Precalculus with Review 1: Unit 6

November 29, 2022

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Analyzing Functions

7.1 Composition of Functions

Learning Objectives

- Composition of Functions
  - What does it mean to compose functions?
  - Identify a function as the result of a composition of functions
- Domains of Composite Functions
  - How to find the domain of a composite function
  - How to find the range of a composite function
  - Results of composing $f(x) = x^2$ and $g(x) = \sqrt{x}$

7.1.1 Famous Functions, Updated

Now that we know about the domain and range, we can update our list of famous functions.

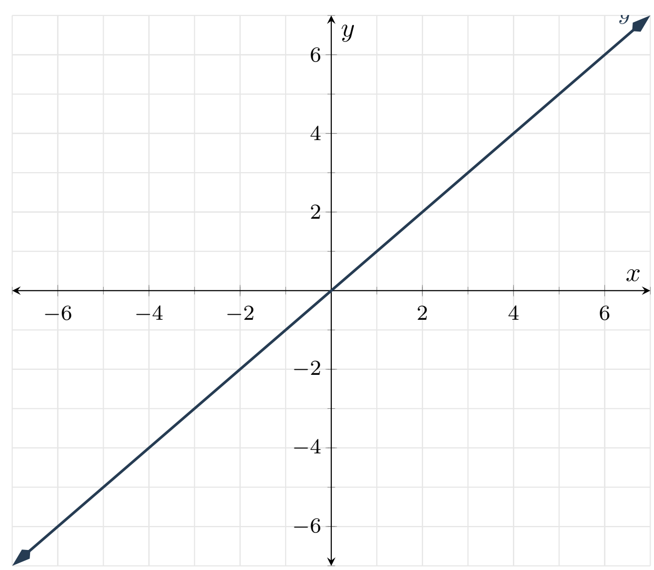
In Section 1-2, you saw a variety of famous functions. Now that we have learned more about properties of functions, we can update our knowledge of those famous functions. We will go through the list of famous functions from before and point out where each function might have properties we've discussed.

Linear Functions

Recall that the graph of a linear function is a line.

Example 1. *A prototypical example of a linear function is*

$$y = x.$$



Important Values of $y = x$

x	y
-2	-2
-1	-1
0	0
1	1
2	2

In general, linear functions can be written as $y = mx + b$ where m and b can be any numbers. We learned that m represents the slope, and b is the y-coordinate of the y-intercept. You can play with changing the values of m and b on the graph using Desmos and see how that changes the line.

Desmos link: `https://www.desmos.com/calculator/japnhapzvn`

Note that a linear function f defined by $f(x) = mx + b$ with $b = 0$ is odd. If $m = 0$, then f is periodic, since it is constant. Furthermore, constant functions are always even.

Additionally, if $m \neq 0$, then a linear function is one-to-one, and therefore invertible. We summarize this information in the table below.

Note that any real number can be plugged into $f(x) = mx+b$, so the domain of linear functions is $(-\infty, \infty)$. Unless $m = 0$, we can find a y such that $y = mx+b$, so the range of linear functions with $m \neq 0$ is $(-\infty, \infty)$. If $m = 0$, then the only output of the linear function is b, so its range is $\{b\}$.

Properties of Linear Functions

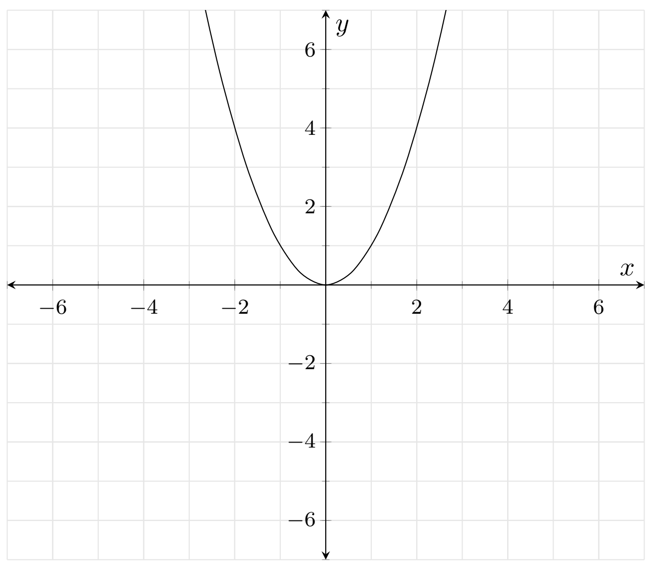
Periodic?	If $m = 0$
Odd?	If $b = 0$
Even?	If $m = 0$
One-to-one/invertible?	Yes
Domain	$(-\infty, \infty)$
Range	$(-\infty, \infty)$

Quadratic Functions

Recall that the graph of a quadratic function is a parabola.

Example 2. *A prototypical example of a quadratic function is*

$$y = x^2.$$



Important Values of $y = x^2$

x	y
-2	4
-1	1
0	0
1	1
2	4

In general, quadratic functions can be written as $y = ax^2 + bx + c$ where a, b, and c can be any numbers. You can play with changing the values of a, b, and c on the graph using Desmos and see how that changes the parabola.

Desmos link: `https://www.desmos.com/calculator/nmlghfrws9`

Note that for a quadratic function f defined by $f(x) = ax^2 + bx + c$, if $b = 0$, then f is even. In general, quadratic functions are not one-to-one, odd, or periodic, except in cases where $a = 0$, in which we're actually dealing with a linear function.

Note that any real number can be plugged into $f(x) = ax^2+bx+c$, so the domain of quadratic functions is $(-\infty, \infty)$. In Chapter 4, we saw that all quadratic functions have a vertex form $f(x) = a(x-h)^2 + k$, where the vertex is at (h, k). If $a > 0$, all points above the vertex, that is $[k, \infty)$ are in the range of the quadratic, and if $a < 0$, all points below the vertex, that is $(\infty, k]$ are in the range of the quadratic.

We summarize this information in the table below.

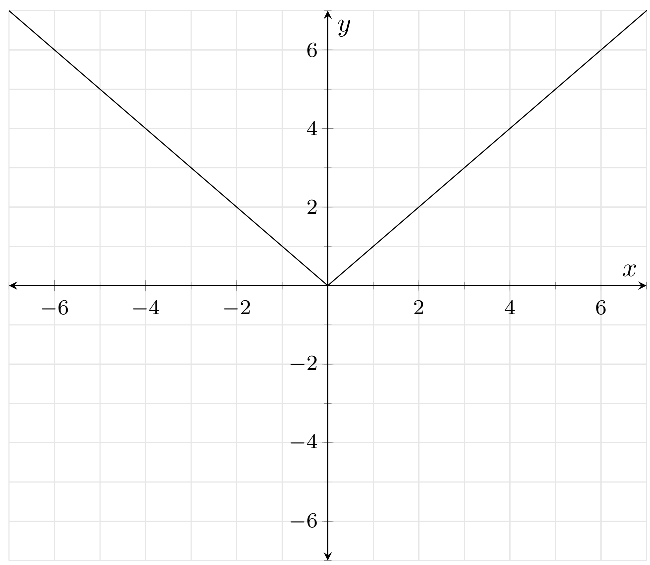
Properties of Quadratic Functions $y = ax^2 + bx + c, a \neq 0$

Periodic?	No
Odd?	No
Even?	If $b = 0$
One-to-one/invertible?	No
Domain	$(-\infty, \infty)$
Range	If $a > 0$, $[k, \infty)$, if $a < 0$, $(-\infty, k]$

Absolute Value Function

Another important type of function is the absolute value function. This is the function that takes all y-values and makes them positive. The absolute value function is written as

$$y = |x|.$$



Important Values of $y = |x|$

x	y
-2	2
-1	1
0	0
1	1
2	2

Notice that the absolute value function is even. Is it one-to-one? The fact that it's even tells us that it is not, since $|-x| = |x|$ for all x. We summarize this information in the table below.

Note that any real number has an absolute value, so the domain of the absolute value function is $(-\infty, \infty)$. Furthermore, by looking at the graph, we can see that all non-negative numbers are in the range of the absolute value function.

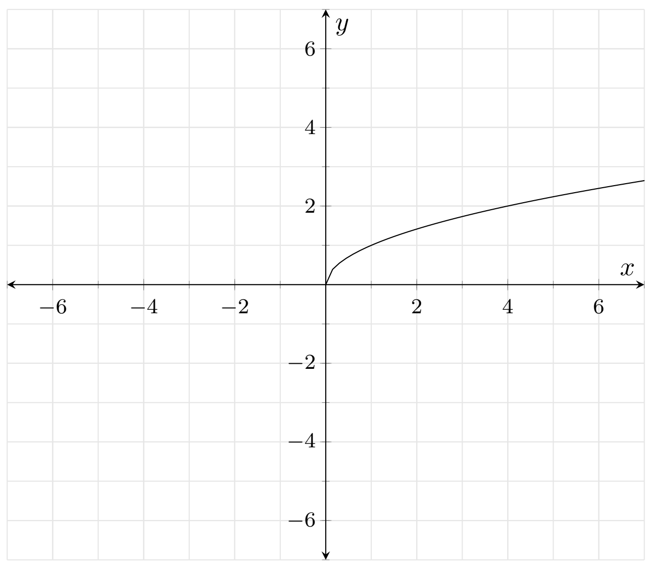
Properties of the Absolute Value Function $y = |x|$

Periodic?	No
Odd?	No
Even?	Yes
One-to-one/invertible?	No
Domain	$(-\infty, \infty)$
Range	$[0, \infty)$

Square Root Function

Another famous function is the square root function,

$$y = \sqrt{x}.$$



Important Values of $y = \sqrt{x}$

x	y
0	0
1	1
4	2
9	3
25	5

The square root function is one-to-one. Negative inputs are not valid for the square root function, so it is neither even, odd, nor periodic. We summarize this information in the table below.

Note that only non-negative numbers have square roots, so the domain of the square root function is $[0, \infty)$. Furthermore, by looking at the graph, we can see that all non-negative numbers are in the range of the square root function. Algebraically, we can say that for any non-negative y, $\sqrt{(y^2)} = y$, so y is in the range of the square root function.

Properties of the Square Root Function $y = \sqrt{x}$

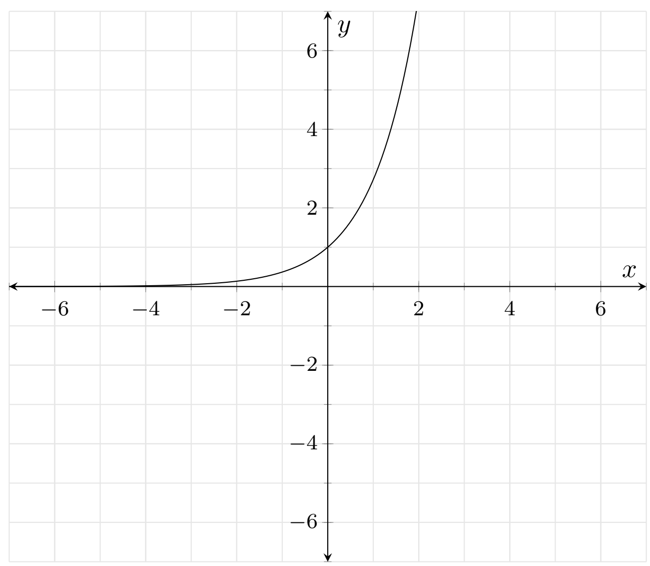
Periodic?	No
Odd?	No
Even?	No
One-to-one/invertible?	Yes
Domain	$[0, \infty)$
Range	$[0, \infty)$

Exponential Functions

Another famous function is the exponential growth function,

$$y = e^x.$$

Here e is the mathematical constant known as Euler's number. $e \approx 2.71828..$



Important Values of $y = e^x$

x	y
0	1
1	e
-1	$e^{-1} = \frac{1}{e}$

In general, we can talk about exponential functions of the form $y = b^x$ where b is a positive number not equal to 1. You can play with changing the values of b on the graph using Desmos and see how that changes the graph. Pay particular attention to the difference between $b > 1$ and $0 < b < 1$.

Desmos link: https://www.desmos.com/calculator/qsmvb7tiex

Notice that exponential functions are one-to-one, and therefore invertible. However, they are neither even, odd, nor periodic.

Note that the domain of the exponential functions is $(-\infty, \infty)$. Furthermore, by looking at the graph, we can see that all non-negative numbers are in the range of the exponential functions.

We summarize this information in the table below.

Properties of the Exponential Functions $y = b^x$

Periodic?	No
Odd?	No
Even?	No
One-to-one/invertible?	Yes
Domain	$(-\infty, \infty)$
Range	$(0, \infty)$

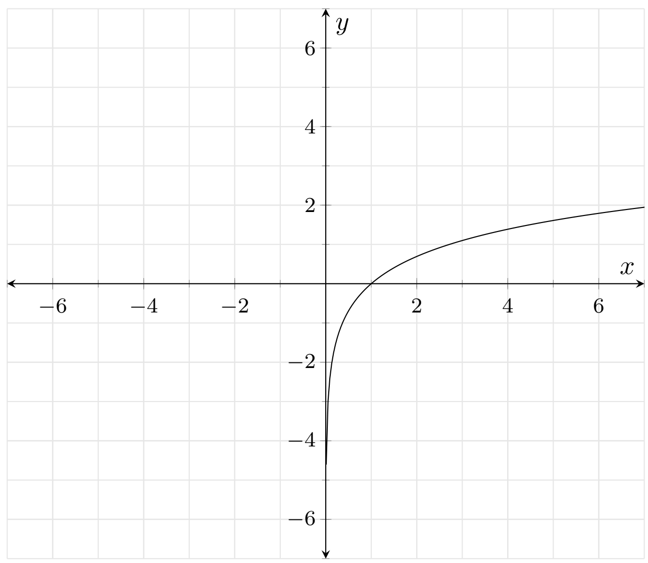
Logarithm Functions

Another group of famous functions are logarithms.

Example 3. *The most famous logarithm function is*

$$y = \ln(x) = \log_e(x).$$

Here e is the mathematical constant known as Euler's number. $e \approx 2.71828$.



Important Values of $y = \ln(x)$

x	y
0	*undefined*
$\frac{1}{e}$	-1
1	0
e	1

In general, we can talk about logarithmic functions of the form $y = \log_b(x)$ where b is a positive number not equal to 1. You can play with changing the values of b on the graph using Desmos and see how that changes the graph. Pay particular attention to the difference between $b > 1$ and $0 < b < 1$.

Desmos link: `https://www.desmos.com/calculator/lxllnpdi6w`

Notice that logarithms are neither even, odd, nor periodic. However, they are one-to-one, and therefore invertible. It turns out that the inverse of a logarithm is an exponential function, and vice versa!

Note that since the logarithm is the inverse of the exponential, the domain of the logarithms is the range of the exponentials: $[0, \infty)$. Furthermore, the range of the logarithms is the range of the exponentials: $(-\infty, \infty)$.

We summarize this information in the table below.

Properties of the Logarithm Functions $y = \log_b(x)$

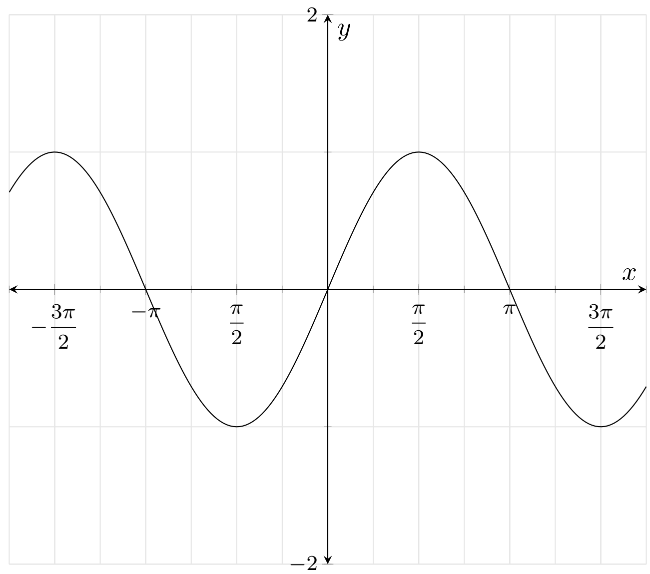
Periodic?	No
Odd?	No
Even?	No
One-to-one/invertible?	Yes
Domain	$(0, \infty)$
Range	$(-\infty, \infty)$

Sine

Another important function is the sine function,

$$y = \sin(x).$$

This function comes from trigonometry. In the table below we will use another mathematical constant, π ("pi" pronounced pie). $\pi \approx 3.14159$.



Important Values of $y = \sin(x)$

x	y
$-\pi$	0
$-\frac{\pi}{2}$	-1
0	0
$\frac{\pi}{2}$	1
π	0
$\frac{3\pi}{2}$	-1
2π	0

As mentioned earlier, the sine function is odd and periodic with period 2π. Since it is periodic, however, it cannot be one-to-one, since its values repeat.

Note that the domain of the sine function is $(-\infty, \infty)$. Furthermore, by looking at the graph, we can see that its range is $[-1, 1]$.

We summarize this information in the table below.

Properties of the Sine Function $y = \sin(x)$

Periodic?	Yes, with period 2π
Odd?	Yes
Even?	No
One-to-one/invertible?	No
Domain	$(-\infty, \infty)$
Range	$[-1, 1]$

In general, we can consider $y = a\sin(bx)$. You can play with changing the values of a and b on the graph using Desmos and see how that changes the graph.

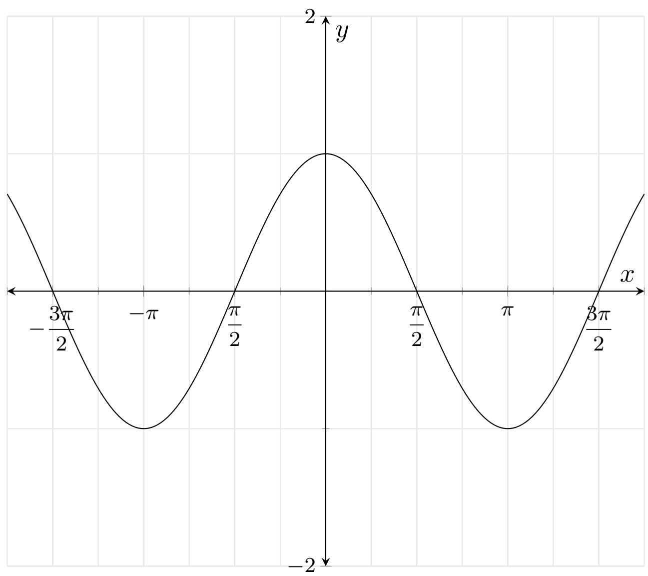
Desmos link: `https://www.desmos.com/calculator/vkxzcfv2aq`

Cosine

A function introduced in Section 3-2 is the cosine function,

$$y = \cos(x).$$

As with sine, the cosine function comes from trigonometry. In the table below we will again use π.



Important Values of $y = \cos(x)$

x	y
$-\pi$	-1
$-\frac{\pi}{2}$	0
0	1
$\frac{\pi}{2}$	0
π	-1
$\frac{3\pi}{2}$	0
2π	1

As mentioned earlier, the cosine function is even and periodic with period 2π. Since it is periodic, however, it cannot be one-to-one, since its values repeat.

Note that the domain of the cosine function is $(-\infty, \infty)$. Furthermore, by looking at the graph, we can see that its range is $[-1, 1]$.

We summarize some information in the table below.

Properties of the Cosine Function $y = \cos(x)$

Periodic?	Yes, with period 2π
Odd?	No
Even?	Yes
One-to-one/invertible?	No
Domain	$(-\infty, \infty)$
Range	$[-1, 1]$

In general, we can consider $y = a\cos(bx)$. You can play with changing the values of a and b on the graph using Desmos and see how that changes the graph.

Desmos link: `https://www.desmos.com/calculator/kvmz1kt19n`

7.1.2 Composition of Functions

Motivating Questions

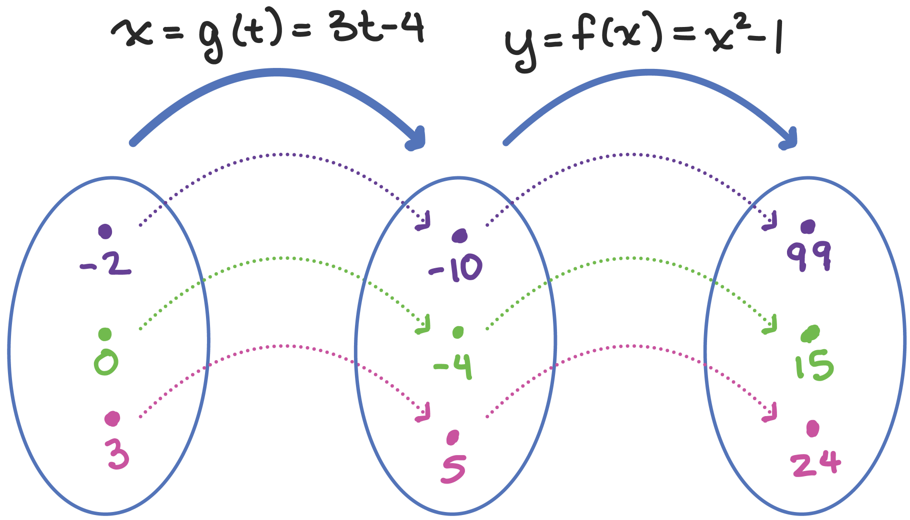
- How does the process of function composition produce a new function from two other functions?
- In the composite function $h(x) = f(g(x))$, what do we mean by the "inner" and "outer" function?
- How does the expression for $\text{AROC}_{[a,a+h]}$ involve a composite function?

Introduction

Recall that a function, by definition, is a process that takes a collection of inputs and produces a corresponding collection of outputs in such a way that the process produces one and only one output value for any single input value. Because every function is a process, it makes sense to think that it may be possible to take two function processes and do one of the processes first, and then apply the second process to the result.

Example 4. *Suppose we know that y is a function of x according to the process defined by $y = f(x) = x^2 - 1$ and, in turn, x is a function of t via $x = g(t) = 3t - 4$. Is it possible to combine these processes to generate a new function so that y is a function of t?*

Explanation Since y depends on x and x depends on t, it follows that we can also think of y depending directly on t. Let's looks at this as an arrow diagram with a few sample points.



Notice that if we take a point such as $t = 2$, we can put that value in for t in the function $x = g(t) = 3t - 4$. This will give

$$x = g(-2) = 3(-2) - 4 = -6 - 4 - 10.$$

Now we have an x-value of -10. $g(x)$ takes in x-values so we can put -10 into $f(x) = x^2 - 1$. This will give

$$f(-10) = (-10)^2 - 1 = 100 - 1 = 99.$$

You should verify that the arrow diagram above gives the correct values of y that corresponds to $t = 0$ and $t = 3$.

Now, we would like to create a new function that will directly take in any t value and give us the corresponding y value. We can use substitution and the notation of functions to determine this function.

First, it's important to realize what the rule for f tells us. In words, f says "to generate the output that corresponds to an input, take the input and square it, and then subtract 1." In symbols, we might express f more generally by writing "$f(\square) = \square^2 - 1$."

Now, observing that $y = f(x) = x^2 - 1$ and that $x = g(t) = 3t - 4$, we can substitute the expression $g(t)$ for x in f. Doing so,

$$\begin{aligned} y &= f(x) \\ &= f(g(t)) \\ &= f(3t - 4). \end{aligned}$$

Applying the process defined by the function f to the input $3t - 4$, we see that

$$y = (3t - 4)^2 - 1,$$

which defines y as a function of t.

One way to think about the substitution above is that we are putting the entire expression $3t-4$ inside the input box in "$f(\square) = \square^2 - 1$." That is, $f\left(\boxed{3t-4}\right) = \left(\boxed{3t-4}\right)^2 - 1$. For the substitution, we are thinking of $3t-4$ as a single object!

When we have a situation such as in the example above where we use the output of one function as the input of another, we often say that we have **composed** two functions. In addition, we use the notation $h(t) = f(g(t))$ to denote that a new function, h, results from composing the two functions f and g.

Exploration

a. Let $y = p(x) = 3x - 4$ and $x = q(t) = t^2$. Determine a formula for r that depends only on t and not on p or q. What is the biggest difference between your work in this problem compared to the example above?

b. Let $t = s(z) = \dfrac{1}{t+4}$ and recall that $x = q(t) = t^2$. Determine a formula for $x = q(s(z))$ that depends only on z.

c. Suppose that $h(t) = \sqrt{2t^2 + 5}$. Determine formulas for two related functions, $y = f(x)$ and $x = g(t)$, so that $h(t) = f(g(t))$.

Composing Two Functions

Whenever we have two functions, g and f, where the outputs of g match inputs of f, it is possible to link the two processes together to create a new process that we call the *composition* of f and g.

Definition If f and g are functions, we define the **composition of** f **and** g to be the new function h given by

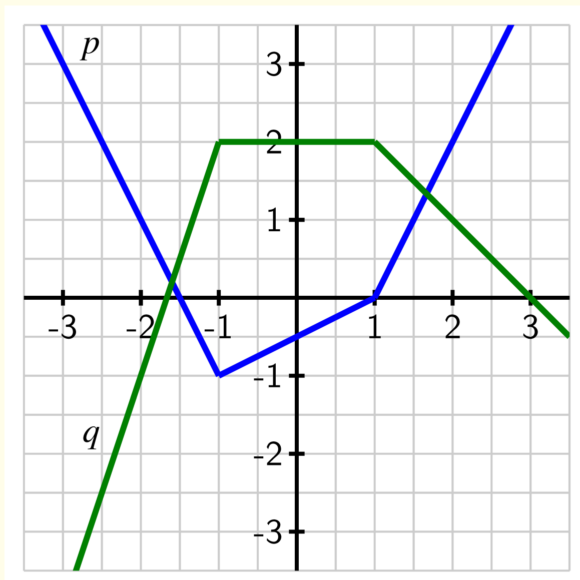
$$h(t) = f(g(t)).$$

This composition is denoted by $h = f \circ g$, where $f \circ g$ means the single function defined by $(f \circ g)(t) = f(g(t))$.

We sometimes call g the "inner function" and f the "outer function". It is important to note that the inner function is actually the first function that gets applied to a given input, and then the outer function is applied to the output of the inner function. In addition, in order for a composite function to make sense, we need to ensure that the outputs of the inner function are values that it makes sense to put into the outer function so that the resulting composite function is defined.

In addition to the possibility that functions are given by formulas, functions can be given by tables or graphs. We can think about composite functions in these settings as well, and the following activities prompt us to consider functions given in this way.

Exploration Let functions p and q be given by the graphs below (which are each piecewise linear - that is, parts that look like straight lines are straight lines) and let f and g be given by the table below.



x	$f(x)$	$g(x)$
0	6	1
1	4	3
2	3	0
3	4	4
4	6	2

Compute each of the following quantities or explain why they are not defined.

a. $p(q(0))$

b. $q(p(0))$

c. $(p \circ p)(-1)$

d. $(f \circ g)(2)$

e. $(g \circ f)(3)$

f. $g(f(0))$

g. For what value(s) of x is $f(g(x)) = 4$?

h. For what value(s) of x is $q(p(x)) = 1$?

Composing functions in content

In the late 1800s, the physicist Amos Dolbear was listening to crickets chirp and noticed a pattern: how frequently the crickets chirped seemed to be connected to the outside temperature. If we let T represent the temperature in degrees Fahrenheit and N the number of chirps per minute, we can summarize Dolbear's observations with the following function, $T = D(N) = 40 + 0.25T$. Scientists who made many additional cricket chirp observations following Dolbear's initial counts found that this formauls holds with remarkable accuracy for the snowy tree cricket in temperatures ranging from $50°$ to $85°$. This function is called Dolbear's Law.

In what follows, we replace T with F to emphasize that temperature is measured in Fahrenheit degrees.

The Celsius and Fahrenheit temperature scales are connected by a linear function. Indeed, the function that converts Fahrenheit to Celsius is

$$C = G(F) = \frac{5}{9}(F - 32).$$

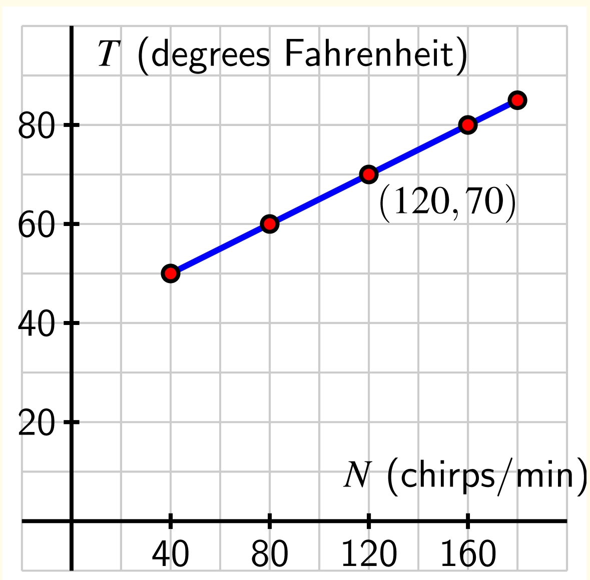
For instance, a Fahrenheit temperature of 32 degrees corresponds to $C = G(32) = \frac{5}{9}(32 - 32) = 0$ degrees Celsius.

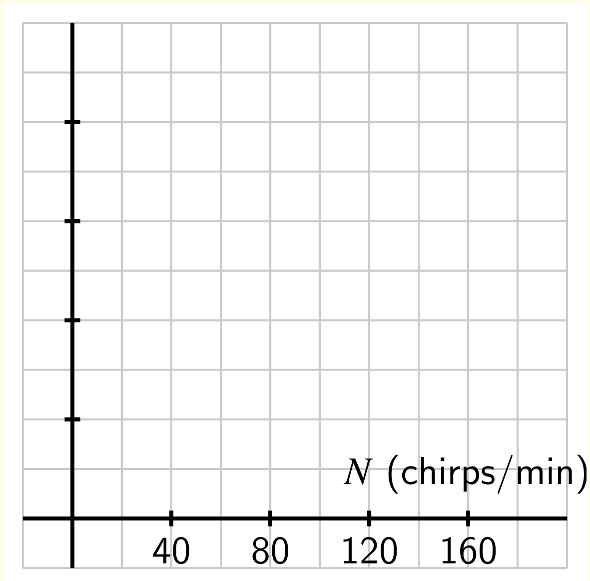
Exploration Let $D(N) = 40 + 0.25N$ be Dolbear's function that converts an input of number of chirps per minute to degrees Fahrenheit, and let $G(F) = \frac{5}{9}(F - 32)$ be the function that converts an input of degrees Fahrenheit to an output of degrees Celsius.

a. Determine a formula for the new function $(G \circ D)(N)$ that depends only on the variable N.

b. What is the meaning of the function you found in (a)?

c. Let $H = G \circ D$. How does a plot of the function H compare to the that of Dolbear's function? Sketch a plot of H on the blank axes to the right of the plot of Dolbear's function, and discuss the similarities and differences between them. Be sure to label the vertical scale on your axes.



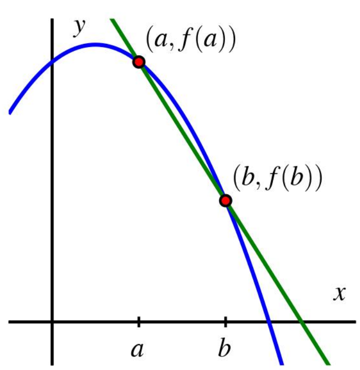


Function Composition and Average Rate of Change

Recall that the average rate of change of a function f on the interval $[a, b]$ is given by

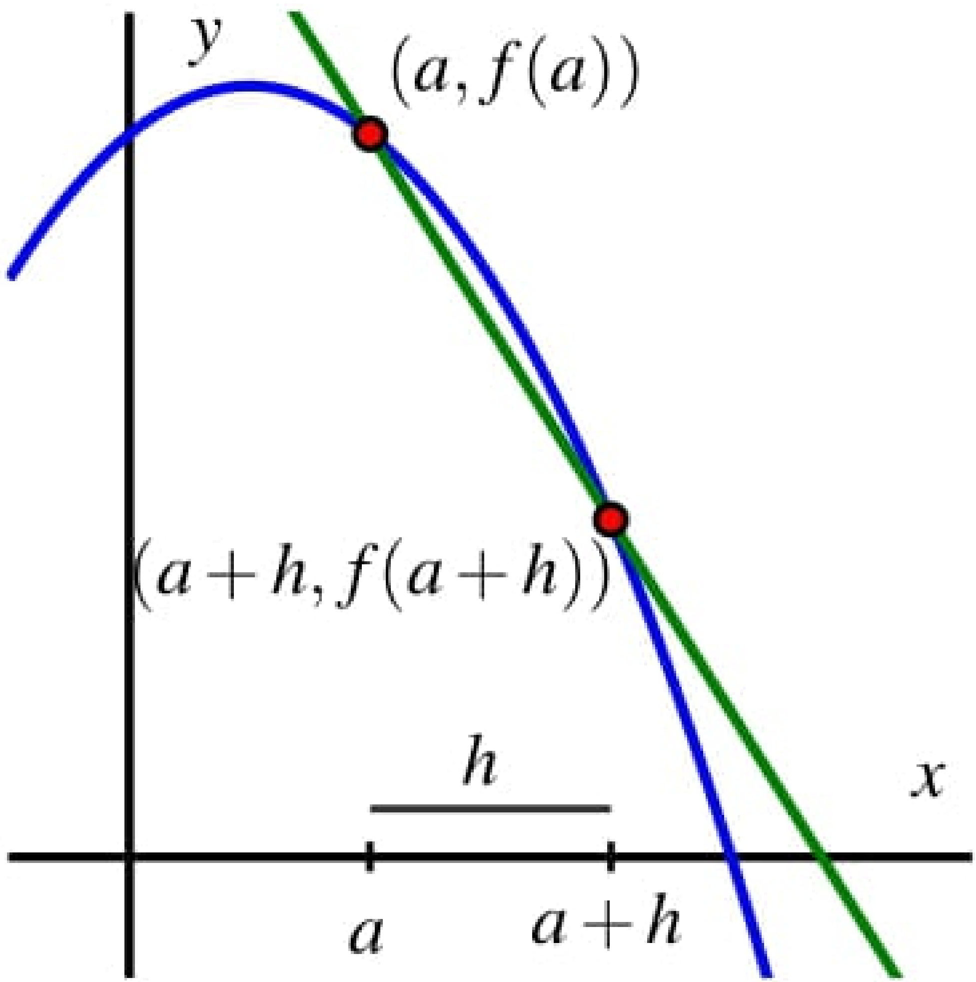
$$\text{AROC}_{[a,b]} = \frac{f(b) - f(a)}{b - a}.$$

In the graph below, we see the familiar representation of $\text{AROC}_{[a,b]}$ as the slope of the line joining the points $(a, f(a))$ and $(b, f(b))$ on the graph of f.



In the study of calculus, we progress from the *average rate of change on an interval* to the *instantaneous rate of change of a function at a single value*; the core idea that allows us to move from an *average* rate to an *instantaneous* one is letting the interval $[a, b]$ shrink in size.

To think about the interval $[a, b]$ shrinking while a stays fixed, we often change our perspective and think of b as $b = a + h$, where h measures the horizontal difference from b to a.



This allows us to eventually think about h getting closer and closer to 0, and in that context we consider the equivalent expression

$$\text{AROC}_{[a,a+h]} = \frac{f(a+h) - f(a)}{a+h-a} = \frac{f(a+h) - f(a)}{h}$$

for the average rate of change of f on $[a, a+h]$.

Example 5. *Suppose that* $f(x) = x^2$. *Determine the simplest possible expression you can find for* $\text{AROC}_{[3,3+h]}$, *the average rate of change of* f *on the interval* $[3, 3+h]$.

Explanation By definition, we know that

$$\text{AROC}_{[3,3+h]} = \frac{f(3+h) - f(3)}{h}.$$

Using the formula for f, we see that

$$\text{AROC}_{[3,3+h]} = \frac{(3+h)^2 - (3)^2}{h}.$$

Expanding the numerator and combining like terms, it follows that

$$\begin{aligned} \text{AROC}_{[3,3+h]} &= \frac{(9 + 6h + h^2) - 9}{h} \\ &= \frac{6h + h^2}{h}. \end{aligned}$$

Removing a factor of h in the numerator and observing that $h \neq 0$, we can simplify and find that

$$\begin{aligned} \text{AROC}_{[3,3+h]} &= \frac{h(6+h)}{h} \\ &= 6 + h. \end{aligned}$$

Hence, $\text{AROC}_{[3,3+h]} = 6 + h$, which is the average rate of change of $f(x) = x^2$ on the interval $[3, 3+h]$.

Exploration Let $f(x) = 2x^2 - 3x + 1$ and $g(x) = \dfrac{5}{x}$.

a. Compute $f(1+h)$ and expand and simplify the result as much as possible by combining like terms.

b. Determine the most simplified expression you can for the average rate of change of f on the interval $[1, 1+h]$. That is, determine $\text{AROC}_{[1,1+h]}$ for f and simplify the result as much as possible.

c. Compute $g(1+h)$. Is there any valid algebra you can do to write $g(1+h)$ more simply?

d. Determine the most simplified expression you can for the average rate of change of g on the interval $[1, 1+h]$. That is, determine $\text{AROC}_{[1,1+h]}$ for g and simplify the result.

Summary

- When defined, the composition of two functions f and g produces a single new function $f \circ g$ according to the rule $(f \circ g)(x) = f(g(x))$. We note that g is applied first to the input x, and then f is applied to the output $g(x)$ that results from g.

- In the composite function $h(x) = f(g(x))$, the "inner" function is g and the *outer* function is f. Note that the inner function gets applied to x first, even though the outer function appears first when we read from left to right.

- Because the expression $\text{AROC}_{[a,a+h]}$ is defined by

$$\text{AROC}_{[a,a+h]} = \frac{f(a+h) - f(a)}{h}$$

and this includes the quantity $f(a+h)$, the average rate of change of a function on the interval $[a, a+h]$ always involves the evaluation of a composite function expression. This idea plays a crucial role in the study of calculus.

7.1.3 Domains and Ranges of Composite Functions

> **Motivating Questions**
>
> - How does the process of function composition effect the domain of the function?
> - How does the process of function composition effect the range of the function?

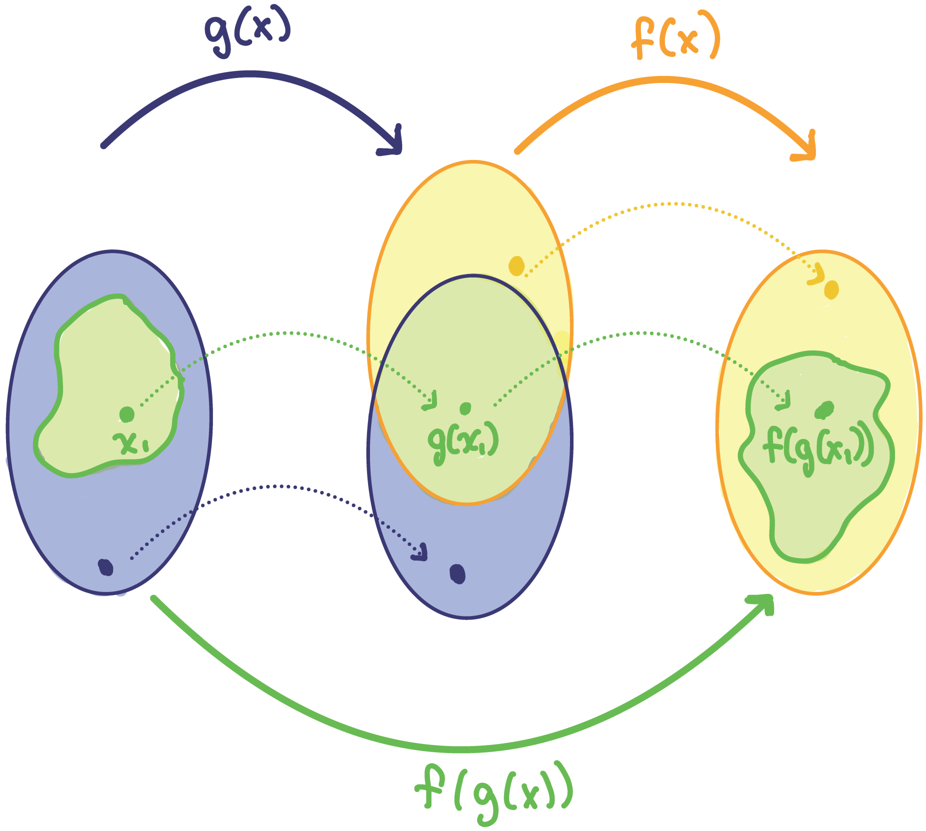
Domains of Composite Functions

The domain of a composite function such as $f \circ g$ is dependent on the domain of g and the domain of f. It is important to know when we can apply a composite function and when we cannot, that is, to know the domain of a function such as $f \circ g$.

Let us assume we know the domains of the functions f and g separately. If we write the composite function for an input x as $f(g(x))$, we can see right away that x must be a member of the domain of g in order for the expression to be meaningful, because otherwise we cannot complete the inner function evaluation. However, we also see that $g(x)$ must be a member of the domain of f, otherwise the second function evaluation in $f(g(x))$ cannot be completed, and the expression is still undefined. Thus the domain of $f \circ g$ consists of only those inputs in the domain of g that produce outputs from g belonging to the domain of f. Note that the domain of f composed with g is the set of all x such that x is in the domain of g and $g(x)$ is in the domain of f.

> The domain of a composite function $f(g(x))$ is the set of those inputs x in the domain of g for which $g(x)$ is in the domain of f.

Acknowledgements: OpenStax College Algebra (https://openstax.org/details/books/college-algebra)



To find the domain of a composite function, $f \circ g$, you can follow these three steps:

1) Find the domain of g.
2) Find the domain of f.
3) Find those inputs x in the domain of g for which $g(x)$ is in the domain of f. That is, exclude those inputs x from the domain of g for which $g(x)$ is not in the domain of f. The resulting set is the domain of $f \circ g$.

Example 6. *Find the domain of* $f \circ g$ *where* $f(x) = \dfrac{5}{x-1}$ *and* $g(x) = \dfrac{4}{3x-2}$.

Explanation The domain of g consists of all real numbers except $x = \dfrac{2}{3}$, since that input value would cause us to divide by 0. Likewise, the domain of f consists of all real numbers except 1. We need to exclude from the domain of g any value of x for which $g(x) = 1$.

Acknowledgements: OpenStax College Algebra (https://openstax.org/details/books/college-algebra)

$$\begin{aligned}\frac{4}{3x-2} &= 1\\ 4 &= 3x-2\\ 6 &= 3x\\ x &= 2\end{aligned}$$

So the domain of $f \circ g$ is the set of all real numbers except $\frac{2}{3}$ and 2. This means that

$$x \neq \frac{2}{3} \text{ or } x \neq 2$$

We can write this in interval notation as

$$\left(-\infty, \frac{2}{3}\right) \cup \left(\frac{2}{3}, 2\right) \cup (2, \infty)$$

Example 7. *Find the domain of $f \circ g$ where $f(x) = \sqrt{x+2}$ and $g(x) = \sqrt{3-x}$.*

Explanation

Because we cannot take the square root of a negative number, the domain of g is $(-\infty, 3]$. Now we check the domain of the composite function

$$(f \circ g)(x) = \sqrt{\sqrt{3-x}+2}$$

For $(f \circ g)(x) = \sqrt{\sqrt{3-x}+2}$, we need $\sqrt{3-x}+2 \geq 0$, since the inside of a square root cannot be negative. Since square roots are non-negative, $\sqrt{3-x} \geq 0$ sp $\sqrt{3-x}+2 \geq 0$ as long as $\sqrt{3-x}$ exists. That means $3-x \geq 0$, which gives a domain of $(-\infty, 3]$.

This example shows that knowledge of the range of functions (specifically the inner function) can also be helpful in finding the domain of a composite function. It also shows that the domain of $f \circ g$ can contain values that are not in the domain of f, though they must be in the domain of g.

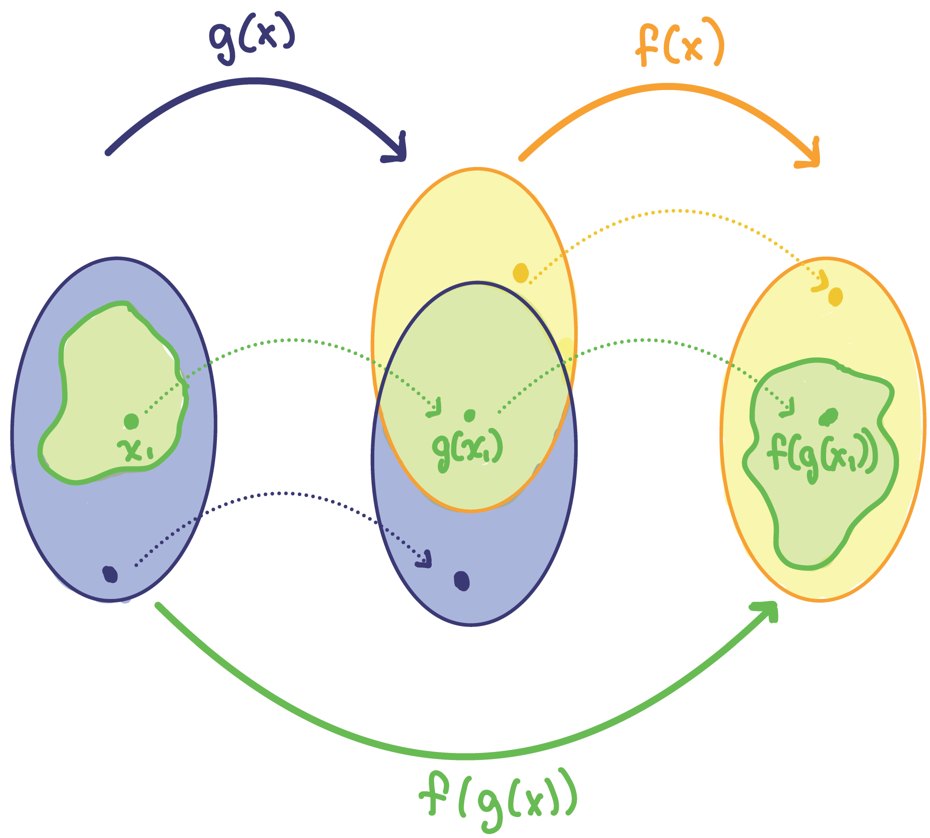
Exploration Find the domain of $f \circ g$ where $f(x) = \frac{1}{x-2}$ and $g(x) = \sqrt{x+4}$.

Acknowledgements: OpenStax College Algebra (https://openstax.org/details/books/college-algebra)

Ranges of Composite Functions

The range of a composite function such as $f \circ g$ is dependent on the range of g and the range of f. It is important to know what values can result from a composite function, that is, to know the range of a function such as $f \circ g$.

Let us assume we know the ranges of the functions f and g separately. If we write the composite function for an input x as $f(g(x))$, we can see right away that $f(g(x))$ must be a member of the range of f since we will input the value $g(x)$ into f. However, we also see that it is possible that not all values in the range of f are in the range of $f(g(x))$.



From the image above, we can see that there might be values in the yellow region which are in the range of f but for which there are no x values for which $f(g(x))$ gives that output.

> The range of a composite function $f \circ g$ is a subset of the range of f.

Acknowledgements: OpenStax College Algebra (https://openstax.org/details/books/college-algebra)

To find the domain of a composite function, $f \circ g$, you can follow these three steps:

1) Find the range of g.

2) Find the range of f.

3) Restrict the domain of f to the *range* of g and then determine the outputs of f of these values.

Example 8. *Find the range of $f \circ g$ where $f(x) = \sqrt{x+2}$ and $g(x) = \sqrt{3-x}$.*

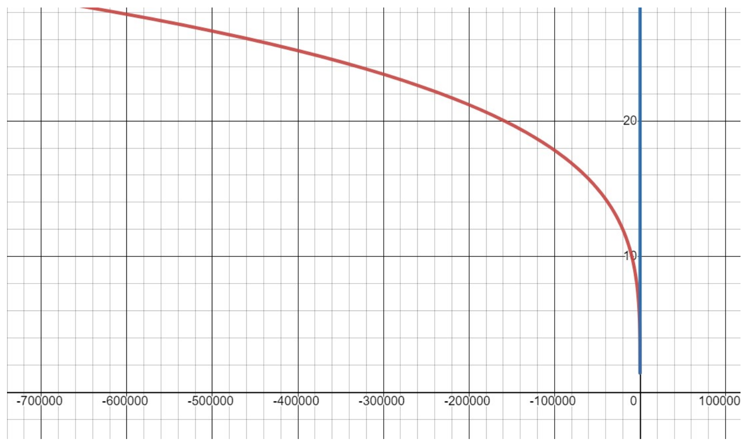
Explanation Because the output of a square root is always a positive number, the range of g is $[0, \infty)$. Similarly, the range of f is $[0, \infty)$. But now we must think about what happens when we restrict the input of f to values in the range of g, $[0, \infty)$. If $x \geq 0$, then $x + 2 \geq 2$. Taking the square root of both sides, we see that possible outputs of $f(g(x))$ will be $\sqrt{x+2} \geq \sqrt{2}$. That is, the range of $f \circ g$ is $[\sqrt{2}, \infty)$.

If we look at this function in Desmos, we can confirm graphically that this answer makes sense. What we want to do is think about collapsing the graph unto the y-axis. The range of the function will be the y-values that correspond to a point (x, y) on the curve.

First, we graph the function using a standard window.

This allows us to see the domain pretty well. In the previous example, we found the domain to be $(-\infty, 3]$ and if we collapse this function to the x-axis, it looks like the x-values that correspond to points on this curve are exactly the x in $(-\infty, 3]$. If might be difficult to tell the domain from this graph, though. Let's zoom out some.

 Acknowledgements: OpenStax College Algebra (https://openstax.org/details/books/college-algebra)



Here is the same graph in Desmos, so you can zoom in and out yourself.

Desmos link: `https://www.desmos.com/calculator/0wf1e4yyhf`

You can now see that the blue line is showing this graph collapsed to the y-axis. We can tell that the range will be positive numbers above some value between 1 and 2. This corresponds with our result above of $[\sqrt{2}, \infty)$. In order to find the exact point $\sqrt{2}$ where the interval begins or to confirm that the interval really goes to infinity, we need to do the reasoning above.

Summary

- For a composite function $f \circ g$ to be defined, we need outputs of g to be among the allowed inputs for f. In particular, if the range of g is a subset of the domain of f, we can say that if $g : A \to B$ and $f : B \to C$, then $f \circ g : A \to C$. In this case, the domain of the composite function is the domain of the inner function, and the range of the composite function is the codomain of the outer function.

- In general, the domain of a composite function $f \circ g$ is the set of those inputs x in the domain of g for which $g(x)$ is in the domain of f.

- In general, the range of a composite function $f \circ g$ is a subset of the range of f.

7.2 What are the Zeros of Functions?

Learning Objectives

- Zeros of Functions
  - Definition of Zeros
  - Compare and contrast zeros, solutions, roots, and x-intercepts
  - Examples of why we might want to find zeros
  - Identify a zero on a graph
  - Computing the zero of a function (early examples)
- The Importance of Equals
  - Compare and contrast expressions, equations, and functions
  - Appreciate the importance of using the equals sign appropriately

7.2.1 Zeros of Functions

Motivating Questions

- What does it mean to find the zero of a function?
- What are other terms used for zeros of functions?
- Why might we want to find the zero of a function?

Introduction

In this section, we will study zeros of functions. Let's start with a classic example of when we might want to find the zero of a function.

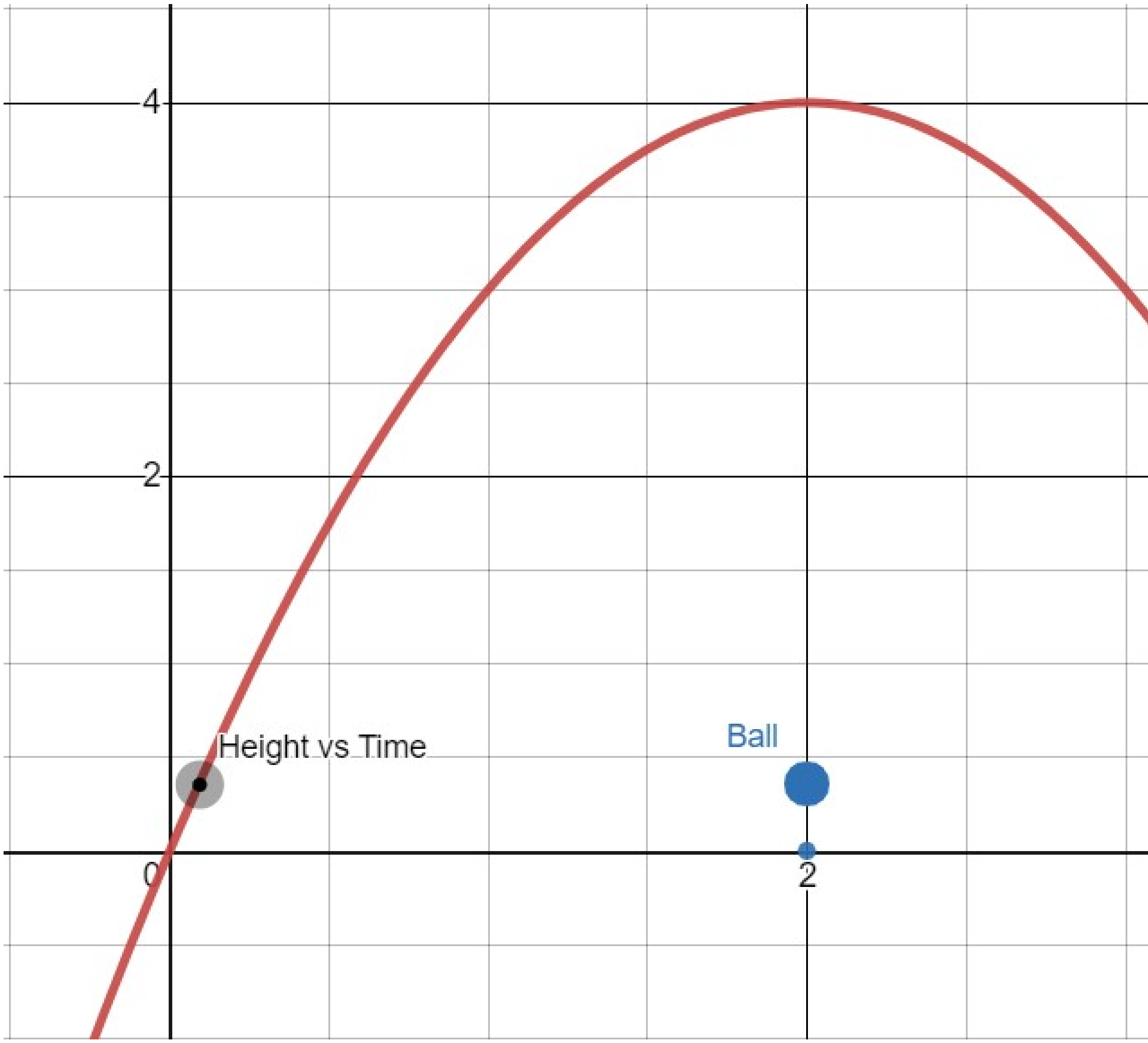
Example 9. *A ball is thrown straight upward from the ground. The distance the ball is from the ground is given by the function* $f(x) = 4-(x-2)^2$ *where* x *is the time measured in seconds and* $f(x)$ *is the distance from the ground measured in feet. What time will the ball hit the ground?*

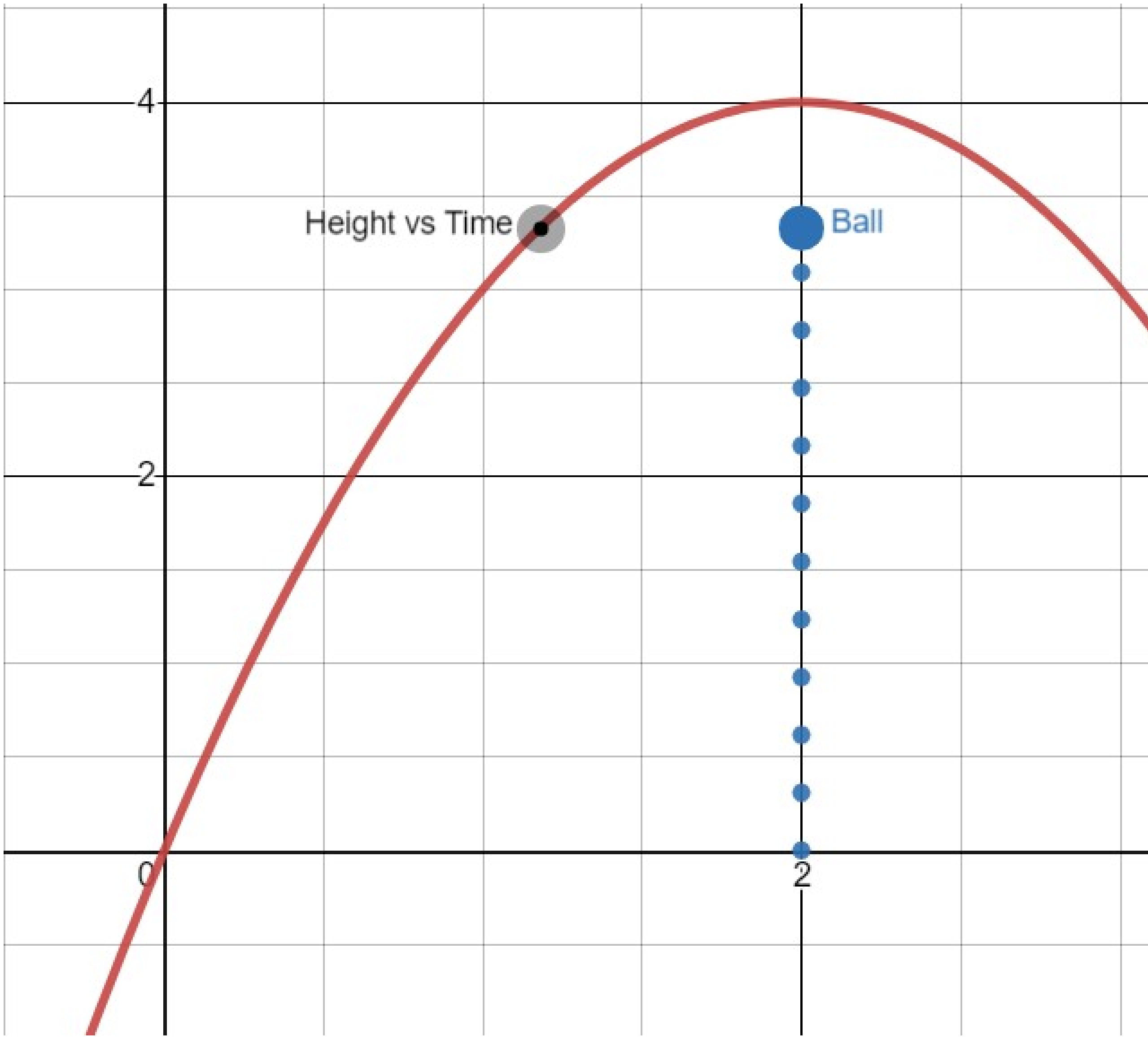
Explanation First, it is important to understand what this problem is saying. Consider this model of the situation. Click the play button next to the a to see an animation of the ball being thrown up.

Desmos link: https://www.desmos.com/calculator/f9d7ngsznm

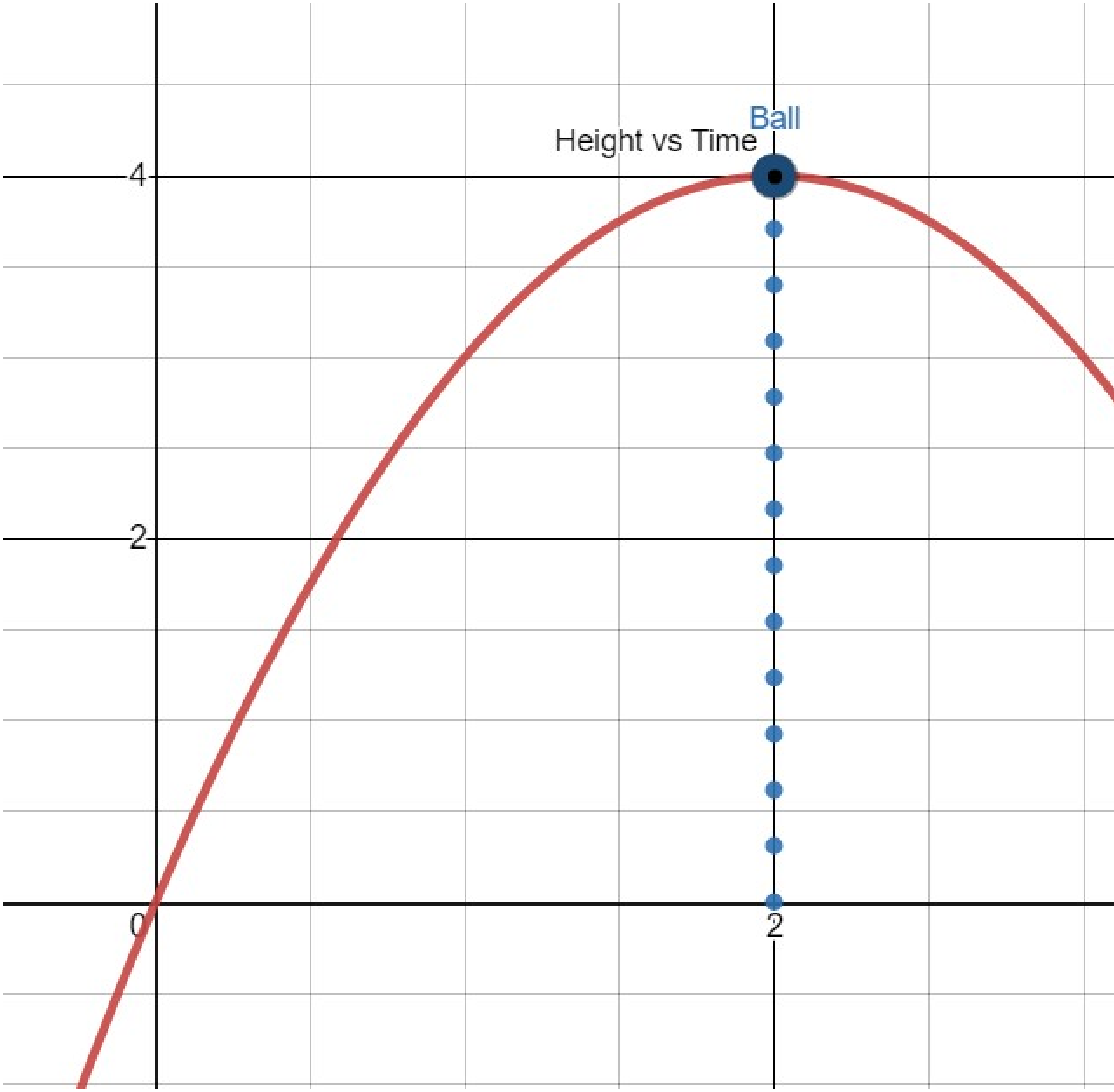
Notice that while the ball goes straight up and down, the graph of the distance from the ground vs. time makes an upside down parabola.

As the ball goes up, we see:

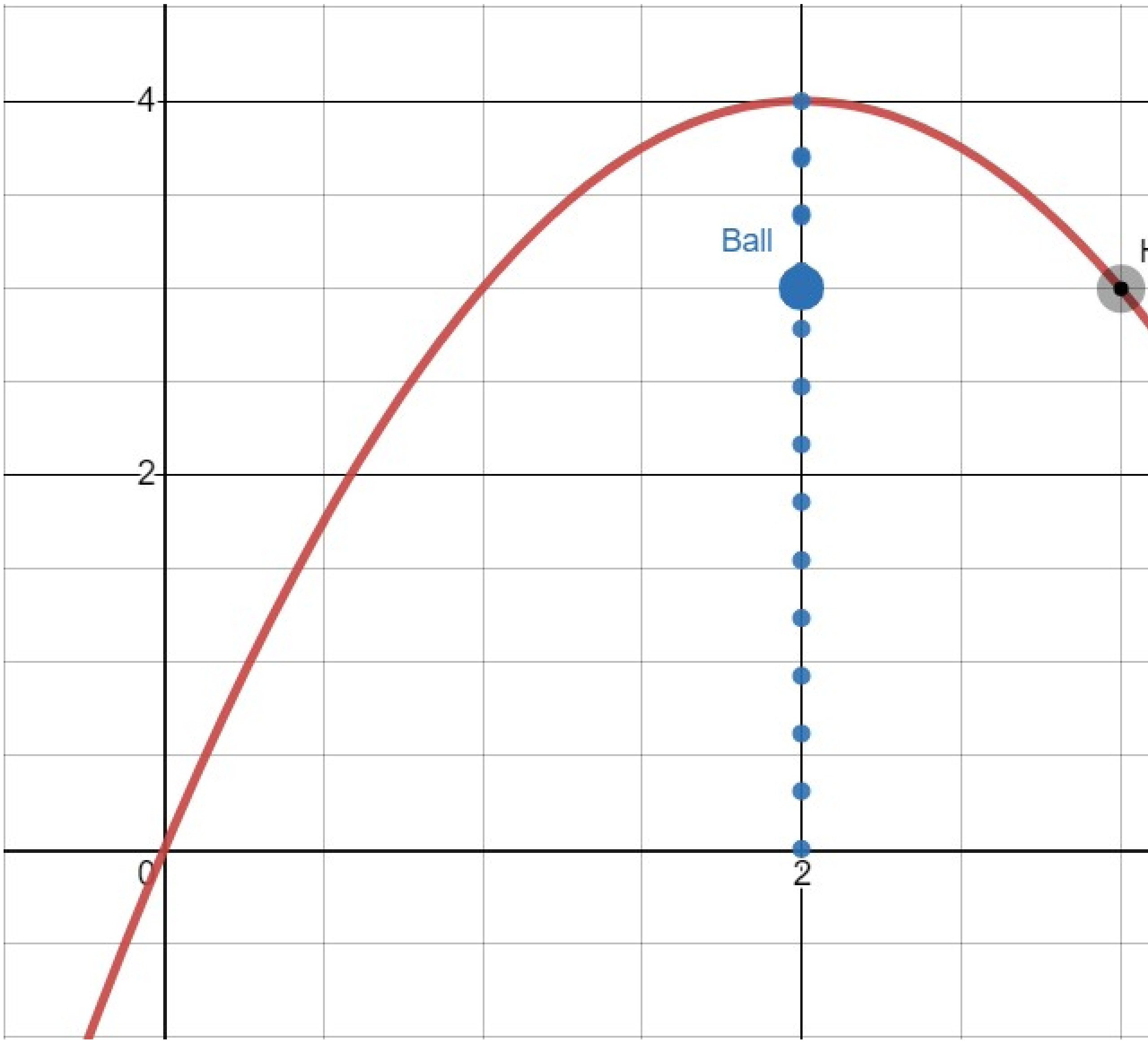

4
2
0
2
Height vs Time
Ball

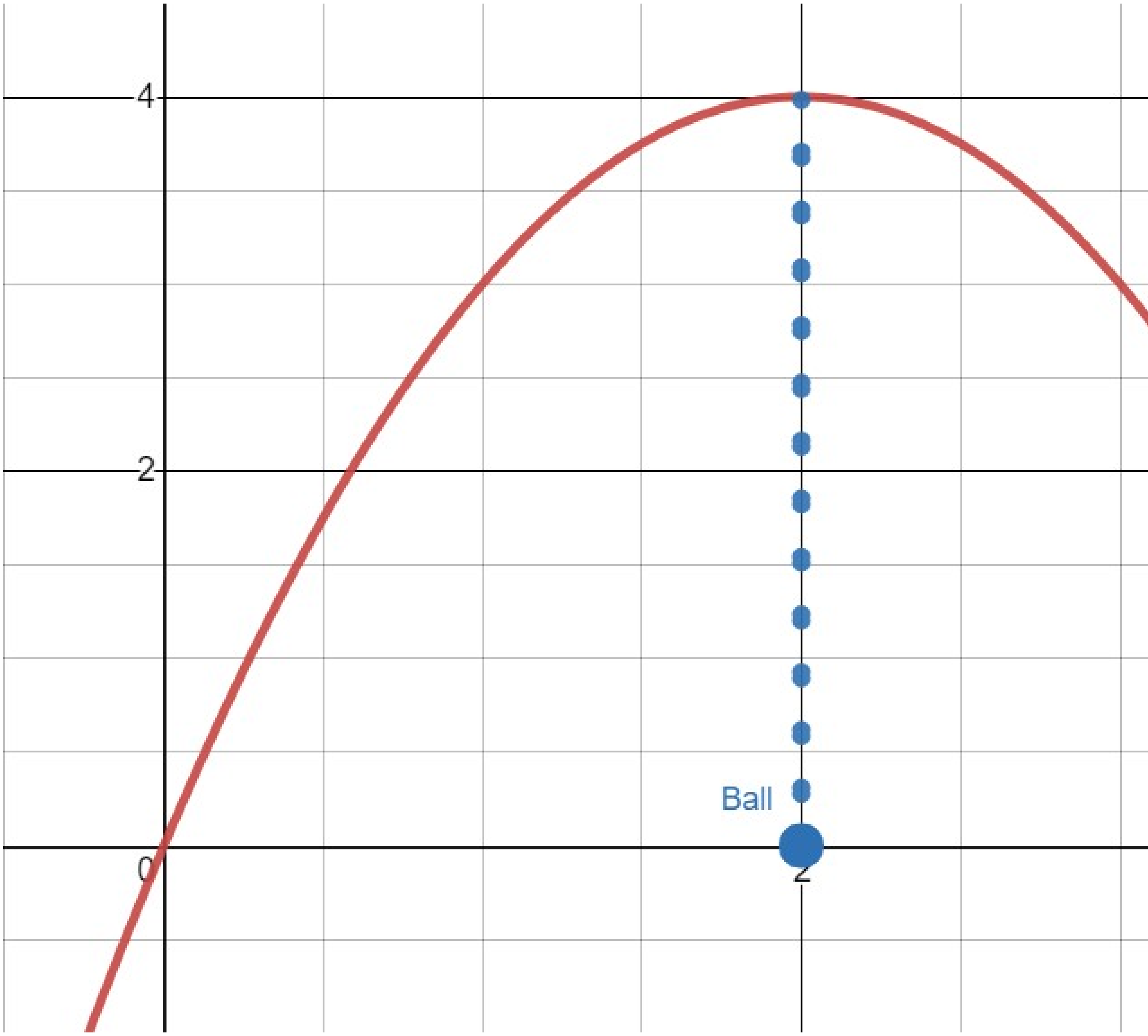


At the ball's highest point, we have:



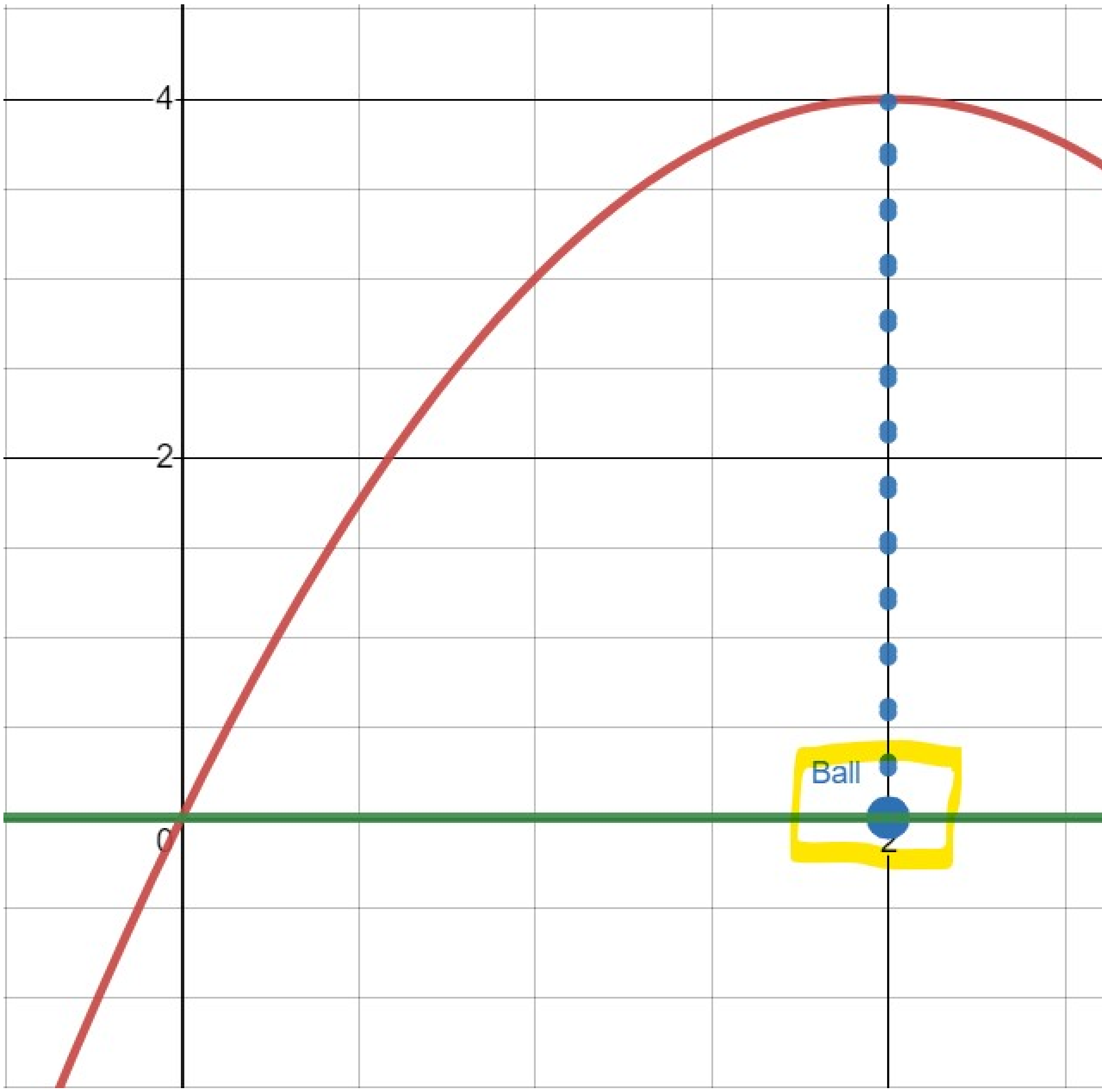
And as the ball goes down, we have:


4
2
0
2
Ball



It is also important to note that in this situation, we know this model only makes sense from the time we throw the ball to the time it hits the ground. Having a negative distance would correspond with the ball going underground, which is not what we want to model. That means our domain will be from $x = 0$ to the x-value where the ball hits the ground.

This means, once again, we are back to wanting to know when the ball will hit the ground. Looking at the graph, we can focus on the time when the ball seems to hit the ground.



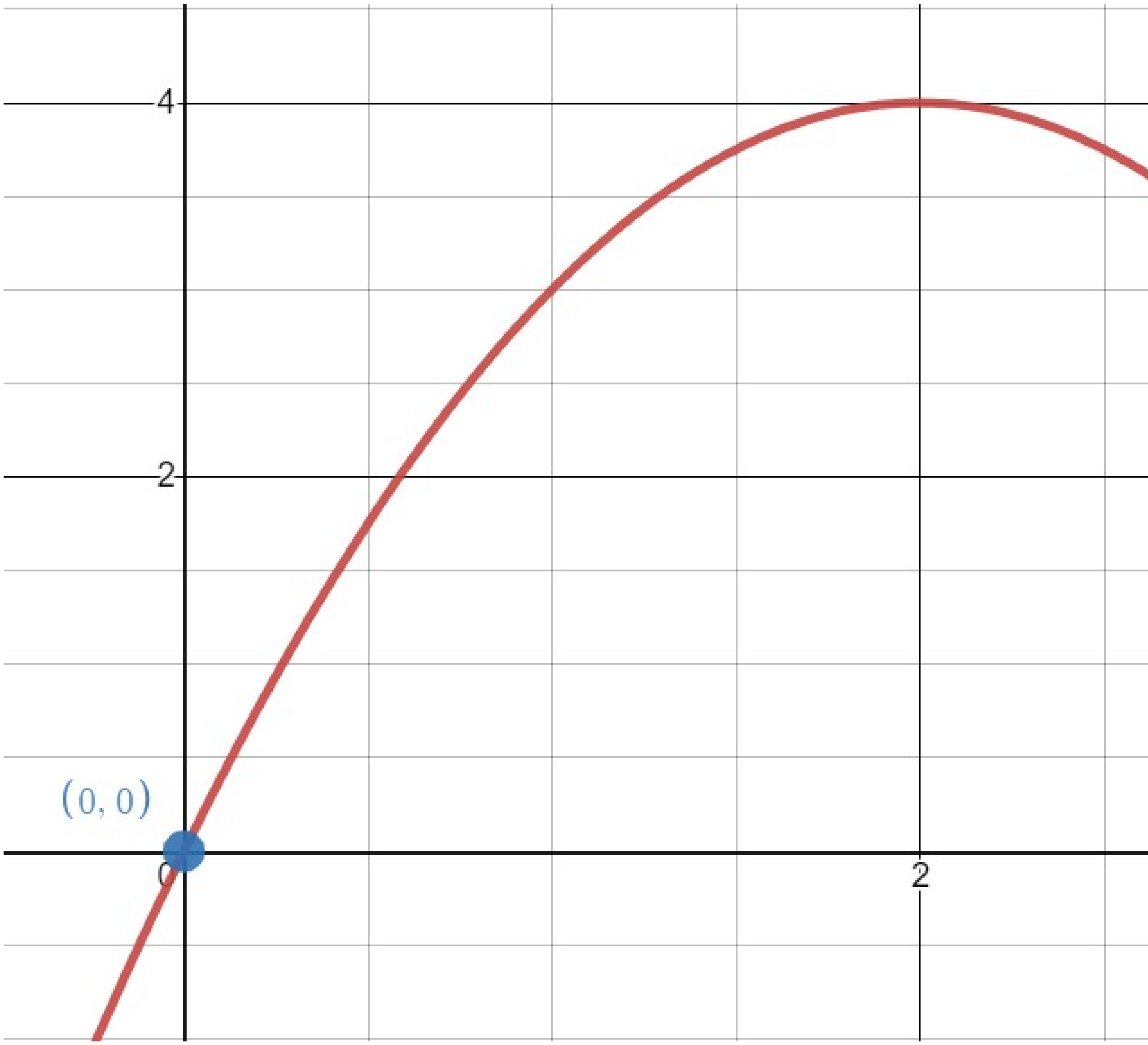
Notice that at the time the ball hits the ground, the function giving the distance vs. time graph, $f(x) = 4 - (x-2)^2$, is crossing the x-axis. This means we are looking for **the x-value of the x-intercept**. That is, we are looking for when this function equals 0. We call an x-value where a function equals zero the

zero of a function. It can also be called the **root of a function** when the function is a polynomial. Occasionally, people will also call it the **solution of a function** but techincally they should say **the solution of** $f(x) = 0$.

Let's formalize this with a definition.

> **Definition** We call an x-value where a function equals zero the **zero of a function**. It can also be called the **root of a function** when the function is a polynomial. Another equivalent expression is **the solution of** $f(x) = 0$. Another label for the same value is **the x-coordinate of the x-intercept**.

Finding Zeros Graphically One method for finding zeros or roots of functions is to read them off the graph. Since the zero is the x-coordinate of the x-intercept, we are looking for the places where the graph crosses the x-axis. This is where the y-value or output will be zero. In the image below, the two roots are colored blue and green.



The root $x = 0$ cooresponds to the blue dot at $(0, 0)$ and represents when the ball is first thrown. The second root, labeled in green, is where $x = 4$. This root is the one we are looking for. The time when the ball hits the ground is 4 seconds after the ball is thrown.

Finding Zeros Algebraically Reading zeros graphically can be useful, but it is not very precise. The root in our example could actually be at $x = 3.99$ and we would not know from the graph. When possible, it is best to find zeros algebraically. To do this, we want to set $f(x) = 0$ and solve for x. This algebra can get tricky. In this section, we will stick to relatively straightforward examples and in the next couple sections we will explore some more involved methods for solving equations where one side equals 0. It is not always possible to solve for the zeros of a function algebraically in this manner. In calculus, you will also learn methods to approximate zeros when it is not possible to solve for them exactly.

In our current example, we will have:

$$\begin{aligned} f(x) &= 0 \\ 4-(x-2)^2 &= 0 \\ 4 &= (x-2)^2 \\ \sqrt{4} &= \sqrt{(x-2)^2} \end{aligned}$$

Recall that $\sqrt{x^2} = |x|$ from the section on domains and ranges of composite functions.

Continuing, we have:

$$\begin{aligned} \sqrt{4} &= |x-2| \\ \pm 2 &= x-2 \\ 2 \pm 2 &= x \end{aligned}$$

That is, the zeros or roots of $f(x) = 4-(x-2)^2$ are:

$$x = 2-2 = 0 \text{ or } x = 2+2 = 4$$

Considering our problem in context, we know that the root at time $x = 0$ is when the ball was initially thrown, so the root at $x = 4$ must be when the ball hits the ground.

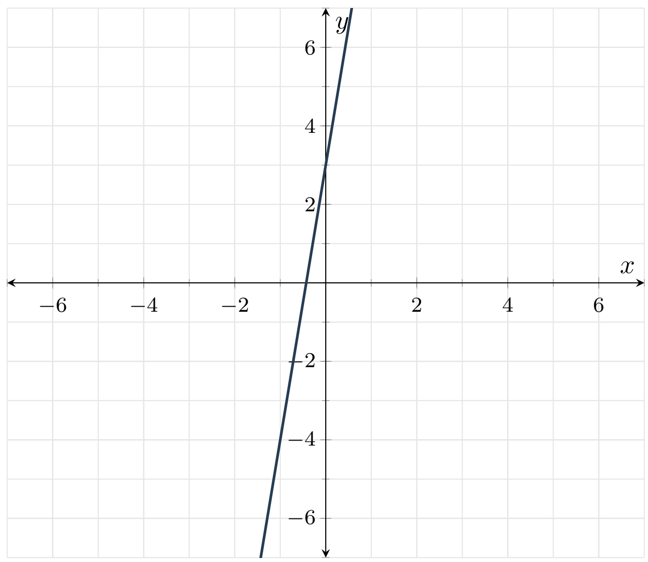
Let's look at a couple more examples:

Example 10. *Find the zeros of the linear function $f(x) = 7x + 3$.*

Explanation We need to set $f(x) = 7x + 3 = 0$

$$\begin{aligned} 7x+3 &= 0 \\ 7x &= -3 \\ x &= \tfrac{-3}{7} \end{aligned}$$

This linear function only has one zero. This should make sense if we think about the graph of a line, as a line will only have one x-intercept.



In fact, all linear functions except constant functions will have one zero. Constant functions will have no zeros except for the linear function $x = 0$ in which every point is a zero.

Example 11. *Write $g(x) = |7x + 3|$ as a piecewise function and find its zeros.*
Explanation Recall that the absolute value makes all the outputs positive. Multiplying a negative number by a -1 makes the value positive. Therefore, $g(x) = |7x + 3|$ is a piecewise function where

$$g(x) = |7x + 3| = \begin{cases} 7x + 3 & \text{if } 7x + 3 \geq 0 \\ -(7x + 3) & \text{if } 7x + 3 < 0 \end{cases}$$

To simplify this expression, notice that we need to know when $7x+3 = 0$. This is what we did in the last example. We know this happens at $x = \frac{-3}{7}$. Now, we also need to know when $f(x) = 7x + 3$ is positive and when it is negative.

Since we have the graph of the function above, we could look at it and see that $7x + 4 > 0$ when $x > \frac{-3}{7}$ and $7x + 4 < 0$ when $x < \frac{-3}{7}$.

Alternatively, if we wanted to figure this out algebraically (without the graph), we can plug values into the function $f(x) = 7x + 3$ on either side of $x = \frac{-3}{7}$. This will work because we know a property about lines. We know that it cannot switch from positive to negative without being equal to 0 in the middle. This is a property that you will study more in calculus.

Let's choose to look at $x = 0$ as a representative of values of $x > \frac{-3}{7}$ and $x = -1$ as a represenitive of values for $x < \frac{-3}{7}$.

$$f(0) = 7(0) + 3 = 3 > 0$$

This means that $f(x) = 7x + 3 > 0$ when $x > \frac{-3}{7}$.

$$f(-1) = 7(-1) + 3 = -4 < 0$$

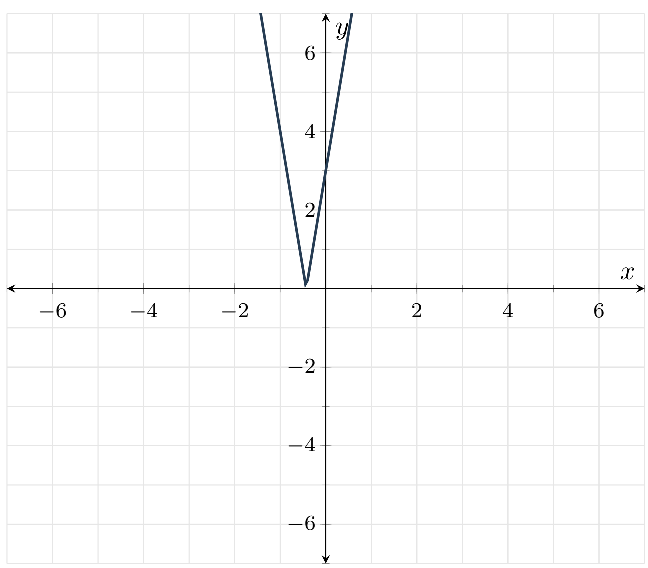
This means that $f(x) = 7x + 3 < 0$ when $x < \frac{-3}{7}$.

Putting this all together, we can now write $g(x) = |7x + 3|$ as a piecewise function:

$$g(x) = |7x + 3| = \begin{cases} 7x + 3 & \text{if } x \geq \frac{-3}{7} \\ -(7x + 3) & \text{if } x < \frac{-3}{7} \end{cases}$$

Now we can find the zeros of this function. We know that an absolute value function is only zero where it switches from the positive branch to the negative branch of the piecewise function so the zero is when $x = \frac{-3}{7}$.

Let's look at the graph of this function to verify that this makes sense.



By looking at this graph, we can see that the zero, that is the x-intercept on $g(x)s = |7x + 3|$ is the same as on $f(x) = 7x + 3$.

Summary

- We call an x-value where a function equals zero the **zero of a function**. It can also be called the **root of a function** if the function is a polynomial. Another equivalent expression is **the**

solution of $f(x) = 0$. Another label for the same value is **the** x**-coordinate of the** x**-intercept**.

- You can find this value graphically by looking for the x-value where the function crosses the x-axis.
- You can find this value algebraically by setting $f(x) = 0$ and then solving for x. This can sometimes be difficult (or even impossible!) to do.

7.2.2 The Importance of the Equals Sign

Motivating Questions

- What are some similarities and differences between mathematical expressions, equations, and functions?
- How is solving an equation related to finding the zero of a function?
- When should we and when shouldn't we use an equals sign?

Introduction

Now that we are exploring the zeros of functions, one issue that often comes up for students (and for teachers reading students' work!) is when you should and should not use an equals sign. We are going to review when is and is not ok to use equals.

First, it is helpful to review a few important terms. In mathematics, it is important to use these and other terms precisely so that you are communicating clearly and saying what you intend to say. Speaking precisely using mathematical terms can be difficult to learn and takes some practice!

Expressions

Definition An **algebraic expression** is any combination of variables and numbers using arithmetic operations such as addition, subtraction, multiplication, division, and exponentiation.

Here are some examples of algebraic expressions:

$$5x^2 - 17 \qquad \frac{56x}{\sqrt{17x}} \qquad 2x + 3y + 7z$$

The important thing to notice is that there are no equals signs in an expression. There are also no inequality signs.

Definition An **mathematical expression**, or just an **expression**, is similar to an algebraic expression, but can contain other mathematical objects such as $\sin(x)$ or $\ln(x)$ or similar objects that you will learn about in future classes. In particular, it does not contain an equals sign or an inequality sign.

Here are some examples of mathematical expressions:

$$\frac{\sin(x)}{\cos(x)} \qquad 5x + \ln(x) - 12 \qquad 2x + 3y + 7z$$

Every algebraic expression is also a mathematical expression.

> **Definition** **Evaluating an expression** is substituting in a particular value for the variable in a mathematical expression.

Here is an example of evaluating an expression. Consider the expression $5x^2 - 17$. Let's evaluate that expression at $x = 1$.

$$5(1)^2 - 17 = 5 - 17 = -12$$

Notice that when evaluating this expression at a particular point, we can use an equals sign. This is a good use of the equals sign and shows us simplifying. But, we should not put an equals sign between $5x^2 - 17$ and $5(1)^2 - 17$ as these two expressions are only equal when $x = 1$.

Equations

When we use an equals sign to say that two different mathematical expressions give the same value, we are creating an equation.

> **Definition** An **equation** is a statement that two mathematical expressions are equal.

Here are some examples of equations:

$$5x^2 - 17 = -12 \qquad \frac{56x}{\sqrt{17x}} = 12x \qquad 2x + 3y + 7z = \frac{x}{y+z}$$

When we are given an equation in a problem, we often want to know what value of the variable will make the equation true. That is, what value of the variable will make both sides give the same value.

> **Definition** **Solving an equation** is the process of determining precisely what value of a variable makes the equation true.

Here is an example of solving an equation. Let's solve $5x^2 - 17 = -12$.

$$
\begin{aligned}
5x^2 - 17 &= -12 \\
5x^2 &= 5 \\
x^2 &= 1 \\
x = 1 &\text{ or } x = -1
\end{aligned}
$$

Notice that this is the reverse process of evaluating the expression $5x^2 - 17$. When evaluating the expression, we knew the x-value and substituted it in. When solving the equation, we knew what the output should be and had to find the x-value that would produce that output. In fact, we found two such values!

> **Remark** Notice that when solving an equation, we don't put equals between the steps. This is very important. In many cases, the steps are not equal!

This is a key observation. Notice that if we naively wrote

$$5x^2 - 17 = -12 = 5x^2 = 5 = ...,$$

we would be saying something not true. In particular, we would be claiming that $-12 = 5$!

The best thing is to do when solving an equation is to make a new line for each step, but if you need to write your steps on a single line, you can use an arrow to show the next step. For example, we could write

$$5x^2 - 17 = -12 \to 5x^2 = 5 \to x^2 = 1 \to x = 1 \text{ or } x = -1.$$

We could also use connecting words between equations. For example, we could write:

$$5x^2 - 17 = -12 \text{ so } 5x^2 = 5 \text{ thus } x^2 = 1 \text{ therefore } x = 1 \text{ or } x = -1.$$

Zeros of Functions Revisited

Notice that when we are working with functions, we are also working with equations and expressions.

- When we write f, we are referencing the function by name.
- When we write $f(x)$, this is an expression for the output of the function at x.
- When we write $f(x) = 5x^2 - 17$, we are defining the way the function produces outputs.

- When we want to find the zeros of this function, we set up the equation $f(x) = 0$. In our case, this would mean solving the equation $5x^2 - 17 = 0$.

$$\begin{aligned} 5x^2 - 17 &= 0 \\ 5x^2 &= 17 \\ x^2 &= \frac{17}{5} \\ x = \sqrt{\frac{17}{5}} &\text{ or } x = -\sqrt{\frac{17}{5}} \end{aligned}$$

  Notice that this is solving an equation so we do not write equals signs between the steps.

Another important connection between finding zeros of functions and solving equations is that every equation can be thought of as the zero of a function. Consider the following example.

Example 12. *Rewrite the equation* $5x + 7 = 6 - x^2$ *as the zero of a function. You do not need to find the zero.*

Explanation In order to rewrite this problem so that solving this equation is equivalent to finding the zero of a function, we want to move all the terms to the same side and combine like terms. For our example, this means

$$\begin{aligned} 5x + 7 &= 6 - x^2 \\ 5x + 7 - (6 - x^2) &= 0 \\ -6x^2 + 5x + 1 &= 0 \end{aligned}$$

Now we let

$$f(x) = -6x^2 + 5x + 1$$

Now, the x values which are zeros of f will be the same x-values that solve $5x + 7 = 6 - x^2$. We will learn to find zeros of quadratic equations in the the next section.

Summary

- You should not write an equals sign between two things which do not have the same value. Equals does not mean “next step”! Instead, to indicate a next step, you may use an arrow, a new line, or connecting words like “so” and“thus”.

- Every equation can be thought of as the zero of a function by moving all the terms to one side and then defining a function to be the output of that side.

7.3 Function Transformations

Learning Objectives

- Vertical and Horizontal Shifts
  - How to shift a function vertically
  - How to shift a function horizontally
  - Combining shifts and properties of quadratics (vertex, completing the square)
- Stretching Functions
  - Vertical stretch
  - Horizontal stretch
- Reflections of Functions
  - Reflections across the x-axis, the y-axis, the origin, $y = x$
  - Connect reflections to inverses, even, and odd functions

7.3.1 Reflections of Functions

Putting it Together

Transformations may be performed one after another. If the transformations include stretches, shrinks, or reflections, the order in which the transformations are performed may make a difference. In those cases, be sure to pay particular attention to the order.

Example 13. (a) *The graph of $y = x^2$ undergoes the following transformations, in order. Find the equation of the graph that results.*

- *a horizontal shift 2 units to the right*
- *a vertical stretch by a factor of 3*
- *a vertical translation 5 units up*

(b) *Apply the transformations above in the opposite order and find the equations of the graph that results.*

Explanation

(a) Applying the transformations in order we have

$y = x^2$	Original function
$y = (x-2)^2$	Horizontal shift
$y = 3(x-2)^2$	Vertical stretch
$y = 3(x-2)^2 + 5$	Vertical translation
$y = 3x^2 - 12x + 17$	Expanded form

(b) Applying the transformations in the opposite order we have

$y = x^2$	Original function
$y = x^2 + 5$	Vertical translation
$y = 3(x^2 + 5)$	Vertical stretch
$y = 3((x-2)^2 + 5)$	Horizontal translation
$y = 3x^2 - 12x + 27$	Expanded form

7.4 Solving Inequalities

Learning Objectives

- Solving Inequalities Graphically
  - Motivating solutions to inequalities
  - Definition of a solution to an inequality
  - Review finding zeros of equations
- Solving Inequalities without a Graph
  - Famous functions are continuous except at... IVT
  - Solving with a sign chart
  - Using signs of famous functions

7.4.1 Solving Inequalities Graphically

Motivating Questions

- What is a solution to an inequality?
- How can we use graphs to solve inequalities?

Introduction

Dabin and Melina are having a walking race. Dabin can walk 1 meter per second, but Melina can walk 2 meters per second. Since Melina is the faster walker, she gives Dabin a head start of 5 meters. At this point, we can ask a few questions about the race. Two questions we'll focus on are "When is Dabin in the lead?" and "When is Melina in the lead?" both of which can be answered by considering inequalities.

To start, let's define some relevant functions. The function D defined by $D(t) = 5 + t$ represents how far (in meters) Dabin has walked t seconds after the start of the race. Similarly, the function M defined by $M(t) = 2t$ represents how far Melina has walked t seconds after the start of the race. In our framework, asking when Dabin is in the lead is the same as asking for all t such that $D(t) > M(t)$. In the vocabulary that we'll use, we want to solve the inequality $D(t) > M(t)$.

Definition Say f and g are functions. A **solution** to the inequality $f(x) < g(x)$ is the set of x values where $f(x) < g(x)$. Similarly, a solution to the inequality $f(x) > g(x)$ is the set of x values where $f(x) > g(x)$.

Note that we define a solution to an inequality as a set. We will often write the sets in interval notation.

Solving inequalities graphically

Example 14. *Let D be defined by $D(t) = 5+t$, and M be defined by $M(t) = 2t$. Find a solution to the inequality $D(t) < M(t)$.*

Explanation This example asks us to find the set of t values where $D(t) < M(t)$. One approach to inequalities of this form is to look at the graphs of the equations involved. The following figure shows the graphs of $y = D(t)$ and $y = M(t)$.

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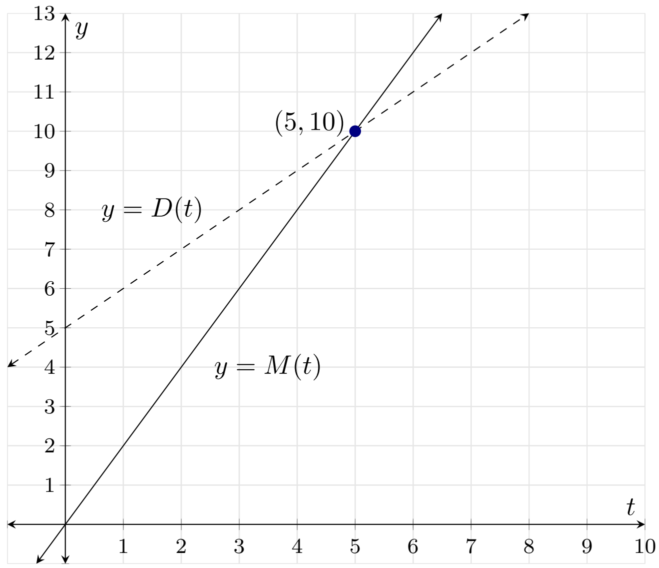

Because of the way we draw the graphs of functions, if $D(x) < M(x)$ for some x if and only if the graph of D lies below the graph of M at the point x. Using this information, we can see that if $t > 5$, then the graph of D lies below the graph of M. Therefore, the set of all t such that $t > 5$ is the solution to $D(t) < M(t)$. Writing this in interval notation, the solution is $(5, \infty)$.

Putting this in terms of the scenario described at the beginning of the section, Melinda is in the lead after 5 seconds.

Exploration Find a solution to the inequality $D(t) < M(t)$.

Example 15. *Let f and g be functions whose graphs are shown below. Assume all important behavior of the functions is shown in the figure.*

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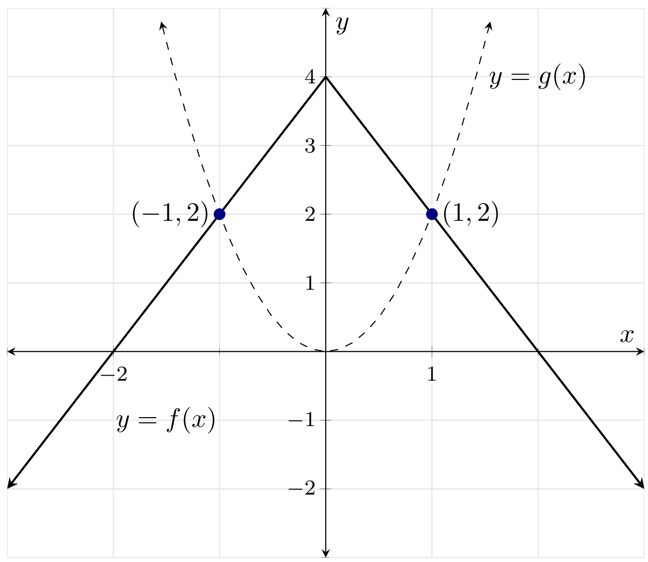

(a) *Solve the inequality* $f(x) < g(x)$.

(b) *Solve the inequality* $f(x) \geq g(x)$.

Explanation

(a) To solve $f(x) < g(x)$, we look for where the graph of f is below the graph of g. This appears to happen for the x values less than -1 and greater than 1. Our solution is $(-\infty, -1) \cup (1, \infty)$.

(b) To solve $f(x) \geq g(x)$, we look for solutions to $f(x) = g(x)$ as well as $f(x) > g(x)$. To solve the former equation we can look at the x-coordinates of the intersection points. This yields $x = \pm 1$. To solve $f(x) > g(x)$, we look for where the graph of f is above the graph of g. This appears to happen between $x = -1$ and $x = 1$, on the interval $(-1, 1)$. Hence, our solution to $f(x) \geq g(x)$ is $[-1, 1]$.

Now let's turn our attention to inequalities involving absolute values, which are often a source of confusion. The following theorem provides the complete story.

Let c be a real number.

- For $c > 0$, $|x| < c$ is equivalent to $-c < x < c$.
- For $c > 0$, $|x| \leq c$ is equivalent to $-c \leq x \leq c$.
- For $c \leq 0$, $|x| < c$ has no solution, and for $c < 0$, $|x| \leq c$ has no solution.
- For $c \geq 0$, $|x| > c$ is equivalent to $x < -c$ or $x > c$.

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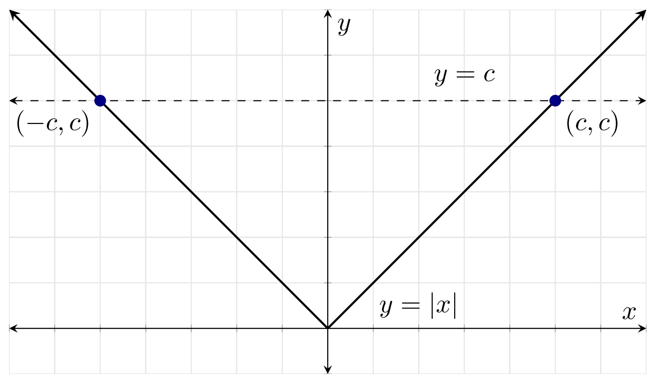
- For $c \geq 0$, $|x| \geq c$ is equivalent to $x \leq -c$ or $x \geq c$.
- For $c < 0$, $|x| > c$ and $|x| \geq c$ are true for all real numbers.

In light of what we have developed in this section, we can understand these statements graphically. For instance, if $c > 0$, the graph of $y = c$ is a horizontal line which lies above the x-axis through $(0, c)$. To solve $|x| < c$, we are looking for the x values where the graph of $y = |x|$ is below the graph of $y = c$. We know that the graphs intersect when $|x| = c$, which we know happens when $x = c$ or $x = -c$. Graphing, we get



We see that the graph of $y = |x|$ is below $y = c$ for x between $-c$ and c, and hence we get $|x| < c$ is equivalent to $-c < x < c$. The other properties in the theorem can be shown similarly. You can try changing the value of c using Desmos.

Desmos link: `https://www.desmos.com/calculator/dbpb01aybm`

Example 16. *Solve the inequality* $4 - 3|2x + 1| > -2$.

Explanation Let's start by graphing both sides of the inequality on the same axes.

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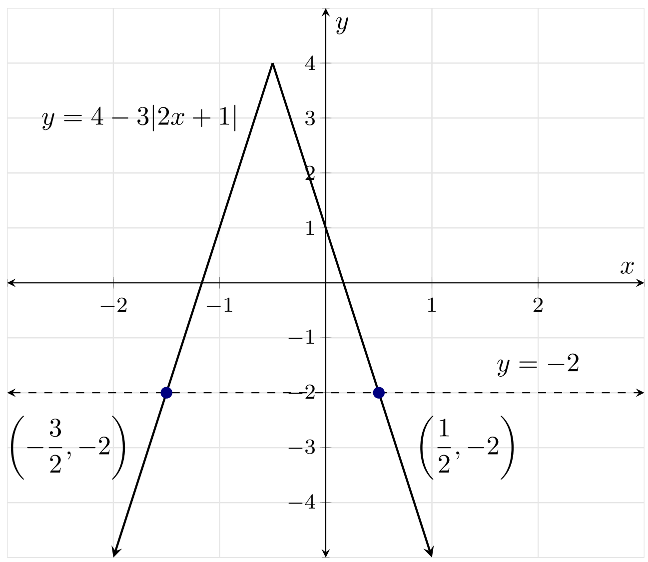
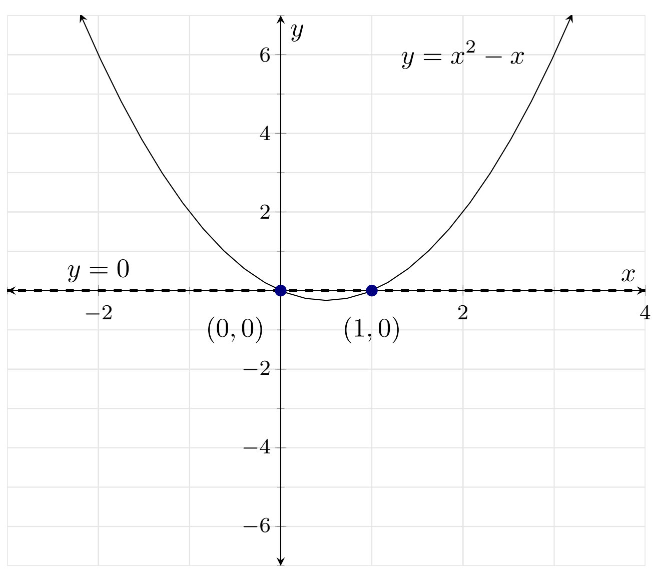


We see that the graph of $y = 4 - 3|2x+1|$ is above $y = -2$ for x values between $-\frac{3}{2}$ and $\frac{1}{2}$. Therefore, the solution in interval notation is $\left(-\frac{3}{2}, \frac{1}{2}\right)$.

Example 17. *Solve the inequality* $x^2 \leq x$.

Explanation We could start by graphing both sides of the inequality on the same graph, but here, we'll demonstrate another possible approach. Note that $x^2 \leq x$ is equivalent to the inequality $x^2 - x \leq 0$, by subtracting x from both sides. Now, let's graph $y = x^2 - x$ and $y = 0$ on the same axes.



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Notice that the two points of intersection are $(0, 0)$ and $(1, 0)$, so $x^2 - x = 0$ for $x = 0$ and $x = 1$. To find the solution to $x^2 - x < 0$, we can see that the graph of $y = x^2 - x$ lies below the graph of $y = 0$ between 0 and 1. Therefore, the solution is $[0, 1]$.

The above example illustrates a common technique. Rather than considering two functions f and g and asking when one is greater than, less than, or equal to the other, we can move one function to the other side, and consider the function $f - g$. Now, the problem becomes one of finding when the function $f - g$ is positive, negative, or zero.

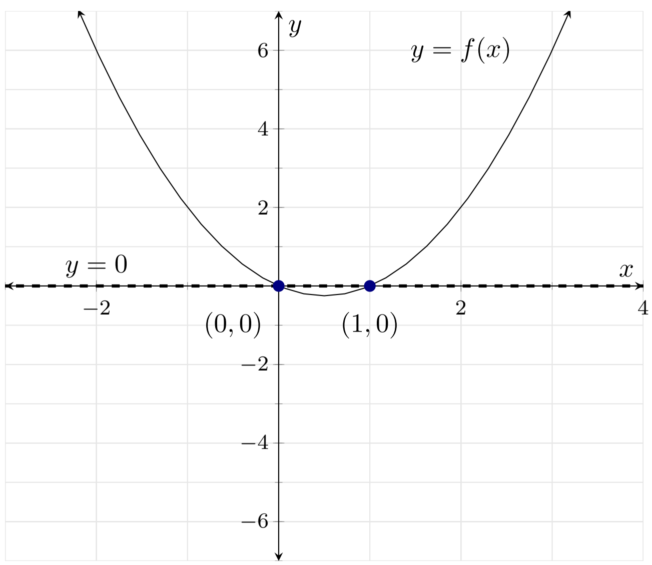
7.4.2 Solving Inequalities without a Graph

Motivating Questions

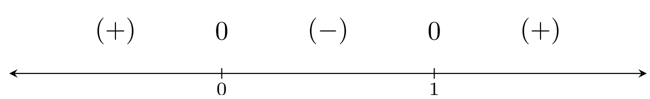
- How can we find solutions to inequalities without using a graph?
- How can we use the zeros of functions to solve inequalities?

Introduction and the Importance of Zeros

In the previous section, we constructed the following graph to solve the inequality $f(x) \leq 0$, where $f(x) = x^2 - x$.



We can see that the graph of f does dip below the x-axis between its two x-intercepts. The zeros of f are $x = 0$ and $x = 1$ in this case and they divide the domain (the x-axis) into three intervals: $(-\infty, 0)$, $(0, 1)$ and $(1, \infty)$. For every number in $(-\infty, 0)$, the graph of f is above the x-axis; in other words, $f(x) > 0$ for all x in $(-\infty, 0)$. Similarly, $f(x) < 0$ for all x in $(0, 1)$, and $f(x) > 0$ for all x in $(1, \infty)$. We can schematically represent this with the *sign diagram* below.



Here, the $(+)$ above a portion of the number line indicates $f(x) > 0$ for those values of x; the $(-)$ indicates $f(x) < 0$ there. The numbers labeled on the

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number line are the zeros of f, so we place 0 above them. We see at once that the solution to $f(x) < 0$ is $(0, 1)$. Adding in the zeros, the solution to $f(x) \leq 0$ is $[0, 1]$.

Our next goal is to establish a procedure by which we can generate the sign diagram without graphing the function.

Continuity and the Intermediate Value Theorem

An important property of quadratic functions is that if the function is positive at one point and negative at another, the function must have at least one zero in between. Graphically, this means that a parabola can't be above the x-axis at one point and below the x-axis at another point without crossing the x-axis.

This is a special case of a theorem called the Intermediate Value Theorem, or IVT for short. To talk about the IVT, we first need to discuss what it means for a function to be *continuous.*

> **Definition** *(Informal.)* We say a function f on an interval is **continuous** if the graph of f has no 'breaks' or 'holes' on that interval.

In further courses, you will learn a more formal definition of continuity, but for now, this will suffice.

Example 18. (a) *Linear and quadratic functions are continuous.*

(b) *In fact, all of our famous functions are continuous where they are defined.*

(c) *All polynomials are continuous.*

(d) *Rational functions are continuous where they are defined. In particular,* $\frac{1}{x}$ *is continuous on its domain,* $(-\infty, 0) \cup (0, \infty)$.

(e) *If* f *and* g *are continuous functions, so are* $f + g$ *and* $f \cdot g$.

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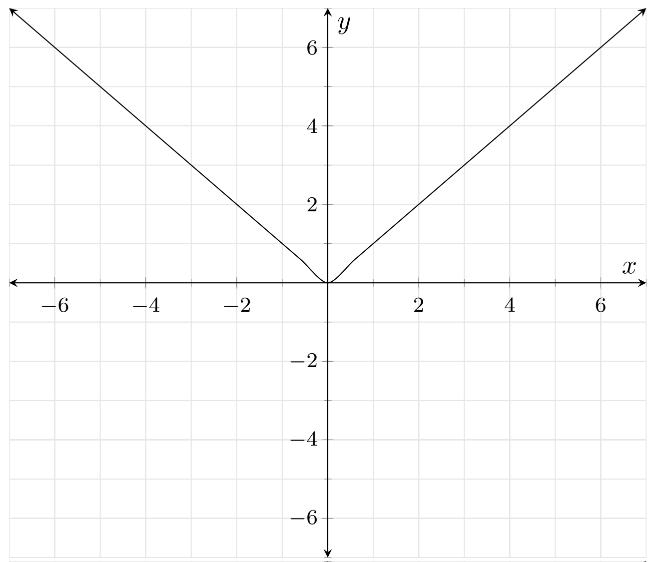

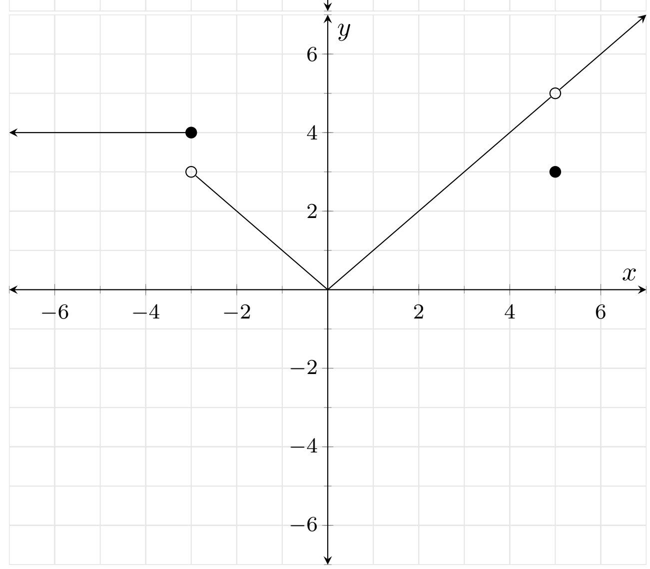

The function whose graph is shown on the left above is continuous, while the function whose graph is shown on the right above is not, since it has breaks in its graph.

One way to think about continuous functions is that they are the functions whose graphs you could draw on an infinite piece of paper without ever taking your pencil off the paper (except where they aren't defined). You will encounter and learn about continuous functions more in-depth in calculus, but for now, familiarity at this level will be enough.

Now that we know about continuous functions, we can state our version of the

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IVT.

[Intermediate Value Theorem (Zero Version)] Suppose f is a continuous function on an interval containing $x = a$ and $x = b$ with $a < b$. If $f(a)$ and $f(b)$ have different signs, then f has at least one zero between $x = a$ and $x = b$; that is, for at least one real number c such that $a < c < b$, we have $f(c) = 0$.

Reinterpreted, this means that the graph of a continuous function can't be above the x-axis at one point and below the x-axis at another point without crossing the x-axis.

Here's how we'll use the IVT to solve inequalities of the form $f(x) > 0$, where f is a continuous function. If a given interval does not contain a zero of f, then by the IVT either all the function values on the interval are positive or they're all negative. In this way, the IVT allows us to determine the sign of *all* of the function values on the interval by testing the function at just *one* value in the interval, which we're free to choose.

This gives us the following steps for solving an inequality involving a continuous function.

(a) Rewrite the inequality, if necessary, as a continuous function $f(x)$ on one side of the inequality and 0 on the other.

(b) Find the zeros of f and place them on the number line with the number 0 above them.

(c) Choose a real number, called a *test value*, in each of the intervals determined in step 2.

(d) Determine the sign of $f(x)$ for each test value in step 3, and write that sign above the corresponding interval.

(e) Choose the intervals which correspond to the correct sign to solve the inequality.

As you can see, the zeros of continuous functions are important, so in the examples that follow, we'll highlight the techniques we use to find zeros. It may also be useful to review methods for finding zeros that you've seen before.

Solving Inequalities Algebraically

Example 19. *Solve the inequality* $3x^2 + x < 6x - 2$.

Explanation To start, let's put the inequality in a nice form, with a continuous function on one side and 0 on the other. It doesn't matter which side the 0 is

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on, so we'll choose to rewrite the inequality as $3x^2 - 5x + 2 < 0$. Since quadratic functions are continuous, we can use the steps outlined in the previous section to solve the inequality.

First, we find the zeros of f, where $f(x) = 3x^2 - 5x + 2$. We could do this by using the quadratic formula, but let's factor. Factoring gives us $f(x) = (3x-2)(x-1)$. In order for $f(x) = 0$ to be true, we need $3x - 2 = 0$ or $x - 1 = 0$. This tells us that the zeros of f are $x = \dfrac{2}{3}$ and $x = 1$. This gives us a good start to our sign diagram:

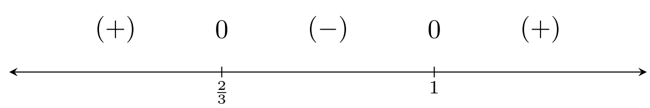
This sign diagram tells us that we have to check three intervals: $\left(-\infty, \dfrac{2}{3}\right)$, $\left(\dfrac{2}{3}, 1\right)$, and $(1, \infty)$. However, thanks to the IVT, we only need to check one test value per interval. Be careful not to choose $x = \dfrac{2}{3}$ or $x = 1$ as your test values!

For the interval $\left(-\infty, \dfrac{2}{3}\right)$, we choose $x = 0$ to be our test value and see that $f(0) = 3(0)^2 - 5(0) + 2 = 2$, which is positive.

For the interval $\left(\dfrac{2}{3}, 1\right)$, we choose $x = \dfrac{5}{6}$ to be our test value and see that $f(0) = 3\left(\dfrac{5}{6}\right)^2 - 5\left(\dfrac{5}{6}\right) + 2 = \dfrac{25}{12} - \dfrac{25}{6} + 2 = -\dfrac{1}{12}$, which is negative.

For the interval $(1, \infty)$, we choose $x = 2$ to be our test value and see that $f(2) = 3(2)^2 - 5(2) + 2 = 4$, which is positive.

We can now update our sign diagram to the following:



Since we wanted to find where f was negative, we choose $\left(\dfrac{2}{3}, 1\right)$ as the solution to the inequality.

Example 20. *Solve the inequality* $xe^x \geq -x$.

Explanation Rewriting our inequality, we have $xe^x + x \geq 0$. Since linear and exponential functions are continuous and products and sums of continuous

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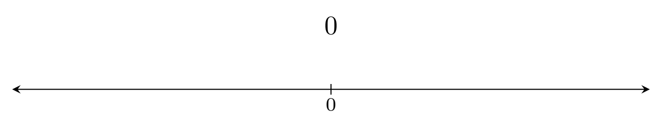
functions are continuous, we know that the function f defined by $f(x) = xe^x + x$ is a continuous function.

First, we find the zeros of f. We start by noticing that each term in $xe^x + x$ contains a factor of x, so we can factor that out and find $f(x) = x(e^x + 1)$. In order to solve the equation $f(x) = 0$, we need to solve $x = 0$ and $e^x + 1 = 0$. The first equation is already solved, and tells us that $x = 0$ is one zero of f. To solve the second equation, we calculate

$$\begin{aligned} e^x + 1 &= 0 \\ e^x &= -1, \end{aligned}$$

and note that the exponential function is never negative, so there are no solutions. Therefore, the only zero of f is $x = 0$. We now begin to construct the sign diagram.

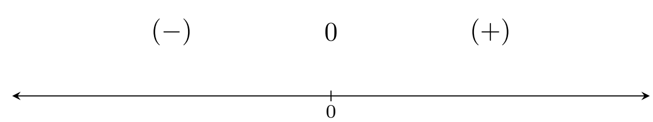


This sign diagram tells us that we have to check two intervals: $(-\infty, 0)$ and $(0, \infty)$. Again, thanks to the IVT, we only need to check one test value per interval.

For the interval $(-\infty, 0)$, we choose $x = -1$ to be our test value and see that $f(-1) = e^{-1} - 1$, which is negative. To see that $e^{-1} - 1$ is negative, notice that $e^{-1} = \dfrac{1}{e}$, and since $e > 1$, $\dfrac{1}{e} < 1$. Therefore, when we subtract 1 from e^{-1}, we obtain a negative number.

For the interval $(0, \infty)$, we choose $x = 1$ to be our test value and see that $f(1) = e^1 + 1$, which is positive.

We can now update our sign diagram to the following:



Since we wanted to find where f was non-negative, we choose $[0, \infty)$ as the solution to the inequality. Remember to include 0 in the interval, since zeros of f are also points where f is non-negative.

Example 21. *Solve the inequality* $2^{x^2-3x} \geq 16$*.*

Explanation We set $r(x) = 2^{x^2-3x} - 16$ and solve the equivalent inequality $r(x) \geq 0$. The domain of r is all real numbers, so in order to construct our sign

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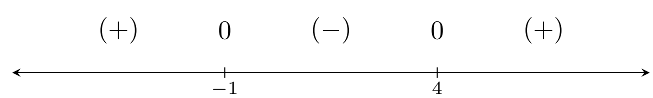
diagram, we need to find the zeros of r. Setting $r(x) = 0$ gives $2^{x^2-3x} - 16 = 0$ or $2^{x^2-3x} = 16$. Since $16 = 2^4$ we have $2^{x^2-3x} = 2^4$, so by taking logarithms, $x^2 - 3x = 4$. Solving $x^2 - 3x - 4 = 0$ gives $x = 4$ and $x = -1$. Therefore, the intervals in which we need to find test values are $(-\infty, -1)$, $(-1, 4)$, and $(4, \infty)$.

For the interval $(-\infty, -1)$, we choose $x = -2$ to be our test value. We see that $r(-2) = 2^{4+6} - 16 = 2^{10} - 2^4 > 0$, so r is positive on the interval.

For the interval $(-1, 4)$, we choose $x = 0$ to be our test value. We see that $r(0) = 2^0 - 16 = 2^0 - 2^4 < 0$, so r is negative on the interval.

For the interval $(4, \infty)$, we choose $x = 5$ to be our test value. We see that $r(5) = 2^{25-15} - 16 = 2^{10} - 2^4 > 0$, so r is positive on the interval.

We can now construct a sign diagram.



From the sign diagram, we see $r(x) \geq 0$ on $(-\infty, -1] \cup [4, \infty)$, which corresponds to where the graph of $y = r(x) = 2^{x^2-3x} - 16$ is on or above the x-axis.

Dealing with Difficult Denominators

Even after we feel comfortable with the procedure for solving inequalities involving continuous functions, you might still wonder about functions which aren't defined on all real numbers, such as rational functions or more generally, functions with denominators that could potentially evaluate to 0. The good news is that if f and g are continuous functions, the function $\frac{f}{g}$ is continuous wherever it is defined. Therefore, we can adapt our technique from before, but remembering that a change of sign *could* happen around a point where a function is undefined, so we need to add any places our functions are undefined to our sign diagram.

Example 22. *Solve the inequality* $\dfrac{x^3 - 2x + 1}{x - 1} \geq \dfrac{1}{2}x - 1.$

Explanation To solve the inequality, it may be tempting to begin by clearing denominators. The problem is that, depending on x, $(x - 1)$ may be positive (which doesn't affect the inequality) or $(x - 1)$ could be negative (which would reverse the inequality). Instead of working by cases, we collect all of the terms on one side of the inequality with 0 on the other and begin to make a sign diagram.

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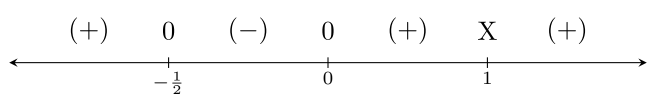
$$\begin{aligned}
\frac{x^3-2x+1}{x-1} &\geq \frac{1}{2}x-1 \\
\frac{x^3-2x+1}{x-1}-\frac{1}{2}x+1 &\geq 0 \\
\frac{2\left(x^3-2x+1\right)-x(x-1)+1(2(x-1))}{2(x-1)} &\geq 0 \qquad \text{get a common denominator} \\
\frac{2x^3-x^2-x}{2x-2} &\geq 0 \qquad \text{expand}
\end{aligned}$$

Viewing the left hand side as a rational function $r(x)$ we make a sign diagram. The candidates for zeros of r are the solutions to $2x^3-x^2-x=0$, which we can find by factoring.

$$\begin{aligned}
2x^3-x^2-x &= 0 \\
x(2x^2-x-1) &= 0 \\
x(2x+1)(x-1) &= 0.
\end{aligned}$$

Therefore, the candidates for zeros of r are $x=0$, $x=-\frac{1}{2}$ and $x=1$. However, $x=1$ is not in the domain of r, since it is the solution to $2x-2=0$, which is the equation we get by setting the denominator equal to 0. However, x-values for which the function is undefined are also possible places where the sign of the function might change, so we should include them on the sign diagram. Since r is a rational function, it is continuous everywhere it is defined, so when constructing the sign diagram, we only need to consider the intervals between zeros or places where it is undefined. For us, these intervals will be $\left(-\infty,-\frac{1}{2}\right)$, $\left(-\frac{1}{2},0\right)$, $(0,1)$, and $(1,\infty)$.

Choosing test values in each test interval (we encourage you to check the calculation), we can construct the sign diagram below.



We used an X to denote that r is not defined at $x=1$.

We are interested in where $r(x) \geq 0$. We find $r(x)$ is positive on the intervals $\left(-\infty,-\frac{1}{2}\right)$, $(0,1)$ and $(1,\infty)$. We add to these intervals the zeros of r, $x=-\frac{1}{2}$, and $x=0$, to get our final solution: $\left(-\infty,-\frac{1}{2}\right] \cup [0,1) \cup (1,\infty)$.

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Example 23. *Solve the inequality* $\dfrac{e^x}{e^x-4} \leq 3$.

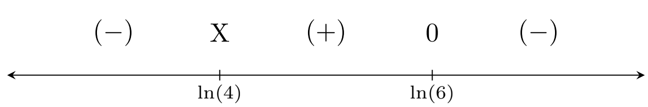
Explanation The first step we need to take to solve $\dfrac{e^x}{e^x-4} \leq 3$ is to get 0 on one side of the inequality. To that end, we subtract 3 from both sides and get a common denominator

$$\begin{aligned}
\frac{e^x}{e^x-4} &\leq 3 \\
\frac{e^x}{e^x-4} - 3 &\leq 0 \\
\frac{e^x}{e^x-4} - \frac{3\left(e^x-4\right)}{e^x-4} &\leq 0 \quad \text{Common denomintors.} \\
\frac{12-2e^x}{e^x-4} &\leq 0
\end{aligned}$$

We set $r(x) = \dfrac{12-2e^x}{e^x-4}$. We note that r is undefined when its denominator $e^x - 4 = 0$, or when $e^x = 4$. Solving this by taking logarithms gives $x = \ln(4)$, so the domain of r is $(-\infty, \ln(4)) \cup (\ln(4), \infty)$. To find the zeros of r, we set the numerator equal to zero and obtain $12 - 2e^x = 0$. Solving for e^x, we find $e^x = 6$, or $x = \ln(6)$. When we build our sign diagram, finding test values may be a little tricky since we need to check values around $\ln(4)$ and $\ln(6)$. Recall that the function $\ln(x)$ is increasing[1] which means $\ln(3) < \ln(4) < \ln(5) < \ln(6) < \ln(7)$. This indicates that we might want to use $\ln(3)$, $\ln(5)$, and $\ln(7)$ as our test values. While the prospect of determining the sign of $r\left(\ln(3)\right)$ may be very unsettling, remember that $e^{\ln(3)} = 3$, so

$$r\left(\ln(3)\right) = \frac{12-2e^{\ln(3)}}{e^{\ln(3)}-4} = \frac{12-2(3)}{3-4} = -6$$

We determine the signs of $r\left(\ln(5)\right)$ and $r\left(\ln(7)\right)$ similarly and construct the sign diagram.



From the sign diagram, we find our answer to be $(-\infty, \ln(4)) \cup [\ln(6), \infty)$.

[1] This is because the base of $\ln(x)$ is $e > 1$. If the base b were in the interval $0 < b < 1$, then $\log_b(x)$ would decreasing.

Acknowledgements: Stitz Zeager Open Source Mathematics (https://www.stitz-zeager.com/)

Conclusion

We hope that the specific examples we've gone through illustrate a general principle when it comes to solving inequalities. First, we want to rewrite the inequality in a form where 0 is on one side and a nice-enough function is on the other side. Then, we use the fact that our functions are continuous on their domains to narrow down where possible sign changes can occur. From there, we can use test values to compute the sign of the function on intervals, and finish by putting our solution in interval notation.

Part 8

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