Precalculus with Review 2

January 17, 2023

Contents

Part 1

Variables and CoVariation - See Unit 1 PDF

Part 2

Comparing Lines and Exponentials - See Unit 2 PDF

Part 3

Functions - See Unit 3 PDF

Part 4

Building New Functions - See Unit 4 PDF

Part 5

Exponential Functions Revisited - See Unit 5 PDF

Part 6

Rational Functions - See Unit 6 PDF

Part 7

Analyzing Functions

7.1 Composition of Functions

Learning Objectives

- Composition of Functions
  - What does it mean to compose functions?
  - Identify a function as the result of a composition of functions
- Domains of Composite Functions
  - How to find the domain of a composite function
  - How to find the range of a composite function
  - Results of composing $f(x) = x^2$ and $g(x) = \sqrt{x}$

7.1.1 Famous Functions, Updated

Now that we know about the domain and range, we can update our list of famous functions.

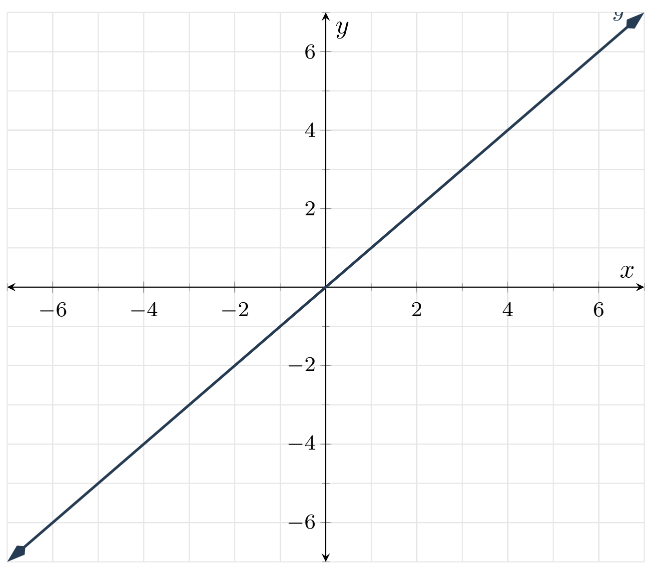
In Section 1-2, you saw a variety of famous functions. Now that we have learned more about properties of functions, we can update our knowledge of those famous functions. We will go through the list of famous functions from before and point out where each function might have properties we've discussed.

Linear Functions

Recall that the graph of a linear function is a line.

Example 1. *A prototypical example of a linear function is*

$$y = x.$$



Important Values of $y = x$

x	y
-2	-2
-1	-1
0	0
1	1
2	2

In general, linear functions can be written as $y = mx + b$ where m and b can be any numbers. We learned that m represents the slope, and b is the y-coordinate of the y-intercept. You can play with changing the values of m and b on the graph using Desmos and see how that changes the line.

Desmos link: `https://www.desmos.com/calculator/japnhapzvn`

Note that a linear function f defined by $f(x) = mx + b$ with $b = 0$ is odd. If $m = 0$, then f is periodic, since it is constant. Furthermore, constant functions are always even.

Additionally, if $m \neq 0$, then a linear function is one-to-one, and therefore invertible. We summarize this information in the table below.

Note that any real number can be plugged into $f(x) = mx + b$, so the domain of linear functions is $(-\infty, \infty)$. Unless $m = 0$, we can find a y such that $y = mx+b$, so the range of linear functions with $m \neq 0$ is $(-\infty, \infty)$. If $m = 0$, then the only output of the linear function is b, so its range is $\{b\}$.

Properties of Linear Functions

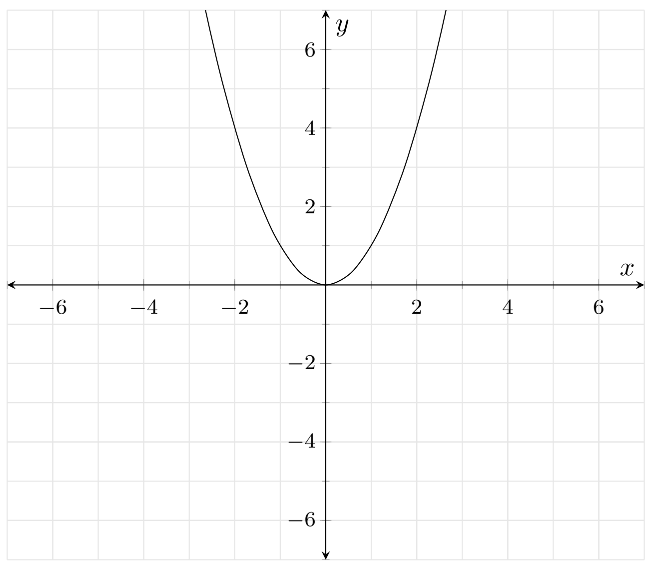
Periodic?	If $m = 0$
Odd?	If $b = 0$
Even?	If $m = 0$
One-to-one/invertible?	Yes
Domain	$(-\infty, \infty)$
Range	$(-\infty, \infty)$

Quadratic Functions

Recall that the graph of a quadratic function is a parabola.

Example 2. *A prototypical example of a quadratic function is*

$$y = x^2.$$



Important Values of $y = x^2$

x	y
-2	4
-1	1
0	0
1	1
2	4

In general, quadratic functions can be written as $y = ax^2 + bx + c$ where a, b, and c can be any numbers. You can play with changing the values of a, b, and c on the graph using Desmos and see how that changes the parabola.

Desmos link: https://www.desmos.com/calculator/nmlghfrws9

Note that for a quadratic function f defined by $f(x) = ax^2 + bx + c$, if $b = 0$, then f is even. In general, quadratic functions are not one-to-one, odd, or periodic, except in cases where $a = 0$, in which we're actually dealing with a linear function.

Note that any real number can be plugged into $f(x) = ax^2+bx+c$, so the domain of quadratic functions is $(-\infty, \infty)$. In Chapter 4, we saw that all quadratic functions have a vertex form $f(x) = a(x-h)^2 + k$, where the vertex is at (h, k). If $a > 0$, all points above the vertex, that is $[k, \infty)$ are in the range of the quadratic, and if $a < 0$, all points below the vertex, that is $(\infty, k]$ are in the range of the quadratic.

We summarize this information in the table below.

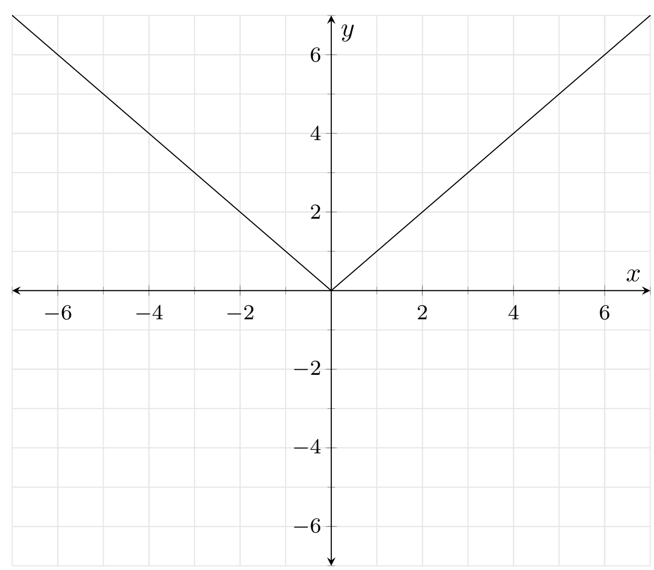
Properties of Quadratic Functions $y = ax^2 + bx + c, a \neq 0$

Periodic?	No
Odd?	No
Even?	If $b = 0$
One-to-one/invertible?	No
Domain	$(-\infty, \infty)$
Range	If $a > 0$, $[k, \infty)$, if $a < 0$, $(-\infty, k]$

Absolute Value Function

Another important type of function is the absolute value function. This is the function that takes all y-values and makes them positive. The absolute value function is written as

$$y = |x|.$$



Important Values of $y = |x|$

x	y
-2	2
-1	1
0	0
1	1
2	2

Notice that the absolute value function is even. Is it one-to-one? The fact that it's even tells us that it is not, since $|-x| = |x|$ for all x. We summarize this information in the table below.

Note that any real number has an absolute value, so the domain of the absolute value function is $(-\infty, \infty)$. Furthermore, by looking at the graph, we can see that all non-negative numbers are in the range of the absolute value function.

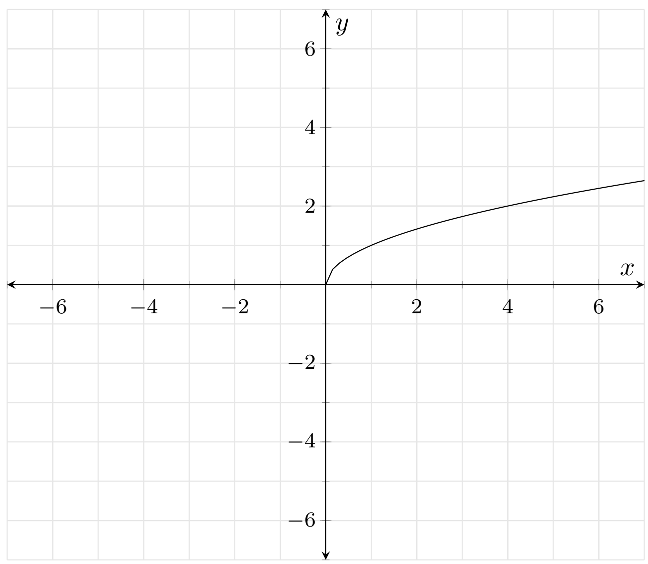
Properties of the Absolute Value Function $y = |x|$

Periodic?	No
Odd?	No
Even?	Yes
One-to-one/invertible?	No
Domain	$(-\infty, \infty)$
Range	$[0, \infty)$

Square Root Function

Another famous function is the square root function,

$$y = \sqrt{x}.$$



Important Values of $y = \sqrt{x}$

x	y
0	0
1	1
4	2
9	3
25	5

The square root function is one-to-one. Negative inputs are not valid for the square root function, so it is neither even, odd, nor periodic. We summarize this information in the table below.

Note that only non-negative numbers have square roots, so the domain of the square root function is $[0, \infty)$. Furthermore, by looking at the graph, we can see that all non-negative numbers are in the range of the square root function. Algebraically, we can say that for any non-negative y, $\sqrt{(y^2)} = y$, so y is in the range of the square root function.

Properties of the Square Root Function $y = \sqrt{x}$

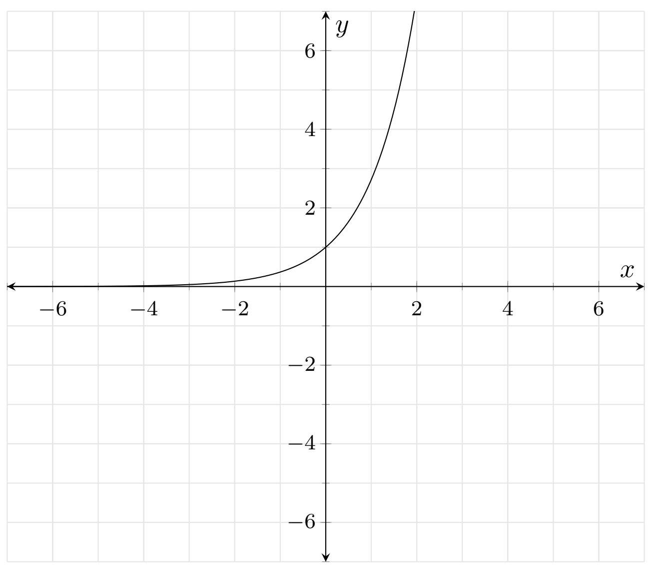
Periodic?	No
Odd?	No
Even?	No
One-to-one/invertible?	Yes
Domain	$[0, \infty)$
Range	$[0, \infty)$

Exponential Functions

Another famous function is the exponential growth function,

$$y = e^x.$$

Here e is the mathematical constant known as Euler's number. $e \approx 2.71828..$



Important Values of $y = e^x$

x	y
0	1
1	e
-1	$e^{-1} = \frac{1}{e}$

In general, we can talk about exponential functions of the form $y = b^x$ where b is a positive number not equal to 1. You can play with changing the values of b on the graph using Desmos and see how that changes the graph. Pay particular attention to the difference between $b > 1$ and $0 < b < 1$.

Desmos link: https://www.desmos.com/calculator/qsmvb7tiex

Notice that exponential functions are one-to-one, and therefore invertible. However, they are neither even, odd, nor periodic.

Note that the domain of the exponential functions is $(-\infty, \infty)$. Furthermore, by looking at the graph, we can see that all non-negative numbers are in the range of the exponential functions.

We summarize this information in the table below.

Properties of the Exponential Functions $y = b^x$

Periodic?	No
Odd?	No
Even?	No
One-to-one/invertible?	Yes
Domain	$(-\infty, \infty)$
Range	$(0, \infty)$

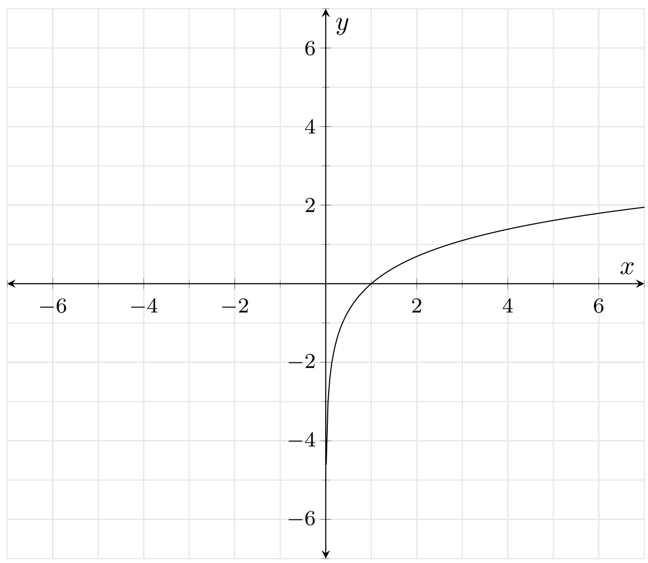
Logarithm Functions

Another group of famous functions are logarithms.

Example 3. *The most famous logarithm function is*

$$y = \ln(x) = \log_e(x).$$

Here e is the mathematical constant known as Euler's number. $e \approx 2.71828$.



Important Values of $y = \ln(x)$

x	y
0	*undefined*
$\frac{1}{e}$	-1
1	0
e	1

In general, we can talk about logarithmic functions of the form $y = \log_b(x)$ where b is a positive number not equal to 1. You can play with changing the values of b on the graph using Desmos and see how that changes the graph. Pay particular attention to the difference between $b > 1$ and $0 < b < 1$.

Desmos link: https://www.desmos.com/calculator/lxllnpdi6w

Notice that logarithms are neither even, odd, nor periodic. However, they are one-to-one, and therefore invertible. It turns out that the inverse of a logarithm is an exponential function, and vice versa!

Note that since the logarithm is the inverse of the exponential, the domain of the logarithms is the range of the exponentials: $[0, \infty)$. Furthermore, the range of the logarithms is the range of the exponentials: $(-\infty, \infty)$.

We summarize this information in the table below.

Properties of the Logarithm Functions $y = \log_b(x)$

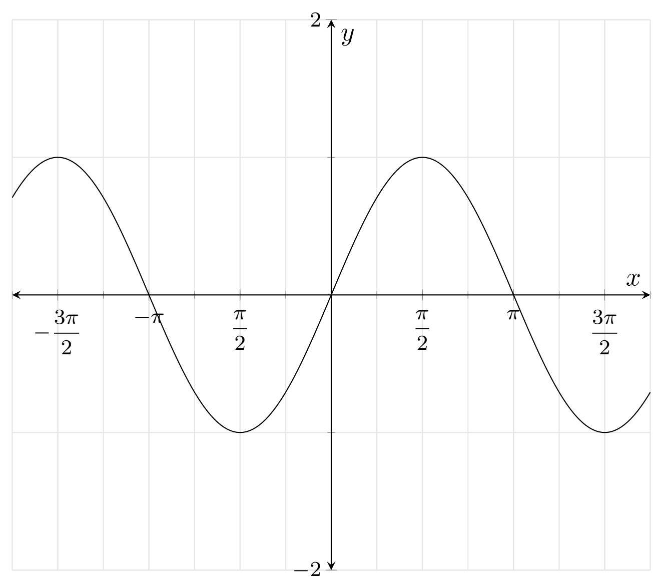
Periodic?	No
Odd?	No
Even?	No
One-to-one/invertible?	Yes
Domain	$(0, \infty)$
Range	$(-\infty, \infty)$

Sine

Another important function is the sine function,

$$y = \sin(x).$$

This function comes from trigonometry. In the table below we will use another mathematical constant, π ("pi" pronounced pie). $\pi \approx 3.14159$.



Important Values of $y = \sin(x)$

x	y
$-\pi$	0
$-\dfrac{\pi}{2}$	-1
0	0
$\dfrac{\pi}{2}$	1
π	0
$\dfrac{3\pi}{2}$	-1
2π	0

As mentioned earlier, the sine function is odd and periodic with period 2π. Since it is periodic, however, it cannot be one-to-one, since its values repeat.

Note that the domain of the sine function is $(-\infty, \infty)$. Furthermore, by looking at the graph, we can see that its range is $[-1, 1]$.

We summarize this information in the table below.

Properties of the Sine Function $y = \sin(x)$

Periodic?	Yes, with period 2π
Odd?	Yes
Even?	No
One-to-one/invertible?	No
Domain	$(-\infty, \infty)$
Range	$[-1, 1]$

In general, we can consider $y = a\sin(bx)$. You can play with changing the values of a and b on the graph using Desmos and see how that changes the graph.

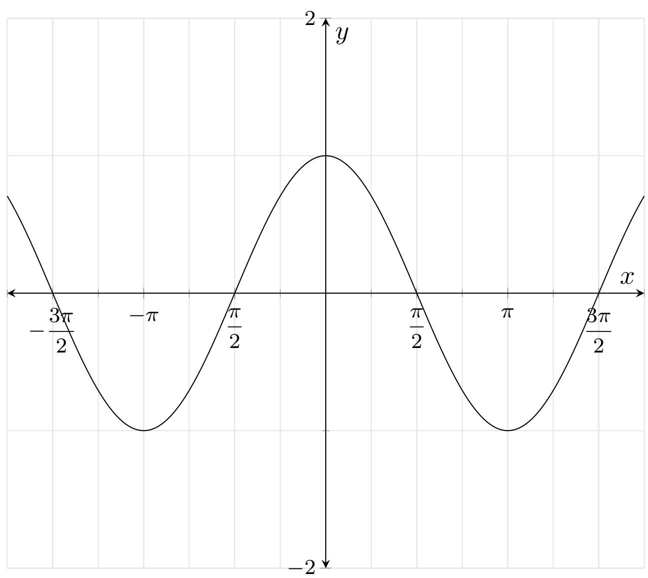
Desmos link: `https://www.desmos.com/calculator/vkxzcfv2aq`

Cosine

A function introduced in Section 3-2 is the cosine function,

$$y = \cos(x).$$

As with sine, the cosine function comes from trigonometry. In the table below we will again use π.



Important Values of $y = \cos(x)$

x	y
$-\pi$	-1
$-\dfrac{\pi}{2}$	0
0	1
$\dfrac{\pi}{2}$	0
π	-1
$\dfrac{3\pi}{2}$	0
2π	1

As mentioned earlier, the cosine function is even and periodic with period 2π. Since it is periodic, however, it cannot be one-to-one, since its values repeat.

Note that the domain of the cosine function is $(-\infty, \infty)$. Furthermore, by looking at the graph, we can see that its range is $[-1, 1]$.

We summarize some information in the table below.

Properties of the Cosine Function $y = \cos(x)$

Periodic?	Yes, with period 2π
Odd?	No
Even?	Yes
One-to-one/invertible?	No
Domain	$(-\infty, \infty)$
Range	$[-1, 1]$

In general, we can consider $y = a\cos(bx)$. You can play with changing the values of a and b on the graph using Desmos and see how that changes the graph.

Desmos link: `https://www.desmos.com/calculator/kvmz1kt19n`

7.1.2 Composition of Functions

Motivating Questions

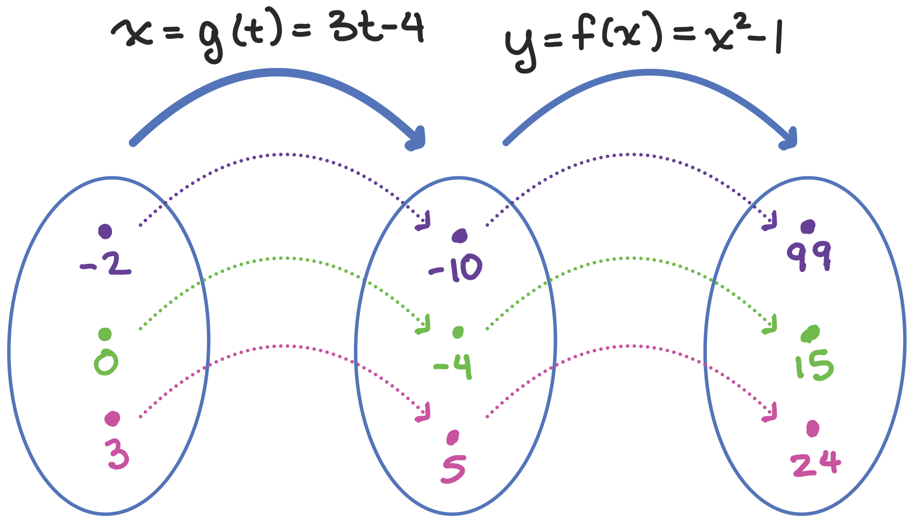
- How does the process of function composition produce a new function from two other functions?
- In the composite function $h(x) = f(g(x))$, what do we mean by the "inner" and "outer" function?
- How does the expression for $\text{AROC}_{[a,a+h]}$ involve a composite function?

Introduction

Recall that a function, by definition, is a process that takes a collection of inputs and produces a corresponding collection of outputs in such a way that the process produces one and only one output value for any single input value. Because every function is a process, it makes sense to think that it may be possible to take two function processes and do one of the processes first, and then apply the second process to the result.

Example 4. *Suppose we know that y is a function of x according to the process defined by $y = f(x) = x^2 - 1$ and, in turn, x is a function of t via $x = g(t) = 3t - 4$. Is it possible to combine these processes to generate a new function so that y is a function of t?*

Explanation Since y depends on x and x depends on t, it follows that we can also think of y depending directly on t. Let's looks at this as an arrow diagram with a few sample points.



Notice that if we take a point such as $t = 2$, we can put that value in for t in the function $x = g(t) = 3t - 4$. This will give

$$x = g(-2) = 3(-2) - 4 = -6 - 4 - 10.$$

Now we have an x-value of -10. $g(x)$ takes in x-values so we can put -10 into $f(x) = x^2 - 1$. This will give

$$f(-10) = (-10)^2 - 1 = 100 - 1 = 99.$$

You should verify that the arrow diagram above gives the correct values of y that corresponds to $t = 0$ and $t = 3$.

Now, we would like to create a new function that will directly take in any t value and give us the corresponding y value. We can use substitution and the notation of functions to determine this function.

First, it's important to realize what the rule for f tells us. In words, f says "to generate the output that corresponds to an input, take the input and square it, and then subtract 1." In symbols, we might express f more generally by writing "$f(\square) = \square^2 - 1$."

Now, observing that $y = f(x) = x^2 - 1$ and that $x = g(t) = 3t - 4$, we can substitute the expression $g(t)$ for x in f. Doing so,

$$\begin{aligned} y &= f(x) \\ &= f(g(t)) \\ &= f(3t-4). \end{aligned}$$

Applying the process defined by the function f to the input $3t - 4$, we see that

$$y = (3t - 4)^2 - 1,$$

which defines y as a function of t.

One way to think about the substitution above is that we are putting the entire expression $3t-4$ inside the input box in "$f(\square) = \square^2 - 1$." That is, $f\left(\boxed{3t-4}\right) = \left(\boxed{3t-4}\right)^2 - 1$. For the substitution, we are thinking of $3t-4$ as a single object!

When we have a situation such as in the example above where we use the output of one function as the input of another, we often say that we have **composed** two functions. In addition, we use the notation $h(t) = f(g(t))$ to denote that a new function, h, results from composing the two functions f and g.

Exploration

a. Let $y = p(x) = 3x - 4$ and $x = q(t) = t^2$. Determine a formula for r that depends only on t and not on p or q. What is the biggest difference between your work in this problem compared to the example above?

b. Let $t = s(z) = \dfrac{1}{t+4}$ and recall that $x = q(t) = t^2$. Determine a formula for $x = q(s(z))$ that depends only on z.

c. Suppose that $h(t) = \sqrt{2t^2 + 5}$. Determine formulas for two related functions, $y = f(x)$ and $x = g(t)$, so that $h(t) = f(g(t))$.

Composing Two Functions

Whenever we have two functions, g and f, where the outputs of g match inputs of f, it is possible to link the two processes together to create a new process that we call the *composition* of f and g.

Definition If f and g are functions, we define the **composition of** f **and** g to be the new function h given by

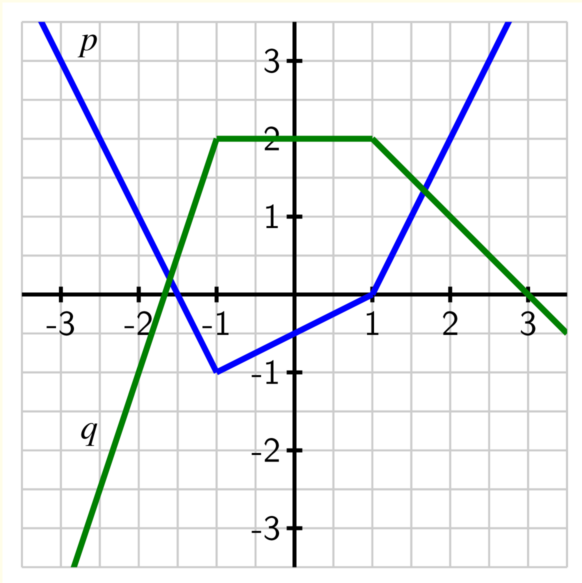
$$h(t) = f(g(t)).$$

This composition is denoted by $h = f \circ g$, where $f \circ g$ means the single function defined by $(f \circ g)(t) = f(g(t))$.

We sometimes call g the "inner function" and f the "outer function". It is important to note that the inner function is actually the first function that gets applied to a given input, and then the outer function is applied to the output of the inner function. In addition, in order for a composite function to make sense, we need to ensure that the outputs of the inner function are values that it makes sense to put into the outer function so that the resulting composite function is defined.

In addition to the possibility that functions are given by formulas, functions can be given by tables or graphs. We can think about composite functions in these settings as well, and the following activities prompt us to consider functions given in this way.

Exploration Let functions p and q be given by the graphs below (which are each piecewise linear - that is, parts that look like straight lines are straight lines) and let f and g be given by the table below.



x	$f(x)$	$g(x)$
0	6	1
1	4	3
2	3	0
3	4	4
4	6	2

Compute each of the following quantities or explain why they are not defined.

a. $p(q(0))$

b. $q(p(0))$

c. $(p \circ p)(-1)$

d. $(f \circ g)(2)$

e. $(g \circ f)(3)$

f. $g(f(0))$

g. For what value(s) of x is $f(g(x)) = 4$?

h. For what value(s) of x is $q(p(x)) = 1$?

Composing functions in content

In the late 1800s, the physicist Amos Dolbear was listening to crickets chirp and noticed a pattern: how frequently the crickets chirped seemed to be connected to the outside temperature. If we let T represent the temperature in degrees Fahrenheit and N the number of chirps per minute, we can summarize Dolbear's observations with the following function, $T = D(N) = 40 + 0.25T$. Scientists who made many additional cricket chirp observations following Dolbear's initial counts found that this formauls holds with remarkable accuracy for the snowy tree cricket in temperatures ranging from 50° to 85°. This function is called Dolbear's Law.

In what follows, we replace T with F to emphasize that temperature is measured in Fahrenheit degrees.

The Celsius and Fahrenheit temperature scales are connected by a linear function. Indeed, the function that converts Fahrenheit to Celsius is

$$C = G(F) = \frac{5}{9}(F - 32).$$

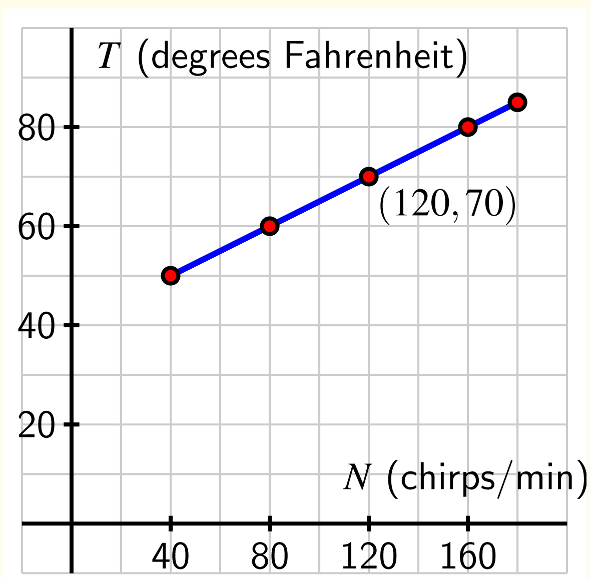
For instance, a Fahrenheit temperature of 32 degrees corresponds to $C = G(32) = \frac{5}{9}(32 - 32) = 0$ degrees Celsius.

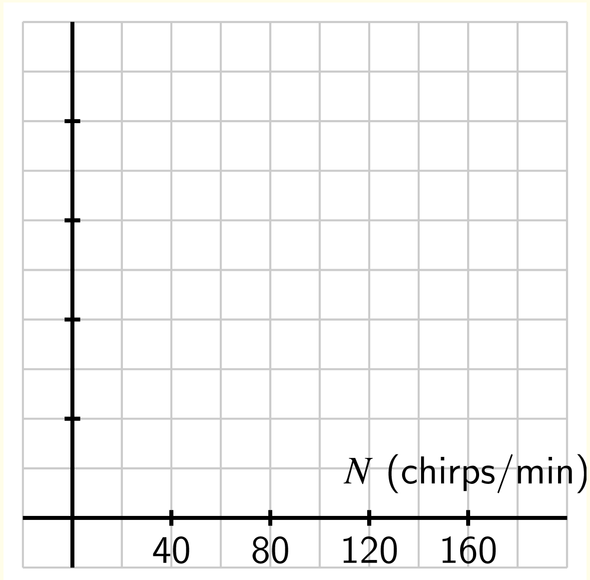
Exploration Let $D(N) = 40 + 0.25N$ be Dolbear's function that converts an input of number of chirps per minute to degrees Fahrenheit, and let $G(F) = \frac{5}{9}(F - 32)$ be the function that converts an input of degrees Fahrenheit to an output of degrees Celsius.

a. Determine a formula for the new function $(G \circ D)(N)$ that depends only on the variable N.

b. What is the meaning of the function you found in (a)?

c. Let $H = G \circ D$. How does a plot of the function H compare to the that of Dolbear's function? Sketch a plot of H on the blank axes to the right of the plot of Dolbear's function, and discuss the similarities and differences between them. Be sure to label the vertical scale on your axes.



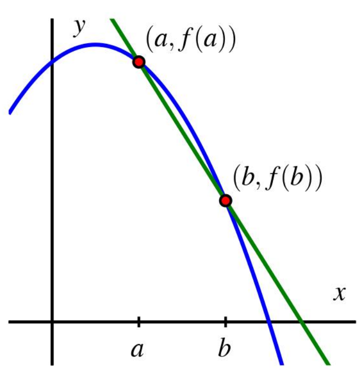


Function Composition and Average Rate of Change

Recall that the average rate of change of a function f on the interval $[a, b]$ is given by

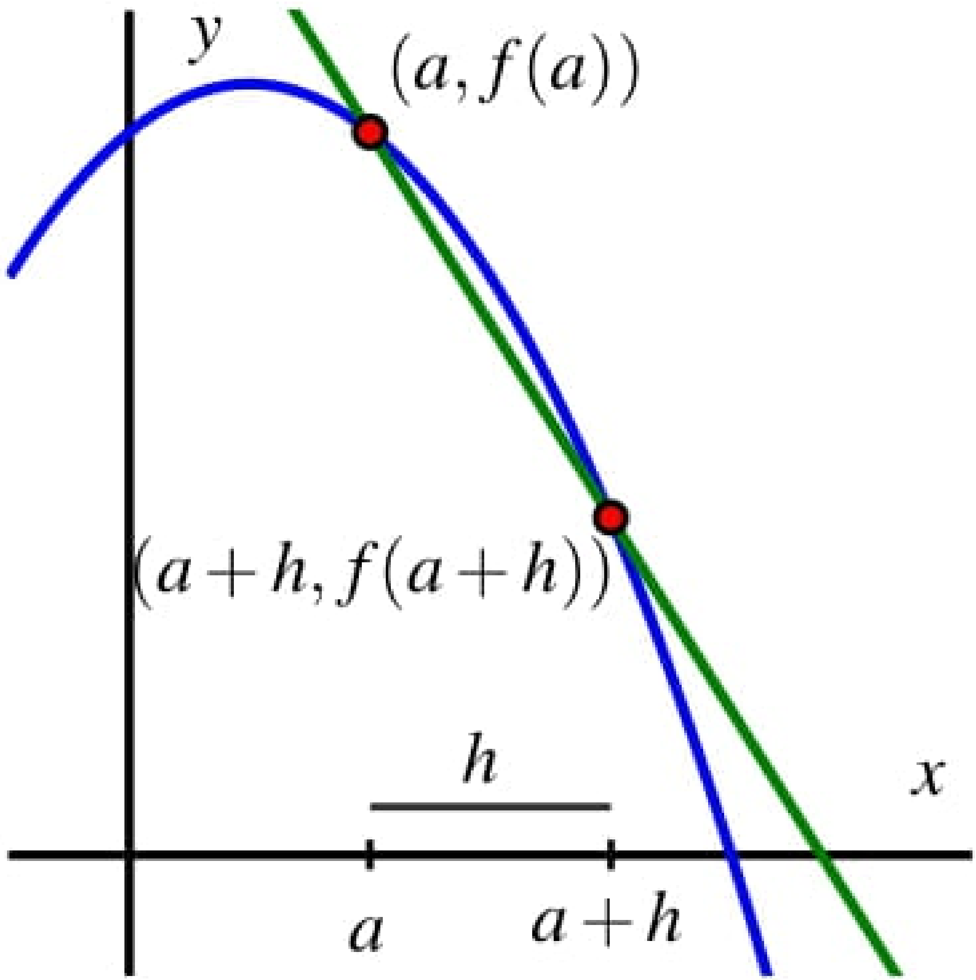
$$\mathrm{AROC}_{[a,b]} = \frac{f(b) - f(a)}{b - a}.$$

In the graph below, we see the familiar representation of $\mathrm{AROC}_{[a,b]}$ as the slope of the line joining the points $(a, f(a))$ and $(b, f(b))$ on the graph of f.



In the study of calculus, we progress from the *average rate of change on an interval* to the *instantaneous rate of change of a function at a single value*; the core idea that allows us to move from an *average* rate to an *instantaneous* one is letting the interval $[a, b]$ shrink in size.

To think about the interval $[a, b]$ shrinking while a stays fixed, we often change our perspective and think of b as $b = a + h$, where h measures the horizontal difference from b to a.



This allows us to eventually think about h getting closer and closer to 0, and in that context we consider the equivalent expression

$$\text{AROC}_{[a,a+h]} = \frac{f(a+h) - f(a)}{a+h-a} = \frac{f(a+h) - f(a)}{h}$$

for the average rate of change of f on $[a, a+h]$.

Example 5. *Suppose that* $f(x) = x^2$. *Determine the simplest possible expression you can find for* $\text{AROC}_{[3,3+h]}$*, the average rate of change of* f *on the interval* $[3, 3+h]$.

Explanation By definition, we know that

$$\text{AROC}_{[3,3+h]} = \frac{f(3+h) - f(3)}{h}.$$

Using the formula for f, we see that

$$\text{AROC}_{[3,3+h]} = \frac{(3+h)^2 - (3)^2}{h}.$$

Expanding the numerator and combining like terms, it follows that

$$\begin{aligned} \text{AROC}_{[3,3+h]} &= \frac{(9 + 6h + h^2) - 9}{h} \\ &= \frac{6h + h^2}{h}. \end{aligned}$$

Removing a factor of h in the numerator and observing that $h \neq 0$, we can simplify and find that

$$\begin{aligned} \text{AROC}_{[3,3+h]} &= \frac{h(6+h)}{h} \\ &= 6 + h. \end{aligned}$$

Hence, $\text{AROC}_{[3,3+h]} = 6 + h$, which is the average rate of change of $f(x) = x^2$ on the interval $[3, 3+h]$.

Exploration Let $f(x) = 2x^2 - 3x + 1$ and $g(x) = \dfrac{5}{x}$.

a. Compute $f(1+h)$ and expand and simplify the result as much as possible by combining like terms.

b. Determine the most simplified expression you can for the average rate of change of f on the interval $[1, 1+h]$. That is, determine $\text{AROC}_{[1,1+h]}$ for f and simplify the result as much as possible.

c. Compute $g(1+h)$. Is there any valid algebra you can do to write $g(1+h)$ more simply?

d. Determine the most simplified expression you can for the average rate of change of g on the interval $[1, 1+h]$. That is, determine $\text{AROC}_{[1,1+h]}$ for g and simplify the result.

Summary

- When defined, the composition of two functions f and g produces a single new function $f \circ g$ according to the rule $(f \circ g)(x) = f(g(x))$. We note that g is applied first to the input x, and then f is applied to the output $g(x)$ that results from g.

- In the composite function $h(x) = f(g(x))$, the "inner" function is g and the *outer* function is f. Note that the inner function gets applied to x first, even though the outer function appears first when we read from left to right.

- Because the expression $\text{AROC}_{[a,a+h]}$ is defined by

$$\text{AROC}_{[a,a+h]} = \frac{f(a+h) - f(a)}{h}$$

  and this includes the quantity $f(a+h)$, the average rate of change of a function on the interval $[a, a+h]$ always involves the evaluation of a composite function expression. This idea plays a crucial role in the study of calculus.

7.1.3 Domains and Ranges of Composite Functions

Motivating Questions

- How does the process of function composition affect the domain of the function?
- How does the process of function composition affect the range of the function?

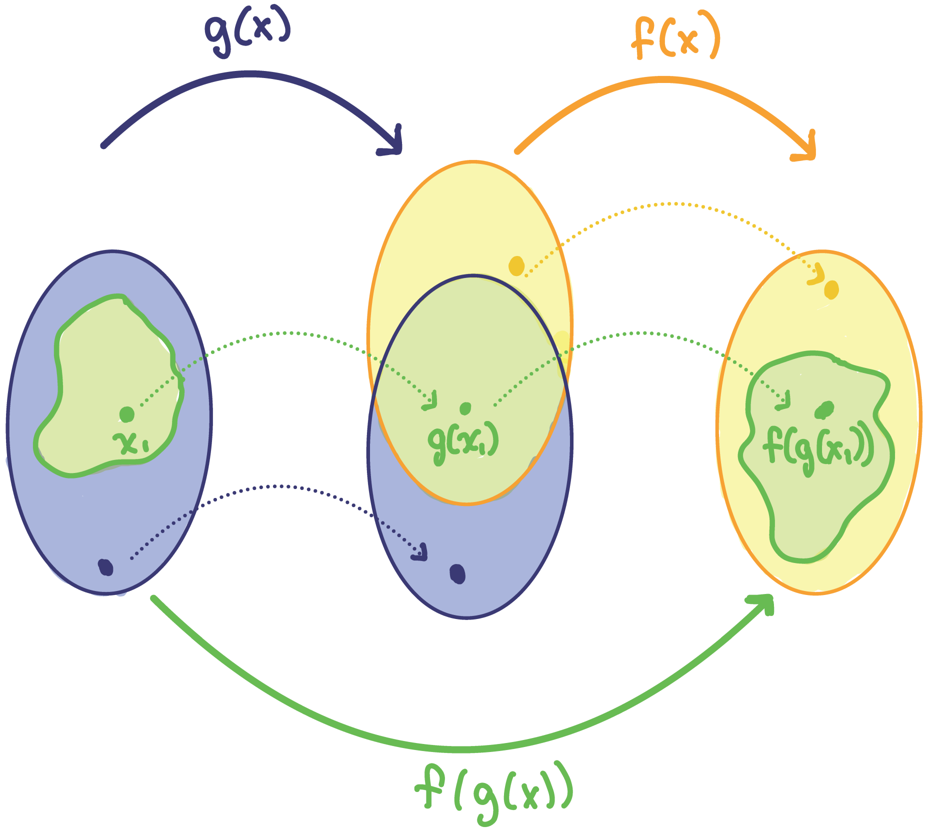
Domains of Composite Functions

The domain of a composite function, such as $f \circ g$, is dependent on the domain of g and the domain of f. The domain of $f \circ g$ is important because it tells us when we can apply a composite function and when we cannot.

Let us assume we know the domains of the functions f and g separately. We can write the composite function $f \circ g$ for an input x as $f(g(x))$. Using the figure in Remark 2 below, we can see that x must be a member of the domain of g in order for the expression to be meaningful, because otherwise we cannot complete the inner function evaluation. However, we also see that $g(x)$ must be a member of the domain of f, otherwise the second function evaluation in $f(g(x))$ cannot be completed, and the expression is still undefined. Thus the domain of $f \circ g$ consists of only those inputs in the domain of g that produce outputs from g belonging to the domain of f. Note that the domain of f composed with g is the set of all x such that x is in the domain of g and $g(x)$ is in the domain of f.

The domain of a composite function $f(g(x))$ is the set of those inputs x in the domain of g for which $g(x)$ is in the domain of f.

Acknowledgements: OpenStax College Algebra (https://openstax.org/details/books/college-algebra)



To find the domain of a composite function, $f \circ g$, you can follow these three steps:

1) Find the domain of g.

2) Find the domain of f.

3) Find those inputs x in the domain of g for which $g(x)$ is in the domain of f. That is, exclude those inputs x from the domain of g for which $g(x)$ is not in the domain of f. The resulting set is the domain of $f \circ g$.

Example 6. *Find the domain of $f \circ g$ where $f(x) = \dfrac{5}{x-1}$ and $g(x) = \dfrac{4}{3x-2}$.*

Explanation The domain of g consists of all real numbers except $x = \dfrac{2}{3}$, since that input value would cause us to divide by 0. Likewise, the domain of f consists of all real numbers except 1. We need to exclude from the domain of g any value of x for which $g(x) = 1$.

Acknowledgements: OpenStax College Algebra (https://openstax.org/details/books/college-algebra)

$$
\begin{aligned}
\frac{4}{3x-2} &= 1 \\
4 &= 3x-2 \\
6 &= 3x \\
x &= 2
\end{aligned}
$$

So the domain of $f \circ g$ is the set of all real numbers except $\frac{2}{3}$ and 2. This means that

$$x \neq \frac{2}{3} \text{ or } x \neq 2$$

We can write this in interval notation as

$$\left(-\infty, \frac{2}{3}\right) \cup \left(\frac{2}{3}, 2\right) \cup (2, \infty)$$

Example 7. *Find the domain of $f \circ g$ where $f(x) = \sqrt{x+2}$ and $g(x) = \sqrt{3-x}$.*

Explanation

Because we cannot take the square root of a negative number, the domain of g is $(-\infty, 3]$. Now we check the domain of the composite function

$$(f \circ g)(x) = \sqrt{\sqrt{3-x}+2}$$

For $(f \circ g)(x) = \sqrt{\sqrt{3-x}+2}$, we need $\sqrt{3-x}+2 \geq 0$, since the inside of a square root cannot be negative. Since square roots are non-negative, $\sqrt{3-x} \geq 0$ and $\sqrt{3-x}+2 \geq 0$ as long as $\sqrt{3-x}$ exists. That means $3-x \geq 0$, which gives a domain of $(-\infty, 3]$.

This example shows that knowledge of the range of functions (specifically the inner function, which in this case the range is $\sqrt{3-x} \geq 0$) can also be helpful in finding the domain of a composite function. It also shows that the domain of $f \circ g$ can contain values that are not in the domain of f, though they must be in the domain of g. In this example, the domain of f is $(-2, \infty)$ but the domain of $f \circ g$ is $(-\infty, 3]$.

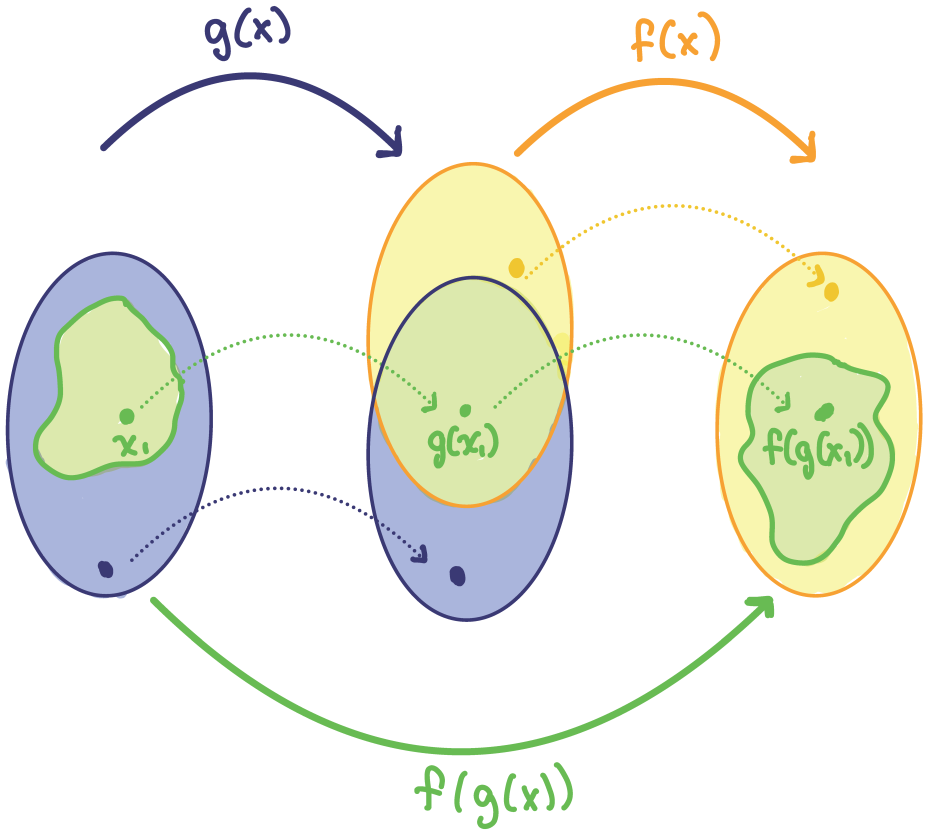
Exploration Find the domain of $f \circ g$ where $f(x) = \frac{1}{x-2}$ and $g(x) = \sqrt{x+4}$.

 Acknowledgements: OpenStax College Algebra (https://openstax.org/details/books/college-algebra)

Ranges of Composite Functions

The range of a composite function such as $f \circ g$ is dependent on the range of g and the range of f. It is important to know what values can result from a composite function, that is, to know the range of a function such as $f \circ g$.

Let us assume we know the ranges of the functions f and g separately. If we write the composite function for an input x as $f(g(x))$, we can see that $f(g(x))$ must be a member of the range of f since we will input the value $g(x)$ into f. However, we also see that it is possible that not all values in the range of f are in the range of $f(g(x))$.



From the image above, we can see that there might be values in the yellow region which are in the range of f but for which there are no x values for which $f(g(x))$ gives that output.

> The range of a composite function $f \circ g$ is a subset of the range of f.

Acknowledgements: OpenStax College Algebra (https://openstax.org/details/books/college-algebra)

To find the range of a composite function, $f \circ g$, you can follow these three steps:

1) Find the range of g.
2) Find the range of f.
3) Restrict the domain of f to the *range* of g and then determine the outputs of f of these values.

Example 8. *Find the range of* $f \circ g$ *where* $f(x) = \sqrt{x+2}$ *and* $g(x) = \sqrt{3-x}$.

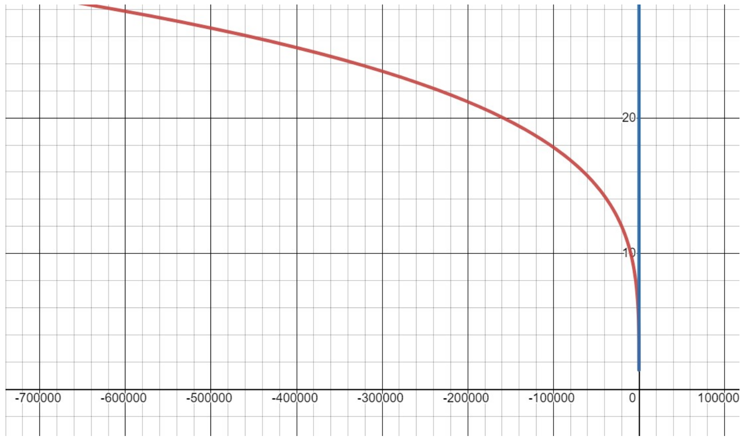
Explanation Because the output of a square root is always a positive number, the range of g is $[0, \infty)$. Similarly, the range of f is $[0, \infty)$. But now we must think about what happens when we restrict the input of f to values in the range of g, $[0, \infty)$. If $x \geq 0$, then $x + 2 \geq 2$. Taking the square root of both sides, we see that possible outputs of $f(g(x))$ will be $\sqrt{x+2} \geq \sqrt{2}$. That is, the range of $f \circ g$ is $[\sqrt{2}, \infty)$.

If we look at this function in Desmos, we can confirm graphically that this answer makes sense. What we want to do is think about collapsing the graph unto the y-axis. The range of the function will be the y-values that correspond to a point (x, y) on the curve.

First, we graph the function using a standard window.

This allows us to see the domain pretty well. In the previous example, we found the domain to be $(-\infty, 3]$ and if we collapse this function to the x-axis, it looks like the x-values that correspond to points on this curve are exactly the x in $(-\infty, 3]$. If might be difficult to tell the domain from this graph, though. Let's zoom out some.

Acknowledgements: OpenStax College Algebra (https://openstax.org/details/books/college-algebra)



Here is the same graph in Desmos, so you can zoom in and out yourself.

Desmos link: `https://www.desmos.com/calculator/0wf1e4yyhf`

You can now see that the blue line is showing this graph collapsed to the y-axis. We can tell that the range will be positive numbers above some value between 1 and 2. This corresponds with our result above of $[\sqrt{2}, \infty)$. In order to find the exact point $\sqrt{2}$ where the interval begins or to confirm that the interval really goes to infinity, we need to do the reasoning above.

Summary

- For a composite function $f \circ g$ to be defined, we need outputs of g to be among the allowed inputs for f. In particular, if the range of g is a subset of the domain of f, we can say that if $g : A \to B$ and $f : B \to C$, then $f \circ g : A \to C$. In this case, the domain of the composite function is the domain of the inner function, and the range of the composite function is the codomain of the outer function.
- In general, the domain of a composite function $f \circ g$ is the set of those inputs x in the domain of g for which $g(x)$ is in the domain of f.
- In general, the range of a composite function $f \circ g$ is a subset of the range of f.

7.2 What are the Zeros of Functions?

Learning Objectives

- Zeros of Functions
  - Definition of Zeros
  - Compare and contrast zeros, solutions, roots, and x-intercepts
  - Examples of why we might want to find zeros
  - Identify a zero on a graph
  - Computing the zero of a function (early examples)
- The Importance of Equals
  - Compare and contrast expressions, equations, and functions
  - Appreciate the importance of using the equals sign appropriately

7.2.1 Zeros of Functions

Motivating Questions

- What does it mean to find the zero of a function?
- What are other terms used for zeros of functions?
- Why might we want to find the zero of a function?

Introduction

In this section, we will study zeros of functions. Let's start with a classic example of when we might want to find the zero of a function.

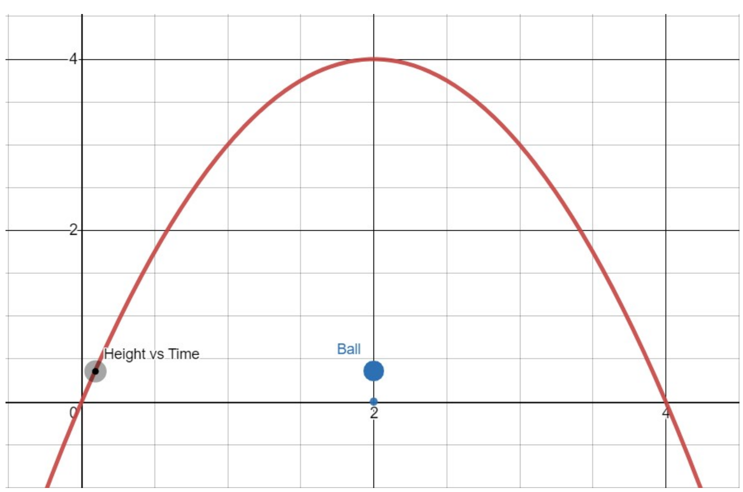
Example 9. *A ball is thrown straight upward from the ground. The distance the ball is from the ground is given by the function $f(x) = 4-(x-2)^2$ where x is the time measured in seconds and $f(x)$ is the distance from the ground measured in feet. What time will the ball hit the ground?*

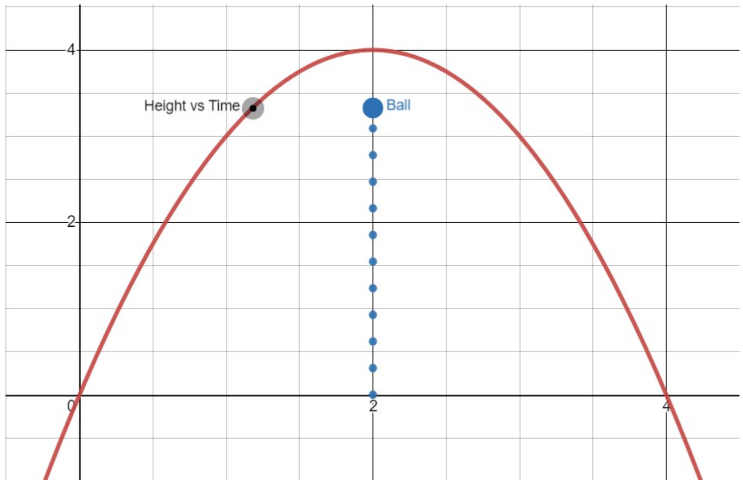
Explanation First, it is important to understand what this problem is saying. Consider this model of the situation. Click the play button next to the a to see an animation of the ball being thrown up.

Desmos link: https://www.desmos.com/calculator/f9d7ngsznm

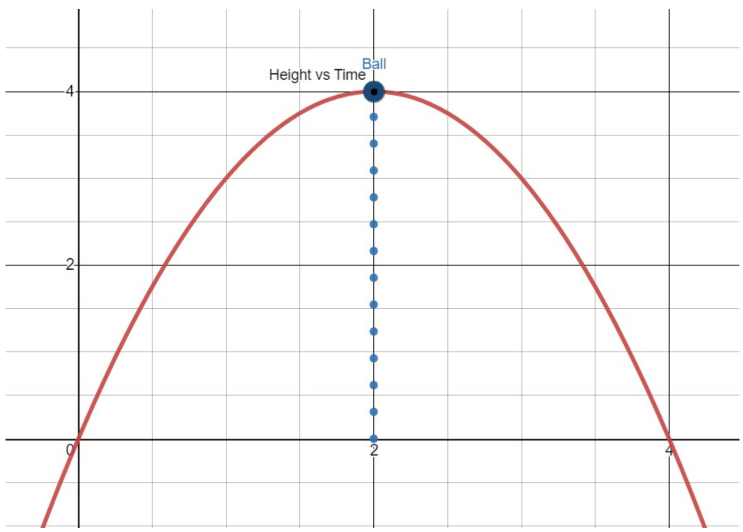
Notice that while the ball goes straight up and down, the graph of the distance from the ground vs. time makes an upside down parabola.

As the ball goes up, we see:

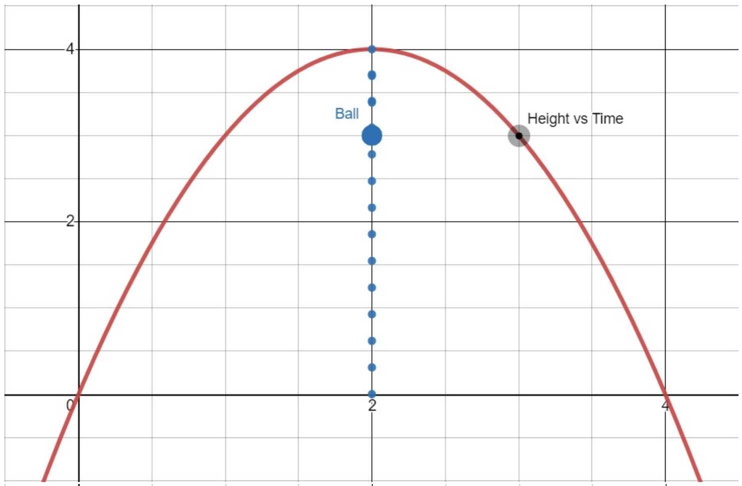


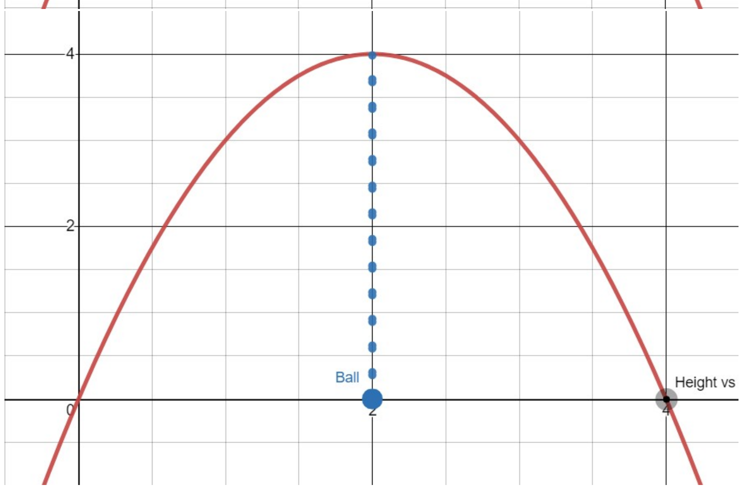


At the ball's highest point, we have:



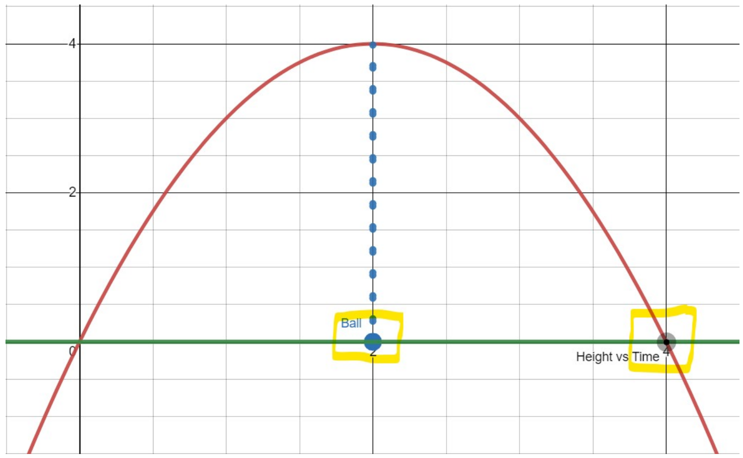
And as the ball goes down, we have:





It is also important to note that in this situation, we know this model only makes sense from the time we throw the ball to the time it hits the ground. Having a negative distance would correspond with the ball going underground, which is not what we want to model. That means our domain will be from $x = 0$ to the x-value where the ball hits the ground.

This means, once again, we are back to wanting to know when the ball will hit the ground. Looking at the graph, we can focus on the time when the ball seems to hit the ground.

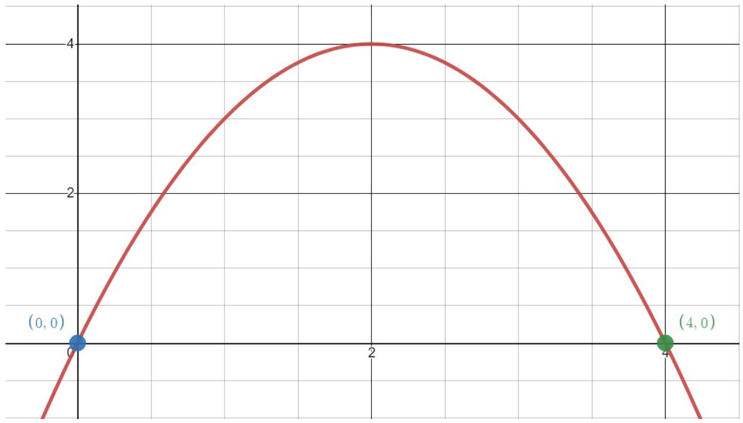


Notice that at the time the ball hits the ground, the function giving the distance vs. time graph, $f(x) = 4 - (x-2)^2$, is crossing the x-axis. This means we are looking for **the x-value of the x-intercept**. That is, we are looking for when this function equals 0. We call an x-value where a function equals zero the **zero of a function**. It can also be called the **root of a function** when the function is a polynomial. Occasionally, people will also call it the **solution of a function** but techincally they should say **the solution of** $f(x) = 0$.

Let's formalize this with a definition.

> **Definition** We call an x-value where a function equals zero the **zero of a function**. It can also be called the **root of a function** when the function is a polynomial. Another equivalent expression is **the solution of** $f(x) = 0$. Another label for the same value is **the x-coordinate of the x-intercept**.

Finding Zeros Graphically One method for finding zeros or roots of functions is to read them off the graph. Since the zero is the x-coordinate of the x-intercept, we are looking for the places where the graph crosses the x-axis. This is where the y-value or output will be zero. In the image below, the two roots are colored blue and green.



The root $x = 0$ cooresponds to the blue dot at $(0, 0)$ and represents when the ball is first thrown. The second root, labeled in green, is where $x = 4$. This root is the one we are looking for. The time when the ball hits the ground is 4 seconds after the ball is thrown.

Finding Zeros Algebraically Reading zeros graphically can be useful, but it is not very precise. The root in our example could actually be at $x = 3.99$ and we would not know from the graph. When possible, it is best to find zeros algebraically. To do this, we want to set $f(x) = 0$ and solve for x. This algebra can get tricky. In this section, we will stick to relatively straightforward examples and in the next couple sections we will explore some more involved methods for solving equations where one side equals 0. It is not always possible to solve for the zeros of a function algebraically in this manner. In calculus, you will also learn methods to approximate zeros when it is not possible to solve for them exactly.

In our current example, we will have:

$$\begin{aligned} f(x) &= 0 \\ 4 - (x-2)^2 &= 0 \\ 4 &= (x-2)^2 \\ \sqrt{4} &= \sqrt{(x-2)^2} \end{aligned}$$

Recall that $\sqrt{x^2} = |x|$ from the section on domains and ranges of composite functions.

Continuing, we have:

$$\begin{aligned}\sqrt{4} &= |x-2| \\ \pm 2 &= x-2 \\ 2 \pm 2 &= x\end{aligned}$$

That is, the zeros or roots of $f(x) = 4 - (x-2)^2$ are:

$$x = 2 - 2 = 0 \text{ or } x = 2 + 2 = 4$$

Considering our problem in context, we know that the root at time $x = 0$ is when the ball was initially thrown, so the root at $x = 4$ must be when the ball hits the ground.

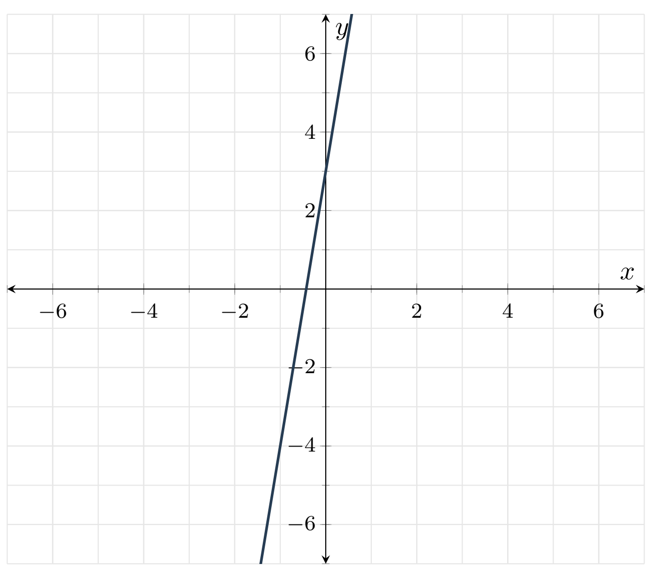
Let's look at a couple more examples:

Example 10. *Find the zeros of the linear function* $f(x) = 7x + 3$.

Explanation We need to set $f(x) = 7x + 3 = 0$

$$\begin{aligned}7x + 3 &= 0 \\ 7x &= -3 \\ x &= \tfrac{-3}{7}\end{aligned}$$

This linear function only has one zero. This should make sense if we think about the graph of a line, as a line will only have one x-intercept.



In fact, all linear functions except constant functions will have one zero. Constant functions will have no zeros except for the linear function $x = 0$ in which every point is a zero.

Example 11. *Write $g(x) = |7x+3|$ as a piecewise function and find its zeros.*
Explanation Recall that the absolute value makes all the outputs positive. Multiplying a negative number by -1 makes the value positive. Therefore, $g(x) = |7x+3|$ is a piecewise function where

$$g(x) = |7x+3| = \begin{cases} 7x+3 & \text{if } 7x+3 \geq 0 \\ -(7x+3) & \text{if } 7x+3 < 0 \end{cases}$$

To simplify this expression, notice that we need to know when $7x+3 = 0$. This is what we did in the last example. We know this happens at $x = \frac{-3}{7}$. Now, we also need to know when $f(x) = 7x+3$ is positive and when it is negative.

Since we have the graph of the function above, we could look at it and see that $7x+4 > 0$ when $x > \frac{-3}{7}$ and $7x+4 < 0$ when $x < \frac{-3}{7}$.

Alternatively, if we wanted to figure this out algebraically (without the graph), we can plug values into the function $f(x) = 7x+3$ on either side of $x = \frac{-3}{7}$. This will work because we know a property about lines. We know that it cannot switch from positive to negative without being equal to 0 in the middle. You'll learn about this later in the course.

Let's choose to look at $x = 0$ as a representative of values of $x > \frac{-3}{7}$ and $x = -1$ as a representive of values for $x < \frac{-3}{7}$.

$$f(0) = 7(0) + 3 = 3 > 0$$

This means that $f(x) = 7x+3 > 0$ when $x > \frac{-3}{7}$.

$$f(-1) = 7(-1) + 3 = -4 < 0$$

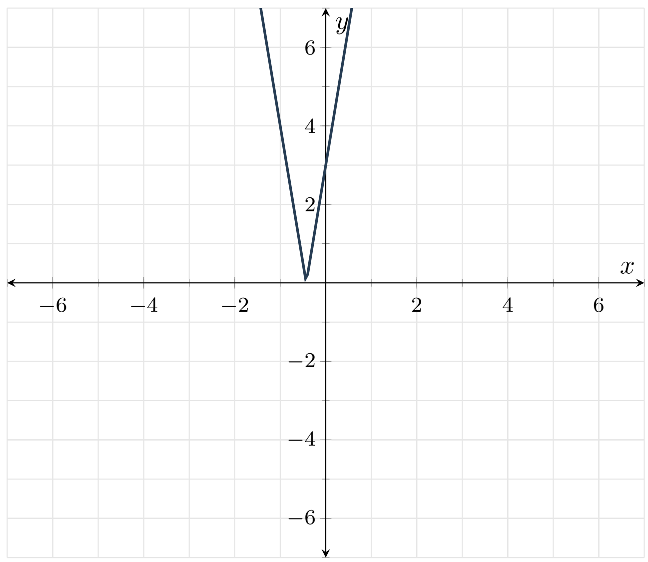
This means that $f(x) = 7x+3 < 0$ when $x < \frac{-3}{7}$.

Putting this all together, we can now write $g(x) = |7x+3|$ as a piecewise function:

$$g(x) = |7x+3| = \begin{cases} 7x+3 & \text{if } x \geq \frac{-3}{7} \\ -(7x+3) & \text{if } x < \frac{-3}{7} \end{cases}$$

Now we can find the zeros of this function. We know that an absolute value function is only zero where it switches from the positive branch to the negative branch of the piecewise function so the zero is when $x = \frac{-3}{7}$.

Let's look at the graph of this function to verify that this makes sense.



By looking at this graph, we can see that the zero, that is the x-intercept on $g(x)s = |7x + 3|$ is the same as on $f(x) = 7x + 3$.

Summary

- We call an x-value where a function equals zero the **zero of a function**. It can also be called the **root of a function** if the function is a polynomial. Another equivalent expression is **the solution of** $f(x) = 0$. Another label for the same value is **the x-coordinate of the x-intercept**.
- You can find this value graphically by looking for the x-value where the function crosses the x-axis.
- You can find this value algebraically by setting $f(x) = 0$ and then solving for x. This can sometimes be difficult (or even impossible!) to do.

7.2.2 The Importance of the Equals Sign

Motivating Questions

- What are some similarities and differences between mathematical expressions, equations, and functions?
- How is solving an equation related to finding the zero of a function?
- When should we and when shouldn't we use an equals sign?

Introduction

Now that we are exploring the zeros of functions, one issue that often comes up for students (and for teachers reading students' work!) is when you should and should not use an equals sign. We are going to review when it is and is not ok to use an equals sign.

First, it is helpful to review a few important terms. In mathematics, it is important to use these and other terms precisely so that you are communicating clearly and saying what you intend to say. Speaking precisely using mathematical terms can be difficult to learn and takes some practice!

Expressions

Definition An **algebraic expression** is any combination of variables and numbers using arithmetic operations such as addition, subtraction, multiplication, division, and exponentiation.

Here are some examples of algebraic expressions:

$$5x^2 - 17 \qquad \frac{56x}{\sqrt{17x}} \qquad 2x + 3y + 7z$$

The important thing to notice is that there are no equals signs in an expression. There are also no inequality signs.

Definition An **mathematical expression**, or just an **expression**, is similar to an algebraic expression, but can contain other mathematical objects such as $\sin(x)$ or $\ln(x)$ or similar objects that you will learn about in future classes. In particular, it does not contain an equals sign or an inequality sign.

Here are some examples of mathematical expressions:

$$\frac{\sin(x)}{\cos(x)} \qquad 5x + \ln(x) - 12 \qquad 2x + 3y + 7z$$

Every algebraic expression is also a mathematical expression.

> **Definition** **Evaluating an expression** is substituting in a particular value for the variable in a mathematical expression.

Here is an example of evaluating an expression. Consider the expression $5x^2 - 17$. Let's evaluate that expression at $x = 1$.

$$5(1)^2 - 17 = 5 - 17 = -12$$

Notice that when evaluating this expression at a particular point, we can use an equals sign. This is a good use of the equals sign and shows us simplifying. But, we should not put an equals sign between $5x^2 - 17$ and $5(1)^2 - 17$ as these two expressions are only equal when $x = 1$.

Equations

When we use an equals sign to say that two different mathematical expressions give the same value, we are creating an equation.

> **Definition** An **equation** is a statement that two mathematical expressions are equal.

Here are some examples of equations:

$$5x^2 - 17 = -12 \qquad \frac{56x}{\sqrt{17x}} = 12x \qquad 2x + 3y + 7z = \frac{x}{y+z}$$

When we are given an equation in a problem, we often want to know what value of the variable will make the equation true. That is, what value of the variable will make both sides give the same value.

> **Definition** **Solving an equation** is the process of determining precisely what value of a variable makes the equation true.

Here is an example of solving an equation. Let's solve $5x^2 - 17 = -12$.

$$\begin{aligned} 5x^2 - 17 &= -12 \\ 5x^2 &= 5 \\ x^2 &= 1 \\ x = 1 &\text{ or } x = -1 \end{aligned}$$

Notice that this is the reverse process of evaluating the expression $5x^2 - 17$. When evaluating the expression, we knew the x-value and substituted it in. When solving the equation, we knew what the output should be and had to find the x-value that would produce that output. In fact, we found two such values!

> **Remark** Notice that when solving an equation, we don't put equals between the steps. This is very important. In many cases, the steps are not equal!

This is a key observation. Notice that if we wrote

$$5x^2 - 17 = -12 = 5x^2 = 5 = ...,$$

we would be saying something not true. In particular, we would be claiming that $-12 = 5$!

The best thing is to do when solving an equation is to make a new line for each step, but if you need to write your steps on a single line, you can use an arrow to show the next step. For example, we could write

$$5x^2 - 17 = -12 \to 5x^2 = 5 \to x^2 = 1 \to x = 1 \text{ or } x = -1.$$

We could also use connecting words between equations. For example, we could write:

$$5x^2 - 17 = -12 \text{ so } 5x^2 = 5 \text{ thus } x^2 = 1 \text{ therefore } x = 1 \text{ or } x = -1.$$

Zeros of Functions Revisited

Notice that when we are working with functions, we are also working with equations and expressions.

- When we write f, we are referencing the function by name.
- When we write $f(x)$, this is an expression for the output of the function at x.
- When we write $f(x) = 5x^2 - 17$, we are defining the way the function produces outputs.

- When we want to find the zeros of this function, we set up the equation $f(x) = 0$. In our case, this would mean solving the equation $5x^2 - 17 = 0$.

$$\begin{aligned} 5x^2 - 17 &= 0 \\ 5x^2 &= 17 \\ x^2 &= \frac{17}{5} \\ x = \sqrt{\frac{17}{5}} &\text{ or } x = -\sqrt{\frac{17}{5}} \end{aligned}$$

  Notice that this is solving an equation so we do not write equals signs between the steps.

Another important connection between finding zeros of functions and solving equations is that every equation can be thought of as the zero of a function. Consider the following example.

Example 12. *Rewrite the equation* $5x + 7 = 6 - x^2$ *as the zero of a function. You do not need to find the zero.*

Explanation In order to rewrite this problem so that solving this equation is equivalent to finding the zero of a function, we want to move all the terms to the same side and combine like terms. For our example, this means

$$\begin{aligned} 5x + 7 &= 6 - x^2 \\ 5x + 7 - (6 - x^2) &= 0 \\ -6x^2 + 5x + 1 &= 0 \end{aligned}$$

Now we let

$$f(x) = -6x^2 + 5x + 1$$

Now, the x values which are zeros of f will be the same x-values that solve $5x + 7 = 6 - x^2$. We will learn to find zeros of quadratic equations in the the next section.

Summary

- You should not write an equals sign between two things which do not have the same value. Equals does not mean "next step"! Instead, to indicate a next step, you may use an arrow, a new line, or connecting words like "so" and"thus".
- Every equation can be thought of as the zero of a function by moving all the terms to one side and then defining a function to be the output of that side.

7.2.3 Finding Zeros

Motivating Questions

- How can we find the zeros of functions?

Introduction

In this section, we will explore how to find zeros of more complicated functions.

Finding Zeros of Compositions

Example 13. *Find the zeros of the function* $f(x) = \ln(x^2 - 8)$.

Explanation To start, we need to recognize that this is a composition of two functions, so we will be drawing on our knowledge of two different types of functions. The inner function is defined by $g(x) = x^2 - 7$, and the outer function is the natural logarithm.

Since the outer function is the last function applied, to find the zeros of f, we must find the zeros of ln. Recall that ln has only one zero, when $x = 1$, since $\ln(1) = 0$.

We might be tempted to stop here and say that $x = 1$ is our zero, but we have to remember we're working with a composition of functions. We need to find the x such that $\ln(x^2 - 8) = 0$, not the x such that $\ln(x) = 0$. However, the work we've done is helpful: we know that plugging 1 into ln gives 0, so we need to find the numbers we can plug into $x^2 - 8$ to get 1, since we're plugging $x^2 - 8$ into ln. To do this, we set $x^2 - 8 = 1$ and solve:

$$\begin{aligned} x^2 - 8 &= 1 \\ x^2 &= 9 \\ x &= \pm 3 \end{aligned}$$

Therefore, the zeros of f are -3 and 3.

Let's look closer at what we've found. Recall that to be a zero, we need $f(x) = 0$. Let's check that -3 is in fact a zero.

$$f(-3) = \ln((-3)^2 - 8) = \ln(9 - 8) = \ln(1) = 0,$$

which is what we wanted.

Example 14. *Write* $f(x) = e^{\sqrt{x}-2} - 3$ *as a composition of functions and find its zeros.* **Explanation** We can write $f(x) = (g \circ h)(x)$, where $g(x) = e^x - 3$ and $h(x) = \sqrt{x} - 2$.

To find the zeros of f, we could set $f(x) = 0$ and solve, but we'll show another approach using the composition. First, we find the zeros of the outer function g. To do this, we set $g(x) = 0$ and solve. This gives us

$$\begin{aligned} e^x - 3 &= 0 \\ e^x &= 3 \\ x &= \ln(3). \end{aligned}$$

Next, since $f(x) = g(h(x))$, to find the zeros of f, we need to find when $h(x) = \ln(3)$. This gives us

$$\begin{aligned} \sqrt{x} - 2 &= \ln(3) \\ \sqrt{x} &= \ln(3) + 2 \\ x &= (\ln(3) + 2)^2, \end{aligned}$$

which is our zero.

Example 15. *Find the zeros of* $f(x) = (\log_2(x))^2 - 4$. **Explanation** First, we decompose f as $f(x) = g(h(x))$, where $g(x) = x^2 - 4$ and $h(x) = \log_2(x)$.

Next, we can set $g(x) = 0$ and solve:

$$\begin{aligned} x^2 - 4 &= 0 \\ x^2 &= 4 \\ x &= \pm 2. \end{aligned}$$

Now, we'll set $h(x) = -2$ and $h(x) = 2$.

$$\begin{aligned} \log_2(x) &= -2 \\ x &= 2^{-2} \\ x &= \frac{1}{4} \end{aligned}$$

and

$$\begin{aligned} \log_2(x) &= 2 \\ x &= 2^2 \\ x &= 4, \end{aligned}$$

so our zeros are $\frac{1}{4}$ and 4.

To find the zeros of compositions of functions, we can find the zeros of the outer function, then, for each zero we've found, find the input to the inner function whose output is that zero.

7.3 Function Transformations

Learning Objectives

- Vertical and Horizontal Shifts
  - How to shift a function vertically
  - How to shift a function horizontally
  - Combining shifts and properties of quadratics (vertex, completing the square)
- Stretching Functions
  - Vertical stretch
  - Horizontal stretch
- Reflections of Functions
  - Reflections across the x-axis, the y-axis, the origin, $y = x$
  - Connect reflections to inverses, even, and odd functions

7.3.1 Function Transformations Revisited

Introduction

In this section, we will review our earlier discussion about function transformations. In addition, we'll explore what happens when multiple function transformations occur.

Review of Single Transformations

The following table gives the formulas and descriptions of all the function transformations we learned about. If f is the parent function, then the formula in the left column gives the function that corresponds to the transformation of the graph given in the middle column. The right column gives the new location of the point (x, y) on the graph of f.

Formula	Description	Transformed Point
$f(x)+k$	shift up k, for $k>0$	$(x, y+k)$
$f(x)-k$	shift down k, for $k>0$	$(x, y-k)$
$af(x)$	vertical stretch by a, for $a>1$	(x, ay)
$\frac{1}{a}f(x)$	vertical compression by a, for $a>1$	$\left(x, \frac{y}{a}\right)$
$f(x-h)$	shift right by h, for $h>0$	$(x+h, y)$
$f(x+h)$	shift left by h, for $h>0$	$(x-h, y)$
$f(bx)$	horizontal compression by b, for $b>1$	$\left(\frac{x}{b}, y\right)$
$f\left(\frac{x}{b}\right)$	horizontal stretch by b, for $b>1$	(bx, y)
$-f(x)$	vertical flip (flip over x-axis)	$(x, -y)$
$f(-x)$	horizontal flip (flip over y-axis)	$(-x, y)$
$-f(-x)$	$180°$ rotation (flip over x-axis and y-axis)	$(-x, -y)$

Putting it Together

Transformations may be performed one after another. If the transformations include stretches, shrinks, or reflections, the order in which the transformations are performed may make a difference. In those cases, be sure to pay particular attention to the order.

Example 16. (a) *The graph of $y = x^2$ undergoes the following transformations, in order. Find the equation of the graph that results.*

- *a horizontal shift 2 units to the right*
- *a vertical stretch by a factor of 3*

- *a vertical shift 5 units up*

(b) *Apply the transformations above in the opposite order and find the equation of the graph that results.*

Explanation

(a) We will use f_0 to denote the parent function and $f_1, f_2, \cdots$ to denote the intermediate functions. Applying the transformations in order we have

$f_0(x) = x^2$	Parent function
$f_1(x) = f_0(x-2) = (x-2)^2$	Horizontal shift
$f_2(x) = 3f_1(x) = 3(x-2)^2$	Vertical stretch
$f_3(x) = f_2(x) + 5 = 3(x-2)^2 + 5$	Vertical shift
$f(x) = 3x^2 - 12x + 17$	Expanded form

(b) Applying the transformations in the opposite order we have

$f_0(x) = x^2$	Parent function
$f_1(x) = f_0(x) + 5 = x^2 + 5$	Vertical shift
$f_2(x) = 3f_1(x) = 3(x^2 + 5)$	Vertical stretch
$f_3(x) = f_2(x-2) = 3((x-2)^2 + 5)$	Horizontal shift
$f(x) = 3x^2 - 12x + 27$	Expanded form

This shows that changing the order in which the transformations are applied may end up changing the resulting function.

The previous example shows how to construct the formula of a function given a sequence of transformations. One might wonder how to find the transformations applied to a parent function's graph given a complicated formula. That is, if we know a formula $af(bx + h) + k$, can we reconstruct the sequence of transformations? There are two parts to this: finding the transformations and finding the order of those transformations. The following example gives some of the reasoning behind the process.

Example 17. *Given a function f defined by $f(x) = |2x + 7|$, find its parent function and list the transformations, in order, applied to the parent function's graph to produce the graph of f.* **Explanation** The parent function f_0 is given by $f_0(x) = |x|$. We will use subscripts to denote the intermediate steps between the parent function and the function f.

Now we have to think about what transformations are being applied to the graph $y = |x|$. We can see from the table in the previous section that based on the changes made to the formula, there are

- a horizontal compression by a factor of 2 and
- a horizontal shift 7 units to the left

Note that this is not necessarily the order in which these transformations occur, but an unordered list of them.

It remains for us to find the order in which these transformations were applied to produce the function f. It's a good rule of thumb to start with the transformations closer to the variable x, since those are applied to the function first. Let's try starting with the horizontal compression by a factor of 2. Applying this transformation to $f_0(x)$ yields $f_1(x) = f_0(2x) = |(2x)| = |2x|$, since a horizontal compression adds a factor to the variable x in the previous function. This seems to be a good start. Next, let's take care of the horizontal shift by 7 units to the left. Applying this transformation to $f_1(x)$ yields $f_2(x) = f_1(x+7) = |2(x+7)|$, since shifting 7 to the left adds 7 to the variable x in the previous function. Note that since it adds 7 only to the x, we have to use parentheses here. However, we've gotten off track: $2(x + 7) = 2x + 14$, which is not what we wanted to appear inside the absolute value symbols. Let's try the other order.

This time, begin with the horizontal shift 7 units to the left. Applying this to the parent function $f_0(x)$ yields $f_1(x) = f_0(x + 7) = |x + 7|$. Then, move on to the horizontal compression by a factor of 2. Applying this transformation to $f_1(x)$ yields $f_2(x) = f_1(2x) = |(2x) + 7| = |2x + 7|$, since according to the table, we multiply only the variable x by 2 for this transformation.

This tells us that the correct order for our transformations is

(a) a horizontal shift 7 units to the left and

(b) a horizontal compression by a factor of 2

Let's see how this fits into a more complicated example.

Example 18. *Given a function f defined by $f(x) = 3|2x+7|-2$, find its parent function and list the transformations, in order, applied to the parent function's graph to produce the graph of f.* **Explanation** The parent function f_0 is given by $f_0(x) = |x|$.

Now we have to think about what transformations are being applied to the graph $y = |x|$. We can see from the table in the previous section that based on the changes made to the formula, there are

- a vertical stretch by a factor of 3,
- a horizontal compression by a factor of 2,
- a horizontal shift 7 units to the left, and
- a vertical shift 2 units down.

Note that this is not necessarily the order in which these transformations occur, but an unordered list of them.

As in the previous example, we begin with the horizontal shift 7 units to the left. Applying this to the parent function $f_0(x)$ yields $f_1(x) = f_0(x+7) = |x+7|$. Then, move on to the horizontal compression by a factor of 2. Applying this transformation to $f_1(x)$ yields $f_2(x) = f_1(2x) = |(2x)+7| = |2x+7|$, since according to the table, we multiply only the variable x by 2 for this transformation. Next, let's try taking care of the vertical stretch by a factor of 3. Applying this transformation to $f_2(x)$ gives us $f_3(x) = 3f_2(x) = 3|2x+7|$, since vertical stretches correspond to multiplying the entire previous function by the factor. Last, we can incorporate the vertical shift 2 units down. Since that corresponds to subtracting 2 from the previous function $f_3(x)$, we obtain $f_4(x) = f_3(x) - 2 = 3|2x+7| - 2$, which is exactly the function we wanted to reconstruct.

This tells us that the correct order for our transformations is

(a) a horizontal shift 7 units to the left,

(b) a horizontal compression by a factor of 2,

(c) a vertical stretch by a factor of 3, and

(d) a vertical shift 2 units down.

Note that if we applied transformations 3 and 4 in a different order, we would have obtained a different function.

Given a function f, note that we can apply the same reasoning to $af(bx+h)+k$ to find the order of the transformations applied to the graph of f:

(a) horizontal shift,

(b) horizontal stretch or compression,

(c) vertical stretch or compression, and

(d) vertical shift.

One might wonder where reflections fit into all this. Let's see with another example.

Example 19. *Given a function f defined by $f(x) = -\frac{1}{2}\sin\left(-\frac{x}{3}+\pi\right)+1$, find its parent function and list the transformations, in order, applied to the parent function's graph to produce the graph of f.* **Explanation** The parent function of f is f_0 defined by $f_0(x) = \sin(x)$.

Let's see what happens if we do the reflections after the stretches or compressions, but before the vertical shift. Following our order, we first shift left by π units to obtain $f_1(x) = f_0(x+\pi) = \sin(x+\pi)$, then stretch horizontally by a factor of 3 to obtain $f_2(x) = f_1\left(\frac{x}{3}\right) = \sin\left(\frac{x}{3} + \pi\right)$, then compress vertically by a factor of 2 to obtain $f_3(x) = \frac{1}{2}f_2(x) = \frac{1}{2}\sin\left(\frac{x}{3} + \pi\right)$. Flipping vertically puts a minus sign on the outside of the function, giving us $f_4(x) = -f_3(x) = -\frac{1}{2}\sin\left(\frac{x}{3} + \pi\right)$. Next, flipping horizontally puts a minus sign on the x, giving us $f_5(x) = f_4(-x) = -\frac{1}{2}\sin\left(\frac{-x}{3} + \pi\right)$. Finally, shifting up by 1 gives us $f_6(x) = f_5(x) + 1 = -\frac{1}{2}\sin\left(-\frac{x}{3} + \pi\right) + 1$, which is what we wanted in the end.

Note that we could have done the horizontal reflection before the vertical reflection and achieved the same result. Therefore, as long as you do the reflections after the stretches or compressions, it doesn't matter which order each reflection comes in.

We can summarize all this information as follows. Given a function f, the graph of $af(bx + h) + k$ can be found using the following transformations in order:

(a) horizontal shifts given by h,

(b) horizontal stretches or compressions given by b,

(c) vertical stretches or compressions given by a,

(d) reflections given by the sign of a and b, and

(e) vertical shifts given by k.

Note that this is not the only order you can use to graph transformations of functions. This is just one order that works every time, provided you start with a function of the form $af(bx + h) + k$.

Example 20. *Given a function f defined by $f(x) = -\frac{1}{2}|3x - 5| + 1$, find its parent function and list the transformations, in order, applied to the parent function's graph to produce the graph of f. Then graph f.* **Explanation** The parent function is given by $|x|$, and the transformations in order are

(a) a horizontal shift right by 5 units,

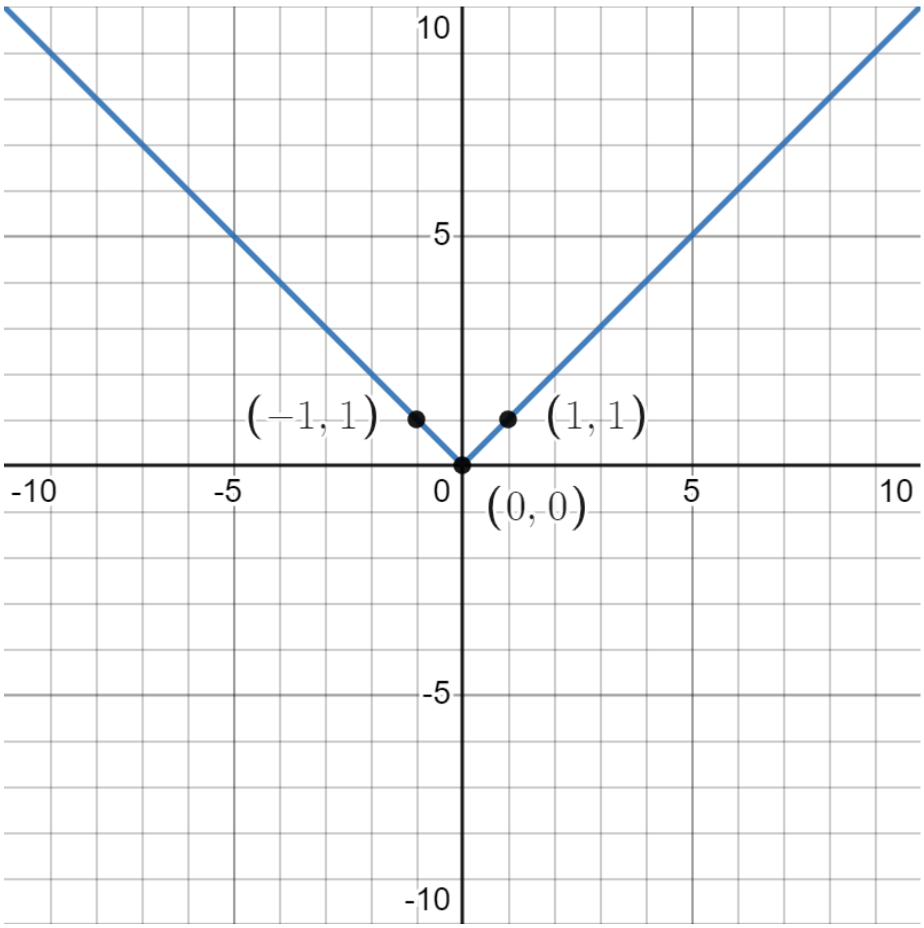
(b) a horizontal compression by a factor of 3,

(c) a vertical compression by a factor of 2,

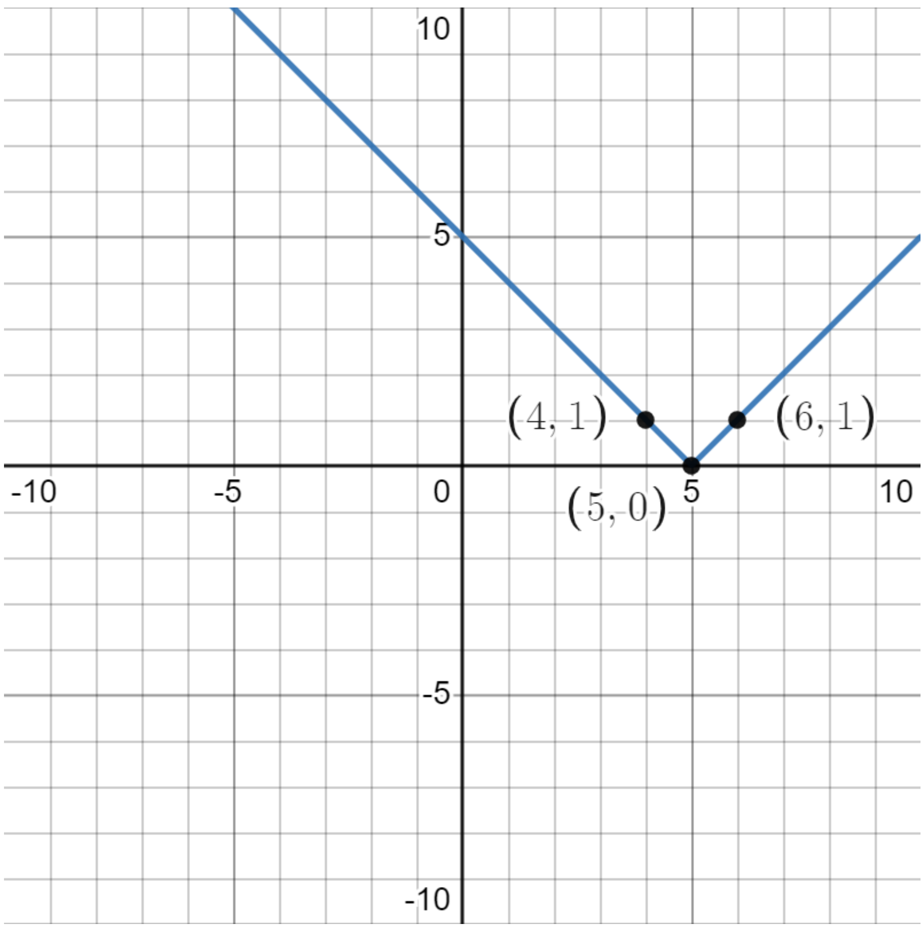
(d) a reflection across the x-axis, and

(e) a vertical shift up by 1 unit.

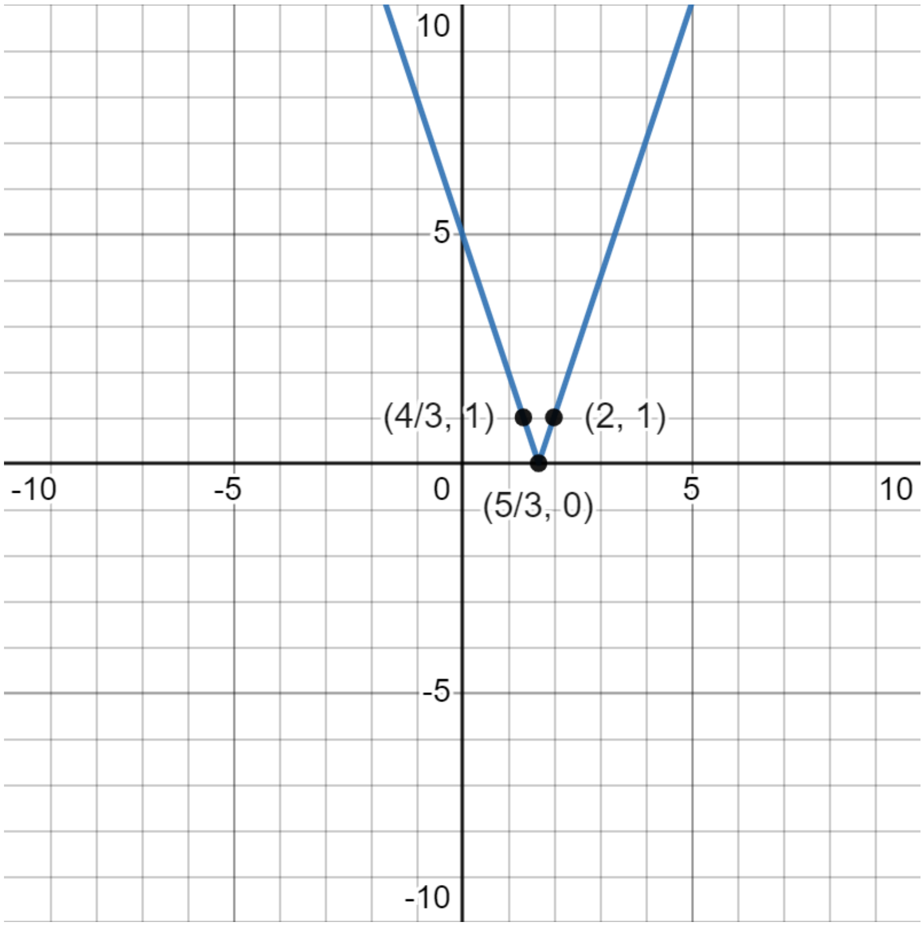
To graph f, we'll start with the graph of $f_0(x) = |x|$, and by keeping track of the transformations of a few points, $(0, 0)$, $(1, 1)$, and $(-1, 1)$, we'll draw the graph after applying each transformation. To start with, here's the graph of $f_0(x) = |x|$.



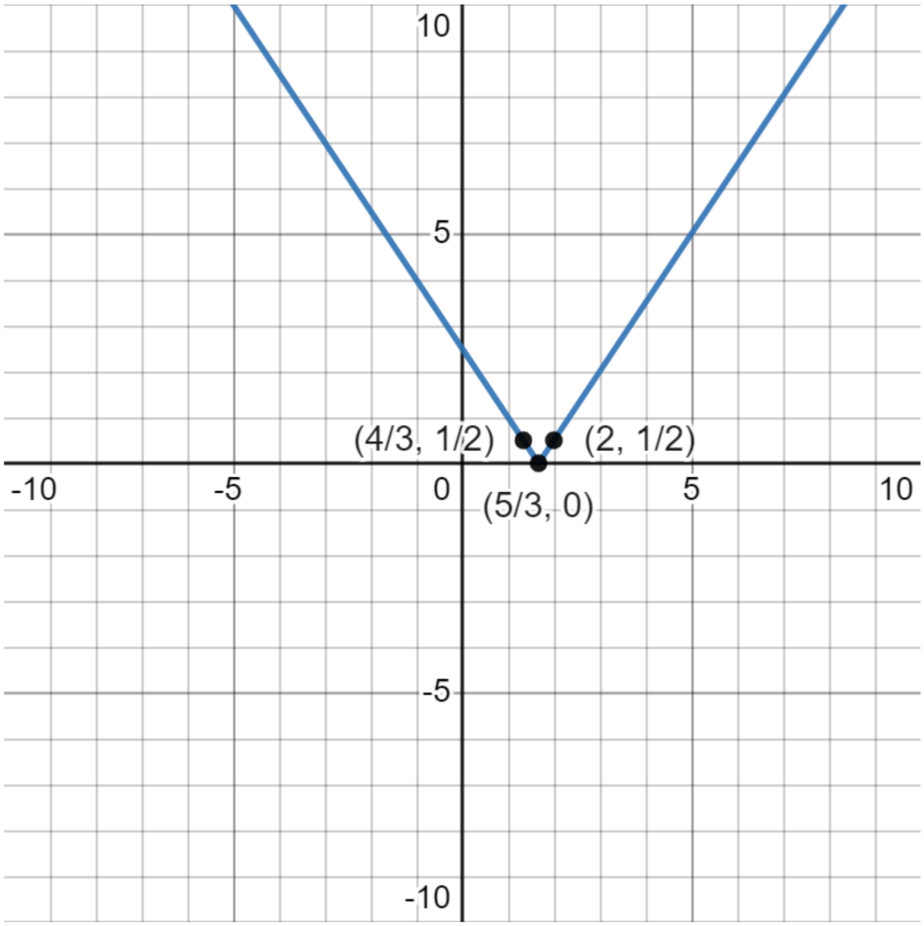
After horizontally shifting right by 5 units, our points are $(5, 0)$, $(6, 1)$, and $(4, 1)$, and the graph of $f_1(x) = f_0(x - 5) = |x - 5|$ looks like this:



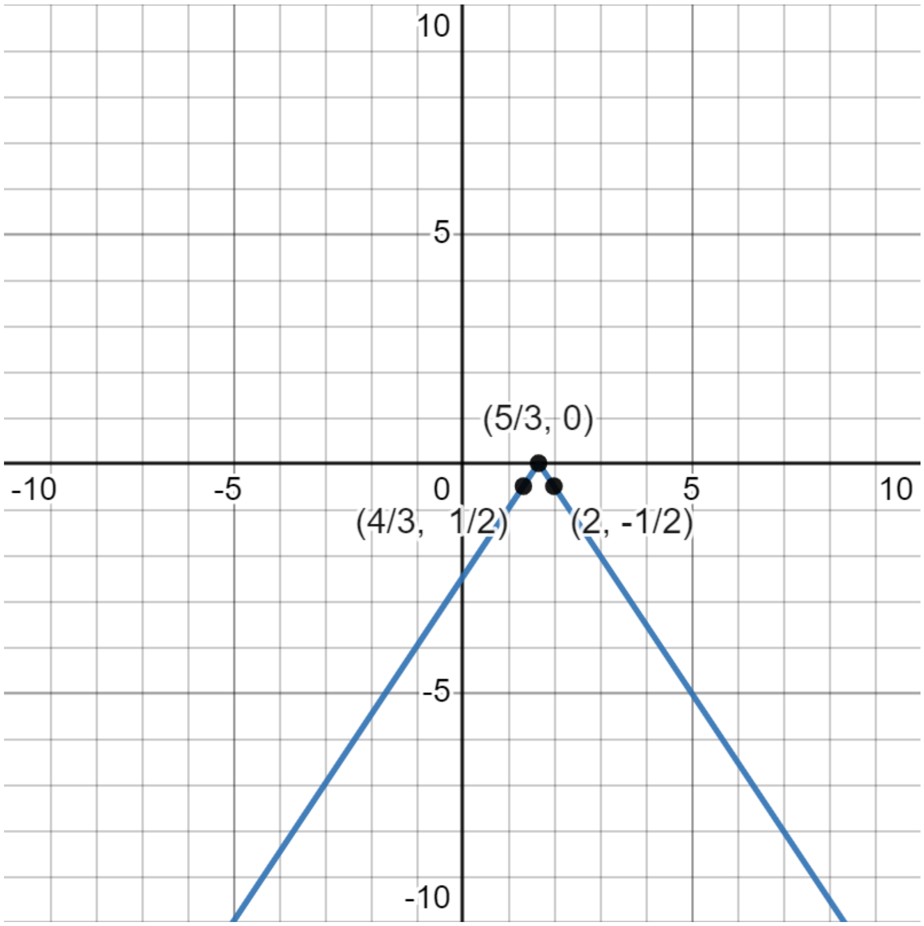
Compressing horizontally by a factor of 3 gives us $f_2(x) = f_1(3x) = |3x - 5|$, and our points are $\left(\frac{5}{3}, 0\right)$, $(2, 1)$, and $\left(\frac{4}{3}, 1\right)$:



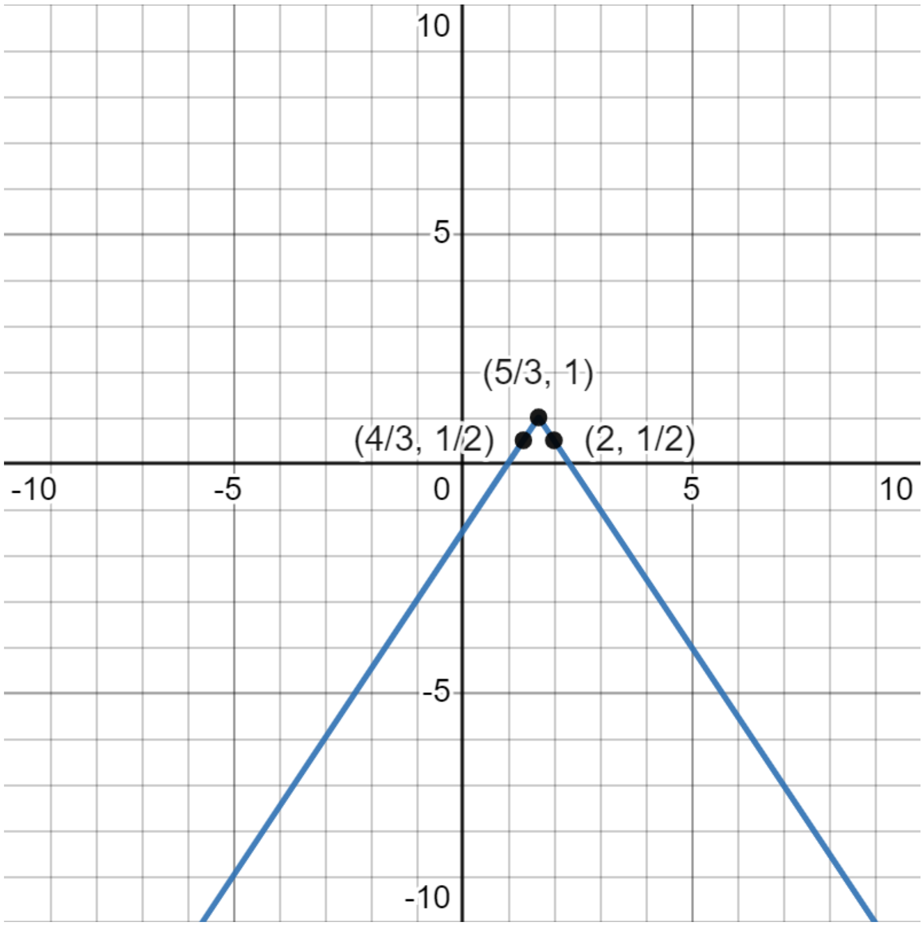
Vertically compressing by a factor of 2 gives us the points $\left(\frac{5}{3}, 0\right)$, $\left(2, \frac{1}{2}\right)$, and $\left(\frac{4}{3}, \frac{1}{2}\right)$ and the function $f_3(x) = \frac{1}{2} f_2(x) = \frac{1}{2}|3x - 5|$:



Reflecting across the x-axis gives us the points $\left(\frac{5}{3}, 0\right)$, $\left(2, -\frac{1}{2}\right)$, and $\left(\frac{4}{3}, -\frac{1}{2}\right)$ and the function $f_4(x) = -f_3(x) = -\frac{1}{2}|3x - 5|$:



Finally, vertically shifting up by 1 gives us the points $\left(\frac{5}{3}, 1\right)$, $\left(2, \frac{1}{2}\right)$, and $\left(\frac{4}{3}, \frac{1}{2}\right)$ and the function $f(x) = f_4(x) + 1 = -\frac{1}{2}|3x - 5| + 1$:



This last graph is our final result. While graphing each step along the way may seem like an awful lot of work, it is often easier than trying to work with multiple transformations at once in your head.

Below is a link to a Desmos graph containing $af(bx + h) + k$ for a function f where you can adjust the values of a, b, h, and k to see how they affect the graph. The function f in the link below is the formula for a half-circle, but don't worry too much about where it came from.

Desmos link: `https://www.desmos.com/calculator/wf5aevwjhk`

Summary

- Multiple transformations can be applied to a parent function.
- The order in which transformations are applied can change the

resulting function.

- Given a formula of a transformed function, we can infer the order of transformations that produced it, as well as graph the function.

7.4 Solving Inequalities

Learning Objectives

- Solving Inequalities Graphically
  - Motivating solutions to inequalities
  - Definition of a solution to an inequality
  - Review finding zeros of equations
- Solving Inequalities without a Graph
  - Famous functions are continuous except at... IVT
  - Solving with a sign chart
  - Using signs of famous functions

7.4.1 Solving Inequalities Graphically

Motivating Questions

- What is a solution to an inequality?
- How can we use graphs to solve inequalities?

Introduction

Dabin and Melina are having a walking race. Dabin can walk 1 meter per second, but Melina can walk 2 meters per second. Since Melina is the faster walker, she gives Dabin a head start of 5 meters. At this point, we can ask a few questions about the race. Two questions we'll focus on are "When is Dabin in the lead?" and "When is Melina in the lead?" both of which can be answered by considering inequalities.

To start, let's define some relevant functions. The function D defined by $D(t) = 5 + t$ represents how far (in meters) Dabin has walked t seconds after the start of the race. Similarly, the function M defined by $M(t) = 2t$ represents how far Melina has walked t seconds after the start of the race. Here, asking when Dabin is in the lead is the same as asking for all t such that $D(t) > M(t)$. In the vocabulary that we'll use, we want to solve the inequality $D(t) > M(t)$.

Definition Say f and g are functions. A **solution** to the inequality $f(x) < g(x)$ is the set of x values where $f(x) < g(x)$. Similarly, a solution to the inequality $f(x) > g(x)$ is the set of x values where $f(x) > g(x)$.

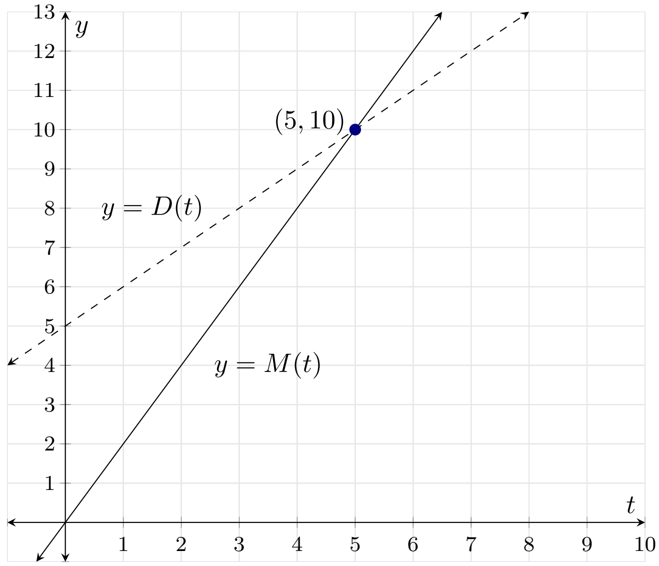
Note that we define a solution to an inequality as a set. We will often write the sets in interval notation.

Solving inequalities graphically

Example 21. *Let D be defined by $D(t) = 5+t$, and M be defined by $M(t) = 2t$. Find a solution to the inequality $D(t) < M(t)$.*

Explanation This example asks us to find the set of t values where $D(t) < M(t)$. One approach to inequalities of this form is to look at the graphs of the equations involved. The following figure shows the graphs of $y = D(t)$ and $y = M(t)$.

 Acknowledgements: Stitz Zeager Open Source Mathematics (https://www.stitz-zeager.com/)



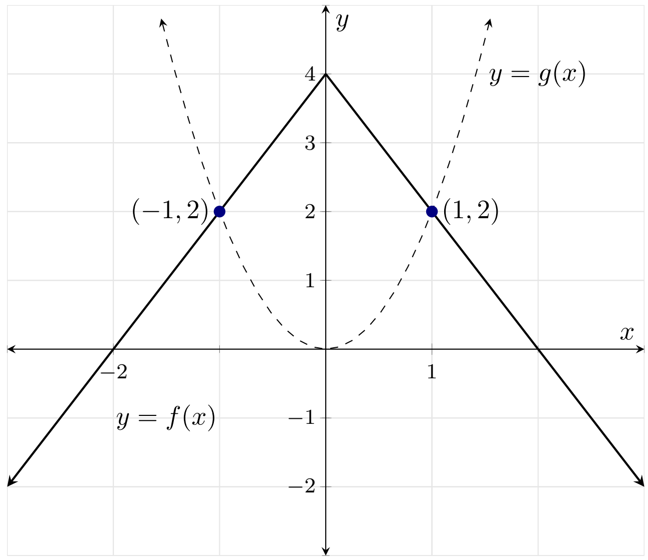
Because of the way we draw the graphs of functions, if $D(t) < M(t)$ for some y if and only if the graph of D lies below the graph of M at the point t. Using this information, we can see that if $t > 5$, then the graph of D lies below the graph of M. Therefore, the set of all t such that $t > 5$ is the solution to $D(t) < M(t)$. Writing this in interval notation, the solution is $(5, \infty)$.

Putting this in terms of the scenario described at the beginning of the section, Melinda is in the lead after 5 seconds.

Exploration Find a solution to the inequality $D(t) < M(t)$.

Example 22. *Let f and g be functions whose graphs are shown below. Assume all important behavior of the functions is shown in the figure.*

Acknowledgements: Stitz Zeager Open Source Mathematics (https://www.stitz-zeager.com/)



(a) *Solve the inequality* $f(x) < g(x)$.

(b) *Solve the inequality* $f(x) \geq g(x)$.

Explanation

(a) To solve $f(x) < g(x)$, we look for where the graph of f is below the graph of g. This appears to happen for the x values less than -1 and greater than 1. Our solution is $(-\infty, -1) \cup (1, \infty)$.

(b) To solve $f(x) \geq g(x)$, we look for solutions to $f(x) = g(x)$ as well as $f(x) > g(x)$. To solve the former equation we can look at the x-coordinates of the intersection points. This yields $x = \pm 1$. To solve $f(x) > g(x)$, we look for where the graph of f is above the graph of g. This appears to happen between $x = -1$ and $x = 1$, on the interval $(-1, 1)$. Hence, our solution to $f(x) \geq g(x)$ is $[-1, 1]$.

Now let's turn our attention to inequalities involving absolute values, which are often a source of confusion. The following theorem provides the complete story. As you read through the theorem, use the interactive figure below to help you interpret and make sense of each bullet point statement.

Let c be a real number.

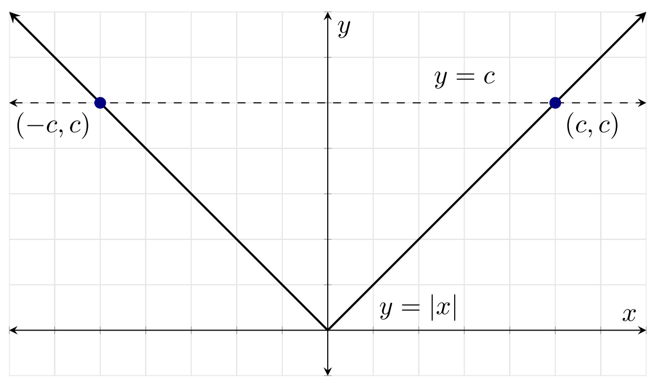
- For $c > 0$, $|x| < c$ is equivalent to $-c < x < c$.
- For $c > 0$, $|x| \leq c$ is equivalent to $-c \leq x \leq c$.
- For $c \leq 0$, $|x| < c$ has no solution, and for $c < 0$, $|x| \leq c$ has no

Acknowledgements: Stitz Zeager Open Source Mathematics (https://www.stitz-zeager.com/)

solution.

- For $c \geq 0$, $|x| > c$ is equivalent to $x < -c$ or $x > c$.
- For $c \geq 0$, $|x| \geq c$ is equivalent to $x \leq -c$ or $x \geq c$.
- For $c < 0$, $|x| > c$ and $|x| \geq c$ are true for all real numbers.

In light of what we have developed in this section, we can understand these statements graphically. For instance, if $c > 0$, the graph of $y = c$ is a horizontal line which lies above the x-axis through $(0, c)$. To solve $|x| < c$, we are looking for the x values where the graph of $y = |x|$ is below the graph of $y = c$. We know that the graphs intersect when $|x| = c$, which we know happens when $x = c$ or $x = -c$. Graphing, we get



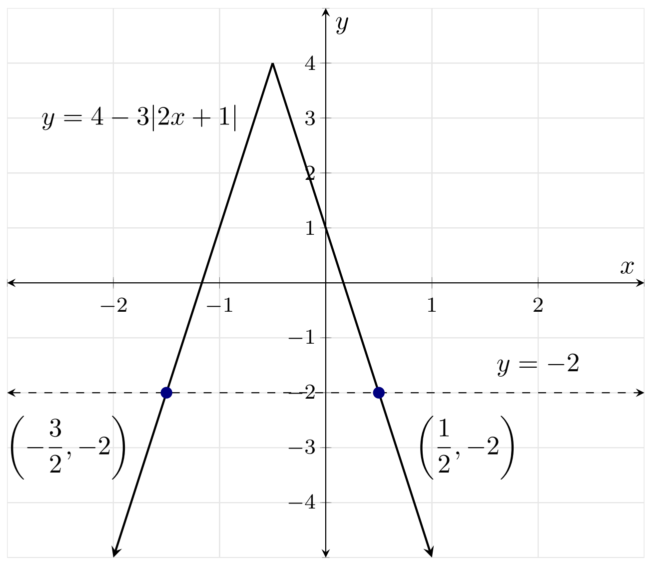
We see that the graph of $y = |x|$ is below $y = c$ for x between $-c$ and c, and hence we get $|x| < c$ is equivalent to $-c < x < c$. The other properties in the theorem can be shown similarly. You can try changing the value of c using Desmos.

Desmos link: `https://www.desmos.com/calculator/dbpb01aybm`

Example 23. *Solve the inequality* $4 - 3|2x + 1| > -2$.

Explanation Let's start by graphing both sides of the inequality on the same axes.

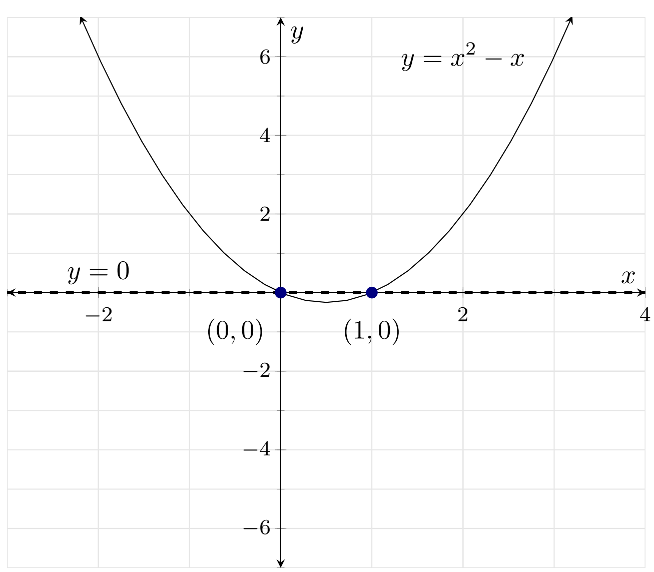
 Acknowledgements: Stitz Zeager Open Source Mathematics (https://www.stitz-zeager.com/)



We see that the graph of $y = 4 - 3|2x + 1|$ is above $y = -2$ for x values between $-\frac{3}{2}$ and $\frac{1}{2}$. Therefore, the solution in interval notation is $\left(-\frac{3}{2}, \frac{1}{2}\right)$.

Example 24. *Solve the inequality* $x^2 \leq x$.

Explanation We could start by graphing both sides of the inequality on the same graph, but here, we'll demonstrate another possible approach. Note that $x^2 \leq x$ is equivalent to the inequality $x^2 - x \leq 0$, by subtracting x from both sides. Now, let's graph $y = x^2 - x$ and $y = 0$ on the same axes.



 Acknowledgements: Stitz Zeager Open Source Mathematics (https://www.stitz-zeager.com/)

Notice that the two points of intersection are $(0, 0)$ and $(1, 0)$, so $x^2 - x = 0$ for $x = 0$ and $x = 1$. To find the solution to $x^2 - x \leq< 0$, we can see that the graph of $y = x^2 - x$ lies below the graph of $y = 0$ between 0 and 1. Therefore, the solution is $[0, 1]$.

The above example illustrates a common technique. Rather than considering two functions f and g and asking when one is greater than, less than, or equal to the other, we can move one function to the other side, and consider the function $f - g$. Now, the problem becomes one of finding when the function $f - g$ is positive, negative, or zero.

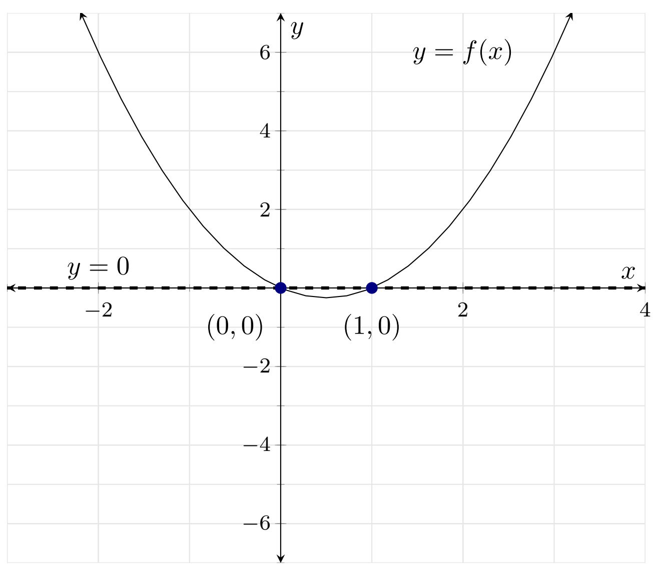
7.4.2 Solving Inequalities without a Graph

Motivating Questions

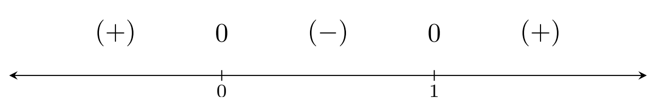
- How can we find solutions to inequalities without using a graph?
- How can we use the zeros of functions to solve inequalities?

Introduction and the Importance of Zeros

In the previous section, we constructed the following graph to solve the inequality $f(x) \leq 0$, where $f(x) = x^2 - x$.



We can see that the graph of f dips below the x-axis between its two x-intercepts. The zeros of f are $x = 0$ and $x = 1$ in this case and they divide the domain (the x-axis) into three intervals: $(-\infty, 0)$, $(0, 1)$ and $(1, \infty)$. For every number in $(-\infty, 0)$, the graph of f is above the x-axis; in other words, $f(x) > 0$ for all x in $(-\infty, 0)$. Similarly, $f(x) < 0$ for all x in $(0, 1)$, and $f(x) > 0$ for all x in $(1, \infty)$. We can schematically represent this with the *sign diagram* below.



Here, the $(+)$ above a portion of the number line indicates $f(x) > 0$ for those values of x; the $(-)$ indicates $f(x) < 0$ there. The numbers labeled on the number line are the zeros of f, so we place 0 above them. We see at once that the solution to $f(x) < 0$ is $(0, 1)$. Adding in the zeros, the solution to $f(x) \leq 0$ is $[0, 1]$.

Our next goal is to establish a procedure by which we can generate the sign diagram without graphing the function.

Continuity and the Intermediate Value Theorem

An important property of quadratic functions is that if the function is positive at one point and negative at another, the function must have at least one zero in between. Graphically, this means that a parabola can't be above the x-axis at one point and below the x-axis at another point without crossing the x-axis.

This is a special case of a theorem called the Intermediate Value Theorem, or IVT for short. To talk about the IVT, we first need to discuss what it means for a function to be *continuous*.

> **Definition** *(Informal.)* We say a function f on an interval is **continuous** if the graph of f has no 'breaks' or 'holes' on that interval.

In further courses, you will learn a more formal definition of continuity, but for now, this will suffice.

Example 25. (a) *Linear and quadratic functions are continuous.*

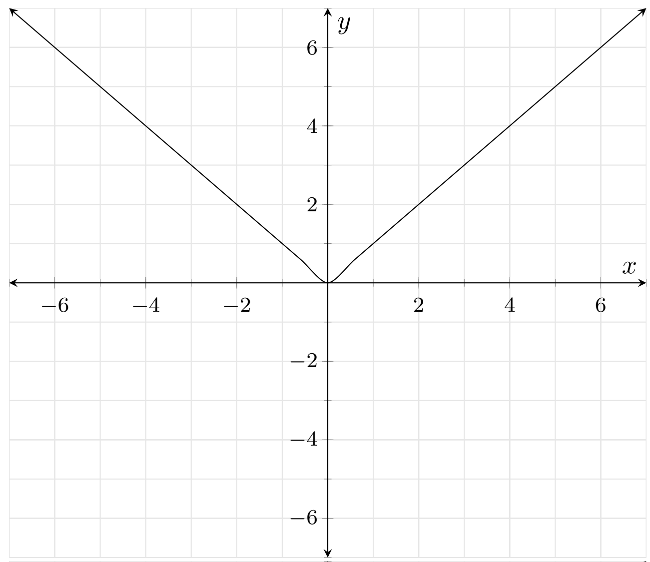
(b) *In fact, all of our famous functions are continuous where they are defined.*

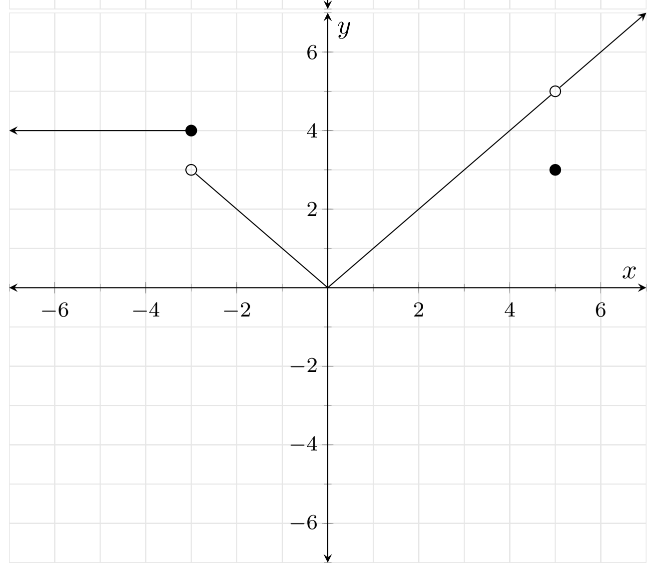
(c) *All polynomials are continuous.*

(d) *Rational functions are continuous where they are defined. In particular,* $\dfrac{1}{x}$ *is continuous on its domain,* $(-\infty, 0) \cup (0, \infty)$.

(e) *If* f *and* g *are continuous functions, so are* $f + g$ *and* $f \cdot g$.

 Acknowledgements: Stitz Zeager Open Source Mathematics (https://www.stitz-zeager.com/)





The function whose graph is shown as the first of the two graphs above is continuous, while the function whose graph is shown second above is not, since it has breaks in its graph.

One way to think about continuous functions is that they are the functions whose graphs you could draw on an infinite piece of paper without ever taking your pencil off the paper (except where they aren't defined). You will encounter and learn about continuous functions more in-depth in calculus, but for now, familiarity at this level will be enough.

Now that we know about continuous functions, we can state our version of the

Acknowledgements: Stitz Zeager Open Source Mathematics (https://www.stitz-zeager.com/)

IVT.

[Intermediate Value Theorem (Zero Version)] Suppose f is a continuous function on an interval containing $x = a$ and $x = b$ with $a < b$. If $f(a)$ and $f(b)$ have different signs, then f has at least one zero between $x = a$ and $x = b$; that is, for at least one real number c such that $a < c < b$, we have $f(c) = 0$.

Reinterpreted, this means that the graph of a continuous function can't be above the x-axis at one point and below the x-axis at another point without crossing the x-axis.

Here's how we'll use the IVT to solve inequalities of the form $f(x) > 0$, where f is a continuous function. If a given interval does not contain a zero of f, then by the IVT either all the function values on the interval are positive or they're all negative. In this way, the IVT allows us to determine the sign of *all* of the function values on the interval by testing the function at just *one* value in the interval, which we're free to choose.

This gives us the following steps for solving an inequality involving a continuous function.

(a) Rewrite the inequality, if necessary, as a continuous function $f(x)$ on one side of the inequality and 0 on the other.

(b) Find the zeros of f and place them on the number line with the number 0 above them.

(c) Choose a real number, called a *test value*, in each of the intervals determined in step 2.

(d) Determine the sign of $f(x)$ for each test value in step c, and write that sign above the corresponding interval.

(e) Choose the intervals which correspond to the correct sign to solve the inequality.

As you can see, the zeros of continuous functions are important, so in the examples that follow, we'll highlight the techniques we use to find zeros. It may also be useful to review methods for finding zeros that you've seen before.

Solving Inequalities Algebraically

Example 26. *Solve the inequality* $3x^2 + x < 6x - 2$.

Explanation To start, let's put the inequality in a nice form, with a continuous function on one side and 0 on the other. It doesn't matter which side the 0 is

 Acknowledgements: Stitz Zeager Open Source Mathematics (https://www.stitz-zeager.com/)

on, so we'll choose to rewrite the inequality as $3x^2 - 5x + 2 < 0$. Since quadratic functions are continuous, we can use the steps outlined in the previous section to solve the inequality.

First, we find the zeros of f, where $f(x) = 3x^2 - 5x + 2$. We could do this by using the quadratic formula, but let's factor. Factoring gives us $f(x) = (3x - 2)(x - 1)$. In order for $f(x) = 0$ to be true, we need $3x - 2 = 0$ or $x - 1 = 0$. This tells us that the zeros of f are $x = \dfrac{2}{3}$ and $x = 1$. This gives us a good start to our sign diagram:

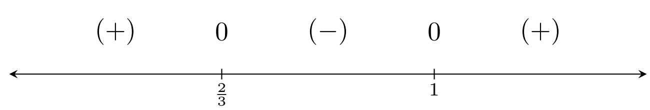
This sign diagram tells us that we have to check three intervals: $\left(-\infty, \dfrac{2}{3}\right)$, $\left(\dfrac{2}{3}, 1\right)$, and $(1, \infty)$. However, thanks to the IVT, we only need to check one test value per interval. Be careful not to choose $x = \dfrac{2}{3}$ or $x = 1$ as your test values since they are zeros.

For the interval $\left(-\infty, \dfrac{2}{3}\right)$, we choose $x = 0$ to be our test value and see that $f(0) = 3(0)^2 - 5(0) + 2 = 2$, which is positive.

For the interval $\left(\dfrac{2}{3}, 1\right)$, we choose $x = \dfrac{5}{6}$ to be our test value and see that $f(0) = 3\left(\dfrac{5}{6}\right)^2 - 5\left(\dfrac{5}{6}\right) + 2 = \dfrac{25}{12} - \dfrac{25}{6} + 2 = -\dfrac{1}{12}$, which is negative.

For the interval $(1, \infty)$, we choose $x = 2$ to be our test value and see that $f(2) = 3(2)^2 - 5(2) + 2 = 4$, which is positive.

We can now update our sign diagram to the following:



Since we wanted to find where f was negative, we choose $\left(\dfrac{2}{3}, 1\right)$ as the solution to the inequality.

Example 27. *Solve the inequality* $xe^x \geq -x$.

Explanation Rewriting our inequality, we have $xe^x + x \geq 0$. Since linear and exponential functions are continuous and products and sums of continuous

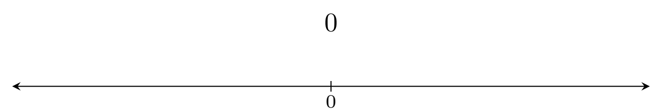
Acknowledgements: Stitz Zeager Open Source Mathematics (https://www.stitz-zeager.com/)

functions are continuous, we know that the function f defined by $f(x) = xe^x + x$ is a continuous function.

First, we find the zeros of f. We start by noticing that each term in $xe^x + x$ contains a factor of x, so we can factor that out and find $f(x) = x(e^x + 1)$. In order to solve the equation $f(x) = 0$, we need to solve $x = 0$ and $e^x + 1 = 0$. The first equation is already solved, and tells us that $x = 0$ is one zero of f. To solve the second equation, we calculate

$$\begin{aligned} e^x + 1 &= 0 \\ e^x &= -1, \end{aligned}$$

and note that the exponential function is never negative, so there are no solutions. Therefore, the only zero of f is $x = 0$. We now begin to construct the sign diagram.

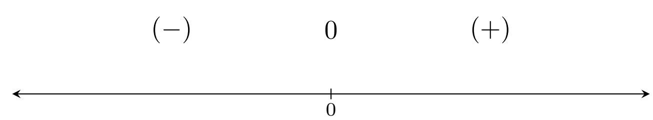


This sign diagram tells us that we have to check two intervals: $(-\infty, 0)$ and $(0, \infty)$. Again, thanks to the IVT, we only need to check one test value per interval.

For the interval $(-\infty, 0)$, we choose $x = -1$ to be our test value and see that $f(-1) = e^{-1} - 1$, which is negative. To see that $e^{-1} - 1$ is negative, notice that $e^{-1} = \dfrac{1}{e}$, and since $e > 1$, $\dfrac{1}{e} < 1$. Therefore, when we subtract 1 from e^{-1}, we obtain a negative number.

For the interval $(0, \infty)$, we choose $x = 1$ to be our test value and see that $f(1) = e^1 + 1$, which is positive.

We can now update our sign diagram to the following:



Since we wanted to find where f was non-negative, we choose $[0, \infty)$ as the solution to the inequality. Remember to include 0 in the interval, since zeros of f are also points where f is non-negative.

Example 28. *Solve the inequality* $2^{x^2 - 3x} \geq 16$.

Explanation We set $r(x) = 2^{x^2-3x} - 16$ and solve the equivalent inequality $r(x) \geq 0$. The domain of r is all real numbers, so in order to construct our sign

Acknowledgements: Stitz Zeager Open Source Mathematics (https://www.stitz-zeager.com/)

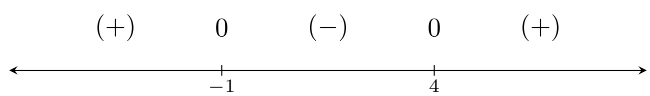
diagram, we need to find the zeros of r. Setting $r(x) = 0$ gives $2^{x^2-3x} - 16 = 0$ or $2^{x^2-3x} = 16$. Since $16 = 2^4$ we have $2^{x^2-3x} = 2^4$, so by taking logarithms, $x^2 - 3x = 4$. Solving $x^2 - 3x - 4 = 0$ gives $x = 4$ and $x = -1$. Therefore, the intervals in which we need to find test values are $(-\infty, -1)$, $(-1, 4)$, and $(4, \infty)$.

For the interval $(-\infty, -1)$, we choose $x = -2$ to be our test value. We see that $r(-2) = 2^{4+6} - 16 = 2^{10} - 2^4 > 0$, so r is positive on the interval.

For the interval $(-1, 4)$, we choose $x = 0$ to be our test value. We see that $r(0) = 2^0 - 16 = 2^0 - 2^4 < 0$, so r is negative on the interval.

For the interval $(4, \infty)$, we choose $x = 5$ to be our test value. We see that $r(5) = 2^{25-15} - 16 = 2^{10} - 2^4 > 0$, so r is positive on the interval.

We can now construct a sign diagram.



From the sign diagram, we see $r(x) \geq 0$ on $(-\infty, -1] \cup [4, \infty)$, which corresponds to where the graph of $y = r(x) = 2^{x^2-3x} - 16$ is on or above the x-axis.

Dealing with Difficult Denominators

Even after we feel comfortable with the procedure for solving inequalities involving continuous functions, you might still wonder about functions which aren't defined on all real numbers, such as rational functions or more generally, functions with denominators that could potentially evaluate to 0. The good news is that if f and g are continuous functions, the function $\dfrac{f}{g}$ is continuous wherever it is defined. Therefore, we can adapt our technique from before, but remembering that a change of sign *could* happen around a point where a function is undefined, so we need to add any places our functions are undefined to our sign diagram.

Example 29. *Solve the inequality* $\dfrac{x^3 - 2x + 1}{x - 1} \geq \dfrac{1}{2}x - 1.$

Explanation To solve the inequality, it may be tempting to begin by clearing denominators. The problem is that, depending on x, $(x - 1)$ may be positive (which doesn't affect the inequality) or $(x - 1)$ could be negative (which would reverse the inequality). Instead of working by cases, we collect all of the terms on one side of the inequality with 0 on the other and begin to make a sign diagram.

 Acknowledgements: Stitz Zeager Open Source Mathematics (https://www.stitz-zeager.com/)

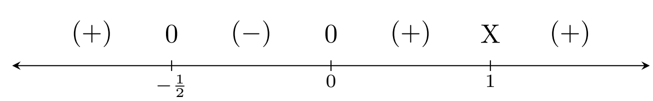
$$
\begin{aligned}
\frac{x^3-2x+1}{x-1} &\geq \frac{1}{2}x-1 \\
\frac{x^3-2x+1}{x-1}-\frac{1}{2}x+1 &\geq 0 \\
\frac{2\left(x^3-2x+1\right)-x(x-1)+1(2(x-1))}{2(x-1)} &\geq 0 && \text{get a common denominator} \\
\frac{2x^3-x^2-x}{2x-2} &\geq 0 && \text{expand}
\end{aligned}
$$

Viewing the left hand side as a rational function $r(x)$ we make a sign diagram. The candidates for zeros of r are the solutions to $2x^3-x^2-x=0$, which we can find by factoring.

$$
\begin{aligned}
2x^3-x^2-x&=0 \\
x(2x^2-x-1)&=0 \\
x(2x+1)(x-1)&=0.
\end{aligned}
$$

Therefore, the candidates for zeros of r are $x=0$, $x=-\dfrac{1}{2}$ and $x=1$. However, $x=1$ is not in the domain of r, since it is the solution to $2x-2=0$, which is the equation we get by setting the denominator equal to 0. However, x-values for which the function is undefined are also possible places where the sign of the function might change, so we should include them on the sign diagram. Since r is a rational function, it is continuous everywhere it is defined, so when constructing the sign diagram, we only need to consider the intervals between zeros or places where it is undefined. For us, these intervals will be $\left(-\infty,-\dfrac{1}{2}\right)$, $\left(-\dfrac{1}{2},0\right)$, $(0,1)$, and $(1,\infty)$.

Choosing test values in each test interval (we encourage you to check the calculation), we can construct the sign diagram below.



We used an X to denote that r is not defined at $x=1$.

We are interested in where $r(x)\geq 0$. We find $r(x)$ is positive on the intervals $\left(-\infty,-\dfrac{1}{2}\right)$, $(0,1)$ and $(1,\infty)$. We add to these intervals the zeros of r, $x=-\dfrac{1}{2}$, and $x=0$, to get our final solution: $\left(-\infty,-\dfrac{1}{2}\right]\cup[0,1)\cup(1,\infty)$.

Acknowledgements: Stitz Zeager Open Source Mathematics (https://www.stitz-zeager.com/)

Example 30. *Solve the inequality* $\dfrac{e^x}{e^x - 4} \leq 3$.

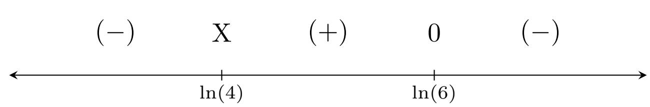
Explanation The first step we need to take to solve $\dfrac{e^x}{e^x-4} \leq 3$ is to get 0 on one side of the inequality. To that end, we subtract 3 from both sides and get a common denominator

$$\begin{aligned} \frac{e^x}{e^x-4} &\leq 3 \\ \frac{e^x}{e^x-4} - 3 &\leq 0 \\ \frac{e^x}{e^x-4} - \frac{3\left(e^x-4\right)}{e^x-4} &\leq 0 \quad \text{Common denomintors.} \\ \frac{12-2e^x}{e^x-4} &\leq 0 \end{aligned}$$

We set $r(x) = \dfrac{12-2e^x}{e^x-4}$. We note that r is undefined when its denominator $e^x - 4 = 0$, or when $e^x = 4$. Solving this by taking logarithms gives $x = \ln(4)$, so the domain of r is $(-\infty, \ln(4)) \cup (\ln(4), \infty)$. To find the zeros of r, we set the numerator equal to zero and obtain $12 - 2e^x = 0$. Solving for e^x, we find $e^x = 6$, or $x = \ln(6)$. When we build our sign diagram, finding test values may be a little tricky since we need to check values around $\ln(4)$ and $\ln(6)$. Recall that the function $\ln(x)$ is increasing[1] which means $\ln(3) < \ln(4) < \ln(5) < \ln(6) < \ln(7)$. This indicates that we might want to use $\ln(3)$, $\ln(5)$, and $\ln(7)$ as our test values. While the prospect of determining the sign of $r\left(\ln(3)\right)$ may be very unsettling, remember that $e^{\ln(3)} = 3$, so

$$r\left(\ln(3)\right) = \frac{12-2e^{\ln(3)}}{e^{\ln(3)}-4} = \frac{12-2(3)}{3-4} = -6$$

We determine the signs of $r\left(\ln(5)\right)$ and $r\left(\ln(7)\right)$ similarly and construct the sign diagram.



From the sign diagram, we find our answer to be $(-\infty, \ln(4)) \cup [\ln(6), \infty)$.

[1]This is because the base of $\ln(x)$ is $e > 1$. If the base b were in the interval $0 < b < 1$, then $\log_b(x)$ would decreasing.

Acknowledgements: Stitz Zeager Open Source Mathematics (https://www.stitz-zeager.com/)

Conclusion

We hope that the specific examples we've gone through illustrate a general principle when it comes to solving inequalities. First, we want to rewrite the inequality in a form where 0 is on one side and a nice-enough function is on the other side. Then, we use the fact that our functions are continuous on their domains to narrow down where possible sign changes can occur. From there, we can use test values to compute the sign of the function on intervals, and finish by putting our solution in interval notation.

Part 8

Origins of Trig

8.1 Right Triangle Trig

Learning Objectives

- Sine, Cosine, and Tangent
  - Similar triangles, and trig functions as ratios of triangle sides
  - Famous right triangles and deducing famous values
  - Find missing side
  - $\sin^2\theta + \cos^2\theta = 1$
- Secant, Cosecant, and Cotangent
- Definitions and famous values of the secant, cosecant, and cotangent functions in terns of a right triangle
- All From One, One From All
  - How to find values of all trig functions for an acute angle, given only one of such values.

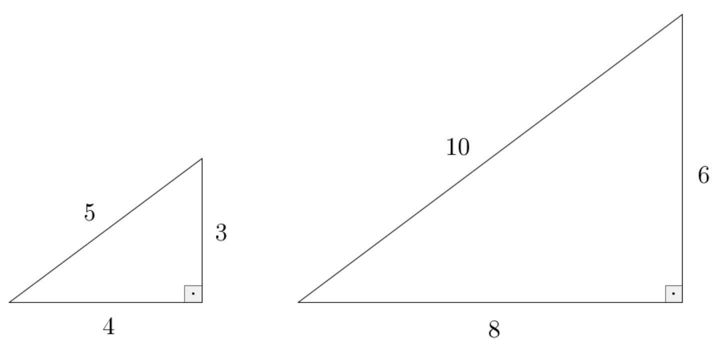
8.1.1 Sine, Cosine, and Tangent

Motivating Questions

- How to study, in a systematic way, ratios between the sides of a right triangle?
- What are the values of sine, cosine, and tangent, for the most frequent angles of 30°, 45° and 60°? And why?

Introduction

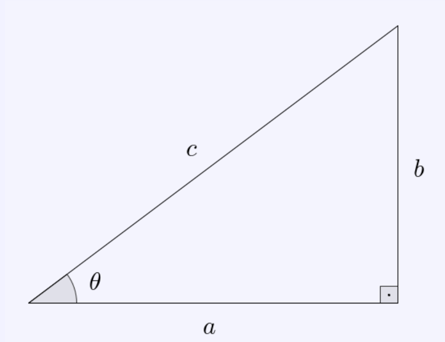
Recall that two triangles are called **similar** if one of them can be obtained by rescaling and moving around the other. Here's an example:



The dotted square symbol is a shorthand for "90° degrees". Triangles which have a 90° angle are called **right triangles**, and will be the main focus of our discussion. What do similar triangles actually have in common? Certainly angles, but not necessarily the lengths of the sides. However, the **ratios** between any two sides of a triangle will remain the same, no matter how the triangle gets rescaled. Such ratios ultimately give us so much information about the given right triangle that they deserve special names: sine, cosine, and tangent.

Definitions and examples

Definition (right triangle trig): Consider the following right triangle, with one angle θ (this is the lowercase greek letter "theta") indicated, and sides labeled a, b and c.



Then:

- The side labeled with a is called the **adjacent** side to θ.
- The side labeled with b is called the **opposite** side to θ.
- The side labeled with c is called the **hypotenuse** of the triangle.

With this in place, we define the **sine**, **cosine**, and **tangent** of θ, by

$$\sin\theta = \frac{b}{c}\left(=\frac{\text{opp.}}{\text{hyp.}}\right), \quad \cos\theta = \frac{a}{c}\left(=\frac{\text{adj.}}{\text{hyp.}}\right), \text{ and } \tan\theta = \frac{b}{a}\left(=\frac{\text{opp.}}{\text{adj.}}\right).$$

Remark The hypotenuse of a right triangle is always the side opposite to the right angle. Also note that $\tan\theta = \sin\theta/\cos\theta$. You might have seen the mnemonic "SOH CAH TOA" before: for example, "SOH" means "sine equals opposite over hypotenuse", and so on.

Example 31. *For each of the following triangles with given angle θ, identify the adjacent (adj.), opposite (opp.) and hypotenuse (hyp.), and compute $\sin\theta$, $\cos\theta$ and $\tan\theta$.*

a. **Explanation** We have opp. = 12, adj. = 5 and hyp. = 13. This means that

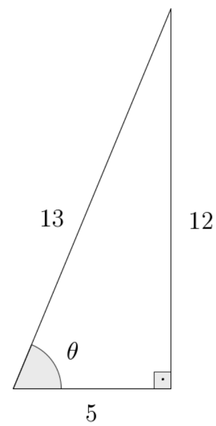
$$\sin\theta = \frac{12}{13}, \quad \cos\theta = \frac{5}{13}, \quad \text{and} \quad \tan\theta = \frac{12}{5}.$$

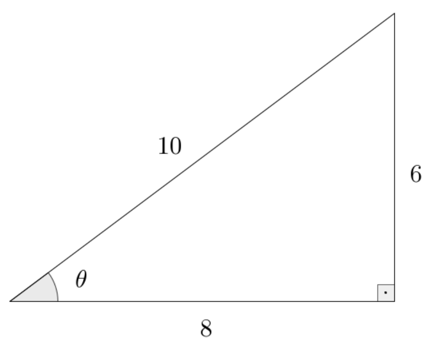
b. **Explanation** We have opp. = 8, adj. = 6 and hyp. = 10. This means that

$$\sin\theta = \frac{8}{10} = \frac{4}{5}, \quad \cos\theta = \frac{6}{10} = \frac{3}{5}, \quad \text{and} \quad \tan\theta = \frac{8}{10} = \frac{4}{3}.$$

c. **Explanation** We have opp. = 24, adj. = 18 and hyp. = 30. This means that

$$\sin\theta = \frac{24}{30} = \frac{4}{5}, \quad \cos\theta = \frac{18}{30} = \frac{3}{5}, \quad \text{and} \quad \tan\theta = \frac{24}{18} = \frac{4}{3}.$$





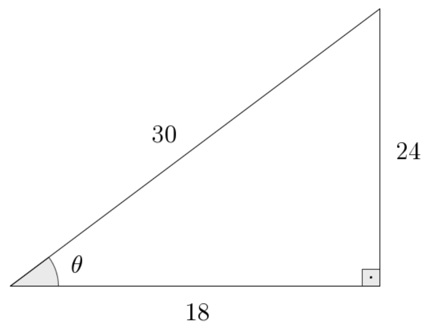
Note that the values were the same values as in the previous item. This was expected, as the triangle there is similar to the triangle given here (the scaling factor is 3).

Remark Note that in all of the above examples, the values of $\sin\theta$ and $\cos\theta$ were always less than 1. This is always true, and a general consequence of the fact that the hypotenuse is always bigger than either of the other two sides.

Often, one has information about the angles, but not about all the sides. Knowing $\sin\theta$, $\cos\theta$ and $\tan\theta$ helps us find out missing sides of a given right triangle. For that, the following fact is extremely important:

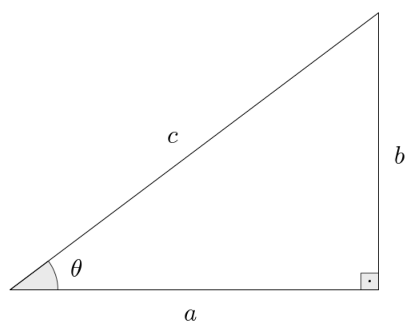
Theorem (Fundamental Identity): For any given angle θ, we have that

$$\sin^2\theta + \cos^2\theta = 1.$$



Here, $\sin^2\theta$ means $(\sin\theta)^2$, and similarly for $\cos^2\theta$.

Why is this true? Consider again a right triangle like below:



Then we know that $\sin\theta = b/c$ and $\cos\theta = a/c$. But the Pythagorean theorem also says that $a^2 + b^2 = c^2$. Putting all of this together, we have that

$$\sin^2\theta + \cos^2\theta = \left(\frac{b}{c}\right)^2 + \left(\frac{a}{c}\right)^2 = \frac{b^2}{c^2} + \frac{a^2}{c^2} = \frac{a^2+b^2}{c^2} = \frac{c^2}{c^2} = 1,$$

as required.

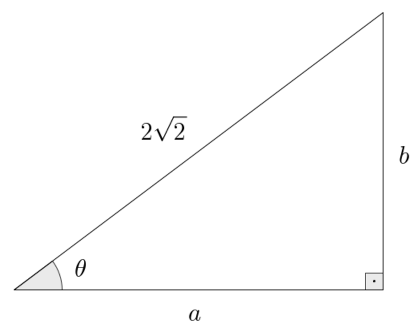
Let's see how to apply this.

Example 32. *For each of the following triangles, given the value of a trigonometric function at the indicated angle θ, find the lengths of the missing sides.*

a. *Given:* $\sin\theta = \sqrt{2}/6$ *on*

Explanation From the given information, we know that

$$\frac{\sqrt{2}}{6} = \sin\theta = \frac{b}{2\sqrt{2}} \implies b = \frac{\sqrt{2}\times(2\sqrt{2})}{6} = \frac{2}{3}.$$



Now we use the Pythagorean theorem: the relation $a^2 + (2/3)^2 = (2\sqrt{2})^2$ gives us that

$$a^2 + \frac{4}{9} = 8 \implies a^2 = 8 - \frac{4}{9} = \frac{68}{9} \implies a = \frac{2\sqrt{17}}{3}.$$

Alternatively, to find the value of a, we can also use the fundamental identity $\sin^2\theta + \cos^2\theta = 1$ to find $\cos\theta$ first — which then yields a. Here's how this goes:

$$\left(\frac{\sqrt{2}}{6}\right)^2 + \cos^2\theta = 1 \implies \cos^2\theta = 1 - \frac{2}{36} \implies \cos^2\theta = \frac{34}{36},$$

and so $\cos\theta = \sqrt{34}/6$. Thus

$$\frac{\sqrt{34}}{6} = \cos\theta = \frac{a}{2\sqrt{2}} \implies a = \frac{2\sqrt{2} \times \sqrt{34}}{6} = \frac{2\sqrt{17}}{3},$$

as it should be. This is not something particular to this example: usually there is more than one strategy to solve this sort of problem. Which one is the best? You'll be the judge.

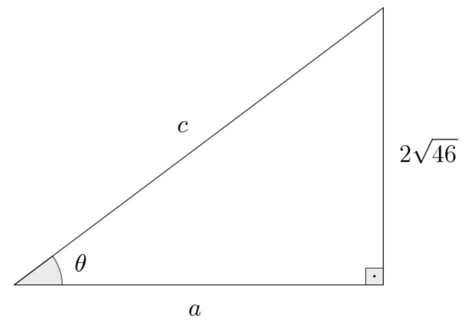
b. Given: $\cos\theta = \sqrt{3}/7$ *on*

Explanation Since this time we were given $\cos\theta$, but also the opposite side to θ, which does not appear on the expression for $\cos\theta$, we must rely more on the Pythagorean theorem instead. In any case, we know that

$$\frac{\sqrt{3}}{7} = \cos\theta = \frac{a}{c} \implies c = \frac{7a}{\sqrt{3}}.$$

Now, the Pythagorean relation reads $a^2 + (2\sqrt{46})^2 = (7a/\sqrt{3})^2$, and so:

$$a^2 + 184 = \frac{49a^2}{3} \implies 184 = \frac{49a^2}{3} - a^2$$



Continuing to manipulate this, we see that

$$184 = \frac{46a^2}{3} \implies a^2 = \frac{184 \times 3}{46} \implies a^2 = 12 \implies a = 2\sqrt{3}.$$

It remains to find the value of c. So we go back to the beginning and compute

$$c = \frac{7a}{\sqrt{3}} \implies c = \frac{7(2\sqrt{3})}{\sqrt{3}} \implies c = 14.$$

Values of trig functions for standard angles

We know that the sum of the inner angles of a triangle is always 180°. For right triangles, one of the angles is 90°, which means that the sum of the remaining two angles must also be 90°. Frequently we encounter triangles whose angles are 30°, 60° and 90°, and also triangles whose angles are 45°, 45° and 90°.

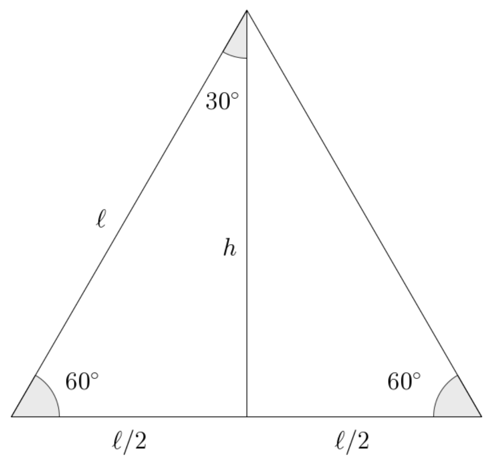
[figure]

These triangles have a special type of symmetry, which we'll exploit to find the values of sine, cosine, and tangent, for 30°, 45° and 60°. Finding the values of these trig functions for arbitrary angles, by hand, is a very difficult task. We will see later some trigonometric identities that may help us find such values for other angles but, in general, using a calculator (paying close attention to whether it is set to right "units") is the way to go.

For 30° **and** 60° Consider an equilateral triangle of side length ℓ. Equilateral means that all the sides have the same length. This implies that all the inner angles must be equal and, since they must add up to 180°, each of them equals 60°. But also draw a height h:

By the Pythagorean theorem, we know that

$$\ell^2 = \left(\frac{\ell}{2}\right)^2 + h^2,$$



and so we may compute:

$$\ell^2 = \frac{\ell^2}{4} + h^2 \implies \frac{3\ell^2}{4} = h^2 \implies h = \frac{\ell\sqrt{3}}{2}.$$

Now, relative to the 60° angle, we recognize

$$\text{opp.} = h = \frac{\ell\sqrt{3}}{2}, \quad \text{adj.} = \frac{\ell}{2}, \quad \text{and} \quad \text{hyp.} = \ell.$$

This means that

$$\sin(60^\circ) = \frac{h}{\ell} = \frac{\left(\frac{\ell\sqrt{3}}{2}\right)}{\ell} = \frac{\sqrt{3}}{2},$$

as well as

$$\cos(60^\circ) = \frac{\ell/2}{\ell} = \frac{1}{2} \quad \text{and} \quad \tan(60^\circ) = \frac{\sin(60^\circ)}{\cos(60^\circ)} = \frac{\sqrt{3}/2}{1/2} = \sqrt{3}.$$

To find the values of $\sin(30^\circ)$, $\cos(30^\circ)$, and $\tan(30^\circ)$, we can use the same triangle, noting that the opposite side to 30° is the adjacent side to 60°, and that the adjacent side to 30° is the opposite side to 60°. Since the hypotenuse is always the side opposite to the right angle, we conclude that

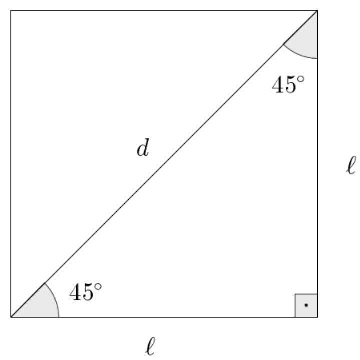
$$\sin(30^\circ) = \cos(60^\circ) = \frac{1}{2}, \quad \cos(30^\circ) = \sin(60^\circ) = \frac{\sqrt{3}}{2}$$

and, finally, that

$$\tan(30^\circ) = \frac{\sin(30^\circ)}{\cos(30^\circ)} = \frac{1/2}{\sqrt{3}/2} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}.$$

Remark This is a general phenomenon: two acute angles are called **complementary** if they add up to 90°. In other words, the complementary angle to θ is always $90^\circ - \theta$, and $\sin\theta = \cos(90^\circ - \theta)$, as well as $\cos\theta = \sin(90^\circ - \theta)$. In particular, this justifies the name "cosine": it is the sine of the complement. We will discuss "coterminal angles" and "cofunctions" in more generality later.

For 45° Consider a square of side length ℓ, and draw a diagonal d.



By the Pythagorean theorem, $d^2 = \ell^2 + \ell^2 = 2\ell^2$ implies that $d = \ell\sqrt{2}$. Relative to either of the 45° angles, we have

$$\text{opp.} = \ell, \quad \text{adj.} = \ell, \quad \text{and} \quad \text{hyp.} = d = \ell\sqrt{2}.$$

Hence

$$\sin(45^\circ) = \cos(45^\circ) = \frac{\ell}{\ell\sqrt{2}} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} \quad \text{and} \quad \tan(45^\circ) = \frac{\sin(45^\circ)}{\cos(45^\circ)} = 1.$$

Remark It is convenient to write $\sqrt{2}/2$ instead of $1/\sqrt{2}$ (similarly for $\sqrt{3}/3$ versus $1/\sqrt{3}$), even though the latter is mathematically acceptable, because it makes it easier to estimate. Namely, knowing that $\sqrt{2} \approx 1.414$, we know that $\sqrt{2}/2 \approx 0.707$, but when looking at $1/\sqrt{2}$, what does it mean to divide 1 by 1.414? This is the general reason why rationalizing fractions is useful.

Standard values

We can summarize what we have discovered here in a table. Besides our standard angles of 30°, 45°, and 60°, we can also consider 0° and 90° as extreme cases. Let's do a quick thought experiment to understand this: if a right triangle had an angle of 0°, this triangle would in fact collapse to a line segment, and we would have opp. $= 0$, while hyp. $=$ adj., suggesting we set $\sin(0^\circ) = 0$ and $\cos(0^\circ) = 1$. Since 0° and 90° are complementary, we're forced to set $\sin(90^\circ) = 1$ and $\cos(90^\circ) = 0$. But while

$$\tan(0^\circ) = \frac{\sin(0^\circ)}{\cos(0^\circ)} = \frac{0}{1} = 0,$$

computing $\tan(90^\circ)$ does not make sense, as we would have a division by $\cos(90^\circ) = 0$. We say that $\tan(90^\circ)$ is **undefined**, or that it **does not exist** ("DNE" for short, as usual). So, we have:

	0°	30°	45°	60°	90°
$\sin\theta$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
$\tan\theta$	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	DNE

Those values should be committed to heart, but it's easier than what it seems. Here's how you can think about it:

- No need to memorize values for tangent: if you know $\sin\theta$ and $\cos\theta$, you can just compute $\tan\theta = \sin\theta/\cos\theta$.
- No need to memorize the values for cosine: recall that the cosine of an angle is the sine of the complement. So if you know values for sine, you're in business.
- How to memorize values for sine? The one thing you should remember here is that the values 0, $1/2$, $\sqrt{2}/2$, $\sqrt{3}/2$ and 1 will appear. What is their order? Simple: write them in increasing order, just like the angles from 0° to 90°. So

$$\sin(0^\circ) = 0,\ \sin(30^\circ) = \frac{1}{2},\ \sin(45^\circ) = \frac{\sqrt{2}}{2},\ \sin(60^\circ) = \frac{\sqrt{3}}{2},\ \sin(90^\circ) = 1.$$

Summary

- We have defined sine, cosine, and tangent, as ratios between sides of a right triangle. For each angle θ, the fundamental identity $\sin^2\theta + \cos^2\theta = 1$ holds. It can be used together with the Pythagorean Theorem to get information about all sides of a given triangle, when some of them might be missing, provided you have some information about the angles.

- We have established the standard values of sine, cosine, and tangent, for the most frequent angles of $30°$, $45°$, and $60°$. Those values have been organized in a table. They are so frequent that knowing the values there by heart is useful, but exaggerated efforts into memorizing the table should not be wasted — understanding how the values are deduced pays off more in the long run.

8.1.2 Secant, Cosecant and Cotangent

Motivating Questions

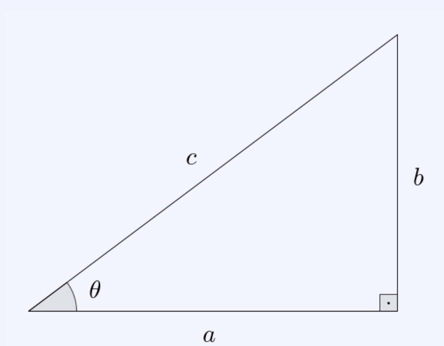
- How to use the reciprocal ratios of $\sin\theta$, $\cos\theta$, and $\tan\theta$, to also obtain information about a given right triangle?
- What are the values of such reciprocal rations for the standard angles of $30°$, $45°$ and $60°$?

Introduction

Briefly speaking, we have met three fundamental ratios between sides of a right triangle: sine, cosine, and tangent. But their reciprocals are also relevant ratios between the sides of the given triangle. Now, while such reciprocals ratios turn out to carry the same information as sine, cosine, and tangent, it is useful to know how to manipulate them as well. Later, when we study trigonometric functions as actual functions of a real parameter, discussing their graphs, symmetries, etc., more differences will become apparent.

Definitions and examples

Definition (right triangle trig – bis): Consider the following right triangle, with one angle θ indicated, and sides labeled a, b and c.



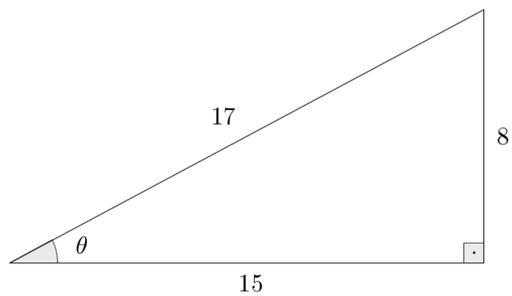
We define the **secant**, **cosecant**, and **cotangent** of θ, by

- $\sec\theta = \dfrac{c}{b} = \dfrac{1}{\cos\theta}\left(= \dfrac{\text{hyp.}}{\text{adj.}}\right)$;
- $\csc\theta = \dfrac{c}{a} = \dfrac{1}{\sin\theta}\left(= \dfrac{\text{hyp.}}{\text{opp.}}\right)$, and;

- $\cot\theta = \dfrac{a}{b} = \dfrac{1}{\tan\theta} \left(= \dfrac{\text{adj.}}{\text{opp.}}\right)$.

Note that since for acute angles we always have $\sin\theta$ and $\cos\theta$ between 0 and 1, the reciprocals $\csc\theta$ and $\sec\theta$ will always be bigger than 1.

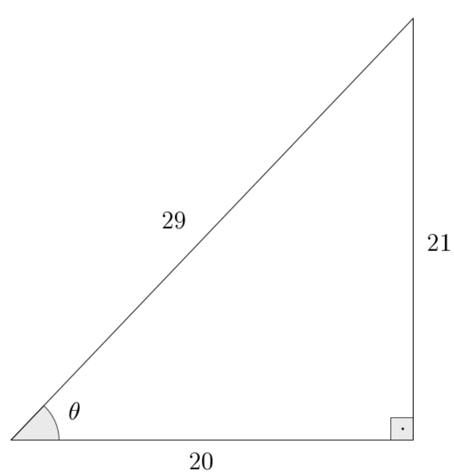
Example 33. *For each of the following triangles with a given angle θ, identify the adjacent (adj.), opposite (opp.) and hypotenuse (hyp.), and compute* $\sec\theta$*,* $\csc\theta$*, and* $\cot\theta$*.*



a. **Explanation** We have opp. = 8, adj. = 5 and hyp. = 17. This means that

$$\sec\theta = \frac{17}{15}, \quad \csc\theta = \frac{17}{8}, \quad \text{and} \quad \cot\theta = \frac{15}{8}.$$

Of course, you can find $\cos\theta$, $\sin\theta$, and $\tan\theta$ first, and then just flip all the fractions.



b. **Explanation** This time, we have opp. = 21, adj. = 20 and hyp. = 29.

This means that

$$\sec\theta = \frac{29}{20}, \quad \csc\theta = \frac{29}{21}, \quad \text{and} \quad \cot\theta = \frac{20}{21}.$$

Next, we had the fundamental identity $\sin^2\theta + \cos^2\theta = 1$. It turns out that with this, we may obtain two extra useful identities.

Theorem (Fundamental Identities – bis): For any given angle θ, we have that

$$1 + \cot^2\theta = \csc^2\theta \quad \text{and} \quad \tan^2\theta + 1 = \sec^2\theta,$$

where $\cot^2\theta$ means $(\cot\theta)^2$, and similarly for all other functions.

You should *not* think of those two extra identities as something more to be memorized. The only identity worth the trouble is $\sin^2\theta + \cos^2\theta = 1$. The following strategy is something you can quickly reproduce on a scrap paper if you need to recall these formulas, once you have understood the idea once:

- Divide both sides of $\sin^2\theta + \cos^2\theta = 1$ by $\sin^2\theta$:

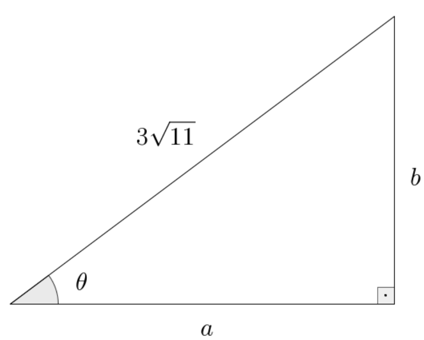
$$\frac{\sin^2\theta + \cos^2\theta}{\sin^2\theta} = \frac{1}{\sin^2\theta} \implies 1 + \frac{\cos^2\theta}{\sin^2\theta} = \csc^2\theta \implies 1 + \cot^2\theta = \csc^2\theta.$$

- Divide both sides of $\sin^2\theta + \cos^2\theta = 1$ by $\cos^2\theta$:

$$\frac{\sin^2\theta + \cos^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta} \implies \frac{\sin^2\theta}{\cos^2\theta} + 1 = \sec^2\theta \implies \tan^2\theta + 1 = \sec^2\theta.$$

Example 34. *For each of the given triangles, given the value of a trigonometric function at the indicated angle θ, find the lengths of the missing sides.*

a. *Given:* $\csc\theta = \sqrt{11}/2$ *on*



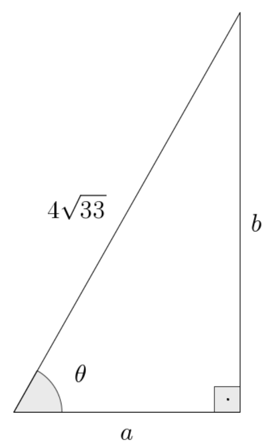
Explanation We start with

$$\frac{3\sqrt{11}}{a} = \csc\theta = \frac{\sqrt{11}}{2} \implies 6\sqrt{11} = a\sqrt{11} \implies a = 6.$$

It remains to find the value of b. This can be done with the Pythagorean Theorem, as follows: $a^2 + b^2 = c^2$ becomes

$$36 + b^2 = (3\sqrt{11})^2 \implies 36 + b^2 = 99 \implies b^2 = 63 \implies b = 3\sqrt{7}.$$

b. Given: $\cot\theta = 2\sqrt{2}/5$ *on*



Explanation Let's start again using the trigonometric function we were given:

$$\frac{a}{b} = \cot\theta = \frac{2\sqrt{2}}{5} \implies a = \frac{2b\sqrt{2}}{5}.$$

We cannot conclude anything else about a and b just from this, so we must resort to the Pythagorean Theorem again. The relation $a^2 + b^2 = c^2$ gives us that

$$\left(\frac{2b\sqrt{2}}{5}\right)^2 + b^2 = (4\sqrt{33})^2 \implies \frac{8b^2}{25} + b^2 = 528 \implies \frac{33b^2}{25} = 528.$$

Simplifying this, we have that

$$b^2 = 25 \times \frac{528}{33} = 25 \times 16 \implies b = 5 \times 4 \implies b = 20.$$

Now, we may go back and find a:

$$a = \frac{2b\sqrt{2}}{5} = \frac{40\sqrt{2}}{5} \implies b = 8\sqrt{2}.$$

Values of trig functions for standard angles – bis

Previously, we have obtained the following table of standard values for sine, cosine, and tangent:

	$0°$	$30°$	$45°$	$60°$	$90°$
$\sin\theta$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
$\tan\theta$	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	DNE

By simply inverting all of those values, we naturally obtain a similar table with standard values for secant, cosecant and cotangent. Of course, this is *not* another table you have to memorize, but we'll list it here for completeness:

	$0°$	$30°$	$45°$	$60°$	$90°$
$\csc\theta$	DNE	2	$\sqrt{2}$	$\frac{2\sqrt{3}}{3}$	1
$\sec\theta$	1	$\frac{2\sqrt{3}}{3}$	$\sqrt{2}$	2	DNE
$\cot\theta$	DNE	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$	0

Note the undefined "extreme" values: $\csc(0°)$ is undefined, because that would be $1/\sin(0°)$, but $\sin(0°) = 0$ and we cannot divide by zero. Similarly for the other ones.

Summary

- We have defined secant, cosecant, and cotangent, as the reciprocal ratios of cosine, sine, and tangent. For each angle θ, we have the associated fundamental identities $1 + \tan^2\theta = \sec^2\theta$ and $\cot^2\theta + 1 = \csc^2\theta$, which can be easily deduced from the good old $\sin^2\theta + \cos^2\theta = 1$. Again, such identities can be used together with the Pythagorean Theorem to obtain information about sides of a right triangle.

- We have summarized (again in a table) the standard values of secant, cosecant, and cotangent, for the most frequent angles of $30°$, $45°$, and $60°$.

8.1.3 All From One, One From All

Motivating Questions

- Do all the trigonometric functions we have seen so far carry the same information?
- How to find all trigonometric functions, given a single one of them?

Introduction

We have encountered six trigonometric functions of an acute angle θ so far: $\sin\theta$, $\cos\theta$, $\tan\theta$, $\sec\theta$, $\csc\theta$, and $\cot\theta$. They all help up get information about right triangles having θ as one of the inner angles. But here is the thing: at this stage, they all carry the same information. All of these quantities are positive real numbers, and we have not only the Pythagorean Theorem, but also the fundamental relations

$$\sin^2\theta + \cos^2\theta = 1$$
$$1 + \cot^2\theta = \csc^2\theta$$
$$\tan^2\theta + 1 = \sec^2\theta$$

Recall that while the first one is the most important one, the second two are immediate consequences of the first, by dividing it by $\sin^2\theta$ and $\cos^2\theta$, respectively. With all of this in place, once you have one of the six trigonometric values at θ, you can in fact find all of them. We'll explore this in this section, with several examples.

How to find all trigonometric functions, given one of them?

There are a few main facts one should keep in mind here.

- You know $\sin\theta$ if and only if you know $\csc\theta$.
- You know $\cos\theta$ if and only if you know $\sec\theta$.
- You know $\tan\theta$ if and only if you know $\cot\theta$.
- If you know $\sin\theta$ and $\cos\theta$, you know $\tan\theta$.

And there are two strategies: using just the trigonometric identities and proceeding algebraically (let's call this "strategy 1"), or drawing a suitable right triangle and thinking of opp., adj. and hyp. (let's call this "strategy 2"). We'll illustrate both of them with several examples, but in the end of the day, you may choose whichever strategy you'd like (unless specifically instructed otherwise).

Example 35. *Let θ be an acute angle. In all of the following problems, given the value of a certain trigonometric function at the value θ, find the remaining five.*

a. *Given:* $\sin\theta = 1/4$.

Explanation

- **Strategy 1:** Let's find $\cos\theta$ first, using the first fundamental identity $\sin^2\theta + \cos^2\theta = 1$. We have

$$\left(\frac{1}{4}\right)^2 + \cos^2\theta = 1 \implies \frac{1}{16} + \cos^2\theta = 1 \implies \cos^2\theta = 1 - \frac{1}{16},$$

so

$$\cos^2\theta = \frac{15}{16} \implies \cos\theta = \frac{\sqrt{15}}{4}.$$

With this, we have that

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{1/4}{\sqrt{15}/4} = \frac{1}{\sqrt{15}} = \frac{\sqrt{15}}{15}.$$

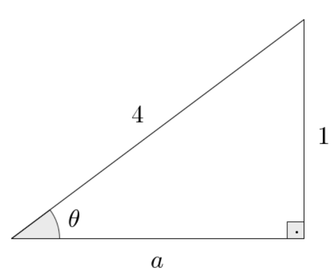
Flipping the fractions

$$\sin\theta = \frac{1}{4}, \quad \cos\theta = \frac{\sqrt{15}}{4} \quad \text{and} \quad \tan\theta = \frac{\sqrt{15}}{15},$$

respectively, we obtain

$$\csc\theta = 4, \quad \sec\theta = \frac{4}{\sqrt{15}} = \frac{4\sqrt{15}}{15}, \quad \text{and} \quad \cot\theta = \sqrt{15}.$$

- **Strategy 2:** We start drawing the following triangle: So, we find the



missing side a, using the Pythagorean relation $a^2 + b^2 = c^2$, which reads and gives:

$$a^2 + 1^2 = 4^2 \implies a^2 + 1 = 16 \implies a^2 = 15,$$

so that $a = \sqrt{15}$. Now we have all the sides, so finding all the ratios is immediate:

$$\sin\theta = \frac{1}{4}, \quad \cos\theta = \frac{\sqrt{15}}{4}, \quad \text{and} \quad \tan\theta = \frac{1}{\sqrt{15}} = \frac{\sqrt{15}}{15}.$$

Flipping all the fractions, we obtain

$$\csc\theta = 4, \quad \sec\theta = \frac{4\sqrt{15}}{15}, \quad \text{and} \quad \cot\theta = \sqrt{15}.$$

b. Given: $\cos\theta = 2/3$.

Explanation

- **Strategy 1:** Let's find $\sin\theta$ first, using the first fundamental identity $\sin^2\theta + \cos^2\theta = 1$. We have

  $$\sin^2\theta + \left(\frac{2}{3}\right)^2 = 1 \implies \sin^2\theta + \frac{4}{9} = 1 \implies \sin^2\theta = 1 - \frac{4}{9},$$

  so

  $$\sin^2\theta = \frac{5}{9} \implies \sin\theta = \frac{\sqrt{5}}{3}.$$

  With this, we have that

  $$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\sqrt{5}/3}{2/3} = \frac{\sqrt{5}}{2}.$$

  Flipping the fractions

  $$\sin\theta = \frac{\sqrt{5}}{3}, \quad \cos\theta = \frac{2}{3} \quad \text{and} \quad \tan\theta = \frac{\sqrt{5}}{2},$$

  respectively, we obtain

  $$\csc\theta = \frac{3}{\sqrt{5}} = \frac{3\sqrt{5}}{5} \quad \sec\theta = \frac{3}{2}, \quad \text{and} \quad \cot\theta = \frac{2}{\sqrt{5}} = \frac{2\sqrt{5}}{5}.$$

- **Strategy 2:** This time, here's the triangle we'll use: We'll use the Pythagorean relation $a^2+b^2 = c^2$ to find the missing side b, as follows:

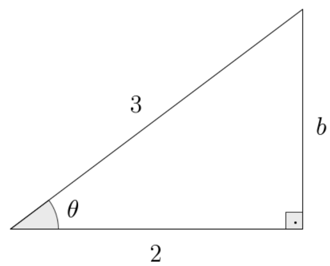
  $$2^2 + b^2 = 3^2 \implies 4 + b^2 = 9 \implies b^2 = 5,$$

  so $b = \sqrt{5}$. Again, with all the sides, we can read all the main ratios:

  $$\sin\theta = \frac{\sqrt{5}}{3}, \quad \cos\theta = \frac{2}{3}, \quad \text{and} \quad \tan\theta = \frac{\sqrt{5}}{3}.$$

  Taking reciprocals, we get the rest:

  $$\csc\theta = \frac{3\sqrt{5}}{5} \quad \sec\theta = \frac{3}{2}, \quad \text{and} \quad \cot\theta = \frac{2\sqrt{5}}{5}.$$



c. *Given:* $\tan\theta = 5/4$.

Explanation

- **Strategy 1:** The only fundamental identity we have involving $\tan\theta$ is $\tan^2\theta + 1 = \sec^2\theta$, so we might as well use it. It reads

$$\left(\frac{5}{4}\right)^2 + 1 = \sec^2\theta \implies \sec^2\theta = \frac{25}{16} + 1 \implies \sec^2\theta = \frac{41}{16},$$

and so

$$\sec\theta = \frac{\sqrt{41}}{4} \implies \cos\theta = \frac{4}{\sqrt{41}} = \frac{4\sqrt{41}}{41}.$$

With this, we could in principle find $\sin\theta$, by using $\sin^2\theta + \cos^2\theta = 1$ as usual. But there is a simpler way. Namely, we use that

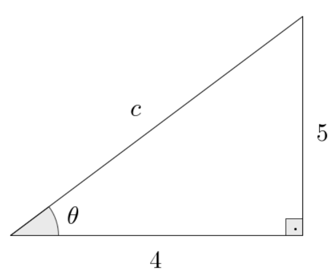
$$\tan\theta = \frac{\sin\theta}{\cos\theta} \implies \sin\theta = \tan\theta\cos\theta = \frac{5}{4}\cdot\frac{4\sqrt{41}}{41} \implies \sin\theta = \frac{5\sqrt{41}}{41}.$$

Meaning that

$$\csc\theta = \frac{\sqrt{41}}{5} \quad \text{and} \quad \cot\theta = \frac{4}{5},$$

and we are done.

- **Strategy 2:** Now, we set a right triangle with legs 4 and 5, like below:



Since the hypotenuse c is missing, applying the Pythagorean Theorem is even easier:

$$c^2 = 4^2 + 5^2 = 16 + 25 = 41 \implies c = \sqrt{41}.$$

With this in place, we read from the triangle the main ratios as

$$\sin\theta = \frac{5\sqrt{41}}{41}, \quad \cos\theta = \frac{4\sqrt{41}}{41} \quad \text{and} \quad \tan\theta = \frac{5}{4}.$$

And taking reciprocals:

$$\csc\theta = \frac{\sqrt{41}}{5}, \quad \sec\theta = \frac{\sqrt{41}}{4} \quad \text{and} \quad \cot\theta = \frac{4}{5}.$$

d. *Given:* $\sec\theta = 7/3$.

Explanation

- **Strategy 1:** We immediately know that $\cos\theta = 3/7$, so let's find $\sin\theta$ next, using the first fundamental identity $\sin^2\theta + \cos^2\theta = 1$. We have

$$\sin^2\theta + \left(\frac{3}{7}\right)^2 = 1 \implies \sin^2\theta + \frac{9}{49} = 1 \implies \sin^2\theta = 1 - \frac{9}{49},$$

so

$$\sin^2\theta = \frac{40}{49} \implies \sin\theta = \frac{2\sqrt{10}}{7}.$$

With this, we have that

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{2\sqrt{10}/7}{3/7} = \frac{2\sqrt{10}}{3}.$$

Flipping the remaining fractions

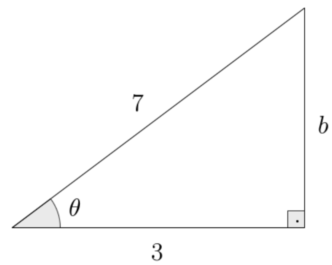
$$\sin\theta = \frac{2\sqrt{10}}{7}, \quad \text{and} \quad \tan\theta = \frac{2\sqrt{10}}{3},$$

respectively, we obtain

$$\csc\theta = \frac{7}{2\sqrt{10}} = \frac{7\sqrt{10}}{20} \quad \text{and} \quad \cot\theta = \frac{3}{2\sqrt{10}} = \frac{3\sqrt{10}}{20}.$$

- **Strategy 2:** Let's draw a triangle with hypotenuse 7 and adjacent side to θ having length 3: Let's use, as usual, the Pythagorean relation $a^2 + b^2 = c^2$ to find b. It gives us that

$$3^2 + b^2 = 7^2 \implies 9 + b^2 = 49 \implies b^2 = 40 \implies b = 2\sqrt{10}.$$



So we have that

$$\sin\theta = \frac{2\sqrt{10}}{7}, \quad \cos\theta = \frac{3}{7}, \quad \text{and} \quad \tan\theta = \frac{2\sqrt{10}}{3}.$$

Taking reciprocals and rationalizing each of them, we also get

$$\csc\theta = \frac{7\sqrt{10}}{20}, \quad \sec\theta = \frac{7}{3}, \quad \text{and} \quad \cot\theta = \frac{3\sqrt{10}}{20}.$$

e. Given: $\csc\theta = 8/7$.

Explanation

- **Strategy 1:** We immediately know that $\sin\theta = 7/8$, so let's find $\cos\theta$ next, using the first fundamental identity $\sin^2\theta + \cos^2\theta = 1$. We have

$$\left(\frac{7}{8}\right)^2 + \cos^2\theta = 1 \implies \frac{49}{64} + \cos^2\theta = 1 \implies \cos^2\theta = 1 - \frac{49}{64},$$

so

$$\cos^2\theta = \frac{15}{64} \implies \cos\theta = \frac{\sqrt{15}}{8}.$$

With this, we have that

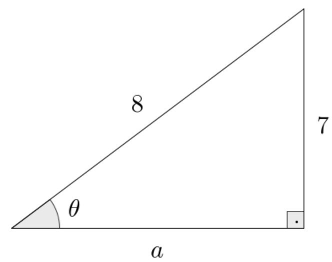
$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{7/8}{\sqrt{15}/8} = \frac{7}{\sqrt{15}} = \frac{7\sqrt{15}}{15}.$$

Flipping the remaining fractions

$$\cos\theta = \frac{\sqrt{15}}{8}, \quad \text{and} \quad \tan\theta = \frac{7}{\sqrt{15}},$$

respectively, we obtain

$$\sec\theta = \frac{8}{\sqrt{15}} = \frac{8\sqrt{15}}{15} \quad \text{and} \quad \cot\theta = \frac{\sqrt{15}}{7}.$$



- **Strategy 2:** Consider the following triangle: Let's find a using the Pythagorean relation $a^2 + b^2 = c^2$, which becomes

$$a^2 + 7^2 = 8^2 \implies a^2 + 49 = 64 \implies a^2 = 15,$$

so that $a = \sqrt{15}$. Having all the sides of the triangle, we read that

$$\sin\theta = \frac{7}{8}, \quad \cos\theta = \frac{\sqrt{15}}{8}, \quad \text{and} \quad \tan\theta = \frac{7\sqrt{15}}{15}.$$

Taking reciprocals, we get

$$\csc\theta = \frac{8}{7}, \quad \sec\theta = \frac{8\sqrt{15}}{15}, \quad \text{and} \quad \cot\theta = \frac{\sqrt{15}}{7}$$

as well.

f. Given: $\cot\theta = 2/9$.

Explanation

- **Strategy 1:** We immediately know that $\tan\theta = 9/2$ and, again, the only fundamental identity we have involving $\tan\theta$ is $\tan^2\theta + 1 = \sec^2\theta$. It reads

$$\left(\frac{9}{2}\right)^2 + 1 = \sec^2\theta \implies \sec^2\theta = \frac{81}{4} + 1 \implies \sec^2\theta = \frac{85}{4},$$

and so

$$\sec\theta = \frac{\sqrt{85}}{2} \implies \cos\theta = \frac{2}{\sqrt{85}} = \frac{2\sqrt{85}}{85}.$$

Again, instead of using $\sin^2\theta + \cos^2\theta = 1$ to find $\sin\theta$, we can just argue that

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \implies \sin\theta = \tan\theta\cos\theta = \frac{9}{2}\cdot\frac{2\sqrt{85}}{85} \implies \sin\theta = \frac{9\sqrt{85}}{85}.$$

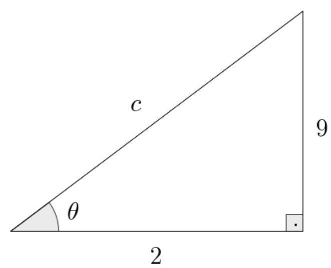
Meaning that

$$\csc\theta = \frac{\sqrt{85}}{9},$$

and so we are done.

As an aside, try to solve this problem again without immediately using that $\tan\theta = 9/2$, start only with $\cot\theta = 2/9$, use the identity $1 + \cot^2\theta = \csc^2\theta$ and go from there, it is instructive.

- **Strategy 2:** Again, let's set up a convenient triangle: Here, in par-



ticular, note that the scale and proportions in this picture are completely off. This is ok, since drawing triangles is just meant to help us organize what is the opposite side to θ, what is the adjacent side, and what is the hypotenuse. It doesn't matter how bad your picture looks, as long as the "positions" are correct. In any case, we immediately find c with

$$c^2 = a^2 + b^2 = 2^2 + 9^2 = 4 + 81 = 85,$$

so $c = \sqrt{85}$. Now we can read all the ratios and rationalize them to obtain:

$$\sin\theta = \frac{9\sqrt{85}}{85}, \quad \cos\theta = \frac{2\sqrt{85}}{85}, \quad \text{and} \quad \tan\theta = \frac{9}{2}.$$

Take reciprocals and rationalize whatever is needed to get

$$\csc\theta = \frac{\sqrt{85}}{9}, \quad \sec\theta = \frac{\sqrt{85}}{2}, \quad \text{and} \quad \cot\theta = \frac{2}{9}$$

as well.

Let's summarize the highlights of the strategy, from the algebraic perspective.

(a) If you're given sin or cos, use the fundamental identity to find the other one. Then find tan = sin / cos, and flip all the fractions to get csc, sec and tan.

(b) If you're given csc or sec, flip it to get sin or cos, and proceed as (a).

(c) If you're given tan, use $\tan^2\theta + 1 = \sec^2\theta$ to find $\sec\theta$. Once you have $\sec\theta$, you have $\cos\theta$. Then proceed as (a).

(d) If you're given cot, flip it to get tan, and proceed as (c).

Note that we're employing a mathematician's general philosophy here: take a problem and reduce it to something which you already know how to solve (namely, we're arguing that — morally — if you know how to solve the problem when you were given either sin or cos, then you in fact know how to solve it when given *any* of the six trigonometric functions). And also from the geometric perspective, the strategy is even easier to describe: recognize the trigonometric function you were given in terms of opp., adj. and hyp., then draw a right triangle with this information. You will be missing one side, which can be found with the Pythagorean Theorem. Once you have all sides, you can find all the ratios between sides.

Of course, the two above ways to go about this are not the only ones, but they're as good a recipe as any. In any case, you have room for creativity here. And even if one method seems easier than the other, it is useful to be comfortable with both, as this is already a good chance to start getting acquainted with trigonometry identities, which will be indispensable later.

We will see later how to define and deal with trigonometric functions for angles which are not necessarily acute. Then, everything we did here becomes slightly more subtle, as one must now pay attention to signs (for example, we'll have that $\cos(120°) = -1/2$). But the overall program of using the fundamental trigonometric identities and the relations between the main trigonometric functions (sin, cos, and tan) with their reciprocals (csc, sec, and cot) will always be useful.

Summary

- We have illustrated, with several examples, two ways to find all the values of the trigonometric functions at an acute angle θ, once we know one of the values. This can be done algebraically by exploring trigonometric identities, or geometrically by drawing the "correct" triangle and applying the Pythagorean Theorem to find the missing side – to then read all ratios directly from the triangle itself.

8.2 The Unit Circle

Learning Objectives

- The Unit Circle
  - Degrees and Radians
  - Reference Angles
  - The Definition of trigonometric functions in terms of the Unit Circle
  - Evaluating trigonometric functions at standard angles

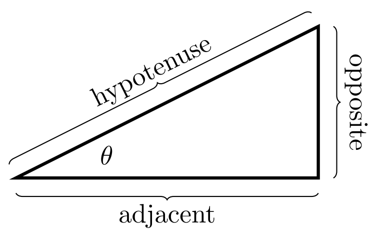
8.2.1 Unit Circle

Motivating Questions

- How can we define trigonometric functions for angles that do not come from triangles?

Introduction

In the previous sections, you were introduced to the basic trigonometric functions sine and cosine, and saw how they relate measures of angles to measurements of triangles. Given a right triangle



[2in]

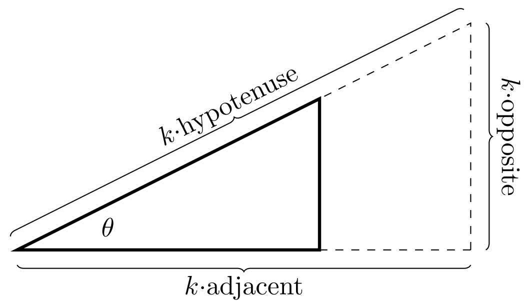
we define

$$\cos(\theta) = \frac{\text{adjacent}}{\text{hypotenuse}} \qquad \text{and} \qquad \sin(\theta) = \frac{\text{opposite}}{\text{hypotenuse}}.$$

There is a limitation in this, which you may have noticed. We can only build a triangle with a base angle θ if θ is between 0° and 90°. We work now to rectify this deficiency.

The Unit Circle

First, note that the values of sine and cosine do not depend on the scale of the triangle. Being very explicit, if we take our triangle and scale it up by a factor of k (multiplying each side length by k) we obtain



[2in]

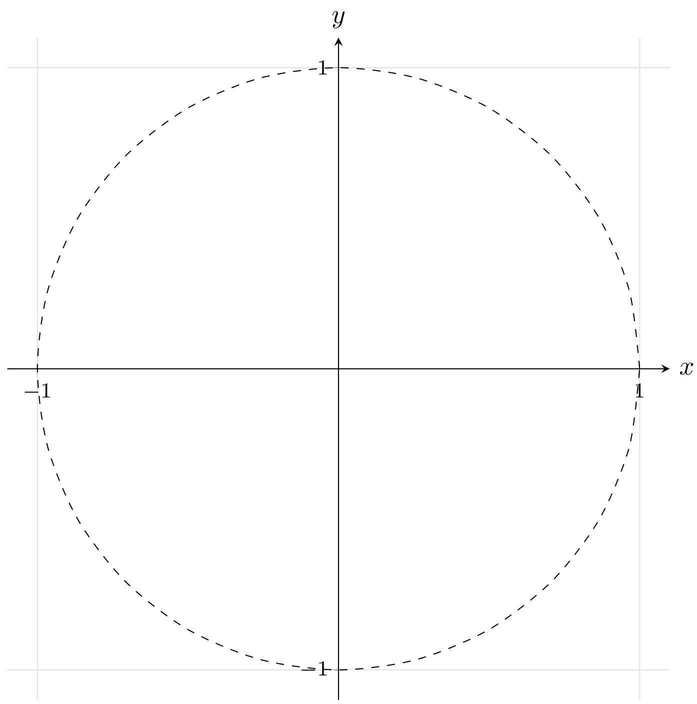
$$\cos(\theta) = \frac{k \cdot \text{adjacent}}{k \cdot \text{hypotenuse}} = \frac{\text{adjacent}}{\text{hypotenuse}}$$

and

$$\sin(\theta) = \frac{k \cdot \text{opposite}}{k \cdot \text{hypotenuse}} = \frac{\text{opposite}}{\text{hypotenuse}}.$$

Notice that the *ratios* of the corresponding side lengths are not changed. The individual side lengths are changed, but the ratios are preserved.

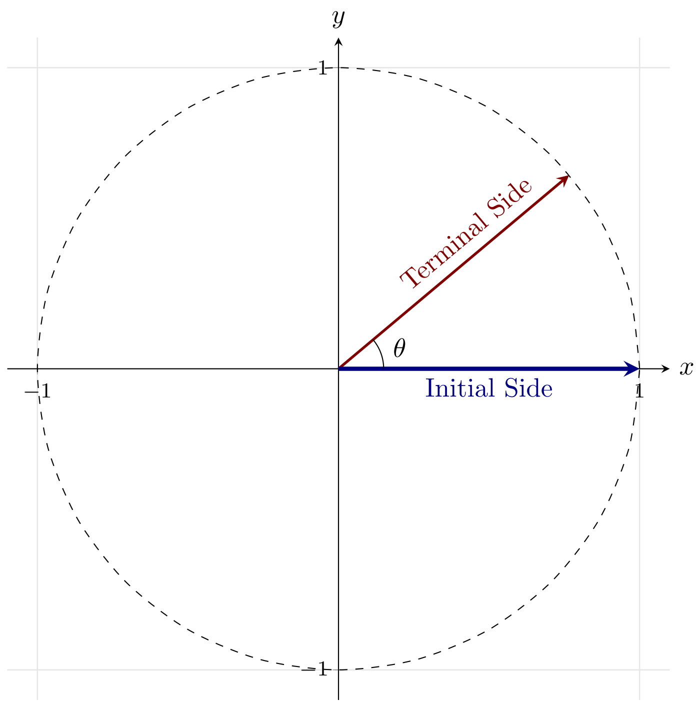
Because of this we could simply assume that whenever we draw a triangle for computing sine and cosine, that the hypotenuse will have length 1 (by dividing each side by the length of the hypotenuse). We can do this because we are simply scaling the triangle, and as we see above, this makes absolutely no difference when computing sine and cosine. When the hypotenuse is 1, we find that a convenient way to think about sine and cosine is via a circle.

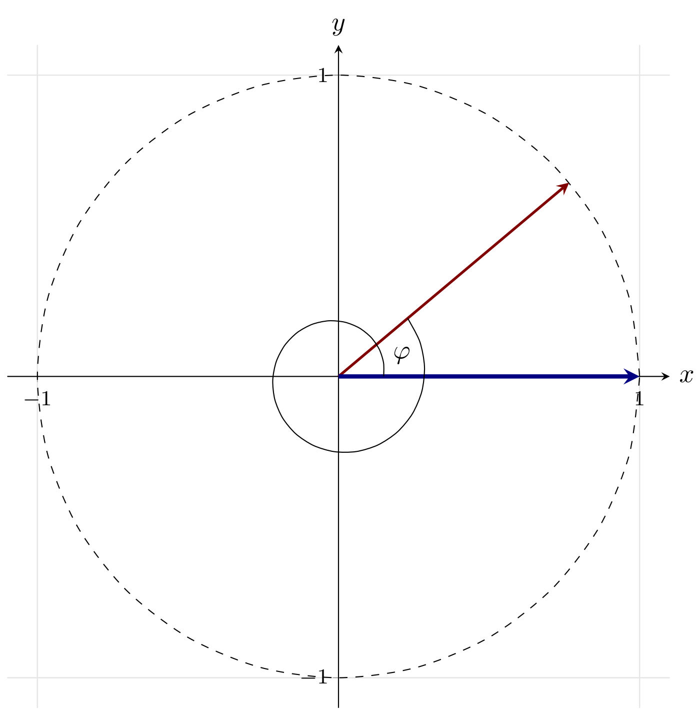


We call this the *unit circle*.

> **Definition** The **Unit Circle** is the circle of radius 1, with center at the origin. It is the graph of the equation $x^2 + y^2 = 1$.

An angle θ is in *standard position* if the vertex of the angle is at the origin and one side oriented along the positive x-axis. The ray along the positive x-axis is called the *initial side* of the angle, and the other ray is called the *terminal side* of the angle. The angle can be thought of as the counter-clockwise rotation necessary to spin the initial side to the terminal side.


y
1
Terminal Side
θ
x
−1
Initial Side
1
−1


y
1
φ
x
−1
1
−1

Notice that the angles θ and φ from the two images above both rotate the initial side of the angle to the same terminal side, but the angle φ wraps around the origin first. Two angles are *coterminal* if they have the same terminal side. The angles θ and φ are coterminal.

You can also think about an angle wrapping two, three, or four times before getting to the terminal side. You can also think about rotating clockwise instead of counter-clockwise. We consider counter-clockwise the positive direction, and clockwise the negative direction for angles.

Example 36. *Find two angles that are coterminal with* $30°$*, one positive and one negative.*

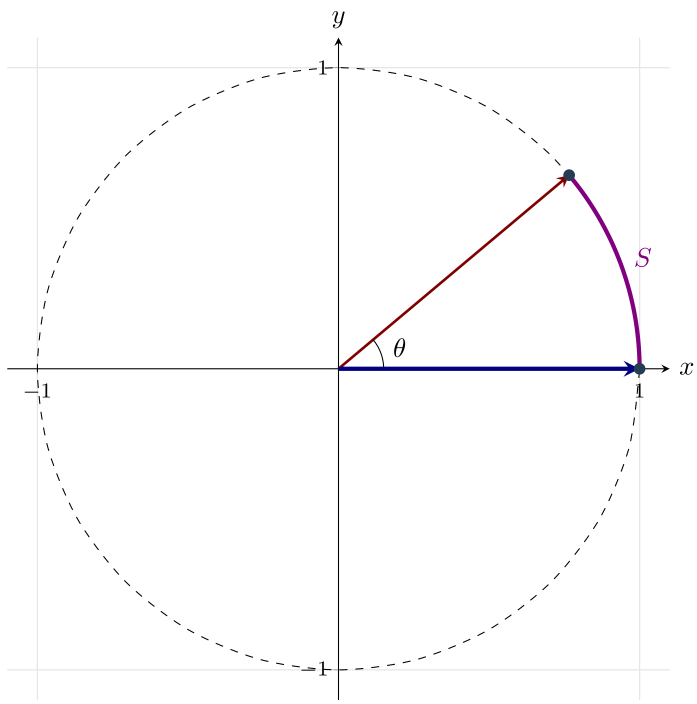
Explanation

If we start with $30°$, and then take another complete counter-clockwise rotation ($360°$), we end up at $390°$. That means $390°$ is coterminal with $30°$.

If we start with $30°$, and then take another complete clockwise rotation ($-360°$), we end up at $30° - 360° = -330°$. That means $-330°$ is coterminal with $30°$.

Radians

In everyday life, we typically measure angles in degrees. You will see in Calculus that using degrees can lead to a lot of complications. There is a better choice, more closely related to the geometry of the circle. Notice that an angle identifies an arc along the circumerence of the unit circle? We'll call the arc constructed this way the *subtended arc*.



Notice that as θ grows (counter-clockwise) from $0°$ the length of this arc, called S, also grows.

Example 37. *If θ is a right angle, calculate the length S.*

Explanation

If $\theta = 90°$, the arc is a quarter of the circle. The circumference of the circle is $C = 2\pi r = 2\pi(1) = 2\pi$. Since S will be a quarter of that, $S = \frac{1}{4}(2\pi) = \frac{\pi}{2}$.

The units we will be measuring angles, *radians*, is actually based on arc lengths like this.

> **Definition** One **radian** is the angle which, in standard position, subtends an arc of length 1.

That is, an angle measuring 1 radian has $S = 1$. Let us suppose that the radius of the circle, and therefore the length of the subtended arc, has units. This means that an angle θ of 1 radian, in a circle of radius 1 unit, subtends an arclength of $S = 1$ unit. We know that the formula for arclength is given by $S = R\theta$, so that $\theta = \frac{S}{R}$. That means 1 radian is equal to $\frac{1\text{unit}}{1\text{unit}}$. Notice that the "units" cancel out? That means **radians are a unitless unit**. When our angle is measured in radians, that angle is really just a number.

In one complete revolution (360°) we have subtended the entire circle so $S = 2\pi$. Based on this and the definition of the radian above, one complete revolution measures 2π radians. That is an angle measuring 360° measures 2π radians. This gives us a way to convert between degrees and radians! Note that $\frac{360}{2\pi}$ can be reduced to $\frac{180}{\pi}$.

- To convert from radians to degrees, multiply by the factor $\frac{180}{\pi}$.
- To convert from degrees to radians, multiply by the factor $\frac{\pi}{180}$.

Here we're thinking about $\frac{180}{\pi}$ as having units $\frac{\text{degrees}}{\text{radians}}$. What units do you think $\frac{\pi}{180}$ should have?

Example 38. (a) *Convert* 30°, 45°, *and* 90° *to radians.*

(b) *Convert* $\frac{\pi}{3}$ *radians,* π *radians, and* $-\frac{\pi}{10}$ *radians to degrees.*

Explanation

(a) $30 \cdot \left(\frac{\pi}{180}\right) = \frac{30\pi}{180} = \frac{\pi}{6}$. That means 30° is equivalent to $\frac{\pi}{6}$ radians.

$45 \cdot \left(\frac{\pi}{180}\right) = \frac{45\pi}{180} = \frac{\pi}{4}$. That means 45° is equivalent to $\frac{\pi}{4}$ radians.

$90 \cdot \left(\frac{\pi}{180}\right) = \frac{90\pi}{180} = \frac{\pi}{2}$. That means 90° is equivalent to $\frac{\pi}{2}$ radians.

(b) $\frac{\pi}{3} \cdot \left(\frac{180}{\pi}\right) = \frac{180\pi}{3\pi} = 60$. That means $\frac{\pi}{3}$ radians is equivalent to 60°.

$\pi \cdot \left(\frac{180}{\pi}\right) = \frac{180\pi}{\pi} = 180$. That means π radians is equivalent to 180°.

$-\frac{\pi}{10} \cdot \left(\frac{180}{\pi}\right) = -\frac{180\pi}{10\pi} = -18$. That means $-\frac{\pi}{10}$ radians is equivalent to -18°.

Frequently we will describe angles by their quadrants. An angle will be called a *first quadrant angle* if its terminal side lies in the first quadrant. Any angle in the interval $\left(0, \frac{\pi}{2}\right)$ will be a first quadrant angle, but there are others. For example, $\frac{9\pi}{4}$ is a first quadrant angle since it is coterminal with $\frac{\pi}{4} = \frac{9\pi}{4} - 2\pi$. Similarly we will call an angle a *second quadrant angle* if its terminal side lies in the second quadrant. These angles are coterminal to angles with measures $\left(\frac{\pi}{2}, \pi\right)$. *Third quadrant angles* and *fourth quadrant angles* are defined similarly.

The radian measure of some standard angles are given in the chart below.

Degrees	Radians
$0°$	0
$30°$	$\frac{\pi}{6}$
$45°$	$\frac{\pi}{4}$
$60°$	$\frac{\pi}{3}$
$90°$	$\frac{\pi}{2}$

Triangles in the Unit Circle

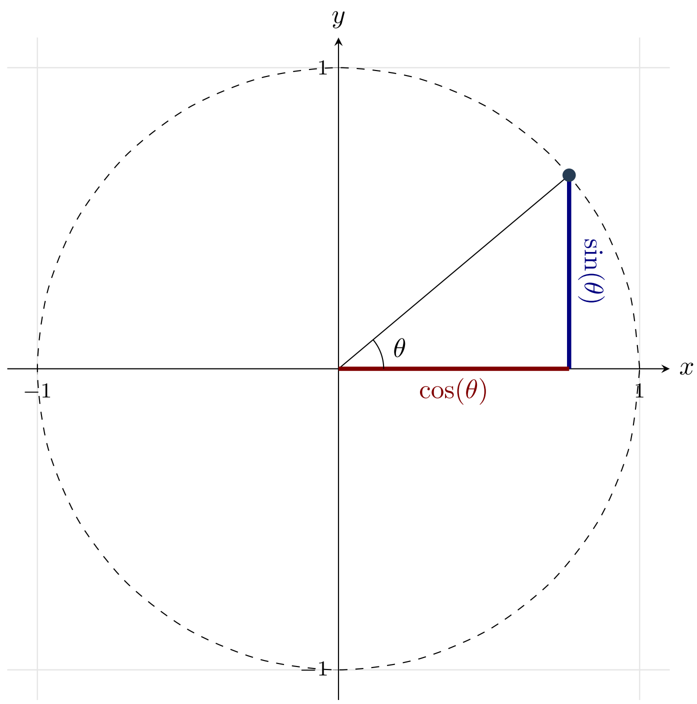
Let's draw our right triangle from before with the base angle θ in standard position, and scaled to have hypotenuse of length 1. Remember that since the hypotenuse has length 1, we know that

$$\sin(\theta) = \frac{\text{opp}}{\text{hyp}} = \text{opp}$$

and

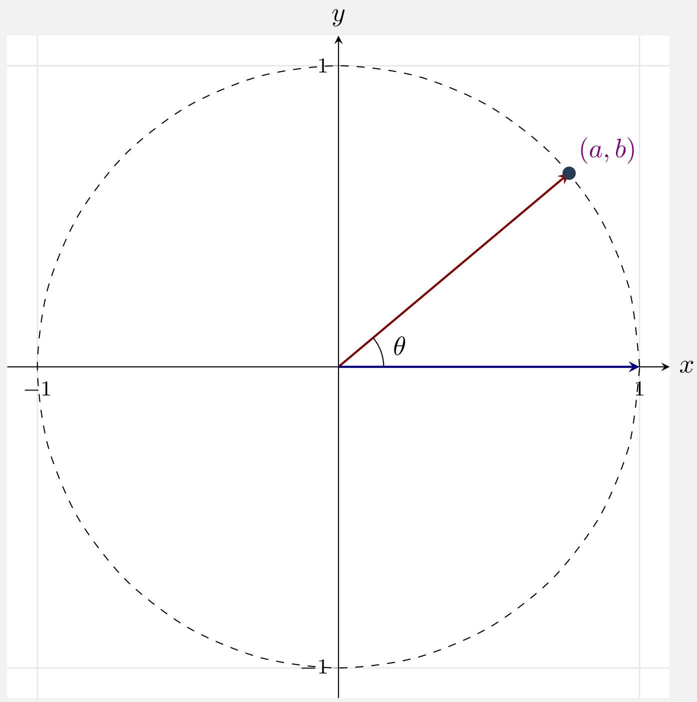
$$\cos(\theta) = \frac{\text{adj}}{\text{hyp}} = \text{adj}.$$

When we scale our triangle to have hypotenuse of length 1, sine and cosine measure the lengths of the opposite and adjacent sides. The triangle in the figure below has its side lengths labeled with this in mind.



If we consider the hypotenuse of this triangle as terminal side of θ, the point where this terminal side intersects the unit circle has coordinates $(\cos(\theta), \sin(\theta))$. This has given us our method to extend trigonometric functions to all angles, instead of just triangles.

Definition Suppose θ is an angle in standard position in the unit circle, and denote by (a, b) the coordinates of the point where the terminal side of θ intersects the unit circle.



$\sin(\theta) = b$ $\qquad$ $\cos(\theta) = a$

$\tan(\theta) = \dfrac{b}{a}$, if $a \neq 0$ $\qquad$ $\sec(\theta) \quad = \dfrac{1}{a}$, if $a \neq 0$ $\qquad$ $\cot(\theta) = \dfrac{a}{b}$, if $b \neq 0.$ $\csc(\theta) \quad = \dfrac{1}{b}$, if $b \neq 0$

The domain of sine and cosine is all real numbers, and the other trigonometric functions are defined precisely when their denominators are nonzero.

Example 39. *Which of the following expressions are equal to* $\sec(\theta)$*?*

(a) $\dfrac{1}{\cos(\theta)}$

(b) $\dfrac{1}{\sin(\theta)}$

(c) $\dfrac{adj}{hyp}$

(d) $\dfrac{hyp}{adj}$

(e) $\dfrac{\tan(\theta)}{\sin(\theta)}$

(f) $\dfrac{1}{\sin(\theta)\cdot\cot(\theta)}$

Explanation Given the angles and intersection point (a, b) from the definition above:

(a) $\cos(\theta) = a$ so $\sec(\theta) = \dfrac{1}{a} = \dfrac{1}{\cos(\theta)}$, provided that $a \neq 0$. This one is correct.

(b) $\sin(\theta) = b$ so $\dfrac{1}{\sin(\theta)} = \dfrac{1}{b}$, provided that $b \neq 0$. This is NOT $\sec(\theta)$.

(c) $\dfrac{\text{adj}}{\text{hyp}} = \dfrac{a}{1} = a$. This is $\cos(\theta)$, not $\sec(\theta)$.

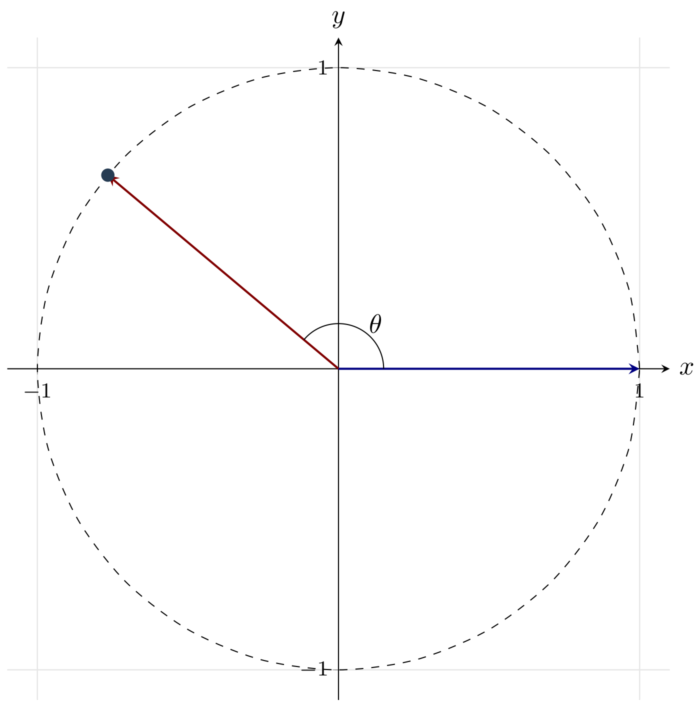
(d) $\dfrac{\text{hyp}}{\text{adj}} = \dfrac{1}{a}$, provided that $a \neq 0$. This one is also correct.

(e) $\dfrac{\tan(\theta)}{\sin(\theta)} = \dfrac{\left(\frac{b}{a}\right)}{b} = \dfrac{1}{a}$, provided that BOTH $a \neq 0$ AND $b \neq 0$. For example, when $\theta = 0$ this fraction is undefined but $\sec(0) = \dfrac{1}{1} = 1$. That means $\dfrac{\tan(\theta)}{\sin(\theta)}$ is not always the same as $\sec(\theta)$.

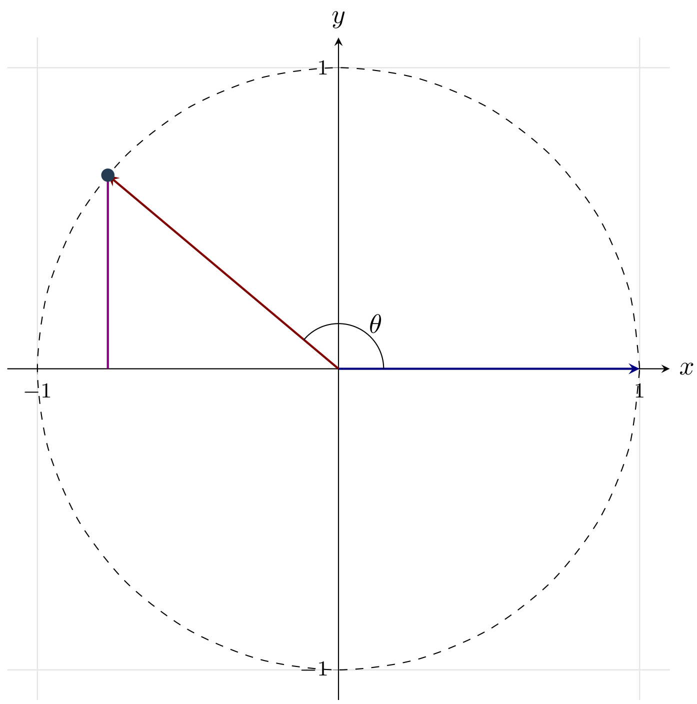
(f) $\dfrac{1}{\sin(\theta)\cdot\cot(\theta)} = \dfrac{1}{b\left(\frac{a}{b}\right)} = \dfrac{1}{a}$, provided that BOTH $a \neq 0$ AND $b \neq 0$. For example, when $\theta = 0$ this fraction is undefined but $\sec(0) = \dfrac{1}{1} = 1$. That means $\dfrac{\tan(\theta)}{\sin(\theta)}$ is not always the same as $\sec(\theta)$.

Reference Angles

We've seen above how to draw a (scaled version of) a right triangle inside the unit circle, with it's base angle in standard position. How about the other way around? If we have an angle that isn't necessarily an acute angle (one whose terminal side lies within the first quadrant), would it be possible to relate it to a triangle? Consider the second quadrant angle θ in the following image.



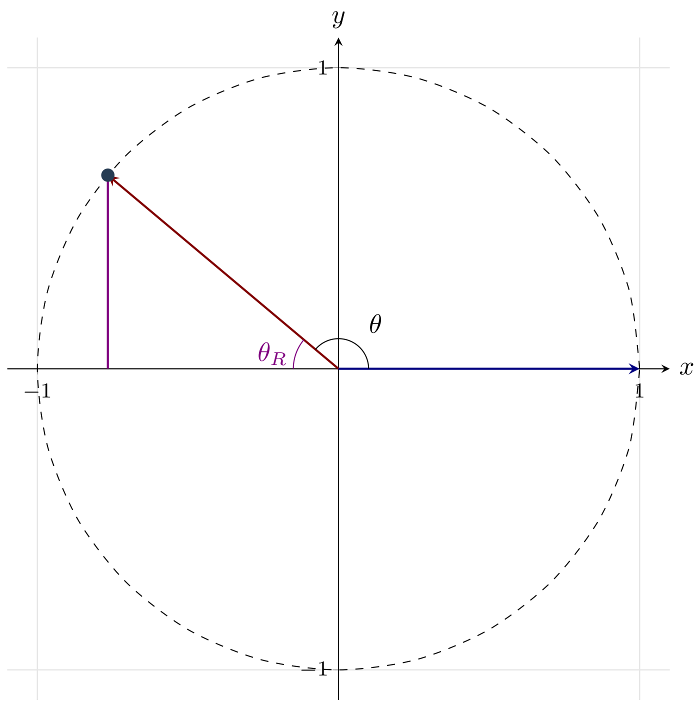
As before, we can draw a vertical line from the point where the terminal side of θ intersects the unit circle to the x-axis.



No matter the quandrant θ lies in, we can always construct a triangle by drawing this vertical side between the x-axis and the intersection point. Notice that this triangle has an acute angle with vertex at the origin.

> **Definition** Suppose θ is an angle in standard position. The **reference angle**, θ_R, is the *acute* angle between the terminal side and the x-axis.

If the terminal side of θ is along the x-axis (in either direction), in which case we will have $\theta_R = 0$. However if the terminal side of θ lies along the y-axis (in either direction) we will have $\theta_R = \dfrac{\pi}{2}$. A reference angle is never less than 0, nor greater than $\dfrac{\pi}{2}$.



Exercise 1 *Find the reference angle for each of the following angles.*

(a) $\alpha = \dfrac{5\pi}{9}$

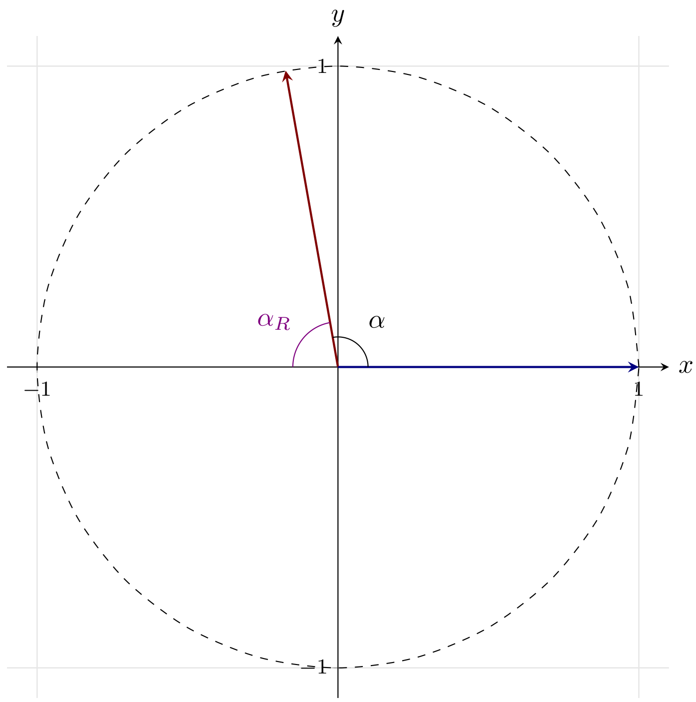
(b) $\varphi = \dfrac{7\pi}{5}$

(c) $\theta = \dfrac{23\pi}{3}$

(d) $\gamma = 7$

Explanation

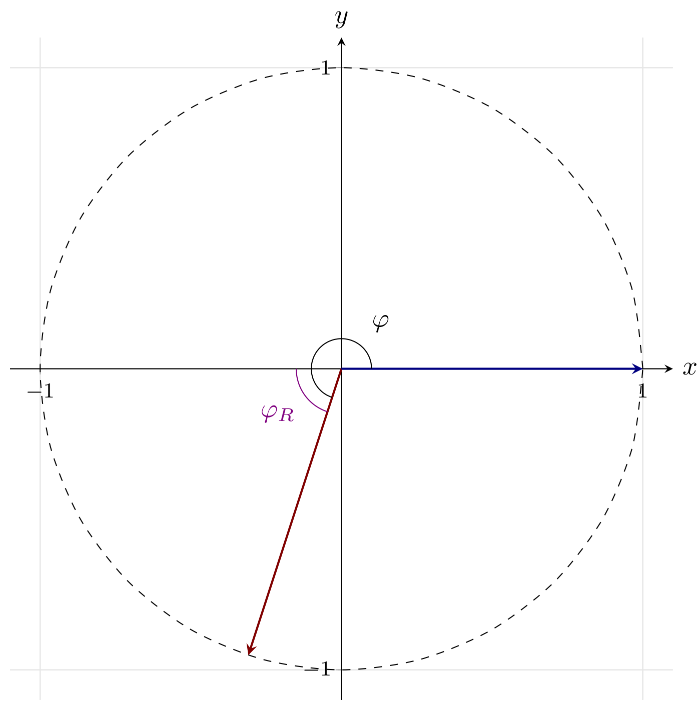
(a) The angle $\dfrac{5\pi}{9}$ is between $\dfrac{\pi}{2}$ and π, so α is in the second quadrant.



Since π is further (in the counter-clockwise direction) than α, we have

$$\begin{aligned}\alpha_R &= \pi - \alpha \\ &= \pi - \frac{5\pi}{9} \\ &= \frac{9\pi}{9} - \frac{5\pi}{9} \\ &= \frac{4\pi}{9}.\end{aligned}$$

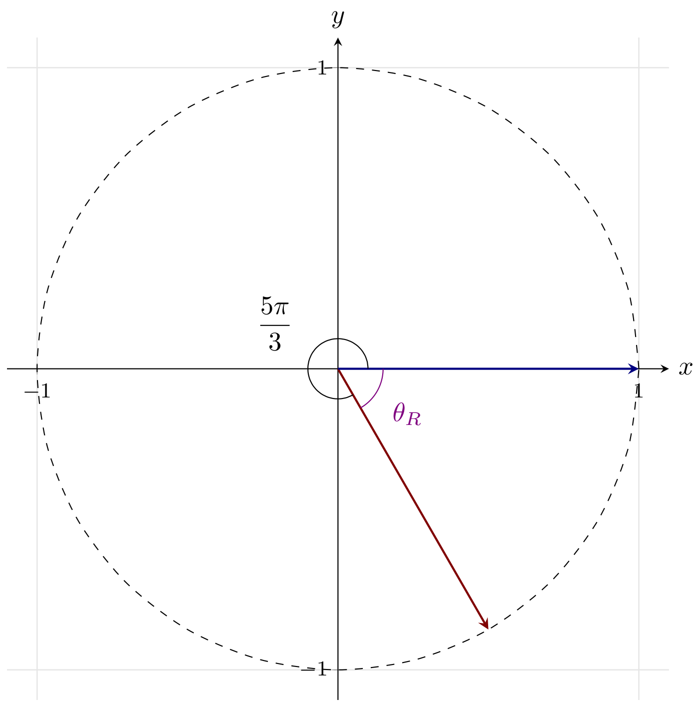
(b) The angle $\frac{7\pi}{5}$ is between π and $\frac{3\pi}{2}$, so φ is in the third quadrant.



In this case, φ is further (in the counter-clockwise direction) than π. That means

$$\begin{aligned}\varphi_R &= \varphi - \pi \\ &= \frac{7\pi}{5} - \pi \\ &= \frac{7\pi}{5} - \frac{5\pi}{5} \\ &= \frac{2\pi}{5}.\end{aligned}$$

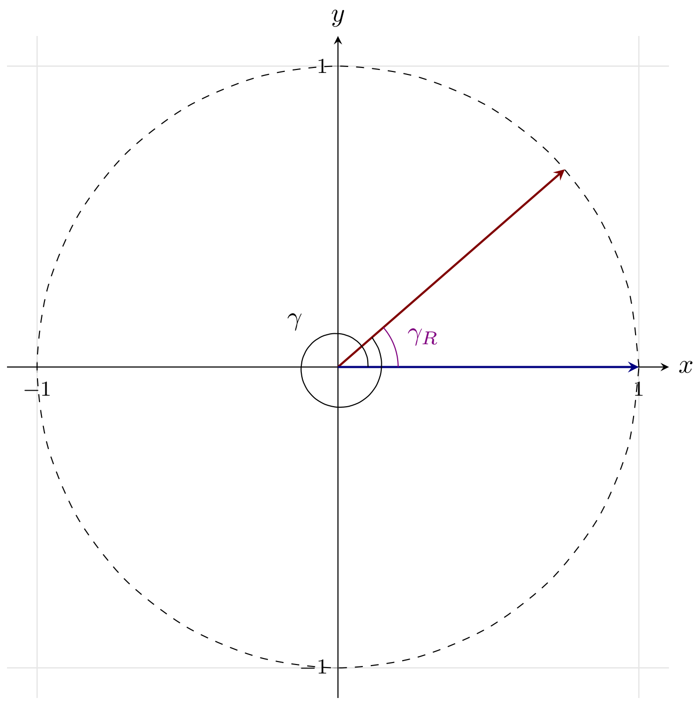
(c) The angle $\dfrac{23\pi}{3}$ is coterminal with $\dfrac{23\pi}{3} - 6\pi = \dfrac{5\pi}{3}$. Since $\dfrac{3}{2} < \dfrac{5}{3} < 2$ we see that θ is a third quadrant angle. Rather than working with θ, we work with the coterminal angle $\dfrac{5\pi}{3}$.



In this case, 2π is further (in the counter-clockwise direction) than $\frac{5\pi}{3}$. That means

$$\begin{aligned}\theta_R &= 2\pi - \frac{5\pi}{3} \\ &= \frac{6\pi}{3} - \frac{5\pi}{3} \\ &= \frac{\pi}{3}.\end{aligned}$$

(d) Don't let this one fool you. There is no degree symbol. This angle is not written as a fraction and it does not seem to include π, but that just means γ is not one of our standard angles. This measurement is still in radians, though. We are asked about $\gamma = 7$ radians, not degrees. Since 2π is a little more than 6, we know $2\pi < 7 < \frac{5\pi}{2}$, so γ is a first quadrant angle.

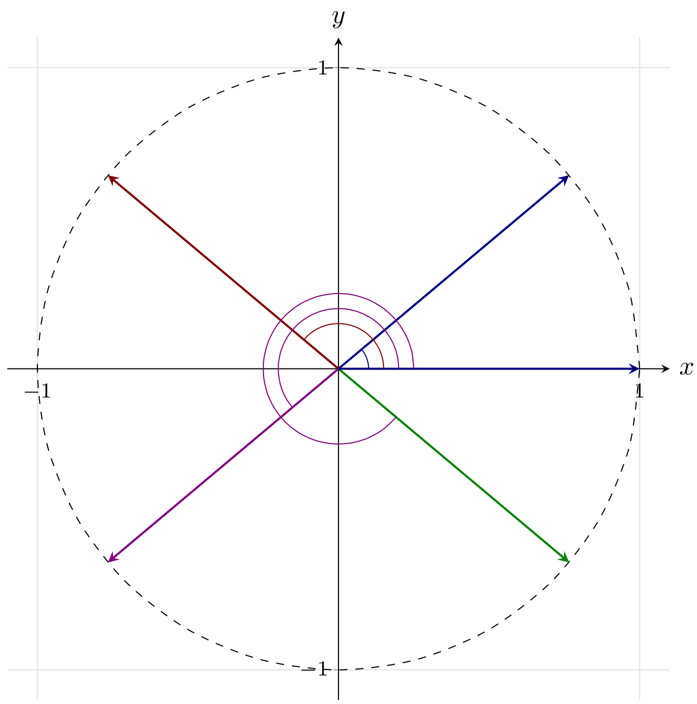


In this case, γ is further (in the counter-clockwise direction) than 2π. That means

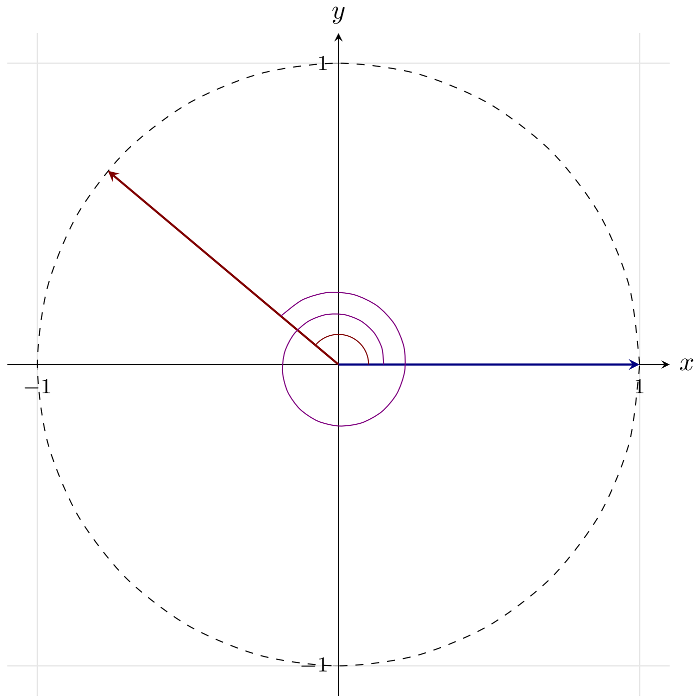
$$\begin{aligned}\gamma_R &= \gamma - 2\pi\\ &= 7 - 2\pi.\end{aligned}$$

To think about this in a different way, the angle γ is coterminal with $\gamma - 2\pi = 7 - 2\pi$. Since $0 < 7 - 2\pi < \dfrac{\pi}{2}$, it must be its own reference angle. That would mean $7 - 2\pi$ is the reference angle of any angles which are coterminal with it, including γ.

If we know the reference angle, θ_R, can we determine θ? Not exactly. In the graph below are four different angles each having the same reference angle. What's different about these angles? They are in *different quadrants*.



If we know both the reference angle *and* the quadrant, can we determine the angle? Not quite. Two angles can have the same quadrant and reference angle if they are coterminal angles.



We can't determine the angle exactly, but we can determine the angle's *terminal side*. Since trigonometric functions are given in terms of the coordinates on the terminal side of the angle, knowing the reference angle and quadrant is enough for us.

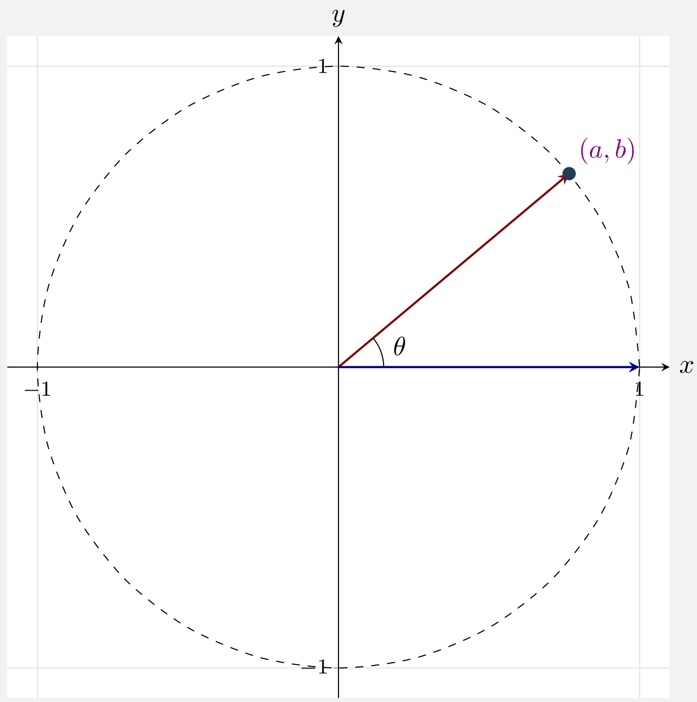
Let's examine the effects of the quadrants on the trigonometric function values we discussed earlier.

- Quadrant I: x and y coordinates are both positive so sine values and cosine values will be positive for these angles.
- Quadrant II: x is negative and y is positive, so cosine values will be negative and sine values will be positive for these angles.
- Quadrant III: x and y coordinates are both negative so sine values and cosine values will be negative for these angles.
- Quadrant IV: x is positive and y is negative, so cosine values will be positive and sine values will be negative for these angles.

Evaluating Trigonometric Functions at Standard Angles

Recall the definition from above.

Definition Suppose θ is an angle in standard position in the unit circle, and denote by (a, b) the coordinates of the point where the terminal side of θ intersects the unit circle.



$$\cos(\theta) = a \qquad \sin(\theta) = b$$

$$\sec(\theta) = \frac{1}{a}, \text{ if } a \neq 0 \qquad \csc(\theta) = \frac{1}{b}, \text{ if } b \neq 0$$

$$\tan(\theta) = \frac{b}{a}, \text{ if } a \neq 0 \qquad \cot(\theta) = \frac{a}{b}, \text{ if } b \neq 0.$$

The domain of sine and cosine is all real numbers, and the other trigonometric functions are defined precisely when their denominators are nonzero.

We are now in a position to evaluate these trig functions for any standard angle.

Exercise 2 *Calculate the values of the six trigonometric functions for each of the following angles.*

(a) $\dfrac{2\pi}{3}$

(b) $\dfrac{9\pi}{4}$

(c) $\dfrac{7\pi}{6}$

(d) $\dfrac{15\pi}{2}$

Explanation

(a) $\dfrac{2\pi}{3}$ is a second quadrant angle with reference angle $\pi - \dfrac{2\pi}{3} = \dfrac{\pi}{3}$ radians, which is equivalent to 60°. We know the trig values of 60°: $\sin(60^\circ) = \dfrac{\sqrt{3}}{2}$, $\cos(60^\circ) = \dfrac{1}{2}$. In the Quadrant II, x-values are negative and y-values are positive so we are looking for a negative cosine value and a positive sine value. This gives $\sin\left(\dfrac{2\pi}{3}\right) = \dfrac{\sqrt{3}}{2}$ and $\cos\left(\dfrac{2\pi}{3}\right) = -\dfrac{1}{2}$. Using the definitions of the other trig functions:

$$\begin{aligned}
\sin\left(\frac{2\pi}{3}\right) &= \frac{\sqrt{3}}{2} & \cos\left(\frac{2\pi}{3}\right) &= -\frac{1}{2}\\
\tan\left(\frac{2\pi}{3}\right) &= \frac{\sin\left(\frac{2\pi}{3}\right)}{\cos\left(\frac{2\pi}{3}\right)} & \cot\left(\frac{2\pi}{3}\right) &= \frac{\cos\left(\frac{2\pi}{3}\right)}{\sin\left(\frac{2\pi}{3}\right)}\\
&= -\sqrt{3} & &= -\frac{1}{\sqrt{3}}\\
\sec\left(\frac{2\pi}{3}\right) &= \frac{1}{\cos\left(\frac{2\pi}{3}\right)} & \csc\left(\frac{2\pi}{3}\right) &= \frac{1}{\sin\left(\frac{2\pi}{3}\right)}\\
&= -2 & &= \frac{2}{\sqrt{3}}.
\end{aligned}$$

(b) $\dfrac{9\pi}{4}$ is a first quadrant angle with reference angle $\dfrac{9\pi}{4} - 2\pi = \dfrac{\pi}{4}$ radians, which is equivalent to 45°. We know the trig values of 45°: $\sin(45^\circ) = \dfrac{\sqrt{2}}{2}$, $\cos(45^\circ) = \dfrac{\sqrt{2}}{2}$. In the Quadrant I, both the x-values and y-values are positive so all trig function values will be positive. This gives $\sin\left(\dfrac{9\pi}{4}\right) =$

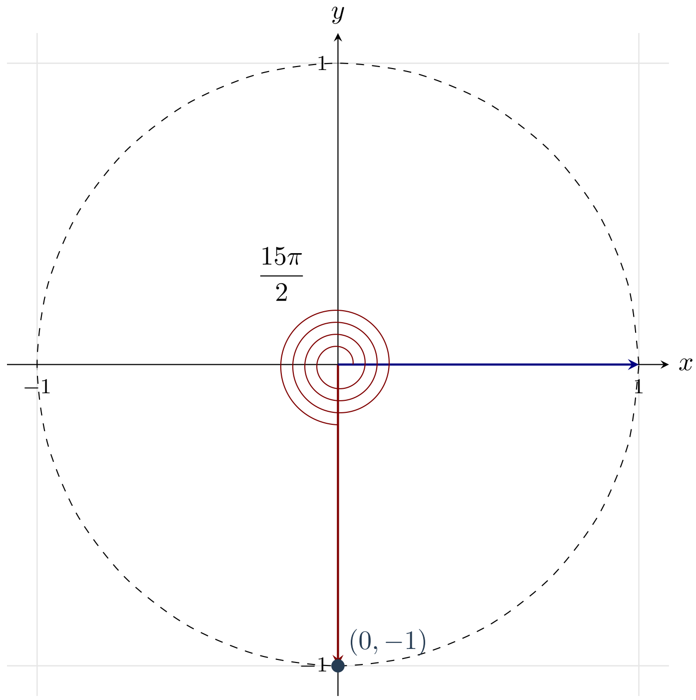
$\dfrac{\sqrt{2}}{2}$ and $\cos\left(\dfrac{9\pi}{4}\right) = \dfrac{\sqrt{2}}{2}$. Using the definitions of the other trig functions:

$$\begin{aligned}
\sin\left(\frac{9\pi}{4}\right) &= \frac{\sqrt{2}}{2} & \cos\left(\frac{9\pi}{4}\right) &= \frac{\sqrt{2}}{2} \\
\tan\left(\frac{9\pi}{4}\right) &= \frac{\sin\left(\frac{9\pi}{4}\right)}{\cos\left(\frac{9\pi}{4}\right)} & \cot\left(\frac{9\pi}{4}\right) &= \frac{\cos\left(\frac{9\pi}{4}\right)}{\sin\left(\frac{9\pi}{4}\right)} \\
&= 1 & &= 1 \\
\sec\left(\frac{9\pi}{4}\right) &= \frac{1}{\cos\left(\frac{9\pi}{4}\right)} & \csc\left(\frac{9\pi}{4}\right) &= \frac{1}{\sin\left(\frac{9\pi}{4}\right)} \\
&= \sqrt{2} & &= \sqrt{2}.
\end{aligned}$$

(c) $\dfrac{7\pi}{6}$ is a third quadrant angle with reference angle $\dfrac{7\pi}{6} - \pi = \dfrac{\pi}{6}$ radians, which is equivalent to $30°$. We know the trig values of $30°$: $\sin(30°) = \dfrac{1}{2}$, $\cos(30°) = \dfrac{\sqrt{3}}{2}$. In the Quadrant III, both the x-values and y-values are negative so we will have a negative sine and cosine values. This gives $\sin\left(\dfrac{7\pi}{6}\right) = -\dfrac{1}{2}$ and $\cos\left(\dfrac{7\pi}{6}\right) = -\dfrac{\sqrt{3}}{2}$. Using the definitions of the other trig functions:

$$\begin{aligned}
\sin\left(\frac{7\pi}{6}\right) &= -\frac{1}{2} & \cos\left(\frac{7\pi}{6}\right) &= -\frac{\sqrt{3}}{2} \\
\tan\left(\frac{7\pi}{6}\right) &= \frac{\sin\left(\frac{7\pi}{6}\right)}{\cos\left(\frac{7\pi}{6}\right)} & \cot\left(\frac{7\pi}{6}\right) &= \frac{\cos\left(\frac{7\pi}{6}\right)}{\sin\left(\frac{7\pi}{6}\right)} \\
&= \frac{1}{\sqrt{3}} & &= \sqrt{3} \\
\sec\left(\frac{7\pi}{6}\right) &= \frac{1}{\cos\left(\frac{7\pi}{6}\right)} & \csc\left(\frac{7\pi}{6}\right) &= \frac{1}{\sin\left(\frac{7\pi}{6}\right)} \\
&= -\frac{2}{\sqrt{3}} & &= -2.
\end{aligned}$$

(d) $\dfrac{15\pi}{2}$ has terminal side along the negative y-axis (it is coterminal with the angle $\dfrac{3\pi}{2}$). The terminal side intersects the unit circle at the point $(0, -1)$.



The coordinates of the intersection give the values of the trig functions $\cos\left(\frac{15\pi}{2}\right) = 0$ and $\sin\left(\frac{15\pi}{2}\right) = -1$. Using the definitions of the other trig functions:

$$\sin\left(\frac{15\pi}{2}\right) = -1 \qquad \cos\left(\frac{15\pi}{2}\right) = 0$$

$$\tan\left(\frac{15\pi}{2}\right) \text{ does not exist} \qquad \cot\left(\frac{15\pi}{2}\right) = \frac{\cos\left(\frac{15\pi}{2}\right)}{\sin\left(\frac{15\pi}{2}\right)} = 0$$

$$\sec\left(\frac{15\pi}{2}\right) \text{ does not exist} \qquad \csc\left(\frac{15\pi}{2}\right) = \frac{1}{\sin\left(\frac{15\pi}{2}\right)} = -1.$$

Typically we have $\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$ and $\sec(\theta) = \frac{1}{\cos(\theta)}$, but for the angle $\frac{15\pi}{2}$ we have $\cos\left(\frac{15\pi}{2}\right) = 0$.

8.3 Trig Identities

Learning Objectives

- Trig Identities
  - What expressions composed of trig functions are identically equal for all values of x?
- Trig Identity Applications
  - How can applying trig identities help us solve problems?

8.3.1 Trigonometric Identities

Introduction to Identities

From the previous section, we have found some identities. We will now summarize what we have already found and begin to introduce new identities. These will help us to breakdown and simplify trigonometric equations that will hopefully make our lives easier.

Remember that an identity is an equation that is true for all possible values of x for which the involved quantities are defined. An example of a non-trigonometric identity is

$$(x+1)^2 = x^2 + 2x + 1,$$

since this equation is true for every value of x, and the left and right sides of the equation are simply two different-looking but entirely equivalent expressions.

Trigonometric identities are simply identities that involve trigonometric functions. While there are a large number of such identities one can study, we choose to focus on those that turn out to be useful in the study of calculus. The most important trigonometric identity is the fundamental trigonometric identity, which is a trigonometric restatement of the Pythagorean Theorem.

For any real angle θ,

$$\cos^2(\theta) + \sin^2(\theta) = 1$$

Identities are important because they enable us to view the same idea from multiple perspectives. For example, the fundamental trigonometric identity allows us to think of $\cos^2(\theta) + \sin^2(\theta)$ as simply 1, or alternatively, to view $\cos^2(\theta)$ as the same quantity as $1 - \sin^2(\theta)$.

There are two related Pythagorean identities that involve the tangent, secant, cotangent, and cosecant functions, which we previously derived from the fundamental trigonometric identity by dividing both sides by either $\cos^2(\theta)$ or $\sin^2(\theta)$. We will take another look at this identity before going deeper. If we divide both sides of $\cos^2(\theta) + \sin^2(\theta) = 1$ by $\cos^2(\theta)$ (and assume that $\cos(\theta) \neq 0$, we see that

$$1 + \frac{\sin^2(\theta)}{\cos^2(\theta)} = \frac{1}{\cos^2(\theta)},$$

or equivalently,

$$1 + \tan^2(\theta) = \sec^2(\theta).$$

A similar argument dividing by $\sin^2(\theta)$ (while assuming $\sin(\theta) \neq 0$) shows that

$$\cot^2(\theta) + 1 = \csc^2(\theta).$$

These identities prove useful in calculus when we develop the formulas for the derivatives of the tangent and cotangent functions.

Sums and Differences of Angles

In calculus, it is also beneficial to know a couple of other standard identities for sums of angles or double angles.

Sum and Difference Identities for Cosine: For all angles α and β,

- $\cos(\alpha+\beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$
- $\cos(\alpha-\beta) = \cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta)$

Double Angle Formula:

- θ, $\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta)$

Sum and Difference Identities for Sine: For all angles α and β,

- $\sin(\alpha+\beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta)$
- $\sin(\alpha-\beta) = \sin(\alpha)\cos(\beta) - \cos(\alpha)\sin(\beta)$

Double Angle Formula:

- θ, $\sin(2\theta) = 2\sin(\theta)\cos(\theta)$

Example 40.

(a) *Find the exact value of* $\cos(15°)$.

(b) *Verify the identity:* $\cos\left(\frac{\pi}{2} - \theta\right) = \sin(\theta)$.

Solution.

(a) *In order to use the theorem to find* $\cos(15°)$*, we need to write* $15°$ *as a sum or difference of angles whose cosines and sines we know. One way to do so is to write* $15° = 45° - 30°$.

$$\begin{aligned}
\cos(15°) &= \cos(45° - 30°) \\
&= \cos(45°)\cos(30°) + \sin(45°)\sin(30°) \\
&= \left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{3}}{2}\right) + \left(\frac{\sqrt{2}}{2}\right)\left(\frac{1}{2}\right) \\
&= \frac{\sqrt{6}+\sqrt{2}}{4}
\end{aligned}$$

(b) *In a straightforward application of the theorem, we find*

$$\begin{aligned}\cos\left(\frac{\pi}{2}-\theta\right) &= \cos\left(\frac{\pi}{2}\right)\cos(\theta)+\sin\left(\frac{\pi}{2}\right)\sin(\theta)\\ &= (0)(\cos(\theta))+(1)(\sin(\theta))\\ &= \sin(\theta)\end{aligned}$$

■

Example 41.

(a) *Find the exact value of* $\sin\left(\frac{19\pi}{12}\right)$

(b) *If* α *is a Quadrant II angle with* $\sin(\alpha)=\frac{5}{13}$, *and* β *is a Quadrant III angle with* $\tan(\beta)=2$, *find* $\sin(\alpha-\beta)$.

(c) *Derive a formula for* $\tan(\alpha+\beta)$ *in terms of* $\tan(\alpha)$ *and* $\tan(\beta)$.

Solution.

(a) *As in the earlier example, we need to write the angle* $\frac{19\pi}{12}$ *as a sum or difference of common angles. The denominator of* 12 *suggests a combination of angles with denominators* 3 *and* 4. *One such combination is* $\frac{19\pi}{12}=\frac{4\pi}{3}+\frac{\pi}{4}$. *Applying what we know about sum and difference with Sine, we get*

$$\begin{aligned}\sin\left(\frac{19\pi}{12}\right) &= \sin\left(\frac{4\pi}{3}+\frac{\pi}{4}\right)\\ &= \sin\left(\frac{4\pi}{3}\right)\cos\left(\frac{\pi}{4}\right)+\cos\left(\frac{4\pi}{3}\right)\sin\left(\frac{\pi}{4}\right)\\ &= \left(-\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{2}}{2}\right)+\left(-\frac{1}{2}\right)\left(\frac{\sqrt{2}}{2}\right)\\ &= \frac{-\sqrt{6}-\sqrt{2}}{4}\end{aligned}$$

(b) *In order to find* $\sin(\alpha-\beta)$ *usingthe theorem, we need to find* $\cos(\alpha)$ *and both* $\cos(\beta)$ *and* $\sin(\beta)$. *To find* $\cos(\alpha)$, *we use the Pythagorean Identity* $\cos^2(\alpha)+\sin^2(\alpha)=1$. *Since* $\sin(\alpha)=\frac{5}{13}$, *we have* $\cos^2(\alpha)+\left(\frac{5}{13}\right)^2=1$, *or* $\cos(\alpha)=\pm\frac{12}{13}$. *Since* α *is a Quadrant II angle,* $\cos(\alpha)=-\frac{12}{13}$. *We now set about finding* $\cos(\beta)$ *and* $\sin(\beta)$. *We have several ways to proceed, but the Pythagorean Identity* $1+\tan^2(\beta)=\sec^2(\beta)$ *is a quick way to get* $\sec(\beta)$, *and hence,* $\cos(\beta)$. *With* $\tan(\beta)=2$, *we get* $1+2^2=\sec^2(\beta)$ *so that*

$\sec(\beta) = \pm\sqrt{5}$. *Since* β *is a Quadrant III angle, we choose* $\sec(\beta) = -\sqrt{5}$ *so* $\cos(\beta) = \dfrac{1}{\sec(\beta)} = \dfrac{1}{-\sqrt{5}} = -\dfrac{\sqrt{5}}{5}$. *We now need to determine* $\sin(\beta)$. *We could use The Pythagorean Identity* $\cos^2(\beta) + \sin^2(\beta) = 1$, *but we opt instead to use a quotient identity. From* $\tan(\beta) = \dfrac{\sin(\beta)}{\cos(\beta)}$, *we have* $\sin(\beta) = \tan(\beta)\cos(\beta)$ *so we get* $\sin(\beta) = (2)\left(-\dfrac{\sqrt{5}}{5}\right) = -\dfrac{2\sqrt{5}}{5}$. *We now have all the pieces needed to find* $\sin(\alpha - \beta)$*:*

$$\begin{aligned}
\sin(\alpha - \beta) &= \sin(\alpha)\cos(\beta) - \cos(\alpha)\sin(\beta) \\
&= \left(\frac{5}{13}\right)\left(-\frac{\sqrt{5}}{5}\right) - \left(-\frac{12}{13}\right)\left(-\frac{2\sqrt{5}}{5}\right) \\
&= -\frac{29\sqrt{5}}{65}
\end{aligned}$$

(c) *We can start expanding* $\tan(\alpha + \beta)$ *using a quotient identity and our sum formulas*

$$\begin{aligned}
\tan(\alpha + \beta) &= \frac{\sin(\alpha + \beta)}{\cos(\alpha + \beta)} \\
&= \frac{\sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta)}{\cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)}
\end{aligned}$$

Since $\tan(\alpha) = \dfrac{\sin(\alpha)}{\cos(\alpha)}$ *and* $\tan(\beta) = \dfrac{\sin(\beta)}{\cos(\beta)}$, *it looks as though if we divide both numerator and denominator by* $\cos(\alpha)\cos(\beta)$ *we will have what we want*

$$
\begin{aligned}
\tan(\alpha+\beta) &= \frac{\sin(\alpha)\cos(\beta)+\cos(\alpha)\sin(\beta)}{\cos(\alpha)\cos(\beta)-\sin(\alpha)\sin(\beta)} \cdot \frac{\dfrac{1}{\cos(\alpha)\cos(\beta)}}{\dfrac{1}{\cos(\alpha)\cos(\beta)}} \\
&= \frac{\dfrac{\sin(\alpha)\cos(\beta)}{\cos(\alpha)\cos(\beta)} + \dfrac{\cos(\alpha)\sin(\beta)}{\cos(\alpha)\cos(\beta)}}{\dfrac{\cos(\alpha)\cos(\beta)}{\cos(\alpha)\cos(\beta)} - \dfrac{\sin(\alpha)\sin(\beta)}{\cos(\alpha)\cos(\beta)}} \\
&= \frac{\dfrac{\sin(\alpha)\cancel{\cos(\beta)}}{\cos(\alpha)\cancel{\cos(\beta)}} + \dfrac{\cancel{\cos(\alpha)}\sin(\beta)}{\cancel{\cos(\alpha)}\cos(\beta)}}{\dfrac{\cancel{\cos(\alpha)}\cancel{\cos(\beta)}}{\cancel{\cos(\alpha)}\cancel{\cos(\beta)}} - \dfrac{\sin(\alpha)\sin(\beta)}{\cos(\alpha)\cos(\beta)}} \\
&= \frac{\tan(\alpha)+\tan(\beta)}{1-\tan(\alpha)\tan(\beta)}
\end{aligned}
$$

Naturally, this formula is limited to those cases where all of the tangents are defined. ■

Example 42.

(a) *If* $\sin(\theta) = x$ *for* $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$, *find an expression for* $\sin(2\theta)$ *in terms of* x.

(b) *Verify the identity:* $\sin(2\theta) = \dfrac{2\tan(\theta)}{1+\tan^2(\theta)}$.

(c) *Express* $\cos(3\theta)$ *as a polynomial in terms of* $\cos(\theta)$.

Solution.

(a) *If your first reaction to '*$\sin(\theta) = x$*' is 'No it's not,* $\cos(\theta) = x$*!' then you have indeed learned something, and we take comfort in that. However, context is everything. Here, '*x*' is just a variable - it does not necessarily represent the* x*-coordinate of the point on The Unit Circle which lies on the terminal side of* θ*, assuming* θ *is drawn in standard position. Here,* x *represents the quantity* $\sin(\theta)$*, and what we wish to know is how to express* $\sin(2\theta)$ *in terms of* x*. We will see more of this kind of thing in future sections, and, as usual, this is something we need for Calculus. Since* $\sin(2\theta) = 2\sin(\theta)\cos(\theta)$*, we need to write* $\cos(\theta)$ *in terms of* x *to finish the problem. We substitute* $x = \sin(\theta)$ *into the Pythagorean Identity,*

$\cos^2(\theta) + \sin^2(\theta) = 1$, *to get* $\cos^2(\theta) + x^2 = 1$, *or* $\cos(\theta) = \pm\sqrt{1-x^2}$. *Since* $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$, $\cos(\theta) \geq 0$, *and thus* $\cos(\theta) = \sqrt{1-x^2}$. *Our final answer is* $\sin(2\theta) = 2\sin(\theta)\cos(\theta) = 2x\sqrt{1-x^2}$.

(b) *We start with the right hand side of the identity and note that* $1+\tan^2(\theta) = \sec^2(\theta)$. *From this point, we use the Reciprocal and Quotient Identities to rewrite* $\tan(\theta)$ *and* $\sec(\theta)$ *in terms of* $\cos(\theta)$ *and* $\sin(\theta)$*:*

$$\begin{aligned}
\frac{2\tan(\theta)}{1+\tan^2(\theta)} &= \frac{2\tan(\theta)}{\sec^2(\theta)} = \frac{2\left(\frac{\sin(\theta)}{\cos(\theta)}\right)}{\frac{1}{\cos^2(\theta)}} = 2\left(\frac{\sin(\theta)}{\cos(\theta)}\right)\cos^2(\theta) \\
&= 2\left(\frac{\sin(\theta)}{\cancel{\cos(\theta)}}\right)\cancel{\cos(\theta)}\cos(\theta) = 2\sin(\theta)\cos(\theta) = \sin(2\theta)
\end{aligned}$$

(c) *In our list of identities, one of the formulas for* $\cos(2\theta)$, *namely* $\cos(2\theta) = 2\cos^2(\theta) - 1$, *expresses* $\cos(2\theta)$ *as a polynomial in terms of* $\cos(\theta)$. *We are now asked to find such an identity for* $\cos(3\theta)$. *Using the sum formula for cosine, we begin with*

$$\begin{aligned}
\cos(3\theta) &= \cos(2\theta + \theta) \\
&= \cos(2\theta)\cos(\theta) - \sin(2\theta)\sin(\theta)
\end{aligned}$$

Our ultimate goal is to express the right hand side in terms of $\cos(\theta)$ *only. We substitute* $\cos(2\theta) = 2\cos^2(\theta) - 1$ *and* $\sin(2\theta) = 2\sin(\theta)\cos(\theta)$ *which yields*

$$\begin{aligned}
\cos(3\theta) &= \cos(2\theta)\cos(\theta) - \sin(2\theta)\sin(\theta) \\
&= \left(2\cos^2(\theta) - 1\right)\cos(\theta) - (2\sin(\theta)\cos(\theta))\sin(\theta) \\
&= 2\cos^3(\theta) - \cos(\theta) - 2\sin^2(\theta)\cos(\theta)
\end{aligned}$$

Finally, we exchange $\sin^2(\theta)$ *for* $1 - \cos^2(\theta)$ *courtesy of the Pythagorean Identity, and get*

$$\begin{aligned}
\cos(3\theta) &= 2\cos^3(\theta) - \cos(\theta) - 2\sin^2(\theta)\cos(\theta) \\
&= 2\cos^3(\theta) - \cos(\theta) - 2\left(1 - \cos^2(\theta)\right)\cos(\theta) \\
&= 2\cos^3(\theta) - \cos(\theta) - 2\cos(\theta) + 2\cos^3(\theta) \\
&= 4\cos^3(\theta) - 3\cos(\theta)
\end{aligned}$$

and we are done. ■

Lists of Identities

The Pythagorean Identities:

(a) $\cos^2(\theta) + \sin^2(\theta) = 1$.

Common Alternate Forms:

- $1 - \sin^2(\theta) = \cos^2(\theta)$
- $1 - \cos^2(\theta) = \sin^2(\theta)$

(b) $1 + \tan^2(\theta) = \sec^2(\theta)$, provided $\cos(\theta) \neq 0$.

Common Alternate Forms:

- $\sec^2(\theta) - \tan^2(\theta) = 1$
- $\sec^2(\theta) - 1 = \tan^2(\theta)$

(c) $1 + \cot^2(\theta) = \csc^2(\theta)$, provided $\sin(\theta) \neq 0$.

Common Alternate Forms:

- $\csc^2(\theta) - \cot^2(\theta) = 1$
- $\csc^2(\theta) - 1 = \cot^2(\theta)$

Reciprocal and Quotient Identities:

- $\sec(\theta) = \dfrac{1}{\cos(\theta)}$, provided $\cos(\theta) \neq 0$;
  if $\cos(\theta) = 0$, $\sec(\theta)$ is undefined.
- $\csc(\theta) = \dfrac{1}{\sin(\theta)}$, provided $\sin(\theta) \neq 0$;
  if $\sin(\theta) = 0$, $\csc(\theta)$ is undefined.
- $\tan(\theta) = \dfrac{\sin(\theta)}{\cos(\theta)}$, provided $\cos(\theta) \neq 0$;
  if $\cos(\theta) = 0$, $\tan(\theta)$ is undefined.
- $\cot(\theta) = \dfrac{\cos(\theta)}{\sin(\theta)}$, provided $\sin(\theta) \neq 0$;
  if $\sin(\theta) = 0$, $\cot(\theta)$ is undefined.

Pythagorean Conjugates

- $1 - \cos(\theta)$ and $1 + \cos(\theta)$:
  $(1 - \cos(\theta))(1 + \cos(\theta)) = 1 - \cos^2(\theta) = \sin^2(\theta)$

- $1 - \sin(\theta)$ and $1 + \sin(\theta)$:
  $(1 - \sin(\theta))(1 + \sin(\theta)) = 1 - \sin^2(\theta) = \cos^2(\theta)$
- $\sec(\theta) - 1$ and $\sec(\theta) + 1$:
  $(\sec(\theta) - 1)(\sec(\theta) + 1) = \sec^2(\theta) - 1 = \tan^2(\theta)$
- $\sec(\theta) - \tan(\theta)$ and $\sec(\theta) + \tan(\theta)$:
  $(\sec(\theta) - \tan(\theta))(\sec(\theta) + \tan(\theta)) = \sec^2(\theta) - \tan^2(\theta) = 1$
- $\csc(\theta) - 1$ and $\csc(\theta) + 1$:
  $(\csc(\theta) - 1)(\csc(\theta) + 1) = \csc^2(\theta) - 1 = \cot^2(\theta)$
- $\csc(\theta) - \cot(\theta)$ and $\csc(\theta) + \cot(\theta)$:
  $(\csc(\theta) - \cot(\theta))(\csc(\theta) + \cot(\theta)) = \csc^2(\theta) - \cot^2(\theta) = 1$

Exploration In this activity, we investigate how a sum of two angles identity for the sine function helps us gain a different perspective on the average rate of change of the sine function.
Recall that for any function f on an interval $[a, a + h]$, its average rate of change is

$$\mathrm{AROC}_{[a,a+h]} = \frac{f(a + h) - f(a)}{h}.$$

a. Let $f(x) = \sin(x)$. Use the definition of $\mathrm{AROC}_{[a,a+h]}$ to write an expression for the average rate of change of the sine function on the interval $[a + h, a]$.

b. Apply the sum of two angles identity for the sine function, $\sin(\alpha + \beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta)$, to the expression $\sin(a + h)$.

c. Explain why your work in (a) and (b) together with some algebra shows that

$$\mathrm{AROC}_{[a,a+h]} = \sin(a) \cdot \frac{\cos(h) - 1}{h} - \cos(a)\frac{\sin(h)}{h}.$$

d. In calculus, we move from *average* rate of change to *instantaneous* rate of change by letting h approach 0 in the expression for average rate of change. Using a computational device in radian mode, investigate the behavior of

$$\frac{\cos(h) - 1}{h}$$

as h gets close to 0. What happens? Similarly, how does $\frac{\sin(h)}{h}$ behave for small values of h? What does this tell us about $\mathrm{AROC}_{[a,a+h]}$ as h approaches 0?

More Identities

In the previous sections, we saw the utility of the Pythagorean Identities. Not only did these identities help us compute the values of the circular functions for angles, they were also useful in simplifying expressions involving the circular functions. We will introduce this set of identities as the 'Even / Odd' identities and we will discuss them further with trigonometric transformations in a later section.

Even / Odd Identities: For all applicable angles θ,

- $\cos(-\theta) = \cos(\theta)$
- $\sec(-\theta) = \sec(\theta)$
- $\sin(-\theta) = -\sin(\theta)$
- $\csc(-\theta) = -\csc(\theta)$
- $\tan(-\theta) = -\tan(\theta)$
- $\cot(-\theta) = -\cot(\theta)$

8.3.2 Application of Trig Identities

Verifying Identities

In this section, we will look at strategies to verify identities.

Remark **Strategies for Verifying Identities**

- Try working on the more complicated side of the identity.
- Use the Reciprocal and Quotient Identities to write functions on one side of the identity in terms of the functions on the other side of the identity. Simplify the resulting complex fractions.
- Add rational expressions with unlike denominators by obtaining common denominators.
- Use the Pythagorean Identities to 'exchange' sines and cosines, secants and tangents, cosecants and cotangents, and simplify sums or differences of squares to one term.
- Multiply numerator **and** denominator by Pythagorean Conjugates in order to take advantage of the Pythagorean Identities.
- If you find yourself stuck working with one side of the identity, try starting with the other side of the identity and see if you can find a way to bridge the two parts of your work.

Example 43. *Verify the following:*

$$\tan(\theta)\cos(\theta) = \sin(\theta)$$

Explanation Let's start with the more complicated side. We know that we want to end up with $\sin(\theta)$ in the end, so using the Quotient identity to replace $\tan(\theta)$ with $\dfrac{\sin(\theta)}{\cos(\theta)}$ is a reasonable place to start.

$$\tan(\theta)\cos(\theta) = \left(\frac{\sin(\theta)}{\cos(\theta)}\right)\cos(\theta)$$

We can now cancel our $\cos(\theta)$ terms.

$$\tan(\theta)\cos(\theta) = \left(\frac{\sin(\theta)}{\cancel{\cos(\theta)}}\right)\cancel{\cos(\theta)}$$

This, thankfully, leaves us with our original equation, so we have verified this identity.

$$\tan(\theta)\cos(\theta) = \sin(\theta)$$

Example 44. *Verify the following:*

$$\frac{\sec(x) - cos(x)}{\sec(x)} = sin^2(x)$$

Explanation Let's start with the more complicated side. Since $\sec(x)$ is the reciprocal of $\cos(x)$, rewriting this side completely in terms of $\cos(x)$ could help us verify this identity.

$$\frac{\sec(x) - \cos(x)}{\sec(x)} = \frac{\frac{1}{\cos(x)} - \cos(x)}{\frac{1}{\cos(x)}}$$

We can now simplify the fractional division by inverting and multiplying.

$$\frac{\frac{1}{\cos(x)} - \cos(x)}{\frac{1}{\cos(x)}} = \left(\frac{1}{\cos(x)} - \cos(x)\right) \cdot \cos(x)$$

We continue by distributing our $\cos(x)$ term and then simplify.

$$\left(\frac{1}{\cos(x)} - \cos(x)\right) \cdot \cos(x) = \frac{\cos(x)}{\cos(x)} - \cos^2(x) = 1 - \cos^2(x)$$

Using an alternate form of the Pythagorean Identity, we can make the following substitution.

$$1 - \cos^2(x) = sin^2(x)$$

We have now verified our original equation.

$$\frac{\sec(x) - cos(x)}{\sec(x)} = sin^2(x)$$

Example 45. *Verify the following:*

$$\tan(\theta) + \cot(\theta) = \csc(\theta)\sec(\theta)$$

Explanation Let's start with the left side. By using the Quotient identities we can change both of our terms to be in the form of $\sin(\theta)$ and $\cos(\theta)$

$$\tan(\theta) + \cot(\theta) = \frac{\sin(\theta)}{\cos(\theta)} + \frac{\cos(\theta)}{\sin(\theta)}$$

We can find a common demoninator to begin combining our terms.

$$\frac{\sin(\theta)}{\cos(\theta)} + \frac{\cos(\theta)}{\sin(\theta)} = \frac{\sin^2(\theta)}{\sin(\theta)\cos(\theta)} + \frac{\cos^2(\theta)}{\sin(\theta)\cos(\theta)}$$

Now we combine our terms and simplify.

$$\frac{\sin^2(\theta)}{\sin(\theta)\cos(\theta)} + \frac{\cos^2(\theta)}{\sin(\theta)\cos(\theta)} = \frac{\sin^2(\theta) + \cos^2(\theta)}{\sin(\theta)\cos(\theta)} = \frac{1}{\sin(\theta)\cos(\theta)}$$

We are now left with two terms being multiplied together. We will split them up to better show the next step.

$$\frac{1}{\sin(\theta)\cos(\theta)} = \frac{1}{\sin(\theta)} \cdot \frac{1}{\cos(\theta)}$$

We can use the Reciprocal identies for these two terms and we have found what we wanted.

$$\frac{1}{\sin(\theta)} \cdot \frac{1}{\cos(\theta)} = \csc(\theta)\sec(\theta)$$

The identity is verified.

$$\tan(\theta) + \cot(\theta) = \csc(\theta)\sec(\theta)$$

Example 46. *Verify the following:*

$$\sin(x) = \tan(x) + \cos(x)$$

Explanation Let's start with the more complicated side. By using the Quotient identities we can change both of our terms to be in the form of $\sin(\theta)$ and $\cos(\theta)$

$$\tan(x) + \cos(x) = \frac{\sin(x)}{\cos(x)} + \cos(x)$$

We can find a common demoninator to begin combining our terms.

$$\frac{\sin(x)}{\cos(x)} + \cos(x) = \frac{\sin(x)}{\cos(x)} + \frac{\cos^2(x)}{\cos(x)}$$

We combine our terms.

$$\frac{\sin(x)}{\cos(x)} + \frac{\cos^2(x)}{\cos(x)} = \frac{\sin(x) + \cos^2(x)}{\cos(x)}$$

We use a Pythagorean identity.

$$\frac{\sin(x) + \cos^2(x)}{\cos(x)} = \frac{\sin(x) + 1 - \sin^2(x)}{\cos(x)}$$

We are now at a difficult point. The equation does not seem to be simplifying and we are not making any progress. It is possible that this is **NOT** equal. In order, to prove that we have to go back and use a test value in our original equation. Let's try $\frac{\pi}{6}$.

$$\sin(\frac{\pi}{6}) = \tan(\frac{\pi}{6}) + \cos(\frac{\pi}{6})$$

Since this is a trig value, we can go ahead and evaluate it.

$$\frac{1}{2} = \frac{\sqrt{3}}{3} + \frac{\sqrt{3}}{2}$$

We again find a common denominator

$$\frac{1}{2} = \frac{2\sqrt{3}}{6} + \frac{3\sqrt{3}}{6}$$

With a little simplifying, we have the following equation.

$$\frac{1}{2} = \frac{5\sqrt{3}}{6}$$

$\frac{5\sqrt{3}}{6}$ is definitely larger than $\frac{1}{2}$ (verify with a calculator) and thus not equal.

$$\frac{1}{2} \neq \frac{5\sqrt{3}}{6}$$

We have proved that this is equation **NOT** equal.

$$\sin(x) \neq \tan(x) + \cos(x)$$

Part 9

Trigonometric Functions

9.1 The Unit Circle to the Function Graph

Learning Objectives

- –
  –
  –

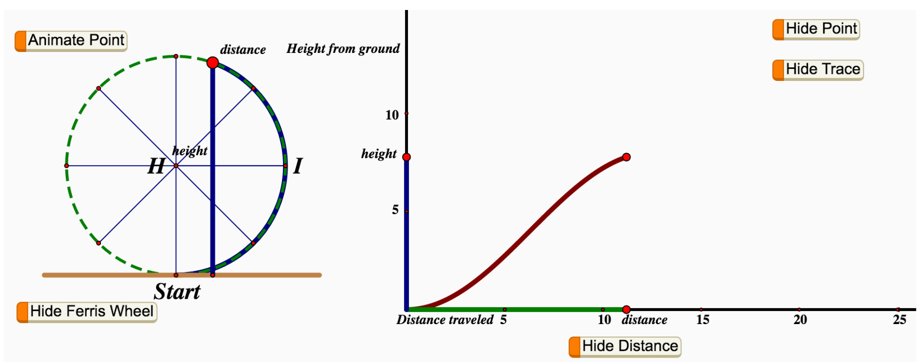
9.1.1 Traversing A Circle

Motivating Questions

- How does a point traversing a circle naturally generate a function?
- What are some important properties that characterize a function generated by a point traversing a circle?

Certain naturally occurring phenomena eventually repeat themselves, especially when the phenomenon is somehow connected to a circle. You may recall from when we first studied periodic function that we considered the case of taking a ride on a ferris wheel. We considered your height, h, above the ground and how your height changed in tandem with the distance, d, that you have traveled around the wheel. We saw snapshot of this situation, which is available as a full animation at http://gvsu.edu/s/0Dt .

A snapshot of the motion of a cab moving around a ferris wheel. Reprinted with permission from Illuminations by the National Council of Teachers of Mathematics. All rights reserved.



Because we have two quantities changing in tandem, it is natural to wonder if it is possible to represent one as a function of the other.

Exploration In the context of the ferris wheel pictured in above, assume that the height, h, of the moving point (the cab in which you are riding), and the distance, d, that the point has traveled around the circumference of the ferris wheel are both measured in meters. Further, assume that the circumference of the ferris wheel is 150 meters. In addition, suppose that after getting in your cab at the lowest point on the wheel, you traverse the full circle several times.

a. Recall that the circumference, C, of a circle is connected to the

 Acknowledgements: Active Prelude to Calculus (https://activecalculus.org/prelude)

circle's radius, r, by the formula $C = 2\pi r$. What is the radius of the ferris wheel? How high is the highest point on the ferris wheel?

b. How high is the cab after it has traveled $1/4$ of the circumference of the circle?

c. How much distance along the circle has the cab traversed at the moment it first reaches a height of $\frac{150}{\pi} \approx 47.75$ meters?

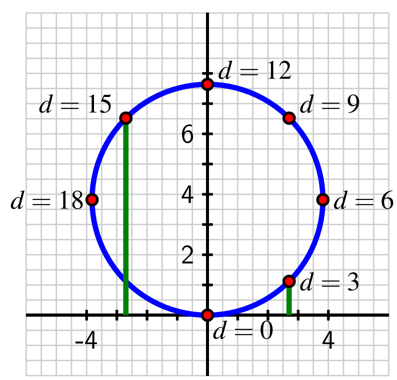
d. Can h be thought of as a function of d? Why or why not?

e. Can d be thought of as a function of h? Why or why not?

f. Why do you think the curve shown above has the shape that it does? Write several sentences to explain.

Circular Functions

The natural phenomenon of a point moving around a circle leads to interesting relationships. Let's consider a point traversing a circle of circumference 24 and examine how the point's height, h, changes as the distance traversed, d, changes. Note particularly that each time the point traverses $\frac{1}{8}$ of the circumference of the circle, it travels a distance of $24 \cdot \frac{1}{8} = 3$ units, as seen below where each noted point lies 3 additional units along the circle beyond the preceding one.



Note that we know the exact heights of certain points. Since the circle has circumference $C = 24$, we know that $24 = 2\pi r$ and therefore $r = \frac{12}{\pi} \approx 3.82$. Hence, the point where $d = 6$ (located $1/4$ of the way along the circle) is at a height of $h = \frac{12}{\pi} \approx 3.82$. Doubling this value, the point where $d = 12$ has

 Acknowledgements: Active Prelude to Calculus (https://activecalculus.org/prelude)

height $h = \frac{24}{\pi} \approx 7.64$. Other heights, such as those that correspond to $d = 3$ and $d = 15$ (identified on the figure by the green line segments) are not obvious from the circle's radius, but can be estimated from the grid in the graph above as $h \approx 1.1$ (for $d = 3$) and $h \approx 6.5$ (for $d = 15$). Using all of these observations along with the symmetry of the circle, we can determine the other entries in the table below.

Data for height, h, as a function of distance traversed, d.

d	0	3	6	9	12	15	18	21	24
h	0	1.1	3.82	6.5	7.64	6.5	3.82	1.1	0

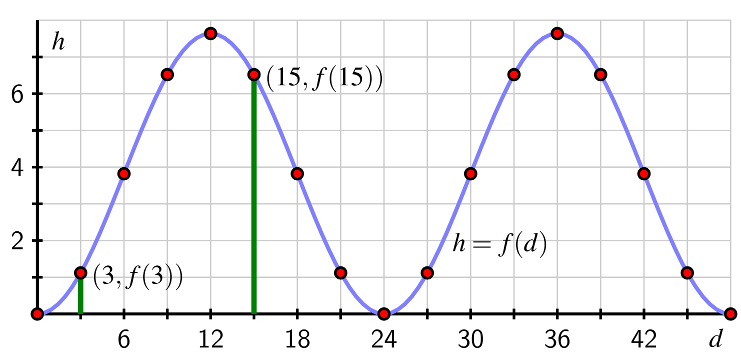
Moreover, if we now let the point continue traversing the circle, we observe that the d-values will increase accordingly, but the h-values will repeat according to the already-established pattern, resulting in the data in the table below.

Additional data for height, h, as a function of distance traversed, d.

d	24	27	30	33	36	39	42	45	48
h	0	1.1	3.82	6.5	7.64	6.5	3.82	1.1	0

It is apparent that each point on the circle corresponds to one and only one height, and thus we can view the height of a point as a function of the distance the point has traversed around the circle, say $h = f(d)$. Using the data from the two tables and connecting the points in an intuitive way, we get the graph shown below.

The height, h, of a point traversing a circle of radius 24 as a function of distance, d, traversed around the circle.



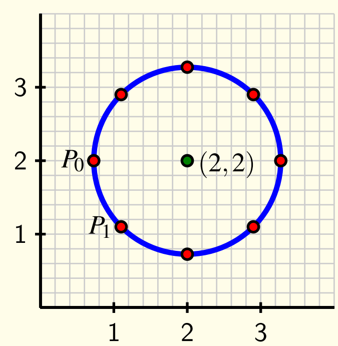
 Acknowledgements: Active Prelude to Calculus (https://activecalculus.org/prelude)

The function $h = f(d)$ we have been discussing is an example of what we will call a **circular function**. Indeed, it is apparent that if we:

- take any circle in the plane,
- choose a starting location for a point on the circle,
- let the point traverse the circle continuously,
- and track the height of the point as it traverses the circle,

the height of the point is a function of distance traversed and the resulting graph will have the same basic shape as the curve shown in the graph above. It also turns out that if we track the location of the x-coordinate of the point on the circle, the x-coordinate is also a function of distance traversed and its curve has a similar shape to the graph of the height of the point (the y-coordinate). Both of these functions are circular functions because they are generated by motion around a circle.

Exploration Consider the circle pictured below that is centered at the point $(2, 2)$ and that has circumference 8. Assume that we track the y-coordinate (that is, the height, h) of a point that is traversing the circle counterclockwise and that it starts at P_0 as pictured.



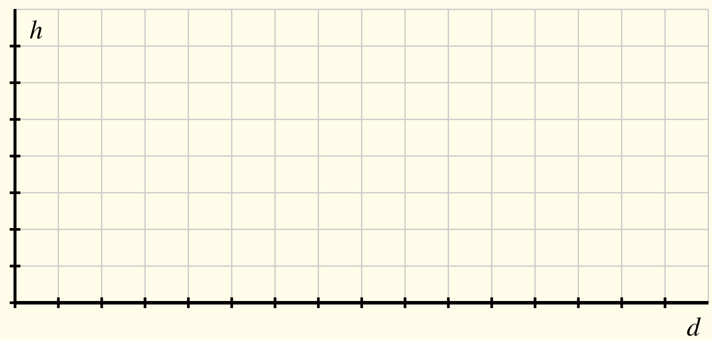
a. How far along the circle is the point P_1 from P_0? Why?

b. Label the subsequent points in the figure P_2, P_3, ... as we move counterclockwise around the circle. What are the exact coordinates of P_2? of P_4? Why?

c. Determine the coordinates of the remaining points on the circle (exactly where possible, otherwise approximately) and hence complete the entries in the table below that track the height, h, of the point traversing the circle as a function of distance traveled, d.

Acknowledgements: Active Prelude to Calculus (https://activecalculus.org/prelude)

d. By plotting the points in the table and connecting them in an intuitive way, sketch a graph of h as a function of d on the axes provided over the interval $0 \leq d \leq 16$. Clearly label the scale of your axes and the coordinates of several important points on the curve.



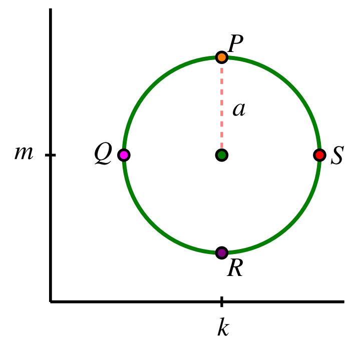
e. What is similar about your graph in comparison to the one in we created at the beginning of this section? What is different?

f. What will be the value of h when $d = 51$? How about when $d = 102$?

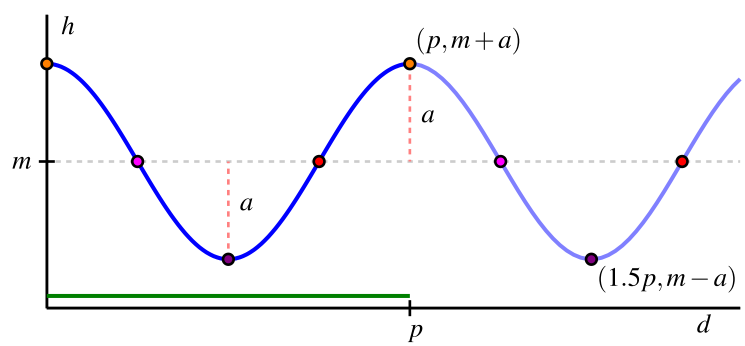
Properties of Circular Functions

Every circular function has several important features that are connected to the circle that defines the function. For the discussion that follows, we focus on circular functions that result from tracking the y-coordinate of a point traversing counterclockwise a circle of radius a centered at the point (k, m). Further, we will denote the circumference of the circle by the letter p.

 Acknowledgements: Active Prelude to Calculus (https://activecalculus.org/prelude)



We assume that the point traversing the circle starts at P in teh graph of the circle above. Its height is initially $y = m + a$, and then its height decreases to $y = m$ as we traverse to Q. Continuing, the point's height falls to $y = m - a$ at R, and then rises back to $y = m$ at S, and eventually back up to $y = m + a$ at the top of the circle. If we plot these heights continuously as a function of distance, d, traversed around the circle, we get the curve shown below.



This curve has several important features for which we introduce important terminology.

Definition The **midline** of a circular function is the horizontal line $y = m$ for which half the curve lies above the line and half the curve lies below. If the circular function results from tracking the y-coordinate of a point traversing a circle, $y = m$ corresponds to the y-coordinate of the center of the circle. In addition, the **amplitude** of a circular function is the maximum deviation of the curve from the midline. Note particularly that the value of the amplitude, a, corresponds to the radius of the circle that generates the curve.

 Acknowledgements: Active Prelude to Calculus (https://activecalculus.org/prelude)

Because we can traverse the circle in either direction and for as far as we wish, the domain of any circular function is the set of all real numbers. From our observations about the midline and amplitude, it follows that the range of a circular function with midline $y = m$ and amplitude a is the interval $[m - a, m + a]$.

This graph is an example of a periodic function. Recall the definition of a periodic function.

Definition Let f be a function whose domain and codomain are each the set of all real numbers. We say that f is **periodic** provided that there exists a real number k such that $f(x+k) = f(x)$ for every possible choice of x. The smallest value p for which $f(x + p) = f(x)$ for every choice of x is called the **period** of f.

For a circular function, the period is always the circumference of the circle that generates the curve. In the graph of the function above, we see how the curve has completed one full cycle of behavior every p units, regardless of where we start on the curve.

Circular functions arise as models for important phenomena in the world around us, such as in a *harmonic oscillator*. Consider a mass attached to a spring where the mass sits on a frictionless surface. After setting the mass in motion by stretching or compressing the spring, the mass will oscillate indefinitely back and forth, and its distance from a fixed point on the surface turns out to be given by a circular function.

The Average Rate of Change of a Circular Function

Just as there are important trends in the values of a circular function, there are also interesting patterns in the average rate of change of the function. These patterns are closely tied to the geometry of the circle.

For the next part of our discussion, we consider a circle of radius 1 centered at $(0, 0)$, and consider a point that travels a distance d counterclockwise around the circle with its starting point viewed as $(1, 0)$. We use this circle to generate the circular function $h = f(d)$ that tracks the height of the point at the moment the point has traversed d units around the circle from $(1, 0)$. Let's consider the average rate of change of f on several intervals that are connected to certain fractions of the circumference.

Remembering that h is a function of distance traversed along the circle, it follows that the average rate of change of h on any interval of distance between two points P and Q on the circle is given by

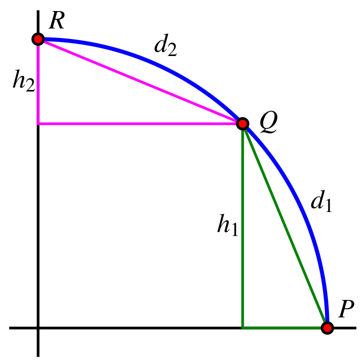
$$\text{AROC}_{[P,Q]} = \frac{\text{change in height}}{\text{distance along the circle}},$$

 Acknowledgements: Active Prelude to Calculus (https://activecalculus.org/prelude)

where both quantities are measured from point P to point Q.

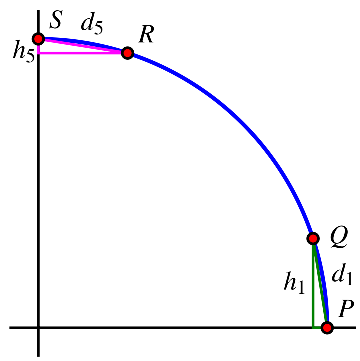
First, we consider points P, Q, and R where Q results from traversing $1/8$ of the circumference from P, and R $1/8$ of the circumference from Q. In particular, we note that the distance d_1 along the circle from P to Q is the same as the distance d_2 along the circle from Q to R, and thus $d_1 = d_2$. At the same time, it is apparent from the geometry of the circle that the change in height h_1 from P to Q is greater than the change in height h_2 from Q to R, so $h_1 > h_2$. Thus, we can say that

$$\mathrm{AROC}_{[P,Q]} = \frac{h_1}{d_1} > \frac{h_2}{d_2} = \mathrm{AROC}_{[Q,R]}.$$



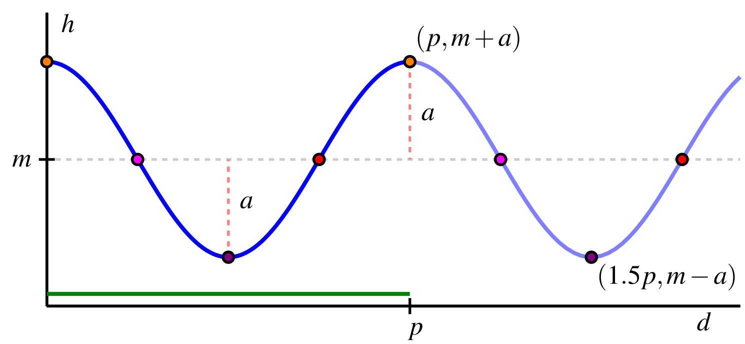
The differences in certain average rates of change appear to become more extreme if we consider shorter arcs along the circle. Next we consider traveling $1/20$ of the circumference along the circle. In the graph below, points P and Q lie $1/20$ of the circumference apart, as do R and S, so here $d_1 = d_5$. In this situation, it is the case that $h_1 > h_5$ for the same reasons as above, but we can say even more. From the green triangle, we see that $h_1 \approx d_1$ (while $h_1 < d_1$), so that $\mathrm{AROC}_{[P,Q]} = \dfrac{h_1}{d_1} \approx 1$. At the same time, in the magenta triangle in the figure we see that h_5 is very small, especially in comparison to d_5, and thus $\mathrm{AROC}_{[R,S]} = \dfrac{h_5}{d_5} \approx 0$. Hence, in this graph,

$$\mathrm{AROC}_{[P,Q]} \approx 1 \text{ and } \mathrm{AROC}_{[R,S]} \approx 0.$$



This information tells us that a circular function appears to change most rapidly for points near its midline and to change least rapidly for points near its highest and lowest values.

We can study the average rate of change not only on the circle itself, but also on a generic circular function graph, and thus make conclusions about where the function is increasing, decreasing, concave up, and concave down.



Summary

- When a point traverses a circle, a corresponding function can be generated by tracking the height of the point as it moves around the circle, where height is viewed as a function of distance traveled around the circle. We call such a function a *circular function*.
- Circular functions have several standard features. The function has a *midline* that is the line for which half the points on the curve lie above the line and half the points on the curve lie below. A circular function's *amplitude* is the maximum deviation of the

 Acknowledgements: Active Prelude to Calculus (https://activecalculus.org/prelude)

function value from the midline; the amplitude corresponds to the radius of the circle that generates the function. Circular functions also repeat themselves, and we call the smallest value of p for which $f(x+p) = f(x)$ for all x the period of the function. The period of a circular function corresponds to the circumference of the circle that generates the function.

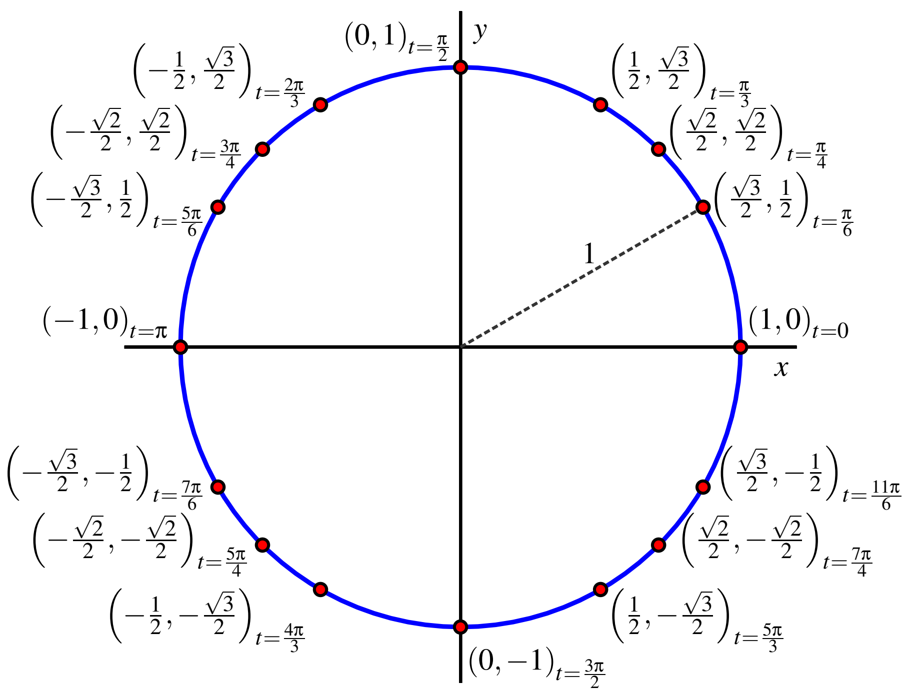
- Non-constant linear functions are either always increasing or always decreasing; quadratic functions are either always concave up or always concave down. Circular functions are sometimes increasing and sometimes decreasing, plus sometimes concave up and sometimes concave down. These behaviors are closely tied to the geometry of the circle.

9.1.2 The Sine and Cosine Functions

Motivating Questions

- What are the sine and cosine functions and how do they arise from a point traversing the unit circle?
- What important properties do the sine and cosine functions share?

In the last section, we saw how tracking the height of a point that is traversing a circle generates a periodic function. Previously, we also identified a collection of special points on the unit circle.

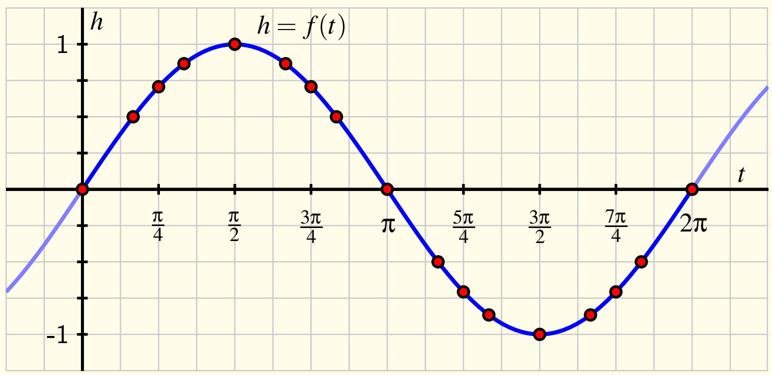


You can also use the *Desmos* file:

Desmos link: `https://www.desmos.com/calculator/jgddn7tzxg`

 Acknowledgements: Active Prelude to Calculus (https://activecalculus.org/prelude)

Exploration
If we consider the unit circle, start at $t = 0$, and traverse the circle counterclockwise, we may view the height, h, of the traversing point as a function of the angle, t, in radians. From there, we can plot the resulting (t, h) ordered pairs and connect them to generate the circular function pictured below.



a. What is the exact value of $h\left(\frac{\pi}{4}\right)$? of $h\left(\frac{\pi}{3}\right)$?

b. Complete the following table with the exact values of h that correspond to the stated inputs.

t	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
h									

t	π	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{7\pi}{4}$	$\frac{11\pi}{6}$	2π
h									

c. What is the exact value of $h\left(\frac{11\pi}{4}\right)$? of $h\left(\frac{14\pi}{3}\right)$?

d. Give four different values of t for which $h(t) = -\frac{\sqrt{3}}{2}$.

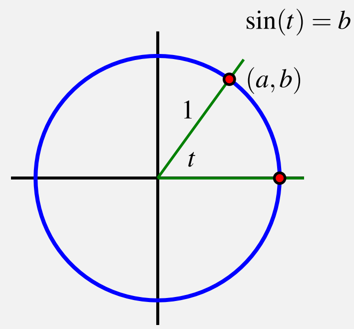
The Definition of the Sine Function

The circular function that tracks the height of a point on the unit circle traversing counterclockwise from $(1, 0)$ as a function of the corresponding central angle

 Acknowledgements: Active Prelude to Calculus (https://activecalculus.org/prelude)

(in radians) is one of the most important functions in mathematics. As such, we give the function a name: the **sine** function.

Definition



Given a central angle in the unit circle that measures t radians and that intersects the circle at both $(1, 0)$ and (a, b), we define the **sine of** t, denoted $\sin(t)$, by the rule

$$\sin(t) = b.$$

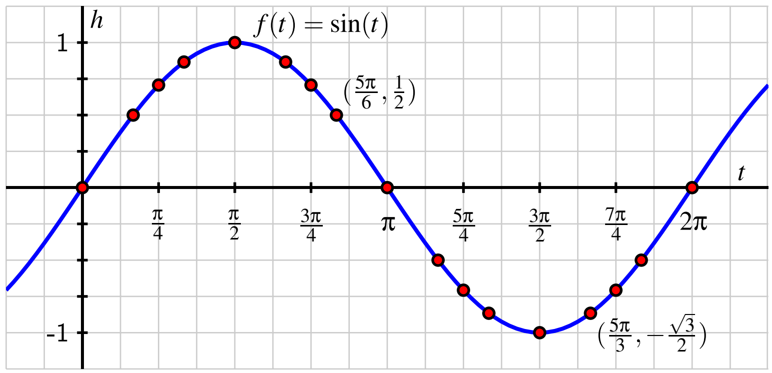
Because of the correspondence between radian angle measure and distance traversed on the unit circle, we can also think of $\sin(t)$ as identifying the y-coordinate of the point after it has traveled t units counterclockwise along the circle from $(1, 0)$. Note particularly that we can consider the sine of negative inputs: for instance, $\sin(-\frac{\pi}{2}) = -1$.

Based on our earlier work with the unit circle, we know many different exact values of the sine function, and summarize these in in the table below:

t	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
h	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0

t	π	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{7\pi}{4}$	$\frac{11\pi}{6}$	2π
h	0	$-\frac{1}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{3}}{2}$	-1	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{1}{2}$	0

Moreover, if we now plot these points in the usual way, we get the familiar circular wave function that comes from tracking the height of a point traversing a circle. We often call this graph the **sine wave**.



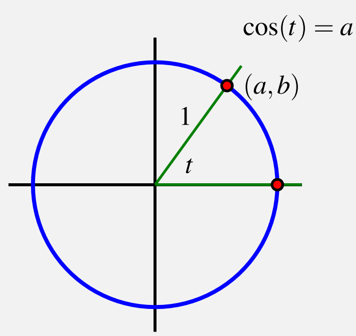
At https://www.desmos.com/calculator/f9foqx24cthttps://www.desmos.com/calculator/f9foqx24ct you can explore and investigate a helpful Desmos animation that shows how this motion around the circle generates the sine graph.

Desmos link: https://www.desmos.com/calculator/f9foqx24ct

The Definition of the Cosine Function

Given any central angle of radian measure t in the unit circle with one side passing through the point $(1, 0)$, the angle generates a unique point (a, b) that lies on the circle. Just as we can view the y-coordinate as a function of t, the x-coordinate is likewise a function of t. We therefore make the following definition.

Definition



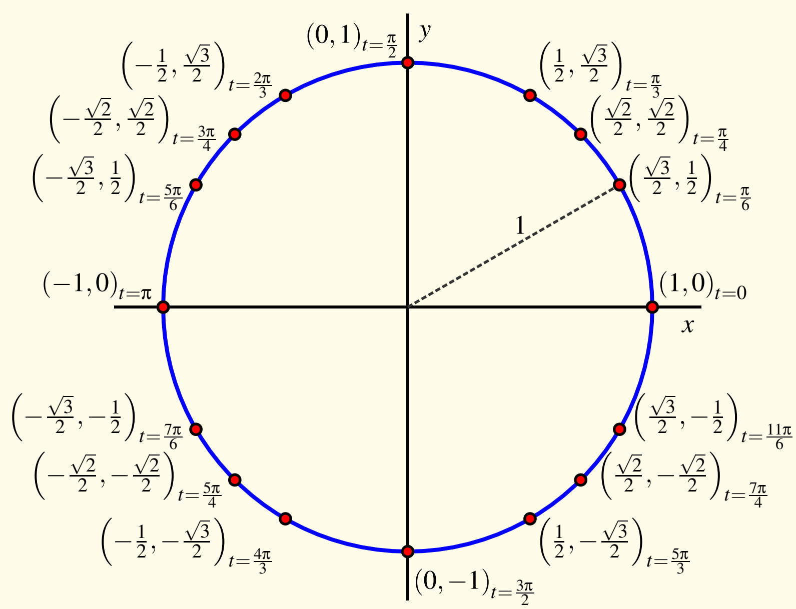
Given a central angle in the unit circle that measures t radians and that intersects the circle at both $(1, 0)$ and (a, b), we define the **cosine of** t, denoted $\cos(t)$, by the rule

$$\cos(t) = a.$$

Acknowledgements: Active Prelude to Calculus (https://activecalculus.org/prelude)

Again because of the correspondence between the radian measure of an angle and arc length along the unit circle, we can view the value of $\cos(t)$ as tracking the x-coordinate of a point traversing the unit circle clockwise a distance of t units along the circle from $(1, 0)$. We now use the data and information we have developed about the unit circle to build a table of values of $\cos(t)$ as well as a graph of the curve it generates.

Exploration Let $k = g(t)$ be the function that tracks the x-coordinate of a point traversing the unit circle counterclockwise from $(1, 0)$. That is, $g(t) = \cos(t)$. Use the information we know about the unit circle to respond to the following questions.



a. What is the exact value of $\cos(\frac{\pi}{6})$? of $\cos(\frac{5\pi}{6})$? $\cos(-\frac{\pi}{3})$?

b. Complete the following table with the exact values of k that cor-

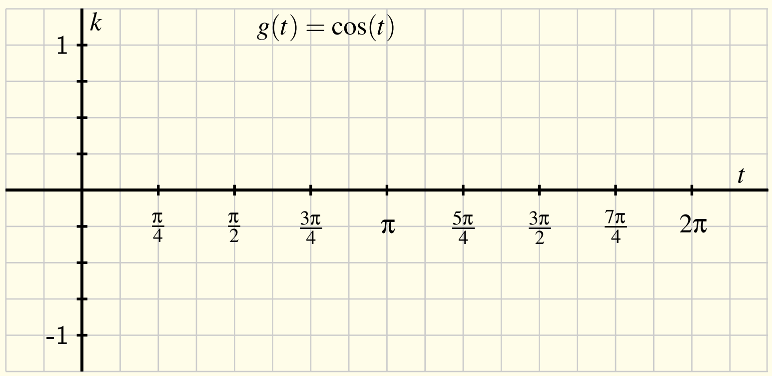
Acknowledgements: Active Prelude to Calculus (https://activecalculus.org/prelude)

respond to the stated inputs.

t	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
h									

t	π	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{7\pi}{4}$	$\frac{11\pi}{6}$	2π
h									

c. On the axes provided, sketch an accurate graph of $k = \cos(t)$. Label the exact location of several key points on the curve.



d. What is the exact value of $\cos(\frac{11\pi}{4})$? of $\cos(\frac{14\pi}{3})$?

e. Give four different values of t for which $\cos(t) = -\frac{\sqrt{3}}{2}$.

f. How is the graph of $k = \cos(t)$ different from the graph of $h = \sin(t)$? How are the graphs similar?

As we work with the sine and cosine functions, it's always helpful to remember their definitions in terms of the unit circle and the motion of a point traversing the circle. At `https://www.desmos.com/calculator/9s1ms0nlyf`https://www.desmos.com/calculator/9s1ms you can explore and investigate a helpful Desmos animation that shows how this motion around the circle generates the cosine graph.

Desmos link: `https://www.desmos.com/calculator/9s1ms0nlyf`

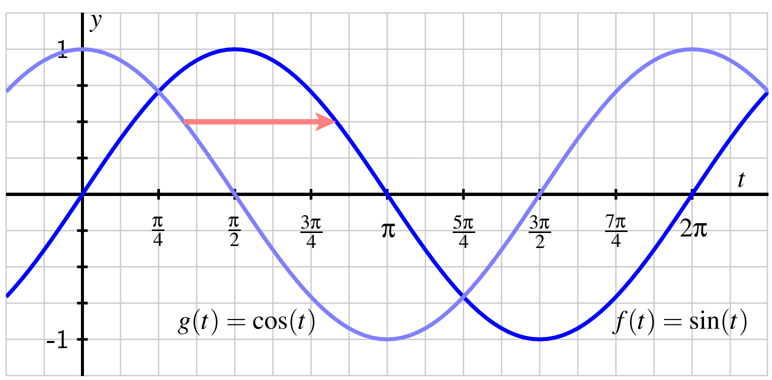
Properties of Sine and Cosine

Because the sine function results from tracking the y-coordinate of a point traversing the unit circle and the cosine function from the x-coordinate, the two functions have several shared properties of circular functions.

For both $f(t) = \sin(t)$ and $g(t) = \cos(t)$,

- the domain of the function is all real numbers;
- the range of the function is $[-1, 1]$;
- the midline of the function is $y = 0$;
- the amplitude of the function is $a = 1$;
- the period of the function is $p = 2\pi$.

It is also insightful to juxtapose the sine and cosine functions' graphs on the same coordinate axes. When we do, as seen in Figure **??**, we see that the curves can be viewed as horizontal translations of one another.



In particular, since the sine graph can be viewed as the cosine graph shifted $\frac{\pi}{2}$ units to the right, it follows that for any value of t,

$$\sin(t) = \cos(t - \frac{\pi}{2}).$$

Similarly, since the cosine graph can be viewed as the sine graph shifted left,

$$\cos(t) = \sin(t + \frac{\pi}{2}).$$

Because each of the two preceding equations hold for every value of t, they are often referred to as *identities*.

 Acknowledgements: Active Prelude to Calculus (https://activecalculus.org/prelude)

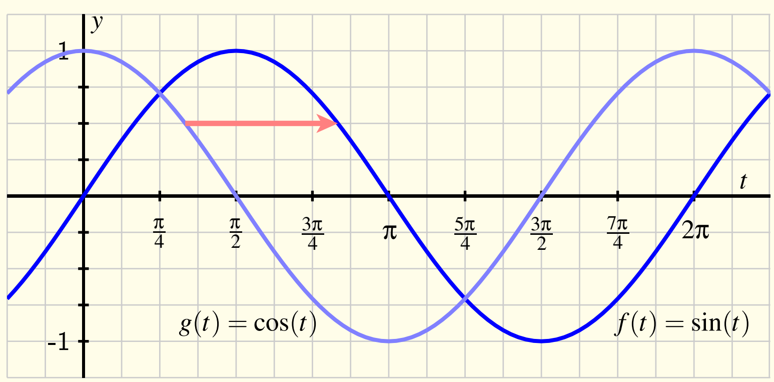
In light of the definitions of the sine and cosine functions, we can now view any point (x, y) on the unit circle as being of the form $(\cos(t), \sin(t))$, where t is the measure of the angle whose vertices are $(1, 0)$, $(0, 0)$, and (x, y). Note particularly that since $x^2 + y^2 = 1$, it is also true that $\cos^2(t) + \sin^2(t) = 1$. We call this fact the Fundamental Trigonometric Identity.

For any real number t,

$$\cos^2(t) + \sin^2(t) = 1.$$

There are additional trends and patterns in the two functions' graphs that we explore further in the following activity.

Exploration Use the figure below to assist in answering the following questions.



a. Give an example of the largest interval you can find on which $f(t) = \sin(t)$ is decreasing.

b. Give an example of the largest interval you can find on which $f(t) = \sin(t)$ is decreasing and concave down.

c. Give an example of the largest interval you can find on which $g(t) = \cos(t)$ is increasing.

d. Give an example of the largest interval you can find on which $g(t) = \sin(t)$ is increasing and concave up.

e. Without doing any computation, on which interval is the average rate of change of $g(t) = \cos(t)$ greater: $[\pi, \pi + 0.1]$ or $\left[\frac{3\pi}{2}, \frac{3\pi}{2} + 0.1\right]$? Why?

 Acknowledgements: Active Prelude to Calculus (https://activecalculus.org/prelude)

f. In general, how would you characterize the locations on the sine and cosine graphs where the functions are increasing or decreasingly most rapidly?

g. For which quadrants of the x-y plane is $\cos(t)$ negative for an angle in that quadrant?

Using Computing Technology

We have established that we know the exact value of $\sin(t)$ and $\cos(t)$ for any of the t-values labeled on the unit circle, as well as for any such $t \pm 2j\pi$, where j is a whole number, due to the periodicity of the functions. But what if we want to know $\sin(1.35)$ or $\cos(\frac{\pi}{5})$ or values for other inputs not in the table?

Any standard computing device a scientific calculator, *Desmos*, *Geogebra*, *WolframAlpha*, etc. has the ability to evaluate the sine and cosine functions at any input we desire. Because the input is viewed as an angle, each computing device has the option to consider the angle in radians or degrees. *It is always essential that you are sure which type of input your device is expecting.* Our computational device of choice is *Desmos*. In *Desmos*, you can change the input type between radians and degrees by clicking the wrench icon in the upper right and choosing the desired units. Radians is the default, and radians is what we will primarily use in both this class and calculus.

It take substantial and sophisticated mathematics to enable a computational device to evaluate the sine and cosine functions at any value we want; the algorithms involve an idea from calculus known as an infinite series. While your computational device is powerful, it's both helpful and important to understand the meaning of these values on the unit circle and to remember the special points for which we know the outputs of the sine and cosine functions exactly.

Exploration
Answer the following questions exactly wherever possible. If you estimate a value, do so to at least 5 decimal places of accuracy.

a. The x coordinate of the point on the unit circle that lies in the third quadrant and whose y-coordinate is $y = -\frac{3}{4}$.

b. The y-coordinate of the point on the unit circle generated by a central angle in standard position that measures $t = 2$ radians.

c. The x-coordinate of the point on the unit circle generated by a central angle in standard position that measures $t = -3.05$ radians.

d. The value of $\cos(t)$ where t is an angle in Quadrant II that satisfies

 Acknowledgements: Active Prelude to Calculus (https://activecalculus.org/prelude)

$\sin(t) = \frac{1}{2}$.

e. The value of $\sin(t)$ where t is an angle in Quadrant III for which $\cos(t) = -0.7$.

f. The average rate of change of $f(t) = \sin(t)$ on the intervals $[0.1, 0.2]$ and $[0.8, 0.9]$.

g. The average rate of change of $g(t) = \cos(t)$ on the intervals $[0.1, 0.2]$ and $[0.8, 0.9]$.

Summary

- The sine and cosine functions result from tracking the y- and x-coordinates of a point traversing the unit circle counterclockwise from $(1, 0)$. The value of $\sin(t)$ is the y-coordinate of a point that has traversed t units along the circle from $(1, 0)$ (or equivalently that corresponds to an angle of t radians), while the value of $\cos(t)$ is the x-coordinate of the same point.

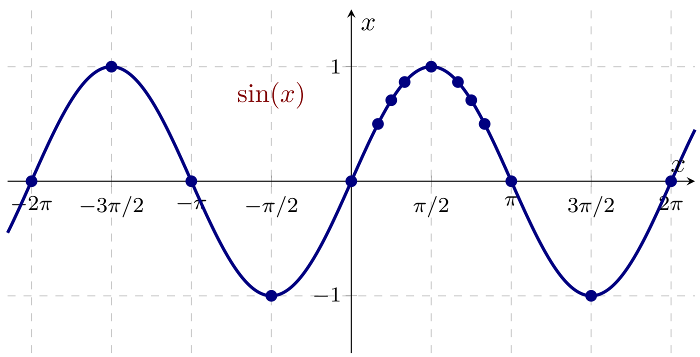
- The sine and cosine functions are both periodic functions that share the same domain (the set of all real numbers), range (the interval $[-1, 1]$), midline ($y = 0$), amplitude ($a = 1$), and period ($P = 2\pi$). In addition, the sine function is horizontal shift of the cosine function by $\frac{\pi}{2}$ units to the right, so $\sin(t) = \cos(t - \frac{\pi}{2})$ for any value of t.

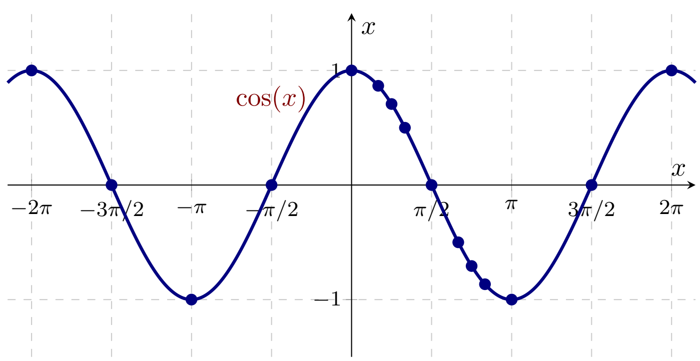
- If t corresponds to one of the special angles that we know on the unit circle, we can compute the values of $\sin(t)$ and $\cos(t)$ exactly. For other values of t, we can use a computational device to estimate the value of either function at a given input; when we do so, we must take care to know whether we are computing in terms of radians or degrees.

9.1.3 Creating a New Function: Tangent

Introduction

We are now going to determine the graph of the tangent function by analyzing what we now know about the sine and cosine functions. As a reminder, here is a graph of those functions with some important points marked. Specifically the points at all multiples of $\frac{\pi}{2}$ have been marked, as well as at the standard points $x = \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{2\pi}{3}, \frac{3\pi}{4}$, and $\frac{5\pi}{6}$.



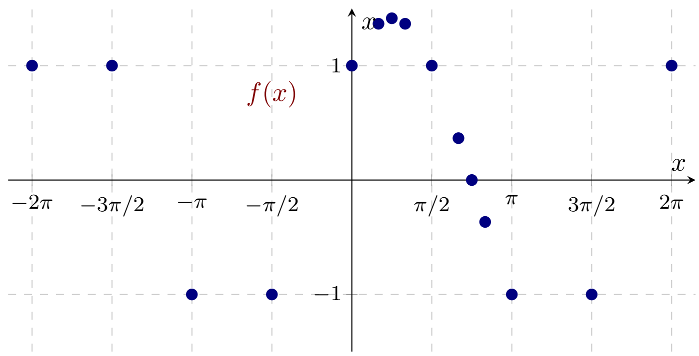


Graph of $f(x) = \sin(x) + \cos(x)$

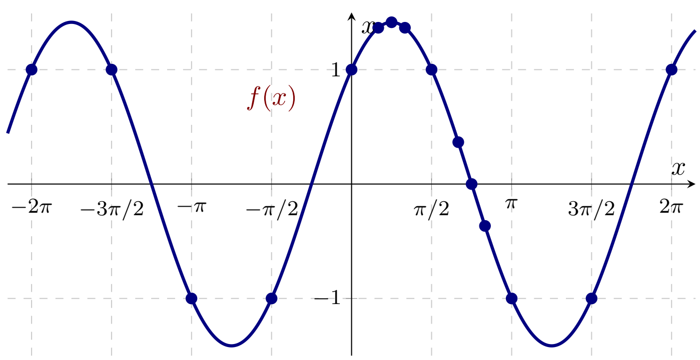
Before considering the function $\tan(x) = \dfrac{\sin(x)}{\cos(x)}$, lets consider the (possibly) more straightforward $f(x) = \sin(x) + \cos(x)$. Let's practice with this function creating a graph of a sum of two known functions.

What is the value of $f(0)$? We know $f(0) = \sin(0) + \cos(0) = 0 + 1 = 1$.

We can easily calculate the values of f at all of the important points marked in the graphs above. Let us plot the points of f corresponding to them.



Those extra points plotted between $x = 0$ and $x = \pi$ show us the behavior of this function f. Notice that between $x = 0$ and $x = \dfrac{\pi}{2}$, the graph increases to a peak, then decreases in a very sinusoidal manner. If we continue this with standard values and "connect the dots", we end up with the following graph.



We've ended up with another periodic function that looks like a stretched and shifted version of sine or cosine.

Now, let's try again using division instead of addition.

Determining the Graph of Tangent

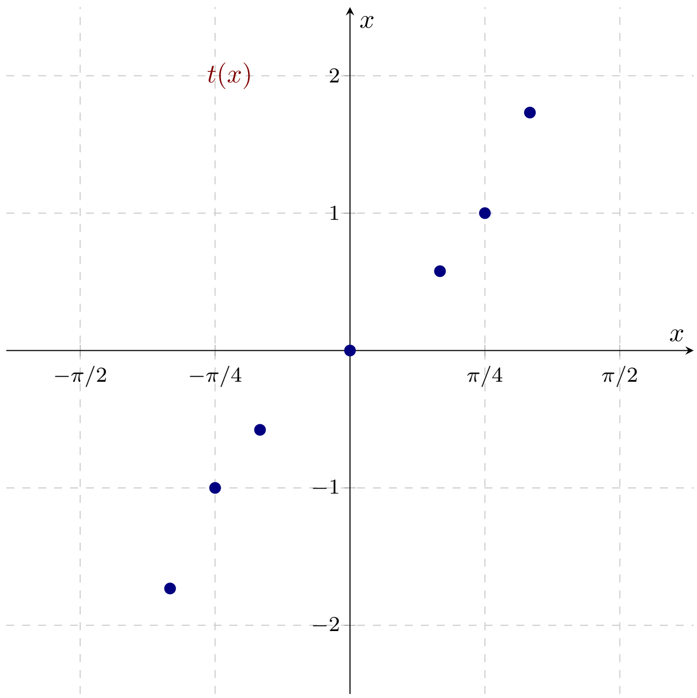
Recall that $\tan(x) = \dfrac{\sin(x)}{\cos(x)}$.

Notice that this function is undefined at all x-values with $\cos(x) = 0$. That means the function $\tan(x)$ is not defined for $x = \pm\dfrac{\pi}{2}, \pm\dfrac{3\pi}{2}, \pm\dfrac{5\pi}{2}, \ldots$.

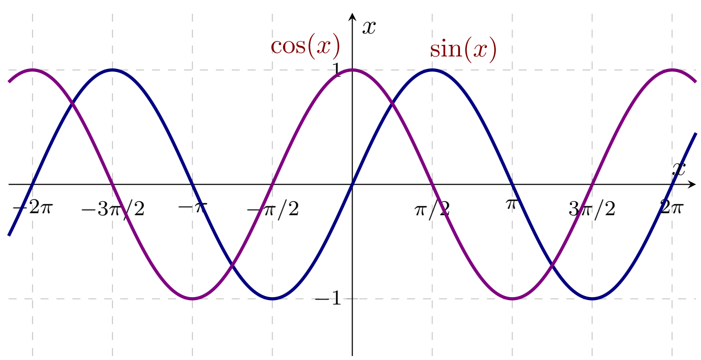
We can calculate values of $\tan(x)$ for other inputs. $\tan(0) = \dfrac{\sin(0)}{\cos(0)} = \dfrac{0}{1} = 0$. The following table lists some of the other values arising from this division. Notice that in this table we have chosen to rationalize the denominators of the fractions that have appeared. That is, we have written $\dfrac{1}{\sqrt{2}}$ as $\dfrac{\sqrt{2}}{2}$, by multiplying the fraction by 1 written in the form $\dfrac{\sqrt{2}}{\sqrt{2}}$. Similarly $\dfrac{1}{\sqrt{3}}$ is written as $\dfrac{\sqrt{3}}{3}$.

x	$\sin(x)$	$\cos(x)$	$t(x) = \dfrac{\sin(x)}{\cos(x)}$
$-\dfrac{\pi}{3}$	$-\dfrac{\sqrt{3}}{2}$	$\dfrac{1}{2}$	$-\sqrt{3}$
$-\dfrac{\pi}{4}$	$-\dfrac{\sqrt{2}}{2}$	$\dfrac{\sqrt{2}}{2}$	-1
$-\dfrac{\pi}{6}$	$-\dfrac{1}{2}$	$\dfrac{\sqrt{3}}{2}$	$-\dfrac{\sqrt{3}}{3}$
0	0	1	0
$\dfrac{\pi}{6}$	$\dfrac{1}{2}$	$\dfrac{\sqrt{3}}{2}$	$\dfrac{\sqrt{3}}{3}$
$\dfrac{\pi}{4}$	$\dfrac{\sqrt{2}}{2}$	$\dfrac{\sqrt{2}}{2}$	1
$\dfrac{\pi}{3}$	$\dfrac{\sqrt{3}}{2}$	$\dfrac{1}{2}$	$\sqrt{3}$

If we plot these points, we find the following graph.



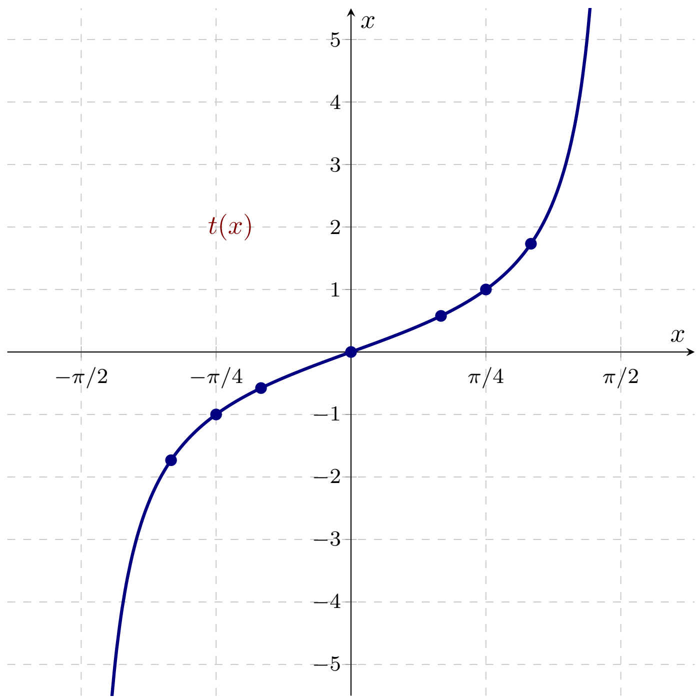
Let's think about what happens if x is a number really close to $\dfrac{\pi}{2}$, but just a little bit smaller than $\dfrac{\pi}{2}$. Notice from the graphs that the value of $\sin(x)$ will be a positive number that is really close to 1 and the value of $\cos(x)$ will be really close to 0 but still positive.



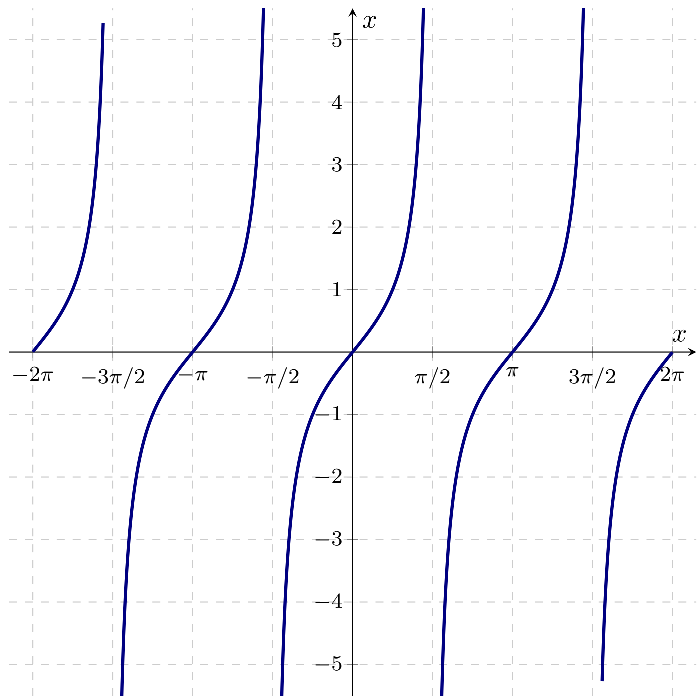
What happens if we take 1 and divide it by a small positive number? Let's look at a table of values to see.

z	$\frac{1}{z}$
1	1
$\frac{1}{2}$	2
$\frac{1}{3}$	3
$\frac{1}{10}$	10
$\frac{1}{100}$	100
$\frac{1}{1000}$	1000

Notice that as the numbers $z = 1,\ 1/2,\ 1/3,\ \ldots$ got smaller and smaller, the values of $\frac{1}{z} = 1,\ 2,\ 3,\ \ldots$ got larger and larger? That is the same thing we are noticing in the graph of the function t we are building above. For values of x really close to $\pi/2$, but still less than $\pi/2$, the value of $t(x)$ is basically 1 divided by a very small positive number. This table of values tells us that the smaller that denominator gets, the larger the fraction becomes. Adding this behavior to the graph of t gives the following.



By repeating similar calculations for other standard inputs, we arrive at the following graph.



As you can see from the graph, $\tan(x)$ is an odd, periodic function with period π (not 2π like sine and cosine).

9.1.4 Graphs of Secant, Cosecant, and Cotangent

Motivating Questions

- What do the graphs of secant, cosecant, and cotangent look like?
- What are some important properties of these graphs? Where do they have asymptotes?

Like the tangent function, the secant, cosecant, and cotangent functions are defined in terms of the sine and cosine functions, so we can determine the exact values of these functions at each of the special points on the unit circle. In addition, we can use our understanding of the unit circle and the properties of the sine and cosine functions to determine key properties of these other trigonometric functions.

The Secant Function

We begin by investigating the secant function. Using the fact that $\sec(t) = \frac{1}{\cos(t)}$, we note that anywhere $\cos(t) = 0$, the value of $\sec(t)$ is undefined. We record such instances in the following table by writing "u". At all other points, the value of the secant function is simply the reciprocal of the cosine function's value. Since $|\cos(t)| \leq 1$ for all t, it follows that $|\sec(t)| \geq 1$ for all t (for which the secant's value is defined).

Values of Secant in Quadrant I

t	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$\cos(t)$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
$\sec(t)$	1	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2	u

 Acknowledgements: Active Prelude to Calculus (https://activecalculus.org/prelude)

Values of Secant in Quadrant II

t	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
$\cos(t)$	0	$-\frac{1}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{3}}{2}$	-1
$\sec(t)$	u	-2	$-\sqrt{2}$	$-\frac{2}{\sqrt{3}}$	-1

Values of Secant in Quadrant III

t	π	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$
$\cos(t)$	-1	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{1}{2}$	0
$\sec(t)$	-1	$-\frac{2}{\sqrt{3}}$	$-\sqrt{2}$	-2	u

Values of Secant in Quadrant IV

t	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{7\pi}{4}$	$\frac{11\pi}{6}$	2π
$\cos(t)$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
$\sec(t)$	u	2	$\sqrt{2}$	$\frac{2}{\sqrt{3}}$	1

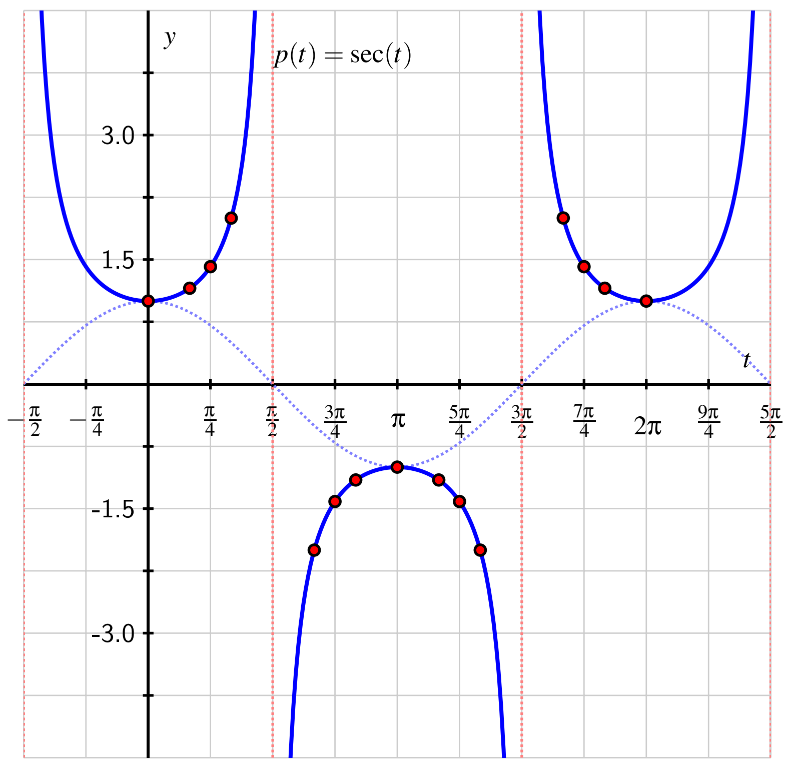
These tables help us identify trends in the secant function. The sign of $\sec(t)$ matches the sign of $\cos(t)$ and thus is positive in Quadrant I, negative in Quadrant II, negative in Quadrant III, and positive in Quadrant IV.

In addition, we observe that as t-values in the first quadrant get closer to $\frac{\pi}{2}$, $\cos(t)$ gets closer to 0 (while being always positive). Since the numerator of the secant function is always 1, having its denominator approach 0 means that $\sec(t)$ increases without bound as t approaches $\frac{\pi}{2}$ from the left side. Once t

 Acknowledgements: Active Prelude to Calculus (https://activecalculus.org/prelude)

is slightly greater than $\frac{\pi}{2}$ in Quadrant II, the value of $\cos(t)$ is negative (and close to zero). This makes the value of $\sec(t)$ decrease without bound (negative and getting further away from 0) for t approaching $\frac{\pi}{2}$ from the right side, and results in $p(t) = \sec(t)$ having a vertical asymptote at $t = \frac{\pi}{2}$. The periodicity and sign behavior of $\cos(t)$ mean this asymptotic behavior of the secant function will repeat.

Plotting the data in the table along with the expected asymptotes and connecting the points intuitively, we see the graph of the secant function.



We see from both the table and the graph that the secant function has period $P = 2\pi$. We summarize our recent work as follows.

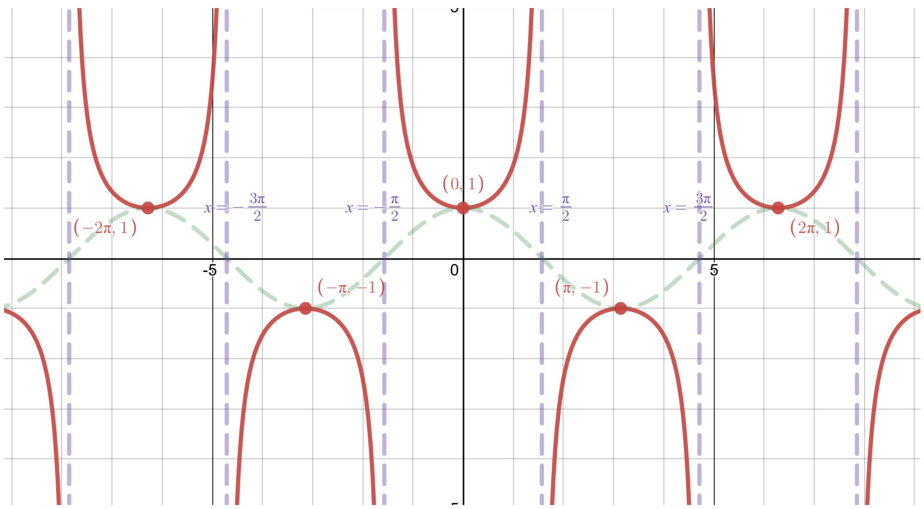
Properties of the secant function.
For the function $p(t) = \sec(t)$,

- its domain is the set of all real numbers except $t = \frac{\pi}{2} \pm k\pi$ where k is any whole number;

 Acknowledgements: Active Prelude to Calculus (https://activecalculus.org/prelude)

- its range is the set of all real numbers y such that $|y| \geq 1$;
- its period is $P = 2\pi$.

We can see the secant function in Desmos as well.



Try playing with the secant graph yourself.

Desmos link: `https://www.desmos.com/calculator/gmsanlrjza`

The Cosecant Function

Graphing the cosecant function is extremely similar to graphing the secant function, except we use $\csc(t) = \dfrac{1}{\sin(t)}$ so we are flipping over the values of sine instead of the values of cosine. Since the sine and cosine graphs look very similar except they are are shifted by $\dfrac{\pi}{2}$, the secant and cosecant graphs will also look very similar but be shifted from one another in the exact same way.

Let's create the table of famous values for cosecant.

 Acknowledgements: Active Prelude to Calculus (https://activecalculus.org/prelude)

Values of Cosecant in Quadrant I

t	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$\sin(t)$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
$\csc(t)$	u	2	$\sqrt{2}$	$\frac{2}{\sqrt{3}}$	1

Values of Cosecant in Quadrant II

t	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
$\sin(t)$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
$\csc(t)$	1	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2	u

Values of Cosecant in Quadrant III

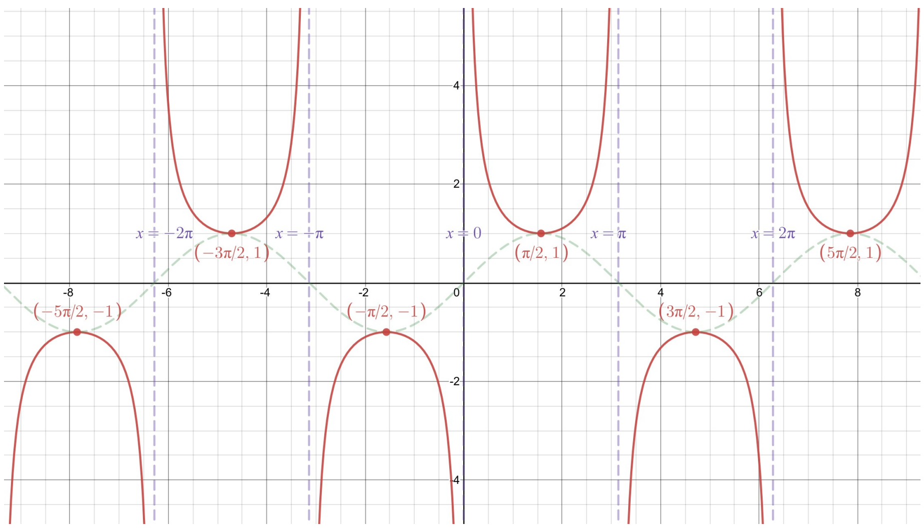
t	π	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$
$\sin(t)$	0	$-\frac{1}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{3}}{2}$	-1
$\csc(t)$	u	-2	$-\sqrt{2}$	$-\frac{2}{\sqrt{3}}$	-1

 Acknowledgements: Active Prelude to Calculus (https://activecalculus.org/prelude)

Values of Cosecant in Quadrant IV

t	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{7\pi}{4}$	$\frac{11\pi}{6}$	2π
$\sin(t)$	-1	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{1}{2}$	0
$\csc(t)$	-1	$-\frac{2}{\sqrt{3}}$	$-\sqrt{2}$	-2	u

Plotting the data in the table along with the expected asymptotes and connecting the points intuitively, we see the graph of the cosecant function.



We see from both the table and the graph that the cosecant function has period $P = 2\pi$. We summarize our recent work as follows.

Properties of the cosecant function.
For the function $p(t) = \csc(t)$,

- its domain is the set of all real numbers except $t = \pm k\pi$ where k is any whole number;
- its range is the set of all real numbers y such that $|y| \geq 1$;
- its period is $P = 2\pi$.

Try playing with the cosecant graph yourself.

 Acknowledgements: Active Prelude to Calculus (https://activecalculus.org/prelude)

Desmos link: https://www.desmos.com/calculator/norjxi7z4r

The Cotangent Function

Graphing the cotangent function is similarto graphing the secant and cosecant functions, except we use $\cot(t) = \dfrac{1}{\tan(t)}$ so we are flipping over the values of tangent. Since the tangent graph has a period of π, the graph of cotangent will also have a period of π. Therefore, we only need to calculate tables of values for the first two quadrants.

Let's create the table of famous values for cotangent.

Values of Cotangent in Quadrant I

t	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$\tan(t)$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	u
$\cot(t)$	u	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$	1

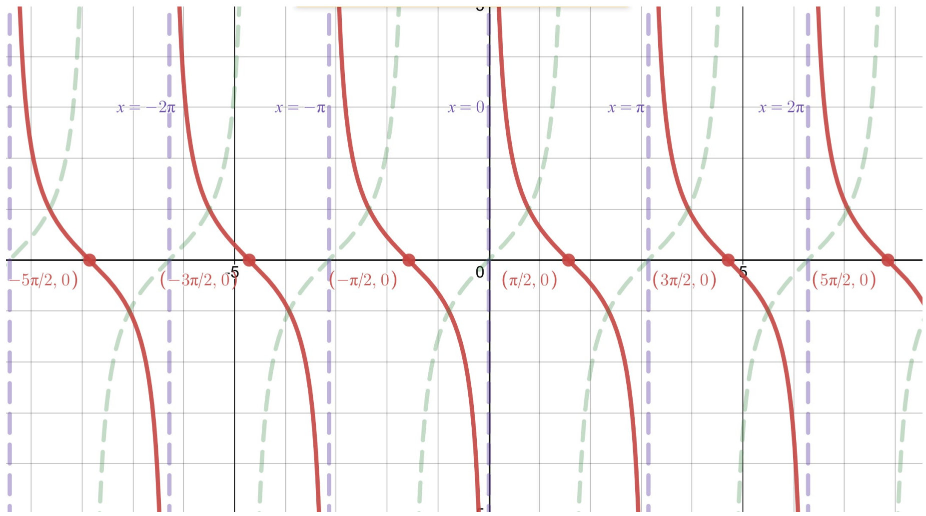
Notice that something unusual happened here. Even though tangent is undefined at $t = \dfrac{\pi}{2}$, we have that $\cot(\dfrac{\pi}{2}) = 0$. This is because $\cot(t) = \dfrac{\cos(t)}{\sin(t)}$ and so we can define $\cot(\dfrac{\pi}{2}) = \dfrac{\cos(\frac{\pi}{2})}{\sin(\frac{\pi}{2})} = \dfrac{0}{1} = 0$.

Values of Cotangent in Quadrant II

t	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
$\tan(t)$	u	$-\sqrt{3}$	-1	$-\frac{1}{\sqrt{3}}$	0
$\cot(t)$	0	$-\frac{1}{\sqrt{3}}$	-1	$-\sqrt{3}$	u

Plotting the data in the table along with the expected asymptotes and connecting the points intuitively, we see the graph of the cotangent function.

 Acknowledgements: Active Prelude to Calculus (https://activecalculus.org/prelude)



We summarize our recent work as follows.

Properties of the cotangent function.
For the function $p(t) = \csc(t)$,

- its domain is the set of all real numbers except $t = \pm k\dfrac{\pi}{2}$ where k is any whole number;
- its range is the set of all real numbers;
- its period is $P = \pi$.

Try playing with the cotangent graph yourself.

Desmos link: `https://www.desmos.com/calculator/yhmzxaq7ep`

9.2 Trig Functions

Learning Objectives

-
    -
    -
    -

9.2.1 Trig Functions as Functions

Motivating Questions

- How do trigonometric functions interact with other functions?
- How do we find zeros of trigonometric functions?
- How does average rate of change look like with trigonometric functions?

Trig Function Compositions

Trigonometric functions can be composed with any of the types of functions that we have already seen. Just as with other function compostions, we need to be mindful of the domains and ranges of our functions.

Example 47. *Let's consider the following functions:* $f(x) = \sin(x)$ *and* $g(x) = 3x^2$.
Find the function below and state its domain and range

(a) $f(g(x))$

(b) $g(f(x))$

(c) $f(f(x))$

Explanation First let's find the domain and range of $f(x)$ and $g(x)$.
The domain for both $f(x)$ and $g(x)$ is $(-\inf, \inf)$. The range for $f(x)$ is $[-1, 1]$ and the range for $g(x)$ is $[0, \inf)$. Now we can look at the compositions.

(a) $f(g(x)) = \sin(3x^2)$
The domain is $(-\infty, \infty)$ and the range is $[-1, 1]$.

(b) $g(f(x)) = 3\sin(x^2)$
The domain is $(-\infty, \infty)$. The range is $[0, 3]$

(c) $f(f(x)) = \sin(\sin(x))$
The domain is $(-\infty, \infty)$. The range is $[-\sin(1), \sin(1)]$.

Finding Zeros of Trigonometric Funtions.

Example 48. $f(x) = \sin^2(x) - 1$
Find the zeros of f *for* $0 \leq x \leq 2\pi$
Explanation First we need to set f to equal 0

$$\sin^2(x) - 1 = 0$$

Now we can recognize that we have a difference of squares, so we have the following:

$$\sin^2(x) - 1 = (\sin(x) + 1)(\sin(x) - 1)$$

Now we can set each part equal to zero. So we have

$$\sin(x) + 1 = 0 \text{ and } \sin(x) - 1 = 0$$

After simplifying them a bit we have

$$\sin(x) = -1 \text{ and } \sin(x) = 1$$

Because these are famous values of $\sin(x)$, we can find values for x without using inverse trigonometry.
$x = -\frac{\pi}{2}, \frac{\pi}{2}$

Example 49. $f(x) = \sin^2(x) + \frac{5}{2}\sin(x) - \frac{3}{2}$
Find the zeros of f for $0 \le x \le 2\pi$
Explanation First we need to set f to equal 0
$\sin^2(x) + \frac{5}{2}\sin(x) - \frac{3}{2} = 0$
Thankfully f can factor nicely, so we have

$$\sin^2(x) + \frac{5}{2}\sin(x) - \frac{3}{2} = (\sin(x) - \frac{1}{2})(\sin(x) + 3)$$

Now we set each part equal to zero.

$$\sin(x) - \frac{1}{2} = 0 \text{ and } \sin(x) + 3 = 0$$

After simplifying them a bit we have

$$\sin(x) = \frac{1}{2} \text{ and } \sin(x) = -3$$

-3 is outside of the range of $\sin(x)$ so we can disregard that term. Our only concern is the famous value where $sin(x)$ is $\frac{1}{2}$. This happens to occur in two places for $0 \le x \le 2\pi$, so our answer is $x = \frac{\pi}{6}, \frac{5\pi}{6}$

Average Rate of Change with Trigonometry

We can still find average rate of change with trigonometric functions, but because they are periodic there can be some interesting results.

Example 50. *Let* $f(x) = \sin(x)$

(a) $\text{AROC}_{[\frac{\pi}{6}, \frac{3\pi}{4}]}$

(b) $\text{AROC}_{[\frac{\pi}{3},\frac{2\pi}{3}]}$

Explanation

(a) $\text{AROC}_{[\frac{\pi}{6},\frac{3\pi}{4}]} = \dfrac{\sin(\frac{3\pi}{4}) - \sin(\frac{\pi}{6})}{\frac{3\pi}{4} - \frac{\pi}{6}}$

First we substitute our trig values

$\dfrac{\sin(\frac{3\pi}{4}) - \sin(\frac{\pi}{6})}{\frac{3\pi}{4} - \frac{\pi}{6}} = \dfrac{\frac{\sqrt{2}}{2} - \frac{1}{2}}{\frac{3\pi}{4} - \frac{\pi}{6}}$

Now we simplify our fractions

$\dfrac{\frac{\sqrt{2}}{2} - \frac{1}{2}}{\frac{3\pi}{4} - \frac{\pi}{6}} = \dfrac{\frac{\sqrt{2}-1}{2}}{\frac{9\pi}{12} - \frac{2\pi}{12}} = \dfrac{\frac{\sqrt{2}-1}{2}}{\frac{7\pi}{12}}$

$\dfrac{\frac{\sqrt{2}-1}{2}}{\frac{7\pi}{12}} = \dfrac{6(\sqrt{2}-1)}{7\pi}$

$\text{AROC}_{[\frac{\pi}{6},\frac{3\pi}{4}]} = \dfrac{6\sqrt{2}-6}{7\pi}$

(b) $\text{AROC}_{[\frac{\pi}{3},\frac{2\pi}{3}]} = \dfrac{\sin(\frac{2\pi}{3}) - \sin(\frac{\pi}{3})}{\frac{2\pi}{3} - \frac{\pi}{3}}$

First we substitute our trig values

$\dfrac{\sin(\frac{2\pi}{3}) - \sin(\frac{\pi}{3})}{\frac{2\pi}{3} - \frac{\pi}{3}} = \dfrac{\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2}}{\frac{2\pi}{3} - \frac{\pi}{3}}$

Now we simplify

$\dfrac{\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2}}{\frac{2\pi}{3} - \frac{\pi}{3}} = \dfrac{0}{\frac{\pi}{3}}$

We can see that we have 0 in the numerator, which means that our average rate of change will be 0. This is because we are working with a periodic function and picked values in similar positions. Even though we had positive and negative rates of change at some point between $\dfrac{\pi}{3}$ and $\dfrac{2\pi}{3}$, we ended up with the same y value, so *on average* there was no change.

9.2.2 Trig Functions Transformations

Motivating Questions

- How do the three standard transformations (vertical translation, horizontal translation, and vertical scaling) affect the midline, amplitude, range, and period of sine and cosine curves?
- What algebraic transformation results in horizontal stretching or scaling of a function?
- How can we determine a formula involving sine or cosine that models any circular periodic function for which the midline, amplitude, period, and an anchor point are known?

Recall our previous work with transformations, where we studied how the graph of the function g defined by $g(x) = af(x-b)+c$ is related to the graph of f, where a, b, and c are real numbers with $a \neq 0$. Because such transformations can shift and stretch a function, we are interested in understanding how we can use transformations of the sine and cosine functions to fit formulas to circular functions.

Example 51. *Let $f(t) = \cos(t)$. First, answer all of the questions below* without *using* Desmos*; then use* Desmos *to confirm your conjectures. For each prompt, describe the graphs of g and h as transformations of f and, in addition, state the amplitude, midline, and period of both g and h.*

a. $g(t) = 3\cos(t)$ *and* $h(t) = -\dfrac{1}{4}\cos(t)$

b. $g(t) = \cos(t-\pi)$ *and* $h(t) = \cos\left(t+\dfrac{\pi}{2}\right)$

c. $g(t) = \cos(t)+4$ *and* $h(t) = \cos(t)-2$

d. $g(t) = 3\cos(t-\pi)+4$ *and* $h(t) = -\dfrac{1}{4}\cos\left(t+\dfrac{\pi}{2}\right)-2$

Explanation

a. $g(t)$ Has a vertical stretch by 3 and an amplitude of 3. $h(t)$ has a reflection across the x-axis with a vertical shrink of $\dfrac{1}{4}$ and an amplitude of $\dfrac{1}{4}$. $g(t)$ and $h(t)$ both have a midline at $y=0$ and a period of 2π.

b. $g(t)$ Has a horizontal shift to the right of π. $h(t)$ has a horizontal shift to the left of $\dfrac{\pi}{2}$. $g(t)$ and $h(t)$ both have a midline at $y=0$, a period of 2π, and an amplitude of 1.

c. $g(t)$ Has a vertical shift up of 4 and a midline at $y = 4$. $h(t)$ has a vertical shift down of 2 and a midline of $y = -2$. $g(t)$ and $h(t)$ both have a period of 2π and an amplitude of 1.

d. $g(t)$ Has a vertical stretch by 3, a horizontal shift to the right of π,and a vertical shift up of 4. $g(t)$ has an amplitude of 3 and a midline at $y = 4$. $h(t)$ has a reflection across the x-axis with a vertical shrink of $\frac{1}{4}$, a horizontal shift to the left of $\frac{\pi}{2}$, and a vertical shift down of 2. $h(t)$has an amplitude of $\frac{1}{4}$ and a midline of $y = -2$. $g(t)$ and $h(t)$ both have a period of 2π.

Shifts and vertical stretches of the sine and cosine functions

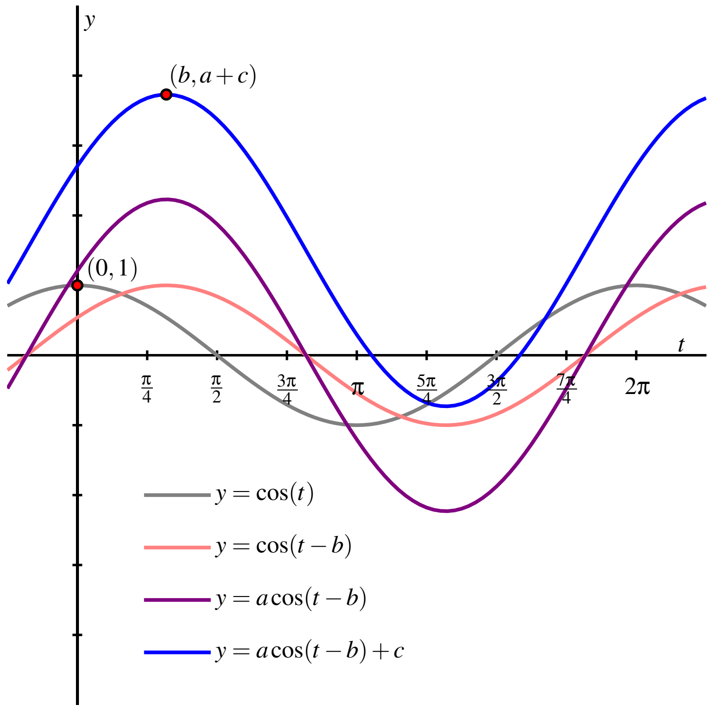
We know that the standard functions $f(t) = \sin(t)$ and $g(t) = \cos(t)$ are circular functions that each have midline $y = 0$, amplitude $a = 1$, period $p = 2\pi$, and range $[-1, 1]$. This suggests the following general principles.

Transformations of sine and cosine with shifts and vertical stretches.
Given real numbers a, c, and d with $a \neq 0$, the functions

$$f(t) = a\cos(t - c) + d \text{ and } g(t) = a\sin(t - c) + d$$

each represent a horizontal shift by c units to the right, followed by a vertical stretch by a units, followed by a vertical shift of d units, applied to the parent function ($\cos(t)$ or $\sin(t)$, respectively). The resulting circular functions have midline $y = d$, amplitude a, range $[d-a, d+a]$, and period $p = 2\pi$. In addition, the point $(c, a + d)$ lies on the graph of f and the point (c, d) lies on the graph of g.

In the figure below, we see how the overall transformation $k(t) = a\cos(t-b)+c$ comes from executing a sequence of simpler ones. The original parent function $y = \cos(t)$ (in dark gray) is first shifted b units right to generate the light red graph of $y = \cos(t - b)$. In turn, that graph is then scaled vertically by a to generate the purple graph of $y = a\cos(t-b)$. Finally, the purple graph is shifted c units vertically to result in the final graph of $y = a\cos(t - b) + c$ in blue.



It is often useful to follow one particular point through a sequence of transformations. In the above figure, we see the red point that is located at $(0, 1)$ on the original function $y = \cos(t)$, as well as the point $(c, a + d)$ that is the corresponding point on $k(t) = a\cos(t-c)+d$ under the overall transformation. Note that the point $(b, a + c)$ results from the input, $t = c$, that makes the argument of the cosine function zero: $k(c) = a\cos(c - c) + d = a\cos(0) + d$.

While the sine and cosine functions extend infinitely in either direction, it's natural to think of the point $(0, 1)$ as the "starting point" of the cosine function, and similarly the point $(0, 0)$ as the starting point of the sine function. We will refer to the corresponding points $(c, a + d)$ and (c, d) on $k(t) = a\cos(t - c) + d$ and $h(t) = a\sin(t - c) + d$ as anchor points. Anchor points, along with other information about a circular function's amplitude, midline, and period help us to determine a formula for a function that fits a given situation.

Exploration Consider a spring-mass system where the weight resting on a frictionless table. We let $s(t)$ denote the distance from the wall (where the spring is attached) to the weight at time t in seconds and know that the weight oscillates periodically with a minimum value of $s(t) = 2$ feet and a maximum value of $s(t) = 7$ feet with a period of 2π. We also know that $s(0) = 4.5$ and $s\left(\dfrac{\pi}{2}\right) = 2$.

Determine a formula for $s(t)$ in the form $s(t) = a\cos(t-b)+c$ or $s(t) = a\sin(t-c)+d$. Is it possible to find two different formulas that work? For any formula you find, identify the anchor point.

Subsection 2.4.2 Horizontal scaling

There is one more very important transformation of a function that we've not yet explored. Given a function $y = f(x)$, we want to understand the related function $g(x) = f(bx)$, where b is a positive real number. The sine and cosine functions are ideal functions with which to explore these effects; moreover, this transformation is crucial for being able to use the sine and cosine functions to model phenomena that oscillate at different frequencies.

By using a graphing utility such as Desmos, we can explore the effect of the transformation $g(t) = f(bt)$, where $f(t) = \sin(t)$.

By experimenting with various values or a slider, we gain an intuitive sense for how the value of b affects the graph of $h(t) = f(bt)$ in comparision to the graph of $f(t)$. When $b = 2$, we see that the graph of h is oscillating twice as fast as the graph of f since $h(t) = f(2t)$ completes two full cycles over an interval in which f completes one full cycle. In contrast, when $b = \frac{1}{2}$, the graph of h oscillates half as fast as the graph of f, as $h(t) = f(\frac{1}{2}t)$ completes only half of one cycle over an interval where $f(t)$ completes a full one.

We can also understand this from the perspective of function composition. To evaluate $h(t) = f(2t)$, at a given value of t, we first multiply the input t by a factor of 2, and then evaluate the function f at the result. An important observation is that

$$h\left(\frac{1}{2}t\right) = f\left(2 \cdot \frac{1}{2}t\right) = f(t).$$

This tells us that the point $(\frac{1}{2}t, f(t))$ lies on the graph of h since an input of $\frac{1}{2}t$ in h results in the value $f(t)$. At the same time, the point $(t, f(t))$ lies on the graph of f. Thus we see that the correlation between points on the graphs of f and h (where $h(t) = f(2t)$) is

$$(t, f(t)) \to \left(\frac{1}{2}t, f(t)\right).$$

We can therefore think of the transformation $h(t) = f(2t)$ as achieving the output values of f twice as fast as the original function $f(t)$ does. Analogously, the transformation $h(t) = f(\frac{1}{2}t)$ will achieve the output values of f only half as quickly as the original function.

Given a function $y = f(t)$ and a real number $b > 0$, the transformed function $y = h(t) = f(bt)$ is a *horizontal stretch* of the graph of f. Every point $(t, f(t))$ on the graph of f gets stretched horizontally to the corresponding point $(\frac{1}{b}t, f(t))$ on the graph of h. If $0 < b < 1$, the graph of v is a stretch of f away from the y-axis by a factor of $\frac{1}{b}$; if $b > 1$, the graph of h is a compression of f toward the y-axis by a factor of $\frac{1}{b}$. The only point on the graph of f that is unchanged by the transformation is $(0, f(0))$.

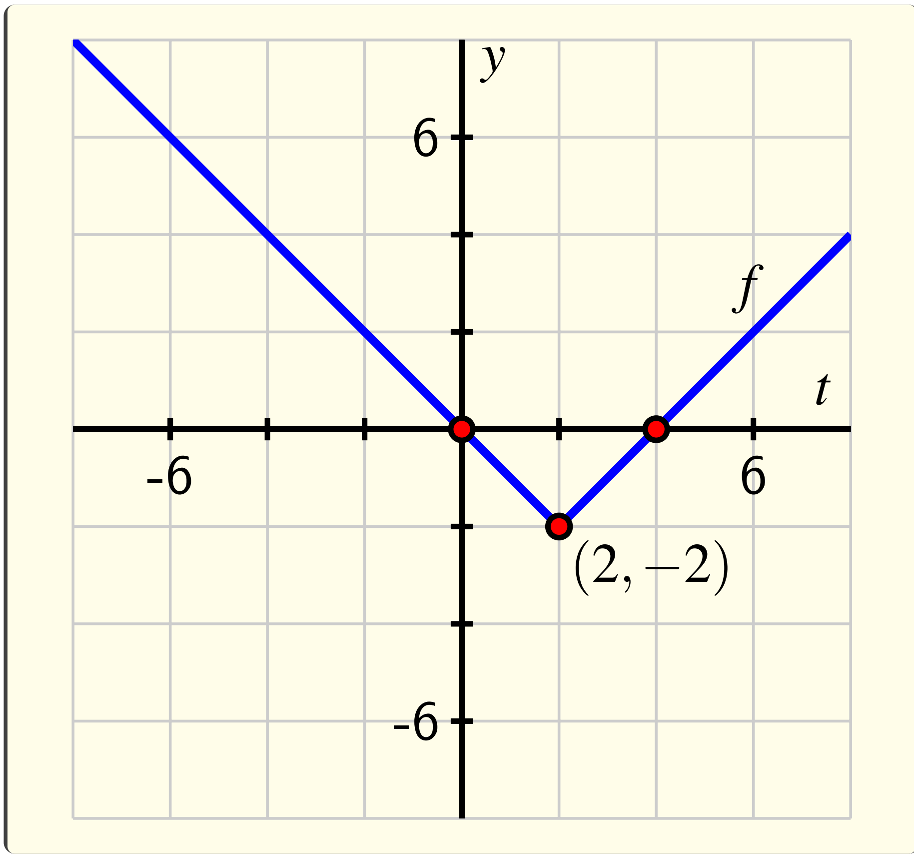
Transformations of sine and cosine with any shift or stretch. Given real numbers a, b, c, and d with $a \neq 0$ and $b > 0$, the functions

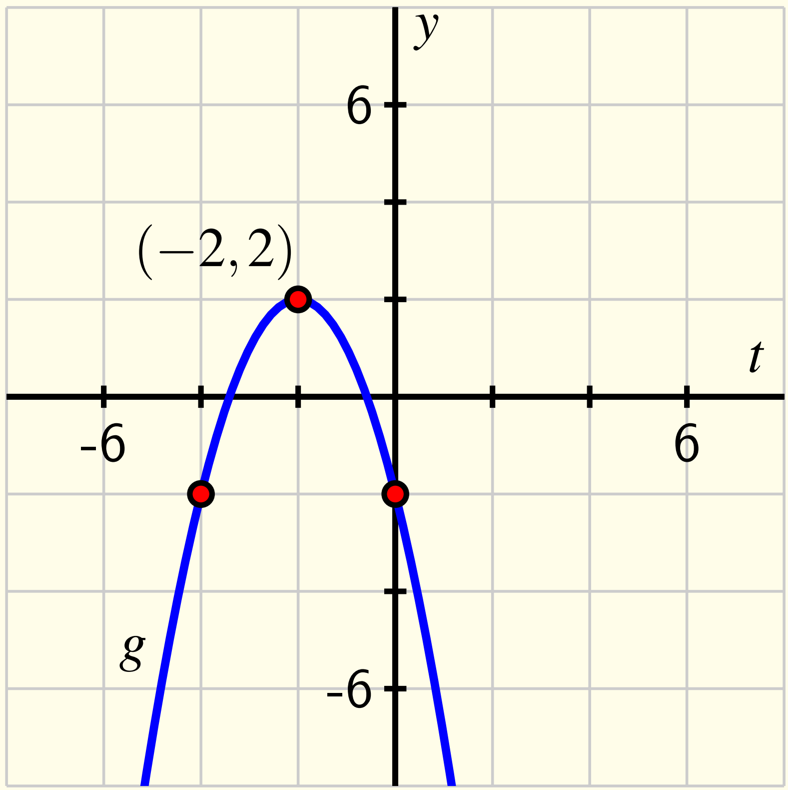
$$f(t) = a\cos(b(t-c)) + d \text{ or } g(t) = a\sin(b(t-c)) + d.$$

each represent a horizontal shift by c units to the right, followed by a vertical stretch by a units, followed by a vertical shift of d units, applied to the parent function ($\cos(t)$ or $\sin(t)$, respectively). They also contain a horizontal scaling of f by a factor of $\frac{1}{b}$. The resulting circular functions have midline $y = d$, amplitude a, range $[d-a, d+a]$, and period $p = 2\pi$. In addition, the point $(c, a+d)$ lies on the graph of f and the point (c, d) lies on the graph of g.

While we will soon focus on horizontal stretches of the sine and cosine functions for the remainder of this section, it's important to note that horizontal scaling follows the same principles for any function we choose.

Exploration Consider the functions f and g given in the following figures.

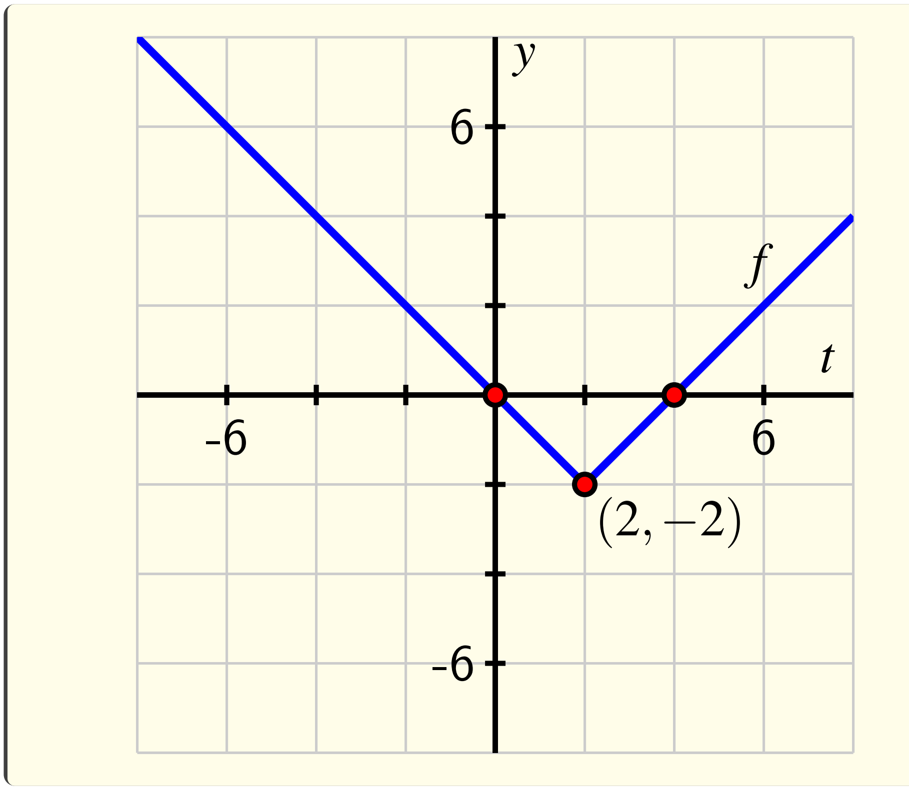

y
6
f
t
-6
6
(2, −2)
-6

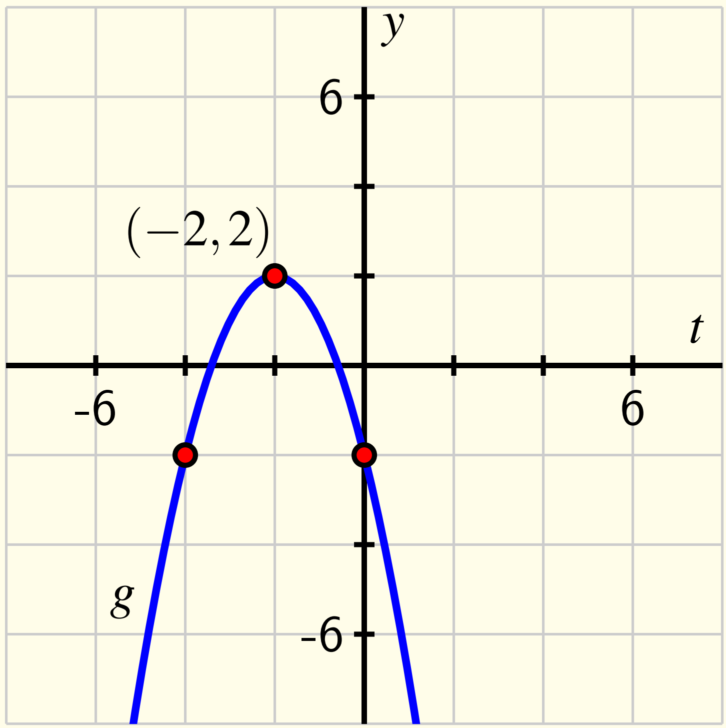


a. On the same axes as the plot of $y = f(t)$, sketch the following graphs: $y = h(t) = f(\frac{1}{3}t)$ and $y = j(t) = r = f(4t)$. Be sure to label several points on each of f, h, and k with arrows to indicate their correspondence. In addition, write one sentence to explain the overall transformations that have resulted in h and j from f.

b. On the same axes as the plot of $y = g(t)$, sketch the following graphs: $y = k(t) = g(2t)$ and $y = m(t) = g(\frac{1}{2}t)$. Be sure to label several points on each of g, k, and m with arrows to indicate their correspondence. In addition, write one sentence to explain the overall transformations that have resulted in k and m from g.

c. On the additional copies of the two figures below, sketch the graphs of the following transformed functions: $y = r(t) = 2f(\frac{1}{2}t)$ and $y = s(t) = \frac{1}{2}g(2t)$. As above, be sure to label several points on each graph and indicate their correspondence to points on the original parent function.


y
6
-6
f
t
-6
6
(2, −2)



d. Describe in words how the function $y = r(t) = 2f(\frac{1}{2}t)$ is the result of two elementary transformations of $y = f(t)$. Does the order in which these transformations occur matter? Why or why not?

Circular functions with different periods

Because the circumference of the unit circle is 2π, the sine and cosine functions each have period 2π. Of course, as we think about using transformations of the sine and cosine functions to model different phenomena, it is apparent that we will need to generate functions with different periods than 2π. For instance, if a ferris wheel makes one revolution every 5 minutes, we'd want the period of the function that models the height of one car as a function of time to be $P = 5$. Horizontal scaling of functions enables us to generate circular functions with any period we desire.

We begin by considering two basic examples. First, let $f(t) = \sin(t)$ and $g(t) = f(2t) = \sin(2t)$. We know from our most recent work that this transformation results in a horizontal compression of the graph of $\sin(t)$ by a factor of $\frac{1}{2}$ toward

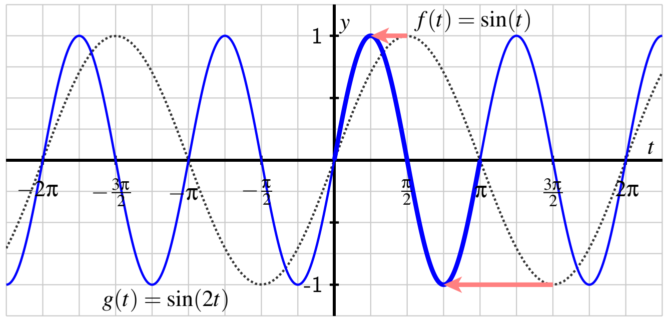


Figure 1: A plot of the parent function, $f(t) = \sin(t)$ (dashed, in gray), and the transformed function $g(t) = f(2t) = \sin(2t)$ (in blue).

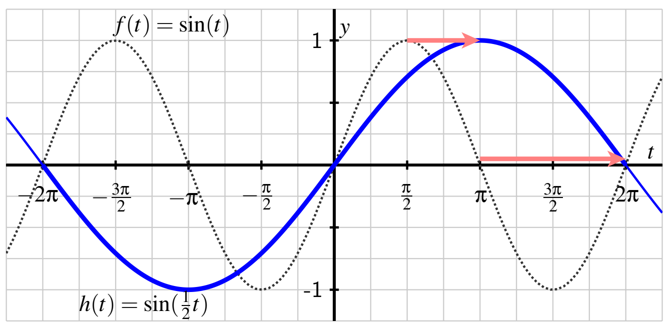


Figure 2: A plot of the parent function, $f(t) = \sin(t)$ (dashed, in gray), and the transformed function $h(t) = f(\frac{1}{2}t) = \sin(\frac{1}{2}t)$ (in blue).

the y-axis. If we plot the two functions on the same axes, it becomes apparent how this transformation affects the period of f.

From the graph, we see that $g(t) = \sin(2t)$ oscillates twice as frequently as $f(t) = \sin(t)$, and that g completes a full cycle on the interval $[0, \pi]$, which is half the length of the period of f. Thus, the "2" in $f(2t)$ causes the period of f to be $\frac{1}{2}$ as long; specifically, the period of g is $P = \frac{1}{2}(2\pi) = \pi$.

On the other hand, if we let $h(t) = f(\frac{1}{2}t) = \sin(\frac{1}{2}t)$, the transformed graph h is stretched away from the y-axis by a factor of 2. This has the effect of doubling the period of f, so that the period of h is $P = 2 \cdot 2\pi = 4\pi$, as seen in the previous figure.

Our observations generalize for any positive constant $b > 0$. In the case where

$b = 2$, we saw that the period of $g(t) = \sin(2t)$ is $P = \frac{1}{2} \cdot 2\pi$, whereas in the case where $b = \frac{1}{2}$, the period of $h(t) = \sin(\frac{1}{2}t)$ is $P = 2 \cdot 2\pi = \frac{1}{\frac{1}{2}} \cdot 2\pi$. Identical reasoning holds if we are instead working with the cosine function. In general, we can say the following.

For any constant $b > 0$, the period of the functions $\sin(bt)$ and $\cos(bt)$ is

$$P = \frac{2\pi}{b}.$$

Thus, if we know the b-value from the given function, we can deduce the period. If instead we know the desired period, we can determine b by the rule $b = \frac{2\pi}{P}$.

Example 52. *Determine the exact period, amplitude, and midline of each of the following functions. In addition, state the range of each function and any horizontal shift that has been introduced to the graph. Make your conclusions without consulting* Desmos*, and then use the program to check your work.*

a. $p(x) = \sin(10x) + 2$

b. $q(x) = -3\cos(0.25x) - 4$

c. $r(x) = 2\sin\left(\frac{\pi}{4}x\right) + 5$

d. $w(x) = 2\cos\left(\frac{\pi}{2}(x-3)\right) + 5$

e. $u(x) = -0.25\sin(3x - 6) + 5$

Explanation

a. $p(x)$ has a period of $\frac{\pi}{5}$, an amplitude of 1 and a midline of $y = 2$.
The range of $p(x)$ is $[1, 3]$.

b. $q(x)$ has a period of 8π, an amplitude of 3 and a midline of $y = -4$.
The range of $q(x)$ is $[-7, -1]$.

c. $r(x)$ has a period of 8, an amplitude of 2 and a midline of $y = 5$.
The range of $r(x)$ is $[3, 7]$.

d. $w(x)$ has a period of 4, an amplitude of 2 and a midline of $y = 5$.
The range of $w(x)$ is $[3, 7]$.
There is a horizontal shift of 3 to the right.

e. $u(x)$ has a period of $\frac{2\pi}{3}$, an amplitude of 0.25 and a midline of $y = 5$.
The range of $u(x)$ is $[4.75, 5.25]$.
There is a horizontal shift of 2 to the right.

Exploration Consider a spring-mass system where the weight is hanging from the ceiling in such a way that the following is known: we let $d(t)$ denote the distance from the ceiling to the weight at time t in seconds and know that the weight oscillates periodically with a minimum value of $d(t) = 1.5$ and a maximum value of $d(t) = 4$, with a period of 3, and you know $d(0.5) = 2.75$ and $d\,(1.25) = 4$.
State the midline, amplitude, range, and an anchor point for the function, and hence determine a formula for $d(t)$ in the form $a\cos(k(t+b)) + c$ or $a\sin(k(t+b)) + c$. Show your work and thinking, and use *Desmos* appropriately to check that your formula generates the desired behavior.

Summary

- Given real numbers a, c, and d with $a \neq 0$, the functions

  $$f(t) = a\cos(b(t-c)) + d \text{ and } g(t) = a\sin(b(t-c)) + d$$

  each represent a horizontal shift by c units to the right, followed by a vertical stretch by a units, followed by a vertical shift of d units, applied to the parent function ($\cos(t)$ or $\sin(t)$, respectively). The resulting circular functions have midline $y = d$, amplitude a, range $[d-a, d+a]$, and period $p = 2\pi$. In addition, the anchor point $(c, a+d)$ lies on the graph of f and the anchor point (c, d) lies on the graph of h.

- Given a function f and a constant $b > 0$, the algebraic transformation $h(t) = f(bt)$ results in horizontal scaling of f by a factor of $\frac{1}{b}$. In particular, when $b > 1$, the graph of f is compressed toward the y-axis by a factor of $\frac{1}{b}$ to create the graph of h, while when $0 < b < 1$, the graph of f is stretched away from the y-axis by a factor of $\frac{1}{b}$ to create the graph of h.

- Given any circular periodic function for which the midline, amplitude, period, and an anchor point are known, we can find a corresponding formula for the function of the form

  $$f(t) = a\cos(b(t-c)) + d \text{ or } g(t) = a\sin(b(t-c)) + d.$$

  Each of these functions has period midline $y = d$, amplitude a, and period $P = \frac{2\pi}{b}$. The point $(c, a+d)$ lies on f and the point (c, d) lies on g.

9.3 Some Applications of Trigonometry

Learning Objectives

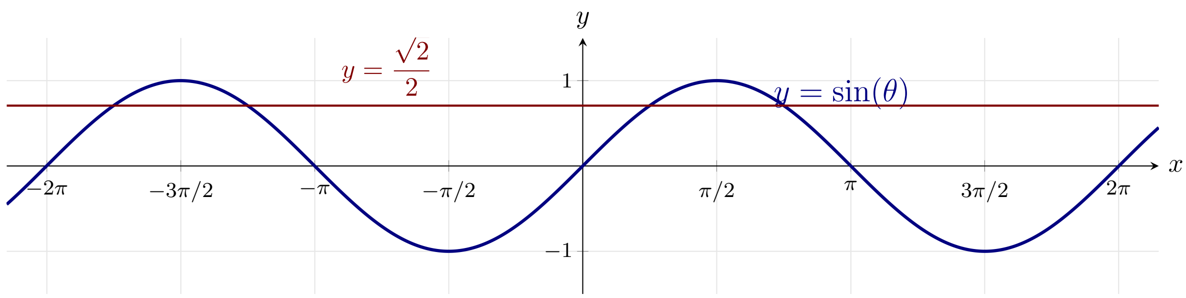
- Solving Trigonometric Equations
  - Solving elementary trigonometric equations.
  - Solving trigonometric equations using identities.
  - FInding solutions on restricted domains.
- Applications of Trigonometric Functions
  - Applications involving trigonometric functions

9.3.1 Solving Trigonometric Equations

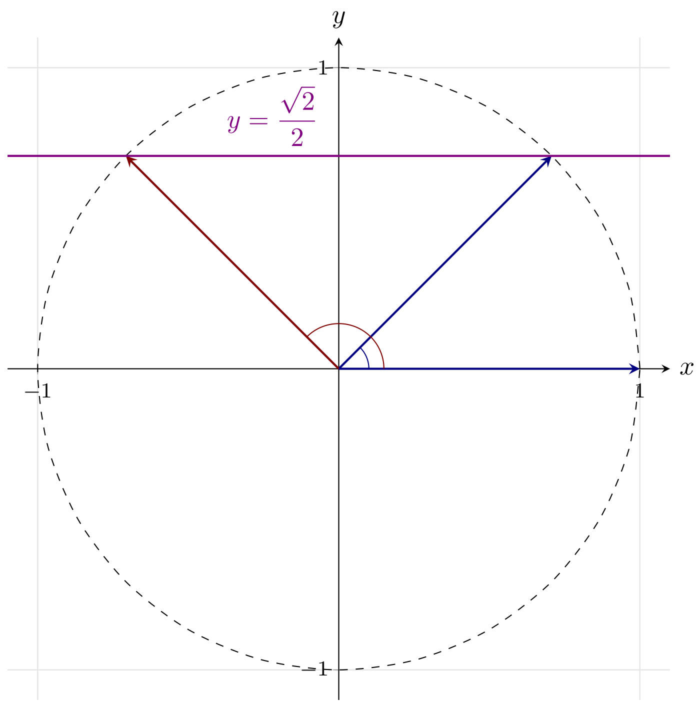
Introduction

Frequently we are in the situation of having to determine precisely which angles satisfy a particular equation. Something like $\sin(\theta) = \frac{\sqrt{2}}{2}$. We know that $\sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$, meaning that $\theta = \frac{\pi}{4}$ is a solution of this equation, but is that the only solution or *are there more*?

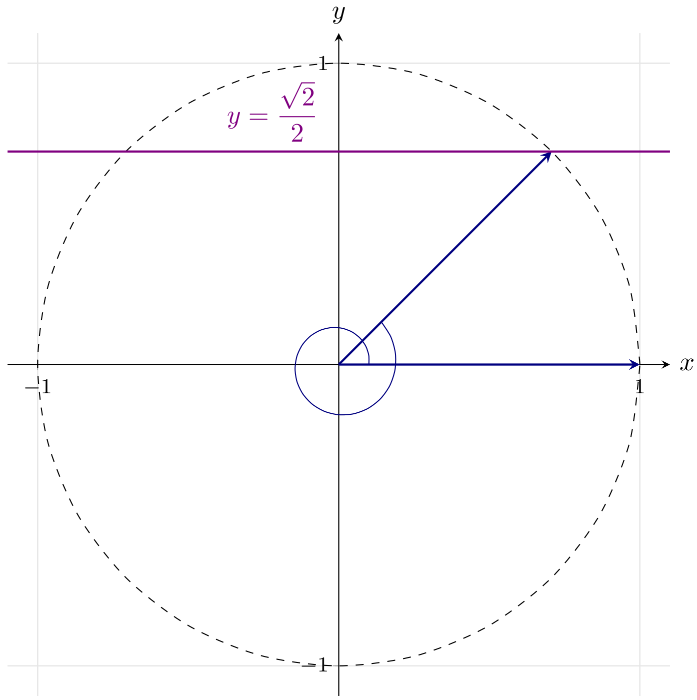
Let's look at the graph of the sine function.



Notice that the graph of $\sin(\theta)$ and the graph of the constant function $y = \frac{\sqrt{2}}{2}$ intersect many times, not just once. In fact, since sine is a periodic function, these graphs intersect infinitely many times. Each of these intersections represents a single solution of the equation $\sin(\theta) = \frac{\sqrt{2}}{2}$. We need a process to identify and write down each of these solutions. Let's start by looking at the unit circle. Remember that sine values correspond to the y-coordinate of points on the unit circle. This equation is asking us to find all the points on the unit circle with a y-coordinate of $\frac{\sqrt{2}}{2}$.



You see that there are two locations on the unit circle with y-coordinate equal to $\frac{\sqrt{2}}{2}$, one in the first quadrant and another in the second. As we mentioned earlier, the first quadrant angle is $\theta = \frac{\pi}{4}$. The angle in the second quadrant has reference angle $\frac{\pi}{4}$, which means the angle is $\frac{3\pi}{4}$. Those are the only two points on the circle with that y-coordinate, but remember that there are many other angles which are coterminal with those. For instance:



The only solutions are the angles $\frac{\pi}{4}$, $\frac{3\pi}{4}$, and *all the angles coterminal with them*. Since the sine function has period 2π, that means any other solution has to be an integer multiple of 2π away from one of these first two solutions. Putting that together, our solutions are:

$$\theta = \frac{\pi}{4} + 2\pi k,\ \frac{3\pi}{4} + 2\pi k,\ k \text{ any integer.}$$

The steps we've followed are summarized in the following.

To solve a triginometric equation:

(a) Find the reference angle of the solutions. Typically the standard values will help identify this.

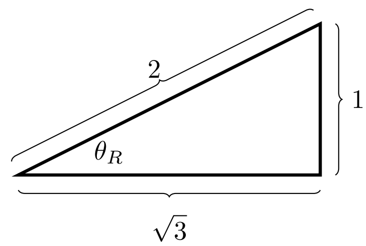
(b) Find all solutions on a single period of the function. Use the graph, the unit circle, and the reference angle to identify these.

(c) Find all solutions. Use the period of the function to find all requested solutions.

Example 53. *Solve the equation:*

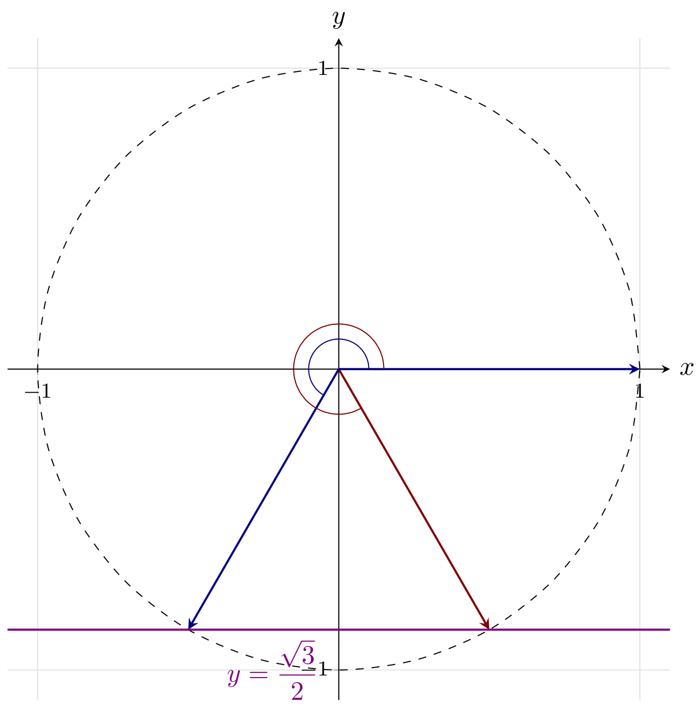
$$\sin(\theta) = -\frac{1}{2}.$$

Explanation We'll start by finding the reference angle, θ_R, the acute angle between the terminal side of θ and the x-axis. The reference angle satisfies $\sin(\theta_R) = \dfrac{1}{2}$ and the negative sign will be used to indicate the quadrant of the angle.



[2in]

From the picture we see $\theta_R = \dfrac{\pi}{6}$. Let's look at the unit circle.



In one period $[0, 2\pi)$, there are two angles that have reference angle $\dfrac{\pi}{6}$ and have negative sine value. One is in quadrant 3, and one in quadrant 4. That means the solutions in the interval $[0, 2\pi)$ are $\dfrac{5\pi}{6}$ and $\dfrac{11\pi}{6}$.

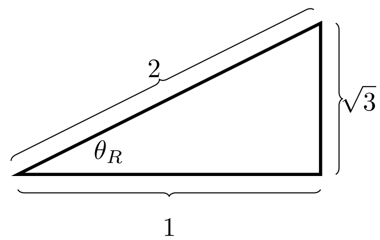
To find all solutions, we have to add all multiples of 2π to these. The solutions are then

$$\theta = \frac{5\pi}{6} + 2\pi k,\ \frac{11\pi}{6} + 2\pi k,\ k \text{ any integer.}$$

Example 54. *Solve the equation:*

$$\tan(\theta) = -\sqrt{3}.$$

Explanation We'll start by finding the reference angle, θ_R, the acute angle between the terminal side of θ and the x-axis. The reference angle satisfies $\tan(\theta_R) = \sqrt{3}$ and the negative sign will be used to indicate the quadrant of the angle. Since tangent is opposite over adjacent, we have the following triangle.



[2in]

From the picture we see $\theta_R = \frac{\pi}{3}$. Let's look at the unit circle.

The tangent function has one period $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. In the interval $\left(-\frac{\pi}{2}, 0\right)$ (which is Quadrant IV), tangent is negative while in $\left(0, \frac{\pi}{2}\right)$ (which is Quadrant I), tangent is positive. For $\tan(\theta)$ to be negative in this interval, we need θ to be in $\left(-\frac{\pi}{2}, 0\right)$. The only angle in that interval with reference angle $\frac{\pi}{3}$ is $\theta = -\frac{\pi}{3}$. This is the only solution on this period.

> Remember that the tangent function has period π, unlike sine and cosine which have period 2π. On the period $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, tangent is one-to-one, so there is exactly one angle which gives the desired output value. Sine and cosine are not one-to-one across a full period.

To find all solutions, we have to add all multiples of π to this. The solutions are then

$$\theta = \frac{2\pi}{3} + \pi k,\ k \text{ any integer.}$$

Let's try one a bit more complicated.

Example 55. *Solve the equation:*

$$\cos(\theta)\left(\cos(\theta) + 1\right) = \sin^2(\theta).$$

Explanation We'll start by simplifying a bit.

$$\begin{aligned}
\cos(\theta)\left(\cos(\theta)+1\right) &= \sin^2(\theta)\\
\cos^2(\theta)+\cos(\theta) &= \sin^2(\theta)\\
\cos^2(\theta)-\sin^2(\theta)+\cos(\theta) &= 0\\
\cos^2(\theta)-\left(1-\cos^2\theta\right)+\cos(\theta) &= 0\\
2\cos^2(\theta)+\cos(\theta)-1 &= 0.
\end{aligned}$$

Notice that this equation is quadratic in $\cos(\theta)$. We can factor it like we try to do to solve any other quadratic equation:

$$\left(\cos(\theta)+1\right)\left(2\cos(\theta)-1\right)=0.$$

On the interval $[0, 2\pi)$, $\cos(\theta) = -1$ has only one solution, $\theta = \pi$. For $\cos(\theta) = \frac{1}{2}$, we see that the reference angle $\theta_R = \frac{\pi}{3}$. Since cosine is positive in Quadrants I and IV, we find solutions $\theta = \frac{\pi}{3}$ and $\frac{5\pi}{3}$.

All solutions are:

$$\theta = \pi + 2\pi k,\ \frac{\pi}{3} + 2\pi k,\ \frac{5\pi}{3} + 2\pi k,\ k \text{ any integer.}$$

9.3.2 Applications of Trigonometry

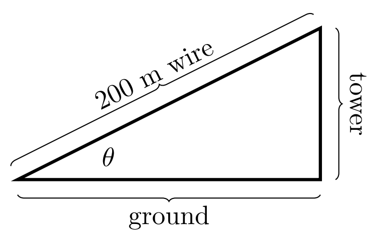
Applications of Trigonometry

In the previous sections, you have been learning about trigonometric functions in the abstract. In this section, we wish to apply them.

Example 56. *A wire 200 meters long is attached to the top of a tower. When pulled taut, it makes a* 60° *angle with the ground. How tall is the tower? How far away from the base of the tower does the wire hit the ground?*

Explanation

In application problems, we are often given data about angles measured in degrees. It is up to us to translate this into radians for our calculations. Let's call the angle we're given as θ, so $\theta = \dfrac{\pi}{3}$ radians. Let's draw a diagram of this situation.



[2in]

Based on this image, the height of the tower is the opposite the angle we know, and the distance along the ground is adjacent. That means the tower height is related to the angle by $\sin(\theta) = \dfrac{\text{tower}}{200}$ and the distance across the ground is given by $\cos(\theta) = \dfrac{\text{ground}}{200}$.

Since $\sin\left(\dfrac{\pi}{3}\right) = \dfrac{\sqrt{3}}{2}$, the tower height can be computed by:

$$\begin{aligned}
\sin\left(\frac{\pi}{3}\right) &= \frac{\text{tower}}{200} \\
\text{tower} &= 200\sin\left(\frac{\pi}{3}\right) \\
&= 200\left(\frac{\sqrt{3}}{2}\right) \\
&= 100\sqrt{3}.
\end{aligned}$$

The tower has a height of $100\sqrt{3}$ m, which is approximately 173.21 m.

Acknowledgements: Stitz Zeager Open Source Mathematics (https://www.stitz-zeager.com/)

The EXACT VALUE of the height is $100\sqrt{3}$ m. Saying that this is approximately 173.21 m is provided to give us an indication of scale. Our actual answer is the exact value, not this approximation.

We'll follow a similar calculation to find the distance from the base of the tower to the wire along the ground.

$$\begin{aligned}
\cos\left(\frac{\pi}{3}\right) &= \frac{\text{ground}}{200} \\
\text{ground} &= 200\cos\left(\frac{\pi}{3}\right) \\
&= 200\left(\frac{1}{2}\right) \\
&= 100.
\end{aligned}$$

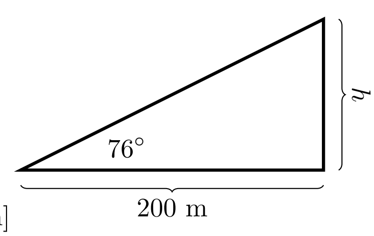
The wire hits the ground 100 m from the base of the tower.

Example 57. *A camera is setup 200 m from the base of a building, pointed at the top of the building. If the angle-of-elevation is measured as* 76°*, find the height of the building.*

Explanation

The angle-of-elevation means the angle between the camera's line-of-sight and horizontal. Since the camera is setup 200 meters from the building, this gives us a right triangle where we know the base angle and the length of the adjacent side.

If we call the height of the building as h, then we have a triangle



[2in]

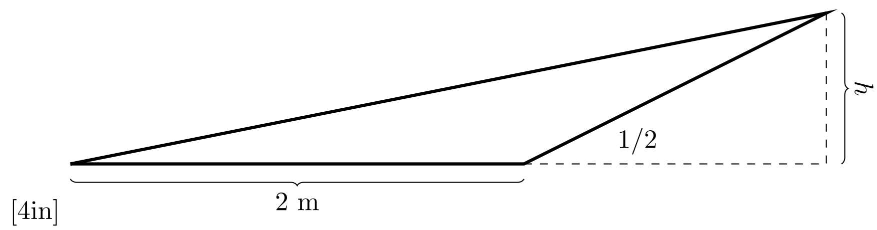
That means $\tan(76^\circ) = \dfrac{h}{200}$, so $h = 200\tan(76^\circ)$. The exact height is $200\tan(76^\circ)$ meters. This is approximately 802.16 meters.

Example 58. *A 4 meter long piece of wire is going to be bent at its midpoint. The right side of the wire is bent up through an angle of* $\dfrac{1}{2}$*. The two ends of the wire are joined by a piece of string, creating an obtuse triangle. What is the area of the resulting triangle?*

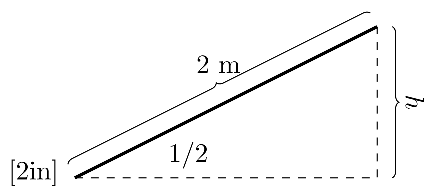
Explanation

 Acknowledgements: Stitz Zeager Open Source Mathematics (https://www.stitz-zeager.com/)

That the wire is bent at its midpoint, means the resulting triangle will have two sides of length 2 m. Call the height of the triangle h.



Let us focus on the dotted triangle on the right side



Notice that the height of this right triangle is the same as the height of the obtuse triangle above. Since we know the hypotenuse and base angle of this right triangle, we can find the height (the opposite side) using sine.

$$\begin{aligned} \sin\left(\frac{1}{2}\right) &= \frac{h}{2} \\ 2\sin\left(\frac{1}{2}\right) &= h. \end{aligned}$$

Be careful here! That angle $\frac{1}{2}$ is not in degrees, it's in radians. (You can tell, because there is no "degrees symbol".) If you are going to approximate this value, make sure you are using radians.

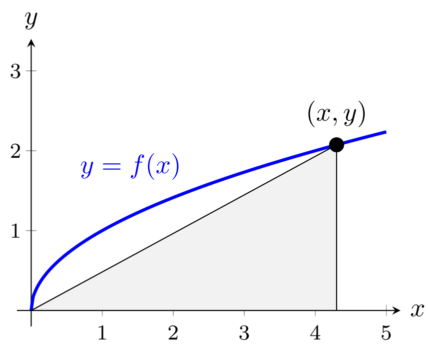
Now the area of the whole triangle is:

$$\begin{aligned} A &= \frac{1}{2}bh \\ &= \frac{1}{2}(2)\left(2\sin\left(\frac{1}{2}\right)\right) \\ &= 2\sin\left(\frac{1}{2}\right). \end{aligned}$$

The exact value of the area is $2\sin\left(\frac{1}{2}\right)\, m^2$. Using a calculator, this is approximately $0.959\, m^2$. (Remember that m^2 is the abbreviation for "square meters".)

 Acknowledgements: Stitz Zeager Open Source Mathematics (https://www.stitz-zeager.com/)

Example 59. *A right triangle is constructed by taking a point (x, y) on the graph of the function $f(x) = \sqrt{x}$, drawing a line vertically downward to the x-axis, then connecting both of those points to the origin as in the picture below.*



For one particular point (x, y), the acute angle between the hypotenuse of the triangle and the positive x-axis is found to measure $\frac{\pi}{6}$ radians. Find the coordinates of the point (x, y).

Explanation

Since the hypotenuse of the right triangle runs from the origin, which has coordinates $(0, 0)$, to the point (x, y), the horizontal side of the triangle has length x and vertical side has length y. We know that the value of tangent is given by the ratio of the opposite side length divided by the adjacent side length. Using the given angle of $\frac{\pi}{6}$, this means $\tan\left(\frac{\pi}{6}\right) = \frac{1}{\sqrt{3}} = \frac{y}{x}$. That is, $x = y\sqrt{3}$

We also know that the point lies on the graph of $f(x) = \sqrt{x}$, which means $y = \sqrt{x}$.

This gives us a nonlinear system of two equations:

$$\begin{cases} x = y\sqrt{3} \\ y = \sqrt{x} \end{cases}$$

Squaring the bottom equation yields $y^2 = x$. When substituting the top equation into the bottom equation, we arrive at:.

$$\begin{aligned} y^2 &= y\sqrt{3} \\ y^2 - y\sqrt{3} &= 0 \\ y(y - \sqrt{3}) &= 0 \end{aligned}$$

so either $y = 0$ or $y = \sqrt{3}$.

The value $y = 0$ corresponds to the point $(0, 0)$ on the graph, which does not yield any angle. This is an extraneous solution, which is discarded.

Acknowledgements: Stitz Zeager Open Source Mathematics (https://www.stitz-zeager.com/)

Substituting the value $y = \sqrt{3}$ into $x = y\sqrt{3}$ gives $x = (\sqrt{3})\sqrt{3} = 3$. The point is $\left(3, \sqrt{3}\right)$.

Part 10

Inverse Functions In Depth

10.1 Review of Inverse Functions

Learning Objectives

- –
  - –
  - –
- –

10.1.1 Review of Inverse Functions

In Section 3-2-2, we briefly introduced the concept of *inverse functions*. Recall that for a one-to-one function f, we can define the inverse function f^{-1}. If we think of f as a process that takes some input x and produces some output $f(x)$, then providing $f(x)$ as an input to f^{-1} produces the original input x, and vice versa. Symbolically, we wrote that $f^{-1}(f(x)) = x$ and $f(f^{-1}(x)) = x$.

We learned several important principles, which we summarize below.

- A function f has an inverse function if and only if there exists a function g that undoes the work of f: that is, there is some function g for which $g(f(x)) = x$ for each x in the domain of f, and $f(g(y)) = y$ for each y in the range of f. We call g the inverse of f, and write $g = f^{-1}$.
- A function f has an inverse function if and only if the graph of f passes the *Horizontal Line Test*.
- A function f has an inverse function if and only if f is a *one-to-one* function.
- When f has an inverse, we know that writing "$y = f(t)$" and "$t = f^{-1}(y)$" are two different perspectives on the same statement.
- If $(a, f(a))$ is a point on the graph of f, then $(f(a), a)$ is a point on the graph of f^{-1}.
- The graph of f^{-1} is the graph of f reflected across the line $y = x$.
- The domain of f is the range of f^{-1} and the range of f is the domain of f^{-1}.
- If f^{-1} is the inverse of f, then f is the inverse of f^{-1}.

In this section, we'll explore inverse functions more in-depth.

10.2 Logarithms

Learning Objectives

- –
  - –
  - –
- –

10.2.1 Definition of Logarithms

Motivating Questions

- How is the base-10 logarithm defined?
- What is the "natural logarithm" and how is it different from the base-10 logarithm?
- How can we solve an equation that involves e to some unknown quantity?

In previous sections, we introduced the idea of an inverse function. The fundamental idea is that f has an inverse function if and only if there exists another function g such that f and g "undo" one another's respective processes. In other words, the process of the function f is reversible to generate a related function g.

More formally, recall that a function $y = f(x)$ (where $f : A \to B$) has an inverse function if and only if there exists another function $g : B \to A$ such that $g(f(x)) = x$ for every x in the domain of f and $f(g(y)) = y$. We know that given a function f, we can use the Horizontal Line Test to determine whether or not f has an inverse function. Finally, whenever a function f has an inverse function, we call its inverse function f^{-1} and know that the two equations $y = f(x)$ and $x = f^{-1}(y)$ say the same thing from different perspectives.

Exploration
Let $P(t)$ be the "powers of 10" function, which is given by $P(t) = 10^t$.

a. Completethe following table to generate certain values of P.

t	-3	-2	-1	0	1	2	3
$y = P(t) = 10^t$							

b. Why does P have an inverse function?

c. Since P has an inverse function, we know there exists some other function, say L, such that writing "$y = P(t)$" says the exact same thing as writing "$t = L(y)$". In words, where P produces the result of raising 10 to a given power, the function L reverses this process and instead tells us the power to which we need to raise 10, given a desired result. Complete the table to generate a collection of values of L.

y	10^{-3}	10^{-2}	10^{-1}	10^0	10^1	10^2	10^3
$L(y)$							

d. What are the domain and range of the function P? What are the domain and range of the function L?

The base-10 logarithm

The powers-of-10 function $P(t) = 10^t$ is an exponential function with base $b > 1$. As such, P is always increasing, and thus its graph passes the Horizontal Line Test, so P has an inverse function. We therefore know there exists some other function, L, such that writing $y = P(t)$ is equivalent to writing $t = L(y)$. For instance, we know that $P(2) = 100$ and $P(-3) = \dfrac{1}{1000}$, so it's equivalent to say that $L(100) = 2$ and $L(\dfrac{1}{1000}) = -3$. This new function L we call the *base* 10 *logarithm*, which is formally defined as follows.

Given a positive real number y, the *base*-10 *logarithm of* y is the power to which we raise 10 to get y. We use the notation "$\log_{10}(y)$" to denote the base-10 logarithm of y.

The base-10 logarithm is therefore the inverse of the powers of 10 function. Whereas $P(t) = 10^t$ takes an input whose value is an exponent and produces the result of taking 10 to that power, the base-10 logarithm takes an input number we view as a power of 10 and produces the corresponding exponent such that 10 to that exponent is the input number.

In the notation of logarithms, we can now update our earlier observations with the functions P and L and see how exponential equations can be written in two equivalent ways. For instance,

$$10^2 = 100 \text{ and } \log_{10}(100) = 2 \tag{1}$$

each say the same thing from two different perspectives. The first says 100 is 10 to the power 2 , while the second says 2 is the power to which we raise 10 to get 100. Similarly,

$$10^{-3} = \frac{1}{1000} \text{ and } \log_{10}\left(\frac{1}{1000}\right) = -3. \tag{2}$$

If we rearrange the statements of the facts, we can see yet another important relationship between the powers of 10 and base-10 logarithm function. Noting that $\log_{10}(100) = 2$ and $100 = 10^2$ are equivalent statements, and substituting the former equation into the latter shows, we see that

$$\log_{10}(10^2) = 2. \tag{3}$$

In words, the equation says that "the power to which we raise 10 to get 10^2, is 2". That is, the base-10 logarithm function undoes the work of the powers of 10 function.

In a similar way, we can observe that by replacing -3 with $\log_{10}(\dfrac{1}{1000})$ we have

$$10^{\log_{10}(\frac{1}{1000})} = \frac{1}{1000}. \tag{4}$$

In words, this says that "when 10 is raised to the power to which we raise 10 in order to get $\frac{1}{1000}$, we get $\frac{1}{1000}$".

We summarize the key relationships between the powers-of-10 function and its inverse, the base-10 logarithm function, more generally as follows.$P(t) = 10^t$ and $L(y) = \log_{10}(y)$.

- The domain of P is the set of all real numbers and the range of P is the set of all positive real numbers.
- The domain of L is the set of all positive real numbers and the range of L is the set of all real numbers.
- For any real number t, $\log_{10}(10^t) = t$. That is, $L(P(t)) = t$.
- For any positive real number y, $10^{\log_{10}(y)} = y$. That is, $P(L(y)) = y$.
- $10^0 = 1$ and $\log_{10}(1) = 0$.

The base-10 logarithm function is like the sine or cosine function in this way: for certain special values, it's easy to know by heart the value of the logarithm function. While for sine and cosine the familiar points come from specially placed points on the unit circle, for the base-10 logarithm function, the familiar points come from powers of 10. In addition, like sine and cosine, for all other input values, (a) calculus ultimately determines the value of the base-10 logarithm function at other values, and (b) we use computational technology in order to compute these values. For most computational devices, the command $\log(y)$ produces the result of the base-10 logarithm of y.

It's important to note that the logarithm function produces exact values. For instance, if we want to solve the equation $10^t = 5$, then it follows that $t = \log_{10}(5)$ is the exact solution to the equation. Like $\sqrt{2}$ or $\cos(1)$, $\log_{10}(5)$ is a number that is an exact value. A computational device can give us a decimal approximation, and we normally want to distinguish between the exact value and the approximate one. For the three different numbers here, $\sqrt{2} \approx 1.414$, $\cos(1) \approx 0.540$, and $\log_{10}(5) \approx 0.699$.

Exploration

For each of the following equations, determine the exact value of the unknown variable. If the exact value involves a logarithm, use a computational device to also report an approximate value. For instance, if the exact value is $y = \log_{10}(2)$, you can also note that $y \approx 0.301$.

a. $10^t = 0.00001$

b. $\log_{10}(1000000) = t$

c. $10^t = 37$

d. $\log_{10}(y) = 1.375$

e. $10^t = 0.04$

f. $3 \cdot 10^t + 11 = 147$

g. $2\log_{10}(y) + 5 = 1$

The natural logarithm

The base-10 logarithm is a good starting point for understanding how logarithmic functions work because powers of 10 are easy to mentally compute. We could similarly consider the powers of 2 or powers of 3 function and develop a corresponding logarithm of base 2 or 3. But rather than have a whole collection of different logarithm functions, in the same way that we now use the function e^t and appropriate scaling to represent any exponential function, we develop a single logarithm function that we can use to represent any other logarithmic function through scaling. In correspondence with the natural exponential function, e^t, we now develop its inverse function, and call this inverse function the *natural logarithm*.

Given a positive real number y, the *natural logarithm of* y is the power to which we raise e to get y. We use the notation "$\ln(y)$" to denote the natural logarithm of y.

We can think of the natural logarithm, $\ln(y)$, as the "base-e logarithm". For instance,

$$\ln(e^2) = 2$$

and

$$e^{\ln(-1)} = -1.$$

The former equation is true because "the power to which we raise e to get e^2 is 2"; the latter equation is true since "when we raise e to the power to which we raise e to get -1, we get -1".

Exploration Let $E(t) = e^t$ and $N(y) = \ln(y)$ be the natural exponential function and the natural logarithm function, respectively.

a. What are the domain and range of E?

b. What are the domain and range of N?

c. What can you say about $\ln(e^t)$ for every real number t?

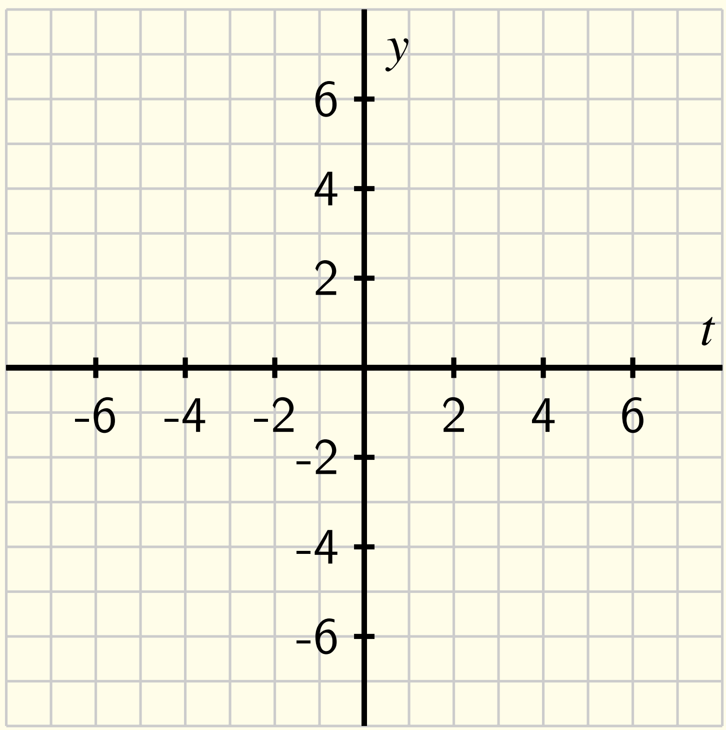
d. What can you say about $e^{\ln(y)}$ for every positive real number y?

e. Complete the following tables with both exact and approximate of E and N. Then, plot the corresponding ordered pairs from each table on the axes below and connect the points in an intuitive way.

When you plot the ordered pairs on the axes, in both cases view the first line of the table as generating values on the horizontal axis and the second line of the table as producing values on the vertical axis label each ordered pair you plot appropriately.

t	-2	-1	0	1	2
$E(t) = e^t$	$e^{-2} \approx 0.135$				

y	e^{-2}	e^{-1}	1	e^1	e^2
$N(y) = \ln(y)$	-2				



$\log_b$ or logarithms in general

In the previous sections, we looked at two specific (and the most common) types of logarithms, base-10 and natural log. In order to fully discuss logarithms, we need to talk about logarithms in general with any base. Let $b > 1$. Because the function $y = f(t) = b^t$ has an inverse function, it makes sense to define its inverse like we did when $b = 10$ or $b = e$. The base-b logarithm, denoted $\log_b(y)$ is defined to be the power to which we raise b to get y.

$$\log_b(y) = t$$
$$y = f(t) = b^t$$

Example 60. *Evaluate the following base-b logarithms.*

(a) $\log_2(8)$

(b) $\log_5(25)$

Explanation

(a)

$$\log_2(8) = \log_2(2^3)\log_2(8) \qquad = 3$$

(b)

$$\log_5(25) = \log_5(5^2)\log_5(25) \qquad = 2$$

Revisiting $f(t) = b^t$

In earlier sections, we saw that that function $f(t) = b^t$ plays a key role in modeling exponential growth and decay, and that the value of b not only determines whether the function models growth ($b > 1$) or decay ($0 < b < 1$), but also how fast the growth or decay occurs. Furthermore, once we introduced the natural base e, we realized that we could write every exponential function of form $f(t) = b^t$ as a horizontal scaling of the function $E(t) = e^t$ by writing

$$b^t = f(t) = E(kt) = e^{kt}$$

for some value k. Our development of the natural logarithm function in the current section enables us to now determine k exactly.

Example 61. *Determine the exact value of k for which $f(t) = 3^t = e^{kt}$.*

Explanation Since we want $3^t = e^{kt}$ to hold for every value of t and $e^{kt} = (e^k)^t$, we need to have $3^t = (e^k)^t$, and thus $3 = e^k$. Therefore, k is the power to which we raise e to get 3, which by definition means that $k = \ln(3)$.

In modeling important phenomena using exponential functions, we will frequently encounter equations where the variable is in the exponent, like in the example where we had to solve $e^k = 3$. It is in this context where logarithms find one of their most powerful applications.

Example 62. *Solve each of the following equations for the exact value of the unknown variable. If there is no solution to the equation, explain why not.*

a. $e^t = \dfrac{1}{10}$

b. $5e^t = 7$

c. $\ln(t) = -\dfrac{1}{3}$

d. $e^{1-3t} = 4$

e. $2\ln(t) + 1 = 4$

f. $4 - 3e^{2t} = 2$

g. $4 + 3e^{2t} = 2$

h. $\ln(5 - 6t) = -2$

Explanation

a.

$$\begin{aligned} e^t &= \frac{1}{10} \\ \ln(e^t) &= \ln\left(\frac{1}{10}\right) \\ t &= \ln\left(\frac{1}{10}\right) \end{aligned}$$

b.

$$\begin{aligned} 5e^t &= 7 \\ e^t &= \frac{7}{5} \\ \ln\left(e^t\right) &= \ln\left(\frac{7}{5}\right) \\ t &= \ln\left(\frac{7}{5}\right) \end{aligned}$$

c.

$$\begin{aligned} \ln(t) &= -\frac{1}{3} \\ e^{\ln(t)} &= e^{-\frac{1}{3}} \\ t &= e^{-\frac{1}{3}} \end{aligned}$$

d.

$$\begin{aligned}
e^{1-3t} &= 4 \\
\ln(e^{1-3t}) &= \ln(4) \\
1-3t &= \ln(4) \\
-3t &= \ln(4)-1 \\
t &= \frac{\ln(4)-1}{-3}
\end{aligned}$$

e.

$$\begin{aligned}
2\ln(t)+1 &= 4 \\
2\ln(t) &= 3 \\
\ln(t) &= \frac{3}{2} \\
e^{ln(t)} &= e^{\frac{3}{2}} \\
t &= e^{\frac{3}{2}}
\end{aligned}$$

f.

$$\begin{aligned}
4-3e^{2t} &= 2 \\
-3e^{2t} &= -2 \\
e^{2t} &= \frac{2}{3} \\
\ln(e^{2t}) &= \ln\left(\frac{2}{3}\right) \\
2t &= \ln\left(\frac{2}{3}\right) \\
t &= \frac{\ln\left(\frac{2}{3}\right)}{2}
\end{aligned}$$

g.

$$\begin{aligned}
4+3e^{2t} &= 2 \\
3e^{2t} &= -2 \\
e^{2t} &= \frac{-2}{3}
\end{aligned}$$

No solution, because $\frac{-2}{3}$ is outside of the range of e^{2t}

h.

$$
\begin{aligned}
\ln(5-6t) &= -2 \\
\ln(5-6t) &= -2 \\
e^{\ln(5-6t)} &= e^{-2} \\
5-6t &= e^{-2} \\
t &= \frac{e^{-2}-5}{-6}
\end{aligned}
$$

Summary

(a) The base-10 logarithm of y, denoted $\log_{10}(y)$ is defined to be the power to which we raise 10 to get y. For instance, $\log_{10}(1000) = 3$, since $10^3 = 1000$. The function $L(y) = \log_{10}(y)$ is thus the inverse of the powers-of-10 function, $P(t) = 10^t$.

(b) The natural logarithm $N(y) = \ln(y)$ differs from the base-10 logarithm in that it is the logarithm with base e instead of 10, and thus $\ln(y)$ is the power to which we raise e to get y. The function $N(y) = \ln(y)$ is the inverse of the natural exponential function $E(t) = e^t$.

(c) The natural logarithm often enables us solve an equation that involves e to some unknown quantity. For instance, to solve $2e^{3t-4} + 5 = 13$, we can first solve for e^{3t-4} by subtracting 5 from each side and dividing by 2 to get

$$e^{3t-4} = 4.$$

This last equation says "e to some power is 4". We know that it is equivalent to say

$$\ln(4) = 3t - 4.$$

Since $\ln(4)$ is a number, we can solve this most recent linear equation for t. In particular, $3t = 4 + \ln(4)$, so

$$t = \frac{1}{3}(4 + \ln(4)).$$

10.2.2 Properties of Logarithms

The key to understanding logarithms is through their relationship with exponential functions. Since $f(x) = \log_b(x)$ is the inverse function to $g(x) = b^x$, many of the properties of exponential functions can be translated into properties of logarithms. In this section, we'll try to discover these and find several other interesting properties of logarithms along with way.

We highlight several important principles from our previous discussion of inverse functions:

- A function f has an inverse function if and only if there exists a function g that undoes the work of f: that is, there is some function g for which $g(f(x)) = x$ for each x in the domain of f, and $f(g(y)) = y$ for each y in the range of f. We call g the inverse of f, and write $g = f^{-1}$.
- When f has an inverse, we know that writing "$y = f(t)$" and "$t = f^{-1}(y)$" are two different perspectives on the same statement.

Inverse Property of Logarithms

An important fact to recall is that the range of the function $g(x) = b^x$ is $(0, \infty)$, the set of all positive real numbers. This means that any positive real number can be written as the output of the exponential function with base b. Let's fix $b = 10$ and try to write the number 17 as an output of the function $g(x) = 10^x$. If 17 is an output of g, then $17 = 10^x$ for some real number x. Taking log of both sides of this equation, we find that $\log(17) = \log(10^x)$.

Now we use the most important property of logarithms: the logarithms and exponential of the same base are inverses. With our base being set to 10, this tells us that $\log(10^x) = x$. It is important to remember that even though our notation for the exponential function writes its input as an exponent, and not by wrapping it in parenthesis, x is the input to the exponential function in 10^x.

Returning to our original quest to write 17 as an output of the exponential with base 10, we use the inverse property of logarithms to say that $\log(17) = x$, and therefore,

$$17 = 10^{\log(17)}.$$

Another way to see this is by using the fact that the function $g(x) = 10^x$ is the inverse of $f(x) = \log(x)$.

There was nothing special about 10 and 17 in what we just showed, so this allows us to arrive at a very general way to write positive real numbers as exponentials.

> If x and b are positive real numbers, we can write $x = b^{\log_b(x)}$.

Another way to understand this is to remember the definition of the logarithm. $\log_b(x)$ is precisely the power to which you have to raise b in order to obtain x.

Finally, this can also be viewed as a statement about inverse functions. If $f(x) = \log_b(x)$, then $f^{-1}(x) = b^x$. In this setup, the statement $f^{-1}(f(x)) = x$ becomes $b^{\log_b(x)} = x$.

Product Property of Logarithms

You might think that the method in the previous section of writing positive real numbers as exponentials unnecessarily complicates things, but we can use it to adapt properties of exponents into properties of logarithms.

Recall that multiplying exponential expressions of the same base results in another exponential expression: in symbols,

$$b^u \cdot b^v = b^{u+v}$$

for any real numbers u and v.

Let's see if we can use this fact, again restricting our attention to $b = 10$. Since 2 and 3 are positive real numbers, we can write $2 = 10^{\log(2)}$ and $3 = 10^{\log(3)}$. Then,

$$\log(6) = \log(2\cdot 3) = \log(10^{\log(2)} \cdot 10^{\log(3)}) = \log(10^{\log(2)+\log(3)}) = \log(2)+\log(3).$$

Notice again how we used the fact that the logarithm and exponential with base 10 are inverses! There's nothing special about 2 and 3, so for any positive real numbers x and y, $\log(xy) = \log(x) + \log(y)$. Even more, there's nothing special about base 10, allowing us to come up with a general rule.

If x, y, and b are positive real numbers, then $\log_b(xy) = \log_b(x)+\log_b(y)$.

Quotient Property of Logarithms

Now that we've dealt with multiplication, it makes sense to deal with division. If x and y are positive real numbers, we can think about the quotient x/y as a product: $x \cdot (1/y)$. What's more, we can write $1/y$ as a power of y: $1/y = y^{-1}$. Using the product property of logarithms from the previous section, we can conclude that $\log_b(x/y) = \log_b(x) + \log_b(y^{-1})$.

It would be really nice if there was a nice relationship between $\log_b(y^{-1})$ and $\log_b(y)$. Indeed, there is! Using the definition of the logarithm, $\log_b(y)$ is the power to which you have to raise b to obtain y, but to obtain y^{-1}, we can use the negative power. As an example, note that $\log(1000) = \log(10^3) = 3$, but

$\log\left(\frac{1}{1000}\right) = \log(10^{-3}) = -3$. In general,

$$\log_b(y^{-1}) = -\log_b(y).$$

Combining this with our previous work, we obtain the following quotient property of logarithms.

> If x, y, and b are positive real numbers, then $\log_b\left(\frac{x}{y}\right) = \log_b(x) - \log_b(y)$.

Power Property of Logarithms

Something else you might remember about exponents is that repeated exponentiation is the same thing as multiplying exponents. For example, $(7^3)^2 = 7^{(3\cdot 2)} = 7^6$ (check this yourself!). In words, this says that raising 7 to the 3rd power, then raising that result to the 2nd power is the same as raising 7 to the $3 \cdot 2 = 6$th power. Since $7^3 = 343$, $\log_7(343) = 3$. So in the language of logarithms, the above says that $\log_7(343^2) = 2 \cdot \log_7(343)$.

In general,

$$(b^u)^v = b^{u\cdot v}$$

for all real numbers b, u, and v.

Let's see if this fact has any consequences for logarithms! Recall that for positive b and x, $\log_b(x^u)$ is the power to which we need to raise b in order to obtain x^u. However, another way to obtain x^u is to raise b to the power $\log_b(x)$ (yielding x) and then raise that result to the power u. Since repeated exponentiation is the same thing as multiplying exponents, this amounts to raising b to the power $u \log_b(x)$. In symbols, we've shown that

> If x and b are positive real numbers, and u is a real number, then $\log_b(x^u) = u \log_b(x)$.

In essence, taking the logarithm of a power of x is the same thing as multiplying the logarithm of x by the power. An intuitive way to think about this property is in the context of the product property from above. Since logarithms "turn multiplication into addition" and exponentiation is repeated multiplication, logarithms should "turn exponentiation into repeated addition", that is, multiplication. As an example, notice that

$$\begin{aligned}
\log_2(3^4) &= \log_2(3^2 \cdot 3^2) \\
&= \log_2(3^2) + \log_2(3^2) \\
&= \log_2(3 \cdot 3) + \log_2(3 \cdot 3) \\
&= \log_2(3) + \log_2(3) + \log_2(3) + \log_2(3) \\
&= 4\log_2(3).
\end{aligned}$$

The above calculation uses the product property to arrive at the same conclusion as the power property.

Change-of-Base Formula

One important thing to recognize is that logarithms can have any positive number as their base. Sometimes, when doing calculations, it may be preferable to use one base over another. The good news is that any logarithm can be computed using this preferred base.

As an example, consider the quantity $\log_3(7)$. Many calculators are unable to directly calculate logarithms with a base other than e or 10, so let's convert this into a natural logarithm (logarithm with base e). Rewriting 7 as $3^{\log_3(7)}$ using the inverse property of logarithms, we see that $\ln(7) = \ln(3^{\log_3(7)})$. Now, using the power property of logarithms, we see that $\ln(3^{\log_3(7)}) = \log_3(7) \cdot \ln(3)$. This gives us the equality $\ln(7) = \log_3(7) \cdot \ln(3)$, so dividing both sides by $\ln(3)$, $\log_3(7) = \dfrac{\ln(7)}{\ln(3)}$. If you have an aversion to $\log_3$ and a fondness for ln, then this allows you to calculate $\ln(7)/\ln(3)$ instead of $\log_3(7)$.

Of course, there's nothing special about 3, 7, and the natural logarithm. In general, we have the following formula.

> If a, b, and x are positive real numbers, then $\log_b(x) = \dfrac{\log_a(x)}{\log_a(b)}$.

Logarithm Properties in Action

Example 63. *Say* $\log_b(3)$ *is approximately* 0.388 *and* $\log_b(2)$ *is approximately* 0.245*. Using the properties of logarithms, approximate* $\log_b(108)$*.*

Explanation To use the properties of logarithms, we can make use of the factorization of 108: $108 = 4 \cdot 27 = 2^2 \cdot 3^3$. Using the product property of logarithms, $\log_b(108) = \log_b(2^2 \cdot 3^3) = \log_b(2^2) + \log_b(3^3)$. Now we can apply the product property of logarithms to simplify each term. We conclude that $\log_b(2^2) + \log_b(3^3) = 2\log_b(2) + 3\log_b(3) = 2(0.388) + 3(0.245) = 1.511$.

Therefore, $\log_b(108)$ is approximately 1.511.

Example 64. *Use the properties of logarithms to write* $5\log_5(u) - \dfrac{1}{3}\log_5(v) + \log_5(v)$ *as a single logarithm with coefficient 1. Simplify as much as possible.*

Explanation We can first use the power property to rewrite $5\log_5(u) = \log_5(u^5)$ and $\dfrac{1}{3}\log_5(v) = \log_5(v^{1/3})$. Then we can use the product and quotient properties to combine the terms of the expression.

$$\begin{aligned}
\log_5(u^5) - \log_5(v^{1/3}) + \log_5(v) &= \log_5\left(\frac{u^5}{v^{1/3}}\right) + \log_5(t) \\
&= \log_5\left(\frac{u^5 v}{v^{1/3}}\right) \\
&= \log_5(u^5 v^{2/3})
\end{aligned}$$

There are other ways to approach this problem as well. See if you can find another way to do this problem!

Summary

If x, y, and b are positive real numbers,

- $\log_b(xy) = \log_b(x) + \log_b(y)$
- $\log_b\left(\dfrac{x}{y}\right) = \log_b(x) - \log_b(y)$
- $\log_b(x^u) = u\log_b(x)$ for all real numbers u.
- $\log_b(x) = \dfrac{\log_a(x)}{\log_a(b)}$ for all positive real numbers a.

10.2.3 Solving Logarithmic Equations

Using Inverses to Solve Equations

Now that we have an understanding of the properties of logarithms, we're prepared to solve equations involving logarithms and exponential functions. Before we do that, however, let's discuss a method of solving equations that you're already familiar with.

Consider the equation

$$x + 2 = 7.$$

You may have already found that the solution is $x = 5$, but let's think about the process of finding the solution.

Our general plan when solving equations is to isolate the variable we're solving for. In this case, we'd like to isolate x by itself on one side of the equation. However, x is not by itself: it's contained in a sum! Naturally, to undo the addition of 2, we subtract 2 from both sides and obtain $x = 5$. The key here is that it was stuck in some operation, and in order to "access" the x, we had to undo that operation.

We can also view this process in the context of functions. Let f be a function defined by $f(x) = x+2$. Then, our equation becomes $f(x) = 7$. In the language of functions, "undoing" f corresponds to applying the inverse function f^{-1}. In this case, $f^{-1}(x) = x - 2$. By applying f^{-1} to both sides of our original equation, we find that

$$\begin{aligned} f(x) &= 7 \\ f^{-1}(f(x)) &= f^{-1}(7) \\ x &= 7 - 2 \\ x &= 5. \end{aligned}$$

This may seem like an awfully strange way to subtract 2, but it has the benefit of being usable for any invertible function.

For example, say we want to solve the equation $\dfrac{x+1}{x} = 4$. If we define a function g by $g(x) = \dfrac{x+1}{x}$, our equation becomes $g(x) = 4$. We can find that the inverse is defined by $g^{-1}(x) = \dfrac{1}{x-1}$. Therefore,

$$\begin{aligned} g(x) &= 4 \\ g^{-1}(g(x)) &= g^{-1}(4) \\ x &= \frac{1}{4-1} \\ x &= \frac{1}{3} \end{aligned}$$

yields the solution to the equation.

Since we had to do quite a bit of work to find the equation for g^{-1} in the above scenario, this method may not be useful in that context. However, there are many functions for which we already know the inverse! For example, the inverse function of $h(x) = x^3$ is $h^{-1}(x) = \sqrt[3]{x}$. Therefore, if we want to solve $h(x) = 343$, we can apply h^{-1} on both sides to find that

$$\begin{aligned} h^{-1}(h(x)) &= h^{-1}(343) \\ x &= 7. \end{aligned}$$

Another important example of inverse functions that we know instantly comes from logarithms! If $f(x) = b^x$, then we know from our previous discussion that $f^{-1}(x) = \log_b(x)$. This is the definition of the logarithm, and looking at solving equations from the point of view of applying inverses is key to solving logarithmic and exponential equations.

For example, if we want to solve the equation $\log(2t-5) = 7$, we can define $f(x) = \log(x)$, so $f^{-1}(x) = 10^x$. This means our equation is $f(2t-5) = 7$. Therefore,

$$\begin{aligned} f(2t-5) &= 7 \\ f^{-1}(f(2t-5)) &= f^{-1}(7) \\ 2t-5 &= 10^7 \\ 2t &= 1000000 + 5 \\ t &= \frac{1000005}{2} \end{aligned}$$

yields the solution to the equation.

Exponential Equations

Example 65. *Solve the equation* $-4^{x-1} + 6 = 3$.

Explanation Notice that the variable we're solving for in this equation is located in the exponent of the exponential expression 4^{x-1}. Whenever this occurs, we call the equation an *exponential equation*.

If we define a function by $f(x) = 4^x$, then our equation becomes $-f(x-1)+6 = 2$. In order to solve this equation, we must use the inverse function: $f^{-1}(x) = \log_4(x)$. However, before we can apply this to both sides of the equation, we need to isolate $f(x-1)$ like so:

$$\begin{aligned} -f(x-1) + 6 &= 3 \\ -f(x-1) &= -3 \\ f(x-1) &= 3. \end{aligned}$$

Now we can take f^{-1} of both sides of the equation and obtain

$$\begin{aligned} f^{-1}(f(x-1)) &= f^{-1}(3) \\ x - 1 &= \log_4(3) \\ x &= \log_4(3) + 1. \end{aligned}$$

Therefore, the solution to the equation $-4^x + 6 = 3$ is $\log_4(3) + 1$. Your first instinct might be that this doesn't seem like a solution, since there's still a logarithm in our expression! However, there is no nicer way to write the number $\log_4(3)$. If you were to plug this into a calculator, you would get a decimal approximation to the value of $\log_4(3)$, but any decimal approximation loses some information, so the exact value of the solution is $\log_4(3) + 1$.

The process of writing out a function $f(x) = 4^x$ and then taking inverses may seem unnecessary, and indeed, there's no need to actually be so explicit when doing your own calculations. For example, the work

$$\begin{aligned} -4^{x-1} + 6 &= 3 \\ -4^{x-1} &= -3 \\ 4^{x-1} &= 3 \\ \log_4(4^{x-1}) &= \log_4(3) \\ x - 1 &= \log_4(3) \\ x &= \log_4(3) + 1 \end{aligned}$$

would be perfectly sufficient, and is usually how work for this kind of problem would be written. However, it must be emphasized that solving exponential equations involves more than just the basic operations of addition, subtraction, multiplication, and division. We now need to involve the process of taking logarithms of both sides of the equation.

Example 66. *Solve the equation* $3^x = 5^{2-x}$.

Explanation At first glance, this problem seems fundamentally different from the previous example. Instead of dealing with an exponential function with one base, we're dealing with two different bases: 3 and 5.

However, recall from the previous section that any positive real number can be written as a power of any number we want. In this case, 3 can be written as $3 = 5^{\log_5(3)}$. Therefore, our equation becomes

$$5^{x\log_5(3)} = 5^{2-x}.$$

Dividing both sides by 5^{2-x} yields

$$\begin{aligned}\frac{5^{x\log_5(3)}}{5^{2-x}} &= 1\\ 5^{x\log_5(3)-(2-x)} &= 1.\end{aligned}$$

Now, since our variable is trapped in the exponent, we're in the situation from before! If $f(x) = 5^x$, then our equation has become $f(x\log_5(3) - (2-x)) = 1$. To solve this, we can take $f^{-1} = \log_5$ of both sides of the equation and do some more algebra to isolate x.

$$\begin{aligned}\log_5(5^{x\log_5(3)-(2-x)}) &= \log_5(1)\\ x\log_5(3) - (2-x) &= 0\\ x\log_5(3) + x - 2 &= 0\\ x\log_5(3) + x &= 2\\ x(\log_5(3) + 1) &= 2\\ x &= \frac{2}{\log_5(3)+1}.\end{aligned}$$

Logarithmic Equations

Example 67. *Solve the equation* $5\log_2(x+3) = -2$.

Explanation To start off, divide both sides by 5 to isolate the $\log_2(x+3)$ on the left-hand side.

$$\log_2(x+3) = -\frac{2}{5}$$

Next, we apply the inverse of $\log_2$ to both sides of the equation, obtaining

$$\begin{aligned}2^{\log_2(x+3)} &= 2^{-2/5}\\ x+3 &= \frac{1}{\sqrt[5]{4}}\\ x &= \frac{1}{\sqrt[5]{4}} - 3.\end{aligned}$$

Since logarithms are not always defined (their domain is only positive real numbers), we should check that plugging in our solution for x does not result in any part of our original equation being undefined. In our case, this amounts to checking that $\log_2(x+3)$ is defined, that is, that $x+3$ is positive. Since $x+3 = \dfrac{1}{\sqrt[5]{4}} > 0$, our solution is $\dfrac{1}{\sqrt[5]{4}} - 3$.

Example 68. *Solve the equation* $\log_6(x) = 1 - \log_6(x-1)$.

Explanation As in Example 2, there appear to be too many functions going on here. However, if we add $\log_6(x-1)$ to both sides of the equation and use the product property of logarithms, we obtain:

$$\begin{aligned} \log_6(x) + \log_6(x-1) &= 1 \\ \log_6(x(x-1)) &= 1. \end{aligned}$$

Next, we can apply the inverse function of $\log_6$, which is given by $f(x) = 6^x$. Doing so, we see that

$$\begin{aligned} 6^{\log_6(x(x-1))} &= 6^1 \\ x(x-1) &= 6 \\ x^2 - x - 6 &= 0. \end{aligned}$$

This results in a quadratic equation! This is something we know how to solve. By using our preferred method, we find that $x = 3$ or $x = -2$.

We're not done yet, however! We need to check that these x-values don't cause any logarithms in our original equation to be undefined. Note that $\log_6(-2)$ is undefined, since -2 is negative, so $x = -2$ is not a solution to our equation. Since $\log_6(3)$ and $\log_6(3-1)$ are both defined, $x = 3$ is our only solution.

Summary

- When solving exponential equations, our strategy is to isolate a single exponential on one side of the equation, then apply a logarithm to both sides to undo the exponential.
- When solving logarithmic equations, our strategy is to isolate a single logarithm on one side of the equation, then apply an exponential function to both sides to undo the logarithm.
- Since the domain of logarithms is only $(0, \infty)$, we need to check that our solutions do not make our original logarithmic equations undefined.

10.3 Inverse Trigonometric Functions

Learning Objectives

- –
  –
  –
- –

10.3.1 Inverse Cosine

Motivating Questions

- Is it possible for a periodic function that fails the Horizontal Line Test to have an inverse?
- For the restricted cosine function, how do we define the corresponding arccosine function?
- What are the key properties of arccosine?

Introduction

In our prior work with inverse functions, we learned several important principles, including

- A function f has an inverse function if and only if there exists a function g that undoes the work of f: that is, there is some function g for which $g(f(x)) = x$ for each x in the domain of f, and $f(g(y)) = y$ for each y in the range of f. We call g the inverse of f, and write $g = f^{-1}$.
- A function f has an inverse function if and only if the graph of f passes the *Horizontal Line Test*.
- When f has an inverse, we know that writing "$y = f(t)$" and "$t = f^{-1}(y)$" are two different perspectives on the same statement.

The trigonometric function $g(t) = \cos(t)$ is periodic, so it fails the horizontal line test. Hence, considering this function on its full domain, it does not have an inverse function. At the same time, it is reasonable to think about changing perspective and viewing angles as outputs in certain restricted settings. For instance, we may want to say both

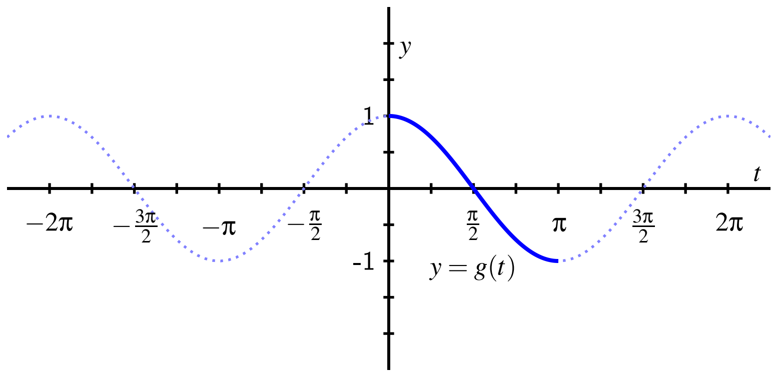
$$\frac{\sqrt{3}}{2} = \cos\left(\frac{\pi}{6}\right) \quad \text{and} \quad \frac{\pi}{6} = \cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$$

depending on the context in which we are considering the relationship between the angle and side length.

It is also helpful to contextualize the importance of finding an angle in terms of a known value of a trigonometric function. Suppose we know the following information about a right triangle: one leg has length 2.5, and the hypotenuse has length 4. If we let θ be the angle adjacent to the side of length 2.5, it follows that $\cos(\theta) = \frac{2.5}{4}$. We naturally want to use the inverse of the cosine function to solve the most recent equation for θ. But the cosine function does not have an inverse function, so how can we address this situation?

While the original trigonometric function $g(t) = \cos(t)$ does not have an inverse function, we can instead consider a restricted version of the function that does. We thus investigate how we can think differently about the trigonometric functions so that we can discuss inverses in a meaningful way.

Consider the plot of the standard cosine function on $\left[-\frac{5\pi}{2}, \frac{5\pi}{2}\right]$ with the portion on $[0, \pi]$ emphasized below.



Exploration Let g be the function whose domain is $0 \le t \le \pi$ and whose outputs are determined by the rule $g(t) = \cos(t)$.
The key observation here is that g is defined in terms of the cosine function, but because it has a different domain, it is *not* the cosine function.

a. What is the domain of g?

b. What is the range of g?

c. Does g pass the horizontal line test? Why or why not?

d. Explain why g has an inverse function, g^{-1}, and state the domain and range of g^{-1}.

e. We know that $g\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$. What is the exact value of $g^{-1}\left(\frac{\sqrt{2}}{2}\right)$? How about the exact value of $g^{-1}\left(-\frac{\sqrt{2}}{2}\right)$?

f. Determine the exact values of $g^{-1}\left(-\frac{1}{2}\right)$, $g^{-1}\left(\frac{\sqrt{3}}{2}\right)$, $g^{-1}(0)$, and $g^{-1}(-1)$. Use proper notation to label your results.

The Arccosine Function

For the cosine function restricted to the domain $[0, \pi]$ that we considered above, the function is strictly decreasing on its domain and thus passes the Horizontal Line Test. Therefore, this restricted version of the cosine function has an inverse function; we will call this inverse function the *arccosine* function.

> **Definition** Let $y = g(t) = \cos(t)$ be defined on the domain $[0, \pi]$, and observe $g : [0, \pi] \to [-1, 1]$. For any real number y that satisfies $-1 \le y \le 1$, the **arccosine of y**, denoted
>
> $$\arccos(y)$$
>
> is the angle t satisfying $0 \le t \le \pi$ such that $\cos(t) = y$. Note that we use $t = \cos^{-1}(y)$ interchangeably with with $t = \arccos(y)$.

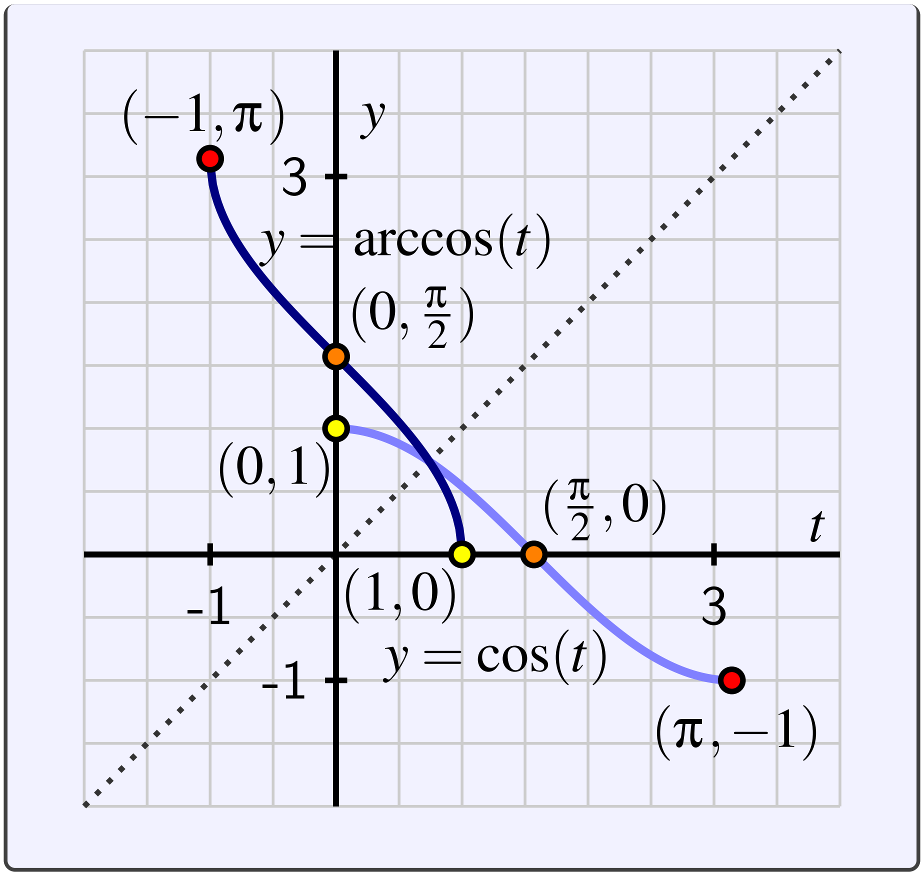
In particular, we note that the output of the arccosine function is an angle. Recall that in the context of the unit circle, an angle measured in radians and the corresponding arc length along the unit circle are numerically equal. This is the origin of the "arc" in "arccosine": given a value $-1 \le y \le 1$, the arccosine function produces the corresponding *arc* (measured counterclockwise from $(1, 0)$) such that the cosine of that arc is y.

For any function with an inverse function, the inverse function reverses the process of the original function. Thus, given $y = \cos(t)$, we can read this statement as saying "y is the cosine of the angle t". Changing perspective and writing the equivalent statement, $t = \arccos(y)$, we read this statement as "t is the angle whose cosine is y". Just as $y = f(t)$ and $t = f^{-1}(y)$ mean the same thing for a function and its inverse in general. To summarize, both expressions

$$y = \cos(t) \text{ and } t = \arccos(y)$$

mean the same thing for any angle t that satisfies $0 \le t \le \pi$. We read $t = \cos^{-1}(y)$ as "t is the angle whose cosine is y" or "t is the inverse cosine of y". Key properties of the arccosine function can be summarized as follows.

> **Properties of the arccosine function.**
>
> - The restricted cosine function, $y = g(t) = \cos(t)$, is defined on the domain $[0, \pi]$ with range $[-1, 1]$. This function has an inverse function that we call the arccosine function, denoted $t = g^{-1}(y) = \arccos(y)$.
> - The domain of $y = g^{-1}(t) = \arccos(t)$ is $[-1, 1]$ with range $[0, \pi]$.
> - The arccosine function is always decreasing on its domain.
> - Below we have a plot of the restricted cosine function (in light blue) and its corresponding inverse, the arccosine function (in dark blue).



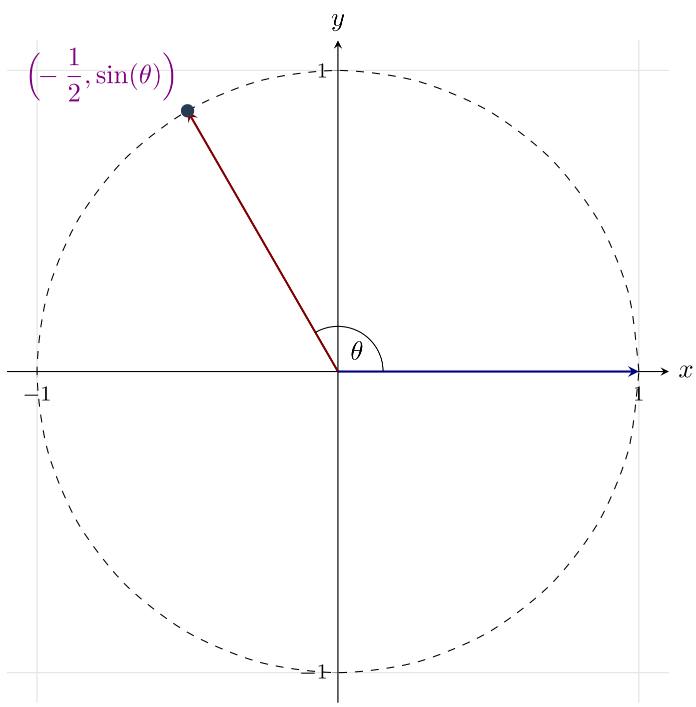
Just as the natural logarithm function allowed us to rewrite exponential equations in an equivalent way (for instance, $y = e^t$ and $t = \ln(y)$ give the same information), the arccosine function allows us to do likewise for certain angles and cosine outputs. For instance, saying $\cos\left(\frac{\pi}{2}\right) = 0$ is the same as writing $\frac{\pi}{2} = \arccos(0)$, which reads "$\frac{\pi}{2}$ is the angle whose cosine is 0". Indeed, these relationships are reflected in the plot above, where we see that any point (a, b) that lies on the graph of $y = \cos(t)$ corresponds to the point (b, a) that lies on the graph of $y = \arccos(t)$.

Exploring Arccosine

Example 69. *Use the special points on the unit circle to determine the exact values of each of the following numerical expressions. Do so without using a computational device.*

(a) $\cos\left(\arccos\left(-\frac{1}{2}\right)\right)$

Explanation We start by finding $\arccos\left(-\frac{1}{2}\right)$. Remember that for x in $[-1,1]$, $\arccos(x)$ is the angle θ in $[0,\pi]$ such that $\cos(\theta)=x$. Hence we are looking for the value of θ corresponding to the point on the upper hemisphere of the unit circle with x-value $-\frac{1}{2}$.



Hence, θ is $\frac{2\pi}{3}$, and we now see that

$$\cos\left(\arccos\left(-\frac{1}{2}\right)\right)=\cos\left(\frac{2\pi}{3}\right)=-\frac{1}{2}.$$

Now, if you're thinking, "Hey, we didn't need that extra step!" Then you would be correct. But ***why*** didn't we need that final step?

Let's recall how we defined arccosine. Since cosine is a periodic function, it fails the horizontal line test. However, if we *restrict* cosine to a portion of its domain on which it is only decreasing, $[0,\pi]$, then we may define a function g on this domain such that $g(x)=\cos(x)$ for x in $[0,\pi]$. Arccosine is defined as the inverse of this function g. Therefore, g is the inverse of arccosine. Thus, in practice, cosine is the inverse of arccosine.

A word of caution: arccosine is only the inverse of restricted cosine, as we will demonstrate with the next example.

(b) $\arccos\left(\cos\left(\frac{7\pi}{6}\right)\right)$

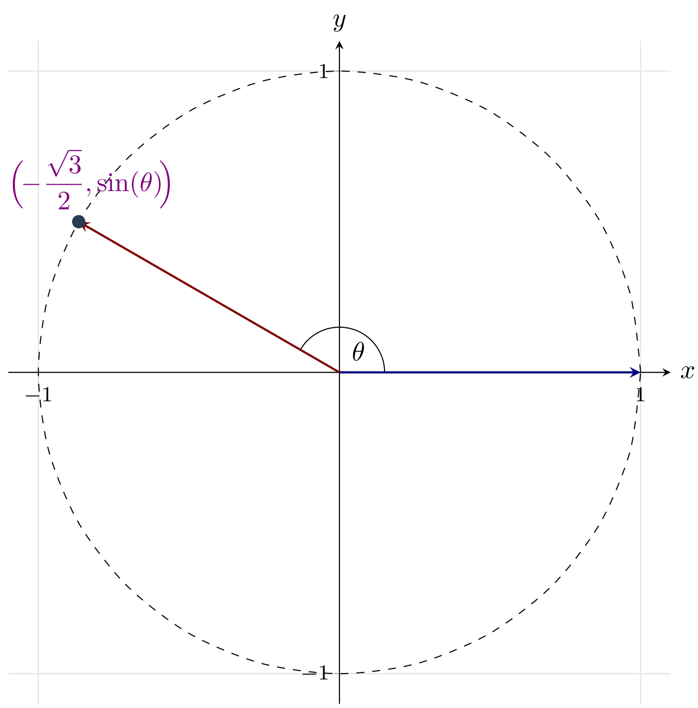
Explanation It may be tempting to take a look at this expression and conclude that the solution is $\frac{7\pi}{6}$ since arccosine is the inverse of cosine.

But, wait!

Remember, we had to restrict the domain of cosine in order to define an inverse function, which we called arccosine. Arccosine is the inverse of the *restricted* cosine function, whose domain is $[0, \pi]$. $\frac{7\pi}{6}$ is larger than π, so it is not within the domain of this restricted cosine.

Thus, we begin by simplifying $\cos\left(\frac{7\pi}{6}\right) = -\frac{\sqrt{3}}{2}$.

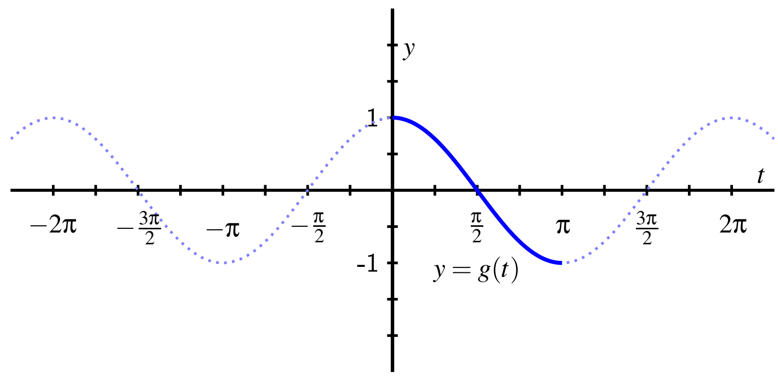
Now, when we consider $\arccos\left(-\frac{\sqrt{3}}{2}\right)$, we will once again recall the unit circle. We are looking at the upper hemisphere, but this time we want to find the angle θ in $[0, \pi]$ that corresponds to the point with x-value $-\frac{\sqrt{3}}{2}$.



Hence, θ is $\frac{5\pi}{6}$, and we now see that

$$\arccos\left(\cos\left(\frac{7\pi}{6}\right)\right) = \arccos\left(-\frac{\sqrt{3}}{2}\right) = \frac{5\pi}{6}.$$

Now, let's look again at the graph of cosine. Here we highlight $g : [0, \pi] \to [0, \pi]$ defined by $y = g(x) = \cos(x)$, the restricted cosine function. We may use the symmetry of the graph of cosine to help find the appropriate values for arccosine.



(c) *We can also solve trig equations as in Section 10-3 Some Applications of Trig Functions:* $4\arccos(x) - 3\pi = 0$.

Explanation We start by isolating the arccosine term so that our equation is now

$$\arccos(x) = \frac{3\pi}{4}.$$

We observe that $\frac{3\pi}{4}$ is in the range of arccosine, so we may use the fact that cosine is the inverse of arccosine. Thus, $\arccos(x) = \frac{3\pi}{4}$ is equivalent to

$$\cos(\arccos(x)) = \cos\left(\frac{3\pi}{4}\right).$$

This is further equivalent to $x = -\frac{2}{2}$.

Summary

- Any function that fails the Horizontal Line Test cannot have an inverse function. However, for a periodic function that fails the horizontal line test, if we restrict the domain of the function to an interval that is the length of a single period of the function, we then determine a related function that does, in fact, have an inverse function. This makes it possible for us to develop the inverse function of the restricted cosine function.

- We choose to define the restricted cosine function on the domain

$[0, \pi]$. On this interval, the restricted function is strictly decreasing, and thus has an inverse function. The restricted cosine function has range $[-1, 1]$.

10.3.2 Other Inverse Trig Functions

Motivating Questions

- For the restricted sine, tangent, and secant functions, how do we define the corresponding arcsine, arctangent, and arcsecant functions?
- What are the key properties of arcsine, arctangent, and arcsecant?

Introduction

In the last section we defined *arccosine*, the inverse for cosine restricted to a single period. In this section we will explore the definition of similar inverse functions on restricted domains of sine, tangent, and secant.

As we recalled last time,

- A function f has an inverse function if and only if there exists a function g that undoes the work of f: that is, there is some function g, the inverse of f, for which $g(f(x)) = x$ for each x in the domain of f, and $f(g(y)) = y$ for each y in the range of f.
- A function f has an inverse function if and only if the graph of f passes the *Horizontal Line Test*.
- When f has an inverse, we know that "$y = f(t)$" and "$t = f^{-1}(y)$" are two different perspectives on the same statement.

As with the cosine function, the trigonometric functions $f(t) = \sin(t)$, $h(t) = \tan(t)$, and $k(t) = \sec(t)$ are periodic, so they fail the horizontal line test. Hence, considering these functions on their full domains, neither has an inverse function. At the same time, it is reasonable to think about changing perspective and viewing angles as outputs in certain restricted settings, as we did with cosine.

The Arcsine Function

We can develop an inverse function for a restricted version of the sine function in a similar way. As with the cosine function, we need to choose an interval on which the sine function is always increasing or always decreasing in order to have the function pass the horizontal line test. The standard choice is the domain $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ on which $f(t) = \sin(t)$ is increasing and attains all of the values in the range of the sine function. Thus, we consider $f(t) = \sin(t)$ so

Acknowledgements: Active Prelude to Calculus (https://activecalculus.org/prelude) and Stitz Zeager Open Source Mathematics (https://www.stitz-zeager.com/)

that $f : \left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right] \to \left[-1, 1\right]$ and use this restricted function to define the corresponding arcsine function.

Definition Let $y = f(t) = \sin(t)$ be defined on the domain $\left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$, and observe $f : \left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right] \to \left[-1, 1\right]$. For any real number y that satisfies $-1 \leq y \leq 1$, the **arcsine of y**, denoted

$$\arcsin(y)$$

is the angle t satisfying $-\dfrac{\pi}{2} \leq t \leq \dfrac{\pi}{2}$ such that $\sin(t) = y$. Note that we use $t = \sin^{-1}(y)$ interchangeably with with $t = \arcsin(y)$.

Problem 3 *The goal of this activity is to understand key properties of the arcsine function in a way similar to our discussion of the arccosine function in the previous section. We will use our deductive reasoning skills a la Sherlock Holmes to build off our discussion from the last section.*

(a) *Using the definition of arcsine given above, what are the domain and range of the arcsine function?*

- *The domain of arcsine is* $\left[\boxed{?}, \boxed{?}\right]$.
- *The range of arcsine is* $[\boxed{?}, \boxed{?}]$.

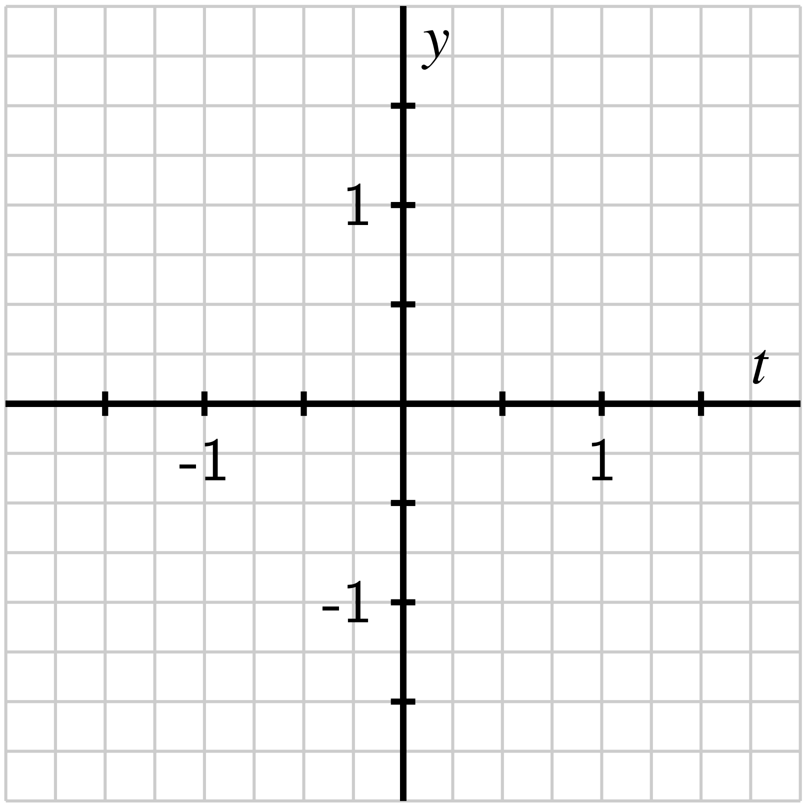
(b) *Determine the following values exactly:*

- $\arcsin(-1) = \boxed{?}$
- $\arcsin\left(-\dfrac{\sqrt{2}}{2}\right) = \boxed{?}$
- $\arcsin(0) = \boxed{?}$
- $\arcsin\left(\dfrac{1}{2}\right) = \boxed{?}$
- $\arcsin\left(\dfrac{\sqrt{3}}{2}\right) = \boxed{?}$

(c) *On the axes provided below, sketch a careful plot of the restricted sine function on the interval* $\left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$ *along with its corresponding inverse, the arcsine function. Label at least three points on each curve so that each point on the sine graph corresponds to a point on the arcsine graph. In addition, sketch the line* $y = t$ *to demonstrate how the graphs are reflections of one another across this line.*

Acknowledgements: Active Prelude to Calculus (https://activecalculus.org/prelude) and Stitz Zeager Open Source Mathematics (https://www.stitz-zeager.com/)

(d) *True or false:* $\arcsin(\sin(5\pi)) = 5\pi$? *(true/false)*
Write a complete sentence to explain your reasoning.

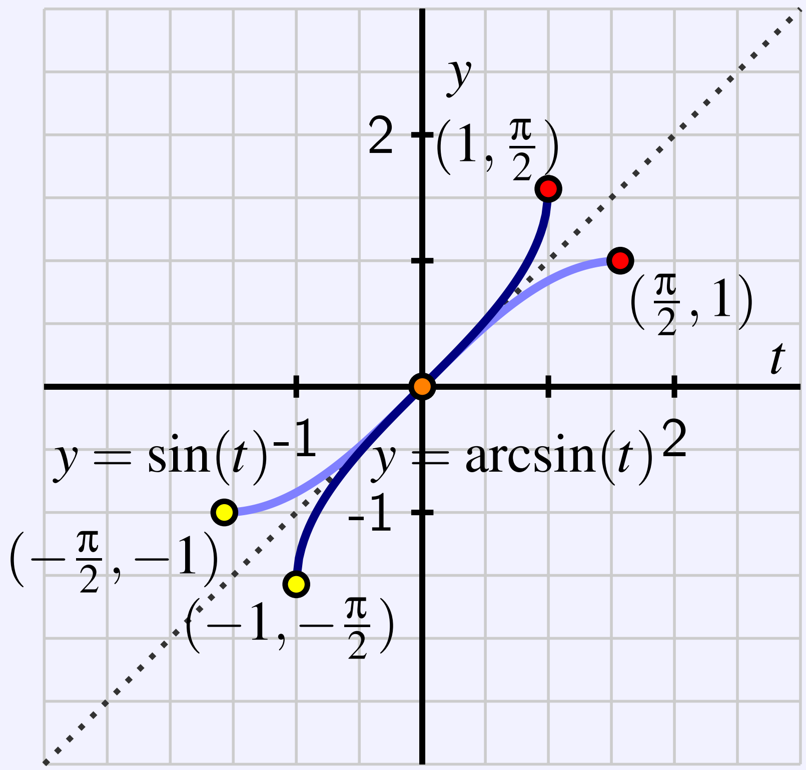


Properties of the arcsine function.

- The restricted sine function, $y = f(t) = \sin(t)$, is defined on the domain $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ with range $[-1, 1]$. This function has an inverse function that we call the arcsine function, denoted $t = f^{-1}(y) = \arcsin(y)$.
- The domain of $y = f^{-1}(t) = \arcsin(t)$ is $[-1, 1]$ with range $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.
- The arcsine function is always increasing on its domain.

Acknowledgements: Active Prelude to Calculus (https://activecalculus.org/prelude) and Stitz Zeager Open Source Mathematics (https://www.stitz-zeager.com/)

- Below we have a plot of the restricted sine function (in light blue) and its corresponding inverse, the arcsine function (in dark blue).



Exploring Arcsine

Example 70. *Let's solve the following equations analytically, then we can consider the graph of arcsine.*

(a) $\sin\left(\arcsin\left(-\frac{\sqrt{2}}{2}\right)\right)$

Explanation We start by finding $\arcsin\left(-\frac{\sqrt{2}}{2}\right)$. Remember that for x in $[-1,1]$, $\arcsin(x)$ is the value y in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ such that $\sin(y) = x$. Hence, y is $-\frac{\pi}{4}$, and we now see that

$$\sin\left(\arcsin\left(-\frac{\sqrt{2}}{2}\right)\right) = \sin\left(-\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}.$$

Acknowledgements: Active Prelude to Calculus (https://activecalculus.org/prelude) and Stitz Zeager Open Source Mathematics (https://www.stitz-zeager.com/)

Now, if you're thinking, "Hey, we didn't need that extra step!" Then you would be correct. But ***why*** didn't we need that final step?

Let's recall how we defined arcsine. Since sine is a periodic function, it fails the horizontal line test. However, if we *restrict* sine to a portion of its domain on which it is only increasing, $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, then we may define a function f on this domain such that $f(x) = \sin(x)$ for x in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. Arcsine then is defined as the inverse of this function f. Therefore, f is the inverse of arcsine. Thus, in practice, sine is the inverse of arcsine.

A word of caution: As was the case with arccosine and cosine, arcsine is only the inverse of restricted sine. We will illustrate this with the next example.

(b) $\arcsin\left(\sin\left(\frac{5\pi}{4}\right)\right)$

Explanation It may be tempting to take a look at this expression and conclude that the solution is $\frac{5\pi}{4}$ since arcsine is the inverse of sine.

Hold those horses!

Remember, we had to restrict the domain of sine in order to define an inverse function, which we called arcsine. Arcsine is the inverse of the *restricted* sine function, whose domain is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. $\frac{5\pi}{4}$ is larger than $\frac{\pi}{2}$, so it is not within the domain of this restricted sine function.

Thus, we begin by simplifying $\sin\left(\frac{5\pi}{4}\right) = -\frac{\sqrt{2}}{2}$.

Now, let's consider $\arcsin\left(-\frac{\sqrt{2}}{2}\right)$, recalling again the *range* of arcsine. We are looking for the value of y in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ such that $\sin(y) = -\frac{\sqrt{2}}{2}$.

Hence, y is $-\frac{\pi}{4}$, and we now see that

$$\arcsin\left(\sin\left(\frac{5\pi}{4}\right)\right) = \arcsin\left(-\frac{\sqrt{2}}{2}\right) = -\frac{\pi}{4}.$$

[graph for arcsine?]

(c) $\arcsin(2x) = \frac{\pi}{3}$

Explanation First, we observe that $\frac{\pi}{3}$ is in the range of arcsine, so there should be a solution. We will now use the fact that sine is the inverse of

Acknowledgements: Active Prelude to Calculus (https://activecalculus.org/prelude) and Stitz Zeager Open Source Mathematics (https://www.stitz-zeager.com/)

arcsine to reduce this to a linear equation.

$$\arcsin(2x) = \frac{\pi}{3}$$
$$\sin(\arcsin(2x)) = \sin(\frac{\pi}{3})$$

Thus, we have

$$2x = \frac{\sqrt{3}}{2},$$

which is equivalent to $x = \dfrac{\sqrt{3}}{4}$.

[Insert graph here?]

The Arctangent Function

Finally, we develop an inverse function for a restricted version of the tangent function. We choose the domain $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ on which $h(t) = \tan(t)$ is increasing and attains all of the values in the range of the tangent function.

Definition Let $y = h(t) = \tan(t)$ be defined on the domain $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, and observe $h : \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \to \big(-\infty, \infty\big)$. For any real number y, the **arctangent of y**, denoted

$$\arctan(y)$$

is the angle t satisfying $-\frac{\pi}{2} < t < \frac{\pi}{2}$ such that $\tan(t) = y$. Note that we use $t = \tan^{-1}(y)$ interchangeably with with $t = \arctan(y)$.

Problem 4 *Let us once again channel our inner Sherlock Holmes to understand key properties of the arctangent function.*

(a) *Using the definition given above, what are the domain and range of the arctangent function?*

- *The domain of arctangent is* $\left(\boxed{?}, \boxed{?}\right)$.
- *The range of arctangent is* $(\boxed{?}, \boxed{?})$.

(b) *Determine the following values exactly:*

- $\arctan(-\sqrt{3}) = \boxed{?}$

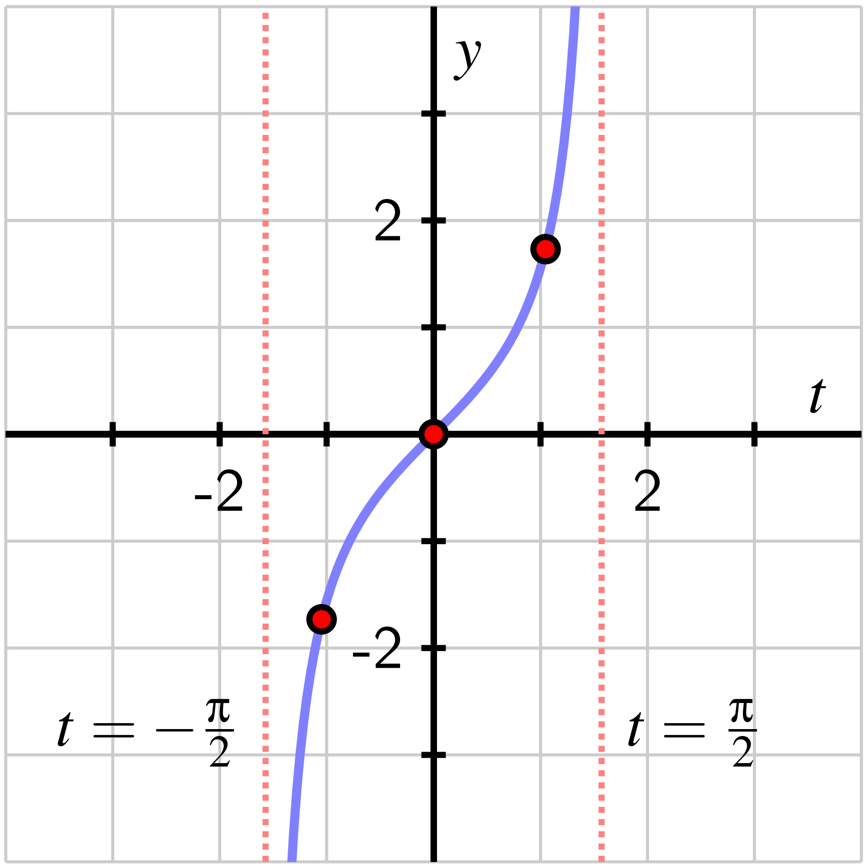
Acknowledgements: Active Prelude to Calculus (https://activecalculus.org/prelude) and Stitz Zeager Open Source Mathematics (https://www.stitz-zeager.com/)

- $\arctan(-1) = \boxed{?}$
- $\arctan(0) = \boxed{?}$
- $\arctan\left(\dfrac{1}{\sqrt{3}}\right) = \boxed{?}$.

(c) *The restricted tangent function on the interval* $\left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$ *is plotted below. On the same axes, sketch its corresponding inverse function (arctangent). Label at least three points on each curve so that each point on the tangent graph corresponds to a point on the arctangent graph. In addition, sketch the line* $y = t$ *to demonstrate how the graphs are reflections of one another across this line.*

(d) *Complete the following sentence: "as* t *increases without bound,* $\arctan(t)$ ..." (*increases without bound*/*decreases without bound*/*increases toward* $\dfrac{\pi}{2}$/*decreases toward* $-\dfrac{\pi}{2}$)

Acknowledgements: Active Prelude to Calculus (https://activecalculus.org/prelude) and Stitz Zeager Open Source Mathematics (https://www.stitz-zeager.com/)

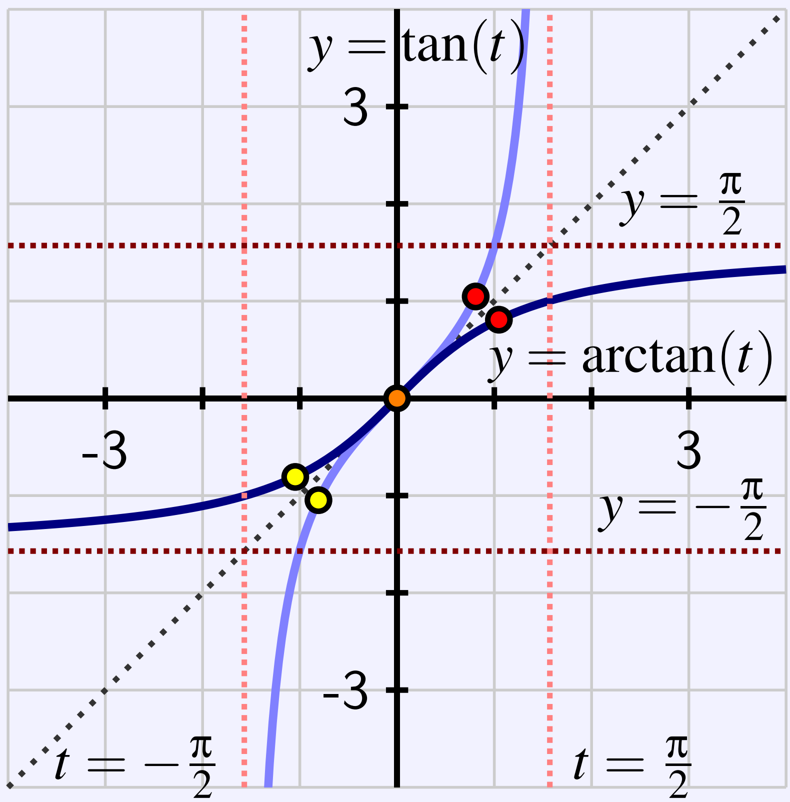


Properties of the arctangent function.

- The restricted tangent function, $y = h(t) = \tan(t)$, is defined on the domain $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ with range $(-\infty, \infty)$. This function has an inverse function that we call the arctangent function, denoted $t = h^{-1}(y) = \arctan(y)$.
- The domain of $y = h^{-1}(t) = \arctan(t)$ is $(-\infty, \infty)$ with range $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.
- The arctangent function is always increasing on its domain.
- Below we have a plot of the restricted tangent function (in light

Acknowledgements: Active Prelude to Calculus (https://activecalculus.org/prelude) and Stitz Zeager Open Source Mathematics (https://www.stitz-zeager.com/)

blue) and its corresponding inverse, the arctangent function (in dark blue).



Exploring Arctangent

Example 71. *Let's solve the following equations analytically, then we can consider the graph of arctangent.*

(a) $\tan(\arctan(-\sqrt{3}))$
Explanation We start by finding $\arctan(-\sqrt{3})$. Remember that for x in $(-\infty, \infty)$, $\arctan(x)$ is the value y in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ such that $\tan(y) = x$.

Hence, y is $-\frac{\pi}{3}$, and we now see that

$$\tan(\arctan(-\sqrt{3})) = \tan\left(-\frac{\pi}{3}\right) = -\sqrt{3}.$$

Now, I know you're thinking, "Hey, why do you keep making us do an

 Acknowledgements: Active Prelude to Calculus (https://activecalculus.org/prelude) and Stitz Zeager Open Source Mathematics (https://www.stitz-zeager.com/)

extra step?" It's because it is imperative that you ***consider the range*** of the arc trig functions. These are considerations you will also need to make when we start combining different trig functions with the inverses of others (say sine of arctangent of a value).

Let's recall how we defined arctangent. Since tangent is a periodic function, it fails the horizontal line test. However, if we *restrict* tangent to a single period (note tangent only increasing), $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, then we may define a function h on this domain such that $h(x) = \tan(x)$ for x in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Arctangent then is defined as the inverse of this function h. Therefore, h is the inverse of arctangent. Thus, in practice, tangent is the inverse of arctangent.

A word of caution: As was the case with the previous two trig functions and their respective inverses, arctangent is only the inverse of restricted tangent. We will illustrate this with the next example.

(b) $\arctan\left(\tan\left(\frac{5\pi}{3}\right)\right)$

Explanation It may be tempting to take a look at this expression and conclude that the solution is $\frac{5\pi}{3}$ since arctangent is the inverse of tangent.

But, wait!

Remember, we had to restrict the domain of tangent in order to define an inverse function, which we called arctangent. Arctangent is the inverse of the *restricted* tangent function, whose domain is $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. $\frac{5\pi}{3}$ is larger than $\frac{\pi}{2}$, so it is not within the domain of this restricted tangent function.

Thus, we begin by simplifying $\tan\left(\frac{5\pi}{3}\right) = -\sqrt{3}$.

Now, let's consider $\arctan(-\sqrt{3})$, recalling again the *range* of arctangent. We are looking for the value of y in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ such that $\tan(y) = -\sqrt{3}$.

Hence, y is $-\frac{\pi}{3}$, and we now see that

$$\arctan\left(\tan\left(\frac{5\pi}{3}\right)\right) = \arctan(-\sqrt{3}) = -\frac{\pi}{3}.$$

[graph for arctangent?]

(c) $4\arctan^2(x) - 3\pi\arctan(x) - \pi^2 = 0$

Explanation We will treat this like a quadratic equation to begin, as we did in Section 10-3 Some Applications of Trig Functions.

Let $y = \arctan(x)$, then we have a standard quadratic equation: $4y^2 - 3\pi y - \pi^2 = 0$. Factoring, we see that this is equivalent to

$$(4y + \pi)(y - \pi) = 0.$$

 Acknowledgements: Active Prelude to Calculus (https://activecalculus.org/prelude) and Stitz Zeager Open Source Mathematics (https://www.stitz-zeager.com/)

This has two solutions: $y = -\dfrac{\pi}{4}$ and $y = \pi$. In other words, we now simply solve (a) $\arctan(x) = -\dfrac{\pi}{4}$ and (b) $\arctan(x) = \pi$. π is not in the range of arctangent, so (b) does not have a solution. Hence, this cannot be a solution to our equation, and we must look at (a). $-\dfrac{\pi}{4}$ is in the range of arctangent, so the solution to (a) will be a solution to our original equation.

Since tangent is the inverse to arctangent, the equation (a) is equivalent to

$$\tan(\arctan(x)) = \tan\Big(-\frac{\pi}{4}\Big),$$

which is further equivalent to $x = -1$

The Arcsecant Function

We will also consider the inverse function for a restricted version of the secant function. As with the cosine and sine functions, we need to choose an interval on which the secant function is always increasing or always decreasing in order to have the function pass the horizontal line test. In the case of secant, this means choosing two distinct intervals. A word of caution in working with the restricted secant function and its associated inverse, there is not a "standard" choice for the domain of restricted secant. *However*, we will establish a convention in this course.

We Restrict the domain of the function $k(t) = \sec(t)$ to $[0, \pi/2) \cup (\pi/2, \pi]$, where secant is increasing on each interval and attains all the values within the range of the secant function. By reflecting across the line $y = t$ and switching the t and y coordinates we are able to define the function $k^{-1}(t) = \operatorname{arcsec}(t)$ as follows.

Definition Let $y = k(t) = \sec(t)$ be defined on the domain $[0, \pi/2) \cup (\pi/2, \pi]$, and observe that

$$k : \Big[0, \frac{\pi}{2}\Big) \cup \Big(\frac{\pi}{2}, \pi\Big] \to (-\infty, -1] \cup [1, \infty).$$

For any real number y, the **arcsecant of y**, denoted

$$\operatorname{arcsec}(y)$$

is the angle t satisfying $0 \leq t < \dfrac{\pi}{2}$ or $\dfrac{\pi}{2} < t \leq \pi$. Note that we use $t = \sec^{-1}(y)$ interchangeably with $t = \operatorname{arcsec}(y)$.

Problem 5 *Take the lead Watson, and we will deduce the key properties of the arcsecant function as we did above for arcsine and arctangent.*

 Acknowledgements: Active Prelude to Calculus (https://activecalculus.org/prelude) and Stitz Zeager Open Source Mathematics (https://www.stitz-zeager.com/)

(a) *Using the definition of arcsecant given above, what are the domain and range of the arcsecant function?*

- *The domain of arcsecant is* $(\boxed{?}, \boxed{?}] \cup [\boxed{?}, \boxed{?})$.
- *The range of arcsecant is* $\left[\boxed{?}, \boxed{?}\right) \cup \left(\boxed{?}, \boxed{?}\right]$.

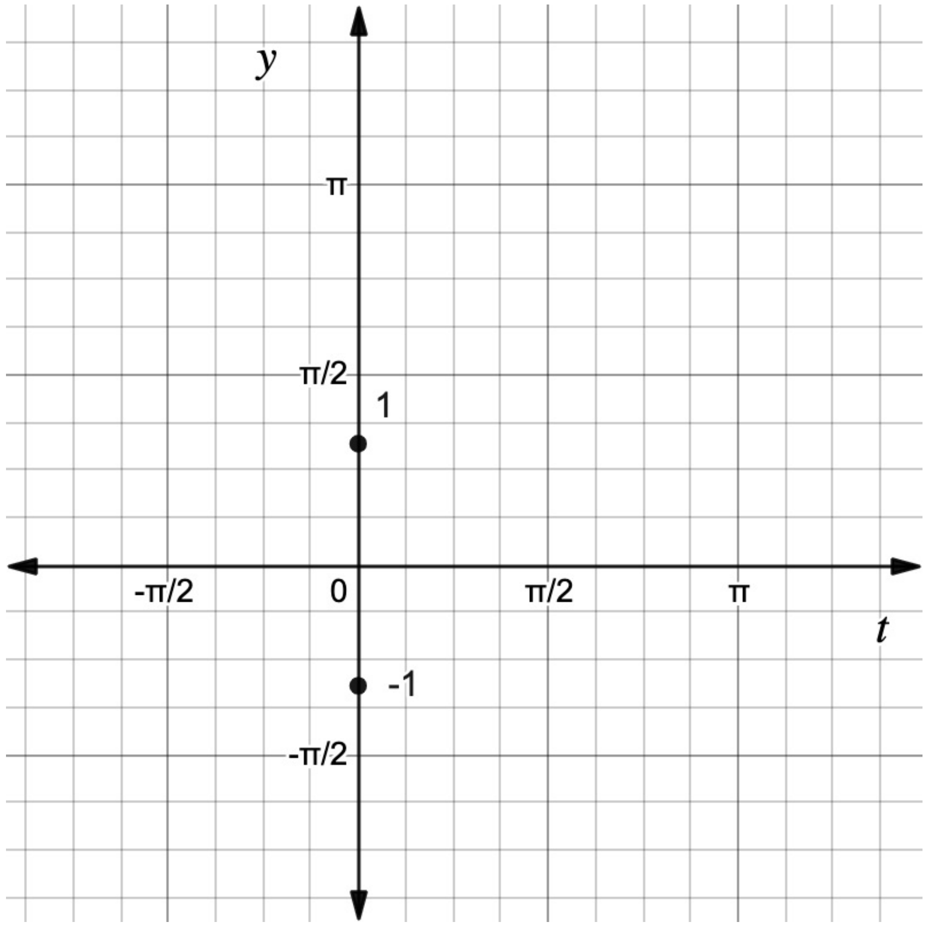
(b) *Determine the following values exactly:*

- $\operatorname{arcsec}(1) = \boxed{?}$
- $\operatorname{arcsec}(-1) = \boxed{?}$
- $\operatorname{arcsec}(2) = \boxed{?}$

(c) *On the axes provided below, sketch a careful plot of the restricted secant function on the intervals* $\left[0, \frac{\pi}{2}\right)$ *and* $\left(\frac{\pi}{2}, \pi\right]$ *along with its corresponding inverse, the arcsecant function. Label at least three points on each curve so that each point on the secant graph corresponds to a point on the arcsecant graph.*

(d) *True or false:* $\operatorname{arcsec}(\sec(2\pi)) = 2\pi$*? (true/false)*
Write a complete sentence to explain your reasoning.

Acknowledgements: Active Prelude to Calculus (https://activecalculus.org/prelude) and Stitz Zeager Open Source Mathematics (https://www.stitz-zeager.com/)

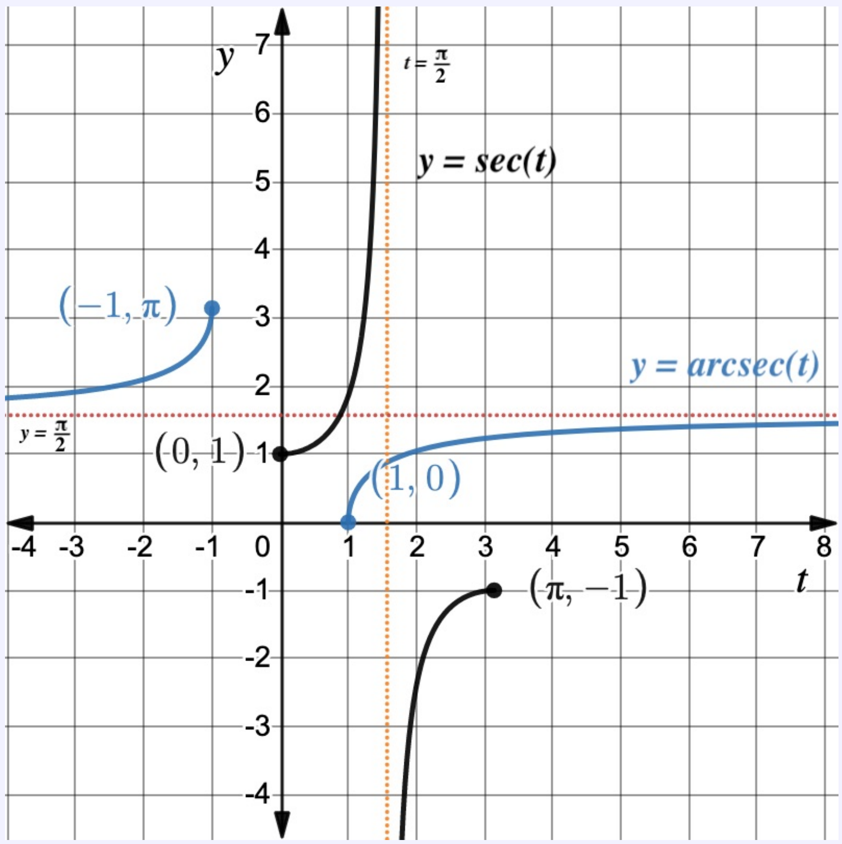


Properties of the arcsecant function.

- The restricted secant function, $y = k(t) = \sec(t)$, is defined on the domain $\left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$ with range $(-\infty, -1] \cup [1, \infty)$. This function has an inverse function that we call the arcsecant function, denoted $t = k^{-1}(y) = \text{arcsec}(y)$.
- The domain of $y = k^{-1}(t) = \text{arcsec}(t)$ is $(-\infty, -1] \cup [1, \infty)$ with range $\left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$.
- The arcsecant function is always increasing on each interval in its domain.

Acknowledgements: Active Prelude to Calculus (https://activecalculus.org/prelude) and Stitz Zeager Open Source Mathematics (https://www.stitz-zeager.com/)

- Recall that $\sec(\theta) = \dfrac{1}{\cos(\theta)}$. Arcsecant and arccosine maintain a relationship as well, though they are *not* reciprocals:
  For t in the domain of arcsecant, $\operatorname{arcsec}(t) = \arccos\left(\dfrac{1}{t}\right)$.



Exploring Arcsecant

Example 72. *Sometimes we must rely on other properties of these functions and their relations to more familiar functions to find solutions. In the following examples, we wish to find x in the range of arcsecant such that*

(a) $x = \operatorname{arcsec}(-\sqrt{2})$
Explanation We may use the relationship between arcsecant and arccosine to rewrite this equation in terms of arccosine. In other words, since

 Acknowledgements: Active Prelude to Calculus (https://activecalculus.org/prelude) and Stitz Zeager Open Source Mathematics (https://www.stitz-zeager.com/)

$\operatorname{arcsec}(y) = \arccos\left(\dfrac{1}{y}\right)$, for y in the domain of arcsecant,

$$\operatorname{arcsec}(-\sqrt{2}) = \arccos\left(-\frac{\sqrt{2}}{2}\right) = \frac{3\pi}{4}.$$

(b) $x = \operatorname{arcsec}\left(-\dfrac{2\sqrt{3}}{3}\right)$

Explanation Again, we use the relationship $\operatorname{arcsec}(y) = \arccos\left(\dfrac{1}{y}\right)$, for y in the domain of arcsecant:

$$\operatorname{arcsec}\left(-\frac{2\sqrt{3}}{3}\right) = \arccos\left(-\frac{\sqrt{3}}{2}\right),$$

since the reciprocal of $\dfrac{2\sqrt{3}}{3} = \dfrac{3}{2\sqrt{3}} = \dfrac{\sqrt{3}}{2}$. Thus, we have

$$\operatorname{arcsec}\left(-\frac{2\sqrt{3}}{3}\right) = \frac{5\pi}{6}.$$

Let's consider a couple more traditional problems combining secant and arcsecant. Remember that we must be cautious of their respective domains and ranges as with combinations of sine and arcsine and tangent and cotangent explored above.

(c) $\sec(\operatorname{arcsec}(-\sqrt{2}))$

Explanation Recall from part (a), that we already solved the equation $x = \operatorname{arcsec}(-\sqrt{2})$, and found that $x = \dfrac{3\pi}{4}$. Hence, we can now plug that in to solve our current equation:

$$\sec(\operatorname{arcsec}(-\sqrt{2})) = \sec\left(\frac{3\pi}{4}\right) = \sqrt{2}.$$

As we have observed previously with other trig inverses, we have $\sec(\operatorname{arcsec}(x)) = x$ for x in the domain of arcsecant. However, we must be careful in our application of this, as exemplified by the next example.

(d) $\operatorname{arcsec}\left(\sec\left(\dfrac{5\pi}{4}\right)\right)$

Explanation Remember that we had to restrict the domain of secant in order to define the inverse function, arcsecant. Arcsecant is thus the inverse of the *restricted* secant function, which has domain $[0, \pi/2) \cup (\pi/2, \pi]$. Observing that $\dfrac{5\pi}{4} > \pi$, and is therefore not in the domain of the restricted secant function, we cannot simply treat arcsecant as the inverse.

Instead, we begin by finding $\sec\left(\dfrac{5\pi}{4}\right)$, which is equal to $-\sqrt{2}$.

 Acknowledgements: Active Prelude to Calculus (https://activecalculus.org/prelude) and Stitz Zeager Open Source Mathematics (https://www.stitz-zeager.com/)

We now have $\operatorname{arcsec}\left(\sec\left(\dfrac{5\pi}{4}\right)\right) = \operatorname{arcsec}(-\sqrt{2})$, which we found to be equal to $\dfrac{3\pi}{4}$ in part (a). Thus, we may conclude that

$$\operatorname{arcsec}\left(\sec\left(\frac{5\pi}{4}\right)\right) = \frac{3\pi}{4}.$$

Summary

- We choose to define the restricted cosine, sine, tangent, and secant functions on the respective domains $[0, \pi]$, $[-\pi/2, \pi/2]$, $(-\pi/2, \pi/2)$, and $[0, \pi/2) \cup (\pi/2, \pi]$. On each such interval, the restricted function is strictly decreasing (cosine) or strictly increasing (sine, tangent, and secant), and thus has an inverse function. The restricted sine and cosine functions each have range $[-1, 1]$, while the restricted tangent's range is the set of all real numbers, and the restricted secant's range is $(-\infty, 1] \cup [1, \infty)$. We thus define the inverse function of each as follows:

  i. For any y such that $-1 \le y \le 1$, the arccosine of y (denoted $\arccos(y)$) is the angle t in the interval $[0, \pi]$ such that $\cos(t) = y$. That is, t is the angle whose cosine is y.

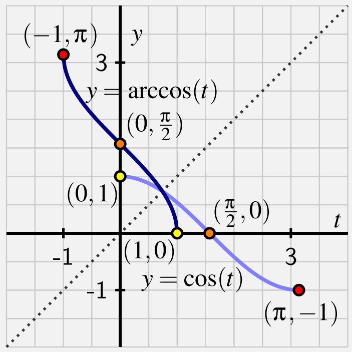
  ii. For any y such that $-1 \le y \le 1$, the arcsine of y (denoted $\arcsin(y)$) is the angle t in the interval $[-\pi/2, \pi/2]$ such that $\sin(t) = y$. That is, t is the angle whose sine is y.

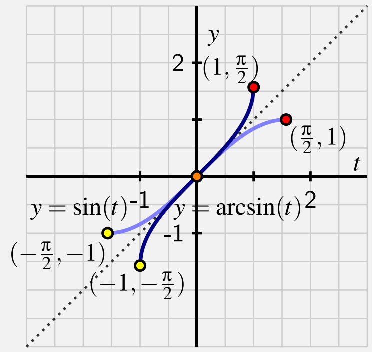
  iii. For any real number y, the arctangent of y (denoted $\arctan(y)$) is the angle t in the interval $(-\pi/2, \pi/2)$ such that $\tan(t) = y$. That is, t is the angle whose tangent is y.

  iv. For any real number y, the arcsecant of y (denoted $\operatorname{arcsec}(y)$) is the angle t in the interval $[0, \pi/2) \cup (\pi/2, \pi]$ such that $\sec(t) = y$. That is, t is the angle whose secant is y.

- The domain of $y = g^{-1}(t) = \arccos(t)$ is $[-1, 1]$ with corresponding range $[0, \pi]$, and the arccosine function is always decreasing. These facts correspond to the domain and range of the restricted cosine function and the fact that the restricted cosine function is decreasing on $[0, \pi]$.

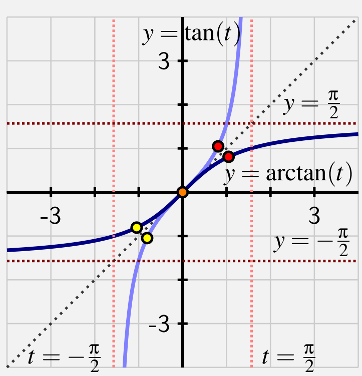
Acknowledgements: Active Prelude to Calculus (https://activecalculus.org/prelude) and Stitz Zeager Open Source Mathematics (https://www.stitz-zeager.com/)

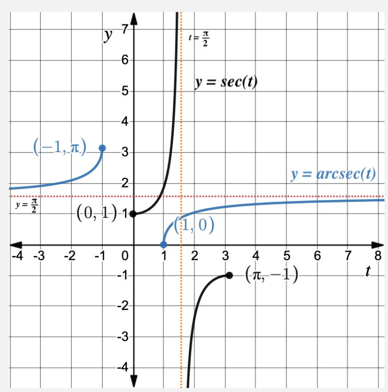




The domain of $y = f^{-1}(t) = \arcsin(t)$ is $[-1, 1]$ with corresponding range $[-\pi/2, \pi/2]$, and the arcsine function is always increasing. These facts correspond to the domain and range of the restricted sine function and the fact that the restricted sine function is increasing on $[-\pi/2, \pi/2]$.

The domain of $y = h^{-1}(t) = \arctan(t)$ is the set of all real numbers with corresponding range $(-\pi/2, \pi/2)$, and the arctangent function is always increasing. These facts correspond to the domain and range of the restricted tangent function and the fact that the restricted tangent function is increasing on $(-\pi/2, \pi/2)$.





The domain of $y = k^{-1}(t) = \text{arcsec}(t)$ is $(-\infty, -1] \cup [1, \infty)$,[a] with corresponding range $[0, \pi/2) \cup (\pi/2, \pi]$. These facts correspond to the domain and range of the restricted secant function.

[a]We note that this may also be written as $\{x : |x| \geq 1\}$.

10.3.3 Applications of Inverse Trigonometry

Motivating Questions

- How can we use inverse trigonometric functions to determine missing angles in right triangles?
- What other situations may require us to use inverse trignometric functions?

Introduction

When we learned about trig functions in Section 10, we observed that in any right triangle, if we know the measure of one additional angle and the length of one additional side, we can determine all of the other parts of the triangle. With the inverse trigonometric functions that we developed in the last two sections, we are now also able to determine the missing angles in any right triangle where we know the lengths of two sides.

While the original trigonometric functions take a particular angle as input and provide an output that can be viewed as the ratio of two sides of a right triangle, the inverse trigonometric functions take an input that can be viewed as a ratio of two sides of a right triangle and produce the corresponding angle as output. Indeed, it's imperative to remember that statements such as

$$\arccos(x) = \theta \text{ and } \cos(\theta) = x$$

say the exact same thing from two different perspectives, and that we read "$\arccos(x)$" as "the angle whose cosine is x".

Exploration Consider a right triangle that has one leg of length 3 and another leg of length $\sqrt{3}$. Let θ be the angle that lies opposite the shorter leg. Sketch a labeled picture of the triangle.

(a) What is the exact length of the triangle's hypotenuse?

(b) What is the exact value of $\sin(\theta)$?

(c) Rewrite your equation from (b) using the arcsine function in the form $\arcsin(\square) = \Delta$, where $\square$ and Δ are numerical values.

(d) What special angle from the unit circle is θ?

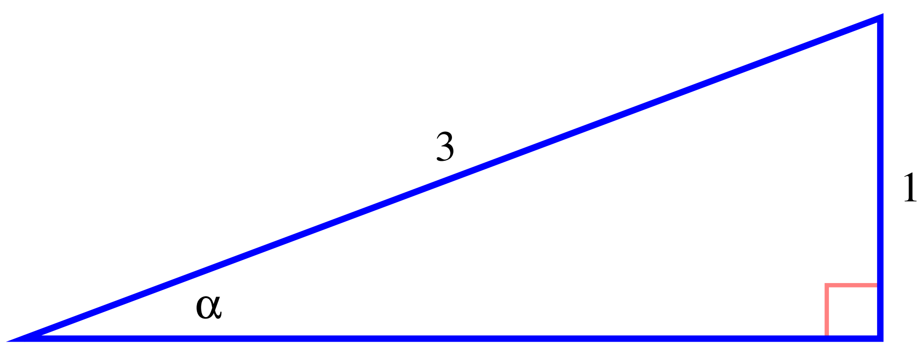
Acknowledgements: Active Prelude to Calculus (https://activecalculus.org/prelude) and Stitz Zeager Open Source Mathematics (https://www.stitz-zeager.com/)

Evaluating Inverse Trigonometric Functions

Like the trigonometric functions themselves, there are a handful of important values of the inverse trigonometric functions that we can determine exactly without the aid of a computer. For instance, we know from the unit circle that $\arcsin\left(-\frac{\sqrt{3}}{2}\right) = -\frac{\pi}{3}$, $\arccos\left(-\frac{\sqrt{3}}{2}\right) = \frac{5\pi}{6}$, and $\arctan\left(-\frac{1}{\sqrt{3}}\right) = -\frac{\pi}{6}$. In these evaluations, we have to be careful to remember that the range of the arccosine function is $[0, \pi]$, while the range of the arcsine function is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and the range of the arctangent function is $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, in order to ensure that we choose the appropriate angle that results from the inverse trigonometric function. This is why our emphasis is now turning to the *graphs* of these functions.

In addition, there are many other values at which we may wish to know the angle that results from an inverse trigonometric function. To determine such values, one can use a computational device (such as *Desmos*) in order to evaluate the function; however, in this class we leave it in the form $\arccos(a)$, as this is the exact value.

Example 73. *Consider the right triangle pictured below and assume we know that the vertical leg has length* 1 *and the hypotenuse has length* 3*. Let* α *be the angle opposite the known leg. Determine exact values for all of the remaining parts of the triangle.*



Explanation Because we know the hypotenuse and the side opposite α, we observe that $\sin(\alpha) = \frac{1}{3}$. Rewriting this statement using inverse function notation, we have equivalently that $\alpha = \arcsin\left(\frac{1}{3}\right)$, which is the exact value of α. Since this is not one of the known special angles on the unit circle, we leave it in this form.

We can now find the remaining leg's length and the remaining angle's measure. If we let x represent the length of the horizontal leg, by the Pythagorean

Acknowledgements: Active Prelude to Calculus (https://activecalculus.org/prelude) and Stitz Zeager Open Source Mathematics (https://www.stitz-zeager.com/)

Theorem we know that

$$x^2 + 1^2 = 3^2,$$

and thus $x^2 = 8$ so $x = \sqrt{8}$. Calling the remaining angle β, since $\alpha + \beta = \dfrac{\pi}{2}$, it follows that

$$\beta = \frac{\pi}{2} - \arcsin\left(\frac{1}{3}\right).$$

Example 74. *Let's consider the composite function* $h(x) = \cos(\arcsin(x))$. *Does it makes sense to consider this function? Let's think ...*
This function makes sense to consider since the arcsine function has range $[-1, 1]$*, on which we may evaluate the cosine function. In the questions that follow, we investigate how to express* h *without using trigonometric functions at all.*

(a) *What is the domain of* h*? The range of* h*?*
Explanation The domain of h is the domain of the inner function, $\arcsin(x)$, which produces values within the domain of the outer function, $\cos(z)$. As noted at the beginning, since the range of $\arcsin(x)$ is $[-1, -1]$ contained in $(-\infty, \infty)$, the domain of $\cos(z)$, the domain of h is simply the domain of $\arcsin(x)$.
The domain of h is therefore $[-1, 1]$.

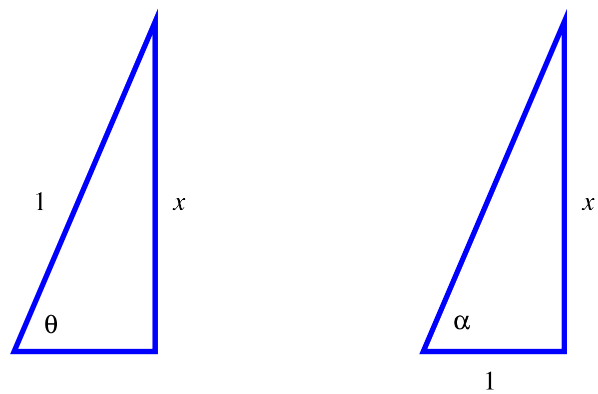
Now, the range of h will be the output of the outer function when the input is the range of the inner function. In other words, we are looking for the values that $\cos(z)$ attains on the interval $[-1, 1]$. Since cosine is symmetric about the y-axis, this is the same as the values attained by $\cos(z)$ on the interval $[0, 1]$. Thus, we have a range of $[\cos(1), 1]$.

(b) *Since the arcsine function produces a value we can consider as an angle, let's say that* $\theta = \arcsin(x)$*, so that* θ *is the angle whose sine is* x*. By definition, we can picture* θ *as an angle in a right triangle with hypotenuse* 1 *and a vertical leg of length* x*, as shown in the image on the left below. Use the Pythagorean Theorem to determine the length of the horizontal leg as a function of* x*.*
Explanation First we recall the Pythagorean Theorem, $a^2 + b^2 = c^2$, where c is the hypotenuse of a right triangle with legs of lengths a, b. Hence, in this instance, let's denote the length of the horizontal leg by y, so we have $y^2 + x^2 = 1^2$. In other words

$$y = \sqrt{1 - x^2},$$

since a triangle leg will have positive length.

Acknowledgements: Active Prelude to Calculus (https://activecalculus.org/prelude) and Stitz Zeager Open Source Mathematics (https://www.stitz-zeager.com/)



The right triangle on the left corresponds to the angle $\theta = \arcsin(x)$. *The right triangle on the right corresponds to the angle* $\alpha = \arctan(x)$.

(c) *What is the value of* $\cos(\theta)$ *as a function of* x*? What have we shown about* $h(x) = \cos(\arcsin(x))$*?*
Explanation Here, we use the results of part (b). Since we know that $\cos(\theta)$ is $\dfrac{adj}{hyp}$ and $\theta = \arcsin(x)$,

$$\cos(\theta) = \frac{y}{1} = \sqrt{1 - x^2}.$$

From this we see that

$$h(x) = \cos(\arcsin(x)) = \sqrt{1 - x^2}.$$

(d) *How about the function* $p(x) = \cos(\arctan(x))$*? How can you reason similarly to write* p *in a way that doesn't involve any trigonometric functions at all? (Hint: let* $\alpha = \arctan(x)$ *and consider the right triangle on the right above.)*
Explanation We can now use a similar approach to determine p as an algebraic function of x. Let $\alpha = \arctan(x)$, so that $p(x) = \cos(\arctan(x)) = \cos(\alpha)$.

In the second triangle we must find the value of the hypotenuse, call it y. Then

$$y^2 = 1^2 + x^2 \text{ which implies } y = \sqrt{1 + x^2}.$$

Now, $\cos(\alpha) = \dfrac{1}{y} = \dfrac{1}{\sqrt{1 + x^2}}$. Therefore,

$$p(x) = \cos(\arctan(x)) = \frac{1}{\sqrt{1 + x^2}}.$$

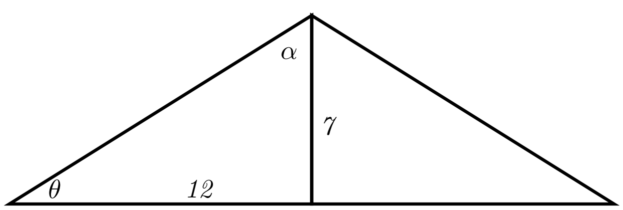
 Acknowledgements: Active Prelude to Calculus (https://activecalculus.org/prelude) and Stitz Zeager Open Source Mathematics (https://www.stitz-zeager.com/)

Using Inverse Trig in Applied Contexts

Now that we have developed the (restricted) sine, cosine, tangent, and secant functions and their respective inverses, in any setting in which we have a right triangle together with one side length and any one additional piece of information (another side length or a non-right angle measurement), we can determine all of the remaining pieces of the triangle. In the example that follows and the homework, we explore these possibilities in a variety of different applied contexts.

Example 75. *A roof is being built with a "7-12 pitch." This means that the roof rises* 7 *inches vertically for every* 12 *inches of horizontal span; in other words, the slope of the roof is* $\frac{7}{12}$.

(a) *What is the exact measure of the angle the roof makes with the horizontal?*
Explanation Looking at a side view of the house, we may divide the triangle of the roof in half to get a right triangle with legs 7 and 12 feet long. We want to find the angle of inclination (θ in the diagram below), which satisfies the equation $\tan(\theta) = \frac{7}{12}$. In other words, we wish to find $\theta = \arctan\left(\frac{7}{12}\right)$. As in Example 1, this does not match one of our known values, so we leave it in this form since this is the *exact* value.



The image above is a side-view of the roof.

(b) *What is the exact measure of the angle at the peak of the roof (made by the front and back portions of the roof that meet to form the ridge)?*
Explanation This will be double the angle at the top of the right triangle we used for part (a), since we had bisected this angle to form the right triangle. We now wish to find the angle α satisfying $\tan(\alpha) = \frac{12}{7}$. In other words, we are looking for $\alpha = \arctan\left(\frac{12}{7}\right)$. Once again, this is not a common angle, so it is the *exact* value. We need double this angle, so $2\alpha = 2\arctan\left(\frac{12}{7}\right)$ is our solution.

Acknowledgements: Active Prelude to Calculus (https://activecalculus.org/prelude) and Stitz Zeager Open Source Mathematics (https://www.stitz-zeager.com/)

Exploration On a baseball diamond (which is a square with 90-foot sides), the third baseman fields the ball right on the line from third base to home plate and 10 feet away from third base (towards home plate). Give exact solutions without using a computational device.

(a) When he throws the ball to first base, what angle does the line the ball travels make with the first base line?

(b) What angle does it make with the third base line? Draw a well-labeled diagram.

(c) What angles arise if he throws the ball to second base instead?

Exploration Give exact solutions without using a computational device. A camera is tracking the launch of a SpaceX rocket. The camera is located 4000' from the rocket's launching pad, and the camera elevates in order to keep the rocket in focus.

(a) At what angle is the camera tilted when the rocket is 3000' off the ground?

Now, rather than considering the rocket at a fixed height of 3000 ft, let its height vary and call the rocket's height h.

(b) Determine the camera's angle, θ as a function of h, and compute the average rate of change of θ on the intervals $[3000, 3500]$, $[5000, 5500]$, and $[7000, 7500]$.

(c) What do you observe about how the camera angle is changing?

Further Exploration

When composing trigonometric functions with inverse trigonometric functions, the expressions can often be rewritten as algebraic expressions of x. We will see two examples of this below.

Example 76. *Rewrite the following values as algebraic expressions of x and give the domain on which these equivalences are valid.*

(a) $\cos(\arctan(x))$.
Explanation Recall that we found this expression in Example 2, part (d), to be
$$\cos(\arctan(x)) = \frac{1}{\sqrt{1+x^2}}.$$
Now, we must find the domain for which this is true.

Acknowledgements: Active Prelude to Calculus (https://activecalculus.org/prelude) and Stitz Zeager Open Source Mathematics (https://www.stitz-zeager.com/)

We start by checking the domain of the outer function, $\cos(y)$. Since the domain of the cosine function is all real numbers, we do not have any restrictions to consider here. Thus, our only concern is the domain of the arctangent function, which is also the real line. Thus, we see that

$$\cos(\arctan(x)) = \frac{1}{\sqrt{1+x^2}} \text{ for all } x \text{ in } (-\infty, \infty).$$

(b) $\sin(\arccos(2x))$.

Explanation $\sin(\arccos(2x)) = \sqrt{1-4x^2}$ given x in $\left[-\frac{1}{2}, \frac{1}{2}\right]$

Again, to see this, we begin by setting $t = \arccos(2x)$, so that $\cos(t) = 2x$ for t in the domain of restricted cosine, $[0, \pi]$. In other words, we have $\cos(t) = 2x$ for t in $[0, \pi]$, and must find a formula for $\sin(t)$. Now, we must relate sine and cosine, for which we use the well-known trigonometric identity $\sin^2(t) + \cos^2(t) = 1$. Re-writing this to solve our equation, we see that we have $\sin^2(t) + (2x)^2 = 1$, which is equivalent to

$$\sin(t) = \pm\sqrt{1-4x^2}.$$

Since sine is positive on the interval $[0, \pi]$, where we defined t, we choose the positive square root, and observe that $\sin(\arccos(2x)) = \sqrt{1-4x^2}$, as desired.

Finally, to establish the domain on which this equivalence holds, we recall that the domain of arccosine is $[-1, 1]$. Since we consider $\arccos(2x)$, we want $2x$ in $[-1, 1]$. This is equivalent to x in $\left[-\frac{1}{2}, \frac{1}{2}\right]$, and so our work is done.

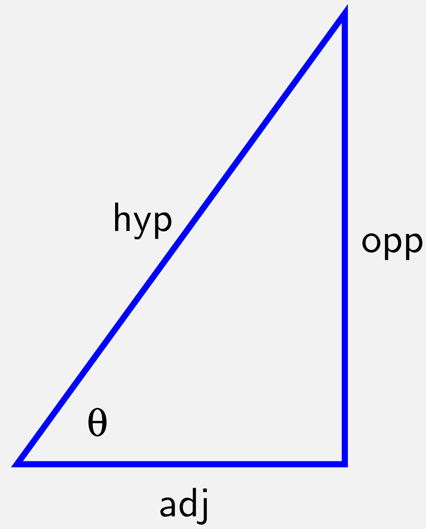
A Note on Triangles We can now use trigonometry to find angles of right triangles if we know the side lengths and side lengths of right triangles if we know the angles. You might be wondering, "What about triangles that are not right triangles? Can we use trig to learn anything about those?" It turns out that the Law of Sines and the Law of Cosines gives use a way to analyze other triangles beyond just right triangles using trig functions. For more information about this topic, see Laws of Sines and Cosines by Katherine Yoshiwara .

Summary Anytime we know two side lengths in a right triangle, we can use one of the inverse trigonometric functions to determine the measure of one of the non-right angles. For instance, if we know the values of opp and adj in the triangle pictured below, then since

$$\tan(\theta) = \frac{\text{opp}}{\text{adj}},$$

it follows that $\theta = \arctan\left(\frac{\text{opp}}{\text{adj}}\right)$.

Acknowledgements: Active Prelude to Calculus (https://activecalculus.org/prelude) and Stitz Zeager Open Source Mathematics (https://www.stitz-zeager.com/)



If we instead know the hypotenuse and one of the two legs, we can use either the arcsine or arccosine function accordingly.

Similarly, we may use this relationship along with the Pythagorean Theorem to find algebraic expressions for compositions of trig functions with trig inverses (see Example 2). The trig identities we learned in Section 10 are also useful to rewrite the compositions of functions as algebraic expressions (see Example 4).

Part 11

Preparing for Calculus

11.1 Linear Systems of Equations

Learning Objectives

- From Systems to Solutions
  - What is a system of equations?
  - What is a solution to a system?
  - Solving systems via graphs
- Solving Systems Algebraically
  - Eliminating Variables
  - Substitution Method
- Applications of Systems of Equations
  - Word problems
  - Mixture Problems

11.1.1 From Systems to Solutions

Motivating Questions

- What is a system of equations?
- What is a solution to a system?
- How can we solve systems of equations using graphs?

Introduction

We have already seen many techniques for solving equations. Until now, however, we have only solved equations of the form $f(x) = 0$ for the variable x. In this section, we will consider equations with more than one variable and discuss how to solve them.

Consider a peculiar grocery store where the prices of all the items for sale are not listed, and you only find out the total cost of your purchase. Say you buy 6 mangos and 3 bananas, and your total cost is 9 dollars. Assume all the mangos cost the same amount and all the bananas cost the same amount. Without making any more purchases, is it possible find out how much a mango and a banana cost on their own?

Let's create an equation to describe this situation. Let x be a variable representing the cost of a mango, and let y be a variable representing the cost of a banana. Then, the equation

$$6x + 3y = 9$$

represents that buying 6 mangos at a cost of x dollars and 3 bananas at a cost of y dollars yields a total cost of 9 dollars.

You might have noticed that plugging $x = 1$ and $y = 1$ into the equation gives us a true statement, so you might conclude that mangos and bananas both cost 1 dollar. However, notice that plugging $x = 1.20$ and $y = 0.60$ into the equation also gives us a true statement, so it's also possible that mangos cost \$1.20 and bananas cost \$0.60. Even more worrying is that $x = 0$ and $y = 3$ also gives us a solution to the equation: is this store peculiar enough to be giving away mangos for free and charging \$3 per banana?

Examining the equation we set up can give us more insight. Let's rearrange the equation to solve for y in terms of x:

$$\begin{aligned} 6x + 3y &= 9 \\ 3y &= 9 - 6x \\ y &= 3 - 2x. \end{aligned}$$

Now it becomes clearer what's going on. Whatever x is, we can find a value of y that satisfies our original equation. No matter the cost of a single mango, there's a way to price the bananas so that our equation is true! This means it's impossible to find the price of a single mango or a single banana with the information you've been given. We need more data!

Systems of linear equations

In order to collect more information, you go back to the store and buy 2 mangos and 2 banana for a total cost of \$3.20. This can be modeled by the equation

$$2x + 2y = 3.2.$$

Keep in mind that this x and y are the same x and y from before, so in order to find the cost of a mango and a banana, we must find x and y that satisfy both equations

$$\begin{cases} 6x + 3y = 9 \\ 2x + 2y = 3.2 \end{cases}$$

at the same time. This coupling of two (or more) linear equations is called a *system of linear equations*.

Definition A **linear equation of two variables** is an equation of the form

$$a_1x + a_2y = c,$$

where a_1, a_2, and c are real numbers and at least one of a_1 and a_2 is nonzero.

A **system of linear equations of two variables** is a collection of two or more linear equations of two variables.

We say a **solution** to a system of linear equations of two variables is a point (x, y) satisfying all equations in the system.

It is clear that some systems of equations have solutions, and some do not. Those which have solutions are called *consistent*, those with no solution are called *inconsistent*.

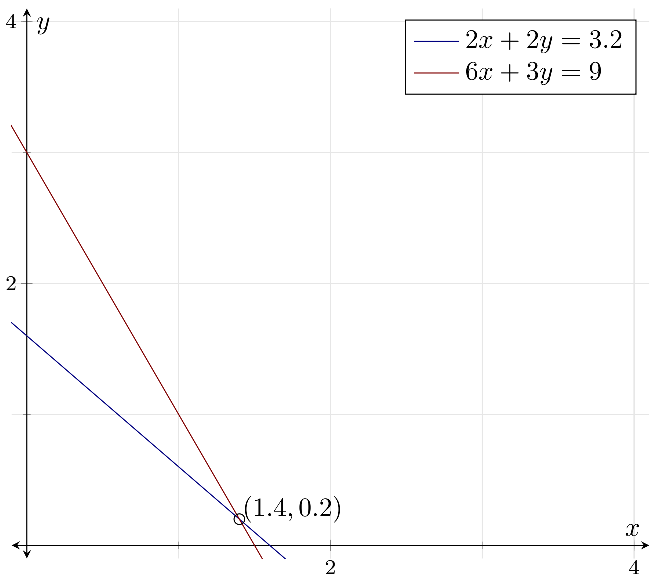
The key to identifying linear equations is to note that the variables involved are to the first power and that the coefficients of the variables are numbers. Some examples of equations which are non-linear are $x^2 + y = 1$, $xy = 5$ and $e^{2x} + \ln(y) = 1$. Note that we can still have systems of non-linear equations, but they can be much more difficult to solve.

Finding solutions graphically

Let's return to our example from earlier and try to find a solution to the system of linear equation:

$$\begin{cases} 6x + 3y = 9 \\ 2x + 2y = 3.2 \end{cases}.$$

We want to find x and y satisfying both equations in the system. If x and y satisfy $6x + 3y = 9$, then the point (x, y) lies on the graph of $6x + 3y = 9$. Similarly, if x and y satisfy $2x+2y = 3.2$, then the point (x, y) lies on the graph of $2x + 2y = 3.2$. Therefore, to find any solutions, we can look at the graphs of $6x + 3y = 9$ and $2x + 2y = 3.2$, and see if there are any points that lie at the intersection of the two graphs:



By inspecting the graph, we see that these two lines intersect only at $(1.4, 0.2)$, so the only solution to the system is $x = 1.4$ and $y = 0.2$.

In context, this means that mangos cost \$1.40 each and bananas cost \$0.20 each. Note that in order to have exactly one solution to our system of linear equations in two variables, we needed the system to have two equations.

Note that not every system of linear equations will have one solution. If the graphs of the two equations are parallel, they will never intersect, so there won't be any solutions. Additionally, if the two equations are represented by the same graph, there will be infinitely many intersection points, and therefore, infinitely many solutions.

Next, we will see some methods for solving systems of equations algebraically.

11.1.2 Solving Systems of Equations Algebraically

Substitution

In the previous section, we focused on solving systems of equations by graphing. In addition to being time consuming, graphing can be an awkward method to determine the exact solution when the solution has large numbers, fractions, or decimals. There are two symbolic methods for solving systems of linear equations, and in this section we will use one of them: substitution.

Example 77. *In 2014, the New York Times posted the following about the movie, "The Interview":*

> *"The Interview" generated roughly* \$15 *million in online sales and rentals during its first four days of availability, Sony Pictures said on Sunday. Sony did not say how much of that total represented* \$6 *digital rentals versus* \$15 *sales. The studio said there were about two million transactions overall.*

A few days later, Joey Devilla cleverly pointed out in his blog, that there is enough information given to find the amount of sales versus rentals.

Explanation Using algebra, we can write a system of equations and solve it to find the two quantities. Although since the given information uses approximate values, the solutions we will find will only be approximations too.

First, we will define variables. We need two variables, because there are two unknown quantities: how many sales there were and how many rentals there were. Let r be the number of rental transactions and let s be the number of sales transactions.

If you are unsure how to write an equation from the background information, use the units to help you. The units of each term in an equation must match because we can only add like quantities. Both r and s are in transactions. The article says that the total number of transactions is 2 million. So our first equation will add the total number of rental and sales transactions and set that equal to 2 million. Our equation is:

$$(r\,\text{transactions}) + (s\,\text{transactions}) = 2{,}000{,}000\,\text{transactions}$$

Without the units:

$$r + s = 2{,}000{,}000$$

The price of each rental was \$6. That means the problem has given us a *rate* of $6\,\dfrac{\text{dollars}}{\text{transaction}}$ to work with. The rate unit suggests this should be multiplied by something measured in transactions. It makes sense to multiply by r, and then the number of dollars generated from rentals was $6r$. Similarly, the price of each sale was \$15, so the revenue from sales was $15s$. The total revenue was \$15 million, which we can represent with this equation:

$$\left(6\,\frac{\text{dollars}}{\text{transaction}}\right)(r\text{ transactions}) + \left(15\,\frac{\text{dollars}}{\text{transaction}}\right)(s\text{ transactions}) = \$15{,}000{,}000$$

Without the units:

$$6r + 15s = 15{,}000{,}000$$

Here is our system of equations:

$$\begin{array}{rcrcr} r & + & s & = & 2{,}000{,}000 \\ 6r & + & 15s & = & 15{,}000{,}000 \end{array}$$

To solve the system, we will use the **substitution** method. The idea is to use *one* equation to find an expression that is equal to r but, cleverly, does not use the variable "r." Then, substitute this for r into the *other* equation. This leaves you with *one* equation that only has *one* variable.

The first equation from the system is an easy one to solve for r:

$$\begin{aligned} r + s &= 2{,}000{,}000 \\ r &= 2{,}000{,}000 - s \end{aligned}$$

This tells us that the expression $2{,}000{,}000 - s$ is equal to r, so we can *substitute* it for r in the second equation:

$$\begin{aligned} 6r + 15s &= 15{,}000{,}000 \\ 6({\color{blue}2{,}000{,}000 - s}) + 15s &= 15{,}000{,}000 \end{aligned}$$

Now we have an equation with only one variable, s, which we will solve for:

$$\begin{aligned} 6(2{,}000{,}000 - s) + 15s &= 15{,}000{,}000 \\ 12{,}000{,}000 - 6s + 15s &= 15{,}000{,}000 \\ 12{,}000{,}000 + 9s &= 15{,}000{,}000 \\ 9s &= 3{,}000{,}000 \\ \frac{9s}{\color{blue}9} &= \frac{3{,}000{,}000}{\color{blue}9} \\ s &= 333{,}333.\overline{3} \end{aligned}$$

At this point, we know that $s = 333{,}333.\overline{3}$. This tells us that out of the 2 million transactions, roughly 333,333 were from online sales. Recall that we solved the first equation for r, and found $r = 2{,}000{,}000 - s$.

$$\begin{aligned} r &= 2{,}000{,}000 - s \\ r &= 2{,}000{,}000 - {\color{blue}333{,}333.\overline{3}} \\ r &= 1{,}666{,}666.\overline{6} \end{aligned}$$

To check our answer, we will see if $s = 333{,}333.\overline{3}$ and $r = 1{,}666{,}666.\overline{6}$ make the original equations true:

$$\begin{aligned}
r + s &= 2{,}000{,}000 \\
1{,}666{,}666.\overline{6} + 333{,}333.\overline{3} &= 2{,}000{,}000 \\
2{,}000{,}000 &= 2{,}000{,}000 \\
6r + 15s &= 15{,}000{,}000 \\
6\left(1{,}666{,}666.\overline{6}\right) + 15\left(333{,}333.\overline{3}\right) &= 15{,}000{,}000 \\
10{,}000{,}000 + 5{,}000{,}000 &= 15{,}000{,}000
\end{aligned}$$

In summary, there were roughly 333,333 copies sold and roughly 1,666,667 copies rented.

Elimination

We just learned how to solve a system of linear equations using substitution above. Now, we will learn a second symbolic method for solving systems of linear equations.

Example 78. *Alicia has* \$1000 *to give to her two grandchildren for New Year's. She would like to give the older grandchild* \$120 *more than the younger grandchild, because that is the cost of the older grandchild's college textbooks this term. How much money should she give to each grandchild?*

Explanation To answer this question, we will demonstrate a new technique. You may have a very good way for finding how much money Alicia should give to each grandchild, but right now we will try to see this new method.

Let A be the dollar amount she gives to her older grandchild, and B be the dollar amount she gives to her younger grandchild. (As always, we start solving a word problem like this by defining the variables, including their units.) Since the total she has to give is \$1000, we can say that $A + B = 1000$. And since she wants to give \$120 more to the older grandchild, we can say that $A - B = 120$. So we have the system of equations:

$$\begin{aligned}
A + B &= 1000 \\
A - B &= 120
\end{aligned}$$

We could solve this system by substitution as we learned previously but there is an easier method. If we add together the *left* sides from the two equations, it should equal the sum of the *right* sides:

$$\begin{array}{rcl}
A + B & = & 1000 \\
+A - B & & +120
\end{array}$$

So we have:

$$2A = 1120$$

Note that the variable B is eliminated. This happened because the $+B$ and the $-B$ perfectly cancel each other out when they are added. With only one variable left, it doesn't take much to finish:

$$\begin{aligned} 2A &= 1120 \\ A &= 560 \end{aligned}$$

To finish solving this system of equations, we need the value of B. For now, an easy way to find B is to substitute in our value of A into one of the original equations:

$$\begin{aligned} A + B &= 1000 \\ {\color{blue}560} + B &= 1000 \\ B &= 440 \end{aligned}$$

To check our work, substitute $A = 560$ and $B = 440$ into the original equations:

$$\begin{aligned} A + B &= 1000 \\ {\color{blue}560} + {\color{blue}440} &= 1000 \\ 1000 &= 1000 \\ A - B &= 120 \\ {\color{blue}560} - {\color{blue}440} &= 120 \\ 120 &= 120 \end{aligned}$$

This confirms that our solution is correct. In summary, Alicia should give \$560 to her older grandchild, and \$440 to her younger grandchild.

This method for solving the system of equations in the example above worked because B and $-B$ add to zero. Once the B-terms were eliminated we were able to solve for A. This method is called the **elimination method**. Some people call it the **addition method**, because we added the corresponding sides from the two equations to eliminate a variable.

If neither variable can be immediately eliminated, we can still use this method but it will require that we first adjust one or both of the equations. Let's look at an example where we need to adjust one of the equations.

Example 79. *Solve the system of equations using the elimination method.*

$$\begin{array}{rcrcr} 3x & - & 4y & = & 2 \\ 5x & + & 8y & = & 18 \end{array}$$

Explanation To start, we want to see whether it will be easier to eliminate x or y. We see that the coefficients of x in each equation are 3 and 5, and the coefficients of y are -4 and 8. Because 8 is a multiple of 4 and the coefficients already have opposite signs, the y variable will be easier to eliminate.

To eliminate the y terms, we will multiply each side of the first equation by 2 so that we will have $-8y$. We can call this process scaling the first equation by 2.

$$\begin{array}{rcrcr} 2 \cdot (3x & - & 4y) & = & 2 \cdot (2) \\ 5x & + & 8y & = & 18 \end{array}$$

$$\begin{array}{rcrcr} 6x & - & 8y & = & 4 \\ 5x & + & 8y & = & 18 \end{array}$$

We now have an equivalent system of equations where the y-terms can be eliminated:

$$\begin{array}{rcr} 6x - 8y & = & 4 \\ +5x + 8y & & +18 \end{array}$$

So we have:

$$\begin{aligned} 11x &= 22 \\ x &= 2 \end{aligned}$$

To solve for y, we can substitute 2 for x into either of the original equations or the new one. We use the first original equation, $3x - 4y = 2$:

$$\begin{aligned} 3x - 4y &= 2 \\ 3(2) - 4y &= 2 \\ 6 - 4y &= 2 \\ -4y &= -4 \\ y &= 1 \end{aligned}$$

Our solution is $x = 2$ and $y = 1$. We will check this in both of the original equations:

$$\begin{aligned} 5x + 8y &= 18 \\ 5(2) + 8(1) &= 18 \\ 10 + 8 &= 18 \\ 3x - 4y &= 2 \\ 3(2) - 4(1) &= 2 \\ 6 - 4 &= 2 \end{aligned}$$

The solution to this system is $(2, 1)$ and the solution set is $\{(2, 1)\}$.

11.1.3 Applications of Systems of Equations

Example 80. *Two Different Interest Rates Notah made some large purchases with his two credit cards one month and took on a total of* \$8,400 *in debt from the two cards. He didn't make any payments the first month, so the two credit card debts each started to accrue interest. That month, his Visa card charged* 2% *interest and his Mastercard charged* 2.5% *interest. Because of this, Notah's total debt grew by* \$178*. How much money did Notah charge to each card?*
Explanation To start, we will define two variables based on our two unknowns. Let v be the amount charged to the Visa card (in dollars) and let m be the amount charged to the Mastercard (in dollars).

To determine our equations, notice that we are given two different totals. We will use these to form our two equations. The total amount charged is \$8,400 so we have:

$$(v\,\text{dollars}) + (m\,\text{dollars}) = \$8400$$

Or without units:

$$v + m = 8400$$

The other total we were given is the total amount of interest, \$178, which is also in dollars. The Visa had v dollars charged to it and accrues 2% interest. So $0.02v$ is the dollar amount of interest that comes from using this card. Similarly, $0.025m$ is the dollar amount of interest from using the Mastercard. Together:

$$0.02(v\,\text{dollars}) + 0.025(m\,\text{dollars}) = \$178$$

Or without units:

$$0.02v + 0.025m = 178$$

As a system, we write:

$$\begin{array}{ccccc} v & + & m & = 8400 \\ 0.02v & + & 0.025m & = 178 \end{array}$$

To solve this system by substitution, notice that it will be easier to solve for one of the variables in the first equation. We'll solve that equation for v:

$$\begin{aligned} v + m &= 8400 \\ v &= 8400 - m \end{aligned}$$

Now we will substitute $8400 - m$ for v in the second equation:

$$\begin{aligned}
0.02v + 0.025m &= 178\\
0.02(8400 - m) + 0.025m &= 178\\
168 - 0.02m + 0.025m &= 178\\
168 + 0.005m &= 178\\
0.005m &= 10\\
\frac{0.005m}{0.005} &= \frac{10}{0.005}\\
m &= 2000
\end{aligned}$$

Lastly, we can determine the value of v by using the earlier equation where we isolated v:

$$\begin{aligned}
v &= 8400 - m\\
v &= 8400 - 2000\\
v &= 6400
\end{aligned}$$

In summary, Notah charged \$6400 to the Visa and \$2000 to the Mastercard. We should check that these numbers work as solutions to our original system ¡em¿and¡/em¿ that they make sense in context. (For instance, if one of these numbers were negative, or was something small like \$0.50, they wouldn't make sense as credit card debt.)

Mixture Problems

The next two examples are called **mixture problems**, because they involve mixing two quantities together to form a combination and we want to find out how much of each quantity to mix.

Example 81. *Mixing Solutions with Two Different Concentrations LaVonda is a meticulous bartender and she needs to serve* 600 *milliliters of Rob Roy, an alcoholic cocktail that is* 34% *alcohol by volume. The main ingredients are scotch that is* 42% *alcohol and vermouth that is* 18% *alcohol. How many milliliters of each ingredient should she mix together to make the concentration she needs?*
Explanation The two unknowns are the quantities of each ingredient. Let s be the amount of scotch (in mL) and let v be the amount of vermouth (in mL).

One quantity given to us in the problem is 600 mL. Since this is the total volume of the mixed drink, we must have:

$$(s\,\mathrm{mL}) + (v\,\mathrm{mL}) = 600\,\mathrm{mL}$$

Or without units:

$$s + v = 600$$

To build the second equation, we have to think about the alcohol concentrations for the scotch, vermouth, and Rob Roy. It can be tricky to think about percentages like these correctly. One strategy is to focus on the *amount* (in mL) of *alcohol* being mixed. If we have s milliliters of scotch that is 42% alcohol, then $0.42s$ is the actual *amount* (in mL) of alcohol in that scotch. Similarly, $0.18v$ is the amount of alcohol in the vermouth. And the final cocktail is 600 mL of liquid that is 34% alcohol, so it has $0.34(600) = 204$ milliliters of alcohol. All this means:

$$0.42(s\,\text{mL}) + 0.18(v\,\text{mL}) = 204\,\text{mL}$$

Or without units:

$$0.42s + 0.18v = 204$$

So our system is:

$$\begin{aligned} s \quad &+ \quad v &= 600 \\ 0.42s \quad &+ \quad 0.18v &= 204 \end{aligned}$$

To solve this system, we'll solve for s in the first equation:
$$\begin{aligned} s + v &= 600 \\ s &= 600 - v \end{aligned}$$

And then substitute s in the second equation with $600 - v$:

$$\begin{aligned} 0.42s + 0.18v &= 204 \\ 0.42(600 - v) + 0.18v &= 204 \\ 252 - 0.42v + 0.18v &= 204 \\ 252 - 0.24v &= 204 \\ -0.24v &= -48 \\ \frac{-0.24v}{-0.24} &= \frac{-48}{-0.24} \\ v &= 200 \end{aligned}$$

As a last step, we will determine s using the equation where we had isolated s:

$$\begin{aligned} s &= 600 - v \\ s &= 600 - 200 \\ s &= 400 \end{aligned}$$

In summary, LaVonda needs to combine 400 mL of scotch with 200 mL of vermouth to create 600 mL of Rob Roy that is 34% alcohol by volume.

As a check for the previous example, we can use estimation to see that our solution is reasonable. Since LaVonda is making a 34% solution, she would need to use more of the 42% concentration than the 18% concentration, because 34% is closer to 42% than to 18%. This agrees with our answer because we found that she needed 400 mL of the 42% solution and

200 mL of the 18% solution. This is an added check that we have found reasonable answers.

Example 82. *Mixing a Coffee Blend Desi owns a coffee shop and they want to mix two different types of coffee beans to make a blend that sells for* \$12.50 *per pound. They have some coffee beans from Columbia that sell for* \$9.00 *per pound and some coffee beans from Honduras that sell for* \$14.00 *per pound. How many pounds of each should they mix to make* 30 *pounds of the blend?*
Explanation Before we begin, it may be helpful to try to estimate the solution. Let's compare the three prices. Since \$12.50 is between the prices of \$9.00 and \$14.00, this mixture is possible. Now we need to estimate the amount of each type needed. The price of the blend (\$12.50 per pound) is closer to the higher priced beans (\$14.00 per pound) than the lower priced beans (\$9.00 per pound). So we will need to use more of that type. Keeping in mind that we need a total of 30 pounds, we roughly estimate 20 pounds of the \$14.00 Honduran beans and 10 pounds of the \$9.00 Columbian beans. How good is our estimate? Next we will solve this exercise exactly.

To set up our system of equations we define variables, letting C be the amount of Columbian coffee beans (in pounds) and H be the amount of Honduran coffee beans (in pounds).

The equations in our system will come from the total amount of beans and the total cost. The equation for the total amount of beans can be written as:

$$(C\,\mathrm{lb}) + (H\,\mathrm{lb}) = 30\,\mathrm{lb}$$

Or without units:

$$C + H = 30$$

To build the second equation, we have to think about the cost of all these beans. If we have C pounds of Columbian beans that cost \$9.00 per pound, then $9C$ is the cost of those beans in dollars. Similarly, $14H$ is the cost of the Honduran beans. And the total cost is for 30 pounds of beans priced at \$12.50 per pound, totaling $12.5(30) = 37.5$ dollars. All this means:

$$\left(9\,\tfrac{\text{dollars}}{\text{lb}}\right)(C\,\mathrm{lb}) + \left(14\,\tfrac{\text{dollars}}{\text{lb}}\right)(H\,\mathrm{lb}) = \left(12.50\,\tfrac{\text{dollars}}{\text{lb}}\right)(30\,\mathrm{lb})$$

Or without units and carrying out the multiplication on the right:

$$9C + 14H = 37.5$$

Now our system is:

$$\begin{aligned} C \quad + \quad H \quad &= 30 \\ 9C \quad + \quad 14H \quad &= 37.50 \end{aligned}$$

To solve the system, we'll solve the first equation for C:

$$\begin{aligned} C + H &= 30 \\ C &= 30 - H \end{aligned}$$

Next, we'll substitute C in the second equation with $30 - H$:

$$\begin{aligned} 9C + 14H &= 375 \\ 9(30 - H) + 14H &= 375 \\ 270 - 9H + 14H &= 375 \\ 270 + 5H &= 375 \\ 5H &= 105 \\ H &= 21 \end{aligned}$$

Since $H = 21$, we can conclude that $C = 9$.

In summary, Desi needs to mix 21 pounds of the Honduran coffee beans with 9 pounds of the Columbian coffee beans to create this blend. Our estimate at the beginning was pretty close, so we feel this answer is reasonable.

11.2 Non-linear Systems

Learning Objectives

- Famous Formulas
  - Reviewing famous functions
  - Introducing conic sections and their formulas
- Solving Non-linear Systems Graphically
  - What is a non-linear system?
  - Finding solutions graphically
  - What can we say about when solutions exist?
- Eliminating Variables
  - Algebra of reducing multivariable systems to a single equation
  - Systems created from functions
  - Reviewing some algebra and misconceptions

11.2.1 Famous Formulas

Introduction

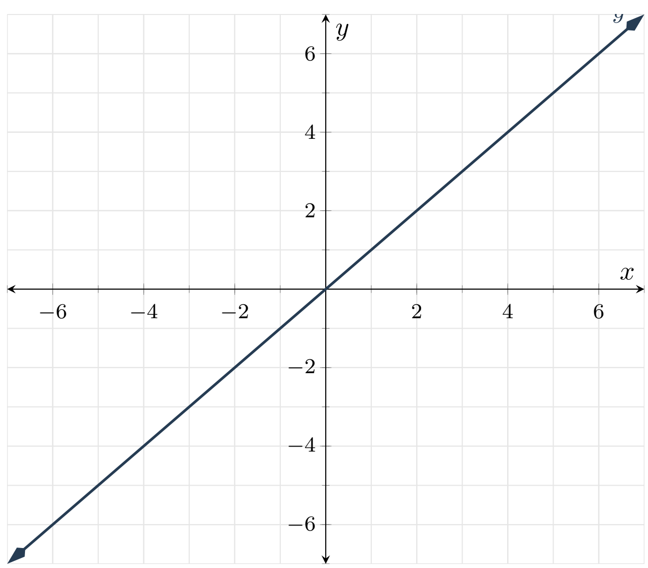
As a review, we go over the list of famous functions from earlier. Then, we move to a discussion of conic sections.

Linear Functions

Recall that the graph of a linear function is a line.

Example 83. *A prototypical example of a linear function is*

$$y = x.$$



Important Values of $y = x$	
x	y
-2	-2
-1	-1
0	0
1	1
2	2

In general, linear functions can be written as $y = mx + b$ where m and b can be any numbers. We learned that m represents the slope, and b is the y-coordinate of the y-intercept. You can play with changing the values of m and b on the graph using Desmos and see how that changes the line.

Desmos link: `https://www.desmos.com/calculator/japnhapzvn`

Table 1: Properties of Linear Functions $y = mx + b$

Periodic?	If $m = 0$
Odd?	If $b = 0$
Even?	If $m = 0$
One-to-one/invertible	If $b = 0$

Note that any real number can be plugged into $f(x) = mx + b$, so the domain of linear functions is $(-\infty, \infty)$. Unless $m = 0$, we can find a y such that $y = mx+b$, so the range of linear functions with $m \neq 0$ is $(-\infty, \infty)$. If $m = 0$, then the only output of the linear function is b, so its range is $\{b\}$.

Table 2: Domain and Range of Linear Functions $y = mx + b$

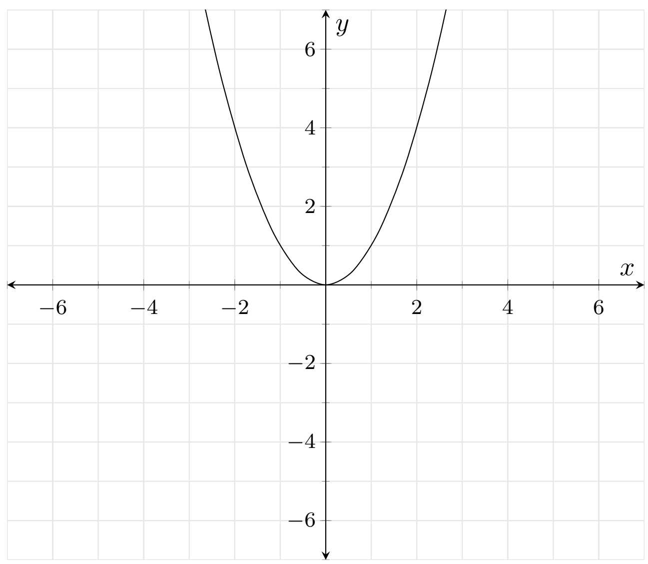
Domain	$(-\infty, \infty)$
Range	If $m \neq 0$, $(-\infty, \infty)$; if $m = 0$, $\{b\}$

Quadratic Functions

Recall that the graph of a quadratic function is a parabola.

Example 84. *A prototypical example of a quadratic function is*

$$y = x^2.$$



Important Values of $y = x^2$	
x	y
-2	4
-1	1
0	0
1	1
2	4

In general, quadratic functions can be written as $y = ax^2 + bx + c$ where a, b, and c can be any numbers. You can play with changing the values of a, b, and c on the graph using Desmos and see how that changes the parabola.

Desmos link: `https://www.desmos.com/calculator/nmlghfrws9`

Note that any real number can be plugged into $f(x) = ax^2+bx+c$, so the domain of quadratic functions is $(-\infty, \infty)$. In Chapter 4, we saw that all quadratic

Table 3: Properties of Quadratic Functions $y = ax^2 + bx + c$

Periodic?	If $a = 0$ and $b = 0$
Odd?	If $a = 0$, $b = 0$, and $c = 0$
Even?	If $b = 0$
One-to-one/invertible	If $a = 0$ and $c = 0$

functions have a vertex form $f(x) = d(x-h)^2 + k$, where the vertex is at (h, k). If $d > 0$, all points above the vertex, that is $[k, \infty)$ are in the range of the quadratic, and if $d < 0$, all points below the vertex, that is $(\infty, k]$ are in the range of the quadratic.

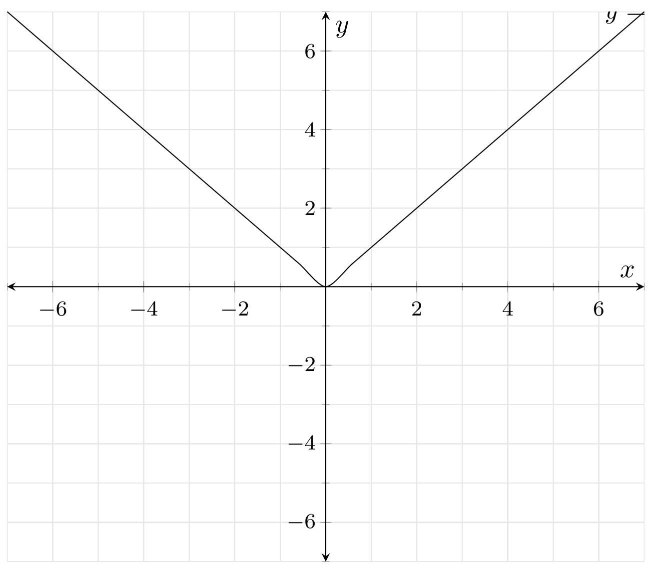
Table 4: Domain and Range of Quadratic Functions $y = d(x - h)^2 + k$

Domain	$(-\infty, \infty)$
Range	If $d > 0$, $[k, \infty)$; if $d < 0$, $(\infty, k]$

Absolute Value

Another important type of function is the absolute value function. This is the function that takes all y-values and makes them positive. The absolute value function is written as

$$y = |x|.$$



Important Values of $y = \|x\|$	
x	y
-2	2
-1	1
0	0
1	1
2	2

Table 5: Properties of The Absolute Value Function $y = |x|$

Periodic?	No
Odd?	No
Even?	Yes
One-to-one/invertible	No

Note that any real number has an absolute value, so the domain of the absolute value function is $(-\infty, \infty)$. Furthermore, by looking at the graph, we can see that all non-negative numbers are in the range of the absolute value function.

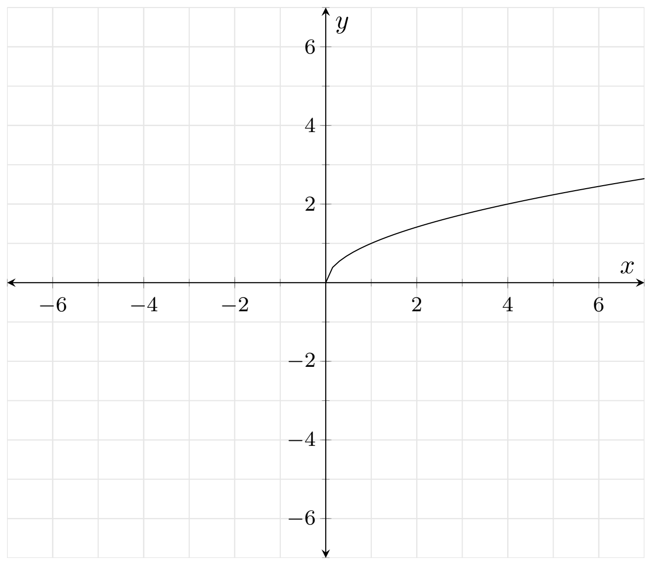
Table 6: Domain and Range of The Absolute Value Function $y = |x|$

Domain	$(-\infty, \infty)$
Range	$[0, \infty)$

Square Root

Another famous function is the square root function,

$$y = \sqrt{x}.$$



Important Values of $y = \sqrt{x}$	
x	y
0	0
1	1
4	2
9	3
25	5

Table 7: Properties of The Square Root Function $y = \sqrt{x}$

Periodic?	No
Odd?	No
Even?	No
One-to-one/invertible	Yes

Note that only non-negative numbers have square roots, so the domain of the square root function is $[0, \infty)$. Furthermore, by looking at the graph, we can see that all non-negative numbers are in the range of the square root function. Algebraically, we can say that for any non-negative y, $\sqrt{(y^2)} = y$, so y is in the range of the square root function.

Table 8: Domain and Range of The Square Root Function $y = \sqrt{x}$

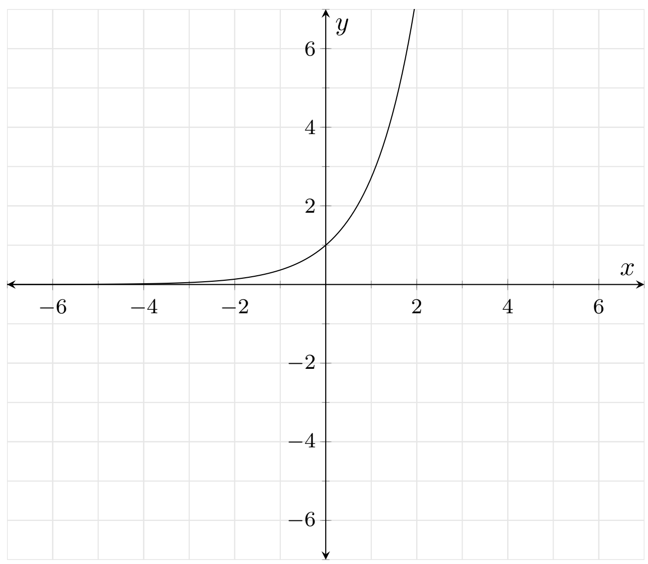
Domain	$[0, \infty)$
Range	$[0, \infty)$

Exponential

Another famous function is the exponential growth function,

$$y = e^x.$$

Here e is the mathematical constant known as Euler's number. $e \approx 2.71828..$



Important Values of $y = e^x$	
x	y
0	1
1	e
−1	$\frac{1}{e}$

In general, we can talk about exponential functions of the form $y = b^x$ where b is a positive number not equal to 1. You can play with changing the values of b on the graph using Desmos and see how that changes the graph. Pay particular attention to the difference between $b > 1$ and $0 < b < 1$.

Desmos link: `https://www.desmos.com/calculator/qsmvb7tiex`

Note that the domain of the exponential functions is $(-\infty, \infty)$. Furthermore, by looking at the graph, we can see that all non-negative numbers are in the range of the exponential functions.

Table 9: Properties of The Exponential Functions $y = b^x$

Periodic?	No
Odd?	No
Even?	No
One-to-one/invertible	Yes

Table 10: Domain and Range of The Exponential Functions $y = b^x$

Domain	$(-\infty, \infty)$
Range	$[0, \infty)$

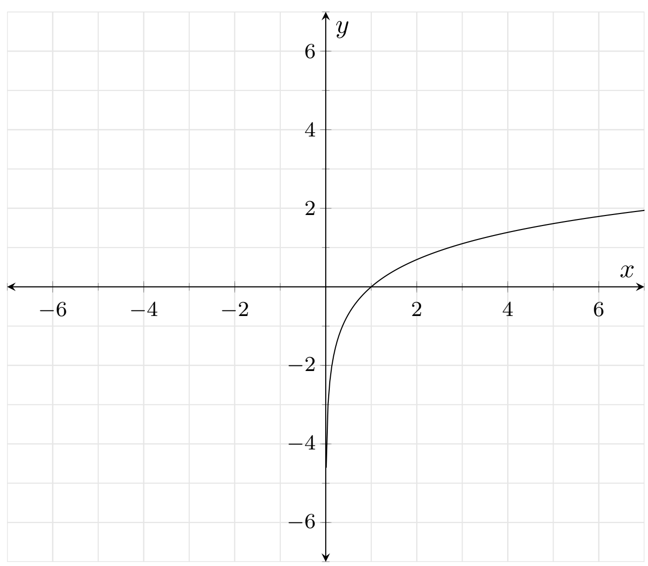
Logarithm

Another group of famous functions are logarithms.

Example 85. *The most famous logarithm function is*

$$y = \ln(x) = \log_e(x).$$

Here e is the mathematical constant known as Euler's number. $e \approx 2.71828$*.*



Important Values of $y = ln(x)$	
x	y
0	undefined
$\frac{1}{e}$	-1
1	0
e	1

You may notice that the table of values for $y = \ln(x)$ and $y = e^x$ are similiar. This is becase these two functions are interconnected. We will explore this more later in the course.

In general, we can talk about logarithmic functions of the form $y = \log_b(x)$ where b is a positive number not equal to 1. You can play with changing the values of b on the graph using Desmos and see how that changes the graph. Pay particular attention to the difference between $b > 1$ and $0 < b < 1$.

Desmos link: https://www.desmos.com/calculator/lxllnpdi6w

Table 11: Properties of The Logarithm Functions $y = \log_b(x)$

Periodic?	No
Odd?	No
Even?	No
One-to-one/invertible	Yes

Note that since the logarithm is the inverse of the exponential, the domain of the logarithms is the range of the exponentials: $[0, \infty)$. Furthermore, the range of the logarithms is the range of the exponentials: $(-\infty, \infty)$.

Table 12: Domain and Range of The Logarithms $y = \log_b(x)$

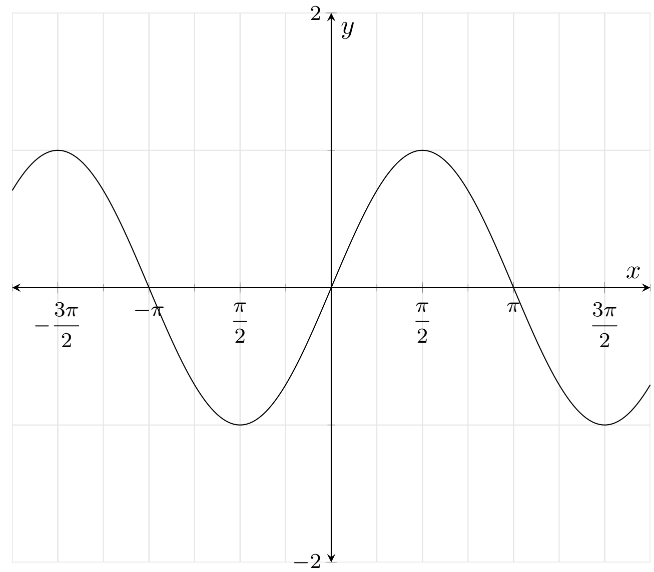
Domain	$[0, \infty)$
Range	$(-\infty, \infty)$

Sine

Another important function is the sine function,

$$y = \sin(x).$$

This function comes from trigonometry. In the table below we will use another mathematical constant, π ("pi" pronounced pie). $\pi \approx 3.14159$.



Important Values of $y = \sin(x)$	
x	y
$-\pi$	0
$-\frac{\pi}{2}$	-1
0	0
$\frac{\pi}{2}$	1
π	0
$\frac{3\pi}{2}$	-1
2π	0

Note that the domain of the sine function is $(-\infty, \infty)$. Furthermore, by looking at the graph, we can see that its range is $[-1, 1]$

Table 13: Properties of The Sine Function $y = \sin(x)$

Periodic?	Yes, with period 2π
Odd?	Yes
Even?	No
One-to-one/invertible	No

Table 14: Domain and Range of The Sine Function $y = \sin(x)$

Domain	$(-\infty, \infty)$
Range	$[-1, 1]$

In general, we can consider $y = a\sin(bx)$. You can play with changing the values of a and b on the graph using Desmos and see how that changes the graph.

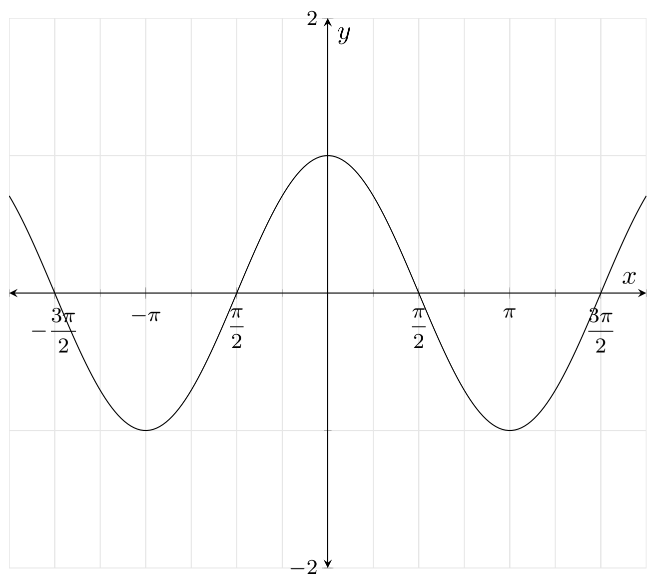
Desmos link: https://www.desmos.com/calculator/vkxzcfv2aq

Cosine

A function introduced in Section 3-2 is the cosine function,

$$y = \cos(x).$$

As with sine, the cosine function comes from trigonometry. In the table below we will again use π.



Important Values of $y = \cos(x)$

x	y
$-\pi$	-1
$-\frac{\pi}{2}$	0
0	1
$\frac{\pi}{2}$	0
π	-1
$\frac{3\pi}{2}$	0
2π	1

As mentioned earlier, the cosine function is even and periodic with period 2π. Since it is periodic, however, it cannot be one-to-one, since its values repeat. We summarize some information in Table 15.

Table 15: Properties of The Cosine Function $y = \cos(x)$

Periodic?	Yes, with period 2π
Odd?	No
Even?	Yes
One-to-one/invertible	No

Note that the domain of the cosine function is $(-\infty, \infty)$. Furthermore, by looking at the graph, we can see that its range is $[-1, 1]$

Table 16: Domain and Range of The Cosine Function $y = \cos(x)$

Domain	$(-\infty, \infty)$
Range	$[-1, 1]$

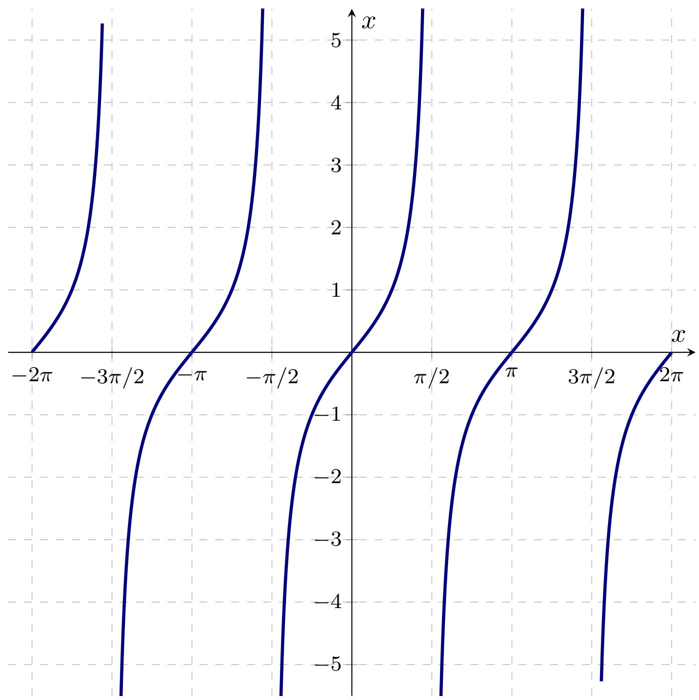
In general, we can consider $y = a\cos(bx)$. You can play with changing the values of a and b on the graph using Desmos and see how that changes the graph.

Desmos link: https://www.desmos.com/calculator/kvmz1kt19n

Tangent

A function introduced in Section 4-1 is the tangent function,

$$y = \tan(x).$$



Important Values of $y = \tan(x)$	
x	y
$-\pi$	0
$-\dfrac{\pi}{2}$	undefined
0	0
$\dfrac{\pi}{2}$	undefined
π	0
$\dfrac{3\pi}{2}$	undefined
2π	0

As mentioned earlier, the tangent function is odd and periodic with period π.

Since it is periodic, however, it cannot be one-to-one, since its values repeat. We summarize some information in Table 17.

Table 17: Properties of The Tangent Function $y = \tan(x)$

Periodic?	Yes, with period π
Odd?	Yes
Even?	No
One-to-one/invertible	No

Note that the domain of the tangent function is all real numbers except for odd multiples of $\frac{\pi}{2}$, since tangent is undefined at those places. Furthermore, by looking at the graph, we can see that its range is $(-\infty, \infty)$.

Table 18: Domain and Range of The Tangent Function $y = \tan(x)$

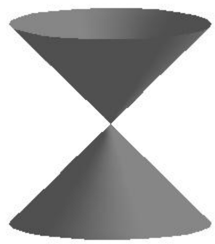
Domain	$\cdots \cup \left(-\frac{5\pi}{2}, -\frac{3\pi}{2}\right) \cup \left(-\frac{3\pi}{2}, -\frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \frac{3\pi}{2}\right) \cup \left(\frac{3\pi}{2}, \frac{5\pi}{2}\right) \cup \cdots$
Range	$(-\infty, \infty)$

In general, we can consider $y = a\tan(bx)$. You can play with changing the values of a and b on the graph using Desmos and see how that changes the graph.

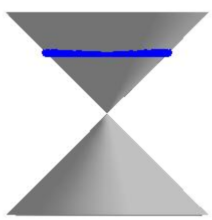
Desmos link: `https://www.desmos.com/calculator/1je3xt6hag`

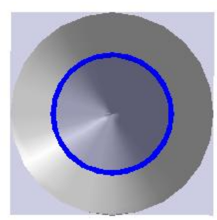
Conic Sections

In this section, we study the **Conic Sections** - literally 'sections of a cone'. Imagine a double-napped cone as seen below being 'sliced' by a plane.

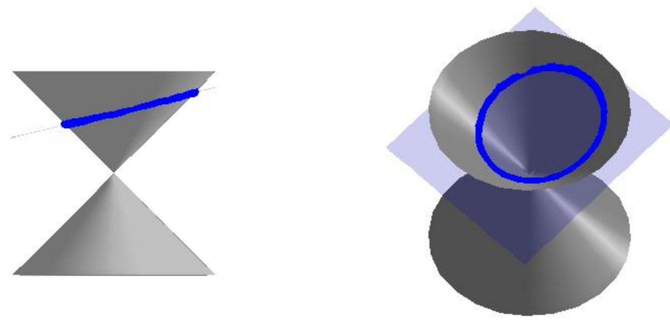


If we slice the cone with a horizontal plane the resulting curve is a **circle**.

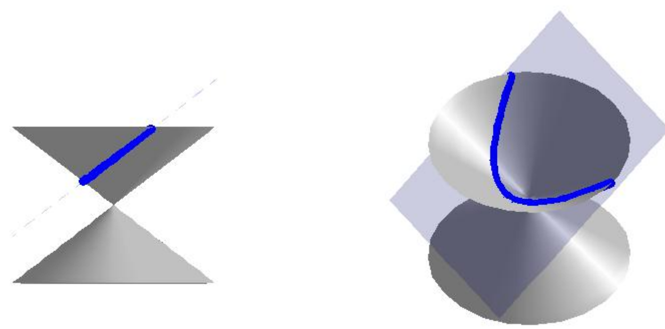




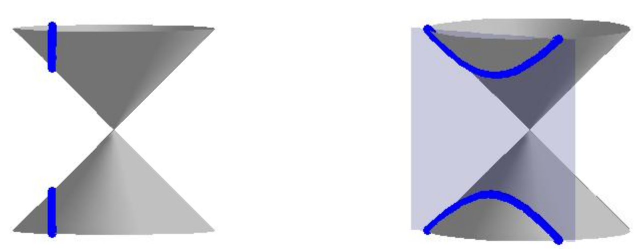
Tilting the plane ever so slightly produces an **ellipse**.



If the plane cuts parallel to the cone, we get a **parabola**.

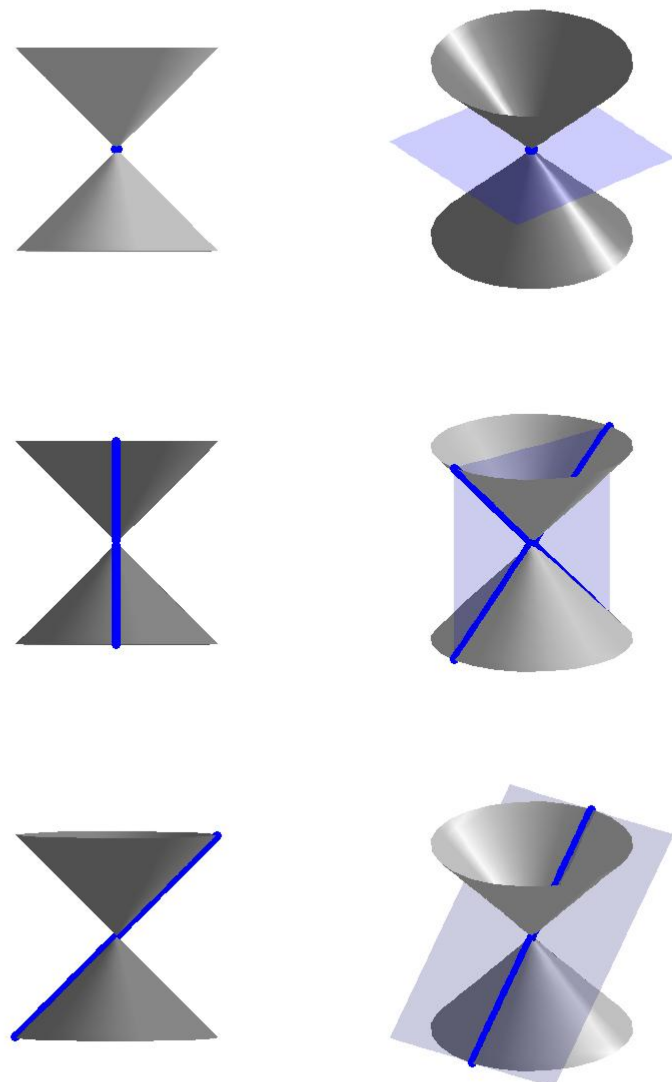


If we slice the cone with a vertical plane, we get a **hyperbola**.



For a wonderful animation describing the conics as intersections of planes and cones, see Dr. Louis Talman's Mathematics Animated Website.

If the slicing plane contains the vertex of the cone, we get the so-called 'degenerate' conics: a point, a line, or two intersecting lines.

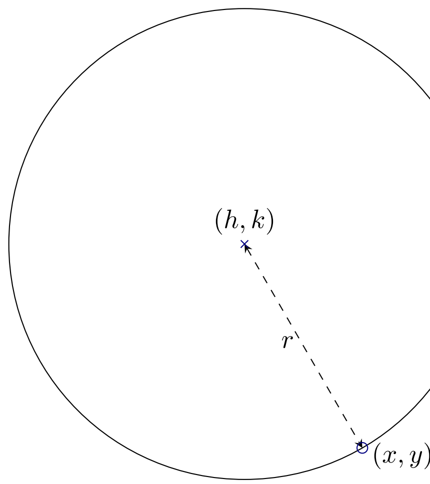


We will focus the discussion on the non-degenerate cases: circles, parabolas, ellipses, and hyperbolas, in that order. It's not necessary to memorize the description of each conic section. We'd just like you to understand that each conic section is the graph of an equation which can be rearranged into a certain *standard equation*. This standard equation is useful, because it allows us to say something about various geometric properties of the graph. In addition, we will only discuss conic sections centered at the origin.

Circles

Recall from Geometry that a circle can be determined by fixing a point (called the *center*) and a positive number (called the *radius*) as follows.

> **Definition** A **circle** with center (h, k) and radius $r > 0$ is the set of all points (x, y) in the plane whose distance to (h, k) is r.



We express this relationship algebraically using the Distance Formula as

$$r = \sqrt{(x-h)^2 + (y-k)^2}$$

By squaring both sides of this equation, we get an equivalent equation (since $r > 0$) which gives us the standard equation of a circle.

> **Definition** **The standard equation of a circle** with center (h, k) and radius $r > 0$ is $(x-h)^2 + (y-k)^2 = r^2$.

This is the first example of a standard equation. If we are given a standard equation of a circle, we can easily find its center and its radius, which is all that we need to be able to draw the circle in the xy-plane. In other courses, we would spend a lot of time taking an equation, converting it into a standard equation, recognizing it as the standard equation of a conic section, and then using the information provided by the standard equation to graph the relation. For our purposes, we only need to know that this process can be done.

We close this section with the most important circle in all of mathematics: the *unit circle*.

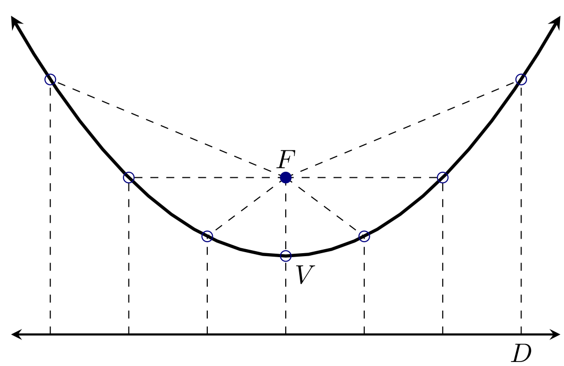
Definition The **unit circle** is the circle centered at $(0, 0)$ with a radius of 1. The standard equation of the unit circle is $x^2 + y^2 = 1$.

As you will soon see, the unit circle is central to the study of trigonometry.

Parabolas

We know that parabolas are the graphs of quadratic functions. To our surprise and delight, we may also define parabolas in terms of distance.

Definition Let F be a point in the plane and D be a line not containing F. A **parabola** is the set of all points equidistant from F and D. The point F is called the **focus** of the parabola and the line D is called the **directrix** of the parabola. The **vertex** is the point on the parabola closest to the focus.



Each dashed line from the point F to a point on the curve has the same length as the dashed line from the point on the curve to the line D. The point suggestively labeled V is, as you should expect, the vertex. Notice that the focus F is not actually a point on the parabola, but only serves to help in its construction.

As with circles, there is a standard equation for parabolas.

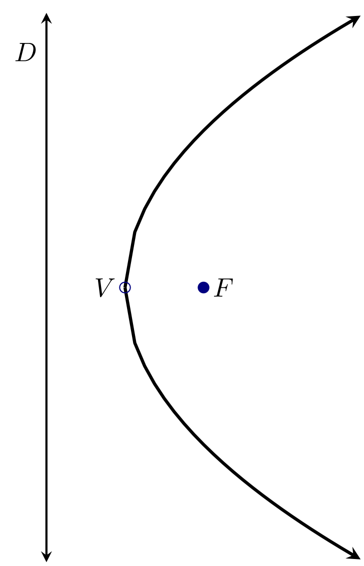
Definition The **standard equation of a parabola** which opens up or down with vertex (h, k) and focal length $|p|$ is

$$(x - h)^2 = 4p(y - k)$$

If $p > 0$, the parabola opens upwards; if $p < 0$, it opens downwards. The **focal length** of the parabola is the distance from the focus to the vertex.

Notice that in the standard equation of the parabola above, only one of the variables, x, is squared. This is a quick way to distinguish an equation of a parabola from that of a circle because in the equation of a circle, both variables are squared.

Recall from our earlier discussion of inverse functions that interchanging the roles of x and y results in reflecting the graph across the line $y = x$. Therefore, if we interchange the roles of x and y, we can produce 'horizontal' parabolas: parabolas which open to the left or to the right. The directrices (plural of 'directrix') of such animals would be vertical lines and the focus would either lie to the left or to the right of the vertex, as seen below.



Definition **The standard equation of a parabola** that opens to the left or right with vertex (h, k) and focal length $|p|$ is

$$(y - k)^2 = 4p(x - h)$$

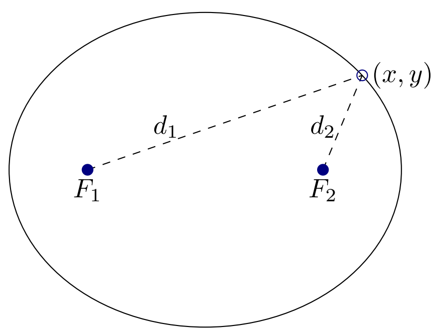
If $p > 0$, the parabola opens to the right; if $p < 0$, it opens to the left.

Ellipses

In the definition of a circle, we fixed a point called the *center* and considered all of the points which were a fixed distance r from that one point. For our next conic section, the ellipse, we fix two distinct points and a distance d to use in our definition.

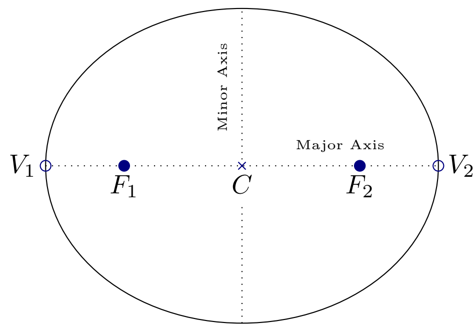
> **Definition** Given two distinct points F_1 and F_2 in the plane and a fixed distance d, an **ellipse** is the set of all points (x, y) in the plane such that the sum of each of the distances from F_1 and F_2 to (x, y) is d. The points F_1 and F_2 are called the **foci** (the plural of 'focus') of the ellipse.

In the figure below, d_1 is the distance from (x, y) to F_1, and d_2 is the distance from (x, y) to F_2. Since (x, y) is on the ellipse, $d_1 + d_2 = d$ for some fixed d.



We may imagine taking a length of string and anchoring it to two points on a piece of paper. The curve traced out by taking a pencil and moving it so the string is always taut is an ellipse. Notice again that the foci are not actually points on the ellipse, but only serve to help in its construction.

The *center* of the ellipse is the midpoint of the line segment connecting the two foci. The *major axis* of the ellipse is the line segment connecting two opposite ends of the ellipse which also contains the center and foci. The *minor axis* of the ellipse is the line segment connecting two opposite ends of the ellipse which contains the center but is perpendicular to the major axis. The major axis is always the longer of the two segments. The *vertices* of an ellipse are the points of the ellipse which lie on the major axis. Notice that the center is also the midpoint of the major axis, hence it is the midpoint of the vertices. In pictures we have,



There is also a standard equation for ellipses.

Definition For positive unequal numbers a and b, **the standard equation of an ellipse** with center (h, k) is

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

First note that the values a and b determine how far in the x and y directions, respectively, one counts from the center to arrive at points on the ellipse. Also take note that if $a > b$, then we have an ellipse whose major axis is horizontal, and hence, the foci lie to the left and right of the center. In this case, the distance from the center to the focus, c, can be found by $c = \sqrt{a^2 - b^2}$. If $b > a$, the roles of the major and minor axes are reversed, and the foci lie above and below the center. In this case, $c = \sqrt{b^2 - a^2}$. In either case, c is the distance from the center to each focus, and $c = \sqrt{\text{bigger denominator} - \text{smaller denominator}}$. Finally, it is worth mentioning that if we take the standard equation of a circle and divide both sides by r^2, we get

Definition **The alternate standard equation of a circle** with center (h, k) and radius $r > 0$ is

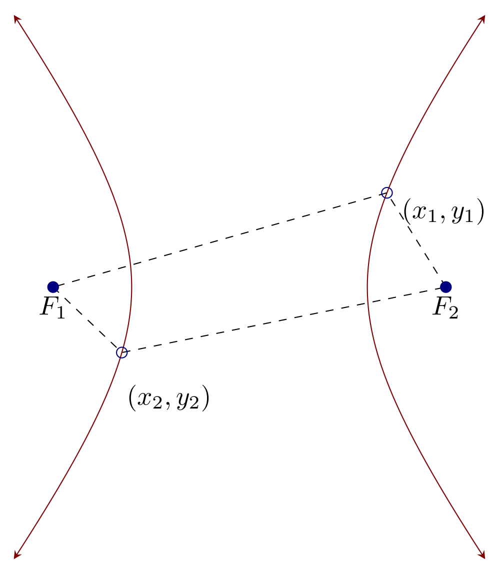
$$\frac{(x-h)^2}{r^2} + \frac{(y-k)^2}{r^2} = 1$$

Notice the similarity between the two equations. Both involve a sum of squares equal to 1; the difference is that with a circle, the denominators are the same, and with an ellipse, they are different. If we take a transformational approach, we can consider both equations as shifts and stretches of the unit circle $x^2 + y^2 = 1$. Replacing x with $(x - h)$ and y with $(y - k)$ causes the usual horizontal and vertical shifts. Replacing x with $\frac{x}{a}$ and y with $\frac{y}{b}$ causes the usual vertical and horizontal stretches. In other words, it is perfectly fine to think of an ellipse as the deformation of a circle in which the circle is stretched farther in one direction than the other.

Hyperbolas

In the definition of an ellipse, we fixed two points called foci and looked at points whose distances to the foci always *added* to a constant distance d. Those prone to syntactical tinkering may wonder what, if any, curve we'd generate if we replaced *added* with *subtracted*. The answer is a hyperbola.

> **Definition** Given two distinct points F_1 and F_2 in the plane and a fixed distance d, a **hyperbola** is the set of all points (x, y) in the plane such that the difference of each of the distances from F_1 and F_2 to (x, y) is d. The points F_1 and F_2 are called the **foci** of the hyperbola.



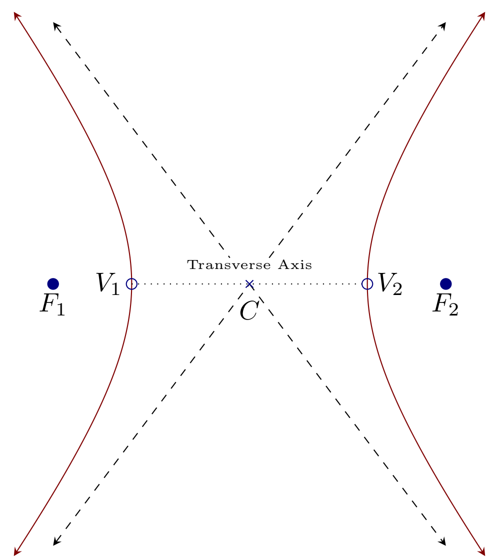
In the figure above:

$$\text{the distance from } F_1 \text{ to } (x_1, y_1) - \text{the distance from } F_2 \text{ to } (x_1, y_1) \quad = \quad d$$

and

$$\text{the distance from } F_2 \text{ to } (x_2, y_2) - \text{the distance from } F_1 \text{ to } (x_2, y_2) \quad = \quad d$$

Note that the hyperbola has two parts, called *branches*. The *center* of the hyperbola is the midpoint of the line segment connecting the two foci. The *transverse axis* of the hyperbola is the line segment connecting two opposite ends of the hyperbola which also contains the center and foci. The *vertices* of a hyperbola are the points of the hyperbola which lie on the transverse axis. In addition, there are lines called *asymptotes* which the branches of the hyperbola approach for large x and y values. They serve as guides to the graph. In pictures,



The above hyperbola has center C, foci F_1 and F_2, and vertices V_1 and V_2. The asymptotes are represented by dashed lines.

The *conjugate axis* of a hyperbola is the line segment through the center which is perpendicular to the transverse axis and has the same length as the line segment through a vertex which connects the asymptotes.

As with all the other conic sections, we have a standard equation for hyperbolas.

Definition For positive numbers a and b, **the equation of a hyperbola** opening left and right with center (h, k) is

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$

If the roles of x and y were interchanged, then the hyperbola's branches would open upwards and downwards and we would get a 'vertical' hyperbola.

Definition For positive numbers a and b, **the equation of a hyperbola** opening upwards and downwards with center (h, k) is

$$\frac{(y-k)^2}{b^2} - \frac{(x-h)^2}{a^2} = 1$$

The values of a and b determine how far in the x and y directions, respectively, one counts from the center to determine the rectangle through which the asymptotes pass. In both cases, the distance from the center to the foci, c, can be found by the formula $c = \sqrt{a^2 + b^2}$. Lastly, note that we can quickly distinguish the equation of a hyperbola from that of a circle or ellipse because the hyperbola formula involves a *difference* of squares where the circle and ellipse formulas both involve the *sum* of squares.

11.2.2 Solving Non-linear Systems Graphically

Motivating Questions

- What is a non-linear system of equations?
- How can we find solutions graphically?
- What can we say about when solutions exist?

Introduction

In this section, we study systems of non-linear equations. Unlike the systems of linear equations for which we have developed several algorithmic solution techniques, there is no general algorithm to solve systems of non-linear equations. Moreover, all of the usual hazards of non-linear equations like extraneous solutions and unusual function domains are once again present. Along with the tried and true techniques of substitution and elimination, we shall often need equal parts tenacity and ingenuity to see a problem through to the end. You may find it necessary to review topics throughout the text which pertain to solving equations involving the various functions we have studied thus far.

What are non-linear systems of equations?

The key to identifying non-linear equations is to note that the variables involved are not necessarily to the first power, and the coefficients of the variables may not just be real numbers. Some examples of equations which are non-linear are $x^2 + y = 1$, $xy = 5$ and $e^{2x} + \ln(y) = 1$. An example of a non-linear system of equations is given by

$$\begin{cases} x^2 + y^2 & = 4 \\ 4x^2 - 9y^2 & = 36 \end{cases}.$$

Note that this system is non-linear because the variables x and y are raised to the second power.

Another example of a non-linear system of equations is given by

$$\begin{cases} x^2 + y^2 & = 4 \\ y - 2x & = 0 \end{cases}.$$

Even though y and x are both raised to the first power in the second equation above, the first equation still contains second powers of variables, so this is a non-linear system.

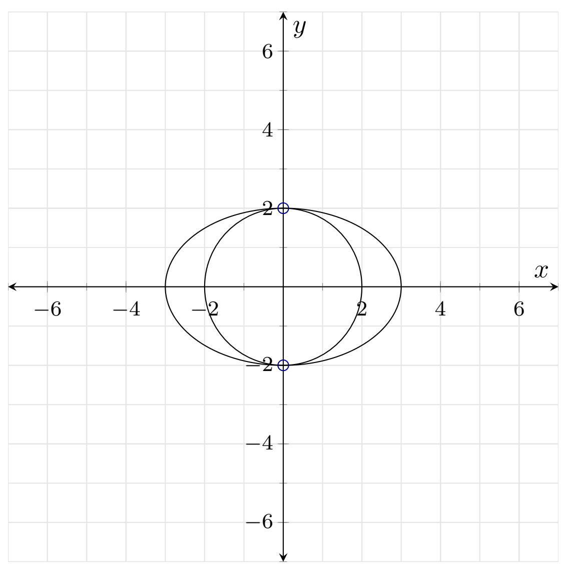
Solving systems graphically

Finding solutions to non-linear systems is the same concept as finding solutions to linear systems. This means that we can also think about finding solutions as finding intersections points of the graphs of the equations in our system.

Example 86. *Find all solutions to the following system of equations:*

$$\begin{cases} x^2 + y^2 &= 4 \\ 4x^2 + 9y^2 &= 36 \end{cases}.$$

Explanation We sketch the graphs of both equations and look for their points of intersection. The graph of $x^2 + y^2 = 4$ is a circle centered at $(0, 0)$ with a radius of 2, whereas the graph of $4x^2 + 9y^2 = 36$, when written in the standard form $\dfrac{x^2}{9} + \dfrac{y^2}{4} = 1$ can be recognized as an ellipse centered at $(0, 0)$ with a major axis along the x-axis of length 6 and a minor axis along the y-axis of length 4. This is illustrated in the figure below.



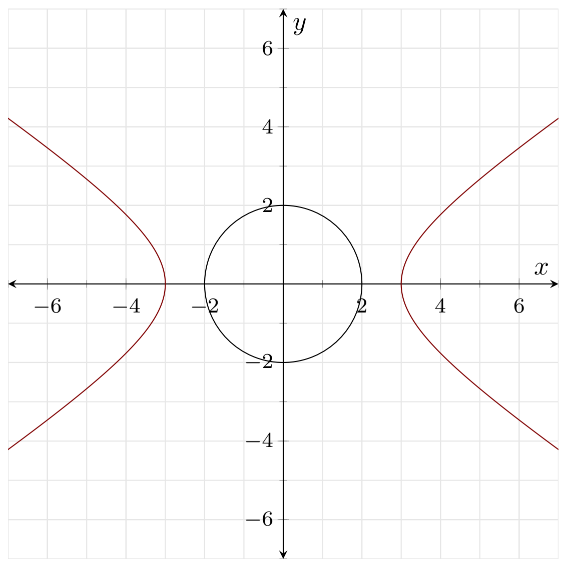
We see from the figure that the two graphs intersect at their y-intercepts only, $(0, 2)$ and $(0, -2)$. Recalling that points of intersection correspond to solutions to the system of equations, $(0, 2)$ and $(0, -2)$ are the only solutions to the system.

Example 87. *Find all solutions to the following system of equations:*

$$\begin{cases} x^2 + y^2 &= 4 \\ 4x^2 - 9y^2 &= 36 \end{cases}.$$

Explanation First, notice that this system only differs from the previous example in that it has a minus sign in front of the $9y^2$ in the bottom equation.

We again sketch the graphs of both equations and look for their points of intersection. The graph of $x^2 + y^2 = 4$ is a circle centered at $(0, 0)$ with a radius of 2, as in the previous example. However, the graph of $4x^2 - 9y^2 = 36$, when written in the standard form $\dfrac{x^2}{9} - \dfrac{y^2}{4} = 1$ can be recognized as a hyperbola centered at $(0, 0)$ opening to the left and right with a transverse axis of length 6 and a conjugate axis of length 4. This is illustrated in the figure below.



We see that the circle and the hyperbola have no points in common. Recalling that points of intersection correspond to solutions to the system of equations, we say that the system has no solutions.

Note that we can characterize systems of nonlinear equations as being consistent or inconsistent, just like their linear counterparts. Unlike systems of linear equations, however, it is possible for a system of non-linear equations to have more than one solution without having infinitely many solutions. Secondly, as we have seen above, sometimes making a quick sketch of the problem situation can save a lot of time and effort. While in general the graphs of equations in a non-linear system may not be easily visualized, it sometimes pays to take advantage of visualization when you are able.

11.2.3 Eliminating Variables

Solving Non-linear Systems Algebraically

Algebraically, we can use the methods of substitution and elimination outlined in Section 8.1 to solve non-linear systems of equations. However, we need to exercise care when solving non-linear systems, especially since the operations involved may not always result in valid solutions!

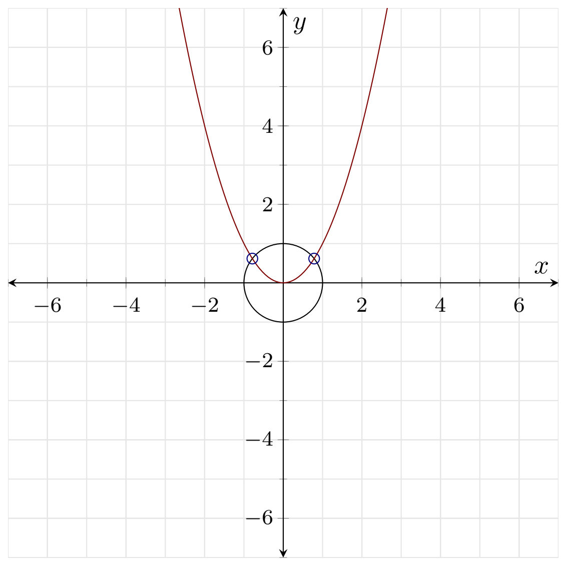
For example, consider the system given by

$$\begin{cases} y - x^2 &= 0 \\ x^2 + y^2 &= 1 \end{cases}.$$

Let's try to use substitution here. From the top equation, we can see that $y = x^2$. Substituting this into the bottom equation results in $x^2 + (x^2)^2 = 1$, or $x^4 + x^2 = 1$, which we can rewrite as $x^4 + x^2 - 1 = 0$. We can now use the quadratic formula on $x^4 + x^2 - 1$ to find that $x^2 = \dfrac{-1 \pm \sqrt{5}}{2}$. Taking a square root, we find that $x = \pm\sqrt{\dfrac{-1 \pm \sqrt{5}}{2}}$ are possible values of x. Note that there are actually *four* separate possible values of x, one for each choice of plus or minus in the expression above: $\sqrt{\dfrac{-1+\sqrt{5}}{2}}, \sqrt{\dfrac{-1-\sqrt{5}}{2}}, -\sqrt{\dfrac{-1+\sqrt{5}}{2}}$, and $-\sqrt{\dfrac{-1-\sqrt{5}}{2}}$.

However, $\dfrac{-1-\sqrt{5}}{2}$ is actually negative! Since the square root of a negative number is not a real number, $\sqrt{\dfrac{-1-\sqrt{5}}{2}}$ and $-\sqrt{\dfrac{-1-\sqrt{5}}{2}}$ are not valid x-values of a solution to this system. The other two solutions are fine. Therefore, keeping in mind that $y = x^2$, the solutions to our system are given by $\left(\sqrt{\dfrac{-1+\sqrt{5}}{2}}, \dfrac{-1+\sqrt{5}}{2}\right)$ and $\left(-\sqrt{\dfrac{-1+\sqrt{5}}{2}}, \dfrac{-1+\sqrt{5}}{2}\right)$. We can plug these back into the original equations to make sure that they satisfy both.

Taking a look at the graphs of these equations should shed some light on what's happening here.



The only intersection points of the two graphs have a positive y-coordinate. This could have tipped us off earlier that some of the x-values we got wouldn't be valid. Indeed, from the first equation, we have that $x = \sqrt{y}$, and this ensures that y must be positive.

The above example illustrates the importance of always checking that the solutions you find are real numbers, and also checking that the solutions you find are actually solutions to the system.

Eliminating Variables

Now we illustrate the method of elimination, which can be used when you notice that the equations in the system have like terms. The difference from before is that we now may have non-linear terms that we can eliminate.

Let's apply this technique to a system we saw previously.

Example 88. *Find all solutions to the following system of equations:*

$$\begin{cases} x^2 + y^2 & = 4 \\ 4x^2 + 9y^2 & = 36 \end{cases}.$$

Explanation We can multiply the top equation by -4, so we get the equivalent system of equations

$$\begin{cases} -4x^2 - 4y^2 & = -16 \\ 4x^2 + 9y^2 & = 36 \end{cases}.$$

Now we can eliminate the x^2 terms to obtain $5y^2 = 20$. From here, we see that $y^2 = 4$, so $y = \pm 2$. To find the associated x values, we substitute each value of y into one of the equations to find the resulting value of x. Choosing $x^2 + y^2 = 4$, we find that for both $y = -2$ and $y = 2$, we get $x = 0$. Our solution set is thus $\{(0, 2), (0, -2)\}$.

Example 89. *Find all solutions to the following system of equations:*

$$\begin{cases} x^2 + y^2 & = 4 \\ 4x^2 - 9y^2 & = 36 \end{cases}.$$

Explanation We proceed as before to eliminate one of the variables. We can multiply the top equation by -4, so we get the equivalent system of equations

$$\begin{cases} -4x^2 - 4y^2 & = -16 \\ 4x^2 - 9y^2 & = 36 \end{cases}.$$

Now we can eliminate the x^2 terms to obtain $-13y^2 = 20$. From here, we see that $y^2 = -\dfrac{20}{13}$. Since the square root of a negative number is not a real number, we see that there are no real values of y that solve this equation. Therefore, we conclude that this system has no solution. Recall that a system that has no solution is called inconsistent.

Example 90. *Find all solutions to the following system of equations:*

$$\begin{cases} x^2 + 2xy - 16 & = 0 \\ y^2 + 2xy - 16 & = 0 \end{cases}.$$

Explanation At first glance, it doesn't appear as though elimination will do us any good since it's clear that we cannot completely eliminate one of the variables. The alternative, solving one of the equations for one variable and substituting it into the other, is full of unpleasantness. Returning to elimination, we note that it is possible to eliminate the troublesome xy term, and the constant term as well, by elimination and doing so we get a more tractable relationship between x and y. We can multiply the top equation by -1, so we get the equivalent system of equations

$$\begin{cases} -x^2 - 2xy + 16 & = 0 \\ y^2 + 2xy - 16 & = 0 \end{cases}.$$

Eliminating, we find that $y^2 - x^2 = 0$, so $y^2 = x^2$, and $y = \pm x$. Substituting $y = x$ into the top equation, we get $x^2 + 2x^2 - 16 = 0$, so that $x^2 = \dfrac{16}{3}$ or $x = \pm\dfrac{4\sqrt{3}}{3}$. On the other hand, when we substitute $y = -x$ into the top equation, we get $x^2 - 2x^2 - 16 = 0$ or $x^2 = -16$, which gives no real solutions. Substituting each of $x = \pm\frac{4\sqrt{3}}{3}$ into the substitution equation $y = x$ yields the solution set $\left\{\left(\frac{4\sqrt{3}}{3}, \frac{4\sqrt{3}}{3}\right), \left(-\frac{4\sqrt{3}}{3}, -\frac{4\sqrt{3}}{3}\right)\right\}$. Try plugging these into the original system to see that they are actually solutions. Verifying this graphically would be a fun exercise, but we leave that up to you.

Some Common Issues and Techniques

Example 91. *Find all solutions to the following system of equations:*

$$\begin{cases} x^2 + y &= 12 \\ 3xy &= 0 \end{cases}.$$

Explanation Notice that this is, in fact, a non-linear system, since the second equation contains an xy term. Since we can't see any like terms in the two equations, it makes sense to try to use substitution. We might be tempted to divide both sides of the bottom equation by $3x$, so as to isolate y, but as always with division, we need to be careful! Indeed, $x = 0$ is still a possibility, so we cannot divide through by $3x$, since we'd then be dividing by 0.

Instead, it helps to think about what it means for the product of two numbers to equal 0. In fact, the product of two nonzero numbers can never be 0. In our situation, we know that $3xy = 0$, so either $3x = 0$ or $y = 0$.

If $3x = 0$, then dividing by 3 (since $3 \neq 0$) gives us $x = 0$. We can plug that into the top equation and find that $0^2 + y = 12$, so $y = 12$. We can then check that $(0, 12)$ is a solution to our original system.

If $y = 0$, we can plug that into the top equation to find that $x^2 + 0 = 12$. Solving for x yields $x = \pm\sqrt{12} = \pm 2\sqrt{3}$. We can then check that $(2\sqrt{3}, 0)$ and $(-2\sqrt{3}, 0)$ are solutions to the system.

Our final solution set is $\{(2\sqrt{3}, 0), (-2\sqrt{3}, 0), (0, 12)\}$.

Example 92. *Find all solutions to the following system of equations:*

$$\begin{cases} \frac{4}{x} + \frac{3}{y} &= 1 \\ \\ \frac{3}{x} + \frac{2}{y} &= -1 \end{cases}.$$

Explanation Notice that this is, in fact, a non-linear system, since both equations divide by the variables we're using.

If we define new (but related) variables by letting $u = \dfrac{1}{x}$ and $v = \dfrac{1}{y}$ then the system becomes

$$\begin{cases} 4u + 3v &= 1 \\ 3u + 2v &= -1 \end{cases}.$$

This associated system of linear equations can then be solved using any of the techniques you've learned earlier to find that $u = -5$ and $v = 7$. Therefore, $x = \dfrac{1}{u} = -\dfrac{1}{5}$ and $y = \dfrac{1}{v} = \dfrac{1}{7}$, and our solution set is $\left\{\left(-\dfrac{1}{5}, \dfrac{1}{7}\right)\right\}$.

We say that the original system is linear in form because its equations are not linear, but a few substitutions reveal a structure that we can treat like a system of linear equations. However, the substitutions may introduce some complexity, as seen in the following example.

Example 93. *Find all solutions to the following system of equations:*

$$\begin{cases} 5e^x + 3e^{2y} &= 1 \\ \\ 3e^x + e^{2y} &= 2 \end{cases}.$$

Explanation Notice that this is, in fact, a non-linear system, since both equations divide by the variables we're using.

If we define new (but related) variables by letting $u = e^x$ and $v = e^{2y}$ then the system becomes

$$\begin{cases} 5u + 3v &= 1 \\ 3u + v &= 2 \end{cases}.$$

This associated system of linear equations can then be solved using any of the techniques you've learned earlier to find that $u = \dfrac{3}{4}$ and $v = -\dfrac{1}{4}$. Therefore, $x = \ln(u) = \ln\left(\dfrac{3}{4}\right)$ and $y = \dfrac{\ln(v)}{2} = \dfrac{\ln\left(-\frac{1}{4}\right)}{7}$. However, the nonlinearity of the system throws us a wrench! Logarithms are not defined on negative numbers, so $\dfrac{\ln\left(-\frac{1}{4}\right)}{7}$ does not exist, and there is actually no value of y that satisfies both equations. Therefore, this system does not have a solution.

Exploration Consider the following system.

$$\begin{cases} 4\ln(x) + 3y^2 &= 1 \\ 3\ln(x) + 2y^2 &= -1 \end{cases}.$$

(a) Is the system linear in form?

(b) If so, make substitutions by defining variables u and v so that the system in terms of u and v is linear. What is u? What is v? What is our new associated linear system?

(c) What is the solution set to our associated linear system?

(d) What is the solution set to our original system?

11.3 Applications of Systems

Learning Objectives

- Applications of Systems
  - Word problems with extraneous variables
  - Word problems similar to related rates or optimization

11.3.1 Applications of Systems

Introduction

Suppose a rectangle has width w and length l, with area 24 and perimeter 20. The area of the rectangle is wl and the perimeter is given by $2w+2l$, giving the following system of equations

$$\begin{cases} wl &= 24 \\ 2w+2l &= 20. \end{cases}$$

Since the first equation here is not a linear equation, this is a nonlinear system of equations. If we want to find the dimensions of the corresponding rectangle, we must solve this system. Since calculations of areas and volumes are nonlinear in general, situations involving geometric shapes often result in nonlinear systems of equations.

To solve this system, we will start by dividing both sides of the second equation by 2, to obtain the following equivalent system.

$$\begin{cases} wl &= 24 \\ w+l &= 10. \end{cases}$$

If this second equation is satisfied, that means $l = 10 - w$, which can be substituted into the top equation to eliminate the variable l.

$$\begin{aligned} wl &= 24 \\ w(10-w) &= 24 \\ 10w - w^2 &= 24 \\ w^2 - 10w + 24 &= 0 \\ (w-6)(w-4) &= 0. \end{aligned}$$

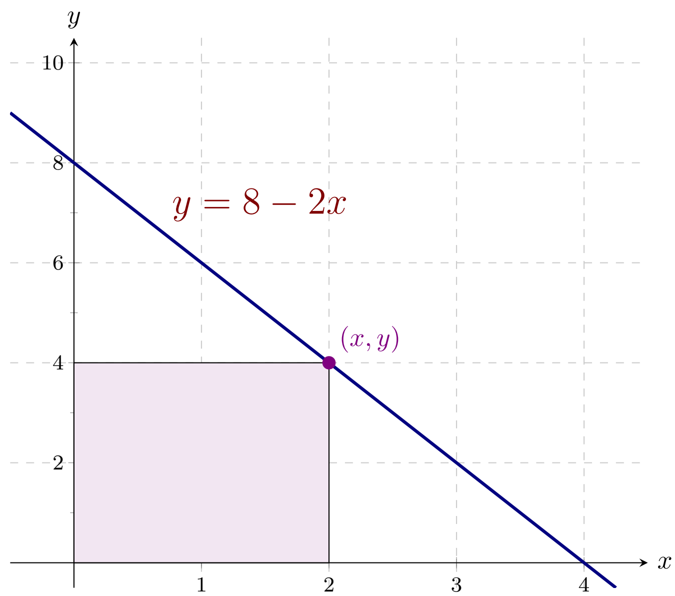
The $w-6$ factor gives a solution of $w = 6$, and the $w-4$ factor gives a solution of $w = 4$.

Looking back at $l = 10 - w$ we see that if $w = 6$, then $l = 10 - 6 = 4$ and if $w = 4$ then $l = 10 - 4 = 6$.

There are two possible rectangles: One with width 6 and length 4, and the other with width 4 and length 6.

Applications of Systems

Exercise 6 *A rectangle is drawn in the first quadrant with one side along the x-axis, one side along the y-axis, the lower left corner at the origin, and* *upper right corner on the graph of the equation $y = 8 - 2x$. Denote this* upper right *vertex as (x, y). Find the coordinates of the point (x, y) if the area of the rectangle is $\frac{15}{2}$.*



Explanation Since (x, y) are the coordinates of the upper right vertex, this tells us that x and y are both positive. It also tells us that the distance from the x-axis is y, and the distance from the y-axis is x. In other words, the height of the rectangle is just y, and the width of the rectangle is x. In terms of x and y, the area is given by xy, giving us one equation $xy = \frac{15}{2}$.

Since the upper right corner (x, y) is on the graph of the line, we also know that $y = 8 - 2x$. This leaves us with the following system: This gives a system of nonlinear equations

$$\begin{cases} xy &= \frac{15}{2} \\ y &= 8 - 2x. \end{cases}$$

This bottom equation is already solved for y, so the easiest way to eliminate a variable would be to substitute it into the y in the top equation.

$$\begin{aligned}
xy &= \frac{15}{2} \\
x(8-2x) &= \frac{15}{2} \\
8x - 2x^2 &= \frac{15}{2} \\
2x^2 - 8x &= -\frac{15}{2} \\
x^2 - 4x &= -\frac{15}{4}.
\end{aligned}$$

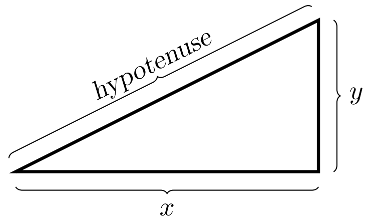
We'll solve this equation by completing the square.

$$\begin{aligned}
x^2 - 4x &= -\frac{15}{4} \\
x^2 - 4x + 4 &= -\frac{15}{4} + 4 \\
x^2 - 4x + 4 &= \frac{1}{4} \\
(x-2)^2 &= \frac{1}{4} \\
x - 2 &= \pm\sqrt{\frac{1}{4}} \\
x - 2 &= \pm\frac{1}{2} \\
x &= 2 \pm \frac{1}{2} \\
x &= \frac{3}{2}, \frac{5}{2}
\end{aligned}$$

If $x = \frac{3}{2}$ then $y = 8 - 2\left(\frac{3}{2}\right) = 5$, and if $x = \frac{5}{2}$ then $y = 8 - 2\left(\frac{5}{2}\right) = 3$.

There are two possibilities. One has coordinates $\left(\frac{3}{2}, 5\right)$ and the other has coordinates $\left(\frac{5}{2}, 3\right)$.

Exercise 7 *A right triangle has hypotenuse of length* $13m$ *and area* $30m^2$*. Find the lengths of the two legs of the triangle.*



[2in]

Explanation

Since this is a right triangle, the Pythagorean Theorem tells us that $x^2 + y^2 = 13^2 = 169$. The area of a triangle is given by $\frac{1}{2} \times \text{base} \times \text{height}$ which means $\frac{1}{2}xy = 30$, or equivalently $xy = 60$.

This gives a system of nonlinear equations

$$\begin{cases} x^2 + y^2 & = 169 \\ xy & = 60. \end{cases}$$

A direct way to solve this system of equations would be to solve the bottom equation for y, giving $y = \frac{60}{x}$, and substitute this into the top equation eliminating the y variable. After simplification that will yield a degree 4 polynomial equation to solve for x. Instead of following that method, we will make use of a different algebraic trick.

We know that there is a difference between $x^2 + y^2$ and $(x + y)^2$. If we multiply out $(x + y)^2$ we get $x^2 + 2xy + y^2$. That means if we add $2xy$ to $x^2 + y^2$, it becomes $(x + y)^2$. To make use of that, we need to know the value of $2xy$ so we can add it to the other side of our top equation. Notice that the bottom equation of our system $xy = 60$ means that $2xy = 2(60) = 120$. If the bottom equation is satisfied, the top equation can be rewritten as:

$$\begin{aligned} x^2 + y^2 &= 169 \\ x^2 + y^2 + 2xy &= 169 + 2xy \\ x^2 + 2xy + y^2 &= 169 + 120 \\ (x + y)^2 &= 289. \end{aligned}$$

Taking square roots of both sides gives $|x + y| = \sqrt{289} = 17$. That is, $x + y = \pm 17$. Neither x nor y can be negative (since they denote lengths of the sides of this triangle), this results in $x + y = 17$. Our system of equations is equivalent to:

$$\begin{cases} x + y & = 17 \\ xy & = 60. \end{cases}$$

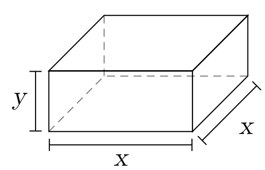
Now that we've been able to simplify the first equation, we will proceed with substitution as mentioned above. If $y = \dfrac{60}{x}$ is satisfied, then the top equation gives:

$$\begin{aligned} x + y &= 17 \\ x + \frac{60}{x} &= 17 \\ x\left(x + \frac{60}{x}\right) &= x(17) \\ x^2 + 60 &= 17x \\ x^2 - 17x + 60 &= 0 \\ (x-12)(x-5) &= 0. \end{aligned}$$

The $x-12$ factor yields a solution of $x = 12$, and the $x-5$ factor gives a solution of $x = 5$. Using $y = \dfrac{60}{x}$ we find that if $x = 12$ then $y = \dfrac{60}{12} = 5$, and if $x = 5$ then $y = \dfrac{60}{5} = 12$.

The two legs of the triangle have lengths $5m$ and $12m$.

Exercise 8 *Suppose we have a box with square base, as illustrated below, constructed to have volume $8cm^3$ and surface area $24cm^2$. Call the side-lengths of the base as x, and the height of the box as y.*



Find the dimensions of the box.

Explanation We know that the volume of the box is given by x^2y (length $\times$ width $\times$ height), giving the nonlinear equation $x^2y = 8$. The surface of the box consists of six rectangles. The top and bottom each have area x^2, and the four sides each have area xy. The full surface area of the box is given by $2x^2 + 4xy$. This setup gives us a system of two nonlinear equations with two unknowns:

$$\begin{cases} x^2y &= 8 \\ 2x^2 + 4xy &= 24. \end{cases}$$

If we want to find the dimensions of the box, we will have to solve this system of equations.

As you have seen in the previous section, solving systems of nonlinear equations involves finding a way to eliminate one of the variables by performing operations on the two equations and/or using substutition. In the case of these equations, notice that the variable y only occurs in a single term in each equation. In the top equation there is an x^2y term, while in the bottom equation there is an xy term. These are not like terms, so we will need to deal with that. Let us begin by multiplying both sides of the second equation by x. This gives the system

$$\begin{cases} x^2y & = 8 \\ 2x^3 + 4x^2y & = 24x. \end{cases}$$

Since no solution has x-coordinate equal to 0, this system is equivalent to the original one. This modification has given that the y variable appears in like terms in boh equations. Substituting $x^2y = 8$ into the new bottom equation gives:

$$\begin{aligned} 2x^3 + 4x^2y &= 24x \\ 2x^3 + 4\left(x^2y\right) &= 24x \\ 2x^3 + 4\,(8) &= 24x \\ 2x^3 + 32 &= 24x \\ 2x^3 - 24x + 32 &= 0 \\ x^3 - 12x + 16 &= 0. \end{aligned}$$

That is, if the $x^2y = 8$ equation is satisfied, then the bottom equation of the system is equivalent to $x^3 - 12x + 16 = 0$. This is a polynomial equation in the single variable, x. (Notice that if we had taken the original top equation $x^2y = 8$, solved it for y to obtain $y = \dfrac{8}{x^2}$, and substituted that into the original bottom equation, we would have arrived at this exact same result.)

Notice that $2^3 - 12(2) + 16 = 8 - 24 + 16 = 0$. That means $x = 2$ is a solution to this cubic equation, and that $x - 2$ is a factor of the polynomial $x^3 - 12x + 16$. By long-division we can find that $x^3 - 12x + 16 = (x - 2)(x^2 + 2x - 8)$. Since $x^2 + 2x - 8 = (x + 4)(x - 2)$ we see that $x^3 - 12x + 16 = (x - 2)^2(x + 4)$. The zeroes of this polymial are $x = 2$ and $x = -4$. Since x represents a length of the side of the box, the $x = -4$ solution is extraneous and should be dropped.

The only solution to the system has $x = 2$. Looking back at the first equation of the system:

$$\begin{aligned} x^2y &= 8 \\ (2)^2y &= 8 \\ 4y &= 8 \\ y &= 2. \end{aligned}$$

The solution is for $(x, y) = (2, 2)$. Since the question asks us to find the dimensions of the box, we say that the box is $2cm \times 2cm \times 2cm$. That is, it's a cube with side length $2cm$.

11.4 Average Rate of Change: Difference Quotients

Learning Objectives

- Secant Lines
  - Definition
  - Finding secant lines
  - Applications
- Difference Quotients
  - Average rate of change when one or both points are given as letters
  - Simplify with algebra (early examples)
  - Finding slopes of secant lines

11.4.1 Average Rate of Change and Secant Lines

Motivating Questions

- What does a line passing through two points of a function represent?
- How does this inform our understanding of the function?

Introduction

We begin by recalling the definitions of *average rate of change* of a function and *secant line* to the graph of a function.

Definition For a function f defined on an interval $[a, b]$,

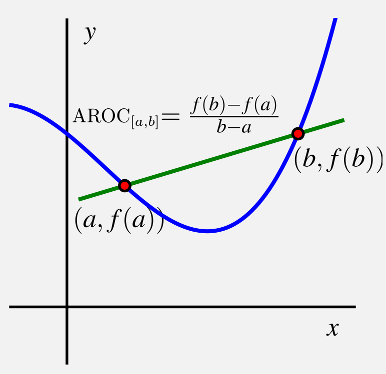
- the **average rate of change of** f **on** $[a, b]$ is the quantity

$$\text{AROC}_{[a,b]} = \frac{f(b) - f(a)}{b - a}.$$

- a **secant line** to the graph of f is a line passing through two points $(a, f(a))$ and $(b, f(b))$, with $a \neq b$.

  Recall: The slope of a secant line is the average rate of change of the function on the interval $[a, b]$.

This is illustrated in the figure below, where the green line (between the red points on the graph) is the secant line of f from $(a, f(a))$ to $(b, f(b))$.



Recall that given two points (x_0, y_0) and (x_1, y_1) in the plane, with $x_0 \neq x_1$, we can find the equation of the line passing through them by using the slope ("rise-over-run"):

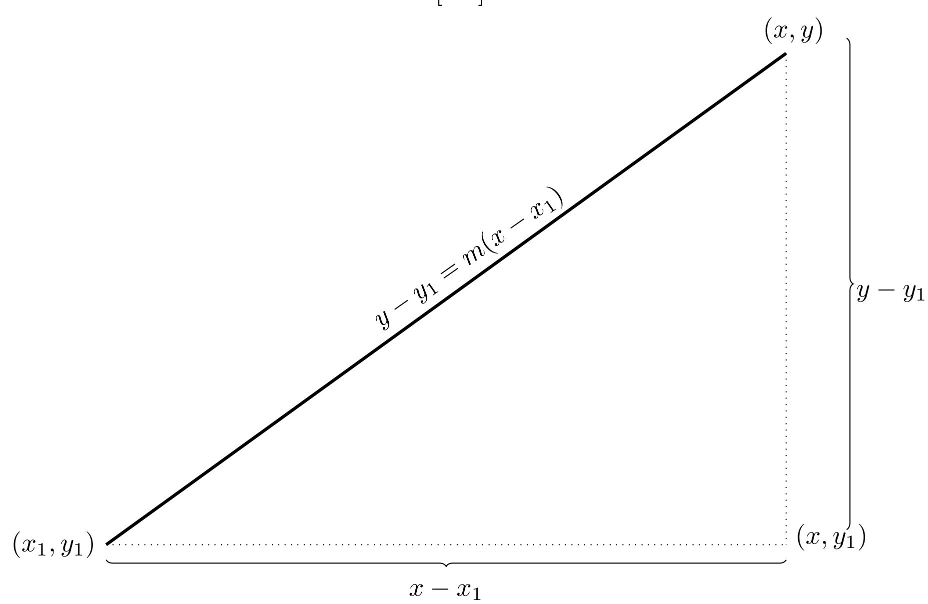
$$m = \frac{y_1 - y_0}{x_1 - x_0}.$$

Then the line equation is given by $y - y_0 = m(x - x_0)$, simply because any given point (x, y) in such line must realize the *same* slope:

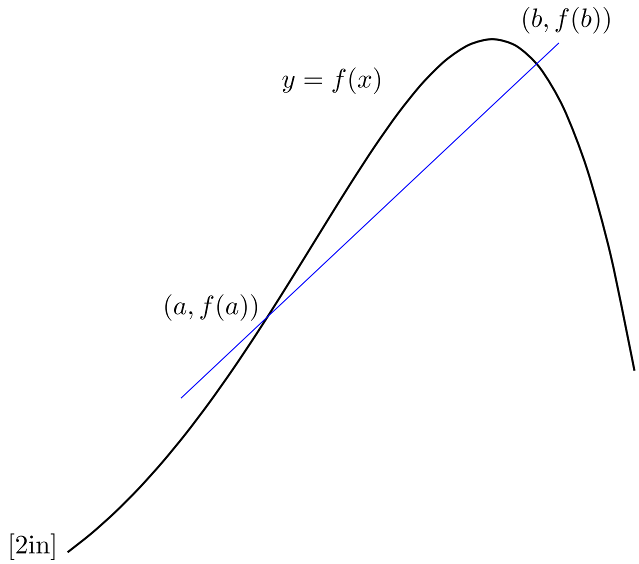
$$m = \frac{y - y_0}{x - x_0}.$$

Of course, one may also use the point (x_1, y_1) instead of (x_0, y_0) and consider the equation $y - y_1 = m(x - x_1)$, as it describes the same line.

[3in]



With this in place, we'll focus on the situation where two such points lie in the graph of some function $y = f(x)$.



Definitions and examples

Definition: Consider a function $y = f(x)$. A line passing through two points $(a, f(a))$ and $(b, f(b))$, with $a \neq b$, in the graph of $y = f(x)$, is called a **secant line** to the graph.
Recall: The slope of a secant line is the average rate of change of the function on the interval $[a, b]$.

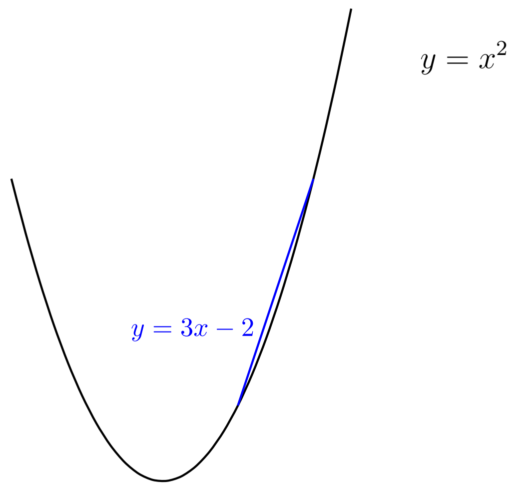
Example 94. *On the following situations, given a function $y = f(x)$ and two points in the graph, find the equation of the secant line they determine.*

a. $f(x) = x^2$, *points* $(1, f(1))$ *and* $(2, f(2))$.

Explanation First, we have that $f(1) = 1^2 = 1$ and $f(2) = 2^2 = 4$, so the points given are actually $(1, 1)$ and $(2, 4)$. So

$$m = \frac{4 - 1}{2 - 1} = 3$$

means that the line equation we're looking for is $y - 1 = 3(x - 1)$, which may be rewritten simply as $y = 3x - 2$.



[2in]

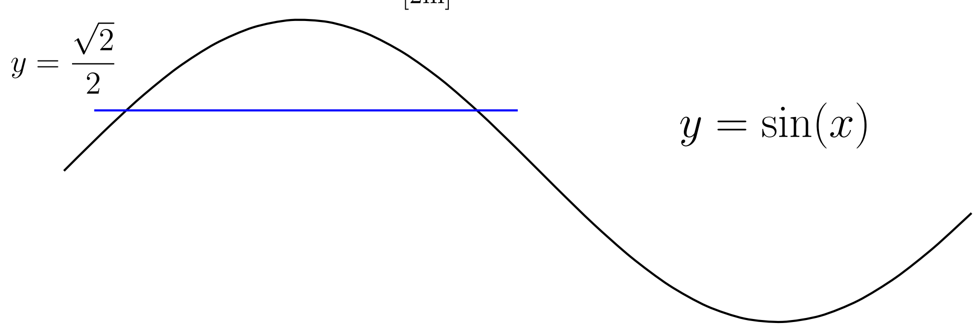
b. $f(x) = \sin x$, *points* $(\pi/4, f(\pi/4))$ *and* $(3\pi/4, f(3\pi/4))$.

Explanation This time, we have that $f(\pi/4) = \sin(\pi/4) = \sqrt{2}/2$ and also that $f(3\pi/4) = \sin(3\pi/4) = \sqrt{2}/2$, so the points given were in fact $(\pi/4, \sqrt{2}/2)$ and $(3\pi/4, \sqrt{2}/2)$. Hence the slope of the secant line is

$$m = \frac{\sqrt{2}/2 - \sqrt{2}/2}{3\pi/4 - \pi/4} = \frac{0}{\pi/2} = 0,$$

so the line equation $y - \sqrt{2}/2 = 0(x - \pi/4)$ boils down to $y = \sqrt{2}/2$. Again, you should see $y = \sqrt{2}/2$ not as a single value of y, but as a line equation for which it just happens that x does not appear — thus describing a horizontal line.

[2in]



c. $f(x) = 2x + 3$, *points* $(-1, f(-1))$ *and* $(3, f(3))$.

Explanation Now, we have $f(-1) = 2(-1) + 3 = 1$ and $f(3) = 2 \cdot 3 + 3 = 9$, so the points given were $(-1, 1)$ and $(3, 9)$. The slope these points determine is

$$m = \frac{9-1}{3-(-1)} = \frac{8}{4} = 2,$$

so we obtain $y - 1 = 2(x - (-1))$, which can be rewritten as $y = 2x + 3$. This is not a coincidence! The secant line to the graph of a line must be the line itself. This is because a line is determined by two points, and since both the original line and the secant line must share the two given points, they must be, in fact, equal.

11.4.2 Slopes of Secant Lines as a Function of h

Motivating Questions

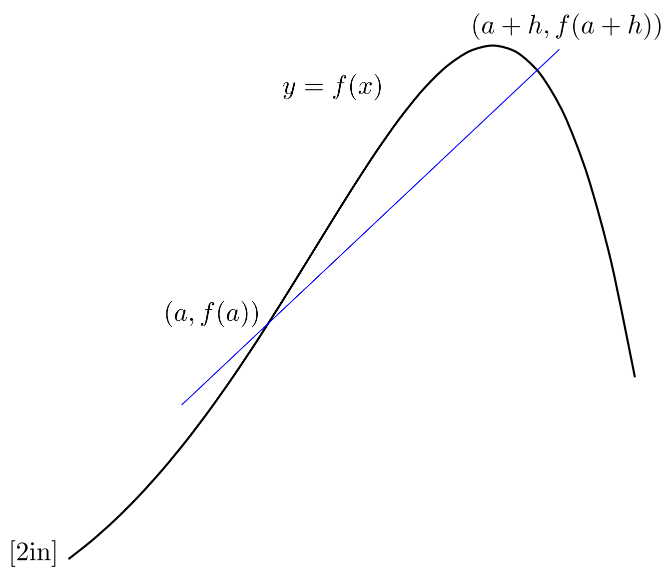
- How can we rewrite the average rate of change of a function in terms of the horizontal distance, h, between the points? Why would we want to do that?
- What are some algebra techniques that allow us to simplify the average rate of change for an arbitrary h?
- What does it tell us when we put in small values for h?

Introduction

We have discussed a secant line to the graph of a function $y = f(x)$ from $(a, f(a))$ to $(b, f(b))$, and the fact that the slope, m, of this line is the average rate of change of the function f on the interval $[a, b]$, $\text{AROC}_{[a,b]}$. Furthermore, recall that we can let $h = b - a$, so that the slope expression becomes

$$m = \frac{f(a+h) - f(a)}{h},$$

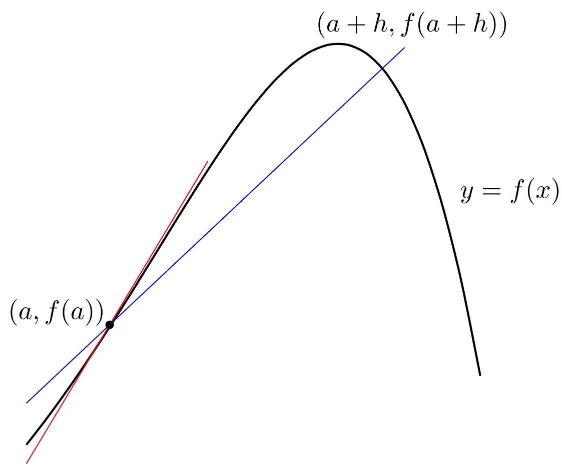
where h is the horizontal distance between the points.



One of the main objectives of Calculus is to understand instantaneous rates of change, as opposed to average rates of change. Namely, what is the behavior of the expression

$$\frac{f(a+h)-f(a)}{h}$$

when h gets very small? Geometrically, making h become very small is making the secant line through $(a, f(a))$ and $(a+h, f(a+h))$ approach a certain line — the tangent line to the graph of $y = f(x)$ at the point a. This is demonstrated in the figure below, where the secant line is blue, and the line tangent to the graph at a is in red.



[2.5in]

Follow the link below to an example Desmos graph where you can see the effect of changing the value of h on the secant line in real-time. Desmos link: https://www.desmos.com/calculator/f6fh2wkrrn

The slope of such a tangent line, when it exists, is called the derivative of f at a. Here, we'll discuss difference quotients and several examples, to prepare you to learn those things in more detail in a future Calculus class.

Definitions and examples

Definition: The difference quotient of a function $y = f(x)$ at a point a of its domain is the quantity

$$\frac{f(a+h)-f(a)}{h},$$

i.e., the average rate of change of f on the interval $[a, a+h]$.

Example 95. *Find the difference quotients of the following functions, at the given point.*

a. $f(x) = x^2$, $a = 2$.

Explanation Let's evaluate it directly:

$$\frac{f(2+h)-f(2)}{h} = \frac{(2+h)^2-2^2}{h} = \frac{2^2+4h+h^2-2^2}{h}$$
$$= \frac{4h+h^2}{h} = h+4.$$

We thus have an equation for the slope of the secant line from $(2, f(2))$ to $(2+h, f(2+h))$:

$$\text{AROC}_{[2,2+h]} = m = h+4.$$

Recall from Example 1(a) of Section 12-3-1, that we calculated the slope of the secant line from $(1, f(1))$ to $(2, f(2))$. Letting $h = -1$, we see that this gives the same answer for the slope of that secant line.

Furthermore, if we now let $h \to 0$, this expression, $\dfrac{f(2+h)-f(2)}{h} \to 4$.

b. $f(x) = \sin(x)$, $a = \dfrac{\pi}{3}$.

Explanation Again, we evaluate directly:

$$\frac{f(\frac{\pi}{3}+h)-f(\frac{\pi}{3})}{h} = \frac{\sin(\frac{\pi}{3}+h)-\sin(\frac{\pi}{3})}{h}$$

Recognizing that we cannot further simplify this expression in its current form, we replace $\sin((\pi/3)+h)$ using the sine sum expression:

$$\frac{\sin(\frac{\pi}{3}+h)-\sin(\frac{\pi}{3})}{h} = \frac{\sin(\frac{\pi}{3})\cos(h)+\cos(\frac{\pi}{3})\sin(h)-\frac{\sqrt{3}}{2}}{h}$$
$$= \frac{1}{2h}\big(\sqrt{3}(\cos(h)-1)+\sin(h)\big)$$

This gives a less easy to visualize equation for the slope of the secant line from $(a, f(a))$ to $(a+h, f(a+h))$, for $a = \dfrac{\pi}{3}$:

$$\text{AROC}_{[\frac{\pi}{3},\frac{\pi}{3}+h]} = m = \frac{1}{2h}\big(\sqrt{3}(\cos(h)-1)+\sin(h)\big).$$

However, consider $h = \dfrac{\pi}{2}$, so that we are looking at the secant line from $(\pi/3, f(\pi/3))$ to $(5\pi/6, f(5\pi/6))$. We then see that

$$\frac{f(\frac{\pi}{3}+h)-f(\frac{\pi}{3})}{h} = \frac{1}{2}\cdot\frac{\pi}{2}\big(\sqrt{3}(\cos\left(\frac{\pi}{2}\right)-1)+\sin\left(\frac{\pi}{2}\right)\big) = 0,$$

as before.

Furthermore, what happens as we let $h \to 0$. Consider $h = \dfrac{\pi}{6}$, then we have

$$\begin{aligned}\frac{f(\frac{\pi}{3}+h)-f(\frac{\pi}{3})}{h} &= \frac{1}{2}\cdot\frac{6}{\pi}\left(\sqrt{3}(\cos\left(\frac{\pi}{6}\right)-1)+\sin\left(\frac{\pi}{6}\right)\right)\\ &= \frac{3}{\pi}\left(\sqrt{3}\left(\frac{\sqrt{3}}{2}-1\right)+\frac{1}{2}\right)\\ &= \frac{6-3\sqrt{3}}{\pi},\end{aligned}$$

Note that this is greater than 0. Think about the graph of $y = sin(x)$. It is increasing on the interval $\left[\frac{\pi}{3}, \frac{\pi}{2}\right]$.

Follow the Desmos link to explore more initial values of x and see what happens as you adjust h smaller and smaller to zero. Desmos link: `https://www.desmos.com/calculator/xbi081bx3w`

c. $f(x) = 2x+3,\ a = -1.$
Explanation Evaluate directly:

$$\frac{f(-1+h)-f(-1)}{h} = \frac{2(h-1)+3-(2(-1)+3)}{h} = \frac{2h-2+2}{h} = 2$$

Observe that the equation for the slope of the secant line from $(-1, f(-1))$ to $(-1+h, f(-1+h))$ is simply

$$\text{AROC}_{[-1,-1+h]} = m = 2.$$

Recall from Example 1(c) of Section 12-3-1, that the equation of the secant line from $(-1, f(-1))$ to $(3, f(3))$ was simply $f(x)$. Why was this?

Now, this tells us that regardless of the points we choose, the secant line between them will have the same slope and equation as the line itself.

11.4.3 Algebra of Secant Lines

Motivating Questions

- What are some algebra techniques that allow us to simplify the equation of a secant line?
- Why is this important?

Introduction

Given the graph of a function $y = f(x)$, we have discussed methods to determine the slope of the secant line between two points, $(a, f(a))$ and $(b, f(b))$, on the graph. We know that this slope represents the average rate of change of the function f on the interval $[a, b]$, denoted by $\text{AROC}_{[a,b]}$. Both of these can be rewritten by letting $b = h - a$, so that we have the value h representing the horizontal distance between the points. This means that as $h \to 0$, the secant line, or the average rate of change of the function, approaches a value known as the slope of the tangent line of f at a. This will be discussed extensively in future calculus courses, but in this section we will focus on tools to simplify the expression $\text{AROC}_{[a,a+h]}$, as they are essential to calculating this limit.

Definitions and examples

Recall the special formula for difference of squares, $a^2 - b^2 = (a - b)(a + b)$. For non-square values of a and b we can use the same idea to rationalize differences (or sums) of square roots through multiplication by the corresponding sum (or difference), which we call the *conjugate*. Given any expression $\sqrt{a} \pm \sqrt{b}$, a, b real numbers, the conjugate of this expression is $\sqrt{a} \mp \sqrt{b}$. Multiplying such an expression by its conjugate rationalizes it through the distributive property: $(\sqrt{a} + \sqrt{b}) \cdot (\sqrt{a} - \sqrt{b}) = (\sqrt{a})^2 + \sqrt{a}\sqrt{b} - \sqrt{a}\sqrt{b} - (\sqrt{b})^2 = a - b$

Definition: Given any difference of positive values $a - b$, we know from the difference of squares, that $a - b = (\sqrt{a} - \sqrt{b})(\sqrt{a} + \sqrt{b})$.
The sum $\sqrt{a} + \sqrt{b}$ is the *conjugate* of the difference $\sqrt{a} - \sqrt{b}$. Likewise, the difference $\sqrt{a} - \sqrt{b}$ is the *conjugate* of the sum $\sqrt{a} + \sqrt{b}$.
Multiplying such an expression by its conjugate will rationalize the expression.

Note that this is one of the most important tools in your simplification toolbox. Other tools include simplifying polynomials and fractions (finding the common denominator), moving coefficients inside or outside the square root, and the trigonometric identities introduced in Section 10-2.

Example 96. *For the following, find the difference quotient. Simplify as much as possible*

(a) $f(x) = \sqrt{x}$, $x \geq 0$

Explanation We consider $h > 0$ to avoid any potential undefined values plugged into our function f since its domain is $[0, \infty)$.

$$\frac{f(x+h) - f(x)}{h} = \frac{\sqrt{x+h} - \sqrt{x}}{h} \tag{5}$$

Observe that we cannot combine any terms in (5), but the numerator is of the form $\sqrt{a} - \sqrt{b}$. Hence, we will multiply by the conjugate to rationalize the numerator:

$$\frac{\sqrt{x+h} - \sqrt{x}}{h} = \frac{(\sqrt{x+h} - \sqrt{x})}{h} \cdot \frac{(\sqrt{x+h} + \sqrt{x})}{(\sqrt{x+h} + \sqrt{x})} \tag{6}$$

$$= \frac{(\sqrt{x+h})^2 - (\sqrt{x})^2}{h(\sqrt{x+h} + \sqrt{x})} \tag{7}$$

Remember, in (6), that in order to avoid changing the value of the expression, we must multiply by the conjugate over itself, i.e., multiply by 1. Then (7) has a difference of squares in the numerator and is equal to

$$\frac{(x+h) - x}{h(\sqrt{x+h} + \sqrt{x})} = \frac{h}{h(\sqrt{x+h} + \sqrt{x})}$$
$$= \frac{1}{\sqrt{x+h} + \sqrt{x}},$$

by cancelling out the h in the numerator and the denominator.

This expression that we have found is now in a form that allows us to consider what happens when $h \to 0$ by removing the h from the denominator.

(b) $g(x) = \sqrt{8 - 2x}$, $x \geq 4$.

Explanation We consider $h > 0$ to avoid any potential undefined values plugged into our function g since its domain is $[4, \infty)$.

$$\frac{g(x+h) - g(x)}{h} = \frac{\sqrt{8 - 4(x+h)} - \sqrt{8 - 4x}}{h} \tag{8}$$

Once again, we cannot combine any terms in the numerator of (8), so we will multiply by the conjugate to rationalize the numerator, hoping we will be able to simplify the equation. (8) is equal to

$$\begin{aligned}
\frac{(\sqrt{8-4(x+h)}-\sqrt{8-4x})}{h} \cdot \frac{(\sqrt{8-4(x+h)}+\sqrt{8-4x})}{(\sqrt{8-4(x+h)}+\sqrt{8-4x})} &= \frac{(\sqrt{8-4(x+h)})^2-(\sqrt{8-4x})^2}{h(\sqrt{8-4(x+h)}+\sqrt{8-4x})} \\
&= \frac{(8-4x-4h)-(8-4x)}{h(\sqrt{4(2-(x+h))}+\sqrt{4(2-x)})} \\
&= \frac{-4h}{2h(\sqrt{2-(x+h)}+\sqrt{2-x})}
\end{aligned}$$

Now, we simply cancel the $2h$ in the numerator and the denominator, giving

$$\frac{g(x+h)-g(x)}{h} = \frac{-2}{\sqrt{2-(x+h)}+\sqrt{2-x}}.$$

(c) $f(x) = \cos(2x)$
Explanation Note that $\cos(z)$ is defined for all real numbers z, so we need not worry about the values of x and h plugged into the difference quotient formula.

$$\frac{f(x+h)-f(x)}{h} = \frac{\cos(2(x+h))-\cos(2x)}{h}$$

Now, we can expand out using the summation formula for cosine:

$$\begin{aligned}
\frac{\cos(2x)\cos(2h)-\sin(2x)\sin(2h)-\cos(2x)}{h} &= \frac{\cos(2x)(\cos(2h)-1)-\sin(2x)\sin(2h)}{h} \\
&= \cos(2x)\frac{\cos(2h)-1}{h} - \sin(2x)\frac{\sin(2h)}{h}
\end{aligned}$$

Here we can plug in decreasing values of h for cosine and sine and start to notice a pattern..
Explore this further by changing the x and h values in the following Desmos graph: Desmos link: https://www.desmos.com/calculator/lf9wkgurwz

(d) $f(x) = |x-1|$
Explanation We will consider two regions and ranges of h: (1) $x \in (-\infty, 1)$ with $h < 0$ and (2) $x \in (1, \infty)$ with $h > 0$.

Let's start with region (1), where $x < 1$ and $h < 0$. From this, we know that $x + h - 1 < x - 1 < 0$, so $|x-1| = -(x-1)$. Hence we have

$$\begin{aligned}
\frac{f(x+h)-f(x)}{h} &= \frac{|x+h-1|-|x-1|}{h} \\
&= \frac{-x-h+1+(x-1)}{h} \\
&= \frac{-h}{h} = -1
\end{aligned}$$

Alternatively, if we consider region (2), where $x > 1$ and $h > 0$, then we have $x + h - 1 > x - 1 > 0$, so that

$$\begin{aligned}\frac{|x+h+1| - |x+1|}{h} &= \frac{x+h-1-(x-1)}{h}\\ &= \frac{h}{h} = 1\end{aligned}$$

Notice that this is not as clear-cut if we consider say $x < 1$ and $h > 0$. Then we would need to consider if h is large enough that $x + h > 1$. Let's explore this some more.

Let $x < 1$ and $h > 0$. Further, assume $h > 1 - x$, then

$$\begin{aligned}\frac{f(x+h) - f(x)}{h} &= \frac{|x+h-1| - |x-1|}{h}\\ &= \frac{x+h+1+(x-1)}{h}\\ &= \frac{2x+h}{h}\end{aligned}$$

Alternatively, if $h < 1 - x$, then

$$\begin{aligned}\frac{f(x+h) - f(x)}{h} &= \frac{|x+h-1| - |x-1|}{h}\\ &= \frac{-x-h+1+(x-1)}{h}\\ &= \frac{-h}{h} = -1\end{aligned}$$

As when $x < 1$ and $h < 0$.

(e) $g(x) = \dfrac{2x}{x^2+3}$

Explanation First note that the denominator of our function g is greater than zero for all real values of x, so the function is defined for all real numbers. Thus, we may calculate the difference quotient without concern for input values of x and h.

$$\frac{g(x+h) - g(x)}{h} = \frac{\frac{2(x+h)}{(x+h)^2+3} - \frac{2x}{x^2+3}}{h}$$

This expression for the difference quotient looks rather messy, so let's find the common denominator and see if we can cancel out some terms in the numerator by combining the fractions. We will leave the terms in the denominator in their current format, but multiply out the $(x + h)^2$ in the numerator for ease of simplification.

Note that the common denominator is $((x+h)^2+3)(x^2+3)$. Then we have

$$\begin{aligned}\frac{g(x+h)-g(x)}{h} &= \frac{\frac{(2x+2h)(x^2+3)-2x(x^2+2xh+h^2+3)}{((x+h)^2+3)(x^2+3)}}{h} \\ &= \frac{2x^3+2x^2h+6x+6h-(2x^3+4x^2h+2xh^2+6x)}{h\big((x+h)^2+3\big)(x^2+3)}\end{aligned}$$

Observe that we have combined the denominators now and have many common terms in the numerator that can be subtracted from each other, so that

$$\frac{g(x+h)-g(x)}{h} = \frac{-2x^2h+6h-2xh^2}{h\big((x+h)^2+3\big)(x^2+3)}.$$

Now, all the terms in the numerator have a factor of h, so we can cancel the h in the numerator and denominator for a final, simplified difference quotient of

$$\frac{-2x^2-2xh+6}{\big((x+h)^2+3\big)(x^2+3)}.$$

Summary Useful tools for simplification:

- Simplifying polynomials.
- Simplifying fractions by finding common denominators.
- Multiplying by the conjugate to rationalize the numerator.
- Considering regions for absolute value functions.

11.5 Functions: The Big Picture

Learning Objectives

- Functions: A Summary
  -
  -
  -
- What is Calculus?
  -
  -
  -

11.5.1 Functions: A Summary

Motivating Questions

- What have we learned about functions in this course?

Introduction

Over the past two semesters, you've learned quite a bit about functions. When we started, we didn't even say what a function was, but we've now talked about many functions and discussed their properties, as well as how to work with them.

Here are some questions you should be able to answer.

- What is a function?
- What are some properties that all functions share?
- What are the domain and range of a function, and how can they be calculated?
- What are zeros of functions, and how can they be found?
- How can functions be built out of other functions?
- Which functions have inverses, and how can they be found?
- What kinds of symmetries can the graphs of functions show?
- What are some famous kinds of functions? What do their graphs look like? Why are they important? What do they model?
- How can we describe the average rate of change of a function?
- How can we go back and forth between different representations of a function?

Using Functions to Solve Problems

Analyzing a Function

Once we have a function that models some phenomenon, we can ask all sorts of questions about our function. In this section, we'll take a particular function, and see what kinds of interesting things we can discover. Our hope is to demonstrate how you can use the tools we have developed in this course to gain information about complicated functions.

A model used in many fields is the *logistic function*. The standard logistic function is a function f defined by $f(x) = \dfrac{1}{1+e^{-x}}$. Looking at this function can be intimidating, but we have all the tools at our disposal to be able to analyze this function.

Domain and Range Let's start by finding the domain and range of f. To find the domain, notice that the only possible obstruction to $f(x)$ being defined is if the denominator were to equal zero. This tells us that to find the domain, we need to solve the equation $1 + e^{-x} = 0$. To do this, we take the following steps:

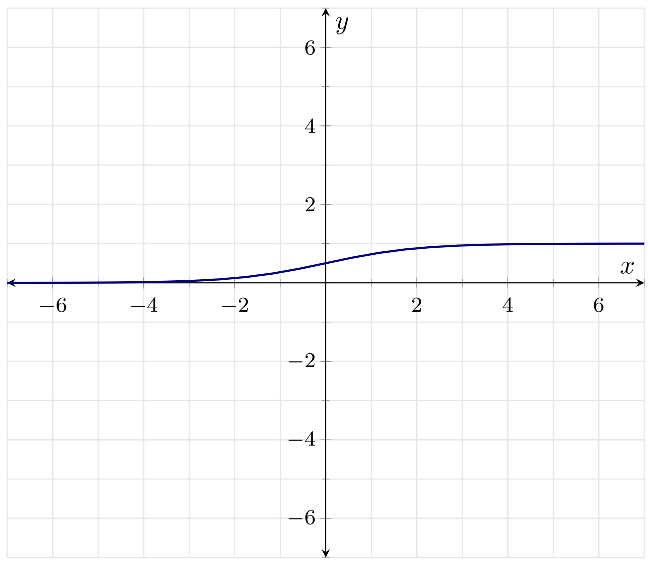
$$\begin{aligned} 1 + e^{-x} &= 0 \\ 1 &= -e^{-x} \\ -1 &= e^{-x} \\ \ln(-1) &= \ln(e^{-x}) \\ \ln(-1) &= -x \\ -\ln(-1) &= x \end{aligned}$$

However, as we learned, the domain of the natural logarithm is $(0, \infty)$, so -1 is not in the domain of the natural logarithm, and therefore, this equation does not have a solution. Since the equation does not have a solution, there are no obstructions to $f(x)$ being defined, and the domain of f is $(-\infty, \infty)$.

To find the range of f, we first must recall that the range of e^{-x} is $(0, \infty)$. Let's consider what happens as e^{-x} gets arbitrarily large. In this case, the denominator of $\dfrac{1}{1+e^{-x}}$ becomes arbitrarily large, and therefore, $f(x)$ becomes arbitrarily close to 0, but always remains positive.

As e^{-x} gets arbitrarily close to 0, the denominator of $\dfrac{1}{1+e^{-x}}$ becomes arbitrarily close to 1, but is always greater than 1, therefore, $f(x)$ can be arbitrarily close to 1, but never greater than or equal to 1.

Combining these two statements tells us that the range of f is $(0, 1)$. The above arguments use reasoning that you will develop further in calculus: the idea of getting arbitrarily close to a point is a major topic in that subject. Another way to get an idea of the range is to graph the function on a graphing utility such as Desmos.



This makes it easier to see what's going on, but being able to understand how to find the range using reasoning about the range of e^{-x} is an important skill to develop.

Average Rate of Change Notice that the graph of f has an S-shape. It flattens out as the absolute value of x becomes large. We will make this more concrete in the following exploration.

Exploration In this exploration, we will learn about the average rate of change of f over various intervals. Use a calculator to get a sense of how large or small the rates of change are.

a. Find the average rate of change of f over the interval $[0, 1]$.

b. Find the average rate of change of f over the interval $[2, 3]$.

c. Find the average rate of change of f over the interval $[5, 6]$.

d. Find the average rate of change of f over the interval $[-7, -6]$.

e. Of the intervals above, which had the highest rate of change? The lowest?

Adapting Models One application of the logistic function is to model population growth. At time x, the logistic growth model says that the population is $f(x)$. The rationale behind this is that various factors (space, resources, etc.) put a limit, or "carrying capacity" on how many individuals can survive in a population. Therefore, population growth should slow down based on how close the population is to the carrying capacity. That is, the closer $f(x)$ is to the carrying capacity, the slower the rate of change of f should be. Population should also be slower when $f(x)$ is close to 0, since there are fewer individuals to reproduce.

A very reasonable question to ask would be "How can this be used to model populations if its range is $(0, 1)$?" The answer is that function transformations allow us to fit the function to our specific need. For example, if the carrying capacity is 5000, instead of using $f(x)$ to model the population, we would use $5000f(x)$. We can use horizontal stretches and compressions to adjust how steep the growth is and use horizontal shifts to adjust the starting population.

A more general form of the logistic function would then be something of the form

$$Kf(r(x-h)) = \frac{K}{1+e^{-r(x-h)}}.$$

The value K adjusts the vertical stretch and therefore the carrying capacity. The value r corresponds to a horizontal compression or stretch. For the logistic function, this affects the steepness of the graph. As usual, h represents a horizontal shift.

Exploration Here is a table containing Columbus population data from Wikipedia.

Year	Population (in thousands)
1812	0.300
1820	1.450
1830	2.435
1840	6.048
1850	17.882
1860	18.554
1870	31.274
1880	51.647
1890	88.150
1900	125.560
1910	181.511
1920	237.031
1930	290.564
1940	306.087
1950	375.901
1960	471.316
1970	539.677
1980	564.871
1990	632.910
2000	711.470
2010	787.033

Use the following Desmos link to answer the following questions.

Desmos link: `https://www.desmos.com/calculator/ymiuslnbqw`

a. Experiment with the sliders to find values of K, r, and h that make $\dfrac{K}{1+e^{-r(x-h)}}$ a model for the data above that is as suitable as possible. Answers may vary.

b. Based on your values of K, r, and h, what is the carrying capacity of the population of Columbus?

c. Use your model to estimate the population of Columbus in 2020.

d. The actual population of Columbus in 2020 was 905,748. Does this agree with the model you found? Why do you think this is the case?

Inverse Function Another fun fact about the function f is that it is one-to-one, and therefore, invertible. What's more, we can use the tools we developed during the sections on inverse functions to be able to find an inverse for the logistic function. Since the logistic function takes a time as an input and returns the population at that time, its inverse takes a number and returns the time when the population has reached that number.

Exploration Use your model from the previous section. Call it f.

a. Find a formula for $f^{-1}(x)$.

b. Use your formula to estimate when the population of Columbus will be 1,000,000.

c. What is the domain of f^{-1}?

11.5.2 What is Calculus?

Motivating Questions

- What are the main ideas of calculus?

Introduction

This course aims to provide you with a background in all the tools you'll need to be successful in calculus. In this section, we'll provide a brief overview of some of the types of problems you'll be able to solve with calculus.

Speed

Say you're running, and the number of miles you run in t minutes can be given by $f(t) = \sqrt{\frac{t}{6}}$. What is your speed at the start of mile 2?

Calculus allows us to talk about instantaneous speeds and rates of change, rather than just average rates of change.

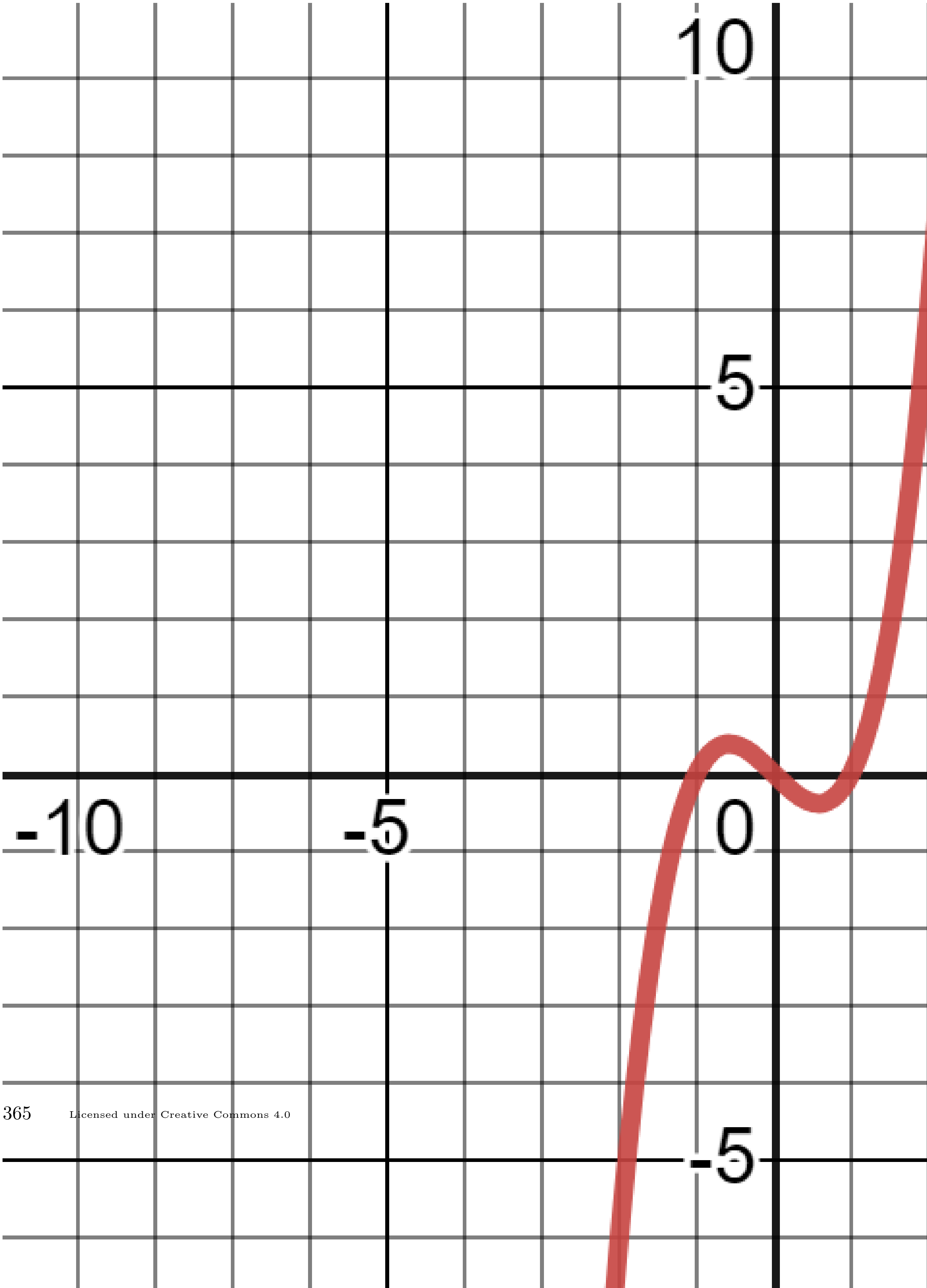
Optimization

Say you want to build a fence, and one side of the fenced-in area is already bounded by a building. If your budget only allows for the purchase of 1000 feet of fencing material, what is the largest area which can be enclosed by the fence?

Calculus can answer many questions about maximizing or minimizing certain values.

Finding Maxima and Minima

When we talked about rational functions and polynomials, there was certain information about their graphs that we couldn't provide. For example, the graph of the function defined by $f(x) = x^3 - x$ is shown below.


10
5
-10
-5
0
-5

We can answer questions about its end behavior with the concepts learned in this course. However, we notice that there are a peak and a valley on the graph of f. Calculus will give us the tools to be able to find the exact location of those points.

Calculus helps us find relative maxima and minima.

Volume of Solids

Say you have a bowl whose silhouette viewed from the side can be described by a parabola. Assume its cross-sections are perfect circles. What is the volume of the bowl?

Calculus can help us find the volume of many different kinds of 3D solids.

Related Rates

Say you're filling a cone with base radius 100 centimeters and height 100 centimeters with water at a rate of 4 centimeters cubed per minute. How quickly is the depth of water in the cone is changing when the water is at a height of 50 centimeters?

Calculus can allow us to describe rates of change in relation to other rates of change.

Newton's Method

Say you're looking at a really complicated function, like $x^6-2x^5-x^4+x^2-\pi x+e$. Finding the roots of this function algebraically is impossible, but if you start with a good estimate of the root, a tool from calculus called Newton's Method can refine your solution into a better estimate. What's more, you can then use Newton's method on that better estimate to obtain an even better estimate!

Calculus can help us estimate roots of functions.

Approximations

We know that the sine function is extremely complicated. We saw in the section on inverse trigonometric functions, that sometimes, we can rewrite functions involving trig functions in purely algebraic terms. For example, you can show that $\sin(\arctan(x)) = \dfrac{x}{\sqrt{x^2+1}}$. However, we haven't seen a way to rewrite sin by itself. However, in certain situations, we can come up with an approximation to functions, and this approximation is linear, which is often much easier to work with! There is a famous linear approximation of $\sin(x)$, but it only works when x is very close to zero. Part of calculus will be learning when certain approximations are good substitutes for the actual function.

Also, there's no reason to stop at linear functions! We can use higher and higher degree polynomials to get better and better approximations of trig functions, as well as other functions.

Calculus can allow us to approximate complicated functions with polynomials, but these approximations are only good for certain values of x.

Average Rates of Change

Say you're driving in an area where the speed limit is 60 miles per hour. If you drive 6 miles in 5 minutes, your average speed is greater than 60 miles per hour, but your speed at any given point on your trip need not be greater than 60 miles per hour: you may slow down and speed up. Is it possible to go 6 miles in 5 minutes without speeding? In other words, if your average rate of change is greater than 60 miles per hour, is it possible for your instantaneous rate of change to be below 60 miles per hour for the entire trip?

Calculus can give us important information on the relationship between average rates of change and instantaneous rates of change.

Finding the Area

In the project this semester, you wrote equations and inequalities that defined the shapes of letters of the alphabet in the plane. However, if you were using these as a font, a reasonable question would be, "How much ink does each letter use?" To answer this question, we need to find the area of each letter. With the tools we have now, this isn't possible. However, in calculus, you will learn a way to calculate the exact area of each of your letters.

Calculus provides a method for finding the area bounded by curves in the plane.

Part 12

Back Matter

Index