Precalculus with Review 2: Unit 8

January 17, 2023

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Origins of Trig

8.1 Right Triangle Trig

Learning Objectives

- Sine, Cosine, and Tangent
  - Similar triangles, and trig functions as ratios of triangle sides
  - Famous right triangles and deducing famous values
  - Find missing side
  - $\sin^2\theta + \cos^2\theta = 1$
- Secant, Cosecant, and Cotangent
- All From One, One From All
  - How to find values of all trig functions for an acute angle, given only one of such values.

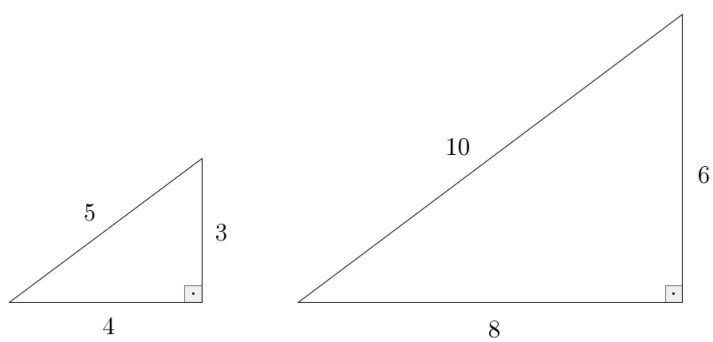
8.1.1 Sine, Cosine, and Tangent

Motivating Questions

- How to study, in a systematic way, ratios between the sides of a right triangle?
- What are the values of sine, cosine, and tangent, for the most frequent angles of $30°$, $45°$ and $60°$? And why?

Introduction

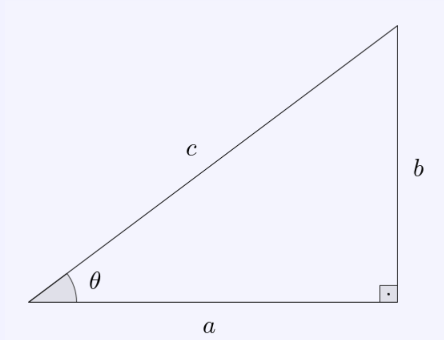
Recall that two triangles are called **similar** if one of them can be obtained by rescaling and moving around the other. Here's an example:



The dotted square symbol is a shorthand for "$90°$ degrees". Triangles which have a $90°$ angle are called **right triangles**, and will be the main focus of our discussion. What do similar triangles actually have in common? Certainly angles, but not necessarily the lengths of the sides. However, the **ratios** between any two sides of a triangle will remain the same, no matter how the triangle gets rescaled. Such ratios ultimately give us so much information about the given right triangle that they deserve special names: sine, cosine, and tangent.

Definitions and examples

Definition (right triangle trig): Consider the following right triangle, with one angle θ (this is the lowercase greek letter "theta") indicated, and sides labeled a, b and c.



Then:

- The side labeled with a is called the **adjacent** side to θ.
- The side labeled with b is called the **opposite** side to θ.
- The side labeled with c is called the **hypotenuse** of the triangle.

With this in place, we define the **sine**, **cosine**, and **tangent** of θ, by

$$\sin\theta = \frac{b}{c}\left(= \frac{\text{opp.}}{\text{hyp.}}\right), \quad \cos\theta = \frac{a}{c}\left(= \frac{\text{adj.}}{\text{hyp.}}\right), \text{ and } \tan\theta = \frac{b}{a}\left(= \frac{\text{opp.}}{\text{adj.}}\right).$$

Remark The hypotenuse of a right triangle is always the side opposite to the right angle. Also note that $\tan\theta = \sin\theta/\cos\theta$. You might have seen the mnemonic "SOH CAH TOA" before: for example, "SOH" means "sine equals opposite over hypotenuse", and so on.

Example 1. *For each of the following triangles with given angle θ, identify the adjacent (adj.), opposite (opp.) and hypotenuse (hyp.), and compute $\sin\theta$, $\cos\theta$ and $\tan\theta$.*

a. **Explanation** We have opp. = 12, adj. = 5 and hyp. = 13. This means that

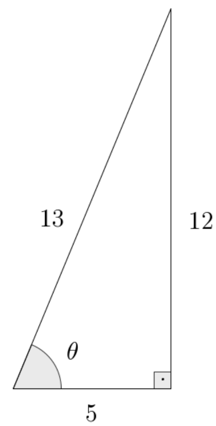
$$\sin\theta = \frac{12}{13}, \quad \cos\theta = \frac{5}{13}, \quad \text{and} \quad \tan\theta = \frac{12}{5}.$$

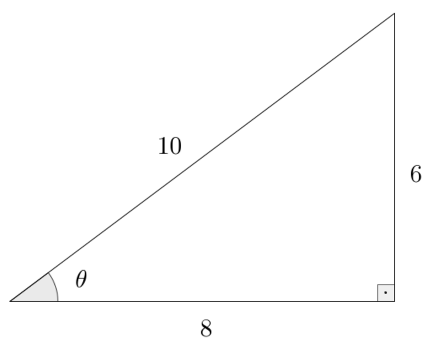
b. **Explanation** We have opp. = 8, adj. = 6 and hyp. = 10. This means that

$$\sin\theta = \frac{8}{10} = \frac{4}{5}, \quad \cos\theta = \frac{6}{10} = \frac{3}{5}, \quad \text{and} \quad \tan\theta = \frac{8}{10} = \frac{4}{3}.$$

c. **Explanation** We have opp. = 24, adj. = 18 and hyp. = 30. This means that

$$\sin\theta = \frac{24}{30} = \frac{4}{5}, \quad \cos\theta = \frac{18}{30} = \frac{3}{5}, \quad \text{and} \quad \tan\theta = \frac{24}{18} = \frac{4}{3}.$$





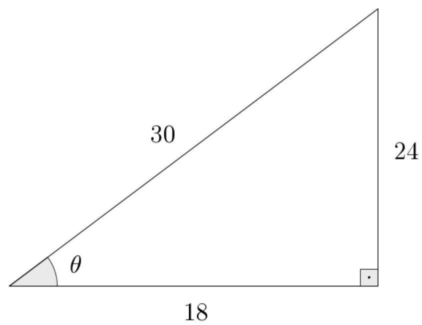
Note that the values were the same values as in the previous item. This was expected, as the triangle there is similar to the triangle given here (the scaling factor is 3).

Remark Note that in all of the above examples, the values of $\sin\theta$ and $\cos\theta$ were always less than 1. This is always true, and a general consequence of the fact that the hypotenuse is always bigger than either of the other two sides.

Often, one has information about the angles, but not about all the sides. Knowing $\sin\theta$, $\cos\theta$ and $\tan\theta$ helps us find out missing sides of a given right triangle. For that, the following fact is extremely important:

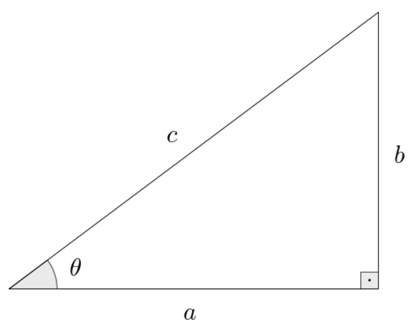
Theorem (Fundamental Identity): For any given angle θ, we have that

$$\sin^2\theta + \cos^2\theta = 1.$$



Here, $\sin^2\theta$ means $(\sin\theta)^2$, and similarly for $\cos^2\theta$.

Why is this true? Consider again a right triangle like below:



Then we know that $\sin\theta = b/c$ and $\cos\theta = a/c$. But the Pythagorean theorem also says that $a^2 + b^2 = c^2$. Putting all of this together, we have that

$$\sin^2\theta + \cos^2\theta = \left(\frac{b}{c}\right)^2 + \left(\frac{a}{c}\right)^2 = \frac{b^2}{c^2} + \frac{a^2}{c^2} = \frac{a^2+b^2}{c^2} = \frac{c^2}{c^2} = 1,$$

as required.

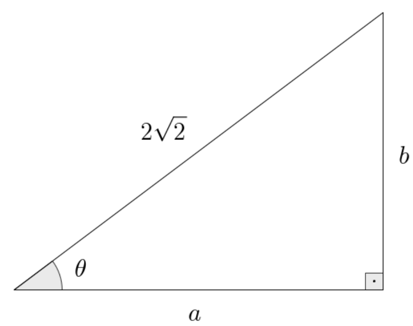
Let's see how to apply this.

Example 2. *For each of the following triangles, given the value of a trigonometric function at the indicated angle θ, find the lengths of the missing sides.*

a. *Given:* $\sin\theta = \sqrt{2}/6$ *on*

Explanation From the given information, we know that

$$\frac{\sqrt{2}}{6} = \sin\theta = \frac{b}{2\sqrt{2}} \implies b = \frac{\sqrt{2}\times(2\sqrt{2})}{6} = \frac{2}{3}.$$



Now we use the Pythagorean theorem: the relation $a^2 + (2/3)^2 = (2\sqrt{2})^2$ gives us that

$$a^2 + \frac{4}{9} = 8 \implies a^2 = 8 - \frac{4}{9} = \frac{68}{9} \implies a = \frac{2\sqrt{17}}{3}.$$

Alternatively, to find the value of a, we can also use the fundamental identity $\sin^2\theta + \cos^2\theta = 1$ to find $\cos\theta$ first — which then yields a. Here's how this goes:

$$\left(\frac{\sqrt{2}}{6}\right)^2 + \cos^2\theta = 1 \implies \cos^2\theta = 1 - \frac{2}{36} \implies \cos^2\theta = \frac{34}{36},$$

and so $\cos\theta = \sqrt{34}/6$. Thus

$$\frac{\sqrt{34}}{6} = \cos\theta = \frac{a}{2\sqrt{2}} \implies a = \frac{2\sqrt{2} \times \sqrt{34}}{6} = \frac{2\sqrt{17}}{3},$$

as it should be. This is not something particular to this example: usually there is more than one strategy to solve this sort of problem. Which one is the best? You'll be the judge.

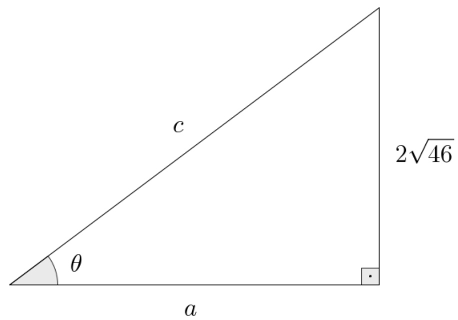
b. *Given:* $\cos\theta = \sqrt{3}/7$ *on*

Explanation Since this time we were given $\cos\theta$, but also the opposite side to θ, which does not appear on the expression for $\cos\theta$, we must rely more on the Pythagorean theorem instead. In any case, we know that

$$\frac{\sqrt{3}}{7} = \cos\theta = \frac{a}{c} \implies c = \frac{7a}{\sqrt{3}}.$$

Now, the Pythagorean relation reads $a^2 + (2\sqrt{46})^2 = (7a/\sqrt{3})^2$, and so:

$$a^2 + 184 = \frac{49a^2}{3} \implies 184 = \frac{49a^2}{3} - a^2$$



Continuing to manipulate this, we see that

$$184 = \frac{46a^2}{3} \implies a^2 = \frac{184 \times 3}{46} \implies a^2 = 12 \implies a = 2\sqrt{3}.$$

It remains to find the value of c. So we go back to the beginning and compute

$$c = \frac{7a}{\sqrt{3}} \implies c = \frac{7(2\sqrt{3})}{\sqrt{3}} \implies c = 14.$$

Values of trig functions for standard angles

We know that the sum of the inner angles of a triangle is always 180°. For right triangles, one of the angles is 90°, which means that the sum of the remaining two angles must also be 90°. Frequently we encounter triangles whose angles are 30°, 60° and 90°, and also triangles whose angles are 45°, 45° and 90°.

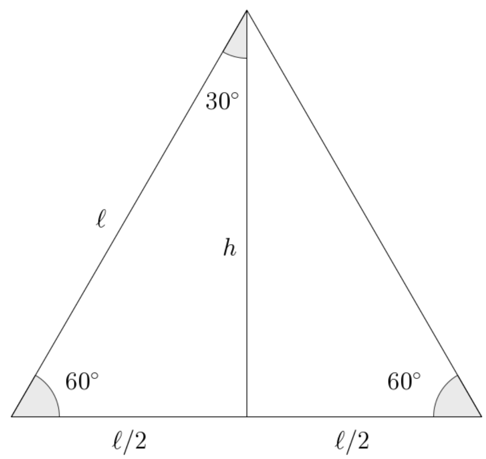
[figure]

These triangles have a special type of symmetry, which we'll exploit to find the values of sine, cosine, and tangent, for 30°, 45° and 60°. Finding the values of these trig functions for arbitrary angles, by hand, is a very difficult task. We will see later some trigonometric identities that may help us find such values for other angles but, in general, using a calculator (paying close attention to whether it is set to right "units") is the way to go.

For 30° and 60° Consider an equilateral triangle of side length ℓ. Equilateral means that all the sides have the same length. This implies that all the inner angles must be equal and, since they must add up to 180°, each of them equals 60°. But also draw a height h:

By the Pythagorean theorem, we know that

$$\ell^2 = \left(\frac{\ell}{2}\right)^2 + h^2,$$



and so we may compute:

$$\ell^2 = \frac{\ell^2}{4} + h^2 \implies \frac{3\ell^2}{4} = h^2 \implies h = \frac{\ell\sqrt{3}}{2}.$$

Now, relative to the 60° angle, we recognize

$$\text{opp.} = h = \frac{\ell\sqrt{3}}{2}, \quad \text{adj.} = \frac{\ell}{2}, \quad \text{and} \quad \text{hyp.} = \ell.$$

This means that

$$\sin(60^\circ) = \frac{h}{\ell} = \frac{\left(\frac{\ell\sqrt{3}}{2}\right)}{\ell} = \frac{\sqrt{3}}{2},$$

as well as

$$\cos(60^\circ) = \frac{\ell/2}{\ell} = \frac{1}{2} \quad \text{and} \quad \tan(60^\circ) = \frac{\sin(60^\circ)}{\cos(60^\circ)} = \frac{\sqrt{3}/2}{1/2} = \sqrt{3}.$$

To find the values of $\sin(30^\circ)$, $\cos(30^\circ)$, and $\tan(30^\circ)$, we can use the same triangle, noting that the opposite side to 30° is the adjacent side to 60°, and that the adjacent side to 30° is the opposite side to 60°. Since the hypotenuse is always the side opposite to the right angle, we conclude that

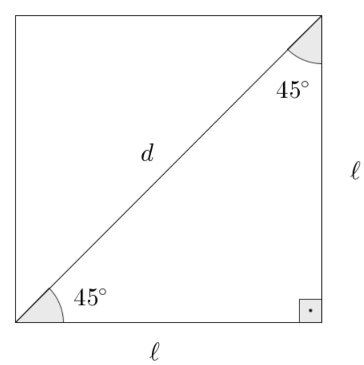
$$\sin(30^\circ) = \cos(60^\circ) = \frac{1}{2}, \quad \cos(30^\circ) = \sin(60^\circ) = \frac{\sqrt{3}}{2}$$

and, finally, that

$$\tan(30^\circ) = \frac{\sin(30^\circ)}{\cos(30^\circ)} = \frac{1/2}{\sqrt{3}/2} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}.$$

Remark This is a general phenomenon: two acute angles are called **complementary** if they add up to 90°. In other words, the complementary angle to θ is always $90^\circ - \theta$, and $\sin\theta = \cos(90^\circ - \theta)$, as well as $\cos\theta = \sin(90^\circ - \theta)$. In particular, this justifies the name "cosine": it is the sine of the complement. We will discuss "coterminal angles" and "cofunctions" in more generality later.

For 45° Consider a square of side length ℓ, and draw a diagonal d.



By the Pythagorean theorem, $d^2 = \ell^2 + \ell^2 = 2\ell^2$ implies that $d = \ell\sqrt{2}$. Relative to either of the 45° angles, we have

$$\text{opp.} = \ell, \quad \text{adj.} = \ell, \quad \text{and} \quad \text{hyp.} = d = \ell\sqrt{2}.$$

Hence

$$\sin(45^\circ) = \cos(45^\circ) = \frac{\ell}{\ell\sqrt{2}} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} \quad \text{and} \quad \tan(45^\circ) = \frac{\sin(45^\circ)}{\cos(45^\circ)} = 1.$$

Remark It is convenient to write $\sqrt{2}/2$ instead of $1/\sqrt{2}$ (similarly for $\sqrt{3}/3$ versus $1/\sqrt{3}$), even though the latter is mathematically acceptable, because it makes it easier to estimate. Namely, knowing that $\sqrt{2} \approx 1.414$, we know that $\sqrt{2}/2 \approx 0.707$, but when looking at $1/\sqrt{2}$, what does it mean to divide 1 by 1.414? This is the general reason why rationalizing fractions is useful.

Standard values

We can summarize what we have discovered here in a table. Besides our standard angles of 30°, 45°, and 60°, we can also consider 0° and 90° as extreme cases. Let's do a quick thought experiment to understand this: if a right triangle had an angle of $0°$, this triangle would in fact collapse to a line segment, and we would have opp. $= 0$, while hyp. $=$ adj., suggesting we set $\sin(0°) = 0$ and $\cos(0°) = 1$. Since $0°$ and $90°$ are complementary, we're forced to set $\sin(90°) = 1$ and $\cos(90°) = 0$. But while

$$\tan(0°) = \frac{\sin(0°)}{\cos(0°)} = \frac{0}{1} = 0,$$

computing $\tan(90°)$ does not make sense, as we would have a division by $\cos(90°) = 0$. We say that $\tan(90°)$ is **undefined**, or that it **does not exist** ("DNE" for short, as usual). So, we have:

	$0°$	$30°$	$45°$	$60°$	$90°$
$\sin\theta$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
$\tan\theta$	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	DNE

Those values should be committed to heart, but it's easier than what it seems. Here's how you can think about it:

- No need to memorize values for tangent: if you know $\sin\theta$ and $\cos\theta$, you can just compute $\tan\theta = \sin\theta / \cos\theta$.
- No need to memorize the values for cosine: recall that the cosine of an angle is the sine of the complement. So if you know values for sine, you're in business.
- How to memorize values for sine? The one thing you should remember here is that the values 0, $1/2$, $\sqrt{2}/2$, $\sqrt{3}/2$ and 1 will appear. What is their order? Simple: write them in increasing order, just like the angles from $0°$ to $90°$. So

  $$\sin(0°) = 0, \ \sin(30°) = \frac{1}{2}, \ \sin(45°) = \frac{\sqrt{2}}{2}, \ \sin(60°) = \frac{\sqrt{3}}{2}, \ \sin(90°) = 1.$$

Summary

- We have defined sine, cosine, and tangent, as ratios between sides of a right triangle. For each angle θ, the fundamental identity $\sin^2\theta + \cos^2\theta = 1$ holds. It can be used together with the Pythagorean Theorem to get information about all sides of a given triangle, when some of them might be missing, provided you have some information about the angles.

- We have established the standard values of sine, cosine, and tangent, for the most frequent angles of $30°$, $45°$, and $60°$. Those values have been organized in a table. They are so frequent that knowing the values there by heart is useful, but exaggerated efforts into memorizing the table should not be wasted — understanding how the values are deduced pays off more in the long run.

8.1.2 Secant, Cosecant and Cotangent

Motivating Questions

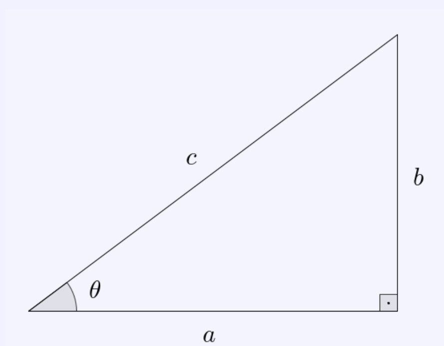
- How to use the reciprocal ratios of $\sin\theta$, $\cos\theta$, and $\tan\theta$, to also obtain information about a given right triangle?
- What are the values of such reciprocal rations for the standard angles of $30°$, $45°$ and $60°$?

Introduction

Briefly speaking, we have met three fundamental ratios between sides of a right triangle: sine, cosine, and tangent. But their reciprocals are also relevant ratios between the sides of the given triangle. Now, while such reciprocals ratios turn out to carry the same information as sine, cosine, and tangent, it is useful to know how to manipulate them as well. Later, when we study trigonometric functions as actual functions of a real parameter, discussing their graphs, symmetries, etc., more differences will become apparent.

Definitions and examples

Definition (right triangle trig – bis): Consider the following right triangle, with one angle θ indicated, and sides labeled a, b and c.



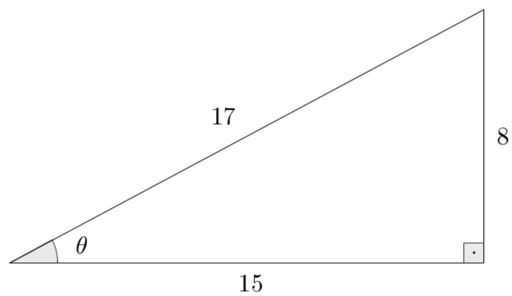
We define the **secant**, **cosecant**, and **cotangent** of θ, by

- $\sec\theta = \dfrac{c}{b} = \dfrac{1}{\cos\theta}\left(= \dfrac{\text{hyp.}}{\text{adj.}}\right)$;
- $\csc\theta = \dfrac{c}{a} = \dfrac{1}{\sin\theta}\left(= \dfrac{\text{hyp.}}{\text{opp.}}\right)$, and;

- $\cot\theta = \dfrac{a}{b} = \dfrac{1}{\tan\theta} \left(= \dfrac{\text{adj.}}{\text{opp.}} \right)$.

Note that since for acute angles we always have $\sin\theta$ and $\cos\theta$ between 0 and 1, the reciprocals $\csc\theta$ and $\sec\theta$ will always be bigger than 1.

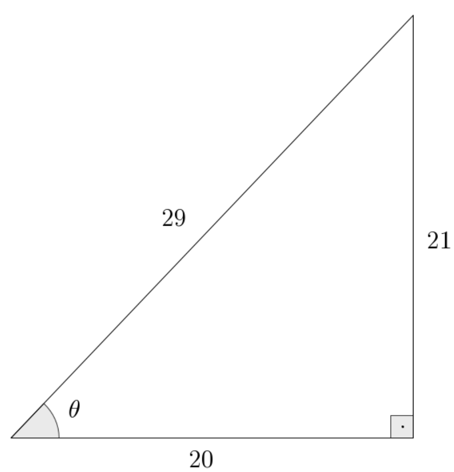
Example 3. *For each of the following triangles with a given angle θ, identify the adjacent (adj.), opposite (opp.) and hypotenuse (hyp.), and compute* $\sec\theta$*,* $\csc\theta$*, and* $\cot\theta$*.*



a. **Explanation** We have opp. = 8, adj. = 5 and hyp. = 17. This means that

$$\sec\theta = \frac{17}{15}, \quad \csc\theta = \frac{17}{8}, \quad \text{and} \quad \cot\theta = \frac{15}{8}.$$

Of course, you can find $\cos\theta$, $\sin\theta$, and $\tan\theta$ first, and then just flip all the fractions.



b. **Explanation** This time, we have opp. = 21, adj. = 20 and hyp. = 29.

This means that

$$\sec\theta = \frac{29}{20}, \quad \csc\theta = \frac{29}{21}, \quad \text{and} \quad \cot\theta = \frac{20}{21}.$$

Next, we had the fundamental identity $\sin^2\theta + \cos^2\theta = 1$. It turns out that with this, we may obtain two extra useful identities.

Theorem (Fundamental Identities – bis): For any given angle θ, we have that

$$1 + \cot^2\theta = \csc^2\theta \quad \text{and} \quad \tan^2\theta + 1 = \sec^2\theta,$$

where $\cot^2\theta$ means $(\cot\theta)^2$, and similarly for all other functions.

You should *not* think of those two extra identities as something more to be memorized. The only identity worth the trouble is $\sin^2\theta + \cos^2\theta = 1$. The following strategy is something you can quickly reproduce on a scrap paper if you need to recall these formulas, once you have understood the idea once:

- Divide both sides of $\sin^2\theta + \cos^2\theta = 1$ by $\sin^2\theta$:

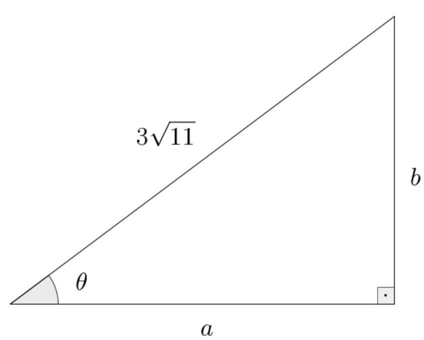
$$\frac{\sin^2\theta + \cos^2\theta}{\sin^2\theta} = \frac{1}{\sin^2\theta} \implies 1 + \frac{\cos^2\theta}{\sin^2\theta} = \csc^2\theta \implies 1 + \cot^2\theta = \csc^2\theta.$$

- Divide both sides of $\sin^2\theta + \cos^2\theta = 1$ by $\cos^2\theta$:

$$\frac{\sin^2\theta + \cos^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta} \implies \frac{\sin^2\theta}{\cos^2\theta} + 1 = \sec^2\theta \implies \tan^2\theta + 1 = \sec^2\theta.$$

Example 4. *For each of the given triangles, given the value of a trigonometric function at the indicated angle θ, find the lengths of the missing sides.*

a. *Given:* $\csc\theta = \sqrt{11}/2$ *on*



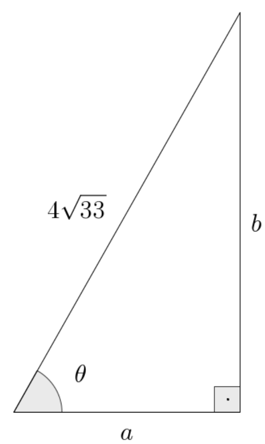
Explanation We start with

$$\frac{3\sqrt{11}}{a} = \csc\theta = \frac{\sqrt{11}}{2} \implies 6\sqrt{11} = a\sqrt{11} \implies a = 6.$$

It remains to find the value of b. This can be done with the Pythagorean Theorem, as follows: $a^2 + b^2 = c^2$ becomes

$$36 + b^2 = (3\sqrt{11})^2 \implies 36 + b^2 = 99 \implies b^2 = 63 \implies b = 3\sqrt{7}.$$

b. Given: $\cot\theta = 2\sqrt{2}/5$ *on*



Explanation Let's start again using the trigonometric function we were given:

$$\frac{a}{b} = \cot\theta = \frac{2\sqrt{2}}{5} \implies a = \frac{2b\sqrt{2}}{5}.$$

We cannot conclude anything else about a and b just from this, so we must resort to the Pythagorean Theorem again. The relation $a^2 + b^2 = c^2$ gives us that

$$\left(\frac{2b\sqrt{2}}{5}\right)^2 + b^2 = (4\sqrt{33})^2 \implies \frac{8b^2}{25} + b^2 = 528 \implies \frac{33b^2}{25} = 528.$$

Simplifying this, we have that

$$b^2 = 25 \times \frac{528}{33} = 25 \times 16 \implies b = 5 \times 4 \implies b = 20.$$

Now, we may go back and find a:

$$a = \frac{2b\sqrt{2}}{5} = \frac{40\sqrt{2}}{5} \implies b = 8\sqrt{2}.$$

Values of trig functions for standard angles – bis

Previously, we have obtained the following table of standard values for sine, cosine, and tangent:

	$0°$	$30°$	$45°$	$60°$	$90°$
$\sin\theta$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
$\tan\theta$	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	DNE

By simply inverting all of those values, we naturally obtain a similar table with standard values for secant, cosecant and cotangent. Of course, this is *not* another table you have to memorize, but we'll list it here for completeness:

	$0°$	$30°$	$45°$	$60°$	$90°$
$\csc\theta$	DNE	2	$\sqrt{2}$	$\frac{2\sqrt{3}}{3}$	1
$\sec\theta$	1	$\frac{2\sqrt{3}}{3}$	$\sqrt{2}$	2	DNE
$\cot\theta$	DNE	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$	0

Note the undefined "extreme" values: $\csc(0°)$ is undefined, because that would be $1/\sin(0°)$, but $\sin(0°) = 0$ and we cannot divide by zero. Similarly for the other ones.

Summary

- We have defined secant, cosecant, and cotangent, as the reciprocal ratios of cosine, sine, and tangent. For each angle θ, we have the associated fundamental identities $1 + \tan^2\theta = \sec^2\theta$ and $\cot^2\theta + 1 = \csc^2\theta$, which can be easily deduced from the good old $\sin^2\theta + \cos^2\theta = 1$. Again, such identities can be used together with the Pythagorean Theorem to obtain information about sides of a right triangle.

- We have summarized (again in a table) the standard values of secant, cosecant, and cotangent, for the most frequent angles of $30°$, $45°$, and $60°$.

8.1.3 All From One, One From All

Motivating Questions

- Do all the trigonometric functions we have seen so far carry the same information?
- How to find all trigonometric functions, given a single one of them?

Introduction

We have encountered six trigonometric functions of an acute angle θ so far: $\sin\theta$, $\cos\theta$, $\tan\theta$, $\sec\theta$, $\csc\theta$, and $\cot\theta$. They all help up get information about right triangles having θ as one of the inner angles. But here is the thing: at this stage, they all carry the same information. All of these quantities are positive real numbers, and we have not only the Pythagorean Theorem, but also the fundamental relations

$$\sin^2\theta + \cos^2\theta = 1$$
$$1 + \cot^2\theta = \csc^2\theta$$
$$\tan^2\theta + 1 = \sec^2\theta$$

Recall that while the first one is the most important one, the second two are immediate consequences of the first, by dividing it by $\sin^2\theta$ and $\cos^2\theta$, respectively. With all of this in place, once you have one of the six trigonometric values at θ, you can in fact find all of them. We'll explore this in this section, with several examples.

How to find all trigonometric functions, given one of them?

There are a few main facts one should keep in mind here.

- You know $\sin\theta$ if and only if you know $\csc\theta$.
- You know $\cos\theta$ if and only if you know $\sec\theta$.
- You know $\tan\theta$ if and only if you know $\cot\theta$.
- If you know $\sin\theta$ and $\cos\theta$, you know $\tan\theta$.

And there are two strategies: using just the trigonometric identities and proceeding algebraically (let's call this "strategy 1"), or drawing a suitable right triangle and thinking of opp., adj. and hyp. (let's call this "strategy 2"). We'll illustrate both of them with several examples, but in the end of the day, you may choose whichever strategy you'd like (unless specifically instructed otherwise).

Example 5. *Let θ be an acute angle. In all of the following problems, given the value of a certain trigonometric function at the value θ, find the remaining five.*

a. *Given:* $\sin\theta = 1/4$.

Explanation

- **Strategy 1:** Let's find $\cos\theta$ first, using the first fundamental identity $\sin^2\theta + \cos^2\theta = 1$. We have

$$\left(\frac{1}{4}\right)^2 + \cos^2\theta = 1 \implies \frac{1}{16} + \cos^2\theta = 1 \implies \cos^2\theta = 1 - \frac{1}{16},$$

so

$$\cos^2\theta = \frac{15}{16} \implies \cos\theta = \frac{\sqrt{15}}{4}.$$

With this, we have that

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{1/4}{\sqrt{15}/4} = \frac{1}{\sqrt{15}} = \frac{\sqrt{15}}{15}.$$

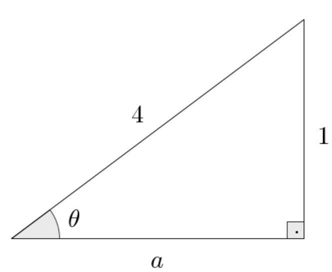
Flipping the fractions

$$\sin\theta = \frac{1}{4}, \quad \cos\theta = \frac{\sqrt{15}}{4} \quad \text{and} \quad \tan\theta = \frac{\sqrt{15}}{15},$$

respectively, we obtain

$$\csc\theta = 4, \quad \sec\theta = \frac{4}{\sqrt{15}} = \frac{4\sqrt{15}}{15}, \quad \text{and} \quad \cot\theta = \sqrt{15}.$$

- **Strategy 2:** We start drawing the following triangle: So, we find the



missing side a, using the Pythagorean relation $a^2 + b^2 = c^2$, which reads and gives:

$$a^2 + 1^2 = 4^2 \implies a^2 + 1 = 16 \implies a^2 = 15,$$

so that $a = \sqrt{15}$. Now we have all the sides, so finding all the ratios is immediate:

$$\sin\theta = \frac{1}{4}, \quad \cos\theta = \frac{\sqrt{15}}{4}, \quad \text{and} \quad \tan\theta = \frac{1}{\sqrt{15}} = \frac{\sqrt{15}}{15}.$$

Flipping all the fractions, we obtain

$$\csc\theta = 4, \quad \sec\theta = \frac{4\sqrt{15}}{15}, \quad \text{and} \quad \cot\theta = \sqrt{15}.$$

b. Given: $\cos\theta = 2/3$.

Explanation

- **Strategy 1:** Let's find $\sin\theta$ first, using the first fundamental identity $\sin^2\theta + \cos^2\theta = 1$. We have

$$\sin^2\theta + \left(\frac{2}{3}\right)^2 = 1 \implies \sin^2\theta + \frac{4}{9} = 1 \implies \sin^2\theta = 1 - \frac{4}{9},$$

so

$$\sin^2\theta = \frac{5}{9} \implies \sin\theta = \frac{\sqrt{5}}{3}.$$

With this, we have that

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\sqrt{5}/3}{2/3} = \frac{\sqrt{5}}{2}.$$

Flipping the fractions

$$\sin\theta = \frac{\sqrt{5}}{3}, \quad \cos\theta = \frac{2}{3} \quad \text{and} \quad \tan\theta = \frac{\sqrt{5}}{2},$$

respectively, we obtain

$$\csc\theta = \frac{3}{\sqrt{5}} = \frac{3\sqrt{5}}{5} \quad \sec\theta = \frac{3}{2}, \quad \text{and} \quad \cot\theta = \frac{2}{\sqrt{5}} = \frac{2\sqrt{5}}{5}.$$

- **Strategy 2:** This time, here's the triangle we'll use: We'll use the Pythagorean relation $a^2+b^2 = c^2$ to find the missing side b, as follows:

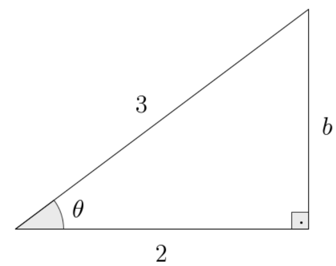
$$2^2 + b^2 = 3^2 \implies 4 + b^2 = 9 \implies b^2 = 5,$$

so $b = \sqrt{5}$. Again, with all the sides, we can read all the main ratios:

$$\sin\theta = \frac{\sqrt{5}}{3}, \quad \cos\theta = \frac{2}{3}, \quad \text{and} \quad \tan\theta = \frac{\sqrt{5}}{3}.$$

Taking reciprocals, we get the rest:

$$\csc\theta = \frac{3\sqrt{5}}{5} \quad \sec\theta = \frac{3}{2}, \quad \text{and} \quad \cot\theta = \frac{2\sqrt{5}}{5}.$$



c. *Given:* $\tan\theta = 5/4$.

Explanation

- **Strategy 1:** The only fundamental identity we have involving $\tan\theta$ is $\tan^2\theta + 1 = \sec^2\theta$, so we might as well use it. It reads

$$\left(\frac{5}{4}\right)^2 + 1 = \sec^2\theta \implies \sec^2\theta = \frac{25}{16} + 1 \implies \sec^2\theta = \frac{41}{16},$$

and so

$$\sec\theta = \frac{\sqrt{41}}{4} \implies \cos\theta = \frac{4}{\sqrt{41}} = \frac{4\sqrt{41}}{41}.$$

With this, we could in principle find $\sin\theta$, by using $\sin^2\theta + \cos^2\theta = 1$ as usual. But there is a simpler way. Namely, we use that

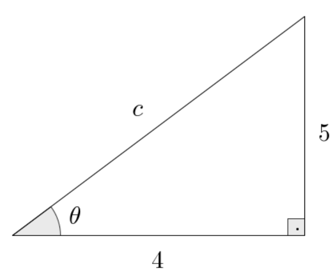
$$\tan\theta = \frac{\sin\theta}{\cos\theta} \implies \sin\theta = \tan\theta\cos\theta = \frac{5}{4}\cdot\frac{4\sqrt{41}}{41} \implies \sin\theta = \frac{5\sqrt{41}}{41}.$$

Meaning that

$$\csc\theta = \frac{\sqrt{41}}{5} \quad \text{and} \quad \cot\theta = \frac{4}{5},$$

and we are done.

- **Strategy 2:** Now, we set a right triangle with legs 4 and 5, like below:



Since the hypotenuse c is missing, applying the Pythagorean Theorem is even easier:

$$c^2 = 4^2 + 5^2 = 16 + 25 = 41 \implies c = \sqrt{41}.$$

With this in place, we read from the triangle the main ratios as

$$\sin\theta = \frac{5\sqrt{41}}{41}, \quad \cos\theta = \frac{4\sqrt{41}}{41} \quad \text{and} \quad \tan\theta = \frac{5}{4}.$$

And taking reciprocals:

$$\csc\theta = \frac{\sqrt{41}}{5}, \quad \sec\theta = \frac{\sqrt{41}}{4} \quad \text{and} \quad \cot\theta = \frac{4}{5}.$$

d. *Given:* $\sec\theta = 7/3$.

Explanation

- **Strategy 1:** We immediately know that $\cos\theta = 3/7$, so let's find $\sin\theta$ next, using the first fundamental identity $\sin^2\theta + \cos^2\theta = 1$. We have

$$\sin^2\theta + \left(\frac{3}{7}\right)^2 = 1 \implies \sin^2\theta + \frac{9}{49} = 1 \implies \sin^2\theta = 1 - \frac{9}{49},$$

so

$$\sin^2\theta = \frac{40}{49} \implies \sin\theta = \frac{2\sqrt{10}}{7}.$$

With this, we have that

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{2\sqrt{10}/7}{3/7} = \frac{2\sqrt{10}}{3}.$$

Flipping the remaining fractions

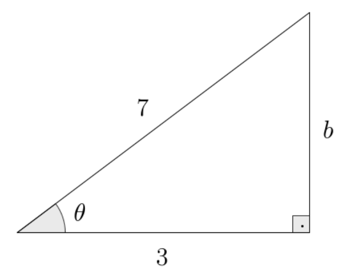
$$\sin\theta = \frac{2\sqrt{10}}{7}, \quad \text{and} \quad \tan\theta = \frac{2\sqrt{10}}{3},$$

respectively, we obtain

$$\csc\theta = \frac{7}{2\sqrt{10}} = \frac{7\sqrt{10}}{20} \quad \text{and} \quad \cot\theta = \frac{3}{2\sqrt{10}} = \frac{3\sqrt{10}}{20}.$$

- **Strategy 2:** Let's draw a triangle with hypotenuse 7 and adjacent side to θ having length 3: Let's use, as usual, the Pythagorean relation $a^2 + b^2 = c^2$ to find b. It gives us that

$$3^2 + b^2 = 7^2 \implies 9 + b^2 = 49 \implies b^2 = 40 \implies b = 2\sqrt{10}.$$



So we have that

$$\sin\theta = \frac{2\sqrt{10}}{7}, \quad \cos\theta = \frac{3}{7}, \quad \text{and} \quad \tan\theta = \frac{2\sqrt{10}}{3}.$$

Taking reciprocals and rationalizing each of them, we also get

$$\csc\theta = \frac{7\sqrt{10}}{20}, \quad \sec\theta = \frac{7}{3}, \quad \text{and} \quad \cot\theta = \frac{3\sqrt{10}}{20}.$$

e. Given: $\csc\theta = 8/7$.

Explanation

- **Strategy 1:** We immediately know that $\sin\theta = 7/8$, so let's find $\cos\theta$ next, using the first fundamental identity $\sin^2\theta + \cos^2\theta = 1$. We have

  $$\left(\frac{7}{8}\right)^2 + \cos^2\theta = 1 \implies \frac{49}{64} + \cos^2\theta = 1 \implies \cos^2\theta = 1 - \frac{49}{64},$$

  so

  $$\cos^2\theta = \frac{15}{64} \implies \cos\theta = \frac{\sqrt{15}}{8}.$$

  With this, we have that

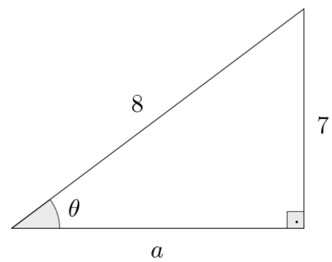
  $$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{7/8}{\sqrt{15}/8} = \frac{7}{\sqrt{15}} = \frac{7\sqrt{15}}{15}.$$

  Flipping the remaining fractions

  $$\cos\theta = \frac{\sqrt{15}}{8}, \quad \text{and} \quad \tan\theta = \frac{7}{\sqrt{15}},$$

  respectively, we obtain

  $$\sec\theta = \frac{8}{\sqrt{15}} = \frac{8\sqrt{15}}{15} \quad \text{and} \quad \cot\theta = \frac{\sqrt{15}}{7}.$$



- **Strategy 2:** Consider the following triangle: Let's find a using the Pythagorean relation $a^2 + b^2 = c^2$, which becomes

$$a^2 + 7^2 = 8^2 \implies a^2 + 49 = 64 \implies a^2 = 15,$$

so that $a = \sqrt{15}$. Having all the sides of the triangle, we read that

$$\sin\theta = \frac{7}{8}, \quad \cos\theta = \frac{\sqrt{15}}{8}, \quad \text{and} \quad \tan\theta = \frac{7\sqrt{15}}{15}.$$

Taking reciprocals, we get

$$\csc\theta = \frac{8}{7}, \quad \sec\theta = \frac{8\sqrt{15}}{15}, \quad \text{and} \quad \cot\theta = \frac{\sqrt{15}}{7}$$

as well.

f. *Given:* $\cot\theta = 2/9$.

Explanation

- **Strategy 1:** We immediately know that $\tan\theta = 9/2$ and, again, the only fundamental identity we have involving $\tan\theta$ is $\tan^2\theta + 1 = \sec^2\theta$. It reads

$$\left(\frac{9}{2}\right)^2 + 1 = \sec^2\theta \implies \sec^2\theta = \frac{81}{4} + 1 \implies \sec^2\theta = \frac{85}{4},$$

and so

$$\sec\theta = \frac{\sqrt{85}}{2} \implies \cos\theta = \frac{2}{\sqrt{85}} = \frac{2\sqrt{85}}{85}.$$

Again, instead of using $\sin^2\theta + \cos^2\theta = 1$ to find $\sin\theta$, we can just argue that

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \implies \sin\theta = \tan\theta\cos\theta = \frac{9}{2}\cdot\frac{2\sqrt{85}}{85} \implies \sin\theta = \frac{9\sqrt{85}}{85}.$$

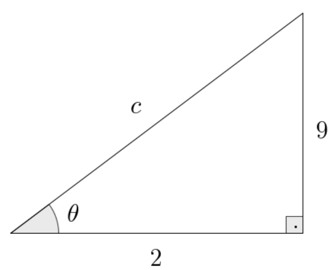
Meaning that

$$\csc\theta = \frac{\sqrt{85}}{9},$$

and so we are done.

As an aside, try to solve this problem again without immediately using that $\tan\theta = 9/2$, start only with $\cot\theta = 2/9$, use the identity $1 + \cot^2\theta = \csc^2\theta$ and go from there, it is instructive.

- **Strategy 2:** Again, let's set up a convenient triangle: Here, in par-



ticular, note that the scale and proportions in this picture are completely off. This is ok, since drawing triangles is just meant to help us organize what is the opposite side to θ, what is the adjacent side, and what is the hypotenuse. It doesn't matter how bad your picture looks, as long as the "positions" are correct. In any case, we immediately find c with

$$c^2 = a^2 + b^2 = 2^2 + 9^2 = 4 + 81 = 85,$$

so $c = \sqrt{85}$. Now we can read all the ratios and rationalize them to obtain:

$$\sin\theta = \frac{9\sqrt{85}}{85}, \quad \cos\theta = \frac{2\sqrt{85}}{85}, \quad \text{and} \quad \tan\theta = \frac{9}{2}.$$

Take reciprocals and rationalize whatever is needed to get

$$\csc\theta = \frac{\sqrt{85}}{9}, \quad \sec\theta = \frac{\sqrt{85}}{2}, \quad \text{and} \quad \cot\theta = \frac{2}{9}$$

as well.

Let's summarize the highlights of the strategy, from the algebraic perspective.

(a) If you're given sin or cos, use the fundamental identity to find the other one. Then find tan = sin / cos, and flip all the fractions to get csc, sec and tan.

(b) If you're given csc or sec, flip it to get sin or cos, and proceed as (a).

(c) If you're given tan, use $\tan^2\theta + 1 = \sec^2\theta$ to find $\sec\theta$. Once you have $\sec\theta$, you have $\cos\theta$. Then proceed as (a).

(d) If you're given cot, flip it to get tan, and proceed as (c).

Note that we're employing a mathematician's general philosophy here: take a problem and reduce it to something which you already know how to solve (namely, we're arguing that — morally — if you know how to solve the problem when you were given either sin or cos, then you in fact know how to solve it when given *any* of the six trigonometric functions). And also from the geometric perspective, the strategy is even easier to describe: recognize the trigonometric function you were given in terms of opp., adj. and hyp., then draw a right triangle with this information. You will be missing one side, which can be found with the Pythagorean Theorem. Once you have all sides, you can find all the ratios between sides.

Of course, the two above ways to go about this are not the only ones, but they're as good a recipe as any. In any case, you have room for creativity here. And even if one method seems easier than the other, it is useful to be comfortable with both, as this is already a good chance to start getting acquainted with trigonometry identities, which will be indispensable later.

We will see later how to define and deal with trigonometric functions for angles which are not necessarily acute. Then, everything we did here becomes slightly more subtle, as one must now pay attention to signs (for example, we'll have that $\cos(120°) = -1/2$). But the overall program of using the fundamental trigonometric identities and the relations between the main trigonometric functions (sin, cos, and tan) with their reciprocals (csc, sec, and cot) will always be useful.

Summary

- We have illustrated, with several examples, two ways to find all the values of the trigonometric functions at an acute angle θ, once we know one of the values. This can be done algebraically by exploring trigonometric identities, or geometrically by drawing the "correct" triangle and applying the Pythagorean Theorem to find the missing side – to then read all ratios directly from the triangle itself.

8.2 The Unit Circle

Learning Objectives

- The Unit Circle
  - Degrees and Radians
  - Reference Angles
  - The Definition of trigonometric functions in terms of the Unit Circle
  - Evaluating trigonometric functions at standard angles

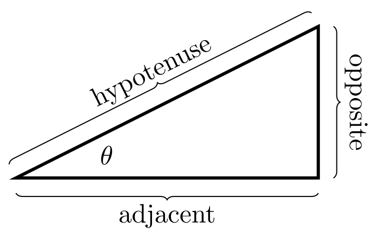
8.2.1 Unit Circle

Motivating Questions

- How can we define trigonometric functions for angles that do not come from triangles?

Introduction

In the previous sections, you were introduced to the basic trigonometric functions sine and cosine, and saw how they relate measures of angles to measurements of triangles. Given a right triangle



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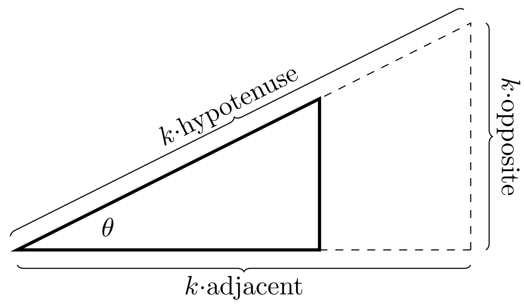
we define

$$\cos(\theta) = \frac{\text{adjacent}}{\text{hypotenuse}} \qquad \text{and} \qquad \sin(\theta) = \frac{\text{opposite}}{\text{hypotenuse}}.$$

There is a limitation in this, which you may have noticed. We can only build a triangle with a base angle θ if θ is between 0° and 90°. We work now to rectify this deficiency.

The Unit Circle

First, note that the values of sine and cosine do not depend on the scale of the triangle. Being very explicit, if we take our triangle and scale it up by a factor of k (multiplying each side length by k) we obtain



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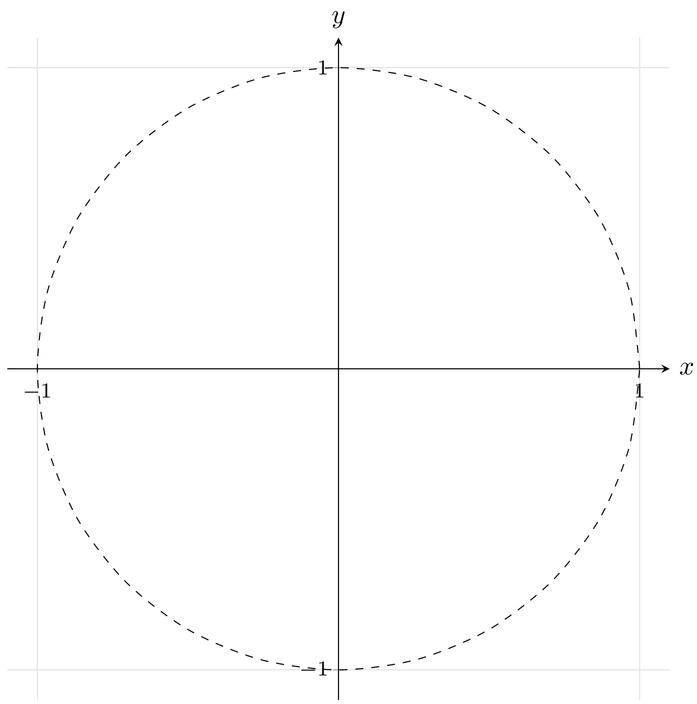
$$\cos(\theta) = \frac{k \cdot \text{adjacent}}{k \cdot \text{hypotenuse}} = \frac{\text{adjacent}}{\text{hypotenuse}}$$

and

$$\sin(\theta) = \frac{k \cdot \text{opposite}}{k \cdot \text{hypotenuse}} = \frac{\text{opposite}}{\text{hypotenuse}}.$$

Notice that the *ratios* of the corresponding side lengths are not changed. The individual side lengths are changed, but the ratios are preserved.

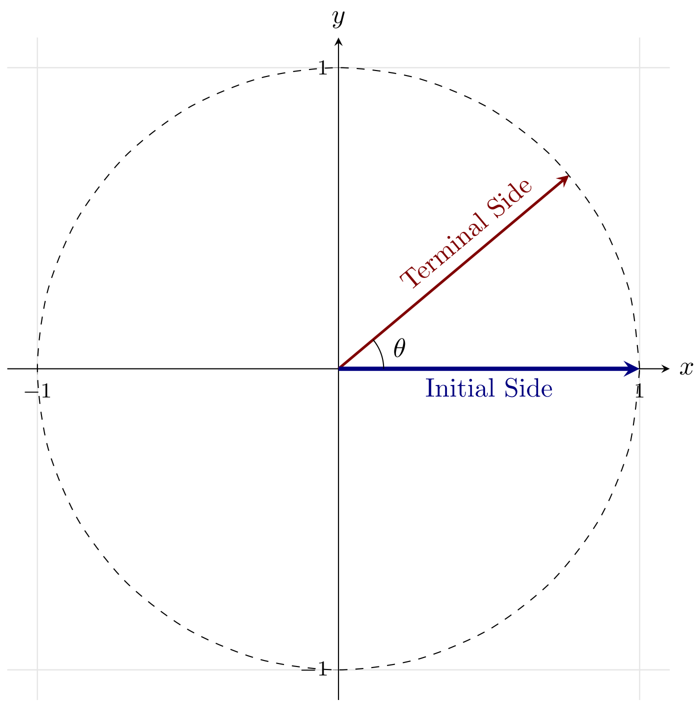
Because of this we could simply assume that whenever we draw a triangle for computing sine and cosine, that the hypotenuse will have length 1 (by dividing each side by the length of the hypotenuse). We can do this because we are simply scaling the triangle, and as we see above, this makes absolutely no difference when computing sine and cosine. When the hypotenuse is 1, we find that a convenient way to think about sine and cosine is via a circle.

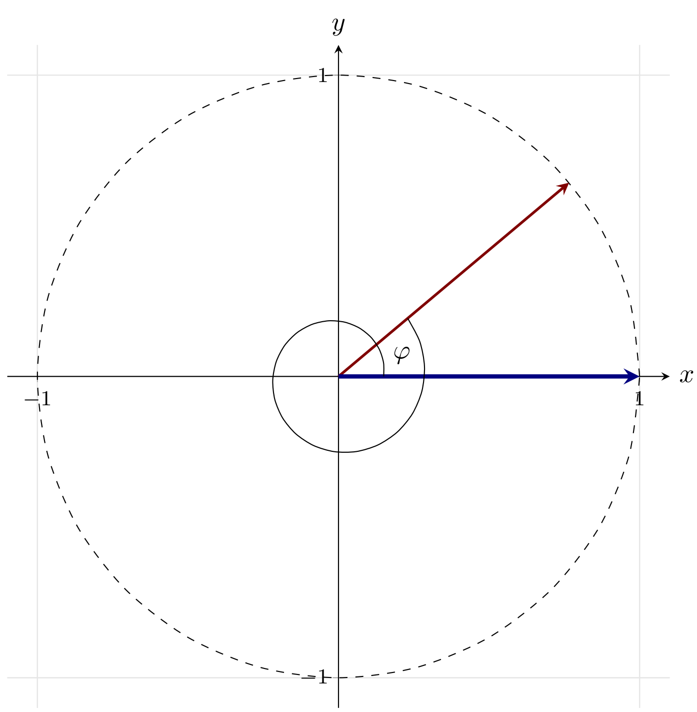


We call this the *unit circle*.

Definition The **Unit Circle** is the circle of radius 1, with center at the origin. It is the graph of the equation $x^2 + y^2 = 1$.

An angle θ is in *standard position* if the vertex of the angle is at the origin and one side oriented along the positive x-axis. The ray along the positive x-axis is called the *initial side* of the angle, and the other ray is called the *terminal side* of the angle. The angle can be thought of as the counter-clockwise rotation necessary to spin the initial side to the terminal side.


y
1
Terminal Side
θ
x
−1
Initial Side
1
−1


y
1
φ
x
−1
1
−1

Notice that the angles θ and φ from the two images above both rotate the initial side of the angle to the same terminal side, but the angle φ wraps around the origin first. Two angles are *coterminal* if they have the same terminal side. The angles θ and φ are coterminal.

You can also think about an angle wrapping two, three, or four times before getting to the terminal side. You can also think about rotating clockwise instead of counter-clockwise. We consider counter-clockwise the positive direction, and clockwise the negative direction for angles.

Example 6. *Find two angles that are coterminal with* 30°*, one positive and one negative.*

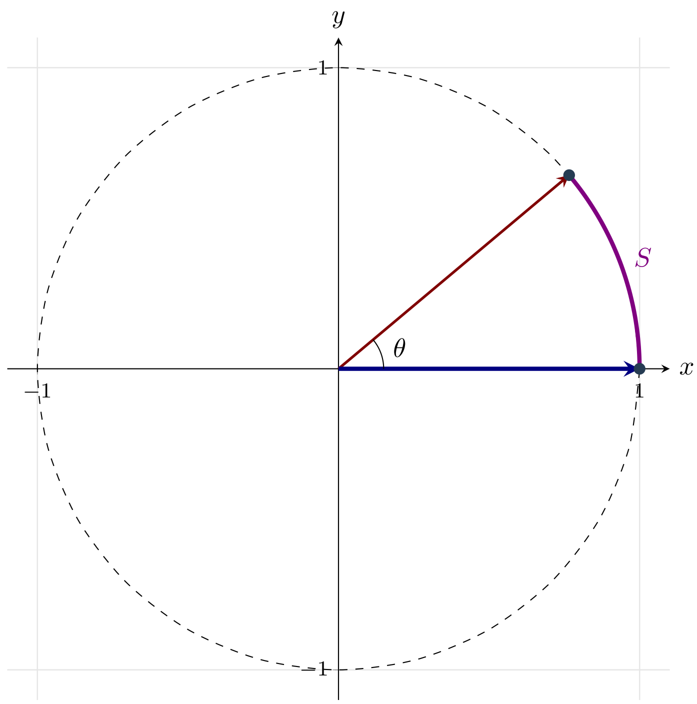
Explanation

If we start with 30°, and then take another complete counter-clockwise rotation (360°), we end up at 390°. That means 390° is coterminal with 30°.

If we start with 30°, and then take another complete clockwise rotation (-360°), we end up at $30^\circ - 360^\circ = -330^\circ$. That means -330° is coterminal with 30°.

Radians

In everyday life, we typically measure angles in degrees. You will see in Calculus that using degrees can lead to a lot of complications. There is a better choice, more closely related to the geometry of the circle. Notice that an angle identifies an arc along the circumerence of the unit circle? We'll call the arc constructed this way the *subtended arc*.



Notice that as θ grows (counter-clockwise) from 0° the length of this arc, called S, also grows.

Example 7. *If θ is a right angle, calculate the length S.*

Explanation

If $\theta = 90^\circ$, the arc is a quarter of the circle. The circumference of the circle is $C = 2\pi r = 2\pi(1) = 2\pi$. Since S will be a quarter of that, $S = \frac{1}{4}(2\pi) = \frac{\pi}{2}$.

The units we will be measuring angles, *radians*, is actually based on arc lengths like this.

> **Definition** One **radian** is the angle which, in standard position, subtends an arc of length 1.

That is, an angle measuring 1 radian has $S = 1$. Let us suppose that the radius of the circle, and therefore the length of the subtended arc, has units. This means that an angle θ of 1 radian, in a circle of radius 1 unit, subtends an arclength of $S = 1$ unit. We know that the formula for arclength is given by $S = R\theta$, so that $\theta = \frac{S}{R}$. That means 1 radian is equal to $\frac{1\text{unit}}{1\text{unit}}$. Notice that the "units" cancel out? That means **radians are a unitless unit**. When our angle is measured in radians, that angle is really just a number.

In one complete revolution (360°) we have subtended the entire circle so $S = 2\pi$. Based on this and the definition of the radian above, one complete revolution measures 2π radians. That is an angle measuring 360° measures 2π radians. This gives us a way to convert between degrees and radians! Note that $\dfrac{360}{2\pi}$ can be reduced to $\dfrac{180}{\pi}$.

- To convert from radians to degrees, multiply by the factor $\dfrac{180}{\pi}$.
- To convert from degrees to radians, multiply by the factor $\dfrac{\pi}{180}$.

Here we're thinking about $\dfrac{180}{\pi}$ as having units $\dfrac{\text{degrees}}{\text{radians}}$. What units do you think $\dfrac{\pi}{180}$ should have?

Example 8. (a) *Convert* 30°, 45°, *and* 90° *to radians.*

(b) *Convert* $\dfrac{\pi}{3}$ *radians,* π *radians, and* $-\dfrac{\pi}{10}$ *radians to degrees.*

Explanation

(a) $30 \cdot \left(\dfrac{\pi}{180}\right) = \dfrac{30\pi}{180} = \dfrac{\pi}{6}$. That means 30° is equivalent to $\dfrac{\pi}{6}$ radians.

$45 \cdot \left(\dfrac{\pi}{180}\right) = \dfrac{45\pi}{180} = \dfrac{\pi}{4}$. That means 45° is equivalent to $\dfrac{\pi}{4}$ radians.

$90 \cdot \left(\dfrac{\pi}{180}\right) = \dfrac{90\pi}{180} = \dfrac{\pi}{2}$. That means 90° is equivalent to $\dfrac{\pi}{2}$ radians.

(b) $\dfrac{\pi}{3} \cdot \left(\dfrac{180}{\pi}\right) = \dfrac{180\pi}{3\pi} = 60$. That means $\dfrac{\pi}{3}$ radians is equivalent to 60°.

$\pi \cdot \left(\dfrac{180}{\pi}\right) = \dfrac{180\pi}{\pi} = 180$. That means π radians is equivalent to 180°.

$-\dfrac{\pi}{10} \cdot \left(\dfrac{180}{\pi}\right) = -\dfrac{180\pi}{10\pi} = -18$. That means $-\dfrac{\pi}{10}$ radians is equivalent to -18°.

Frequently we will describe angles by their quadrants. An angle will be called a *first quadrant angle* if its terminal side lies in the first quadrant. Any angle in the interval $\left(0, \dfrac{\pi}{2}\right)$ will be a first quadrant angle, but there are others. For example, $\dfrac{9\pi}{4}$ is a first quadrant angle since it is coterminal with $\dfrac{\pi}{4} = \dfrac{9\pi}{4} - 2\pi$. Similarly we will call an angle a *second quadrant angle* if its terminal side lies in the second quadrant. These angles are coterminal to angles with measures $\left(\dfrac{\pi}{2}, \pi\right)$. *Third quadrant angles* and *fourth quadrant angles* are defined similarly.

The radian measure of some standard angles are given in the chart below.

Degrees	Radians
0°	0
30°	$\frac{\pi}{6}$
45°	$\frac{\pi}{4}$
60°	$\frac{\pi}{3}$
90°	$\frac{\pi}{2}$

Triangles in the Unit Circle

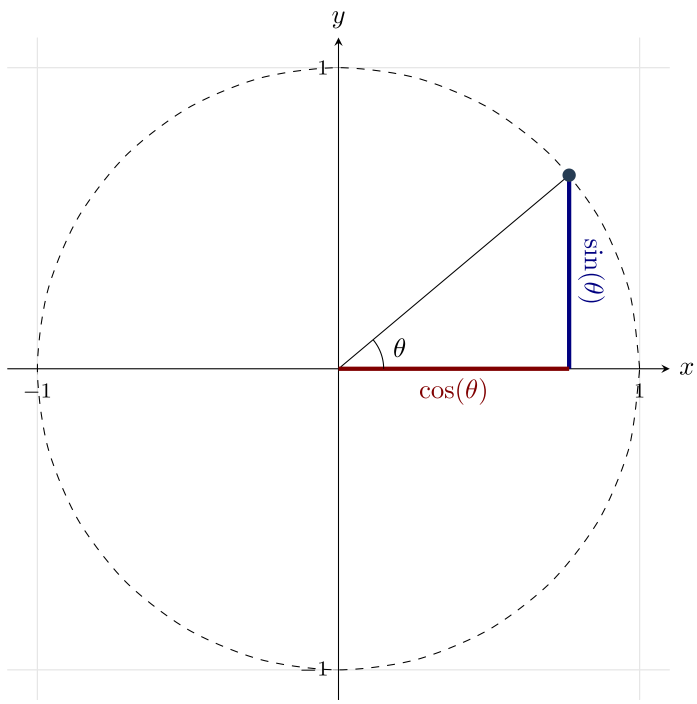
Let's draw our right triangle from before with the base angle θ in standard position, and scaled to have hypotenuse of length 1. Remember that since the hypotenuse has length 1, we know that

$$\sin(\theta) = \frac{\text{opp}}{\text{hyp}} = \text{opp}$$

and

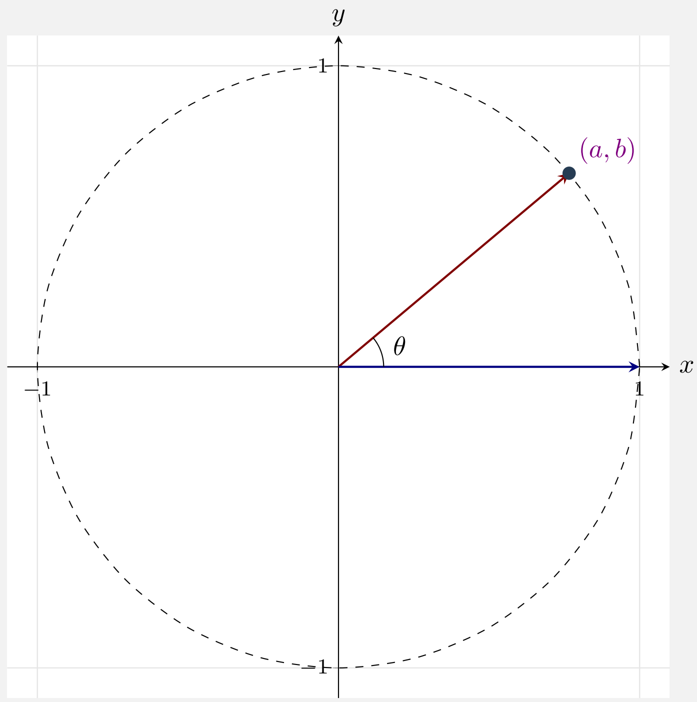
$$\cos(\theta) = \frac{\text{adj}}{\text{hyp}} = \text{adj}.$$

When we scale our triangle to have hypotenuse of length 1, sine and cosine measure the lengths of the opposite and adjacent sides. The triangle in the figure below has its side lengths labeled with this in mind.



If we consider the hypotenuse of this triangle as terminal side of θ, the point where this terminal side intersects the unit circle has coordinates $(\cos(\theta), \sin(\theta))$. This has given us our method to extend trigonometric functions to all angles, instead of just triangles.

Definition Suppose θ is an angle in standard position in the unit circle, and denote by (a, b) the coordinates of the point where the terminal side of θ intersects the unit circle.



$\sin(\theta) = b$ $\qquad$ $\cos(\theta) = a$

$\tan(\theta) = \dfrac{b}{a}$, if $a \neq 0$ $\qquad$ $\sec(\theta) = \dfrac{1}{a}$, if $a \neq 0$ $\qquad$ $\cot(\theta) = \dfrac{a}{b}$, if $b \neq 0$. $\csc(\theta) = \dfrac{1}{b}$, if $b \neq 0$

The domain of sine and cosine is all real numbers, and the other trigonometric functions are defined precisely when their denominators are nonzero.

Example 9. *Which of the following expressions are equal to* $\sec(\theta)$*?*

(a) $\dfrac{1}{\cos(\theta)}$

(b) $\dfrac{1}{\sin(\theta)}$

(c) $\dfrac{adj}{hyp}$

(d) $\dfrac{hyp}{adj}$

(e) $\dfrac{\tan(\theta)}{\sin(\theta)}$

(f) $\dfrac{1}{\sin(\theta)\cdot\cot(\theta)}$

Explanation Given the angles and intersection point (a, b) from the definition above:

(a) $\cos(\theta) = a$ so $\sec(\theta) = \dfrac{1}{a} = \dfrac{1}{\cos(\theta)}$, provided that $a \neq 0$. This one is correct.

(b) $\sin(\theta) = b$ so $\dfrac{1}{\sin(\theta)} = \dfrac{1}{b}$, provided that $b \neq 0$. This is NOT $\sec(\theta)$.

(c) $\dfrac{\text{adj}}{\text{hyp}} = \dfrac{a}{1} = a$. This is $\cos(\theta)$, not $\sec(\theta)$.

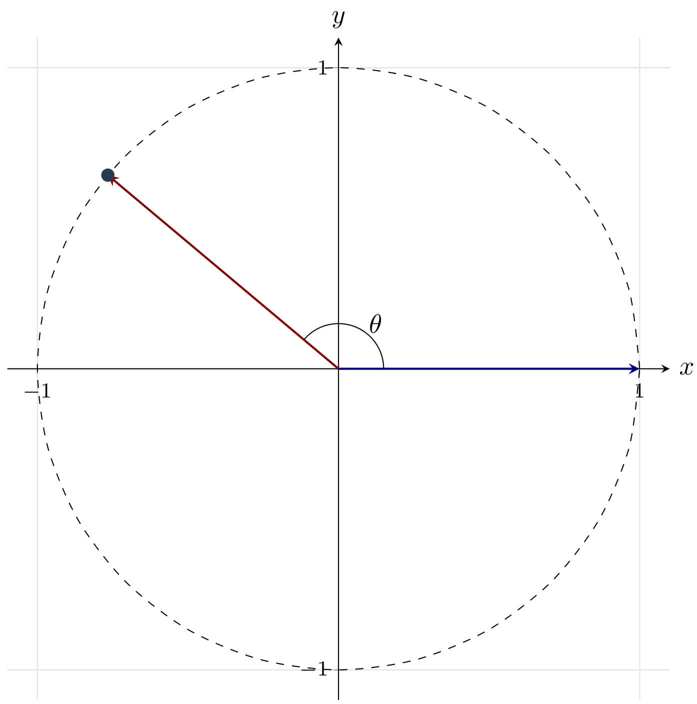
(d) $\dfrac{\text{hyp}}{\text{adj}} = \dfrac{1}{a}$, provided that $a \neq 0$. This one is also correct.

(e) $\dfrac{\tan(\theta)}{\sin(\theta)} = \dfrac{\left(\frac{b}{a}\right)}{b} = \dfrac{1}{a}$, provided that BOTH $a \neq 0$ AND $b \neq 0$. For example, when $\theta = 0$ this fraction is undefined but $\sec(0) = \dfrac{1}{1} = 1$. That means $\dfrac{\tan(\theta)}{\sin(\theta)}$ is not always the same as $\sec(\theta)$.

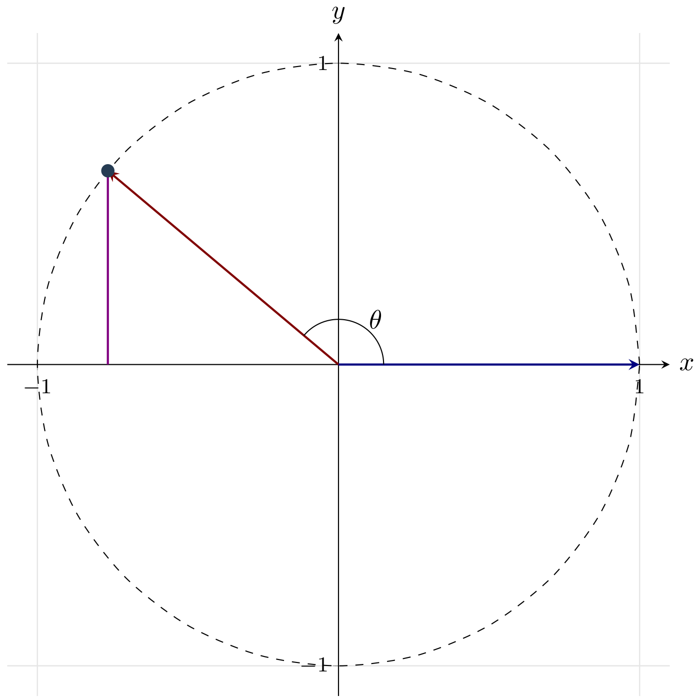
(f) $\dfrac{1}{\sin(\theta)\cdot\cot(\theta)} = \dfrac{1}{b\left(\frac{a}{b}\right)} = \dfrac{1}{a}$, provided that BOTH $a \neq 0$ AND $b \neq 0$. For example, when $\theta = 0$ this fraction is undefined but $\sec(0) = \dfrac{1}{1} = 1$. That means $\dfrac{\tan(\theta)}{\sin(\theta)}$ is not always the same as $\sec(\theta)$.

Reference Angles

We've seen above how to draw a (scaled version of) a right triangle inside the unit circle, with it's base angle in standard position. How about the other way around? If we have an angle that isn't necessarily an acute angle (one whose terminal side lies within the first quadrant), would it be possible to relate it to a triangle? Consider the second quadrant angle θ in the following image.



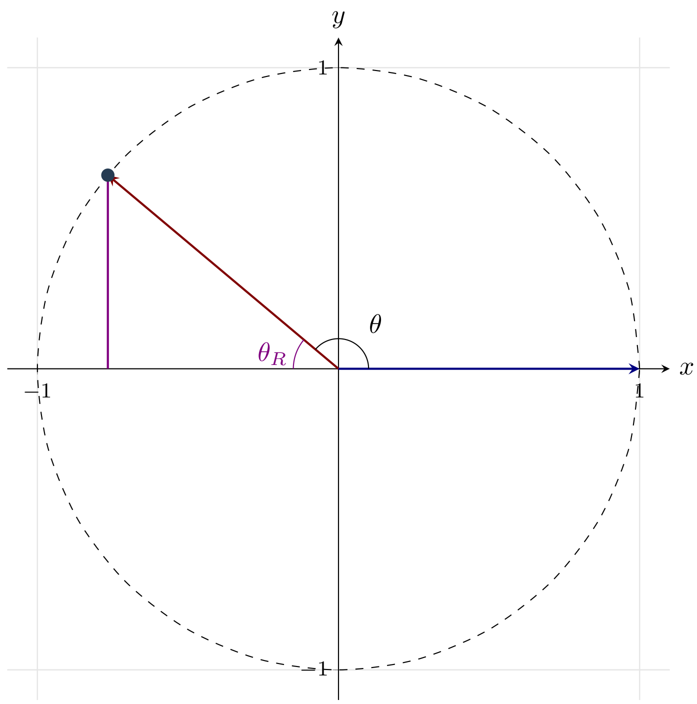
As before, we can draw a vertical line from the point where the terminal side of θ intersects the unit circle to the x-axis.



No matter the quandrant θ lies in, we can always construct a triangle by drawing this vertical side between the x-axis and the intersection point. Notice that this triangle has an acute angle with vertex at the origin.

> **Definition** Suppose θ is an angle in standard position. The **reference angle**, θ_R, is the *acute* angle between the terminal side and the x-axis.

If the terminal side of θ is along the x-axis (in either direction), in which case we will have $\theta_R = 0$. However if the terminal side of θ lies along the y-axis (in either direction) we will have $\theta_R = \dfrac{\pi}{2}$. A reference angle is never less than 0, nor greater than $\dfrac{\pi}{2}$.



Exercise 1 *Find the reference angle for each of the following angles.*

(a) $\alpha = \dfrac{5\pi}{9}$

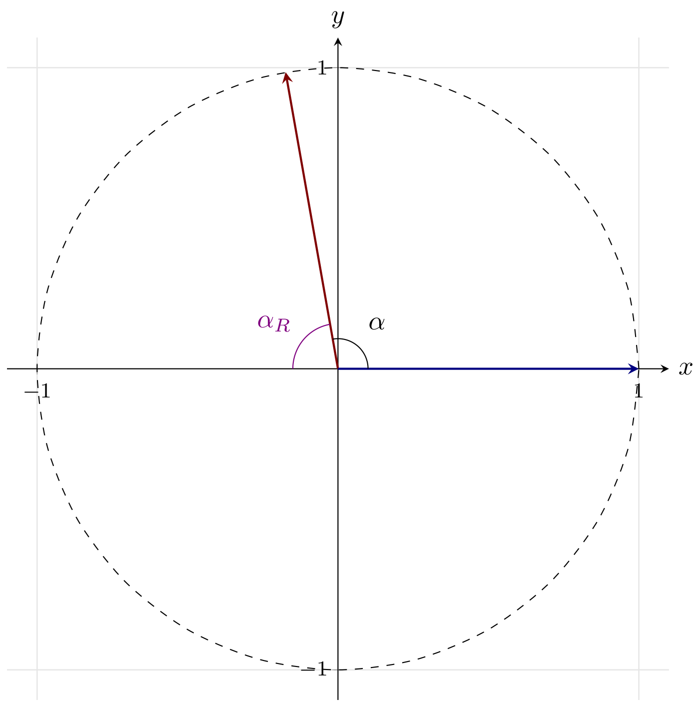
(b) $\varphi = \dfrac{7\pi}{5}$

(c) $\theta = \dfrac{23\pi}{3}$

(d) $\gamma = 7$

Explanation

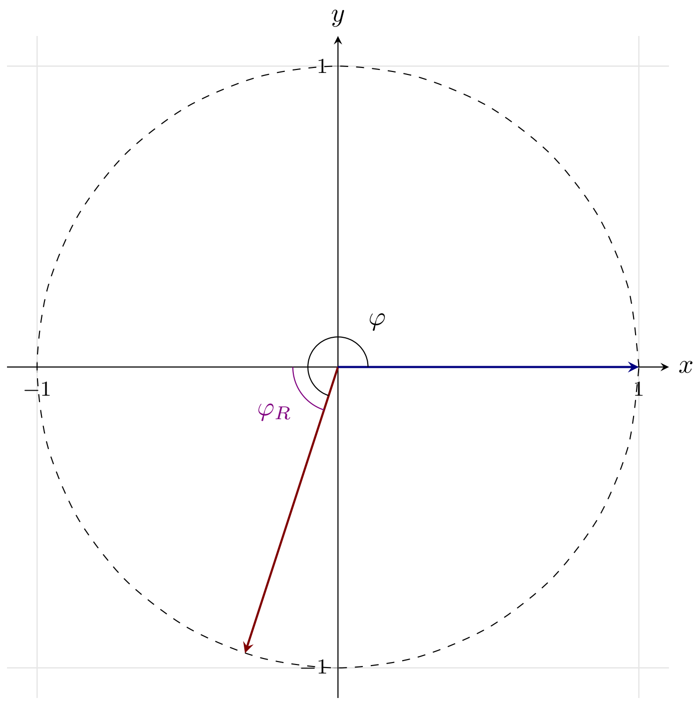
(a) The angle $\dfrac{5\pi}{9}$ is between $\dfrac{\pi}{2}$ and π, so α is in the second quadrant.



Since π is further (in the counter-clockwise direction) than α, we have

$$\begin{aligned}\alpha_R &= \pi - \alpha \\ &= \pi - \frac{5\pi}{9} \\ &= \frac{9\pi}{9} - \frac{5\pi}{9} \\ &= \frac{4\pi}{9}.\end{aligned}$$

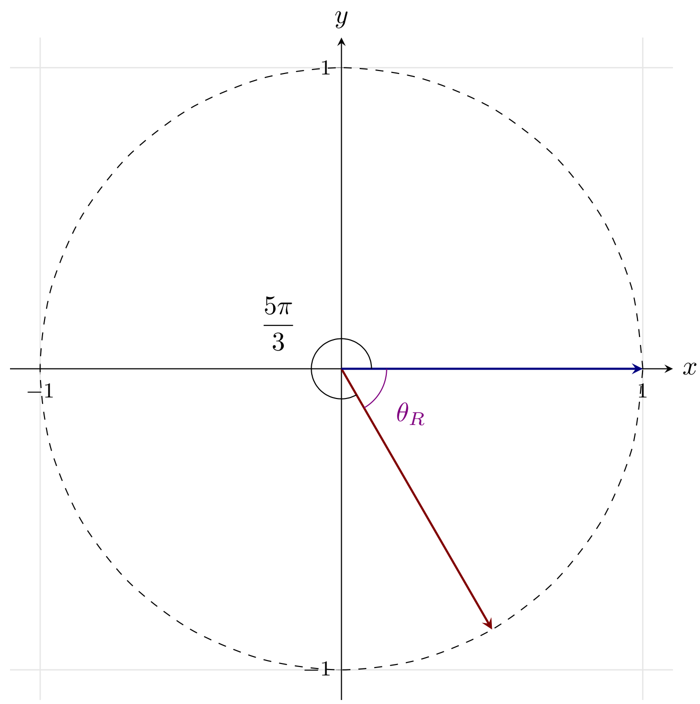
(b) The angle $\dfrac{7\pi}{5}$ is between π and $\dfrac{3\pi}{2}$, so φ is in the third quadrant.



In this case, φ is further (in the counter-clockwise direction) than π. That means

$$\begin{aligned}
\varphi_R &= \varphi - \pi \\
&= \frac{7\pi}{5} - \pi \\
&= \frac{7\pi}{5} - \frac{5\pi}{5} \\
&= \frac{2\pi}{5}.
\end{aligned}$$

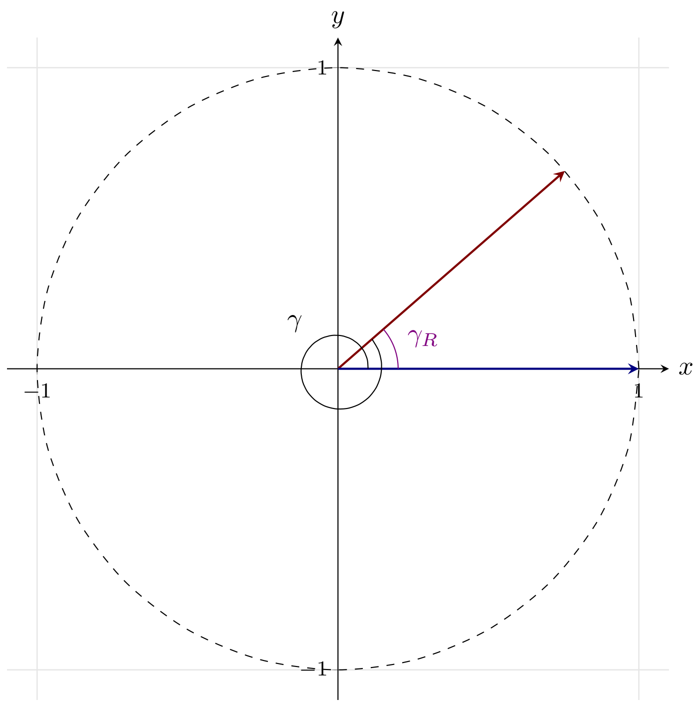
(c) The angle $\dfrac{23\pi}{3}$ is coterminal with $\dfrac{23\pi}{3} - 6\pi = \dfrac{5\pi}{3}$. Since $\dfrac{3}{2} < \dfrac{5}{3} < 2$ we see that θ is a third quadrant angle. Rather than working with θ, we work with the coterminal angle $\dfrac{5\pi}{3}$.



In this case, 2π is further (in the counter-clockwise direction) than $\frac{5\pi}{3}$. That means

$$\begin{aligned}\theta_R &= 2\pi - \frac{5\pi}{3}\\ &= \frac{6\pi}{3} - \frac{5\pi}{3}\\ &= \frac{\pi}{3}.\end{aligned}$$

(d) Don't let this one fool you. There is no degree symbol. This angle is not written as a fraction and it does not seem to include π, but that just means γ is not one of our standard angles. This measurement is still in radians, though. We are asked about $\gamma = 7$ radians, not degrees. Since 2π is a little more than 6, we know $2\pi < 7 < \frac{5\pi}{2}$, so γ is a first quadrant angle.

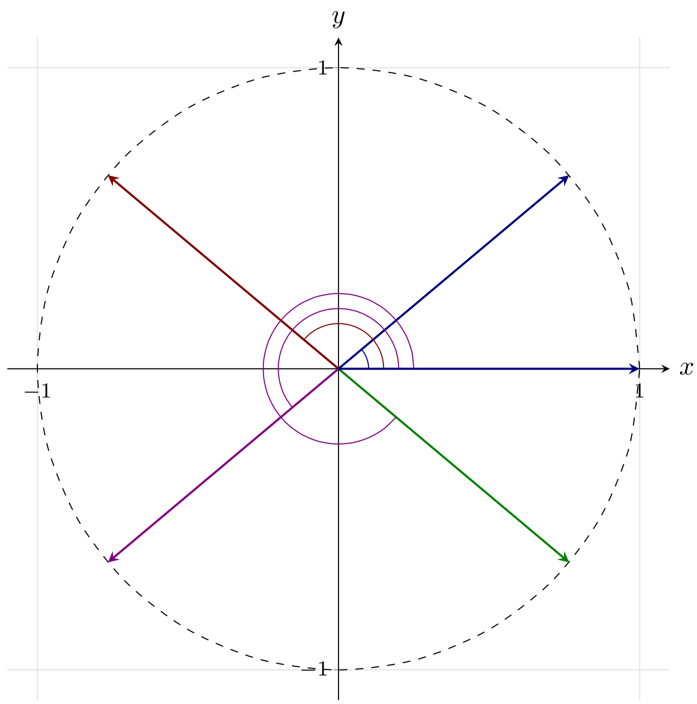


In this case, γ is further (in the counter-clockwise direction) than 2π. That means

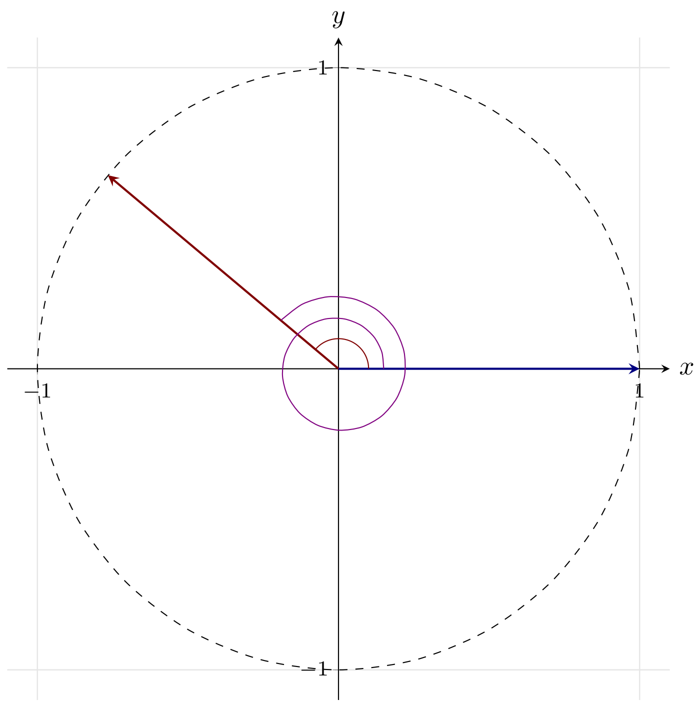
$$\begin{aligned}\gamma_R &= \gamma - 2\pi \\ &= 7 - 2\pi.\end{aligned}$$

To think about this in a different way, the angle γ is coterminal with $\gamma - 2\pi = 7 - 2\pi$. Since $0 < 7 - 2\pi < \dfrac{\pi}{2}$, it must be its own reference angle. That would mean $7 - 2\pi$ is the reference angle of any angles which are coterminal with it, including γ.

If we know the reference angle, θ_R, can we determine θ? Not exactly. In the graph below are four different angles each having the same reference angle. What's different about these angles? They are in *different quadrants*.



If we know both the reference angle *and* the quadrant, can we determine the angle? Not quite. Two angles can have the same quadrant and reference angle if they are coterminal angles.



We can't determine the angle exactly, but we can determine the angle's *terminal side*. Since trigonometric functions are given in terms of the coordinates on the terminal side of the angle, knowing the reference angle and quadrant is enough for us.

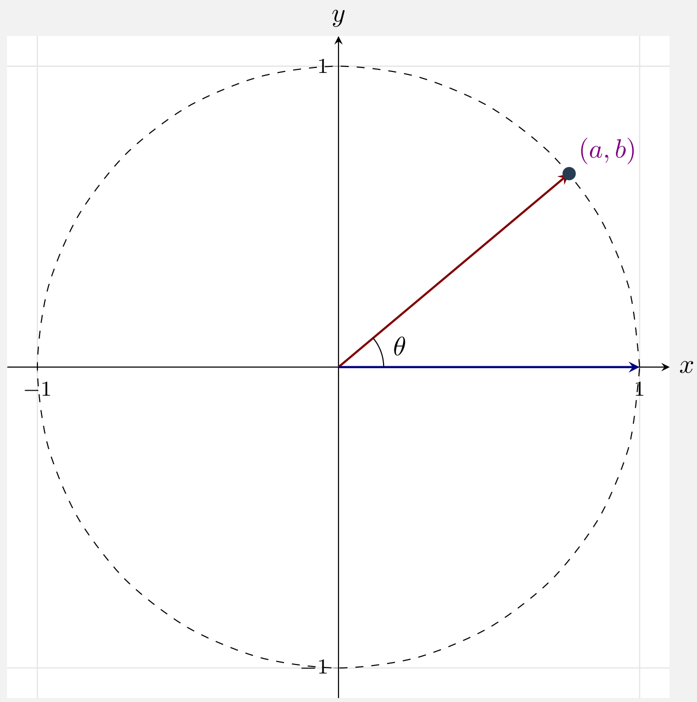
Let's examine the effects of the quadrants on the trigonometric function values we discussed earlier.

- Quadrant I: x and y coordinates are both positive so sine values and cosine values will be positive for these angles.
- Quadrant II: x is negative and y is positive, so cosine values will be negative and sine values will be positive for these angles.
- Quadrant III: x and y coordinates are both negative so sine values and cosine values will be negative for these angles.
- Quadrant IV: x is positive and y is negative, so cosine values will be positive and sine values will be negative for these angles.

Evaluating Trigonometric Functions at Standard Angles

Recall the definition from above.

Definition Suppose θ is an angle in standard position in the unit circle, and denote by (a, b) the coordinates of the point where the terminal side of θ intersects the unit circle.



$$\cos(\theta) = a \qquad \sin(\theta) = b$$

$$\sec(\theta) = \frac{1}{a}, \text{ if } a \neq 0 \qquad \csc(\theta) = \frac{1}{b}, \text{ if } b \neq 0$$

$$\tan(\theta) = \frac{b}{a}, \text{ if } a \neq 0 \qquad \cot(\theta) = \frac{a}{b}, \text{ if } b \neq 0.$$

The domain of sine and cosine is all real numbers, and the other trigonometric functions are defined precisely when their denominators are nonzero.

We are now in a position to evaluate these trig functions for any standard angle.

Exercise 2 *Calculate the values of the six trigonometric functions for each of the following angles.*

(a) $\dfrac{2\pi}{3}$

(b) $\dfrac{9\pi}{4}$

(c) $\dfrac{7\pi}{6}$

(d) $\dfrac{15\pi}{2}$

Explanation

(a) $\dfrac{2\pi}{3}$ is a second quadrant angle with reference angle $\pi - \dfrac{2\pi}{3} = \dfrac{\pi}{3}$ radians, which is equivalent to 60°. We know the trig values of 60°: $\sin(60^\circ) = \dfrac{\sqrt{3}}{2}$, $\cos(60^\circ) = \dfrac{1}{2}$. In the Quadrant II, x-values are negative and y-values are positive so we are looking for a negative cosine value and a positive sine value. This gives $\sin\left(\dfrac{2\pi}{3}\right) = \dfrac{\sqrt{3}}{2}$ and $\cos\left(\dfrac{2\pi}{3}\right) = -\dfrac{1}{2}$. Using the definitions of the other trig functions:

$$\begin{aligned}
\sin\left(\frac{2\pi}{3}\right) &= \frac{\sqrt{3}}{2} & \cos\left(\frac{2\pi}{3}\right) &= -\frac{1}{2}\\
\tan\left(\frac{2\pi}{3}\right) &= \frac{\sin\left(\frac{2\pi}{3}\right)}{\cos\left(\frac{2\pi}{3}\right)} & \cot\left(\frac{2\pi}{3}\right) &= \frac{\cos\left(\frac{2\pi}{3}\right)}{\sin\left(\frac{2\pi}{3}\right)}\\
&= -\sqrt{3} & &= -\frac{1}{\sqrt{3}}\\
\sec\left(\frac{2\pi}{3}\right) &= \frac{1}{\cos\left(\frac{2\pi}{3}\right)} & \csc\left(\frac{2\pi}{3}\right) &= \frac{1}{\sin\left(\frac{2\pi}{3}\right)}\\
&= -2 & &= \frac{2}{\sqrt{3}}.
\end{aligned}$$

(b) $\dfrac{9\pi}{4}$ is a first quadrant angle with reference angle $\dfrac{9\pi}{4} - 2\pi = \dfrac{\pi}{4}$ radians, which is equivalent to 45°. We know the trig values of 45°: $\sin(45^\circ) = \dfrac{\sqrt{2}}{2}$, $\cos(45^\circ) = \dfrac{\sqrt{2}}{2}$. In the Quadrant I, both the x-values and y-values are positive so all trig function values will be positive. This gives $\sin\left(\dfrac{9\pi}{4}\right) =$

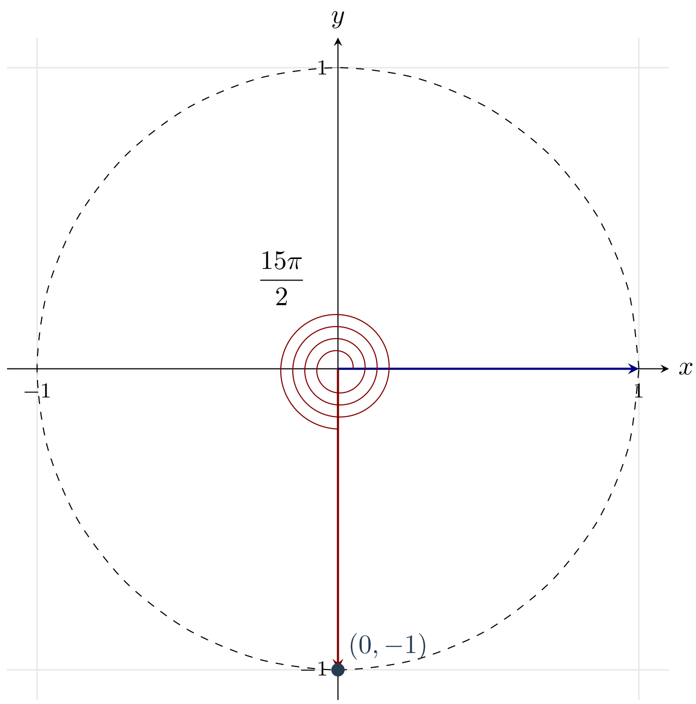
$\dfrac{\sqrt{2}}{2}$ and $\cos\left(\dfrac{9\pi}{4}\right) = \dfrac{\sqrt{2}}{2}$. Using the definitions of the other trig functions:

$$\begin{aligned}
\sin\left(\frac{9\pi}{4}\right) &= \frac{\sqrt{2}}{2} & \cos\left(\frac{9\pi}{4}\right) &= \frac{\sqrt{2}}{2} \\
\tan\left(\frac{9\pi}{4}\right) &= \frac{\sin\left(\frac{9\pi}{4}\right)}{\cos\left(\frac{9\pi}{4}\right)} & \cot\left(\frac{9\pi}{4}\right) &= \frac{\cos\left(\frac{9\pi}{4}\right)}{\sin\left(\frac{9\pi}{4}\right)} \\
&= 1 & &= 1 \\
\sec\left(\frac{9\pi}{4}\right) &= \frac{1}{\cos\left(\frac{9\pi}{4}\right)} & \csc\left(\frac{9\pi}{4}\right) &= \frac{1}{\sin\left(\frac{9\pi}{4}\right)} \\
&= \sqrt{2} & &= \sqrt{2}.
\end{aligned}$$

(c) $\dfrac{7\pi}{6}$ is a third quadrant angle with reference angle $\dfrac{7\pi}{6} - \pi = \dfrac{\pi}{6}$ radians, which is equivalent to $30°$. We know the trig values of $30°$: $\sin(30°) = \dfrac{1}{2}$, $\cos(30°) = \dfrac{\sqrt{3}}{2}$. In the Quadrant III, both the x-values and y-values are negative so we will have a negative sine and cosine values. This gives $\sin\left(\dfrac{7\pi}{6}\right) = -\dfrac{1}{2}$ and $\cos\left(\dfrac{7\pi}{6}\right) = -\dfrac{\sqrt{3}}{2}$. Using the definitions of the other trig functions:

$$\begin{aligned}
\sin\left(\frac{7\pi}{6}\right) &= -\frac{1}{2} & \cos\left(\frac{7\pi}{6}\right) &= -\frac{\sqrt{3}}{2} \\
\tan\left(\frac{7\pi}{6}\right) &= \frac{\sin\left(\frac{7\pi}{6}\right)}{\cos\left(\frac{7\pi}{6}\right)} & \cot\left(\frac{7\pi}{6}\right) &= \frac{\cos\left(\frac{7\pi}{6}\right)}{\sin\left(\frac{7\pi}{6}\right)} \\
&= \frac{1}{\sqrt{3}} & &= \sqrt{3} \\
\sec\left(\frac{7\pi}{6}\right) &= \frac{1}{\cos\left(\frac{7\pi}{6}\right)} & \csc\left(\frac{7\pi}{6}\right) &= \frac{1}{\sin\left(\frac{7\pi}{6}\right)} \\
&= -\frac{2}{\sqrt{3}} & &= -2.
\end{aligned}$$

(d) $\dfrac{15\pi}{2}$ has terminal side along the negative y-axis (it is coterminal with the angle $\dfrac{3\pi}{2}$). The terminal side intersects the unit circle at the point $(0, -1)$.



The coordinates of the intersection give the values of the trig functions $\cos\left(\frac{15\pi}{2}\right) = 0$ and $\sin\left(\frac{15\pi}{2}\right) = -1$. Using the definitions of the other trig functions:

$$\sin\left(\frac{15\pi}{2}\right) = -1 \qquad \cos\left(\frac{15\pi}{2}\right) = 0$$

$$\tan\left(\frac{15\pi}{2}\right) \text{ does not exist} \qquad \cot\left(\frac{15\pi}{2}\right) = \frac{\cos\left(\frac{15\pi}{2}\right)}{\sin\left(\frac{15\pi}{2}\right)} = 0$$

$$\sec\left(\frac{15\pi}{2}\right) \text{ does not exist} \qquad \csc\left(\frac{15\pi}{2}\right) = \frac{1}{\sin\left(\frac{15\pi}{2}\right)} = -1.$$

Typically we have $\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$ and $\sec(\theta) = \frac{1}{\cos(\theta)}$, but for the angle $\frac{15\pi}{2}$ we have $\cos\left(\frac{15\pi}{2}\right) = 0$.

8.3 Trig Identities

Learning Objectives

- Trig Identities
  - What expressions composed of trig functions are identically equal for all values of x?
- Trig Identity Applications
  - How can applying trig identities help us solve problems?

8.3.1 Trigonometric Identities

Introduction to Identities

From the previous section, we have found some identities. We will now summarize what we have already found and begin to introduce new identities. These will help us to breakdown and simplify trigonometric equations that will hopefully make our lives easier.

Remember that an identity is an equation that is true for all possible values of x for which the involved quantities are defined. An example of a non-trigonometric identity is

$$(x+1)^2 = x^2 + 2x + 1,$$

since this equation is true for every value of x, and the left and right sides of the equation are simply two different-looking but entirely equivalent expressions.

Trigonometric identities are simply identities that involve trigonometric functions. While there are a large number of such identities one can study, we choose to focus on those that turn out to be useful in the study of calculus. The most important trigonometric identity is the fundamental trigonometric identity, which is a trigonometric restatement of the Pythagorean Theorem.

For any real angle θ,

$$\cos^2(\theta) + \sin^2(\theta) = 1$$

Identities are important because they enable us to view the same idea from multiple perspectives. For example, the fundamental trigonometric identity allows us to think of $\cos^2(\theta) + \sin^2(\theta)$ as simply 1, or alternatively, to view $\cos^2(\theta)$ as the same quantity as $1 - \sin^2(\theta)$.

There are two related Pythagorean identities that involve the tangent, secant, cotangent, and cosecant functions, which we previously derived from the fundamental trigonometric identity by dividing both sides by either $\cos^2(\theta)$ or $\sin^2(\theta)$. We will take another look at this identity before going deeper. If we divide both sides of $\cos^2(\theta) + \sin^2(\theta) = 1$ by $\cos^2(\theta)$ (and assume that $\cos(\theta) \neq 0$, we see that

$$1 + \frac{\sin^2(\theta)}{\cos^2(\theta)} = \frac{1}{\cos^2(\theta)},$$

or equivalently,

$$1 + \tan^2(\theta) = \sec^2(\theta).$$

A similar argument dividing by $\sin^2(\theta)$ (while assuming $\sin(\theta) \neq 0$) shows that

$$\cot^2(\theta) + 1 = \csc^2(\theta).$$

These identities prove useful in calculus when we develop the formulas for the derivatives of the tangent and cotangent functions.

Sums and Differences of Angles

In calculus, it is also beneficial to know a couple of other standard identities for sums of angles or double angles.

Sum and Difference Identities for Cosine: For all angles α and β,

- $\cos(\alpha + \beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$
- $\cos(\alpha - \beta) = \cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta)$

Double Angle Formula:

- θ, $\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta)$

Sum and Difference Identities for Sine: For all angles α and β,

- $\sin(\alpha + \beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta)$
- $\sin(\alpha - \beta) = \sin(\alpha)\cos(\beta) - \cos(\alpha)\sin(\beta)$

Double Angle Formula:

- θ, $\sin(2\theta) = 2\sin(\theta)\cos(\theta)$

Example 10.

(a) *Find the exact value of* $\cos(15^\circ)$.

(b) *Verify the identity:* $\cos\left(\frac{\pi}{2} - \theta\right) = \sin(\theta)$.

Solution.

(a) *In order to use the theorem to find* $\cos(15^\circ)$*, we need to write* 15° *as a sum or difference of angles whose cosines and sines we know. One way to do so is to write* $15^\circ = 45^\circ - 30^\circ$.

$$\begin{aligned}
\cos(15^\circ) &= \cos(45^\circ - 30^\circ) \\
&= \cos(45^\circ)\cos(30^\circ) + \sin(45^\circ)\sin(30^\circ) \\
&= \left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{3}}{2}\right) + \left(\frac{\sqrt{2}}{2}\right)\left(\frac{1}{2}\right) \\
&= \frac{\sqrt{6} + \sqrt{2}}{4}
\end{aligned}$$

(b) *In a straightforward application of the theorem, we find*

$$\begin{aligned}
\cos\left(\frac{\pi}{2}-\theta\right) &= \cos\left(\frac{\pi}{2}\right)\cos(\theta)+\sin\left(\frac{\pi}{2}\right)\sin(\theta) \\
&= (0)(\cos(\theta))+(1)(\sin(\theta)) \\
&= \sin(\theta)
\end{aligned}$$

■

Example 11.

(a) *Find the exact value of* $\sin\left(\frac{19\pi}{12}\right)$

(b) *If* α *is a Quadrant II angle with* $\sin(\alpha)=\frac{5}{13}$*, and* β *is a Quadrant III angle with* $\tan(\beta)=2$*, find* $\sin(\alpha-\beta)$*.*

(c) *Derive a formula for* $\tan(\alpha+\beta)$ *in terms of* $\tan(\alpha)$ *and* $\tan(\beta)$*.*

Solution.

(a) *As in the earlier example, we need to write the angle* $\frac{19\pi}{12}$ *as a sum or difference of common angles. The denominator of* 12 *suggests a combination of angles with denominators* 3 *and* 4*. One such combination is* $\frac{19\pi}{12}=\frac{4\pi}{3}+\frac{\pi}{4}$*. Applying what we know about sum and difference with Sine, we get*

$$\begin{aligned}
\sin\left(\frac{19\pi}{12}\right) &= \sin\left(\frac{4\pi}{3}+\frac{\pi}{4}\right) \\
&= \sin\left(\frac{4\pi}{3}\right)\cos\left(\frac{\pi}{4}\right)+\cos\left(\frac{4\pi}{3}\right)\sin\left(\frac{\pi}{4}\right) \\
&= \left(-\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{2}}{2}\right)+\left(-\frac{1}{2}\right)\left(\frac{\sqrt{2}}{2}\right) \\
&= \frac{-\sqrt{6}-\sqrt{2}}{4}
\end{aligned}$$

(b) *In order to find* $\sin(\alpha-\beta)$ *usingthe theorem, we need to find* $\cos(\alpha)$ *and both* $\cos(\beta)$ *and* $\sin(\beta)$*. To find* $\cos(\alpha)$*, we use the Pythagorean Identity* $\cos^2(\alpha)+\sin^2(\alpha)=1$*. Since* $\sin(\alpha)=\frac{5}{13}$*, we have* $\cos^2(\alpha)+\left(\frac{5}{13}\right)^2=1$*, or* $\cos(\alpha)=\pm\frac{12}{13}$*. Since* α *is a Quadrant II angle,* $\cos(\alpha)=-\frac{12}{13}$*. We now set about finding* $\cos(\beta)$ *and* $\sin(\beta)$*. We have several ways to proceed, but the Pythagorean Identity* $1+\tan^2(\beta)=\sec^2(\beta)$ *is a quick way to get* $\sec(\beta)$*, and hence,* $\cos(\beta)$*. With* $\tan(\beta)=2$*, we get* $1+2^2=\sec^2(\beta)$ *so that*

$\sec(\beta) = \pm\sqrt{5}$. *Since* β *is a Quadrant III angle, we choose* $\sec(\beta) = -\sqrt{5}$ *so* $\cos(\beta) = \dfrac{1}{\sec(\beta)} = \dfrac{1}{-\sqrt{5}} = -\dfrac{\sqrt{5}}{5}$. *We now need to determine* $\sin(\beta)$. *We could use The Pythagorean Identity* $\cos^2(\beta) + \sin^2(\beta) = 1$, *but we opt instead to use a quotient identity. From* $\tan(\beta) = \dfrac{\sin(\beta)}{\cos(\beta)}$, *we have* $\sin(\beta) = \tan(\beta)\cos(\beta)$ *so we get* $\sin(\beta) = (2)\left(-\dfrac{\sqrt{5}}{5}\right) = -\dfrac{2\sqrt{5}}{5}$. *We now have all the pieces needed to find* $\sin(\alpha - \beta)$*:*

$$\begin{aligned}
\sin(\alpha - \beta) &= \sin(\alpha)\cos(\beta) - \cos(\alpha)\sin(\beta) \\
&= \left(\frac{5}{13}\right)\left(-\frac{\sqrt{5}}{5}\right) - \left(-\frac{12}{13}\right)\left(-\frac{2\sqrt{5}}{5}\right) \\
&= -\frac{29\sqrt{5}}{65}
\end{aligned}$$

(c) *We can start expanding* $\tan(\alpha + \beta)$ *using a quotient identity and our sum formulas*

$$\begin{aligned}
\tan(\alpha + \beta) &= \frac{\sin(\alpha + \beta)}{\cos(\alpha + \beta)} \\
&= \frac{\sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta)}{\cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)}
\end{aligned}$$

Since $\tan(\alpha) = \dfrac{\sin(\alpha)}{\cos(\alpha)}$ *and* $\tan(\beta) = \dfrac{\sin(\beta)}{\cos(\beta)}$, *it looks as though if we divide both numerator and denominator by* $\cos(\alpha)\cos(\beta)$ *we will have what we want*

$$\begin{aligned}
\tan(\alpha+\beta) &= \frac{\sin(\alpha)\cos(\beta)+\cos(\alpha)\sin(\beta)}{\cos(\alpha)\cos(\beta)-\sin(\alpha)\sin(\beta)} \cdot \frac{\dfrac{1}{\cos(\alpha)\cos(\beta)}}{\dfrac{1}{\cos(\alpha)\cos(\beta)}} \\
&= \frac{\dfrac{\sin(\alpha)\cos(\beta)}{\cos(\alpha)\cos(\beta)}+\dfrac{\cos(\alpha)\sin(\beta)}{\cos(\alpha)\cos(\beta)}}{\dfrac{\cos(\alpha)\cos(\beta)}{\cos(\alpha)\cos(\beta)}-\dfrac{\sin(\alpha)\sin(\beta)}{\cos(\alpha)\cos(\beta)}} \\
&= \frac{\dfrac{\sin(\alpha)\cancel{\cos(\beta)}}{\cos(\alpha)\cancel{\cos(\beta)}}+\dfrac{\cancel{\cos(\alpha)}\sin(\beta)}{\cancel{\cos(\alpha)}\cos(\beta)}}{\dfrac{\cancel{\cos(\alpha)}\cancel{\cos(\beta)}}{\cancel{\cos(\alpha)}\cancel{\cos(\beta)}}-\dfrac{\sin(\alpha)\sin(\beta)}{\cos(\alpha)\cos(\beta)}} \\
&= \frac{\tan(\alpha)+\tan(\beta)}{1-\tan(\alpha)\tan(\beta)}
\end{aligned}$$

Naturally, this formula is limited to those cases where all of the tangents are defined. ■

Example 12.

(a) *If* $\sin(\theta) = x$ *for* $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$, *find an expression for* $\sin(2\theta)$ *in terms of* x.

(b) *Verify the identity:* $\sin(2\theta) = \dfrac{2\tan(\theta)}{1+\tan^2(\theta)}$.

(c) *Express* $\cos(3\theta)$ *as a polynomial in terms of* $\cos(\theta)$.

Solution.

(a) *If your first reaction to '*$\sin(\theta) = x$*' is 'No it's not,* $\cos(\theta) = x$*!' then you have indeed learned something, and we take comfort in that. However, context is everything. Here, '*x*' is just a variable - it does not necessarily represent the* x*-coordinate of the point on The Unit Circle which lies on the terminal side of* θ*, assuming* θ *is drawn in standard position. Here,* x *represents the quantity* $\sin(\theta)$*, and what we wish to know is how to express* $\sin(2\theta)$ *in terms of* x*. We will see more of this kind of thing in future sections, and, as usual, this is something we need for Calculus. Since* $\sin(2\theta) = 2\sin(\theta)\cos(\theta)$*, we need to write* $\cos(\theta)$ *in terms of* x *to finish the problem. We substitute* $x = \sin(\theta)$ *into the Pythagorean Identity,*

$\cos^2(\theta) + \sin^2(\theta) = 1$, *to get* $\cos^2(\theta) + x^2 = 1$, *or* $\cos(\theta) = \pm\sqrt{1-x^2}$. *Since* $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$, $\cos(\theta) \geq 0$, *and thus* $\cos(\theta) = \sqrt{1-x^2}$. *Our final answer is* $\sin(2\theta) = 2\sin(\theta)\cos(\theta) = 2x\sqrt{1-x^2}$.

(b) *We start with the right hand side of the identity and note that* $1+\tan^2(\theta) = \sec^2(\theta)$. *From this point, we use the Reciprocal and Quotient Identities to rewrite* $\tan(\theta)$ *and* $\sec(\theta)$ *in terms of* $\cos(\theta)$ *and* $\sin(\theta)$*:*

$$\begin{aligned}
\frac{2\tan(\theta)}{1+\tan^2(\theta)} &= \frac{2\tan(\theta)}{\sec^2(\theta)} = \frac{2\left(\frac{\sin(\theta)}{\cos(\theta)}\right)}{\frac{1}{\cos^2(\theta)}} = 2\left(\frac{\sin(\theta)}{\cos(\theta)}\right)\cos^2(\theta) \\
&= 2\left(\frac{\sin(\theta)}{\cancel{\cos(\theta)}}\right)\cancel{\cos(\theta)}\cos(\theta) = 2\sin(\theta)\cos(\theta) = \sin(2\theta)
\end{aligned}$$

(c) *In our list of identities, one of the formulas for* $\cos(2\theta)$, *namely* $\cos(2\theta) = 2\cos^2(\theta) - 1$, *expresses* $\cos(2\theta)$ *as a polynomial in terms of* $\cos(\theta)$. *We are now asked to find such an identity for* $\cos(3\theta)$. *Using the sum formula for cosine, we begin with*

$$\begin{aligned}
\cos(3\theta) &= \cos(2\theta + \theta) \\
&= \cos(2\theta)\cos(\theta) - \sin(2\theta)\sin(\theta)
\end{aligned}$$

Our ultimate goal is to express the right hand side in terms of $\cos(\theta)$ *only. We substitute* $\cos(2\theta) = 2\cos^2(\theta) - 1$ *and* $\sin(2\theta) = 2\sin(\theta)\cos(\theta)$ *which yields*

$$\begin{aligned}
\cos(3\theta) &= \cos(2\theta)\cos(\theta) - \sin(2\theta)\sin(\theta) \\
&= \left(2\cos^2(\theta) - 1\right)\cos(\theta) - (2\sin(\theta)\cos(\theta))\sin(\theta) \\
&= 2\cos^3(\theta) - \cos(\theta) - 2\sin^2(\theta)\cos(\theta)
\end{aligned}$$

Finally, we exchange $\sin^2(\theta)$ *for* $1 - \cos^2(\theta)$ *courtesy of the Pythagorean Identity, and get*

$$\begin{aligned}
\cos(3\theta) &= 2\cos^3(\theta) - \cos(\theta) - 2\sin^2(\theta)\cos(\theta) \\
&= 2\cos^3(\theta) - \cos(\theta) - 2\left(1 - \cos^2(\theta)\right)\cos(\theta) \\
&= 2\cos^3(\theta) - \cos(\theta) - 2\cos(\theta) + 2\cos^3(\theta) \\
&= 4\cos^3(\theta) - 3\cos(\theta)
\end{aligned}$$

and we are done. ■

Lists of Identities

The Pythagorean Identities:

(a) $\cos^2(\theta) + \sin^2(\theta) = 1.$

Common Alternate Forms:

- $1 - \sin^2(\theta) = \cos^2(\theta)$
- $1 - \cos^2(\theta) = \sin^2(\theta)$

(b) $1 + \tan^2(\theta) = \sec^2(\theta)$, provided $\cos(\theta) \neq 0$.

Common Alternate Forms:

- $\sec^2(\theta) - \tan^2(\theta) = 1$
- $\sec^2(\theta) - 1 = \tan^2(\theta)$

(c) $1 + \cot^2(\theta) = \csc^2(\theta)$, provided $\sin(\theta) \neq 0$.

Common Alternate Forms:

- $\csc^2(\theta) - \cot^2(\theta) = 1$
- $\csc^2(\theta) - 1 = \cot^2(\theta)$

Reciprocal and Quotient Identities:

- $\sec(\theta) = \dfrac{1}{\cos(\theta)}$, provided $\cos(\theta) \neq 0$;
  if $\cos(\theta) = 0$, $\sec(\theta)$ is undefined.
- $\csc(\theta) = \dfrac{1}{\sin(\theta)}$, provided $\sin(\theta) \neq 0$;
  if $\sin(\theta) = 0$, $\csc(\theta)$ is undefined.
- $\tan(\theta) = \dfrac{\sin(\theta)}{\cos(\theta)}$, provided $\cos(\theta) \neq 0$;
  if $\cos(\theta) = 0$, $\tan(\theta)$ is undefined.
- $\cot(\theta) = \dfrac{\cos(\theta)}{\sin(\theta)}$, provided $\sin(\theta) \neq 0$;
  if $\sin(\theta) = 0$, $\cot(\theta)$ is undefined.

Pythagorean Conjugates

- $1 - \cos(\theta)$ and $1 + \cos(\theta)$:
  $(1 - \cos(\theta))(1 + \cos(\theta)) = 1 - \cos^2(\theta) = \sin^2(\theta)$

- $1 - \sin(\theta)$ and $1 + \sin(\theta)$:
  $(1 - \sin(\theta))(1 + \sin(\theta)) = 1 - \sin^2(\theta) = \cos^2(\theta)$
- $\sec(\theta) - 1$ and $\sec(\theta) + 1$:
  $(\sec(\theta) - 1)(\sec(\theta) + 1) = \sec^2(\theta) - 1 = \tan^2(\theta)$
- $\sec(\theta) - \tan(\theta)$ and $\sec(\theta) + \tan(\theta)$:
  $(\sec(\theta) - \tan(\theta))(\sec(\theta) + \tan(\theta)) = \sec^2(\theta) - \tan^2(\theta) = 1$
- $\csc(\theta) - 1$ and $\csc(\theta) + 1$:
  $(\csc(\theta) - 1)(\csc(\theta) + 1) = \csc^2(\theta) - 1 = \cot^2(\theta)$
- $\csc(\theta) - \cot(\theta)$ and $\csc(\theta) + \cot(\theta)$:
  $(\csc(\theta) - \cot(\theta))(\csc(\theta) + \cot(\theta)) = \csc^2(\theta) - \cot^2(\theta) = 1$

Exploration In this activity, we investigate how a sum of two angles identity for the sine function helps us gain a different perspective on the average rate of change of the sine function.
Recall that for any function f on an interval $[a, a+h]$, its average rate of change is

$$\text{AROC}_{[a,a+h]} = \frac{f(a+h) - f(a)}{h}.$$

a. Let $f(x) = \sin(x)$. Use the definition of $\text{AROC}_{[a,a+h]}$ to write an expression for the average rate of change of the sine function on the interval $[a+h, a]$.

b. Apply the sum of two angles identity for the sine function, $\sin(\alpha + \beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta)$, to the expression $\sin(a+h)$.

c. Explain why your work in (a) and (b) together with some algebra shows that

$$\text{AROC}_{[a,a+h]} = \sin(a) \cdot \frac{\cos(h) - 1}{h} - \cos(a)\frac{\sin(h)}{h}.$$

d. In calculus, we move from *average* rate of change to *instantaneous* rate of change by letting h approach 0 in the expression for average rate of change. Using a computational device in radian mode, investigate the behavior of

$$\frac{\cos(h) - 1}{h}$$

as h gets close to 0. What happens? Similarly, how does $\frac{\sin(h)}{h}$ behave for small values of h? What does this tell us about $\text{AROC}_{[a,a+h]}$ as h approaches 0?

More Identities

In the previous sections, we saw the utility of the Pythagorean Identities. Not only did these identities help us compute the values of the circular functions for angles, they were also useful in simplifying expressions involving the circular functions. We will introduce this set of identities as the 'Even / Odd' identities and we will discuss them further with trigonometric transformations in a later section.

Even / Odd Identities: For all applicable angles θ,

- $\cos(-\theta) = \cos(\theta)$
- $\sec(-\theta) = \sec(\theta)$
- $\sin(-\theta) = -\sin(\theta)$
- $\csc(-\theta) = -\csc(\theta)$
- $\tan(-\theta) = -\tan(\theta)$
- $\cot(-\theta) = -\cot(\theta)$

8.3.2 Application of Trig Identities

Verifying Identities

In this section, we will look at strategies to verify identities.

Remark **Strategies for Verifying Identities**

- Try working on the more complicated side of the identity.
- Use the Reciprocal and Quotient Identities to write functions on one side of the identity in terms of the functions on the other side of the identity. Simplify the resulting complex fractions.
- Add rational expressions with unlike denominators by obtaining common denominators.
- Use the Pythagorean Identities to 'exchange' sines and cosines, secants and tangents, cosecants and cotangents, and simplify sums or differences of squares to one term.
- Multiply numerator **and** denominator by Pythagorean Conjugates in order to take advantage of the Pythagorean Identities.
- If you find yourself stuck working with one side of the identity, try starting with the other side of the identity and see if you can find a way to bridge the two parts of your work.

Example 13. *Verify the following:*

$$\tan(\theta)\cos(\theta) = \sin(\theta)$$

Explanation Let's start with the more complicated side. We know that we want to end up with $\sin(\theta)$ in the end, so using the Quotient identity to replace $\tan(\theta)$ with $\dfrac{\sin(\theta)}{\cos(\theta)}$ is a reasonable place to start.

$$\tan(\theta)\cos(\theta) = \left(\frac{\sin(\theta)}{\cos(\theta)}\right)\cos(\theta)$$

We can now cancel our $\cos(\theta)$ terms.

$$\tan(\theta)\cos(\theta) = \left(\frac{\sin(\theta)}{\cancel{\cos(\theta)}}\right)\cancel{\cos(\theta)}$$

This, thankfully, leaves us with our original equation, so we have verified this identity.

$$\tan(\theta)\cos(\theta) = \sin(\theta)$$

Example 14. *Verify the following:*

$$\frac{\sec(x) - cos(x)}{\sec(x)} = sin^2(x)$$

Explanation Let's start with the more complicated side. Since $\sec(x)$ is the reciprocal of $\cos(x)$, rewriting this side completely in terms of $\cos(x)$ could help us verify this identity.

$$\frac{\sec(x) - \cos(x)}{\sec(x)} = \frac{\frac{1}{\cos(x)} - \cos(x)}{\frac{1}{\cos(x)}}$$

We can now simplify the fractional division by inverting and multiplying.

$$\frac{\frac{1}{\cos(x)} - \cos(x)}{\frac{1}{\cos(x)}} = \left(\frac{1}{\cos(x)} - \cos(x)\right) \cdot \cos(x)$$

We continue by distributing our $\cos(x)$ term and then simplify.

$$\left(\frac{1}{\cos(x)} - \cos(x)\right) \cdot \cos(x) = \frac{\cos(x)}{\cos(x)} - \cos^2(x) = 1 - \cos^2(x)$$

Using an alternate form of the Pythagorean Identity, we can make the following substitution.

$$1 - \cos^2(x) = sin^2(x)$$

We have now verified our original equation.

$$\frac{\sec(x) - cos(x)}{\sec(x)} = sin^2(x)$$

Example 15. *Verify the following:*

$$\tan(\theta) + \cot(\theta) = \csc(\theta)\sec(\theta)$$

Explanation Let's start with the left side. By using the Quotient identities we can change both of our terms to be in the form of $\sin(\theta)$ and $\cos(\theta)$

$$\tan(\theta) + \cot(\theta) = \frac{\sin(\theta)}{\cos(\theta)} + \frac{\cos(\theta)}{\sin(\theta)}$$

We can find a common demoninator to begin combining our terms.

$$\frac{\sin(\theta)}{\cos(\theta)} + \frac{\cos(\theta)}{\sin(\theta)} = \frac{\sin^2(\theta)}{\sin(\theta)\cos(\theta)} + \frac{\cos^2(\theta)}{\sin(\theta)\cos(\theta)}$$

Now we combine our terms and simplify.

$$\frac{\sin^2(\theta)}{\sin(\theta)\cos(\theta)} + \frac{\cos^2(\theta)}{\sin(\theta)\cos(\theta)} = \frac{\sin^2(\theta) + \cos^2(\theta)}{\sin(\theta)\cos(\theta)} = \frac{1}{\sin(\theta)\cos(\theta)}$$

We are now left with two terms being multiplied together. We will split them up to better show the next step.

$$\frac{1}{\sin(\theta)\cos(\theta)} = \frac{1}{\sin(\theta)} \cdot \frac{1}{\cos(\theta)}$$

We can use the Reciprocal identies for these two terms and we have found what we wanted.

$$\frac{1}{\sin(\theta)} \cdot \frac{1}{\cos(\theta)} = \csc(\theta)\sec(\theta)$$

The identity is verified.

$$\tan(\theta) + \cot(\theta) = \csc(\theta)\sec(\theta)$$

Example 16. *Verify the following:*

$$\sin(x) = \tan(x) + \cos(x)$$

Explanation Let's start with the more complicated side. By using the Quotient identities we can change both of our terms to be in the form of $\sin(\theta)$ and $\cos(\theta)$

$$\tan(x) + \cos(x) = \frac{\sin(x)}{\cos(x)} + \cos(x)$$

We can find a common demoninator to begin combining our terms.

$$\frac{\sin(x)}{\cos(x)} + \cos(x) = \frac{\sin(x)}{\cos(x)} + \frac{\cos^2(x)}{\cos(x)}$$

We combine our terms.

$$\frac{\sin(x)}{\cos(x)} + \frac{\cos^2(x)}{\cos(x)} = \frac{\sin(x) + \cos^2(x)}{\cos(x)}$$

We use a Pythagorean identity.

$$\frac{\sin(x) + \cos^2(x)}{\cos(x)} = \frac{\sin(x) + 1 - \sin^2(x)}{\cos(x)}$$

We are now at a difficult point. The equation does not seem to be simplifying and we are not making any progress. It is possible that this is **NOT** equal. In order, to prove that we have to go back and use a test value in our original equation. Let's try $\frac{\pi}{6}$.

$$\sin(\frac{\pi}{6}) = \tan(\frac{\pi}{6}) + \cos(\frac{\pi}{6})$$

Since this is a trig value, we can go ahead and evaluate it.

$$\frac{1}{2} = \frac{\sqrt{3}}{3} + \frac{\sqrt{3}}{2}$$

We again find a common denominator

$$\frac{1}{2} = \frac{2\sqrt{3}}{6} + \frac{3\sqrt{3}}{6}$$

With a little simplifying, we have the following equation.

$$\frac{1}{2} = \frac{5\sqrt{3}}{6}$$

$\frac{5\sqrt{3}}{6}$ is definitely larger than $\frac{1}{2}$ (verify with a calculator) and thus not equal.

$$\frac{1}{2} \neq \frac{5\sqrt{3}}{6}$$

We have proved that this is equation **NOT** equal.

$$\sin(x) \neq \tan(x) + \cos(x)$$

Part 9

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