Summation Notation

Motivating Questions

- What is the sum of the the first n integers, $1 + 2 + 3 + \cdots + n$?
- Is there an efficient way of dealing with sums of lots of numbers, without having to write them out one by one?
- How to make sense of complicated mathematical expressions like $\displaystyle\sum_{k=1}^{n}(3k - 2^k + 1)$, and what does $\sum$ mean?

Introduction

Germany, 1787. Imagine a very rowdy middle school classroom. The math teacher, out of patience, tells the students to add all integers from 1 to 100, in the hopes that he would finally be able get some silence and hear his own thoughts. Just a few seconds after the problem was posed, Carl, one of his students, simply walks to the board and writes "5050" as the answer. The teacher, completely surprised, asked Carl to explain his reasoning. And the trick, in hindsight, was rather simple. Writing

$$S = 1 + 2 + 3 + \cdots + 98 + 99 + 100$$

for the desired sum, Carl rewrote the sum backwards:

$$S = 100 + 99 + 98 + \cdots + 3 + 2 + 1,$$

and noted that twice the sum S was equal to a sum of 100 pairs, which added to 101 each:

$$2S = (1 + 100) + (2 + 99) + (3 + 98) + \cdots + (98 + 3) + (99 + 1) + (100 + 1).$$

So $2S = 100 \times 101 = 10100$, and hence $S = 5050$.

Johann Carl Friedrich Gauss (1777–1855) grew up to be one of the greatest mathematicians to have ever lived. In this unit, we will learn how *summation notation* – a powerful device to deal more easily with sums of a large number of terms – works, and we'll apply it to understand how to obtain actual formulas giving the result of adding all integers from 1 to 100, and more.

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Basic Properties Assume that $a_1, a_2, \dots, a_9$ and a_{10} are all real numbers, and say we want to consider their sum. Instead of painstakingly writing every single term, we write

$$\sum_{i=1}^{10} a_i = a_1 + a_2 + \cdots + a_9 + a_{10}.$$

Here's how to read this: the symbol $\sum$ stands for the capital Greek letter *sigma* (so "s" for "sum") and i is a *dummy index* which will take the values from 1 to 10, as indicated in the notation. In other words, it's the "sum of the a_i's as i ranges from 1 to 10".

There is no reason to stop at 10, however. For any positive integer n, we write

$$\sum_{i=1}^{n} a_i = a_1 + \cdots + a_n.$$

Let's elaborate further on what we meant by saying that i is a dummy index: the value of the actual sum doesn't depend on what we actually named the index i. More precisely, how does writing $\displaystyle\sum_{i=1}^{n} a_i$ in full works?

- Substitute i with 1, so we have a_1.
- Then substitute i with 2, and add to the previous result, so we have a_1+a_2.
- Then subsitute i with 3, and add to the previous result, so we have $a_1 + a_2 + a_3$.
- Keep going until we reach $i = n$, and then stop.

The index i itself does not appear in the full sum $a_1 + \cdots + a_n$, as it only worked as a placeholder to indicate the integers we would plug for i in a_i. We could have used a different index, as in

$$\sum_{i=1}^{n} a_i = \sum_{j=1}^{n} a_j = \sum_{k=1}^{n} a_k = \sum_{\ell=1}^{n} a_\ell = \sum_{m=1}^{n} a_m = \cdots = a_1 + \cdots + a_n.$$

What really matters here is the integer n, which tells us when we should stop adding. The "bounds of summation" 1 and n could have been some other numbers, so writing things such as

$$\sum_{j=3}^{n-2} a_j = a_3 + \cdots + a_{n-2}$$

also makes perfect sense.

Example 1. *Write the sum* $\sum_{i=1}^{4}(2^i + 3i)$ *in full and evaluate it.*

Explanation We have that

$$\sum_{i=1}^{4}(2^i + 3i) = (2^1 + 3 \cdot 1) + (2^2 + 3 \cdot 2) + (2^3 + 3 \cdot 3) + (2^4 + 3 \cdot 4) = 60.$$

Example 2. *Write the sum* $\sum_{j=1}^{5} \frac{3}{j+1}$ *in full and evaluate it.*

Explanation We have that

$$\sum_{j=1}^{5} \frac{3}{j+1} = \frac{3}{1+1} + \frac{3}{2+1} + \frac{3}{3+1} + \frac{3}{4+1} + \frac{3}{5+1} = \frac{87}{20}$$

Observe that this is the same thing as computing the sum $\sum_{k=2}^{6} \frac{3}{k}$, as if we let $k = j + 1$, then $k = 2$ when $j = 1$, and $k = 6$ when $j = 5$. At this point, k is again a dummy index, which we may just as well rename as j, so that

$$\sum_{j=1}^{5} \frac{3}{j+1} = \sum_{j=2}^{6} \frac{3}{j}.$$

We will see later how renaming indices, as silly as it may seem, will turn out to be a useful technique for deriving some nice closed formulas for certain sums.

Example 3. *Write*

$$\sum_{k=4}^{6} \frac{2k}{k-3}$$

in full and evaluate it.

Explanation We have that

$$\sum_{k=4}^{6} \frac{2k}{k-3} = \frac{2 \cdot 4}{4-3} + \frac{2 \cdot 5}{5-3} + \frac{2 \cdot 6}{6-3} = 17.$$

Exploration Write the following sums in full and evaluate them:

(a) $\sum_{k=3}^{6}(k+1)$

(b) $\displaystyle\sum_{\ell=2}^{5} 3^{k-2}$

(c) $\displaystyle\sum_{j=1}^{4} j^2$

There are some basic properties of the summation notation which will allow us to manipulate those symbols more easily.

Theorem: Let n be a positive integer, and $a_1, \ldots, a_n, b_1, \ldots, b_n, c$ be all real numbers. Then we have that

- $\displaystyle\sum_{i=1}^{n}(a_i + b_i) = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i$;
- $\displaystyle\sum_{i=1}^{n} ca_i = c\sum_{i=1}^{n} a_i$;
- $\displaystyle\sum_{i=1}^{n} 1 = n$.

Let's justify these properties properly. For the first one, we have:

$$\begin{aligned}
\sum_{i=1}^{n}(a_i + b_i) &= (a_1 + b_1) + \cdots + (a_n + b_n) \\
&= a_1 + b_1 + \cdots + a_n + b_n \\
&\overset{(*)}{=} a_1 + \cdots + a_n + b_1 + \cdots + b_n \\
&= \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i.
\end{aligned}$$

where in $(*)$ we have used that addition is "commutative", that is, the order in which we add the numbers doesn't matter. For the second one, we just have to literally factor c out of all the terms in a sum:

$$\sum_{i=1}^{n} ca_i = ca_1 + \cdots + ca_n = c(a_1 + \cdots + a_n) = c\sum_{i=1}^{n} a_i.$$

As for the last property stated, it might seem a bit unnerving to consider $\displaystyle\sum_{i=1}^{n} 1$ as i doesn't appear in the expression "1" being added. But don't let this throw you off your game: for $i = 1$, we have 1. For $i = 2$, we have 1 to be added to the previous result, so $1+1$. Then for $i=3$, we have 1 yet again to be added to the previous result, so $1+1+1$. Going on and on, we see that $\sum_{i=1}^{n} 1$ is just a contrived way to add 1 to itself n times, so the result is (not surprisingly), n.

Example 4. *Write the sum $\sum_{i=1}^{n}(3a_i - 4b_i + 5)$ in terms of the sums $\sum_{i=1}^{n} a_i$ and $\sum_{i=1}^{n} b_i$.*

Explanation We just have to use all properties just established together:

$$\begin{aligned}\sum_{i=1}^{n}(3a_i - 4b_i + 5) &= \sum_{i=1}^{n}(3a_i) + \sum_{i=1}^{n}(-4b_i) + \sum_{i=1}^{n} 5\\ &= 3\sum_{i=1}^{n} a_i - 4\sum_{i=1}^{n} b_i + 5\sum_{i=1}^{n} 1\\ &= 3\sum_{i=1}^{n} a_i - 4\sum_{i=1}^{n} b_i + 5n.\end{aligned}$$

Arithmetic and geometric sequences

Sum of consecutive integers, squares, and cubes Let's compute the sum $S = 1 + 2 + \cdots + (n-1) + n$, where n is a fixed positive integer. We have already seen how Gauss solved the case $n = 100$ in the introduction to this unit. Repeating his argument, we have that

$$S = n + (n-1) + \cdots + 2 + 1,$$

so twice the sum S equals to a sum of n pairs, which added to $n+1$ each:

$$2S = (1+n) + (2+n-1) + \cdots + (n-1+2) + (n+1),$$

so $2S = n(n+1)$, and so $S = n(n+1)/2$. Understanding how this works with summation notation will help us obtain other formulas of the sort later (for which a direct calculation like the one did here may not be so simple). So, we want to compute the sum

$$S = \sum_{k=1}^{n} k.$$

Start writing

$$\sum_{k=1}^{n}(k+1)^2 = \sum_{k=1}^{n}(k^2 + 2k + 1) = \sum_{k=1}^{n} k^2 + 2\sum_{k=1}^{n} k + \sum_{k=1}^{n} 1.$$

We wish to solve for the second term in the very last right hand side, but we have no idea what is the value of the sum of squares. Note, however, that k on the left side is a dummy index. So let, say, $m = k+1$. Then $m = 2$ when $k = 1$ and $m = n+1$ when $k = n$, meaning that

$$\sum_{k=1}^{n}(k+1)^2 = \sum_{m=2}^{n+1} m^2 = -1 + \sum_{m=1}^{n} m^2 + (n+1)^2.$$

On the summation of the right side, we may rename m back to k, after all, it is a dummy index. Putting everything together leads to

$$-1 + \sum_{k=1}^{n} k^2 + (n+1)^2 = \sum_{k=1}^{n} k^2 + 2S + n,$$

as $\sum_{k=1}^{n} 1 = 1 + \cdots + 1$ (n times) equals n. Now, we may solve for S:

$$\begin{aligned} 2S &= (n+1)^2 - n - 1 \\ &= n^2 + 2n + 1 - n - 1 \\ &= n^2 + n = n(n+1) \end{aligned}$$

Hence, we conclude that:

The sum of the first n integers equals

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}.$$

Example 5. *Evaluate the sum* $\sum_{i=1}^{6}(4i-2)$.

Explanation This time, we have that

$$\begin{aligned} \sum_{i=1}^{6}(4i-2) &= \sum_{i=1}^{6} 4i + \sum_{i=1}^{6}(-2) \\ &= 4\sum_{i=1}^{n} i - 2\sum_{i=1}^{6} 1 \\ &= 4\frac{6(6-1)}{2} - 2 \cdot 6 \\ &= 60 - 12 \\ &= 48. \end{aligned}$$

Exploration Evaluate the sum $\sum_{j=1}^{15}(6j+5)$.

And what about the sum of squares

$$\sum_{k=1}^{n} k^2 = 1^2 + 2^2 + \cdots + (n-1)^2 + n^2,$$

where a positive integer n is fixed, which appeared in the previous example? There, we managed to do an "index substitution" to cancel it, but let's agree that it doesn't feel too great to simply not know what that sum would turn out to be. We will morally repeat the trick used last time, of doing a binomial expansion with a higher power. Namely, we start with

$$\sum_{k=1}^{n}(k+1)^3 = \sum_{k=1}^{n}(k^3+3k^2+3k+1) = \sum_{k=1}^{n} k^3 + 3\sum_{k=1}^{n} k^2 + 3\sum_{k=1}^{n} k + \sum_{k=1}^{n} 1.$$

We're looking for the sum of squares, but a sum of cubes has appeared! And we will eliminate it with the same index substituion as before: let $m = k+1$, so that $m = 2$ when $k = 1$ and $m = n+1$ when $k = n$, leading to

$$\sum_{k=1}^{n}(k+1)^3 = \sum_{m=2}^{n+1} m^3 = -1 + \sum_{m=1}^{n} m^3 + (n+1)^3.$$

Rename back m to k, and use the formula previously obtained for the sum of the first n integers, $\sum_{k=1}^{n} k = n(n+1)/2$, to get

$$-1 + \sum_{m=1}^{n} k^3 + (n+1)^3 = \sum_{k=1}^{n} k^3 + 3\sum_{k=1}^{n} k^2 + 3\frac{n(n+1)}{2} + n.$$

Then the sum of cubes gets cancelled on both sides, so that

$$\begin{aligned}
3\sum_{k=1}^{n} k^2 &= (n+1)^3 - 3\frac{n(n+1)}{2} - n - 1 \\
&= (n+1)^3 - \frac{3n}{2}(n+1) - (n+1) \\
&= (n+1)^3 - \left(\frac{3n}{2}+1\right)(n+1) \\
&= (n+1)^2\,()
\end{aligned}$$

[CORRECT AND FINISH PROPERLY]