The Definition of a Rational Function

Introduction

We have previously discussed the function $1/x$. Note that both the numerator 1 and the denominator x are polynomials (1 is a *constant* polynomial). We can study what happens when we replace those with arbitrary polynomials.

Definition 1. *A* rational function *is a function defined as a ratio* $r(x) = \dfrac{p(x)}{q(x)}$ *of two polynomials* $p(x)$ *and* $q(x)$*, and this ratio makes sense for all real values of* x*,* except *for those such that* $q(x) = 0$.

Example 1. *Are the following functions rational? For which values of* x *is the function undefined?*

(a) $f(x) = \dfrac{x^2 - 2}{x + 1}$.

Explanation. *It is a rational function. It is defined for all values of* x*, except for* $x = -1$*, because this makes the denominator* $x + 1$ *be zero.*

(b) $f(x) = \dfrac{x^4 - 3x + 1}{x^2 - 5x + 6}$.

Explanation. *It is a rational function. It is defined for all values of* x*, except for* $x = 2$ *and* $x = 3$*, because* $x^2 - 5x + 6 = (x - 2)(x - 3)$.

(c) $f(x) = \dfrac{x - 1}{\sqrt{x^4 + 1}}$.

Explanation. *It is* not *a rational function, because the denominator* $\sqrt{x^4 + 1}$ *is* not a polynomial. *Recall that a square root cannot distribute over addition or subtraction, so* $\sqrt{x^4 + 1} \neq \sqrt{x^4} + \sqrt{1} = x^2 + 1$*. This is a common error to make when first learning about rational functions, so be sure to watch out for it! Note that even though this does not define a rational function, it is defined for all possible values of* x*, since* $\sqrt{x^4 + 1} \geq 1 > 0$ *for all* x.

Learning outcomes:

Author(s): Ivo Terek

Acknowledgements: Stitz Zeager Open Source Mathematics (https://www.stitz-zeager.com/)

(d) *A polynomial function $p(x)$.*

Explanation. *Any polynomial function is a rational function, simply because we can write $p(x) = p(x)/1$, and 1 is a polynomial. And it is defined for every real value of x.*

(e) $f(x) = x^2 - 1 + \dfrac{x^3}{x^5 - 1}$.

Explanation. *It is a rational function, which is defined for all x except for $x = 1$. To see that this is a rational function, you can either say that it is the sum of the rational functions $x^2 - 1$ and $x^3/(x^5 - 1)$, or rewrite it as*

$$f(x) = \frac{(x^2 - 1)(x^5 - 1) + x^3}{x^5 - 1} = \frac{x^7 - x^5 - x^2 + x^3 - 1}{x^5 - 1},$$

which is manifestly rational.

Domains of Rational Functions

While we said it already, it is worth emphasizing that a rational functions is not defined when the polynomial in the denominator is equal to zero. That is, if $r(x) = \dfrac{p(x)}{q(x)}$ is a rational function and both $p(x)$ and $q(x)$ are polynomials, than the domain of $r(x)$ is all values of x except those where $q(x) = 0$. Notice, that finding the domain of a rational function is going to require finding the x-intercepts of a polynomial!

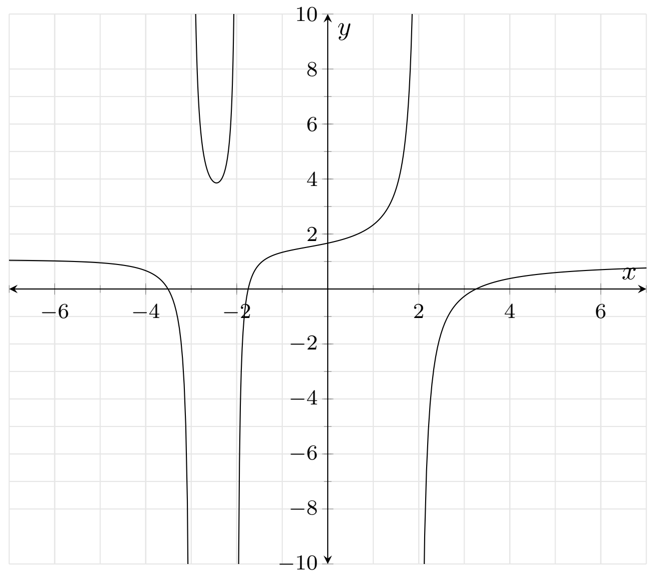
Example 2. *Find the domain of the rational function $f(x) = \dfrac{x^3 + 2x^2 - 11x - 20}{x^3 + 3x^2 - 4x - 12}$.*

Explanation. *Notice that our discussion of finding the domain of a rational function does not involve the numerator at all. Since the numerator is a polynomial, it is defined for all values of x. What we need to do is look for where the denominator is equal to zero.*

$$x^3 + 3x^2 - 4x - 12 = 0 \tag{1}$$
$$x^2(x + 3) - 4(x + 3) = 0 \tag{2}$$
$$(x + 3)(x^2 - 4) = 0 \tag{3}$$
$$(x + 3)(x - 2)(x + 2) = 0 \tag{4}$$
$$x + 3 = 0 \quad OR \quad x - 2 = 0 \quad OR \quad x + 2 = 0 \tag{5}$$
$$x = -3 \quad OR \quad x = 2 \quad OR \quad x = -2 \tag{6}$$

Acknowledgements: Stitz Zeager Open Source Mathematics (https://www.stitz-zeager.com/)

Therefore, $x = -3, -3$, and 2 are not in the domain of our function f. We can write this domain in interval notation as $(-\infty, -3) \cup (-3, -2) \cup (-2, 2) \cup (2, \infty)$. From the fact that the the domain comes in four pieces, we can know that the graph of the rational function f will also have four pieces. Here is what this function looks like when graphed:



That looks quite different from the graphs of functions we have studied thus far! We will continue to explore the behavior of rational functions in this unit so that we can understand and predict this graph shape.

Combining Rational Functions

When we add, subtract, multiply, or divide rational functions, we get another rational function.

Acknowledgements: Stitz Zeager Open Source Mathematics (https://www.stitz-zeager.com/)