Precalculus with Review 1: Unit 3

September 20, 2022

Contents

Part 1

Variables and CoVariation (See Unit 1 PDF)

Part 2

Comparing Lines and Exponentials (See Unit 2 PDF)

Part 3

Functions

3.1 What is a Function?

Learning Objectives

- What is a Function?
  - Deep understanding of multiple representations of functions, including graphs, tables, equations, function notation, words, arrow diagrams, applications, data, and a function machine
  - Recognize functions in everyday life and appreciate the ubiquity of functions
  - Use definition of function to argue whether something is a function using all different types of representations
  - Understand the difference between a function, an expression, and an equation

3.1.1 What is a Function?

Motivating Questions

- What is a function?

Mathematical Models

A mathematical model is an abstract concept through which we use mathematical language and notation to describe a phenomenon in the world around us. One example of a mathematical model is found in Dolbear's Law, which has proven to be remarkably accurate for the behavior of snowy tree crickets. For even more of the story, including a reference to this phenomenon on the popular show The Big Bang Theory, see https://priceonomics.com/how-to-tell-the-temperature-using-crickets/. In the late 1800s, the physicist Amos Dolbear was listening to crickets chirp and noticed a pattern: how frequently the crickets chirped seemed to be connected to the outside temperature.

If we let T represent the temperature in degrees Fahrenheit and N the number of chirps per minute, we can summarize Dolbear's observations in the following table.

Acknowledgements: Source material includes: Active Prelude to Calculus (https://activecalculus.org/prelude), Stitz Zeager Open Source Mathematics (https://www.stitz-zeager.com/), ORCCA: Original source material, products with readable and accessible math content, and other information freely available at pcc.edu/orcca.

N(chirps per minute)	T(degrees Fahrenheit)
40	50
80	60
120	70
160	80

For a mathematical model, we often seek an algebraic formula that captures observed behavior accurately and can be used to predict behavior not yet observed. For the data in the table above, we observe that each of the ordered pairs in the table make the equation

$$T = 40 + 0.25N$$

true. For instance, $70 = 40 + 0.25(120)$. Indeed, scientists who made many additional cricket chirp observations following Dolbear's initial counts found that the formula above holds with remarkable accuracy for the snowy tree cricket in temperatures ranging from about 50° F to 85° F.

This model captures a pattern that is found in the world, and can be used to predict the temperature if only the number of chirps per minute is known. Not all phenomenon in the world that can be measured mathematically occur in a predictable pattern. In this section, we will study functions which are mathematical ways of formally studying situations where for a given input, such as the number of chirps above, there is one consistant output. For situations where a given input might give a variety of outputs, we encourage you to study statistics! Also note that this relationship is not causal. Even though the number of chirps is considered out "input" the increase in chirps does not cause the temperature to increase.

Functions

The mathematical concept of a function is one of the most central ideas in all of mathematics, in part since functions provide an important tool for representing and explaining patterns. At its core, a function is a repeatable process that takes a collection of input values and generates a corresponding collection of output values with the property that if we use a particular single input, the process always produces exactly the same single output.

For instance, Dolbear's Law provides a process that takes a given number of chirps between 40 and 180 per minute and reliably produces the corresponding temperature that corresponds to the number of chirps, and thus this equation generates a function. We often give functions shorthand names; using "D" for the "Dolbear" function, we can represent the process of taking inputs (observed

Acknowledgements: Source material includes: Active Prelude to Calculus (https://activecalculus.org/prelude), Stitz Zeager Open Source Mathematics (https://www.stitz-zeager.com/), ORCCA: Original source material, products with readable and accessible math content, and other information freely available at pcc.edu/orcca.

chirp rates) to outputs (corresponding temperatures) using arrows:

$$80 \xrightarrow{D} 60$$
$$120 \xrightarrow{D} 70$$
$$N \xrightarrow{D} 40 + 0.25N$$

Alternatively, for the relationship "$80 \xrightarrow{D} 60$" we can also use the equivalent notation "$D(80) = 60$" to indicate that Dolbear's Law takes an input of 80 chirps per minute and produces a corresponding output of 60 degrees Fahrenheit. More generally, we write "$T = D(N) = 40 + 0.25N$" to indicate that a certain temperature, T, is determined by a given number of chirps per minute, N, according to the process $D(N) = 40 + 0.25N$.

We will define a function informally and formally. The informal definition corresponds to the way we will most often think of functions, as a process with inputs and outputs.

> **Definition** [Informal Definition of a Function] A **function** is a process that may be applied to a collection of input values to produce a corresponding collection of output values in such a way that the process produces one and only one output value for any single input value.

The formal definition of a function will establish a function as a special type of relation. Recall that a *relation* is a collection of points of the form (x, y). If the point (x_0, y_0) is in the relation, then we say x_0 and y_0 are *related*.

> **Definition** [Formal Definition of a Function] A **function** is a collection of ordered pairs (x, y) such that any particular value of x is paired with at most one value for y. That is, a relation in which each x-coordinate is matched with only one y-coordinate is said to describe y as a **function** of x.

How is this definition consistent with the informal definition, which describes a function as a process? Well, if you have a collection of ordered pairs (x, y), you can choose to view the left number as an input, and the right value as the output. For a function f, $f(x)$ is defined as the unique y value that x is paired with. If x does not appear among the first coordinates of the ordered pairs, then x is not a possible input to f, since there is no way to determine $f(x)$.

Domain and Range

We now give some important definitions that allow us to talk about the inputs and outputs of functions.

Acknowledgements: Source material includes: Active Prelude to Calculus (https://activecalculus.org/prelude), Stitz Zeager Open Source Mathematics (https://www.stitz-zeager.com/), ORCCA: Original source material, products with readable and accessible math content, and other information freely available at pcc.edu/orcca.

> **Definition** Let f be a function from A to B. The set A of possible inputs to f is called the **domain** of f. The set B is called the **codomain** of f.

Sometimes, when we are given a function as a formula, we are not told the domain. In these circumstances we use the *implied domain*.

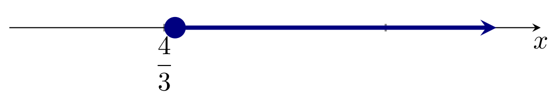
> **Definition** Let f be a function whose inputs are real numbers. The **implied domain** of f is the collection of all real numbers x for which $f(x)$ is a real number.

> **Definition** Let f be a function from A to B. The **range** of f is the collection of the outputs of f.

It is important to note that the definition of a function includes its domain and range. A function needs a rule and a domain to precisely determine a set of points, the range.

Interval Notation

Intervals (contiguous sections of the number line) play an important role in expressing the domains of many types of functions. As a standard way of writing these solutions, we rely on *interval notation*. Interval notation is a short-hand way of representing the intervals as they appear when sketched on a number line. As an example, consider $x \geq \frac{4}{3}$ which, when sketched on a number line, is given by



This sketch consists of a single interval with left-hand endpoint at $\frac{4}{3}$ and no right-hand endpoint (it keeps going). In interval notation, this would be written as $\left[\frac{4}{3}, \infty\right)$. This is an example of a *closed infinite interval*, "closed" because the point at $\frac{4}{3}$ (the only endpoint) is included and "infinite" because it has infinite width. The solid dot at $\frac{4}{3}$ indicates that the point is included in the interval.

There are four different types of infinite intervals, two are closed infinite intervals (which contain their respective endpoint) and the other two are open infinite intervals (which do not contain the endpoint). For a a fixed real number, these

Acknowledgements: Source material includes: Active Prelude to Calculus (https://activecalculus.org/prelude), Stitz Zeager Open Source Mathematics (https://www.stitz-zeager.com/), ORCCA: Original source material, products with readable and accessible math content, and other information freely available at pcc.edu/orcca.

are:

(a) $[a, \infty)$ represents $x \geq a$,

(b) $(-\infty, a]$ represents $x \leq a$,

(c) (a, ∞) represents $x > a$, and

(d) $(-\infty, a)$ represents $x < a$.

The notation indicates uses the square bracket to indicate that the endpoint is included and the round parenthesis to indicate that the endpoint is not included.

Not every interval is infinite, however. Consider the interval in the following sketch

which consists of all x with $-2 < x \leq 3$. It is not an infinite interval, having endpoints at -2 and 3. The endpoint at -2 is not included, but the endpoint at 3 is included. In interval notation this would be written as $(-2, 3]$. As with the infinite intervals, the square bracket indicates that the right-hand endpoint is included and the round parenthesis endicates that the left-hand endpoint is not included. (This is an example of a "half-open interval".)

For a bounded intervals (ones that are not infinite), there are also four possibilities. For a and b both fixed real numbers, these are:

(a) $[a, b]$ represents $a \leq x \leq b$,

(b) $[a, b)$ represents $a \leq x < b$,

(c) $(a, b]$ represents $a < x \leq b$ and

(d) (a, b) represents $a < x < b$.

Practically, this amounts to writing the left-hand endpoint, the right-hand endpoint, then indicating which endpoints are included in the interval. When neither endpoint is included, (a, b) can be mistaken for a point on a graph. You will need to use the context to know which is meant.

Summary

- Informally, a function is a process that may be applied to a collection of input values to produce a corresponding collection of output values in such a way that the process produces one and only one output value for any single input value.

- Formally, a function is a collection of ordered pairs (x, y) such that

Acknowledgements: Source material includes: Active Prelude to Calculus (https://activecalculus.org/prelude), Stitz Zeager Open Source Mathematics (https://www.stitz-zeager.com/), ORCCA: Original source material, products with readable and accessible math content, and other information freely available at pcc.edu/orcca.

any particular value of x is paired with at most one value for y. That is, a relation in which each x-coordinate is matched with only one y-coordinate is said to describe y as a **function** of x.

- Functions can be used to model many phenomena in the real world, as long as for a given input there is one predictable output.

3.1.2 Representations of Functions

Motivating Questions

- How can functions be represented?
- When is a relation not a function?

Respresentations of Functions

While the formal definition of a function is a set of ordered pairs, there are many ways that we represent functions when studying them. Each representation has advantages and disadvantages and being able to change between different representations of the same function is an important skill. Let's look at some different types of representations.

Tables Functions on a finite set of points are often represented by tables. One advantage of a table is that you can easily see all the information for a function. One disadvantage is if you have too many input values, it can be difficult to analyze them all in a table format. The exploration below shows and example of a function given by a table.

Exploration The function f is defined by the table below. Note: this means the table gives all the values of the function.
Answer the questions about the function below.

x	$f(x)$
banana	.5
apple	1.59
pear	2.50
orange	1

a. Is f a function?

b. What is $f(\text{apple})$?

c. What are the inputs of f?

d. What are the outputs of f?

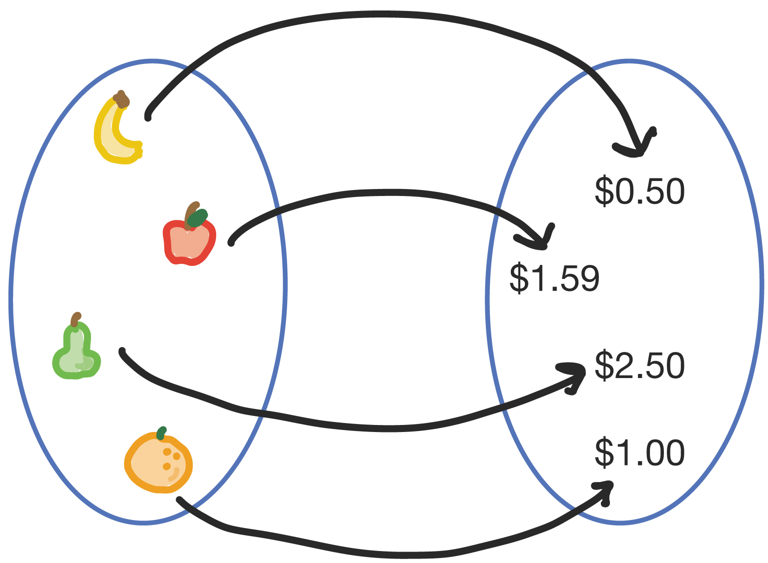
e. Rewrite f as a set of ordered pairs.

f. Assume that f gives the price of a single item of a given fruit at a grocery store. Interpret $f(\text{orange}) = 1$ in this context.

Acknowledgements: Source material includes: Active Prelude to Calculus (https://activecalculus.org/prelude), Stitz Zeager Open Source Mathematics (https://www.stitz-zeager.com/), ORCCA: Original source material, products with readable and accessible math content, and other information freely available at pcc.edu/orcca.

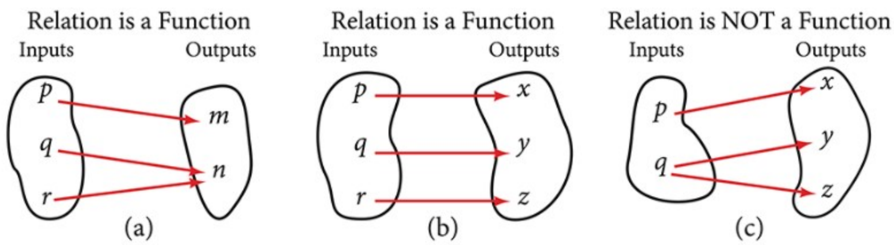
If the set of ordered pairs that makes up your function is an infinite set, you cannot represent your entire function as a table because it would go on forever! Instead, you might see a table with a sampling of points from a function. In that case, unless you have additional information about your function, you cannot know what the outputs are for inputs not listed in the table. In fact, without additional information, you cannot even say which values are allowed to be inputs for the function!

Arrow Diagrams Arrow diagrams are another tool used to represent functions. Here is an arrow diagram that corresponds to the function f in the exploration above.



Arrow diagrams can help make it easier to see when multiple inputs go to the same output. Arrow diagrams can also be used to represent relations, and can make it easier to see if the relation is a function. The three arrow diagrams below show relations. The first two relations are functions, but the last relation is not a function because the same input goes to two different outputs.

Acknowledgements: Source material includes: Active Prelude to Calculus (https://activecalculus.org/prelude), Stitz Zeager Open Source Mathematics (https://www.stitz-zeager.com/), ORCCA: Original source material, products with readable and accessible math content, and other information freely available at pcc.edu/orcca.

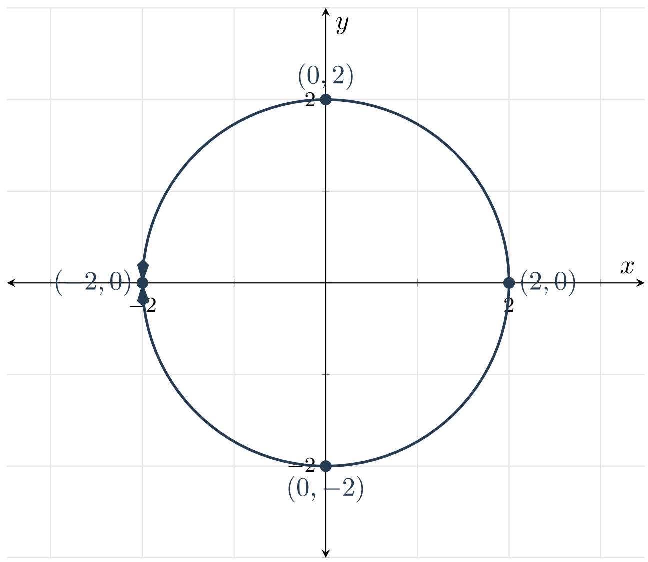


Remark Remember: A function is always a relation, but not every relation is a function!

Graphs Graphing points on a coordinate plane is a great way to represent a function and one which is used often. Graphs are most often used for functions where both the inputs and outputs are numbers. When graphing functions, typically the horizontal axis represents the input values and the vertical axis represents the output values.

Example 1. *Here are some graphs of relations. Can you idenfity which of the relations represented here are functions?*

Acknowledgements: Source material includes: Active Prelude to Calculus (https://activecalculus.org/prelude), Stitz Zeager Open Source Mathematics (https://www.stitz-zeager.com/), ORCCA: Original source material, products with readable and accessible math content, and other information freely available at pcc.edu/orcca.



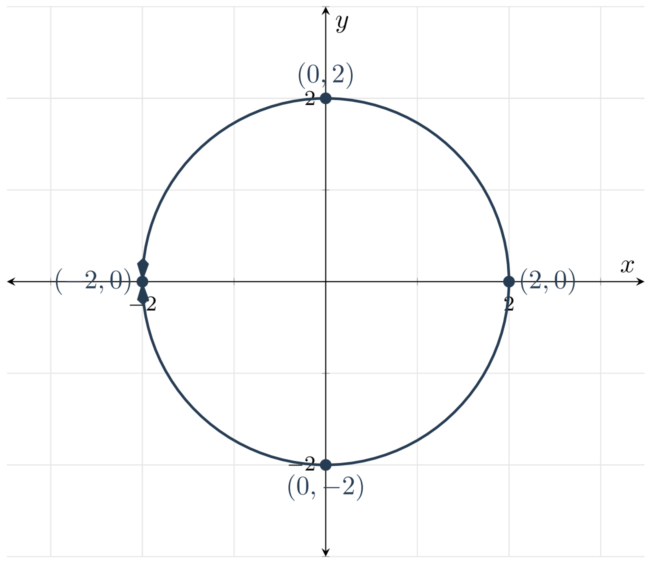
Explanation

a. A graph of a (nonvertical) line is the graph of a function. This is because, for each x-value, there is only one y-value that corresponds to it on the line.

The line in the graph above is the set of all points of the form $(x, 3)$ for any real number value of x.

Acknowledgements: Source material includes: Active Prelude to Calculus (https://activecalculus.org/prelude), Stitz Zeager Open Source Mathematics (https://www.stitz-zeager.com/), ORCCA: Original source material, products with readable and accessible math content, and other information freely available at pcc.edu/orcca.

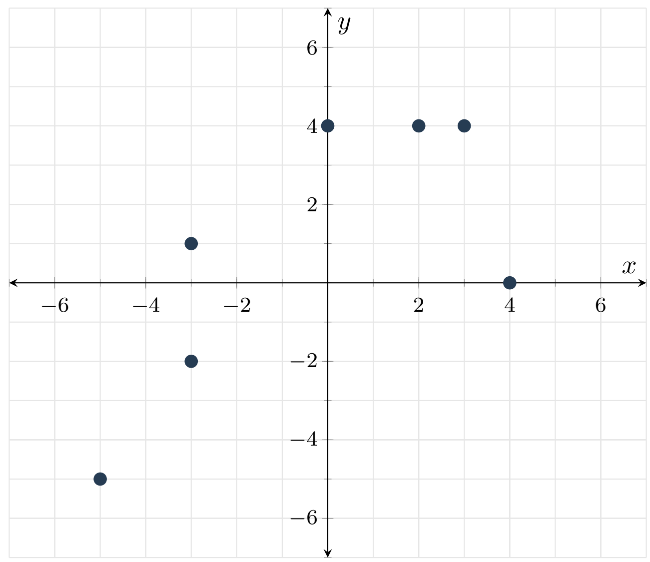
b. This graph of a circle is not the graph of a function. There exists an input value (many of them, actually) for which there are multiple different outputs.

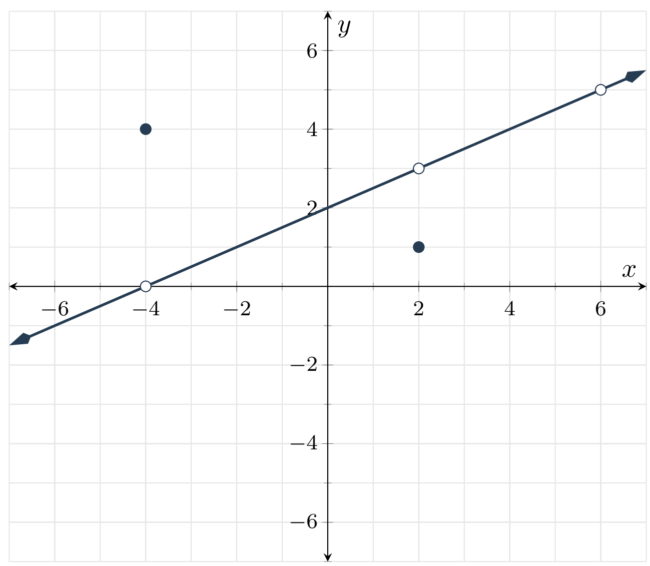


For example, one the graph above, the points $(0, 2)$ and $(0, -2)$ are both on the graph. This tells us that for the input of $x = 0$, we have two outputs, $y = 2$ and $y = -2$. This circle represents a relation but not a function.

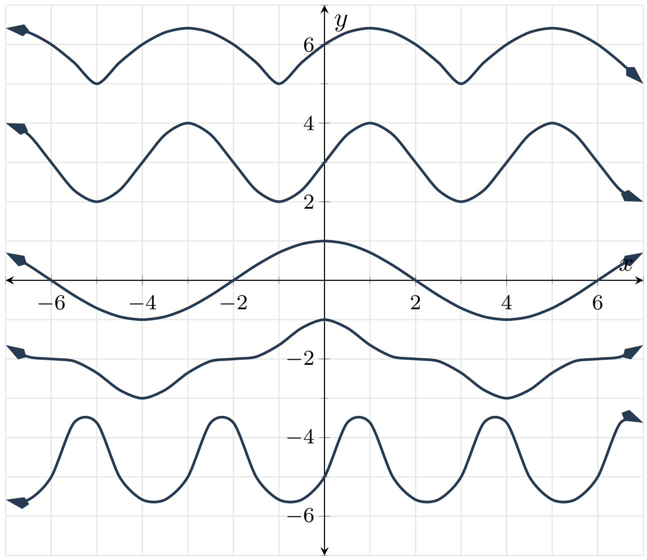
Example 2. *Here are some more graphs of relations. Can you idenfity which of the relations represented here are functions?*

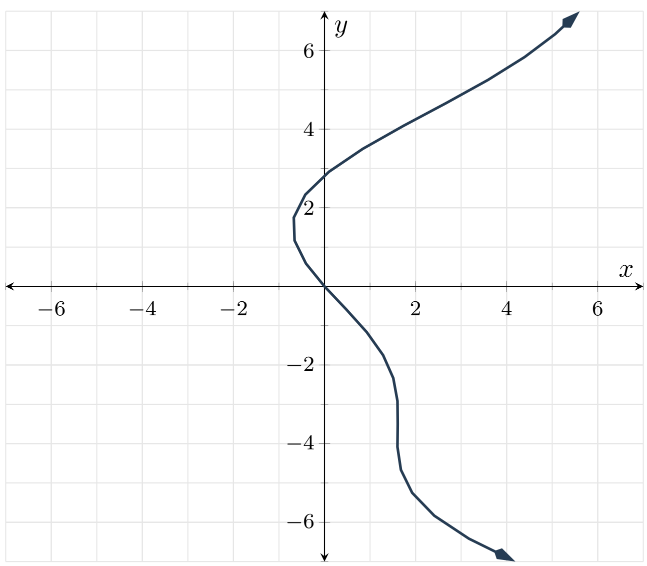
Acknowledgements: Source material includes: Active Prelude to Calculus (https://activecalculus.org/prelude), Stitz Zeager Open Source Mathematics (https://www.stitz-zeager.com/), ORCCA: Original source material, products with readable and accessible math content, and other information freely available at pcc.edu/orcca.





Acknowledgements: Source material includes: Active Prelude to Calculus (https://activecalculus.org/prelude), Stitz Zeager Open Source Mathematics (https://www.stitz-zeager.com/), ORCCA: Original source material, products with readable and accessible math content, and other information freely available at pcc.edu/orcca.





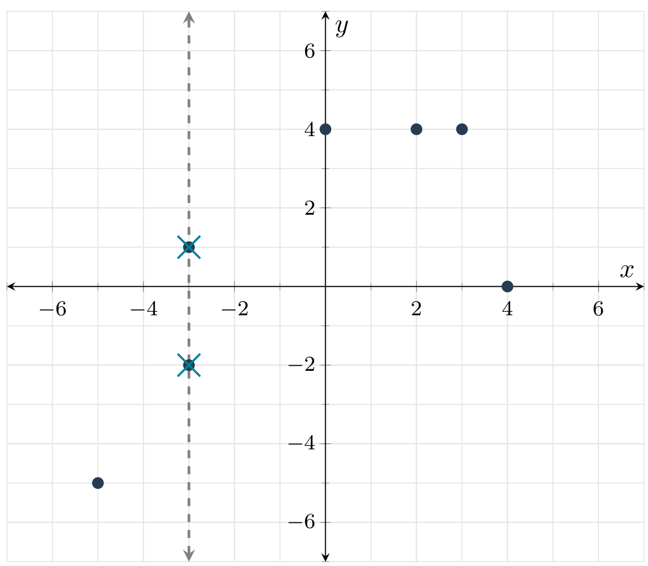
Explanation As in the previous example, we are looking to determine if there are any values of x for which there are multiple outputs. Visually, what that means is there are places on the graph that are directly above/below each other. Thinking about this leads to a quick visual "test" to determine if a graph gives y as a function of x.

Acknowledgements: Source material includes: Active Prelude to Calculus (https://activecalculus.org/prelude), Stitz Zeager Open Source Mathematics (https://www.stitz-zeager.com/), ORCCA: Original source material, products with readable and accessible math content, and other information freely available at pcc.edu/orcca.

Vertical Line "Test" Given a graph in the xy-plane, if there exists a vertical line that touchs the graph in more than one place, the graph does not represent a function. If no such vertical line exists, then the relation represented by the graph is also a function.

Let's use this test to analyze each of the graphs in this example.

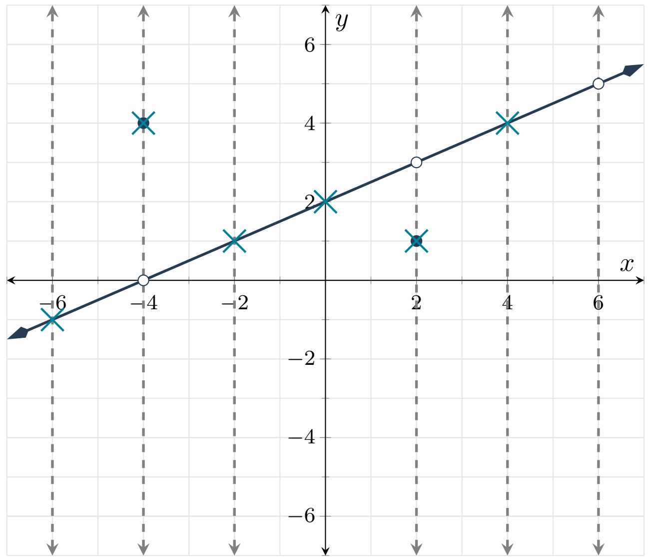
a. We notice that it is possible to draw a vertical line that touches this graph in two places.



Therefore, by the vertical line test, this graph is not the graph of a function. Really, the vertical line we found at $x = -3$ is helping us to quickly identify that both $(-3, -2)$ and $(-3, 1)$ are points on this graph so that for the same input, 3, there are two different outputs, -2 and 1.

b. This graph is the graph of a function. Here are some examples of some vertical lines we might consider.

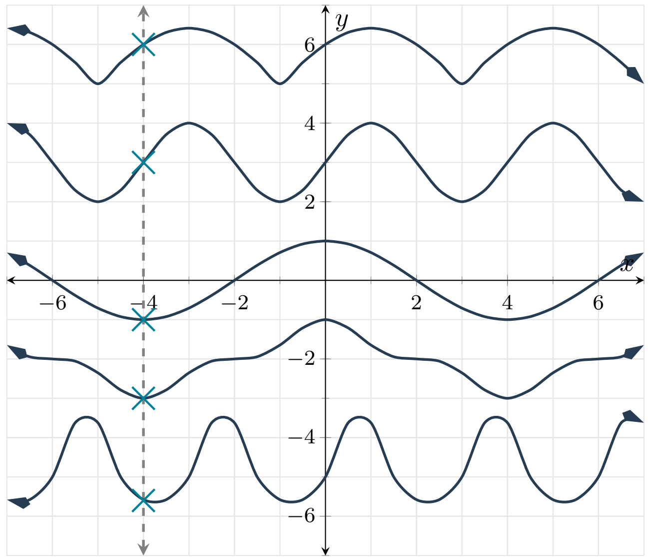
Acknowledgements: Source material includes: Active Prelude to Calculus (https://activecalculus.org/prelude), Stitz Zeager Open Source Mathematics (https://www.stitz-zeager.com/), ORCCA: Original source material, products with readable and accessible math content, and other information freely available at pcc.edu/orcca.



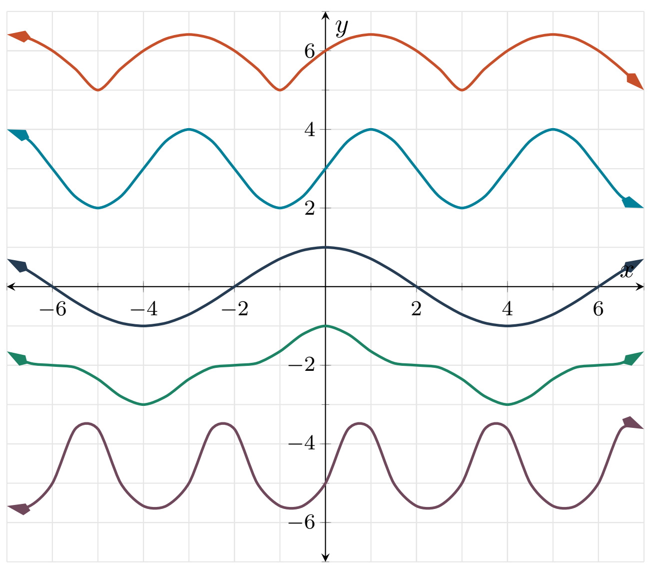
Note that we cannot draw all possible vertical lines. Really, we only need to look at the points on this graph which are not a line. We already said the graphs of lines (without any other points or curves above or below that line) are functions. So on this graph, the interesting points we want to look at more closely are $x = -4$, $x = 2$, and $x = 6$. In each of these cases, there is either one or no outputs. In particular, for $x = -4$ the corresponding output is $y = 4$. There is an open dot at $(-4, 0)$ so this point is not in the relation. Similarly, $(2, 1)$ is a point on this graph but $(2, 3)$ is not. At $x = 6$, there are no corresponds outputs. We say are function is not defined at $x = 6$.

c. This graph is not a function. Any vertical line we draw will cross the graph multiple times. Here is an example of a vertical line at $x = -4$.

Acknowledgements: Source material includes: Active Prelude to Calculus (https://activecalculus.org/prelude), Stitz Zeager Open Source Mathematics (https://www.stitz-zeager.com/), ORCCA: Original source material, products with readable and accessible math content, and other information freely available at pcc.edu/orcca.

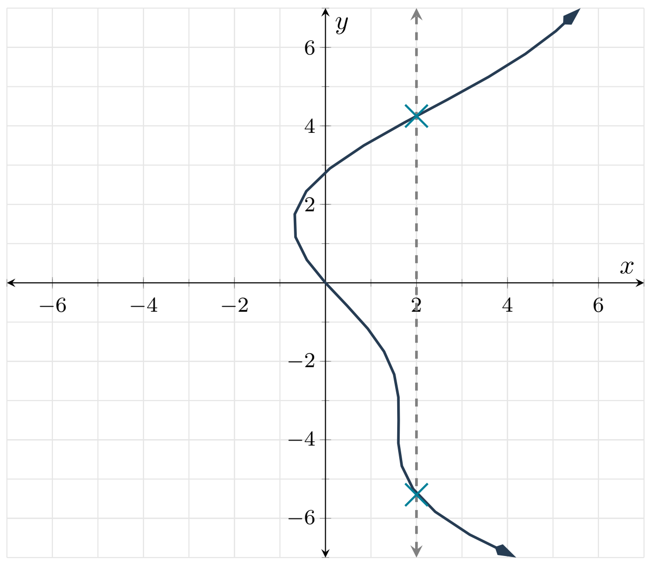


You could consider this a graph of five separate functions all graphed on the same coordinate axes, but that is a different question from the one being asked.



d. This graph is also not a function. Even though it is a single curve, it has input values, such as $x = 2$ with multiple corresponding output values.

Acknowledgements: Source material includes: Active Prelude to Calculus (https://activecalculus.org/prelude), Stitz Zeager Open Source Mathematics (https://www.stitz-zeager.com/), ORCCA: Original source material, products with readable and accessible math content, and other information freely available at pcc.edu/orcca.



Formula Representation Another common way to represent functions is using a formula. In the example of Dolbear's Law, the function which models this law is given as a formula by

$$D(N) = 40 + 0.25N \text{ for } 50 \leq x \leq 85$$

When we give functions as a formula, we also need to say which input values are allowed, that is, what the domain of the function is. In this case, the domain is $50 \leq x \leq 85$. If the domain is not given, the domain is assumed to be all the values for which the formula used to defined the function makes sense, that is, the implied domain.

Here are some examples of functions represented by a formula.

- $f(x) = x^2$
- $g(x) = 5x - 7$
- $h(x) = \sin(x)$

  Recall that this is the famous function named sine.
- $z(x) = \dfrac{x^2 \sin(x) - 2}{72 - x}$

Here is an example of working with the formula representation of a function.

Example 3. *Let $f(x) = 5x - 2$. Find $f(1)$. Then, find $f(m)$, $f(k+1)$, $f(2h)$, and $f(\bigstar)$.*

Acknowledgements: Source material includes: Active Prelude to Calculus (https://activecalculus.org/prelude), Stitz Zeager Open Source Mathematics (https://www.stitz-zeager.com/), ORCCA: Original source material, products with readable and accessible math content, and other information freely available at pcc.edu/orcca.

Explanation Recall that the notation $f(1)$ means to evaluate the function f at the input $x = 1$. That is, $f(1) = 5(1) - 2 = 3$.

Similarly, $f(m)$ means to evaluate the function f at the input $x = m$. That is, $f(m) = 5m - 2$. This expression contains no like terms, so it cannot be simplified further.

$f(k+1) = 5(k+1)-2$. Distributing, we find that $5(k+1)-2 = 5k+5-2 = 5k+3$, so $f(k+1) = 5k+3$.

$f(2h) = 5(2h) - 2$, and multiplying $5(2h)$ yields $10h$, so $f(2h) = 10h - 2$.

Although $f(\bigstar)$ involves an unusual symbol, we can treat the symbol like a number. Thus, $f(\bigstar) = 5\bigstar - 2$.

Intercepts

Zero is a very important number, and as we will see later, knowing where a function's x- or y-value equals zero can be powerful information.

Definition [Intercepts] Say f is a function.
An x**-intercept** of f is a point $(x, 0)$ such that $f(x) = 0$. That is, a point in which the graph of the function touches the x-axis.
The y**-intercept** of f is a point $(0, y)$ such that $f(0) = y$. That is, a point in which the graph of the function touches the y-axis. Unlike x-intercepts, there can be only one y-intercept for each function.

Summary

- Functions can be represented in many ways including as:
  - sets of ordered pairs
  - tables
  - arrow diagrams
  - graphs
  - formula with domain and range

3.2 Function Properties

Learning Objectives

- Function Properties
  - Periodic
  - Even and Odd
- Inverse Functions
  - Definition of an Inverse Function
  - One-to-one functions and why they are necessary for inverse functions to exist
  - Draw or identify the graph of an inverse function
- Famous Function Properties
  - Updated list of famous functions and their properties

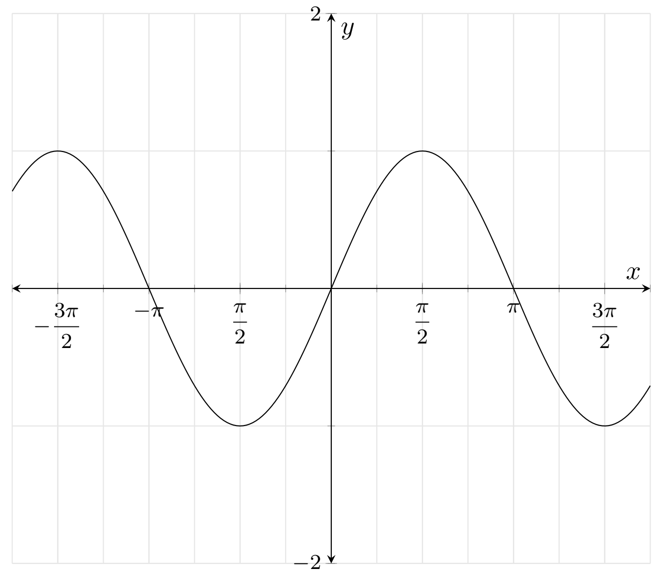
3.2.1 Even and Odd Functions

Motivating Questions

- What do we mean when we say a function is even or odd? How can we identify even and odd functions?

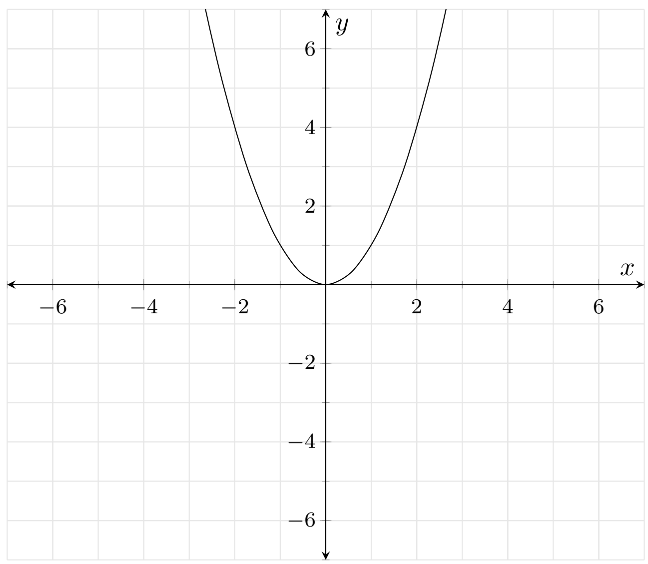
When working with functions and looking at their graphs, we might notice some interesting patterns or behaviors. For example, a function like the sine function appears to repeat itself over and over again, and the quadratic function defined by $y = x^2$ appears to be symmetric about the y-axis.

$y = \sin(x)$



Acknowledgements: Source material includes: Active Prelude to Calculus (https://activecalculus.org/prelude), Stitz Zeager Open Source Mathematics (https://www.stitz-zeager.com/), CSCC PreCalculus and Calculus texts (https://ximera.osu.edu/csccmathematics)

$y = x^2$



In this section, we'll discuss new vocabulary we can use to describe these behaviors as well as how to show analytically that a function has a certain behavior.

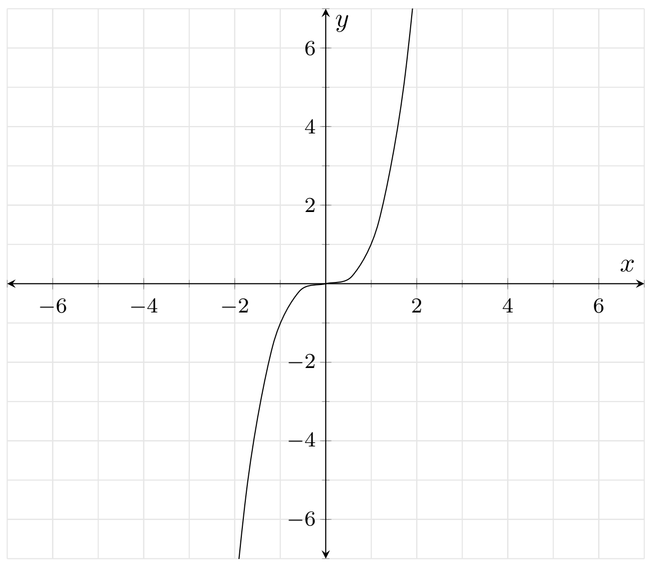
We'll also discuss the important concept of inverse functions, which can provide a way to "undo" functions.

Odd and even functions

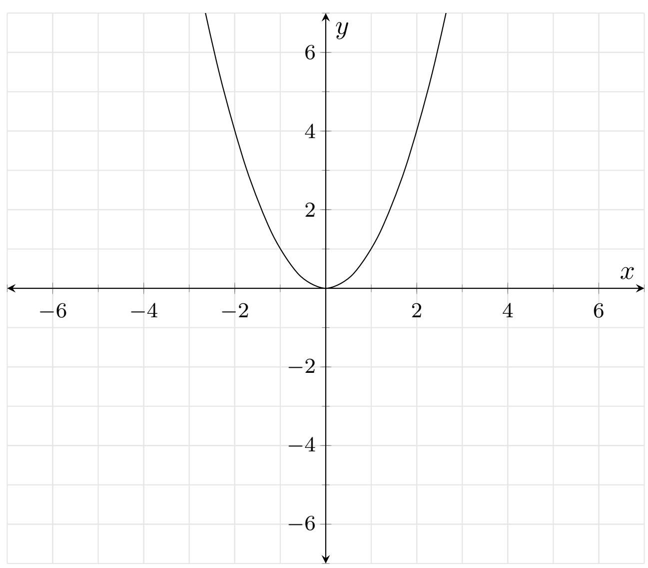
Consider the two functions, $g(x) = x^3$ and $h(x) = x^2$, whose graphs are shown below.

Acknowledgements: Source material includes: Active Prelude to Calculus (https://activecalculus.org/prelude), Stitz Zeager Open Source Mathematics (https://www.stitz-zeager.com/), CSCC PreCalculus and Calculus texts (https://ximera.osu.edu/csccmathematics)

$g(x) = x^3$



$h(x) = x^2$



Note that the graph of g seems to be symmetric about the origin, meaning that when we rotate the graph a half-turn, we get the same graph. Also, the graph of h seems to be symmetric about the y-axis, meaning that when we flip the graph across the y-axis, we get the same graph.

Let's first consider the case of g by looking at a few test points.

Acknowledgements: Source material includes: Active Prelude to Calculus (https://activecalculus.org/prelude), Stitz Zeager Open Source Mathematics (https://www.stitz-zeager.com/), CSCC PreCalculus and Calculus texts (https://ximera.osu.edu/csccmathematics)

$$g(2) = (2)^3 = 8$$
$$g(-2) = (-2)^3 = -8$$
$$g(3) = (3)^3 = 27$$
$$g(-3) = (-3)^3 = -27$$

We see that $(2, 8)$, $(-2, 8)$, $(3, 27)$, and $(-3, -27)$ are points on the graph which indicates $g(x)$ is an odd function. We can check this for all point, showing that $(-x, -y)$ is on the graph whenever (x, y) is. In other words, we need to show $(-x, -y)$ satisfies the equation $y = x^3$ whenever (x, y) does. Substituting $(-x, -y)$ into the equation gives

$$g(-x) = (-x)^3 = -x \cdot -x \cdot -x = -x^3 = -g(x)$$

This shows that g is an odd function.

Exploration Consider the function h defined by $h(x) = x^2$. We'll try to prove that h is symmetric about the y-axis.

a. Assume (x, y) is a generic point on the graph of h, so $y = x^2$. What point is symmetric to (x, y) about the y-axis?

b. Show your answer to part a is on the graph of h whenever (x, y) is. Conclude that h is symmetric about the y-axis.

Notice that to test an equation's graph for symmetry about the origin, we replaced x and y with $-x$ and $-y$, respectively. Doing this substitution in the equation $y = f(x)$ results in $-y = f(-x)$. Solving the latter equation for y gives $y = -f(-x)$. In order for this equation to be equivalent to the original equation $y = f(x)$ we need $-f(-x) = f(x)$, or, equivalently, $f(-x) = -f(x)$. In the exploration, you checked whether the graph of an equation was symmetric about the y-axis by replacing x with $-x$ and checking to see if an equivalent equation results. If we are graphing the equation $y = f(x)$, substituting $-x$ for x results in the equation $y = f(-x)$. In order for this equation to be equivalent to the original equation $y = f(x)$ we need $f(-x) = f(x)$. This leads us to the definition of an even function and an odd function.

Definition
A function f is **even** if $f(-x) = f(x)$ for all x in the domain of f. That is, if (x, y) is a point of f, so is $(-x, y)$.

Acknowledgements: Source material includes: Active Prelude to Calculus (https://activecalculus.org/prelude), Stitz Zeager Open Source Mathematics (https://www.stitz-zeager.com/), CSCC PreCalculus and Calculus texts (https://ximera.osu.edu/csccmathematics)

Definition
A function f is **odd** if $f(-x) = -f(x)$ for all x in the domain of f. That is, if (x, y) is a point of f, so is $(-x, -y)$.

A function is even if and only if its graph is symmetric about the y-axis. A function is odd if and only if its graph is symmetric about the origin.

Example 4. *Determine if the following functions are even, odd, or neither even nor odd using the definition of even and odd functions.*

(a) $f(x) = \dfrac{5}{2 - x^2}$

(b) $g(x) = \dfrac{5x}{2 - x^2}$

(c) $h(x) = \dfrac{5x}{2 - x^3}$

Explanation The defintion of even and odd functions gives information about $f(-x)$, the first step in all these problems is to replace input $-x$ into the function and simplify.

(a) Here, $f(x) = \dfrac{5}{2 - x^2}$. Inputting $-x$ into f, we find that

$$f(-x) = \frac{5}{2 - (-x)^2}$$
$$f(-x) = \frac{5}{2 - (-1 \cdot x)^2}$$
$$f(-x) = \frac{5}{2 - (-1)^2 \cdot x^2}$$
$$f(-x) = \frac{5}{2 - x^2},$$

so $f(-x) = f(x)$. This shows that f is *even*.

(b) Here, $g(x) = \dfrac{5x}{2 - x^2}$. Inputting $-x$ into g, we find that

$$g(-x) = \frac{5(-x)}{2 - (-x)^2}$$
$$g(-x) = \frac{-5x}{2 - x^2}.$$

Acknowledgements: Source material includes: Active Prelude to Calculus (https://activecalculus.org/prelude), Stitz Zeager Open Source Mathematics (https://www.stitz-zeager.com/), CSCC PreCalculus and Calculus texts (https://ximera.osu.edu/csccmathematics)

It doesn't appear that $g(-x)$ is equal to $g(x)$. To prove this, we check with an x value. After some trial and error, we see that $g(1) = 5$ whereas $g(-1) = -5$. This proves that g is not even, but it doesn't rule out the possibility that g is odd. (Why not?) To check if g is odd, we compare $g(-x)$ with $-g(x)$:

$$
\begin{aligned}
-g(x) &= -\frac{5x}{2 - x^2} \\
-g(x) &= \frac{-5x}{2 - x^2} \\
-g(x) &= g(-x).
\end{aligned}
$$

Since $-g(x) = g(-x)$, g is *odd*.

(c) Here, $h(x) = \dfrac{5x}{2 - x^3}$. Inputting $-x$ into h, we find that

$$
\begin{aligned}
h(-x) &= \frac{5(-x)}{2 - (-x)^3} \\
h(-x) &= \frac{-5x}{2 + x^3}.
\end{aligned}
$$

Once again, $h(-x)$ doesn't appear to be equal to $h(x)$. We check with an x value. For example, $h(1) = 5$, but $h(-1) = -\dfrac{5}{3}$. This proves that h is not even and it also shows h is not odd. You may be wondering how we konw that h is not odd. Recall that when h is odd, then if (x_1, y_1) is a point on h, then $(-x_1, -y_1)$ must also be a point on h. But that would mean that for h to be odd, $h(-1) = -h(1)$. But $-h(1) = -5$ and $h(-1) = -\dfrac{5}{3} \neq -5$.

Completing the Graph of Even or Odd Functions

If we know a function is either even or odd, we can determine one half of the graph or function values if we are giving the other half. This is because we know that:

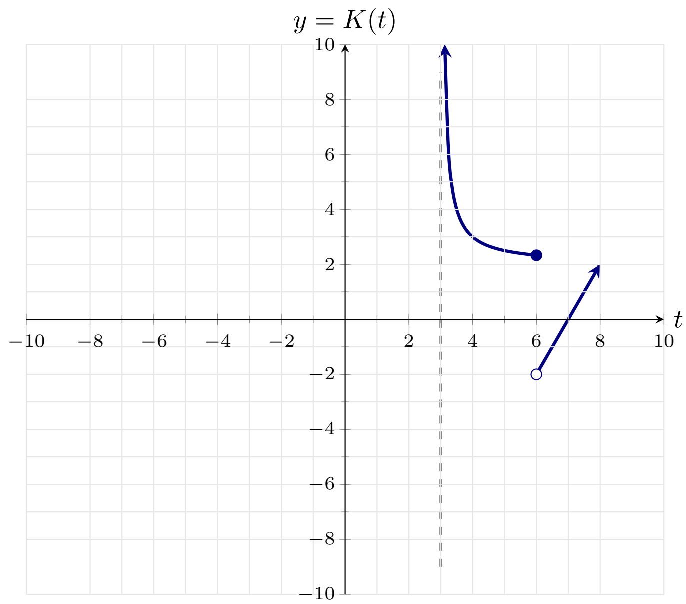
- If a function is **even** and the point (x_1, y_1) is on the function graph, then the point $(-x_1, y_1)$ is also on the function graph.
- If a function is **odd** and the point (x_1, y_1) is on the function graph, then the point $(-x_1, -y_1)$ is also on the function graph.

Try the two examples below and then compare your work with the solution.

Acknowledgements: Source material includes: Active Prelude to Calculus (https://activecalculus.org/prelude), Stitz Zeager Open Source Mathematics (https://www.stitz-zeager.com/), CSCC PreCalculus and Calculus texts (https://ximera.osu.edu/csccmathematics)

Example 5. *Even*

Half of the graph of $y = K(t)$ is displayed below. If $K(t)$ is an even function, then think of what the other half of the graph would look like.

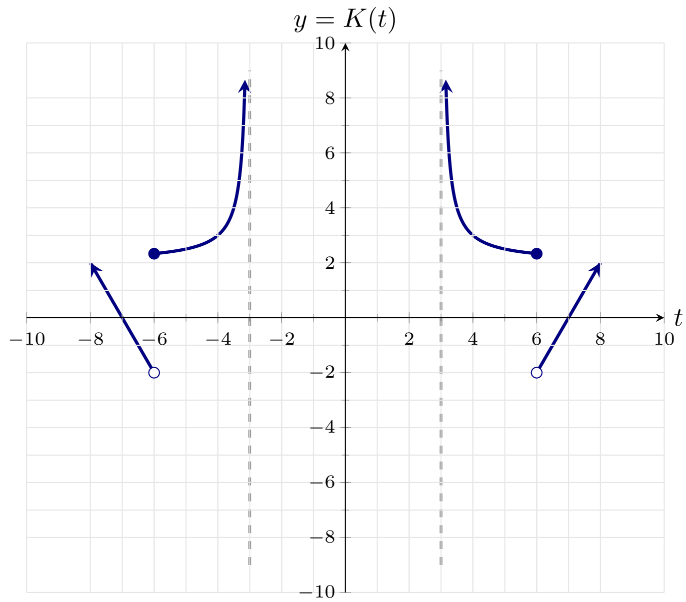


Visualize what the full graph looks like, then click the arrow to compare with the solution below.

Explanation

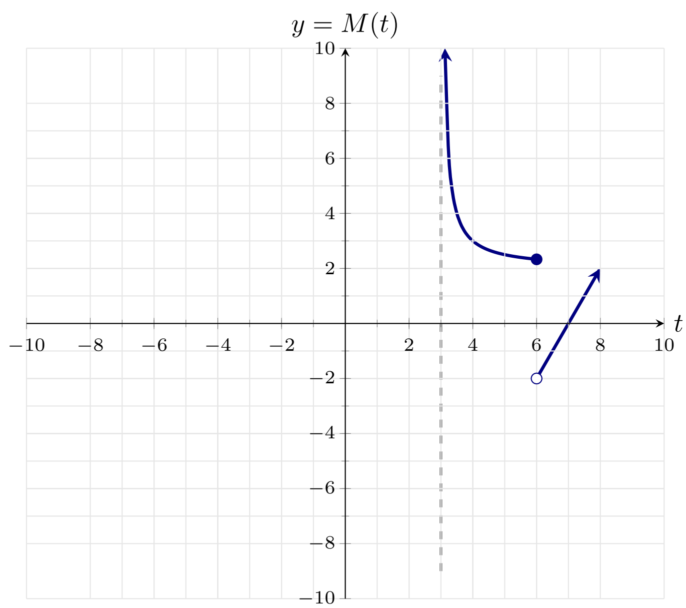
To sketch our graph, we'll use the following fact: If a function is **even** and the point (x_1, y_1) is on the function graph, then the point $(-x_1, y_1)$ is also on the function graph. First, let's look at the linear portion of the graph. This is a line with a hole at $(6, -2)$ and an x-intercept at $(7, 0)$. This tells us that we have a hole at $(-6, -2)$ and an x-intercept at $(-7, 0)$. We then draw a line from $(-6, -2)$, through the point $(-7, 0)$. Next, we see a point at approximately $(6, 2.25)$ so we plot the point $(-6, 2.25)$. There's a vertical asymptote at $x = 3$ so we'll sketch the vertical asymptote $x = -3$. Last, we'll sketch the function from $(6, 2.25)$, curving up toward the vertical asymptote $x = -3$.

Acknowledgements: Source material includes: Active Prelude to Calculus (https://activecalculus.org/prelude), Stitz Zeager Open Source Mathematics (https://www.stitz-zeager.com/), CSCC PreCalculus and Calculus texts (https://ximera.osu.edu/csccmathematics)



Example 6. *Odd*

Half of the graph of $y = M(t)$ is displayed below. If $M(t)$ is an odd function, then think of what the other half of the graph would look like.



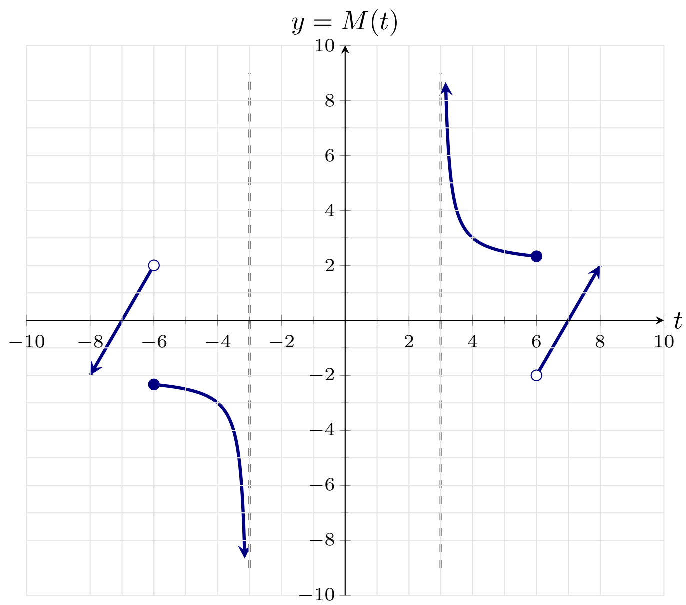
Visualize what the full graph looks like, then click the arrow to compare with the

Acknowledgements: Source material includes: Active Prelude to Calculus (https://activecalculus.org/prelude), Stitz Zeager Open Source Mathematics (https://www.stitz-zeager.com/), CSCC PreCalculus and Calculus texts (https://ximera.osu.edu/csccmathematics)

solution below.

Explanation

If a function is **odd** and the point (x_1, y_1) is on the function graph, then the point $(-x_1, -y_1)$ is also on the function graph. First, let's look at the linear portion of the graph. This is a line with a hole at $(6, -2)$ and an x-intercept at $(7, 0)$. This tells us that we have a hole at $(-6, 2)$ and an x-intercept at $(-7, 0)$. We then draw a line from $(-6, 2)$, through the point $(-7, 0)$. Next, we see a point at approximately $(6, 2.25)$ so we plot the point $(-6, -2.25)$. There's a vertical asymptote at $x = 3$ so we'll sketch the vertical asymptote $x = -3$. Last, we'll sketch the function from $(-6, -2.25)$, curving up toward the vertical asymptote $x = -3$.



Summary

- A function f is called
  - even if $f(-x) = f(x)$ for all possible choices of x. Even functions are symmetric about the y-axis.
  - odd if $f(-x) = -f(x)$ for all possible choices of x. Odd functions are symmetric about the origin.

3.2.2 Periodic Functions

Motivating Questions

- What do we mean when we say a function is periodic?

Periodic functions

Certain naturally occurring phenomena eventually repeat themselves, especially when the phenomenon is somehow connected to a circle. For example, suppose that you are taking a ride on a Ferris wheel and we consider your height, h, above the ground and how your height changes in tandem with the distance, d, that you have traveled around the wheel. We can see a full animation of this situation at `http://gvsu.edu/s/0Dt`.

Because we have two quantities changing in tandem, it is natural to wonder if it is possible to represent one as a function of the other.

Exploration In the context of the ferris wheel mentioned above, assume that the height, h, of the moving point (the cab in which you are riding), and the distance, d, that the point has traveled around the circumference of the ferris wheel are both measured in meters.
Further, assume that the circumference of the ferris wheel is 24 meters (it's a pretty short ferris wheel). In addition, suppose that after getting in your cab at the lowest point on the wheel, you traverse the full circle several times.

(a) Recall that the circumference, C, of a circle is connected to the circle's radius, r, by the formula $C = 2\pi r$. What is the radius of the ferris wheel? How high is the highest point on the ferris wheel?

(b) How high is the cab after it has traveled $\frac{1}{4}$ of the circumference of the circle?

(c) How much distance along the circle has the cab traversed at the moment it first reaches a height of $\frac{12}{\pi} \approx 3.82$ meters?

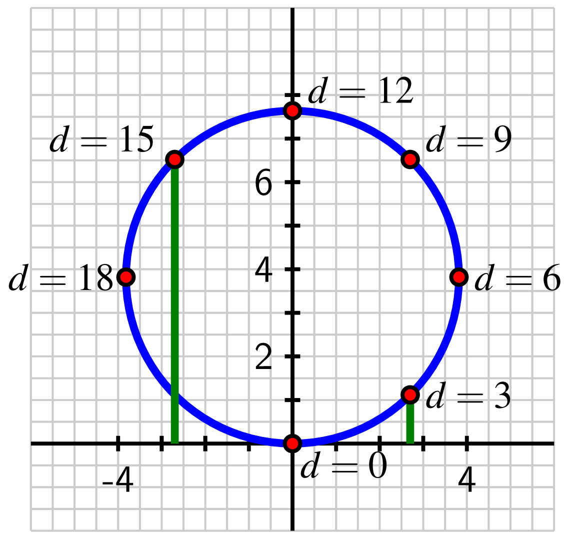
(d) Can h be thought of as a function of d? Why or why not?

(e) Can d be thought of as a function of h? Why or why not?

Let's consider a point traversing a circle of circumference 24 and examine how the point's height, h, changes as the distance traversed, d, changes. Note particularly that each time the point traverses $\frac{1}{8}$ of the circumference of the circle,

Acknowledgements: Source material includes: Active Prelude to Calculus (https://activecalculus.org/prelude), Stitz Zeager Open Source Mathematics (https://www.stitz-zeager.com/), CSCC PreCalculus and Calculus texts (https://ximera.osu.edu/csccmathematics)

it travels a distance of $24 \cdot \frac{1}{8} = 3$ units, as seen below, where each noted point lies 3 additional units along the circle beyond the preceding one.



Note that we know the exact heights of certain points. Since the circle has circumference $C = 24$, we know that $24 = 2\pi r$ and therefore $r = \frac{12}{\pi} \approx 3.82$. Hence, the point where $d = 6$ (located 1/4 of the way along the circle) is at a height of $h = \frac{12}{\pi} \approx 3.82$. Doubling this value, the point where $d = 12$ has height $h = \frac{24}{\pi} \approx 7.64$. Other heights, such as those that correspond to $d = 3$ and $d = 15$ (identified on the figure by the green line segments) are not obvious from the circle's radius, but can be estimated from the grid in the figure above as $h \approx 1.1$ (for $d = 3$) and $h \approx 6.5$ (for $d = 15$). Using all of these observations along with the symmetry of the circle, we can construct a table..

Acknowledgements: Source material includes: Active Prelude to Calculus (https://activecalculus.org/prelude), Stitz Zeager Open Source Mathematics (https://www.stitz-zeager.com/), CSCC PreCalculus and Calculus texts (https://ximera.osu.edu/csccmathematics)

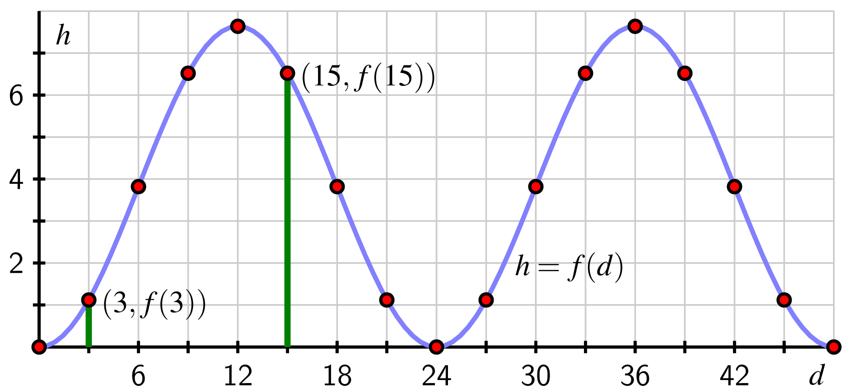
d	h
0	0
3	1.1
6	3.82
9	6.5
12	7.64
15	6.5
18	3.82
21	1.1
24	0

Moreover, if we now let the point continue traversing the circle, we observe that the d-values will increase accordingly, but the h-values will repeat according to the already-established pattern, resulting in the data in the table below.

d	h
24	0
27	1.1
30	3.82
33	6.5
36	7.64
39	6.5
42	3.82
45	1.1
48	0

It is apparent that each point on the circle corresponds to one and only one height, and thus we can view the height of a point as a function of the distance the point has traversed around the circle, say $h = f(d)$. Using the data from the two tables and connecting the points in an intuitive way, we get the graph shown below

Acknowledgements: Source material includes: Active Prelude to Calculus (https://activecalculus.org/prelude), Stitz Zeager Open Source Mathematics (https://www.stitz-zeager.com/), CSCC PreCalculus and Calculus texts (https://ximera.osu.edu/csccmathematics)



Notice that the graph above resembles the graph of the sine function. As it turns out, the sine function exhibits some of the same oscillatory behavior as f. This shared property turns out to be very important, especially when looking at functions that are related to circles.

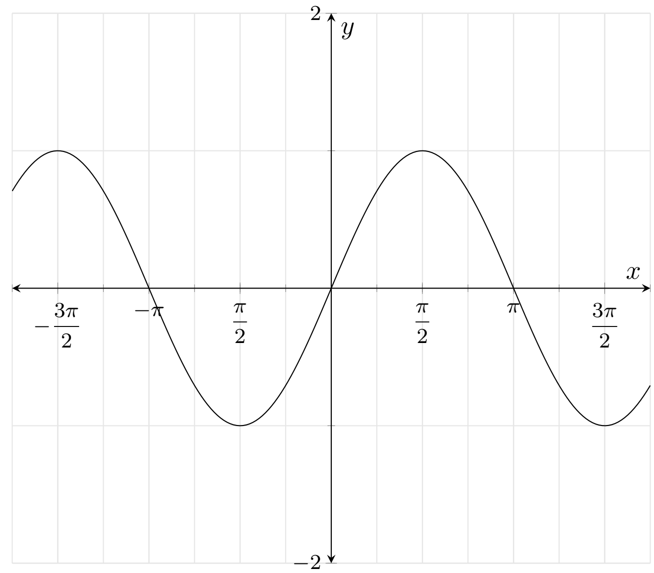
> **Definition** Let f be a function whose domain and codomain are each the set of all real numbers. We say that f is **periodic** provided that there exists a real number k such that $f(x+k) = f(x)$ for every possible choice of x. The smallest positive value p for which $f(x+p) = f(x)$ for every choice of x is called the **period** of f.

For our ferris wheel example above, the period is the circumference of the circle that generates the curve. In the graph, we see how the curve has completed one full cycle of behavior every 24 units, regardless of where we start on the curve.

Two important periodic functions are the sine function, which you have seen, and the cosine function, which resembles the sine function. You will study these functions and learn about their relationship with circles in trigonometry.

As a reminder, here is a graph of the sine function along with a table listing some of its values.

Acknowledgements: Source material includes: Active Prelude to Calculus (https://activecalculus.org/prelude), Stitz Zeager Open Source Mathematics (https://www.stitz-zeager.com/), CSCC PreCalculus and Calculus texts (https://ximera.osu.edu/csccmathematics)



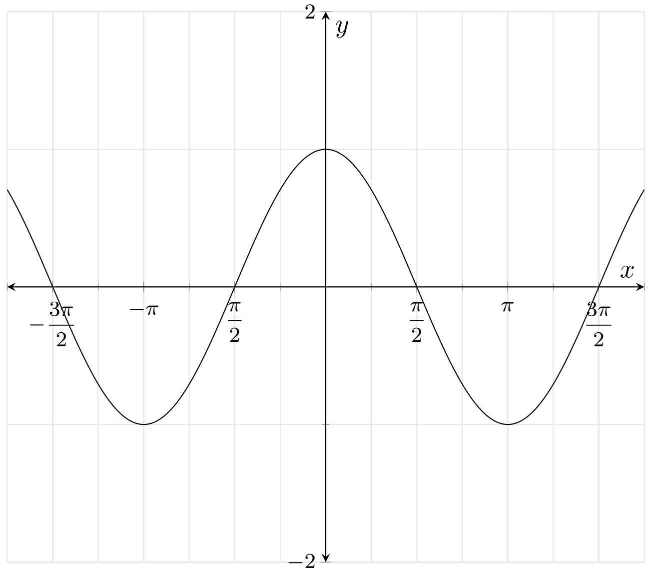
Important Values of $y = \sin(x)$

x	y
$-\pi$	0
$-\frac{\pi}{2}$	-1
0	0
$\frac{\pi}{2}$	1
π	0
$\frac{3\pi}{2}$	-1
2π	0

Acknowledgements: Source material includes: Active Prelude to Calculus (https://activecalculus.org/prelude), Stitz Zeager Open Source Mathematics (https://www.stitz-zeager.com/), CSCC PreCalculus and Calculus texts (https://ximera.osu.edu/csccmathematics)

Notice that $\sin\left(-\dfrac{\pi}{2}\right) = -1$ and $\sin\left(\dfrac{3\pi}{2}\right) = -1$ as well. In fact, the sine function is periodic with period 2π.

Now, here is a graph of the cosine function along with a table listing some of its values.



Acknowledgements: Source material includes: Active Prelude to Calculus (https://activecalculus.org/prelude), Stitz Zeager Open Source Mathematics (https://www.stitz-zeager.com/), CSCC PreCalculus and Calculus texts (https://ximera.osu.edu/csccmathematics)

Important Values of $y = \cos(x)$	
x	y
$-\pi$	-1
$-\frac{\pi}{2}$	0
0	1
$\frac{\pi}{2}$	0
π	-1
$\frac{3\pi}{2}$	0
2π	1

Notice that $\cos(-\pi) = -1$ and $\cos(\pi) = -1$ as well. In fact, the cosine function is also periodic with period 2π.

Summary

- For a function f defined on the real numbers, we say f is periodic if there exists some k such that

$$f(x+k) = f(x)$$

for all possible choices of x. The smallest value of k for which $f(x+k) = f(x)$ for all possible choices of x is called the period of f.

3.2.3 Inverse Functions

Motivating Questions

- What does it mean to say that a function has an inverse?
- How can we identify when we can find inverse functions?
- What are the properties of an inverse function compared to the original function?

Because every function is a process that converts a collection of inputs to a corresponding collection of outputs, a natural question is: for a particular function, can we find a function that "undoes" the first functions? That is, given an output of the function, can we determine the input? If we phrase this question algebraically, it is analogous to asking: given an equation that defines y is a function of x, is it possible to find a corresponding equation where x is a function of y?

Inverse functions

Let's think about the problem in a more concrete way. Consider a situation in which Jessica is running a candle company. Say she starts the day with \$15, but makes \$4 for every candle she sells. A linear function f representing the amount of money in dollars she has after selling x candles is given by $f(x) = 4x + 15$. To find out how much money she has after selling 20 candles, we can plug in 20 to the equation above:

$$f(20) = 4 \cdot 20 + 15 = 95.$$

Now suppose that Jessica tells us that at the end of the day, she ended up with \$135. Would it be possible to figure out how many candles she sold? This type of "inverse question" is common in math. Notice that above, when we started with an amount of candles and wanted to find an amount of money, we first multiplied by 4, then added 15. Now, since we're starting with an amount of money, we need to undo the processes we did before: first we subtract 15, and then divide by 4. We can represent this process by a function g, which represents the number of candles sold if Jessica has y dollars:

$$g(y) = \frac{1}{4}(y - 15).$$

When we plug in \$135 to this function, we find that Jessica has sold

$$g(135) = \frac{1}{4}(135 - 15) = 30$$

candles.

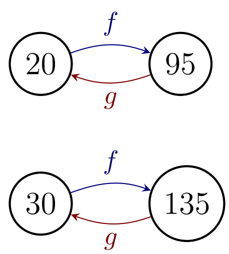
Acknowledgements: Active Prelude to Calculus (https://activecalculus.org/prelude) and Stitz Zeager Open Source Mathematics (https://www.stitz-zeager.com/)

Exploration Let f be a function defined by $f(x) = 4x + 15$. Let g be a function defined by $g(y) = \frac{1}{4}(y - 15)$.

a. Recall that $f(20) = 95$. What is $g(95)$?

b. Recall that $g(135) = 30$. What is $f(30)$?

After plugging in, we see that when we plug 20 into f, we get $f(20) = 95$, and when we plug 95 into g (that is, we plug $f(20)$ into g), we get back 20. Similarly, when we plug in the value $g(135)$ into f, we get back 135. The following diagram illustrates the situation:



In the above exploration, f and g are examples of *inverse functions*.

Definition Let f be a function. If there exists a function g such that

$$g(f(a)) = a \text{ and } f(g(b)) = b$$

for each a and b, then we say that f has an **inverse function** and that the function g is the **inverse** of f. We also say that f is **invertible**.

Note particularly what the equation $g(f(a)) = a$ says: for any input a, the function g will reverse the process of f (which converts a to $f(a)$) because g converts $f(a)$ back to a.

When a given function f has a corresponding inverse function g, we usually rename g as f^{-1}, which we read aloud as "f-inverse". The equation $g(f(a)) = a$ now reads as $f^{-1}(f(a)) = a$, which we interpret as saying "f-inverse converts $f(a)$ back to a". We similarly write that $f(f^{-1}(b)) = b$.

Exploration Dolbear's function $F = D(N) = 40 + \frac{1}{4}N$ is used to model the number N of snowy tree cricket chirps per minute to a corresponding Fahrenheit temperature.

a. Solve the equation $F = 40 + \frac{1}{4}N$ for N in terms of F. Call the

Acknowledgements: Active Prelude to Calculus (https://activecalculus.org/prelude) and Stitz Zeager Open Source Mathematics (https://www.stitz-zeager.com/)

resulting function $N = E(F)$.

b. Explain in words the process or effect of the function $N = E(F)$. What does it take as input? What does it generate as output?

c. Use the function E that you found above to compute $j(N) = E(D(N))$. Simplify your result as much as possible. Do likewise for $k(F) = D(E(F))$. What do you notice about these two functions j and k?

d. Consider the equations $F = 40 + \frac{1}{4}N$ and $N = 4(F - 40)$. Do these equations express different relationships between F and N, or do they express the same relationship in two different ways? Explain.

When a given function has an inverse function, it allows us to express the same relationship from two different points of view. For instance, if $y = f(t) = 2t + 1$, we can show that the function $t = g(y) = \frac{y-1}{2}$ reverses the effect of f (and vice versa), and thus $g = f^{-1}$. We observe that

$$y = f(t) = 2t + 1 \text{ and } t = f^{-1}(y) = \frac{y-1}{2}$$

are equivalent forms of the same equation, and thus they say the same thing from two different perspectives. The first version of the equation is solved for y in terms of t, while the second equation is solved for t in terms of y. This important principle holds in general whenever a function has an inverse function.

If $y = f(t)$ has an inverse function, then the equations

$$y = f(t) \text{ and } t = f^{-1}(y)$$

say the exact same thing but from two different perspectives.

When can we find inverses?

It's important to note in the above definition of inverse functions that we say "*If* there exists …". That is, we don't guarantee that an inverse function exists for a given function. Thus, we might ask: how can we determine whether or not a given function has a corresponding inverse function? As with many questions about functions, there are often many different possible ways to explore such a question. Let's consider this question using a table, through a graph, or through an algebraic formula.

Consider the functions f and g given in the following tables.

Acknowledgements: Active Prelude to Calculus (https://activecalculus.org/prelude) and Stitz Zeager Open Source Mathematics (https://www.stitz-zeager.com/)

x	$f(x)$
0	6
1	4
2	3
3	4
4	6

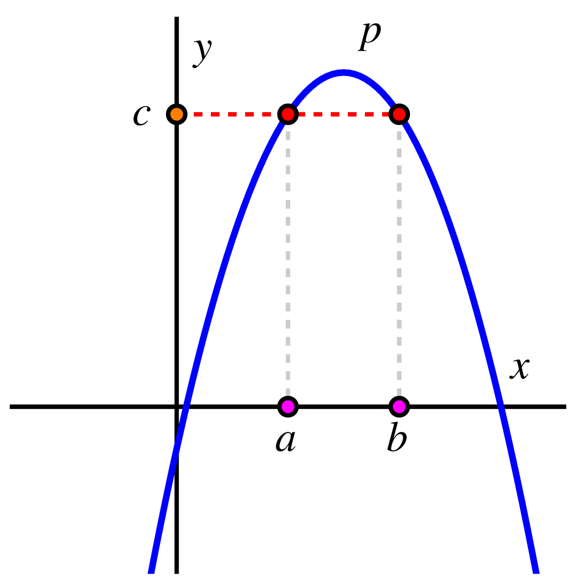
x	$g(x)$
0	3
1	1
2	4
3	2
4	0

For any function, the question of whether or not it has an inverse comes down to whether or not the process of the function can be reliably reversed. For functions given in table form such as f and g, we essentially ask if it's possible to switch the input and output columns and have the new resulting table also represent a function.

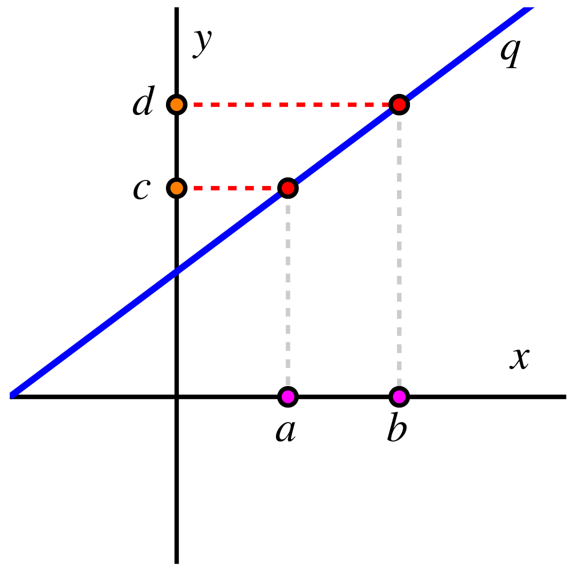
The function f does not have an inverse function because there are two different inputs that lead to the same output: $f(0) = 6$ and $f(4) = 6$. If we attempt to reverse this process, we have a situation where the input 6 would correspond to *two* potential outputs, 4 and 6, which is not allowed for a function.

However, the function g does have an inverse function because when we reverse the columns in the table each input (in order, 3, 1, 4, 2, 0) indeed corresponds to one and only one output (in order, 0, 1, 2, 3, 4). We can thus make observations such as $g^{-1}(4) = 2$, which is the same as saying that $g(2) = 4$, just from a different perspective.

Now, consider the functions p and q represented by the following graphs.



Acknowledgements: Active Prelude to Calculus (https://activecalculus.org/prelude) and Stitz Zeager Open Source Mathematics (https://www.stitz-zeager.com/)



Recall that when a point such as (a, c) lies on the graph of a function p, this means that the input $x = a$, which represents a value on the horizontal axis, corresponds with the output $y = c$ that is represented by a value on the vertical axis. In this situation, we write $p(a) = c$. We note explicitly that p is a function because its graph passes the vertical line test: any vertical line intersects the graph of p once, and thus each input corresponds to one output.

If we attempt to change perspective and use the graph of p to view x as a function of y, we see that this fails because the output value c is associated with two different inputs, a and b. Said differently, because the horizontal line $y = c$ intersects the graph of p at both (a, c) and (b, c) (as shown in the figure), we cannot view y as the input to a function process that produces the corresponding x-value. Therefore, p does not have an inverse function. Note: We can have a relation that switches the x- and y-values of p, but it won't be a function!

On the other hand, provided that the behavior seen in the figure continues, the function q does have an inverse because we can view x as a function of y via the graph. This is because for any choice of y, there corresponds one x that results from y. We can think of this visually by starting at a value such as $y = c$ on the y-axis, moving horizontally to where the line intersects the graph of p, and then moving down to the corresponding location (here $x = a$) on the horizontal axis. From the behavior of the graph of q (a straight line that is always increasing), we see that this correspondence will hold for any choice of y, and thus indeed x is a function of y. From this, we can say that q indeed has an inverse function. We thus can write that $q^{-1}(c) = a$, which is a different way to express the equivalent fact that $q(a) = c$.

Acknowledgements: Active Prelude to Calculus (https://activecalculus.org/prelude) and Stitz Zeager Open Source Mathematics (https://www.stitz-zeager.com/)

The two examples above illustrate an important requirement for a function to have an inverse function. For a function to have an inverse, different inputs must go to different outputs, or else we will run into the same problems as we did in the examples of f and p above.

Definition A function f is said to be **one-to-one** if f matches unique inputs to unique outputs. Equivalently, f is one-to-one if and only if whenever $f(a) = f(b)$, then $a = b$. If a function is one-to-one, it has an inverse.

In particular, the graphical observations that we made for the function q in the last example provide a general test for whether or not a function given by a graph has a corresponding inverse function.

Exploration Consider the functions r and s defined by

$$y = r(t) = 3 - \frac{1}{5}(t-1)^3 \text{ and } y = s(t) = 3 - \frac{1}{5}(t-1)^2.$$

a. Solve the equation $y = r(t)$ for t. Can t be expressed as a single function of y? What could go wrong?

b. Is r one-to-one?

c. Find $s(2)$ and $s(0)$. Compare them; is s is one-to-one?

d. Solve the equation $y = s(t)$ for t. Can t be expressed as a single function of y? What could go wrong?

To check your answers for (b) and (b) above, you could try graphing r and s on Desmos and applying the horizontal line test.

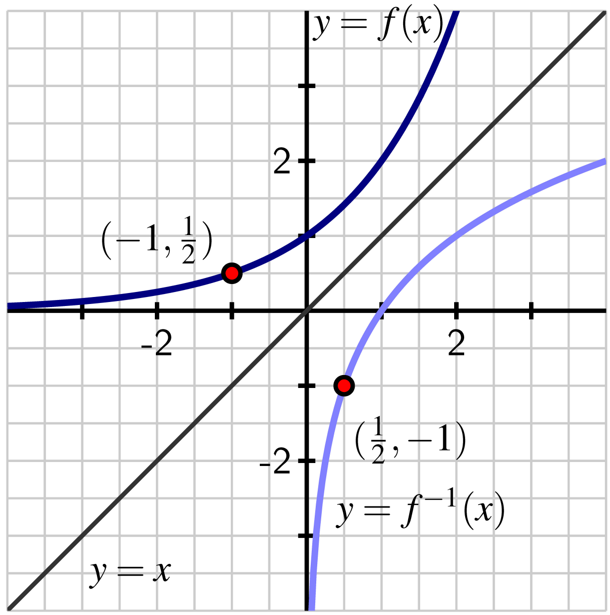
Remark A function whose graph lies in the x-y plane is one-to-one if and only if every horizontal line intersects the graph at most once. When the graph passes this test, the horizontal coordinate of each point on the graph can be viewed as a function of the vertical coordinate of the point. This is sometimes referred to as *the horizontal line test.*

Graphical properties of inverse functions

Finally, we mention an important relationship between the graph of a function and the graph of its inverse function. If f is one-to-one, then recall that a point (x, y) lies on the graph of f if and only if $y = f(x)$. From this, since f is one-to-one, we can equivalently say that $x = f^{-1}(y)$. Hence, the point (y, x) lies on the graph of $x = f^{-1}(y)$.

Acknowledgements: Active Prelude to Calculus (https://activecalculus.org/prelude) and Stitz Zeager Open Source Mathematics (https://www.stitz-zeager.com/)

The last item above leads to a special relationship between the graphs of f and f^{-1} when viewed on the same coordinate axes. In that setting, we need to view x as the input of each function (since it's the horizontal coordinate) and y as the output. If we know a particular input-output relationship for f, say $f(-1) = \frac{1}{2}$, then it follows that $f^{-1}\left(\frac{1}{2}\right) = -1$. We observe that the points $\left(-1, \frac{1}{2}\right)$ and $\left(\frac{1}{2}, -1\right)$ are reflections of each other across the line $y = x$. Because such a relationship holds for every point (x, y) on the graph of f, this means that the graphs of f and f^{-1} are reflections of one another across the line $y = x$, as seen in the figure below.



Summary

- We say a function f has an inverse function g if $g(f(a)) = a$ and $f(g(b)) = b$ for all a and b. We often use the notation f^{-1} for g.
- We say a function is one-to-one if it matches different inputs to different outputs. We can check graphically if a function is one-to-one with the horizontal line test. All one-to-one functions have inverses, and vice versa.
- The graph of f^{-1} is the graph of f reflected across the line $y = x$.

3.2.4 Famous Function Properties

In Section 1-2, you saw a variety of famous functions. Now that we have learned more about properties of functions, we can update our knowledge of those famous functions. We will go through the list of famous functions from before and point out where each function might have properties we've discussed.

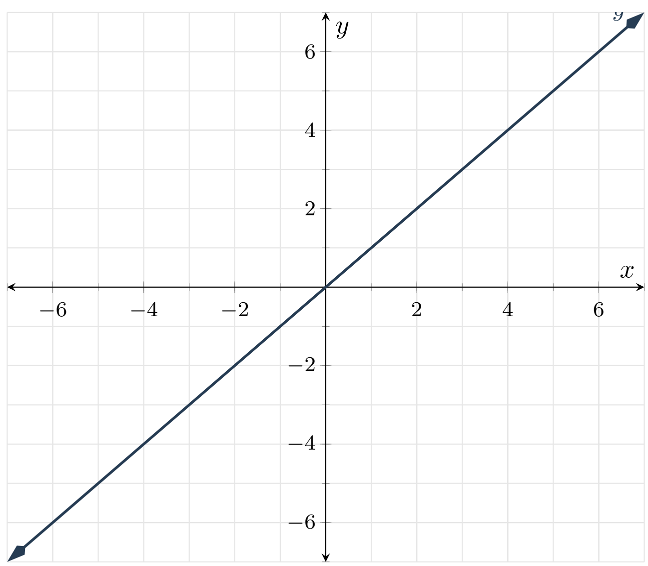
 Licensed under Creative Commons 4.0

Linear Functions

Recall that the graph of a linear function is a line.

Example 7. *A prototypical example of a linear function is*

$$y = x.$$



Important Values of $y = x$

x	y
-2	-2
-1	-1
0	0
1	1
2	2

In general, linear functions can be written as $y = mx + b$ where m and b can be any numbers. We learned that m represents the slope, and b is the y-coordinate of the y-intercept. You can play with changing the values of m and b on the graph using Desmos and see how that changes the line.

Desmos link: `https://www.desmos.com/calculator/japnhapzvn`

 Licensed under Creative Commons 4.0

Note that a linear function f defined by $f(x) = mx + b$ with $b = 0$ is odd. If $m = 0$, then f is periodic, since it is constant. Furthermore, constant functions are always even.

Additionally, if $m \neq 0$, then a linear function is one-to-one, and therefore invertible. We summarize this information in the table below.

Properties of Linear Functions

Periodic?	If $m = 0$
Odd?	If $b = 0$
Even?	If $m = 0$
One-to-one/invertible?	Yes

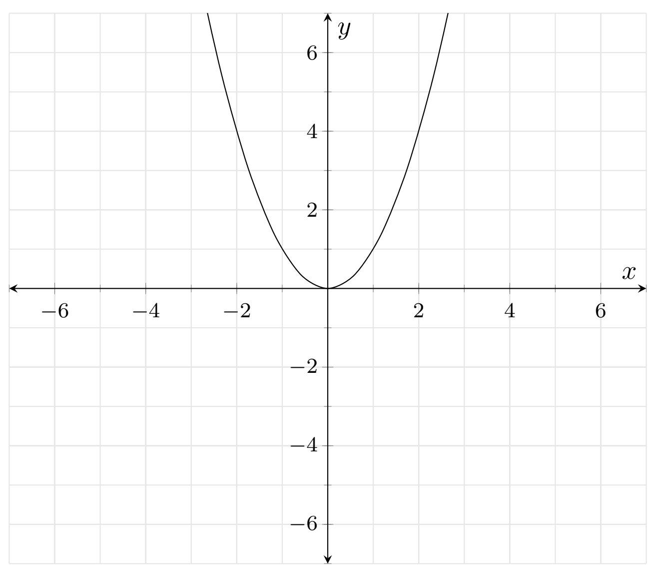
 Licensed under Creative Commons 4.0

Quadratic Functions

Recall that the graph of a quadratic function is a parabola.

Example 8. *A prototypical example of a quadratic function is*

$$y = x^2.$$



Important Values of $y = x^2$

x	y
-2	4
-1	1
0	0
1	1
2	4

In general, quadratic functions can be written as $y = ax^2 + bx + c$ where a, b, and c can be any numbers. You can play with changing the values of a, b, and c on the graph using Desmos and see how that changes the parabola.

 Licensed under Creative Commons 4.0

Desmos link: https://www.desmos.com/calculator/nmlghfrws9

Note that for a quadratic function f defined by $f(x) = ax^2 + bx + c$, if $b = 0$, then f is even. In general, quadratic functions are not one-to-one, odd, or periodic, except in cases where $a = 0$, in which we're actually dealing with a linear function. We summarize this information in the table below.

Properties of Quadratic Functions $y = ax^2 + bx + c, a \neq 0$

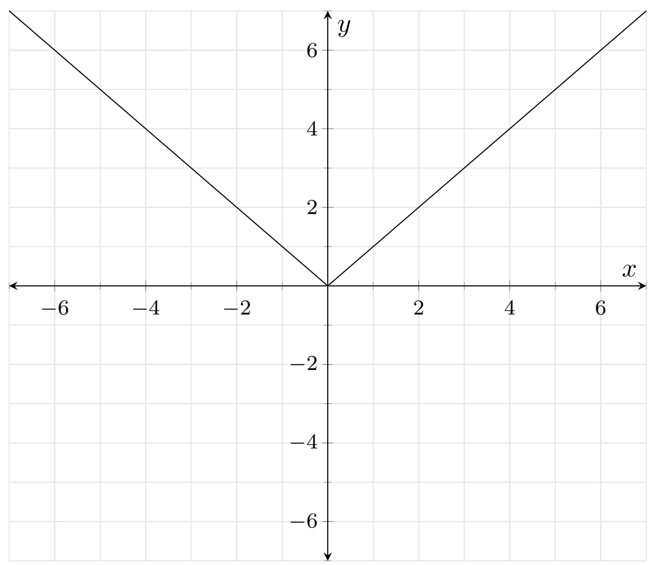
Periodic?	No
Odd?	No
Even?	If $b = 0$
One-to-one/invertible?	No

 Licensed under Creative Commons 4.0

Absolute Value Function

Another important type of function is the absolute value function. This is the function that takes all y-values and makes them positive. The absolute value function is written as

$$y = |x|.$$



Important Values of $y = |x|$

x	y
-2	2
-1	1
0	0
1	1
2	2

Notice that the absolute value function is even. Is it one-to-one? The fact that it's even tells us that it is not, since $|-x| = |x|$ for all x. We summarize this information in the table below.

Licensed under Creative Commons 4.0

Properties of the Absolute Value Function $y = |x|$

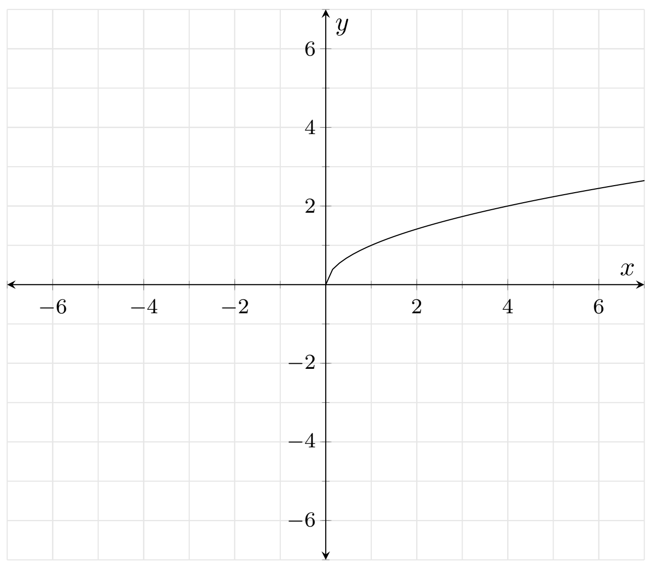
Periodic?	No
Odd?	No
Even?	Yes
One-to-one/invertible?	No

 Licensed under Creative Commons 4.0

Square Root Function

Another famous function is the square root function,

$$y = \sqrt{x}.$$



Important Values of $y = \sqrt{x}$

x	y
0	0
1	1
4	2
9	3
25	5

The square root function is one-to-one. Negative inputs are not valid for the square root function, so it is neither even, odd, nor periodic. We summarize this information in the table below.

 Licensed under Creative Commons 4.0

Properties of the Square Root Function $y = \sqrt{x}$

Periodic?	No
Odd?	No
Even?	No
One-to-one/invertible?	Yes

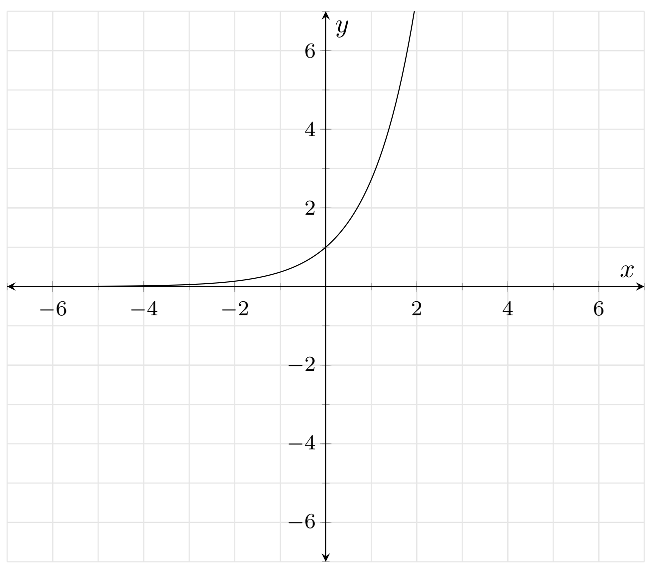
 Licensed under Creative Commons 4.0

Exponential Functions

Another famous function is the exponential growth function,

$$y = e^x.$$

Here e is the mathematical constant known as Euler's number. $e \approx 2.71828..$



Important Values of $y = e^x$

x	y
0	1
1	e
−1	$e^{-1} = \frac{1}{e}$

In general, we can talk about exponential functions of the form $y = b^x$ where b is a positive number not equal to 1. You can play with changing the values of b on the graph using Desmos and see how that changes the graph. Pay particular attention to the difference between $b > 1$ and $0 < b < 1$.

 Licensed under Creative Commons 4.0

Desmos link: https://www.desmos.com/calculator/qsmvb7tiex

Notice that exponential functions are one-to-one, and therefore invertible. However, they are neither even, odd, nor periodic. We summarize this information in the table below.

Properties of the Exponential Functions $y = b^x$

Periodic?	No
Odd?	No
Even?	No
One-to-one/invertible?	Yes

 Licensed under Creative Commons 4.0

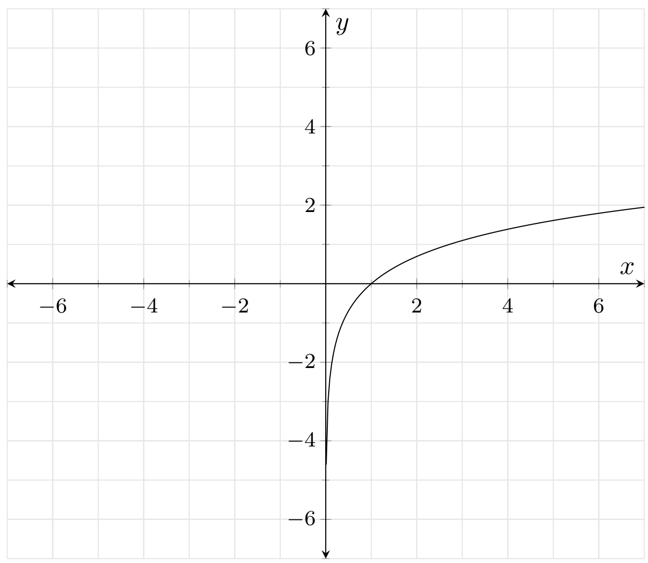
Logarithm Functions

Another group of famous functions are logarithms.

Example 9. *The most famous logarithm function is*

$$y = \ln(x) = \log_e(x).$$

Here e is the mathematical constant known as Euler's number. $e \approx 2.71828$*.*



Important Values of $y = \ln(x)$

x	y
0	*undefined*
$\frac{1}{e}$	-1
1	0
e	1

In general, we can talk about logarithmic functions of the form $y = \log_b(x)$ where b is a positive number not equal to 1. You can play with changing the values of b on the graph using Desmos and see how that changes the graph. Pay particular attention to the difference between $b > 1$ and $0 < b < 1$.

 Licensed under Creative Commons 4.0

Desmos link: https://www.desmos.com/calculator/lxllnpdi6w

Notice that logarithms are neither even, odd, nor periodic. However, they are one-to-one, and therefore invertible. It turns out that the inverse of a logarithm is an exponential function, and vice versa! We summarize this information in the table below.

Properties of the Logarithm Functions $y = \log_b(x)$

Periodic?	No
Odd?	No
Even?	No
One-to-one/invertible?	Yes

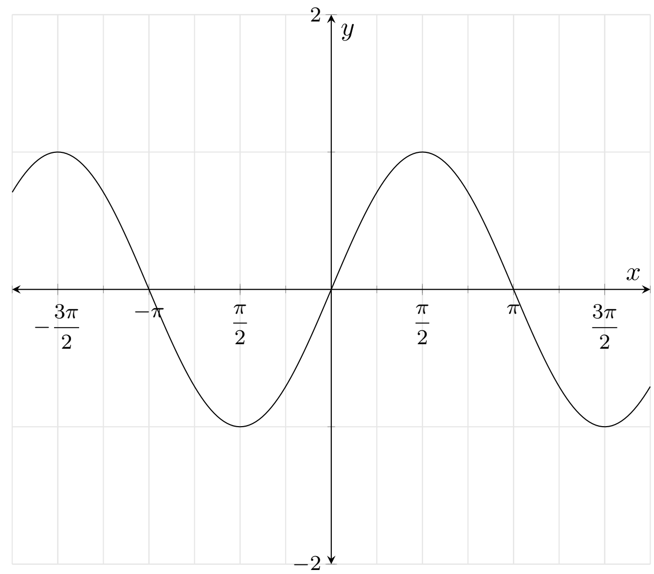
 Licensed under Creative Commons 4.0

Sine

Another important function is the sine function,

$$y = \sin(x).$$

This function comes from trigonometry. In the table below we will use another mathematical constant, π ("pi" pronounced pie). $\pi \approx 3.14159$.



 Licensed under Creative Commons 4.0

Important Values of $y = \sin(x)$

x	y
$-\pi$	0
$-\frac{\pi}{2}$	-1
0	0
$\frac{\pi}{2}$	1
π	0
$\frac{3\pi}{2}$	-1
2π	0

As mentioned earlier, the sine function is odd and periodic with period 2π. Since it is periodic, however, it cannot be one-to-one, since its values repeat. We summarize this information in the table below.

Properties of the Sine Function $y = \sin(x)$

Periodic?	Yes, with period 2π
Odd?	Yes
Even?	No
One-to-one/invertible?	No

In general, we can consider $y = a\sin(bx)$. You can play with changing the values of a and b on the graph using Desmos and see how that changes the graph.

Desmos link: `https://www.desmos.com/calculator/vkxzcfv2aq`

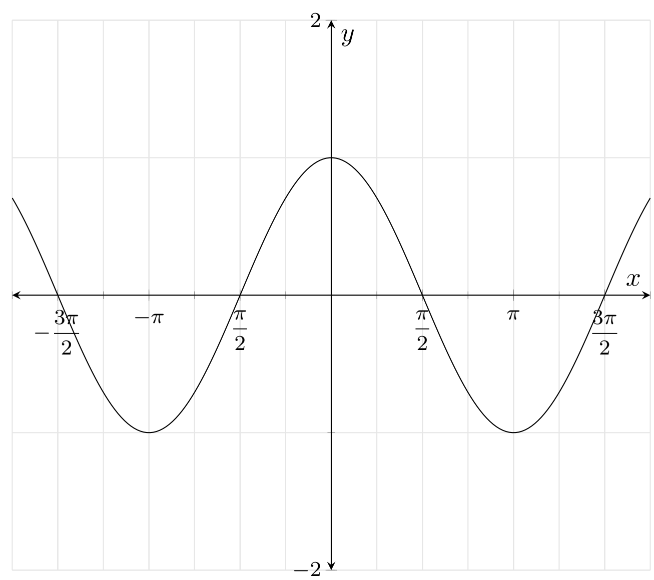
 Licensed under Creative Commons 4.0

Cosine

A function introduced in Section 3-2 is the cosine function,

$$y = \cos(x).$$

As with sine, the cosine function comes from trigonometry. In the table below we will again use π.



 Licensed under Creative Commons 4.0

Important Values of $y = \cos(x)$

x	y
$-\pi$	-1
$-\dfrac{\pi}{2}$	0
0	1
$\dfrac{\pi}{2}$	0
π	-1
$\dfrac{3\pi}{2}$	0
2π	1

As mentioned earlier, the cosine function is even and periodic with period 2π. Since it is periodic, however, it cannot be one-to-one, since its values repeat. We summarize some information in the table below.

Properties of the Cosine Function $y = \cos(x)$

Periodic?	Yes, with period 2π
Odd?	No
Even?	Yes
One-to-one/invertible?	No

In general, we can consider $y = a\cos(bx)$. You can play with changing the values of a and b on the graph using Desmos and see how that changes the graph.

Desmos link: `https://www.desmos.com/calculator/kvmz1kt19n`

3.3 Average Rate of Change of Functions

Learning Objectives

- Average Rate of Change
  - What is average rate of change?
  - What does it measure?
  - Connecting Average Rate of Change to slope of the line between the two points

3.3.1 Average Rate of Change

Motivating Questions

- What do we mean by the average rate of change of a function on an interval?
- What does the average rate of change of a function measure? How do we interpret its meaning in context?
- How is the average rate of change of a function connected to a line that passes through two points on the curve?

Given a function that models a certain phenomenon, it's natural to ask such questions as "how is the function changing on a given interval" or "on which interval is the function changing more rapidly?" The concept of *average rate of change* enables us to make these questions more mathematically precise. Initially, we will focus on the average rate of change of an object moving along a straight-line path.

First, let's define some notation for the intervals we will be referring to in this section and going forward.

Definition $[a, b]$ represents the values of x such that $a \leq x \leq b$. We call this the **closed interval from a to b**. (a, b) represents the values of x such that $a < x < b$. We call this the **open interval from a to b**. Notice that the major difference between these intervals is that $x = a$ and $x = b$ are included in the closed interval but not in the open interval.

For a function s that tells the location of a moving object along a straight path at time t, we define the average rate of change of s between $(a, s(a))$ and $(b, s(b))$ to be the quantity

$$\text{AROC}_{[a,b]} = \frac{s(b) - s(a)}{b - a}.$$

Note particularly that the average rate of change of s between $(a, s(a))$ and $(b, s(b))$ is measuring the *change in position* divided by the *change in time*.

Exploration Let the height function for a ball tossed vertically be given by $s(t) = 64 - 16(t-1)^2$, where t is measured in seconds and s is measured in feet above the ground.

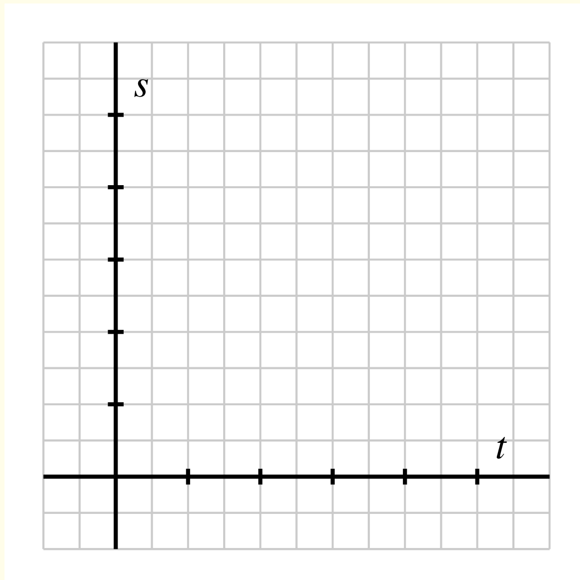
a. Compute the value of $\text{AROC}_{[1.5,2.5]}$

b. What are the units on the quantity $\text{AROC}_{[1.5,2.5]}$? What is the meaning of this number in the context of the rising/falling ball?

c. In *Desmos*, plot the function $s(t) = 64 - 16(t-1)^2$ along with the

Licensed under Creative Commons 4.0

points $(1.5, s(1.5))$ and $(2.5, s(2.5))$. Make a copy of your plot on the axes below, labeling key points as well as the scale on your axes. What is the domain of the model? The range? Why?



d. Work by hand to find the equation of the line through the points $(1.5, s(1.5))$ and $(2.5, s(2.5))$. Write the line in the form $y = mt + b$ and plot the line in *Desmos*, as well as on the axes above.

e. What is a geometric interpretation of the value $\text{AROC}_{[1.5,2.5]}$ in light of your work in the preceding questions?

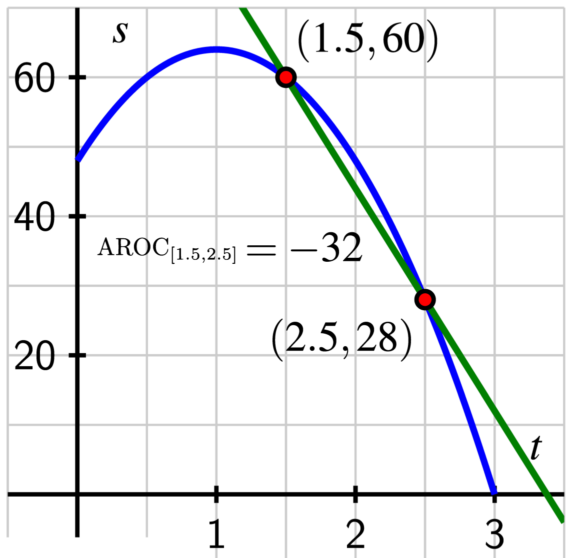
f. How do your answers in the preceding questions change if we instead consider the interval $[0.25, 0.75]$? $[0.5, 1.5]$? $[1, 3]$?

Defining and interpreting the average rate of change of a function

In the context of a function that measures height or position of a moving object at a given time, the meaning of the average rate of change of the function on a given interval is the *average velocity of the moving object* because it is the ratio of *change in position* to *change in time*. For example, in the exploration above, the units on $\text{AROC}_{[1.5,2.5]} = -32$ are "feet per second" since the units on the numerator are "feet" and on the denominator "seconds". Morever, -32 is numerically the same value as the slope of the line that connects the two corresponding points on the graph of the position function, as seen below. The

 Licensed under Creative Commons 4.0

fact that the average rate of change is negative in this example indicates that the ball is falling.

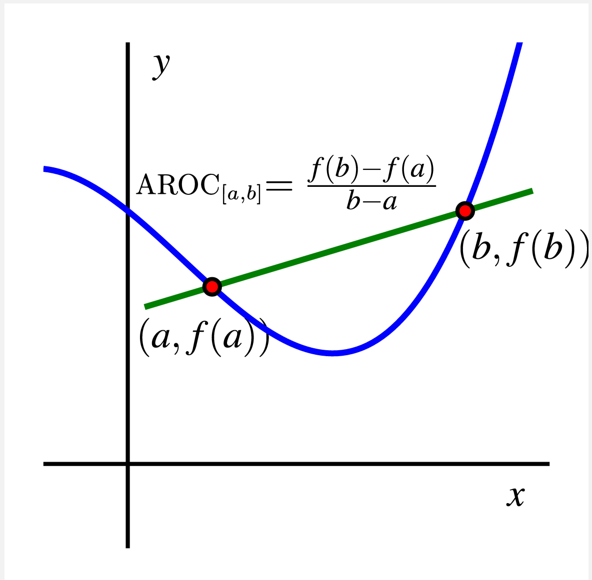


While the average rate of change of a position function tells us the moving object's average velocity, in other contexts, the average rate of change of a function can be similarly defined and has a related interpretation. We make the following formal definition.

Definition For a function f defined on an interval $[a,b]$, the **average rate of change of f between $(a, f(a))$ and $(b, f(b))$** is the quantity

$$\text{AROC}_{[a,b]} = \frac{f(b) - f(a)}{b - a}.$$

 Licensed under Creative Commons 4.0



In every situation, the units on the average rate of change help us interpret its meaning, and those units are always "units of output per unit of input." Moreover, the average rate of change of f on $[a, b]$ always corresponds to the slope of the line between the points $(a, f(a))$ and $(b, f(b))$. Before we explore this concept further, we note that this line has a special name, it is the *secant line* to the graph.

Definition: Consider a function $y = f(x)$. A line passing through two points $(a, f(a))$ and $(b, f(b))$, with $a \neq b$, in the graph of $y = f(x)$, is called a **secant line** to the graph. The slope of a secant line is the average rate of change of the function on the interval $[a, b]$.

Exploration According to the US census, the populations of Kent and Ottawa Counties (Grand Rapids is in Kent, Allendale in Ottawa) from 1960 to 2010 measured in 10-year intervals are given in the following tables.

Kent County Population data

1960	1970	1980	1990	2000	2010
363,187	411,044	444,506	500,631	574,336	602,622

Ottawa County Population data

Licensed under Creative Commons 4.0

1960	1970	1980	1990	2000	2010
98,719	128,181	157,174	187,768	238,313	263,801

Let $K(Y)$ represent the population of Kent County in year Y and $W(Y)$ the population of Ottawa County in year Y.

a. Compute $\text{AROC}_{[1990,2010]}$ for both K and O.

b. What are the units on each of the quantities you computed in (a.)?

c. Write a careful sentence that explains the meaning of the average rate of change of the Ottawa county population on the time interval $[1990, 2010]$. Your sentence should begin something like "In an average year between 1990 and 2010, the population of Ottawa County was ..."

d. Which county had a greater average rate of change during the time interval $[2000, 2010]$? Were there any intervals in which one of the counties had a negative average rate of change?

e. Using the given data, what do you predict will be the population of Ottawa County in 2018? Why?

The average rate of change of a function on an interval gives us an excellent way to describe how the function behaves, on average. For instance, if we compute $\text{AROC}_{[1970,2000]}$ for Kent County, we find that

$$\text{AROC}_{[1970,2000]} = \frac{573,336 - 411,044}{30} \approx 5409.73,$$

which tells us that in an average year from 1970 to 2000, the population of Kent County increased by about 5410 people. Said differently, we could also say that from 1970 to 2000, Kent County was growing at an average rate of 5410 people per year. These ideas also afford the opportunity to make comparisons over time. Since

$$\text{AROC}_{[1990,2000]} = \frac{573,336 - 500,631}{30} = 7270.5,$$

we can not only say that the county's population increased by about 7270 in an average year between 1990 and 2000, but also that the population was growing faster from 1990 to 2000 than it did from 1970 to 2000.

Finally, we can even use the average rate of change of a function to predict future behavior. Since the population was changing on average by 7270.5 people per year from 1990 to 2000, we can estimate that the population in 2002 is

$$K(2002) \approx K(2000) + 2 \cdot 7270.5 = 573,336 + 14,541 = 587,877.$$

 Licensed under Creative Commons 4.0

How average rate of change indicates function trends

We have already seen that it is natural to use words such as "increasing" and "decreasing" to describe a function's behavior. For instance, for the tennis ball whose height is modeled by $s(t) = 64 - 16(t-1)^2$, we computed that $\text{AROC}_{[1.5,2.5]} = -32$, which indicates that on the interval $[1.5, 2.5]$, the tennis ball's height is decreasing at an average rate of 32 feet per second. Similarly, for the population of Kent County, since $\text{AROC}_{[1990,2000]} = 7270.5$, we know that on the interval $[1990, 2000]$ the population is increasing at an average rate of 7270.5 people per year.

We make the following formal definitions to clarify what it means to say that a function is increasing or decreasing.

Definition Let f be a function defined on an interval (a, b) (that is, on the set of all x for which $a < x < b$). We say that f is **increasing on** (a, b) provided that the function is always rising as we move from left to right. That is, for any x and y in (a, b), if $x < y$, then $f(x) < f(y)$.

Similarly, we say that f is **decreasing on** (a, b) provided that the function is always falling as we move from left to right. That is, for any x and y in (a, b), if $x < y$, then $f(x) > f(y)$.

If a function is increasing, its average rate of change will be positive. If a function is decreasing, its average rate of change will be negative. However the reverse doesn't necessarily hold true. If we compute the average rate of change of a function on an interval, we can decide if the function is increasing or decreasing *on average* on the interval, but it takes more work[1] to decide if the function is increasing or decreasing *always* on the interval.

Exploration Let's consider two different functions and see how different computations of their average rate of change tells us about their respective behavior. Plots of q and h are shown below.

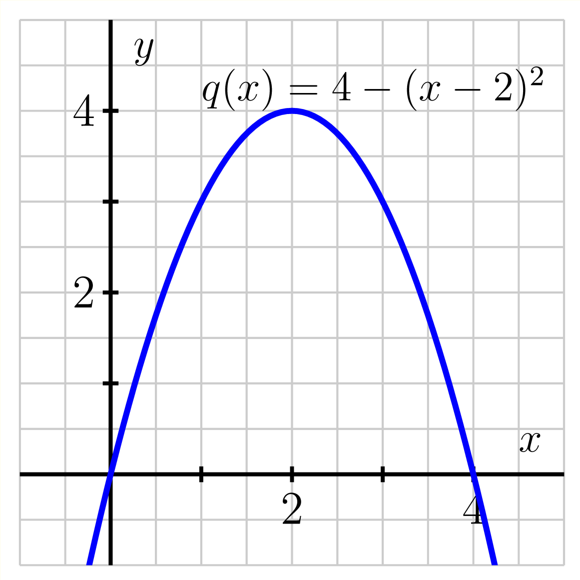
a. Consider the function $q(x) = 4 - (x-2)^2$. Compute $\text{AROC}_{[0,1]}$, $\text{AROC}_{[1,2]}$, $\text{AROC}_{[2,3]}$, and $\text{AROC}_{[3,4]}$. What do your last two computations tell you about the behavior of the function q on $[2, 4]$?

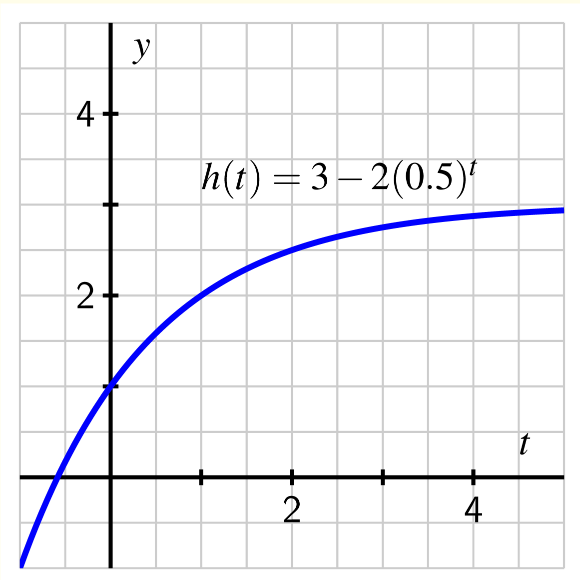
b. Consider the function $h(t) = 3 - 2(0.95)^t$. Compute $\text{AROC}_{[-1,1]}$, $\text{AROC}_{[1,3]}$, and $\text{AROC}_{[3,5]}$. What do your computations tell you about the behavior of the function h on $[-1, 5]$?

c. On the graphs below, plot the line segments whose respective slopes are the average rates of change you computed in (a) and (b).

[1] Calculus offers one way to justify that a function is always increasing or always decreasing on an interval.

Licensed under Creative Commons 4.0





d. True or false: Since $\text{AROC}_{[0,3]} = 1$, the function q is increasing on the interval $(0, 3)$. Justify your decision.

e. Give an example of a function that has the same average rate of change no matter what interval you choose. You can provide your example through a table, a graph, or a formula; regardless of your choice, write a sentence to explain.

 Licensed under Creative Commons 4.0

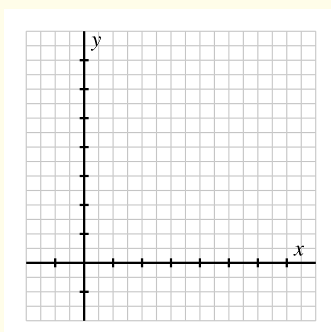
It is helpful be able to connect information about a function's average rate of change and its graph. For instance, if we have determined that $\text{AROC}_{[-3,2]} = 1.75$ for some function f, this tells us that, on average, the function rises between the points $x = -3$ and $x = 2$ and does so at an average rate of 1.75 vertical units for every horizontal unit. Moreover, we can even determine that the difference between $f(2)$ and $f(-3)$ is

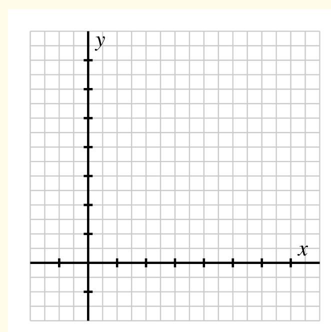
$$f(2) - f(-3) = 1.75 \cdot 5 = 8.75$$

since $\dfrac{f(2) - f(-3)}{2 - (-3)} = 1.75$.

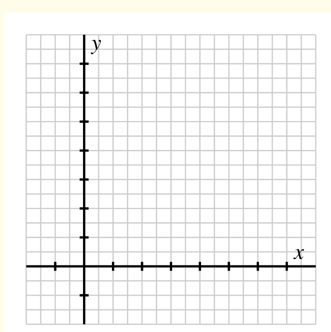
Exploration Sketch at least two different possible graphs that satisfy the criteria for the function stated in each part. Make your graphs as significantly different as you can. If it is impossible for a graph to satisfy the criteria, explain why.

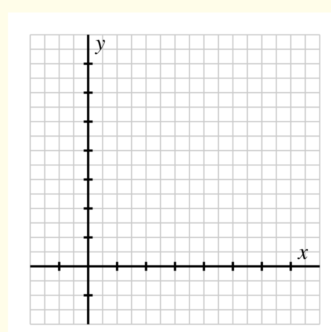
a. f is a function defined on $[-1, 7]$ such that $f(1) = 4$ and $\text{AROC}_{[1,3]} = -2$.





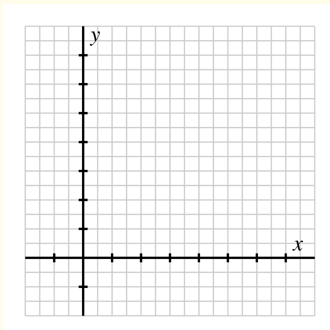
b. g is a function defined on $[-1, 7]$ such that $g(4) = 3$, $\text{AROC}_{[0,4]}$, and g is not always increasing on $(0, 4)$.

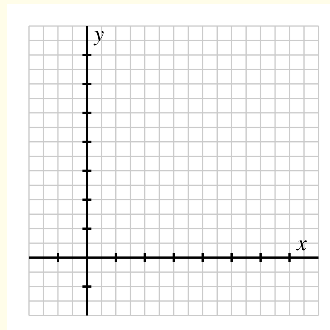




 Licensed under Creative Commons 4.0

c. h is a function defined on $[-1,7]$ such that $h(2)=5$, $h(4)=3$ and $\text{AROC}_{[2,4]}=-2$.





Summary

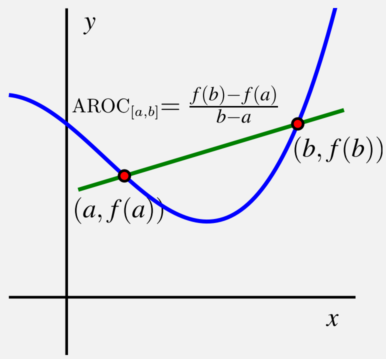
- For a function f defined on an interval $[a,b]$, the average rate of change of f on $[a,b]$ is the quantity

$$\text{AROC}_{[a,b]} = \frac{f(b)-f(a)}{b-a}.$$

- The value of $\text{AROC}_{[a,b]} = \dfrac{f(b)-f(a)}{b-a}$ tells us how much the function rises or falls, on average, for each additional unit we move to the right on the graph. For instance, if $\text{AROC}_{[3,7]} = 0.75$, this means that for additional 1-unit increase in the value of x on the interval $[3,7]$, the function increases, on average, by 0.75 units. In applied settings, the units of $\text{AROC}_{[a,b]}$ are "units of output per unit of input".

- The value of $\text{AROC}_{[a,b]} = \dfrac{f(b)-f(a)}{b-a}$ is also the slope of the line that passes through the points $(a,f(a))$ and $(b,f(b))$ on the graph of f, as shown in the graph below.

 Licensed under Creative Commons 4.0



- This line passing through these two points $(a, f(a))$ and $(b, f(b))$, is called a secant line to the graph, and its slope is equal to the average rate of change of the function on the interval $[a, b]$.

Part 4

Back Matter

Index