Precalculus with Review 1: Unit 4

October 27, 2022

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4.1 Building New Functions

Learning Objectives

- Algebra of Functions
  - Add, subtract, multiply, and divide functions
  - Think of complicated functions as objects in their own right
  - Evaluate complicated functions
  - Break more complicated functions into famous function
  - Understand functions via graphs, tables, algebraically, and abstractly
- Composition
  - Compose two functions
  - See how composition can be used in context
  - Understand finding average rate of change as involving composition

4.1.1 Algebra of Functions

Motivating Questions

- We know that we can add, subtract, multiply, and divide numbers. What kinds of operations can we perform on functions?

Introduction

In arithmetic, we execute processes where we take two numbers to generate a new number. For example, $2+3=5$. The number 5 results from adding the numbers 2 and 3. Similarly, we can multiply two numbers to generate a new one: $2 \cdot 3 = 6$.

We can work similarly with functions. Just as we can add, subtract, multiply, and divide numbers, we can also add, subtract, multiply, and divide functions to create a new function from two or more given functions.

Algebra of Functions

In most mathematics up until calculus, the main object we study is *numbers*. We ask questions such as

- "What number(s) form solutions to the equation $x^2 - 4x - 5 = 0$?"
- "What number is the slope of the line represented by $3x - 4y = 7$?"
- "What number is generated as output by the function $f(x) = \sqrt{x^2+1}$ by the input $x = -2$?"

Certainly we also study overall patterns as seen in functions and equations, but this usually occurs through an examination of numbers themselves, and we think of numbers as the main objects being acted upon.

This changes in calculus. In calculus, the fundamental objects being studied are functions themselves. A function is a much more sophisticated mathematical object than a number, in part because a function can be thought of in terms of its graph, which is an infinite collection of ordered pairs of the form $(x, f(x))$.

It is often helpful to look at a function's formula and observe algebraic structure. For instance, given the quadratic function

$$q(x) = -3x^2 + 5x - 7$$

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we might benefit from thinking of this as the sum of three simpler functions: the constant function $c(x) = -7$, the linear function $s(x) = 5x$ that passes through the point $(0, 0)$ with slope $m = 5$, and the concave down basic quadratic function $w(x) = -3x^2$. Indeed, each of the simpler functions c, s, and w contribute to making q be the function that it is. Likewise, if we were interested in the function $p(x) = (3x^2 + 4)(9 - 2x^2)$, it might be natural to think about the two simpler functions $f(x) = 3x^2 + 4$ and $g(x) = 9 - 2x^2$ that are being multiplied to produce p.

We thus naturally arrive at the ideas of adding, subtracting, multiplying, or dividing two or more functions, and hence introduce the following definitions and notation.

Definition Let f and g be functions.

- The **sum of** f **and** g is the function $f + g$ defined by $(f + g)(x) = f(x) + g(x)$.
- The **difference of** f **and** g is the function $f - g$ defined by $(f - g)(x) = f(x) - g(x)$.
- The **product of** f **and** g is the function $f \cdot g$ defined by $(f \cdot g)(x) = f(x) \cdot g(x)$.
- The **quotient of** f **and** g is the function $\frac{f}{g}$ defined by $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$ for all x such that $g(x) \neq 0$.

We are thinking here about f and g being functions with real numbers as outputs. Performing these operations on the functions means applying the corresponding operation to the output values of the functions.

Example 1. *Consider the functions f and g defined by the table of values below.*

x	$f(x)$
2	0
4	3
6	7
8	-2

x	$g(x)$
1	5
2	9
3	-1
4	4

(a) *Determine the value of* $(f + g)(2)$.

(b) *Determine the value of* $(f - g)(4)$.

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(c) *Determine the value of* $(f \cdot g)(2)$.

(d) *Determine the value of* $\left(\frac{f}{g}\right)(4)$.

(e) *What can we say about the value of* $(f+g)(3)$*?*

Explanation

(a) We know that $(f+g)(2) = f(2)+g(2)$. From the tables above $f(2) = 0$ and $g(2) = 9$, so $(f+g)(2) = 0+9 = 9$.

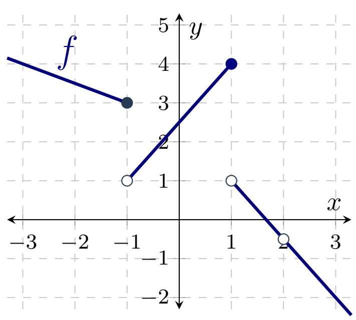
(b) Since $f(4) = 3$ and $g(4) = 4$, we know $(f-g)(4) = f(4)-g(4) = 3-4 = -1$.

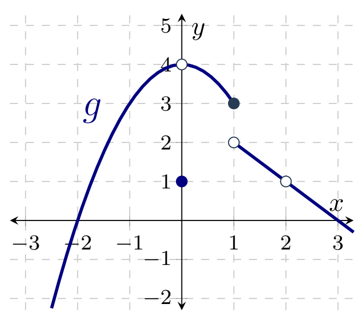
(c) $(f \cdot g)(2) = f(2) \cdot g(2) = 0 \cdot 9 = 0$.

(d) $\left(\frac{f}{g}\right)(4) = \frac{f(4)}{g(4)} = \frac{3}{4}$.

(e) The value of $(f+g)(3)$ would be given by $f(3)+g(3)$. we are given the value of $g(3)$ in the table above, but there is no listed value for $f(3)$. That means $f(3)$ is undefined, since 3 is not a valid input. Therefore, $(f+g)(3)$ is undefined.

Example 2. *Consider the functions* f *and* g *defined by*





(a) *Determine the exact value of* $(f+g)(0)$.

(b) *Determine the exact value of* $(g-f)(1)$.

(c) *Determine the exact value of* $(f \cdot g)(-1)$.

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(d) *Are there any values of x for which $\left(\frac{f}{g}\right)(x)$ is undefined? If not, explain why. If so, determine the values and justify your answer.*

(e) *For what values of x is $(f \cdot g)(x) = 0$? Why?*

Explanation

(a) The notation $(f + g)(0)$ means we are plugging the input 0 into both functions f and g, then *adding* the results. That is, $(f+g)(0) = f(0)+g(0)$. From the graph above we see $g(0) = 1$. Computing the slope of the segment in the middle of the graph of f yields $m = 3/2$. Using point slope form to find the equation of the line containing that segment, we find that the line is given by $y = 3/2x+5/2$. That means $(f+g)(0) = f(0)+g(0) = \frac{5}{2} + 1 = \frac{7}{2}$.

(b) The notation $(g - f)(1)$ means we are plugging the input 1 into both functions g and f, then *subtracting* the results.That is, $(g-f)(1) = g(1)-f(1)$. From the graphs above we see $f(1) = 4$ and $g(1) = 3$. That means $(g - f)(1) = 3 - 4 = -1$.

(c) The notation $(f \cdot g)(-1)$ means we are plugging the input -1 into both functions f and g, then *multiplying* the results. That is, $(f \cdot g)(-1) = f(-1) \cdot g(-1)$. From the graphs above we see $f(-1) = 3$ and $g(-1) = 3$, which tells us $(f \cdot g)(-1) = 3 \cdot 3 = 9$.

(d) For any valid value of the input x, $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$. In order for that fraction to be defined $f(x)$ has to exist, $g(x)$ has to exist, and $g(x) \neq 0$ since division by zero is undefined. From the graphs above, f is defined for all x-values except $x = 2$, and g is defined for all x-values except $x = 2$. That tells us that $\left(\frac{f}{g}\right)(2)$ is undefined. Notice that $g(-2) = 0$ and $g(3) = 0$? That means $\left(\frac{f}{g}\right)(x)$ is undefined at $x = -2$ and $x = 3$ as well

(e) Since $(f \cdot g)(x) = f(x) \cdot g(x)$, if an x-value makes $(f \cdot g)(x) = 0$, then $f(x) \cdot g(x) = 0$. The only way a product of two real numbers can be zero is if at least one of the factors is itself zero. That means we are looking for all of the x-values satisfying either $f(x) = 0$ or $g(x) = 0$. (In other words, we're looking for the x-intercepts of these graphs.)

From the graph of g we see that $g(-2) = 0$ and $g(3) = 0$. The graph of f crosses the x-axis somewhere between the points $(1, 0)$ and $(2, 0)$, but we

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will have to be more careful to find the exact value we are looking for.

Notice that the graph of f looks to be a straight line if we only look at those x-values with $x > 1$. The straight line that f follows travels through the point $(1,1)$ and $(3,-2)$. Its slope is given by $m = \frac{-2-1}{3-1} = -\frac{3}{2}$. Since the line contains the point $(1,1)$, the point-slope form of the equation of the line can be written as $y - 1 = -\frac{3}{2}(x-1)$. This line crosses the x-axis when its y-coordinate is zero. Solving for the corresponding x-value gives us:

$$\begin{aligned}
0 - 1 &= -\frac{3}{2}(x-1) \\
-1 &= -\frac{3}{2}(x-1) \\
-\frac{2}{3}\cdot(-1) &= -\frac{2}{3}\cdot\left(-\frac{3}{2}(x-1)\right) \\
\frac{2}{3} &= x - 1 \\
\frac{2}{3} + 1 &= x \\
\frac{5}{3} &= x.
\end{aligned}$$

That means the point $\left(\frac{5}{3}, 0\right)$ is on the graph of f, so $f\left(\frac{5}{3}\right) = 0$.

The x-values with $(f \cdot g)(x) = 0$ are $x = -2$, $x = \frac{5}{3}$, and $x = 3$.

Remark The only way a product of two real numbers can be zero is if at least one of the factors is itself zero. That is, if $ab = 0$ for two numbers a and b, then $a = 0$ or $b = 0$. This technique does not work for numbers other than zero.

Consider the functions $f(x) = 2x$ and $g(x) = x$. These are functions whose graphs are straight lines with slopes 2 and 1 respectively.

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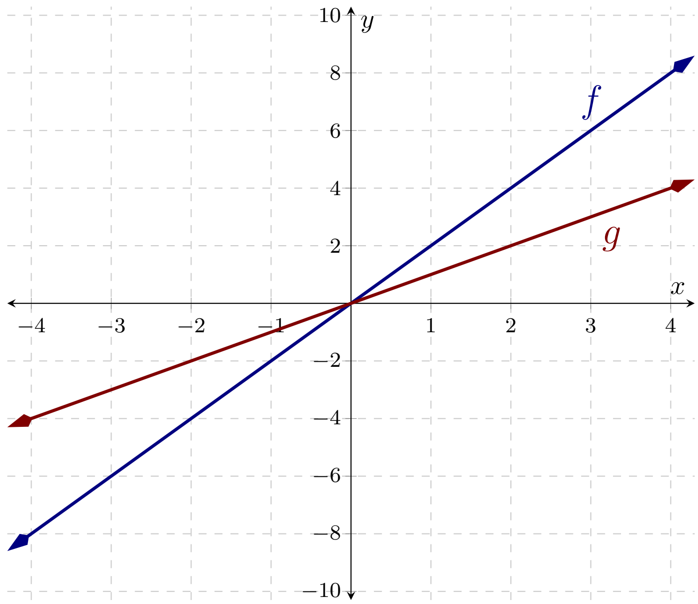
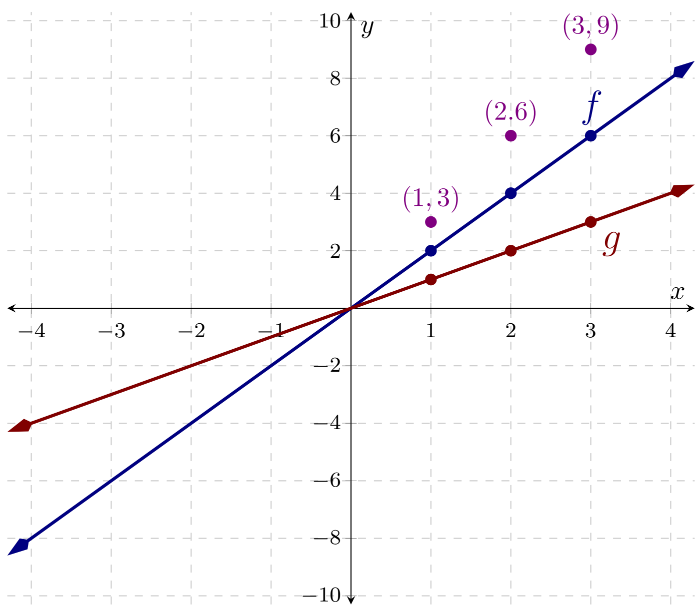


Let's look at a few points on these graphs. Since $f(1) = 2$ and $g(1) = 1$, the sum of those output values is 3, so we'll mark the point $(1, 3)$. Similarly $f(2) = 4$ and $g(2) = 2$, with $4 + 2 = 6$ so we'll mark the point $(2, 6)$. As $f(3) = 6$ and $g(3) = 3$, we'll also mark $(3, 9)$.



Notice that the points we've marked $(1, 3)$, $(2, 6)$, and $(3, 9)$ are starting to form

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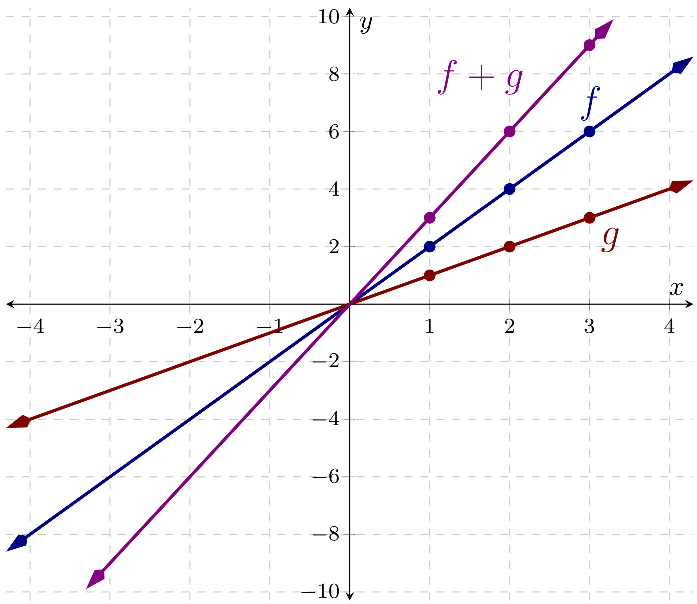
a straight line. Let's connect those dots to examine the line constructed this way.



The graph obtained this way is the graph of $f+g$. This graph is a straight line passing through the points $(0,0)$ and $(1,3)$, so the line has equation $y=3x$.

For a given value of x, we know $(f+g)(x)=f(x)+g(x)$. This means $(f+g)(x)=2x+x=3x$ by combining the like terms. Notice that this aligns with the graph we found above. This example shows that we can work with these operations through formulas for our functions as well.

Example 3. *Let f and g be the functions given by the formulas $f(x)=2\sin(x)$ and $g(x)=5x-4$.*

(a) *Find the value of* $(f-g)\left(\frac{\pi}{3}\right)$.

(b) *Find a formula for* $(f+g)(x)$.

(c) *Find a formula for* $(f\cdot g)(x)$.

(d) *Find a formula for* $\left(\frac{f}{g}\right)(x)$.

Explanation

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(a) $(f-g)\left(\frac{\pi}{3}\right) = f\left(\frac{\pi}{3}\right) - g\left(\frac{\pi}{3}\right)$. Sine is one of our famous functions, which has $\sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$, so $f\left(\frac{\pi}{3}\right) = 2\sin\left(\frac{\pi}{3}\right) = 2\frac{\sqrt{3}}{2} = \sqrt{3}$. $g\left(\frac{\pi}{3}\right) = 5\left(\frac{\pi}{3}\right) - 4 = \frac{5\pi}{3} - 4$.

Together this gives $(f-g)\left(\frac{\pi}{3}\right) = \sqrt{3} - \frac{5\pi}{3} + 4$

(b) $(f+g)(x) = f(x) + g(x) = 2\sin(x) + 5x - 4$.

(c) $(f \cdot g)(x) = 2\sin(x)\,(5x-4)$.

(d) $\left(\frac{f}{g}\right)(x) = \frac{2\sin(x)}{5x-4}$.

When we work in applied settings with functions that model phenomena in the world around us, it is often useful to think carefully about the units of various quantities. Analyzing units can help us both understand the algebraic structure of functions and the variables involved, as well as assist us in assigning meaning to quantities we compute. We have already seen this with the notion of average rate of change: if a function $P(t)$ measures the population in a city in year t and we compute $\text{AROC}_{[5,11]}$, then the units on $\text{AROC}_{[5,11]}$ are "people per year," and the value of $\text{AROC}_{[5,11]}$ is telling us the average rate at which the population changes in people per year on the time interval from year 5 to year 11.

Example 4. *Say that an investor is regularly purchasing stock in a particular company. Let $N(t)$ represent the number of shares owned on day t, where $t = 0$ represents the first day on which shares were purchased. Let $S(t)$ give the value of one share of the stock on day t; note that the units on $S(t)$ are dollars per share. How is the total value, $V(t)$, of the held stock on day t determined?*
Explanation Observe that the units on $N(t)$ are "shares" and the units on $S(t)$ are "dollars per share". Thus when we compute the product

$$N(t)\,\text{shares} \cdot S(t)\,\text{dollars per share},$$

it follows that the resulting units are "dollars", which is the total value of held stock. Hence,

$$V(t) = N(t) \cdot S(t).$$

Exploration Let f be a function that measures a car's fuel economy in the following way. Given an input velocity v in miles per hour, $f(v)$ is the number of gallons of fuel that the car consumes per mile (i.e., "gallons per mile"). We know that $f(60) = 0.04$.

 Acknowledgements: Active Prelude to Calculus (https://activecalculus.org/prelude)

(a) What is the meaning of the statement "$f(60) = 0.04$" in the context of the problem? That is, what does this say about the car's fuel economy? Write a complete sentence.

(b) Consider the function $g(v) = \dfrac{1}{f(v)}$. What is the value of $g(60)$? What are the units on g? What does g measure?

(c) Consider the function $h(v) = v \cdot f(v)$. What is the value of $h(60)$? What are the units on h? What does h measure?

(d) Do $f(60)$, $g(60)$, and $h(60)$ tell us fundamentally different information, or are they all essentially saying the same thing? Explain.

(e) Suppose we also know that $f(70) = 0.045$. Find the average rate of change of f on the interval $[60, 70]$. What are the units on the average rate of change of f? What does this quantity measure? Write a complete sentence to explain.

Taking a complicated function and determining how it is constructed out of simpler ones is an important skill to develop. At the beginning of this section we split the function $q(x) = -3x^2 + 5x - 7$ into the sum/difference of three simple functions, $-3x^2$, $5x$, and 7. Let us experiment with splitting a few more complicated functions.

Example 5. (a) *Find functions h and k so that $f(x) = 4x^2 \sin(x)$ can be written as $(h \cdot k)(x)$.*

(b) *Find functions f and g so that $h(x) = \dfrac{2x+3}{x-1}$ can be written as $\left(\dfrac{f}{g}\right)(x)$.*

(c) *Find functions r and s so that $t(x) = \dfrac{x+1}{3x-1} + xe^x$ can be written as $(r+s)(x)$.*

(d) *Find functions f, g, and h so that $k(x) = \dfrac{\sin(x)\sqrt{2x+3}}{1+\ln(x)}$ can be written as $\left(\dfrac{f \cdot g}{h}\right)(x)$.*

Explanation

Before working through these questions, we want to remind you that the answers we give are not unique. There are many different, equally valid choices for the simpler functions requested. For the first question, we will mention multiple possibilities. We leave it to you to find other answers for the remaining questions.

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(a) Notice that if $h(x) = 4x^2$ and $k(x) = \sin(x)$, then $(h \cdot k)(x) = 4x^2 \sin(x)$, so this choice of h and k is one valid answer. What if we had chosen $h(x) = 4x$ and $k(x) = x \sin(x)$ instead? Then $(h \cdot k)(x) = (4x)(x \sin(x)) = 4x^2 \sin(x)$, so this choice would also be a valid answer. Another possibility would have been to choose $h(x) = 4$ and $k(x) = x^2 \sin(x)$.

(b) If we set $f(x) = 2x + 3$ and $g(x) = x - 1$, then $\left(\dfrac{f}{g}\right)(x) = \dfrac{2x+3}{x-1}$.

(c) Since $t(x) = \dfrac{x+1}{3x-1} + xe^x$ is written as a sum of two terms, take $r(x) = \dfrac{x+1}{3x-1}$ to be the first term and $s(x) = xe^x$ to be the seond term. Then $(r + s)(x) = t(x)$.

(d) We are trying to identify $\dfrac{\sin(x)\sqrt{2x+3}}{1+\ln(x)}$ as a fraction with the numerator a product. The denominator is $1 + \ln(x)$ and the numerator is a product of $\sin(x)$ and $\sqrt{2x+3}$. We will choose $f(x) = \sin(x)$, $g(x) = \sqrt{2x+3}$, and $h(x) = 1 + \ln(x)$.

Exploration

(a) Find functions f and g so that $h(x) = \dfrac{5e^x}{1+\sin(x)}$ can be written as $(f \cdot g)(x)$.

(b) Find functions h and k so that $f(x) = 3x^2 - \sqrt{x+1}$ can be written as $(h + k)(x)$. Find two other choices for h and k.

(c) If f and g are functions, we know that $f + g$ is the function given by $(f + g)(x) = f(x) + g(x)$. What function do you think the notation $f + 3g$ means? Find functions f and g so that $h(x) = 4x^2 - 5\sqrt{x} + 7 \cdot 2^x$ can be written as $(f + 3g)(x)$.

4.1.2 Composition of Functions

Motivating Questions

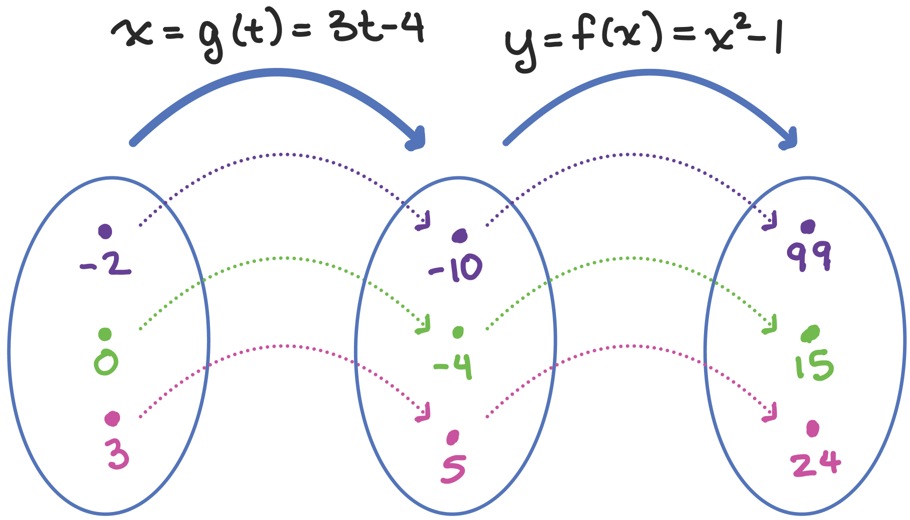
- How does the process of function composition produce a new function from two other functions?
- In the composite function $h(x) = f(g(x))$, what do we mean by the "inner" and "outer" function?
- How does the expression for $\text{AROC}_{[a,a+h]}$ involve a composite function?

Introduction

Recall that a function, by definition, is a process that takes a collection of inputs and produces a corresponding collection of outputs in such a way that the process produces one and only one output value for any single input value. Because every function is a process, it makes sense to think that it may be possible to take two function processes and do one of the processes first, and then apply the second process to the result.

Example 6. *Suppose we know that y is a function of x according to the process defined by $y = f(x) = x^2 - 1$ and, in turn, x is a function of t via $x = g(t) = 3t - 4$. Is it possible to combine these processes to generate a new function so that y is a function of t?*

Explanation Since y depends on x and x depends on t, it follows that we can also think of y depending directly on t. Let's looks at this as an arrow diagram with a few sample points.



Notice that if we take a point such as $t = 2$, we can put that value in for t in the function $x = g(t) = 3t - 4$. This will give

$$x = g(-2) = 3(-2) - 4 = -6 - 4 - 10.$$

Now we have an x-value of -10. $g(x)$ takes in x-values so we can put -10 into $f(x) = x^2 - 1$. This will give

$$f(-10) = (-10)^2 - 1 = 100 - 1 = 99.$$

You should verify that the arrow diagram above gives the correct values of y that corresponds to $t = 0$ and $t = 3$.

Now, we would like to create a new function that will directly take in any t value and give us the corresponding y value. We can use substitution and the notation of functions to determine this function.

First, it's important to realize what the rule for f tells us. In words, f says "to generate the output that corresponds to an input, take the input and square it, and then subtract 1." In symbols, we might express f more generally by writing "$f(\square) = \square^2 - 1$."

Now, observing that $y = f(x) = x^2 - 1$ and that $x = g(t) = 3t - 4$, we can substitute the expression $g(t)$ for x in f. Doing so,

$$\begin{aligned} y &= f(x) \\ &= f(g(t)) \\ &= f(3t - 4). \end{aligned}$$

Applying the process defined by the function f to the input $3t - 4$, we see that

$$y = (3t - 4)^2 - 1,$$

which defines y as a function of t.

One way to think about the substitution above is that we are putting the entire expression $3t-4$ inside the input box in "$f(\square) = \square^2 - 1$." That is, $f\left(\boxed{3t-4}\right) = \left(\boxed{3t-4}\right)^2 - 1$. For the substitution, we are thinking of $3t-4$ as a single object!

When we have a situation such as in the example above where we use the output of one function as the input of another, we often say that we have **composed** two functions. In addition, we use the notation $h(t) = f(g(t))$ to denote that a new function, h, results from composing the two functions f and g.

Exploration

a. Let $y = p(x) = 3x - 4$ and $x = q(t) = t^2$. Determine a formula for r that depends only on t and not on p or q. What is the biggest difference between your work in this problem compared to the example above?

b. Let $t = s(z) = \dfrac{1}{t+4}$ and recall that $x = q(t) = t^2$. Determine a formula for $x = q(s(z))$ that depends only on z.

c. Suppose that $h(t) = \sqrt{2t^2 + 5}$. Determine formulas for two related functions, $y = f(x)$ and $x = g(t)$, so that $h(t) = f(g(t))$.

Composing Two Functions

Whenever we have two functions, g and f, where the outputs of g match inputs of f, it is possible to link the two processes together to create a new process that we call the *composition* of f and g.

Definition If f and g are functions, we define the **composition of** f **and** g to be the new function h given by

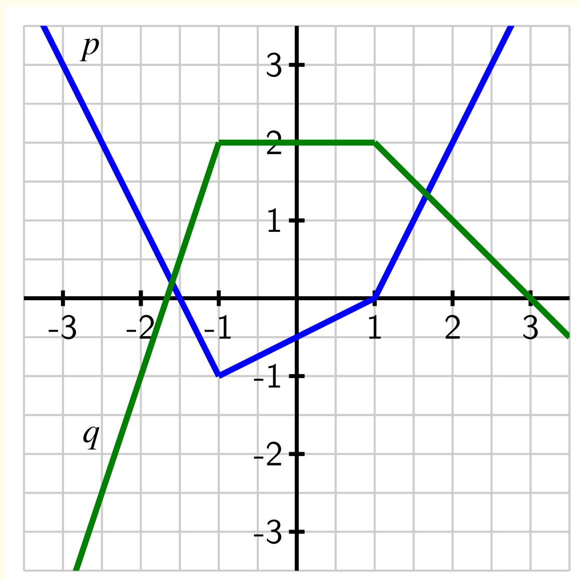
$$h(t) = f(g(t)).$$

This composition is denoted by $h = f \circ g$, where $f \circ g$ means the single function defined by $(f \circ g)(t) = f(g(t))$.

We sometimes call g the "inner function" and f the "outer function". It is important to note that the inner function is actually the first function that gets applied to a given input, and then the outer function is applied to the output of the inner function. In addition, in order for a composite function to make sense, we need to ensure that the outputs of the inner function are values that it makes sense to put into the outer function so that the resulting composite function is defined.

In addition to the possibility that functions are given by formulas, functions can be given by tables or graphs. We can think about composite functions in these settings as well, and the following activities prompt us to consider functions given in this way.

Exploration Let functions p and q be given by the graphs below (which are each piecewise linear - that is, parts that look like straight lines are straight lines) and let f and g be given by the table below.



x	$f(x)$	$g(x)$
0	6	1
1	4	3
2	3	0
3	4	4
4	6	2

Compute each of the following quantities or explain why they are not defined.

a. $p(q(0))$

b. $q(p(0))$

c. $(p \circ p)(-1)$

d. $(f \circ g)(2)$

e. $(g \circ f)(3)$

f. $g(f(0))$

g. For what value(s) of x is $f(g(x)) = 4$?

h. For what value(s) of x is $q(p(x)) = 1$?

Composing functions in content

In the late 1800s, the physicist Amos Dolbear was listening to crickets chirp and noticed a pattern: how frequently the crickets chirped seemed to be connected to the outside temperature. If we let T represent the temperature in degrees Fahrenheit and N the number of chirps per minute, we can summarize Dolbear's observations with the following function, $T = D(N) = 40 + 0.25T$. Scientists who made many additional cricket chirp observations following Dolbear's initial counts found that this formauls holds with remarkable accuracy for the snowy tree cricket in temperatures ranging from $50°$ to $85°$. This function is called Dolbear's Law.

In what follows, we replace T with F to emphasize that temperature is measured in Fahrenheit degrees.

The Celsius and Fahrenheit temperature scales are connected by a linear function. Indeed, the function that converts Fahrenheit to Celsius is

$$C = G(F) = \frac{5}{9}(F - 32).$$

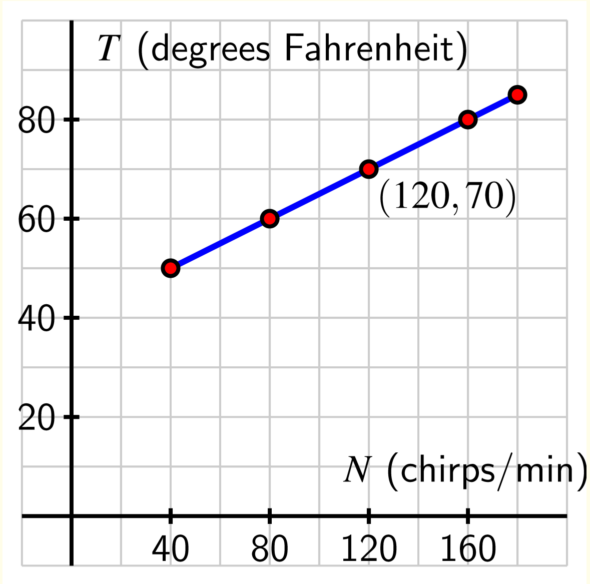
For instance, a Fahrenheit temperature of 32 degrees corresponds to $C = G(32) = \frac{5}{9}(32 - 32) = 0$ degrees Celsius.

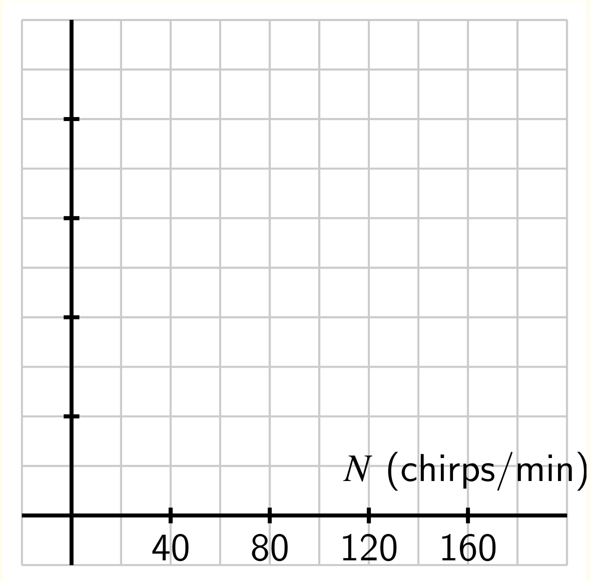
Exploration Let $D(N) = 40 + 0.25N$ be Dolbear's function that converts an input of number of chirps per minute to degrees Fahrenheit, and let $G(F) = \frac{5}{9}(F - 32)$ be the function that converts an input of degrees Fahrenheit to an output of degrees Celsius.

a. Determine a formula for the new function $(G \circ D)(N)$ that depends only on the variable N.

b. What is the meaning of the function you found in (a)?

c. Let $H = G \circ D$. How does a plot of the function H compare to the that of Dolbear's function? Sketch a plot of H on the blank axes to the right of the plot of Dolbear's function, and discuss the similarities and differences between them. Be sure to label the vertical scale on your axes.



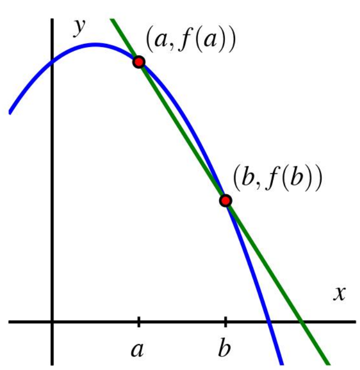


Function Composition and Average Rate of Change

Recall that the average rate of change of a function f on the interval $[a, b]$ is given by

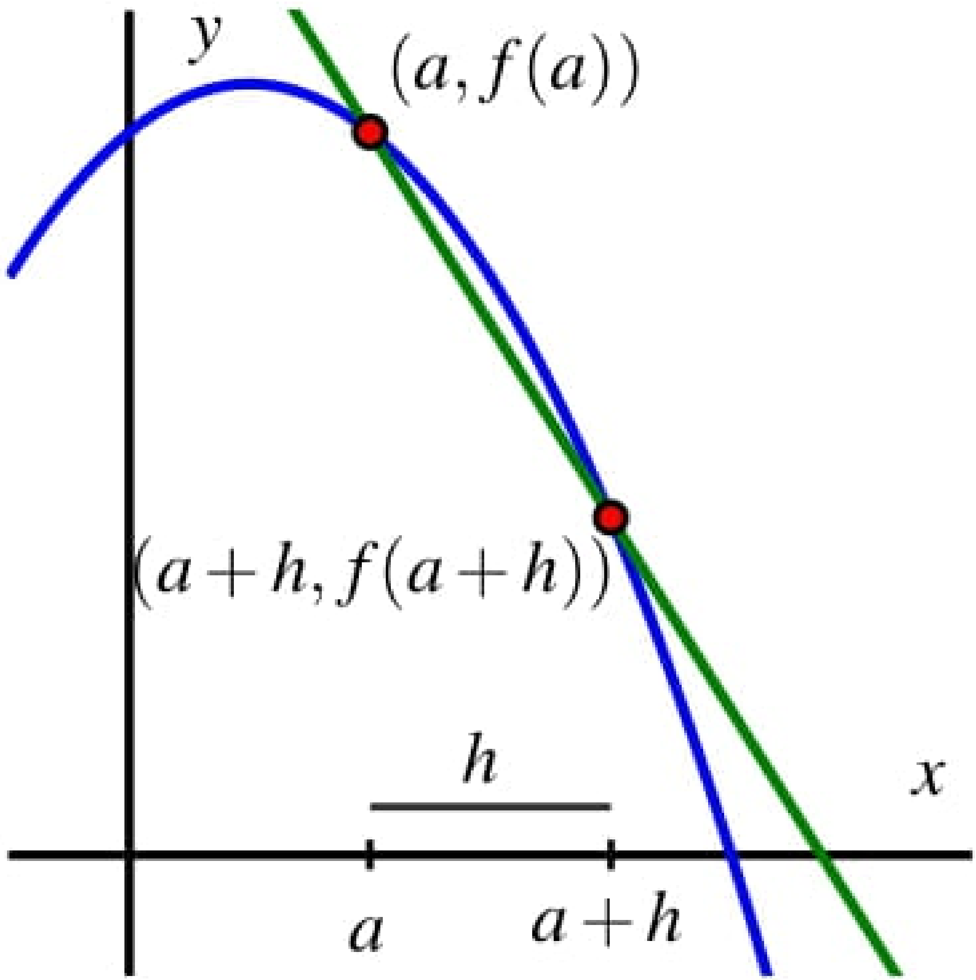
$$\mathrm{AROC}_{[a,b]} = \frac{f(b) - f(a)}{b - a}.$$

In the graph below, we see the familiar representation of $\mathrm{AROC}_{[a,b]}$ as the slope of the line joining the points $(a, f(a))$ and $(b, f(b))$ on the graph of f.



In the study of calculus, we progress from the *average rate of change on an interval* to the *instantaneous rate of change of a function at a single value*; the core idea that allows us to move from an *average* rate to an *instantaneous* one is letting the interval $[a, b]$ shrink in size.

To think about the interval $[a, b]$ shrinking while a stays fixed, we often change our perspective and think of b as $b = a + h$, where h measures the horizontal difference from b to a.



This allows us to eventually think about h getting closer and closer to 0, and in that context we consider the equivalent expression

$$\text{AROC}_{[a,a+h]} = \frac{f(a+h) - f(a)}{a+h-a} = \frac{f(a+h) - f(a)}{h}$$

for the average rate of change of f on $[a, a+h]$.

Example 7. *Suppose that $f(x) = x^2$. Determine the simplest possible expression you can find for $\text{AROC}_{[3,3+h]}$, the average rate of change of f on the interval $[3, 3+h]$.*

Explanation By definition, we know that

$$\text{AROC}_{[3,3+h]} = \frac{f(3+h) - f(3)}{h}.$$

Using the formula for f, we see that

$$\text{AROC}_{[3,3+h]} = \frac{(3+h)^2 - (3)^2}{h}.$$

Expanding the numerator and combining like terms, it follows that

$$\begin{aligned}\text{AROC}_{[3,3+h]} &= \frac{(9+6h+h^2) - 9}{h} \\ &= \frac{6h+h^2}{h}.\end{aligned}$$

Removing a factor of h in the numerator and observing that $h \neq 0$, we can simplify and find that

$$\begin{aligned}\text{AROC}_{[3,3+h]} &= \frac{h(6+h)}{h} \\ &= 6+h.\end{aligned}$$

Hence, $\text{AROC}_{[3,3+h]} = 6+h$, which is the average rate of change of $f(x) = x^2$ on the interval $[3, 3+h]$.

Exploration Let $f(x) = 2x^2 - 3x + 1$ and $g(x) = \dfrac{5}{x}$.

a. Compute $f(1+h)$ and expand and simplify the result as much as possible by combining like terms.

b. Determine the most simplified expression you can for the average rate of change of f on the interval $[1, 1+h]$. That is, determine $\text{AROC}_{[1,1+h]}$ for f and simplify the result as much as possible.

c. Compute $g(1+h)$. Is there any valid algebra you can do to write $g(1+h)$ more simply?

d. Determine the most simplified expression you can for the average rate of change of g on the interval $[1, 1+h]$. That is, determine $\text{AROC}_{[1,1+h]}$ for g and simplify the result.

Summary

- When defined, the composition of two functions f and g produces a single new function $f \circ g$ according to the rule $(f \circ g)(x) = f(g(x))$. We note that g is applied first to the input x, and then f is applied to the output $g(x)$ that results from g.

- In the composite function $h(x) = f(g(x))$, the "inner" function is g and the *outer* function is f. Note that the inner function gets applied to x first, even though the outer function appears first when we read from left to right.

- Because the expression $\text{AROC}_{[a,a+h]}$ is defined by

$$\text{AROC}_{[a,a+h]} = \frac{f(a+h) - f(a)}{h}$$

and this includes the quantity $f(a+h)$, the average rate of change of a function on the interval $[a, a+h]$ always involves the evaluation of a composite function expression. This idea plays a crucial role in the study of calculus.

4.2 Quadratics

Learning Objectives

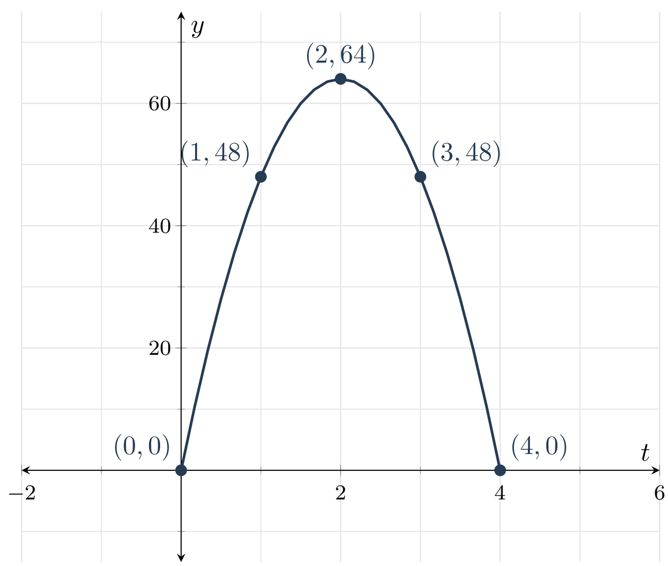
- Definition of Quadratics
  - What is a quadratic function?
  - What does the graph of a quadratic function look like?
  - Recognize a quadratic in standard form
  - Become familiar with standard vocabulary about quadratic functions.
- Vertex Form
  - What is the vertex form of a quadratic?
  - Use completing the square to convert a quadratic in standard form to a quadratic in vertex form.
  - Recognize a quadratic in standard form
  - Become familiar with standard vocabulary about quadratic functions.
- Factored Form
  - What is Factored Form or Root Form of a quadratic? How is it the same or different from saying a quadratic is factored?
  - Use factoring to convert quadratics from standard form to Factored Form.
  - Use Factored Form to find the roots of a quadratic
  - Use the Quadratic Formula to find the roots of a quadratic and/or to write the quadratic in Factored Form

4.2.1 Definition of Quadratics

Quadratic Graphs

Example 8. *Hannah fired a toy rocket from the ground, which launched into the air with an initial speed of* 64 *feet per second. The height of the rocket can be modeled by the equation* $y = -16t^2 + 64t$*, where* t *is how many seconds had passed since the launch. To see the shape of the graph made by this equation, we make a table of values, plot the points, and connect them with a curve.*

t	$-16t^2 + 64t$	*Point*
0	$-16(0)^2 + 64(0) = 0$	$(0, 0)$
1	$-16(1)^2 + 64(1) = 48$	$(1, 48)$
2	$-16(2)^2 + 64(2) = 64$	$(2, 64)$
3	$-16(3)^2 + 64(3) = 48$	$(3, 48)$
4	$-16(4)^2 + 64(4) = 0$	$(4, 0)$



The function in the above story problem is an example is a quadratic function.

Definition A **quadratic** is a function of the form

$$f(x) = ax^2 + bx + c$$

where a, b, and c are real numbers. We say a quadratic written this way is in **Standard Form**.

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The graph of a quadratic function $f(x) = ax^2 + bx + c$ is called a **parabola**. Notice the symmetry in the table, how the y-values in rows $t = 0$ and $t = 4$ match as well as rows $t = 1$ and $t = 3$? Also notice the symmetry in the shape of the graph, how its left side is a mirror image of its right side. The parabola opens upward or downward according to the sign of the leading coefficient a. If the leading coefficient is positive, the parabola opens upward. If the leading coefficient is negative, the parabola opens downward.

You can play with changing the value of a on the graph using Desmos and see how that changes the parabola.

Desmos link: `https://www.desmos.com/calculator/nmlghfrws9`

The **vertex** of a parabola is the highest or lowest point on the graph, depending uponon whether the graph opens downward or upward. In Example 1, the vertex is $(2, 64)$. This tells us that Hannah's rocket reached its maximum height of 64 feet after 2 seconds. If the parabola opens downward, as in the rocket example, then the y-value of the vertex is the **maximum** y-value. If the parabola opens upward then the y-value of the vertex is the **minimum** y-value. The **axis of symmetry** is a vertical line that passes through the vertex, cutting the parabola into two symmetric halves. We write the axis of symmetry as an equation of a vertical line so it always starts with "$x =$". In Example 1, the equation for the axis of symmetry is $x = 2$.

Definition [Intercepts] Suppose f is a function and set $y = f(x)$.
An x**-intercept** is a point $(a, 0)$ such that $f(a) = 0$. That is, it's a point where the graph of the function intersects the x-axis.
The y**-intercept** is a point $(0, b)$ such that $f(0) = b$. That is, it's a point in which the graph of the function intersects the y-axis. Unlike x-intercepts, a function can only have one y-intercept.

It is possible for a quadratic graph to have zero, one, or two x-intercepts.

In Example 1, first note that this is a function $y = f(t)$. We will have y-intercepts but rather than having x-intercepts, these will be t-intercepts due to our use of the variable t and not x. The point $(0, 0)$ is the starting point of the rocket, and it is where the graph crosses the y-axis, so it is the y-intercept. The y-value of 0 means the rocket was on the ground when the t-value was 0, which was when the rocket launched. The point $(0, 0)$ on the path of the rocket is also a t-intercept. The t-value of 0 indicates the time when the rocket was launched from the ground. There is another t-intercept at the point $(4, 0)$, which means the rocket came back to hit the ground after 4 seconds.

Example 9. *Use technology to graph and make a table of the quadratic function f defined by $f(x) = 2x^2 + 4x - 3$ and find each of the key points or features.*

(a) *Find the vertex.*

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(b) *Find the vertical intercept (i.e. the y-intercept).*

(c) *Find the horizontal or (i.e. the x-intercept(s)).*

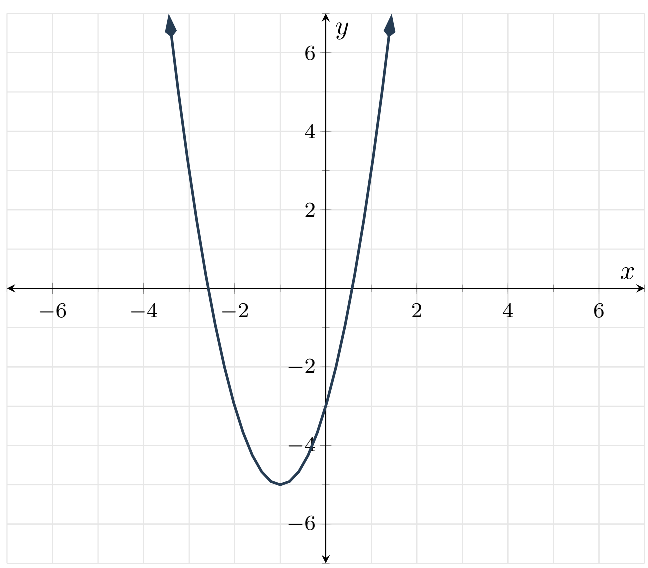
(d) *Find* $f(-2)$*.*

(e) *Solve* $f(x) = 3$ *using the graph.*

(f) *Solve* $f(x) \leq 3$ *using the graph.*

Explanation The specifics of how to use any one particular technology tool vary. Whether you use an app, a physical calculator, or something else, a table and graph should look like:

x	$f(x)$
-2	-3
-1	-5
0	-3
1	3
2	13



Additional features of your technology tool can enhance the graph to help answer these questions. You may be able to make the graph appear like:

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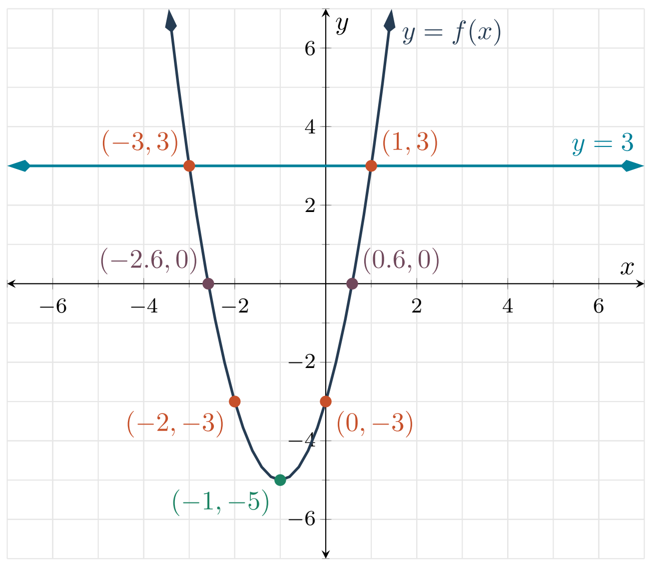


(a) The vertex is $(-1, -5)$.

(b) The vertical intercept is $(0, -3)$.

(c) The horizontal intercepts are approximately $(-2.6, 0)$ and $(0.6, 0)$.

(d) When $x = -2$, $y = -3$, so $f(-2) = -3$.

(e) The solutions to $f(x) = 3$ are the x-values where $y = 3$. We graph the horizontal line $y = 3$ and find the x-values where the graphs intersect. The solution set is $\{-3, 1\}$.

(f) The solutions are all of the x-values where the function's graph is below (or touching) the line $y = 3$. The interval is $[-3, 1]$.

4.2.2 Vertex Form

The Vertex Form of a Quadratic

We have learned the standard form of a quadratic function's formula, which is $f(x) = ax^2 + bx + c$. But quadratic functions also have different forms, similar to linear functions. Here, we will learn another form for quadratic functions called the vertex form.

Vertex Form of a Quadratic Function A quadratic function whose graph has vertex at the point (h, k) is given by

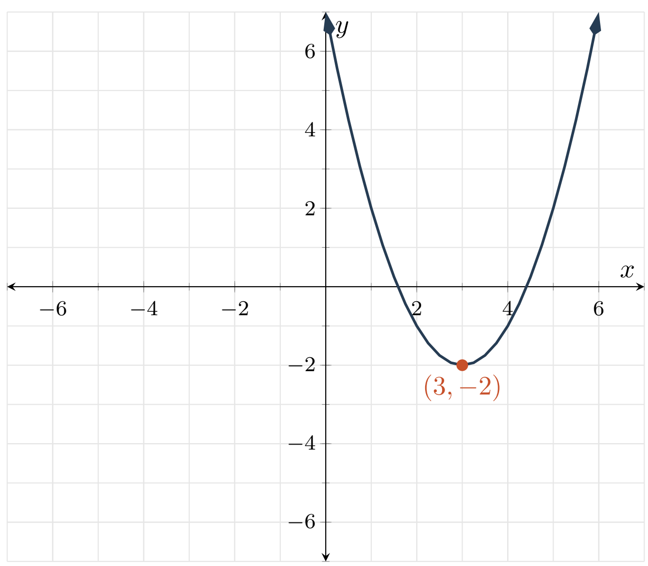
$$f(x) = a(x - h)^2 + k$$

Using graphing technology, consider the graphs of $f(x) = x^2 - 6x + 7$ and $g(x) = (x - 3)^2 - 2$ on the same axes.

We see only one parabola because these are two different forms of the same function. Indeed, if we convert $g(x)$ into standard form:

$$\begin{aligned} g(x) &= (x-3)^2 - 2 \\ &= (x^2 - 6x + 9) - 2 \\ &= x^2 - 6x + 7 \end{aligned}$$

it is clear that f and g are the same function.



Graph of $f(x) = x^2 - 6x + 7$

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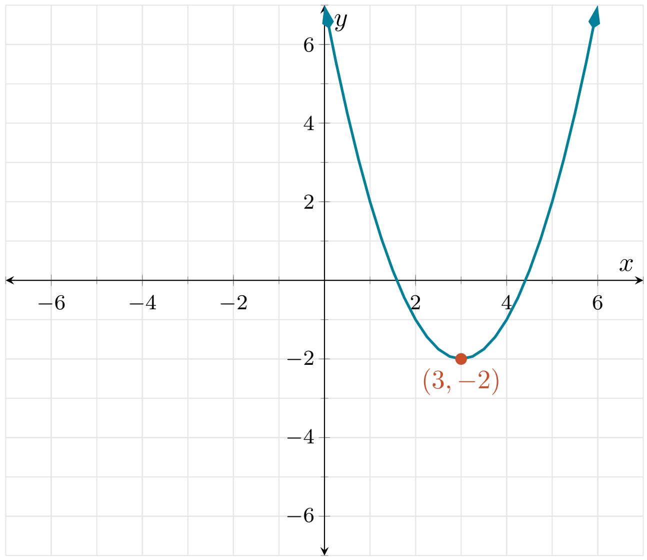
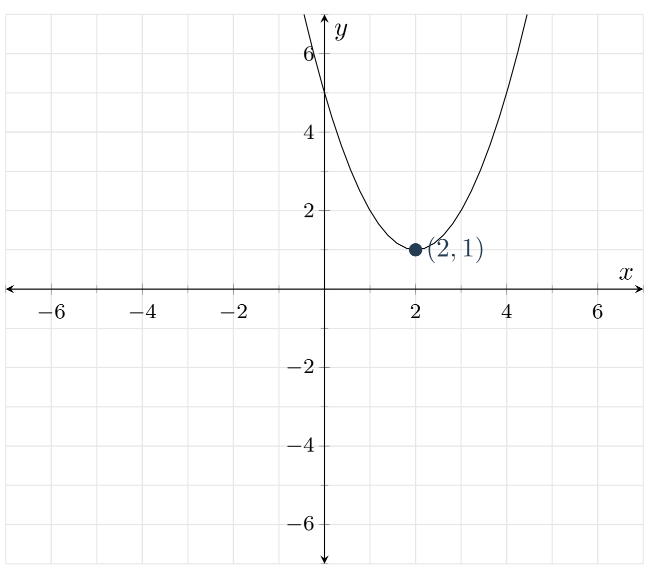


Graph of $g(x) = (x - 3)^2 - 2$

We see that the graphs of the two parabolas are completely identical.

The formula given for g is said to be in vertex form because it allows us to read the vertex without doing any calculations. The vertex of the parabola is $(3, -2)$. We can see those numbers in $g(x) = (x - 3)^2 - 2$. The x-value is the solution to $(x - 3) = 0$, and the y-value is the constant added at the end.

Example 10. *Here are the graphs of three more functions with formulas in vertex form. Compare each function with the vertex of its graph.*



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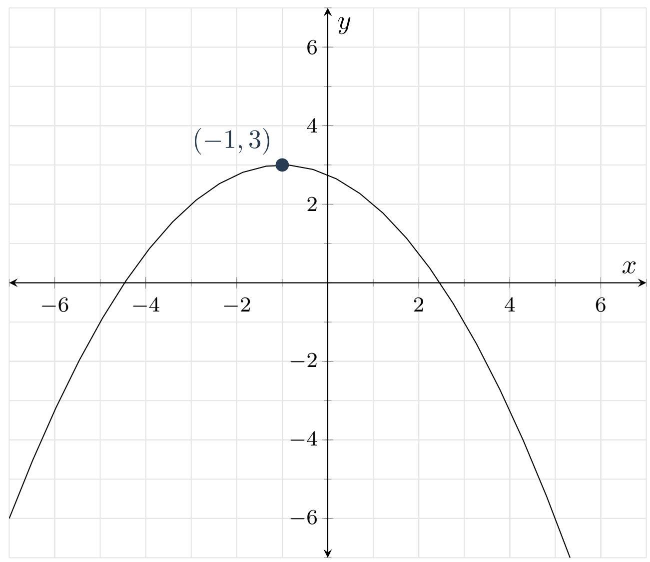
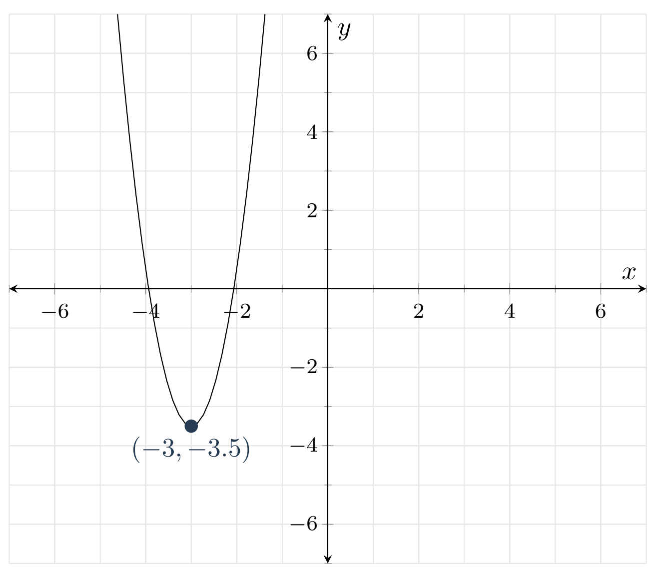
$$r(x) = (x-2)^2 + 1$$



$$s(x) = -\frac{1}{4}(x+1)^2 + 3$$



$$t(x) = 4(x+3)^2 - 3.5$$

Notice that the x-coordinate of the vertex has the opposite sign as the value in the function formula. On the other hand, the y-coordinate of the vertex has

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the same sign as the value in the function formula. Let's look at an example to understand why. We will evaluate $r(2)$.

$r(2) = (2 - 2)^2 + 1 = 1$

The x-value is the solution to $(x - 2) = 0$, which is positive 2. When we substitute 2 for x we get the value $y = 1$. Note that these coordinates create the vertex at $(2, 1)$. Now we can define the vertex form of a quadratic function.

Completing the square

In order to convert a quadratic from standard form to vertex form, we will use a technique called completing the square. First, we'll look at how to use this technique when the coefficient of x^2 is 1.

Consider the following squares of binomials.

Square of binomial $(x+p)^2$	p	$2p$	p^2
$(x+5)^2 = x^2 + 10x + 25$	5	$2(5) = 10$	$5^2 = 25$
$(x-3)^2 = x^2 - 6x + 9$	-3	$2(-3) = -6$	$(-3)^2 = 9$
$(x-12)^2 = x^2 - 24x + 144$	-12	$2(-12) = -24$	$(-12)^2 = 144$

In each case, the square of the binomial is a quadratic trinomial,

$$(x+p)^2 = x^2 + 2px + p^2.$$

Note that the coefficient of the linear term,

$$2p,$$

is twice the constant in the binomial, and the constant term of the trinomial,

$$p^2,$$

is the square of the constant in the binomial.

Question 1 *Question: What is the linear term of* $(x+6)^2$*?*

Multiple Choice:

(a) x^2

(b) $12x$

(c) $6x$

(d) 36

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We would like to reverse the process and write a quadratic expression as the square of a binomial. For example, what constant term can we add to

$$x^2 - 16x$$

to produce a perfect square trinomial? Compare the expression to the formula above:

$$\begin{aligned} x^2 + 2px + p^2 &= (x+p)^2 \\ x^2 - 16x + \underline{\qquad} &= (x + \underline{\qquad})^2. \end{aligned}$$

We see that

$$2p = -16,$$

so

$$p = \frac{1}{2}(-16) = -8,$$

and

$$p^2 = (-8)^2 = 64.$$

We substitute these values for p^2 and p into the equation to find

$$x^2 - 16x + \boxed{?} = (x + \boxed{?})^2.$$

Notice that in the resulting trinomial, the constant term is equal to the square of one-half the coefficient of x. In other words, we can find the constant term by taking one-half the coefficient of x and then squaring the result. Adding a constant term obtained in this way is called completing the square.

Converting to Vertex Form by Completing the Square

Now we will use completing the square to solve quadratic equations. First, we will solve equations in which the coefficient of the squared term is 1. Consider the function

$$f(x) = x^2 - 6x - 7,$$

and follow the steps to find the solutions.

Step 1

Begin by separating the constant term from the other terms of the equation, to get

$$f(x) = (x^2 - 6x \underline{\qquad}) + (-7 \underline{\qquad}).$$

Step 2

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Now complete the square in the parentheses on the left. Because

$$p = \frac{1}{2}(-6) = -3$$

and

$$p^2 = (-3)^2 = 9,$$

we add and subtract 9 in our equation to get

$$f(x) = (x^2 - 6x + 9) + (-7 - 9).$$

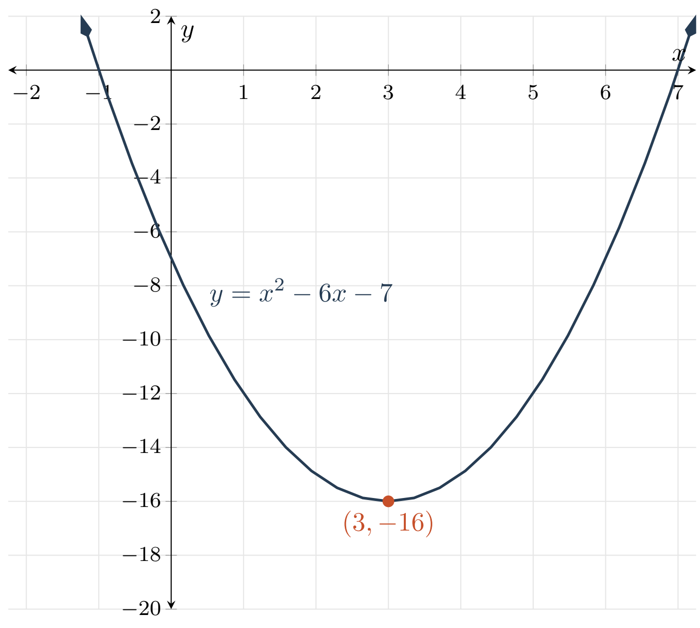
Notice that this doesn't change the value of the function, since we're simply adding $9 - 9 = 0$ and re-grouping.

Step 3

The expression in the left set of parentheses is now the square of a binomial, namely $(x-3)^2$. We write the left side in its square form and simplify the right side, which gives us the final vertex form,

$$f(x) = (x-3)^2 - 16.$$

As before, you can verify that this function is equivalent to the original by expanding the binomial.



Graph of $f(x) = x^2 - 6x - 7$

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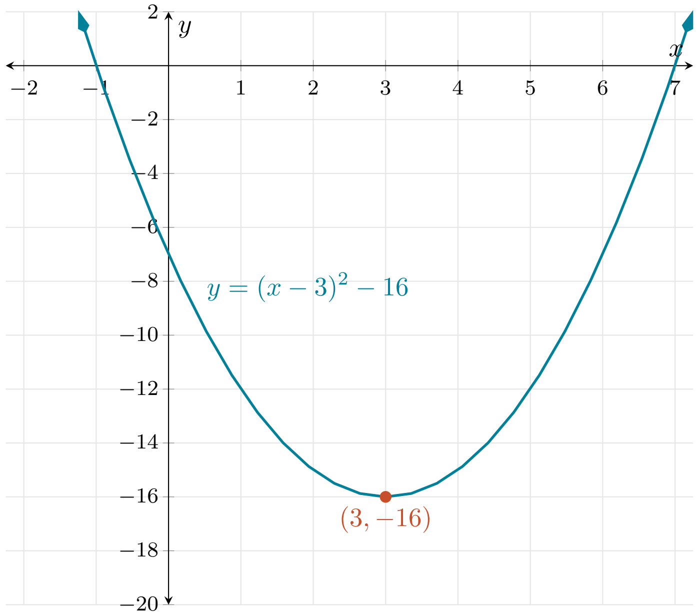

Graph of $g(x) = (x-3)^2 - 16$

Example 11. *Convert $f(x) = x^2 - 4x - 3$ to vertex form by completing the square.*

Explanation

1. Separate the constant term:

$$f(x) = (x^2 - 4x \underline{\qquad}) + (-3 \underline{\qquad}).$$

2. Complete the square in the left parentheses. The coefficient of x is -4, so

$$p = \frac{1}{2}(-4) = -2,$$

and

$$p^2 = (-2)^2 = 4.$$

Add and subtract 4 to the equation:

$$f(x) = (x^2 - 4x + 4) + (-3 - 4)$$

3. Factor the left into a binomial square and simplify the right:

$$f(x) = (x-2)^2 - 7.$$

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Take a moment to think about how to tell if $x^2 + bx + c$ is the square of a binomial.

The General Case What we've done so far only works if the coefficient of x^2 is 1. If we want to put a quadratic with leading coefficient different from 1 into vertex form, we do the same steps, but in addition factor by the leading coefficient as follows.

Example 12. *Put $f(x) = 2x^2 - 6x - 5$ into vertex form.* **Explanation**

1(a). As before, start by separating the constant term from the x and x^2 terms:

$$f(x) = (2x^2 - 6x \underline{\hspace{2em}}) - (5 \underline{\hspace{2em}})$$

1(b). Because the coefficient of x^2 is 2, we must factor the left set of parentheses by 2.

$$f(x) = 2(x^2 - 3x \underline{\hspace{2em}}) - (5 \underline{\hspace{2em}})$$

2. Complete the square in the left parentheses. The coefficient of x is -3, so

$$p = \frac{1}{2}(-3) = -\frac{3}{2},$$

and

$$p^2 = \left(-\frac{3}{2}\right)^2 = \frac{9}{4}.$$

We add $\frac{9}{4}$ to the equation as before, but now we must subtract by $2 \cdot \frac{9}{4}$ instead of just $\frac{9}{4}$ to account for the extra 2 coming from the coefficient of x^2:

$$f(x) = 2\left(x^2 - 3x + \frac{9}{4}\right) + \left(-\frac{5}{2} - 2 \cdot \frac{9}{4}\right).$$

3. Factor the left parentheses into a binomial square and simplify the constants in the right parentheses:

$$f(x) = 2\left(x - \frac{3}{2}\right)^2 - \frac{19}{4}.$$

Summary

1. (a) Write the function in standard form and separate the constant term from the x and x^2 terms.
   (b) Factor the x and x^2 terms by the coefficient of the quadratic

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term.

2. Complete the square for the x and x^2 terms:

   (a) Multiply the coefficient of the first-degree term by one-half, then square the result.

   (b) Add the value obtained in (a) inside the parentheses containing x and x^2, then subtract this value times the original coefficient of x^2 from the constant term.

3. Write the first set of parentheses as the square of a binomial. Simplify the constant term.

4.2.3 Factored Form

We have previously looked at different forms of quadratic functions. We've looked at standard form and vertex form, where characteristics like y-intercept and vertex can be found easily by looking at the function. Another useful way to look at quadratic functions is to have them written out as a product of linear factors. This can help us to quickly determine the x-intercepts of a quadratic function and to get a good idea of the position and shape of the graph. Not all quadratics can be written in factored form, so we will begin by addressing those.

Remark **Irreducible quadratic factors** are quadratic factors that when set equal to zero only have complex roots. As a result they cannot be reduced into factors containing only real numbers, hence the name irreducible.

As seen in the graphs below, the graphs of the functions do not cross the x-axis, so they do not have x-intercepts. The first graph, $y = x^2 + x + 1$ is entirely above the x-axis and the second graph, $y = -x^2 + x - 1$ is entirely below the x-axis. Since neither of them cross the x-axis, they have no x-intercepts and are irreducible.

Factored Form (or Root Form)

Factored (Root) Form of a Quadratic Function
A quadratic function whose graph has x-intercepts (called roots) at the points $(r, 0)$ and $(s, 0)$ can be written as:

$$f(x) = a(x - r)(x - s)$$

This form is called **Root Form** because the roots of the quadratic can be easily read off from this form. It is also sometimes called **Factored Form** because the quadratic is factored into a product of linear terms. We often call quadratics written as $(ax + b)(cx + d)$ a **factored** quadratic. This is not quite the same as Factored Form (Root Form) though because the leading constant is not pulled out to the front. To minimize this confusion, we will typically but not exclusively use the name Root Form.

Factoring from Standard Form when $a = 1$

When $a = 1$, putting a quadratic in Root Form is the same as factoring a quadratic. In general, **factoring** refers to writing as a product of linear terms, but does not necessarily imply that the a term is pulled out front like it is in

Root Form.

Example 13. *Factor the following quadratic into a product of linear factors:*

$$x^2 + 3x + 2$$

Explanation For us to begin factoring this quadratic, we have to look at the b and c terms. We are looking for 2 numbers that muliply to 2 (or c) and add up to 3 (or b). By going through the factors of 2 we can see that the only numbers that satisfy these conditions are 2 and 1.

$$2 + 1 = 3$$

$$2 \cdot 1 = 2$$

This means that we can factor the quadratic the following way:

$$x^2 + 3x + 2 = (x + 2)(x + 1)$$

The quadratic is now written as a product of linear factors and because $a = 1$, these are also our x-intercepts (or roots) for our function.

Factoring from Standard Form when $a > 1$

Example 14. *Rewrite the following quadratic in Root Form.*

$$f(x) = 3x^2 - 4x - 7$$

Explanation First we will start by pulling out a 3 from every term.

$$3\left(x^2 - \frac{4}{3}x - \frac{7}{3}\right)$$

We now have to find factors that add up to $-\frac{4}{3}$ and multiply to $-\frac{7}{3}$. In this particular case, we can see that the difference between the numerators (7&4) is 3, and since $\frac{3}{3} = 1$ our job will be a little easier. This leads us to the following factors:

$$\frac{3}{3} + \frac{-7}{3} = -\frac{4}{3}$$

$$\frac{3}{3} \cdot \frac{-7}{3} = -\frac{7}{3}$$

So, our factored form is as follows:

$$f(x) = 3\left(x + \frac{3}{3}\right)\left(x - \frac{7}{3}\right)$$

Note that in the previous example, it is not necessary to pull at the 3 as the first step. Instead, we could pull out the 3 as the last step and still have the root form.

Explanation This way, we start with

$$f(x) = 3x^2 - 4x - 7$$

We now have to find numbers $m_1, m_2, b_1,$ and b_2 such that:

$$\begin{aligned}(m_1x + b_1)(m_2x + b_2) &= m_1m_2x^2 + m_1b_2x + m_2b_1x + b_1b_2 \\ &= m_1m_2x^2 + (m_1b_2 + m_2b_1)x + b_1b_2 \\ &= 3x^2 - 4x - 7\end{aligned}$$

This means that we need

$$\begin{aligned} m_1m_2 &= 3 \\ m_1b_2 + m_2b_1 &= -4 \\ b_1b_2 &= -7 \end{aligned}$$

because the only way two quadratics in standard form can be equal is if they have the same coefficients for each term.

Through a little trial and error, we find that:

$$\begin{aligned} m_1 &= 1 \\ b_1 &= 1 \\ m_2 &= 3 \\ b_2 &= 7 \end{aligned}$$

will work.

We now have the equation written as a product of linear components

$$f(x) = 3x^2 - 4x - 7 = (x + 1)(3x - 7)$$

Now, to write our answer in Root Form, we just need to factor out both m_1 and m_2. Since in this example, $m_1 = 1$, we don't actually have to do any thing for that one.

$$f(x) = 3x^2 - 4x - 7 = (x+1)(3x-7) = (x+1)\left[3\left(x - \frac{7}{3}\right)\right] = 3(x+1)\left(x - \frac{7}{3}\right)$$

Now, we have our quadratic in Root Form and can read off our roots as $x = 1$ and $x = \frac{-7}{3}$.

Quadratic Formula

When there appears to be no easy way to factor a quadratic, our best option is to use the Quadratic Formula. Let's try the previous example with the Quadratic Formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

when $ax^2 + bx + c = 0$

Example 15. *Find the solutions to the following quadratic equation:*

$$3x^2 - 4x - 7 = 0$$

Explanation

$$\begin{aligned} x &= \frac{-(-4) \pm \sqrt{(-4)^2 - 4(3)(-7)}}{2(3)} \\ &= \frac{4 \pm \sqrt{16 - (-84)}}{6} \\ &= \frac{4 \pm \sqrt{100}}{6} \\ &= \frac{4 \pm 10}{6} \end{aligned}$$

$$x = \frac{4 + 10}{6} \qquad\qquad x = \frac{4 - 10}{6}$$

$$x = \frac{14}{6} \qquad\qquad x = \frac{-6}{6}$$

$$x = -\frac{7}{3} \qquad\qquad x = -1$$

We still get $x = -1$ and $x = \frac{7}{3}$ as our roots. This can be a very useful tool especially with more complicated quadratic equations.

Now that we know the roots, we can use the a-value we see in standard form, $a = 3$ and the two roots $r = 1$ and $s = \frac{-7}{3}$ to plug into our Root Form formula $a(x - r)(x - s)$, so again we get as a final answer that

$$f(x) = 3(x + 1)\left(x - \frac{7}{3}\right)$$

Factoring when missing a term

We've talked about factoring $ax^2 + bx + c$ when all terms are present, but what do we do when one of the terms is missing? Since there are three terms we have three different cases to address.

The first case also happens to be the easiest to solve. How do you a solve a quadratic that is missing the ax^2 term? This is a bit of a trick question, because without an x^2 term, we are no longer dealing with a quadratic.

Example 16. *Solve the following equation*

$$2x - 9 = 0$$

Explanation We can see here that we are only dealing with a linear terms and there are no quadratic (x^2) terms. This means we do not have to factor and we can solve for x directly.

$$\begin{aligned} 2x - 9 &= 0 \\ 2x &= 9 \\ x &= \frac{9}{2} \end{aligned}$$

The second case is when the middle bx term is missing.

Example 17. *Factor the quadratic $f(x) = x^2 - 9$ into linear components.*

Explanation This quadratic is a special case called "difference of squares." There is no "middle" term and the remaining two terms are both perfect squares, so we can use a shortcut when factoring.

> **Difference of Squares**
> When a,b are non zero.
>
> $$a^2 - b^2 = (a + b)(a - b)$$

In our case, we can see that x^2 is a perfect square and 9 is also a perfect square because $9 = 3^2$. This means that our original quadratic will be factored like this:

$$x^2 - 9 = (x + 3)(x - 3)$$

We can also think of it in the same way as factoring other quadratics. Since there is no middle term, we can look at factors of -9 that add up to 0. 3 and -3 add up to 0 and multiply out to -9. The difference of squares is just a useful pattern that helps to speed up our factoring process.

The last case is when there is no constant or c term.

Example 18. *Factor the quadratic $f(x) = x^2 + 2x$ into linear components.*

Explanation Since there is a common x factor in both terms we can pull out that factor and we are left with a product of linear components.

$$\begin{aligned} f(x) &= x^2 + 2x \\ &= x \cdot x + 2x \\ &= x(x+2) \end{aligned}$$

Again, our factoring is already simplified. We do not have go through the whole process of factoring. If we have a quadratic function with only the ax^2 and bx term, then we will always be able to pull out at least an x term when factoring.

4.3 Polynomials

Learning Objectives

- Polynomial Functions
  - Understand polynomials as functions
  - Understand linear functions and quadratic functions as polynomials
  - Recognize polynomials in standard form and factored form
  - Find roots of polynomials in factored form and determine their multiplicity
  - Determine the end behavior of polynomials
  - Sketch a graph of a polynomail based on end behavior and roots with multiplicity

4.3.1 Polynomial Functions

Constant functions, linear functions, and quadratic functions all belong to a much larger group of functions called **polynomials**.

> **Definition** Given real numbers $a_0, a_1, \ldots, a_n$ where $a_n \neq 0$, we say that the function a function P is a **polynomial** of degree n if it can be written in the form
>
> $$P(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_{n-1} x^{n-1} + a_n x^n,$$
>
> When a polynomial is written in this way, we say it is in **standard form**.

Consider $f(x) = 4x^5 - 3x^2 + 2x - 5$. Is this a polynomial function? We can re-write the formula for f as $f(x) = 4x^5 + 0x^4 + 0x^3 + (-3)x^2 + 2x + (-5)$. Comparing this with our definition of Polyomial Functions, we identify $n = 5$, $a_5 = 4$, $a_4 = 0$, $a_3 = 0$, $a_2 = -3$, $a_1 = 2$ and $a_0 = -5$. In other words, a_5 is the coefficient of x^5, a_4 is the coefficient of x^4, and so forth; the subscript on the a's merely indicates to which power of x the coefficient belongs.

Example 19. *Determine if the following functions are polynomials. Explain your reasoning.*

(a) $g(x) = \dfrac{4 + x^3}{x}$ (b) $p(x) = \dfrac{4x + x^3}{x}$ (c) $q(x) = \dfrac{4x + x^3}{x^2 + 4}$

(a) $f(x) = \sqrt[3]{x}$ (b) $h(x) = |x|$ (c) $z(x) = 0$

Explanation

(a) We note directly that the domain of $g(x) = \dfrac{x^3 + 4}{x}$ is $x \neq 0$. By definition, a polynomial has all real numbers as its domain. Hence, g can't be a polynomial.

(b) Even though $p(x) = \dfrac{x^3 + 4x}{x}$ simplifies to $p(x) = x^2 + 4$, which certainly looks like the form given in our definition of polynomials, the domain of p, which, as you may recall, we determine *before* we simplify, excludes 0. Alas, p is not a polynomial function for the same reason g isn't.

(c) After what happened with p in the previous part, you may be a little shy about simplifying $q(x) = \dfrac{x^3 + 4x}{x^2 + 4}$ to $q(x) = x$, which certainly fits our definition of polynomial functions. If we look at the domain of q before we simplified, we see that it is, indeed, all real numbers. A function which can be written in the form of of a polynomial whose domain is all real numbers is, in fact, a polynomial.

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(d) We can rewrite $f(x) = \sqrt[3]{x}$ as $f(x) = x^{\frac{1}{3}}$. Since $\dfrac{1}{3}$ is not a natural number, f is not a polynomial.

(e) The function $h(x) = |x|$ isn't a polynomial, since it can't be written as a combination of powers of x even though it can be written as a piecewise function involving polynomials. As we shall see in this section, graphs of polynomials possess a quality (which relies on Calculus to verify) that the graph of h does not. Polynomials will all be smooth with no sharp corners.

(f) There's nothing in our definition of a polynomial which prevents all the coefficients a_n, etc., from being 0. Hence, $z(x) = 0$, is an honest-to-goodness polynomial.

Definition [Polynomial Vocabulary]
Suppose f is a polynomial function.

- Given $f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_2 x^2 + a_1 x + a_0$ with $a_n \neq 0$, we say
  - The natural number n is called the **degree** of the polynomial f.
  - If $0 \leq k \leq n$, then we call $a_k x^k$ a **term** of the polynomial.
  - We call a_k the **coefficient** of the term $a_k x^k$.
  - The term $a_n x^n$ is called the **leading term** or **highest degree term** of the polynomial f.
  - The real number a_n is called the **leading coefficient** of the polynomial f.
  - The real number a_0 is called the **constant term** of the polynomial f.
  - The x-intercepts of polynomials are also called **roots**. Note that we usually reserve the word roots for talking about the x-intercepts of polynomials and don't use it for the x-intrecepts of other types of functions.
- If $f(x) = a_0$, and $a_0 \neq 0$, we say f has degree 0.
- If $f(x) = 0$, we say f has no degree.

The reader may well wonder why we have chosen to separate off constant functions from the other polynomials. Why not just lump them all together and, instead of forcing n to be a natural number, $n = 1, 2, \ldots$, allow n to be a whole number, $n = 0, 1, 2, \ldots$. We could unify all of the cases, since, after all, isn't $a_0 x^0 = a_0$? The answer is 'yes, as long as $x \neq 0$.' The function $f(x) = 3$

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and $g(x) = 3x^0$ are different, because their domains are different. The number $f(0) = 3$ is defined, whereas $g(0) = 3(0)^0$ is not. (Technically, 0^0 is an indeterminant form, which is a special case of being undefined. You will explore this more in calculus.) Indeed, much of the theory we will develop in this chapter doesn't include the constant functions, so we might as well treat them as outsiders from the start. One good thing that comes from our definition of polynomials is that we can now think of linear functions as degree 1 (or 'first degree') polynomial functions and quadratic functions as degree 2 (or 'second degree') polynomial functions.

Example 20. *Find the degree, leading term, leading coefficient and constant term of the following polynomial functions.*

(a) $f(x) = 4x^5 - 3x^2 + 2x - 5$

(b) $g(x) = 12x + x^3$

(a) $h(x) = \dfrac{4-x}{5}$

(b) $p(x) = (2x-1)^3(x-2)(3x+2)$

Explanation

(a) There are no surprises with $f(x) = 4x^5 - 3x^2 + 2x - 5$. It matches the form of a polynomial given above, and we see that the degree is 5, the leading term is $4x^5$, the leading coefficient is 4 and the constant term is -5.

(b) The form given in has the highest power of x first. To that end, we rewrite $g(x) = 12x + x^3 = x^3 + 12x$, and see that the degree of g is 3, the leading term is x^3, the leading coefficient is 1 and the constant term is 0.

(c) We need to rewrite the formula for h so that it resembles the form given in our definition of polynomials: $h(x) = \frac{4-x}{5} = \frac{4}{5} - \frac{x}{5} = -\frac{1}{5}x + \frac{4}{5}$. The degree of h is 1, the leading term is $-\frac{1}{5}x$, the leading coefficient is $-\frac{1}{5}$ and the constant term is $\frac{4}{5}$.

(d) It may seem that we have some work ahead of us to get p in standard form. However, it is possible to glean the information requested about p without multiplying out the entire expression $(2x-1)^3(x-2)(3x+2)$. The leading term of p will be the term which has the highest power of x. The way to get this term is to multiply the terms with the highest power of x from each factor together - in other words, the leading term of $p(x)$ is the product of the leading terms of the factors of $p(x)$. Hence, the leading term of p is $(2x)^3(x)(3x) = 24x^5$. This means that the degree of p is 5 and the leading coefficient is 24. As for the constant term, we can perform a similar trick. The constant term is obtained by multiplying the constant terms from each of the factors $(-1)^3(-2)(2) = 4$.

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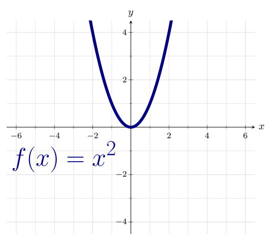
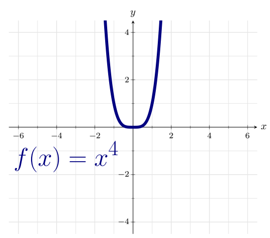
End Behavior of Polynomials

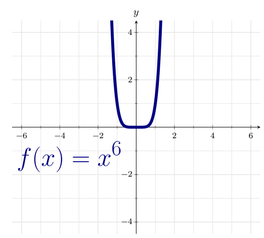
The end behavior of a function is a way to describe what is happening to the function values (the y-values) as the x-values go off the graph on the left and right sides. That is, what happens to y as x becomes large (in the sense of its absolute value) and negative without bound (written $x \to -\infty$) and, on the flip side, as x becomes large and positive without bound (written $x \to \infty$).

For example, given $f(x) = x^2$, as $x \to -\infty$, we imagine substituting $x = -100$, $x = -1000$, etc., into f to get $f(-100) = 10000$, $f(-1000) = 1000000$, and so on. Thus the function values are becoming larger and larger positive numbers (without bound). To describe this behavior, we write: as $x \to -\infty$, $f(x) \to \infty$. If we study the behavior of f as $x \to \infty$, we see that in this case, too, $f(x) \to \infty$.

The same can be said for any function of the form $f(x) = x^n$ where n is an even natural number. For example, the functions $f(x) = x^2$, $f(x) = x^4$, and $f(x) = x^6$ are graphed below.



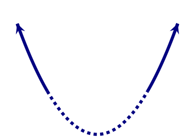




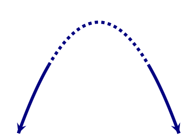
[End Behavior of functions $f(x) = ax^n$, n even.]
Suppose $f(x) = ax^n$ where $a \neq 0$ is a real number and n is an even natural number. The end behavior of the graph of $y = f(x)$ matches one of the following:

- for $a > 0$, as $x \to -\infty$, $f(x) \to \infty$ and as $x \to \infty$, $f(x) \to \infty$
- for $a < 0$, as $x \to -\infty$, $f(x) \to -\infty$ and as $x \to \infty$, $f(x) \to -\infty$

Graphically:



$a > 0, n$ even



$a < 0, n$ even

We now turn our attention to functions of the form $f(x) = x^n$ where $n \geq 3$

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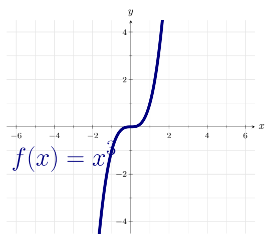
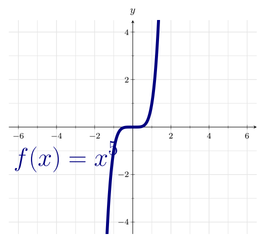
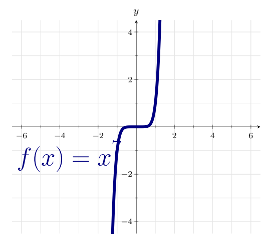
is an odd natural number. (We ignore the case when $n = 1$, since the graph of $f(x) = x$ is a line and doesn't fit the general pattern of higher-degree odd polynomials.) Below we have graphed $y = x^3$, $y = x^5$, and $y = x^7$. The 'flattening' and 'steepening' that we saw with the even powers presents itself here as well, and, it should come as no surprise that all of these functions are odd. The end behavior of these functions is all the same, with $f(x) \to -\infty$ as $x \to -\infty$ and $f(x) \to \infty$ as $x \to \infty$.





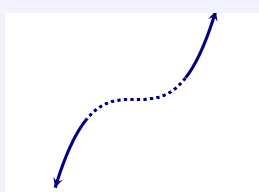


As with the even degreed functions we studied earlier, we can generalize their end behavior.

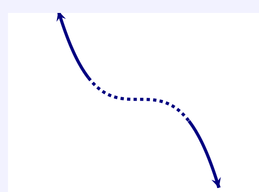
[End Behavior of functions $f(x) = ax^n$, n odd.]
Suppose $f(x) = ax^n$ where $a \neq 0$ is a real number and n is an odd natural number. The end behavior of the graph of $y = f(x)$ matches one of the following:

- for $a > 0$, as $x \to -\infty$, $f(x) \to -\infty$ and as $x \to \infty$, $f(x) \to \infty$
- for $a < 0$, as $x \to -\infty$, $f(x) \to \infty$ and as $x \to \infty$, $f(x) \to -\infty$

Graphically:



$a > 0, n$ odd



$a < 0, n$ odd

Now lets consider the end behavior of polynomials in general. It turns out that the end behavior of a polynomial always matches the end behavior of its leading term.

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[End Behavior of Polynomials] The end behavior of a polynomial $f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_2 x^2 + a_1 x + a_0$ with $a_n \neq 0$ matches the end behavior of $y = a_n x^n$.

To see why this theorem is true, let's first look at a specific example. Consider $f(x) = 4x^3 - x + 5$. If we wish to examine end behavior, we look to see the behavior of f as $x \to \pm\infty$. Since we're concerned with x's far down the x-axis, we are far away from $x = 0$ so can rewrite $f(x)$ for these values of x as

$$f(x) = 4x^3 \left(1 - \frac{1}{4x^2} + \frac{5}{4x^3}\right)$$

As x becomes unbounded (in either direction), the terms $\frac{1}{4x^2}$ and $\frac{5}{4x^3}$ become closer and closer to 0, as the table below indicates.

x	$\frac{1}{4x^2}$	$\frac{5}{4x^3}$
-1000	0.00000025	-0.00000000125
-100	0.000025	-0.00000125
-10	0.0025	-0.00125
10	0.0025	0.00125
100	0.000025	0.00000125
1000	0.00000025	0.00000000125

In other words, as $x \to \pm\infty$, $f(x) \approx 4x^3(1 - 0 + 0) = 4x^3$, which is the leading term of f. The formal proof of this theorem requires calculus, but it works in much the same way. Factoring out the leading term leaves

$$f(x) = a_n x^n \left(1 + \frac{a_{n-1}}{a_n x} + \ldots + \frac{a_2}{a_n x^{n-2}} + \frac{a_1}{a_n x^{n-1}} + \frac{a_0}{a_n x^n}\right)$$

As $x \to \pm\infty$, any term with an x in the denominator becomes closer and closer to 0, and we have $f(x) \approx a_n x^n$.

Geometrically, this theorem says that if we graph $y = f(x)$ using a graphing calculator, and continue to 'zoom out', the graph of it and its leading term become indistinguishable. Below are the graphs of $y = 4x^3 - x + 5$ (the thicker line) and $y = 4x^3$ (the thinner line) in two different windows.

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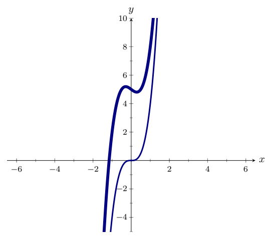
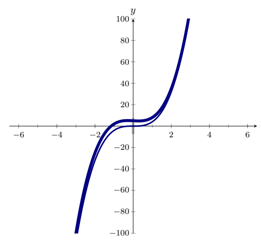
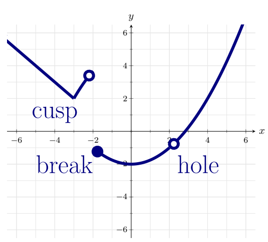
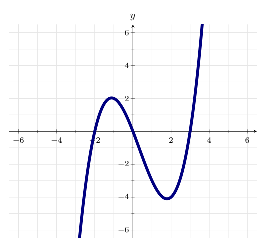
 

Other Properties of Polynomial Graphs

Despite having different end behavior, all functions of the form $f(x) = ax^n$ for natural numbers n share two special function properties: they are **continuous** and **smooth**. While these concepts are formally defined using Calculus, informally, graphs of continuous functions have no 'breaks' or 'holes' in them, and the graphs of smooth functions have no 'sharp turns'. It turns out that these traits are preserved when functions are added together, so general polynomial functions inherit these qualities. Below we find the graph of a function which is neither smooth nor continuous, and to its right we have a graph of a polynomial, for comparison. The function whose graph appears on the left fails to be continuous where it has a 'break' or 'hole' in the graph; everywhere else, the function is continuous. The function is continuous at the 'corner' and the 'cusp', but we consider these 'sharp turns', so these are places where the function fails to be smooth. Apart from these four places, the function is smooth and continuous. Polynomial functions are smooth and continuous everywhere, as exhibited in the graph on the right.





The notion of smoothness is what tells us graphically that, for example, $f(x) = |x|$, whose graph is the characteristic '$\vee$' shape, cannot be a polynomial.

Roots of Polynomials

We will often what to find the x-intercepts or roots of polynomials. To do this, we will use the fact that a product of factors can only equal 0 if one of the factors equals zero. Consider the following example.

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Example 21. *Find the roots of* $f(x) = x^3(x-3)^2(x+2)\left(x^2+1\right)$.

Explanation We are looking for the x-values where

$$x^3(x-3)^2(x+2)\left(x^2+1\right) = 0$$

Because we have a product of factors equal to 0, this will equal 0 exactly when one of the factors equals 0. Therefore, we have:

$$\begin{array}{ccccccc} x^3 = 0 & \text{or} & (x-3)^2 = 0 & \text{or} & (x+2) = 0 & \text{or} & (x^2+1) = 0 \\ x = 0 & & (x-3) = 0 & & (x+2) = 0 & & x^2 = -1 \\ & & x = 3 & & x = -2 & & \text{impossible} \end{array}$$

Notice that the x^2+1 factor does not give any roots. This is because it is an irreducible quadratic.

You may notice that this polynomial was given as a product of linear factors and irreducible quadratic factors. This was not a coincidence. We have the following theorem.

> [Fundemental Theorem of Algebra] Every polynomial can be written as the product of linear factors and irreducible quadratic factors.

This way of writing polynomials is extremely helpful when you want to find the roots. Its is so helpful we give it a name.

> **Definition** A polynomial is written in Factored Form (or Root Form) when it is written as a product of linear and irreducible quadratic factors with the leading coefficient factored out.
>
> $p(x) = a(x-r_1)(x-r_2)...(x-r_k)(x^2+b_1x+c_1)(x^2+b_2x+c_2)...(x^2+b_lx+c_l)$
>
> where $(x^2+b_1x+c_1), (x^2+b_2x+c_2), ..., (x^2+b_lx+c_l)$ are all irreducible quadratics.

It turns out that that $x = r$ is a root of the polynomial $p(x)$ exactly when $(x-r)$ is a factor that appears when $p(x)$ is written in Factored Form, just like in our example above.

> **Remark** Even though all polynomials can be written this way, that does not mean it is easy (or even possible) to take a general polynomial in standard form and write it in factored form! There is no equivalent of the quadratic formula for polynomials of high enough degree.

Acknowledgements: Stitz Zeager Open Source Mathematics (https://www.stitz-zeager.com/)

You may noticed another phenomena in our earlier example. Some factors were raised to a power higher than 1. For example, in $f(x) = x^3(x-3)^2(x+2)\left(x^2+1\right)$, the factor $(x-3)^2$ is raised to a the second power. This does not mean we get any additional roots. Notice that $(x-3)^2 = (x-3)(x-3)$ and so both of these factors give a root of $x = 3$. Instead, the second power has an impact on the way the graph of the polynomial, $f(x)$, looks when it is near the root $x = 3$. On either side of $x = 3$, the graph of $f(x)$ will bounce off the x-axis and turn around just like the graph of $y = x^2$ does as the point $(0, 0)$. In general, it will turn out that if we have $(x-r)^m$ in the factored form of our polynomial, then the graph of the polynomial near $x = r$ will look similar to how the graph of $y = x^m$ looks near $(0, 0)$.

We give this concept a name and solidify it with a theorem.

Definition Suppose f is a polynomial function and m is a natural number. If $(x-c)^m$ is a factor of $f(x)$ but $(x-c)^{m+1}$ is not, then we say $x = c$ is a zero of **multiplicity** m.

Hence, rewriting $f(x) = x^3(x-3)^2(x+2)$ as $f(x) = (x-0)^3(x-3)^2(x-(-2))^1$, we see that $x = 0$ is a zero of multiplicity 3, $x = 3$ is a zero of multiplicity 2 and $x = -2$ is a zero of multiplicity 1.

[Role of Multiplicity] Suppose f is a polynomial function and $x = c$ is a zero of multiplicity m.

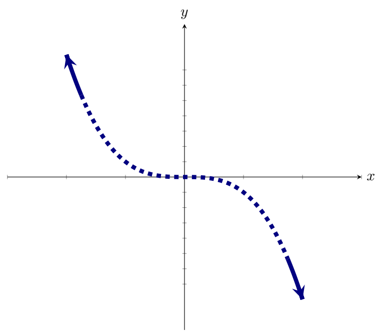
- If m is even, the graph of $y = f(x)$ touches and rebounds from the x-axis at $(c, 0)$.
- If m is odd, the graph of $y = f(x)$ crosses through the x-axis at $(c, 0)$.

Graphing Polynomials

Our last example shows how end behavior and multiplicity allow us to sketch a decent graph without appealing to a sign diagram.

Example 22. *Sketch the graph of $f(x) = -3(2x-1)(x+1)^2$ using end behavior and the multiplicity of its zeros.* **Explanation** The end behavior of the graph of f will match that of its leading term. To find the leading term, we multiply by the leading terms of each factor to get $(-3)(2x)(x)^2 = -6x^3$. This tells us that the graph will start above the x-axis, in Quadrant II, and finish below the x-axis, in Quadrant IV.

Acknowledgements: Stitz Zeager Open Source Mathematics (https://www.stitz-zeager.com/)



Note that we don't yet know what is happening in that dotted region in the middle.

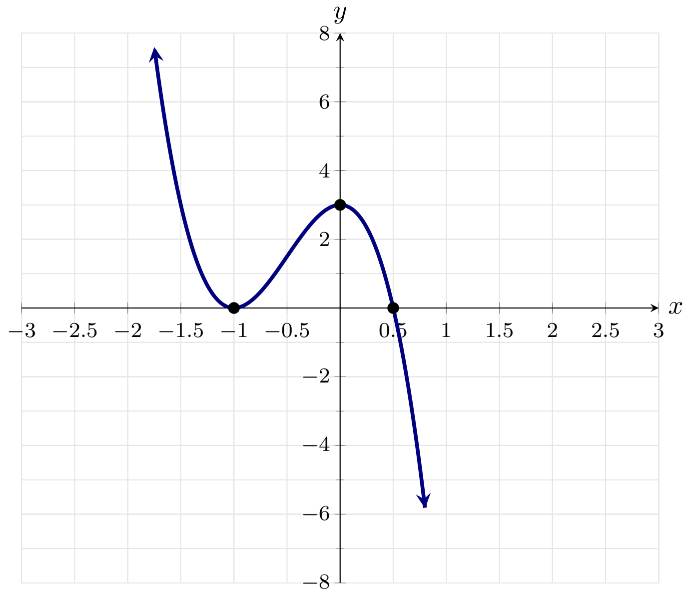
Next, we find the zeros of f. Fortunately for us, f is factored. Setting each factor equal to zero gives is $x = \dfrac{1}{2}$ and $x = -1$ as zeros.

To find the multiplicity of $x = \dfrac{1}{2}$ we note that it corresponds to the factor $(2x - 1)$. This isn't strictly in Factored Form. If we factor out the 2, however, we get $(2x-1) = 2\left(x - \dfrac{1}{2}\right)$, and we see that the multiplicity of $x = \dfrac{1}{2}$ is 1. Since 1 is an odd number, we know from our theorem about multiplicity that the graph of f will cross through the x-axis at $\left(\dfrac{1}{2}, 0\right)$. What's more, we know that the graph will pass right through the x-axis at $x = 0.5$ without flattening out because the graph will look similar to the way $y = x$ looks at $(0, 0)$ when we zoom in around $(0.5, 0)$.

Since the zero $x = -1$ corresponds to the factor $(x+1)^2 = (x-(-1))^2$, we find its multiplicity to be 2 which is an even number. As such, the graph of f will touch and rebound from the x-axis at $(-1, 0)$.

Though we're not asked to, we can find the y-intercept by finding $f(0) = -3(2(0) - 1)(0 + 1)^2 = 3$. Thus $(0, 3)$ is an additional point on the graph. Putting this together gives us the graph below.

Acknowledgements: Stitz Zeager Open Source Mathematics (https://www.stitz-zeager.com/)



Note that while we can say a lot about how graphs of polynomials will look, we cannot say exactly where the turning points (peaks and valleys) will be. That will require calculus.

Factoring

The following section will contain some techniques useful for factoring polynomials.

A common technique is factoring out a common factor. For example, in the polynomial $f(x) = x^4 - x^3$, each term contains a factor of x^3. Therefore, we can factor it out to find $f(x) = x^3(x-1)$.

To factor a polynomial like

$$f(x) = 2x^2 - 11x + 5,$$

we first multiply together the leading coefficient and the constant term to find 10. Our goal is then to find factors of 10 that sum to the linear coefficient, which in this case is -11. -1 and -10 fit the bill, since $(-1)(-10) = 10$ and $(-1) + (-10) = -11$. We can now replace the linear term $-11x$ with $-x - 10x$. This yields

$$2x^2 - x - 10x + 5.$$

We can then group the terms by two:

$$(2x^2 - x) + (-10x + 5).$$

Acknowledgements: Stitz Zeager Open Source Mathematics (https://www.stitz-zeager.com/)

Factoring out a common factor from each yields

$$x(2x - 1) + -5(2x - 1).$$

Now, we factor our a common factor of $2x - 1$ from each term, yielding

$$(x - 5)(2x - 1).$$

This completes the factorization.

To factor a difference of squares $a^2 - b^2$, we can use the formula

$$a^2 - b^2 = (a + b)(a - b),$$

which can be checked by multiplying out the right-hand side.

To factor a difference of cubes, $a^3 - b^3$, we can use the formula

$$a^3 - b^3 = (a - b)(a^2 + ab + b^2),$$

which can similarly be checked by multiplying out the right-hand side.

4.4 Roots

Learning Objectives

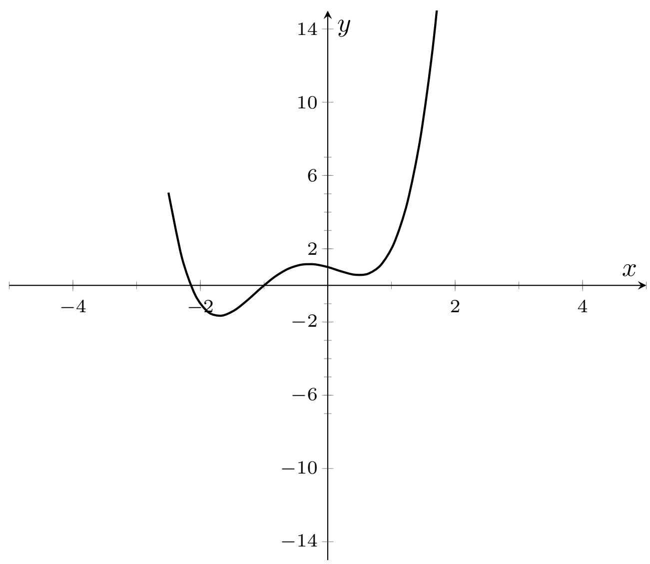
- Roots
  - When in the inverse function for $f(x) = x^n$ defined? What are it's function properties?

4.4.1 Roots

Motivating Questions

- Can we find the inverse of a polynomial?
- What does it mean to take the n^{th} root of a value?

Consider the polynomial $p(x) = x^4 + 2x^3 - x^2 - x + 1$. A good question to ask would be whether the function p is invertible. To help us decide, here is the graph of $p(x)$.

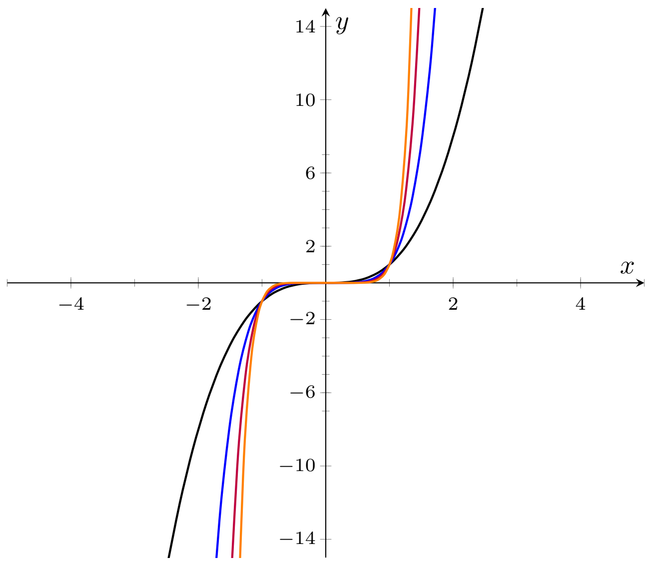


Notice that this graph does not pass the Horizontal Line Test, so the function p is not one-to-one, and therefore not invertible.

Our polynomial $p(x)$ has many terms, so to simplify the situation, we'll look only at polynomials of the form x^n, where n is a positive integer.

Odd Roots

Recall that every polynomial $p(x) = x^n$, where n is odd, has the same basic shape. This is demonstrated in the figure below by the graphs of $y = x^n$ for $n = 3, 5, 7, 9$.



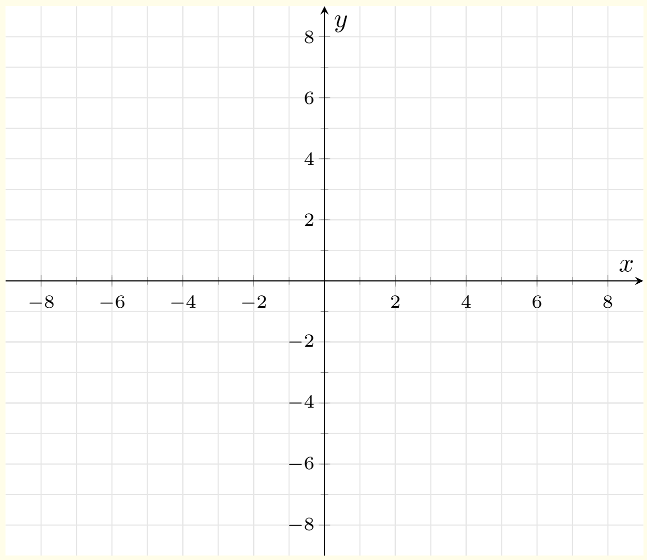
To see more of how these graphs change with n, follow the Desmos link: `https://www.desmos.com/calculator/viuyei80a0`.

Now, are these functions invertible? Looking at the graphs, we see that these functions pass the Horizontal Line Test. Thus, the functions are one-to-one, and therefore invertible.

> **Definition** When n is an odd positive integer, we define **the nth root function** $\sqrt[n]{x}$ to be the inverse of the function defined by x^n. The number n is the **index** of the root, and x is the **radicand**. We call the symbol $\sqrt{}$ the **radical**.

Let's delve more deeply into the $p(x) = x^3$ example. We have now established that it is invertible, and it's inverse is $r(x) = \sqrt[3]{x}$.

> **Exploration** Draw both functions on the axes provided, then answer the following questions about the function $r(x)$.



(a) What is the x-intercept of $r(x)$? $(\boxed{?}, \boxed{?})$

(b) What is the y-intercept of $r(x)$? $(\boxed{?}, \boxed{?})$

(c) What is the domain of $r(x)$? $(\boxed{?}, \boxed{?})$

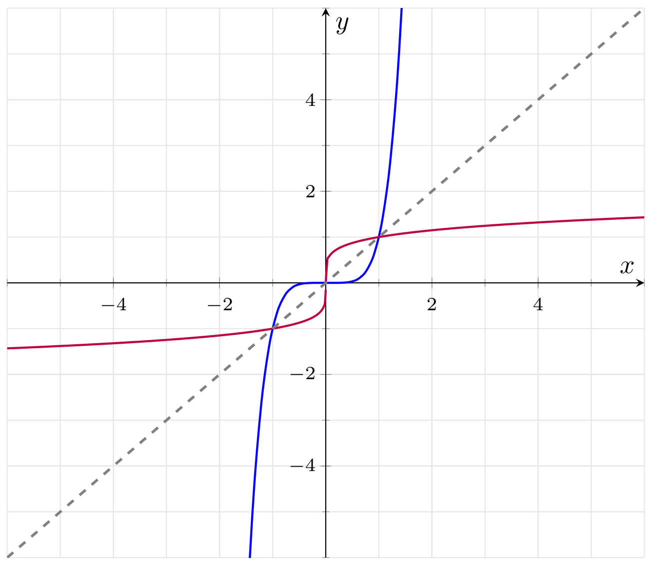
(d) What is the range of $r(x)$? $(\boxed{?}, \boxed{?})$

(e) As x goes to ∞, y goes to $\boxed{?}$.

(f) As x goes to $-\infty$, y goes to $\boxed{?}$.

(g) Does this function have any vertical asymptotes? (yes/ no)

Example 23. *Below, we have an example of the graph of $y = x^5$ and the two functions*

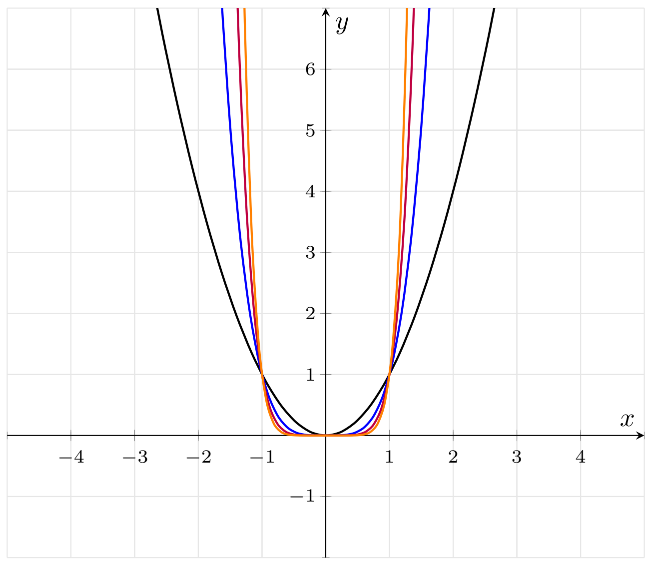


Recall that a function f is *even* if $f(-x) = f(x)$ for all x in its domain, and f is *odd* if $f(-x) = -f(x)$ for all x. Otherwise, the function is neither. Let's consider an example. Given $x = 8$, $r(x) = \sqrt[3]{8} = 2$ and $r(-x) = \sqrt[3]{-8} = -2$, since $(-2) \cdot (-2) \cdot (-2) = (4) \cdot (-2) = -8$. Based on this example, do you think $r(x)$ is even, odd, or neither?

If you guessed odd, then you are correct! All odd-index root functions are odd functions.

Even Roots

We again begin by recalling the general shape of $p(x) = x^n$, but this time for n *even*. These functions also have the same basic shape for all even n. This is demonstrated by the graphs of $y = x^n$ for $n = 2, 4, 6, 8$ given below.

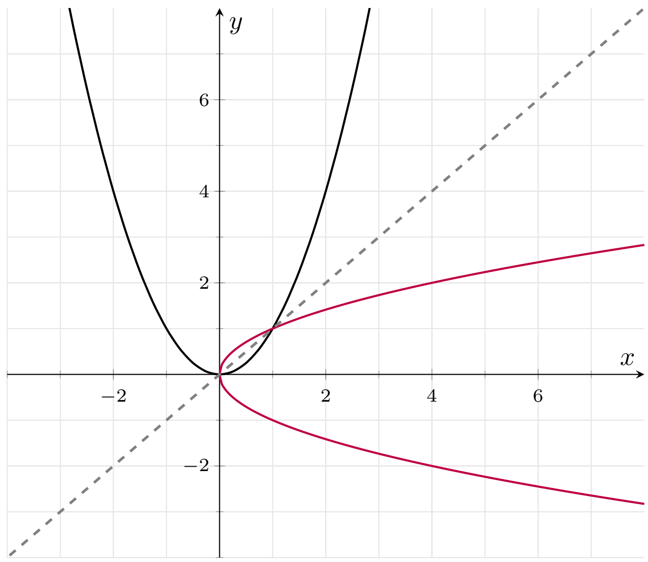


To see more of how these graphs change with n, follow the Desmos link: https://www.desmos.com/calculator/qpqwtrppqt.

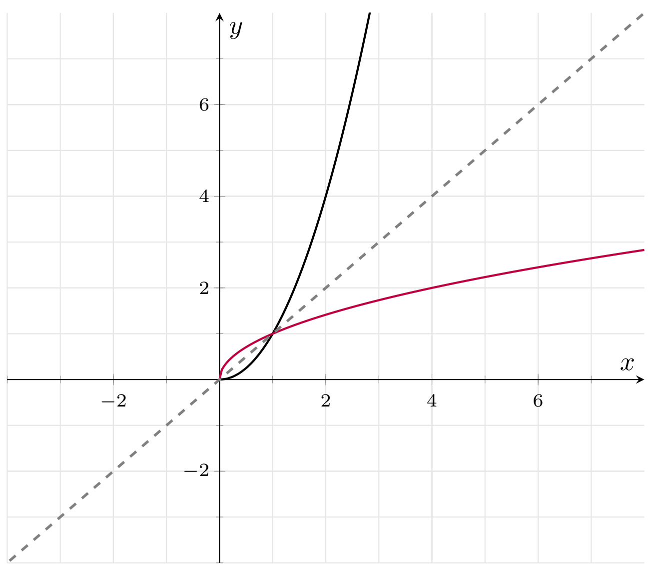
Now, are these functions invertible? All of the graphs in the figure above are symmetric about the y-axis (example $2^2 = 4 = (-2)^2$), so they do *not* pass the horizontal line test. Thus, these functions are *not* one-to-one, and therefore *not* invertible.

So, how can we define an even root function? For example, what does $\sqrt{x}$ really mean, and how is it related to x^2?

Consider the polynomial $p(x) = x^2$, graphed below with its inverse relation $\{(x^2, x) : x \text{ is a real number}\}$.



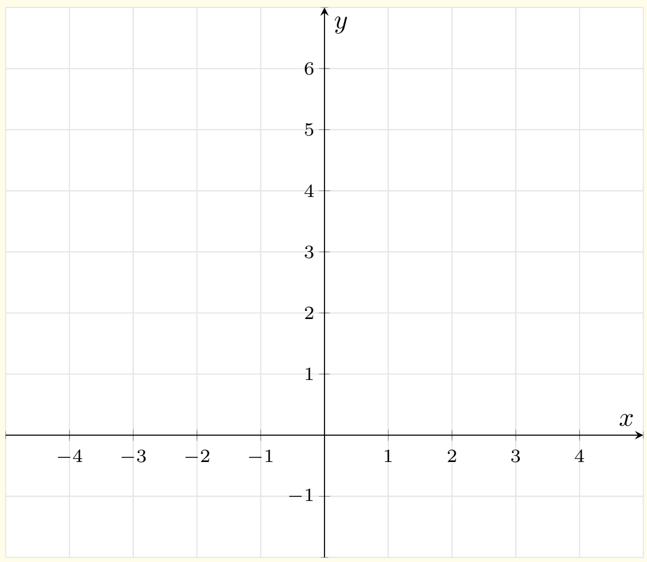
Observe that by restricting the domain of p to $x \geq 0$, we now have a function which passes the horizontal line test, and can thus be inverted. The following picture illustrates the situation.



Now, our inverse relation is actually a function, since it passes the Horizontal Line Test. Therefore, if we let $p(x) = x^2$ for $x \geq 0$, we can then define $r(x) = \sqrt[2]{x} = \sqrt{x}$ as the inverse function of $p(x)$ on this restricted domain.

Definition When n is an even positive integer, we define **the nth root function** $\sqrt[n]{x}$ to be the inverse of the function defined by x^n restricted to the domain $x \geq 0$.

Exploration We now repeat Exploration 1 for $r(x) = \sqrt[4]{x}$. Draw both functions on the axes provided, then answer the following questions about the function $r(x)$.



(a) What is the x-intercept of $r(x)$? $(\boxed{?}, \boxed{?})$

(b) What is the y-intercept of $r(x)$? $(\boxed{?}, \boxed{?})$

(c) What is the domain of $r(x)$? $[\boxed{?}, \boxed{?})$

(d) What is the range of $r(x)$? $[\boxed{?}, \boxed{?})$

(e) As x goes to ∞, y goes to $\boxed{?}$.

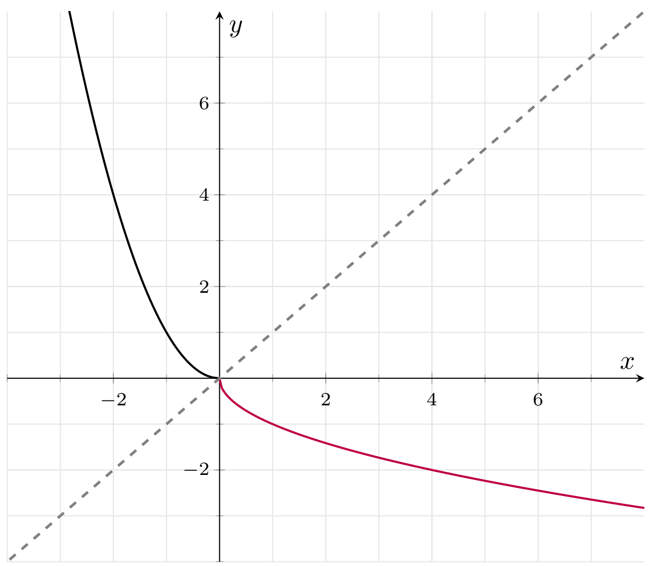
(f) Does this function have any vertical asymptotes? (yes/ no)

One question we might ask is whether $r(x) = \sqrt{x}$ and $p(x) = x^2$ are truly inverses. The answer may seem like an obvious "yes!", but since we restricted the domain of p in order to define r, we need to check. To check whether r and p are inverses, we need to confirm that $r(p(x)) = \sqrt{x^2} = x$ and $p(r(x)) = (\sqrt{x})^2 = x$. That is, when we plug in a number to $\sqrt{x^2}$ and $(\sqrt{x})^2$, we should get the same number as the output. Let's try plugging in -1 to $r(p(x))$. This gives us

$$r(p(-1)) = r((-1)^2) = \sqrt{(-1)^2} = \sqrt{1} = 1,$$

which is not the same as -1. If we repeat this process with a few more numbers, we find that $\sqrt{(-2)^2} = 2$, $\sqrt{1^2} = 1$, $\sqrt{(-45)^2} = 45$, and $\sqrt{98^2} = 98$. We can conclude that $\sqrt{x^2}$ is a function that takes its input and returns its absolute value. That is, $\sqrt{x^2} = |x|$. Since $\sqrt{x^2} = r(p(x))$, we conclude that $r(p(x))$ does not output its input, and therefore, r and p are not inverses. This is something that will be extremely important when solving equations using even roots.

Now, what if we instead restricted our domain to $x \leq 0$? Consider $q(x) = x^2$ defined for $x \leq 0$. The graph of this function is below.



By the Horizontal Line Test, this restriction is one-to-one, and therefore invertible. The inverse of this function as shown above is $s(x) = -\sqrt{x} = -r(x)$.

Example 24. *We demonstrate a few common even and odd n^{th} roots to highlight this distinction.*

(a) $\sqrt[3]{8} = 2$, *since* $2 \cdot 2 \cdot 2 = 8$.

(b) $\sqrt[3]{-8} = -2$, *since* $-2 \cdot (-2) \cdot (-2) = 4 \cdot (-2) = -8$.

(c) $\sqrt[4]{16} = 2$, *since* $2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 4 \cdot 4 = 16$. *However, the* 4^{th} *root of -16 is not defined.*

(d) $\sqrt[4]{0} = 0$, *since zero times any number is always zero. This is* the *example of an even* n^{th} *root that has only* one *solution.*

(e) *Likewise,* $\sqrt[125]{0} = 0$.

Using Roots to Solve Equations

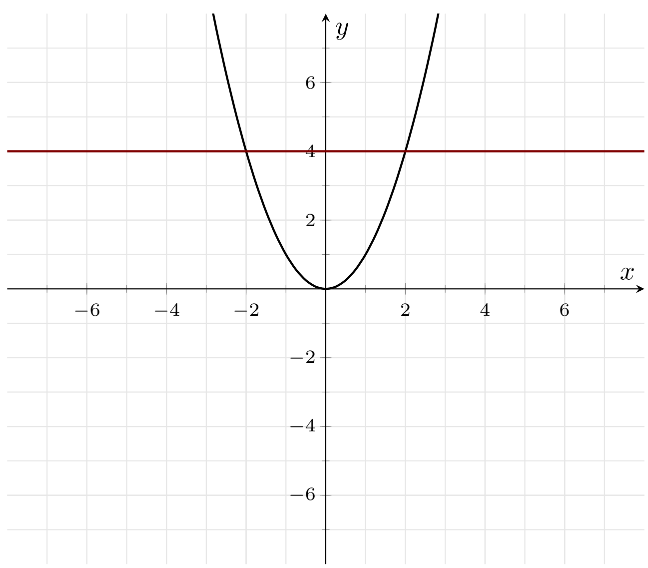
If we are asked to find all values x such that $x^2 = 4$, then the question is asking which values of x multiplied by themselves give 4. In other words, find x such that $x \cdot x$ is equal to 4. It is simple to see that there are two values which make this true:

$$2 \cdot 2 = 4 \text{ and } (-2) \cdot (-2) = 4.$$

In solving an equation, it is common to express this as follows.

$$\begin{aligned} x^2 &= 4 \\ \sqrt{x^2} &= \sqrt{4} \\ |x| &= 2 \\ x &= \pm 2. \end{aligned}$$

Since $f(x) = x^2$ is **not** one-to-one, there are two values of x which make it equal to any positive number, as demonstrated in the following graph.



Example 25. (a) *Solve the equation $x^3 = 8$.*

Taking the cube root on both sides, we find that

$$\begin{aligned} \sqrt[3]{x^3} &= \sqrt[3]{8} \\ x &= 2. \end{aligned}$$

Note that $\sqrt[3]{x^3} = x$, since 3 is odd, and odd roots are really inverses to their corresponding power functions.

(b) *Solve the equation* $x^3 = -8$.

Taking the cube root on both sides, we find that

$$\begin{aligned} \sqrt[3]{x^3} &= \sqrt[3]{-8} \\ x &= -2. \end{aligned}$$

(c) *Solve the equation* $x^2 = 16$.

Taking the square root on both sides, we find that

$$\begin{aligned} \sqrt{x^2} &= \sqrt{16} \\ |x| &= 4 \\ x &= \pm 4. \end{aligned}$$

Therefore, there are two solutions to this equation: -4 *and* 4.

(d) *Solve the equation* $2x^4 - 4 = 28$.

First, rearrange the equation. Add 4 to both sides to find $2x^4 = 32$*. Divide both sides by 2 to find* $x^4 = 16$*. Taking the 4th root on both sides, we find that*

$$\begin{aligned} \sqrt[4]{x^4} &= \sqrt[4]{16} \\ |x| &= 2 \\ x &= \pm 2. \end{aligned}$$

Therefore, there are two solutions to this equation: -2 *and* 2.

Recall that for any even integer n, $\sqrt[n]{x^n} = |x|$.

(e) *Solve the equation* $-3x^4 = 32$.

First, rearrange the equation by dividing both sides by -3. This yields $x^4 = -\frac{32}{3}$*. Taking the 4th root on both sides, we find that*

$$\begin{aligned} \sqrt[4]{x^4} &= \sqrt[4]{-\frac{32}{3}} \\ |x| &= \sqrt[4]{-\frac{32}{3}}. \end{aligned}$$

Since any even-index root of a negative number is not defined, there are no solutions to this equation.

These examples illustrate a general principle that is good to have in your toolbox for solving equations. If $a^3 = b$, then we know that $a = \sqrt[3]{b}$. This is true for any odd powers. However, if $a^2 = b$, then **either** $a = \sqrt{b}$ **or** $a = -\sqrt{b}$. This is true for any even powers.

Finding x-intercepts of a Quadratic in Vertex Form

Now, we can use our understanding of the squareroot function to find the x-intercepts of a quadratic given in vertex form.

Example 26. *Find the x-intercepts of the quadratic $2(x-3)^2-5=0$ which is written in vertex form.*

Explanation First, rearrange the equation by adding 5 to both sides. This yields

$$2(x-3)^2 = 5.$$

Then divide each side by 2, resulting in

$$(x-3)^2 = \frac{5}{2}.$$

Taking the square root on both sides, we find that

$$\begin{aligned}
\sqrt{(x-3)^2} &= \sqrt{\frac{5}{2}} \\
|x-3| &= \sqrt{\frac{5}{2}} \\
x-3 &= \pm\sqrt{\frac{5}{2}} \\
x &= 3 \pm \sqrt{\frac{5}{2}}
\end{aligned}$$

Notice that this gives us a third method for finding the roots (x-intercepts) of a quadratic in general. We can use any of these methods to solve a quadratic.

(a) Factor the quadratic and write it in Root Form

(b) Use the quadratic formula to find the roots

(c) Write the quadratic in vertex form and then solve using a squareroot

Mathematically, these last two methods are actually related. The quadratic formula is just what happens when you rewrite the general quadratic $f(x) = ax^2+bx+c$ in vertex form and then solve for x!

Summary In general, we are not able to simply find the inverse of polynomials.
However, when the polynomial is $p(x) = x^n$ for a positive *odd* integer n, the polynomial is invertible as the n^{th} root function $r(x) = \sqrt[n]{x}$.
When n is *even*, it is possible to define an inverse function $r(x) = \sqrt[n]{x}$ on a restricted domain of $[0,\infty)$. The n^{th} root is defined as the inverse

of $p(x)$ on the restricted domain $[0, \infty)$.

Part 5

Back Matter

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