Precalculus with Review 1: Unit 6

June 23, 2023

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Rational Functions

6.1 Rational Functions

Learning Objectives

- The Famous Function $f(x) = 1/x$
  - Introduce 1/x as a famous function, examine its properties
  - Composing 1/x with other functions (introduce secant, cosecant, and cotangent)
- Definition of Rational Functions
  - What is a rational function?
  - How can one find the domain of a rational function?
  - How can one identify if a function is a rational function?
- Polynomial Long Division
  - Introduce long division of polynomials
  - Understand quotient and remainder in this setting
  - Investigate the relationsships between the degree of the numerator and denominator of a rational function and the resulting degrees after long division

6.1.1 The Famous Function $f(x) = 1/x$

Motivating Questions

- What is a possible explanation, in terms of functions, for the fact that one cannot divide by zero?
- Are sine, cosine, and tangent really the only relevant trigonometric functions? Are there others? If so, how to understand them?

Introduction

We know that if a and b are two real numbers, then a/b makes sense, as long as b is not equal to zero. Let's look at what happens when we make divisions by numbers very close to zero, but not equal to zero. Take $a = 1$ for simplicity.

$$\frac{1}{0.1} = 10$$
$$\frac{1}{0.01} = 100$$
$$\frac{1}{0.001} = 1000$$
$$\frac{1}{0.0001} = 10000$$

This pattern makes us want to say that $1/0$ equals to $+\infty$ (whatever $+\infty$ means, at this point), but this doesn't work. To understand why, let's consider divisions by numbers very close to zero, but this time negative.

$$\frac{1}{-0.1} = -10$$
$$\frac{1}{-0.01} = -100$$
$$\frac{1}{-0.001} = -1000$$
$$\frac{1}{-0.0001} = -10000$$

The same reasoning as before would tempt us to say that $1/0$ equals $-\infty$. And this raises the question of whether ∞ or $-\infty$ is the better choice. While on an instinctive psychological level we could think that $+\infty$ is better than $-\infty$, there's really no way to decide[1] — and this turns out to be related to the concept of *limit*, which you'll learn in Calculus.

Graph and End Behavior

To continue our discussion in a more precise way, let's consider the function f, defined for all real numbers *except for zero*, given by $f(x) = 1/x$. This is a very famous function, particularly useful as the building block for *rational functions*, which we'll discuss soon. Note that essentially what we have just done in the introduction was to consider the values

$$f(0.1), f(0.01), f(0.001), \text{ and } f(0.0001),$$

as well as

$$f(-0.1), f(-0.01), f(-0.001), \text{ and } f(-0.0001).$$

To get a good idea of the behavior a function has, our main strategy so far has been to just consider its graph. Naturally, plugging a handful of values won't cut it. Let's see what happens when we go to the other extreme and make divisions by very large numbers:

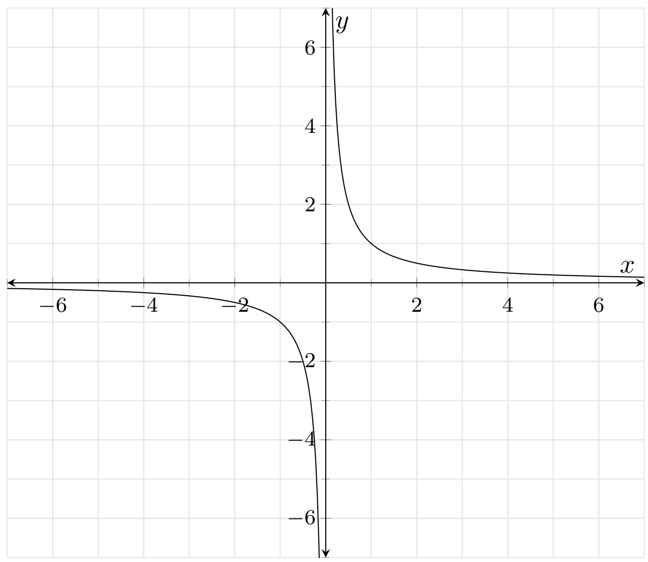
$$\begin{aligned} \frac{1}{10} &= 0.1 \\ \frac{1}{100} &= 0.01 \\ \frac{1}{1000} &= 0.001 \\ \frac{1}{10000} &= 0.0001 \end{aligned}$$

And from the negative side:

$$\begin{aligned} \frac{1}{-10} &= -0.1 \\ \frac{1}{-100} &= -0.01 \\ \frac{1}{-1000} &= -0.001 \\ \frac{1}{-10000} &= -0.0001 \end{aligned}$$

Here's what the graph looks like.

[1] Algebraically, the explanation is simple: if one could make sense of $1/0$ and say that equals some number c, then this would give $1 = 0 \cdot c$, so $1 = 0$ — which is a complete collapse of the number system we have to deal with in our daily lives. But this doesn't give intuition for what is going on.

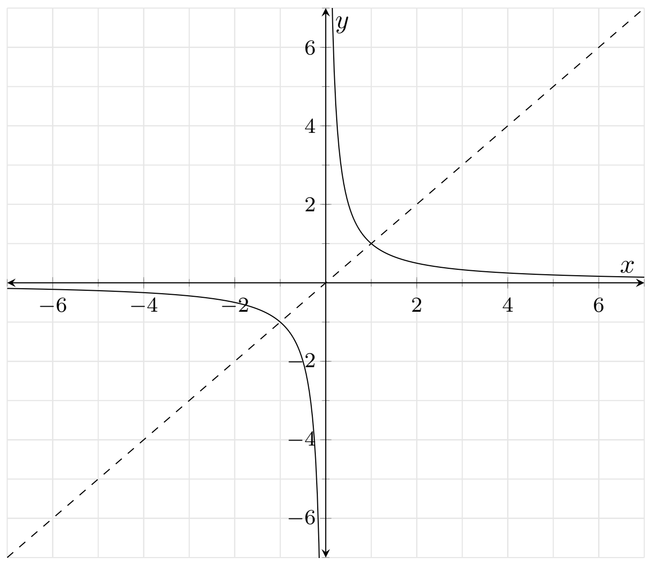


Here's what we can immediately see from the graph, confirming our intuition from the several divisions previously done:

End Behavior of $1/x$.

- If $x \to +\infty$, then $1/x \to 0$ (reads "when x tends to $+\infty$, $1/x$ tends to 0").
- If $x \to 0^+$, then $1/x \to +\infty$ (reads "when x tends to zero from the right, $1/x$ tends to $+\infty$").
- If $x \to 0^-$, then $1/x \to -\infty$ (reads "when x tends to zero from the left, $1/x$ tends to $-\infty$").
- If $x \to -\infty$, then $1/x \to 0$ (reads "when x tends to $-\infty$, $1/x$ tends to 0").

We say that the line $x = 0$ is a *vertical asymptote* for $f(x) = 1/x$, while the line $y = 0$ is a *horizontal asymptote*. We will discuss asymptotes of rational functions in general in the next unit.



Next, as far as symmetries go, we can see that the graph is symmetric about the origin. This indicates that $f(x) = 1/x$ is an odd function. You can also see this algebraically via

$$f(-x) = \frac{1}{-x} = -\frac{1}{x} = -f(x).$$

You may also noticed that the graph of $f(x) = \dfrac{1}{x}$ is symmetric across the line $y = x$. This indicates that $f(x) = 1/x$ is it's own inverse! You can also see this algebraically via

$$f((f(x)) = f\left(\frac{1}{x}\right) = \frac{1}{\frac{1}{x}} = \frac{1}{1} \cdot \frac{x}{1} = x$$

.

By the way, the graph of $f(x) = 1/x$ is called a *hyperbola*.

Summary Here are some of the function properties of the Famous Function $f(x) = \dfrac{1}{x}$.

- Domain: $(-\infty, 0) \cup (0, \infty)$. That is, all real numbers except $x = 0$.
- Range: $(-\infty, 0) \cup (0, \infty)$. That is, all real numbers except $x = 0$.
- Even/Odd/Neither: $f(x) = \dfrac{1}{x}$ is odd.
- Inveerse: The inverse function of $f(x) = \dfrac{1}{x}$ is $f(x) = \dfrac{1}{x}$.

- Intercepts: $f(x) = \dfrac{1}{x}$ has no x-intercepts and no y-intercepts.
- Increasing/Decreasing: $f(x) = \dfrac{1}{x}$ is decreasing everywhere it is defined.
- Concavity: $f(x) = \dfrac{1}{x}$ is concave down from $(-\infty, 0)$ and is concave up on $(0, \infty)$.
- Asympotes: $f(x) = \dfrac{1}{x}$ has a vertical asymptote at $x = 0$ and a horizontal asymptote at $y = 0$.

6.1.2 The Definition of a Rational Function

Introduction

We have previously discussed the function $1/x$. Note that both the numerator 1 and the denominator x are polynomials (1 is a *constant* polynomial). We can study what happens when we replace those with arbitrary polynomials.

> **Definition** A *rational function* is a function defined as a ratio $r(x) = \dfrac{p(x)}{q(x)}$ of two polynomials $p(x)$ and $q(x)$, and this ratio makes sense for all real values of x, *except* for those such that $q(x) = 0$.

Example 1. *Are the following functions rational? For which values of x is the function undefined?*

(a) $f(x) = \dfrac{x^2 - 2}{x + 1}$.

Explanation It is a rational function. It is defined for all values of x, except for $x = -1$, because this makes the denominator $x + 1$ be zero.

(b) $f(x) = \dfrac{x^4 - 3x + 1}{x^2 - 5x + 6}$.

Explanation It is a rational function. It is defined for all values of x, except for $x = 2$ and $x = 3$, because $x^2 - 5x + 6 = (x - 2)(x - 3)$.

(c) $f(x) = \dfrac{x - 1}{\sqrt{x^4 + 1}}$.

Explanation It is *not* a rational function, because the denominator $\sqrt{x^4 + 1}$ is *not a polynomial.* Recall that a square root cannot distribute over addition or subtraction, so $\sqrt{x^4 + 1} \neq \sqrt{x^4} + \sqrt{1} = x^2 + 1$. This is a common error to make when first learning about rational functions, so be sure to watch out for it! Note that even though this does not define a rational function, it is defined for all possible values of x, since $\sqrt{x^4 + 1} \geq 1 > 0$ for all x.

(d) *A polynomial function $p(x)$.*

Explanation Any polynomial function is a rational function, simply because we can write $p(x) = p(x)/1$, and 1 is a polynomial. And it is defined for every real value of x.

(e) $f(x) = x^2 - 1 + \dfrac{x^3}{x^5 - 1}$.

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Explanation It is a rational function, which is defined for all x except for $x = 1$. To see that this is a rational function, you can either say that it is the sum of the rational functions $x^2 - 1$ and $x^3/(x^5 - 1)$, or rewrite it as

$$f(x) = \frac{(x^2 - 1)(x^5 - 1) + x^3}{x^5 - 1} = \frac{x^7 - x^5 - x^2 + x^3 - 1}{x^5 - 1},$$

which is manifestly rational.

Domains of Rational Functions

While we said it already, it is worth emphasizing that a rational functions is not defined when the polynomial in the denominator is equal to zero. That is, if $r(x) = \dfrac{p(x)}{q(x)}$ is a rational function and both $p(x)$ and $q(x)$ are polynomials, than the domain of $r(x)$ is all values of x except those where $q(x) = 0$. Notice, that finding the domain of a rational function is going to require finding the x-intercepts of a polynomial!

Example 2. *Find the domain of the rational function* $f(x) = \dfrac{x^3 + 2x^2 - 11x - 20}{x^3 + 3x^2 - 4x - 12}$.

Explanation Notice that our discussion of finding the domain of a rational function does not involve the numerator at all. Since the numerator is a polynomial, it is defined for all values of x. What we need to do is look for where the denominator is equal to zero.

$$\begin{aligned}
x^3 + 3x^2 - 4x - 12 &= 0 && (1)\\
x^2(x + 3) - 4(x + 3) &= 0 && (2)\\
(x + 3)(x^2 - 4) &= 0 && (3)\\
(x + 3)(x - 2)(x + 2) &= 0 && (4)\\
x + 3 = 0 \quad OR \quad x - 2 &= 0 \quad OR \quad x + 2 = 0 && (5)\\
x = -3 \quad OR \quad x &= 2 \quad OR \quad x = -2 && (6)
\end{aligned}$$

Therefore, $x = -3, -3$, and 2 are not in the domain of our function f. We can write this domain in interval notation as $(-\infty, -3) \cup (-3, -2) \cup (-2, 2) \cup (2, \infty)$. From the fact that the the domain comes in four pieces, we can know that the graph of the rational function f will also have four pieces. Here is what this function looks like when graphed:

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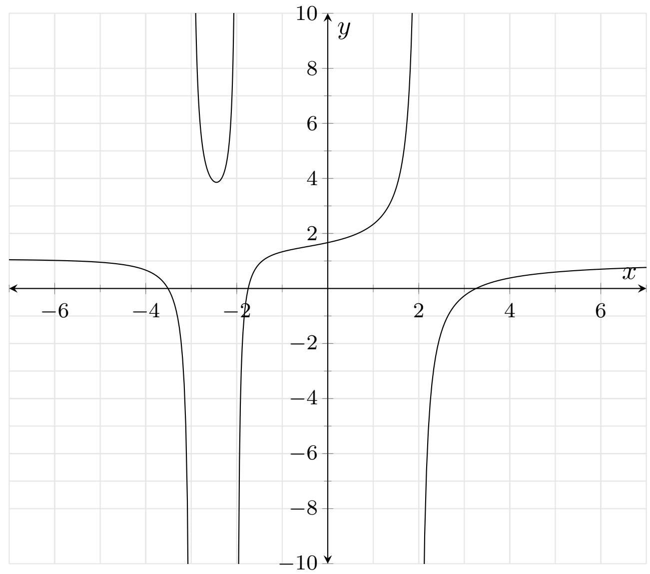

That looks quite different from the graphs of functions we have studied thus far! We will continue to explore the behavior of rational functions in this unit so that we can understand and predict this graph shape.

Combining Rational Functions

When we add, subtract, multiply, or divide rational functions, we get another rational function. Let's see why.

Adding and Subtracting Rational Functions Given the rational functions $f(x) = \dfrac{1}{x-1}$ and $g(x) = \dfrac{1}{x+1}$, we can rewrite them with a common denominator: $f(x) = \dfrac{1}{x-1}\left(\dfrac{x+1}{x+1}\right) = \dfrac{x+1}{(x-1)(x+1)}$ and $g(x) = \dfrac{1}{x+1}\left(\dfrac{x-1}{x-1}\right) = \dfrac{x-1}{(x-1)(x+1)}$. Then,

$$(f+g)(x) = f(x) + g(x) = \frac{x+1}{(x-1)(x+1)} + \frac{x-1}{(x-1)(x+1)} = \frac{2x}{(x-1)(x+1)},$$

yielding another rational function, since both the numerator and denominator are polynomials.

This idea can be expanded to the sum of any two rational functions. Given two rational functions $f(x) = \dfrac{p(x)}{q(x)}$ and $g(x) = \dfrac{r(x)}{s(x)}$, with $p(x)$, $q(x)$, $r(x)$, and $s(x)$

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being polynomials, then

$$(f+g)(x) = \frac{p(x)}{q(x)} + \frac{r(x)}{s(x)} = \frac{p(x)s(x)}{q(x)s(x)} + \frac{r(x)q(x)}{q(x)s(x)} = \frac{p(x)s(x)+r(x)q(x)}{q(x)s(x)},$$

which has polynomials as its numerator and denominator and is therefore a rational function.

Furthermore, by replacing $r(x)$ above with $-r(x)$, we can see that subtracting two rational functions also yields a rational function. Note that the sum and difference of two rational functions are only defined when both rational functions are defined.

Multiplying and Dividing Rational Functions Let's look at multiplication. Given the rational functions $f(x) = \dfrac{1}{x-1}$ and $g(x) = \dfrac{1}{x+1}$, we can write their product $(f \cdot g)(x)$ as $\dfrac{1}{(x-1)(x+1)}$, which has polynomials for its numerator and denominator, and is thus a rational function again.

Given two rational functions $f(x) = \dfrac{p(x)}{q(x)}$ and $g(x) = \dfrac{r(x)}{s(x)}$, with $p(x)$, $q(x)$, $r(x)$, and $s(x)$ being polynomials, then

$$(f \cdot g)(x) = \frac{p(x)}{q(x)} \cdot \frac{r(x)}{s(x)} = \frac{p(x)r(x)}{q(x)s(x)},$$

which has polynomials as its numerator and denominator and is therefore a rational function.

Furthermore, by swapping the roles of $r(x)$ and $s(x)$, we can see that dividing two rational functions also results in a rational function. Note that the product of two rational functions is only defined when both rational functions are defined. In addition, the quotient of two rational functions is only defined when the dividend and the reciprocal of the divisor are defined.

6.1.3 Polynomial Long Division

Motivating Questions

- In previous math courses, we learned how to do long division with numbers, including recognizing quotients and remainders. How do we do this with polynomials?
- Is there a relation between the degrees of the polynomials involved in the division and the degrees of the quotient and remainder?

Introduction

Let's recall some terminology from division of numbers. If we divide 25 by 4, we have that the quotient is 6 and the remainder is 1. In other words, $25 = 4 \cdot 6 + 1$, which can be also written as

$$\frac{25}{4} = 6 + \frac{1}{4}.$$

In general, when a and b are positive integers, performing the division of a by b gives us some quotient q and some remainder r, where the remainder is less than b. We can express this as $a = bq + r$. This sort of idea also works if, instead of numbers, we consider polynomials. If $f(x)$ and $g(x)$ are polynomials, we'll understand how to find polynomials $q(x)$ and $r(x)$ such that

$$f(x) = g(x)q(x) + r(x),$$

with the degree of $r(x)$ less than the degree of $g(x)$, and the degree of $q(x)$ equal to the difference between the degrees of $f(x)$ and $g(x)$. Note that if x is a point where $g(x) \neq 0$, then we can also write this relationship as

$$\frac{f(x)}{g(x)} = q(x) + \frac{r(x)}{g(x)}.$$

This is extremely useful when trying to study rational functions and their asymptotes, since these functions are by definition quotients of polynomials.

Long division of polynomials

The best way to understand this is through guided examples.

Example 3. *Find the quotient and remainder of the division of*

$$f(x) = x^3 + 3x^2 + x + 1 \quad \textit{by} \quad g(x) = x^2 + 1.$$

Explanation We need to find $q(x)$ and $r(x)$. We'll use a diagram similar to those you may have seen for long division of numbers:

$$x^2+1 \overline{\smash{\big)}\, x^3+3x^2+x+1}$$

- First, what do we need to multiply to $g(x) = x^2+1$ in order to get the leading term of $f(x) = x^3+3x^2+x+1$? We can see that x works, since $x \cdot x^2 = x^3$, so we know that $q(x) = x + \cdots$.

- In the diagram, we place x above the x^3 term, then underneath subtract by $x \cdot g(x) = x(x^2+1) = x^3+x$:

$$\begin{array}{r l}
 & \quad x \\
x^2+1 & \overline{\smash{\big)}\, x^3+3x^2+x+1} \\
 & \underline{\;-x^3 \qquad\;\; -x \quad\;} \\
 & \qquad +3x^2 \qquad +1
\end{array}$$

- Now we repeat this with the polynomial on the bottom (namely $3x^2+1$) in place of $f(x)$. So, what do we need to multiply to $g(x) = x^2+1$ in order to get the leading term of $3x^2+1$? This time, we need just 3, so we know that $q(x) = x+3+\cdots$.

- We now add 3 to the polynomial at the top and subtract $3 \cdot g(x) = 3(x^2+1) = 3x^2+3$ from the polynomial at the bottom:

$$\begin{array}{r l}
 & \quad x+3 \\
x^2+1 & \overline{\smash{\big)}\, x^3+3x^2+x+1} \\
 & \underline{\;-x^3 \qquad\;\; -x \quad\;} \\
 & \qquad +3x^2 \qquad +1 \\
 & \qquad \underline{-3x^2 \qquad -3} \\
 & \qquad\qquad\qquad\;\; -2
\end{array}$$

- Now the polynomial at the bottom is -2, which is degree 0. Since the degree is strictly less than the degree of $g(x)$, we're done.

- Our quotient is the polynomial at the top, $q(x) = x+3$, and our remainder is the polynomial at the bottom, $r(x) = -2$. Therefore we may write

$$x^3+3x^2+x+1 = (x^2+1)(x+3) - 2,$$

or equivalently,

$$\frac{x^3+3x^2+x+1}{x^2+1} = x+3-\frac{2}{x^2+1}.$$

Example 4. *Find the quotient and remainder of the division of*

$$f(x) = 6x^5 + 9x^4 + 3x^3 + 10x^2 + 19x + 7 \quad by \quad g(x) = 2x^2 + 3x + 1.$$

Explanation Again, we'll use a diagram:

$$2x^2 + 3x + 1 \,\big)\overline{\; 6x^5 + 9x^4 + 3x^3 + 10x^2 + 19x + 7}$$

- What do we need to multiply to $g(x) = 2x^2 + 3x + 1$ in order to get the leading term of $f(x) = 6x^5+9x^4+3x^3+10x^2+19x+7$? Since $3 \cdot 2 = 6$ and $x^3 \cdot x^2 = x^5$, we should multiply by $3x^3$. So, we know that $q(x) = 3x^3 + \cdots$.
- In the diagram, we write $3x^3$ above and subtract $3x^3 \cdot g(x) = 3x^3(2x^2 + 3x + 1) = 6x^5 + 9x^4 + 3x^3$ from $f(x)$ to get

$$\begin{array}{r l}
 & \;3x^3 \\
2x^2 + 3x + 1 & \big)\overline{\,6x^5 + 9x^4 + 3x^3 + 10x^2 + 19x + 7} \\
 & \;\underline{-6x^5 - 9x^4 - 3x^3 \qquad\qquad\qquad\quad} \\
 & \qquad\qquad\qquad\quad + 10x^2 + 19x + 7
\end{array}$$

- Now we repeat with the bottom polynomial, $10x^2 + 19x + 7$, in place of $f(x)$. What do we need to multiply to $g(x) = 2x^2 + 3x + 1$ in order to get the leading term of $10x^2 + 19x + 7$? This time, 5 will work, so $q(x) = 3x^2 + 5 + \cdots$.
- In the diagram, add 5 to the polynomial at the top, and subtract the bottom by $5 \cdot g(x) = 5(2x^2 + 3x + 1) = 10x^2 + 15x + 5$ to get

$$\begin{array}{r l}
 & \;3x^3 + 5 \\
2x^2 + 3x + 1 & \big)\overline{\,6x^5 + 9x^4 + 3x^3 + 10x^2 + 19x + 7} \\
 & \;\underline{-6x^5 - 9x^4 - 3x^3 \qquad\qquad\qquad\quad} \\
 & \qquad\qquad\qquad\quad + 10x^2 + 19x + 7 \\
 & \qquad\qquad\qquad\quad \underline{-10x^2 - 15x - 5} \\
 & \qquad\qquad\qquad\qquad\qquad\quad 4x + 2
\end{array}$$

- Since the polynomial at the bottom is $4x + 2$ and thus has degree 1, we know we're done since this is strictly less than the degree of $g(x)$. Therefore our quotient is $q(x) = 3x^3 + 5$ and our remainder is $4x + 2$. Written out fully, we have

$$6x^5 + 9x^4 + 3x^3 + 10x^2 + 19x + 7 = (3x^3 + 5)(2x^2 + 3x + 1) + (4x + 2),$$

  or equivalently,

$$\frac{6x^5 + 9x^4 + 3x^3 + 10x^2 + 19x + 7}{2x^2 + 3x + 1} = 3x^3 + 5 + \frac{4x + 2}{2x^2 + 3x + 1}$$

More on Remainders

Polynomial Remainder Theorem (Little Bézout's (Bay-Zoo) Theorem): If f is a polynomial, then $x = r$ is a root of f if and only if one of the following hold:

- $(x - r)$ is a factor of f,
- $(x - r)$ divides evenly into f, or
- the remainder of $\dfrac{f(x)}{x - r}$ is zero.

Notice that each of these points are different ways of saying the same thing.

Example 5. *Check that* -0.5 *is a root of* $f(x) = 6x^5 + 9x^4 + 3x^3 + 10x^2 + 19x + 7$ *using the Polynomial Remainder Theorem.*

Explanation To use the Polynomial Remainder Theorem to check this, we divide $g(x) = x - (-0.5) = x + 0.5$ into $f(x)$.

$$
\begin{array}{r|l}
 & 6x^4 + 6x^3 + 10x + 14 \\
\hline
x + 0.5 & 6x^5 + 9x^4 + 3x^3 + 10x^2 + 19x + 7 \\
 & -6x^5 - 3x^4 \\
\hline
 & + 6x^4 + 3x^3 + 10x^2 + 19x + 7 \\
 & - 6x^4 - 3x^3 \\
\hline
 & + 10x^2 + 19x + 7 \\
 & - 10x^2 - 5x \\
\hline
 & + 14x + 7 \\
 & - 14x - 7 \\
\hline
 & + 0
\end{array}
$$

The remainder is zero, so we can say that -0.5 is indeed a root of $f(x)$.

This can be extended to values r which are not zeros of f as well. Namely, if r is any real number, then $f(r)$ is equal to the remainder of $\dfrac{f(x)}{x - r}$.

Summary

- Long division of polynomials works essentially like long division of numbers.
- When performing the long division of $f(x)$ by $g(x)$ and writing the

relation $f(x) = g(x)q(x) + r(x)$, we know that the degrees of $g(x)$ and $q(x)$ add to the degree of $f(x)$, and the degree of $r(x)$ is strictly less than the degree of $g(x)$ (this tells us when to stop dividing).

- We can detect roots of a polynomial $f(x)$ by dividing by the polynomial $(x - r)$ with no remainder, and conversely, we can detect that $(x - r)$ is a factor of $f(x)$ if r is a root.

6.2 Properties of Rational Functions

Learning Objectives

- End Behavior of Rational Functions
  - When do rational functions have horizontal asymptotes?
  - Are there other kinds of asymptotes for rational functions?
  - How does one find the end behavior and asymptotes of rational functions?
- Zeros of Rational Functions
  - Finding x-intercepts of Rational Functions
  - Realizing the importance of considering the domain of a rational function when finding the zeros
- Graphing Rational Functions
  - What does the graph of a rational function look like near the points which are not in the domain?
  - How does one determine if a rational function has any vertical asymptotes? How does one find them?
  - How does one determine if the graph of a rational function has any holes? How does one find them?
  - How do we put all the information we know about a rational function together to draw the graph?

6.2.1 End Behavior of Rational Functions

As we have with previous functions we have studied, let's look at the end behavior of rational functions. Recall that we noticed that the function $f(x) = \frac{1}{x}$ approached 0 as x approached infinity or negative infinity. In general, if the y-values of a function approach a specific number as x approaches infinity or negative infinity, we call that a horizontal asympote of the function.

Horizontal Asymptotes

Definition

(a) The line $y = c$ is called a *horizontal asymptote* of the graph of a function $y = f(x)$ if as $x \to -\infty$ or $x \to +\infty$, we have $f(x) \to c$.

This means that as x gets bigger and bigger (in the positive or negative direction), the y-values of the function get close to a particular value $y = c$. Anther way to understand this that as x gets bigger and bigger (in the positive or negative direction), the graph of the function approaches the horizontal line $y = c$.

We know from looking at $f(x) = \frac{1}{x}$ that some rational functions have horizontal asymptotes. The question we would like to investigate is which rational functions have horizontal asymptotes, and if a rational function does have a horizontal asymptote, how do we determine it's value? Let's look at a few particular examples.

Example 6. *Consider the rational function $f(x) = \frac{x-1}{x-2}$. To intuitively understand $x \to +\infty$ intuitively, let's plug some big values for x:*

$$f(100) = \frac{99}{98} \approx 1.010...$$
$$f(1000) = \frac{999}{998} \approx 1.001...$$
$$f(10000) = \frac{9999}{9998} \approx 1.000...$$

It seems like $f(x) \to 1$ as $x \to +\infty$. That would mean that the line $y = 1$ is a horizontal asymptote for $f(x)$.

The same strategy shows that $f(x) \to 1$ when $x \to -\infty$ as well, so that $y = 1$ is the only horizontal asymptote for $f(x)$.

Here's a graph of the function $f(x) = \frac{x-1}{x-2}$. Notice how the function gets close to the line $y = 1$ as it goes off the page on the left and right sides.

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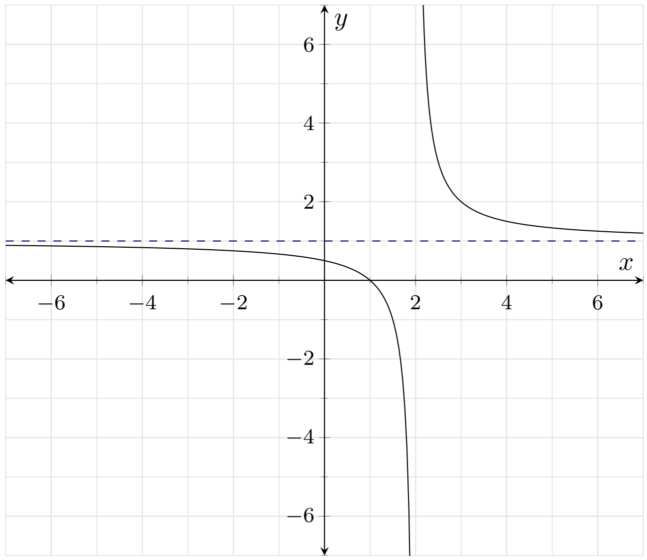

Example 7. *Consider now the rational function* $g(x) = \dfrac{x-1}{x^2-3x+2}$*. Let's do as above, and start looking for horizontal asymptotes.*

$$\begin{aligned} g(100) &\approx 0.010... \\ g(1000) &\approx 0.001... \\ g(10000) &\approx 0.000... \end{aligned}$$

This suggests that $g(x) \to 0$ *as* $x \to +\infty$*. Similarly, you can convince yourself that* $g(x) \to 0$ *as* $x \to -\infty$*, so that* $y = 0$ *is the only horizontal asymptote. Here's the graph of* $g(x) = \dfrac{x-1}{x^2-3x+2}$*. Notice how the function gets close to the line* $y = 0$ *as it goes off the page on the left and right sides.*

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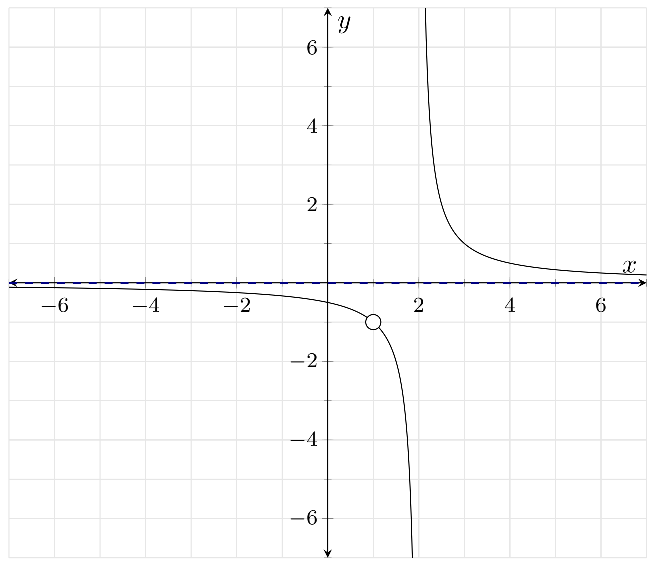

You may wonder why there is a hole in this graph! We will explore that more in the next section.

Now, you must be asking yourself if every time we want to test for vertical or horizontal asymptotes, we need to keep plugging values and guessing. Fortunately, the answer is "no". With a little bit of algebra, it becomes much more apparent what the end behavior of a rational function will be. Formal justificatives require Calculus — we'll be content with getting intuition for now.

We will need the following theorem.

> Functions of the form
> $$f(x) = x^{-n} = \frac{1}{x^n}$$
> where n is a counting number, have horizontal asymptotes of $y = 0$. In particular, as $x \to \infty, y \to 0$ and as $x \to -\infty, y \to 0$.

Hopefully this theorem seems intuitive. If $f(x) = \dfrac{1}{x}$ goes to 0 as x gets big, then raising x to a higher power should just make the function get close to zero more quickly. If we take this theorem as a given, we can use it and some algebra to rewrite other rational functions so that we can find their horizontal asymptotes.

Example 8. *Find the horizontal asympotes of* $f(x) = \dfrac{x-1}{x-2}$.

Explanation Notice that this is the same function we looked at in an earlier example. From plugging in points, we think this function should have a horizontal asymptote of $y = 1$, meaning the y-values of this function get close to 1

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as x gets big (either big and positive or big and negative). Now, lets show this with some algebra. The trick is going to be to **divide the top and bottom of the rational function by the highest power of x in the denominator**.

$$\begin{aligned}
f(x) =& \frac{x-1}{x-2} \\
=& \frac{\frac{1}{x}(x-1)}{\frac{1}{x}(x-2)} \\
=& \frac{\frac{1}{x}\cdot x - \frac{1}{x}}{\frac{1}{x}\cdot x - \frac{1}{x}\cdot 2} \qquad \text{distributing} \\
=& \frac{1-\frac{1}{x}}{1-\frac{2}{x}}
\end{aligned}$$

Now, notice that we have $\frac{1}{x}$ and $\frac{2}{x}$ in our new way of writing this rational function, and we know that both of these functions approach 0 as x goes to infinity and negative infinity. This allows us to say that,

$$\text{as } x \to \infty, \frac{1-\frac{1}{x}}{1-\frac{2}{x}} \to \frac{1-0}{1-0} = \frac{1}{1} = 1$$

and

$$\text{as } x \to -\infty, \frac{1-\frac{1}{x}}{1-\frac{2}{x}} \to \frac{1-0}{1-0} = \frac{1}{1} = 1$$

This tells us that $y = 1$ is a horizontal asymptote of f.

The technical reason why this works require calculus, but for now it is enough to know that if you can rewrite a rational function so it contains terms of $\frac{1}{x^n}$, then as x goes to infinity or negative infinity, those terms will become 0. Let's try another example.

Example 9. *Find the horizontal asympotes of* $f(x) = \dfrac{5x^3 - 3x - 1}{2x^3 - 2x^2 + 8}$.

Explanation Recall that we want to **divide the top and bottom of the rational function by the highest power of x in the denominator**. For

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this function, we want to divide by x^3.

$$\begin{aligned}
f(x) = & \frac{5x^3 - 3x - 1}{2x^3 - 2x^2 + 8} \\
= & \frac{\frac{1}{x^3}\left(5x^3 - 3x - 1\right)}{\frac{1}{x^3}\left(2x^3 - 2x^2 + 8\right)} \\
= & \frac{\frac{1}{x^3} \cdot 5x^3 - \frac{1}{x^3} \cdot 3x - \frac{1}{x^3}}{\frac{1}{x^3} \cdot 2x^3 - \frac{1}{x^3} \cdot 2x^2 + \frac{1}{x^3} \cdot 8} \qquad \text{distributing} \\
= & \frac{5 - \frac{3}{x^2} - \frac{1}{x^3}}{2 - \frac{2}{x} + \frac{8}{x^3}}
\end{aligned}$$

Now, notice that we have $\dfrac{3}{x^2}$, $\dfrac{1}{x^3}$, $\dfrac{2}{x}$ and $\dfrac{8}{x^3}$ in our new way of writing this rational function which all approach 0 as x goes to infinity and negative infinity. This allows us to say that,

$$\text{as } x \to \infty, \frac{5 - \frac{3}{x^2} - \frac{1}{x^3}}{2 - \frac{2}{x} + \frac{8}{x^3}} \to \frac{5 - 0 - 0}{2 - 0 + 0} = \frac{5}{2}$$

and

$$\text{as } x \to -\infty, \frac{5 - \frac{3}{x^2} - \frac{1}{x^3}}{2 - \frac{2}{x} + \frac{8}{x^3}} \to \frac{5 - 0 - 0}{2 - 0 + 0} = \frac{5}{2}$$

This tells us that $y = \dfrac{5}{2}$ is a horizontal asymptote of f.

You may have noticed a pattern. In both examples above, we just ended up with the coefficients of the leading terms from the top and bottom of the rational function. This pattern holds whenever the top and bottom fractions have the same degree. The theorem below gives the pattern for horizontal functions in general.

Locating Horizontal Asymptotes: Assume that $f(x) = p(x)/q(x)$ is a rational function, and that the leading coefficients of $p(x)$ and $q(x)$ are a and b, respectively.

- If the degree of $p(x)$ is the same as the degree of $q(x)$, then $y = a/b$ is the unique horizontal asymptote of the graph of $y = f(x)$.
- If the degree of $p(x)$ is less than the degree of $q(x)$, then $y = 0$ is the unique horizontal asymptote of the graph of $y = f(x)$.
- If the degree of $p(x)$ is greater than the degree of $q(x)$, then the graph of $y = f(x)$ has no horizontal asymptotes.

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The above theorem essentially says that one can detect horizontal asymptotes by looking at degrees and leading coefficients. Only the leading terms of $p(x)$ and $q(x)$ matter, and it makes no difference whether one considers $x \to +\infty$ or $x \to -\infty$. For example, how would you apply this to study horizontal asymptotes for the following function?

$$f(x) = \frac{3x^6 - 5x^4 + 3x^3 - 3x^2 + 10x + 1}{5x^6 + 10000x^5 - 5x + 2}$$

Explanation The degree of the numerator $p(x) = 3x^6 - 5x^4 + 3x^3 - 3x^2 + 10x + 1$ and the degree of the denominator $q(x) = 5x^6 + 10000x^5 - 5x + 2$ are equal, namely, to 6. Thus, since the leading coefficient of $p(x)$ is 3 and the leading coefficient of $q(x)$ is 5, we conclude that the line $y = 3/5$ is the only horizontal asymptote for this rational function.

Slant asymptotes

From the theorem above, we can see that in the third case when the degree of the numerator is bigger than the degree of the denominator, there are no horizontal asymptotes.. It would be natural to ask whether we can still say anything about the rational function in that case. It turns out that we can, and the key is to use **long division of polynomials**. Whenever the degree of the numerator is bigger that the degree of the denominator, we can use long division to rewrite the function as a polynomials plus a rational function where the degree of the numerator is *less* than the degree of the denominator.

Example 10. *For example, lets consider the rational function*

$$h(x) = \frac{x^3 + 3x^2 + x + 1}{x^2 + 1}.$$

In the section on polynomial long division, we determined that that $f(x) = x^3 + 3x^2 + x + 1$ *divided by* $g(x) = x^2 + 1$ *would give us*

$$\frac{x^3 + 3x^2 + x + 1}{x^2 + 1} = x + 3 - \frac{2}{x^2 + 1}.$$

This means we have rewritten $h(x) = \dfrac{x^3 + 3x^2 + x + 1}{x^2 + 1}$ *as the sum of the polynomial* $x + 3$ *and the rational function* $\dfrac{2}{x^2 + 1}$. *By the previous therem, we can say that*

$$\text{as } x \to \pm\infty, \frac{2}{x^2 + 1} \to 0$$

This implies that when x *is really big,* $h(x) = \dfrac{x^3 + 3x^2 + x + 1}{x^2 + 1}$ *behaves very similarly to* $x + 3$ *(because the rest of the function essentially becomes very tiny).*

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Recall that when a graph approaches a line, we call that line an asymptote. We call the line $y = x + 3$ a ***slant asymptote*** *of the function h because, unlike a vertical or horizontal line, the line $y = x + 3$ is slanted.*

> **Definition** The line $y = mx+b$, where $m \neq 0$, is called a *slant asymptote* of the graph of a function $y = f(x)$ if as $x \to -\infty$ or as $x \to +\infty$, we have $f(x) \to mx + b$.

Note that saying that $y = mx + b$ is a slant asymptote for the graph of $y = f(x)$ is the same thing as saying that $y = 0$ is a horizontal asymptote for the graph of the difference function $y = f(x) - (mx + b)$.

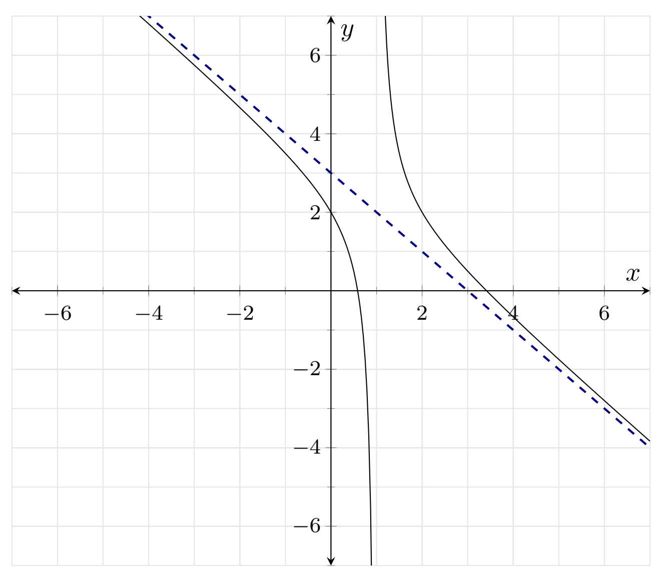
Example 11. *Do the following rational functions have slant asymptotes? If so, what are their line equations?*

(a) $f(x) = \dfrac{x^2 - 4x + 2}{1 - x}$.

Explanation When trying to find slant asymptotes, long division is the way to go. Performing it, we see that

$$\frac{x^2 - 4x + 2}{1 - x} = -x + 3 - \frac{1}{1 - x}.$$

Since $1/(1 - x) \to 0$ as $x \to \infty$ or $x \to -\infty$ and $y = -x + 3$ describes a line, we conclude that $y = -x + 3$ is a slant asymptote for the graph of $y = f(x)$.



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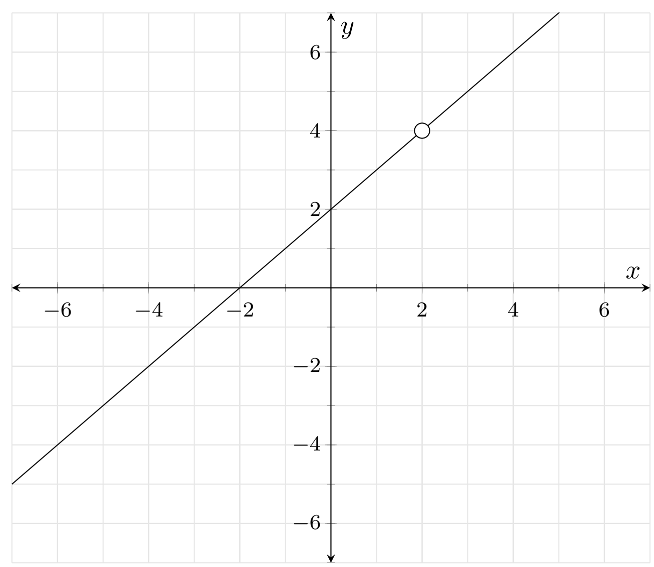
(b) $f(x) = \dfrac{x^2 - 4}{x - 2}$.

Explanation We may just simplify it to $f(x) = x+2$, valid for all $x \neq 2$. We may regard this as a long division for which the remainder is zero, as in

$$f(x) = x + 2 + \frac{0}{x-2},$$

and since $0/(x-2) \to 0$ as $x \to +\infty$ or $x \to -\infty$, it follows that $y = x+2$ is a slant asymptote for the graph of $y = f(x)$, even though the graph is just said line with a hole!



(c) $f(x) = \dfrac{x^3 - 4x^2 + 5x - 1}{x - 1}$.

Explanation Performing the long division, as before, we see that

$$f(x) = \frac{x^3 - 4x^2 + 5x - 1}{x - 1} = x^2 - 3x + 2 + \frac{1}{x-1}.$$

As expected, $1/(x-1) \to 0$ when $x \to +\infty$ or $x \to -\infty$, but $y = x^2 - 3x + 2$ is not a line equation. Hence there are no slant asymptotes for the graph of $f(x)$.

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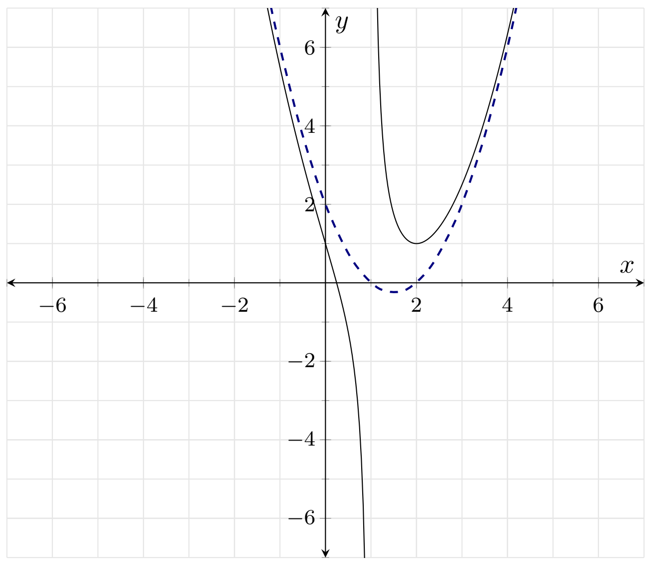

Notice that even though $x^2 - 3x + 2$ is not a slant asymptote (because it is not a straight line), we can see from the graph that it is asymptotic to a parabola. This means that the parabola still tells us about the end behavior of f. Since $x^2 - 3x + 2$ is an even degree polynomial with a positive leading coefficient, we know that

$$\text{as } x \to \pm\infty, x^2 - 3x + 2 \to \infty.$$

Because of this, we can also say that

$$\text{as } x \to \pm\infty, \frac{x^3 - 4x^2 + 5x - 1}{x - 1} \to \infty.$$

(d) $f(x) = \dfrac{x^3 + 2x^2 + x + 1}{x^3 - 2x^2 - x + 1}$.

Explanation Long division shows that:

$$f(x) = \frac{x^3 + 2x^2 + x + 1}{x^3 - 2x^2 - x + 1} = 1 + \underbrace{\frac{4x^2 + 2x}{x^3 - 2x^2 - x + 1}}_{\to 0}.$$

The indicated remainder goes to zero when $x \to +\infty$ or $x \to -\infty$ simply because the degree of the numerator is lower than the degree of the numerator. The remaining quotient does give us the asymptote $y = 1$. But this is not a slant asymptote, it is a horizontal asymptote (as you might have expected). Note that asymptotes are really concerned about the end behavior of the function. In the above example, the line $y = 1$ does intersect the graph of $y = f(x)$, but this is fine — the graph still only approaches said line as $x \to +\infty$ and $x \to -\infty$.

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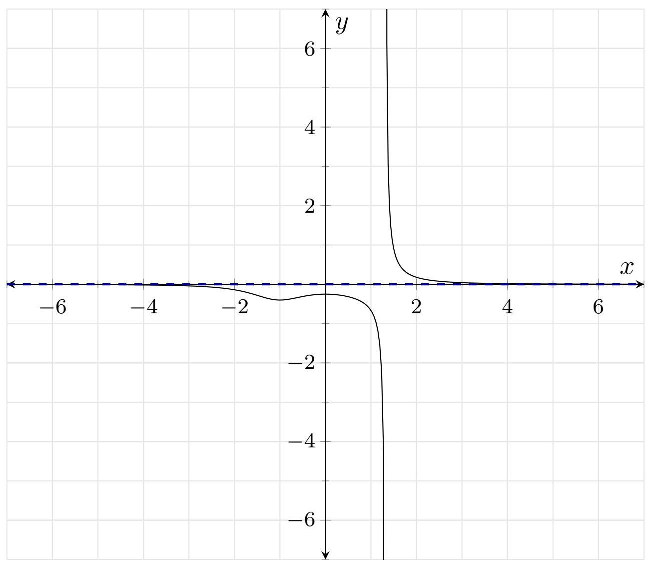
(e) $f(x) = \dfrac{x^2+1}{x^5-4}$.

Explanation Since the degree of the numerator is smaller than the degree of the denominator, you can think of the long division as already having been performed, as in

$$f(x) = \frac{x^2+1}{x^5-4} = 0 + \frac{x^2+1}{x^5-4}.$$

Again, the remainder goes to zero when $x \to +\infty$ or $x \to -\infty$. So, the line $y = 0$ would be an asymptote, but it is horizontal, not slant, as in the previous item.



The above examples suggest that if the degree of the numerator is at least two higher than the degree of the numerator, what survives outside the remainder has degree higher than one, and thus does not describe a line equation — meaning no slant asymptotes. Similarly, if the degree of the numerator is equal or lower to the degree of the denominator, there's "not enough quotient left" to describe a line equation. This is not a coincidence, but a general fact.

Theorem (on slant asymptotes): Let $f(x) = p(x)/q(x)$ be a rational function for which the degree of $p(x)$ is exactly one higher than the degree of $q(x)$. Then the graph of $y = f(x)$ has the slant asymptote $y = L(x)$, where $L(x)$ is the quotient obtained by dividing $p(x)$ by $q(x)$. If the degree of $p(x)$ is not exactly one higher than the degree of $q(x)$, there is no slant asymptote whatsoever.

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Unlike what happened for horizontal and vertical asymptotes, the above theorem does not *immediately* tell you what is the line equation describing the slant asymptote. We must resort to long division.

Summary

- There are three types of asymptotes for rational functions: vertical asymptotes, horizontal asymptotes, and slant asymptotes. The latter occurs when the degree of the numerator is exactly one higher than the degree of the denominator.

6.2.2 Zeros of Rational Functions

Introduction

Suppose Julia is taking her family on a boat trip 12 miles down the river and back. The river flows at a speed of 2 miles per hour and she wants to drive the boat at a constant speed, v miles per hour downstream and back upstream. Due to the current of the river, the actual speed of travel is $v+2$ miles per hour going downstream, and $v-2$ miles per hour going upstream. If Julia plans to spend 8 hours for the whole trip, how fast should she drive the boat?

The time it takes Julia to drive the boat downstream is $\frac{12}{v+2}$ hours and upstream is $\frac{12}{v-2}$ hours. The function to model the whole trip's time is

$$t(v) = \frac{12}{v-2} + \frac{12}{v+2}$$

where t stands for time in hours. The trip will take 8 hours, so we want $t(v)$ to equal 8, and we have:

$$\frac{12}{v-2} + \frac{12}{v+2} = 8.$$

To solve this equation algebraically, we would start by subtracting 8 from both sides to obtain:

$$\frac{12}{v-2} + \frac{12}{v+2} - 8 = 0.$$

This has taken our equation involving rational functions, and converted it into the problem of determining the zeros of a single rational function. Namely, we are really just finding the zeros of $s(v) = \frac{12}{v-2} + \frac{12}{v+2} - 8$. (Notice that the function was changed by subtracting the 8, so we had to use a new name for it.)

In the same way, whenever we are asked to find the solution of a rational equation, it is equivalent to finding the zeros of a rational function instead.

Zeros of Rational Functions

Example 12. *Let us finish the calculation started in the Introduction. Find the zeros of* $s(v) = \frac{12}{v-2} + \frac{12}{v+2} - 8$.

Explanation

We will begin by combining the left-hand side into a single fraction. Notice the

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fractions that appear have a common denominator of $(v-2)(v+2)=v^2-4$.

$$\begin{aligned}
s(v) &= \frac{12}{v-2}+\frac{12}{v+2}-8\\
&= \frac{12}{v-2}\cdot\left(\frac{v+2}{v+2}\right)+\frac{12}{v+2}\cdot\left(\frac{v-2}{v-2}\right)-8\left(\frac{(v+2)(v-2)}{(v+2)(v-2)}\right)\\
&= \frac{12(v+2)}{(v-2)(v+2)}+\frac{12(v-2)}{(v+2)(v-2)}-\frac{8(v^2-4)}{(v+2)(v-2)}\\
&= \frac{12v+24}{(v+2)(v-2)}+\frac{12v-24}{(v+2)(v-2)}-\frac{8v^2-32}{(v+2)(v-2)}\\
&= \frac{(12v+24)+(12v-24)-(8v^2-32)}{(v+2)(v-2)}\\
&= \frac{-8v^2+(12v+12v)+(24-24+32)}{(v+2)(v-2)}\\
&= \frac{-8v^2+24v+32}{(v+2)(v-2)}.
\end{aligned}$$

That means $s(v)=0$ is equivalent to the equation $\dfrac{-8v^2+24v+32}{(v+2)(v-2)}=0$.

Since a fraction is zero if and only if the numerator is zero (and the denominator is nonzero), we need to look at $-8v^2+24v+32=0$. We'll start by factoring, since we see a common factor of 8 in the coefficients. Actually, let's factor out -8 to clean up the sign of the leading term: $-8v^2+24v+32=-8(v^2-3v-4)$. The quadratic factor v^2-3v-4 can be factored to $(v-4)(v+1)$. That means:

$$\begin{aligned}
\frac{-8v^2+24v+32}{(v+2)(v-2)} &= 0\\
-8v^2+24v+32 &= 0\\
-8(v^2-3v-4) &= 0\\
-8(v-4)(v+1) &= 0.
\end{aligned}$$

Setting each factor equal to 0 we see that either $-8=0$ (which is impossible), $v-4=0$ (which gives a possible solution of $v=4$), and $v+1=0$ (which gives a possible solution of $v=-1$.

There are two POSSIBLE solutions, $v=-1$ and $v=4$. The process of solving a rational equation like this can sometimes introduce extraneous solution. That is, a number that appears to be a solution, but doesn't actually satisfy the original equation.

Let's plug both of these possibilities back into our original formula for $s(v)$ to

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verify that they are actually solutions.

$$\begin{aligned} s(-1) &= \frac{12}{(-1)-2} + \frac{12}{(-1)+2} - 8 \\ &= \frac{12}{-3} + \frac{12}{1} - 8 \\ &= -4 + 12 - 8 = 0 \end{aligned}$$

$$\begin{aligned} s(4) &= \frac{12}{(4)-2} + \frac{12}{(4)+2} - 8 \\ &= \frac{12}{2} + \frac{12}{6} - 8 \\ &= 6 + 2 - 8 = 0. \end{aligned}$$

That means both $v = -1$ and $v = 4$ are solutions to the rational equation $\frac{12}{v-2} + \frac{12}{v+2} - 8 = 0$.

Let's remember where this example came from. In the example, v represented a speed, so it cannot be negative. The only solution is $v = 4$ miles per hour.

Example 13. *Let f be the function given by $f(x) = \frac{1}{x-4} + x - \frac{x-3}{x-4}$. Find the zeros of the rational function f.*

Explanation

We are being asked to solve the equation

$$\frac{1}{x-4} + x - \frac{x-3}{x-4} = 0$$

Notice that the only denominators appearing in the fractions are $x-4$, so the common denominator is $x-4$. We start by combining these terms into a single

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fraction with that denominator.

$$\begin{aligned} f(x) &= \frac{1}{x-4} + x - \frac{x-3}{x-4} \\ &= \frac{1}{x-4} + x \cdot \left(\frac{x-4}{x-4}\right) - \frac{x-3}{x-4} \\ &= \frac{1}{x-4} + \frac{x(x-4)}{x-4} - \frac{x-3}{x-4} \\ &= \frac{1}{x-4} + \frac{x^2-4x}{x-4} - \frac{x-3}{x-4} \\ &= \frac{(1)+(x^2-4x)-(x-3)}{x-4} \\ &= \frac{x^2+(-4x-x)+(1+3)}{x-4} \\ &= \frac{x^2-5x+4}{x-4} \end{aligned}$$

Setting the numerator equal to zero and factoring gives the following.

$$\begin{aligned} f(x) &= 0 \\ \frac{x^2-5x+4}{x-4} &= 0 \\ x^2-5x+4 &= 0 \\ (x-4)(x-1) &= 0. \end{aligned}$$

By setting each of these factors equal to 0 we see that either $x - 4 = 0$ (which gives a possible solution of $x = 4$) and $x-1 = 0$ (which gives a possible solution of $x = 1$). The two possible solutions are $x = 4$ and $x = 1$. Let's check them.

$$\begin{aligned} f(1) &= \frac{1}{(1)-4} + (1) - \frac{(1)-3}{(1)-4} \\ &= \frac{1}{-3} + 1 - \frac{-2}{-3} \\ &= -\frac{1}{3} + 1 - \frac{2}{3} = 0. \end{aligned}$$

However, $x = 4$ is not in the domain of f, since it makes the denominators of the first and third terms zero. That is, $x = 4$ is an extraneous solution.

The only solution is $x = 1$.

Example 14. *Let g be the function given by $g(p) = \frac{3}{p-2} + \frac{5}{p+2} - \frac{12}{p^2-4}$. Find the zeros of the rational function g.*

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Explanation

Since $p^2 - 4 = (p+2)(p-2)$, the least common denominator between these three fractions is $(p+2)(p-2) = p^2 - 4$. As before, we start by combining into a single fraction with that denominator.

$$\begin{aligned}
g(p) &= \frac{3}{p-2} + \frac{5}{p+2} - \frac{12}{p^2-4} \\
&= \frac{3}{p-2} \cdot \left(\frac{p+2}{p+2}\right) + \frac{5}{p+2} \cdot \left(\frac{p-2}{p-2}\right) - \frac{12}{p^2-4} \\
&= \frac{3(p+2)}{(p-2)(p+2)} + \frac{5(p-2)}{(p+2)(p-2)} - \frac{12}{p^2-4} \\
&= \frac{3p+6}{(p+2)(p-2)} + \frac{5p-10}{(p+2)(p-2)} - \frac{12}{(p+2)(p-2)} \\
&= \frac{(3p+6)+(5p-10)-(12)}{(p+2)(p-2)} \\
&= \frac{(3p+5p)+(6-10-12)}{(p+2)(p-2)} \\
&= \frac{8p-16}{(p+2)(p-2)}
\end{aligned}$$

Setting the numerator equal to zero gives the following.

$$\begin{aligned}
8p - 16 &= 0 \\
8p &= 16 \\
p &= \frac{16}{8} = 2.
\end{aligned}$$

There is one possible solution, at $p = 2$.

However, $p = 2$ is not in the domain of g, since it makes the denominators of the first and third terms zero. That is, $p = 2$ is an extraneous solution.

The function g does not have any zeros.

Let's look at this last example a bit more. We combined the three terms of $g(p)$

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into a single fraction, but that fraction was not in its reduced form.

$$\begin{aligned} g(p) &= \frac{3}{p-2} + \frac{5}{p+2} - \frac{12}{p^2-4} \\ &= \frac{8p-16}{(p+2)(p-2)} \\ &= \frac{8(p-2)}{(p+2)(p-2)} \\ &= \frac{8\cancel{(p-2)}}{(p+2)\cancel{(p-2)}} \\ &= \frac{8}{(p+2)} \quad \text{, for } p \neq 2. \end{aligned}$$

Why was $p = -2$ not a zero of the function? Because it was also a zero of the denominator. (Notice the common factor of $p - 2$ in both the numerator and denominator.) Rewriting the fraction in lowest terms, we see that the numerator is never zero, since it's a constant 8.

From this example, it may seem that reducing the rational function to lowest terms will always help you bypass the extraneous solutions. That is not the case.

Example 15. *Let r be the function given by $r(t) = \dfrac{(t+3)(t^2-2t+1)}{t^2-1}$. Find the zeros of r.*

Explanation

Since we are already given the formula for r as a single fraction, let us simplify.

$$\begin{aligned} r(t) &= \frac{(t+3)(t^2-2t+1)}{t^2-1} \\ &= \frac{(t+3)(t-1)^2}{(t+1)(t-1)} \\ &= \frac{(t+3)(t-1)(t-1)}{(t+1)(t-1)} \\ &= \frac{(t+3)(t-1)\cancel{(t-1)}}{(t+1)\cancel{(t-1)}} \\ &= \frac{(t+3)(t-1)}{(t+1)} \end{aligned}$$

The fraction will be zero when the numerator is zero. Setting each of the factors of the numerator equal to zero gives $t + 3 = 0$ (which gives a possible solution of $t = -3$), and $t - 1 = 0$ (which gives a possible solution of $t = 1$).

When we cancelled out the common factor of $t - 1$ from the numerator and the denominator, we changed the function without mentioning it. This new fraction $\dfrac{(t+3)(t-1)}{(t+1)}$ has $t = 1$ in its domain, but it is not in the domain of r. To be

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thorough, after that cancellation we should have written

$$r(t) = \frac{(t+3)(t-1)}{(t+1)}\text{, for } t \neq 1$$

to indicate that we're still using the original domain of r.

The function r has a single zero, at $x = -3$.

6.2.3 Graphing Rational Functions

Behavior Near Points Not in the Domain

We are now getting closer to understanding the properties of rational functions. We have discussed how to find the domain, end behavior, and x-intercepts. Finding the y-intercept just involves plugging 0 into the function. But there is one major feature of rational functions we still need to discuss before we can understand them fully. What happens to the graph of a rational function near the points which are not in the domain?

Recall that:

> A rational functions is not defined when the polynomial in the denominator is equal to zero. That is, if $r(x) = \dfrac{p(x)}{q(x)}$ is a rational function and both $p(x)$ and $q(x)$ are polynomials, than the domain of $r(x)$ is all values of x except those where $q(x) = 0$.

Rational functions can have two different kinds of behavior near a point where the denominator is zero.

We know from investigating $f(x) = \dfrac{1}{x}$ that one possibility is a vertical asymptote. Recall that:

> **Definition** The line $x = c$ is called a *vertical asymptote* of the graph of a function $y = f(x)$ if as $x \to c^-$ or $x \to c^+$, either $f(x) \to +\infty$ or $f(x) \to -\infty$.
> Here, as $x \to c^-$ means for points near $x = c$ but less than c and as $x \to c^+$ means for points near $x = c$ but greater than c.

When you put points into a rational function that are close to the point where it is undefined, it will make the bottom of the fraction very small. Usually, that will make the resulting fraction very big, resulting in a vertical asymptote on the graph. The exception is when the top of the fraction is also getting very small. Then, it is not quite clear what is happening to the value of the fraction overall. Let's investigate.

Example 16. *Let's return to the function* $f(x) = \dfrac{x-1}{x^2-3x+2}$. *Earlier, we looked at the end behavior of this function and determined that* $f(x) \to 0$ *as* $x \to +\infty$ *and* $f(x) \to 0$ *as* $x \to -\infty$. *Now, let's investigate the middle behavior of this function, by which I mean what it is doing between the two horizontal asymptotes.*

First, we need to determine the points which are not in the domain of f. *That is, which values of* x *make* $f(x)$ *undefined? Note that we can factor the denominator*

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of this rational function.

$$x^2 - 3x + 2 = (x-1)(x-2).$$

Setting the denominator equal to zero will let us find the points where $f(x)$ is undefined. This means $(x-1) = 0$ or $(x-2) = 0$, giving us that $f(x)$ is undefined for $x = 1$ and $x = 2$. Now lets investigate what happens near each of these points.

Notice that we may write that

$$f(x) = \frac{x-1}{x^2-3x+2} = \frac{(x-1)}{(x-1)(x-2)} = \frac{1}{x-2} \quad \textit{provided} \quad x \neq 1$$

.

This means that for all values of x other than $x = 1$, $f(x)$ gives the same y-value as $\frac{1}{x-2}$. This means that the graph of $f(x)$ and the graph of $\frac{1}{x-2}$ should look identical except for at $x = 1$.

*At $x = 1$, $f(x)$ is not defined, but $x = 1$ is not a special point at all for $\frac{1}{x-2}$. The graph will continue as normal on either side of $x = 1$ and there will be **no asymptote** there. Therefore, the conclusion is that $f(x)$ will have a **hole** at $x = 1$. We can even find the height (y-value) at which to draw the hole by finding out what the y-value of $\frac{1}{x-2}$ is at $x = 1$. Since we keep referencing it, let's give $\frac{1}{x-2}$ a name. It is not the same function as f because it has a different domain. Therefore, it needs a different name. Let's say $g(x) = \frac{1}{x-2}$. Then we have that*

$$g(1) = \frac{1}{1-2} = \frac{1}{-1} = -1.$$

*Our conclusion is that the graph of the function $f(x) = \frac{(x-1)}{(x-1)(x-2)}$ will have a **hole at the point** $(1, -1)$.*

Let's let's compare that with the behavior of $f(x)$ near $x = 2$. Recall that for all values of x other than $x = 1$, $f(x) = \frac{(x-1)}{(x-1)(x-2)}$ gives the same y-value as $g(x) = \frac{1}{x-2}$. Since we are not investigating near one, we can just use the simpler function, $g(x) = \frac{1}{x-2}$. Both f and g will have the same behavior everyone other than at $x = 1$!. But comparing the expression $g(x) = 1/(x-2)$ with what we have previously seen for the function $1/x$, we see that $f(x) \to +\infty$ as $x \to 2^+$ and $f(x) \to -\infty$ as $x \to 2^-$, so that the line $x = 2$ is a vertical asymptote for $f(x)$.

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Finding Vertical Asymptotes, Holes, and Zeros

You may notice that the function g in the previous example played an important role in our ability to determine whether our original function f had a hole or an asymptote at each point which was not in the domain. The function g was special because it had no factors in common between the numerator and denominator of the rational function. We give rational functions like this a special name.

> **Definition** A rational function is said to be in **lowest terms** if the numerator and the denominator have no factors in common.

This allows us to state the following theorem.

> A rational function in lowest terms will have a vertical asymptote at every point which is not in its domain.

For rational functions which are not in lowest terms, we will want to factor the numerator and denominator completely to determine what cancels. We can the write the rational function in lowest terms, being careful to keep track of any points which are not in the domain.

> Every rational function, $f(x) = \dfrac{p(x)}{q(x)}$, can be written in lowest terms $g(x) = \dfrac{r(x)}{s(x)}$ where $r(x)$ and $s(x)$ have no common factors, provided the domain of the function is clearly stated so that any points which were not in the domain of, f, the original function are still excluded from g, the version of the function in lowest terms. Any points which are excluded in this way and for which the $s(x) \neq 0$ will be holes in the graph of the function. Any points where $s(x) = 0$ will be vertical asymptotes.

Let's try applying this theorem to an example.

Example 17. *Let*

$$f(x) = \frac{(x-5)(x^2+3)^2(x+3)(x+2)^3}{x(x+2)(x^2-9)(x-5)^5}$$

Determine where f has vertical asymptotes, holes, and zeros. **Explanation** Let's break this analysis into steps.

Step 1: The first step is to make sure that both the numerator and denominator are completely factored into linear and **irreducible** quadratic terms. Recall that a quadratic is irreducible when the discriminant, $b^2 - 4ac < 0$. Let's look at the two quadratic factor above.

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The factor (x^2+3) is irreducible. It has a disciminant of $0^2-4(1)(3) = -12 < 0$. This means it cannot be factored and never equals zero. Essentially, for the purposes of this problem, this factor is not going to contribute to any zeros, holes, or vertical asymptotes. We can ignore it.

The factor (x^2-9) is a difference of squares. $(x^2-9) = (x-3)(x+3)$. Thus, we want to replace (x^2-9) with $(x+3)(x-3)$ in our formula for $f(x)$.

$$f(x) = \frac{(x-5)(x^2+3)^2(x+3)(x+2)^3}{x(x+2)(x+3)(x-3)(x-5)^5}$$

Step 2: The next step is to determine the points which are not in the domain of f. To do this, we set the denominator equal to zero. $x(x+2)(x+3)(x-3)(x-5)^5 = 0$ whenever one of the factors equals zero. From the Polynomial Remainder Theorem, we can just read off the zeros for each term. The zeros will be $x=0$ coming from the factor of $x=(x-0)$, $x=-2$ which comes from the factor $(x+2)$, $x=-3$ which comes from the factor $(x+3)$, $x=3$ which comes from the factor $(x-3)$ and $x=5$ which comes from the factor $(x-5)^5=0$.

Step 3: Now that we know the domain, we can rewrite $f(x)$ in lowest terms, **provided** we keep track of which points are not in the domain.

$$\begin{aligned} f(x) &= \frac{(x-5)(x^2+3)^2(x+3)(x+2)^3}{x(x+2)(x+3)(x-3)(x-5)^5} \\ &= \frac{(x^2+3)^2(x+3)^0(x+2)^{3-1}}{x(x-3)(x-5)^{5-1}} \quad \text{provided} \quad x \neq -3, -2, 0, 3, \text{or } 5 \\ &= \frac{(x^2+3)^2(x+2)^2}{x(x-3)(x-5)^4} \quad \text{provided} \quad x \neq -3, -2, 0, 3, \text{or } 5 \end{aligned}$$

Step 4: Now that $f(x)$ is written in lowest terms, anywhere the denominator is zero is a vertical asymptote. Therefore, we want to consider $x(x-3)(x-5)^4 = 0$. This gives us vertical asymptotes at $x=0, x=3$, and $x=5$.

Step 5: Now, let's determine the holes. There is a hole at any x-values which are not in the domain but which are not vertical asymptotes. Since the x-values $x=-3,-2,0,3$, and 5 are not in the domain and only $x=0, x=3$, and $x=5$ are vertical asymptotes, we have that $x=-3$ and $x=-2$ will be x-values where holes are in the graph.

Step 6: Finally, let's determine the zeros of f. A rational function equals zero whenever the numerator equals zero, provided those points are in the domain of the function. We can consider the numerator of version of the formula written in lowest terms. We have $(x^2+3)^2(x+2)^2 = 0$. We

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already determined that $(x^2+3)^2 \neq 0$. Thus, we just have $x=-2$ coming from the term $(x+2)^2=0$. We check to see whether $x=-2$ is in the domain of f. It is not. Therefore, it is not a zero. f has no zeros.

In conclusion, we have interesting phenomena happening whenever the numerator or the denominator of the function equal zero. For this function, we have vertical asymptotes at $x=0, x=3$, and $x=5$, holes at $x=-3$ and $x=-2$, and no zeros.

Graphing a Rational Function

Now let's return the previous example and see if we can draw the graph of $f(x)=\dfrac{x-1}{x^2-3x+2}$. We have already determined:

- As $x \to +\infty$ and $f(x) \to 0$ (or a borizontal asymptote at $y=0$)
- As $x \to -\infty$ and $f(x) \to 0$ (or a borizontal asymptote at $y=0$)
- The graph will have a hole at $(1,-1)$
- The graph will have a vertical asymptote at $x=2$.

Now let's find a few additional points to help us graph the function.

To find the y-intercept of f, $f(0)=\dfrac{(0)-1}{(0)^2-3(0)+2}=\dfrac{-1}{2}$. To find any x-intercepts, we set $y=0$, meaning we are looking at

$$0=\frac{x-1}{x^2-3x+2}.$$

A fraction equals zero when the numerator equals zero, so we need

$$\begin{aligned} x-1&=0\\ x&=1 \end{aligned}$$

But, $x=1$ is not in the domain of our function so it is not actually an x-intercept! There are no x-intercepts!

It is usually a good idea to find a point between each place where the function is zero or undefined. This will help us see where the function is. There is a result in calculus that the graph of a rational function can only change from being postive (above the x-axis) to negative (below the x-axis) or vice versa when the function is zero or undefined. We will explore this more later, but this idea motivated the points we are going to choose to plot.

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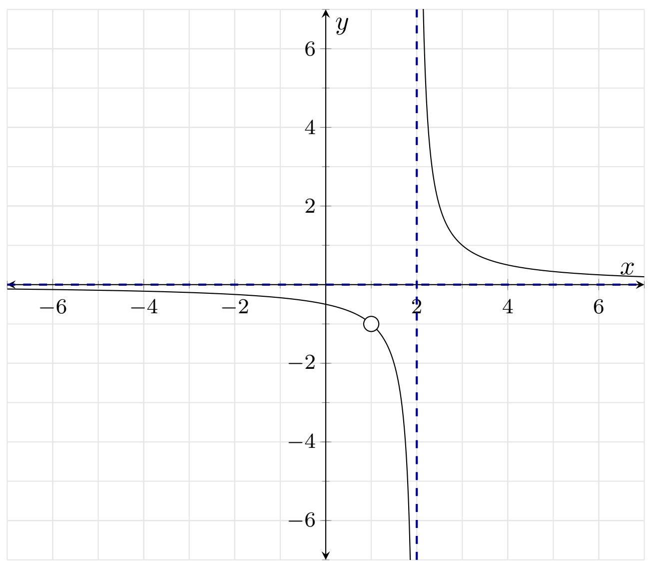
$$f(-1) = \frac{(-1)-1}{(-1)^2-3(-1)+2} = \frac{-2}{6} = \frac{-1}{3}$$

$$f\left(\frac{3}{2}\right) = \frac{\left(\frac{3}{2}\right)-1}{\left(\frac{3}{2}\right)^2-3\left(\frac{3}{2}\right)+2} = \frac{\frac{1}{2}}{\frac{9}{4}-\frac{9}{2}+2} = \frac{\frac{1}{2}}{\frac{-1}{4}} = \frac{1}{2}\cdot\frac{-4}{1} = -2$$

$$f(3) = \frac{(3)-1}{(3)^2-3(3)+2} = \frac{2}{2} = 1$$

Putting all of this together, we are able to draw the following graph:



It is important to note that many graphing calculators such as Desmos will not show the hole in the graph, but it is important to know that it is there.

6.3 Domain

Learning Objectives

- Domain
  - Definition of the Domain
  - Interval Notation
  - The Domains of Famous Functions
  - Spotting Values not in the Domain
  - Piecewise Defined Functions and Restricted Domains
- Range
  - Definition of the Range
  - Ranges of Famous Functions
  - Spotting Values not in the Range
- Famous Functions, Updated

6.3.1 Domain

Motivating Questions

- Are there numbers that cannot be plugged into a given function?
- How do we denote the numbers that can be plugged in?
- What are the allowable inputs for our famous functions?

Introduction

We often think about functions as a process which transforms an input into some output. Sometimes that process is known to us (such as when we have a formula for the function) and sometimes that process is unknown to us (such as when we only have a small table of values).

Exploration

a. Suppose the quadratic function f is given by $f(x) = x^2$. Are there any values that can't be plugged into f?

b. Suppose a square has side length denotes by the variable s, and area denoted by A. The area of the square is a function of the side length, $A(s) = s^2$. Are there any values of s that don't make sense?

c. Suppose that g is the rational function given by $g(x) = \dfrac{x}{x}$ and that h is the constant function given by $h(x) = 1$. Are these the same function? Why or why not?

The Domain of a Function

Definition Let f be a function from A to B. The set A of possible inputs to f is called the **domain** of f. The set B is called the **codomain** of f.

Example 18. *Let P be the population of Columbus, OH as a function of the year. According to Google, the population of Columbus in 1990 was 632,910 and in the year 2010 the population was 787,033. That means we can say:*

$$P(1990) = 632,910 \;\textit{ and }\; P(2010) = 787,033.$$

What if we were asked to find $P(1200)$?

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Explanation This question doesn't really make sense. There were Mound Builder tribes in the area around the year 1200, but the city of Columbus was not incorporated until the year 1816. We say that $P(1200)$ is *undefined*. That is to say, 1200 is not in the domain of P.

Example 19. *Let f be the function given by $f(x) = \frac{1}{x}$. Is there any number that cannot be used as an input to f?*

Explanation There is only one number that is not a valid input, 0. The number 1 can be divided by any nonzero number. For instance $f(7) = \frac{1}{7}$ or $f(-3.7) = \frac{1}{-3.7}$ are perfectly valid outputs. However, if someone attempted to plug $x = 0$ into the formula $\frac{1}{x}$, they would end up with a division-by-zero, which is undefined. The number 0 is not in the domain of f.

When we are given a function, sometimes the domain is given to us explicitly. Consider the function $f(x) = 2x + 1$ for $x \geq 5$. The phrase "for $x \geq 5$" tells us the domain for this function. We may be able to plug any number into the expression $2x + 1$, but it's only when $x \geq 5$ that this gives our function. For instance, $2(0) + 1 = 1$, but $f(0)$ is undefined.

Sometimes, when we are given a function as a formula, we are not told the domain. In these circumstances we use the *implied domain*.

> **Definition** Let f be a function whose inputs are real numbers. The **implied domain** of f is the collection of all real numbers x for which $f(x)$ is a real number.

Example 20. *Let g be the function given by $g(x) = \sqrt{3x - 4}$. Find the domain of g.*

Explanation The only information we are given about g is the formula for $g(x)$. That means we are being asked to find the implied domain. Since the square root only exists (as a real number) when the expression under the root, called the radicand, is non-negative, we need to ensure that:

$$\begin{aligned} 3x - 4 &\geq 0 \\ 3x &\geq 4 \\ x &\geq \frac{4}{3}. \end{aligned}$$

The domain is the set of all x for which $x \geq \frac{4}{3}$.

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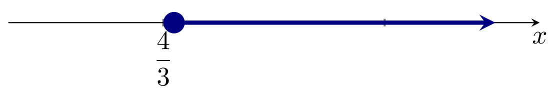
Interval Notation

As in the previous example, solutions of inequalities play an important role in expressing the domains of many types of functions. As a standard way of writing these solutions, we rely on *interval notation*. Interval notation is a short-hand way of representing the intervals as they appear when sketched on a number line. The previous example involved $x \geq \frac{4}{3}$ which, when sketched on a number line, is given by



This sketch consists of a single interval with left-hand endpoint at $\frac{4}{3}$ and no right-hand endpoint (it keeps going). In interval notation, this would be written as $\left[\frac{4}{3}, \infty\right)$. This is an example of a *closed infinite interval*, "closed" because the point at $\frac{4}{3}$ (the only endpoint) is included and "infinite" because it has infinite width. The solid dot at $\frac{4}{3}$ indicates that the point is included in the interval.

There are five different types of infinite intervals: the first two are closed infinite intervals (which contain their respective endpoint) and the other three are open infinite intervals (which do not contain the endpoint). For a fixed real number a, these are:

(a) $[a, \infty)$ represents $x \geq a$,

(b) $(-\infty, a]$ represents $x \leq a$,

(c) (a, ∞) represents $x > a$,

(d) $(-\infty, a)$ represents $x < a$, and

(e) $(-\infty, \infty)$ represents all real numbers.

The notation uses the square bracket to indicate that the endpoint is included and the round parenthesis to indicate that the endpoint is not included.

Not every interval is infinite, however. Consider the interval in the following sketch

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which consists of all x with $-2 < x \leq 3$. It is not an infinite interval, having endpoints at -2 and 3. The endpoint at -2 is not included, but the endpoint at 3 is included. In interval notation this would be written as $(-2, 3]$. As with the infinite intervals, the square bracket indicates that the right-hand endpoint is included and the round parenthesis endicates that the left-hand endpoint is not included. (This is an example of a "half-open interval".)

For a bounded intervals (ones that are not infinite), there are also four possibilities. For a and b both fixed real numbers, these are:

(a) $[a, b]$ represents $a \leq x \leq b$,

(b) $[a, b)$ represents $a \leq x < b$,

(c) $(a, b]$ represents $a < x \leq b$ and

(d) (a, b) represents $a < x < b$.

Practically, this amounts to writing the left-hand endpoint, the right-hand endpoint, then indicating which endpoints are included in the interval.

Metacognitive Moment When neither endpoint is included, (a, b) can be mistaken for a point on a graph. You will need to use the context to know which is meant.

Example 21. *Write the interval notation for* $-\frac{3}{2} \leq x \leq \sqrt{5}$ *and for* $-\frac{3}{2} < x < \sqrt{5}$.

Explanation The interval $-\frac{3}{2} \leq x \leq \sqrt{5}$ has graph

It has one interval with endpoints at $-\frac{3}{2}$ and $\sqrt{5}$, both of which are included. In interval notation it is given by $\left[-\frac{3}{2}, \sqrt{5}\right]$.

The interval $-\frac{3}{2} < x < \sqrt{5}$ has graph

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It has one interval with endpoints at $-\frac{3}{2}$ and $\sqrt{5}$, neither of which are included. In interval notation it is given by $\left(-\frac{3}{2}, \sqrt{5}\right)$.

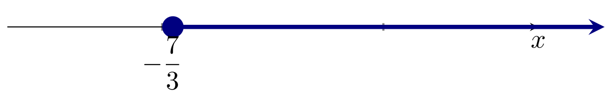
Example 22. *Find the domain of the function f given by $f(x) = \sqrt{3x+7} - \sqrt{5-2x}$.*

Explanation In order for the value of $f(x)$ to exist, we need BOTH $3x+7 \geq 0$ AND $5-2x \geq 0$.

$$\begin{aligned} 3x+7 &\geq 0 \\ 3x &\geq -7 \\ x &\geq -\frac{7}{3} \end{aligned}$$

$$\begin{aligned} 5-2x &\geq 0 \\ -2x &\geq -5 \\ x &\leq \frac{5}{2} \end{aligned}$$

The inequality $x \geq -\frac{7}{3}$ has graph



and the graph of $x \leq \frac{5}{2}$ has graph

The graph of the overlap (the interval where BOTH are true) is

The domain of f is $\left[-\frac{7}{3}, \frac{5}{2}\right]$.

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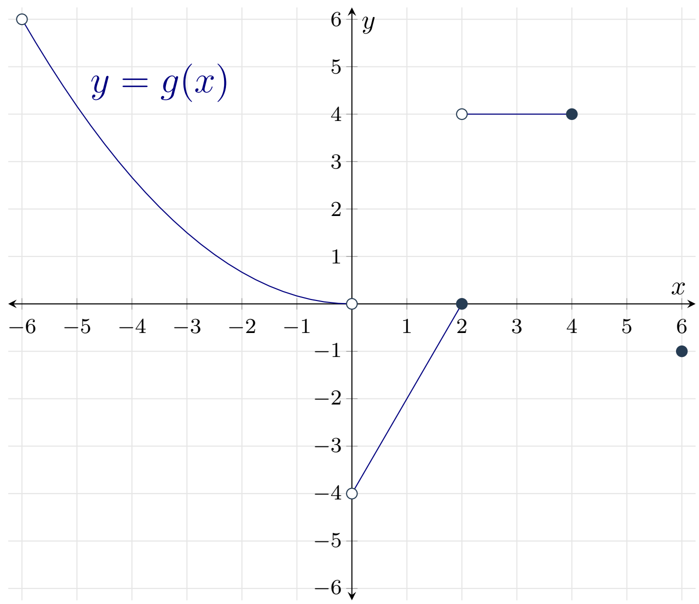
Remark Finally, isolated points are not included in intervals, but are written in the form $\{a\}$, and multiple disjoint intervals are connected using the *Union* symbol $\cup$.

Example 23. *The entire graph of a function g is given in the graph below. Find the domain of g.*

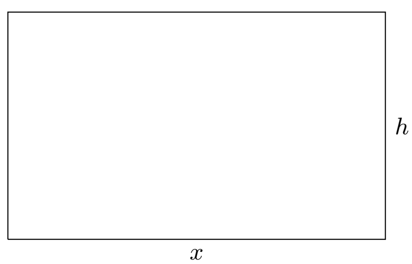


Explanation

Desmos link: `https://www.desmos.com/calculator/re9re7dqew`

Notice that $g(x)$ is defined for all x in $-6 < x < 0$, in $0 < x \le 4$, and at $x = 6$. In interval notation, this is $(-6, 0) \cup (0, 4] \cup \{6\}$.

Example 24. *A piece of wire, 10 meters in length, is folded into a rectangle. Call x the width of the rectangle, as in the image below, and call h the height.*



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Find a formula for the height as a function of x, $h(x)$. What is the domain of h?

Explanation

The wire forms the perimeter of the rectangle. Since the wire has length 10 meters, that means the sum of the lengths of all the edges is 10. Thus, $x + x + h + h = 10$, or $2x + 2h = 10$. Solving this formula for h gives:

$$\begin{aligned} 2x + 2h &= 10 \\ 2h &= 10 - 2x \\ h &= \frac{10 - 2x}{2} \\ &= \frac{10}{2} - \frac{2x}{2} \\ &= 5 - x. \end{aligned}$$

The function h is given by $h(x) = 5 - x$.

Any number can be plugged into the formula $5 - x$, but we have to take into account where these quantities came from in the story. The value x was a length of a side of a rectangle. That means x cannot be negative. For a similar reason, $h(x)$ cannot be negative.

$$\begin{aligned} h(x) &\geq 0 \\ 5 - x &\geq 0 \\ -x &\geq -5 \\ x &\leq 5 \end{aligned}$$

If x has a value larger than 5, it would force $h(x)$ to be negative, which is impossible. The domain of h is $[0, 5]$.

> Think about what it would mean for $x = 0$ or $x = 5$. The value $x = 0$ would correspond to a rectangle with width zero, and $x = 5$ would correspond to a rectangle of height zero (since $h(5) = 0$). For convenience, mathematicians often allow rectangles of width zero or height zero. If you are not comfortable with calling those things rectangles, you can use $(0, 5)$ as your domain instead.

The Domains of Famous Functions

Earlier you were introduced to the graphs of several "Famous Functions". We will revisit these functions over and over again throughout our studies. For now, we will formalize what we have seen with their graphs.

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(a) The Absolute Value function - We can take the absolute value of any number. The Absolute Value function has domain $(-\infty, \infty)$.

(b) Polynomial functions - We can plug any number into a polynomial. All polynomials have domain $(-\infty, \infty)$.

(c) Rational functions - Remember that a rational function is one that can be written as fraction of two polynomials, with the denominator not the zero polynomial. The domain of a rational function consists of all real numbers for which the denominator is nonzero.

(d) The Square Root function - We can take the square root of any non-negative number. The square root function has domain $[0, \infty)$.

(e) Exponential functions - Exponential functions b^x, for $b > 0$ with $b \neq 1$, have domain $(-\infty, \infty)$.

(f) Logarithms - Logarithms have domain $(0, \infty)$. This is similar to the domain of $\sqrt{x}$, except the endpoint is not included.

(g) The Sine function - The sine function $\sin(x)$ has domain $(-\infty, \infty)$.

Spotting Values not in the Domain

Of our list of famous functions, notice that only rational functions, radicals, and logarithms have domain that is not the full set of all real numbers, $(-\infty, \infty)$. When trying to find the domain of a function constructed out of famous functions, this gives us some guidelines to follow. The following list is not exhaustive, but gives a good place to begin.

[Strategies for Finding the Domain]

(a) The input of an even-index radical must be non-negative.

(b) The input of a logarithm must be positive.

(c) The denominator of a fraction cannot be zero.

(d) The real-world context. If a function has a real-world description, this may add additional restrictions on the input values. (You can see this in Example 24 above.)

Remember that the number zero is neither positive nor negative. The non-negative numbers are $[0, \infty)$, while the positive numbers are $(0, \infty)$.

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Example 25. *Find the domain of the function*

$$f(x) = 3|x| - 5x^3 + 7x + \frac{2x+5}{x-1} + \ln(3-x).$$

Explanation Examine the individual terms. The first term is an absolute value function, while the second and third terms are polynomials. There is no restriction on their domain. The last two terms, however, are a fraction and a logarithm.

The denominator of the fraction cannot be zero, so

$$\begin{aligned} x - 1 &\neq 0 \\ x &\neq 1. \end{aligned}$$

The input to the logarithm must be positive, so

$$\begin{aligned} 3 - x &> 0 \\ -x &> -3 \\ x &< 3. \end{aligned}$$

In order for a number to be in the domain of the function, it must be in the domain of every term of the function. That means it must satisfy both $x \neq 1$ and $x < 3$. Altogether, this means the domain is $(-\infty, 1) \cup (1, 3)$.

Example 26. *Find the domain of the function*

$$s(t) = \frac{\ln(2t+3) - \sqrt{5t-1}}{t^2+1}$$

Explanation The denominator of this fraction is t^2+1. The graph of $y = t^2+1$ is an upward-opening parabola with vertex at the point $(0, 1)$. As such, the denominator does not have zero as an output. Our only restrictions will come from the numerator.

The input to the logarithm must be positive, so

$$\begin{aligned} 2t + 3 &> 0 \\ 2t &> -3 \\ t &> -\frac{3}{2}. \end{aligned}$$

That inequality has graph given by

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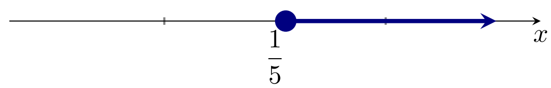
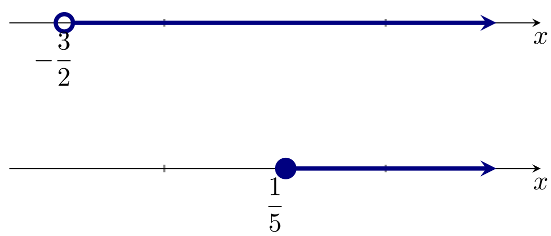
The radicand must be non-negative, so

$$\begin{aligned} 5t - 1 &\geq 0 \\ 5t &\geq 1 \\ t &\geq \frac{1}{5}. \end{aligned}$$

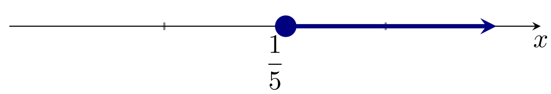
That inequality has graph given by



In order for a number to be in the domain of this function, satisfy both $t > -\frac{3}{2}$ and $t \geq \frac{1}{5}$. The points satisfying both inequalities are given in the graph found by overlaying both graphs above



and taking the graph of all points on both graphs.



Altogether, this means the domain is $\left[\frac{1}{5}, \infty\right)$

Piecewise Defined Functions and Restricted Domains

Consider the function $f(x) = 2|x|+3$ for $x \geq -5$, and the function $g(x) = 2|x|+3$ (given without this restriction). The implied domain of g is $(-\infty, \infty)$, but what can we say about $f(-8)$? The formula $2|x| + 3$ makes sense when $x = -8$, but the function definition for f has the added statement "for $x \geq -5$". This is telling us the domain of f is $[-5, \infty)$. In this case $f(-8)$ is undefined.

We can think of the function f as coming from the function g by deciding that some inputs are not valid. We have *restricted the domain.*

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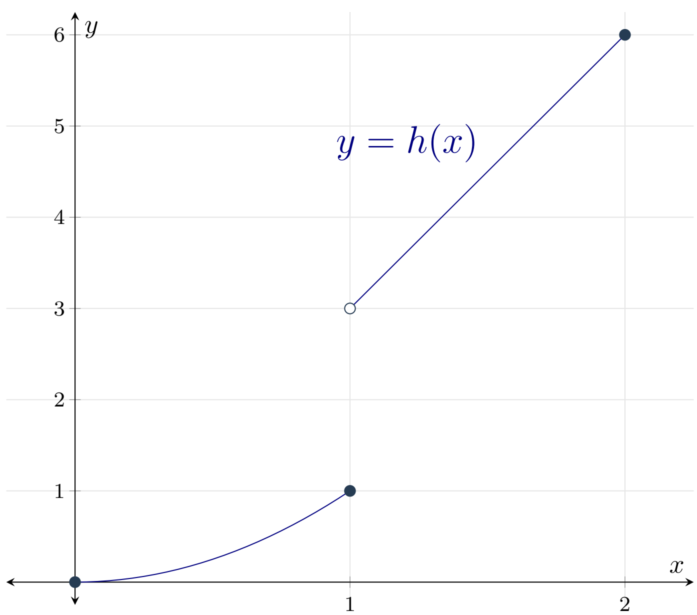
Suppose we have a function f given by $f(x) = x^2$ for $0 \leq x \leq 1$ (which has domain $[0,1]$) and a different function g given by $g(x) = 3x$ for $1 < x \leq 2$ (which has domain $(1,2]$). If we are given an x-value in the interval $[0,2]$, that input can only be plugged into one of these two functions. Let's create a new function h by setting $h(x) = f(x) = x^2$ if $0 \leq x \leq 1$ and by setting $h(x) = g(x) = 3x$ if $1 < x \leq 2$. As a compact way of writing this, we would say:

$$h(x) = \begin{cases} x^2 & \text{for } 0 \leq x \leq 1 \\ 3x & \text{for } 1 < x \leq 2 \end{cases}$$

> **Definition** A **piecewise defined function** is a function that is given by different formulas for different intervals in its domain. This is sometimes shortened to just *piecewise function*.

The function h above is a piecewise defined function. On the interval $[0,1]$ it is given by the formula x^2, and on the interval $(1,2]$ it is given by the formula $3x$. It has two pieces, one piece is quadratic and the other piece is linear. The graph of the function h is given below.



Example 27. *Let f be the piecewise defined function given by*

$$f(x) = \begin{cases} 5 & \textit{for } x \leq -2 \\ \sin(x) & \textit{for } -2 < x < 3 \\ 2^x & \textit{for } x > 4 \end{cases} \cdot$$

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What is the domain of f? Evaluate the following:

(a) $f(-5)$

(b) $f(0)$

(c) $f\left(\frac{\pi}{2}\right)$

(d) $f(4)$

(e) $f(5)$

Explanation

The function f is given as a piecewise defined function with three pieces. The first piece is used $x \leq -2$, the second piece is used when $-2 < x < 3$, and the third piece is used when $x > 4$. This function is defined for all numbers except those between 3 and 4.

The domain of this function is $(-\infty, 3) \cup (4, \infty)$.

(a) Since $-5 \leq -2$, this uses the first piece of the function, so $f(-5) = 5$.

(b) Since $-2 < 0 < 3$, $f(0) = \sin(0) = 0$.

(c) $\frac{\pi}{2}$ is between 1 and 2 (it's approximately 1.57), so $f\left(\frac{\pi}{2}\right) = \sin\left(\frac{\pi}{2}\right) = 1$.

(d) 4 is not in the domain of f, so $f(4)$ is undefined.

(e) Since $5 > 4$, $f(5) = 2^5 = 32$.

Example 28. *Write the absolute value function as a piecewise defined function.*

Explanation

Let's examine the graph of $y = |x|$.

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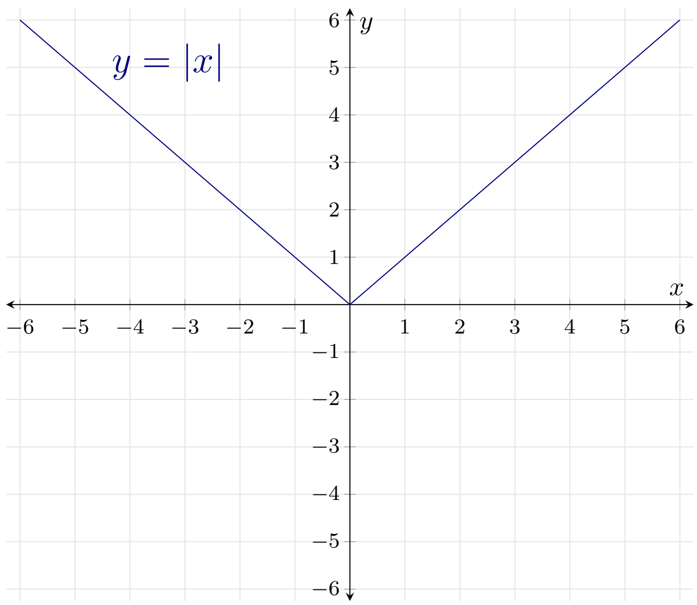

Do you notice that this graph looks like two straight lines, meeting at the origin? Let's focus on the right-hand side first. For $x \geq 0$, this is a line with slope $m = 1$ and y-intercept at the origin $(0, 0)$. This line has equation $y = 1x + 0 = x$. For $x < 0$, this is a line with slope $m = -1$ and y-intercept at the origin $(0, 0)$. This line has equation $y = -1x + 0 = -x$.

That means $|x|$ agrees with x if $x \geq 0$, and agrees with $-x$ if $x < 0$. Putting these together gives us:

$$|x| = \begin{cases} -x & \text{for } x < 0 \\ x & \text{for } x \geq 0. \end{cases}$$

This formula tells us that the absolute value of a positive number is itself, while the absolute value of a negative number changes the sign.

6.3.2 Range

Motivating Questions

- If f is a function from a set A to a set B, does every item in B actually get related to something from A?

Introduction

In the last section, for a function from A to B, we called the set A the *domain* and we called the set B the *codomain*.

Let f be the function defined by $f(x) = x^2$. We can consider this a function from the set of all real numbers $(-\infty, \infty)$ to the set of all real numbers $(-\infty, \infty)$. In this case, the domain is $(-\infty, \infty)$ and the codomain is also $(-\infty, \infty)$. We know that for any real number x, the value of x^2 is never negative. That means there is no input to f that ever gives a negative output.

Let g be the function from the set of capital letters to the set of natural numbers, which assigns each letter to its placement in the alphabet. This means $g(A) = 1$ since 'A' is the first letter of the alphabet. Similarly $g(B) = 2$ and $g(Z) = 26$. In this case the domain is the set of capital letters $\{A, B, C, \ldots, Z\}$ and the codomain is the set of natural numbers $\{1, 2, 3, 4, \ldots\}$. For the function g there are only 26 capital letters in the alphabet, so no number past greater than 26 is ever an output of g.

For both the function f and g just given, not every number in the codomain is actually achieved as the output of the function. There is a difference between the codomain, which measures the "possible outputs" and the actual outputs that are achieved.

Exploration

a. Suppose the quadratic function f is given by $f(x) = x^2$. Are there any values that are never achieved as an output?

b. Explain the difference in finding the domain of a function and finding the range of the function, if you are given the graph of the function. What if you're given a formula for the function instead?

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The Range of a Function

> **Definition** Let f be a function from A to B. The **range** of f is the collection of the outputs of f.

This means the *range* consists of the outputs that are actually achieved. Not everything that is "possible", but only those outputs that actually come out of the function. For each b in the range of the function f, there is actually an a in the domain with $f(a) = b$.

Example 29. *Let f be the function defined by the following table. Find the range of f.*

x	$f(x)$
0	3
1	-2
2	4
3	3
4	0

Explanation

The only outputs of this function are -2, 0, 3, and 4. (Even though the output 3 is identified twice, it only gets counted once here.) There are no intervals, just the separated points. The range of f is $\{-2, 0, 3, 4\}$. Including individual numbers in curly brackets means we are considering the set whose only members are those individual numbers.

Example 30. *Let s be the function given by $s(t) = \frac{1}{t}$. Is $-\frac{1}{2}$ in the range of s? What about $\sqrt{5}$ or 0?*

Explanation

If we want to see if $-\frac{1}{2}$ is in the range of s, that means we need to check whether

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there is a number a in the domain of s with $s(a) = -\frac{1}{2}$.

$$\begin{aligned} s(a) &= -\frac{1}{2} \\ \frac{1}{a} &= -\frac{1}{2} \\ 2a\left(\frac{1}{a}\right) &= 2a\left(-\frac{1}{2}\right) \\ \frac{2a}{a} &= -\frac{2a}{2} \\ 2 &= -a \\ -2 &= a \end{aligned}$$

That means $s(-2) = -\frac{1}{2}$. We found an input that gives the output value $-\frac{1}{2}$, which means $-\frac{1}{2}$ is in the range of s.

Let's try the same calculation for $\sqrt{5}$.

$$\begin{aligned} s(b) &= \sqrt{5} \\ \frac{1}{b} &= \sqrt{5} \\ b\left(\frac{1}{b}\right) &= b\left(\sqrt{5}\right) \\ 1 &= b\sqrt{5} \\ \frac{1}{\sqrt{5}} &= b \end{aligned}$$

That means $s\left(\frac{1}{\sqrt{5}}\right) = \sqrt{5}$, so that $\sqrt{5}$ is in the range of s.

Now for 0:

$$\begin{aligned} s(c) &= 0 \\ \frac{1}{c} &= 0 \\ c\left(\frac{1}{c}\right) &= c\,(0) \\ 1 &= c(0) \end{aligned}$$

This statement is false for all choices of c. That means there is no value of c with $s(c) = 0$. In particular, 0 is not in the range of s.

Example 31. *The entire graph of a function g is given in the graph below. Find the range of g.*

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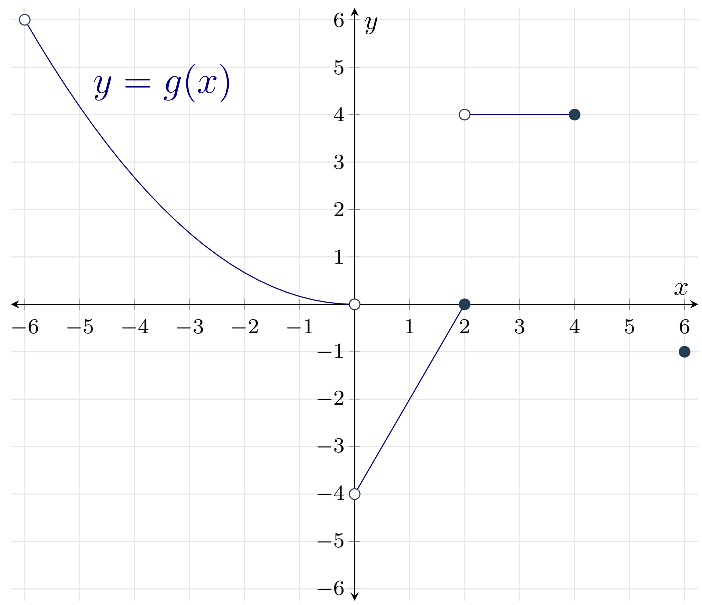

Explanation

Desmos link: `https://www.desmos.com/calculator/392xadl0ab`

Notice that as x changes between -6 to 0, the graph takes all outputs from 6 to 0 (not including the endpoints). As x changes from 0 to 2, all the numbers from -4 to 0 show up as outputs (including 0). Together, this means that every number in the interval $(-4, 6)$ is in the range. The last two pieces of the graph have outputs 4 and -1, which are already included in this interval.

The range is $(-4, 6)$.

Example 32. *Let f be the function given by $f(x) = 3x - 5$. Find the range of f.*

Explanation We know the domain of f is $(-\infty, \infty)$, so any real number is a valid input. We also know that f is a linear polynomial function (a polynomial of degree 1) so we know what its graph is a straight line with slope $m = 3$.

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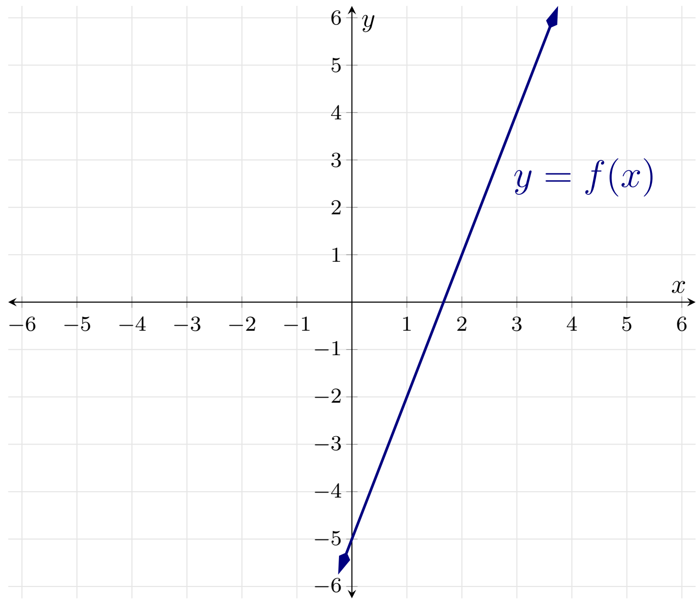

Since this graph goes higher and higher as x travels to the right, and lower as x travels to the left, we believe that every number is eventually an output of this function.

Let's make a calculation to be sure that's true by taking an arbitrary real number and show that it's actually an output of f. If a and b are real numbers with $f(a) = b$, then

$$\begin{aligned} f(a) &= b \\ 3a - 5 &= b \\ 3a &= b + 5 \\ a &= \frac{b+5}{3}. \end{aligned}$$

That means that if b is any arbitrary real number, then $f\left(\frac{b+5}{3}\right) = b$ so that b is achieved as an output of the function, so b is in the range. In other words, this gives us a formula to identify an input that gives b as an output. This means the range of f is $(-\infty, \infty)$.

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The Range of Famous Functions

(a) The Absolute Value function - The average value of a number is never negative. The Absolute Value function has range $[0, \infty)$.

(b) Polynomial functions - This depends on the degree of the polynomial.

  (i) Odd degree - The range is $(-\infty, \infty)$.

  (ii) Even degree - We can only be precise with monomials (polynomials with only one term) like $5x^2$ or $-6x^8$.

    i. If the monomial has positive coefficient, the range is $[0, \infty)$.

    ii. If the monomial has negative coefficient, the range is $(-\infty, 0]$.

    iii. If the monomial is a constant c, the range is $\{c\}$.

(c) The Square Root function - Even-index radicals never have negative outputs. Their range is $[0, \infty)$.

(d) Exponential functions - Exponential functions b^x, for $b > 0$ with $b \neq 1$, have range $(0, \infty)$. Notice that 0 is never an output for these kinds of functions.

(e) Logarithms - Logarithms have range $(-\infty, \infty)$.

(f) The Sine function - The sine function $\sin(x)$ has range $[-1, 1]$.

Remark For polynomials, if we know the turning points or maximum or minimum values of the polynomial (as in the case of a parabola), you can determine the range of the polynomial. Unfortunately, finding these in general requires calculus.

Spotting Values not in the Range

Finding the range of a function is quite a bit more involved than finding the domain. Here are some guidelines if you are given a formula for the function instead of its graph.

(a) The output of an even-index radical is never negative.

(b) The output of x^n is never negative if n is an even natural number.

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(c) The output of an exponential function b^x is always positive.

Example 33. *Find the range of the following functions*

(a) $f(x) = 2 - 4\sqrt{x}$.

(b) $g(x) = e^x + \dfrac{1}{2}$.

(c) $h(x) = 3 + 5\sin(x)$.

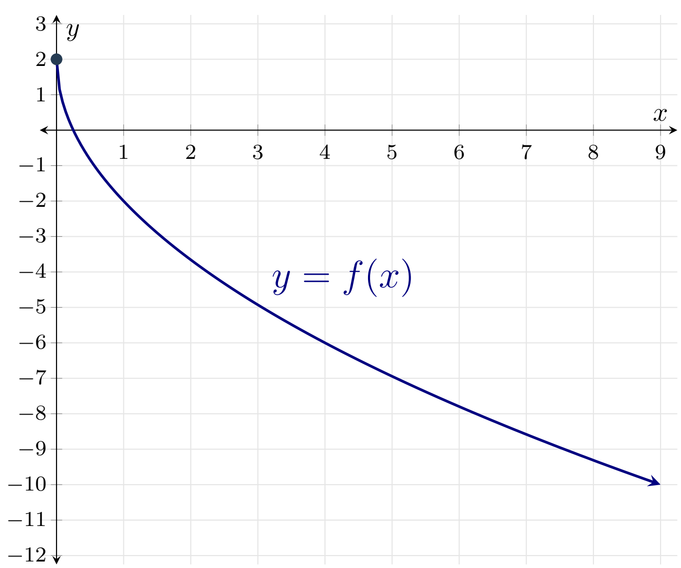
Explanation

You'll notice in these calculations, that finding the range of a function is a great deal more complicated than finding the domain, unless we have an accurate graph of the function, as above.

For each of these calculations, we will follow the same two steps. The idea is that the first step shows that the range is "no more than" the interval we build, and the second step shows that the range is "no less than" that interval. The only possibility left to us is that the range is exactly the interval we have constructed.

More specifically, in the first step will find a bound on the range. We'll determine if any numbers are too big or too small to be a valid output of the function. This will give us an interval that the range will have to be inside. In the second step, we'll see that everything inside that interval is actually attained by the function, by constructing an input value that gets assigned to that output.

(a) Let's start by looking at the graph of f



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Notice that the graph of f looks similar to what we know for $\sqrt{x}$, but upside down, stretched, and moved up. From the graph, we would say that the range should be $(-\infty, 2]$. Let's see how we can verify that with a few calculations.

We know that the range of $\sqrt{x}$ is $[0, \infty)$ so for any x in the domain of f,

$$\begin{aligned} \sqrt{x} &\geq 0 \\ -4\sqrt{x} &\leq 0 \\ 2-4\sqrt{x} &\leq 2 \\ f(x) &\leq 2 \end{aligned}$$

The outputs of f are never larger than 2, so the only numbers in the range are less than or equal to 2. That is, the range must be inside the interval $(-\infty, 2]$.

Metacognitive Moment Notice that this process is just looking at how function transformations are changing the range by undoing the transformations one at a time.

To verify that the range is exactly $(-\infty, 2]$, suppose $b \leq 2$, then:

$$\begin{aligned} f(a) &= b \\ 2-4\sqrt{a} &= b \\ -4\sqrt{a} &= b-2 \\ \sqrt{a} &= \frac{b-2}{-4} \\ \sqrt{a} &= \frac{-(b-2)}{4} \\ \sqrt{a} &= \frac{-b+2}{4} \\ \left(\sqrt{a}\right)^2 &= \left(\frac{-b+2}{4}\right)^2 \\ a &= \left(\frac{-b+2}{4}\right)^2. \end{aligned}$$

For this value of a, we have $f(a) = b$. That means b is in the range of f. That means every number in the interval $(-\infty, 2]$ is in the range of f.

The range of f is $(-\infty, 2]$.

(b) The graph of g looks like this.

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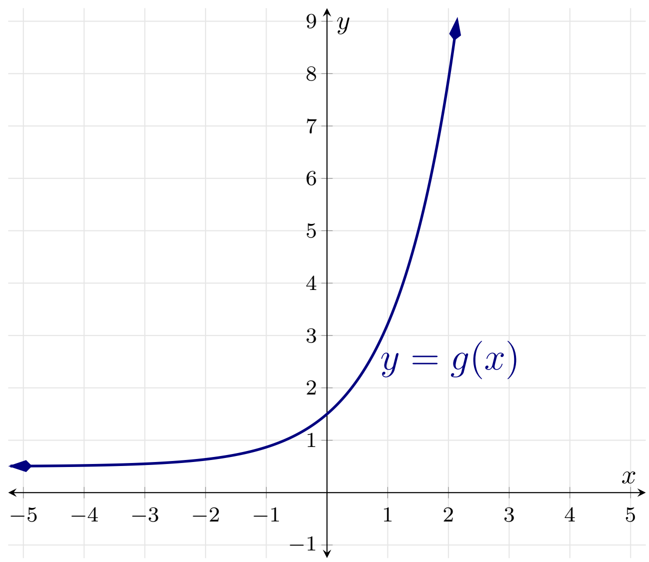

This graph looks exactly like the graph of our famous function e^x, just moved vertically upward by $\frac{1}{2}$. The range of e^x is $(0, \infty)$, so our range should be shifted by $\frac{1}{2}$ to be $\left(\frac{1}{2}, \infty\right)$.

To verify this with a calculation, the fact that e^x has range $(0, \infty)$ means for any value of x,

$$\begin{aligned} e^x &> 0 \\ e^x + \frac{1}{2} &> \frac{1}{2} \end{aligned}$$

The outputs of g are greater than $\frac{1}{2}$, so the range of g is in the interval $\left(\frac{1}{2}, \infty\right)$.

To verify that the range is exactly $\left(\frac{1}{2}, \infty\right)$, suppose $b > \frac{1}{2}$, then:

$$\begin{aligned} g(a) &= b \\ e^a + \frac{1}{2} &= b \\ e^a &= b - \frac{1}{2} \end{aligned}$$

Since the range of e^x is $(0, \infty)$, and $b - \frac{1}{2} > 0$, there is a value for a with $e^a = b - \frac{1}{2}$.

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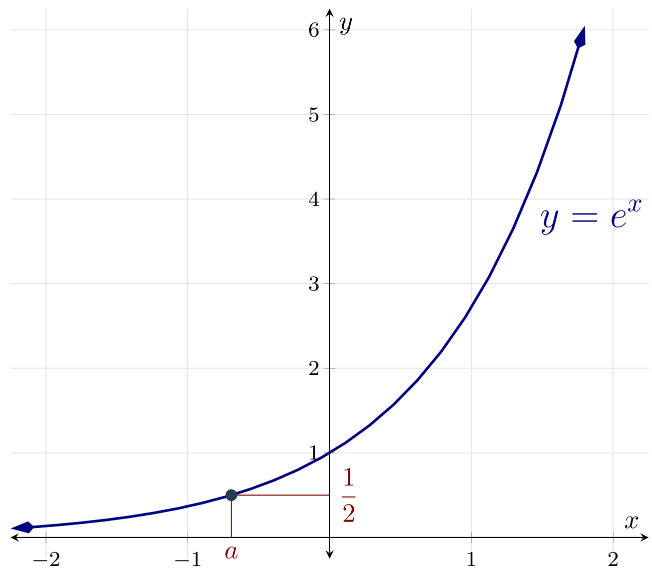
(For instance, if $b = 1$, that would mean $b - \frac{1}{2} = \frac{1}{2}$, so we would be looking to see if there is a value of a with $e^a = \frac{1}{2}$. This is illustrated in the graph below.)



For this value of a, we have $g(a) = b$, meaning that b is in the range of g. The range of g is $\left(\frac{1}{2}, \infty\right)$.

(c) The graph of h is given below, from which it appears the range of h is $[-2, 8]$.

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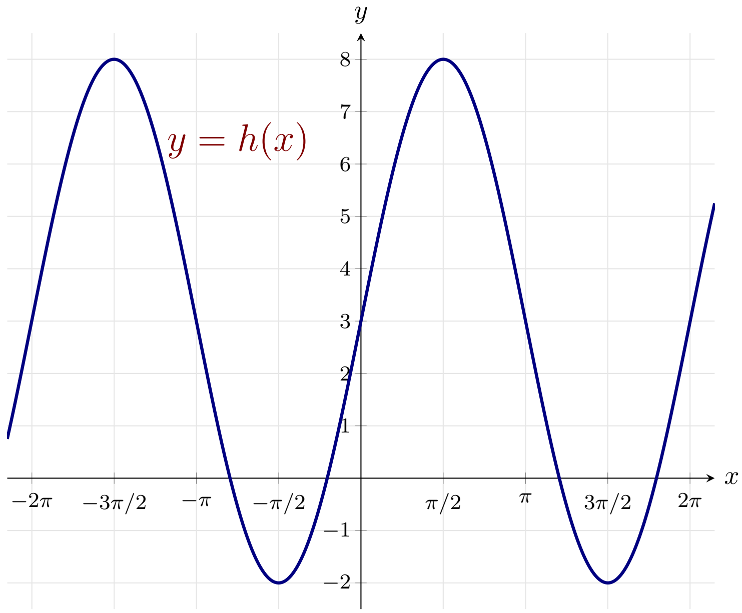


The range of our famous function *sine* is $[-1, 1]$. That means for any x

$$\begin{aligned} -1 &\le \sin(x) \le 1 \\ -5 &\le 5\sin(x) \le 5 \\ 3 + (-5) &\le 3 + 5\sin(x) \le 3 + 5 \\ -2 &\le 3 + 5\sin(x) \le 8 \end{aligned}$$

The outputs of h are in the interval $[-2, 8]$. To verify that the range is exactly $[-2, 8]$, suppose b is a number with $-2 \le b \le 8$. Then:

$$\begin{aligned} h(a) &= b \\ 3 + 5\sin(a) &= b \\ 5\sin(a) &= b - 3 \\ \sin(a) &= \frac{b-3}{5}. \end{aligned}$$

Because $-2 \le b \le 8$, we know $-1 \le \frac{b-3}{5} \le 1$. That means there is a number a with $\sin(a) = \frac{b-3}{5}$, since the range of sine is $[-1, 1]$.

(For instance, if $b = 3$ then $\frac{b-3}{5} = 0$ so we would be looking for a value of a with $\sin(a) = 0$. We know that $\sin(0) = 0$, so this means we'd take $a = 0$.) For this value of a, we have $h(a) = b$, so b is in the range of h. The range of h is then $[-2, 8]$.

6.3.3 Famous Functions, Updated

Now that we know about the domain and range, we can update our list of famous functions.

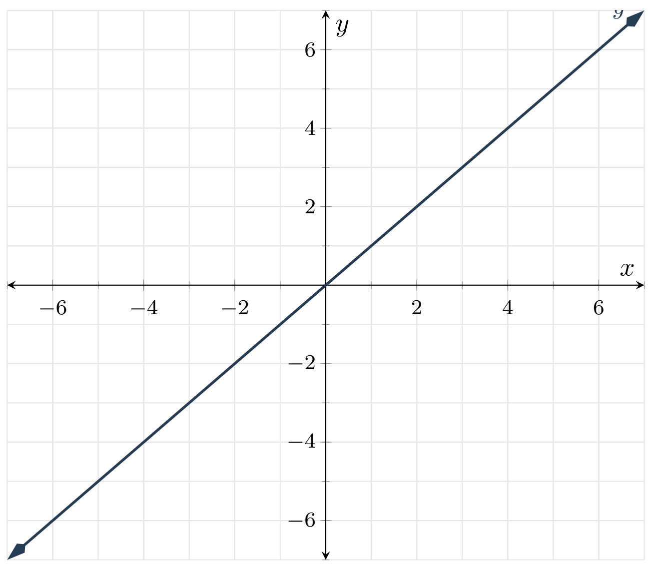
In Section 1-2, you saw a variety of famous functions. Now that we have learned more about properties of functions, we can update our knowledge of those famous functions. We will go through the list of famous functions from before and point out where each function might have properties we've discussed.

Linear Functions

Recall that the graph of a linear function is a line.

Example 34. *A prototypical example of a linear function is*

$$y = x.$$



Important Values of $y = x$

x	y
-2	-2
-1	-1
0	0
1	1
2	2

In general, linear functions can be written as $y = mx + b$ where m and b can be any numbers. We learned that m represents the slope, and b is the y-coordinate of the y-intercept. You can play with changing the values of m and b on the graph using Desmos and see how that changes the line.

Desmos link: `https://www.desmos.com/calculator/japnhapzvn`

Note that a linear function f defined by $f(x) = mx + b$ with $b = 0$ is odd. If $m = 0$, then f is periodic, since it is constant. Furthermore, constant functions are always even.

Additionally, if $m \neq 0$, then a linear function is one-to-one, and therefore invertible. We summarize this information in the table below.

Note that any real number can be plugged into $f(x) = mx + b$, so the domain of linear functions is $(-\infty, \infty)$. Unless $m = 0$, we can find a y such that $y = mx+b$, so the range of linear functions with $m \neq 0$ is $(-\infty, \infty)$. If $m = 0$, then the only output of the linear function is b, so its range is $\{b\}$.

Properties of Linear Functions

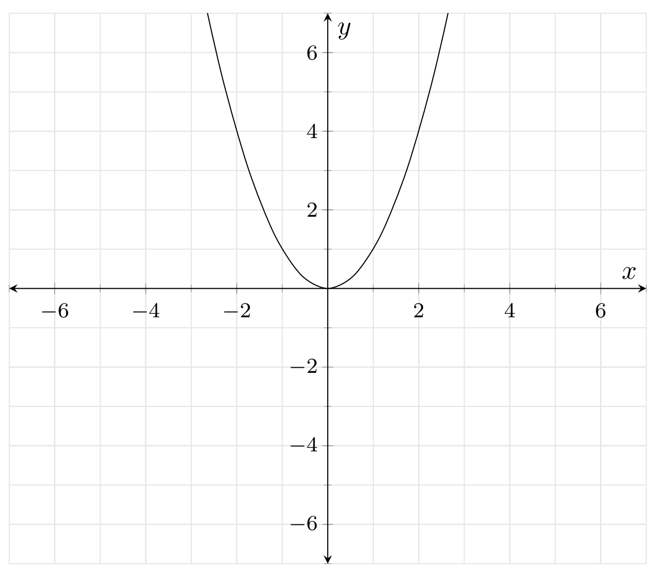
Periodic?	If $m = 0$
Odd?	If $b = 0$
Even?	If $m = 0$
One-to-one/invertible?	Yes
Domain	$(-\infty, \infty)$
Range	$(-\infty, \infty)$

Quadratic Functions

Recall that the graph of a quadratic function is a parabola.

Example 35. *A prototypical example of a quadratic function is*

$$y = x^2.$$



Important Values of $y = x^2$

x	y
-2	4
-1	1
0	0
1	1
2	4

In general, quadratic functions can be written as $y = ax^2 + bx + c$ where a, b, and c can be any numbers. You can play with changing the values of a, b, and c on the graph using Desmos and see how that changes the parabola.

Desmos link: `https://www.desmos.com/calculator/nmlghfrws9`

Note that for a quadratic function f defined by $f(x) = ax^2 + bx + c$, if $b = 0$, then f is even. In general, quadratic functions are not one-to-one, odd, or periodic, except in cases where $a = 0$, in which we're actually dealing with a linear function.

Note that any real number can be plugged into $f(x) = ax^2+bx+c$, so the domain of quadratic functions is $(-\infty, \infty)$. In Chapter 4, we saw that all quadratic functions have a vertex form $f(x) = a(x-h)^2+k$, where the vertex is at (h, k). If $a > 0$, all points above the vertex, that is $[k, \infty)$ are in the range of the quadratic, and if $a < 0$, all points below the vertex, that is $(\infty, k]$ are in the range of the quadratic.

We summarize this information in the table below.

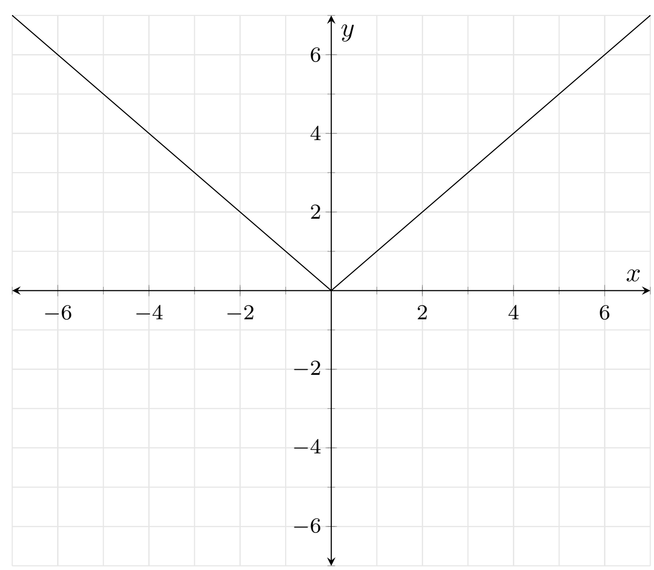
Properties of Quadratic Functions $y = ax^2 + bx + c, a \neq 0$

Periodic?	No
Odd?	No
Even?	If $b = 0$
One-to-one/invertible?	No
Domain	$(-\infty, \infty)$
Range	If $a > 0$, $[k, \infty)$, if $a < 0$, $(-\infty, k]$

Absolute Value Function

Another important type of function is the absolute value function. This is the function that takes all y-values and makes them positive. The absolute value function is written as

$$y = |x|.$$



Important Values of $y = |x|$

x	y
-2	2
-1	1
0	0
1	1
2	2

Notice that the absolute value function is even. Is it one-to-one? The fact that it's even tells us that it is not, since $|-x| = |x|$ for all x. We summarize this information in the table below.

Note that any real number has an absolute value, so the domain of the absolute value function is $(-\infty, \infty)$. Furthermore, by looking at the graph, we can see that all non-negative numbers are in the range of the absolute value function.

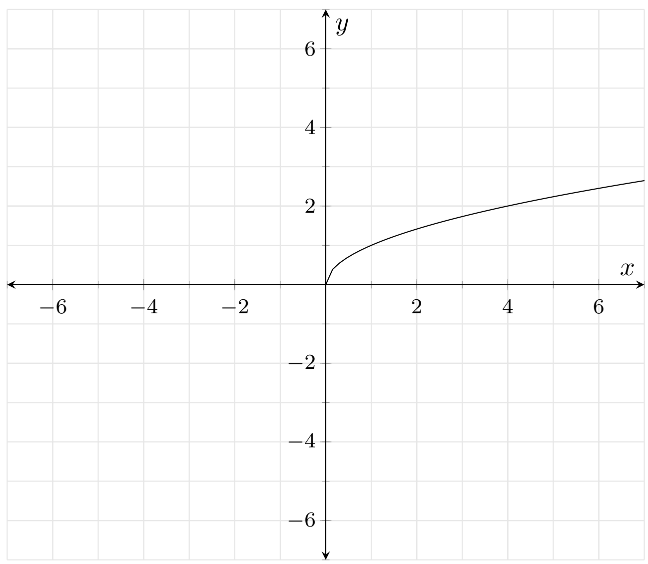
Properties of the Absolute Value Function $y = |x|$

Periodic?	No
Odd?	No
Even?	Yes
One-to-one/invertible?	No
Domain	$(-\infty, \infty)$
Range	$[0, \infty)$

Square Root Function

Another famous function is the square root function,

$$y = \sqrt{x}.$$



Important Values of $y = \sqrt{x}$

x	y
0	0
1	1
4	2
9	3
25	5

The square root function is one-to-one. Negative inputs are not valid for the square root function, so it is neither even, odd, nor periodic. We summarize this information in the table below.

Note that only non-negative numbers have square roots, so the domain of the square root function is $[0, \infty)$. Furthermore, by looking at the graph, we can see that all non-negative numbers are in the range of the square root function. Algebraically, we can say that for any non-negative y, $\sqrt{(y^2)} = y$, so y is in the range of the square root function.

Properties of the Square Root Function $y = \sqrt{x}$

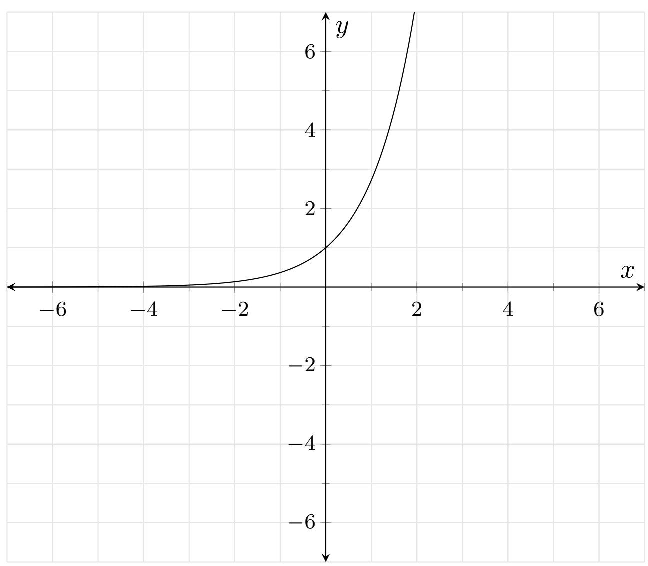
Periodic?	No
Odd?	No
Even?	No
One-to-one/invertible?	Yes
Domain	$[0, \infty)$
Range	$[0, \infty)$

Exponential Functions

Another famous function is the exponential growth function,

$$y = e^x.$$

Here e is the mathematical constant known as Euler's number. $e \approx 2.71828..$



Important Values of $y = e^x$

x	y
0	1
1	e
-1	$e^{-1} = \frac{1}{e}$

In general, we can talk about exponential functions of the form $y = b^x$ where b is a positive number not equal to 1. You can play with changing the values of b on the graph using Desmos and see how that changes the graph. Pay particular attention to the difference between $b > 1$ and $0 < b < 1$.

Desmos link: https://www.desmos.com/calculator/qsmvb7tiex

Notice that exponential functions are one-to-one, and therefore invertible. However, they are neither even, odd, nor periodic.

Note that the domain of the exponential functions is $(-\infty, \infty)$. Furthermore, by looking at the graph, we can see that all non-negative numbers are in the range of the exponential functions.

We summarize this information in the table below.

Properties of the Exponential Functions $y = b^x$

Periodic?	No
Odd?	No
Even?	No
One-to-one/invertible?	Yes
Domain	$(-\infty, \infty)$
Range	$(0, \infty)$

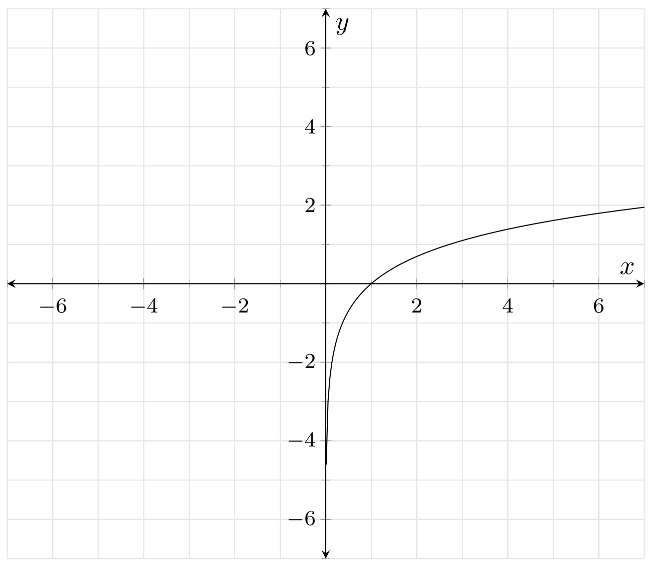
Logarithm Functions

Another group of famous functions are logarithms.

Example 36. *The most famous logarithm function is*

$$y = \ln(x) = \log_e(x).$$

Here e is the mathematical constant known as Euler's number. $e \approx 2.71828$.



Important Values of $y = \ln(x)$

x	y
0	*undefined*
$\frac{1}{e}$	-1
1	0
e	1

In general, we can talk about logarithmic functions of the form $y = \log_b(x)$ where b is a positive number not equal to 1. You can play with changing the values of b on the graph using Desmos and see how that changes the graph. Pay particular attention to the difference between $b > 1$ and $0 < b < 1$.

Desmos link: https://www.desmos.com/calculator/lxllnpdi6w

Notice that logarithms are neither even, odd, nor periodic. However, they are one-to-one, and therefore invertible. It turns out that the inverse of a logarithm is an exponential function, and vice versa!

Note that since the logarithm is the inverse of the exponential, the domain of the logarithms is the range of the exponentials: $[0, \infty)$. Furthermore, the range of the logarithms is the range of the exponentials: $(-\infty, \infty)$.

We summarize this information in the table below.

Properties of the Logarithm Functions $y = \log_b(x)$

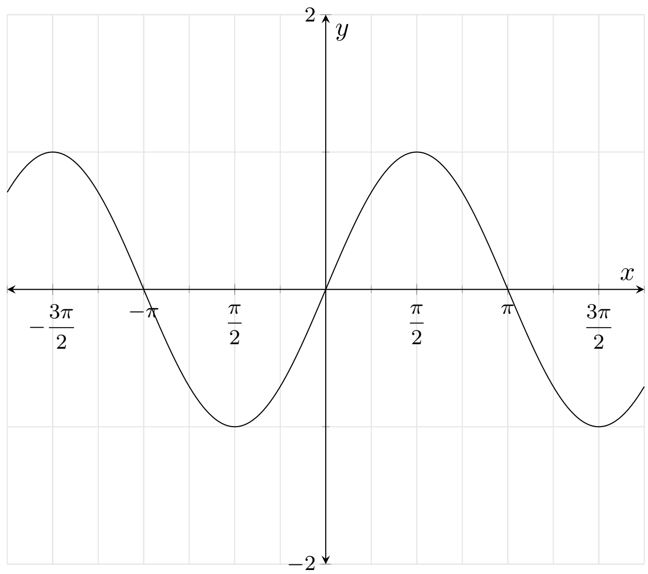
Periodic?	No
Odd?	No
Even?	No
One-to-one/invertible?	Yes
Domain	$(0, \infty)$
Range	$(-\infty, \infty)$

Sine

Another important function is the sine function,

$$y = \sin(x).$$

This function comes from trigonometry. In the table below we will use another mathematical constant, π ("pi" pronounced pie). $\pi \approx 3.14159$.



Important Values of $y = \sin(x)$

x	y
$-\pi$	0
$-\frac{\pi}{2}$	-1
0	0
$\frac{\pi}{2}$	1
π	0
$\frac{3\pi}{2}$	-1
2π	0

As mentioned earlier, the sine function is odd and periodic with period 2π. Since it is periodic, however, it cannot be one-to-one, since its values repeat.

Note that the domain of the sine function is $(-\infty, \infty)$. Furthermore, by looking at the graph, we can see that its range is $[-1, 1]$.

We summarize this information in the table below.

Properties of the Sine Function $y = \sin(x)$

Periodic?	Yes, with period 2π
Odd?	Yes
Even?	No
One-to-one/invertible?	No
Domain	$(-\infty, \infty)$
Range	$[-1, 1]$

In general, we can consider $y = a\sin(bx)$. You can play with changing the values of a and b on the graph using Desmos and see how that changes the graph.

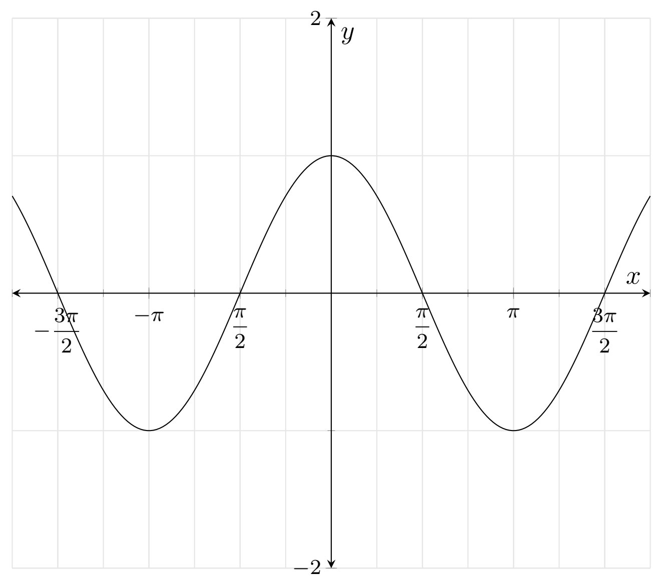
Desmos link: `https://www.desmos.com/calculator/vkxzcfv2aq`

Cosine

A function introduced in Section 3-2 is the cosine function,

$$y = \cos(x).$$

As with sine, the cosine function comes from trigonometry. In the table below we will again use π.



Important Values of $y = \cos(x)$

x	y
$-\pi$	-1
$-\dfrac{\pi}{2}$	0
0	1
$\dfrac{\pi}{2}$	0
π	-1
$\dfrac{3\pi}{2}$	0
2π	1

As mentioned earlier, the cosine function is even and periodic with period 2π. Since it is periodic, however, it cannot be one-to-one, since its values repeat.

Note that the domain of the cosine function is $(-\infty, \infty)$. Furthermore, by looking at the graph, we can see that its range is $[-1, 1]$.

We summarize some information in the table below.

Properties of the Cosine Function $y = \cos(x)$

Periodic?	Yes, with period 2π
Odd?	No
Even?	Yes
One-to-one/invertible?	No
Domain	$(-\infty, \infty)$
Range	$[-1, 1]$

In general, we can consider $y = a\cos(bx)$. You can play with changing the values of a and b on the graph using Desmos and see how that changes the graph.

Desmos link: `https://www.desmos.com/calculator/kvmz1kt19n`

Part 7

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