

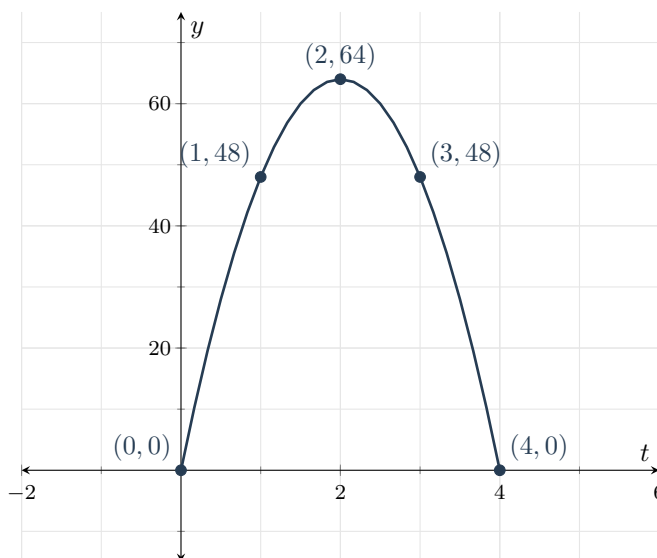
# Parabolas

*We explore polynomial functions.*

## Quadratic Graphs

**Example 1.** *Hannah fired a toy rocket from the ground, which launched into the air with an initial speed of 64 feet per second. The height of the rocket can be modeled by the equation  $y = -16t^2 + 64t$ , where  $t$  is how many seconds had passed since the launch. To see the shape of the graph made by this equation, we make a table of values and plot the points.*

$t$	$-16t^2 + 64t$	Point
0	$-16(0)^2 + 64(0) = 0$	(0, 0)
1	$-16(1)^2 + 64(1) = 48$	(1, 48)
2	$-16(2)^2 + 64(2) = 64$	(2, 64)
3	$-16(3)^2 + 64(3) = 48$	(3, 48)
4	$-16(4)^2 + 64(4) = 0$	(4, 0)



A curve with the shape that we see in the above figure is called a **parabola**.

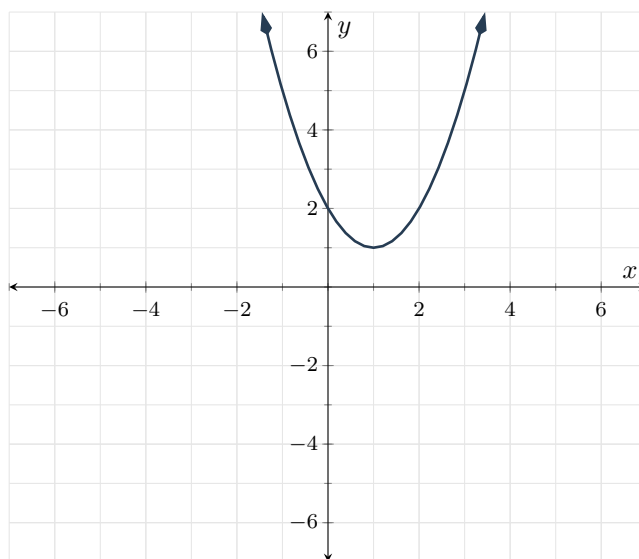
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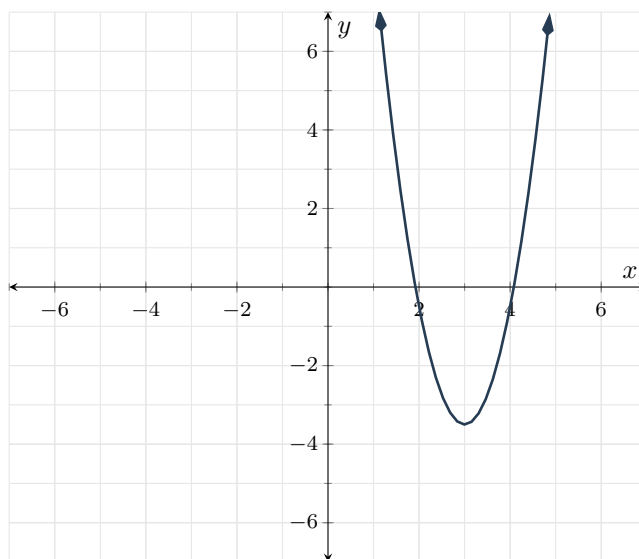
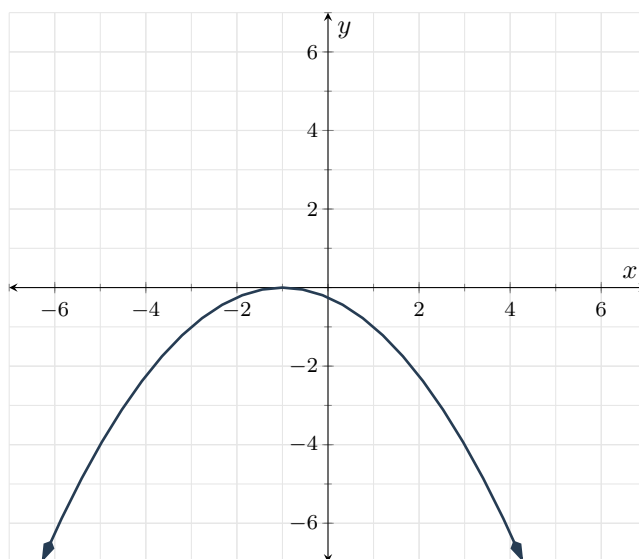
Learning outcomes:  
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Notice the symmetry in figure, how the  $y$ -values in rows above the middle row match those below the middle row. Also notice the symmetry in the shape of the graph, how its left side is a mirror image of its right side.

The first feature that we will talk about is the direction that a parabola opens. All parabolas open either upward or downward. This parabola in the rocket example opens downward because  $a$  is negative. That means that for large values of  $t$ , the  $at^2$  term will be large and negative, and the resulting  $y$ -value will be low on the  $y$ -axis. So the negative leading coefficient causes the arms of the parabola to point downward.

Here are some more quadratic graphs so we can see which way they open.





The graph of a quadratic equation  $y = ax^2 + bx + c$  is a parabola which opens upward or downward according to the sign of the leading coefficient  $a$ . If the leading coefficient is positive, the parabola opens upward. If the leading coefficient is negative, the parabola opens downward.

The **vertex** of a parabola is the highest or lowest point on the graph, depending upon whether the graph opens downward or upward. In Example 1, the vertex is  $(2, 64)$ . This tells us that Hannah's rocket reached its maximum height of 64 feet after 2 seconds. If the parabola opens downward, as in the rocket example,

then the  $y$ -value of the vertex is the **maximum**  $y$ -value. If the parabola opens upward then the  $y$ -value of the vertex is the **minimum**  $y$ -value. The **axis of symmetry** is a vertical line that passes through the vertex, cutting the parabola into two symmetric halves. We write the axis of symmetry as an equation of a vertical line so it always starts with “ $x =$ ”. In Example 1, the equation for the axis of symmetry is  $x = 2$ .

The **vertical intercept** is the point where the parabola crosses the vertical axis. The vertical intercept is the  $y$ -intercept if the vertical axis is labeled  $y$ . In Example 1, the point  $(0, 0)$  is the starting point of the rocket, and it is where the graph crosses the  $y$ -axis, so it is the vertical intercept. The  $y$ -value of 0 means the rocket was on the ground when the  $t$ -value was 0, which was when the rocket launched.

The **horizontal intercept(s)** are the points where the parabola crosses the horizontal axis. They are the  $x$ -intercepts if the horizontal axis is labeled  $x$ . The point  $(0, 0)$  on the path of the rocket is also a horizontal intercept. The  $t$ -value of 0 indicates the time when the rocket was launched from the ground. There is another horizontal intercept at the point  $(4, 0)$ , which means the rocket came back to hit the ground after 4 seconds.

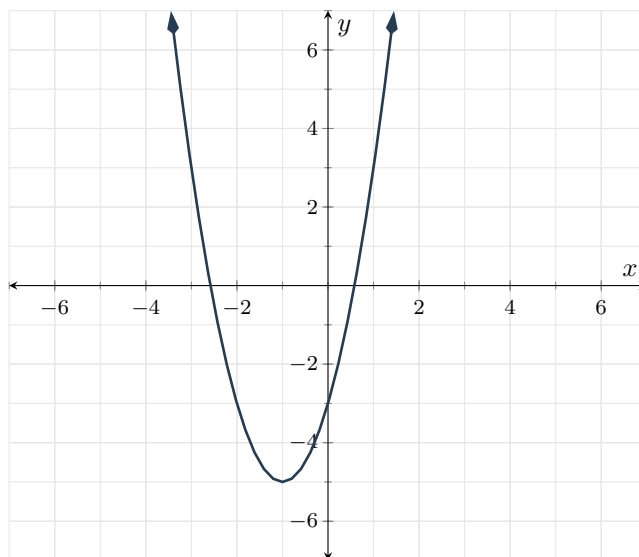
It is possible for a quadratic graph to have zero, one, or two horizontal intercepts. The figures below show an example of each.

**Example 2.** Use technology to graph and make a table of the quadratic function  $f$  defined by  $f(x) = 2x^2 + 4x - 3$  and find each of the key points or features.

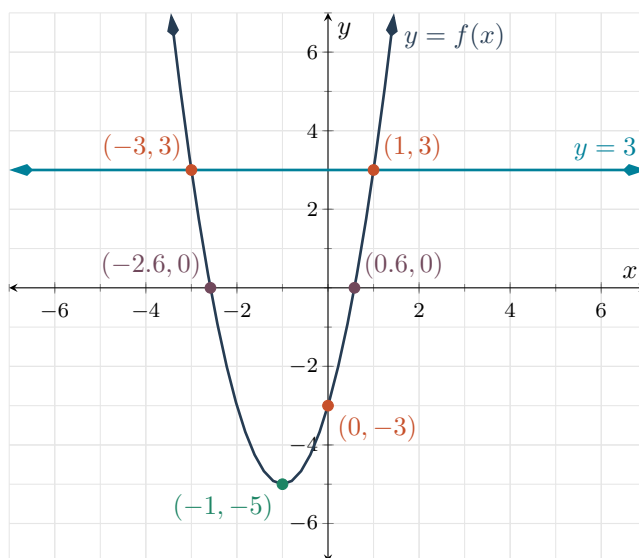
- (a) Find the vertex.
- (b) Find the vertical intercept (i.e. the  $y$ -intercept).
- (c) Find the horizontal or (i.e. the  $x$ -intercept(s)).
- (d) Find  $f(-2)$ .
- (e) Solve  $f(x) = 3$  using the graph.
- (f) Solve  $f(x) \leq 3$  using the graph.

**Explanation** The specifics of how to use any one particular technology tool vary. Whether you use an app, a physical calculator, or something else, a table and graph should look like:

$x$	$f(x)$
-2	-3
-1	-5
0	-3
1	3
2	13



Additional features of your technology tool can enhance the graph to help answer these questions. You may be able to make the graph appear like:



- (a) The vertex is  $(-1, -5)$ .
- (b) The vertical intercept is  $(0, -3)$ .
- (c) The horizontal intercepts are approximately  $(-2.6, 0)$  and  $(0.6, 0)$ .
- (d) When  $x = -2$ ,  $y = -3$ , so  $f(-2) = -3$ .

- (e) The solutions to  $f(x) = 3$  are the  $x$ -values where  $y = 3$ . We graph the horizontal line  $y = 3$  and find the  $x$ -values where the graphs intersect. The solution set is  $\{-3, 1\}$ .
- (f) The solutions are all of the  $x$ -values where the function's graph is below (or touching) the line  $y = 3$ . The interval is  $[-3, 1]$ .

## The Vertex Form of a Quadratic

We have learned the standard form of a quadratic function's formula, which is  $f(x) = ax^2 + bx + c$ . we will learn another form called the vertex form.

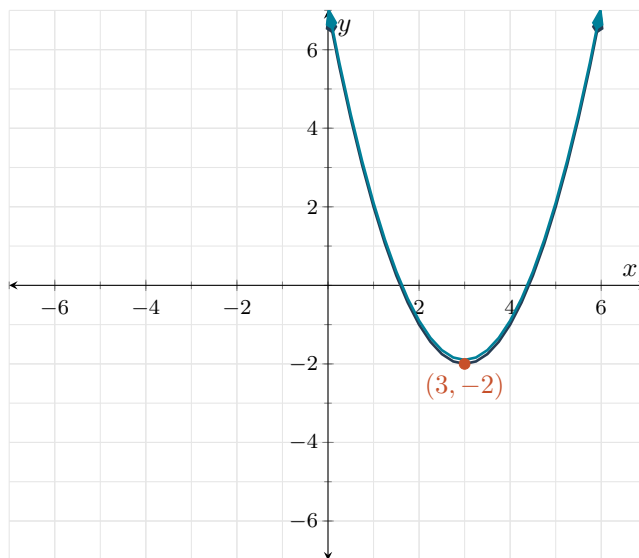
Using graphing technology, consider the graphs of  $f(x) = x^2 - 6x + 7$  and  $g(x) = (x - 3)^2 - 2$  on the same axes.

We see only one parabola because these are two different forms of the same function. Indeed, if we convert  $g(x)$  into standard form:

$$\begin{aligned} g(x) &= (x - 3)^2 - 2 \\ &= (x^2 - 6x + 9) - 2 \\ &= x^2 - 6x + 7 \end{aligned}$$

it is clear that  $f$  and  $g$  are the same function.

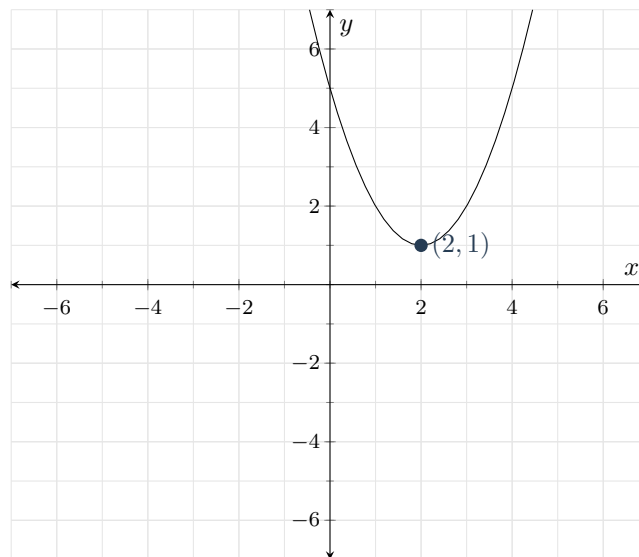
Graph of  $f(x) = x^2 - 6x + 7$  and  $g(x) = (x - 3)^2 - 2$  the graphs of the two parabolas overlap each other completely



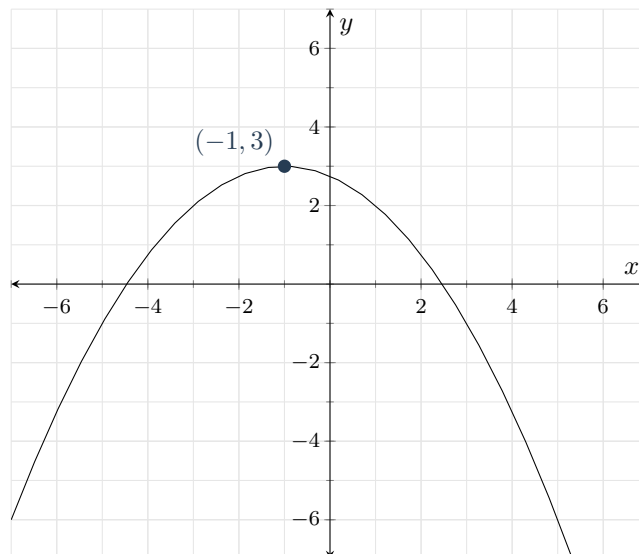
The formula given for  $g$  is said to be in vertex form because it allows us to read the vertex without doing any calculations. The vertex of the parabola is  $(3, -2)$ .

We can see those numbers in  $g(x) = (x - 3)^2 - 2$ . The  $x$ -value is the solution to  $(x - 3) = 0$ , and the  $y$ -value is the constant added at the end.

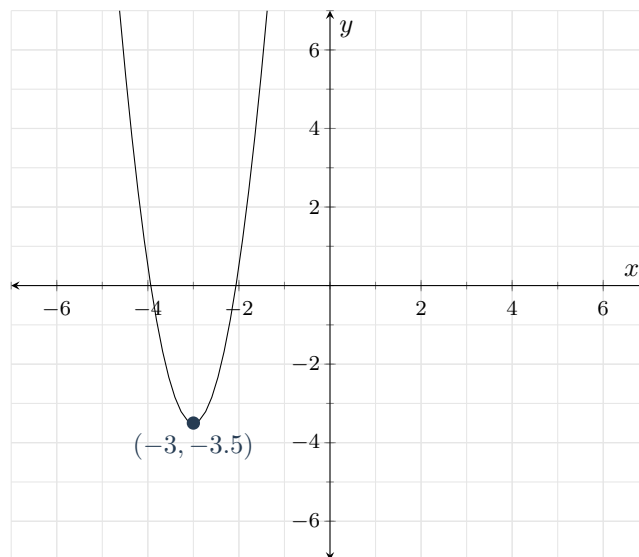
**Example 3.** Here are the graphs of three more functions with formulas in vertex form. Compare each function with the vertex of its graph.



$$r(x) = (x - 2)^2 + 1$$



$$s(x) = -\frac{1}{4}(x + 1)^2 + 3$$



$$t(x) = 4(x + 3)^2 - 3.5$$

Notice that the  $x$ -coordinate of the vertex has the opposite sign as the value in the function formula. On the other hand, the  $y$ -coordinate of the vertex has the same sign as the value in the function formula. Let's look at an example to understand why. We will evaluate  $r(2)$ .

$$r(2) = (2 - 2)^2 + 1 = 1$$

The  $x$ -value is the solution to  $(x - 2) = 0$ , which is positive 2. When we substitute 2 for  $x$  we get the value  $y = 1$ . Note that these coordinates create the vertex at  $(2, 1)$ . Now we can define the vertex form of a quadratic function.

**Vertex Form of a Quadratic Function** A quadratic function whose graph has vertex at the point  $(h, k)$  is given by

$$f(x) = a(x - h)^2 + k$$