What is a Function?

Remark 1. ***Motivating Questions***

- *What is a function?*

Mathematical Models

A mathematical model is an abstract concept through which we use mathematical language and notation to describe a phenomenon in the world around us. One example of a mathematical model is found in Dolbear's Law, which has proven to be remarkably accurate for the behavior of snowy tree crickets. For even more of the story, including a reference to this phenomenon on the popular show The Big Bang Theory, see https://priceonomics.com/how-to-tell-the-temperature-using-crickets/. In the late 1800s, the physicist Amos Dolbear was listening to crickets chirp and noticed a pattern: how frequently the crickets chirped seemed to be connected to the outside temperature.

If we let T represent the temperature in degrees Fahrenheit and N the number

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of chirps per minute, we can summarize Dolbear's observations in the following table.

N(chirps per minute)	T(degrees Fahrenheit)
40	50
80	60
120	70
160	80

For a mathematical model, we often seek an algebraic formula that captures observed behavior accurately and can be used to predict behavior not yet observed. For the data in the table above, we observe that each of the ordered pairs in the table make the equation

$$T = 40 + 0.25N$$

true. For instance, $70 = 40 + 0.25(120)$. Indeed, scientists who made many additional cricket chirp observations following Dolbear's initial counts found that the formula above holds with remarkable accuracy for the snowy tree cricket in temperatures ranging from about 50° F to 85° F.

This model captures a pattern that is found in the world, and can be used to predict the temperature if only the number of chirps per minute is known. Not all phenomenon in the world that can be measured mathematically occur in a predictable pattern. In this section, we will study functions which are mathematical ways of formally studying situations where for a given input, such as the number of chirps above, there is one consistant output. For situations where a given input might give a variety of outputs, we encourage you to study statistics! Also note that this relationship is not causal. Even though the number of chirps is considered out "input" the increase in chirps does not cause the temperature to increase.

Functions

The mathematical concept of a function is one of the most central ideas in all of mathematics, in part since functions provide an important tool for representing and explaining patterns. At its core, a function is a repeatable process that takes a collection of input values and generates a corresponding collection of output values with the property that if we use a particular single input, the process always produces exactly the same single output.

For instance, Dolbear's Law provides a process that takes a given number of chirps between 40 and 180 per minute and reliably produces the corresponding temperature that corresponds to the number of chirps, and thus this equation generates a function. We often give functions shorthand names; using "D" for

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the "Dolbear" function, we can represent the process of taking inputs (observed chirp rates) to outputs (corresponding temperatures) using arrows:

$$80 \xrightarrow{D} 60$$
$$120 \xrightarrow{D} 70$$
$$N \xrightarrow{D} 40 + 0.25N$$

Alternatively, for the relationship "$80 \xrightarrow{D} 60$" we can also use the equivalent notation "$D(80) = 60$" to indicate that Dolbear's Law takes an input of 80 chirps per minute and produces a corresponding output of 60 degrees Fahrenheit. More generally, we write "$T = D(N) = 40 + 0.25N$" to indicate that a certain temperature, T, is determined by a given number of chirps per minute, N, according to the process $D(N) = 40 + 0.25N$.

We will define a function informally and formally. The informal definition corresponds to the way we will most often think of functions, as a process with inputs and outputs.

Definition 1 (Informal Definition of a Function)**.** *A **function** is a process that may be applied to a collection of input values to produce a corresponding collection of output values in such a way that the process produces one and only one output value for any single input value.*

The formal definition of a function will establish a function as a special type of relation. Recall that a *relation* is a collection of points of the form (x, y). If the point (x_0, y_0) is in the relation, then we say x_0 and y_0 are *related*.

Definition 2 (Formal Definition of a Function)**.** *A **function** is a collection of ordered pairs (x, y) such that any particular value of x is paired with at most one value for y. That is, a relation in which each x-coordinate is matched with only one y-coordinate is said to describe y as a **function** of x.*

How is this definition consistent with the informal definition, which describes a function as a process? Well, if you have a collection of ordered pairs (x, y), you can choose to view the left number as an input, and the right value as the output. For a function f, $f(x)$ is defined as the unique y value that x is paired with. If x does not appear among the first coordinates of the ordered pairs, then x is not a possible input to f, since there is no way to determine $f(x)$.

Domain and Range

We now give some important definitions that allow us to talk about the inputs and outputs of functions.

Definition 3. *Let f be a function from A to B. The set A of possible inputs to f is called the **domain** of f. The set B is called the **codomain** of f.*

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Sometimes, when we are given a function as a formula, we are not told the domain. In these circumstances we use the *implied domain*.

Definition 4. *Let f be a function whose inputs are real numbers. The* ***implied domain*** *of f is the collection of all real numbers x for which $f(x)$ is a real number.*

Definition 5. *Let f be a function from A to B. The* ***range*** *of f is the collection of the outputs of f.*

It is important to note that the definition of a function includes its domain and range. A function needs a rule and a domain to precisely determine a set of points, the range.

Interval Notation

Intervals (contiguous sections of the number line) play an important role in expressing the domains of many types of functions. As a standard way of writing these solutions, we rely on *interval notation.* Interval notation is a short-hand way of representing the intervals as they appear when sketched on a number line. As an example, consider $x \geq \frac{4}{3}$ which, when sketched on a number line, is given by

This sketch consists of a single interval with left-hand endpoint at $\frac{4}{3}$ and no right-hand endpoint (it keeps going). In interval notation, this would be written as $\left[\frac{4}{3}, \infty\right)$. This is an example of a *closed infinite interval*, "closed" because the point at $\frac{4}{3}$ (the only endpoint) is included and "infinite" because it has infinite width. The solid dot at $\frac{4}{3}$ indicates that the point is included in the interval.

There are four different types of infinite intervals, two are closed infinite intervals (which contain their respective endpoint) and the other two are open infinite intervals (which do not contain the endpoint). For a a fixed real number, these are:

(a) $[a, \infty)$ represents $x \geq a$,

(b) $(-\infty, a]$ represents $x \leq a$,

(c) (a, ∞) represents $x > a$, and

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(d) $(-\infty, a)$ represents $x < a$.

The notation indicates uses the square bracket to indicate that the endpoint is included and the round parenthesis to indicate that the endpoint is not included.

Not every interval is infinite, however. Consider the interval in the following sketch

which consists of all x with $-2 < x \leq 3$. It is not an infinite interval, having endpoints at -2 and 3. The endpoint at -2 is not included, but the endpoint at 3 is included. In interval notation this would be written as $(-2, 3]$. As with the infinite intervals, the square bracket indicates that the right-hand endpoint is included and the round parenthesis endicates that the left-hand endpoint is not included. (This is an example of a "half-open interval".)

For a bounded intervals (ones that are not infinite), there are also four possibilities. For a and b both fixed real numbers, these are:

(a) $[a, b]$ represents $a \leq x \leq b$,

(b) $[a, b)$ represents $a \leq x < b$,

(c) $(a, b]$ represents $a < x \leq b$ and

(d) (a, b) represents $a < x < b$.

Practically, this amounts to writing the left-hand endpoint, the right-hand endpoint, then indicating which endpoints are included in the interval. When neither endpoint is included, (a, b) can be mistaken for a point on a graph. You will need to use the context to know which is meant.

Intercepts

Zero is a very important number, and as we will see later, knowing where a function's x- or y-value equals zero can be powerful information.

Definition 6 (Intercepts)**.** *Suppose f is a function and set $y = f(x)$.*

*An x-**intercept** is a point $(a, 0)$ such that $f(a) = 0$. That is, it's a point where the graph of the function intersects the x-axis.*

*The y-**intercept** is a point $(0, b)$ such that $f(0) = b$. That is, it's a point in which the graph of the function intersects the y-axis. Unlike x-intercepts, a function can only have one y-intercept.*

Example 1. *Find the x- and y-intercepts of the function f whose graph is given below.*

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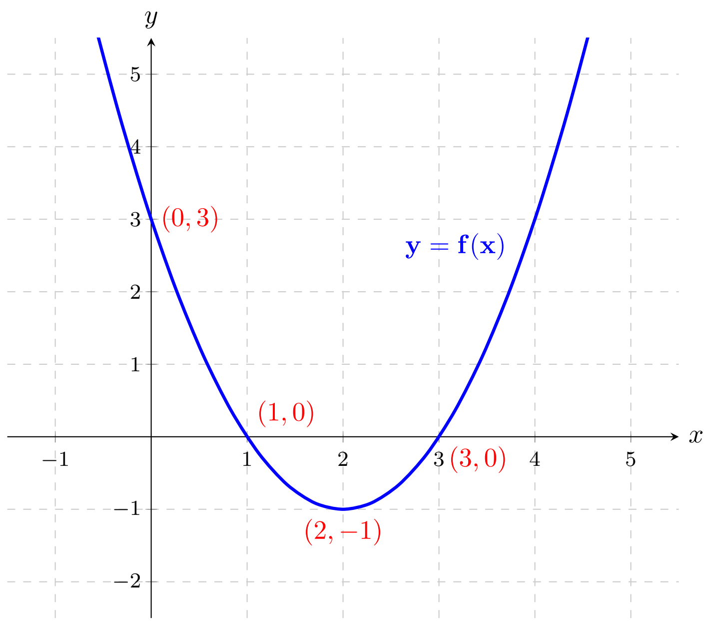

Explanation. *The graph intersects the x-axis at the points $(1, 0)$ and $(3, 0)$, and intersects the y-axis at the point $(0, 3)$. That means the x-intercepts are at $(1, 0)$ and $(3, 0)$, and the y-intercept is at $(0, 3)$.*

Summary 1.

- *Informally, a function is a process that may be applied to a collection of input values to produce a corresponding collection of output values in such a way that the process produces one and only one output value for any single input value.*
- *Formally, a function is a collection of ordered pairs (x, y) such that any particular value of x is paired with at most one value for y. That is, a relation in which each x-coordinate is matched with only one y-coordinate is said to describe y as a **function** of x.*
- *Functions can be used to model many phenomena in the real world, as long as for a given input there is one predictable output.*

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