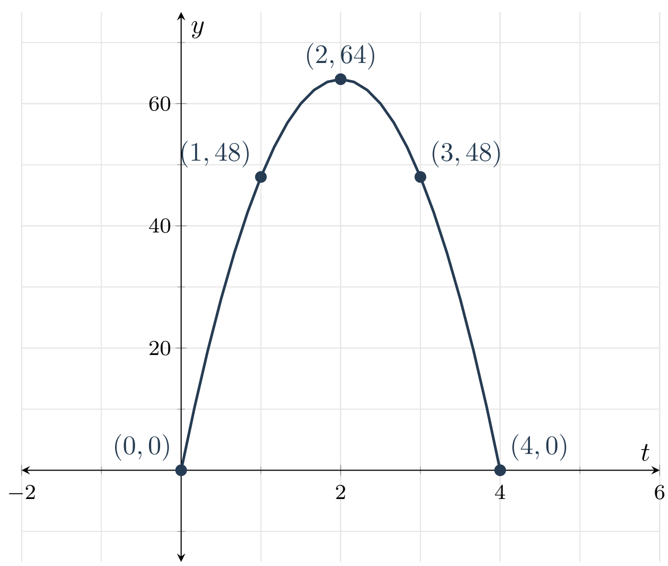
Definition of Quadratics

We explore quadratic functions.

Quadratic Graphs

Example 1. *Hannah fired a toy rocket from the ground, which launched into the air with an initial speed of* 64 *feet per second. The height of the rocket can be modeled by the equation* $y = -16t^2 + 64t$*, where* t *is how many seconds had passed since the launch. To see the shape of the graph made by this equation, we make a table of values and plot the points.*

t	$-16t^2 + 64t$	$Point$
0	$-16(0)^2 + 64(0) = 0$	$(0, 0)$
1	$-16(1)^2 + 64(1) = 48$	$(1, 48)$
2	$-16(2)^2 + 64(2) = 64$	$(2, 64)$
3	$-16(3)^2 + 64(3) = 48$	$(3, 48)$
4	$-16(4)^2 + 64(4) = 0$	$(4, 0)$



A curve with the shape that we see in the above figure is called a **parabola**.

Learning outcomes:
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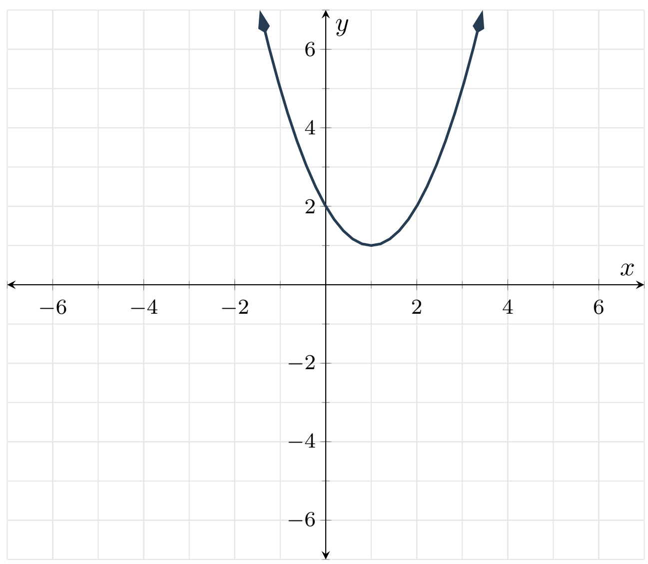
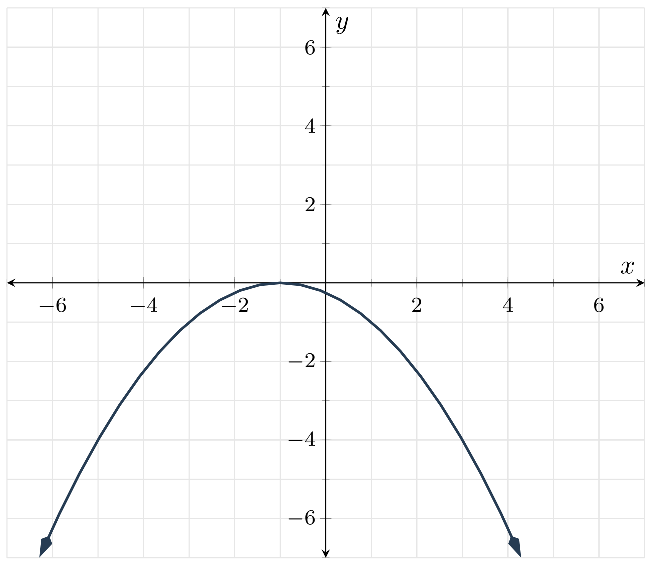
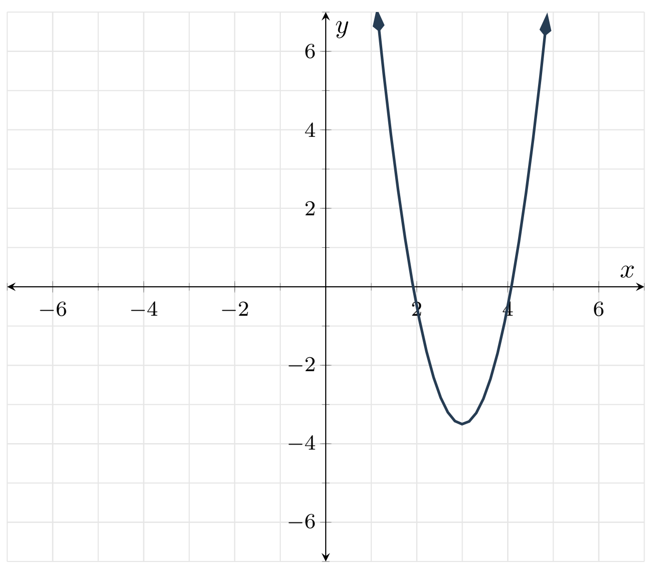
Notice the symmetry in figure, how the y-values in rows above the middle row match those below the middle row. Also notice the symmetry in the shape of the graph, how its left side is a mirror image of its right side.

The first feature that we will talk about is the direction that a parabola opens. All parabolas open either upward or downward. This parabola in the rocket example opens downward because a is negative. That means that for large values of t, the at^2 term will be large and negative, and the resulting y-value will be low on the y-axis. So the negative leading coefficient causes the arms of the parabola to point downward.

Here are some more quadratic graphs so we can see which way they open.







The graph of a quadratic equation $y = ax^2 + bx + c$ is a parabola which opens upward or downward according to the sign of the leading coefficient a. If the leading coefficient is positive, the parabola opens upward. If the leading coefficient is negative, the parabola opens downward.

The **vertex** of a parabola is the highest or lowest point on the graph, depending uponon whether the graph opens downward or upward. In Example 1, the vertex is $(2, 64)$. This tells us that Hannah's rocket reached its maximum height of 64 feet after 2 seconds. If the parabola opens downward, as in the rocket example, then the y-value of the vertex is the **maximum** y-value. If the parabola opens upward then the y-value of the vertex is the **minimum** y-value. The **axis of symmetry** is a vertical line that passes through the vertex, cutting the parabola into two symmetric halves. We write the axis of symmetry as an equation of a vertical line so it always starts with "$x =$". In Example 1, the equation for the axis of symmetry is $x = 2$.

The **vertical intercept** is the point where the parabola crosses the vertical axis. The vertical intercept is the y-intercept if the vertical axis is labeled y. In Example 1, the point $(0, 0)$ is the starting point of the rocket, and it is where the graph crosses the y-axis, so it is the vertical intercept. The y-value of 0 means the rocket was on the ground when the t-value was 0, which was when the rocket launched.

The **horizontal intercept(s)** are the points where the parabola crosses the horizontal axis. They are the x-intercepts if the horizontal axis is labeled x. The point $(0, 0)$ on the path of the rocket is also a horizontal intercept. The t-value of 0 indicates the time when the rocket was launched from the ground. There is another horizontal intercept at the point $(4, 0)$, which means the rocket came back to hit the ground after 4 seconds.

It is possible for a quadratic graph to have zero, one, or two horizontal intercepts. The figures below show an example of each.

Example 2. *Use technology to graph and make a table of the quadratic function f defined by $f(x) = 2x^2 + 4x - 3$ and find each of the key points or features.*

(a) *Find the vertex.*

(b) *Find the vertical intercept (i.e. the y-intercept).*

(c) *Find the horizontal or (i.e. the x-intercept(s)).*

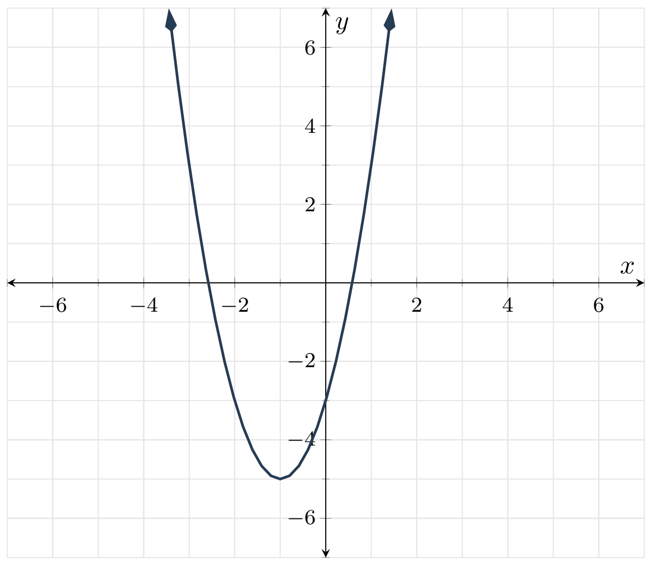
(d) *Find $f(-2)$.*

(e) *Solve $f(x) = 3$ using the graph.*

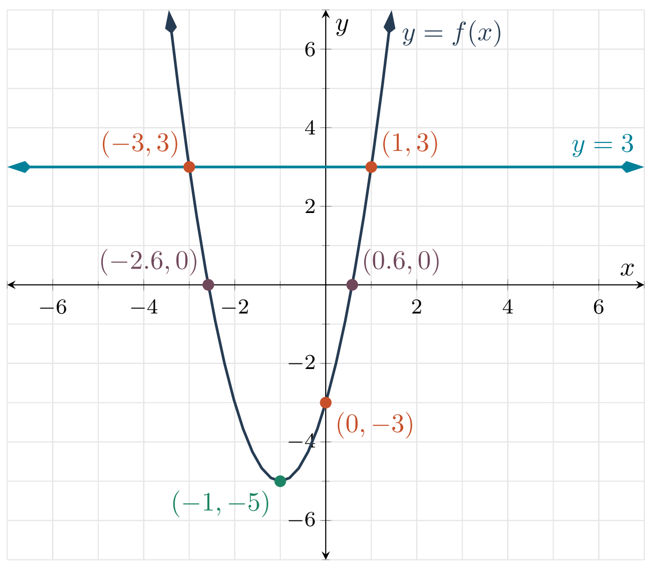
(f) *Solve $f(x) \leq 3$ using the graph.*

Explanation The specifics of how to use any one particular technology tool vary. Whether you use an app, a physical calculator, or something else, a table and graph should look like:

x	$f(x)$
-2	-3
-1	-5
0	-3
1	3
2	13



Additional features of your technology tool can enhance the graph to help answer these questions. You may be able to make the graph appear like:



(a) The vertex is $(-1, -5)$.

(b) The vertical intercept is $(0, -3)$.

(c) The horizontal intercepts are approximately $(-2.6, 0)$ and $(0.6, 0)$.

(d) When $x = -2$, $y = -3$, so $f(-2) = -3$.

(e) The solutions to $f(x) = 3$ are the x-values where $y = 3$. We graph the horizontal line $y = 3$ and find the x-values where the graphs intersect. The solution set is $\{-3, 1\}$.

(f) The solutions are all of the x-values where the function's graph is below (or touching) the line $y = 3$. The interval is $[-3, 1]$.

A polynomal is a particular type of algebraic expression

(a) A company's sales, s (in millions of dollars), can be modeled by $2.2t + 5.8$, where t stands for the number of years since 2010.

(b) The height of an object from the ground, h (in feet), launched upward from the top of a building can be modeled by $-16t^2 + 32t + 300$, where t represents the amount of time (in seconds) since the launch.

(c) The volume of an open-top box with a square base, V (in cubic inches), can be calculated by $30s^2 - \frac{1}{2}s^2$, where s stands for the length of the square base, and the box sides have to be cut from a certain square piece of metal.

Polynomial Vocabulary

A polynomial is an expression with one or more terms summed together. A term of a polynomial must either be a plain number or the product of a number and one or more variables raised to natural number powers. The expression 0 is also considered a polynomial, with zero terms.

Example 3. *Here are some examples of polynomials*

(a) *Here are three polynomials:* $x^2 - 5x + 2$, $t^3 - 1$, $7y$.

(b) *The expression* $3x^4y^3 + 7xy^2 - 12xy$ *is an example of a polynomial in more than one variable.*

(c) *The polynomial* $x^2 - 5x + 3$ *has three terms:* x^2, $-5x$, *and* 3.

(d) *The polynomial* $3x^4 + 7xy^2 - 12xy$ *also has three terms.*

(e) *The polynomial* $t^3 - 1$ *has two terms.*

Definition The coefficient (or numerical coefficient) of a term in a polynomial is the numerical factor in the term.

Example 4. (a) *The coefficient of the term* $\frac{4}{3}x^6$ *is* $\frac{4}{3}$.

(b) *The coefficient of the second term of the polynomial* $x^2 - 5x + 3$ *is* -5.

(c) *The coefficient of the term* $\frac{y^7}{4}$ *is* $\frac{1}{4}$, *because we can rewrite* $\frac{y^7}{4}$ *as* $\frac{1}{4}y^7$.

A term in a polynomial with no variable factor is called a constant term.

Example 5. *The constant term of the polynomial* $x^2 - 5x + 3$ *is* 3.

> **Definition** The degree of a term is one way to measure how large it is. When a term only has one variable, its degree is the exponent on that variable. When a term has more than one variable, its degree is the sum of the exponents on the variables. A nonzero constant term has degree 0.

Example 6. (a) *The degree of* $5x^2$ *is* 2.

(b) *The degree of* $-\frac{4}{7}y^5$ *is* 5.

(c) *The degree of* $-4x^2y^3$ *is* 5.

(d) *The degree of* 17 *is* 0. *Constant terms always have* 0 *degree.*

> **Definition** The **degree** of a nonzero polynomial is the greatest degree that appears amongst its terms

> **Remark** To help us recognize a polynomial's degree, the standard convention at this level is to write a polynomial's terms in order from highest degree to lowest degree. When a polynomial is written in this order, it is written in standard form. For example, it is standard practice to write $7 - 4x - x^2$ as $-x^2 - 4x + 7$ since $-x^2$ is the leading term. By writing the polynomial in standard form, we can look at the first term to determine both the polynomial's degree and leading term.

Adding and Subtracting Polynomials

Bayani started a company that makes one product: one-gallon ketchup jugs for industrial kitchens. The company's production expenses only come from two things: supplies and labor. The cost of supplies, S (in thousands of dollars), can be modeled by $S = 0.05x^2 + 2x + 30$, where x is number of thousands of jugs of ketchup produced. The labor cost for his employees, L (in thousands of dollars), can be modeled by $0.1x^2 + 4x$, where x again represents the number of jugs they produce (in thousands of jugs). Find a model for the company's total production costs.

Evaluating Polynomial Expressions

Recall that evaluating expressions involves replacing the variable(s) in an expression with specific numbers and calculating the result. Here, we will look at evaluating polynomial expressions.

Example 7. *Evaluate the expression*

$$-12y^3 + 4y^2 - 9y + 2 \textit{ for } y = -5$$

Explanation We will replace y with -5 and simplify the result:

$$\begin{aligned} 12y^3 + 4y^2 - 9y + 2 &= -12(-5)^3 + 4(-5)^2 - 9(-5) + 2 \\ &= -12(-125) + 4(25) + 45 + 2 \\ &= 1647 \end{aligned}$$

