Precalculus with Review 1: Unit 2

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2.1 Linear Equations

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- Slope
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  - Slope-intercept form
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2.1.1 Linear Equations: Finding Patterns

Patterns in Tables

Example 1. *What is the missing entry in the table? Can you describe the pattern in words and/or mathematics?*

1	2
2	4
3	6
5	$\boxed{?}$

Explanation We can view the table as assigning each input in the left column a corresponding output in the right column. It takes a number as input, and give twice that number as its output. Mathematically, we can describe the pattern as $y = 2x$, where x represents the input, and y represents the output. Labeling the table mathematically, we have

x (input)	y (output)
1	2
2	4
3	6
5	10
10	20

Pattern: y=2x

For each of the following tables, find an equation that describes the pattern you see. Numerical pattern recognition may or may not come naturally for you. Either way, pattern recognition is an important mathematical skill that anyone can develop. Solutions for these exercises provide some ideas for recognizing patterns.

Problem 1 *Write an equation in the form $y = ...$ suggested by the pattern in the table.*

x	y
0	10
1	11
2	12
3	13

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$y = 10 + x$

Explanation One approach to pattern recognition is to look for a relationship in each row. Here, the y-value in each row is always 10 more than the x-value. So the pattern is described by the equation $y = 10 + x$

Problem 2 *Write an equation in the form $y = \ldots$ suggested by the pattern in the table.*

x	y
0	-1
1	2
2	5
3	8

$y = 3x - 1$

Explanation The relationship between x and y in each row is not as clear here. Another popular approach for finding patterns: in each column, consider how the values change from one row to the next. From row to row, the x-value increases by 1. Also, the y-value increases by 3 from row to row.

	x	y	
	0	-1	
$+1 \rightarrow$	1	2	$\leftarrow +3$
$+1 \rightarrow$	2	5	$\leftarrow +3$
$+1 \rightarrow$	3	8	$\leftarrow +3$

Since row-to-row change is always 1 for x and is always 3 for y the rate of change from one row to another row is always the same: 3 units of y for every 1 unit of x. This suggests that $y = 3x$ might be a good equation for the table pattern. But if we try to make a table with that pattern:

x	$3x$	y
0	0	-1
1	3	2
2	6	5
3	9	8

We find that the values from $y = 3x$ are 1 too large. So now we make an adjustment. The equation $y = 3x - 1$ describes the pattern in the table.

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Rates of Change

For an hourly wage-earner, the amount of money they earn depends on how many hours they work. If a worker earns \$15 per hour, then 10 hours of work corresponds to \$150 of pay. Working one additional hour will change 10 hours to 11 hours; and this will cause the \$150 in pay to rise by fifteen dollars to \$165 in pay. Any time we compare how one amount changes (dollars earned) as a consequence of another amount changing (hours worked), we are talking about a **rate of change**.

Given a table of two-variable data, between any two rows we can compute a **rate of change**.

Example 2. *The following data, given in both table and graphed form, gives the counts of invasive cancer diagnoses in Oregon over a period of time.*
Using the tables, find the ***rate of change*** *in Oregon invasive cancer diagnoses between 2000 and 2010?*

Year	*Invasive Cancer Incidents*
1999	17599
2000	17446
2001	17847
2003	17559
2004	18499
2005	19112
2006	19112
2007	19376
2008	20370
2009	19909
2010	19727
2011	20636
2012	20035
2013	20458

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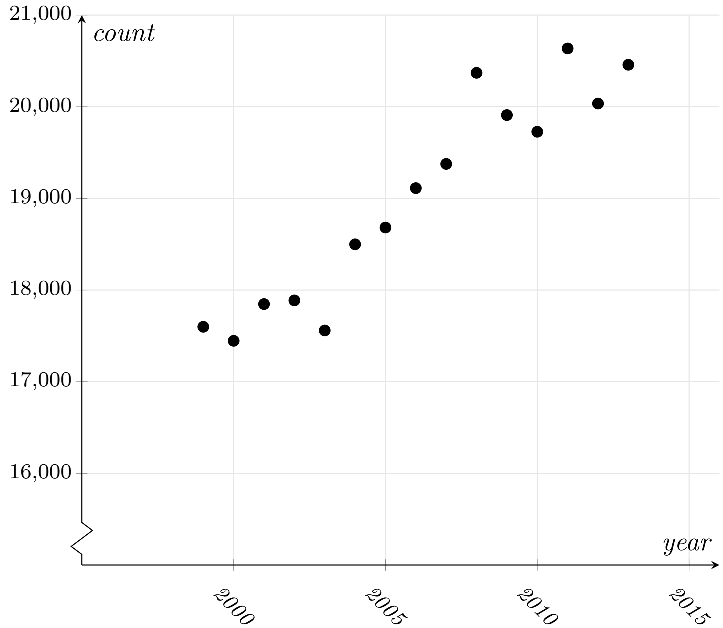

Explanation The total (net) change in diagnoses over that timespan is

$$19727 - 17446 = 2281$$

meaning that there were 2281 more invasive cancer incidents in 2010 than in 2000. Since 10 years passed (which you can calculate as 2010-2000), the rate of change is 2281 diagnoses per 10 years, or

$$\frac{2281 \text{ diagnoses}}{10 \text{ years}} = 228.1 \frac{\text{diagnoses}}{\text{year}}$$

We read that last quantity as “228.1 diagnoses per year.” This rate of change means that between the years 2000 and 2010, there were 228.1 more diagnoses each year, on average. This is just an average over those ten years—it does not mean that the diagnoses grew by exactly this much each year. We dare not interpret why that increase existed, just that it did. If you are interested in examining causal relationships that exist in real life, we strongly recommend a statistics course or two in your future!

Definition If (x_1, y_1) and (x_2, y_2) are two data points with $x_1 \neq x_2$ from a set of two-variable data, then the **rate of change** between them is

$$\frac{\text{change in } y}{\text{change in } x} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

(The Greek letter delta, Δ , is used to represent “change in” since it is

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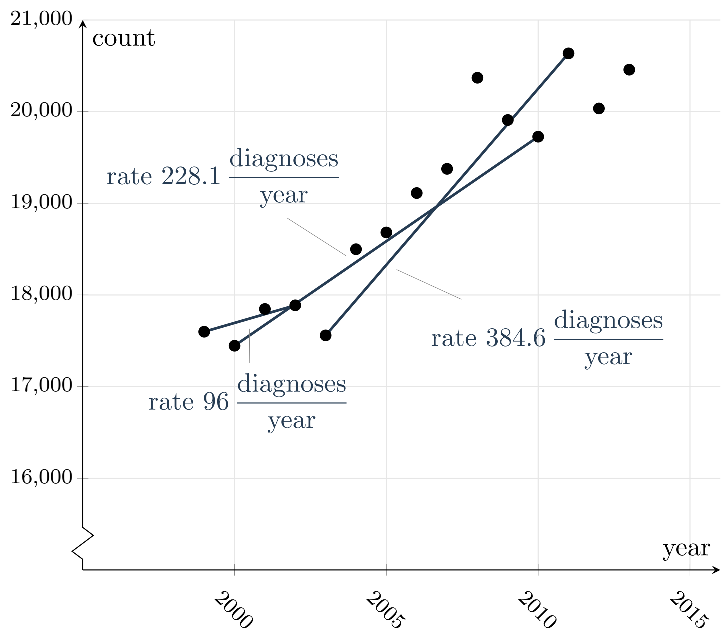
the first letter of the Greek word for "difference.")

Here are some examples of rates of change from our example above.



Note how the larger the numerical rate of change between two points, the steeper the line is that connects them in a graph. This is such an important observation, we'll put it in an official remark.

Remark The rate of change between two data points is intimately related to the steepness of the line segment that connects those points.

(a) The steeper the line, the larger the rate of change, and vice versa.

(b) If one rate of change between two data points equals another rate of change between two different data points, then the corresponding line segments will have the same steepness.

(c) We always measure rate of change from left to right. When a line segment between two data points slants up from left to right, the rate of change between those points will be positive. When a line segment between two data points slants down from left to right, the rate of change between those points will be negative.

Let's revisit the earlier example $y = 3x - 1$

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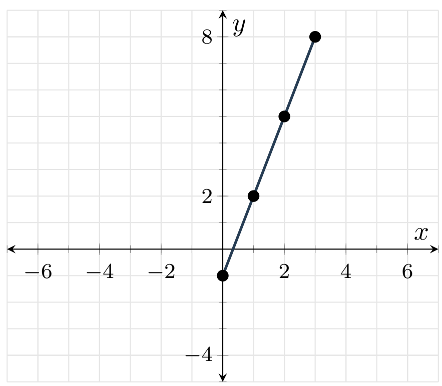
x	y
0	-1
1	2
2	5
3	8

The key observation in this example was that the rate of change from one row to the next was constant: 3 units of increase in y for every 1 unit of increase in x. Graphing this pattern in , we see that every line segment here has the same steepness, so the whole picture is a straight line.



Whenever the rate of change is constant no matter which two (x, y)-pairs (or data pairs) are chosen from a data set, then you can conclude the graph will be a straight line even without making the graph. We call this kind of relationship a **linear relationship**. We'll study linear relationships in more detail throughout this section.

2.1.2 Linear Equations: Slope

We observed that a constant rate of change between points produces a linear relationship, whose graph is a straight line. Such a constant rate of change has a special name, **slope**, and we'll explore slope in more depth here.

> **Definition** When x and y are two variables where the rate of change between any two points is always the same, we call this common rate of change the **slope**. Since having a constant rate of change means the graph will be a straight line, its also called the **slope of the line**.

Considering the definition for **rate of change**, this means that when x and y are two variables where the rate of change between any two points is always the same, then you can calculate slope, m, by finding two distinct data points (x_1, y_1) and (x_2, y_2) , and calculating

$$m = \frac{\text{change in } y}{\text{change in } x} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

A slope is a rate of change. So if there are units for the horizontal and vertical variables, then there will be units for the slope. The slope will be measured in

$$\frac{\text{vertical units}}{\text{horizontal units.}}$$

> **Definition** If the slope is constant, we say that there is a **linear relationship** between x and y.

> **Definition** When the slope is 0, we say that y is **constant** with respect to x, because the y value is the same for all values of x.

Here are some linear scenarios with different slopes. As you read each scenario, note how a slope is more meaningful with units.

- If a tree grows 2.5 feet every year, its rate of change in height is the same from year to year. So the height and time have a linear relationship where the slope is 2.5 ft/yr.
- If a company loses 2 million dollars every year, its rate of change in reserve funds is the same from year to year. So the company's reserve funds and time have a linear relationship where the slope is -2 million dollars per year.
- If Sakura is an adult who has stopped growing, her rate of change in height is the same from year to year—it's zero. So the slope is 0 in/yr. Sakura's height is constant with respect to time.

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> **Remark** A useful phrase for remembering the definition of slope is "rise over run." Here, "rise" refers to "change in y", Δy, and "run" refers to "change in x", Δx. Be careful though. As we have learned, the horizontal direction comes first in mathematics, followed by the vertical direction. The phrase "rise over run" reverses this. (It's a bit awkward to say, but the phrase "run under rise" puts the horizontal change first.)

Example 3. *On Dec. 31, Yara had only \$50 in her savings account. For the the new year, she resolved to deposit \$20 into her savings account each week, without withdrawing any money from the account.*

Yara keeps her resolution, and her account balance increases steadily by \$20 each week. That's a constant rate of change, so her account balance and time have a linear relationship with slope of 20 $\frac{\textit{dollars}}{\textit{week}}$.

Explanation

We can model the balance, y, in dollars, in Yara's savings account x weeks after she started making deposits with an equation. Since Yara started with \$50 and adds \$20 each week, then x weeks after she started making deposits,

$$y = 50 + 20x$$

where y is a dollar amount. Notice that the slope, $20\ \frac{\text{dollars}}{\text{week}}$, serves as the multiplier for x weeks.

We can also consider Yara's savings using a table

	x (weeks since Dec 31)	y (savings account balance in dollars)	
	0	50	
$+1 \rightarrow$	1	70	$\leftarrow +20$
$+1 \rightarrow$	2	90	$\leftarrow +20$
$+2 \rightarrow$	4	130	$\leftarrow +40$
$+3 \rightarrow$	7	190	$\leftarrow +60$
$+5 \rightarrow$	12	290	$\leftarrow +100$

In first few rows of the table, we see that when the number of weeks x increases by 1, the balance y increases by 20. The row-to-row rate of change is

$$\frac{20 \text{ dollars}}{1 \text{ week}} = 20\frac{\text{dollars}}{\text{week}},$$

the slope. In any table for a linear relationship, whenever x increases by 1 unit, y will increase by the slope.

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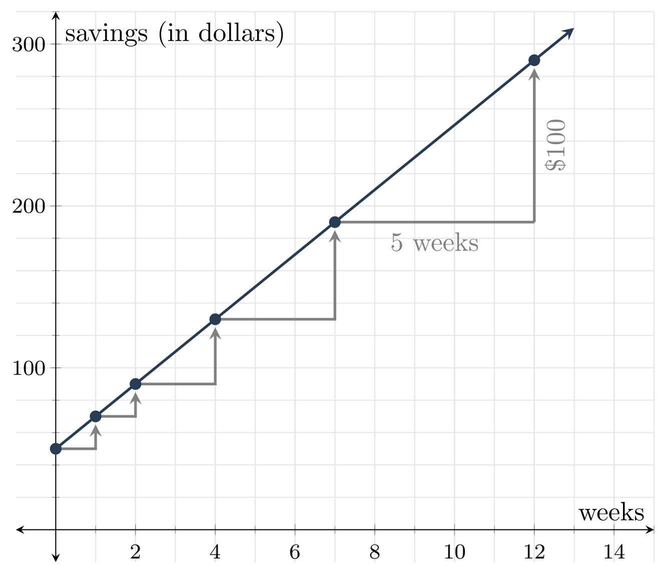
In further rows, notice that as row-to-row change in x increases, row-to-row change in y increases proportionally to preserve the constant rate of change. Looking at the change in the last two rows of the table, we see x increases by 5 and y increases by 100, which gives a rate of change of

$$\frac{100 \text{ dollars}}{5 \text{ week}} = 20\frac{\text{dollars}}{\text{week}},$$

the value of the slope again.

We can see this constant rate of change on the graph by drawing in **slope triangles** between points on the graph, showing the change in x as a horizontal distance and the change in y as a vertical distance.



The Relationship Between Slope and Increase/Decrease

In a linear relationship, as the x-value increases (in other words as you read its graph from left to right):

- if the y-values increase (in other words, the line goes upward), its slope is positive.

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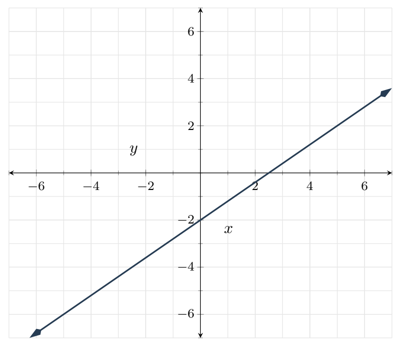
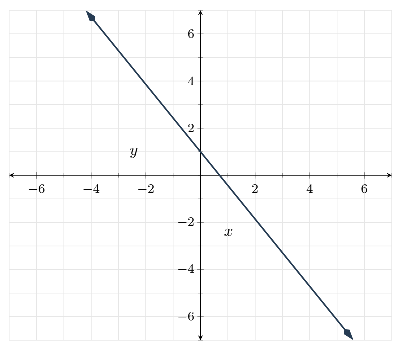
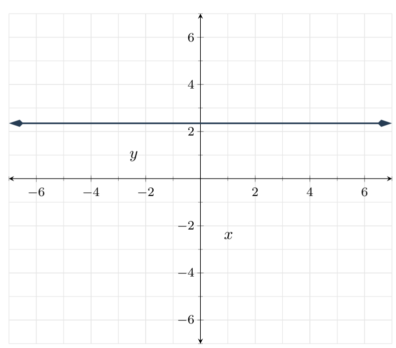


- if the y-values decrease (in other words, the line goes downward), its slope is negative.



- if the y-values don't change (in other words, the line is flat, or horizontal), its slope is 0.



Finding the Slope by Two Given Points

Whenever you know two points on a line, you can find the slope of the line directly from the definition of slope.

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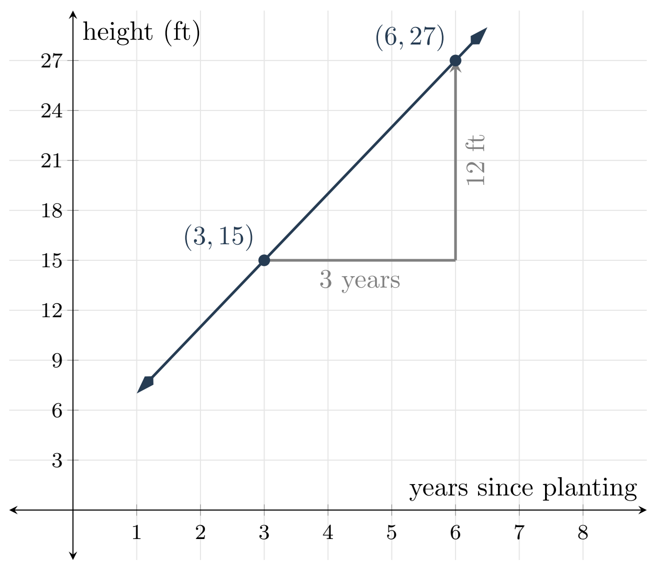
Example 4. *Your neighbor planted a sapling from Portland Nursery in his front yard. Ever since, for several years now, it has been growing at a constant rate. By the end of the third year, the tree was 15 ft tall; by the end of the sixth year, the tree was 27 ft tall. What's the tree's rate of growth (i.e. the slope)?*

Explanation We could sketch a graph for this scenario, and include a slope triangle. If we did that, it would look like:



We don't actually need the picture, though, to find the slope. From the definition of slope, we have that

$$m = \frac{\text{change in } y}{\text{change in } x} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

We know that after 3 yr, the height is 15 ft. As an ordered pair, that information gives us the point (3,15) which we can label as (x_1, y_1) Similarly, the background information tells us to consider (6,27), which we label as (x_2, y_2). Here, x_1 and y_1 represent the first point's x-value and y-value, and x_2 and y_2 represent the second point's x-value and y-value.

Substiuting in our values for $x_1 = 3$, $y_1 = 15$, $x_2 = 6$, and $y_2 = 27$ into our definition of slope, we have

$$m = \frac{\text{change in } y}{\text{change in } x} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\Delta y}{\Delta x} = \frac{27 - 15}{6 - 3} = \frac{12\text{ft}}{3\text{yr}} = 4\frac{\text{ft}}{\text{yr}}$$

2.1.3 Linear Equations: Equations of Lines

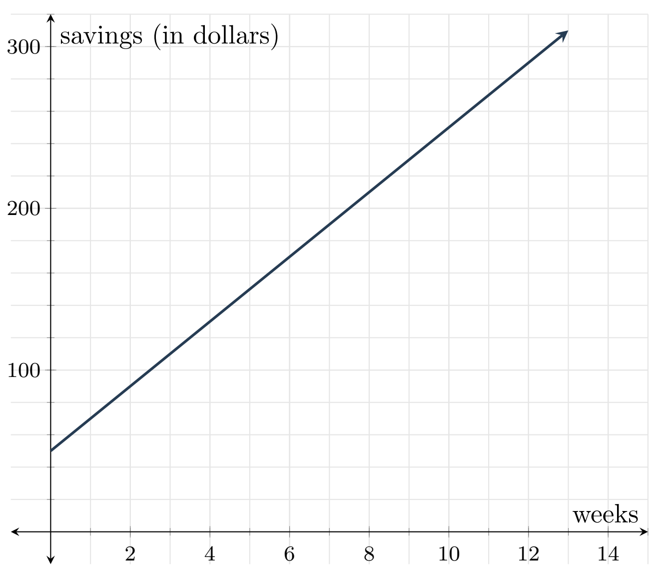
We will explore how to write an equation for a line. The best way to write the equation of a line depends both on what information we have about the line and what we want to do with our equation.

Slope-Intercept Form of a Line

Recall the previous example where Yara had \$50 in her savings account when the year began, and decided to deposit \$20 each week without withdrawing any money. In that example, we model using x to represent how many weeks have passed. After x weeks, Yara has added $20x$ dollars. And since she started with \$50, she has

$$y = 20x + 50$$

in her account after x weeks. In this example, there is a constant rate of change of 20 dollars per week, so we call that the slope. We also saw that plotting Yara's balance over time gives us a straight-line graph.



The graph of Yara's savings has some things in common with almost every straight-line graph. There is a slope, and there is a place where the line crosses the y -axis. We use the symbol, m, for the slope of a line.

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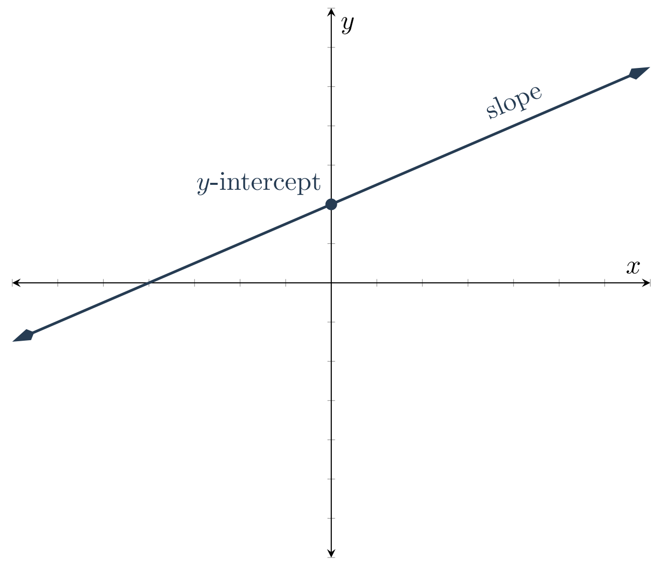

Definition The y-intercept is a point on the y-axis where the line crosses. Since it's on the y-axis, the x-coordinate of this point is 0. It is standard to call the point $(0, b)$ the y-intercept, and call the number b the y-coordinate of the y-intercept.

One way to write the equation for Yara's savings was

$$y = 20x + 50$$

where both $m = 20$ and $b = 50$ are immediately visible in the equation. Now we are ready to generalize this.

Definition When x and y have a linear relationship where m is the slope and $(0, b)$ is the y-intercept, one equation for this relationship is

$$y = mx + b$$

and this equation is called the **slope-intercept form** of the line. It is called this because the slope and y-intercept are immediately discernible from the numbers in the equation.

Problem 3 *What is the slope and y-intercept for the line with the following linear equation?*

$$y = 17x - 14$$

Slope $= 17$ *y-intercept*$=(0, -14)$

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> **Remark** The number b is the y-value when $x = 0$. Therefore it is common to refer to b as the **initial value** or **starting value** of a linear relationship.

Point-Slope Form of a Line

In the previous section, we learned that a linear equation can be written in slope-intercept form, $y = mx + b$. This section covers an alternative that is often more useful, especially in Calculus: point-slope form.

Example 5. *Starting in 1990, the population of the United States has been growing by about 2.865 million people per year. Also, back in 1990, the population was 253 million. Since the rate of growth has been roughly constant, a linear model is appropriate. Let's try to write an equation to model this.*

Explanation We consider using $y = mx + b$, but we would need to know the y-intercept, and nothing in the background tells us that. We'd need to know the population of the United States in the year 0, before there even was a United States.

We could do some side work to calculate the y-intercept, but let's try something else. Here are some things we know:

(a) The slope equation is $m = \dfrac{y_2 - y_1}{x_2 - x_1}$

(b) The slope is $m = 2.865 \dfrac{\text{million people}}{\text{year}}$

(c) One point on the line is $(1990, 253)$ because in 1990, the population was 253 million.

If we use the generic (x, y) to represent a point *somewhere* on this line, then the rate of change between $(1990, 253)$ and (x, y) has to be 2.965. So

$$\frac{y - 253}{x - 1990} = 2.865$$

.

While this is an equation of a line, we might prefer to write the equation without using a fraction. Multiplying both sides by $(x - 1990)$ gives us

$$y - 253 = 2.865(x - 1990)$$

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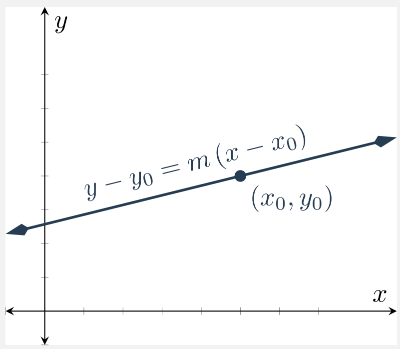
This is a good place to stop. We have isolated y, and three meaningful numbers appear in the equation: the rate of growth, a certain year, and the population in that year. This is a specific example of point-slope form.

Definition When x and y have a linear relationship where m is the slope and (x_0, y_0) is some specific point that the line passes through, one equation for this relationship is

$$y - y_0 = m(x - x_0)$$

and this equation is called the **point-slope form** of the line. It is called this because the slope and one point on the line are immediately discernible from the numbers in the equation.



Sometimes, it is helpful to be able to express our equation as $y = ...$ To do this when working with the Point-Slope form of a line, all you have to do is add y_0 to both sides of the equation. This will give us the Alternate Point-Slope Form.

Definition When x and y have a linear relationship where m is the slope and (x_0, y_0) is some specific point that the line passes through, one equation for this relationship is

$$y = m(x - x_0) + y_0$$

and this equation is called the **(alternate) point-slope form** of the line. It is called this because the slope and one point on the line are immediately discernible from the numbers in the equation.

Note that some people may call this second form the Point-Slope Form of a line. Both ways of writing this form have the advantage that they can be easily written down if you just know a point on the line and the slope of the line.

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Standard Form of a Line

We've seen that a linear relationship can be expressed with an equation in Slope-Intercept form or with an equation in Point-Slope form. There is a third form that you can use to write line equations. It's known as standard form.

Imagine trying to gather donations to pay for a \$10,000 medical procedure you cannot afford. Oversimplifying the mathematics a bit, suppose that there were only two types of donors in the world: those who will donate \$20 and those who will donate \$100.

How many of each, or what combination, do you need to reach the funding goal? As in, if x people donate \$20 and y people donate \$100, what numbers could x and y be? The donors of the first type have collectively donated $20x$ dollars, and the donors of the second type have collectively donated $100y$.

So altogether you'd need

$$20x + 100y = 10000$$

This is an example of a line equation in standard form.

> **Definition** It is always possible to write an equation for a line in the form
> $$Ax + By = C$$
> where A,B, and C are three numbers (each of which might be 0, although at least one of A and B must be nonzero). This form of a line equation is called **standard form**.

In the context of an application, the meaning of A, B, and C depends on that context. This equation is called standard form perhaps because any line can be written this way, even vertical lines (which cannot be written using slope-intercept or point-slope form equations).

Intercept Form of a Line

Intercept form of a line is yet another form used to write line equations. It is useful, because you can immediately pick out the x- and y-intercepts from the equation.

> **Definition** It is sometimes possible to write an equation for a line in the form
> $$\frac{x}{a} + \frac{y}{b} = 1$$
> where a and b are two nonzero numbers. This form of a line equation is called **intercept form**.

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If the x-intercept of a line is located at $(a, 0)$ and the y-intercept of a line is located at $(0, b)$ respectively, and a and b are nonzero, we can write the line in the form $\frac{x}{a} + \frac{y}{b} = 1$. For example, the line with x-intercept $(5, 0)$ and y-intercept $(0, -3)$ has the equation $\frac{x}{5} - \frac{y}{3} = 1$.

It is important to note that intercept form can only be used when the intercepts occur at non-origin points, so it cannot be used to represent a line that goes through the origin, and cannot be used to represent vertical or horizontal lines.

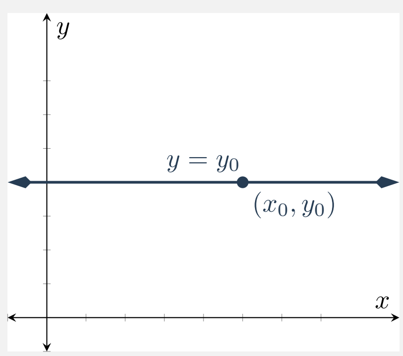
Special Lines

While we can write the equation of a line in different forms, it is important to note that we can easily rearrange a line given in one form to another form using algebra.

There are two special types of lines which it is worth mentioning at this point.

Definition A **horizontal line** is a line where all the y-values of the points are the same. In this case, if the y-value is y_0, then the line can be written as

$$y = y_0$$



Definition A **vertical line** is a line where all the x-values of the points are the same. In this case, if the x-value is x_0, then the line can be written as

$$x = x_0$$

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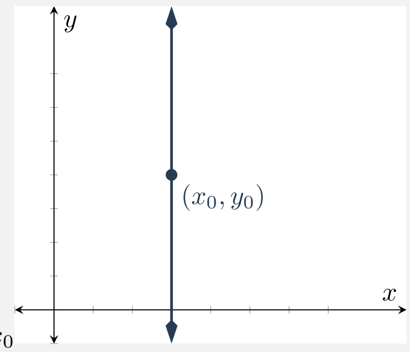

$x = x_0$

2.2 Linear Modeling

Learning Objectives

- Linear Modeling
  - Applications of linear equations to real world situations
  - Emphasis on the constant rate of change
  - Emphasis on sense making (what does this variable or number mean in this context)

2.2.1 Linear Modeling

Let's see how these linear functions can help us in some "real world" contexts.

Example 6. *Your friends are trying to get an idea of how many people they can invite to their wedding. The venue they're looking at costs about \$2,000 to rent and the event coordinator suggested that they budget about \$100 per attendee into the costs.*

(a) *Write a function that represents an estimate for how much this will cost if they have x attendees.*

(b) *Now rewrite your function from part (a) with units for each of the values (including the variables). Do the units make sense? How would you check?*

(c) *Suppose the venue has a maximum capacity of 250 people. What is an estimated maximum cost for using that venue?*

(d) *Suppose your friends are trying to stick to about \$20,000 for their budget. About how many guests can they invite?*

Explanation

(a) Because the rate of change is constant at \$100 dollars per attendee, m must be 100. The cost for 0 attendees is \$2000 so our b value is 2000 this gives us the following equation:

$$y = 100x + 2000$$

where y is the total cost and x is the number of attendees.

(b) Substituting the units in we get the following:

$$y \quad \text{dollars} = \frac{100 \quad \text{dollars}}{1 \quad \text{attendee}} \quad x \quad \text{attendees} + 2000 \quad \text{dollars}$$

To check that the units make sense, we need to make sure that they match on both sides of the equation. The left side only has dollars, so the right side must match up. Since we are multiplying $\frac{\text{dollars}}{\text{attendees}}$ by attendees, the attendees will "cancel out." This leaves us with only dollars on the right side, so both sides agree.

(c) To find the cost for 250 people we just have to plug in 250 for x.
$y = 100 \times 250 + 2000$
$y = 25000 + 2000$
$y = 27000$
The maximum cost is \$27000.

(d) Now we are given the total cost, so we can plug 20000 in for y and find the number of possible attendees.
$20000 = 100x + 2000$
$18000 = 100x$
$180 = x$
The maximum amount of guests they can invite for $20000 would be 180 people.

As we can see, we can use $y = mx + b$ in a variety of situations. These are not just points on a line or values in a table. In this context, they have a specific purpose and thinking about that context can help us understand linear modeling more deeply.

Example 7. *Now let's look closer at how we use $y = mx + b$ for the previous example.*

(a) *What is the "b-value" in your equation? What does it mean in this context?*

(b) *What is the "m-value" in your equation?*

(c) *What are the units for the m-value? Explain why it makes sense that the "m-value" would have these units based on the **x** and **y** values' units.*

(d) *What does this m-value mean in this context?*

Explanation

(a) The "b-value" is 2000. This is also called the y-intercept, because it is the point that the line crosses the y-axis. In the context of this situation, it means that if 0 people attend the wedding the total cost will be $2000.

(b) The "m-value" is 100. This is the slope or rate of change of the line.

(c) In this context, it is $100 per attendee or $\frac{\$100}{1 \text{ attendee}}$. The units for x are attendees and the units for y are dollars. Since slope is defined by the change in y over the change in x, this makes sense that the units for m are dollars per attendee.

(d) In this context, it means that for each person that attends, the final cost will increase by $100.

Now that we can see how to fit our information into a linear model, we can look at a more challenging real world examples. Utility bills, and federal tax returns are both great cases of tricky linear models. Take a look at the following example.

Example 8. *Below is some information for how electricity usage is billed in Columbus*

KWH = Kilowatt Hour **ELECTRICITY RATES Effective January 1, 2018** PCRA = Power Cost Reserve Adjustment

RATE	DESCRIPTION	CHARGES
Residential Schedule A **KW10** (1163.04)	Apartments and dwellings providing domestic accommodations for an individual family. Applicable to residential users with summer usages exceeding 700 KWH in any month.	Customer: \$10.70 Energy (per KWH): .0873 *PCRA varies each month based on actual purchase power costs.
Residential Schedule A-1 *(Small User)* **KW11** (1163.05)	Apartments and dwellings providing domestic accommodations for an individual family. Applicable to residential users with summer usages less than 700 KWH in any month.	Customer: \$10.70 Energy (per KWH): .0724 *PCRA varies each month based on actual purchase power costs.

Reference: https://www.columbus.gov/Templates/Detail.aspx?id=2147500472

(a) *Write a function, or set of functions, which determine how much your electricity bill will be if you use x KWH in a month. Don't worry about the variable PCRA rates. (Note: the 0.0873 and 0.0724 are dollar amounts, i.e. \$0.0873 and \$0.0724)*

(b) *Suppose exactly 700 KWH are used in a month. Use Schedule A information to calculate a cost for the bill.*

(c) *Suppose exactly 700 KWH are used in a month. Now use Schedule A-1 information to calculate a cost for the bill.*

Explanation

(a) Unfortunately, we just have to "know" (assume) that the "Energy (per KWH): .0873" is taking about \$ per KWH (that is what it means) If a household is using less than 700 KWH, then the Schedule A-1 Instructions are used for calculating their energy costs. So for $x < 700$ KWH a month, we use:

$$\$y = \frac{\$0.0724}{\text{KWH}} * xKWH + \$10.70(\text{for monthly bill})$$

If a household is using more than 700 KWH, then the Schedule A Instructions are used for calculating their energy costs. So for x ¿ 700 KWH a month, we use:

$$\$y = \frac{\$0.0873}{\text{KWH}} * xKWH + \$10.70(\text{for monthly bill})$$

It does not seem clear which is used when x = 700 KWH (the wording does not really tell us) but using exactly 700 KWH hours is so rare that it doesn't matter.

(b) x represents KWH, and we want to know the cost for 700 KWH using Schedule A, so we plug $x = 700$ into the Schedule A function:

$$\$y = \frac{\$0.0873}{\text{KWH}} * 700KWH + \$10.70 = \$71.81 \text{ bill}$$

(c) x represents KWH, and we want to know the cost for 700 KWH using Schedule A-1, so we plug $x = 700$ into the Schedule A-1 function:

$$\$y = \frac{\$0.0724}{\text{KWH}} * 700KWH + \$10.70 = \$61.38 \text{ bill}$$

Exploration According to True Car (www.truecar.com), a 2018 Toyota Camry (conventional) (29/41 MPG city/hwy) sells for an average of \$22,030, and a Camry Hybrid (51/53 MPG city/hwy) sells for an average of \$26,247. Currently gas prices are in the upper \$2 per gallon, so let's estimate about \$2.80 per gallon.

(a) Write a linear function that will estimate the cost of driving a conventional Camry x miles, given the information above. (Hint: Think about what the units are for x and what the units should be for y. Then use the units of the information given to help you figure out what should be multiplied and what should be divided in order to give those desired units).

(b) Write a linear function that will estimate the cost of driving a Hybrid Camry x miles, given the information above. (Hint: the function will be very similar to part(a)).

(c) What are the "b-values" in these expressions? What do they represent in this context?

(d) What are the "m-values" in these expressions (write them with their units)? What do these mean in this context?

(e) What other factors could we be considering when comparing the "costs" between these two vehicles?

Summary

- When writing linear equations, consider the units being used in the situation. That can go a long way to properly writing the equation and fully understanding the context.
- The context also helps with making sure that the equations make sense.

2.3 Exponential Modeling

Learning Objectives

- Exponent Rules
  - Properties of Exponents
  - Multiple bases, change of bases
  - y = aR^x to $y = ae^{bx}$
- Early Exponential Modeling
  - Applications of exponential equations to real world situations
  - Emphasis on the proportional change
  - Emphasis on sense making (what does this variable or number mean in this context)
  - Compare and Contrast to Linear Models

2.3.1 Exponential Modeling: Exponent Rules

Exponents

Recall that the notation x^n means to multiple x by itself n times. That is,

Definition

$$x^n = \underbrace{x \cdot x \cdot x ... x \cdot x}_{n \text{ copies of } x}$$

When we write out all the terms in the product instead of using the exponent notation, we call that expanded form.

Example 9. *Write 7^4 in expanded form.* **Explanation**

$$7^4 = 7 \cdot 7 \cdot 7 \cdot 7$$

Notice that we use a dot for multiplication and not a $\times$. This is because it is difficult to distinguish between $7 \times 7 \times 7 \times 7$ and $7x7x7x7 = 7x^3$, especially in handwritten form. It is fine to write $7^4 = (7)(7)(7)(7)$ if you prefer.

The exponent rules that follow below come directly from this definition of exponents.

Product Rule of Exponents

If we write out $3^5 \cdot 3^2$ without using exponents, we'd have:

$$3^5 \cdot 3^2 = (3 \cdot 3 \cdot 3 \cdot 3 \cdot 3) \cdot (3 \cdot 3)$$

If we then count how many 3's are being multiplied together, we find we have $5 + 2 = 7$, a total of seven 3s. So $3^5 \cdot 3^2$ simplifies like this:

$$3^5 \cdot 3^2 = 3^{5+2} = 3^7$$

Example 10. *Simplify $x^2 \cdot x^3$.*
Explanation To simplify $x^2 \cdot x^3$, we write this out in its expanded form, as a product of x's, we have

$$x^2 \cdot x^3 = (x \cdot x)(x \cdot x \cdot x) = x \cdot x \cdot x \cdot x \cdot x = x^5$$

Note that we obtained the exponent of 5 by adding 2 and 3.

This example demonstrates our first exponent rule, the Product Rule:

Theorem 1 (Product Rule of Exponents)**.** *When multiplying two expressions that have the same base, we can simplify the product by adding the exponents.*

$$x^m \cdot x^n = x^{m+n}$$

Recall that $x = x^1$. It helps to remember this when multiplying certain expressions together.

Example 11. *Multiply* $x(x^3 + 2)$ *by using the distributive property.*
Explanation According to the distributive property, $x(x^3 + 2) = x \cdot x^3 + x \cdot 2$ How can we simplify that term $x \cdot x^3$? It's really the same as $x^1 \cdot x^3$, so according to the Product Rule, it is x^4. So we have:

$$x(x^3 + 2) = x \cdot x^3 + x \cdot 2 = x^4 + 2x$$

Power to a Power Rule

If we write out $\left(3^5\right)^2$ without using exponents, we'd have 3^5 multiplied by itself:

$$\left(3^5\right)^2 = \left(3^5\right) \cdot \left(3^5\right) = (3 \cdot 3 \cdot 3 \cdot 3 \cdot 3) \cdot (3 \cdot 3 \cdot 3 \cdot 3 \cdot 3)\,.$$

If we again count how many 3s are being multiplied, we have a total of two groups each with five 3's. So we'd have $2 \cdot 5 = 10$ instances of a 3. So $\left(3^5\right)^2$ simplifies like this:

$$\left(3^5\right)^2 = 3^{2 \cdot 5} = 3^{10}.$$

Example 12. *Simplify* $\left(x^2\right)^3$*.*
Explanation To simplify $\left(x^2\right)^3$, we write this out in its expanded form, as a product of x's, we have $\left(x^2\right)^3 = \left(x^2\right) \cdot \left(x^2\right) \cdot \left(x^2\right) = (x \cdot x) \cdot (x \cdot x) \cdot (x \cdot x) = x^6$.
Note that we obtained the exponent of 6 *by multiplying* 2 *and* 3*.*

This demonstrates our second exponent rule, the Power to a Power Rule:

Theorem 2 (Power to a Power Rule)**.** *when a base is raised to an exponent and that expression is raised to another exponent, we multiply the exponents.*

$$(x^m)^n = x^{m \cdot n}$$

Product to a Power Rule

The third exponent rule deals with having multiplication inside a set of parentheses and an exponent outside the parentheses. If we write out $(3t)^5$ without using an exponent, we'd have $3t$ multiplied by itself five times:

$$(3t)^5 = (3t)(3t)(3t)(3t)(3t)$$

Keeping in mind that there is multiplication between every 3 and t, and multiplication between all of the parentheses pairs, we can reorder and regroup the factors: $(3t)^5 = (3 \cdot t) \cdot (3 \cdot t) \cdot (3 \cdot t) \cdot (3 \cdot t) \cdot (3 \cdot t) = (3 \cdot 3 \cdot 3 \cdot 3 \cdot 3) \cdot (t \cdot t \cdot t \cdot t \cdot t)$ $= 3^5 t^5$ We could leave it written this way if 3^5 feels especially large. But if you are able to evaluate $3^5 = 243$, then perhaps a better final version of this expression is $243t^5$.

We essentially applied the outer exponent to each factor inside the parentheses. It is important to see how the exponent 5 applied to **both** the 3 **and** the t, not just to the t.

Example 13. *Simplify* $(xy)^5$.

To simplify $(xy)^5$, *we write this out in its expanded form, as a product of* x*'s and* y*'s, we have*

$(xy)^5 = (x \cdot y) \cdot (x \cdot y) \cdot (x \cdot y) \cdot (x \cdot y) \cdot (x \cdot y) = (x \cdot x \cdot x \cdot x \cdot x) \cdot (y \cdot y \cdot y \cdot y \cdot y)$ $= x^5 y^5$

Note that the exponent on xy *can simply be applied to both* x *and* y.

This demonstrates our third exponent rule, the Product to a Power Rule:

Theorem 3 (Product to a Power Rule)**.** *When a product is raised to an exponent, we can apply the exponent to each factor in the product.*

$$(x \cdot y)^n = x^n \cdot y^n$$

Summary of the Rules of Exponents for Multiplication

Summary If a and b are real numbers, and m and n are positive integers, then we have the following rules:

Product Rule for Exponents

$$a^m \cdot a^n = a^{m+n}$$

Power to a Power Rule

$$(a^m)^n = a^{m \cdot n}$$

Product to a Power Rule

$$(ab)^m = a^m \cdot b^m$$

Many examples will make use of more than one exponent rule. In deciding which exponent rule to work with first, it's important to remember that the order of operations still applies.

Example 14. *Simplify the following expression.* $\left(3^7r^5\right)^4$
Explanation Since we cannot simplify anything inside the parentheses, we'll begin simplifying this expression using the Product to a Power rule. We'll apply the outer exponent of 4 to each factor inside the parentheses. Then we'll use the Power to a Power Rule to finish the simplification process.

$$\left(3^7r^5\right)^4 = \left(3^7\right)^4 \cdot \left(r^5\right)^4 = 3^{7\cdot4} \cdot r^{5\cdot4} = 3^{28}r^{20}$$

Note that 3^{28} is too large to actually compute, even with a calculator, so we leave it written as 3^{28}.

Example 15. *Simplify the following expression.* $\left(t^3\right)^2 \cdot \left(t^4\right)^5$
Explanation According to the order of operations, we should first simplify any exponents before carrying out any multiplication. Therefore, we'll begin simplifying this by applying the Power to a Power Rule and then finish using the Product Rule.

$$\left(t^3\right)^2 \cdot \left(t^4\right)^5 = t^{3\cdot2} \cdot t^{4\cdot5} = t^6 \cdot t^{20} = t^{6+20} = t^{26}$$

Remark We cannot simplify an expression like x^2y^3 using the Product Rule for Exponents, as the factors x^2 and y^3 do not have the same base.

Quotient to a Power Rule

One rule we have learned is the Product to a Power Rule, as in $(2x)^3 = 2^3x^3$. When two factors are multiplied and the product is raised to a power, we may apply the exponent to each of those factors individually. We can use the rules of fractions to extend this property to a quotient raised to a power.
Let y be a real number, where $y \neq 0$.
Find another way to write $\left(\frac{5}{y}\right)^4$.
Writing the expression without an exponent and then simplifying, we have:

$$\begin{aligned}\left(\frac{5}{y}\right)^4 &= \left(\frac{5}{y}\right)\left(\frac{5}{y}\right)\left(\frac{5}{y}\right)\left(\frac{5}{y}\right)\\ &= \frac{5\cdot5\cdot5\cdot5}{y\cdot y\cdot y\cdot y}\\ &= \frac{5^4}{y^4}\end{aligned}$$

Similar to the Product to a Power Rule, we essentially applied the outer exponent to the factors inside the parentheses to factors of the numerator and factors of the denominator.

Theorem 4 (Quotient to a Power Rule)**.** *For real numbers a and b (with $b \neq 0$) and natural number m,*

$$\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$$

This rule says that when you raise a fraction to a power, you may separately raise the numerator and denominator to that power. In the above example, this means that we can directly calculate $\left(\frac{5}{y}\right)^4$:

$$\begin{aligned}\left(\frac{5}{y}\right)^4 &= \frac{5^4}{y^4}\\ &= \frac{625}{y^4}\end{aligned}$$

Example 16. (a) *Simplify* $\left(\frac{p}{2}\right)^6$.

(b) *Simplify* $\left(\frac{5^6w^7}{5^2w^4}\right)^9$. *If you end up with a large power of a specific number, leave it written that way.*

(c) *Simplify* $\frac{\left(2r^5\right)^7}{\left(2^2r^8\right)^3}$. *If you end up with a large power of a specific number, leave it written that way.*

Explanation

(a) We can use the Quotient to a Power Rule:

$$\begin{aligned}\left(\frac{p}{2}\right)^6 &= \frac{p^6}{2^6}\\ &= \frac{p^6}{64}\end{aligned}$$

(b) If we stick closely to the order of operations, we should first simplify inside the parentheses and then work with the outer exponent. Going this route, we will first use the quotient rule:

$$\begin{aligned}\left(\frac{5^6w^7}{5^2w^4}\right)^9 &= \left(\frac{5^6}{5^2}\cdot\frac{w^7}{w^4}\right)^9\\ &= \left(5^{6-2}w^{7-4}\right)^9\\ &= \left(5^4w^3\right)^9\end{aligned}$$

Now we can apply the outer exponent to each factor inside the parentheses using the Product to a Power Rule:

$= \left(5^4\right)^9 \cdot \left(w^3\right)^9$

To finish, we need to use the Power to a Power Rule:

$$= 5^{4\cdot 9} \cdot w^{3\cdot 9}$$
$$= 5^{36} \cdot w^{27}$$

(c) According to the order of operations, we should simplify inside parentheses first, then apply exponents, then divide. Since we cannot simplify inside the parentheses, we must apply the outer exponents to each factor inside the respective set of parentheses first:

$$\frac{\left(2r^5\right)^7}{\left(2^2r^8\right)^3} = \frac{2^7\left(r^5\right)^7}{\left(2^2\right)^3\left(r^8\right)^3}$$

$$= \frac{2^7r^{5\cdot 7}}{2^{2\cdot 3}r^{8\cdot 3}} \text{ (Power to a Power Rule)}$$

$$= \frac{2^7r^{35}}{2^6r^{24}}$$

$$= 2^{7-6}r^{35-24} \text{ (Quotient Rule)}$$

$$= 2^1r^{11}$$

$$= 2r^{11}$$

Zero as an Exponent

So far, we have been working with exponents that are natural numbers $(1, 2, 3, \ldots)$. Now, we will expand our understanding to include exponents that are any integer, as with 5^0 and 12^{-2}. As a first step, let's explore how 0 should behave as an exponent.

Consider

$$a^5 \cdot a^0$$

for some positive number a. What should this be equal to? Well, based on our Product Rule for Exponents, it should be the case that

$$a^5 \cdot a^0 = a^{5+0} = a^5$$

But if that's the case, the only way $a^5 \cdot a^0 = a^5$ can be true is if $a^0 = 1$. (Try dividing both sides by a^5).

There was nothing special about the a that we used except that we said it was positive. In fact, all we really needed was that $a \neq 0$ so that we could divide both sides by it in that last step. Based on this exploration, we make the following definition.

Definition For all real numbers $a \neq 0$,

$$a^0 = 1.$$

Example 17. *Simplify the following expressions. Assume all variables represent non-zero real numbers*

- $\left(173x^4y^{251}\right)^0$
- $(-8)^0$
- -8^0
- $3x^0$

Explanation To simplify any of these expressions, it is critical that we remember an exponent only applies to what it is touching or immediately next to.

- In the expression $\left(173x^4y^{251}\right)^0$, the exponent 0 applies to everything inside the parentheses. $\left(173x^4y^{251}\right)^0 = 1$
- In the expression $(-8)^0$ the exponent applies to everything inside the parentheses, -8. $(-8)^0 = 1$
- In contrast to the previous example, if we have -8^0, the exponent only applies to the 8. The exponent has a higher priority than negation in the order of operations: $-8^0 = -\left(8^0\right)$, and so $-8^0 = -\left(8^0\right) = -1$.
- In the expression $3x^0$, the exponent 0 only applies to the x: $3x^0 = 3 \cdot x^0$ $= 3 \cdot 1 = 3$

Negative Exponents

We understand what it means for a variable to have a natural number exponent. For example, x^5 means $\overbrace{x \cdot x \cdot x \cdot x \cdot x}^{\text{five times}}$. Now we will try to give meaning to an exponent that is a negative integer, like in x^{-5}.

To consider what it could possibly mean to have a negative integer exponent, let's extend the pattern for powers of 2 in the table below. In this table, each time we move down a row, we reduce the power by 1 and we divide the value by 2. We can continue this pattern in the power and value columns, going all the way down to negative exponents.

Power	*Result*	
2^3	8	
2^2	4	(divide by 2)
2^1	2	(divide by 2)
2^0	1	(divide by 2)
2^{-1}	$\frac{1}{2} = \frac{1}{2^1}$	(divide by 2)
2^{-2}	$\frac{1}{4} = \frac{1}{2^2}$	(divide by 2)
2^{-3}	$\frac{1}{8} = \frac{1}{2^3}$	(divide by 2)

We see a pattern where $2^{\text{negative number}}$ is equal to $\dfrac{1}{2^{\text{positive number}}}$. Note that the choice of base 2 was arbitrary, and this pattern works for all bases except 0, since we cannot divide by 0 in moving from one row to the next.

Theorem 5 (Negative Integers as Exponents)**.** *For any non-zero real number a and any natural number n, we define a^{-n} to mean the reciprocal of a^n. That is,*

$$a^{-n} = \frac{1}{a^n}$$

Fractional Exponents

So far, all exponents we have addressed are integers. Now we will consider fractional exponents.

We want our fractional exponents to still follow the exponent rules we've already defined as well. In particular, we want

$$\left(a^{\frac{1}{n}}\right)^n = a^{\frac{1}{n}\cdot n} = a^1 = a$$

It turns out that is not quite enough to always determine a single value for $a^{\frac{1}{n}}$ but we do only get one number that will work if we also specify that we want $a^{\frac{1}{n}}$ to be a positive, real number. We will explore why we need to state that $a^{\frac{1}{n}}$ must be positive later in this course. This leads us to the following definition.

Definition Fractional Exponents.
For any nonnegative, real number $a \geq 0$ and any natural number n in $\{1, 2, 3, \dots\}$, we define $a^{\frac{1}{n}}$ to mean the number such that

$$a^{\frac{1}{n}} \geq 0$$

and

$$\text{If } a^{\frac{1}{n}} = x \text{, then } x^n = a.$$

That is,

$$\underbrace{a^{\frac{1}{n}} \cdot a^{\frac{1}{n}} \cdot a^{\frac{1}{n}} ... a^{\frac{1}{n}} \cdot a^{\frac{1}{n}}}_{n \text{ copies of } a^{\frac{1}{n}}} = \left(a^{\frac{1}{n}}\right)^n = a$$

We call this number the **nth root of a** and also denote it as

$$a^{\frac{1}{n}} = \sqrt[n]{a}$$

The other exponent rules we have discussed above also apply as expected to fractional powers. In particular, we can use these other exponent rules to define $x^{\frac{m}{n}}$ for any integers m, n.

Definition Let $x \geq 0$, $y, m, n > 0$, then

$$x^{\frac{m}{n}} = \left(x^{\frac{1}{n}}\right)^m$$

Let's summarize all the exponent rules that we have discussed in this section.

Summary [Exponent Rules:]
Let $x \geq 0$, $y, m, n > 0$, then

Product Rule of Exponents:	$x^n \cdot x^m = x^{n+m}$
Quotient Rule:	$\dfrac{x^n}{x^m} = x^{n-m}$
Power to a Power Rule:	$(x^m)^m = x^{mn}$
Product to a Power Rule:	$(x \cdot y)^n = x^n \cdot y^n$
Quotient to a Power Rule:	$\left(\dfrac{x}{y}\right)^n = \dfrac{x^n}{y^n}$
Exponent of Zero:	$y^0 = 1$
Negative Exponents:	$x^{-n} = \dfrac{1}{x^n}$
Fractional Exponents:	$x^{\frac{m}{n}} = \left(x^{\frac{1}{n}}\right)^m = \left(\sqrt[n]{x}\right)^m$

2.3.2 Exponential Modeling: Early Exponentials

Early Exponentials

Example 18. *An athlete signs a contract saying that they will earn \$8.3 million with an increase of 4.8% each year of the 5 year contract. Use a calculator to fill in the following table. Round to the nearest tenth of a million:*

Year of : Contract	*Athlete's Salary*	*Calculate the next year's salary using the previous year's salary*
1	\$8.3 *million*	*N/A*
2	\$8.7 *million*	$8.3 + (8.3 \times .048)$
3	\$9.1 *million*	$8.7 + (8.7 \times .048)$
4	\$9.5 *million*	$\boxed{?} + (9.1 \times .048)$
5	\$10.0 *million*	$\boxed{?} + (9.5 \times .048)$

You may have calculated the salaries for each year by first finding 4.8% of the previous year's salary, and then adding that value to the previous year's salary. Doing it this way, you have to type two calculations into a calculator (calculate 4.8%, then record that value, then enter the addition of that value to the salary). If you did it this way, think about how to write (and do) the computation with just one calculation entry into a calculator by factoring 8.3.

$$8.3 + 8.3 \times \boxed{?} = 8.3 \times (\boxed{?} + \boxed{?}) = 8.3 \times \boxed{?}$$

Here we see how we can simplify the calculation detailed in the table by recognizing that the salary is changing by 4.8% each year. Thus, each year the new salary is 104.8% of the previous year's salary. In other words, the yearly percentage increase is 4.8%, which can be applied to give the new salary by multiplying by 1.048 as demonstrated above.

What if we want to know what the athlete's salary will look like more than one year out? We can use the same process as above without computing each intermediate step to see that the salary after five years will be

$$((((8.3 \cdot 1.048) \cdot 1.048) \cdot 1.048) \cdot 1.048) \cdot 1.048.$$

We can re-write this using the associative property of multiplication as:

$$8.3 \cdot (1.048 \cdot 1.048 \cdot 1.048 \cdot 1.048 \cdot 1.048).$$

Recall from the previous section on exponents, this is the same as

$$8.3(1.048)^5.$$

Here we have demonstrated precisely how we can use an exponential expression to more efficiently calculate (and represent such a calculation) the change in salary represented by the above table.

What we have developed above is an exponential function or model to describe this athlete's salary. Let's look at each value in our model and identify what each piece represents.

$$y = 8.3(1.048)^x$$

y is the athlete's salary for year x of the contract. The athlete has a starting salary of $8.3 million, from which the yearly increase is calculated (as a percentage of the original amount).

Definition In general, an **exponential model** is an equation of the form

$$y = ab^x,$$

where b the **growth factor** and a is the **initial value**. Moreover, if $b = 1+r$, we call r the **growth rate**. Whenever $b > 1$, we often say that the model is exhibiting **exponential growth**, whereas if $0 < b < 1$, we say the model exhibits **exponential decay**. Note that when we have exponential decay the growth rate is negative.

What we have outlined above is an understanding of what each piece of an exponential model in the form $y = a \cdot b^x$ means in terms of a given context. We can now use this understanding to create an exponential model for exponential scenarios without having to go through the "step by step" process as we did in the original table for the athlete salary problem.

Example 19. *Some CDs (a banking investment option) are offering rates that give about 3.10% yield per year as long as a minimum amount, such as $5000, is invested. "3.10% yield" simply means the investment increases (earns) about 3.10%. CDs are only given for a certain amount of time (usually a certain number of years). When a CD "matures" (ends), you usually have the option to renew the CD.*

(a) *Use this scenario to fill out the table below*

After Year:	*Value of investment*	*Calculate the value at the end of the next year using the previous year's value*
0	\$$\boxed{?}$	*N/A*
1	\$$\boxed{?}$	\$5000(1.031)
2	\$5314.81	\$$\boxed{?}(\boxed{?})$
3	\$5479.57	\$$\boxed{?}(\boxed{?})$

(b) *Suppose you invest the minimum amount of $5000 in order to get this 3.10% yield. Write an equation to describe how much the investment will* *be worth after x years. Write the units of each value and identify what each represents.*

(c) *Fill in the first column of the table below with the values you calculated in your pre class work. Then, enter the appropriate values into your model from above to fill in the last column of the table.*

After Year:	*Value of investment from previous table*	*Value of investment from function*
0	\$5000	
1	\$5155	
2	\$5314.81	
3	\$5479.57	

How do these values compare? Are they exactly the same? Should they be? Explain.

(d) *Suppose this is a 7 year CD. How much will your investment be worth at the end of the CD?*

(e) *Suppose you keep renewing this CD with this rate every time it "matures" (comes to the end). About how many years will it take to double the initial investment? Use "Guess and Check" to answer this question:*

(i) *Find the closest whole number that gives us less than we want.*

(ii) *Find the closest whole number that gives us more than we want.*

(iii) *Which of these values is closer to what we want? Use whichever value is closer as your estimated value for the answer.*

Explanation

(a)

After Year:	Value of investment	Calculate the value at the end of the next year using the previous year's value
0	\$5000	N/A
1	\$5155	(5000)(1.031)
2	\$5314.81	(5155)(1.031)
3	\$5479.57	(5314.81)(1.031)

(b)

$$y = 5000(1 + .031)^x \text{ which is equivalent to } y = 5000(1.031)^x$$

y is the amount (in dollars) that the investment is worth after x years.

5000 is the amount of the initial investment (also in dollars).

1.031 is the growth rate of the investment, specifically a 3.1% increase each year, which means we have 103.1% of our initial investment after 1 year.

x is the number of years of investment to reach the value y.

(c)

After Year:	Value of investment from previous table	Value of investment from function
0	\$5000	\$5000
1	\$5155	\$5155
2	\$5314.81	\$5314.81
3	\$5479.57	\$5479.57

The values produced from our function match exactly those given by calculating from the previous year. Think back to our explanation in the previous example. We are performing the same calculation, just via a more efficient method.

(d) To calculate our return after 7 years, we simply plug $x = 7$ into the formula found in part (b):

$$5000(1.031)^7 \approx 6191.28307844$$

We are dealing with money, so we round this to the nearest hundredth, giving a total value of \$6191.28. This is an increase of \$1191.28 over our initial investment.

(e) Doubling our investment, means we want the total value to be \$10000. The process to answer this question is as follows:

(i) Begin "guessing". The idea is to use your calculator to plug in whole numbers for x until the function produces a value less than our goal of \$10000 and such that the next whole number will be more. From the previous part, we know that 7 years only gives about a \$2000 increase, so we should begin with a number at least twice that.

Ultimately, we find that plugging in $x = 22$ will produce a return of

$$\$5000(1.031)^{22} \approx \$9787.24$$

which is less than \$10000, while $x = 23$ results in more (see below).

(ii) As noted above, $x = 23$ will result in slightly more than the \$10000 we are looking for:

$$\$5000(1.031)^{23} \approx \$10090.65$$

The left-hand side is the calculator-ready expression, and the right-hand side is an approximation.

(iii) Finally, we must ask ourselves which of these is closer to the full value of investment that we want?

$$\text{22 years gives: } \$10000 - \$9787.24 \approx \$212.76$$
$$\text{23 years gives: } \$10090.65 - \$10000 \approx \$90.65$$

23 years gets us closer to the goal of $10000. Hence, we must maintain the CD for 23 years to double our investment.

Though it may seem extraneous to write out this subtraction, it is better to develop this habit of checking and concluding your work in a clear and transparent manner. For your success, we recommend you perform such calculations for every problem.

Finally, we give an example where we are looking for a value that decreases over time.

Example 20. *Suppose a patient is given an initial does of 300mg of a medication which degrades by 25% each hour.*

(a) *Write a function describing the degradation of this medication. Identify the units of each value and write a sentence explaining what each piece of the function represents in this context.*

(b) *How much drug concentration is left after one day?*

(c) *Approximately how long until the drug concentration is halved? Estimate this value to the nearest half hour. Use "Guess and Check" to answer this question (as in the previous example).*

(d) *Approximately how long until the drug concentration is reduced to 5mg? Estimate this value to the nearest half hour. Use "Guess and Check" to answer this question (as in the previous example).*

Explanation

(a) $y = 300(1 - 0.25)^x$, which is equivalent to $y = 300(0.75)^x$.

y is the number of milligrams of drug concentration x hours after the initial dose was administered.

300mg is the initial dose of the medicine.

0.75 is the rate of change of the potency of the medication, specifically, a decrease of 25% in the dose each hour.

x is the number of hours since the initial does was administered.

(b) We measure time since the initial dose was administered in hours, so we must plug in $x = 24$ hours (1 day) to the equation found in part (a):

$$y = 300(0.75)^{24} \approx 0.301\text{mg left after 1 day.}$$

(c) The initial administration had a concentration of 300mg, so we are looking for the number of hours until y = 150mg. Since we are looking to the nearest half-hour, we plug in half-values for x as well: 1, 1.5, 2, 2.5 ...
The closest value for x above 150mg is 2, as $300(0.75)^2 \approx 168.75$.

The closest value for x below 150mg is 2.5, as $300(0.75)^{2.5} \approx 146.14$.

We now check to see which value is closer to our target concentration of 150mg:
168.75mg - 150mg = 18.75mg
150mg - 146.14mg = 3.86mg
Thus, the 2.5 hour estimate is closer.

(d) We now wish to find out how long it takes to reduce the concentration to 5mg. Thus, we want to solve $5 = 300(0.75)^x$ for x. Again, we are estimating to the nearest half-hour, so we plug in half-values for x.
The closest value for x that is above 5mg is 14, as $300(0.75)^{14} \approx 5.34$.
The closest value for x that is below 5mg is 14.5, as $300(0.75)^{14.5} \approx 4.63$.

Once again, we check to see which value is closer to our target concentration of 5mg:
5.34mg - 5mg = 0.34mg
5mg - 4.63mg = 0.37mg
0.34mg is barely closer to the target concentration of 5mg, so we use 14.5 hours as an estimate for how long it takes 300mg to degrade to 5mg.

Exploration 4. Suppose there is a new virus that reportedly doubles infection cases in about 20 days. What would be this virus' infection rate (as a percent)? Hint: Set up an exponential function, assuming that there were initially 80 recorded infections. After you have figured out the infection rate, think about/explain why an initial number of infections was not needed in order to answer this question.

Part 3

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