Algebra of Functions

Motivating Questions

- We know that we can add, subtract, multiply, and divide numbers. What kinds of operations can we perform on functions?

Introduction

In arithmetic, we execute processes where we take two numbers to generate a new number. For example, $2 + 3 = 5$. The number 5 results from adding the numbers 2 and 3. Similarly, we can multiply two numbers to generate a new one: $2 \cdot 3 = 6$.

We can work similarly with functions. Just as we can add, subtract, multiply, and divide numbers, we can also add, subtract, multiply, and divide functions to create a new function from two or more given functions.

Algebra of Functions

In most mathematics up until calculus, the main object we study is *numbers*. We ask questions such as

- "What number(s) form solutions to the equation $x^2 - 4x - 5 = 0$?"

- "What number is the slope of the line $3x - 4y = 7$?"

- "What number is generated as output by the function $f(x) = \sqrt{x^2 + 1}$ by the input $x = -2$?"

Certainly we also study overall patterns as seen in functions and equations, but this usually occurs through an examination of numbers themselves, and we think of numbers as the main objects being acted upon.

This changes in calculus. In calculus, the fundamental objects being studied are functions themselves. A function is a much more sophisticated mathematical

Learning outcomes:
Author(s): Bobby Ramsey

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object than a number, in part because a function can be thought of in terms of its graph, which is an infinite collection of ordered pairs of the form $(x, f(x))$.

It is often helpful to look at a function's formula and observe algebraic structure. For instance, given the quadratic function

$$q(x) = -3x^2 + 5x - 7$$

we might benefit from thinking of this as the sum of three simpler functions: the constant function $c(x) = -7$, the linear function $s(x) = 5x$ that passes through the point $(0, 0)$ with slope $m = 5$, and the concave down basic quadratic function $w(x) = -3x^2$. Indeed, each of the simpler functions c, s, and w contribute to making q be the function that it is. Likewise, if we were interested in the function $p(x) = (3x^2 + 4)(9 - 2x^2)$, it might be natural to think about the two simpler functions $f(x) = 3x^2 + 4$ and $g(x) = 9 - 2x^2$ that are being multiplied to produce p.

We thus naturally arrive at the ideas of adding, subtracting, multiplying, or dividing two or more functions, and hence introduce the following definitions and notation.

Definition Let f and g be functions.

- The **sum of f and g** is the function $f + g$ defined by $(f + g)(x) = f(x) + g(x)$.

- The **difference of f and g** is the function $f - g$ defined by $(f - g)(x) = f(x) - g(x)$.

- The **product of f and g** is the function $f \cdot g$ defined by $(f \cdot g)(x) = f(x) \cdot g(x)$.

- The **quotient of f and g** is the function $\frac{f}{g}$ defined by $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$ for all x such that $g(x) \neq 0$.

We are thinking here about f and g being functions with real numbers as outputs. Performing these operations on the functions means applying the corresponding operation to the output values of the functions.

Example 1. *Consider the functions f and g defined by the table of values below.*

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x	$f(x)$
2	0
4	3
6	7
8	-2

x	$g(x)$
1	5
2	9
3	-1
4	4

(a) *Determine the value of* $(f+g)(2)$.

(b) *Determine the value of* $(f-g)(4)$.

(c) *Determine the value of* $(f \cdot g)(2)$.

(d) *Determine the value of* $\left(\dfrac{f}{g}\right)(4)$.

(e) *What can we say about the value of* $(f+g)(3)$*?*

Explanation

(a) We know that $(f+g)(2) = f(2) + g(2)$. From the tables above $f(2) = 0$ and $g(2) = 9$, so $(f+g)(2) = 0 + 9 = 9$.

(b) Since $f(4) = 3$ and $g(4) = 4$, we know $(f-g)(4) = 3 - 4 = -1$.

(c) $(f \cdot g)(2) = f(2) \cdot g(2) = 0 \cdot 9 = 0$.

(d) $\left(\dfrac{f}{g}\right)(4) = \dfrac{f(4)}{g(4)} = \dfrac{3}{4}$.

(e) The value of $(f+g)(3)$ would be given by $f(3) + g(3)$. we are given the value of $g(3)$ in the table above, but there is no listed value for $f(3)$. That means $f(3)$ is undefined, since 3 is not a valid input. $(f+g)(3)$ is undefined.

Example 2. *Consider the functions* f *and* g *defined by*

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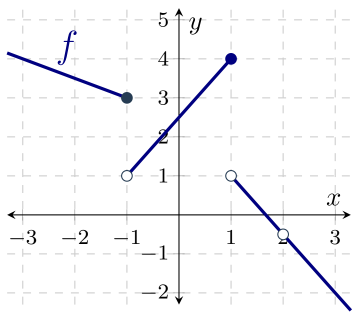

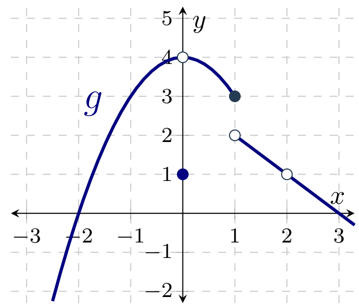

(a) *Determine the exact value of* $(f+g)(0)$.

(b) *Determine the exact value of* $(g-f)(1)$.

(c) *Determine the exact value of* $(f \cdot g)(-1)$.

(d) *Are there any values of* x *for which* $\left(\frac{f}{g}\right)(x)$ *is undefined? If not, explain why. If so, determine the values and justify your answer.*

(e) *For what values of* x *is* $(f \cdot g)(x) = 0$*? Why?*

Explanation

(a) The notation $(f+g)(0)$ means we are plugging the input 0 into both functions f and g, then *adding* the results. That is, $(f+g)(0) = f(0)+g(0)$. From the graphs above we see $f(0) = \frac{5}{2}$ and $g(0) = 1$. That means $(f+g)(0) = \frac{5}{2} + 1 = \frac{7}{2}$.

(b) The notation $(g-f)(1)$ means we are plugging the input 1 into both functions g and f, then *subtracting* the results.That is, $(g-f)(1) = g(1) - f(1)$. From the graphs above we see $f(1) = 4$ and $g(1) = 3$. That means $(g-f)(1) = 3 - 4 = -1$.

(c) The notation $(f \cdot g)(-1)$ means we are plugging the input -1 into both functions f and g, then *multiplying* the results. That is, $(f \cdot g)(-1) = f(-1) \cdot g(-1)$. From the graphs above we see $f(-1) = 3$ and $g(-1) = 3$, which tells us $(f \cdot g)(-1) = 3 \cdot 3 = 9$.

(d) For any valid value of the input x, $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$. In order for that fraction to be defined $f(x)$ has to exist, $g(x)$ has to exist, and $g(x) \neq 0$

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since division by zero is undefined. From the graphs above, f is defined for all x-values except $x = 2$, and g is defined for all x-values except $x = 2$. That tells us that $\left(\frac{f}{g}\right)(2)$ is undefined. Notice that $g(-2) = 0$ and $g(3) = 0$? That means $\left(\frac{f}{g}\right)(x)$ is undefined at $x = -2$ and $x = 3$ as well

(e) Since $(f \cdot g)(x) = f(x) \cdot g(x)$, if an x-value makes $(f \cdot g)(x) = 0$, then $f(x) \cdot g(x) = 0$. The only way a product of two real numbers can be zero is if at least one of the factors is itself zero. That means we are looking for all of the x-values satisfying either $f(x) = 0$ or $g(x) = 0$. (In other words, we're looking for the x-intercepts of these graphs.)

From the graph of g we see that $g(-2) = 0$ and $g(3) = 0$. The graph of f crosses the x-axis somewhere between the points $(1, 0)$ and $(2, 0)$, but we will have to be more careful to find the exact value we are looking for.

Notice that the graph of f looks to be a straight line if we only look at those x-values with $x > 1$. The straight line that f follows travels through the point $(1, 1)$ and $(3, -2)$. Its slope is given by $m = \frac{-2-1}{3-1} = -\frac{3}{2}$. Since the line contains the point $(1, 1)$, the point-slope form of the equation of the line can be written as $y - 1 = -\frac{3}{2}(x - 1)$. This line crosses the x-axis when its y-coordinate is zero. Solving for the corresponding x-value gives us:

$$\begin{aligned}
0 - 1 &= -\frac{3}{2}(x-1) \\
-1 &= -\frac{3}{2}(x-1) \\
-\frac{2}{3} \cdot (-1) &= -\frac{2}{3} \cdot \left(-\frac{3}{2}(x-1)\right) \\
\frac{2}{3} &= x - 1 \\
\frac{2}{3} + 1 &= x \\
\frac{5}{3} &= x.
\end{aligned}$$

That means the point $\left(\frac{5}{3}, 0\right)$ is on the graph of f, so $f\left(\frac{5}{3}\right) = 0$.

The x-values with $(f \cdot g)(x) = 0$ are $x = -2$, $x = \frac{5}{3}$, and $x = 3$.

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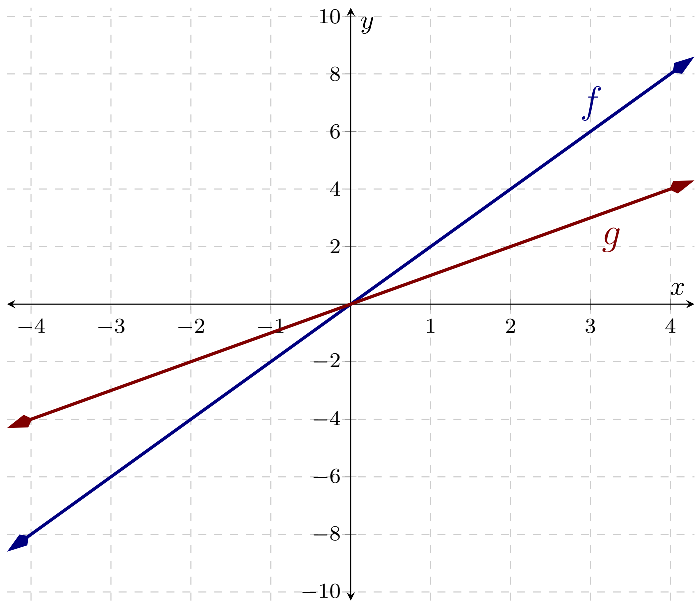
Consider the functions $f(x) = 2x$ and $g(x) = x$. These are functions whose graphs are straight lines with slopes 2 and 1 respectively.



Let's look at a few points on these graphs. Since $f(1) = 2$ and $g(1) = 1$, the sum of those output values is 3, so we'll mark the point $(1, 3)$. Similarly $f(2) = 4$ and $g(2) = 2$, with $4 + 2 = 6$ so we'll mark the point $(2, 6)$. As $f(3) = 6$ and $g(3) = 3$, we'll also mark $(3, 9)$.

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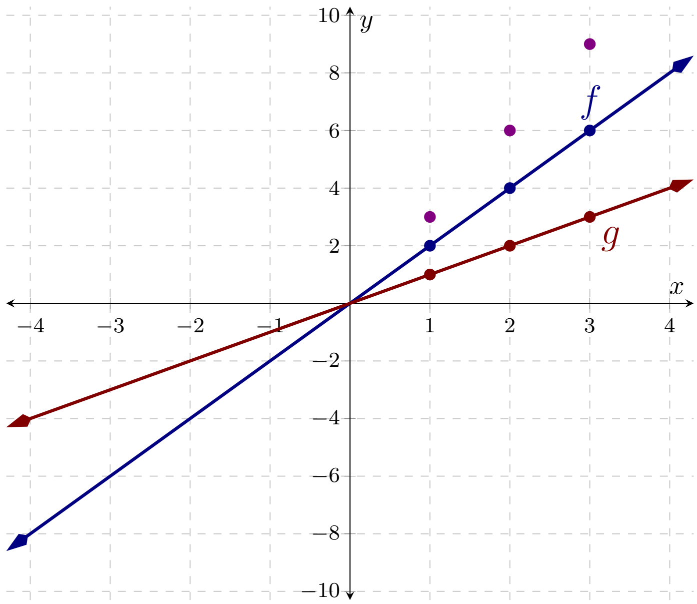
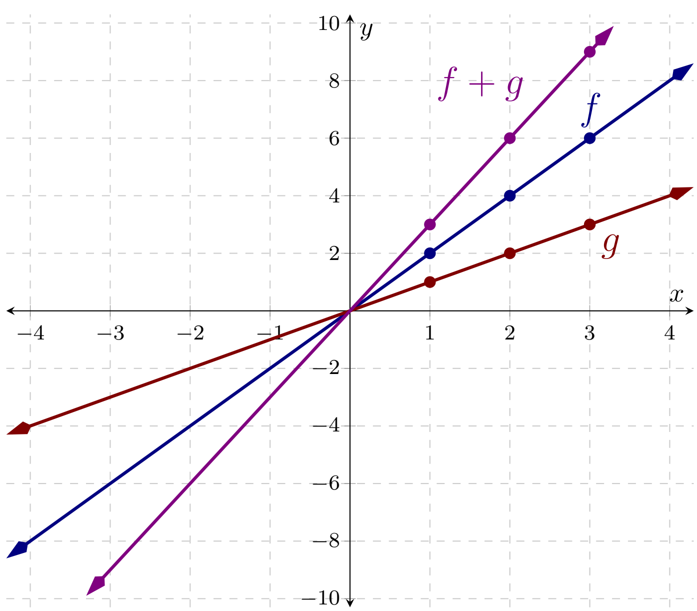


Notice that the points we've marked $(1,3)$, $(2,6)$, and $(3,9)$ are starting to form a straight line. Let's connect those dots to examine the line constructed this way.



The graph obtained this way is the graph of $f+g$. This graph is a straight line passing through the points $(0,0)$ and $(1,3)$, so the line has equation $y = 3x$.

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For a given value of x, we know $(f+g)(x) = f(x)+g(x)$. This means $(f+g)(x) = 2x+x = 3x$ by combining the like terms. Notice that this aligns with the graph we found above. This example shows that we can work with these operations through formulas for our functions as well.

Example 3. *Let f and g be the functions given by the formulas $f(x) = 2\sin(x)$ and $g(x) = 5x - 4$.*

(a) *Find the value of* $(f-g)\left(\frac{\pi}{3}\right)$.

(b) *Find a formula for* $(f+g)(x)$.

(c) *Find a formula for* $(f \cdot g)(x)$.

(d) *Find a formula for* $\left(\frac{f}{g}\right)(x)$.

Explanation

(a) $(f-g)\left(\frac{\pi}{3}\right) = f\left(\frac{\pi}{3}\right) - g\left(\frac{\pi}{3}\right)$. Sine is one of our famous functions, which has $\sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$, so $f\left(\frac{\pi}{3}\right) = 2\sin\left(\frac{\pi}{3}\right) = 2\frac{\sqrt{3}}{2} = \sqrt{3}$. $g\left(\frac{\pi}{3}\right) = 5\left(\frac{\pi}{3}\right) - 4 = \frac{5\pi}{3} - 4$.

Together this gives $(f-g)\left(\frac{\pi}{3}\right) = \sqrt{3} - \frac{5\pi}{3} + 4$

(b) $(f+g)(x) = f(x) + g(x) = 2\sin(x) + 5x - 4$.

(c) $(f \cdot g)(x) = 2\sin(x)\,(5x-4)$.

(d) $\left(\frac{f}{g}\right)(x) = \frac{2\sin(x)}{5x-4}$.

When we work in applied settings with functions that model phenomena in the world around us, it is often useful to think carefully about the units of various quantities. Analyzing units can help us both understand the algebraic structure of functions and the variables involved, as well as assist us in assigning meaning to quantities we compute. We have already seen this with the notion of average rate of change: if a function $P(t)$ measures the population in a city in year t and we compute $\text{AROC}_{[5,11]}$, then the units on $\text{AROC}_{[5,11]}$ are "people per year," and the value of $\text{AROC}_{[5,11]}$ is telling us the average rate at which the population changes in people per year on the time interval from year 5 to year 11.

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Example 4. *Say that an investor is regularly purchasing stock in a particular company. Let $N(t)$ represent the number of shares owned on day t, where $t = 0$ represents the first day on which shares were purchased. Let $S(t)$ give the value of one share of the stock on day t; note that the units on $S(t)$ are dollars per share. How is the total value, $V(t)$, of the held stock on day t determined?*

Explanation Observe that the units on $N(t)$ are "shares" and the units on $S(t)$ are "dollars per share". Thus when we compute the product

$$N(t)\,\text{shares} \cdot S(t)\,\text{dollars per share},$$

it follows that the resulting units are "dollars", which is the total value of held stock. Hence,

$$V(t) = N(t) \cdot S(t).$$

Exploration Let f be a function that measures a car's fuel economy in the following way. Given an input velocity v in miles per hour, $f(v)$ is the number of gallons of fuel that the car consumes per mile (i.e., "gallons per mile"). We know that $f(60) = 0.04$.

(a) What is the meaning of the statement "$f(60) = 0.04$" in the context of the problem? That is, what does this say about the car's fuel economy? Write a complete sentence.

(b) Consider the function $g(v) = \dfrac{1}{f(v)}$. What is the value of $g(60)$? What are the units on g? What does g measure?

(c) Consider the function $h(v) = v \cdot f(v)$. What is the value of $h(60)$? What are the units on h? What does h measure?

(d) Do $f(60)$, $g(60)$, and $h(60)$ tell us fundamentally different information, or are they all essentially saying the same thing? Explain.

(e) Suppose we also know that $f(70) = 0.045$. Find the average rate of change of f on the interval $[60, 70]$. What are the units on the average rate of change of f? What does this quantity measure? Write a complete sentence to explain.

Taking a complicated function and determining how it is constructed out of simpler ones is an important skill to develop. At the beginning of this section we split the function $q(x) = -3x^2 + 5x - 7$ into the sum/difference of three simple functions, $-3x^2$, $5x$, and 7. Let us experiment with splitting a few more complicated functions.

Example 5. (a) *Find functions h and k so that $f(x) = 4x^2 \sin(x)$ can be written as $(h \cdot k)(x)$.*

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(b) *Find functions f and g so that $h(x) = \dfrac{2x+3}{x-1}$ can be written as $\left(\dfrac{f}{g}\right)(x)$.*

(c) *Find functions r and s so that $t(x) = \dfrac{x+1}{3x-1} + xe^x$ can be written as $(r+s)(x)$.*

(d) *Find functions f, g, and h so that $k(x) = \dfrac{\sin(x)\sqrt{2x+3}}{1+\ln(x)}$ can be written as $\left(\dfrac{f \cdot g}{h}\right)(x)$.*

Explanation

Before working through these questions, we want to remind you that the answers we give are not unique. There are many different, equally valid choices for the simpler functions requested. For the first question, we will mention multiple possibilities. We leave it to you to find other answers for the remaining questions.

(a) Notice that if $h(x) = 4x^2$ and $k(x) = \sin(x)$, then $(h \cdot k)(x) = 4x^2 \sin(x)$, so this choice of h and k is one valid answer. What if we had chosen $h(x) = 4x$ and $k(x) = x\sin(x)$ instead? Then $(h \cdot k)(x) = (4x)(x\sin(x)) = 4x^2 \sin(x)$, so this choice would also be a valid answer. Another possibility would have been to choose $h(x) = 4$ and $k(x) = x^2 \sin(x)$.

(b) If we set $f(x) = 2x+3$ and $g(x) = x-1$, then $\left(\dfrac{f}{g}\right)(x) = \dfrac{2x+3}{x-1}$.

(c) Since $t(x) = \dfrac{x+1}{3x-1} + xe^x$ is written as a sum of two terms, take $r(x) = \dfrac{x+1}{3x-1}$ to be the first term and $s(x) = xe^x$ to be the seond term. Then $(r+s)(x) = t(x)$.

(d) We are trying to identify $\dfrac{\sin(x)\sqrt{2x+3}}{1+\ln(x)}$ as a fraction with the numerator a product. The denominator is $1+\ln(x)$ and the numerator is a product of $\sin(x)$ and $\sqrt{2x+3}$. We will choose $f(x) = \sin(x)$, $g(x) = \sqrt{2x+3}$, and $h(x) = 1+\ln(x)$.

Exploration

(a) Find functions f and g so that $h(x) = \dfrac{5e^x}{1+\sin(x)}$ can be written as $(f \cdot g)(x)$.

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(b) Find functions h and k so that $f(x) = 3x^2 - \sqrt{x+1}$ can be written as $(h+k)(x)$. Find two other choices for h and k.

(c) If f and g are functions, we know that $f+g$ is the function given by $(f+g)(x) = f(x) + g(x)$. What function do you think the notation $f+3g$ means? Find functions f and g so that $h(x) = 4x^2 - 5\sqrt{x} + 7 \cdot 2^x$ can be written as $(f+3g)(x)$.

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