = 2.75$. The amplitude is the distance from the midline to the maximum (or minimum): $|2.75 - 4| = 1.25$. Plugging all this information into a function, we obtain $d_1(t) = 1.25\cos\left(\frac{2\pi}{3}t\right) + 2.75$.

Since we started with the cosine function, we know that the maximum of $1.25\cos\left(\frac{2\pi}{3}t\right) + 2.75$ is 4, and occurs at $t = 0$. Since we want $d(1.25) = 4$, we need to shift our graph to the right by 1.25 units. This means $d(t) = d_1(t - 1.25) = 1.25\cos\left(\frac{2\pi}{3}(t - 1.25)\right) + 2.75$. We can then check that $d(0.5) = 2.75$ by plugging into our function.

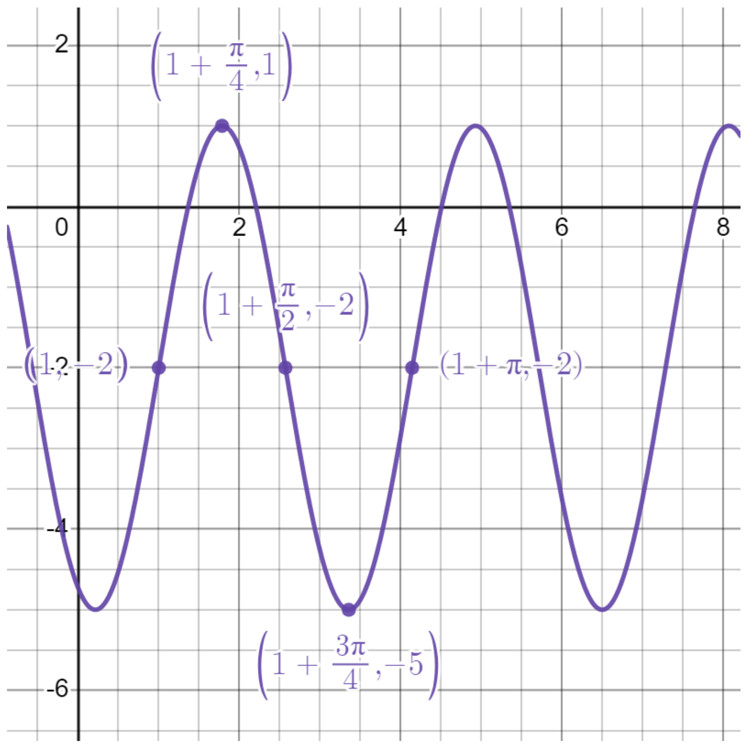
Example 18. *Let f be a circular function with midline π, amplitude $\frac{\pi}{2}$, and period 5, with $f(8) = \frac{3\pi}{2}$. Give a formula for f of the form $a\cos(b(t-h)) + k$ or $a\sin(b(t-h)) + k$.*

Explanation Since a corresponds to the amplitude, $a = \frac{\pi}{2}$. Since k corresponds to the midline, $k = \pi$. Since the period is given by $\frac{2\pi}{b}$, $b = \frac{2\pi}{5}$.

Let's now pay some attention to the choice between sine and cosine. Note that since the midline is π and the amplitude is $\frac{\pi}{2}$, the maximum of our function is $\frac{3\pi}{2}$. This means that the condition that $f(8) = \frac{3\pi}{2}$ tells us where the maximum should be. We have a very easy location for the maximum of cosine: $t = 0$, so we'll choose to use cosine here. Note that choosing sine isn't incorrect, just more difficult.

Plugging our values into the equation, we have $f_1(t) = \frac{\pi}{2}\cos\left(\frac{2\pi}{5}t\right) + \pi$. The maximum for cosine occurs at $t = 0$, as stated above, but we want the maximum to occur at $t = 8$, so we'll shift the graph of f_1 8 units to the right. This gives us $f(t) = f_1(t-8) = \frac{\pi}{2}\cos\left(\frac{2\pi}{5}(t-8)\right) + \pi$.

Example 19. *Give a formula of the form $a\cos(b(t-h))+k$ or $a\sin(b(t-h))+k$ for the graph below.*



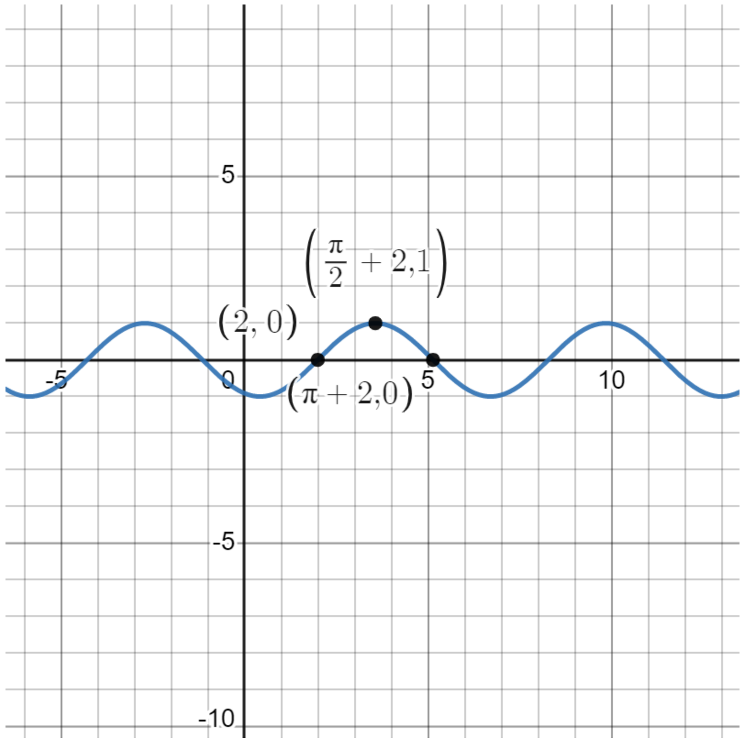
Explanation Although this problem may seem different from the previous problem, we can still collect information about the midline, amplitude, and period from the graph of the function.

Here, the amplitude is 3, the midline is -2, and the period is π. Using the sine function as a parent function, this gives us $f_1(t) = 3\sin(2t) - 2$. Note that $f_1(0) = 3\sin(0) - 2 = 0 - 2 = -2$, and we want our function to have $f(1) = -2$, so we need to shift the graph of f_1 to the right by 1 unit. This gives us $f(t) = f_1(t-1) = 3\sin(2(t-1)) - 2$.

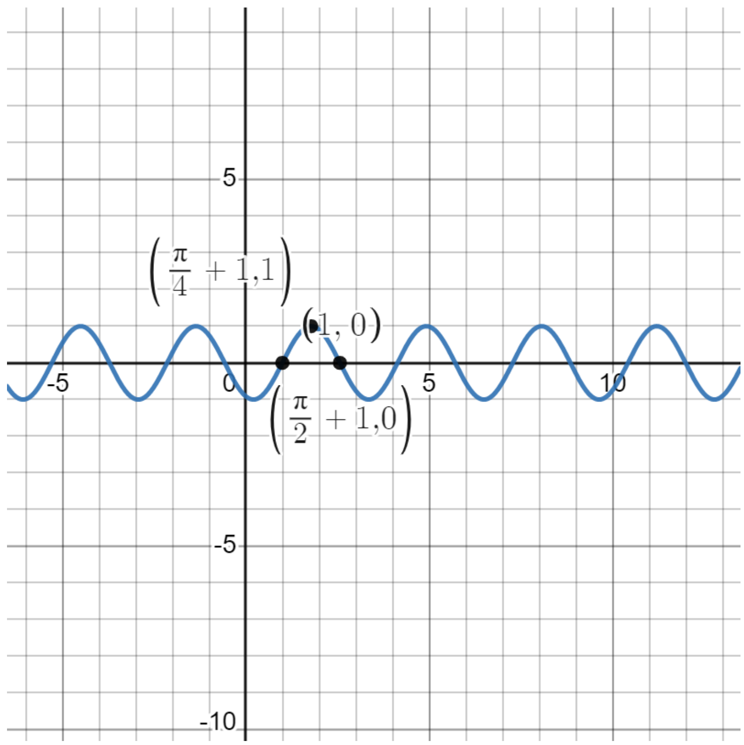
Example 20. *Graph the function f given by $f(x) = -2\sin(2(x-1)) + 3$. Keep track of the points $(0,0)$, $\left(\frac{\pi}{2}, 1\right)$, and $(\pi, 0)$ at each step. Give the midline, amplitude, and period of f.*

Explanation While this may seem like a complicated set of transformations, it's nothing that we couldn't have done in the earlier section on function transformations. The only catch here is that in order to use our order from before, we need to distribute the 2 inside the parentheses to obtain $f(x) = -2\sin(2x-2)+3$. This puts our function in the form needed to use the order.

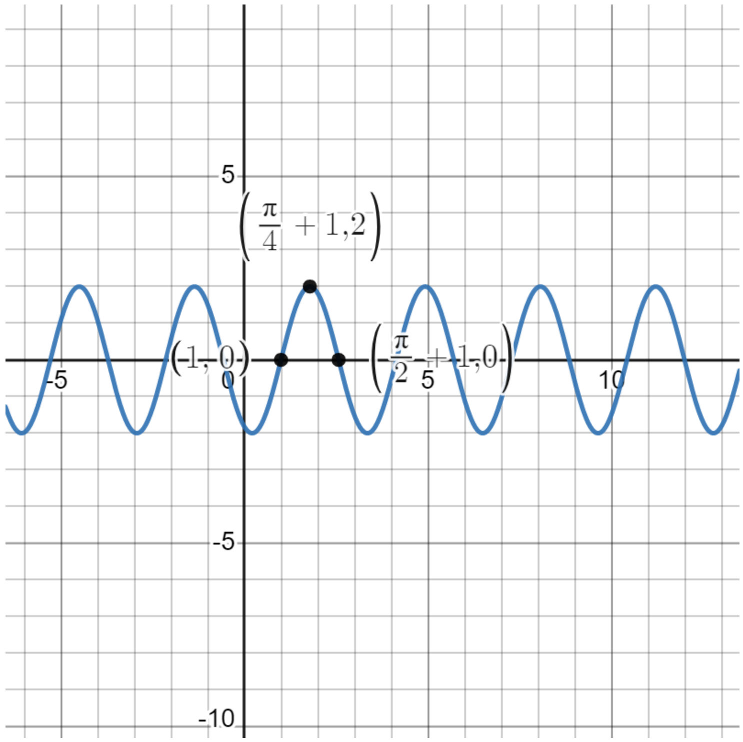
With this change, we can see that our first transformation is a shift right by 2 units, yielding $f_1(x) = \sin(x-2)$. This transforms our points into $(2,0)$, $\left(\frac{\pi}{2}+2, 1\right)$, and $(\pi+2, 0)$. We can see a graph of f_1 below.



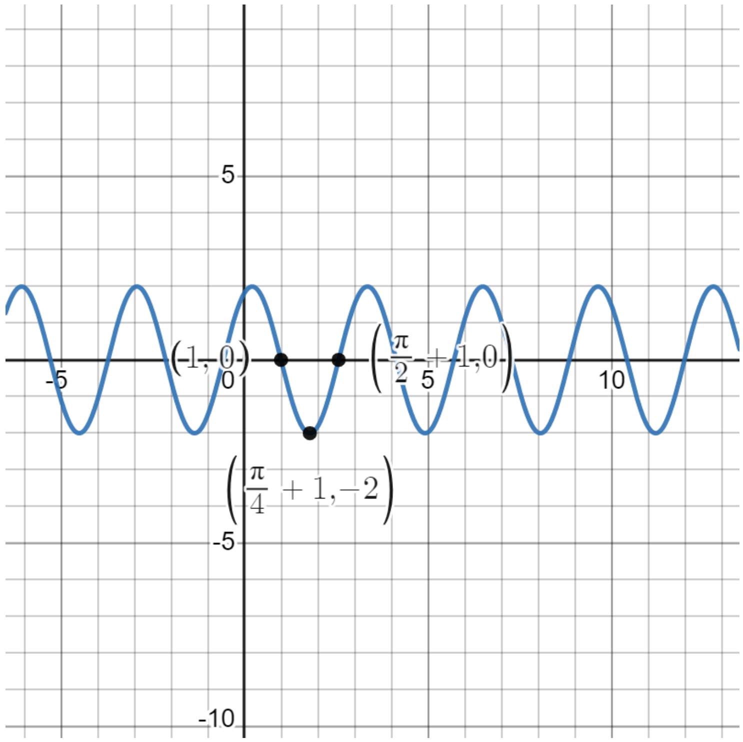
Our next transformation in the order is a horizontal compression by a factor of 2, yielding $f_2(x) = f_1(2x) = \sin(2x - 2)$. This transforms our points into $(2, 0)$, $\left(\frac{\pi}{4} + 1, 1\right)$, and $\left(\frac{\pi}{2} + 1, 0\right)$. We can see a graph of f_2 below.



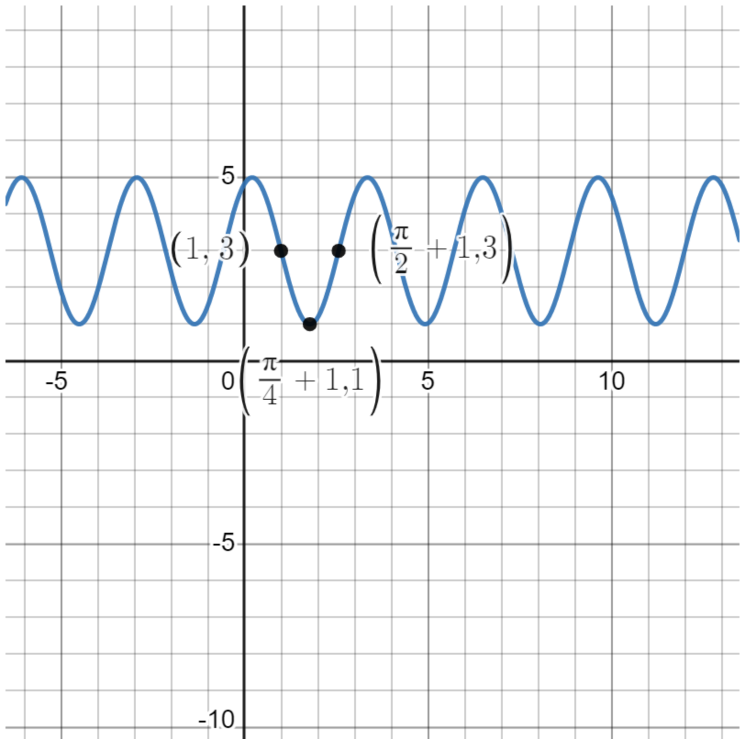
Our next transformation is a vertical stretch by a factor of 2, yielding $f_3(x) = 2f_2(x) = 2\sin(2x-2)$. This transforms our points into $(2,0)$, $\left(\frac{\pi}{4}+1,2\right)$, and $\left(\frac{\pi}{2}+1,0\right)$. We can see a graph of f_3 below.



Our next transformation is a reflection across the x-axis, yielding $f_4(x) = -f_3(x) = -2\sin(2x-2)$. This transforms our points into $(2,0)$, $\left(\frac{\pi}{4}+1,-2\right)$, and $\left(\frac{\pi}{2}+1,0\right)$. We can see a graph of f_4 below.



Our final transformation is a shift up by 3 units, yielding $f_5(x) = f_4(x) + 3 = -2\sin(2x - 2) + 3$. This transforms our points into $(2, 3)$, $\left(\frac{\pi}{4} + 1, 1\right)$, and $\left(\frac{\pi}{2} + 1, 3\right)$. We can see a graph of f_5 below.



The midline of f is 3, the amplitude is 2, and the period is π. This is all readable from the graph, but also from the formula for f, since $|a| = 2$, $b = 2$, and $k = 3$.

You may have noticed that the last example involved a reflection, a transformation we have not talked about in the context of circular functions. This is because reflections of circular functions turn out to be obtainable by shifts, so when we're given graphical information and asked to reconstruct the formula for a function, we don't need to use reflections. Reflections may show up when we are given a formula and asked to provide a graph. But we already have lots of practice producing graphs in this situation and can draw on this prior experience with transformations.

Summary

- Given real numbers a, h, and k with $a > 0$, the functions

$$f(t) = a\cos(t-h)+k \text{ and } g(t) = a\sin(t-h)+k$$

each represent a horizontal shift by h units to the right, followed by a vertical stretch by a units, followed by a vertical shift of k units, applied to the parent function ($\cos(t)$ or $\sin(t)$, respectively). The resulting circular functions have midline $y = k$, amplitude a, range

$[k-a, k+a]$, and period $P = 2\pi$. In addition, the anchor point $(h, a+k)$ lies on the graph of f and the anchor point (h, k) lies on the graph of g.

- Given a function f and a constant $b > 0$, the algebraic transformation $h(t) = f(bt)$ results in horizontal scaling of f by a factor of b. In particular, when $b > 1$, the graph of f is compressed toward the y-axis by a factor of b to create the graph of h, while when $0 < b < 1$, the graph of f is stretched away from the y-axis by a factor of b to create the graph of h.

- Given any circular periodic function for which the midline, amplitude, period, and an anchor point are known, we can find a corresponding formula for the function of the form

  $$f(t) = a\cos(b(t-h)) + k \text{ or } g(t) = a\sin(b(t-h)) + k.$$

  Each of these functions has period midline $y = k$, amplitude a, and period $P = \dfrac{2\pi}{b}$. The point $(h, a+k)$ lies on f and the point (h, k) lies on g.

9.4 Some Applications of Trigonometry

Learning Objectives

- Applications of Trigonometric Functions
  - Applications involving trigonometric functions

9.4.1 Applications of Trigonometry

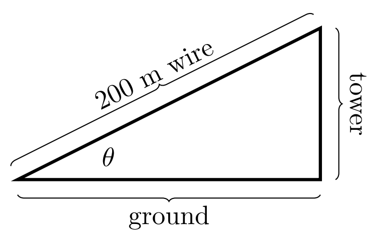
Applications of Trigonometry

In the previous sections, you have been learning about trigonometric functions in the abstract. In this section, we wish to apply them.

Example 21. *A wire 200 meters long is attached to the top of a tower. When pulled taut, it makes a* 60° *angle with the ground. How tall is the tower? How far away from the base of the tower does the wire hit the ground?*

Explanation

In application problems, we are often given data about angles measured in degrees. It is up to us to translate this into radians for our calculations. Let's call the angle we're given as θ, so $\theta = \dfrac{\pi}{3}$ radians. Let's draw a diagram of this situation.



[2in]

Based on this image, the height of the tower is the opposite the angle we know, and the distance along the ground is adjacent. That means the tower height is related to the angle by $\sin(\theta) = \dfrac{\text{tower}}{200}$ and the distance across the ground is given by $\cos(\theta) = \dfrac{\text{ground}}{200}$.

Since $\sin\left(\dfrac{\pi}{3}\right) = \dfrac{\sqrt{3}}{2}$, the tower height can be computed by:

$$\begin{aligned} \sin\left(\frac{\pi}{3}\right) &= \frac{\text{tower}}{200} \\ \text{tower} &= 200 \sin\left(\frac{\pi}{3}\right) \\ &= 200\left(\frac{\sqrt{3}}{2}\right) \\ &= 100\sqrt{3}. \end{aligned}$$

The tower has a height of $100\sqrt{3}$ m, which is approximately 173.21 m.

Acknowledgements: Stitz Zeager Open Source Mathematics (https://www.stitz-zeager.com/)

The EXACT VALUE of the height is $100\sqrt{3}$ m. Saying that this is approximately 173.21 m is provided to give us an indication of scale. Our actual answer is the exact value, not this approximation.

We'll follow a similar calculation to find the distance from the base of the tower to the wire along the ground.

$$
\begin{aligned}
\cos\left(\frac{\pi}{3}\right) &= \frac{\text{ground}}{200} \\
\text{ground} &= 200\cos\left(\frac{\pi}{3}\right) \\
&= 200\left(\frac{1}{2}\right) \\
&= 100.
\end{aligned}
$$

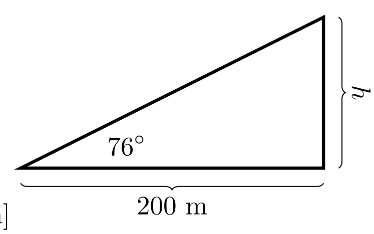
The wire hits the ground 100 m from the base of the tower.

Example 22. *A camera is set up 200 m from the base of a building, pointed at the top of the building. If the angle-of-elevation is measured as* $76°$*, find the height of the building.*

Explanation

The angle-of-elevation means the angle between the camera's line-of-sight and horizontal. Since the camera is setup 200 meters from the building, this gives us a right triangle where we know the base angle and the length of the adjacent side.

If we call the height of the building h, then we have a triangle



[2in]

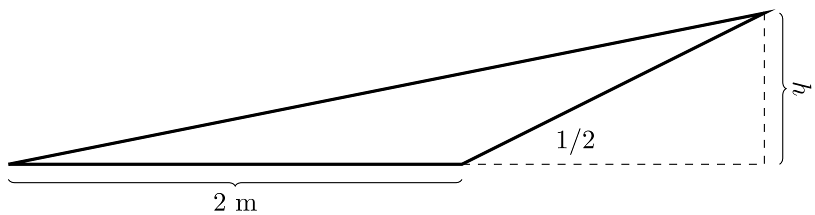
That means $\tan(76°) = \dfrac{h}{200}$, so $h = 200\tan(76°)$. The exact height is $200\tan(76°)$ meters. This is approximately 802.16 meters.

Example 23. *A 4 meter long piece of wire is going to be bent at its midpoint. The right side of the wire is bent up through an angle of* $\frac{1}{2}$*. The two ends of the wire are joined by a piece of string, creating an obtuse triangle. What is the area of the resulting triangle?*

Explanation

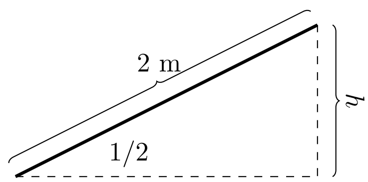
 Acknowledgements: Stitz Zeager Open Source Mathematics (https://www.stitz-zeager.com/)

That the wire is bent at its midpoint, means the resulting triangle will have two sides of length 2 m. Call the height of the triangle h.



[4in]

Let us focus on the dotted triangle on the right side



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Notice that the height of this right triangle is the same as the height of the obtuse triangle above. Since we know the hypotenuse and base angle of this right triangle, we can find the height (the opposite side) using sine.

$$\begin{aligned}\sin\left(\frac{1}{2}\right) &= \frac{h}{2} \\ 2\sin\left(\frac{1}{2}\right) &= h.\end{aligned}$$

Be careful here! That angle $\frac{1}{2}$ is not in degrees, it's in radians. (You can tell, because there is no "degrees symbol".) If you are going to approximate this value, make sure you are using radians.

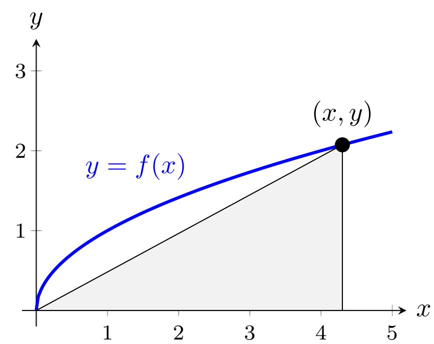
Now the area of the whole triangle is:

$$\begin{aligned}A &= \frac{1}{2}bh \\ &= \frac{1}{2}(2)\left(2\sin\left(\frac{1}{2}\right)\right) \\ &= 2\sin\left(\frac{1}{2}\right).\end{aligned}$$

The exact value of the area is $2\sin\left(\frac{1}{2}\right)\, m^2$. Using a calculator, this is approximately $0.959\, m^2$. (Remember that m^2 is the abbreviation for "square meters".)

Acknowledgements: Stitz Zeager Open Source Mathematics (https://www.stitz-zeager.com/)

Example 24. *A right triangle is constructed by taking a point (x, y) on the graph of the function $f(x) = \sqrt{x}$, drawing a line vertically downward to the x-axis, then connecting both of those points to the origin as in the picture below.*



For one particular point (x, y), the acute angle between the hypotenuse of the triangle and the positive x-axis is found to measure $\frac{\pi}{6}$ radians. Find the coordinates of the point (x, y).

Explanation

Since the hypotenuse of the right triangle runs from the origin, which has coordinates $(0, 0)$, to the point (x, y), the horizontal side of the triangle has length x and vertical side has length y. We know that the value of tangent is given by the ratio of the opposite side length divided by the adjacent side length. Using the given angle of $\frac{\pi}{6}$, this means $\tan\left(\frac{\pi}{6}\right) = \frac{1}{\sqrt{3}} = \frac{y}{x}$. That is, $x = y\sqrt{3}$

We also know that the point lies on the graph of $f(x) = \sqrt{x}$, which means $y = \sqrt{x}$.

This gives us a nonlinear system of two equations:

$$\begin{cases} x = y\sqrt{3} \\ y = \sqrt{x} \end{cases}$$

Squaring the bottom equation yields $y^2 = x$. When substituting the top equation into the bottom equation, we arrive at:.

$$\begin{aligned} y^2 &= y\sqrt{3} \\ y^2 - y\sqrt{3} &= 0 \\ y(y - \sqrt{3}) &= 0 \end{aligned}$$

so either $y = 0$ or $y = \sqrt{3}$.

The value $y = 0$ corresponds to the point $(0, 0)$ on the graph, which does not yield any angle. This is an extraneous solution, which is discarded.

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Substituting the value $y = \sqrt{3}$ into $x = y\sqrt{3}$ gives $x = (\sqrt{3})\sqrt{3} = 3$. The point is $\left(3, \sqrt{3}\right)$.

Part 10

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