Precalculus with Review 1

June 23, 2021

Contents

Part 1

Variables and CoVariation

1.1 Quantitative Reasoning

Learning Objectives

- Estimations
  - Reminders of roles of addition, subtraction, multiplication, division and how to choose each one
- Units
  - Formalize the canceling units as fractions
  - Quick review of properties of fractions, adding, subtracting, multiplying, dividing fractions
- Percentages
  - Formula for finding percent (portion is given by amount*percent)
  - Percent increase and then decrease. What is the whole?

1.1.1 Quantitative Reasoning: Estimates

Tips for Doing Rough Estimates

- Estimations are **NOT** guesses. They can sometimes be educated guesses, but estimations, at least in this course, will always still require some sort of calculation(s).
- Rough estimations are meant to be estimations which can be calculated mentally, meaning without calculators, or even pen and paper. You will still be expected to record/write down your work and thought process for your instructor, but it should be work that you are able to do mentally.
- Everyone has different skill and comfort levels with their mental calculations – you may need to round values a lot to make them something you are comfortable doing in your head. This is completely fine, but the important thing is that in your solutions/write-ups you explain what numbers you rounded and why.

Exploration [In Class Activity]Pizza Party: Now let's give this a try but working with your group to determine what you would buy for a pizza party in the following scenario: You and your roommate are going to have some people over later and so you go to the grocery store to grab some snacks. Everyone has agreed to pitch in $10 to pay for pizza and snacks for the night, and you think about 12 people are coming over. Here are the prices of various snacks from the grocery store:

- Bag of tortilla chips - $3.99
- Salsa - $3.79
- Bag of "normal" chips - $2.99
- Dozen cookies from bakery - $4.99
- Veggie Tray - $14.99
- 12 pack of soda - $5.79

You want to get at least one of each of these items, but you're also going to order some pizza and breadsticks. Here are the prices from your pizza place:

- Cheese:
  - Medium - $9.99
  - Large - $11.99
- Pepperoni:

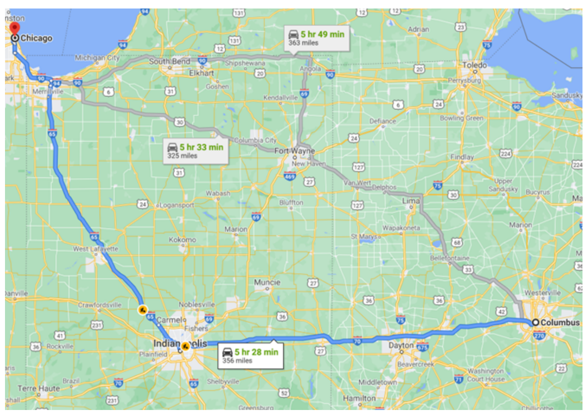
- Medium - $11.49
- Large - $13.49

- Breadsticks:
  - 5 for $4.99

- $2.49 delivery charge. Don't forget tip!

You are a good person and do not plan to pocket any of the money that your friends are going to give you to pay for these pizzas and snacks. Plus, you and your roommate are going to pay for your fairshare (also each pitch in $10). Decide what you're going to get!

Example 1. *Roughly estimate how many gallons of gasoline you might use to drive from here to Chicago, IL.*



Explanation Distance to Chicago, IL: about 350 miles (google maps) My vehicle gets about 30 miles per gallon (30 miles = 1 gallon), so 350 mi $\times \dfrac{1gal}{30mi}$ is about $\dfrac{360}{30} = 12$ gallons

Exploration Roughly estimate how many seconds you've been alive.

- Determine a number that would definitely be way too low for even a rough estimate for each question, but still requires some calculation. Explain why you think this would be unreasonably low. (Still do not use a calculator, use mental arithmetic.)

- Determine a number that would definitely be way too high for even a rough estimate for each question. Explain why you think this would be unreasonably high.

- For each of those problems, do you think your original rough estimation is an underestimate or overestimate? Explain why you think this based off of your calculations. Or if you're not sure if it's under or over, explain why you are unsure.
- If possible, determine a more exact value by using a calculator and not rounding values to get an exact number, or see if google has any estimate(s). Compare this with your original estimations: were you under or over estimating, or can't tell? If you have values to compare, how different are the two values?

1.1.2 Quantitative Reasoning: Units

Units

Is 12 the same as 1? As a mathematical value, no 12 and 1 are not the same value. But if we give these values units, they actually can represent the same thing!
What units can we give to 12 and to 1 so that they are equal?

$$12 \boxed{?} = 1 \boxed{?}$$

Units in everyday life are often used, even if we don't necessarily think of them as units. For example, you wouldn't say "I went to the grocery store and bought a dozen." A dozen what? You've given how many you've bought (the value), but not how many of what you have bought (the units). Instead you would say "I went to the grocery store and bought a dozen eggs."

Example 2. *A typical bottle of wine holds about 24.5 ounces. How much is this in gallons?*
Explanation If we take the original 24.5 ounces in 1 bottle we can set it up as a fraction:

$$\frac{24.5 \quad ounces}{1 \quad bottle}$$

There are 128 ounces in 1 gallon. This can be set up as the following fraction:

$$\frac{1 \quad gallon}{128 \quad ounces}$$

These two fractions have ounces in common. We can set the fractions up in such a way that the ounces will "cancel out."

$$\frac{24.5 \quad ounces}{1 \quad bottle} \times \frac{1 \quad gallon}{128 \quad ounces} = \frac{24.5 \quad \cancel{ounces}}{1 \quad bottle} \times \frac{1 \quad gallon}{128 \quad \cancel{ounces}} = \frac{0.1914 \quad gallon}{1 \quad bottle}$$

So there are 0.1914 gallons in 1 bottle of wine.

Exploration

- "How many ounces are in a yard?" Explain why this question does not make sense.
- "How many feet are in an acre?" Explain why this question does not make sense.

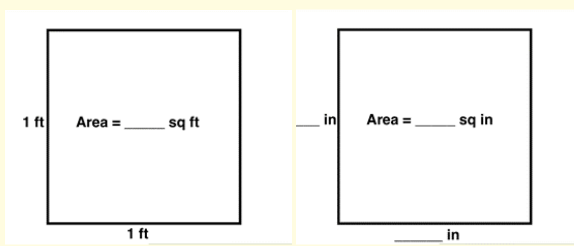
Let's talk a little more about this idea highlighted in the question above (about feet and acre units). It is important to note that feet, square feet, and cubic feet are all different units. They are not measuring the same thing! Feet measures length (one dimension), square feet measure area (two dimensions), and cubic feet measure volume (three dimensions).

Q. We **can** convert between **cubic feet** and which units from previous examples? Why?

We can convert between feet and inches, or feet and miles. Can we convert between square/cubic feet and square/cubic inches, or square/cubic feet and square/cubic miles? To answer this, we have to ask if these pairs of units are measuring the same property. If the answer is yes, then yes we can convert between them!

- Do square feet, square inches, and square miles measure the same property?
- Do cubic feet, cubic inches, and cubic miles measure the same property?

In order to convert between units, we need some sort of equivalence. Just like knowing 4 quarts = 1 gallon, or 16 oz = 1 pound, these are equivalences that we can then use to convert from quarts to gallons or vice versa, or ounces to pounds or vice versa. Let's determine the equivalence for square feet and square inches.



Fill in the blanks above (assume that the squares are the exact same size). Since the squares are the exact same size, we can say that 1 sq ft = $\boxed{?}$ sq in. Use this same reasoning to determine the equivalence between square miles and square feet:
1 sq mi = $\boxed{?}$ sq ft

1.1.3 Quantitative Reasoning: Percents

Percentages

Percentages are everywhere! They are used to describe discounts, markups, commissions, statistics, information, change, and on and on. They are an important topic and so we want to make sure that we really understand them. Let's start by just breaking down the word "percent." "Per" and "cent." There are some specific words that often translate into certain mathematical operations. For example, if a question asks, "how many apples and oranges are there?" what operation are you going to do with the number of apples and oranges? Let's make a short list of which words usually translate into which operation:
If you read a math problem: "There are 5 apples and 6 oranges. How many apples and oranges are there?" What math operation are you going to do with the number of apples (5) and oranges (6)?

"and" often translates to: $\boxed{?}$

Let's change the problem a little bit: "There are 4 apples and 7 oranges. What is the difference between the number of apples and number of oranges?" What math operation are you going to do with the number of apples (4) and oranges (7)?

"difference" often translates to $\boxed{?}$

Let's change it again: "We want each of the 3 people in our group to have 5 apples. How many apples are we going to need?" What math operation are you going to do with the number of people (3) and apples (5)?

"each of" often translates to: $\boxed{?}$

Note: We will also see in the following problems that specifically with percentages, "of" often translates to this same operation.
Another change: "We have a group of 5 people and a total of 15 apples. How many apples per person are there?" What math operation are you going to do with the number of apples (15) and people (5)?

"per" often translates to: $\boxed{?}$

Let's change it one last time: "The total number of apples and oranges is 13. If the number of apples is "x" and the number of oranges is "y," write an equation for the total number of apples and oranges." What math symbol did you put in for the word "is"?

"is" often translates to: $\boxed{?}$

Now "cent." Where have we seen this word before? How many cents are in a dollar? How many years are in a century? How many centimeters are in a meter?

Wherever you see the word "cent", it is likely representing the number $\boxed{?}$

Hence, "percent" can be interpreted as $\boxed{?}$ by $\boxed{?}$

So, for example, $13\% = \boxed{?} = 0.\boxed{?}$

Percent Increase and Decrease

In the problems where percent increases or decreases are calculated, we calculate not only the percent change, but also the "actual value" change as well. There are specific vocabulary terms to describe these two "measures" of changes: absolute change and relative change.

- **Absolute change** is the "actual value" change that occurred. For example, in question 2, the absolute change was \$14.69 – the actual dollar amount that the price changed.
- **Relative change** is the percent change that occurred. For example, again in question 2, the relative change was 30% decrease – the percent or proportion that the dollar amount was changed.

Example 3. *A shop owner raises the price of a \$100 pair of shoes by 50%. After a few weeks, because of falling sales, the owner reduces the price of the shoes by 50%. What is the new price of the shoes (after both percent changes have occurred)?*
Explanation First, we must account for the percent increase. We can find 50% of \$100 and then add it to the original amount

$$0.5 \times \$100 = \$50$$

$$\$100 + \$50 = \$150$$

Now we can deal with the percent decrease. Remember to decrease from the new amount (\$150) and not the original (\$100).

$$0.5 \times \$150 = \$75$$

After subtracting this amount from \$150 we will have the final amount.

$$\$150 - \$75 = \$75$$

After raising the \$100 price by 50% and lowering it be 50% the final price became \$75 an overall decrease of 25%!

Exploration The annual number of burglaries in a town rose by 40% in 2012 and fell by 30% in 2013.

a. What was the total percent change in burglaries over the two years? Let's first try this problem with a specific number of burglaries to start with. In your group, find the % change if there were certain number of burglaries in 2011 (choose a 3 digit number that does NOT end with a 0). Then try to think about how to do this problem without knowing a specific number of burglaries.

b. It might be tempting to say that the change over the two years was a 10% decrease. Why might someone think it is a 10% decrease (i.e., how do you come up with 10% from the numbers in the problem?)

c. Why is 10% incorrect?

1.2 Relations and Graphs

Learning Objectives

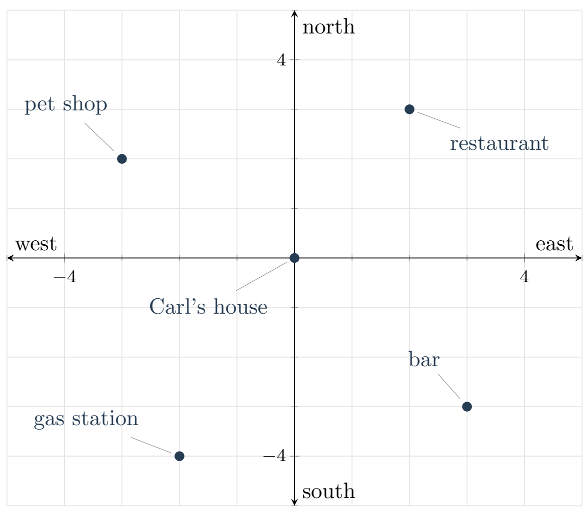
- Cartesian Coordinates
  - What is the Cartesian coordinate system?
  - How are points represented on the Cartesian coordinate system?
- Relations
  - Definition of Relation as a collection of points
  - Identify and graph relations
  - First glimpse at the definition of a function
- Famous Functions
  - Reference sheet of famous functions which will be used through the course and in Calculus

1.2.1 Relations and Graphs: Cartesian Coordinates

Graphing 2-D Cartesian Coordinates

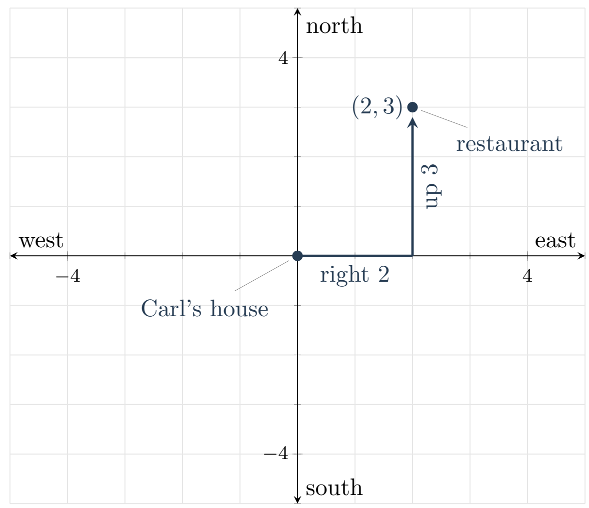
When we model a relationship between two variables visually, we use the Cartesian coordinate system. We begin with the basic vocabulary and ideas that come with the Cartesian coordinate system.

The Cartesian coordinate system identifies the location of every point in a plane. Basically, the system gives every point in a plane its own “address” in relation to a starting point. We’ll use a street grid as an analogy. Here is a map with Carl’s home at the center. The map also shows some nearby businesses. Assume each unit in the grid represents one city block.



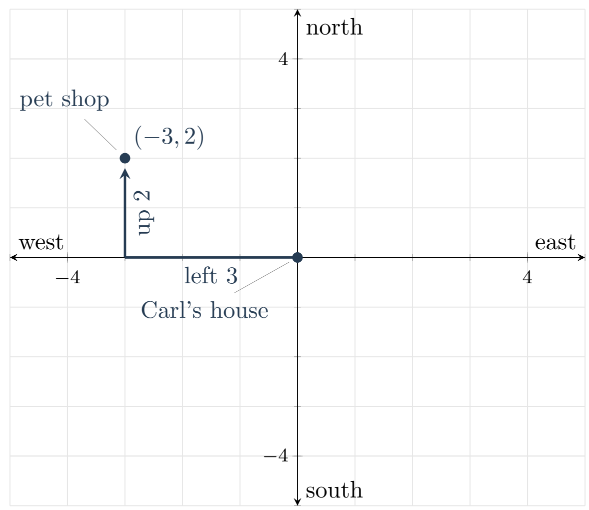
If Carl has an out-of-town guest who asks him how to get to the restaurant, Carl could say, “First go 2 blocks east (to the right on the map), then go 3 blocks north (up on the map).”

Two numbers are used to locate the restaurant. In the Cartesian coordinate system, these numbers are called coordinates and they are written as the ordered pair (2,3). The first coordinate, 2, represents distance traveled from Carl’s house to the east (or to the right horizontally on the graph). The second coordinate, 3, represents distance to the north (up vertically on the graph).



To travel from Carl's home to the pet shop, he would go 3 blocks west, and then 2 blocks north.

In the Cartesian coordinate system, the positive directions are to the right horizontally and up vertically. The negative directions are to the left horizontally and down vertically. So the pet shop's Cartesian coordinates are $(-3, 2)$.

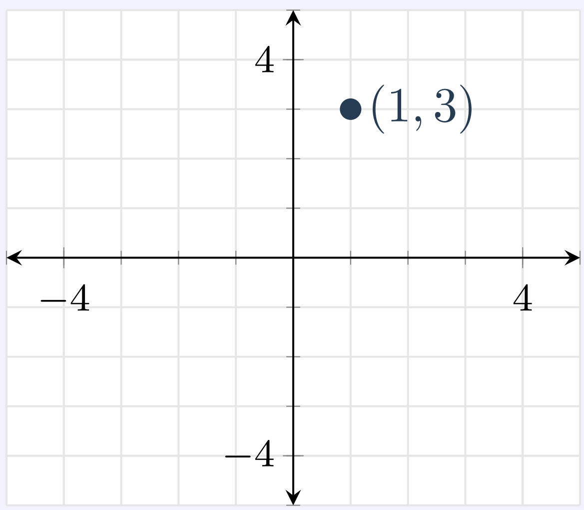


Remark It's important to know that the order of Cartesian coordinates is (horizontal, vertical). This idea of communicating horizontal information before vertical information is consistent throughout most of mathematics.

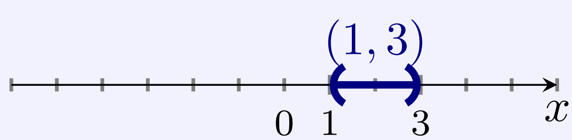
Problem 1 *Use the graph above about Carl's neighborhood to answer the following questions.*

- *What are the coordinates of the bar?* $(3, -3)$
- *What are the coordinates of the gas station?* $(-2, -4)$
- *What are the coordinates of Carl's house?* $(0, 0)$

Remark [Notation Issue] Unfortunately, the notation for an ordered pair looks exactly like interval notation for an open interval. Context will help you understand if (1,3) indicates the point 1 unit right of the origin and 3 units up,

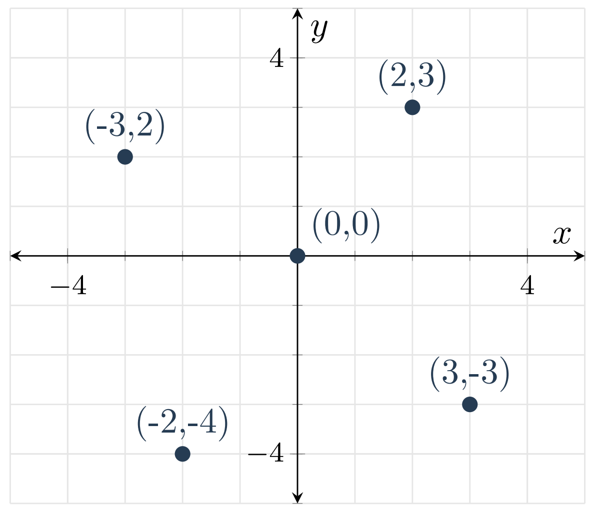


or if (1,3) indicates the interval of all real numbers between 1 and 3.



Traditionally, the variable x represents numbers on the horizontal axis, so it is called the x-axis. The variable y represents numbers on the vertical axis, so it is called the y-axis. The axes meet at the point $(0, 0)$, which is called the origin. Every point in the plane is represented by an ordered pair, (x, y).

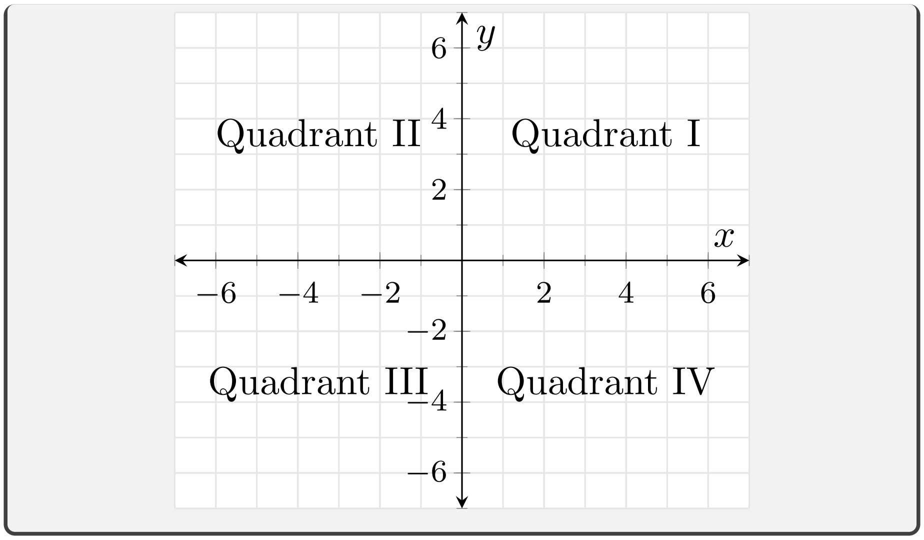
In a Cartesian coordinate system, the map of Carl's neighborhood would look like this:



Definition The **Cartesian coordinate system** is a coordinate system that specifies each point uniquely in a plane by a pair of numerical coordinates, which are the signed (positive/negative) distances to the point from two fixed perpendicular directed lines, measured in the same unit of length. Those two reference lines are called the **horizontal axis** and **vertical axis**, and the point where they meet is the **origin**. The horizontal and vertical axes are often called the x**-axis** and y**-axis**.

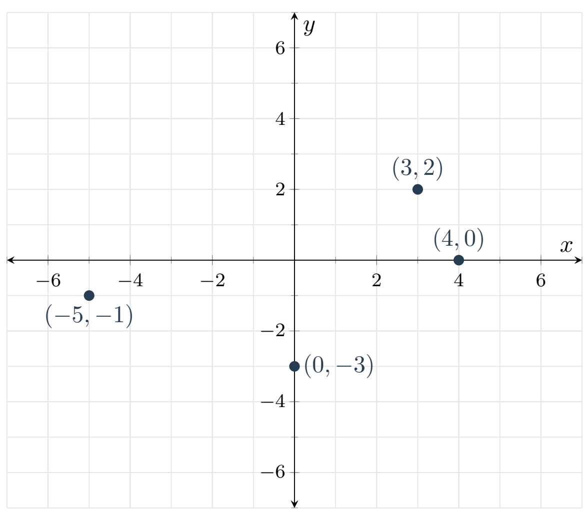
The plane based on the x-axis and y-axis is called a **coordinate plane**. The ordered pair used to locate a point is called the point's **coordinates**, which consists of an x-coordinate and a y-coordinate. For example, the point $(1, 2)$, has x-coordinate 1, and y-coordinate 2. The origin has coordinates $(0, 0)$.

A Cartesian coordinate system is divided into four **quadrants**, as shown below . The quadrants are traditionally labeled with Roman numerals.



Example 4. *On paper, sketch a Cartesian coordinate system with units, and then plot the following points:* $(3, 2)$, $(-5, -1)$, $(0, -3)$, $(4, 0)$.

Explanation



1.2.2 Relations and Graphs: Relations

In the last section, we discussed graphing points using the Cartesian coordinate system. While individual points can be useful, we often want to study collections of points.

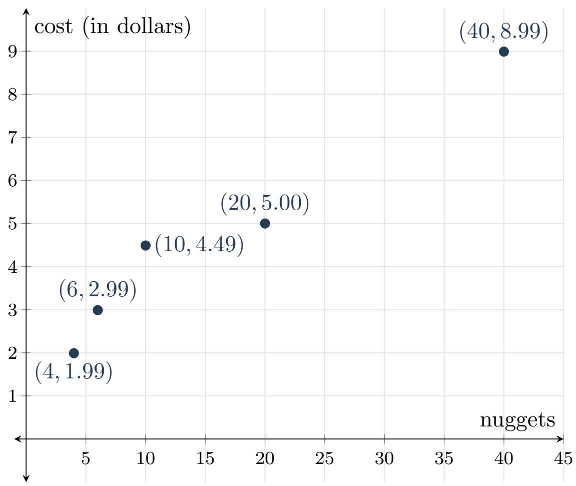
> **Definition** A **relation** is a collection of points of the form (x, y). If the point (x_0, y_0) is in the relation, then we say x_0 and y_0 are **related**.

This might seem like a strange definition, but hopefully a few examples will you see the relationship (pun intended!) between the mathematical definitions of the words "relation" and "related" and the way we often use these words in everyday speech.

Example 5. *Let's look at the relationship between the number of chicken nuggets you can buy in a single container at a local fast food store and the price for that container of nuggets.*

Amount of Nuggets	*Price*
4	1.99
6	2.99
10	4.49
20	5.00
40	8.99

Explanation Now this table defines a relation because we can think of these as the points $(4, 1.99)$, $(6, 2.99)$, $(10, 4.49)$, $(20, 5.00)$, and $(40, 8.99)$. We say that 4 nuggets is related to \$1.99. And 40 nuggets is related to \$8.99. We can also represent this relationship using a graph.



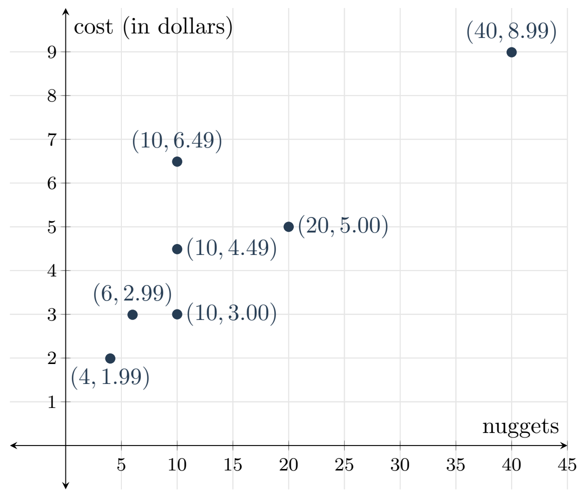
We could ask lots of mathematical questions about this relation. One such example might be, "What's the cheapest way to buy 100 nuggets?" But for now, its enough to know it is a relation and to know you can represent that relation in multiple ways such as a table, a list, or a graph.

Example 6. *It's important to note that nothing about our definition of relation restricts what points can be included. Assume that in our chicken nugget example above, there is a coupon that allows you to buy 10 chicken nuggets for \$3.00. Assume there is also an option to buy a chicken nugget meal which includes 10 chicken nuggets (and fries and a drink but we don't care about those) for \$6.49.*

Explanation Then we could modify the table of our relation to be:

Amount of Nuggets	Price
4	1.99
6	2.99
10	4.49
20	5.00
40	8.99
10	3
10	6.49

Now this table still defines a relation and we can say that 10 nuggets is related to \$4.49 and \$3.00 and \$6.49. Here is the graph of this relation.

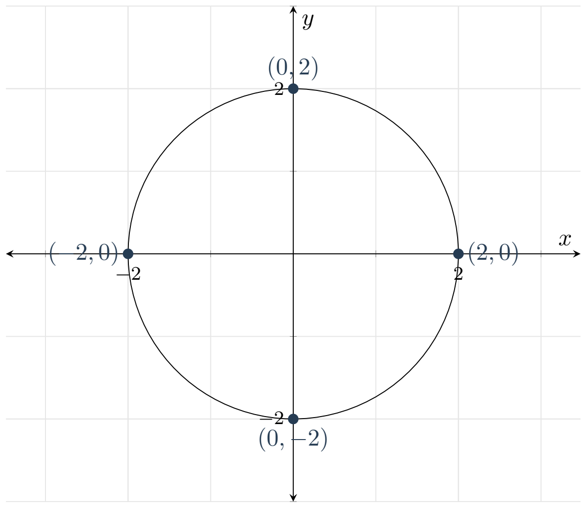


Remark There is a special type of relation called a **function** where each x-coordinate is only allowed to be related to one unique y-coordinate. In the examples above, the first relation is a function but the second relation which includes the coupon and a meal is not a function because 10 nuggets is related to more than one cost. Functions are going to be extremely important and we will come back to them later in the course.

The two examples we have seen so far have been a relations given by a list of points. This does not have to be the case. Relations can contain an infinite number of points. Some of the relations we will be studying the most are given by an equation relating two variables.

Example 7. *Let's consider the relation that is the collection of all points* (x, y) *where* $x^2 + y^2 = 4$.

Explanation Some points contained in this relation are $(2, 0)$, $(0, 2)$, $(-2, 0)$, and $(0, -2)$, but these are not all the points in this relation. Often, one of best ways to think about a relation given by an equation is with a graph. The graph of $x^2 + y^2 = 4$ is the circle of radius 2 centered at the origin.



Example 8. *Verify algebraically that the point* $(2, 0)$ *is on the curve* $x^2+y^2 = 4$.

Explanation Note that this is the same question as, "Verify that the point $(2, 0)$ is a member of the relation given by $x^2 + y^2 = 4$." We want to show that this point satisfies the condition. The point $(2, 0)$ means that the $x = 2$ and $y = 0$. If we plug these values in for x and y in the equation, we want to check that both sides of the equation are equal.

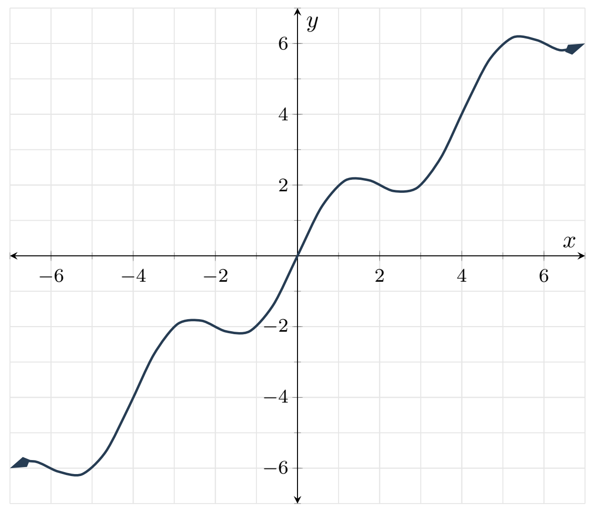
$$x^2 + y^2 = 4 \tag{1}$$

$$(2)^2 + (0)^2 = 4 \tag{2}$$

$$4 + 0 = 4 \tag{3}$$

$$4 = 4 \tag{4}$$

Example 9. *A relation can also be intitally given by a graph. For example, this is a relation.*



Explanation We can list some of the points on this graph. It looks like $(0, 0)$ and $(1, 2)$ are points on this curve, but there are many other points we cannot explicitly list.

1.2.3 Relations and Graphs: Famous Functions

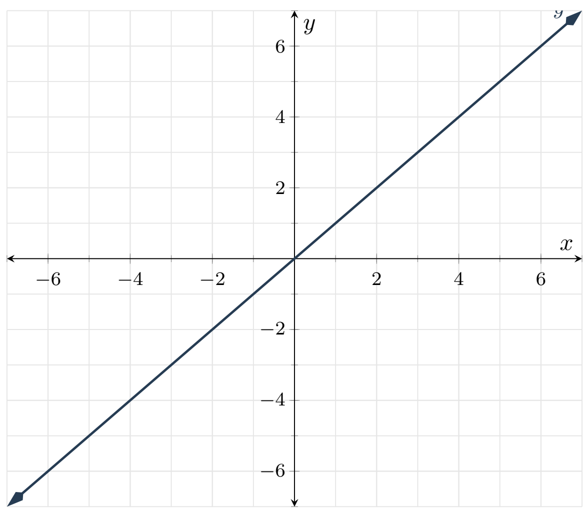
Throughout this course and in Calculus you will study certain relations in depth. These relations are usually functions, and they are the functions that come up often in real world applications. In this section, we will give you a list of some of these functions, including their algebraic equation, their graph, and a table of some of their most important values. You may never have seen some of these functions before and you might not know what their algebraic expressions mean. That is ok. We will learn more about them throughout this course. Remember, though, that a relation or function can be given by a graph. For now, familiarize yourself with these graphs. They will come up in examples throughout the course.

Linear Functions

Of the most important types of functions is a linear function. The graph of a linear function is a line.

Example 10. *A prototypical example of a linear function is*

$$y = x.$$



Important Values of $y = x$	
x	y
-2	-2
-1	-1
0	0
1	1
2	2

In general, linear functions can be written as $y = mx + b$ where m and b can be any numbers. You can play with changing the values of m and b on the graph using Desmos and see how that changes the line.

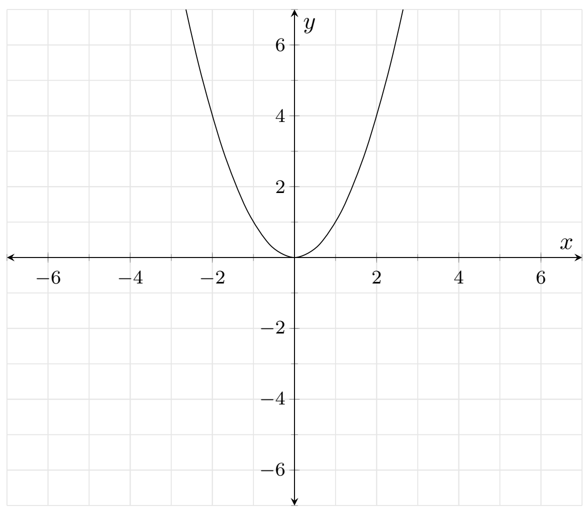
Desmos link: `https://www.desmos.com/calculator/japnhapzvn`

Quadratic Functions

Another very important type of function is the quadratic function. The graph of a quadratic function is a parabola.

Example 11. *A prototypical example of a quadratic function is*

$$y = x^2.$$



Important Values of $y = x^2$	
x	y
-2	4
-1	1
0	0
1	1
2	4

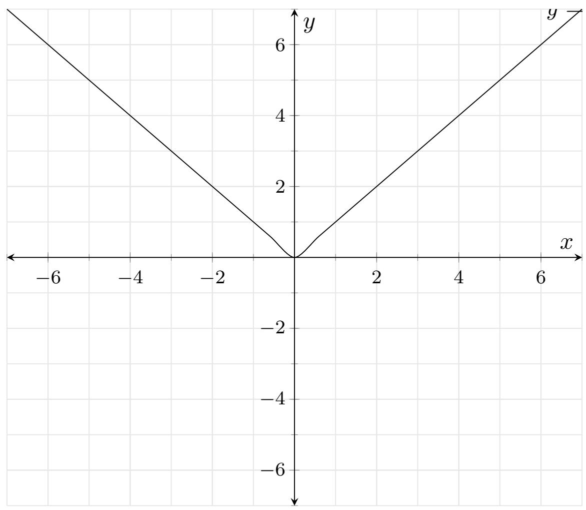
In general, quadratic functinos can be written as $y = ax^2 + bx + c$ where a, b, and c can be any numbers. You can play with changing the values of a, b, and c on the graph using Desmos and see how that changes the parabola.

Desmos link: `https://www.desmos.com/calculator/nmlghfrws9`

Absolute Value

Another important type of function is the absolute value function. This is the function that takes all y-values and makes them positive. The absolute value function is written as

$$y = |x|.$$

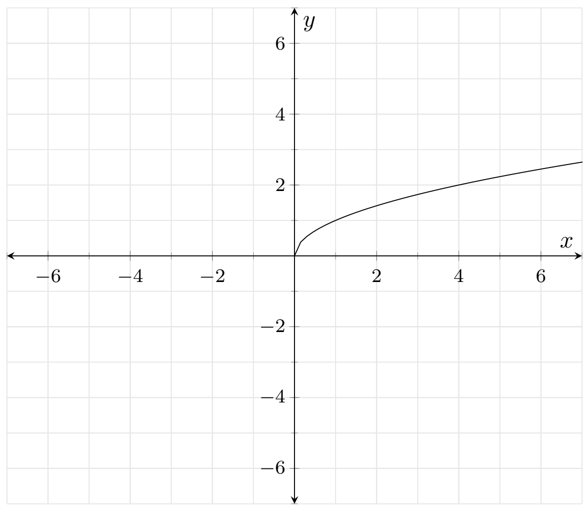


Important Values of $y = \|x\|$	
x	y
-2	2
-1	1
0	0
1	1
2	2

Square Root

Another famous function is the square root function,

$$y = \sqrt{x}.$$



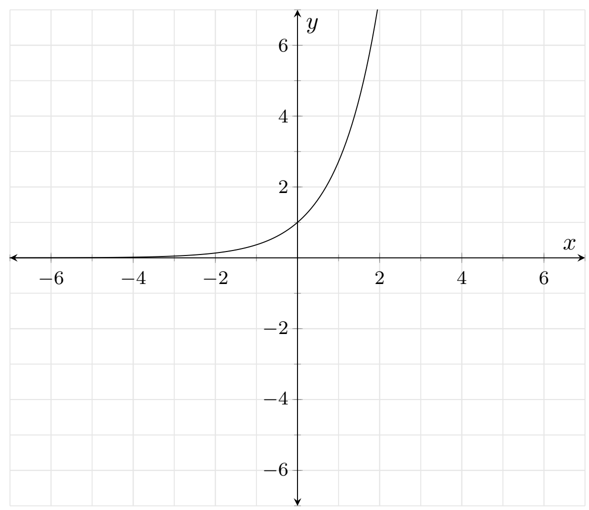
Important Values of $y = \sqrt{x}$	
x	y
0	0
1	1
4	2
9	3
25	5

Exponential

Another famous function is the exponential growth function,

$$y = e^x.$$

Here e is the mathematical constant known as Euler's number. $e \approx 2.71828..$



Important Values of $y = e^x$	
x	y
0	1
1	e
-1	$\frac{1}{e}$

In general, we can talk about exponential functions of the form $y = b^x$ where b is a positive number not equal to 1. You can play with changing the values of b on the graph using Desmos and see how that changes the graph. Pay particular attention to the difference between $b > 1$ and $0 < b < 1$.

Desmos link: `https://www.desmos.com/calculator/qsmvb7tiex`

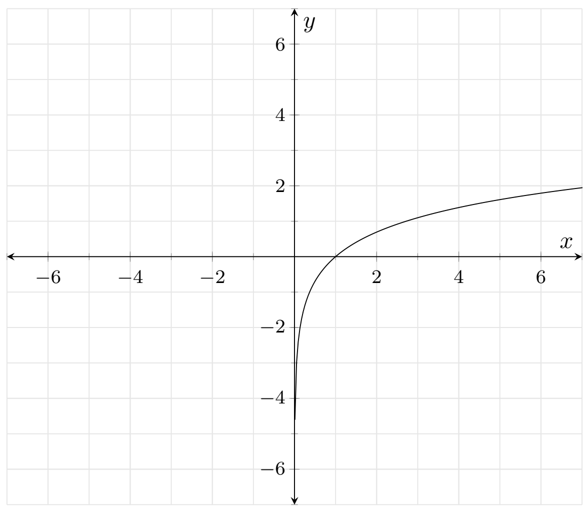
Logarithm

Another group of famous functions are logarithms.

Example 12. *The most famous logarithm function is*

$$y = \ln(x) = \log_e(x).$$

Here e is the mathematical constant known as Euler's number. $e \approx 2.71828$.



Important Values of $y = ln(x)$	
x	y
0	undefined
$\frac{1}{e}$	-1
1	0
e	1

You may notice that the table of values for $y = \ln(x)$ and $y = e^x$ are similiar. This is becase these two functions are interconnected. We will explore this more later in the course.

In general, we can talk about logarithmic functions of the form $y = \log_b(x)$ where b is a positive number not equal to 1. You can play with changing the values of b on the graph using Desmos and see how that changes the graph. Pay particular attention to the difference between $b > 1$ and $0 < b < 1$.

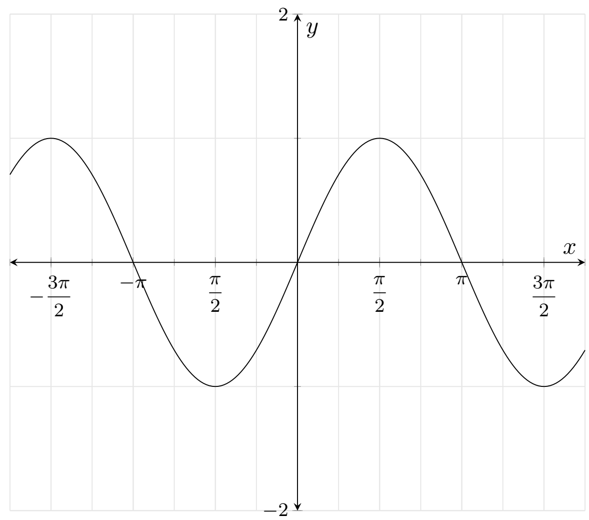
Desmos link: https://www.desmos.com/calculator/lxllnpdi6w

Sine

Another important function is the sine function,

$$y = \sin(x).$$

This function comes from trigonometry. In the table below we will use another mathematical constant, π ("pi" pronounced pie). $\pi \approx 3.14159$.



Important Values of $y = \sin(x)$	
x	y
$-\pi$	0
$-\frac{\pi}{2}$	-1
0	0
$\frac{\pi}{2}$	1
π	0
$\frac{3\pi}{2}$	-1
2π	0

In general, we can consider $y = a\sin(bx)$. You can play with changing the values of a and b on the graph using Desmos and see how that changes the graph.

Desmos link: `https://www.desmos.com/calculator/vkxzcfv2aq`

1.3 Changing in Tandem

Learning Objectives

- Changing in Tandem
  - Introduction to thinking about how changing one variable affects/changes another variable
  - Water filling various shaped vases
  - Working with tables and graphs of the same data

1.3.1 Changing in Tandem

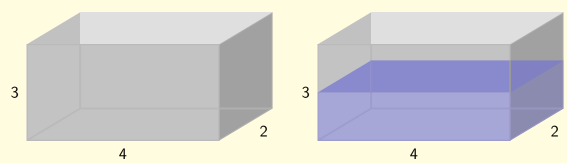
Motivating Questions

- If we have two quantities that are changing in tandem, how can we connect the quantities and understand how change in one affects the other?
- When the amount of water in a tank is changing, what behaviors can we observe?

Introduction

Mathematics is the art of making sense of patterns. One way that patterns arise is when two quantities are changing in tandem. In this setting, we may make sense of the situation by expressing the relationship between the changing quantities through words, through images, through data, or through a formula.

Exploration Suppose that a rectangular aquarium is being filled with water. The tank is 4 feet long by 2 feet wide by 3 feet high, and the hose that is filling the tank is delivering water at a rate of 0.5 cubic feet per minute.



a. What are some different quantities that are changing in this scenario?

b. After 1 minute has elapsed, how much water is in the tank? At this moment, how deep is the water?

c. How much water is in the tank and how deep is the water after 2 minutes? After 3 minutes?

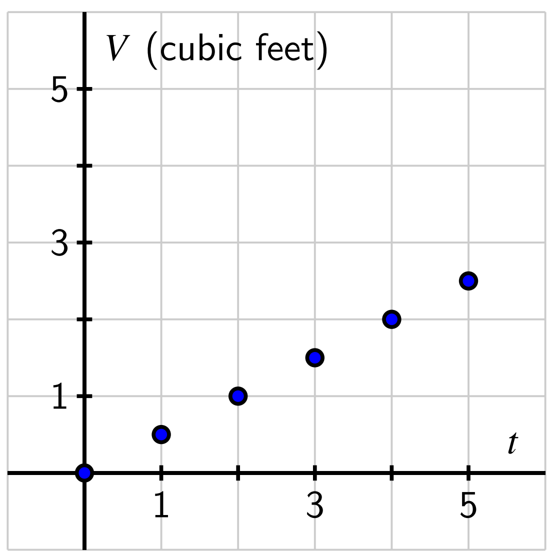
d. How long will it take for the tank to be completely full? Why?

Using Graphs to Represent Relationships

In the previous activity, we saw how several changing quantities were related in the setting of an aquarium filling with water: time, the depth of the water, and the total amount of water in the tank are all changing, and any pair of these quantities changes in related ways. One way that we can make sense of the situation is to record some data in a table. For instance, observing that the tank is filling at a rate of 0.5 cubic feet per minute, this tells us that after 1 minute there will be 0.5 cubic feet of water in the tank, and after 2 minutes there will be 1 cubic foot of water in the tank, and so on. If we let t denote the time in minutes and V the amount of water in the tank at time t, we can represent the relationship between these quantities through a table.

t	V
0	0.0
1	0.5
2	1.0
3	1.5
4	2.0
5	2.5

We can also represent this data in a graph by plotting ordered pairs (t, V) on a system of coordinate axes, where t represents the horizontal distance of the point from the origin, $(0, 0)$, and V represents the vertical distance from $(0, 0)$. The visual representation of the table of values is seen in the graph below.



Sometimes it is possible to use variables and one or more equations to connect quantities that are changing in tandem. In the aquarium example from the preview activity, we can observe that the volume, V, of a rectangular box that has length l, width w, and height h is given by

$$V = l \cdot w \cdot h,$$

and thus, since the water in the tank will always have length $l = 4$ feet and width $w = 2$ feet, the volume of water in the tank is directly related to the depth of water in the tank by the equation

$$V = 4 \cdot 2 \cdot h = 8h.$$

Depending on which variable we solve for, we can either see how V depends on h through the equation $V = 8h$, or how h depends on V via the equation $h = \frac{1}{8}V$. From either perspective, we observe that as depth or volume increases, so must volume or depth correspondingly increase.

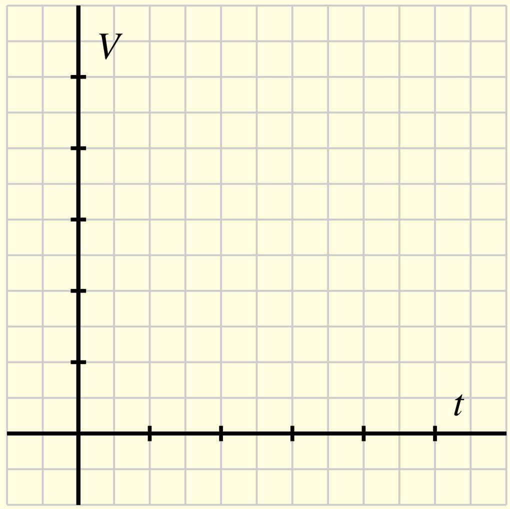
Exploration Consider a tank in the shape of an inverted circular cone (point down) where the tank's radius is 2 feet and its depth is 4 feet. Suppose that the tank is being filled with water that is entering at a constant rate of 0.75 cubic feet per minute.

a. Sketch a labeled picture of the tank, including a snapshot of there being water in the tank prior to the tank being completely full.

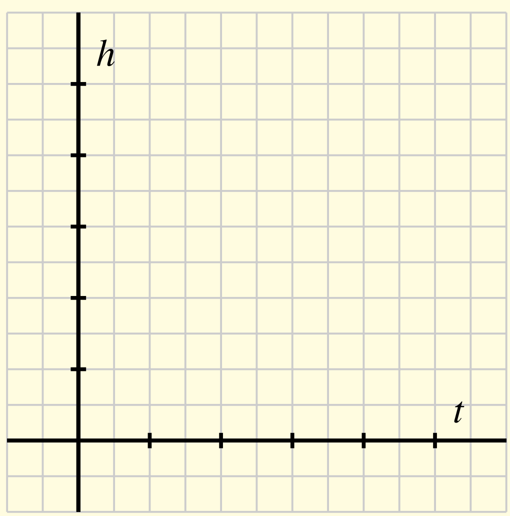
b. What are some quantities that are changing in this scenario? What are some quantities that are not changing?

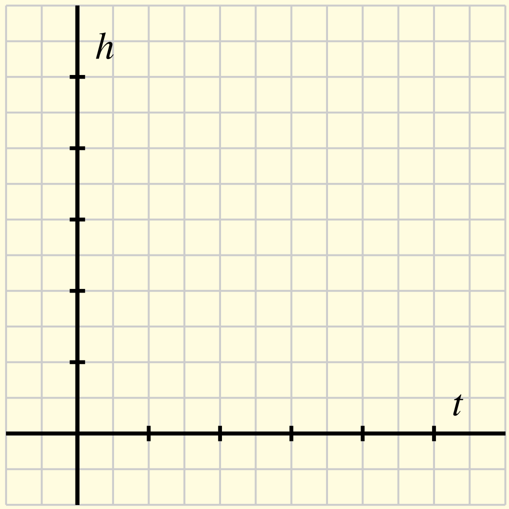
c. Fill in the following table of values to determine how much water, V, is in the tank at a given time in minutes, t, and thus generate a graph of the relationship between volume and time by plotting the data on the provided axes.

t	V
0	
1	
2	
3	
4	
5	



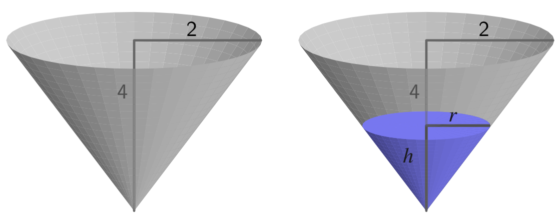
d. Finally, think about how the height of the water changes in tandem with time. Without attempting to determine specific values of h at particular values of t, how would you expect the data for the relationship between h and t to appear? Use the provided axes to sketch at least two possibilities; write at least one sentence to explain how you think the graph should appear.





Using A Table to Add Perspective

One of the ways that we make sense of mathematical ideas is to view them from multiple perspectives. Sometimes we use different means to establish a point of view: words, numerical data, graphs, or symbols. In addition, sometimes by changing our perspective within a particular approach we gain deeper insight.

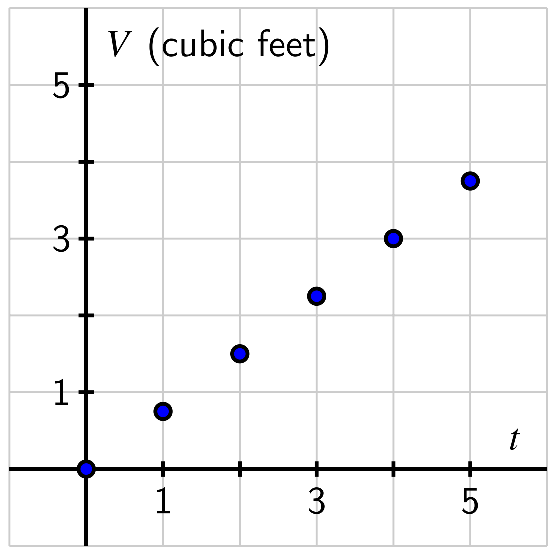


Here is a table of data obtained algebraically of the height of the water in the cone at different times. For more information about how this table was derived, see https://activecalculus.org/prelude/sec-changing-in-tandem.html

Table How time, volume, and height change in a conical tank

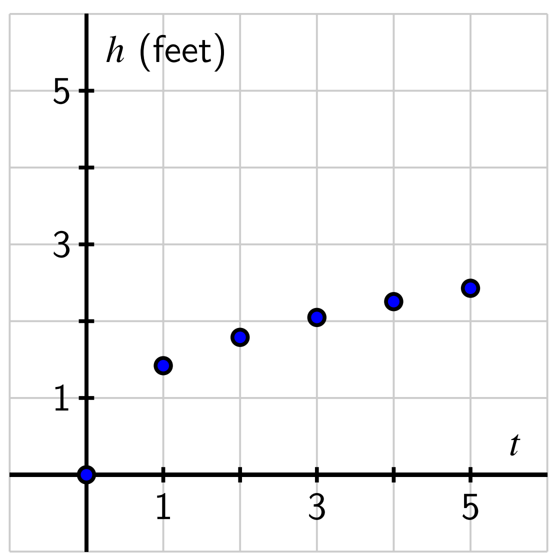
t	0	1	2	3	4	5
V	0.0	0.75	1.5	2.25	3.0	3.75
h	0.0	1.42	1.79	2.05	2.25	2.43

Plotting this data on two different sets of axes allows us to see the different ways that V and h change with t. First we graph how volume changes over time.



Volume increases at a constant rate, as seen by the straight line appearance of the points in the graph above.

Now let's graph how height changes over time.



We observe that the water's height increases in a way that it rises more slowly as time goes on, as shown by the way the curve the points lie on in the graph below "bends down" as time passes.

These different behaviors make sense because of the shape of the tank. Since at first there is less volume relative to depth near the cone's point, as water flows in at a constant rate, the water's height will rise quickly. But as time goes on and more water is added at the same rate, there is more space for the water to fill in order to make the water level rise, and thus the water's heigh rises more and more slowly as time passes.

Exploration

Consider a tank in the shape of a sphere where the tank's radius is 3 feet. Suppose that the tank is initially completely full and that it is being drained by a pump at a constant rate of 1.2 cubic feet per minute.

a. Sketch a labeled picture of the tank, including a snapshot of some water remaining in the tank prior to the tank being completely empty.

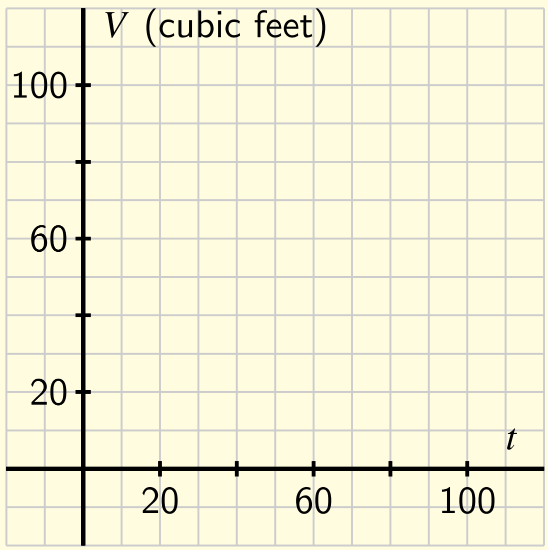
b. What are some quantities that are changing in this scenario? What are some quantities that are not changing?

c. Recall that the volume of a sphere of radius r is $V = \dfrac{4}{3}\pi r^3$. When the tank is completely full at time $t = 0$ right before it starts being drained, how much water is present?

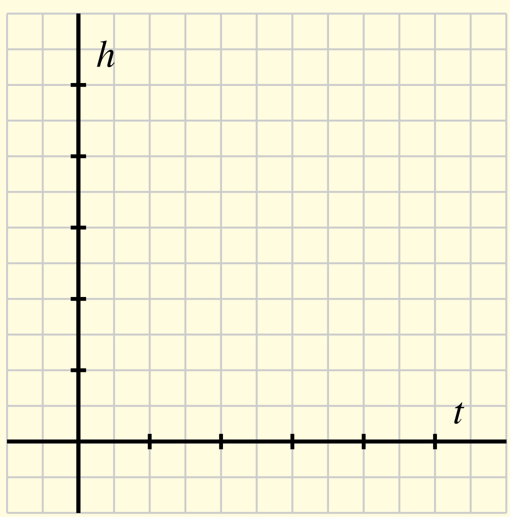
d. How long will it take for the tank to drain completely?

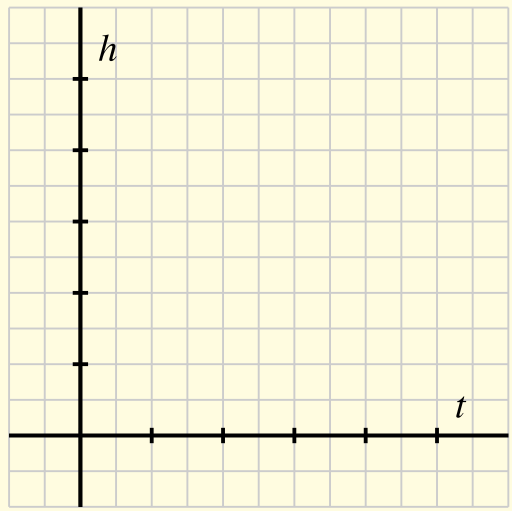
e. Fill in the following table of values to determine how much water, V, is in the tank at a given time in minutes, t, and thus generate a graph of the relationship between volume and time. Write a sentence to explain why the data's graph appears the way that it does.

t	V
0	
20	
40	
60	
80	
94.24	



f. Finally, think about how the height of the water changes in tandem with time. What is the height of the water when $t = 0$? What is the height when the tank is empty? How would you expect the data for the relationship between h and t to appear? Use the provided axes to sketch at least two possibilities; write at least one sentence to explain how you think the graph should appear.





Summary

- When two related quantities are changing in tandem, we can better understand how change in one affects the other by using data, graphs, words, or algebraic symbols to express the relationship between them. See, for instance, Table 1.1.10, Figure1.1.11, and 1.1.12 that together help explain how the height and volume of water in a conical tank change as time changes.

- When the amount of water in a tank is changing, we can observe other quantities that change, depending on the shape of the tank. For instance, if the tank is conical, we can consider both the changing height of the water and the changing radius of the surface of the water. In addition, whenever we think about a quantity that is changing as time passes, we note that time itself is changing.

Part 2

Comparing Lines and Exponentials

2.1 Linear Equations

Learning Objectives

- Finding Patterns
  - Determining a linear equation from a table of points
- Slope
  - Defining and computing Slopes
- Equations of Lines
  - Slope-intercept form
  - Point-slope form
  - Standard form
  - Why we care about each form and how to move between forms

2.1.1 Linear Equations: Finding Patterns

Review Materials

- Combining Like Terms[1]
- Algebraic Properties and Simplifying Expressions[2]

Patterns in Tables

Example 13. *What is the missing entry in each table? Can you describe each pattern in words and/or mathematics?*

1	2
2	4
3	6
5	

Explanation We can view the table as assigning each input in the left column a corresponding output in the right column. It takes a number as input, and give twice that number as its output. Mathematically, we can describe the pattern as $y = 2x$, where x represents the input, and y represents the output. Labeling the table mathematically, we have

x (input)	y (output)
1	2
2	4
3	6
5	10
10	20

Pattern:$y = 2x$

For each of the following tables, find an equation that describes the pattern you see. Numerical pattern recognition may or may not come naturally for you. Either way, pattern recognition is an important mathematical skill that anyone

[1]See Combining Like Terms at https://spot.pcc.edu/math/orcca/ed2/html/section-combining-like-terms.html

[2]See Algebraic Properties and Simplifying Expressions at https://spot.pcc.edu/math/orcca/ed2/html/section-algebraic-properties-and-simplifying-expressions.html

can develop. Solutions for these exercises provide some ideas for recognizing patterns.

Problem 2 *Write an equation in the form $y = ...$ suggested by the pattern in the table.*

x	y
0	10
1	11
2	12
3	13

$y = 10 + x$

Explanation One approach to pattern recognition is to look for a relationship in each row. Here, the y-value in each row is always 10 more than the x-value. So the pattern is described by the equation $y = 10 + x$

Problem 3 *Write an equation in the form $y = ...$ suggested by the pattern in the table.*

x	y
0	-1
1	2
2	5
3	8

$y = 3x - 1$

Explanation The relationship between x and y in each row is not as clear here. Another popular approach for finding patterns: in each column, consider how the values change from one row to the next. From row to row, the x-value increases by 1. Also, the y-value increases by 3 from row to row.

	x	y	
	0	-1	
$+1 \rightarrow$	1	2	$\leftarrow +3$
$+1 \rightarrow$	2	5	$\leftarrow +3$
$+1 \rightarrow$	3	8	$\leftarrow +3$

Since row-to-row change is always 1 for x and is always 3 for y the rate of change from one row to another row is always the same: 3 units of y for every 1 unit of x. This suggests that $y = 3x$ might be a good equation for the table pattern. But if we try to make a table with that pattern:

x	$3x$	y
0	0	-1
1	3	2
2	6	5
3	9	8

We find that the values from $y = 3x$ are 1 too large. So now we make an adjustment. The equation $y = 3x - 1$ describes the pattern in the table.

Rates of Change

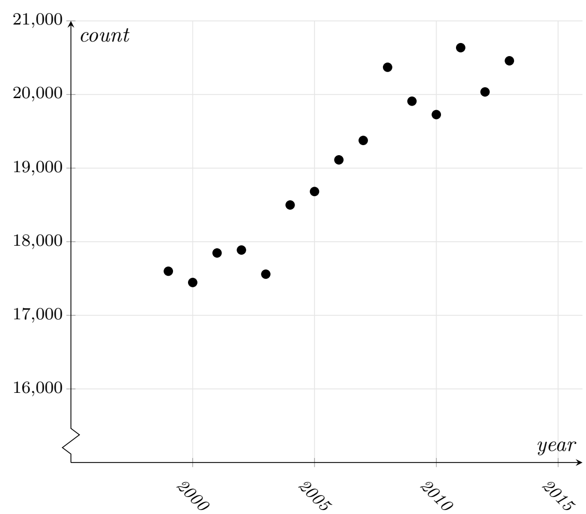
For an hourly wage-earner, the amount of money they earn depends on how many hours they work. If a worker earns \$15per hour, then 10 hours of work corresponds to \$150of pay. Working one additional hour will change 10 hours to 11 hours; and this will cause the \$150 in pay to rise by fifteen dollars to \$165 in pay. Any time we compare how one amount changes (dollars earned) as a consequence of another amount changing (hours worked), we are talking about a **rate of change**.

Given a table of two-variable data, between any two rows we can compute a **rate of change**.

Example 14. *The following data*[3] *, given in both table and graphed form, gives the counts of invasive cancer diagnoses in Oregon over a period of time.*

[3] See data at wonder.cdc.gov

Year	*Invasive Cancer Incidents*
1999	17599
2000	17446
2001	17847
2003	17559
2004	18499
2005	19112
2006	19112
2007	19376
2008	20370
2009	19909
2010	19727
2011	20636
2012	20035
2013	20458



*What is the **rate of change** in Oregon invasive cancer diagnoses between 2000 and 2010?*

Explanation The total (net) change in diagnoses over that timespan is

$$19727 - 17446 = 2281$$

meaning that there were 2281 more invasive cancer incidents in 2010 than in 2000. Since 10 years passed (which you can calculate as 2010-2000), the rate of change is 2281 diagnoses per 10 years, or

$$\frac{2281 \text{ diagnoses}}{10 \text{ years}} = 228.1 \frac{\text{diagnoses}}{\text{year}}$$

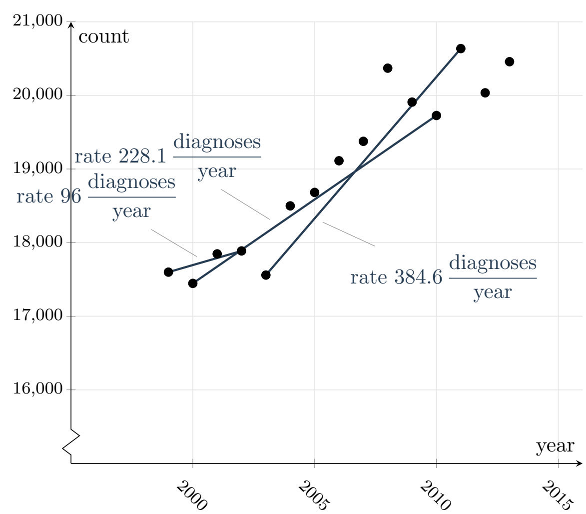
We read that last quantity as "228.1 diagnoses per year." This rate of change means that between the years 2000 and 2010, there were 228.1 more diagnoses each year, on average. This is just an average over those ten years—it does not mean that the diagnoses grew by exactly this much each year. We dare not interpret why that increase existed, just that it did. If you are interested in examining causal relationships that exist in real life, we strongly recommend a statistics course or two in your future!

Definition If (x_1, y_1) and (x_2, y_2) are two data points with $x_1 \neq x_2$ from a set of two-variable data, then the **rate of change** between them is

$$\frac{\text{change in } y}{\text{change in } x} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

(The Greek letter delta, Δ , is used to represent "change in" since it is the first letter of the Greek word for "difference.")

Here are some examples of rates of change from our example above.



Note how the larger the numerical rate of change between two points, the steeper the line is that connects them in a graph. This is such an important observation, we'll put it in an official remark.

Remark The rate of change between two data points is intimately related to the steepness of the line segment that connects those points.

(a) The steeper the line, the larger the rate of change, and vice versa.

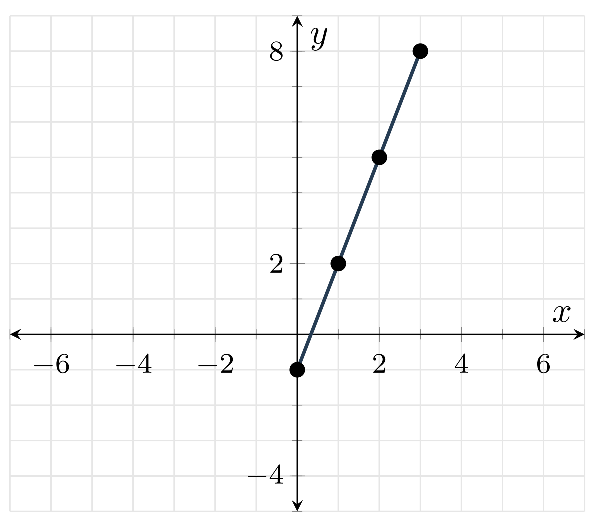
(b) If one rate of change between two data points equals another rate of change between two different data points, then the corresponding line segments will have the same steepness.

(c) We always measure rate of change from left to right. When a line segment between two data points slants up from left to right, the rate of change between those points will be positive. When a line segment between two data points slants down from left to right, the rate of change between those points will be negative.

Let's revisit the earlier example $y = 3x - 1$

x	y
0	-1
1	2
2	5
3	8

The key observation in this example was that the rate of change from one row to the next was constant: 3 units of increase in y for every 1 unit of increase in x. Graphing this pattern in , we see that every line segment here has the same steepness, so the whole picture is a straight line.



Whenever the rate of change is constant no matter which two (x, y)-pairs (or data pairs) are chosen from a data set, then you can conclude the graph will be a straight line even without making the graph. We call this kind of relationship a **linear relationship**. We'll study linear relationships in more detail throughout this section.

2.1.2 Linear Equations: Slope

We observed that a constant rate of change between points produces a linear relationship, whose graph is a straight line. Such a constant rate of change has a special name, **slope**, and we'll explore slope in more depth here.

Definition When x and y are two variables where the rate of change between any two points is always the same, we call this common rate of change the **slope**. Since having a constant rate of change means the graph will be a straight line, its also called the **slope of the line**.

Considering the definition for **rate of change**, this means that when x and y are two variables where the rate of change between any two points is always the same, then you can calculate slope, m, by finding two distinct data points (x_1, y_1) and (x_2, y_2) , and calculating

$$m = \frac{\text{change in } y}{\text{change in } x} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

A slope is a rate of change. So if there are units for the horizontal and vertical variables, then there will be units for the slope. The slope will be measured in

$$\frac{\text{vertical units}}{\text{horizontal units}}$$

.

Definition If the slope is constant and nonzero, we say that there is a **linear relationship** between x and y. When the slope is 0, we say that y is **constant** with respect to x.

Here are some linear scenarios with different slopes. As you read each scenario, note how a slope is more meaningful with units.

- If a tree grows 2.5 feet every year, its rate of change in height is the same from year to year. So the height and time have a linear relationship where the slope is 2.5 ft/yr.
- If a company loses 2 million dollars every year, its rate of change in reserve funds is the same from year to year. So the company's reserve funds and time have a linear relationship where the slope is -2 million dollars per year.
- If Sakura is an adult who has stopped growing, her rate of change in height is the same from year to year—it's zero. So the slope is 0 in/yr. Sakura's height is constant with respect to time.

Remark A useful phrase for remembering the definition of slope is "rise over run." Here, "rise" refers to "change in y", Δy, and "run" refers to "change in x", Δx. Be careful though. As we have learned, the horizontal direction comes first in mathematics, followed by the vertical direction. The phrase "rise over run" reverses this. (It's a bit awkward to say, but the phrase "run under rise" puts the horizontal change first.)

Example 15. *On Dec. 31, Yara had only $50 in her savings account. For the the new year, she resolved to deposit $20 into her savings account each week, without withdrawing any money from the account.*

Yara keeps her resolution, and her account balance increases steadily by $20 each week. That's a constant rate of change, so her account balance and time have a linear relationship with slope of 20 $\dfrac{\textit{dollars}}{\textit{week}}$.

Explanation

We can model the balance, y, in dollars, in Yara's savings account x weeks after she started making deposits with an equation. Since Yara started with $50 and adds $20 each week, then x weeks after she started making deposits,

$$y = 50 + 20x$$

where y is a dollar amount. Notice that the slope,

$$20 \frac{\text{dollars}}{\text{week}}$$

, serves as the multiplier for x weeks.

We can also consider Yara's savings using a table

	x (weeks since Dec 31)	y (savings account balance in dollars)	
	0	50	
$+1 \rightarrow$	1	70	$\leftarrow +20$
$+1 \rightarrow$	2	90	$\leftarrow +20$
$+2 \rightarrow$	4	130	$\leftarrow +40$
$+3 \rightarrow$	7	190	$\leftarrow +60$
$+5 \rightarrow$	12	290	$\leftarrow +100$

In first few rows of the table, we see that when the number of weeks x increases by 1, the balance y increases by 20. The row-to-row rate of change is

$$\frac{20 \text{ dollars}}{1 \text{ week}} = 20 \frac{\text{dollars}}{\text{week}},$$

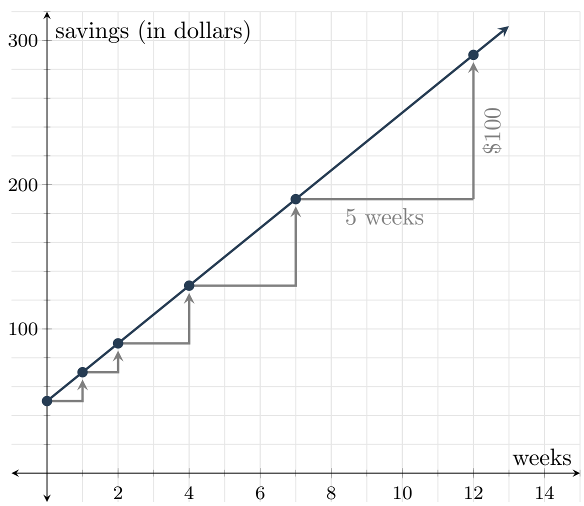
the slope. In any table for a linear relationship, whenever x increases by 1 unit, y will increase by the slope.

In further rows, notice that as row-to-row change in x increases, row-to-row change in y increases proportionally to preserve the constant rate of change. Looking at the change in the last two rows of the table, we see x increases by 5 and y increases by 100, which gives a rate of change of

$$\frac{100 \text{ dollars}}{5 \text{ week}} = 20\frac{\text{dollars}}{\text{week}},$$

the value of the slope again.

We can see this constant rate of change on the graph by drawing in **slope triangles** between points on the graph, showing the change in x as a horizontal distance and the change in y as a vertical distance.

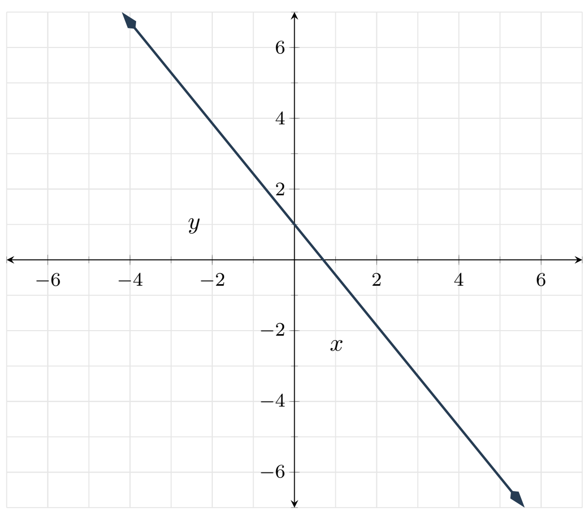


The Relationship Between Slope and Increase/Decrease

In a linear relationship, as the x-value increases (in other words as you read its graph from left to right):

- if the y-values increase (in other words, the line goes upward), its slope is positive.

- if the y-values decrease (in other words, the line goes downward), its slope is negative.



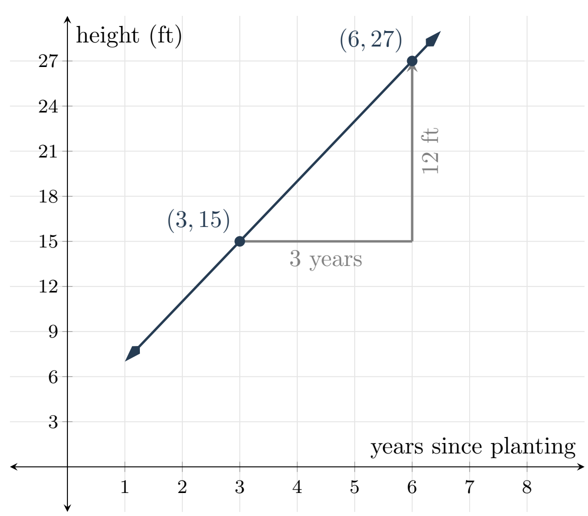
- if the y-values don't change (in other words, the line is flat, or horizontal), its slope is 0.

Finding the Slope by Two Given Points

Whenever you know two points on a line, you can find the slope of the line directly from the definition of slope.

Example 16. *Your neighbor planted a sapling from Portland Nursery in his front yard. Ever since, for several years now, it has been growing at a constant rate. By the end of the third year, the tree was 15 ft tall; by the end of the sixth year, the tree was 27 ft tall. What's the tree's rate of growth (i.e. the slope)?*

Explanation We could sketch a graph for this scenario, and include a slope triangle. If we did that, it would look like:



We don't actually need the picture, though, to find the slope. From the definition of slope, we have that

$$m = \frac{\text{change in } y}{\text{change in } x} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

We know that after 3 yr, the height is 15 ft. As an ordered pair, that information gives us the point (3,15) which we can label as (x_1, y_1) Similarly, the background information tells us to consider (6,27), which we label as (x_2, y_2). Here, x_1 and y_1 represent the first point's x-value and y-value, and x_2 and y_2 represent the second point's x-value and y-value.

Substiuting in our values for $x_1 = 3$, $y_1 = 15$, $x_2 = 6$, and $y_2 = 27$ into our definition of slope, we have

$$m = \frac{\text{change in } y}{\text{change in } x} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\Delta y}{\Delta x} = \frac{27 - 15}{6 - 3} = \frac{12\text{ft}}{3\text{yr}} = 4\frac{\text{ft}}{\text{yr}}$$

2.1.3 Linear Equations: Equations of Lines

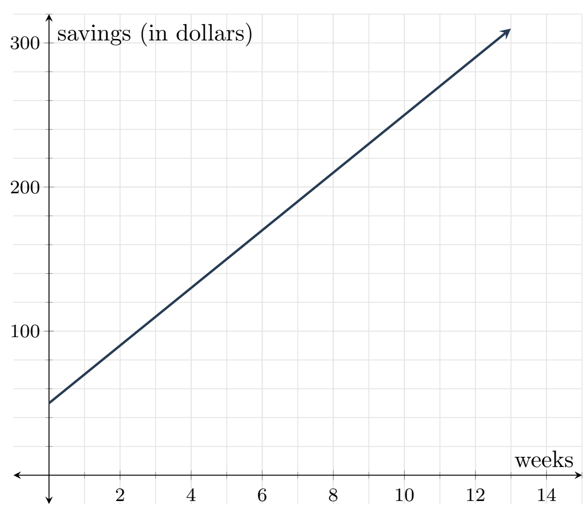
We will explore how to write an equation for a line. The best way to write the equation of a line depends both on what information we have about the line and what we want to do with our equation.

Slope-Intercept Form of a Line

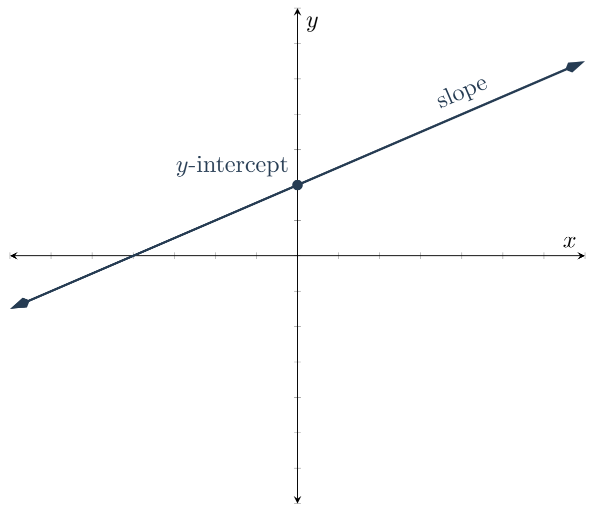
Recall the previous example where Yara had \$50 in her savings account when the year began, and decided to deposit \$20 each week without withdrawing any money. In that example, we model using x to represent how many weeks have passed. After x weeks, Yara has added $20x$ dollars. And since she started with \$50, she has

$$y = 20x + 50$$

in her account after x weeks. In this example, there is a constant rate of change of 20 dollars per week, so we call that the slope. We also saw that plotting Yara's balance over time gives us a straight-line graph.



The graph of Yara's savings has some things in common with almost every straight-line graph. There is a slope, and there is a place where the line crosses the y -axis.



We already have an accepted symbol, m, for the slope of a line. The y-intercept is a point on the y-axis where the line crosses. Since it's on the y-axis, the x-coordinate of this point is 0. It is standard to call the point $(0, b)$ the y-intercept, and call the number b the y-coordinate of the y-intercept.

One way to write the equation for Yara's savings was

$$y = 20x + 50$$

where both $m = 20$ and $b = 50$ are immediately visible in the equation. Now we are ready to generalize this.

Definition When x and y have a linear relationship where m is the slope and $(0, b)$ is the y-intercept, one equation for this relationship is

$$y = mx + b$$

and this equation is called the **slope-intercept form** of the line. It is called this because the slope and y-intercept are immediately discernible from the numbers in the equation.

Problem 4 *What is the slope and y-intercept for the following linear equation?*

$$y = 17x - 14$$

Slope $= 17$ *y-intercept*$=-14$

Remark The number b is the y-value when $x = 0$. Therefore it is common to refer to b as the **initial value** or **starting value** of a linear relationship.

More information about Slope-Intercept Form is available at this link.[4] .

Point-Slope Form of a Line

In the previous section, we learned that a linear equation can be written in slope-intercept form, $y = mx + b$. This section covers an alternative that is often more useful, especially in Calculus: point-slope form.

Example 17. *Starting in 1990, the population of the United States has been growing by about 2.865 million people per year. Also, back in 1990, the population was 253 million. Since the rate of growth has been roughly constant, a linear model is appropriate. Let's try to write an equation to model this.*

Explanation We consider using , but we would need to know the y-intercept, and nothing in the background tells us that. We'd need to know the population of the United States in the year 0, before there even was a United States.

We could do some side work to calculate the y-intercept, but let's try something else. Here are some things we know:

(a) The slope equation is $m = \dfrac{y_2 - y_1}{x_2 - x_1}$

(b) The slope is $m = 2.865 \dfrac{\text{million people}}{\text{year}}$

(c) One point on the line is $(1990, 253)$ because in 1990, the population was 253 million.

If we use the generic (x, y) to represent a point *somewhere* on this line, then the rate of change between $(1990, 253)$ and (x, y) has to be 2.965. So

$$\frac{y - 253}{x - 1990} = 2.865$$

.

While this is an equation of a line, we might prefer to write the equation without using a fraction. Multiplying both sides by $(x - 1990)$ gives us

$$y - 253 = 2.865(x - 1990)$$

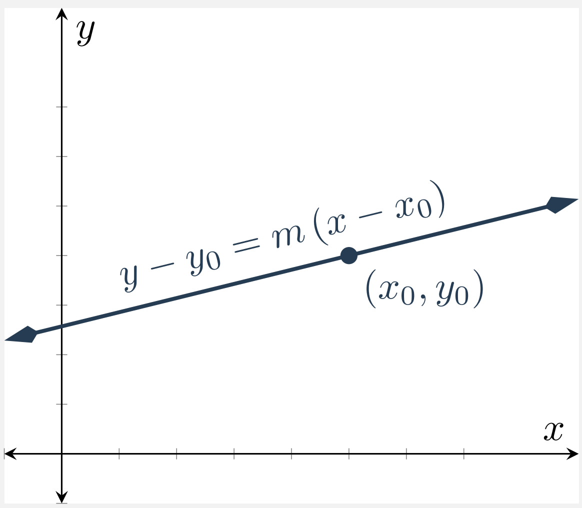
[4]See More information about Slope-Intercept Form is available at this link. at https://spot.pcc.edu/math/orcca/ed2/html/section-slope-intercept-form.html

This is a good place to stop. We have isolated y, and three meaningful numbers appear in the equation: the rate of growth, a certain year, and the population in that year. This is a specific example of point-slope form.

Definition When x and y have a linear relationship where m is the slope and (x_0, y_0) is some specific point that the line passes through, one equation for this relationship is

$$y - y_0 = m(x - x_0)$$

and this equation is called the **point-slope form** of the line. It is called this because the slope and one point on the line are immediately discernible from the numbers in the equation.



Sometimes, it is helpful to be able to express our equation as $y = ...$ To do this when working with the Point-Slope form of a line, all you have to do is add y_0 to both sides of the equation. This will give us the Alternate Point-Slope Form.

Definition When x and y have a linear relationship where m is the slope and (x_0, y_0) is some specific point that the line passes through, one equation for this relationship is

$$y = m(x - x_0) + y_0$$

and this equation is called the **(alternate) point-slope form** of the line. It is called this because the slope and one point on the line are immediately discernible from the numbers in the equation.

Note that some people may call this second form the Point-Slope Form of a line. Both ways of writing this form have the advantage that they can be easily written down if you just know a point on the line and the slope of the line.

More information and examples about Point-Slope Form at this link[5] .

Standard Form of a Line

We've seen that a linear relationship can be expressed with an equation in Slope-Intercept form or with an equation in Point-Slope form. There is a third form that you can use to write line equations. It's known as standard form.

Imagine trying to gather donations to pay for a \$10,000 medical procedure you cannot afford. Oversimplifying the mathematics a bit, suppose that there were only two types of donors in the world: those who will donate \$20 and those who will donate \$100.

How many of each, or what combination, do you need to reach the funding goal? As in, if x people donate \$20 and y people donate \$100, what numbers could x and y be? The donors of the first type have collectively donated $20x$ dollars, and the donors of the second type have collectively donated $100y$.

So altogether you'd need

$$20x + 100y = 10000$$

This is an example of a line equation in standard form.

Definition It is always possible to write an equation for a line in the form

$$Ax + By = C$$

where A,B, and C are three numbers (each of which might be 0, although at least one of A and B must be nonzero). This form of a line equation is called **standard form**.

In the context of an application, the meaning of A, B, and C depends on that context. This equation is called standard form perhaps because any line can be written this way, even vertical lines (which cannot be written using slope-intercept or point-slope form equations).

More information and examples about Point-Slope Form are available at this link.[6] .

[5] See More information and examples about Point-Slope Form at this link at `https://spot.pcc.edu/math/orcca/ed2/html/section-point-slope-form.html`

[6] See More information and examples about Point-Slope Form are available at this link. at `https://spot.pcc.edu/math/orcca/ed2/html/section-standard-form.html`

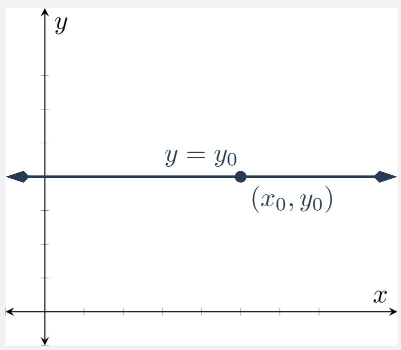
Conclusion

While we can write the equation of a line in different forms, it is important to note that we can easily rearrange a line given in one form to another form using algebra.

There are two special types of lines which it is worth mentioning at this point.

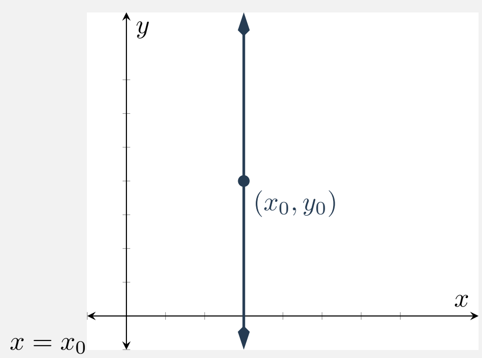
Definition A **horizontal line** is a line where all the y-values of the points are the same. In this case, if the y-value is y_0, then the line can be written as

$$y = y_0$$



Definition A **vertical line** is a line where all the x-values of the points are the same. In this case, if the x-value is x_0, then the line can be written as

$$x = x_0$$



2.2 Linear Modeling

Learning Objectives

- Linear Modeling
  - Applications of linear equations to real world situations
  - Emphasis on the constant rate of change
  - Emphasis on sense making (what does this variable or number mean in this context)

2.2.1 Linear Modeling

Let's see how these linear functions can help us in some "real world" contexts.

Example 18. *Your friends are trying to get an idea of how many people they can invite to their wedding. The venue they're looking at costs about \$2,000 to rent and the event coordinator suggested that they budget about \$100 per attendee into the costs.*

(a) *Write a function that represents an estimate for how much this will cost if they have x attendees.*

(b) *Now rewrite your function from part (a) with units for each of the values (including the variables).*

(c) *Suppose the venue has a maximum capacity of 250 people. What is an estimated maximum cost for using that venue?*

(d) *Suppose your friends are trying to stick to about \$20,000 for their budget. About how many guests can they invite?*

Explanation

(a) $y = 100x + 2000$ where y is the total cost and x is the number of attendees.

(b) $y \quad \text{dollars} = \dfrac{\$100}{1 \quad \text{attendee}} x \quad \text{attendees} + 2000 \quad \text{dollars}$

(c) $y = 100 \times 250 + 2000$
y = 25000 + 2000
y = 27000
The maximum cost is \$27000.

(d) $20000 = 100x + 2000$
18000 = 100x
180 = x
The maximum amount of guests they can invite for \$20000 would be 180 people.

Example 19. *Now let's look closer at how we use $y = mx + b$ for the previous example.*

(a) *What is the "b-value" in your equation? What does it mean in this context?*

(b) *What is the "m-value" in your equation?*

(c) *What are the units for the m-value? Explain why it makes sense that the "m-value" would have these units based on the **x** and **y** values' units.*

(d) *What does this m-value mean in this context?*

Explanation

(a) The "b-value" is 2000. This is also called the y-intercept, because it is the point that the line crosses the y-axis. In the context of this situation, it means that if 0 people attend the wedding the total cost will be \$2000.

(b) The "m-value" is 100. This is the slope or rate of change of the line.

(c) In this context, it is \$100 per attendee or $\frac{\$100}{1 \quad \text{attendee}}$. The units for x are attendees and the units for y are dollars. Since slope is defined by the change in y over the change in x, this makes sense that the units for m are dollars per attendee.

(d) In this context, it means that for each person that attends, the final cost will increase by \$100.

Exploration Below is some information for how electricity usage is billed in Columbus

KWH = Kilowatt Hour **ELECTRICITY RATES Effective January 1, 2018** PCRA = Power Cost Reserve Adjustment

RATE	DESCRIPTION	CHARGES
Residential Schedule A **KW10** (1163.04)	Apartments and dwellings providing domestic accommodations for an individual family. Applicable to residential users with summer usages exceeding 700 KWH in any month.	Customer: \$10.70 Energy (per KWH): .0873 *PCRA varies each month based on actual purchase power costs.
Residential Schedule A-1 *(Small User)* **KW11** (1163.05)	Apartments and dwellings providing domestic accommodations for an individual family. Applicable to residential users with summer usages less than 700 KWH in any month.	Customer: \$10.70 Energy (per KWH): .0724 *PCRA varies each month based on actual purchase power costs.

https://www.columbus.gov/Templates/Detail.aspx?id=2147500472

(a) Write a function, or set of functions, which determine how much your electricity bill will be if you use x KWH in a month. Don't worry about the variable PCRA rates. (Note: the 0.0873 and 0.0724 are dollar amounts, i.e. \$0.0873 and \$0.0724)

(b) Suppose exactly 700 KWH are used in a month. Use Schedule A information to calculate a cost for the bill.

(c) Suppose exactly 700 KWH are used in a month. Now use Schedule A-1 information to calculate a cost for the bill.

Exploration According to True Car (www.truecar.com), a 2018 Toyota Camry (conventional) (29/41 MPG city/hwy) sells for an average of \$22,030, and a Camry Hybrid (51/53 MPG city/hwy) sells for an average of \$26,247. Currently gas prices are in the upper \$2 per gallon, so let's estimate about \$2.80 per gallon.

(a) Write a linear function that will estimate the cost of driving a conventional Camry x miles, given the information above. (Hint:

Think about what the units are for x and what the units should be for y. Then use the units of the information given to help you figure out what should be multiplied and what should be divided in order to give those desired units).

(b) Write a linear function that will estimate the cost of driving a Hybrid Camry x miles, given the information above. (Hint: the function will be very similar to part(a)).

(c) What are the "b-values" in these expressions? What do they represent in this context?

(d) What are the "m-values" in these expressions (write them with their units)? What do these mean in this context?

(e) What other factors could we be considering when comparing the "costs" between these two vehicles?

Summary

- When writing linear equations, consider the units being used in the situation. That can go a long way to properly writing the equation and fully understanding the context.
- context

2.3 Exponential Modeling

Learning Objectives

- Exponent Rules
  - Properties of Exponents
  - Multiple bases, change of bases
  - y $= aR^x$ to $y = ae^{bx}$
- Early Exponential Modeling
  - Applications of exponential equations to real world situations
  - Emphasis on the proportional change
  - Emphasis on sense making (what does this variable or number mean in this context)
  - Compare and Contrast to Linear Models

2.3.1 Exponential Modeling: Exponent Rules

Product Rule

If we write out $3^5 \cdot 3^2$ without using exponents, we'd have:

$$3^5 \cdot 3^2 = (3 \cdot 3 \cdot 3 \cdot 3 \cdot 3) \cdot (3 \cdot 3)$$

If we then count how many 3s are being multiplied together, we find we have $5 + 2 = 7$, a total of seven 3s. So $3^5 \cdot 3^2$ simplifies like this:

$$3^5 \cdot 3^2 = 3^{5+2} = 3^7$$

Example 20. *Simplify $x^2 \cdot x^3$.*
Explanation To simplify $x^2 \cdot x^3$, we write this out in its expanded form, as a product of x's, we have

$$x^2 \cdot x^3 = (x \cdot x)(x \cdot x \cdot x) = x \cdot x \cdot x \cdot x \cdot x = x^5$$

Note that we obtained the exponent of 5 by adding 2 and 3.

This example demonstrates our first exponent rule, the Product Rule:

Product Rule of Exponents

When multiplying two expressions that have the same base, we can simplify the product by adding the exponents.

$$x^m \cdot x^n = x^{m+n}$$

Recall that $x = x^1$. It helps to remember this when multiplying certain expressions together.

Example 21. *Multiply $x(x^3 + 2)$ by using the distributive property.*
Explanation According to the distributive property, $x(x^3 + 2) = x \cdot x^3 + x \cdot 2$ How can we simplify that term $x \cdot x^3$? It's really the same as $x^1 \cdot x^3$, so according to the Product Rule, it is x^4. So we have:

$$x(x^3 + 2) = x \cdot x^3 + x \cdot 2 = x^4 + 2x$$

Power to a Power Rule

If we write out $\left(3^5\right)^2$ without using exponents, we'd have 3^5 multiplied by itself:

$$\left(3^5\right)^2 = \left(3^5\right) \cdot \left(3^5\right) = (3 \cdot 3 \cdot 3 \cdot 3 \cdot 3) \cdot (3 \cdot 3 \cdot 3 \cdot 3 \cdot 3)$$

If we again count how many 3s are being multiplied, we have a total of two groups each with five 3s. So we'd have $2 \cdot 5 = 10$ instances of a 3. So $\left(3^5\right)^2$ simplifies like this:

$\left(3^5\right)^2 = 3^{2\cdot 5} = 3^{10}$

Example 22. *Simplify* $\left(x^2\right)^3$.

Explanation To simplify $\left(x^2\right)^3$, we write this out in its expanded form, as a product of x's, we have $\left(x^2\right)^3 = \left(x^2\right) \cdot \left(x^2\right) \cdot \left(x^2\right) = (x \cdot x) \cdot (x \cdot x) \cdot (x \cdot x)$ $= x^6$ *Note that we obtained the exponent of* 6 *by multiplying* 2 *and* 3.

This demonstrates our second exponent rule, the Power to a Power Rule:

Power to a Power Rule

when a base is raised to an exponent and that expression is raised to another exponent, we multiply the exponents.

$$(x^m)^n = x^{m \cdot n}$$

Product to a Power Rule

The third exponent rule deals with having multiplication inside a set of parentheses and an exponent outside the parentheses. If we write out $(3t)^5$ without using an exponent, we'd have $3t$ multiplied by itself five times:

$$(3t)^5 = (3t)(3t)(3t)(3t)(3t)$$

Keeping in mind that there is multiplication between every 3 and t, and multiplication between all of the parentheses pairs, we can reorder and regroup the factors: $(3t)^5 = (3 \cdot t) \cdot (3 \cdot t) \cdot (3 \cdot t) \cdot (3 \cdot t) \cdot (3 \cdot t) = (3 \cdot 3 \cdot 3 \cdot 3 \cdot 3) \cdot (t \cdot t \cdot t \cdot t \cdot t)$ $= 3^5 t^5$ We could leave it written this way if 3^5 feels especially large. But if you are able to evaluate $3^5 = 243$, then perhaps a better final version of this expression is $243t^5$.

We essentially applied the outer exponent to each factor inside the parentheses. It is important to see how the exponent 5 applied to **both** the 3 **and** the t, not just to the t.

Example 23. *Simplify* $(xy)^5$.

To simplify $(xy)^5$, *we write this out in its expanded form, as a product of* x*'s and* y*'s, we have*

$(xy)^5 = (x \cdot y) \cdot (x \cdot y) \cdot (x \cdot y) \cdot (x \cdot y) \cdot (x \cdot y) = (x \cdot x \cdot x \cdot x \cdot x) \cdot (y \cdot y \cdot y \cdot y \cdot y)$ $= x^5 y^5$

Note that the exponent on xy *can simply be applied to both* x *and* y.

This demonstrates our third exponent rule, the Product to a Power Rule:

Product to a Power Rule

When a product is raised to an exponent, we can apply the exponent to each factor in the product.

$$(x \cdot y)^n = x^n \cdot y^n$$

Summary of the Rules of Exponents for Multiplication

If a and b are real numbers, and m and n are positive integers, then we have the following rules:

Product Rule

$$a^m \cdot a^n = a^{m+n}$$

Power to a Power Rule

$$(a^m)^n = a^{m \cdot n}$$

Product to a Power Rule

$$(ab)^m = a^m \cdot b^m$$

Many examples will make use of more than one exponent rule. In deciding which exponent rule to work with first, it's important to remember that the order of operations still applies.

Example 24. *Simplify the following expression.* $\left(3^7 r^5\right)^4$
Explanation Since we cannot simplify anything inside the parentheses, we'll begin simplifying this expression using the Product to a Power rule. We'll apply the outer exponent of 4 to each factor inside the parentheses. Then we'll use the Power to a Power Rule to finish the simplification process.

$$\left(3^7 r^5\right)^4 = \left(3^7\right)^4 \cdot \left(r^5\right)^4 = 3^{7 \cdot 4} \cdot r^{5 \cdot 4} = 3^{28} r^{20}$$

Note that 3^{28} is too large to actually compute, even with a calculator, so we leave it written as 3^{28}.

Example 25. *Simplify the following expression.* $\left(t^3\right)^2 \cdot \left(t^4\right)^5$
Explanation According to the order of operations, we should first simplify any exponents before carrying out any multiplication. Therefore, we'll begin simplifying this by applying the Power to a Power Rule and then finish using the Product Rule.

$$\left(t^3\right)^2 \cdot \left(t^4\right)^5 = t^{3 \cdot 2} \cdot t^{4 \cdot 5} = t^6 \cdot t^{20} = t^{6+20} = t^{26}$$

> **Remark** We cannot simplify an expression like x^2y^3 using the Product Rule, as the factors x^2 and y^3 do not have the same base.

Quotient to a Power Rule

One rule we have learned is the product to a power rule, as in $(2x)^3 = 2^3x^3$. When two factors are multiplied and the product is raised to a power, we may apply the exponent to each of those factors individually. We can use the rules of fractions to extend this property to a quotient raised to a power.
Let y be a real number, where $y \neq 0$.
Find another way to write $\left(\frac{5}{y}\right)^4$.
Writing the expression without an exponent and then simplifying, we have:

$$\begin{aligned}\left(\frac{5}{y}\right)^4 &= \left(\frac{5}{y}\right)\left(\frac{5}{y}\right)\left(\frac{5}{y}\right)\left(\frac{5}{y}\right)\\ &= \frac{5\cdot5\cdot5\cdot5}{y\cdot y\cdot y\cdot y}\\ &= \frac{5^4}{y^4}\\ &= \frac{625}{y^4}\end{aligned}$$

Similar to the product to a power rule, we essentially applied the outer exponent to the factors inside the parentheses to factors of the numerator and factors of the denominator.

> The general rule is: For real numbers a and b (with $b \neq 0$) and natural number m, $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$

This rule says that when you raise a fraction to a power, you may separately raise the numerator and denominator to that power. In Example, this means that we can directly calculate $\left(\frac{5}{y}\right)^4$:

$$\begin{aligned}\left(\frac{5}{y}\right)^4 &= \frac{5^4}{y^4}\\ &= \frac{625}{y^4}\end{aligned}$$

Example 26. (a) *Simplify* $\left(\frac{p}{2}\right)^6$.

(b) *Simplify* $\left(\frac{5^6 w^7}{5^2 w^4}\right)^9$. *If you end up with a large power of a specific number, leave it written that way.*

(c) *Simplify* $\frac{\left(2r^5\right)^7}{\left(2^2 r^8\right)^3}$. *If you end up with a large power of a specific number, leave it written that way.*

Explanation

(a) We can use the quotient to a power rule:

$$\begin{aligned}\left(\frac{p}{2}\right)^6 &= \frac{p^6}{2^6}\\ &= \frac{p^6}{64}\end{aligned}$$

(b) If we stick closely to the order of operations, we should first simplify inside the parentheses and then work with the outer exponent. Going this route, we will first use the quotient rule:

$$\begin{aligned}\left(\frac{5^6 w^7}{5^2 w^4}\right)^9 &= \left(5^{6-2} w^{7-4}\right)^9\\ &= \left(5^4 w^3\right)^9\end{aligned}$$

Now we can apply the outer exponent to each factor inside the parentheses using theproduct to a power rule.

$$= \left(5^4\right)^9 \cdot \left(w^3\right)^9$$

To finish, we need to use the ¿power to a power rule.

$$\begin{aligned}&= 5^{4\cdot 9} \cdot w^{3\cdot 9}\\ &= 5^{36} \cdot w^{27}\end{aligned}$$

(c) According to the order of operations, we should simplify inside parentheses first, then apply exponents, then divide. Since we cannot simplify inside the parentheses, we must apply the outer exponents to each factor inside the respective set of parentheses first:

$$\frac{\left(2r^5\right)^7}{\left(2^2 r^8\right)^3} = \frac{2^7\left(r^5\right)^7}{\left(2^2\right)^3\left(r^8\right)^3}$$

At this point, we need to use the power-to-a-power rule:

$$\begin{aligned}&= \frac{2^7 r^{5\cdot 7}}{2^{2\cdot 3} r^{8\cdot 3}}\\ &= \frac{2^7 r^{35}}{2^6 r^{24}}\end{aligned}$$

To finish simplifying, we'll conclude with the quotient rule:

$$= 2^{7-6} r^{35-24}$$

$$= 2^1 r^{11}$$
$$= 2r^{11}$$

Zero as an Exponent

So far, we have been working with exponents that are natural numbers $(1, 2, 3, \ldots)$. By the end of this section, we will expand our understanding to include exponents that are any integer, as with 5^0 and 12^{-2}. As a first step, let's explore how 0 should behave as an exponent by considering the pattern of decreasing powers of 2 in ¡xref ref=”figure-descending-powers-of-two”¿Figure¡/xref¿. Simplify the following expressions. Assume all variables represent non-zero real numbers

To simplify any of these expressions, it is critical that we remember an exponent only applies to what it is touching or immediately next to.

In the expression $\left(173x^4y^{251}\right)^0$, the exponent 0 applies to everything inside the parentheses. $\left(173x^4y^{251}\right)^0 = 1$

In the expression $(-8)^0$ the exponent applies to everything inside the parentheses, -8. $(-8)^0 = 1$

In contrast to the previous example, the exponent only applies to the 8. The exponent has a higher priority than negation in the order of operations. We should consider that $-8^0 = -\left(8^0\right)$, and so:

$-8^0 = -\left(8^0\right) = -1$

In the expression $3x^0$, the exponent 0 only applies to the x: $3x^0 = 3 \cdot x^0 = 3 \cdot 1$ $= 3$

Negative Exponents

We understand what it means for a variable to have a natural number exponent. For example, x^5 means $\overbrace{x \cdot x \cdot x \cdot x \cdot x}^{\text{five times}}$. Now we will try to give meaning to an exponent that is a negative integer, like in x^{-5}.

To consider what it could possibly mean to have a negative integer exponent, let's extend the pattern we saw in ¡xref ref=”figure-descending-powers-of-two”¿Figure¡/xref¿. In that table, each time we move down a row, we reduce the power by 1 and we divide the value by 2. We can continue this pattern in the power and value columns, going all the way down into when the exponent is negative.

Power	*Result*	
2^3	8	
2^2	4	(divide by 2)
2^1	2	(divide by 2)
2^0	1	(divide by 2)
2^{-1}	$\frac{1}{2} = \frac{1}{2^1}$	(divide by 2)
2^{-2}	$\frac{1}{4} = \frac{1}{2^2}$	(divide by 2)
2^{-3}	$\frac{1}{8} = \frac{1}{2^3}$	(divide by 2)

We see a pattern where $2^{\text{negative number}}$ is equal to $\frac{1}{2^{\text{positive number}}}$. Note that the choice of base 2 was arbitrary, and this pattern works for all bases except 0, since we cannot divide by 0 in moving from one row to the next.

Remark Negative Integers as Exponents For any non-zero real number a and any natural number n, we define a^{-n} to mean the reciprocal of a^n. That is,

$$a^{-n} = \frac{1}{a^n}$$

2.3.2 Exponential Modeling: Early Exponentials

Early Exponentials

Example 27. *An athlete signs a contract saying that they will earn \$8.3 million with an increase of 4.8% each year of the 5 year contract. Use this scenario to fill in the following table:*

Year of : Contract	*Athlete's Salary*	*Calculate the next year's salary using the previous year's salary*
1	\$8.3 *million*	*N/A*
2	\$$\boxed{?}$ *million*	$8.3 + 8.3 \times \boxed{?}$
3	\$$\boxed{?}$ *million*	$\boxed{?} + \boxed{?} \times \boxed{?}$
4	\$$\boxed{?}$ *million*	$\boxed{?} + \boxed{?} \times \boxed{?}$
5	\$$\boxed{?}$ *million*	$\boxed{?} + \boxed{?} \times \boxed{?}$

You may have calculated the salaries for each year by first finding 4.8% of the previous year's salary, and then adding that value to the previous year's salary. Doing it this way, you have to type two calculations into a calculator (calculate 4.8%, then record that value, then enter the addition of that value to the salary). If you did it this way, think about how to write (and do) the computation with just one calculation entry into a calculator.

$$8.3 + 8.3 \times \boxed{?} = 8.3 \times \boxed{?}$$

What we have developed above is an exponential function to describe this athlete's salary. Let's look at each value in our function and identify what each piece represents.

$$y = 8.3(1.048)^x$$

8.3 1.048 x y What we have outlined above is an understanding of what each piece of an exponential function in the form y=a*Rx means in terms of a given context. So we can now use this understanding to create an exponential function for exponential scenarios without having to go through the "step by step" process as we did in the original table for the athlete salary problem.

Example 28. *Some CDs (a banking investment option) are offering rates that give about 3.10% yield per year as long as a minimum amount, such as \$5,000, is invested. "3.10% yield" simply means the investment increases (earns) about 3.10%. CDs are only given for a certain amount of time (usually a certain number of years). When a CD "matures" (ends), you usually have the option to renew the CD.*

(a) *Use this scenario to fill out the table below*

After Year:	*Value of investment*	*Calculate the value at the end of the next year using the previous year's v*
0	$ $\boxed{?}$	$ $\boxed{?}$ ($\boxed{?}$)
1	$ $\boxed{?}$	$ $\boxed{?}$ ($\boxed{?}$)
2	$5,314.81	$ $\boxed{?}$ ($\boxed{?}$)
3	$5,479.57	$ $\boxed{?}$ ($\boxed{?}$)

(b) *Suppose you invest the minimum amount of $5,000 in order to get this 3.10% yield. Write a function to describe how much the investment will be worth after x years. Write the units of each value and identify what each represents.*

(c) *Fill in the first column of the table below with the values you calculated in your pre class work. Then, enter the appropriate values into your function from above to fill in the last column of the table.*

After Year:	*Value of investment from previous table*	*Value of investment from function*
0	$5,000	
1	$5,155	
2	$5,314.81	
3	$5,479.57	

How do these values compare? Are they exactly the same? Should they be? Explain.

(d) *Suppose this is a 7 year CD. How much will your investment be worth at the end of the CD?*

(e) *Suppose you keep renewing this CD with this rate every time it "matures" (comes to the end). About how many years will it take to double the initial investment? Use "Guess and Check" to answer this question. We want our function to output: Because*

(i) *Find the closest whole number that gives us less than we want:*

(ii) *Find the closest whole number that gives us more than we want:*

(iii) *Which of these values is closer to what we want? Use whichever value is closer as your estimated value for the answer.*

Exploration 4. Suppose there is a new virus that reportedly doubles infection cases in about 20 days. What would be this virus' infection rate (as a percent)? Hint: Set up an exponential function, assuming that there were initially 80 recorded infections. After you have figured out the infection rate, think about/explain why an initial number of infections was not needed in order to answer this question.

Part 3

Functions

3.1 What is a Function?

Learning Objectives

- What is a Function?
    - Deep understanding of multiple representations of functions, including graphs, tables, equations, function notation, words, arrow diagrams, applications, data, and a function machine
    - Recognize functions in everyday life and appreciate the ubiquity of functions
    - Use definition of function to argue whether something is a function using all different types of representations
    - Understand the difference between a function, an expression, and an equation
    - Define x- and y-intercepts of the graphs of functions

3.1.1 What is a Function?

Motivating Questions

- What is a function?
- How can functions be represented?
- When is a relation not a function?

Mathematical Models

A mathematical model is an abstract concept through which we use mathematical language and notation to describe a phenomenon in the world around us. One example of a mathematical model is found in Dolbear's Law[7] , which has proven to be remarkably accurate for the behavior of snowy tree crickets. For even more of the story, including a reference to this phenomenon on the popular showThe Big Bang Theory[8] . In the late 1800s, the physicist Amos Dolbear was listening to crickets chirp and noticed a pattern: how frequently the crickets chirped seemed to be connected to the outside temperature.

If we let T represent the temperature in degrees Fahrenheit and N the number of chirps per minute, we can summarize Dolbear's observations in the following table.

[7] See Dolbear's Law at https://en.wikipedia.org/wiki/Dolbear%27s_law

[8] See The Big Bang Theory at https://priceonomics.com/how-to-tell-the-temperature-using-crickets/

N(chirps per minute)	V(degrees Fahrenheit)
40	50
80	60
120	70
160	80

For a mathematical model, we often seek an algebraic formula that captures observed behavior accurately and can be used to predict behavior not yet observed. For the data in the table above, we observe that each of the ordered pairs in the table make the equation

$$T = 40 + 0.25N \tag{5}$$

true. For instance, $70 = 40 + 0.25(120)$. Indeed, scientists who made many additional cricket chirp observations following Dolbear's initial counts found that the formula above holds with remarkable accuracy for the snowy tree cricket in temperatures ranging from about 50° F to 85° F.

This model captures a pattern that is found in the world, and can be used to predict the temperature if only the number of chirps per minute is known. Not all phenomenon in the world that can be measured mathematically occur in a predictable pattern. In this section, we will study functions which are mathematical ways of formally studying situations where for a given input, such as the number of chirps above, there is one consistant output. For situations where the output is not fixed, we encourage you to study statistics!

Functions

The mathematical concept of a function is one of the most central ideas in all of mathematics, in part since functions provide an important tool for representing and explaining patterns. At its core, a function is a repeatable process that takes a collection of input values and generates a corresponding collection of output values with the property that if we use a particular single input, the process always produces exactly the same single output.

For instance, Dolbear's Law provides a process that takes a given number of chirps between 40 and 180 per minute and reliably produces the corresponding temperature that corresponds to the number of chirps, and thus this equation generates a function. We often give functions shorthand names; using "D" for the "Dolbear" function, we can represent the process of taking inputs (observed chirp rates) to outputs (corresponding temperatures) using arrows:

$$
\begin{aligned}
80 &\xrightarrow{D} 60 \\
120 &\xrightarrow{D} 70 \\
N &\xrightarrow{D} 40 + 0.25N
\end{aligned}
$$

Alternatively, for the relationship "$80 \xrightarrow{D} 60$" we can also use the equivalent notation "$D(80) = 60$" to indicate that Dolbear's Law takes an input of 80 chirps per minute and produces a corresponding output of 60 degrees Fahrenheit. More generally, we write "$T = D(N) = 40 + 0.25N$" to indicate that a certain temperature, T, is determined by a given number of chirps per minute, N, according to the process $D(N) = 40 + 0.25N$.

We will define a function informally and formally. The informal definition corresponds to the way we will most often think of functions, as a process with inputs and outs.

Definition [Informal Definition of a Function] A **function** is a process that may be applied to a collection of input values to produce a corresponding collection of output values in such a way that the process produces one and only one output value for any single input value.

The formal definition of a function will establish a function as a special type of relation. Recall that a *relation* is a collection of points of the form (x, y). If the point (x_0, y_0) is in the relation, then we say x_0 and y_0 are *related*.

Definition [Formal Definition of a Function] A **function** is a collection of ordered pairs (x, y) such that any particular value of x is paired with at most one value for y. That is, a relation in which each x-coordinate is matched with only one y-coordinate is said to describe y as a **function** of x.

How is this definition consistent with the informal , which describes a function as a process? Well, if you have a collection of ordered pairs (x, y), you can choose to view the left number as an input, and the right value as the output. If the function's name is f and you want to find $f(x)$ for a particular number x, look in the collection of ordered pairs to see if x appears among the first coordinates. If it does, then $f(x)$ is the (unique) y-value it was paired with. If it does not, then that x is just not in the domain of f, because you have no way to determine what $f(x)$ would be.

Summary

-

3.2 Function Properties

Learning Objectives

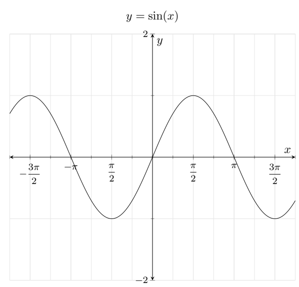
- Function Properties
  - Periodic
  - Even and Odd
- Inverse Functions
  - Definition of an Inverse Function
  - One-to-one functions and why they are necessary for inverse functions to exist
  - Draw or identify the graph of an inverse function

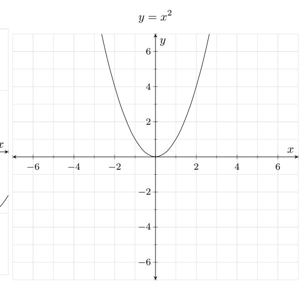
3.2.1 Function Properties

Motivating Questions

- What do we mean when we say a function is periodic?
- What do we mean when we say a function is even or odd? How can we identify even and odd functions?

When working with functions and looking at their graphs, we might notice some interesting patterns or behaviors. For example, a function like the sine function appears to repeat itself over and over again, and the quadratic function defined by $y = x^2$ appears to be symmetric about the y-axis.





In this section, we'll discuss new vocabulary we can use to describe these behaviors as well as how to show analytically that a function has a certain behavior.

We'll also discuss the important concept of inverse functions, which can provide a way to "undo" functions.

Periodic functions

Certain naturally occurring phenomena eventually repeat themselves, especially when the phenomenon is somehow connected to a circle. For example, suppose that you are taking a ride on a ferris wheel and we consider your height, h, above the ground and how your height changes in tandem with the distance, d, that you have traveled around the wheel. We can see a full animation of this situation at `http://gvsu.edu/s/0Dt`.

Because we have two quantities changing in tandem, it is natural to wonder if it is possible to represent one as a function of the other.

Exploration In the context of the ferris wheel mentioned above, assume that the height, h, of the moving point (the cab in which you are riding), and the distance, d, that the point has traveled around the circumference of the ferris wheel are both measured in meters.

Further, assume that the circumference of the ferris wheel is 24 meters (it's a pretty short ferris wheel). In addition, suppose that after getting in your cab at the lowest point on the wheel, you traverse the full circle several times.

(a) Recall that the circumference, C, of a circle is connected to the circle's radius, r, by the formula $C = 2\pi r$. What is the radius of the ferris wheel? How high is the highest point on the ferris wheel?

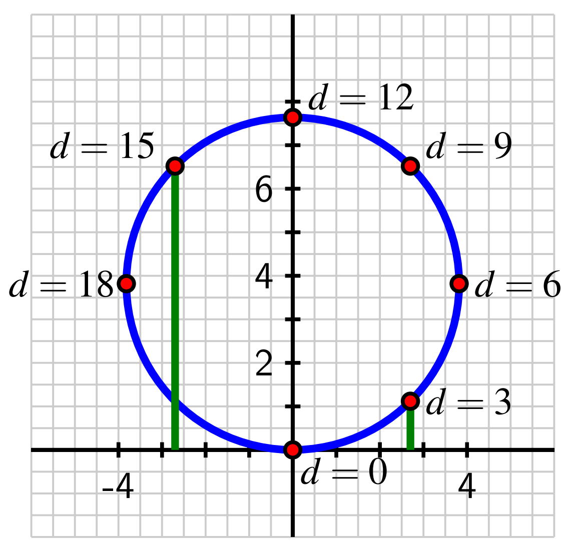
(b) How high is the cab after it has traveled $\frac{1}{4}$ of the circumference of the circle?

(c) How much distance along the circle has the cab traversed at the moment it first reaches a height of $\frac{12}{\pi} \approx 3.82$ meters?

(d) Can h be thought of as a function of d? Why or why not?

(e) Can d be thought of as a function of h? Why or why not?

The natural phenomenon of a point moving around a circle leads to interesting relationships. Let's consider a point traversing a circle of circumference 24 and examine how the point's height, h, changes as the distance traversed, d, changes. Note particularly that each time the point traverses $\frac{1}{8}$ of the circumference of the circle, it travels a distance of $24 \cdot \frac{1}{8} = 3$ units, as seen below, where each noted point lies 3 additional units along the circle beyond the preceding one.



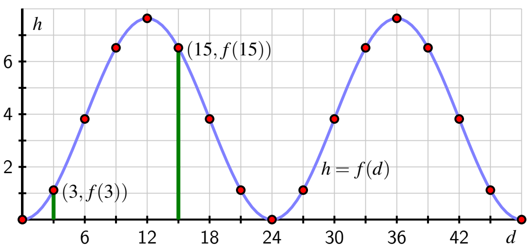
Note that we know the exact heights of certain points. Since the circle has circumference $C = 24$, we know that $24 = 2\pi r$ and therefore $r = \dfrac{12}{\pi} \approx 3.82$. Hence, the point where $d = 6$ (located 1/4 of the way along the circle) is at a height of $h = \dfrac{12}{\pi} \approx 3.82$. Doubling this value, the point where $d = 12$ has height $h = \dfrac{24}{\pi} \approx 7.64$. Other heights, such as those that correspond to $d = 3$ and $d = 15$ (identified on the figure by the green line segments) are not obvious from the circle's radius, but can be estimated from the grid in the figure above as $h \approx 1.1$ (for $d = 3$) and $h \approx 6.5$ (for $d = 15$). Using all of these observations along with the symmetry of the circle, we can construct a table..

d	h
0	0
3	1.1
6	3.82
9	6.5
12	7.64
15	6.5
18	3.82
21	1.1
24	0

Moreover, if we now let the point continue traversing the circle, we observe that the d-values will increase accordingly, but the h-values will repeat according to the already-established pattern, resulting in the data in the table below.

d	h
24	0
27	1.1
30	3.82
33	6.5
36	7.64
39	6.5
42	3.82
45	1.1
48	0

It is apparent that each point on the circle corresponds to one and only one height, and thus we can view the height of a point as a function of the distance the point has traversed around the circle, say $h = f(d)$. Using the data from the two tables and connecting the points in an intuitive way, we get the graph shown below



Notice that the graph above resembles the graph of the sine function. As it turns out, the sine function exhibits some of the same oscillatory behavior as f. This shared property turns out to be very important, especially when looking at functions that are related to circles.

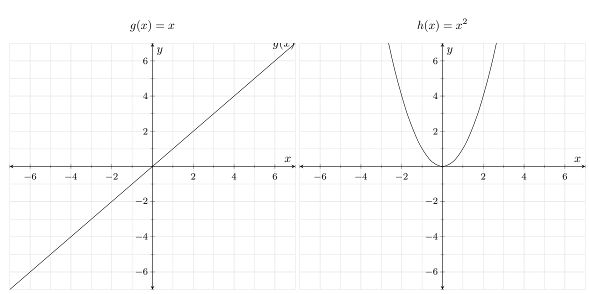
> **Definition** Let f be a function whose domain and codomain are each the set of all real numbers. We say that f is **periodic** provided that there exists a real number k such that $f(x+k) = f(x)$ for every possible choice of x. The smallest value p for which $f(x+p) = f(x)$ for every choice of x is called the **period** of f.

For our ferris wheel example above, the period is the circumference of the circle that generates the curve. In the graph, we see how the curve has completed one full cycle of behavior every 24 units, regardless of where we start on the curve.

Later in the course, you will encounter and study many other examples of periodic functions.

Odd and even functions

Consider the two functions, $g(x) = x$ and $h(x) = x^2$, whose graphs are shown below.



Note that the graph of g seems to be symmetric about the origin, meaning that when we rotate the graph a half-turn, we get the same graph. Also, the graph of h seems to be symmetric about the y-axis, meaning that when we flip the graph across the y-axis, we get the same graph.

Let's first consider the case of g. To actually prove the symmetry about the origin analytically, we assume (x, y) is a generic point on the graph of g. That is, we assume $y = x$. The point symmetric to (x, y) about the origin is $(-x, -y)$. To show the graph is symmetric about the origin, we need to show that $(-x, -y)$ is on the graph whenever (x, y) is. In other words, we need to show $(-x, -y)$ satisfies the equation $y = x$ whenever (x, y) does. Substituting $(-x, -y)$ into the equation gives

$$\begin{aligned} -y &\stackrel{?}{=} -x \\ y &\stackrel{\checkmark}{=} x. \end{aligned}$$

Since we are assuming the original equation $y = x$ is true, we have shown that $(-x, -y)$ satisfies the equation (since it leads to a true result) and hence, is on the graph of g. This shows that g is symmetric about the origin.

Exploration Consider the function h defined by $h(x) = x^2$. We'll try to prove that h is symmetric about the y-axis.

a. Assume (x, y) is a generic point on the graph of h, so $y = x^2$. What point is symmetric to (x, y) about the y-axis?

b. Show your answer to part a is on the graph of h whenever (x, y) is. Conclude that h is symmetric about the y-axis.

Notice that to test an equation's graph for symmetry about the origin, we replaced x and y with $-x$ and $-y$, respectively. Doing this substitution in the equation $y = f(x)$ results in $-y = f(-x)$. Solving the latter equation for y gives $y = -f(-x)$. In order for this equation to be equivalent to the original equation $y = f(x)$ we need $-f(-x) = f(x)$, or, equivalently, $f(-x) = -f(x)$. In the exploration, you checked whether the graph of an equation was symmetric about the y-axis by replacing x with $-x$ and checking to see if an equivalent equation results. If we are graphing the equation $y = f(x)$, substituting $-x$ for x results in the equation $y = f(-x)$. In order for this equation to be equivalent to the original equation $y = f(x)$ we need $f(-x) = f(x)$. These results are summarized below.

The graph of a function f is symmetric

- about the y-axis if and only if $f(-x) = f(x)$ for all x in the domain of f.
- about the origin if and only if $f(-x) = -f(x)$ for all x in the domain of f.

Definition We call a function **even** if its graph is symmetric about the y-axis or **odd** if its graph is symmetric about the origin. Apart from a very specialized family of functions which are both even and odd, functions fall into one of three distinct categories: even, odd, or neither even nor odd.

Example 29. *Determine analytically if the following functions are even, odd, or neither even nor odd.*

(a) $f(x) = \dfrac{5}{2 - x^2}$

(b) $g(x) = \dfrac{5x}{2 - x^2}$

(c) $h(x) = \dfrac{5x}{2 - x^3}$

Explanation The first step in all these problems is to replace x with $-x$ and simplify.

(a) Here, $f(x) = \dfrac{5}{2 - x^2}$. Replacing x with $-x$, we find that

$$f(-x) = \frac{5}{2 - (-x)^2}$$
$$f(-x) = \frac{5}{2 - x^2},$$

so $f(-x) = f(x)$. This shows that f is *even.*

(b) Here, $g(x) = \dfrac{5x}{2 - x^2}$. Replacing x with $-x$, we find that

$$g(-x) = \frac{5(-x)}{2 - (-x)^2}$$
$$g(-x) = \frac{-5x}{2 - x^2}.$$

It doesn't appear that $g(-x)$ is equal to $g(x)$. To prove this, we check with an x value. After some trial and error, we see that $g(1) = 5$ whereas $g(-1) = -5$. This proves that g is not even, but it doesn't rule out the possibility that g is odd. (Why not?) To check if g is odd, we compare $g(-x)$ with $-g(x)$:

$$-g(x) = -\frac{5x}{2-x^2}$$
$$-g(x) = \frac{-5x}{2-x^2}$$
$$-g(x) = g(-x).$$

Since $-g(x) = g(-x)$, g is *odd.*

(c) Here, $h(x) = \dfrac{5x}{2-x^3}$. Replacing x with $-x$, we find that

$$h(-x) = \frac{5(-x)}{2-(-x)^3}$$
$$h(-x) = \frac{-5x}{2+x^3}.$$

Once again, $h(-x)$ doesn't appear to be equal to $h(x)$. We check with an x value. For example, $h(1) = 5$, but $h(-1) = -\dfrac{5}{3}$. This proves that h is not even and it also shows h is not odd. (Why?)

Summary

- For a function f defined on the real numbers, we say f is periodic if there exists some k such that

  $$f(x+k) = f(x)$$

  for all possible choices of x. The smallest value of k for which $f(x+k) = f(x)$ for all possible choices of x is called the period of f.

- A function f is called
  - even if $f(-x) = f(x)$ for all possible choices of x. Even functions are symmetric about the y-axis.
  - odd if $f(-x) = -f(x)$ for all possible choices of x. Odd functions are symmetric about the origin.

3.2.2 Inverse Functions

Motivating Questions

- What does it mean to say that a function has an inverse?
- How can we identify when we can find inverse functions?
- What are the properties of an inverse function compared to the original function?

Because every function is a process that converts a collection of inputs to a corresponding collection of outputs, a natural question is: for a particular function, can we change perspective and think of the original function's outputs as the inputs for a reverse process? If we phrase this question algebraically, it is analogous to asking: given an equation that defines y is a function of x, is it possible to find a corresponding equation where x is a function of y?

Inverse functions

Let's think about the problem in a more concrete way. Consider a situation in which Jessica is running a candle company. Say she starts the day with \$15, but makes \$4 for every candle she sells. A linear function f representing the amount of money in dollars she has after selling x candles is given by $f(x) = 4x + 15$. To find out how much money she has after selling 20 candles, we can plug in 20 to the equation above:

$$f(20) = 4 \cdot 20 + 15 = 95.$$

Now suppose that Jessica tells us that at the end of the day, she ended up with \$135. Would it be possible to figure out how many candles she sold? This type of "inverse question" is common in math. Notice that above, when we started with an amount of candles and wanted to find an amount of money, we first multiplied by 4, then added 15. Now, since we're starting with an amount of money, we need to undo the processes we did before: first we subtract 15, and then divide by 4. We can represent this process by a function g, which represents the number of candles sold if Jessica has y dollars:

$$g(y) = \frac{1}{4}(y - 15).$$

When we plug in \$135 to this function, we find that Jessica has sold

$$g(135) = \frac{1}{4}(135 - 15) = 30$$

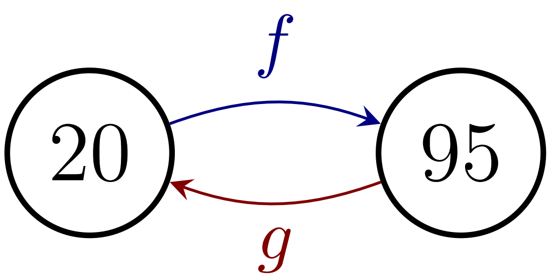
candles.

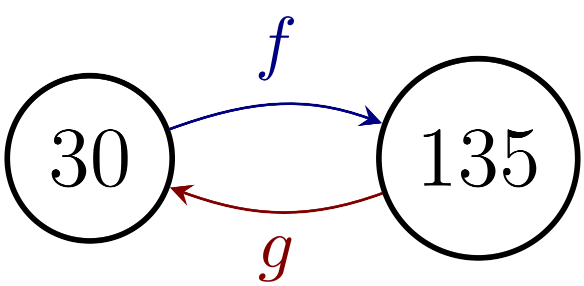
Exploration Let f be a function defined by $f(x) = 4x + 15$. Let g be a function defined by $g(y) = \frac{1}{4}(y - 15)$.

a. Recall that $f(20) = 95$. What is $g(95)$?

b. Recall that $g(135) = 30$. What is $f(30)$?

After plugging in, we see that when we plug 20 into f, we get $f(20) = 95$, and when we plug 95 into g (that is, we plug $f(20)$ into g), we get back 20. Similarly, when we plug in the value $g(135)$ into f, we get back 135. The following diagram illustrates the situation:





In the above exploration, f and g are examples of *inverse functions*.

Definition Let f be a function. If there exists a function g such that

$$g(f(a)) = a \text{ and } f(g(b)) = b$$

for each a and b, then we say that f has an **inverse function** and that the function g is the **inverse** of f. We also say that f is **invertible**.

Note particularly what the equation $g(f(a)) = a$ says: for any input a, the function g will reverse the process of f (which converts a to $f(a)$) because g converts $f(a)$ back to a.

When a given function f has a corresponding inverse function g, we usually rename g as f^{-1}, which we read aloud as "f-inverse". The equation $g(f(a)) = a$ now reads as $f^{-1}(f(a)) = a$, which we interpret as saying "f-inverse converts $f(a)$ back to a". We similarly write that $f(f^{-1}(b)) = b$.

Exploration Dolbear's function $F = D(N) = 40+\dfrac{1}{4}N$ is used to model the number N of snowy tree cricket chirps per minute to a corresponding Fahrenheit temperature.

a. Solve the equation $F = 40 + \dfrac{1}{4}N$ for N in terms of F. Call the resulting function $N = E(F)$.

b. Explain in words the process or effect of the function $N = E(F)$. What does it take as input? What does it generate as output?

c. Use the function E that you found above to compute $j(N) = E(D(N))$. Simplify your result as much as possible. Do likewise for $k(F) = D(E(F))$. What do you notice about these two functions j and k?

d. Consider the equations $F = 40+\dfrac{1}{4}N$ and $N = 4(F-40)$. Do these equations express different relationships between F and N, or do they express the same relationship in two different ways? Explain.

When a given function has an inverse function, it allows us to express the same relationship from two different points of view. For instance, if $y = f(t) = 2t+1$, we can show that the function $t = g(y) = \dfrac{y-1}{2}$ reverses the effect of f (and vice versa), and thus $g = f^{-1}$. We observe that

$$y = f(t) = 2t + 1 \text{ and } t = f^{-1}(y) = \frac{y-1}{2}$$

are equivalent forms of the same equation, and thus they say the same thing from two different perspectives. The first version of the equation is solved for y in terms of t, while the second equation is solved for t in terms of y. This important principle holds in general whenever a function has an inverse function.

If $y = f(t)$ has an inverse function, then the equations

$$y = f(t) \text{ and } t = f^{-1}(y)$$

say the exact same thing but from two different perspectives.

When can we find inverses?

It's important to note in the above definition of inverse functions that we say "*If* there exists . . .". That is, we don't guarantee that an inverse function exists for a given function. Thus, we might ask: how can we determine whether or not a given function has a corresponding inverse function? As with many questions about functions, there are often three different possible ways to explore such a question: through a table, through a graph, or through an algebraic formula.

Consider the functions f and g given in the following tables.

x	$f(x)$
0	6
1	4
2	3
3	4
4	6

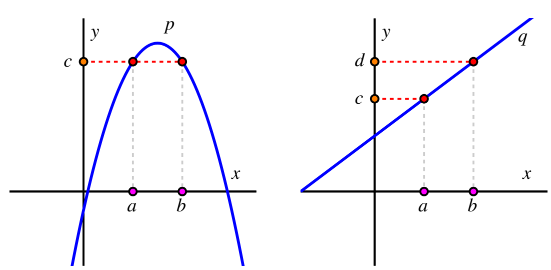
x	$g(x)$
0	3
1	1
2	4
3	2
4	0

For any function, the question of whether or not it has an inverse comes down to whether or not the process of the function can be reliably reversed. For functions given in table form such as f and g, we essentially ask if it's possible to swich the input and output columns and have the new resulting table also represent a function.

The function f does not have an inverse function because there are two different inputs that lead to the same output: $f(0) = 6$ and $f(4) = 6$. If we attempt to reverse this process, we have a situation where the input 6 would correspond to *two* potential outputs, 4 and 6.

However, the function g does have an inverse function because when we reverse the columns in the table each input (in order, 3, 1, 4, 2, 0) indeed corresponds to one and only one output (in order, 0, 1, 2, 3, 4). We can thus make observations such as $g^{-1}(4) = 2$, which is the same as saying that $g(2) = 4$, just from a different perspective.

Now, consider the functions p and q represented by the following graphs.



Recall that when a point such as (a, c) lies on the graph of a function p, this means that the input $x = a$, which represents to a value on the horizontal axis, corresponds with the output $y = c$ that is represented by a value on the vertical axis. In this situation, we write $p(a) = c$. We note explicitly that p is a function because its graph passes the vertical line test: any vertical line intersects the graph of p exactly, and thus each input corresponds to one and only one output.

If we attempt to change perspective and use the graph of p to view x as a function of y, we see that this fails because the output value c is associated with two different inputs, a and b. Said differently, because the horizontal line $y = c$ intersects the graph of p at both (a, c) and (b, c) (as shown in the figure), we cannot view y as the input to a function process that produces the corresending x-value. Therefore, p does not have an inverse function.

On the other hand, provided that the behavior seen in the figure continues, the function q does have an inverse because we can view x as a function of y via the graph. This is because for any choice of y, there corresponds one and only one x that results from y. We can think of this visually by starting at a value such as $y = c$ on the y-axis, moving horizontally to where the line intersects the graph of p, and then moving down to the corresonding location (here $x = a$) on the horizontal axis. From the behavior of the graph of q (a straight line that is always increasing), we see that this correspondence will hold for any choice of y, and thus indeed x is a function of y. From this, we can say that q indeed has an inverse function. We thus can write that $q^{-1}(c) = a$, which is a different way to express the equivalent fact that $q(a) = c$.

The two examples above illustrate an important requirement for a function to have an inverse function. For a function to have an inverse, different inputs must go to different outputs, or else we will run into the same problems as we did in the examples of f and p above.

Definition A function f is said to be **one-to-one** if f matches different inputs to different outputs. Equivalently, f is one-to-one if and only if whenever $f(a) = f(b)$, then $a = b$. If a function is one-to-one, it has an inverse.

In particular, the graphical observations that we made for the function q in the last example provide a general test for whether or not a function given by a graph has a corresponding inverse function.

Remark A function whose graph lies in the x-y plane is one-to-one if and only if every horizontal line intersects the graph at most once. When the graph passes this test, the horizontal coordinate of each point on the graph can be viewed as a function of the vertical coordinate of the point. This is sometimes referred to as *the horizontal line test.*

Exploration Consider the functions r and s defined by

$$y = r(t) = 3 - \frac{1}{5}(t-1)^3 \text{ and } y = s(t) = 3 - \frac{1}{5}(t-1)^2.$$

a. Solve the equation $y = r(t)$ for t. Can t be expressed as a single function of y? What could go wrong?

b. Is r one-to-one?

c. Find $s(2)$ and $s(0)$. Compare them; is s is one-to-one?

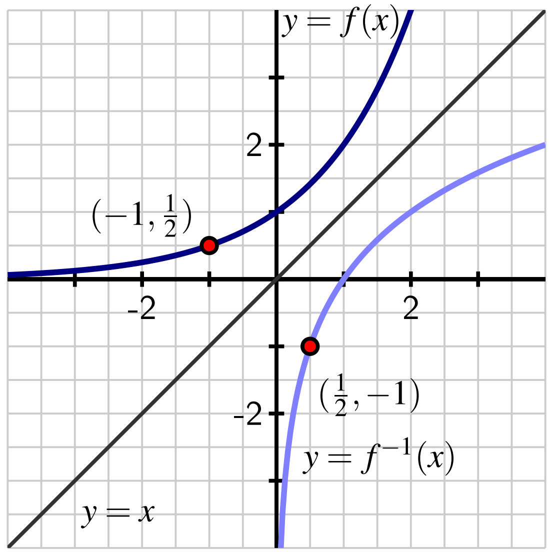
d. Solve the equation $y = s(t)$ for t. Can t be expressed as a single function of y? What could go wrong?

To check your answers for b. and c. above, you could try graphing r and s and applying the horizontal line test.

Graphical properties of inverse functions

Finally, we mention an important relationship between the graph of a function and the graph of its inverse function. If f is one-to-one, then recall that a point (x, y) lies on the graph of f if and only if $y = f(x)$. From this, since f is one-to-one, we can equivalently say that $x = f^{-1}(y)$. Hence, the point (y, x) lies on the graph of $x = f^{-1}(y)$.

The last item above leads to a special relationship between the graphs of f and f^{-1} when viewed on the same coordinate axes. In that setting, we need to view x as the input of each function (since it's the horizontal coordinate) and y as the output. If we know a particular input-output relationship for f, say $f(-1) = \frac{1}{2}$, then it follows that $f^{-1}\left(\frac{1}{2}\right) = -1$. We observe that the points $\left(-1, \frac{1}{2}\right)$ and $\left(\frac{1}{2}, -1\right)$ are reflections of each other across the line $y = x$. Because such a relationship holds for every point (x, y) on the graph of f, this means that the graphs of f and f^{-1} are reflections of one another across the line $y = x$, as seen in the figure below.



Summary

- We say a function f has an inverse function g if $g(f(a)) = a$ and $f(g(b)) = b$ for all a and b. We often use the notation f^{-1} for g.
- We say a function is one-to-one if it matches different inputs to different outputs. We can check graphically if a function is one-to-one with the horizontal line test. All one-to-one functions have inverses, and vice versa.
- The graph of f^{-1} is the graph of f reflected across the line $y = x$.

3.3 Average Rate of Change of Functions

Learning Objectives

- Average Rate of Change
    - What is average rate of change?
    - What does it measure?
    - Connecting Average Rate of Change to slope of the line between the two points

3.3.1 Average Rate of Change

Motivating Questions

- What do we mean by the average rate of change of a function on an interval?
- What does the average rate of change of a function measure? How do we interpret its meaning in context?
- How is the average rate of change of a function connected to a line that passes through two points on the curve?

Given a function that models a certain phenomenon, it's natural to ask such questions as "how is the function changing on a given interval" or "on which interval is the function changing more rapidly?" The concept of *average rate of change* enables us to make these questions more mathematically precise. Initially, we will focus on the average rate of change of an object moving along a straight-line path.

First, let's define some notation for the intervals we will be referring to in this section and going forward.

Definition $[a, b]$ represents the values of x such that $a \le x \le b$. We call this the **closed interval from a to b**. (a, b) represents the values of x such that $a < x < b$. We call this the **open interval from a to b**. Notice that the major difference between these intervals is that $x = a$ and $x = b$ are included in the closed interval but not in the open interval.

For a function s that tells the location of a moving object along a straight path at time t, we define the average rate of change of s on the interval $[a, b]$ to be the quantity

$$AV_{[a,b]} = \frac{s(b) - s(a)}{b - a}.$$

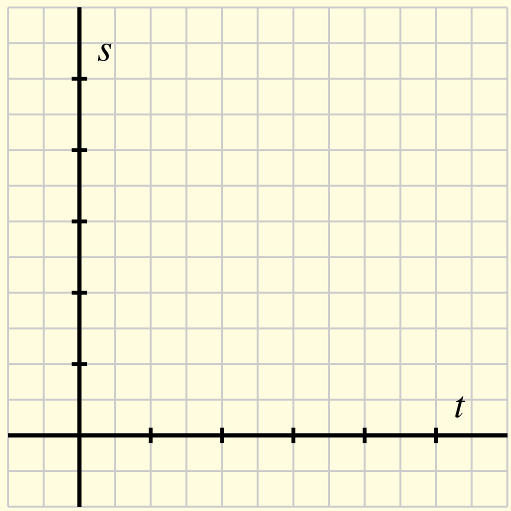
Note particularly that the average rate of change of s on $[a, b]$ is measuring the *change in position* divided by the *change in time*.

Exploration Let the height function for a ball tossed vertically be given by $s(t) = 64 - 16(t - 1)^2$, where t is measured in seconds and s is measured in feet above the ground.

a. Compute the value of $AV_{[1.5,2.5]}$

b. What are the units on the quantity $AV_{[1.5,2.5]}$? What is the meaning of this number in the context of the rising/falling ball?

c. In *Desmos*, plot the function $s(t) = 64 - 16(t - 1)^2$ along with the points $(1.5, s(1.5))$ and $(2.5, s(2.5))$. Make a copy of your plot on the axes below, labeling key points as well as the scale on your axes. What is the domain of the model? The range? Why?



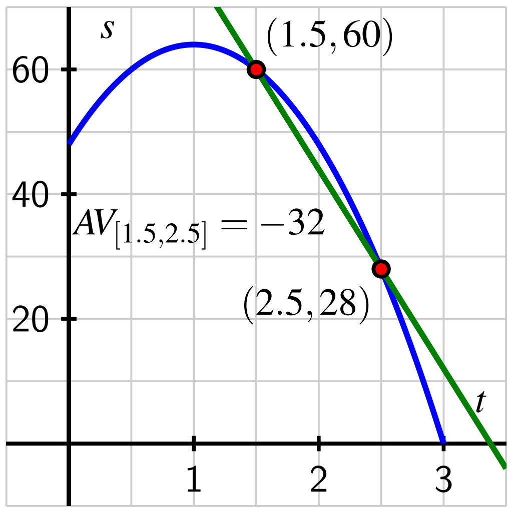
d. Work by hand to find the equation of the line through the points $(1.5, s(1.5))$ and $(2.5, s(2.5))$. Write the line in the form $y = mt + b$ and plot the line in *Desmos*, as well as on the axes above.

e. What is a geometric interpretation of the value $AV_{[1.5,2.5]}$ in light of your work in the preceding questions?

f. How do your answers in the preceding questions change if we instead consider the interval $[0.25, 0.75]$? $[0.5, 1.5]$? $[1, 3]$?

Defining and interpreting the average rate of change of a function

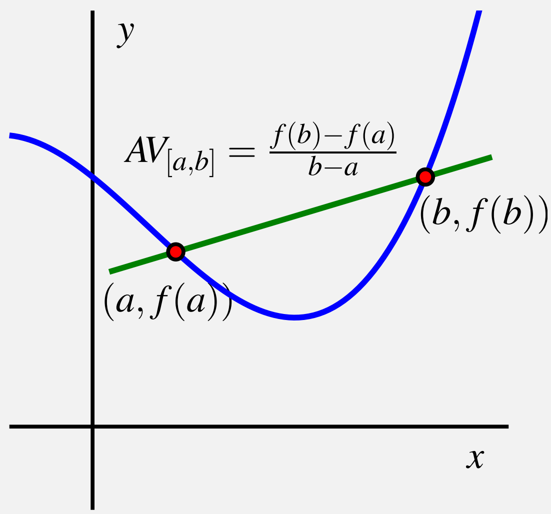
In the context of a function that measures height or position of a moving object at a given time, the meaning of the average rate of change of the function on a given interval is the *average velocity of the moving object* because it is the ratio of *change in position* to *change in time*. For example, in the exploration above, the units on $AV_{[1.5,2.5]} = -32$ are "feet per second" since the units on the numerator are "feet" and on the denominator "seconds". Morever, -32 is numerically the same value as the slope of the line that connects the two corresponding points on the graph of the position function, as seen below. The fact that the average rate of change is negative in this example indicates that the ball is falling.



While the average rate of change of a position function tells us the moving object's average velocity, in other contexts, the average rate of change of a function can be similarly defined and has a related interpretation. We make the following formal definition.

Definition For a function f defined on an interval $[a, b]$, the **average rate of change of** f **on** $[a, b]$ is the quantity

$$AV_{[a,b]} = \frac{f(b) - f(a)}{b - a}.$$



In every situation, the units on the average rate of change help us interpret its meaning, and those units are always "units of output per unit of input." Moreover, the average rate of change of f on $[a, b]$ always corresponds to the slope of the line between the points $(a, f(a))$ and $(b, f(b))$.

Exploration According to the US census, the populations of Kent and Ottawa Counties (Grand Rapids is in Kent, Allenale in Ottawa) from 1960 to 2010 measured in 10-year intervals are given in the following tables.

Kent County Population data

1960	1970	1980	1990	2000	2010
363,187	411,044	444,506	500,631	574,336	602,622

Ottawa County Population data

1960	1970	1980	1990	2000	2010
98,719	128,181	157,174	187,768	238,313	263,801

Let $K(Y)$ represent the population of Kent County in year Y and $W(Y)$ the population of Ottawa County in year Y.

a. Compute $AV_{[1990,2010]}$ for both K and O.

b. What are the units on each of the quantities you computed in (a.)?

c. Write a careful sentence that explains the meaning of the average rate of change of the Ottawa county population on the time interval

[1990, 2010]. Your sentence should begin something like "In an average year between 1990 and 2010, the population of Ottawa County was ..."

d. Which county had a greater average rate of change during the time interval [2000, 2010]? Were there any intervals in which one of the counties had a negative average rate of change?

e. Using the given data, what do you predict will be the population of Ottawa County in 2018? Why?

The average rate of change of a function on an interval gives us an excellent way to describe how the function behaves, on average. For instance, if we compute $AV_{[1970,2000]}$ for Kent County, we find that

$$AV_{[1970,2000]} = \frac{573,336 - 411,044}{30} = 5409.73,$$

which tells us that in an average year from 1970 to 2000, the population of Kent County increased by about 5410 people. Said differently, we could also say that from 1970 to 2000, Kent County was growing at an average rate of 5410 people per year. These ideas also afford the opportunity to make comparisons over time. Since

$$AV_{[1990,2000]} = \frac{573,336 - 500,631}{30} = 7270.5,$$

we can not only say that the county's population increased by about 7270 in an average year between 1990 and 2000, but also that the population was growing faster from 1990 to 2000 than it did from 1970 to 2000.

Finally, we can even use the average rate of change of a function to predict future behavior. Since the population was changing on average by 7270.5 people per year from 1990 to 2000, we can estimate that the population in 2002 is

$$K(2002) \approx K(2000) + 2 \cdot 7270.5 = 573,336 + 14,541 = 587,877.$$

How average rate of change indicates function trends

We have already seen that it is natural to use words such as "increasing" and "decreasing" to describe a function's behavior. For instance, for the tennis ball whose height is modeled by $s(t) = 64 - 16(t-1)^2$, we computed that $AV_{[1.5,2.5]} = -32$, which indicates that on the interval $[1.5, 2.5]$, the tennis ball's height is decreasing at an average rate of 32 feet per second. Similarly, for the population of Kent County, since $AV_{[1990,2000]} = 7270.5$, we know that on the interval $[1990, 2000]$ the population is increasing at an average rate of 7270.5 people per year.

We make the following formal definitions to clarify what it means to say that a function is increasing or decreasing.

Definition Let f be a function defined on an interval (a, b) (that is, on the set of all x for which $a < x < b$). We say that f is **increasing on** (a, b) provided that the function is always rising as we move from left to right. That is, for any x and y in (a, b), if $x < y$, then $f(x) < f(y)$.

Similarly, we say that f is **decreasing on** (a, b) provided that the function is always falling as we move from left to right. That is, for any x and y in (a, b), if $x < y$, then $f(x) > f(y)$.

If we compute the average rate of change of a function on an interval, we can decide if the function is increasing or decreasing *on average* on the interval, but it takes more work[9] to decide if the function is increasing or decreasing *always* on the interval.

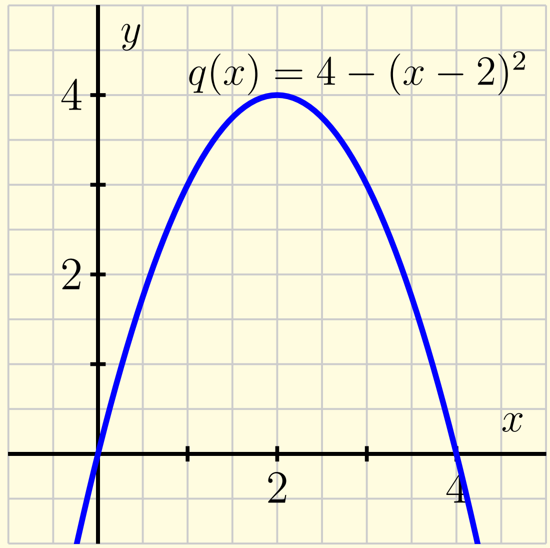
Exploration Let's consider two different functions and see how different computations of their average rate of change tells us about their respective behavior. Plots of q and h are shown below.

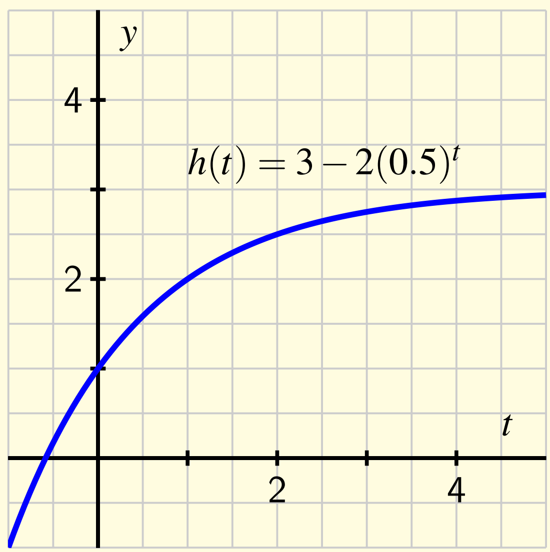
a. Consider the function $q(x) = 4-(x-2)^2$. Compute $AV_{[0,1]}$, $AV_{[1,2]}$, $AV_{[2,3]}$, and $AV_{[3,4]}$. What do your last two computations tell you about the behavior of the function q on $[2, 4]$?

b. Consider the function $h(t) = 3 - 2(0.95)^t$. Compute $AV_{[-1,1]}$, $AV_{[1,3]}$, and $AV_{[3,5]}$. What do your computations tell you about the behavior of the function h on $[-1, 5]$?

c. On the graphs below, plot the line segments whose respective slopes are the average rates of change you computed in (a) and (b).

[9] Calculus offers one way to justify that a function is always increasing or always decreasing on an interval.





d. True or false: Since $AV_{[0,3]} = 1$, the function q is increasing on the interval $(0, 3)$. Justify your decision.

e. Give an example of a function that has the same average rate of change no matter what interval you choose. You can provide your example through a table, a graph, or a formula; regardless of your choice, write a sentence to explain.

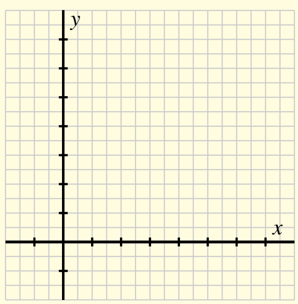
It is helpful be able to connect information about a function's average rate of change and its graph. For instance, if we have determined that $AV_{[-3,2]} = 1.75$ for some function f, this tells us that, on average, the function rises between the points $x = -3$ and $x = 2$ and does so at an average rate of 1.75 vertical units for every horizontal unit. Moreover, we can even determine that the difference between $f(2)$ and $f(-3)$ is

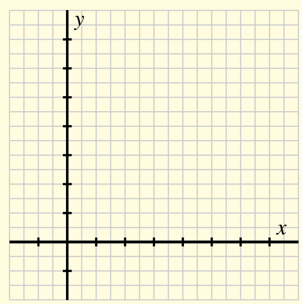
$$f(2) - f(-3) = 1.75 \cdot 5 = 8.75$$

since $\dfrac{f(2) - f(-3)}{2 - (-3)} = 1.75$.

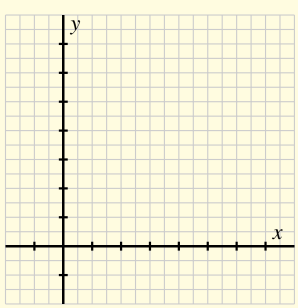
Exploration Sketch at least two different possible graphs that satisfy the criteria for the function stated in each part. Make your graphs as significantly different as you can. If it is impossible for a graph to satisfy the criteria, explain why.

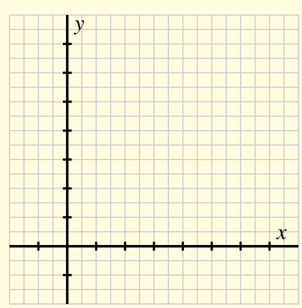
a. f is a function defined on $[-1, 7]$ such that $f(1) = 4$ and $AV_{[1,3]} = -2$.



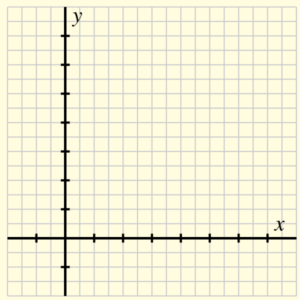


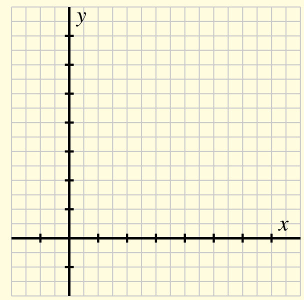
b. g is a function defined on $[-1, 7]$ such that $g(4) = 3$, $AV_{[0,4]}$, and g is not always increasing on $(0, 4)$.





c. h is a function defined on $[-1, 7]$ such that $h(2) = 5$, $h(4) = 3$ and $AV_{[2,4]} = -2$.





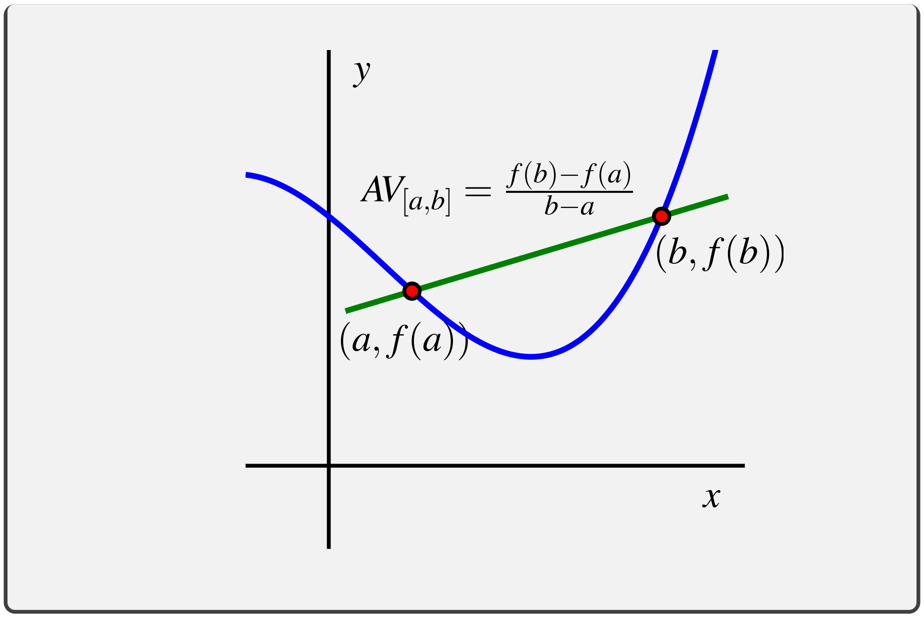
Summary

- For a function f defined on an interval $[a, b]$, the average rate of change of f on $[a, b]$ is the quantity

$$AV_{[a,b]} = \frac{f(b) - f(a)}{b - a}.$$

- The value of $AV_{[a,b]} = \dfrac{f(b) - f(a)}{b - a}$ tells us how much the function rises or falls, on average, for each additional unit we move to the right on the graph. For instance, if $AV_{[3,7]} = 0.75$, this means that for additional 1-unit increase in the value of x on the interval $[3, 7]$, the function increases, on average, by 0.75 units. In applied settings, the units of $AV_{[a,b]}$ are "units of output per unit of input".

- The value of $AV_{[a,b]} = \dfrac{f(b) - f(a)}{b - a}$ is also the slope of the line that passes through the points $(a, f(a))$ and $(b, f(b))$ on the graph of f, as shown in the graph below.


y
$AV_{[a,b]} = \frac{f(b)-f(a)}{b-a}$
$(b, f(b))$
$(a, f(a))$
x

3.4 Exponential Functions Revisited

Learning Objectives

- Exponential Functions
  - Function output changes at a proportional rate
  - Definition as functions of the form $f(x) = ab^x$
  - Finding the formula of an exponential function from a table
  - Increasing (at an increasing rate) vs decreasing (at an increasing rate) exponentials
- Modeling with Exponential Functions Revisited
  - Modeling cooling coffee with $f(x) = ab^x + c$
  - Identifying vertical shifts from $+c$ terms
  - Defining long term (end) behavior, introducing arrows to infinity
  - Discussing the concavity of exponentials
- The Special Number e
  - Varying b in b^t is a horizontal scaling
  - Definition of e
  - Average rate of change of e^t approaches the function value
  - $f(t) = e^t$ is invertible. Name its inverse $f(t) = \ln(t)$

3.4.1 Exponential Functions

Motivating Questions

- What does it mean to say that a function is "exponential"?
- How much data do we need to know in order to determine the formula for an exponential function?
- Are there important trends that all exponential functions exhibit?

Introduction

Linear functions have constant average rate of change and model many important phenomena. In other settings, it is natural for a quantity to change at a rate that is proportional to the amount of the quantity present. For instance, whether you put \$100 or \$100000 or any other amount in a mutual fund, the investment's value changes at a rate proportional the amount present. We often measure that rate in terms of the annual percentage rate of return.

Suppose that a certain mutual fund has a 10% annual return. If we invest \$100, after 1 year we still have the original \$100, plus we gain 10% of \$100, so

$$100 \xrightarrow{\text{year 1}} 100 + 0.1(100) = 1.1(100).$$

If we instead invested \$100000, after 1 year we again have the original \$100000, but now we gain 10% of \$100000, and thus

$$100000 \xrightarrow{\text{year 1}} 100000 + 0.1(100000) = 1.1(100000).$$

We therefore see that regardless of the amount of money originally invested, say P, the amount of money we have after 1 year is $1.1P$.

If we repeat our computations for the second year, we observe that

$$1.1(100) \xrightarrow{\text{year 2}} 1.1(100) + 0.1(1.1(100) = 1.1(1.1(100)) = 1.1^2(100).$$

The ideas are identical with the larger dollar value, so

$$1.1(100000) \xrightarrow{\text{year 2}} 1.1(100000)+0.1(1.1(100000) = 1.1(1.1(100000)) = 1.1^2(100000),$$

and we see that if we invest P dollars, in 2 years our investment will grow to 1.1^2P.

Of course, in 3 years at 10%, the original investment P will have grown to 1.1^3P. Here we see a new kind of pattern developing: annual growth of 10% is leading to *powers* of the base 1.1, where the power to which we raise 1.1 corresponds to the number of years the investment has grown. We often call this phenomenon *exponential growth*.

Exploration Suppose that at age 20 you have \$20000 and you can choose between one of two ways to use the money: you can invest it in a mutual fund that will, on average, earn 8% interest annually, or you can purchase a new automobile that will, on average, depreciate 12% annually. Let's explore how the 20000 changes over time.

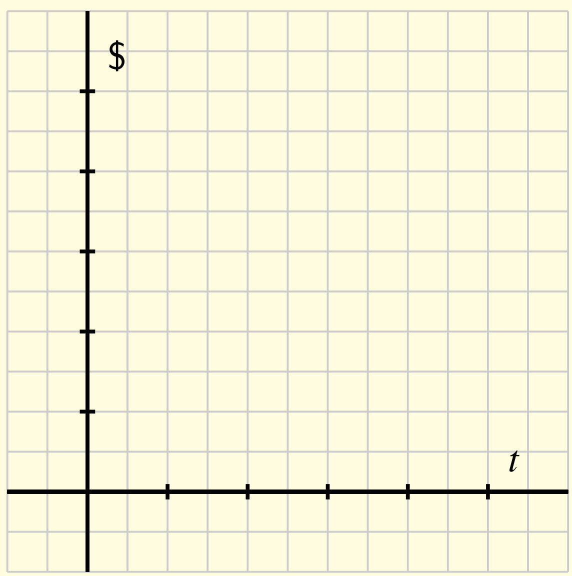
Let $I(t)$ denote the value of the \$20000 after t years if it is invested in the mutual fund, and let $V(t)$ denote the value of the automobile t years after it is purchased.

a. Determine $I(0)$, $I(1)$, $I(2)$, and $I(3)$.

b. Note that if a quantity depreciates 12% annually, after a given year, 88% of the quantity remains. Compute $V(0)$, $V(1)$, $V(2)$, and $V(3)$.

c. Based on the patterns in your computations in (a) and (b), determine formulas for $I(t)$ and $V(t)$.

d. Use *Desmos* to define $I(t)$ and $V(t)$. Plot each function on the interval $0 \leq t \leq 20$ and record your results on the axes below, being sure to label the scale on the axes. What trends do you observe in the graphs? How do $I(20)$ and $V(20)$ compare?



Exponential functions of form $f(t) = ab^t$

In the exploration above, we encountered the functions $I(t)$ and $V(t)$ that had the same basic structure. Each can be written in the form $g(t) = ab^t$ where a and b are positive constants and $b \neq 1$. Based on our earlier work with transformations, we know that the constant a is a vertical scaling factor, and thus the main behavior of the function comes from b^t, which we call an "exponential function".

Definition Let b be a real number such that $b > 0$ and $b \neq 1$. We call the function defined by

$$f(t) = b^t$$

an **exponential function** with **base** b.

For an exponential function $f(t) = b^t$, we note that $f(0) = b^0 = 1$, so an exponential function of this form always passes through $(0, 1)$. In addition, because a positive number raised to any power is always positive (for instance, $2^{10} = 1096$ and $2^{-10} = \frac{1}{2^{10}} = \frac{1}{2096}$), the output of an exponential function is also always positive. In particular, $f(t) = b^t$ is never zero and thus has no x-intercepts.

Because we will be frequently interested in functions such as $I(t)$ and $V(t)$ with the form ab^t, we will also refer to functions of this form as "exponential", understanding that technically these are vertical stretches of exponential functions according to our definition of exponential function. In the exploration above, we found that $I(t) = 20000(1.08)^t$ and $V(t) = 20000(0.88)^t$. It is natural to call 1.08 the "growth factor" of I and similarly 0.88 the growth factor of V. In addition, we note that these values stem from the actual growth rates: 0.08 for I and -0.12 for V, the latter being negative because value is depreciating. In general, for a function of form $f(t) = ab^t$, we call b the **growth factor**. Moreover, if $b = 1 + r$, we call r the **growth rate**. Whenever $b > 1$, we often say that the function f is exhibiting "exponential growth", wherease if $0 < b < 1$, we say f exhibits "exponential decay".

Exploration Suppose that at age 20 you have \$20000 and you can choose between one of two ways to use the money: you can invest it in a mutual fund that will, on average, earn 8% interest annually, or you can purchase a new automobile that will, on average, depreciate 12% annually. Let's explore how the 20000 changes over time.
Let $I(t)$ denote the value of the \$20000 after t years if it is invested in the mutual fund, and let $V(t)$ denote the value of the automobile t years after it is purchased.

a. What is the domain of $g(t) = ab^t$?

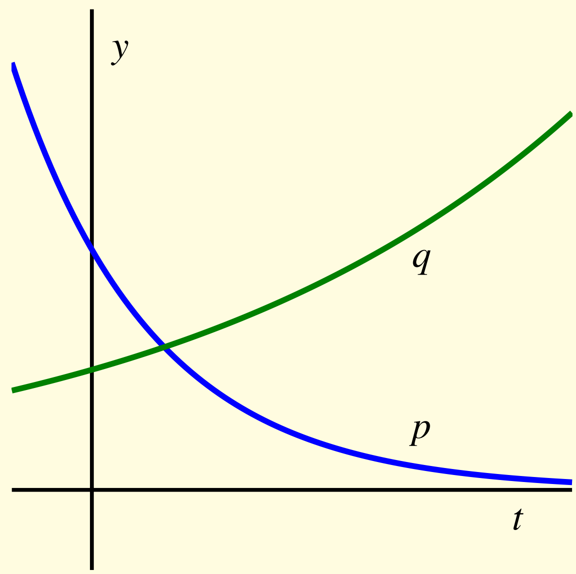
b. What is the range of $g(t) = ab^t$?

c. What is the y-intercept of $g(t) = ab^t$?

d. How does changing the value of b affect the shape and behavior of the graph of $g(t) = ab^t$? Write several sentences to explain.

e. For what values of the growth factor b is the corresponding growth rate positive? For which b-values is the growth rate negative?

f. Consider the graphs of the exponential functions p and q provided in teh figure below. If $p(t) = ab^t$ and $q(t) = cd^t$, what can you say about the values a, b, c, and d (beyond the fact that all are positive and $b \neq 1$ and $d \neq 1$)? For instance, can you say a certain value is larger than another? Or that one of the values is less than 1?



Determining formulas for exponential functions

To better understand the roles that a and b play in an exponential function, let's compare exponential and linear functions. In the tables below, we see output for two different functions r and s that correspond to equally spaced input.

t	$r(t)$
0	12
3	10
6	8
9	6

t	$s(t)$
0	12
3	9
6	6.75
9	5.0625

In the leftside table for $r(t)$, we see a function that exhibits constant average rate of change since the change in output is always $\triangle r = -2$ for any change in input of $\triangle t = 3$. Said differently, r is a linear function with slope $m = -\frac{2}{3}$. Since its y-intercept is $(0, 12)$, the function's formula is $y = r(t) = 12 - \frac{2}{3}t$.

In contrast, the function s given by rightside table for $s(t)$ does not exhibit constant average rate of change. Instead, another pattern is present. Observe that if we consider the ratios of consecutive outputs in the table, we see that

$$\frac{9}{12} = \frac{3}{4}, \frac{6.75}{9} = 0.75 = \frac{3}{4}, \text{ and } \frac{5.0625}{6.75} = 0.75 = \frac{3}{4}.$$

So, where the *differences* in the outputs in the table for $r(t)$ are constant, the *ratios* in the outputs in the table for $s(t)$ are constant. The latter is a hallmark of exponential functions and may be used to help us determine the formula of a function for which we have certain information.

If we know that a certain function is linear, it suffices to know two points that lie on the line to determine the function's formula. It turns out that exponential functions are similar: knowing two points on the graph of a function known to be exponential is enough information to determine the function's formula. In the following example, we show how knowing two values of an exponential function enables us to find both a and b exactly.

Example 30. *Suppose that p is an exponential function and we know that $p(2) = 11$ and $p(5) = 18$. Determine the exact values of a and b for which $p(t) = ab^t$.* **Explanation** Since we know that $p(t) = ab^t$, the two data points give us two equations in the unknowns a and b. First, using $t = 2$,

$$ab^2 = 11, \tag{6}$$

and using $t = 5$ we also have

$$ab^5 = 18. \tag{7}$$

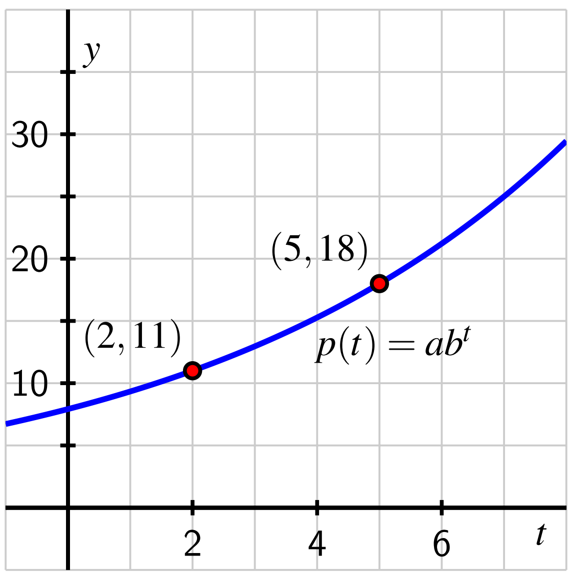
Because we know that the quotient of outputs of an exponential function corresponding to equally-spaced inputs must be constant, we thus naturally consider the quotient $\frac{18}{11}$. Using $ab^2 = 11$ and $ab^5 = 18$, it follows that

$$\frac{18}{11} = \frac{ab^5}{ab^2}.$$

Simplifying the fraction on the right, we see that $\frac{18}{11} = b^3$. Solving for b, we find that $b = \sqrt[3]{\frac{18}{11}}$ is the exact value of b. Substituting this value for b in $ab^2 = 11$, it then follows that $a\left(\sqrt[3]{\frac{18}{11}}\right)^2 = 11$, so $a = \frac{11}{\left(\frac{18}{11}\right)^{2/3}}$. Therefore,

$$p(t) = \frac{11}{\left(\frac{18}{11}\right)^{2/3}} \left(\sqrt[3]{\frac{18}{11}}\right)^t \approx 7.9215 \cdot 1.1784^t,$$

and a plot of $y = p(t)$ confirms that the function indeed passes through $(2, 11)$ and $(5, 18)$ as shown in the figure below.



Exploration The value of an automobile is depreciating. When the car is 3 years old, its value is \$12500; when the car is 7 years old, its value is \$6500.

a. The value of an automobile is depreciating. When the car is 3 years old, its value is \$12500; when the car is 7 years old, its value is \$6500.

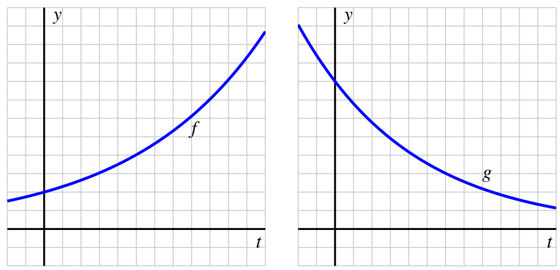
b. The value of an automobile is depreciating. When the car is 3 years old, its value is \$12500; when the car is 7 years old, its value is \$6500.

c. Suppose instead that the car's value is modeled by a linear function

L and satisfies the values stated at the outset of this activity. Find a formula for $L(t)$ and determine both the purchase value of the car and when the car will be worth \$1000.

d. Which model do you think is more realistic? Why?

Recall that a function is increasing on an interval if its value always increasing as we move from left to right. Similarly, a function is decreasing on an interval provided that its value always decreases as we move from left to right.



If we consider an exponential function f with a growth factor $b > 1$, such as the function pictured in the left-hand graph above, then the function is always increasing because higher powers of b are greater than lesser powers (for example, $(1.2)^3 > (1.2)^2$). On the other hand, if $0 < b < 1$, then the exponential function will be decreasing because higher powers of positive numbers less than 1 get smaller (e.g., $(0.9)^3 < (0.9)^2$), as seen for the exponential function in the right-hand graph above.

An additional trend is apparent in the graphs in above. Each graph bends upward and is therefore concave up. We can better understand why this is so by considering the average rate of change of both f and g on consecutive intervals of the same width. We choose adjacent intervals of length 1 and note particularly that as we compute the average rate of change of each function on such intervals,

$$AV_{[t,t+1]} = \frac{f(t+1) - f(t)}{t+1-1} = f(t+1) - f(t).$$

Thus, these average rates of change are also measuring the total change in the function across an interval that is 1-unit wide. We now assume that $f(t) = 2(1.25)^t$ and $g(t) = 8(0.75)^t$ and compute the rate of change of each function on several consecutive intervals.

The average rate of change of $f(t) = 2(1.25)^t$

t	$f(t)$	$AV_{[t,t+1]}$
0	2	0.5
1	2.5	0.625
2	3.125	0.78215
3	3.90625	0.97656

The average rate of change of $g(t) = 8(0.75)^t$

t	$g(t)$	$AV_{[t,t+1]}$
0	8	-2
1	6	-1.5
2	4.5	-1.125
3	3.375	-0.84375

From the data in the first table about $f(t)$ we see that the average rate of change is increasing as we increase the value of t. We naturally say that f appears to be "increasing at an increasing rate". For the function g, we first notice that its average rate of change is always negative, but also that the average rate of change gets *less negative* as we increase the value of t. Said differently, the average rate of change of g is also increasing as we increase the value of t. Since g is always decreasing but its average rate of change is increasing, we say that g appears to be "decreasing at an increasing rate". These trends hold for exponential functions generally according to the conditions given below. It takes calculus to justify this claim fully and rigorously.

Trends in exponential function behavior.

For an exponential function of the form $f(t) = ab^t$ where a and b are both positive with $b \neq 1$,

- if $b > 1$, then f is always increasing and always increases at an increasing rate;
- if $0 < b < 1$, then f is always decreasing and always decreases at an increasing rate.

Observe how a function's average rate of change helps us classify the function's behavior on an interval: whether the average rate of change is always positive or always negative on the interval enables us to say if the function is always increasing or always decreasing, and then how the average rate of change itself changes enables us to potentially say *how* the function is increasing or decreasing through phrases such as "decreasing at an increasing rate".

Exploration
For each of the following prompts, give an example of a function that satisfies the stated characteristics by both providing a formula and sketching a graph.

a. A function p that is always decreasing and decreases at a constant rate.

b. A function q that is always increasing and increases at an increasing rate.

c. A function r that is always increasing for $t < 2$, always decreasing for $t > 2$, and is always changing at a decreasing rate.

d. A function s that is always increasing and increases at a decreasing rate. (Hint: to find a formula, think about how you might use a transformation of a familiar function.)

e. A function u that is always decreasing and decreases at a decreasing rate.

Summary

- We say that a function is exponential whenever its algebraic form is $f(t) = ab^t$ for some positive constants a and b where $b \neq 1$. (Technically, the formal definition of an exponential function is one of form $f(t) = b^t$, but in our everyday usage of the term "exponential" we include vertical stretches of these functions and thus allow a to be any positive constant, not just $a = 1$.)

- To determine the formula for an exponential function of form $f(t) = ab^t$, we need to know two pieces of information. Typically this information is presented in one of two ways.

  – If we know the amount, a, of a quantity at time $t = 0$ and the rate, r, at which the quantity grows or decays per unit time, then it follows $f(t) = a(1 + r)^t$. In this setting, r is often given as a percentage that we convert to a decimal (e.g., if the quantity grows at a rate of 7% per year, we set $r = 0.07$, so $b = 1.07$).

  – If we know any two points on the exponential function's graph, then we can set up a system of two equations in two unknowns and solve for both a and b exactly. In this situation, it is useful to consider the quotient of the two known outputs, as demonstrated in Example 30.

- Exponential functions of the form $f(t) = ab^t$ (where a and b are both positive and $b \neq 1$) exhibit the following important characteristics:
  - The domain of any exponential function is the set of all real numbers and the range of any exponential function is the set of all positive real numbers.
  - The y-intercept of the exponential function $f(t) = ab^t$ is $(0, a)$ and the function has no x-intercepts.
  - If $b > 1$, then the exponential function is always increasing and always increases at an increasing rate. If $0 < b < 1$, then the exponential function is always decreasing and always decreases at an increasing rate.

3.4.2 Modeling with Exponential Functions Revisited

Motivating Questions

- What can we say about the behavior of an exponential function as the input gets larger and larger?
- How do vertical stretches and shifts of an exponontial function affect its behavior?
- Why is the temperature of a cooling or warming object modeled by a function of the form $F(t) = ab^t + c$?

Introduction

If a quantity changes so that its growth or decay occurs at a constant percentage rate with respect to time, the function is exponential. This is because if the growth or decay rate is r, the total amount of the quantity at time t is given by $A(t) = a(1 + r)^t$, where a is the amount present at time $t = 0$. Many different natural quantities change according to exponential models: money growth through compounding interest, the growth of a population of cells, and the decay of radioactive elements

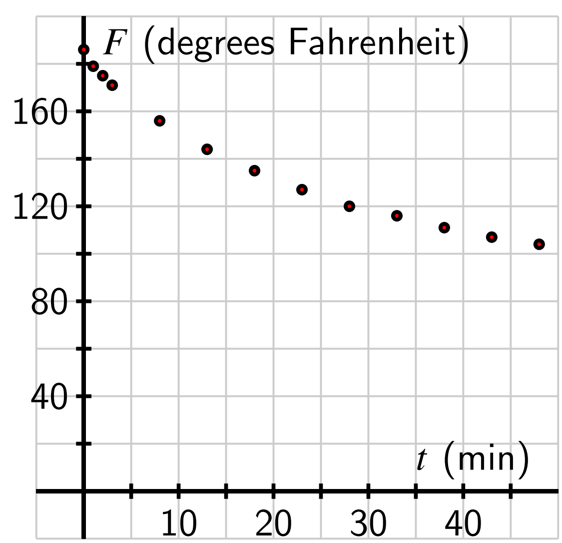
A related situation arises when an object's temperature changes in response to its surroundings. For instance, if we have a cup of coffee at an initial temperature of 186° Fahrenheit and the cup is placed in a room where the surrounding temperature is 71°, our intuition and experience tell us that over time the coffee will cool and eventually tend to the 71° temperature of the surroundings. From an experiment[10] with an actual temperature probe, we have the data in the table below.

[10] See experiment at http://gvsu.edu/s/0SB

Data for cooling coffee
(measured in degrees Fahrenheit at the time t in minutes)

t	$F(t)$
0	186
1	179
2	175
3	171
8	156
13	144
18	135
23	127
28	120
33	116
38	111
43	107
48	104

Here is a graph of these points.



In one sense, the data looks exponential: the points appear to lie on a curve that is always decreasing and decreasing at an increasing rate. However, we know that the function can't have the form $f(t) = ab^t$ because such a function's range is the set of all positive real numbers, and it's impossible for the coffee's temperature to fall below room temperature (71°). It is natural to wonder if a function of the form $g(t) = ab^t + c$ will work. Thus, in order to find a function that fits the data in a situation such as this, we begin by investigating and understanding the roles of a, b, and c in the behavior of $g(t) = ab^t + c$.

Exploration In *Desmos*[a] , define $g(t) = ab^t + c$ and accept the prompt for sliders for both a and b. Edit the sliders so that a has values from $a = 5$ to $a = 50$, b has values from $b = 0.7$ to $b = 1.3$, and c has values from $c = -5$ to $b = 5$ (also with a step-size of 0.01). In addition, in *Desmos* let $P = (0, g(0))$ and check the box to show the label. Finally, zoom out so that the window shows an interval of t-values from $-30 \leq t \leq 30$

a. Set $b = 1.1$ and explore the effects of changing the values of a and c. Write several sentences to summarize your observations.

b. Follow the directions for (a) again, this time with $b = 0.9$

c. Set $a = 5$ and $c = 4$. Explore the effects of changing the value of b; be sure to include values of b both less than and greater than 1. Write several sentences to summarize your observations.

d. When $0 < b < 1$, what happens to the graph of g when we consider positive t-values that get larger and larger?

[a] See *Desmos* at desmos.com

Long-term behavior of exponential functions

We have already established that any exponential function of the form $f(t) = ab^t$ where a and b are positive real numbers with $b \neq 1$ is always concave up and is either always increasing or always decreasing. We next introduce precise language to describe the behavior of an exponential function's value as t gets bigger and bigger. To start, let's consider the two basic exponential functions $p(t) = 2^t$ and $q(t) = \left(\frac{1}{2}\right)^t$ and their respective values at $t = 10$, $t = 20$, and $t = 30$, as displayed below.

t	$p(t)$
10	$2^10 = 1026$
20	$2^20 = 1048576$
30	$2^30 = 1073741824$

t	$q(t)$
10	$\left(\frac{1}{2}\right)^1 0 = \frac{1}{1026} \approx 0.00097656$
10	$\left(\frac{1}{2}\right)^2 0 = \frac{1}{1048576} \approx 0.00000095367$
10	$\left(\frac{1}{2}\right)^3 0 = \frac{1}{1073741824} \approx 0.00000000093192$

For the increasing function $p(t) = 2^t$, we see that the output of the function gets very large very quickly. In addition, there is no upper bound to how large the function can be. Indeed, we can make the value of $p(t)$ as large as we'd like by taking t sufficiently big. We thus say that as t increases, $p(t)$ **increases without bound**.

For the decreasing function $q(t) = \left(\frac{1}{2}\right)^t$, we see that the output $q(t)$ is always positive but getting closer and closer to 0. Indeed, becasue we can make 2^t as large as we like, it follows that we can make its reciprocal $\frac{1}{2^t} = \left(\frac{1}{2}\right)^t$ as small as we'd like. We thus say that as t increases, $q(t)$ **approaches** 0.

To represent these two common phenomena with exponential functions the value increasing without bound or the value approaching 0, we will use shorthand notation. First, it is natural to write "$q(t) \to 0$" as t increases without bound. Moreover, since we have the notion of the infinite to represent quantities without bound, we use the symbol for infinity (∞) and write "$p(t) \to \infty$" as t increases without bound in order to indicate that $p(t)$ increases without bound.

In the exploration above, we saw how the value of b affects the steepness of the graph of $f(t) = ab^t$, as well as how all graphs with $b > 1$ have the similar increasing behavior, and all graphs with $0 < b < 1$ have similar decreasing behavior. For instance, by taking t sufficiently large, we can make $(1.01)^t$ as large as we want; it just takes much larger t to make $(1.01)^t$ big in comparison to 2^t. In the same way, we can make $(0.99)^t$ as close to 0 as we wish by taking t sufficiently big, even though it takes longer for $(0.99)^t$ to get close to 0 in comparison to $\left(\frac{1}{2}\right)^t$. For an arbitrary choice of b, we can say the following.

Long-term behavior of exponential functions.

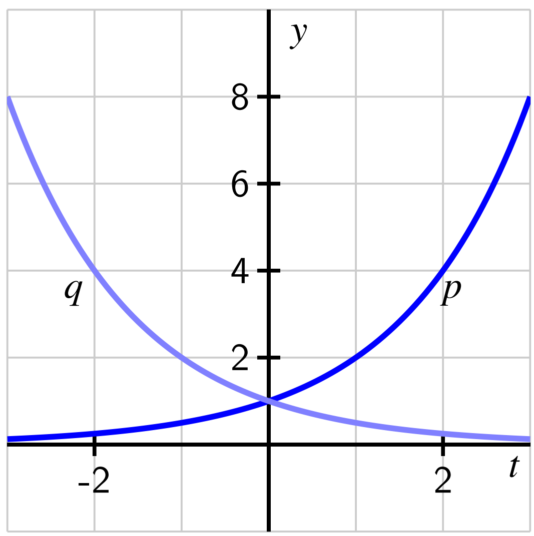
Let $f(t) = b^t$ with $b > 0$ and $b \neq 1$.

- If $0 < b < 1$, then $b^t \to 0$ as $t \to \infty$. We read this notation as "b^t tends to 0 as t increases without bound.
- If $b > 1$, then $b^t \to \infty$ as $t \to \infty$. We read this notation as "b^t increases without bound as t increases without bound.

In addition, we make a key observation about the use of exponents. For the function $q(t) = \left(\frac{1}{2}\right)^t$, there are three equivalent ways we may write the function:

$$\left(\frac{1}{2}\right)^t = \frac{1}{2^t} = 2^{-t}.$$

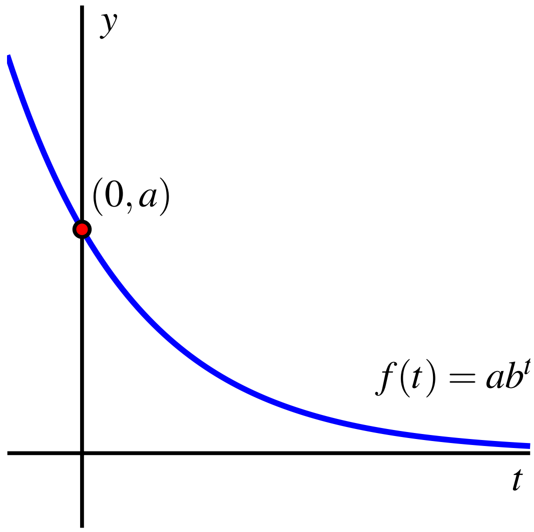
In our work with transformations involving horizontal scaling in Exercise 2.4.5.3, we saw that the graph of $y = h(-t)$ is the reflection of the graph of $y = h(t)$ across the y-axis. Therefore, we can say that the graphs of $p(t) = 2^t$ and $q(t) = \left(\frac{1}{2}\right)^t = 2^{-t}$ are reflections of one another in the y-axis since $p(-t) = 2^{-t} = q(t)$. We see this fact verified in the graph below.

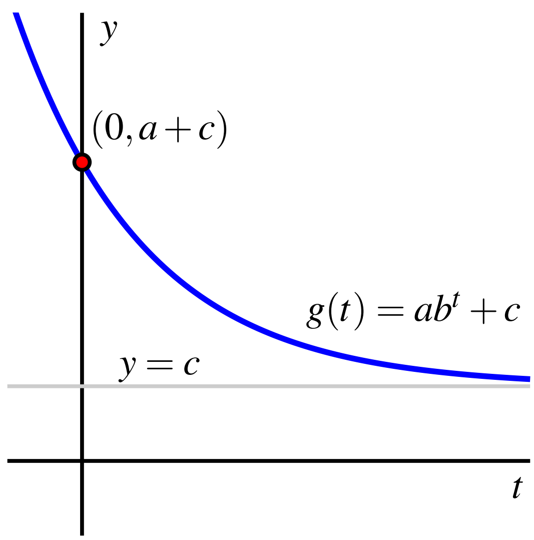


Similar observations hold for the relationship between the graphs of b^t and $\frac{1}{b^t} = b^{-t}$ for any positive $b \neq 1$.

The role of c in $g(t) = ab^t + c$

The function $g(t) = ab^t + c$ is a vertical translation of the function $f(t) = ab^t$. We now have extensive understanding of the behavior of $f(t)$ and how that behavior depends on a and b. Since a vertical translation by c does not change the shape of any graph, we expect that g will exhibit very similar behavior to f. Indeed, we can compare the two functions' graphs as shown in the graphs belowA and then make the following general observations.





Behavior of vertically shifted exponential functions. Let $g(t) = ab^t + c$ with $a > 0$, $b > 0$ and $b \neq 1$, and c any real number.

- If $0 < b < 1$, then $g(t) = ab^t + c \to c$ as $t \to \infty$. The function g is always decreasing, always concave up, and has y-intercept $(0, a+c)$. The range of the function is all real numbers greater than c.

- If $b > 1$, then $g(t) = ab^t + c \to \infty$ as $t \to \infty$. The function g is always increasing, always concave up, and has y-intercept $(0, a+c)$. The range of the function is all real numbers greater than c.

It is also possible to have $a < 0$. In this situation, because $g(t) = ab^t$ is both a reflection of $f(t) = b^t$ across the x-axis and a vertical stretch by $|a|$, the function g is always concave down. If $0 < b < 1$ so that f is always decreasing, then g is always increasing; if instead $b > 1$ so f is increasing, then g is decreasing. Moreover, instead of the range of the function g having a lower bound as when $a > 0$, in this setting the range of g has an upper bound. These ideas are explored further below.

It's an important skill to be able to look at an exponential function of the form $g(t) = ab^t + c$ and form an accurate mental picture of the graph's main features in light of the values of a, b, and c.

Exploration For each of the following functions, *without* using graphing technology, determine whether the function is

i. always increasing or always decreasing;

ii. always concave up or always concave down; and

iii. increasing without bound, decreasing without bound, or increasing/decreasing toward a finite value.

In addition, state the y-intercept and the range of the function. For each function, write a sentence that explains your thinking and sketch a rough graph of how the function appears.

a. $p(t) = 4372(1.000235)^t + 92856$

b. $q(t) = 27931(0.97231)^t + 549786$

c. $r(t) = -17398(0.85234)^t$

d. $s(t) = -17398(0.85234)^t + 19411$

e. $u(t) = -7522(1.03817)^t$

f. $v(t) = -7522(1.03817)^t + 6731$

Modeling temperature data

Newton's Law of Cooling states that the rate that an object warms or cools occurs in direct proportion to the difference between its own temperature and the temperature of its surroundings. If we return to the coffee temperature data and recall that the room temperature in that experiment was 71°, we can see how to use a transformed exponential function to model the data. In the table below, we add a row of information to the table where we compute $F(t) - 71$ to subtract the room temperature from each reading.

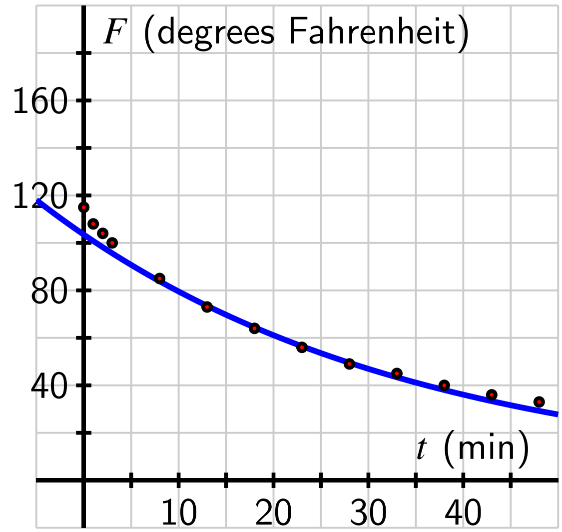
Data for cooling coffee
(measured in degrees Fahrenheit at the time t in minutes)

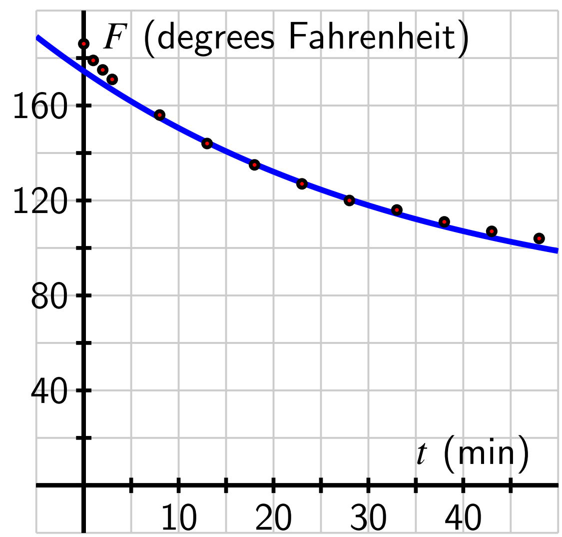
t	$F(t)$	$f(t) = F(t) - 71$
0	186	115
1	179	108
2	175	104
3	171	100
8	156	85
13	144	73
18	135	64
23	127	56
28	120	49
33	116	45
38	111	40
43	107	36
48	104	33

The data in the last row of the table appears exponential, and if we test the data by computing the quotients of output values that correspond to equally-spaced input, we see a nearly constant ratio. In particular,

$$\frac{73}{85} \approx 0.86,\ \frac{64}{73} \approx 0.88,\ \frac{56}{64} \approx 0.88,\ \frac{49}{56} \approx 0.88,\ \frac{45}{49} \approx 0.92, \text{and} \frac{40}{45} \approx 0.89.$$

Of course there is some measurement error in the data (plus it is only recorded to accuracy of whole degrees), so these computations provide convincing evidence that the underlying function is exponential. In addition, we expect that if the data continued in the last of the table, the values would approach 0 because $F(t)$ will approach 71.





If we choose two of the data points, say $(18, 64)$ and $(23, 56)$, and assume that $f(t) = ab^t$, we can determine the values of a and b. Doing so, it turns out that $a \approx 103.503$ and $b \approx 0.974$, so $f(t) = 103.503(0.974)^t$. Since $f(t) = F(t) - 71$, we see that $F(t) = f(t) + 71$, so $F(t) = 103.503(0.974)^t + 71$. Plotting f against the shifted data and F along with the original data in the graphs above, we see that the curves go exactly through the points where $t = 18$ and $t = 23$ as expected, but also that the function provides a reasonable model for the observed behavior at any time t. If our data was even more accurate, we would expect that the curve's fit would be even better.

Our preceding work with the coffee data can be done similarly with data for any cooling or warming object whose temperature initially differs from its surroundings. Indeed, it is possible to show that Newton's Law of Cooling implies that the object's temperature is given by a function of the form $F(t) = ab^t + c$.

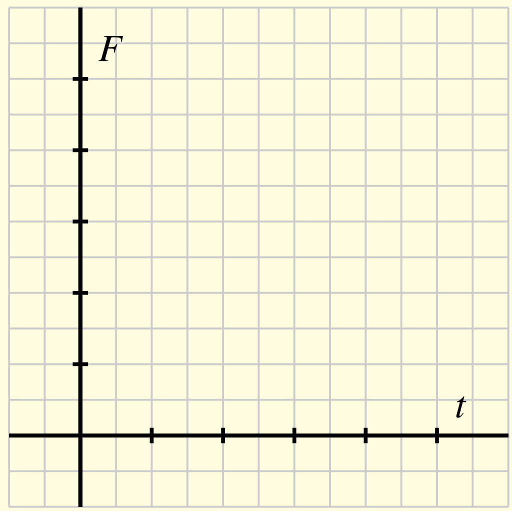
Exploration A can of soda (at room temperature) is placed in a refrigerator at time $t = 0$ (in minutes) and its temperature, $F(t)$, in degrees Fahrenheit, is computed at regular intervals. Based on the data, a model is formulated for the object's temperature, given by

$$F(t) = 42 + 30(0.95)^t.$$

a. Consider the simpler (parent) function $p(t) = (0.95)^t$. How do you expect the graph of this function to appear? How will it behave as time increases? Without using graphing technology, sketch a rough graph of p and write a sentence of explanation.

b. For the slightly more complicated function $r(t) = 30(0.95)^t$, how do you expect this function to look in comparison to p? What is the long-range behavior of this function as t increases? Without using graphing technology, sketch a rough graph of r and write a sentence of explanation.

c. Finally, how do you expect the graph of $F(t) = 42 + 30(0.95)^t$ to appear? Why? First sketch a rough graph without graphing technology, and then use technology to check your thinking and report an accurate, labeled graph on the axes provided.



d. What is the temperature of the refrigerator? What is the room temperature of the surroundings outside the refrigerator? Why?

e. Determine the average rate of change of F on the intervals $[10, 20]$, $[20, 30]$, and $[30, 40]$. Write at least two careful sentences that explain the meaning of the values you found, including units, and discuss any overall trend in how the average rate of change is changing.

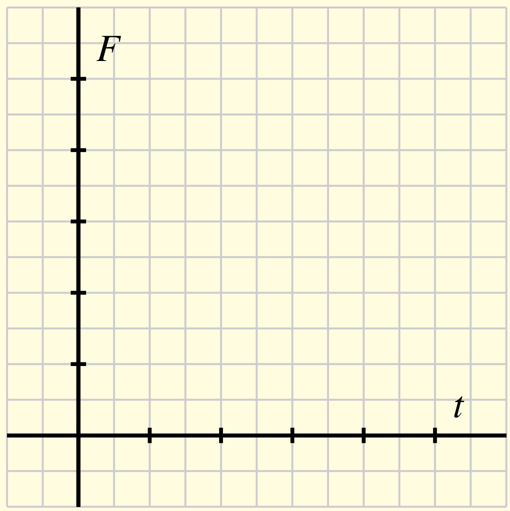
Exploration A potato initially at room temperature (68°) is placed in an oven (at 350°) at time $t = 0$. It is known that the potato's temperature at time t is given by the function $F(t) = a - b(0.98)^t$ for some positive constants a and b, where F is measured in degrees Fahrenheit and t is time in minutes.

a. What is the numerical value of $F(0)$? What does this tell you about the value of $a - b$?

b. Based on the context of the problem, what should be the long-range behavior of the function $F(t)$? Use this fact along with the behavior of $(0.98)^t$ to determine the value of a. Write a sentence to explain your thinking.

c. What is the value of b? Why?

d. Check your work above by plotting the function F using graphing technology in an appropriate window. Record your results on the axes provided, labeling the scale on the axes. Then, use the graph to estimate the time at which the potato's temperature reaches 325 degrees.



e. How can we view the function $F(t) = a - b(0.98)^t$ as a transformation of the parent function $f(t) = (0.98)^t$? Explain.

Summary

- For an exponential function of the form $f(t) = b^t$, the function either approaches zero or grows without bound as the input gets larger and larger. In particular, if $0 < b < 1$, then $f(t) = b^t \to 0$ as $t \to \infty$, while if $b > 1$, then $f(t) = b^t \to \infty$ as $t \to \infty$. Scaling f by a positive value a (that is, the transformed function ab^t) does not affect the long-range behavior: whether the function tends to 0 or increases without bound depends solely on whether b is less than or greater than 1.

- The function $f(t) = b^t$ passes through $(0, 1)$, is always concave up, is either always increasing or always decreasing, and its range is the set of all positive real numbers. Among these properties, a vertical stretch by a positive value a only affects the y-intercept, which is instead $(0, a)$. If we include a vertical shift and write $g(t) = ab^t + c$, the biggest changes is that the range of g is the set of all real numbers greater than c. In addition, the y-intercept of g is $(0, a + c)$.

- In the situation where $a < 0$, several other changes are induced. Here, because $g(t) = ab^t$ is both a reflection of $f(t) = b^t$ across the x-axis and a vertical stretch by $|a|$, the function g is now always concave down. If $0 < b < 1$ so that f is always decreasing, then g (the reflected function) is now always increasing; if instead $b > 1$ so f is increasing, then g is decreasing. Finally, if $a < 0$, then the range of $g(t) = ab^t + c$ is the set of all real numbers c.

- An exponential function can be thought of as a function that changes at a rate proportional to itself, like how money grows with compound interest or the amount of a radioactive quantity decays. Newton's Law of Cooling says that the rate of change of an object's temperature is proportional to the *difference* between its own temperature and the temperature of its surroundings. This leads to the function that measures the difference between the object's temperature and room temperature being exponential, and hence the object's temperature itself is a vertically-shifted exponential function of the form $F(t) = ab^t + c$.

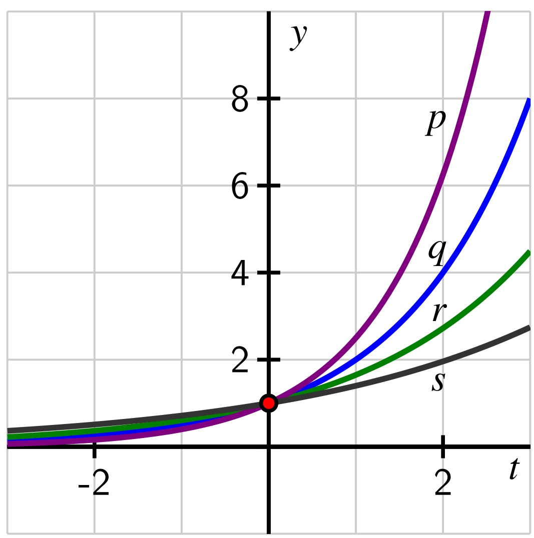
3.4.3 The Special Number e

Motivating Questions

- Why can every exponential function of form $f(t) = b^t$ (where $b > 0$ and $b \neq 1$) be thought of as a horizontal scaling of a single special exponential function?
- What is the natural base e and what makes this number special?

Introduction

We have observed that the behavior of functions of the form $f(t) = b^t$ is very consistent, where the only major differences depend on whether $0 < b < 1$ or $b > 1$. Indeed, if we stipulate that $b > 1$, the graphs of functions with different bases b look nearly identical, as seen in the plots of p, q, r, and s below.



Because the point $(0, 1)$ lies on the graph of each of the four functions in Figure **??**, the functions cannot be vertical scalings of one another. However, it is possible that the functions are *horizontal* scalings of one another. This leads us to a natural question: might it be possible to find a single exponential function with a special base, say e, for which every other exponential function $f(t) = b^t$ can be expressed as a horizontal scaling of $E(t) = e^t$?

Exploration Open a new *Desmos* worksheet and define the following functions: $f(t) = 2^t$, $g(t) = 3^t$, $h(t) = \left(\dfrac{1}{3}\right)^t$, and $p(t) = f(kt)$. After you define p, accept the slider for k, and set the range of the slider to be $-2 \leq k \leq 2$.

a. By experimenting with the value of k, find a value of k so that the graph of $p(t) = f(kt) = 2^{kt}$ appears to align with the graph of $g(t) = 3^t$. What is the value of k?

b. Similarly, experiment to find a value of k so that the graph of $p(t) = f(kt) = 2^{kt}$ appears to align with the graph of $h(t) = (\frac{1}{3})^t$. What is the value of k?

c. For the value of k you determined in (a), compute 2^k. What do you observe?

d. For the value of k you determined in (b), compute 2^k. What do you observe?

e. Given any exponential function of the form b^t, do you think it's possible to find a value of k to that $p(t) = f(kt) = 2^{kt}$ is the same function as b^t? Why or why not?

The natural base e

In the exploration above, we found that it appears possible to find a value of k so that given any base b, we can write the function b^t as the horizontal scaling of 2^t given by

$$b^t = 2^{kt}.$$

It's also apparent that there's nothing particularly special about "2": we could similarly write any function b^t as a horizontal scaling of 3^t or 4^t, albeit with a different scaling factor k for each. Thus, we might also ask: is there a *best* possible single base to use?

Through the central topic of the *rate of change* of a function, calculus helps us decide which base is best to use to represent all exponential functions. While we study *average* rate of change extensively in this course, in calculus there is more emphasis on the *instantaneous* rate of change. In that context, a natural question arises: is there a nonzero function that grows in such a way that its *height* is exactly how *fast* its height is increasing?

Amazingly, it turns out that the answer to this questions is "yes," and the function with this property is **the exponential function with the natural base**, denoted e^t. The number e (named in homage to the great Swiss mathematician

Leonard Euler (1707-1783)) is complicated to define. Like π, e is an irrational number that cannot be represented exactly by a ratio of integers and whose decimal expansion never repeats. Advanced mathematics is needed in order to make the following formal definition of e.

Definition [The natural base, e] The number e is the infinite sum[a]

$$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots$$

From this, $e \approx 2.718281828$.

[a] Infinite sums are usually studied in second semester calculus.

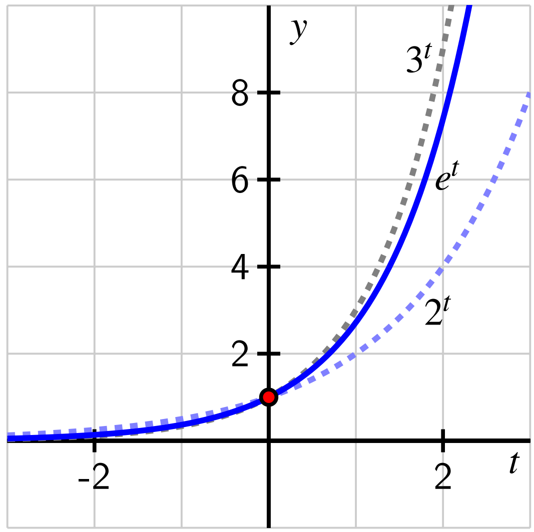
For instance, $1 + \frac{1}{1} + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120} = \frac{163}{60} \approx 2.7167$ is an approximation of e generated by taking the first 6 terms in the infinite sum that defines it. Every computational device knows the number e and we will normally work with this number by using technology appropriately.

Initially, it's important to note that $2 < e < 3$, and thus we expect the function e^t to lie between 2^t and 3^t.

t	-2	-1	0	1	2
2^t	0.25	0.5	1	2	4

t	-2	-1	0	1	2
e^t	0.135	0.368	1	2.718	7.389

t	-2	-1	0	1	2
3^t	0.111	0.333	1	3	9



If we compare the graphs and some selected outputs of each function, as in Table **??** and Figure **??**, we see that the function e^t satisfies the inequality

$$2^t < e^t < 3^t$$

for all positive values of t. When t is negative, we can view the values of each function as being reciprocals of powers of 2, e, and 3. For instance, since $2^2 < e^2 < 3^2$, it follows $\frac{1}{3^2} < \frac{1}{e^2} < \frac{1}{2^2}$, or

$$3^{-2} < e^{-2} < 2^{-2}.$$

Thus, for any $t < 0$,

$$3^t < e^t < 2^t$$

Like 2^t and 3^t, the function e^t passes through $(0, 1)$ is always increasing and always concave up, and its range is the set of all positive real numbers.

Exploration Recall that the average rate of change of a function f on an interval $[a, b]$ is

$$AV_{[a,b]} = \frac{f(b) - f(a)}{b - a}.$$

In this activity we explore the average rate of change of $f(t) = e^t$ near the points where $t = 1$ and $t = 2$.

In a new *Desmos* worksheet, let $f(t) = e^t$ and define the function A by the rule

$$A(t) = \frac{f(t) - f(1)}{t - 1}.$$

a. What is the meaning of $A(1.5)$ in terms of the function f and its graph?

b. Compute the value of $A(t)$ for at least 6 difference values of t that are close to 1, both above and below 1. For instance, one value to try might be $h = 1.0001$. Record a table of your results.

c. What do you notice about the values you found in (b)? How do they compare to an important number?

d. Explain why the following sentence makes sense: "The function e^t is increasing at an average rate that is about the same as its value on small intervals near $t = 1$."

e. Adjust your definition of A in *Desmos* by changing 1 to 2 so that

$$A(t) = \frac{f(t) - f(2)}{t - 2}.$$

How does the value of $A(t)$ for values of t near 2 compare to $f(2)$?

Earlier, we saw graphical evidence that any exponential function $f(t) = b^t$ can be written as a horizontal scaling of the function $g(t) = 2^t$, plus we observed that there wasn't anything particularly special about 2^t. Because of the importance of e^t in calculus, we will choose instead to use the natural exponential function, $E(t) = e^t$ as the function we scale to generate any other exponential function $f(t) = b^t$. We claim that for any choice of $b > 0$ (with $b \neq 1$), there exists a horizontal scaling factor k such that $b^t = f(t) = E(kt) = e^{kt}$.

By the rules of exponents, we can rewrite this last equation equivalently as

$$b^t = (e^k)^t.$$

Since this equation has to hold for every value of t, it follows that $b = e^k$. Thus, our claim that we can scale $E(t)$ to get $f(t)$ requires us to show that regardless of the choice of the positive number b, there exists a single corresponding value of k such that $b = e^k$.

Given $b > 0$, we can always find a corresponding value of k such that $e^k = b$ because the function $f(t) = e^t$ passes the Horizontal Line Test, as seen in the figure below.

t	-2	-1	0	1	2
2^t	0.25	0.5	1	2	4

t	-2	-1	0	1	2
e^t	0.135	0.368	1	2.718	7.389

t	-2	-1	0	1	2
3^t	0.111	0.333	1	3	9

In this figure, we can think of b as a point on the positive vertical axis. From there, we draw a horizontal line over to the graph of $f(t) = e^t$, and then from the (unique) point of intersection we drop a vertical line to the x-axis. At that corresponding point on the x-axis we have found the input value k that corresponds to b. We see that there is always exactly one such k value that corresponds to each chosen b because $f(t) = e^t$ is always increasing, and any always increasing function passes the Horizontal Line Test.

It follows that the function $f(t) = e^t$ has an inverse function, and hence there must be some other function g such that writing $y = f(t)$ is the same as writing $t = g(y)$. This important function g will be be developed more later and will enable us to find the value of k exactly for a given b. For now, we are content to work with these observations graphically and to hence find estimates for the value of k.

Exploration By graphing $f(t) = e^t$ and appropriate horizontal lines, estimate the solution to each of the following equations. Note that in some parts, you may need to do some algebraic work in addition to using the graph.

a. $e^t = 2$

b. $e^{3t} = 5$

c. $2e^t - 4 = 7$

d. $3e^{0.25t} + 2 = 6$

e. $4 - 2e^{-0.7t} = 3$

f. $2e^{1.2t} = 1.5e^{1.6t}$

Let's give a name to the inverse function of $f(x) = e^x$.

Definition The **natural logarithm** is the inverse function of $f(x) = e^x$. This function is written $f(x) = \ln(x)$.

You will notice that this is one of the functions from our Famous Functions list. We will explore this function further later in the course.

Summary

- Any exponential function $f(t) = b^t$ can be viewed as a horizontal scaling of $E(t) = e^t$ because there exists a unique constant k such that $E(kt) = e^{kt} = b^t = f(t)$ is true for every value of t. This holds since the exponential function e^t is always increasing, so given an output b there exists a unique input k such that $e^k = b$, from which it follows that $e^{kt} = b^t$.

- The natural base e is the special number that defines an increasing exponential function whose rate of change at any point is the same as its height at that point, a fact that is established using calculus. The number e turns out to be given exactly by an infinite sum and approximately by $e \approx 2.7182818$.

Part 4

Building New Functions from Famous Functions

4.1 Building New Functions

Learning Objectives

- Algebra of Functions
  - Adding, subtracting, multiplying, and dividing functions
  - Thinking of complicated functions as objects in their own right
  - Evaluating complicated functions
  - Break more complicated functions into famous function
  - Undertand functions via graphs, tables, algebraically, and abstractly
- Creating a new Famous Function: Tangent
  - Pointwise/graphically divide sine and cosine to make a new graph: tangent

4.1.1 Algebra of Functions

Motivating Questions

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-

Introduction

Summary

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-
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4.1.2 Creating a New Function: Tangent

Motivating Questions

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Introduction

Summary

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4.2 Polynomials

Learning Objectives

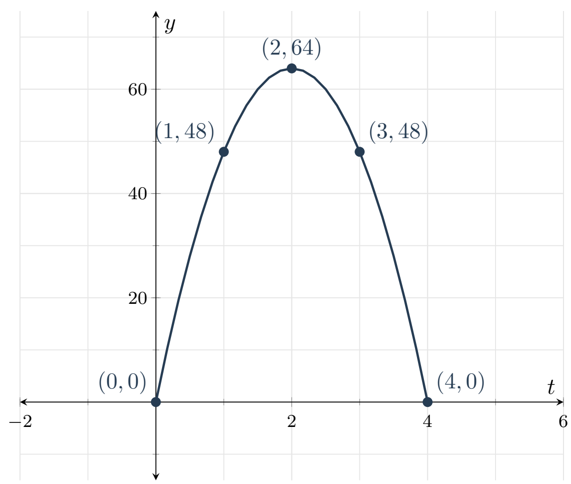
- Parabolas
  - What does a parabola look like? When do they open up versus down?
  - What is the vertex of a parabola and how can one find it?
  - Forms of parabolas, including general polynomial form
- Definition of Polynomials
  - Understand polynomials as functions
  - Understand lines and parabolas as graphs of polynomials
  - Evaluate polynomials
  - Put polynomial expressions into standard form
- Shape of Polynomials
  - Even and Odd polynomials
  - Importance of highest term and sign of highest term
  - End behavior of polynomials

4.2.1 Parabolas

Quadratic Graphs

Example 31. *Hannah fired a toy rocket from the ground, which launched into the air with an initial speed of* 64 *feet per second. The height of the rocket can be modeled by the equation* $y = -16t^2 + 64t$*, where* t *is how many seconds had passed since the launch. To see the shape of the graph made by this equation, we make a table of values and plot the points.*

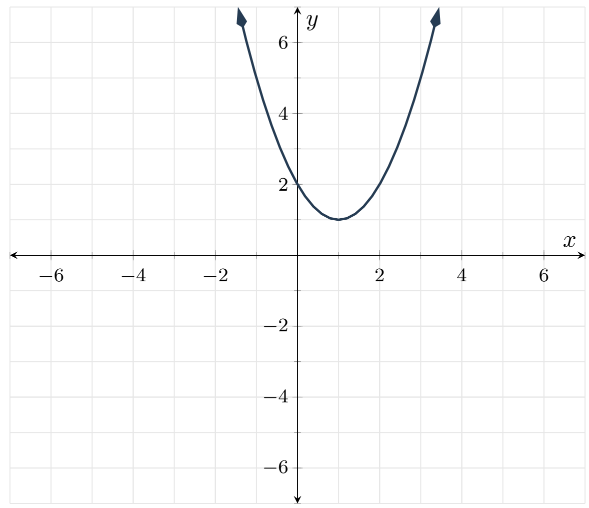
t	$-16t^2 + 64t$	*Point*
0	$-16(0)^2 + 64(0) = 0$	$(0, 0)$
1	$-16(1)^2 + 64(1) = 48$	$(1, 48)$
2	$-16(2)^2 + 64(2) = 64$	$(2, 64)$
3	$-16(3)^2 + 64(3) = 48$	$(3, 48)$
4	$-16(4)^2 + 64(4) = 0$	$(4, 0)$

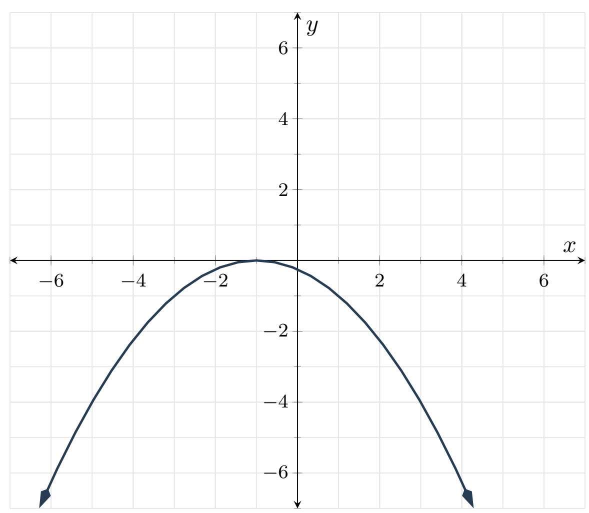


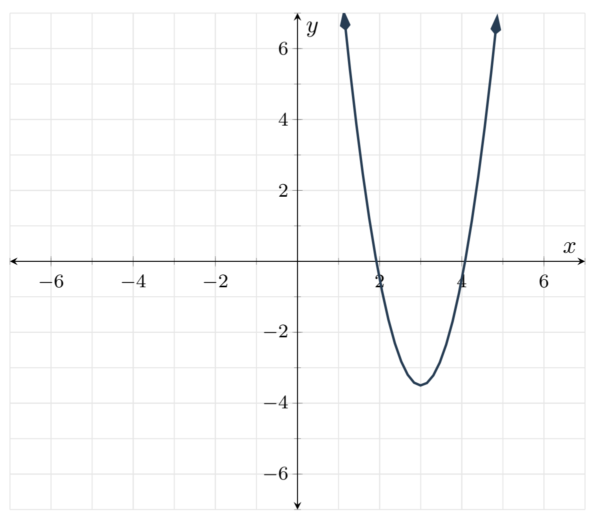
A curve with the shape that we see in the above figure is called a **parabola**. Notice the symmetry in figure, how the y-values in rows above the middle row match those below the middle row. Also notice the symmetry in the shape of the graph, how its left side is a mirror image of its right side.

The first feature that we will talk about is the direction that a parabola opens. All parabolas open either upward or downward. This parabola in the rocket example opens downward because a is negative. That means that for large values of t, the at^2 term will be large and negative, and the resulting y-value will be low on the y-axis. So the negative leading coefficient causes the arms of the parabola to point downward.

Here are some more quadratic graphs so we can see which way they open.







The graph of a quadratic equation $y = ax^2 + bx + c$ is a parabola which opens upward or downward according to the sign of the leading coefficient a. If the leading coefficient is positive, the parabola opens upward. If the leading coefficient is negative, the parabola opens downward.

The **vertex** of a parabola is the highest or lowest point on the graph, depending uponon whether the graph opens downward or upward. In Example 1, the vertex is $(2, 64)$. This tells us that Hannah's rocket reached its maximum height of 64 feet after 2 seconds. If the parabola opens downward, as in the rocket example, then the y-value of the vertex is the **maximum** y-value. If the parabola opens upward then the y-value of the vertex is the **minimum** y-value. The **axis of symmetry** is a vertical line that passes through the vertex, cutting the parabola into two symmetric halves. We write the axis of symmetry as an equation of a vertical line so it always starts with "$x =$". In Example 1, the equation for the axis of symmetry is $x = 2$.

The **vertical intercept** is the point where the parabola crosses the vertical axis. The vertical intercept is the y-intercept if the vertical axis is labeled y. In Example 1, the point $(0, 0)$ is the starting point of the rocket, and it is where the graph crosses the y-axis, so it is the vertical intercept. The y-value of 0 means the rocket was on the ground when the t-value was 0, which was when the rocket launched.

The **horizontal intercept(s)** are the points where the parabola crosses the horizontal axis. They are the x-intercepts if the horizontal axis is labeled x. The point $(0, 0)$ on the path of the rocket is also a horizontal intercept. The t-value of 0 indicates the time when the rocket was launched from the ground. There is another horizontal intercept at the point $(4, 0)$, which means the rocket came back to hit the ground after 4 seconds.

It is possible for a quadratic graph to have zero, one, or two horizontal intercepts.

The figures below show an example of each.

Example 32. *Use technology to graph and make a table of the quadratic function f defined by $f(x) = 2x^2 + 4x - 3$ and find each of the key points or features.*

(a) *Find the vertex.*

(b) *Find the vertical intercept (i.e. the y-intercept).*

(c) *Find the horizontal or (i.e. the x-intercept(s)).*

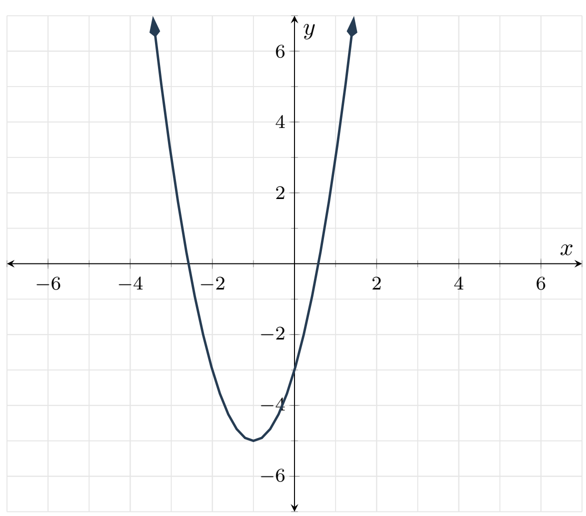
(d) *Find $f(-2)$.*

(e) *Solve $f(x) = 3$ using the graph.*

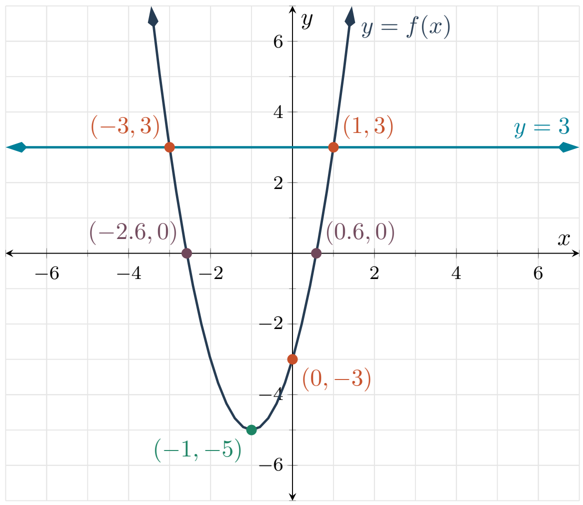
(f) *Solve $f(x) \leq 3$ using the graph.*

Explanation The specifics of how to use any one particular technology tool vary. Whether you use an app, a physical calculator, or something else, a table and graph should look like:

x	$f(x)$
-2	-3
-1	-5
0	-3
1	3
2	13



Additional features of your technology tool can enhance the graph to help answer these questions. You may be able to make the graph appear like:



(a) The vertex is $(-1, -5)$.

(b) The vertical intercept is $(0, -3)$.

(c) The horizontal intercepts are approximately $(-2.6, 0)$ and $(0.6, 0)$.

(d) When $x = -2$, $y = -3$, so $f(-2) = -3$.

(e) The solutions to $f(x) = 3$ are the x-values where $y = 3$. We graph the horizontal line $y = 3$ and find the x-values where the graphs intersect. The solution set is $\{-3, 1\}$.

(f) The solutions are all of the x-values where the function's graph is below (or touching) the line $y = 3$. The interval is $[-3, 1]$.

The Vertex Form of a Quadratic

We have learned the standard form of a quadratic function's formula, which is $f(x) = ax^2 + bx + c$. we will learn another form called the vertex form.

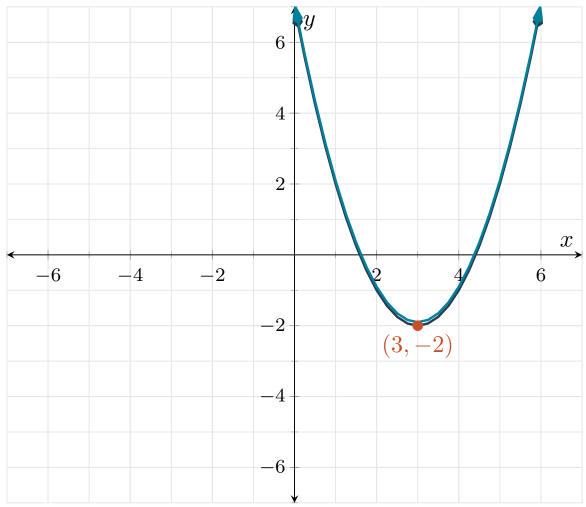
Using graphing technology, consider the graphs of $f(x) = x^2 - 6x + 7$ and $g(x) = (x - 3)^2 - 2$ on the same axes.

We see only one parabola because these are two different forms of the same function. Indeed, if we convert $g(x)$ into standard form:

$$\begin{aligned} g(x) &= (x-3)^2 - 2 \\ &= (x^2 - 6x + 9) - 2 \\ &= x^2 - 6x + 7 \end{aligned}$$

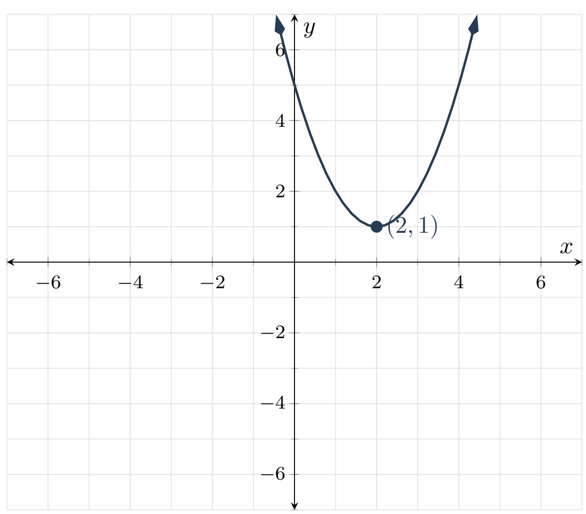
it is clear that f and g are the same function.

Graph of $f(x) = x^2 - 6x + 7$ and $g(x) = (x-3)^2 - 2$ the graphs of the two parabolas overlap each other completely

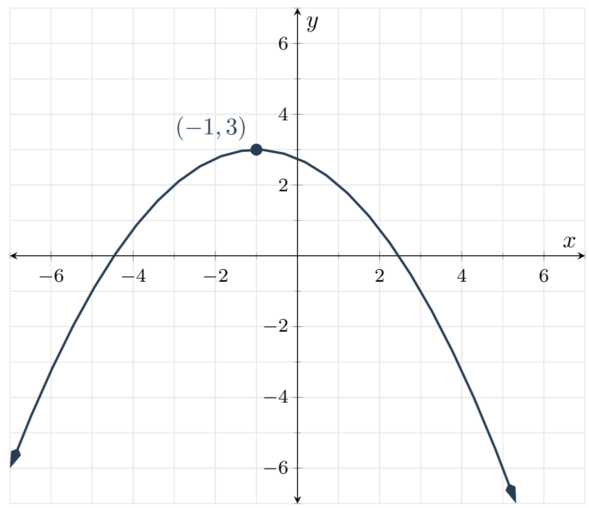


The formula given for g is said to be in vertex form because it allows us to read the vertex without doing any calculations. The vertex of the parabola is $(3, -2)$. We can see those numbers in $g(x) = (x-3)^2 - 2$. The x-value is the solution to $(x-3) = 0$, and the y-value is the constant added at the end.

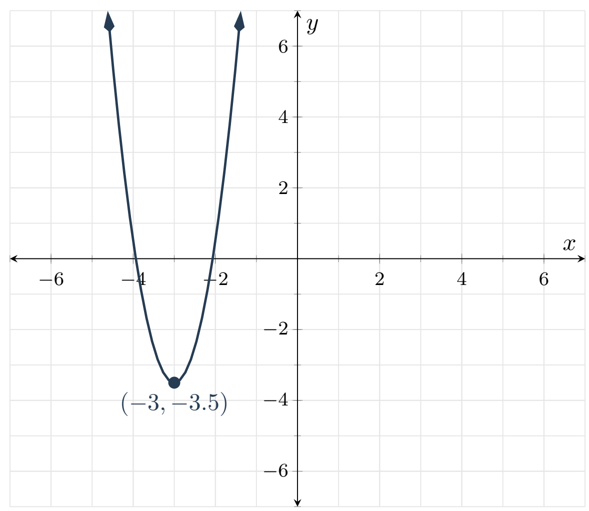
Example 33. *Here are the graphs of three more functions with formulas in vertex form. Compare each function with the vertex of its graph.*



$$r(x) = (x-2)^2 + 1$$



$$s(x) = -\frac{1}{4}(x+1)^2 + 3$$



$$t(x) = 4(x+3)^2 - 3.5$$

Notice that the x-coordinate of the vertex has the opposite sign as the value in the function formula. On the other hand, the y-coordinate of the vertex has the same sign as the value in the function formula. Let's look at an example to understand why. We will evaluate $r(2)$.

$r(2) = (2-2)^2 + 1 = 1$

The x-value is the solution to $(x-2) = 0$, which is positive 2. When we substitute 2 for x we get the value $y = 1$. Note that these coordinates create the vertex at $(2, 1)$. Now we can define the vertex form of a quadratic function.

Vertex Form of a Quadratic Function A quadratic function whose graph has vertex at the point (h, k) is given by

$$f(x) = a(x - h)^2 + k$$

4.2.2 Definition of Polynomials

A polynomal is a particular type of algebraic expression

(a) A company's sales, s (in millions of dollars), can be modeled by $2.2t+5.8$, where t stands for the number of years since 2010.

(b) The height of an object from the ground, h (in feet), launched upward from the top of a building can be modeled by $-16t^2+32t+300$, where t represents the amount of time (in seconds) since the launch.

(c) The volume of an open-top box with a square base, V (in cubic inches), can be calculated by $30s^2-\frac{1}{2}s^2$, where s stands for the length of the square base, and the box sides have to be cut from a certain square piece of metal.

Polynomial Vocabulary

A polynomial is an expression with one or more terms summed together. A term of a polynomial must either be a plain number or the product of a number and one or more variables raised to natural number powers. The expression 0 is also considered a polynomial, with zero terms.

Example 34. *Here are some examples of polynomials*

(a) *Here are three polynomials:* x^2-5x+2, t^3-1, $7y$.

(b) *The expression* $3x^4y^3+7xy^2-12xy$ *is an example of a polynomial in more than one variable.*

(c) *The polynomial* x^2-5x+3 *has three terms:* x^2, $-5x$, *and* 3.

(d) *The polynomial* $3x^4+7xy^2-12xy$ *also has three terms.*

(e) *The polynomial* t^3-1 *has two terms.*

> **Definition** The coefficient (or numerical coefficient) of a term in a polynomial is the numerical factor in the term.

Example 35. (a) *The coefficient of the term* $\frac{4}{3}x^6$ *is* $\frac{4}{3}$.

(b) *The coefficient of the second term of the polynomial* x^2-5x+3 *is* -5.

(c) *The coefficient of the term* $\frac{y^7}{4}$ *is* $\frac{1}{4}$, *because we can rewrite* $\frac{y^7}{4}$ *as* $\frac{1}{4}y^7$.

A term in a polynomial with no variable factor is called a constant term.

Example 36. *The constant term of the polynomial* x^2-5x+3 *is* 3.

Definition The degree of a term is one way to measure how large it is. When a term only has one variable, its degree is the exponent on that variable. When a term has more than one variable, its degree is the sum of the exponents on the variables. A nonzero constant term has degree 0.

Example 37.

The degree of $5x^2$ *is* 2.

The degree of $-\frac{4}{7}y^5$ *is* 5.

The degree of $-4x^2y^3$ *is* 5.

The degree of 17 *is* 0. *Constant terms always have* 0 *degree.*

Definition The **degree** of a nonzero polynomial is the greatest degree that appears amongst its terms

Remark To help us recognize a polynomial's degree, the standard convention at this level is to write a polynomial's terms in order from highest degree to lowest degree. When a polynomial is written in this order, it is written in standard form. For example, it is standard practice to write $7-4x-x^2$ as $-x^2-4x+7$ since $-x^2$ is the leading term. By writing the polynomial in standard form, we can look at the first term to determine both the polynomial's degree and leading term.

Adding and Subtracting Polynomials

Bayani started a company that makes one product: one-gallon ketchup jugs for industrial kitchens. The company's production expenses only come from two things: supplies and labor. The cost of supplies, S (in thousands of dollars), can be modeled by $S = 0.05x^2 + 2x + 30$, where x is number of thousands of jugs of ketchup produced. The labor cost for his employees, L (in thousands of dollars), can be modeled by $0.1x^2 + 4x$, where x again represents the number of jugs they produce (in thousands of jugs). Find a model for the company's total production costs.

Evaluating Polynomial Expressions

Recall that evaluating expressions involves replacing the variable(s) in an expression with specific numbers and calculating the result. Here, we will look at evaluating polynomial expressions.

Example 38. *Evaluate the expression*

$$-12y^3 + 4y^2 - 9y + 2 \textit{ for } y = -5$$

Explanation We will replace y with -5 and simplify the result:

$$\begin{aligned} 12y^3 + 4y^2 - 9y + 2 \quad &= -12(-5)^3 + 4(-5)^2 - 9(-5) + 2 \\ &= -12(-125) + 4(25) + 45 + 2 \\ &= 1647 \end{aligned}$$

4.3 Rational Functions

Learning Objectives

- The Famous Function $f(x) = 1/x$
  - Introduce 1/x as a famous function, examine it's properties
  - Composing 1/x with other functions (introduce secant, cosecant, and cotangent)
- Definition of Rational Functions
  - What is a rational function?
  - What is an asymptotes? When and why does it occur?
  - Properties of rational functions
  - Examine the end behavior of rational functions
- Polynomials in Disguise
  - When can a rational function by simplified to a polynomial?
  - Dividing through by a monomial

4.3.1 The Famous Function $f(x) = 1/x$

Motivating Questions

- What is a possible explanation, in terms of functions, for the fact that one cannot divide by zero?
- Are sin, cos and tan really the only relevant trigonometric functions? Are there others? If so, how to understand them?

Introduction

We know that if a and b are two real numbers, then a/b makes sense, as long as b is not equal to zero. Let's look at what happens when we make divisions by numbers very close to zero, but not equal to zero. Take $a = 1$ for simplicity.

$$\frac{1}{0.1} = 10$$
$$\frac{1}{0.01} = 100$$
$$\frac{1}{0.001} = 1000$$
$$\frac{1}{0.0001} = 10000$$

This pattern makes us want to say that $1/0$ equals to $+\infty$ (whatever $+\infty$ means, at this point), but this doesn't work. To understand why, let's consider divisions by numbers very close to zero, but this time negative.

$$\frac{1}{-0.1} = -10$$
$$\frac{1}{-0.01} = -100$$
$$\frac{1}{-0.001} = -1000$$
$$\frac{1}{-0.0001} = -10000$$

The same reasoning as before would tempt us to say that $1/0$ equals $-\infty$. And this raises the question of whether ∞ or $-\infty$ is the better choice. While on an instinctive psychological level we could think that $+\infty$ is better than $-\infty$, there's really no way to decide[11] — and this turns out to be related to the concept of *limit*, which you'll learn in Calculus.

Graph and asymptotics

To continue our discussion in a more precise way, let's consider the function f, defined for all real numbers *except for zero*, given by $f(x) = 1/x$. This is a very famous function, particularly useful as the building block for *rational functions*, which we'll discuss soon. Note that essentially what we have just done in the introduction was to consider the values

$$f(0.1), f(0.01), f(0.001) \text{ and } f(0.0001),$$

as well as

$$f(-0.1), f(-0.01), f(-0.001), \text{ and } f(-0.0001).$$

To get a good idea of the behavior a function has, our main strategy so far has been to just consider its graph. Naturally, plugging a handful of values won't cut it. Let's see what happens when we go to the other extreme and make divisions by very large numbers:

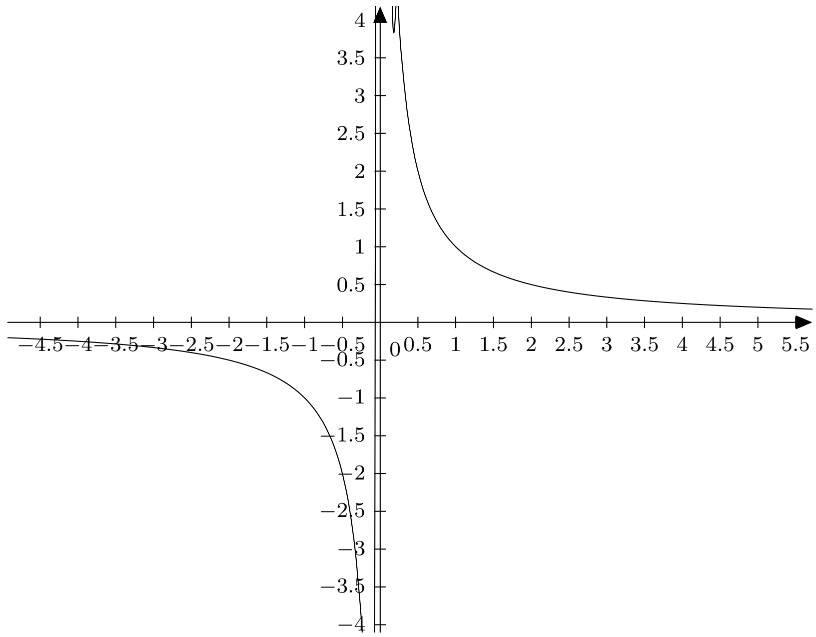
$$\begin{aligned} \frac{1}{10} &= 0.1 \\ \frac{1}{100} &= 0.01 \\ \frac{1}{1000} &= 0.001 \\ \frac{1}{10000} &= 0.0001 \end{aligned}$$

And from the negative side:

$$\begin{aligned} \frac{1}{-10} &= -0.1 \\ \frac{1}{-100} &= -0.01 \\ \frac{1}{-1000} &= -0.001 \\ \frac{1}{-10000} &= -0.0001 \end{aligned}$$

[11] Algebraically, the explanation is simple: if one could make sense of $1/0$ and say that equals some number c, then this would give $1 = 0 \cdot c$, so $1 = 0$ — which is a complete collapse of the number system we have to deal with in our daily lives. But this doesn't give intuition for what is going on.

Here's what the graph looks like.



[MAKE BETTER GRAPH, WILL FIX TIKZ AFTER DRAFTS ARE DONE]

Here's what we can immediately see from the graph, confirming our intuition from the several divisions previously done:

Asymptotics of $1/x$.

- If $x \to +\infty$, then $1/x \to 0$ (reads "when x tends to $+\infty$, $1/x$ tends to 0").
- If $x \to 0^+$, then $1/x \to +\infty$ (reads "when x tends to zero from the right, $1/x$ tends to $+\infty$").
- If $x \to 0^-$, then $1/x \to -\infty$ (reads "when x tends to zero from the left, $1/x$ tends to $-\infty$").
- If $x \to -\infty$, then $1/x \to 0$ (reads "when x tends to $-\infty$, $1/x$ tends to 0").

We say that the line $x = 0$ is a *vertical asymptote* for $f(x) = 1/x$, while the line $y = 0$ is a *horizontal asymptote*. We will discuss asymptotes of rational functions in general in the next unit. Next, as far as symmetries go, we can see that the graph is symmetric about the line $y = x$:

[ADD GRAPH AGAIN WITH ANTI-DIAGONAL INCLUDED]

This indicates that $f(x) = 1/x$ is an odd function. You can also see this algebraically via

$$f(-x) = \frac{1}{-x} = -\frac{1}{x} = -f(x).$$

By the way, the graph of $f(x) = 1/x$ is called a *hyperbola*.

Application: Inverses of trigonometric functions

It turns out that applying f to some famous functions, such as the usual trigonometric functions sin, cos and tan, usually produces new interesting functions which can help to model several situations in a perhaps simpler way.

Definition The *cosecant*, *secant* and *cotangent* functions are defined, respectively, by

$$\csc x = \frac{1}{\sin x}, \quad \sec x = \frac{1}{\cos x} \quad \text{and} \quad \cot x = \frac{1}{\tan x}.$$

Exploration

a. For which values of x do we have that $\sin x = 0$? Draw the graph of sin.

b. For which values of x is csc undefined? Recall that one cannot divide by zero.

c. Repeat items (a) and (b) replacing sin and csc with cos and sec, respectively. What about tan and cot?

[THIS IS PROBABLY NOT VERY ADEQUATE. CHANGE LATER]

Warning: Do not confuse inverses of trigonometric functions, as discussed above, with inverse trigonometric functions such as arcsin, arccos and arctan (as in "inverse functions", as discussed in Section 3-2).

Summary

- The function $f(x) = 1/x$ is defined for all non-zero values of x. It is an odd function, and its asymptotics can be understood by its graph, called a *hyperbola*.
- We can compose $f(x) = 1/x$ with functions we frequently encounter, to produce new functions which may prove useful when modeling certain problems and real life situations. For instance,

doing this to trigonometric functions, one obtains

$$\csc x = \frac{1}{\sin x}, \quad \sec x = \frac{1}{\cos x} \quad \text{and} \quad \cot x = \frac{1}{\tan x}.$$

They are called, respectively, the *cosecant*, *secant* and *cotangent* functions.

4.3.2 The Definition of a Rational Function

> **Motivating Questions**
>
> - By passing from $1/x$ to $p(x)/q(x)$, what changes? What can we say about the behavior of such ratio?
> - Just like $1/x$ had the lines $x = 0$ and $y = 0$ as asymptotes, what happens for an arbitrary rational function? Are vertical and horizontal asymptotes the only possible types?

Introduction

We have previously discussed the function $1/x$. Note that both the numerator 1 and the denominator x are polynomials (the former is a *constant* polynomial). We can study what happens when we replace those with arbitrary polynomials.

> **Definition** A *rational function* is a function defined as a ratio $f(x) = p(x)/q(x)$ of two polynomials $p(x)$ and $q(x)$, and this ratio makes sense for all real values of x, *except* for those such that $q(x) = 0$.

Example 39.

(a) $f(x) = \dfrac{x^2 - 2}{x + 1}$ *is a rational function. It is defined for all values of* x*, except for* $x = -1$*, because this makes the denominator* $x + 1$ *be zero.*

(b) $f(x) = \dfrac{x^4 - 3x + 1}{x^2 - 5x + 6}$ *is a rational function. It is defined for all values of* x*, except for* $x = 2$ *and* $x = 3$*, because* $x^2 - 5x + 6 = (x - 2)(x - 3)$.

(c) $f(x) = \dfrac{x - 1}{\sqrt{x^4 + 1}}$ *is* not *a rational function, because the denominator* $\sqrt{x^4 + 1}$ *is* not a polynomial. *Note that even though this does not define a rational function, it is defined for all possible values of* x*, since* $\sqrt{x^4 + 1} \geq 1 > 0$ *for all* x.

(d) *If* $p(x)$ *is a polynomial function, then it is a rational function, simply because we can write* $p(x) = p(x)/1$*, and* 1 *is a polynomial.*

(e) $f(x) = x^2 - 1 + \dfrac{x^3}{x^5 - 1}$ *is a rational function, which is defined for all* x *except for* $x = 1$*. To see that this is a rational function, you can either say that it is the sum of the rational functions* $x^2 - 1$ *and* $x^3/(x^5 - 1)$*, or* *rewrite it as*

$$f(x) = \frac{(x^2-1)(x^5-1)+x^3}{x^5-1} = \frac{x^7-x^5-x^2+x^3-1}{x^5-1},$$

which is manifestly rational.

Asymptotes

We have seen that the function $1/x$ has the line $x = 0$ as a vertical asymptote, and the line $y = 0$ as a horizontal asymptote. Rational functions, in general, may have not only vertical and horizontal asymptotes, but also *slant asymptotes*. Let's start with the two easier cases:

Definition

(a) The line $x = c$ is called a *vertical asymptote* of the graph of a function $y = f(x)$ if as $x \to c^-$ or $x \to c^+$, either $f(x) \to +\infty$ or $f(x) \to -\infty$.

(b) The line $y = c$ is called a *horizontal asymptote* of the graph of a function $y = f(x)$ if as $x \to -\infty$ or $x \to +\infty$, we have $f(x) \to c$.

Example 40.

(a) *Consider the rational function* $f(x) = \dfrac{x-1}{x-2}$. *To understand* $x \to +\infty$ *intuitively, let's plug some big values for* x*:*

$$f(100) = \frac{99}{98} \approx 1.010...$$
$$f(1000) = \frac{999}{998} \approx 1.001...$$
$$f(10000) = \frac{9999}{9998} \approx 1.000...$$

It seems clear that $f(x) \to 1$ *as* $x \to +\infty$*. This says that the line* $y = 1$ *is a horizontal asymptote for* $f(x)$*. The same strategy shows that* $f(x) \to 1$ *when* $x \to -\infty$ *as well, so that* $y = 1$ *is the only horizontal asymptote for* $f(x)$*. As for vertical asymptotes, we see that the only* x*-value for which* $f(x)$ *is undefined is* $x = 2$*. So that's where we'll look, by choosing values for* x *very close to* 2*, but not equal to* 2*. For example, we have that*

$$f(2.1) = \frac{1.1}{0.1} = 11$$
$$f(2.01) = \frac{1.01}{0.01} = 101$$
$$f(2.001) = \frac{1.001}{0.001} = 1001$$

indicates that $f(x) \to +\infty$ *as* $x \to 2^+$. *Similarly, you can check that* $f(x) \to -\infty$ *as* $x \to 2^-$, *so the line* $x = 2$ *is a vertical asymptote for* $f(x)$ *(in fact, the only one). Here's a graph:*

[ADD GRAPH]

(b) *Consider now the rational function* $f(x) = \dfrac{x-1}{x^2-3x+2}$. *Let's do as above, and start looking for horizontal asymptotes.*

$$f(100) \approx 0.010...$$
$$f(1000) \approx 0.001...$$
$$f(1000) \approx 0.000...$$

This suggests that $f(x) \to 0$ *as* $x \to +\infty$. *Similarly, you can convince yourself that* $f(x) \to 0$ *as* $x \to -\infty$, *so that* $y = 0$ *is the only horizontal asymptote. And for vertical asymptotes, we'll again find the* x*-values for which* $f(x)$ *is undefined, and see whether any of those indicates a vertical asymptote. Noting that* $x^2 - 3x + 2 = (x-1)(x-2)$, *we can see that* $f(x)$ *is undefined for* $x = 1$ *and* $x = 2$. *However, we may write that*

$$f(x) = \frac{x-1}{x^2-3x+2} = \frac{x-1}{(x-1)(x-2)} = \frac{1}{x-2},$$

provided $x \neq 1$. *And* $1/(x-2)$ *does not increase (or decrease) without bound as* x *approaches* 1 *from either side — in fact, it approaches* -1. *So, even though* $f(x)$ *is undefined for the value* $x = 1$, *the line* $x = 1$ *is* not *a vertical asymptote for* $f(x)$. *But comparing the expression* $f(x) = 1/(x-2)$ *(again, valid for* $x \neq 1$*) with what we have previously seen for the function* $1/x$, *we see that* $f(x) \to +\infty$ *as* $x \to 2^+$ *and* $f(x) \to -\infty$ *as* $x \to 2^-$, *so that the line* $x = 2$ *is a vertical asymptote for* $f(x)$. *Here's the graph:*

[ADD GRAPH WITH HOLE]

Warning: *In the above example, the function* $f(x)$ *given is* not *the same thing as the function* $g(x) = 1/(x-2)$. *The function* $f(x)$ *is undefined for* $x = 1$, *but* $g(1)$ *is defined, and it equals* -1. *The domain is a crucial part of the data defining a function. We will address these issues on Unit 5.*

Exploration A mathematical model for the population P, in thousands, of a certain species of bacteria, t days after it is introduced to an environment, is given by $P(t) = \dfrac{200}{(7-t)^2}$, $0 \leq t < 7$.

(a) Find and interprete $P(0)$.

(b) When will the population reach $200{,}000$?

(c) Determine the behavior of P as $t \to 7^-$. Interpret this result graphically and within the context of the problem.

Now, you must be asking yourself if every time we want to test for vertical or horizontal asymptotes, we need to keep plugging values and guessing. Fortunately, the answer is "no". Here's what you need to know:

Theorem (locating horizontal asymptotes): Assume that $f(x) = p(x)/q(x)$ is a rational function, and that the leading coefficients of $p(x)$ and $q(x)$ are a and b, respectively.

- If the degree of $p(x)$ is the same as the degree of $q(x)$, then $y = a/b$ is the unique horizontal asymptote of the graph of $y = f(x)$.
- If the degree of $p(x)$ is less than the degree of $q(x)$, then $y = 0$ is the unique horizontal asymptote of the graph of $y = f(x)$.
- If the degree of $p(x)$ is greater than the degree of $q(x)$, then the graph of $y = f(x)$ has no horizontal asymptotes.

Explanation The above theorem essentially says that one can detect horizontal asymptotes by looking at degrees and leading coefficients. Only the leading terms of $p(x)$ and $q(x)$ matter, and it makes no difference whether one considers $x \to +\infty$ or $x \to -\infty$. For example, as $x \to +\infty$, we have that

$$\frac{3x^6 - 5x^4 + 3x^3 - 3x^2 + 10x + 1}{5x^6 + 10000x^5 - 5x + 2} \approx \frac{3x^6}{5x^6} = \frac{3}{5},$$

which says that $y = 3/5$ is the only horizontal asymptote for this rational function.

Theorem (locating vertical asymptotes): Assume that $f(x) = p(x)/q(x)$ is a rational function written in lowest terms, that is, such that $p(x)$ and $q(x)$ have no common zeros. Let c be a real number for which $f(c)$ is undefined.

- If $q(c) \neq 0$, then the graph of $y = f(x)$ has a hole at the point $(c, f(c))$.
- If $q(c) = 0$, then the line $x = c$ is a vertical asymptote of the graph of $y = f(x)$.

Explanation The above theorem tells us how to distinguish vertical asymptotes and holes in the graph of a rational function. Here's an example: take

$$f(x) = \frac{x^2 - 4x + 3}{x^2 - 3x + 2}.$$

Is $f(x)$ in lowest terms? You can find out by just factoring both numerator and denominator. By factoring the denominator, you will also find out which values of x we have $f(x)$ undefined. Since

$$f(x) = \frac{x^2 - 4x + 3}{x^2 - 3x + 2} = \frac{(x-1)(x-3)}{(x-1)(x-2)} = \frac{x-3}{x-2},$$

we see that the values $f(1)$ and $f(2)$ are undefined, while the above equality holds for all x except for $x = 1$. In this simplified form, we may recognize $q(x) = x - 2$. Since $q(1) = -1 \neq 0$, the graph of $y = f(x)$ has a hole at the point $(1, f(1))$, but since $q(2) = 0$, we have that the line $x = 2$ is a vertical asymptote for the graph of $y = f(x)$.

We are now ready to address the last type of asymptote a rational function may or may not have. And the reasoning is somewhat simple: why should we restrict ourselves to only vertical or horizontal asymptotes? This question itself motivates the name "slant" asymptote. Now, you know that the general equation of a line has the form $y = mx + b$, where m is some slope and b is the y-intercept. When $m = 0$, we have a horizontal line, so when discussing slant asymptotes, we'll always assume that $m \neq 0$.

> **Definition** The line $y = mx+b$, where $m \neq 0$, is called a *slant asymptote* of the graph of a function $y = f(x)$ if as $x \to -\infty$ or as $x \to +\infty$, we have $f(x) \to mx + b$.

Note that saying that $y = mx + b$ is a slant asymptote for the graph of $y = f(x)$ is the same thing as saying that $y = 0$ is a horizontal asymptote for the graph of the difference function $y = f(x) - (mx + b)$.

Example 41.

(a) *Consider the rational function given by* $f(x) = \dfrac{x^2 - 4x + 2}{1 - x}$. *When trying to find slant asymptotes, long division is the way to go. Performing it, we see that*

$$\frac{x^2 - 4x + 2}{1 - x} = -x + 3 - \frac{1}{1 - x}.$$

Since $1/(1 - x) \to 0$ *as* $x \to \infty$ *or* $x \to -\infty$ *and* $y = -x + 3$ *describes a line, we conclude that* $y = -x + 3$ *is a slant asymptote for the graph of* $y = f(x)$. *[GRAPH]*

(b) *Consider now the rational function* $f(x) = \dfrac{x^2 - 4}{x - 2}$. *We may just simplify it to* $f(x) = x + 2$, *valid for all* $x \neq 2$. *We may regard this as a long division for which the remainder is zero, as in*

$$f(x) = x + 2 + \frac{0}{x - 2},$$

and since $0/(x-2) \to 0$ *as* $x \to +\infty$ *or* $x \to -\infty$, *it follows that* $y = x+2$ *is a slant asymptote for the graph of* $y = f(x)$, *even though the graph is just said line with a hole! [GRAPH]*

(c) *Consider the rational function* $f(x) = \dfrac{x^3 - 4x^2 + 5x - 1}{x - 1}$. *Performing the long division, as before, we see that*

$$f(x) = \frac{x^3 - 4x^2 + 5x - 1}{x - 1} = x^2 - 3x + 2 + \frac{1}{x - 1}.$$

As expected, $1/(x-1) \to 0$ *when* $x \to +\infty$ *or* $x \to -\infty$, *but* $y = x^2 - 3x + 2$ *is not a line equation. Hence there are no slant asymptotes for the graph of* $f(x)$ *(loosely speaking, the graph cannot be simultaneously asymptotic to a parabola and to a straight line). [GRAPH]*

(d) *Consider the rational function* $f(x) = \dfrac{x^3 + 2x^2 + x + 1}{x^3 - 2x^2 - x + 1}$. *Long division shows that:*

$$f(x) = \frac{x^3 + 2x^2 + x + 1}{x^3 - 2x^2 - x + 1} = 1 + \underbrace{\frac{4x^2 + 2x}{x^3 - 2x^2 - x + 1}}_{\to 0}.$$

The indicated remainder goes to zero when $x \to +\infty$ *or* $x \to -\infty$ *simply because the degree of the numerator is lower than the degree of the numerator. The remaining quotient does give us the asymptote* $y = 1$. *But this is not a slant asymptote, it is a horizontal asymptote (as you might have expected).*

(e) *Consider the rational function* $f(x) = \dfrac{x^2 + 1}{x^5 - 4}$. *Since the degree of the numerator is smaller than the degree of the denominator, you can think of the long division as already having been performed, as in*

$$f(x) = \frac{x^2 + 1}{x^5 - 4} = 0 + \frac{x^2 + 1}{x^5 - 4}.$$

Again, the remainder goes to zero when $x \to +\infty$ *or* $x \to -\infty$. *So, the line* $y = 0$ *would be an asymptote, but it is horizontal, not slant, as in the previous item.*

The above examples suggest that if the degree of the numerator is at least two higher than the degree of the numerator, what survives outside the remainder has degree higher than one, and thus does not describe a line equation — meaning no slant asymptotes. Similarly, if the degree of the numerator is equal or lower to the degree of the denominator, there's "not enough quotient left" to describe a line equation. This is not a coincidence, but a general fact.

Theorem (on slant asymptotes): Let $f(x) = p(x)/q(x)$ be a rational function for which the degree of $p(x)$ is exactly one higher than the degree of $q(x)$. Then the graph of $y = f(x)$ has the slant asymptote $y = L(x)$, where $L(x)$ is the quotient obtained by dividing $p(x)$ by $q(x)$. If the degree of $p(x)$ is not exactly one higher than the degree of $q(x)$, there is no slant asymptote whatsoever.

Explanation Unlike what happened for horizontal and vertical asymptotes, the above theorem does not *immediately* tell you what is the line equation describing the slant asymptote. We must resort to long division.

Summary

- A *rational* function is, as the name suggests, a function defined as a *ratio* $f(x) = p(x)/q(x)$ of two polynomials $p(x)$ and $q(x)$. It makes sense for all real values of x *except* for those such that $q(x) = 0$, as one cannot divide by zero.
- There are three types of asymptotes for rational functions: vertical asymptotes, horizontal asymptotes, and slant asymptotes. The latter occurs when the degree of the numerator is exactly one higher than the degree of the denominator.

Part 5

Domain and Range

5.1 Domain

Learning Objectives

- Domain
  - Definition of the Domain
  - Interval Notation
  - The Domains of Famous Functions
  - Spotting Values not in the Domain
  - Piecewise Defined Functions and Restricted Domains

5.1.1 Domain

Motivating Questions

- Can a function ever have an input that is not allowed?
- How do we denote the numbers that can be inputs?
- What are the allowable inputs for our famous functions?

Introduction

We often think about functions as a process which transforms an input into some output. Sometimes that process is known to us (such as when we have a formula for the function) and sometimes that process is unknown to us (such as when we only have a small table of values).

Exploration

a. Suppose the quadratic function f is given by $f(x) = x^2$. Are there any input values that can't be plugged into f?

b. Suppose a square has side length denotes by the variable s, and area denoted by A. The area of the square is a function of the side length, $A(s) = s^2$. Are there any values of s that don't make sense?

c. Suppose that g is the rational function given by $g(x) = \dfrac{x}{x}$ and that h is the constant function given by $h(x) = 1$. Are these the same function? Why or why not?

The Domain of a Function

Definition Let f be a function from A to B. The set A of possible inputs to f is called the **domain** of f. The set B is called the **codomain** of f.

Example 42. *Let P be the population of Columbus, OH as a function of the year. According to Google, the population of Columbus in 1990 was 632,910 and in the year 2010 the population was 787,033. That means we can say:*

$$P(1990) = 632,910 \textit{ and } P(2010) = 787,033.$$

What if we were asked to find $P(1200)$?

Explanation This question doesn't really make sense. There were Mound Builder tribes[12] in the area around the year 1200, but the city of Columbus was not incorporated until the year 1816. We say that $P(1200)$ is *undefined*. That is to say, 1200 is not in the domain of P.

Example 43. *Let f be the function given by $f(x) = \frac{1}{x}$. Is there any number that cannot be used as an input to f?*

Explanation There is only one number that is not a valid input, 0. The number 1 can be divided by any nonzero number. For instance $f(7) = \frac{1}{7}$ or $f(-3.7) = \frac{1}{-3.7}$ are perfectly valid outputs. However, if someone attempted to plug $x = 0$ into the formula $\frac{1}{x}$, they would end up with a division-by-zero, which is undefined. The number 0 is not in the domain of f.

When we are given a function, sometimes the domain is given to us explicitly. Consider the function $f(x) = 2x + 1$ for $x \geq 5$. The phrase "for $x \geq 5$" tells us the domain for this function. We may be able to plug any number into the expressio $2x + 1$, but it's only when $x \geq 5$ that this gives our function. For instance, $2(0) + 1 = 1$, but $f(0)$ is undefined.

Sometimes, when we are given a function as a formula, we are not told the domain. In these circumstances we use the *implied domain*.

> **Definition** Let f be a function whose inputs are real numbers. The **implied domain** of f is the collection of all real numbers x for which $f(x)$ is a real number.

Example 44. *Let g be the function given by $g(x) = \sqrt{3x-4}$. Find the domain of g.*

Explanation The only information we are given about g is the formula for $g(x)$. That means we are being asked to find the implied domain. Since the square root only exists (as a real number) when the radicand is non-negative, we need to ensure that:

$$\begin{aligned} 3x - 4 &\geq 0 \\ 3x &\geq 4 \\ x &\geq \frac{4}{3}. \end{aligned}$$

The domain is the set of all x for which $x \geq \frac{4}{3}$.

[12] See Mound Builder tribes at https://en.wikipedia.org/wiki/History_of_Columbus,_Ohio

Interval Notation

As in the previous example, solutions of inequalities play an important role in expressing the domains of many types of functions. As a standard way of writing these solutions, we rely on *interval notation.* Interval notation is a short-hand way of representing the intervals as they appear when sketched on a number line. The previous example involved $x \geq \frac{4}{3}$ which, when sketched on a number line, is given by

This sketch consists of a single interval with left-hand endpoint at $\frac{4}{3}$ and no right-hand endpoint (it keeps going). In interval notation, this would be written as $\left[\frac{4}{3}, \infty\right)$. This is an example of a *closed infinite interval*, "closed" because the point at $\frac{4}{3}$ (the only endpoint) is included and "infinite" because it has infinite width. The solid dot at $\frac{4}{3}$ indicates that the point is included in the interval.

There are four different types of infinite intervals, two are closed infinite intervals (which contain their respective endpoint) and the other two are open infinite intervals (which do not contain the endpoint). For a a fixed real number, these are:

(a) $[a, \infty)$ represents $x \geq a$,

(b) $(-\infty, a]$ represents $x \leq a$,

(c) (a, ∞) represents $x > a$, and

(d) $(-\infty, a)$ represents $x < a$.

The notation indicates uses the square bracket to indicate that the endpoint is included and the round parenthesis to indicate that the endpoint is not included.

Not every interval is infinite, however. Consider the interval in the following sketch

which consists of all x with $-2 < x \leq 3$. It is not an infinite interval, having endpoints at -2 and 3. The endpoint at -2 is not included, but the endpoint at 3 is included. In interval notation this would be written as $(-2, 3]$. As with the infinite intervals, the square bracket indicates that the right-hand endpoint is included and the round parenthesis endicates that the left-hand endpoint is not included. (This is an example of a “half-open interval”.)

For a bounded intervals (ones that are not infinite), there are also four possibilities. For a and b both fixed real numbers, these are:

(a) $[a, b]$ represents $a \leq x \leq b$,

(b) $[a, b)$ represents $a \leq x < b$,

(c) $(a, b]$ represents $a < x \leq b$ and

(d) (a, b) represents $a < x < b$.

Practically, this amounts to writing the left-hand endpoint, the right-hand endpoint, then indicating which endpoints are included in the interval. When neither endpoint is included, (a, b) can be mistaken for a point on a graph. You will need to use the context to know which is meant.

Example 45. *Write the interval notation for* $-\frac{3}{2} \leq x \leq \sqrt{5}$ *and for* $-\frac{3}{2} < x < \sqrt{5}$.

Explanation The interval $-\frac{3}{2} \leq x \leq \sqrt{5}$ has graph

It has one interval with endpoints at $-\frac{3}{2}$ and $\sqrt{5}$, both of which are included. In interval notation it is given by $\left[-\frac{3}{2}, \sqrt{5}\right]$.

The interval $-\frac{3}{2} < x < \sqrt{5}$ has graph

It has one interval with endpoints at $-\frac{3}{2}$ and $\sqrt{5}$, neither of which are included. In interval notation it is given by $\left(-\frac{3}{2}, \sqrt{5}\right)$.

Example 46. *Find the domain of the function f given by* $f(x) = \sqrt{3x + 7} - \sqrt{5 - 2x}$.

Explanation In order for the value of $f(x)$ to exist, we need BOTH $3x+7 \geq 0$ AND $5-2x \geq 0$.

$$\begin{aligned} 3x+7 &\geq 0 \\ 3x &\geq -7 \\ x &\geq -\frac{7}{3} \end{aligned}$$

$$\begin{aligned} 5-2x &\geq 0 \\ -2x &\geq -5 \\ x &\leq \frac{5}{2} \end{aligned}$$

The inequaltiy $x \geq -\dfrac{7}{3}$ has graph

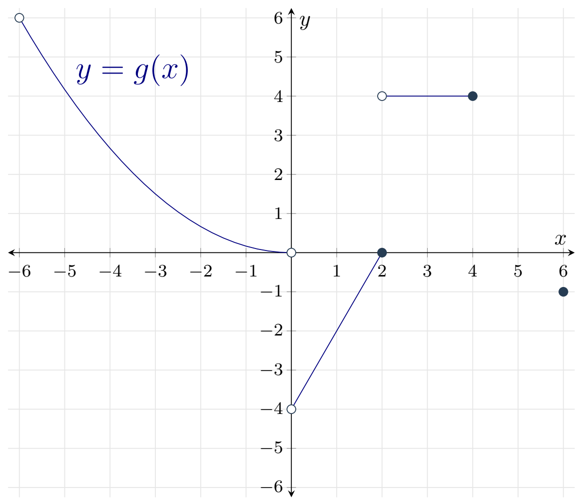
and the graph of $x \leq \dfrac{5}{2}$ has graph

The graph of the overlap (the interval where BOTH are true) is

The domain of f is $\left[-\dfrac{7}{3}, \dfrac{5}{2}\right]$.

Finally, isolated points are not included in intervals, but are written in the form $\{a\}$, and multiple disjoint intervals are connected using the *Union* symbol $\cup$.

Example 47. *The entire graph of a function g is given in the graph below. Find the domain of g.*

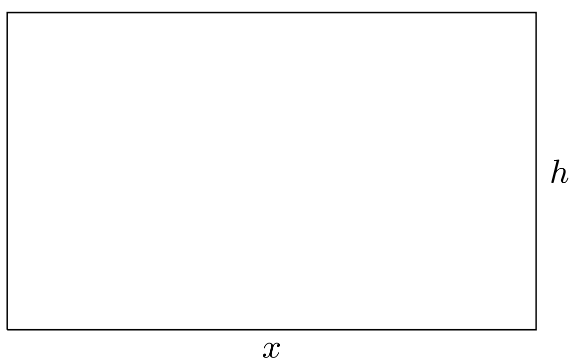


Explanation

Desmos link: https://www.desmos.com/calculator/re9re7dqew

Notice that $g(x)$ is defined for all x in $-6 < x < 0$, in $0 < x \leq 4$, and at $x = 6$. In interval notation, this is $(-6, 0) \cup (0, 4] \cup \{6\}$.

Example 48. *A piece of wire, 10 meters in length, is folded into a rectangle. Call x the width of the rectangle, as in the image below, and call h the height.*



Find a formula for the height as a function of x, $h(x)$. What is the domain of h?

Explanation

The wire forms the perimeter of the rectangle. Since the wire has length 10 meters, that means $2x + 2h = 10$. Solving this formula for h gives:

$$\begin{aligned}
2x + 2h &= 10 \\
2h &= 10 - 2x \\
h &= \frac{10 - 2x}{2} \\
&= \frac{10}{2} - \frac{2x}{2} \\
&= 5 - x.
\end{aligned}$$

The function h is given by $h(x) = 5 - x$.

Any number can be plugged into the formula $5 - x$, but we have to take into account where these quantities came from in the story. The value x was a length of a side of a rectangle. That means x cannot be negative. For a similar reason, $h(x)$ cannot be negative.

$$\begin{aligned}
h(x) &\geq 0 \\
5 - x &\geq 0 \\
-x &\geq -5 \\
x &\leq 5
\end{aligned}$$

If x has a value larger than 5, it would force $h(x)$ to be negative, which is impossible. The domain of h is $[0, 5]$.

Think about what it would mean for $x = 0$ or $x = 5$. The value $x = 0$ would correspond to a rectangle with width zero, and $x = 5$ would correspond to a rectangle of height zero (since $h(5) = 0$). For convenience, mathematicians often allow rectangles of width zero or height zero. If you are not comfortable with calling those things rectangles, you can use $(0, 5)$ as your domain instead.

The Domains of Famous Functions

Earlier you were introduced to the graphs of several "Famous Functions". We will revisit these functions over and over again throughout our studies. For now, we will formalize what we have seen with their graphs.

(a) The Absolute Value function - We can take the absolute value of any number. The Absolute Value function has domain $(-\infty, \infty)$.

(b) Polynomial functions - We can plug any number into a polynomial. All polynomials have domain $(-\infty, \infty)$.

(c) Rational functions - Remember that a rational function is one that can be written as fraction of two polynomials, with the denominator not the zero polynomial. The domain of a rational function consists of all real numbers for which the denominator is nonzero.

(d) The Square Root function - We can take the square root of any non-negative number. The square root function has domain $[0, \infty)$.

(e) Exponential functions - Exponential functions b^x, for $b > 0$ with $b \neq 1$, have domain $(-\infty, \infty)$.

(f) Logarithms - Logarithms have domain $(0, \infty)$. This is similar to the domain of $\sqrt{x}$, except the endpoint is not included.

(g) The Sine function - The sine function $\sin(x)$ has domain $(-\infty, \infty)$.

Spotting Values not in the Domain

Of our list of famous functions, notice that only rational functions, radicals, and logarithms have domain that is not the full set of all real numbers, $(-\infty, \infty)$. When trying to find the domain of a function constructed out of famous functions, this gives us some guidelines to follow. The following list is not exhaustive, but gives a good place to begin.

(a) The input of an even-index radical must be non-negative.

(b) The input of a logarithm must be positive.

(c) The denominator of a fraction cannot be zero.

(d) The real-world context. If a function has a real-world description, this may add additional restrictions on the input values. (You can see this in Example 48 above.)

Remember that the number zero is neither positive nor negative. The non-negative numbers are $[0, \infty)$, while the positive numbers are $(0, \infty)$.

Example 49. *Find the domain of the function*

$$f(x) = 3|x| - 5x^3 + 7x + \frac{2x+5}{x-1} + \ln(3-x).$$

Explanation Examine the individual terms. The first term is an absolute value function, while the second and third terms are polynomials. There is no restriction on their domain. The last two terms, however, are a fraction and a logarithm.

The denominator of the fraction cannot be zero, so

$$\begin{aligned} x - 1 &\neq 0 \\ x &\neq 1. \end{aligned}$$

The input to the logarithm must be positive, so

$$\begin{aligned} 3 - x &> 0 \\ -x &> -3 \\ x &< 3. \end{aligned}$$

In order for a number to be in the domain of the function, it must be in the domain of every term of the function. That means it must satisfy both $x \neq 1$ and $x < 3$. Altogether, this means the domain is $(-\infty, 1) \cup (1, 3)$.

Example 50. *Find the domain of the function*

$$s(t) = \frac{\ln(2t+3) - \sqrt{5t-1}}{t^2+1}$$

Explanation The denominator of this fraction is t^2+1. The graph of $y = t^2+1$ is an upward-opening parabola with vertex at the point $(0, 1)$. As such, the denominator does not have zero as an output. Our only restrictions will come from the numerator.

The input to the logarithm must be positive, so

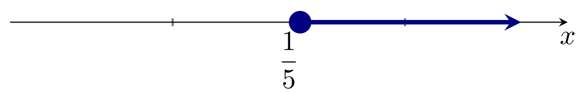
$$\begin{aligned} 2t + 3 &> 0 \\ 2t &> -3 \\ t &> -\frac{3}{2}. \end{aligned}$$

That inequality has graph given by

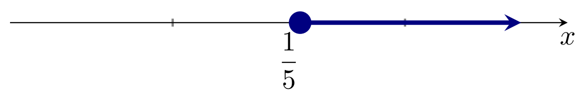
The radicand must be non-negative, so

$$\begin{aligned} 5t - 1 &\geq 0 \\ 5t &\geq 1 \\ t &\geq \frac{1}{5}. \end{aligned}$$

That inequality has graph given by



In order for a number to be in the domain of this function, satisfy both $t > -\dfrac{3}{2}$ and $t \geq \dfrac{1}{5}$. The points satisfying both inequalities are given in the graph



Altogether, this means the domain is $\left[\dfrac{1}{5}, \infty\right)$

Piecewise Defined Functions and Restricted Domains

Consider the function $f(x) = 2|x|+3$ for $x \geq -5$, and the function $g(x) = 2|x|+3$ (given without this restriction). The implied domain of g is $(-\infty, \infty)$, but what can we say about $f(-8)$? The formula $2|x| + 3$ makes sense when $x = -8$, but the function definition for f has the added statement "for $x \geq -5$". This is telling us the domain of f is $[-5, \infty)$. In this case $f(-8)$ is undefined.

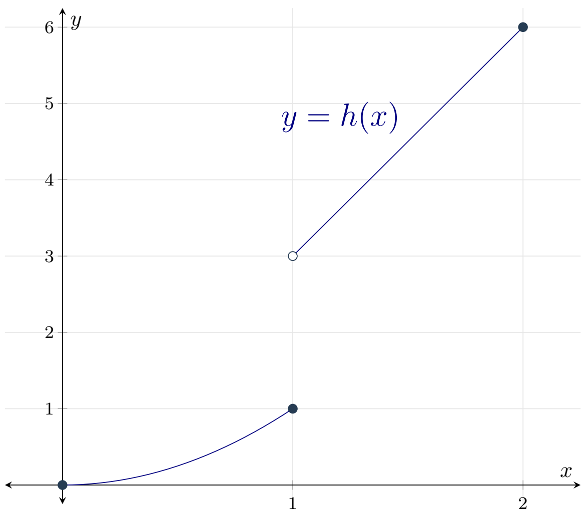
We can think of the function f as coming from the function g by deciding that some inputs are not valid. We have *restricted the domain*.

Suppose we have a function f given by $f(x) = x^2$ for $0 \leq x \leq 1$ (which has domain $[0, 1]$) and a different function g given by $g(x) = 3x$ for $1 < x \leq 2$ (which has domain $(1, 2]$). If we are given an x-value in the interval $[0, 2]$, that input can only be plugged into one of these two functions. Let's create a new function h by setting $h(x) = f(x) = x^2$ if $0 \leq x \leq 1$ and by setting $h(x) = g(x) = 3x$ if $1 < x \leq 2$. As a compact way of writing this, we would say:

$$h(x) = \begin{cases} x^2 & \text{for } 0 \leq x \leq 1 \\ 3x & \text{for } 1 < x \leq 2 \end{cases}$$

> **Definition** A **piecewise defined function** is a function that is given by different formulas for different intervals in its domain. This is sometimes shortened to just *piecewise function*.

The function h above is a piecewise defined function. On the interval $[0, 1]$ it is given by the formula x^2, and on the interval $(1, 2]$ it is given by the formula $3x$. It has two pieces, one piece is quadratic and the other piece is linear. The graph of the function h is given below.



Example 51. *Let f be the piecewise defined function given by*

$$f(x) = \begin{cases} 5 & \text{for } x \leq -2 \\ \sin(x) & \text{for } -2 < x < 3 \\ 2^x & \text{for } x > 4 \end{cases}.$$

What is the domain of f? Evaluate the following:

(a) $f(-5)$

(b) $f(0)$

(c) $f\left(\frac{\pi}{2}\right)$

(d) $f(4)$

(e) $f(5)$

Explanation

The function f is given as a piecewise defined function with three pieces. The first piece is used $x \leq -2$, the second piece is used when $-2 < x < 3$, and the third piece is used when $x > 4$. This function is defined for all numbers except those between 3 and 4.

The domain of this function is $(-\infty, 3) \cup (4, \infty)$.

(a) Since $-5 \leq -2$, this uses the first piece of the function, so $f(-5) = 5$.

(b) Since $-2 < 0 < 3$, $f(0) = \sin(0) = 0$.

(c) $\frac{\pi}{2}$ is between 1 and 2 (it's approximately 1.57), so $f\left(\frac{\pi}{2}\right) = \sin\left(\frac{\pi}{2}\right) = 1$.

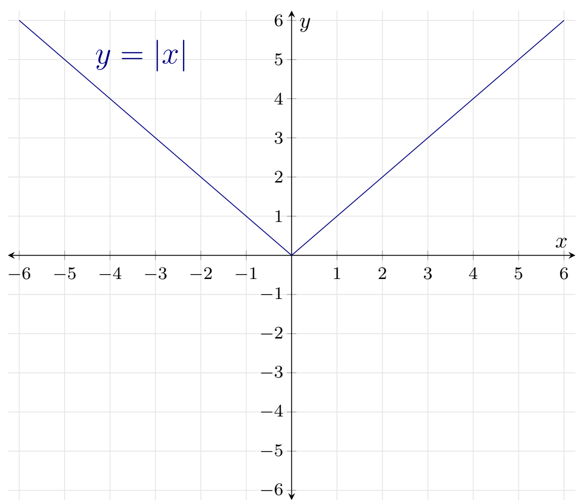
(d) 4 is not in the domain of f, so $f(4)$ is undefined.

(e) Since $5 > 4$, $f(5) = 2^5 = 32$.

Example 52. *Write the absolute value function as a piecewise defined function.*

Explanation

Let's examine the graph of $y = |x|$.



Do you notice that this graph looks like two straight lines, meeting at the origin? Let's focus on the right-hand side first. For $x \geq 0$, this is a line with slope $m = 1$ and y-intercept at the origin $(0, 0)$. This line has equation $y = 1x + 0 = x$. For $x < 0$, this is a line with slope $m = -1$ and y-intercept at the origin $(0, 0)$. This line has equation $y = -1x + 0 = -x$.

That means $|x|$ agrees with x if $x \geq 0$, and agrees with $-x$ if $x < 0$. Putting these together gives us:

$$|x| = \begin{cases} -x & \text{for } x < 0 \\ x & \text{for } x \geq 0. \end{cases}$$

This formula tells us that the absolute value of a positive number is itself, while the absolute value of a negative number changes the sign.

5.2 Domain

Learning Objectives

- Range
  - Definition of the Range
  - Ranges of Famous Functions
  - Spotting Values not in the Range

5.2.1 Range

Motivating Questions

- If f is a function from a set A to a set B, does every item in B actually get related to something from A?

Introduction

In the last section, for a function from A to B, we called the set A the *domain* and we called the set B the *codomain*.

Let f be the function defined by $f(x) = x^2$. We can consider this a function from the set of all real numbers $(-\infty, \infty)$ to the set of all real numbers $(-\infty, \infty)$. In this case, the domain is $(-\infty, \infty)$ and the codomain is also $(-\infty, \infty)$. We know that for any real number x, the value of x^2 is never negative. That means there is no input to f that ever gives a negative output.

Let g be the function from the set of capital letters to the set of natural numbers, which assigns each letter to its placement in the alphabet. This means $g(A) = 1$ since 'A' is the first letter of the alphabet. Similarly $g(B) = 2$ and $g(Z) = 26$. In this case the domain is the set of capital letters $\{A, B, C, \ldots, Z\}$ and the codomain is the set of natural numbers $\{1, 2, 3, 4, \ldots\}$. For the function g there are only 26 capital letters in the alphabet, so no number past greater than 26 is ever an output of g.

For both the function f and g just given, not every number in the codomain is actually achieved as the output of the function. There is a difference between the codomain, which measures the "possible outputs" and the actual outputs that are achieved.

Exploration

a. Suppose the quadratic function f is given by $f(x) = x^2$. Are there any values that are never achieved as an output?

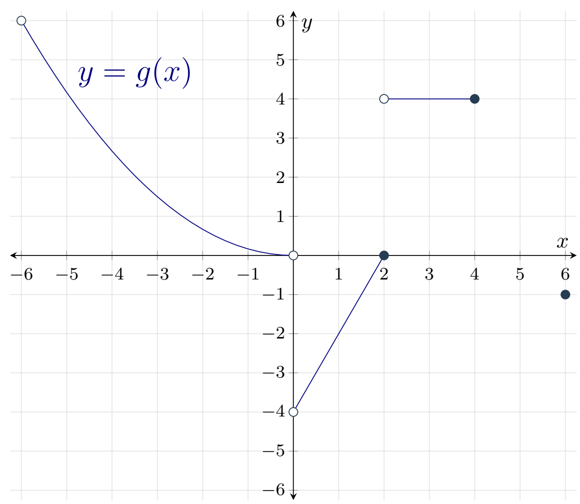
b. Explain the difference in finding the domain of a function and finding the range of the function, if you are given the graph of the function. What if you're given a formula for the function instead?

The Range of a Function

Definition Let f be a function from A to B. The **range** of f is the collection of the outputs of f.

This means the *range* consists of the outputs that are actually achieved. Not everything that is "possible", but only those outputs that actually come out of the function. For each b in the range of the function f, there is actually an a in the domain with $f(a) = b$.

Example 53. *The entire graph of a function g is given in the graph below. Find the range of g.*



Explanation

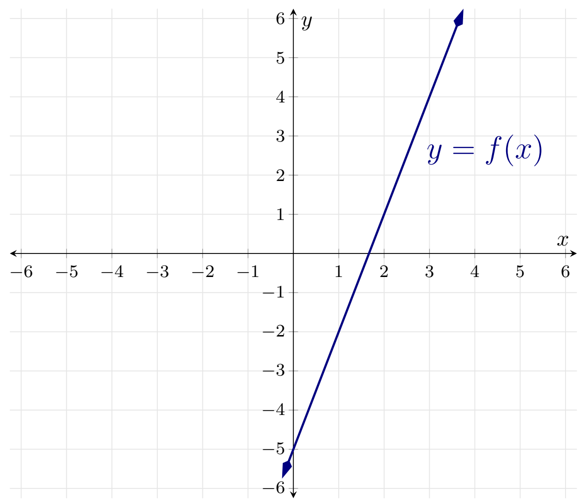
Desmos link: `https://www.desmos.com/calculator/fsdehrzpe6`

Notice that as x changes between -6 to 0, the graph takes all outputs from 6 to 0 (not including the endpoints). As x changes from 0 to 2, all the numbers from -4 to 0 show up as outputs (including 0). Together, this means that every number in the interval $(-4, 6)$ is in the range. The last two pieces of the graph have outputs 4 and -1, which are already included in this interval.

The range is $(-4, 6)$.

Example 54. *Let f be the function given by $f(x) = 3x - 5$. Find the range of f.*

Explanation We know the domain of f is $(-\infty, \infty)$, so any real number is a valid input. We also know that f is a linear polynomial function (a polynomial of degree 1) so we know what its graph is a straight line with slope $m = 3$.



Since this graph goes higher and higher as x travels to the right, and lower as x travels to the left, we believe that every number is eventually an output of this function.

To make that a bit more formal, we'll take an arbitrary real number and show that it's actually an output of f. If a and b are real numbers with $f(a) = b$, then

$$\begin{aligned} f(a) &= b \\ 3a - 5 &= b \\ 3a &= b + 5 \\ a &= \frac{b+5}{3}. \end{aligned}$$

That means that if b is any arbitrary real number, then $f\left(\frac{b+5}{3}\right) = b$ so that b is achieved as an output of the function, so b is in the range.This means the range of f is $(-\infty, \infty)$.

The Range of Famous Functions

(a) The Absolute Value function - The average value of a number is never negative. The Absolute Value function has range $[0, \infty)$.

(b) Polynomial functions - This depends on the degree of the polynomial.

  (i) Odd degree - The range is $(-\infty, \infty)$.

(ii) Even degree - We can only be precise with monomials (polynomials with only one term) like $5x^2$ or $-6x^8$.

i. If the monomial has positive coefficient, the range is $[0, \infty)$.

ii. If the monomial has negative coefficient, the range is $(-\infty, 0]$.

(c) The Square Root function - Even-index radicals never have negative outputs. Their range is $[0, \infty)$.

(d) Exponential functions - Exponential functions b^x, for $b > 0$ with $b \neq 1$, have range $(0, \infty)$. Notice that 0 is never an output for these kinds of functions.

(e) Logarithms - Logarithms have range $(-\infty, \infty)$.

(f) The Sine function - The sine function $\sin(x)$ has range $[-1, 1]$.

Spotting Values not in the Range

Finding the range of a function is quite a bit more involved than finding the domain. Here are some guidelines if you are given a formula for the function instead of its graph.

(a) The output of an even-index radical is never negative.

(b) The output of x^n is never negative if n is an even natural number.

(c) The output of an exponential function b^x is always positive.

Example 55. *Find the range of the following functions*

(a) $f(x) = 2 - 4\sqrt{x}$.

(b) $g(x) = e^x + \frac{1}{2}$.

(c) $h(x) = 3 + 5\sin(x)$.

Explanation

You'll notice in these calculations, that finding the range of a function is a great deal more complicated than finding the domain, unless we have an accurate graph of the function, as above.

For each of these calculations, we will follow the same two steps. The idea is that the first step shows that the range is "no more than" the interval we build, and the second step shows that the range is "no less than" that interval. The only possibility left to us is that the range is exactly the interval we have constructed.

More specifically, in the first step will find a bound on the range. We'll determine if any numbers are too big or too small to be a valid output of the function. This will give us an interval that the range will have to be inside. In the second step, we'll see that everything inside that interval is actually attained by the function, by constructing an input value that gets assigned to that output.

(a) We know that the range of $\sqrt{x}$ is $[0, \infty)$ so for any x in the domain of f,

$$\begin{aligned} \sqrt{x} &\geq 0 \\ -4\sqrt{x} &\leq 0 \\ 2 - 4\sqrt{x} &\leq 2 \\ f(x) &\leq 2 \end{aligned}$$

The outputs of f are never larger than 2, so the only numbers in the range are less than or equal to 2. That is, the range must be inside the interval $(-\infty, 2]$.

To verify that the range is exactly $(-\infty, 2]$, suppose $b \leq 2$, then:

$$\begin{aligned} f(a) &= b \\ 2 - 4\sqrt{a} &= b \\ -4\sqrt{a} &= b - 2 \\ \sqrt{a} &= \frac{b-2}{-4} \\ \sqrt{a} &= \frac{-(b-2)}{4} \\ \sqrt{a} &= \frac{-b+2}{4} \\ \left(\sqrt{a}\right)^2 &= \left(\frac{-b+2}{4}\right)^2 \\ a &= \left(\frac{-b+2}{4}\right)^2. \end{aligned}$$

For this value of a, we have $f(a) = b$. That means b is in the range of f. That means every number in the interval $(-\infty, 2]$ is in the range of f.

The range of f is $(-\infty, 2]$.

(b) The range of e^x is $(0, \infty)$. That means for any value of x,

$$\begin{aligned} e^x &> 0 \\ e^x + \frac{1}{2} &> \frac{1}{2} \end{aligned}$$

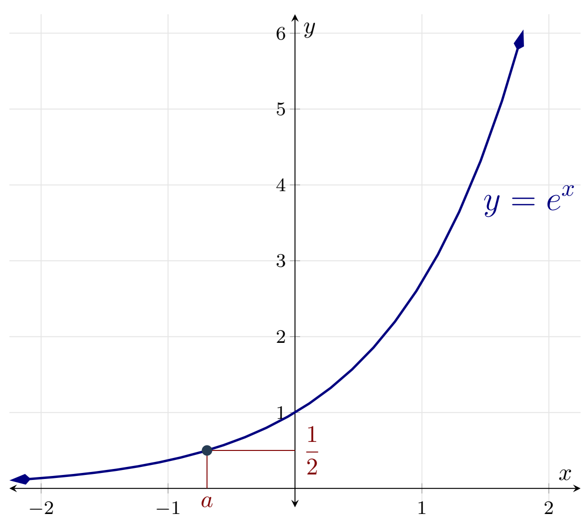
The outputs of g are greater than $\frac{1}{2}$, so the range of g is in the interval $\left(\frac{1}{2}, \infty\right)$.

To verify that the range is exactly $\left(\frac{1}{2}, \infty\right)$, suppose $b > \frac{1}{2}$, then:

$$\begin{aligned} g(a) &= b \\ e^a + \frac{1}{2} &= b \\ e^a &= b - \frac{1}{2} \end{aligned}$$

Since the range of e^x is $(0, \infty)$, and $b - \frac{1}{2} > 0$, there is a value for a with $e^a = b - \frac{1}{2}$.

(For instance, if $b = 1$, that would mean $b - \frac{1}{2} = \frac{1}{2}$, so we would be looking to see if there is a value of a with $e^a = \frac{1}{2}$. This is illustrated in the graph below.)



For this value of a, we have $g\left(a\right) = b$, meaning that b is in the range of g. The range of g is $\left(\frac{1}{2}, \infty\right)$.

(c) The range of sin is $[-1, 1]$. That means for any x

$$\begin{aligned}-1 &\leq \sin(x) \leq 1\\ -5 &\leq 5\sin(x) \leq 5\\ 3+(-5) &\leq 3+5\sin(x) \leq 3+5\\ -2 &\leq 3+5\sin(x) \leq 8\end{aligned}$$

That means the outputs of h are in the interval $[-2, 8]$. To verify that the range is exactly $[-2, 8]$, suppose b is a number with $-2 \leq b \leq 8$. Then:

$$\begin{aligned}h(a) &= b\\ 3+5\sin(a) &= b\\ 5\sin(a) &= b-3\\ \sin(a) &= \frac{b-3}{5}.\end{aligned}$$

Because $-2 \leq b \leq 8$, we know $-1 \leq \frac{b-3}{5} \leq 1$. That means there is a number a with $\sin(a) = \frac{b-3}{5}$, since the range of sin is $[-1, 1]$.

(For instance, if $b = 3$ then $\frac{b-3}{5} = 0$ so we would be looking for a value of a with $\sin(a) = 0$. We know that $\sin(0) = 0$, so this means we'd take $a = 0$.) For this value of a, we have $h(a) = b$, so b is in the range of h. The range of h is then $[-2, 8]$.

5.3 Composition of Functions

Learning Objectives

- Composition of Functions
  - What does it mean to compose functions?
  - Identify a function as the result of a composition of functions
- Domains of Composite Functions
  - How to find the domain of a composite function
  - How to find the range of a composite function
  - Results of composing $f(x) = x^2$ and $g(x) = \sqrt{x}$

5.3.1 Composition of Functions

Motivating Questions

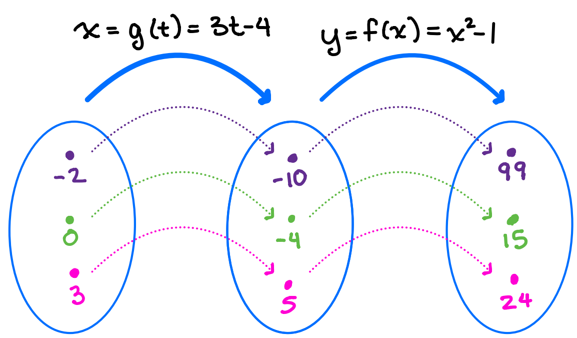
- How does the process of function composition produce a new function from two other functions?
- In the composite function $h(x) = f(g(x))$, what do we mean by the "inner" and "outer" function?
- How does the expression for $AV_{[a,a+h]}$ involve a composite function?

Introduction

Recall that a function, by definition, is a process that takes a collection of inputs and produces a corresponding collection of outputs in such a way that the process produces one and only one output value for any single input value. Because every function is a process, it makes sense to think that it may be possible to take two function processes and do one of the processes first, and then apply the second process to the result.

Example 56. *Suppose we know that y is a function of x according to the process defined by $y = f(x) = x^2 - 1$ and, in turn, x is a function of t via $x = g(t) = 3t - 4$. Is it possible to combine these processes to generate a new function so that y is a function of t?*

Explanation Since y depends on x and x depends on t, it follows that we can also think of y depending directly on t. Let's looks at this as an arrow diagram with a few sample points.



Notice that if we take a point such as $t = 2$, we can put that value in for t in the function $x = g(t) = 3t - 4$. This will give

$$x = g(-2) = 3(-2) - 4 = -6 - 4 - 10.$$

. Now we have an x-value of -10. $g(x)$ take in x-values so we can put -10 into $f(x) = x^2 - 1$. This will give

$$f(-10) = (-10)^2 - 1 = 100 - 1 = 99.$$

You should verify that the arrow diagram above gives the correct values of y that cooresponds to $t = 0$ and $t = 3$.

Now, we would like to create a new function that will directly take in any t value and give us the corresponding y value. We can use substitution and the notation of functions to determine this function.

First, it's important to realize what the rule for f tells us. In words, f says "to generate the output that corresponds to an input, take the input and square it, and then subtract 1." In symbols, we might express f more generally by writing "$f(\square) = \square^2 - 1$."

Now, observing that $y = f(x) = x^2 - 1$ and that $x = g(t) = 3t - 4$, we can substitute the expression $g(t)$ for x in f. Doing so,

$$\begin{aligned} y &= f(x) \\ &= f(g(t)) \\ &= f(3t-4). \end{aligned}$$

Applying the process defined by the function f to the input $3t - 4$, we see that

$$y = (3t-4)^2 - 1,$$

which defines y as a function of t.

One way to think about the substitution above is that we are putting the entire expression $3t-4$ inside the input box in "$f(\square) = \square^2 - 1$." That is, $f\left(\boxed{3t-4}\right) = \left(\boxed{3t-4}\right)^2 - 1$. For the substitution, we are thinking of $3t-4$ as a single object!

When we have a situation such as in the example above where we use the output of one function as the input of another, we often say that we have **composed** two functions. In addition, we use the notation $h(t) = f(g(t))$ to denote that a new function, h, results from composing the two functions f and g.

Exploration

a. Let $y = p(x) = 3x - 4$ and $x = q(t) = t^2$. Determine a formula for r that depends only on t and not on p or q. What is the biggest difference between your work in this problem compared to the example above?

b. Let $t = s(z) = \dfrac{1}{t+4}$ and recall that $x = q(t) = t^2$. Determine a formula for $x = q(s(z))$ that depends only on z.

c. Suppose that $h(t) = \sqrt{2t^2 + 5}$. Determine formulas for two related functions, $y = f(x)$ and $x = g(t)$, so that $h(t) = f(g(t))$.

Composing Two Functions

Whenever we have two functions, g and f, where the outputs of g match inputs of f, it is possible to link the two processes together to create a new process that we call the *composition* of f and g.

Definition If f and g are functions, we define the **composition of** f **and** g to be the new function h given by

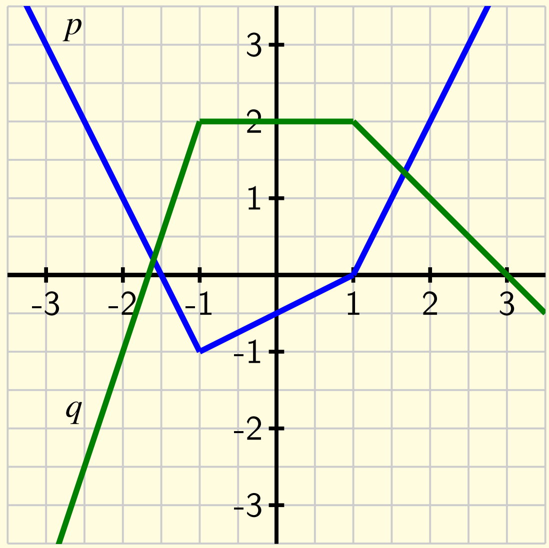
$$h(t) = f(g(t)).$$

This composition is denoted by $h = f \circ g$, where $f \circ g$ means the single function defined by $(f \circ g)(t) = f(g(t))$.

We sometimes call g the "inner function" and f the "outer function". It is important to note that the inner function is actually the first function that gets applied to a given input, and then then outer function is applied to the output of the inner function. In addition, in order for a composite function to make sense, we need to ensure that the outputs of the inner function are values that it makes sense to put into the outer function so that the resulting composite function is defined.

In addition to the possibility that functions are given by formulas, functions can be given by tables or graphs. We can think about composite functions in these settings as well, and the following activities prompt us to consider functions given in this way.

Exploration Let functions p and q be given by the graphs below (which are each piecewise linear - that is, parts that look like straight lines are straight lines) and let f and g be given by the table below.



x	$f(x)$	$g(x)$
0	6	1
1	4	3
2	3	0
3	4	4
4	6	2

Compute each of the following quantities or explain why they are not defined.

a. $p(q(0))$

b. $q(p(0))$

c. $(p \circ p)(-1)$

d. $(f \circ g)(2)$

e. $(g \circ f)(3)$

f. $g(f(0))$

g. For what value(s) of x is $f(g(x)) = 4$?

h. For what value(s) of x is $q(p(x)) = 1$?

Composing functions in content

In the late 1800s, the physicist Amos Dolbear was listening to crickets chirp and noticed a pattern: how frequently the crickets chirped seemed to be connected to the outside temperature. If we let T represent the temperature in degrees Fahrenheit and N the number of chirps per minute, we can summarize Dolbear's observations with the following function, $T = D(N) = 40 + 0.25T$. Scientists who made many additional cricket chirp observations following Dolbear's initial counts found that this formauls holds with remarkable accuracy for the snowy tree cricket in temperatures ranging from 50° to 85°. This function is called Dolbear's Law.

In what follows, we replace T with F to emphasize that temperature is measured in Fahrenheit degrees.

The Celcius and Fahrenheit temperature scales are connected by a linear function. Indeed, the function that converts Fahrenheit to Celcius is

$$C = G(F) = \frac{5}{9}(F - 32).$$

For instance, a Fahrenheit temperature of 32 degrees corresponds to $C = G(32) = \frac{5}{9}(32 - 32) = 0$ degrees Celcius.

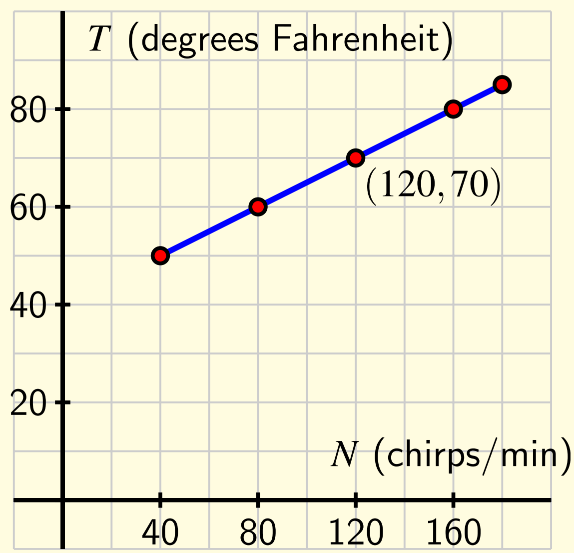
Exploration Let $D(N) = 40 + 0.25N$ be Dolbear's function that converts an input of number of chirps per minute to degrees Fahrenheit, and let $G(F) = \frac{5}{9}(F - 32)$ be the function that converts an input of degrees

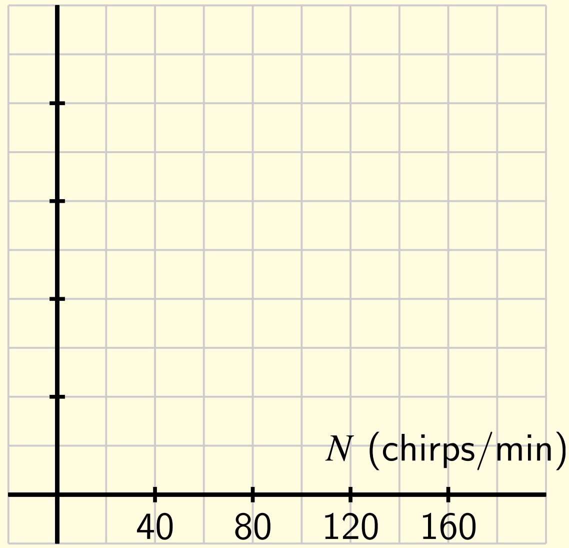
Fahrenheit to an output of degrees Celcius.

a. Determine a formula for the new function $(G \circ D)(N)$ that depends only on the variable N.

b. What is the meaning of the function you found in (a)?

c. Let $H = G \circ D$. How does a plot of the function H compare to the that of Dolbear's function? Sketch a plot of H on the blank axes to the right of the plot of Dolbear's function, and discuss the similarities and differences between them. Be sure to label the vertical scale on your axes.



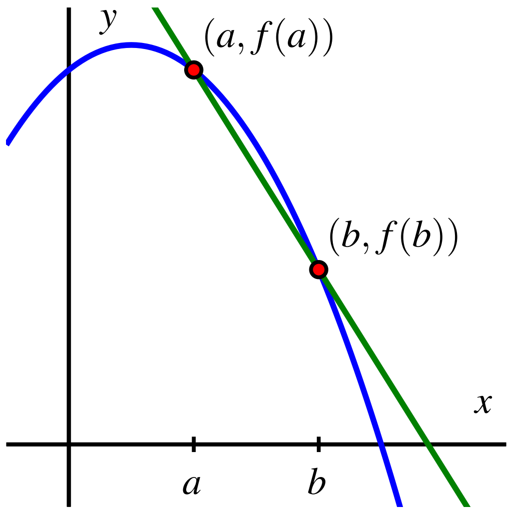


Function Composition and Average Rate of Change

Recall that the average rate of change of a function f on the interval $[a, b]$ is given by

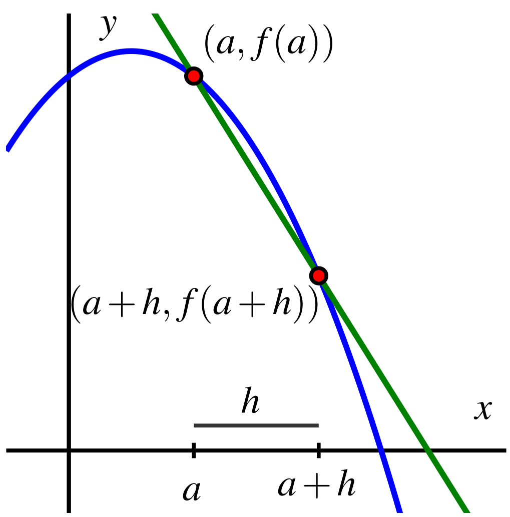
$$AV_{[a,b]} = \frac{f(b) - f(a)}{b - a}.$$

In the graph below, we see the familiar representation of $AV_{[a,b]}$ as the slope of the line joining the points $(a, f(a))$ and $(b, f(b))$ on the graph of f.



In the study of calculus, we progress from the *average rate of change on an interval* to the *instantaneous rate of change of a function at a single value*; the core idea that allows us to move from an *average* rate to an *instantaneous* one is letting the interval $[a, b]$ shrink in size.

To think about the interval $[a, b]$ shrinking while a stays fixed, we often change our perspective and think of b as $b = a + h$, where h measures the horizontal difference from b to a.



This allows us to eventually think about h getting closer and closer to 0, and in that context we consider the equivalent expression

$$AV_{[a,a+h]} = \frac{f(a+h) - f(a)}{a + h - a} = \frac{f(a+h) - f(a)}{h}$$

for the average rate of change of f on $[a, a+h]$.

Example 57. *Suppose that $f(x) = x^2$. Determine the simplest possible expression you can find for $AV_{[3,3+h]}$, the average rate of change of f on the interval $[3, 3+h]$.*

Explanation By definition, we know that

$$AV_{[3,3+h]} = \frac{f(3+h) - f(3)}{h}.$$

Using the formula for f, we see that

$$AV_{[3,3+h]} = \frac{(3+h)^2 - (3)^2}{h}.$$

Expanding the numerator and combining like terms, it follows that

$$\begin{aligned} AV_{[3,3+h]} &= \frac{(9 + 6h + h^2) - 9}{h} \\ &= \frac{6h + h^2}{h}. \end{aligned}$$

Removing a factor of h in the numerator and observing that $h \neq 0$, we can simplify and find that

$$\begin{aligned} AV_{[3,3+h]} &= \frac{h(6+h)}{h} \\ &= 6 + h. \end{aligned}$$

Hence, $AV_{[3,3+h]} = 6 + h$, which is the average rate of change of $f(x) = x^2$ on the interval $[3, 3+h]$.

Exploration Let $f(x) = 2x^2 - 3x + 1$ and $g(x) = \dfrac{5}{x}$.

a. Compute $f(1+h)$ and expand and simplify the result as much as possible by combining like terms.

b. Determine the most simplified expression you can for the average rate of change of f on the interval $[1, 1+h]$. That is, determine $AV_{[1,1+h]}$ for f and simplify the result as much as possible.

c. Compute $g(1+h)$. Is there any valid algebra you can do to write $g(1+h)$ more simply?

d. Determine the most simplified expression you can for the average rate of change of g on the interval $[1, 1+h]$. That is, determine $AV_{[1,1+h]}$ for g and simplify the result.

Summary

- When defined, the composition of two functions f and g produces a single new function $f \circ g$ according to the rule $(f \circ g)(x) = f(g(x))$. We note that g is applied first to the input x, and then f is applied to the output $g(x)$ that results from g.

- In the composite function $h(x) = f(g(x))$, the "inner" function is g and the *outer* function is f. Note that the inner function gets applied to x first, even though the outer function appears first when we read from left to right.

- Because the expression $AV_{[a,a+h]}$ is defined by

  $$AV_{[a,a+h]} = \frac{f(a+h) - f(a)}{h}$$

  and this includes the quantity $f(a+h)$, the average rate of change of a function on the interval $[a, a+h]$ always involves the evaluation of a composite function expression. This idea plays a crucial role in the study of calculus.

5.3.2 Domains and Ranges of Composite Functions

Motivating Questions

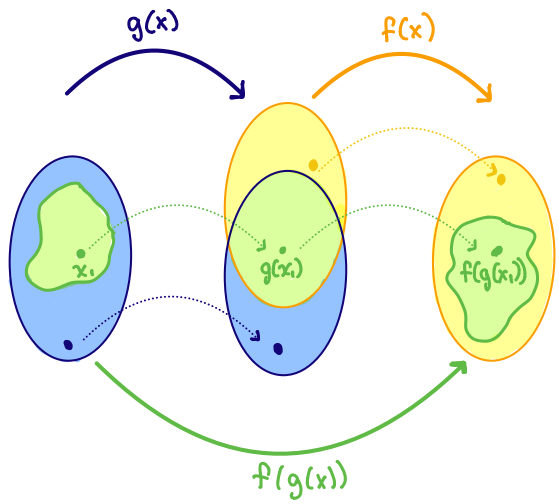
- How does the process of function composition effect the domain of the function?
- How does the process of function composition effect the range of the function?

Domains of Composite Functions

The domain of a composite function such as $f \circ g$ is dependent on the domain of g and the domain of f. It is important to know when we can apply a composite function and when we cannot, that is, to know the domain of a function such as $f \circ g$.

Let us assume we know the domains of the functions f and g separately. If we write the composite function for an input x as $f(g(x))$, we can see right away that x must be a member of the domain of g in order for the expression to be meaningful, because otherwise we cannot complete the inner function evaluation. However, we also see that $g(x)$ must be a member of the domain of f, otherwise the second function evaluation in $f(g(x))$ cannot be completed, and the expression is still undefined. Thus the domain of $f \circ g$ consists of only those inputs in the domain of g that produce outputs from g belonging to the domain of f. Note that the domain of f composed with g is the set of all x such that x is in the domain of g and $g(x)$ is in the domain of f.

The domain of a composite function $f(g(x))$ is the set of those inputs x in the domain of g for which $g(x)$ is in the domain of f.



To find the domain of a composite function, $f \circ g$, you can follow these three steps:

1) Find the domain of g.

2) Find the domain of f.

3) Find those inputs x in the domain of g for which $g(x)$ is in the domain of f. That is, exclude those inputs x from the domain of g for which $g(x)$ is not in the domain of f. The resulting set is the domain of $f \circ g$.

Example 58. *Find the domain of* $f \circ g$ *where* $f(x) = \dfrac{5}{x-1}$ *and* $g(x) = \dfrac{4}{3x-2}$.

Explanation The domain of g consists of all real numbers except $x = \dfrac{2}{3}$, since that input value would cause us to divide by 0. Likewise, the domain of f consists of all real numbers except 1. We need to exclude from the domain of g any value of x for which $g(x) = 1$.

$$\begin{aligned}\frac{4}{3x-2} &= 1\\ 4 &= 3x-2\\ 6 &= 3x\\ x &= 2\end{aligned}$$

So the domain of $f \circ g$ is the set of all real numbers except $\dfrac{2}{3}$ and 2. This means that

$$x \neq \frac{2}{3} \text{ or } x \neq 2$$

We can write this in interval notation as

$$\left(-\infty, \frac{2}{3}\right) \cup \left(\frac{2}{3}, 2\right) \cup (2, \infty)$$

Example 59. *Find the domain of* $f \circ g$ *where* $f(x) = \sqrt{x+2}$ *and* $g(x) = \sqrt{3-x}$.

Explanation

Because we cannot take the square root of a negative number, the domain of g is $(-\infty, 3]$. Now we check the domain of the composite function

$$(f \circ g)(x) = \sqrt{\sqrt{3-x}+2}$$

For $(f \circ g)(x) = \sqrt{\sqrt{3-x}+2}$, we need $\sqrt{3-x}+2 \geq 0$, since the inside of a square root cannot be negative. Since square roots are non-negative, $\sqrt{3-x} \geq 0$ sp $\sqrt{3-x}+2 \geq 0$ as long as $\sqrt{3-x}$ exists. That means $3-x \geq 0$, which gives a domain of $(-\infty, 3]$.

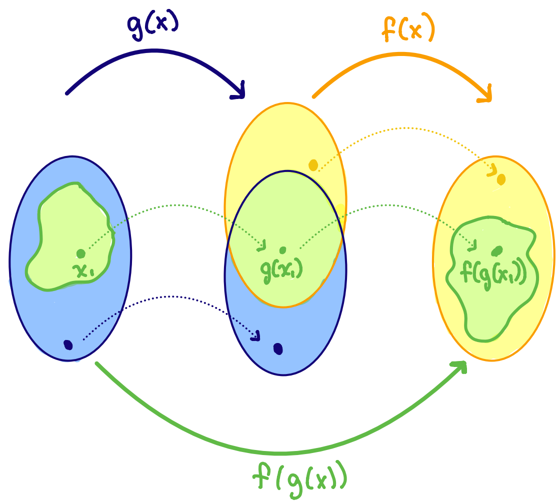
This example shows that knowledge of the range of functions (specifically the inner function) can also be helpful in finding the domain of a composite function. It also shows that the domain of $f \circ g$ can contain values that are not in the domain of f, though they must be in the domain of g.

Exploration Find the domain of $f \circ g$ where $f(x) = \dfrac{1}{x-2}$ and $g(x) = \sqrt{x+4}$.

Ranges of Composite Functions

The range of a composite function such as $f \circ g$ is dependent on the range of g and the range of f. It is important to know what values can result from a composite function, that is, to know the range of a function such as $f \circ g$.

Let us assume we know the ranges of the functions f and g separately. If we write the composite function for an input x as $f(g(x))$, we can see right away that $f(g(x))$ must be a member of the range of f since we will input the value $g(x)$ into f. However, we also see that it is possible that not all values in the range of f are in the range of $f(g(x))$.



From the image above, we can see that there might be values in the yellow region which are in the range of f but for which there are no x values for which $f(g(x))$ gives that output.

> The range of a composite function $f \circ g$ is a subset of the range of f.

To find the domain of a composite function, $f \circ g$, you can follow these three steps:

1) Find the range of g.
2) Find the range of f.
3) Restrict the domain of f to the *range* of g and then determine the outputs of f of these values.

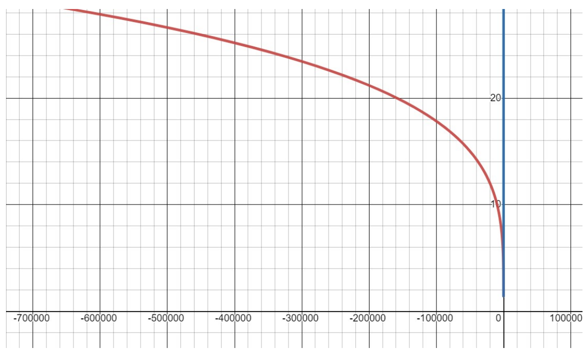
Example 60. *Find the range of $f \circ g$ where $f(x) = \sqrt{x+2}$ and $g(x) = \sqrt{3-x}$.*

Explanation Because the output of a square root is always a positive number, the range of g is $[0, \infty)$. Similarly, the range of f is $[0, \infty)$. But now we must think about what happens when we restrict the input of f to values in the range of g, $[0, \infty)$. If $x \geq 0$, then $x + 2 \geq 2$. Taking the square root of both sides, we see that possible outputs of $f(g(x))$ will be $\sqrt{x+2} \geq \sqrt{2}$. That is, the range of $f \circ g$ is $[\sqrt{2}, \infty)$.

If we look at this function in Desmos, we can confirm graphically that this answer makes sense. What we want to do is think about collapsing the graph unto the y-axis. The range of the function will be the y-values that correspond to a point (x, y) on the curve.

First, we graph the function using a standard window.

This allows us to see the domain pretty well. In the previous example, we found the domain to be $(-\infty, 3]$ and if we collapse this function to the x-axis, it looks like the x-values that correspond to points on this curve are exactly the x in $(-\infty, 3]$. If might be difficult to tell the domain from this graph, though. Let's zoom out some.



Here is the same graph in Desmos, so you can zoom in and out yourself.

Desmos link: `https://www.desmos.com/calculator/0wf1e4yyhf`

You can now see that the blue line is showing this graph collapsed to the y-axis. We can tell that the range will be positive numbers above some value between 1 and 2. This corresponds with our result above of $[\sqrt{2}, \infty)$. In order to find the exact point $\sqrt{2}$ where the interval begins or to confirm that the interval really goes to infinity, we need to do the reasoning above.

Composing $f(x) = x^2$ and $g(x) = \sqrt{x}$

This final example is going to be a very important one that comes up often so we we will give it its own section.

Example 61. *Let $f(x) = x^2$ and let $g(x) = \sqrt{x}$.*

a. *Find the domain and range of $f \circ g$ and compare this function to $id(x) = x$ and $abs(x) = |x|$.*

b. *Find the domain and range of $g \circ f$ and compare this function to $id(x) = x$ and $abs(x) = |x|$.*

Explanation You probably have the idea that the squaring and squarerooting actions undo one another. This is true for nonnegative values of x, but can get tricky when x is allowed to be negative. Let's look at each of these situations closely.

a. First we consider $f \circ g$ and compare this function to $id(x) = x$ and $abs(x) = |x|$. We have $f(x) = x^2$ and $g(x) = \sqrt{x}$ so

$$(f \circ g)(x) = f(g(x)) = (g(x))^2 = (\sqrt{x})^2.$$

Let's consider the domain of this function. Recall that the domain of a composite function $f \circ g$ is the set of those inputs x in the domain of g for which $g(x)$ is in the domain of f. In this case, this means that the domain of $f(g(x)) = (\sqrt{x})^2$ is the set of those inputs x in the domain of $g(x) = \sqrt{x}$ for which $\sqrt{x}$ is in the domain of $f(x) = x^2$. The implied domain of $g(x) = \sqrt{x}$ is $[0, \infty)$ since we cannot take the square root of a negative number. Therefore, since the domain of the composition has to be only values in the domain of $g(x)$, this means the largest our domain can be is $[0, \infty)$. Now, the only additional limiting factor is that the values $\sqrt{x}$ must be in the domain of f but since the domain of f is all real numbers, that will not limit the domain of the composition. Therefore, the domain of $f \circ g$ is $[0, \infty)$.

Now that we know the domain and we know that squaring and squarerooting undo each other for nonnegative values of x, we can conclude that $f \circ g$ is the identity function, $id(x) = x$ but restricted to the domain $[0, \infty)$. That is,

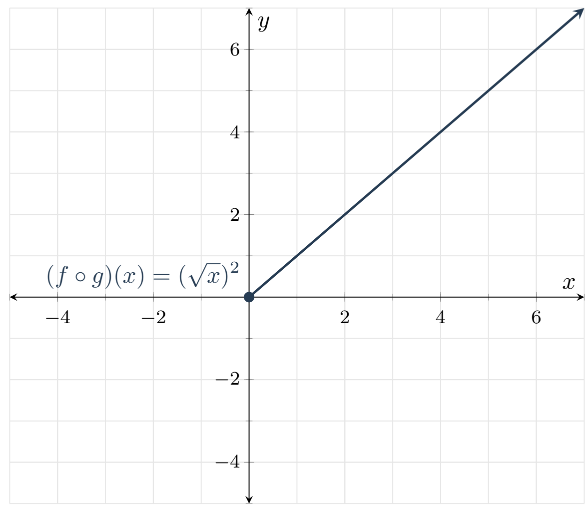
$$(f \circ g)(x) = (\sqrt{x})^2 = x, x \geq 0$$

Since the absolute value function is the same as the identity function when $x \geq 0$. Therefore, we could also say that

$$(f \circ g)(x) = (\sqrt{x})^2 = |x|, x \geq 0$$

From this information, we also know that the range of $f \circ g$ will also be $[0, \infty)$, since $(f \circ g)(x) = (\sqrt{x})^2$.

Here is a graph of $(f \circ g)(x) = (\sqrt{x})^2$.



b. Now we consider $g \circ f$ and compare this function to $id(x) = x$ and $abs(x) = |x|$. The domain of $g(f(x)) = \sqrt{x^2}$ is the set of those inputs x in the domain of $f(x) = x^2$ for which x^2 is in the domain of $g(x) = \sqrt{x}$. The domain of $f(x) = x^2$ is all real numbers, so this does not reduce the domain of the composite function. The range of $f(x) = x^2$ is $[0, \infty)$ since the square of every number will be greater than or equal to zero. The implied domain of $g(x) = \sqrt{x}$ is $[0, \infty)$. Therefore, every output from $f(x) = x^2$ is in the domain of $f(x) = \sqrt{x}$. Therefore, the domain of $g \circ f$ is $(-\infty, \infty)$.

   Now, let's consider the range of $g \circ f$. We know that the range of $g \circ f$ must be contained in the range of $g(x) = \sqrt{x}$. The range of $g(x) = \sqrt{x}$ is $[0, \infty)$, so that is the largest range possible for $g \circ f$. We know that for values of $x \geq 0$, squaring and squarerooting undo one another so we know that all the values of $[0, \infty)$ are contained in the range of $g \circ f$. More precisely, for any value x_0 in $[0, \infty)$, $g(f(x_0)) = x_0$ so x_0 will be in the range of $g \circ f$. Thus, the range of $g \circ f$ is $[0, \infty)$.

   Now, since we know that this function $g \circ f$ only outputs positive numbers, we know it cannot equal the identity function for inputs of $x < 0$. Let's explore what this function does for values of $x < 0$ by considering $x = -2$.

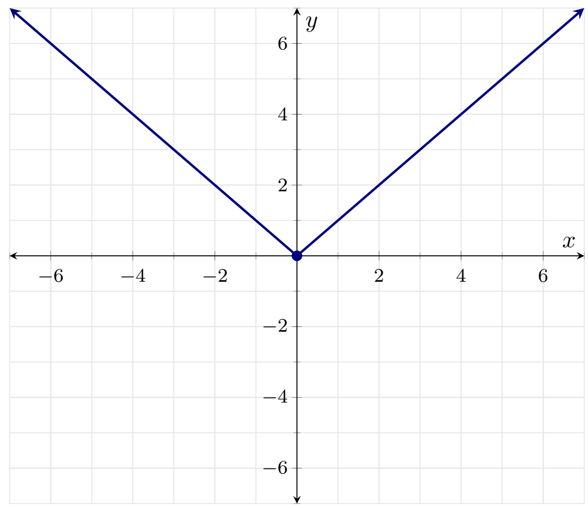
$$g(f(-2)) = \sqrt{(-2)^2} = \sqrt{4} = 2$$

   Notice, that we input $x = -2$ but the output was positive 2. In fact, for all values of $x < 0$, $g(f(x)) = \sqrt{x^2} = -x = |x|$.

   Since the absolute value function is the same as the identity function when $x \geq 0$ but the negative of the idenity function for $x < 0$, we have that

$$(g \circ f)(x) = \sqrt{x^2} = |x|$$

   Here is a graph of $(g \circ f)(x) = \sqrt{x^2} = |x|$.



Summary

- For a composite function $f \circ g$ to be defined, we need outputs of g to be among the allowed inputs for f. In particular, if the range of g is a subset of the domain of f, we can say that if $g : A \to B$ and $f : B \to C$, then $f \circ g : A \to C$. In this case, the domain of the composite function is the domain of the inner function, and the range of the composite function is the codomain of the outer function.

- In general, the domain of a composite function $f \circ g$ is the set of those inputs x in the domain of g for which $g(x)$ is in the domain of f.

- In general, the range of a composite function $f \circ g$ is a subset of the range of f.

Part 6

Zeros

6.1 What are the Zeros of Functions?

Learning Objectives

- Zeros of Functions
  - Definition of Zeros
  - Compare and contrast zeros, solutions, roots, and x-intercepts
  - Examples of why we might want to find zeros
  - Identify a zero on a graph
  - Computing the zero of a function (early examples)
- The Importance of Equals
  - Compare and contrast expressions, equations, and functions
  - Appreciate the importance of using the equals sign appropriately

6.1.1 Zeros of Functions

Motivating Questions

- What does it mean to find the zero of a function?
- What are other terms used for zeros of functions?
- Why might we want to find the zero of a function?

Introduction

In this section, we will study zeros of functions. Let's start with a classic example of when we might want to find the zero of a function.

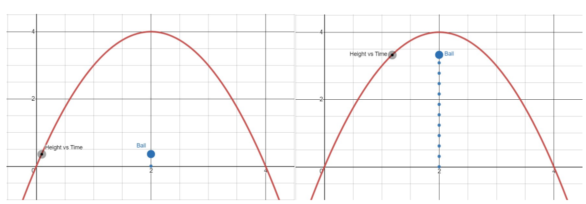
Example 62. *A ball is thrown straight upward from the ground. The distance the ball is from the ground is give by the function $f(x) = 4 - (x-2)^2$ where x is the time measured in seconds and $f(x)$ is the distance from the ground measured in feet. What time will the ball hit the ground?*

Explanation First, it is important to understand what this problem is saying. Consider this model of the situation. Click the play button next to the a to see an animation of the ball being thrown up.

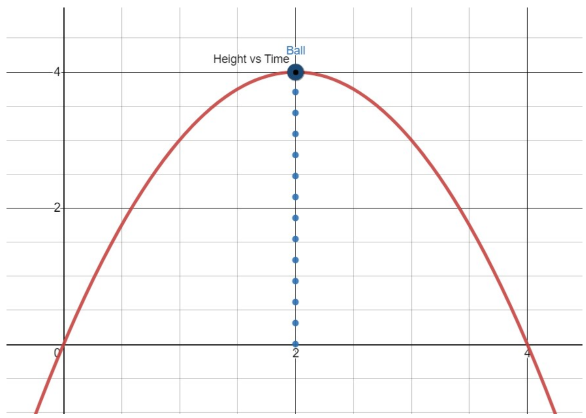
Desmos link: https://www.desmos.com/calculator/f9d7ngsznm

Notice that while the ball goes straight up and down, the graph of the distance from the ground vs. time makes an upside down parabola.

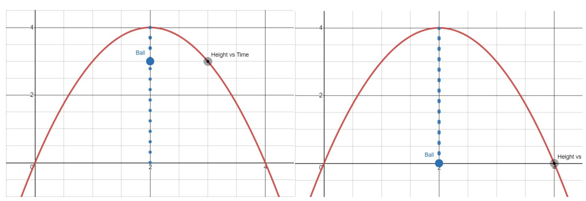
As the ball goes up, we see:



At the ball's highest point, we have:

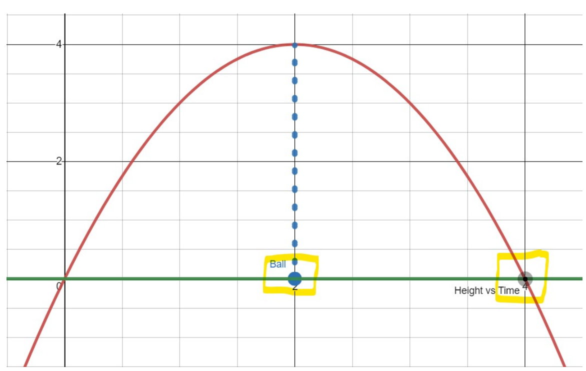


And as the ball goes down, we have:



It is also important to note that in this situation, we know this model only makes sense from the time we throw the ball to the time it hits the ground. Having a negative distance would correspond with the ball going underground, which is not what we want to model. That means our domain will be from $x = 0$ to the x-value where the ball hits the ground.

This means, once again, we are back to wanting to know when the ball will hit the ground. Looking at the graph, we can focus on the time when the ball seems to hit the ground.

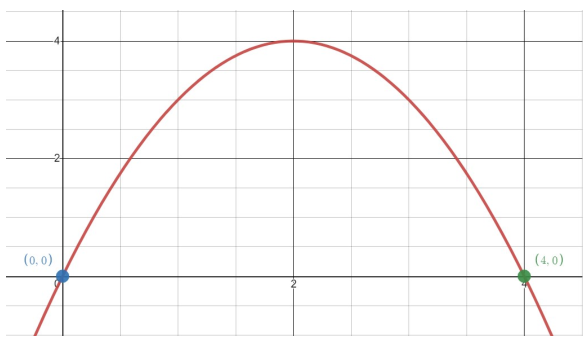


Notice that at the time the ball hits the ground, the function giving the distance vs. time graph, $f(x) = 4 - (x - 2)^2$, is crossing the x-axis. This means we are looking for **the x-value of the x-intercept**. That is, we are looking for when this function equals 0. We call an x-value where a function equals zero the **zero of a function**. It can also be called the **root of a function** when the function is a polynomial. Occasionally, people will also call it the **solution of a function** but techincally they should say **the solution of** $f(x) = 0$.

Let's formalize this with a definition.

> **Definition** We call an x-value where a function equals zero the **zero of a function**. It can also be called the **root of a function** when the function is a polynomial. Another equivalent expression is **the solution of** $f(x) = 0$. Another label for the same value is **the x-coordinate of the x-intercept**.

Finding Zeros Graphically One method for finding zeros or roots of functions is to read them off the graph. Since the zero is the x-coordinate of the x-intercept, we are looking for the places where the graph crosses the x-axis. This is where the y-value or output will be zero. In the image below, the two roots are colored blue and green.



The root $x = 0$ cooresponds to the blue dot at $(0, 0)$ and represents when the ball is first thrown. The second root, labeled in green, is where $x = 4$. This root is the one we are looking for. The time when the ball hits the ground is 4 seconds after the ball is thrown.

Finding Zeros Algebraically Reading zeros graphically can be useful, but it is not very precise. The root in our example could actually be at $x = 3.99$ and we would not know from the graph. When possible, it is best to find zeros algebraically. To do this, we want to set $f(x) = 0$ and solve for x. This algebra can get tricky. In this section, we will stick to relatively straightforward examples and in the next couple sections we will explore some more involved methods for solving equations where one side equals 0. It is not always possible to solve for the zeros of a function algebraically in this manner. In calculus, you will also learn methods to approximate zeros when it is not possible to solve for them exactly.

In our current example, we will have:

$$\begin{aligned} f(x) &= 0 \\ 4-(x-2)^2 &= 0 \\ 4 &= (x-2)^2 \\ \sqrt{4} &= \sqrt{(x-2)^2} \end{aligned}$$

Recall that $\sqrt{x^2} = |x|$ from the section on domains and ranges of composite functions.

Continuing, we have:

$$\begin{aligned} \sqrt{4} &= |x-2| \\ \pm 2 &= x-2 \\ 2 \pm 2 &= x \end{aligned}$$

That is, the zeros or roots of $f(x) = 4-(x-2)^2$ are:

$$x = 2-2 = 0 \text{ or } x = 2+2 = 4$$

Considering our problem in context, we know that the root at time $x = 0$ is when the ball was initially thrown, so the root at $x = 4$ must be when the ball hits the ground.

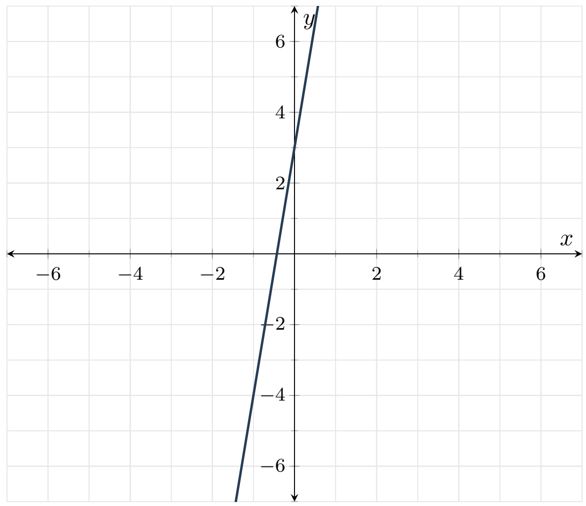
Let's look at a couple more example:

Example 63. *Find the zeros of the linear function $f(x) = 7x+3$.*

Explanation We need to set $f(x) = 7x+3 = 0$

$$\begin{aligned} 7x+3 &= 0 \\ 7x &= -3 \\ x &= \tfrac{-3}{7} \end{aligned}$$

This linear function only has one zero. This should make sense if we think about the graph of a line as a line will only have one x-intercept.



In fact, all linear functions except constant functions will have one zero. Constant functions will have no zeros except for the linear function $x = 0$ in which every point is a zero.

Example 64. *Write $g(x) = |7x + 3|$ as a piecewise function and find its zeros.*
Explanation Recall that the absolute value makes all the outputs positive. Multiplying a negative number by a -1 makes the value positive. Therefore, $g(x) = |7x + 3|$ is a piecewise function where

$$g(x) = |7x + 3| = \begin{cases} 7x + 3 & \text{if } 7x + 3 \geq 0 \\ -(7x + 3) & \text{if } 7x + 3 < 0 \end{cases}$$

To simplify this expression, notice that we need to know when 7x+3=0. This is what we did in the last example. We know this happens at $x = \frac{-3}{7}$. Now, we also need to know when $f(x) = 7x + 3$ is positive and when it is negative.

Since we have the graph of the function above, we could look at it and see that $7x + 4 > 0$ when $x > \frac{-3}{7}$ and $7x + 4 < 0$ when $x < \frac{-3}{7}$.

Alternatively, if we wanted to figure this out algebraically (without the graph), we can plug values into the function $f(x) = 7x + 3$ on either side of $x = \frac{-3}{7}$. This will work because we know a property about lines. We know that it cannot switch from positive to negative without being equal to 0 in the middle. This is a property that you will study more in calculus.

Let's choose to look at $x = 0$ as a representative of values of $x > \frac{-3}{7}$ and $x = -1$ as a representive of values for $x < \frac{-3}{7}$.

$$f(0) = 7(0) + 3 = 3 > 0$$

This means that $f(x) = 7x + 3 > 0$ when $x > \frac{-3}{7}$.

$$f(-1) = 7(-1) + 3 = -4 < 0$$

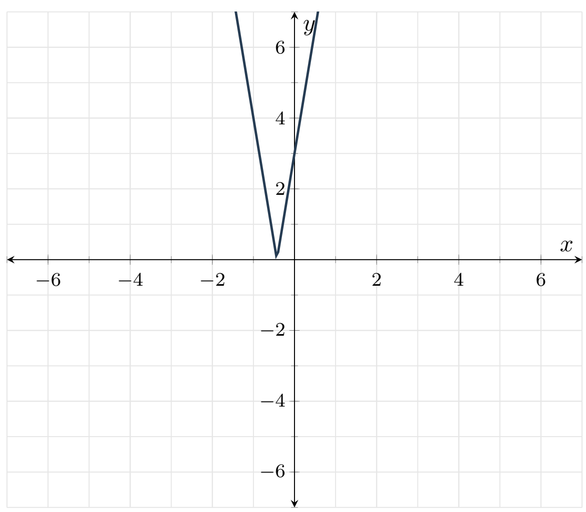
This means that $f(x) = 7x + 3 < 0$ when $x < \frac{-3}{7}$.

Putting this all together, we can now write $g(x) = |7x + 3|$ as a piecewise function:

$$g(x) = |7x + 3| = \begin{cases} 7x + 3 & \text{if } x \geq \frac{-3}{7} \\ -(7x + 3) & \text{if } x < \frac{-3}{7} \end{cases}$$

Now we can find the zeros of this function. We know that an absolute value function is only zero where it switches from the positive branch to the negative branch of the piecewise function so the zero is when $x = \frac{-3}{7}$.

Let's look at the graph of this function to verify that this makes sense.



By looking at this graph, we can see that the zero, that is the x-intercept on $g(x) = |7x + 3|$ is the same as on $f(x) = 7x + 3$.

Summary

- We call an x-value where a function equals zero the **zero of a function**. It can also be called the **root of a function** if the function is a polynomial. Another equivalent expression is **the solution of** $f(x) = 0$. Another label for the same value is **the** x**-coordinate of the** x**-intercept**.

- You can find this value graphically by looking for the x-value where the function crosses the x-axis.
- You can find this value algebraically by setting $f(x) = 0$ and then solving for x. This can sometimes be difficult (or even impossible!) to do.

6.1.2 The Importance of the Equals Sign

Motivating Questions

- What are some similarities and differences between mathematical expressions, equations, and functions?
- How is solving an equation related to finding the zero of a function?
- When should we and when shouldn't we use an equals sign?

Introduction

Now that we are exploring the zeros of functions, one issue that often comes up for students (and for teachers reading students' work!) is when you should and should not use an equals sign. We are going to review when is and is not ok to use equals.

First, it is helpful to review a few important terms. In mathematics, it is important to use these and other terms precisely so that you are communicating clearly and saying what you intend to say. Speaking precisely using mathematical terms can be difficult to learn and takes some practice!

Expressions

Definition An **algebraic expression** is any combination of variables and numbers using arithmetic operations such as addition, subtraction, multiplication, division, and exponentiation.

Here are some examples of algebraic expressions:

$$5x^2 - 17 \qquad \frac{56x}{\sqrt{17x}} \qquad 2x + 3y + 7z$$

The important thing to notice is that there are no equals signs in an expression. There are also no inequality signs.

Definition An **mathematical expression**, or just an **expression**, is similar to an algebraic expression, but can contain other mathematical objects such as $\sin(x)$ or $\ln(x)$ or similar objects that you will learn about in future classes. In particular, it does not contain an equals sign or an inequality sign.

Here are some examples of mathematical expressions:

$$\frac{\sin(x)}{\cos(x)} \qquad 5x + \ln(x) - 12 \qquad 2x + 3y + 7z$$

Every algebraic expression is also a mathematical expression.

> **Definition** **Evaluating an expression** is substituting in a particular value for the variable in a mathematical expression.

Here is an example of evaluating an expression. Consider the expression $5x^2-17$. Let's evaluate that expression at $x = 1$.

$$5(1)^2 - 17 = 5 - 17 = -12$$

Notice that when evaluating this expression at a particular point, we can use an equals sign. This is a good use of the equals sign and shows us simplifying. But, we should not put an equals sign between $5x^2 - 17$ and $5(1)^2 - 17$ as these two expressions are only equal when $x = 1$.

Equations

When we use an equals sign to say that two different mathematical expressions give the same value, we are creating an equation.

> **Definition** An **equation** is a statement that two mathematical expressions are equal.

Here are some examples of equations:

$$5x^2 - 17 = -12 \qquad \frac{56x}{\sqrt{17x}} = 12x \qquad 2x + 3y + 7z = \frac{x}{y+z}$$

When we are given an equation in a problem, we often want to know what value of the variable will make the equation true. That is, what value of the variable will make both sides give the same value.

> **Definition** **Solving an equation** is the process of determining precisely what value of a variable makes the equation true.

Here is an example of solving an equation. Let's solve $5x^2 - 17 = -12$.

$$
\begin{aligned}
5x^2 - 17 &= -12 \\
5x^2 &= 5 \\
x^2 &= 1 \\
x = 1 &\text{ or } x = -1
\end{aligned}
$$

Notice that this is the reverse process of evaluating the expression $5x^2 - 17$. When evaluating the expression, we knew the x-value and substituted it in. When solving the equation, we knew what the output should be and had to find the x-value that would produce that output. In fact, we found two such values!

> **Remark** Notice that when solving an equation, we don't put equals between the steps. This is very important. In many cases, the steps are not equal!

This is a key observation. Notice that if we naively wrote

$$5x^2 - 17 = -12 = 5x^2 = 5 = ...,$$

we would be saying something not true. In particular, we would be claiming that $-12 = 5$!

The best thing is to do when solving an equation is to make a new line for each step, but if you need to write your steps on a single line, you can use an arrow to show the next step. For example, we could write

$$5x^2 - 17 = -12 \to 5x^2 = 5 \to x^2 = 1 \to x = 1 \text{ or } x = -1.$$

We could also use connecting words between equations. For example, we could write:

$$5x^2 - 17 = -12 \text{ so } 5x^2 = 5 \text{ thus } x^2 = 1 \text{ therefore } x = 1 \text{ or } x = -1.$$

Zeros of Functions Revisited

Notice that when we are working with functions, we are also working with equations and expressions.

- When we write f, we are referencing the function by name.
- When we write $f(x)$, this is an expression for the output of the function at x.
- When we write $f(x) = 5x^2 - 17$, we are defining the way the function produces outputs.

- When we want to find the zeros of this function, we set up the equation $f(x) = 0$. In our case, this would mean solving the equation $5x^2 - 17 = 0$.

$$\begin{aligned} 5x^2 - 17 &= 0 \\ 5x^2 &= 17 \\ x^2 &= \frac{17}{5} \\ x = \sqrt{\frac{17}{5}} &\text{ or } x = -\sqrt{\frac{17}{5}} \end{aligned}$$

  Notice that this is solving an equation so we do not write equals signs between the steps.

Another important connection between finding zeros of functions and solving equations is that every equation can be thought of as the zero of a function. Consider the following example.

Example 65. *Rewrite the equation $5x + 7 = 6 - x^2$ as the zero of a function. You do not need to find the zero.*

Explanation In order to rewrite this problem so that solving this equation is equivalent to finding the zero of a function, we want to move all the terms to the same side and combine like terms. For our example, this means

$$\begin{aligned} 5x + 7 &= 6 - x^2 \\ 5x + 7 - (6 - x^2) &= 0 \\ -6x^2 + 5x + 1 &= 0 \end{aligned}$$

Now we let

$$f(x) = -6x^2 + 5x + 1$$

Now, the x values which are zeros of f will be the same x-values that solve $5x + 7 = 6 - x^2$. We will learn to find zeros of quadratic equations in the the next section.

Summary

- You should not write an equals sign between two things which do not have the same value. Equals does not mean "next step"! Instead, to indicate a next step, you may use an arrow, a new line, or connecting words like "so" and"thus".
- Every equation can be thought of as the zero of a function by moving all the terms to one side and then defining a function to be the output of that side.

6.2 Zeros of Polynomials

Learning Objectives

- Finding Zeros of Quadratics
  - Factoring Strategies, including Quadratic Formula
  - Relationship between factoring and roots
- Finding Zeros of Polynomials
  - Polynomial Long-Division
  - Strategies for factoring polynomials, including some cubic tricks, inspection, and graphing
- Interperting Zeros of Polynomials
  - What can the zeros of a polynomial tell us about it's shape?
  - What does it mean to have a zero with multiplicity?
  - How many zeros can a polynomial have?

6.2.1 Finding Zeros of Quadratics

Introduction

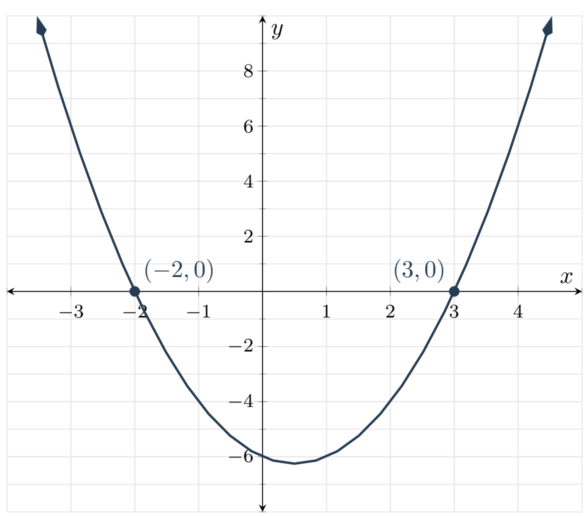
Factoring

We've seen multiplying polynomials before, such as when you start with $(x+2)(x+3)$ and obtain x^2+5x+6. This section, is about the opposite process of factoring. For example, starting with x^2+5x+6 and obtaining $(x+2)(x+3)$. We will start with the simplest kind of factoring: for example starting with x^2+2x and obtaining $x(x+2)$.

When you write x^2+2x, you have an algebraic expression built with two terms that are added together. When you write $x(x+2)$, you have an algebraic expression built with two factors that are multiplied together. Factoring is useful, because sometimes (but not always) having your expression written as parts that are multiplied together makes it easy to simplify the expression.

You've seen this with fractions. To simplify $\frac{15}{35}$, breaking down the numerator and denominator into factors is useful: $\frac{3 \cdot 5}{7 \cdot 5}$. Now you can see that the factors of 5 cancel.

There are other reasons to appreciate the value in factoring. One reason is that there is a relationship between a factored polynomial and the horizontal intercepts of its graph. For example in the graph of $y = (x+2)(x-3)$, the horizontal intercepts are $(-2,0)$ and $(3,0)$. Note the x-values are -2 and 3, and think about what happens when you subsitutue those numbers in for x in $y = (x+2)(x-3)$.



The most basic technique for factoring involves recognizing the **greatest common factor** between two expressions, which is the largest factor that goes in evenly to both expressions. For example, the greatest common factor between 6 and 8 is 2, since 2 divides nicely into both 6 and 8 and no larger number would divide nicely into both 6 and 8.

Similarly, the greatest common factor between $4x$ and $3x^2$ is x. If you write $4x$ as a product of its factors, you have $2 \cdot 2 \cdot x$. And if you fully factor $3x^2$, you have $3 \cdot x \cdot x$. The only factor they have in common is x, so that is the greatest common factor. No larger expression goes in nicely to both expressions.

Example 66. *What is the common factor between $6x^2$ and $70x$? Break down each of these into its factors:*

$$6x^2 = 2 \cdot 3 \cdot x \cdot x \quad 70x = 2 \cdot 5 \cdot 7 \cdot x$$

And identify the common factors:

$$6x^2 = \overset{\downarrow}{2} \cdot 3 \cdot \overset{\downarrow}{x} \cdot x \quad 70x = \overset{\downarrow}{2} \cdot 5 \cdot 7 \cdot \overset{\downarrow}{x}$$

Factoring Trinomials with Leading Coefficient One

We've learned how to multiply binomials like $(x + 2)(x + 3)$ and obtain the trinomial $x^2 + 5x + 6$. In this section, we will learn how to undo that. So we'll be starting with a trinomial like $x^2 + 5x + 6$ and obtaining its factored form $(x + 2)(x + 3)$. The trinomials that we'll factor in this section all have leading coefficient 1, but in a later section will cover some more general trinomials.

Example 67. *Consider the example $x^2 + 5x + 6 = (x + 2)(x + 3)$. There are at least three things that are important to notice:*

- *The leading coefficient of $x^2 + 5x + 6$ is 1.*
- *The two factors on the right use the numbers 2 and 3, and when you multiply these you get the 6.*
- *The two factors on the right use the numbers 2 and 3, and when you add these you get the 5.*

So the idea is that if you need to factor $x^2 + 5x + 6$ and you somehow discover that 2 and 3 are special numbers (because $2 \cdot 3 = 6$ and $2 + 3 = 5$), then you can conclude that $(x + 2)(x + 3)$ is the factored form of the given polynomial.

Factor $x^2 + 13x + 40$. Since the leading coefficient is 1, we are looking to write this polynomial as $(x + ?)(x + ?)$ where the question marks are two possibly different, possibly negative, numbers. We need these two numbers to multiply to 40 and add to 13. How can you track these two numbers down? Since the *numbers need to multiply to* 40*, one method is to list all* ***factor pairs*** *of* 40 *in a table just to see what your options are. We'll write every pair of factors that multiply to* 40*.*

$$\begin{array}{cc} 1 \cdot 40 & -1 \cdot (-40) \\ 2 \cdot 20 & -2 \cdot (-20) \\ 4 \cdot 10 & -4 \cdot (-10) \\ 5 \cdot 8 & -5 \cdot (-8) \end{array}$$

We wanted to find all factor pairs. To avoid missing any, we started using 1 *as a factor, and then slowly increased that first factor. The table skips over using* 3 *as a factor, because* 3 *is not a factor of* 40*. Similarly the table skips using* 6 *and* 7 *as a factor. And there would be no need to continue with* 8 *and beyond, because we already found "large" factors like* 8 *as the partners of "small" factors like* 5*.*

There is an entire second column where the signs are reversed, since these are also ways to multiply two numbers to get 40*. In the end, there are eight factor pairs.*

We need a pair of numbers that also adds to 13*. So we check what each of our factor pairs add up to:*

Factor Pair	*Sum of the Pair*
$1 \cdot 40$	41
$2 \cdot 20$	22
$4 \cdot 10$	14
$5 \cdot 8$	13*(what we wanted)*
$-1 \cdot (-40)$	*(no need to go this far)*
$-2 \cdot (-20)$	*(no need to go this far)*
$-4 \cdot (-10)$	*(no need to go this far)*
$-5 \cdot (-8)$	*(no need to go this far)*

The winning pair of numbers is 5 *and* 8*. Again, what matters is that* $5 \cdot 8 = 40$*, and* $5 + 8 = 13$*. So we can conclude that* $x^2 + 13x + 40 = (x + 5)(x + 8)$*.*

To ensure that we made no mistakes, here are some possible checks.
Multiply it Out Multiplying out our answer $(x + 5)(x + 8)$ should give us $x^2 + 13x + 40$.

$(x + 5)(x + 8) = (x + 5) \cdot x + (x + 5) \cdot 8 = x^2 + 5x + 8x + 40 = x^2 + 13x + 40$
Evaluating If the answer really is $(x + 5)(x + 8)$, then notice how evaluating at -5 would result in 0. So the original expression should also result in 0 if we evaluate at -5. And similarly, if we evaluate it at -8, $x^2 + 13x + 40$ should be 0.

$$\begin{aligned}
(-5)^2 + 13(-5) + 40 &= 0 \\
25 - 65 + 40 &= 0 \\
0 &= 0 \text{ Confirmed} \\
(-8)^2 + 13(-8) + 40 &= 0 \\
64 - 104 + 40 &= 0 \\
0 &= 0 \text{ Confirmed}
\end{aligned}$$

This also gives us evidence that the factoring was correct.

Factor $y^2 - 11y + 24$. The negative coefficient is a small complication from the previous example, but the process is actually still the same.

We need a pair of numbers that multiply to 24 and add to -11. Note that we *do* care to keep track that they sum to a negative total.

Factor Pair	Sum of the Pair
$1 \cdot 24$	25
$2 \cdot 12$	14
$3 \cdot 8$	11(close; wrong sign)
$4 \cdot 6$	10
$-1 \cdot (-24)$	-25
$-2 \cdot (-12)$	-14
$-3 \cdot (-8)$	-11(what we wanted)
$-4 \cdot (-6)$	(no need to go this far)

So $y^2 - 11y + 24 = (y - 3)(y - 8)$. To confirm that this is correct, we should check. Either by multiplying out the factored form:

$$\begin{aligned}
(y-3)(y-8) &= (y-3) \cdot y - (y-3) \cdot 8 \\
&= y^2 - 3y - 8y + 24 \\
&= y^2 - 11y + 24 \text{ Confirmed}
\end{aligned}$$

Or by evaluating the original expression at 3 and 8:

$$\begin{aligned}
3^2 - 11(3) + 24 &= 0 \\
9 - 33 + 24 &= 0 \\
0 &= 0 \\
8^2 - 11(8) + 24 &= 0 \\
64 - 88 + 24 &= 0 \\
0 &= 0
\end{aligned}$$

Our factorization passes the tests.

6.2.2 Finding Zeros of Polynomials

Introduction

This section covers a technique for factoring polynomials like x^3+3x^2+2x+6, which factors as $(x^2+2)(x+3)$. If there are four terms, the technique in this section might help you to factor the polynomial.

Recall that to factor $3x+6$, we factor out the common factor 3:

$$\begin{aligned} 3x+6 &= \overset{\downarrow}{3}x + \overset{\downarrow}{3}\cdot 2 \\ &= 3(x+2) \end{aligned}$$

The "3" here could have been something more abstract, and it still would be valid to factor it out:

$$\begin{aligned} x(a+b)+2(a+b) &= x\overbrace{(a+b)}^{\downarrow} + 2\overbrace{(a+b)}^{\downarrow} \\ &= (a+b)(x+2) \end{aligned}$$

In this last example, we factored out the binomial factor $(a+b)$. Factoring out binomials is the essence of this section, so let's see that a few more times:

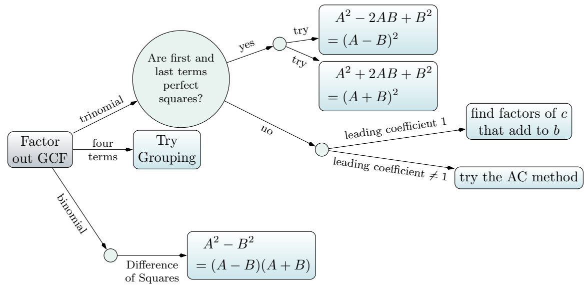
$$\begin{aligned} x(x+2)+3(x+2) &= x\overbrace{(x+2)}^{\downarrow} + 3\overbrace{(x+2)}^{\downarrow} \\ &= (x+2)(x+3) \end{aligned}$$

$$\begin{aligned} z^2(2y+5)+3(2y+5) &= z^2\overbrace{(2y+5)}^{\downarrow} + 3\overbrace{(2y+5)}^{\downarrow} \\ &= (2y+5)(z^2+3) \end{aligned}$$

And even with an expression like $Q^2(Q-3)+Q-3$, if we re-write it in the right way using a 1 and some parentheses, then it too can be factored:

$$\begin{aligned} Q^2(Q-3)+Q-3 &= Q^2(Q-3)+1(Q-3) \\ &= Q^2\overbrace{(Q-3)}^{\downarrow} + 1\overbrace{(Q-3)}^{\downarrow} \\ &= (Q-3)(Q^2+1) \end{aligned}$$

The truth is you are unlikely to come upon an expression like $x(x+2)+3(x+2)$, as in these examples. Why wouldn't someone have multiplied that out already? Or factored it all the way? So far in this section, we have only been looking at a stepping stone to a real factoring technique called **factoring by grouping**.


Factor out GCF
trinomial
four terms
binomial
Are first and last terms perfect squares?
Try Grouping
yes
no
try
try
$A^2 - 2AB + B^2 = (A - B)^2$
$A^2 + 2AB + B^2 = (A + B)^2$
leading coefficient 1
leading coefficient ≠ 1
find factors of c that add to b
try the AC method
Difference of Squares
$A^2 - B^2 = (A - B)(A + B)$

6.2.3 Zeros of Polynomials

Zeros of Polynomials

> **Motivating Questions**
>
> - What properties of a polynomial function can we deduce from its algebraic structure?
> - What is a sign chart and how does it help us understand a polynomial function's behavior?
> - How do zeros of multiplicity other than 1 impact the graph of a polynomial function?

We know that linear functions are the simplest of all funcitons we can consider: their graphs have the simplest shape, their average rate of change is always constant (regardless of the interval chosen), and their formula is elementary. Moreover, computing the value of a linear function only requires multiplication and addition.

If we think of a linear function as having formula $L(x) = b + mx$, and the next-simplest functions, quadratic functions, as having form $Q(x) = c + bx + ax^2$, we can see immediate parallels between their respective forms and realize that it's natural to consider slightly more complicated functions by adding additional power functions.

Indeed, if we instead view linear functions as having form

$$L(x) = a_0 + a_1 x$$

(for some constants a_0 and a_1) and quadratic functions as having form

$$Q(x) = a_0 + a_1 x + a_2 x^2,$$

then it's natural to think about more general functions of this same form, but with additional power functions included.

> **Definition** Given real numbers $a_0, a_1, \ldots, a_n$ where $a_n \neq 0$, we say that the function
>
> $$P(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_{n-1} x^{n-1} + a_n x^n,$$
>
> is a polynomial of degree n. In addition, we say that the value of a_i are the coefficients of the polynomial and the individual power functions $a_i x^i$ are the terms of the polynomial. Any value of x for which $P(x) = 0$ is called a zero of the polynomial.

Example 68. *The polyomial function* $P(x) = 3 - 7x + 4x^2 - 2x^3 + 9x^5$ *has degree* 5*, its constant term is* 3*, and its linear term is* $-7x$*.*

Since a polynomial is simply a sum of constant multiples of various power functions with positive integer powers, we often refer to those individual terms by referring to their individual degrees: the linear term, the quadratic term, and so on. In addition, since the domain of any power function of the form $p(x) = x^n$ where n is a positive whole number is the set of all real numbers, it's also true the the domain of any polynomial function is the set of all real numbers.

Exploration Point your browser to the *Desmos* worksheet at http://gvsu.edu/s/0zy. There you'll find a degree 4 polynomial of the form $p(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4$, where $a_0, \ldots, a_4$ are set up as sliders. In the questions that follow, you'll experiment with different values of $a_0, \ldots, a_4$ to investigate different possible behaviors in a degree 4 polynomial.

a. What is the largest number of distinct points at which $p(x)$ can cross the x-axis?For a polynomial p, we call any value r such that $p(r) = 0$ a zero of the polynomial. Report the values of $a_0, \ldots, a_4$ that lead to that largest number of zeros for $p(x)$.

b. What other numbers of zeros are possible for $p(x)$? Said differently, can you get each possible number of fewer zeros than the largest number that you found in (a)? Why or why not?

c. We say that a function has a turning point if the function changes from decreasing to increasing or increasing to decreasing at the point. For example, any quadratic function has a turning point at its vertex.

   What is the largest number of turning points that $p(x)$ (the function in the *Desmos* worksheet) can have? Experiment with the sliders, and report values of $a_0, \ldots, a_4$ that lead to that largest number of turning points for $p(x)$.

d. What other numbers of turning points are possible for $p(x)$? Can it have no turning points? Just one? Exactly two? Experiment and explain.

e. What long-range behavior is possible for $p(x)$? Said differently, what are the possible results for $\lim_{x\to-\infty} p(x)$ and $\lim_{x\to\infty} p(x)$?

f. What happens when we plot $y = a_4x^4$ in and compare $p(x)$ and a_4x^4? How do they look when we zoom out? (Experiment with different values of each of the sliders, too.)

Our observations in Preview Activity **??** generalize to polynomials of any degree.

In particular, it is possible to prove the following general conclusions regarding the number of zeros, the long-range behavior, and the number of turning points any polynomial of degree n.

For any degree n polynomial $p(x) = a_0 + a_1x + \cdots + a_{n-1}x^{n-1} + a_nx^n$, has at most n real zeros.[13]

We know that each of the power functions x, x^2, ..., x^n grow without bound as $x \to \infty$. Intuitively, we sense that x^5 grows faster than x^4 (and likewise for any comparison of a higher power to a lower one). This means that for large values of x, the most important term in any polynomial is its highest order term, as we saw in Preview Activity **??** when we compared $p(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4$ and $y = a_4x^4$.

For any degree n polynomial $p(x) = a_0 + a_1x + \cdots + a_{n-1}x^{n-1} + a_nx^n$, its long-range behavior is the same as its highest-order term $q(x) = a_nx^n$. Thus, any polynomial of even degree appears "U-shaped" ($\cup$ or $\cap$, like x^2 or $-x^2$) when we zoom way out, and any polynomial of odd degree appears "chair-shaped" (like x^3 or $-x^3$) when we zoom way out.

In Figure **??**, we see how the degree 7 polynomial pictured there (and in Figure **??** as well) appears to look like $q(x) = -x^7$ as we zoom out.

Finally, a key idea from calculus justifies the fact that the maximum number of turning points of a degree n polynomial is $n - 1$, as we conjectured in the degree 4 case in **image**. Moreover, the only possible numbers of turning points must have the same parity as $n-1$; that is, if $n-1$ is even, then the number of turning points must be even, and if instead $n-1$ is odd, the number of turning points must also be odd. For instance, for the degree 7 polynomial in **image**, we know that it is chair-shaped, with one end up and one end down. There could be zero turning points and the function could always decrease. But if there is at least one, then there must be a second, since if there were only one the function would decrease and then increase without turning back, which would force the graph to appear U-shaped.

For any degree n polynomial $p(x) = a_0 + a_1x + \cdots + a_{n-1}x^{n-1} + a_nx^n$, if n is even, its number of turning points is exactly one of $n-1$, $n-3$, ..., 1, and if n is odd, its number of turning points is exactly one of $n-1$, $n-3$, ..., 0.

By experimenting with coefficients in *Desmos*, find a formula for a polynomial function that has the stated properties, or explain why no such polynomial exists. in *Desmos*, you'll get prompted to add sliders that make it easy to explore a degree 5 polynomial.)

a. A polynomial p of degree 5 with exactly 3 real zeros, 4 turning points, and such that $\lim_{x \to -\infty} p(x) = +\infty$ and $\lim_{x \to \infty} p(x) = -\infty$.

[13] We can actually say even more: if we allow the zeros to be complex numbers, then every degree n polynomial has *exactly* n zeros, provided we count zeros according to their multiplicity. For example, the polynomial $p(x) = (x-1)^2 = x^2 - 2x + 1$ because it has a zero of multiplicity two at $x = 1$.

b. A polynomial q of degree 4 with exactly 4 real zeros, 3 turning points, and such that $\lim_{x \to -\infty} p(x) = +\infty$ and $\lim_{x \to \infty} p(x) = -\infty$.

c. A polynomial r of degree 6 with exactly 2 real zeros, 3 turning points, and such that $\lim_{x \to -\infty} p(x) = -\infty$ and $\lim_{x \to \infty} p(x) = -\infty$.

d. A polynomial s of degree 5 with exactly 5 real zeros, 3 turning points, and such that $\lim_{x \to -\infty} p(x) = +\infty$ and $\lim_{x \to \infty} p(x) = -\infty$.

Using zeros and signs to understand polynomial behavior

Just like a quadratic function can be written in different forms (standard: $q(x) = ax^2 + bx + c$, vertex: $q(x) = a(x - h)^2 + k$, and factored: $q(x) = a(x - r_1)(x - r_2)$), it's possible to write a polynomial function in different forms and to gain information about its behavior from those different forms. In particular, if we know all of the zeros of a polynomial function, we can write its formula in factored form, which gives us a deeper understanding of its graph.

The Zero Product Property states that if two or more numbers are multiplied together and the result is 0, then at least one of the numbers must be 0. We use the Zero Product Property regularly with polynomial functions. If we can determine all n zeros of a degree n polynomial, and we call those zeros r_1, r_2, $\ldots$, r_n, we can write

$$p(x) = a_n(x - r_1)(x - r_2) \cdots (x - r_2).$$

Moreover, if we are given a polynomial in this factored form, we can quickly determine its zeros. For instance, if $p(x) = 2(x + 7)(x + 1)(x - 2)(x - 5)$, we know that the only way $p(x) = 0$ is if at least one of the factors $(x + 7)$, $(x + 1)$, $(x - 2)$, or $(x - 5)$ equals 0, which implies that $x = -7$, $x = -1$, $x = 2$, or $x = 5$. Hence, from the factored form of a polynomial, it is straightforward to identify the polynomial's zeros, the x-values at which its graph crosses the x-axis. We can also use the factored form of a polynomial to develop what we call a **sign chart**, which we demonstrate in **image**.

Example 69. *ex-polynomial-signs*

Consider the polynomial function $p(x) = k(x - 1)(x - a)(x - b)$. Suppose we know that $1 < a < b$ and that $k < 0$. Fully describe the graph of p without the aid of a graphing utility.

Since $p(x) = k(x - 1)(x - a)(x - b)$, we immediately know that p is a degree 3 polynomial with 3 real zeros: $x = 1, a, b$. We are given that $1 < a < b$ and in addition that $k < 0$. If we expand the factored form of $p(x)$, it has form $p(x) = kx^3 + \cdots$, and since we know that when we zoom out, $p(x)$ behaves like kx^3, we know that with $k < 0$ it follows $\lim_{x \to -\infty} p(x) = +\infty$ and $\lim_{x \to \infty} p(x) = -\infty$.

Since p is degree 3 and we know it has zeros at $x = 1, a, b$, we know there are no other locations where $p(x) = 0$. Thus, on any interval between two zeros (or to the left of the least or the right of the greatest), the polynomial cannot change sign. We now investigate, interval by interval, the sign of the function.

*When $x < 1$, it follows that $x - 1 < 0$. In addition, since $1 < a < b$, when $x < 1$, x lies to the left of 1, a, and b, which also makes $x - a$ and $x - b$ negative. Moreover, we know that the constant $k < 0$. Hence, on the interval $x < 1$, all four terms in $p(x) = k(x-1)(x-a)(x-b)$ are negative, which we indicate by writing "$- - --$" in that location on the sign chart pictured in **image**.*

In addition, since there are an even number of negative terms in the product, the overall product's sign is positive, which we indicate by the single $+$ beneath "$- - --$", and by writing "POS" below the coordinate axis.

We now proceed to the other intervals created by the zeros. On $1 < x < a$, the term $(x-1)$ has become positive, since $x > 1$. But both $x - a$ and $x - b$ are negative, as is the constant k, and thus we write "$- + --$" for this interval, which has overall sign "$-$", as noted in the figure. Similar reasoning completes the diagram.

*From all of the information we have deduced about p, we conclude that regardless of the locations of a and b, the graph of p must look like the curve shown in **image**.*

Exploration act-poly-polynomials-sign-chart
Consider the polynomial function given by

$$p(x) = 4692(x+1520)(x^2+10000)(x-3471)^2(x-9738).$$

a. What is the degree of p? How can you tell *without* fully expanding the factored form of the function?

b. Explain why the factor $(x^2 + 10000)$ is always positive.

c. What are the zeros of the polynomial p?

d. Construct a sign chart for p by using the zeros you identified in (b) and then analyzing the sign of each factor of p.

e. Without using a graphing utility, construct an approximate graph of p that has the zeros of p carefully labeled on the x-axis.

f. Use a graphing utility to check your earlier work. What is challenging or misleading when using technology to graph p?

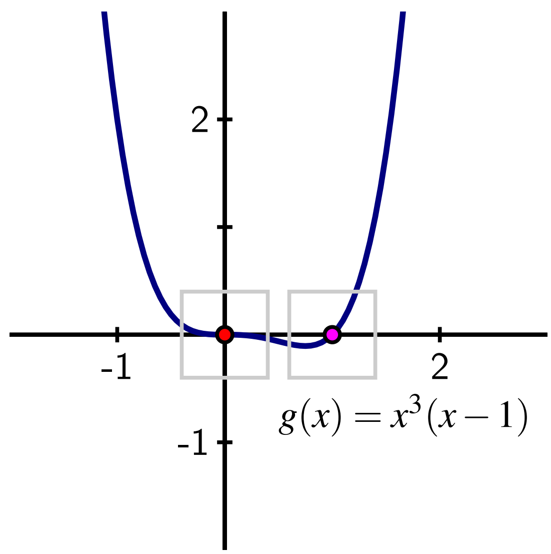
Multiplicity of polynomial zerosMultiplicity of polynomial zerossubsec-poly-polynomials-multiplicity

In **image**, we found that one of the zeros of the polynomial $p(x) = 4692(x + 1520)(x^2 + 10000)(x - 3471)^2(x - 9738)$

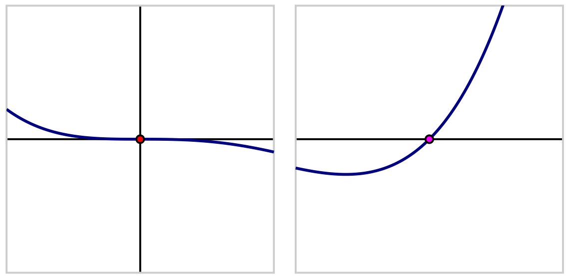
leads to different behavior of the function near that zero than we've seen in other situations. We now consider the more general situation where a polynomial has a repeated factor of the form $(x-r)^n$. When $(x-r)^n$ is a factor of a polynomial p, we say that p has a zero of multiplicity n at $x = r$.

To see the impact of repeated factors, we examine a collection of degree 4 polynomials that each have 4 real zeros. We start with the simplest of all, the function $f(x) = x^4$, whose zeros are $x = 0, 0, 0, 0$. Because the factor "$x - 0$" is repeated 4 times, the zero $x = 0$ has multiplicity 4.

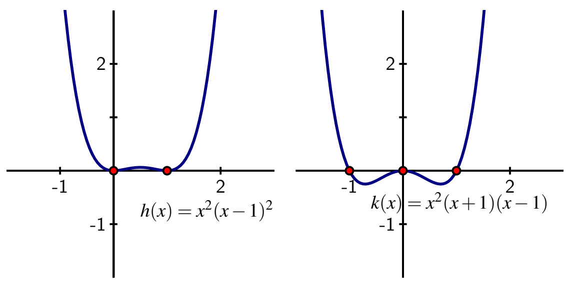
Next we consider the degree 4 polynomial $g(x) = x^3(x - 1)$, which has a zero of multiplicity 3 at $x = 0$ and a zero of multiplicity 1 at $x = 1$.



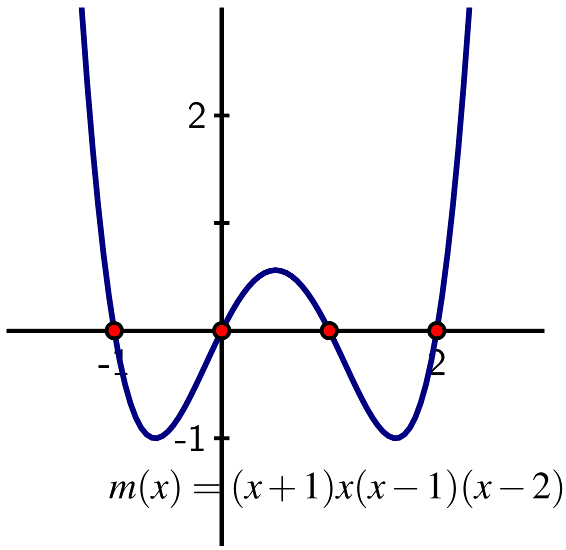
Observe that in**image**, the up-close plot near the zero $x = 0$ of multiplicity 3, the polynomial function g looks similar to the basic cubic polynomial $-x^3$. In addition, in **image**, we observe that if we zoom in even futher on the zero of multiplicity 1, the function g looks roughly linear, like a degree 1 polynomial. This type of behavior near repeated zeros turns out to hold in other cases as well.



If we next let $h(x) = x^2(x-1)^2$, we see that h has two distinct real zeros, each of multiplicity 2. The graph of h in **image** shows that h behaves similar to a basic quadratic function near each of those zeros and thus shows U-shaped behavior nearby. If instead we let $k(x) = x^2(x-1)(x+1)$, we see approximately linear behavior near $x = -1$ and $x = 1$ (the zeros of multiplicity 1), and quadratic (U-shaped) behavior near $x = 0$ (the zero of multiplicity 2), as seen in **image**.



Finally, if we consider $m(x) = (x+1)x(x-1)(x-2)$, which has 4 distinct real zeros each of multiplicity 1, we observe in Figure **??** that zooming in on each zero individually, the function demonstrates approximately linear behavior as it passes through the x-axis.



Our observations with polynomials of degree 4 in the various figures above generalize to polynomials of any degree.

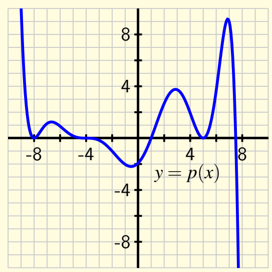
If $(x-r)^n$ is a factor of a polynomial p, then $x = r$ is a zero of p of multiplicity n, and near $x = r$ the graph of p looks like either $-x^n$ or x^n. That is, the shape of the graph near the zero is determined by the multiplicity of the zero.

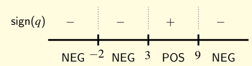
Exploration act-poly-polynomials-multiple-zerosFor each of the following prompts, try to determine a formula for a polynomial that satisfies the given criteria. If no such polynomial exists, explain why.

a. A polynomial f of degree 10 whose zeros are $x = -12$ (multiplicity 3), $x = -9$ (multiplicity 2), $x = 4$ (multiplicity 4), and $x = 10$ (multiplicity 1), and f satisfies $f(0) = 21$. What can you say about the values of $\lim_{x\to-\infty} f(x)$ and $\lim_{x\to\infty} f(x)$.

b. A polynomial p of degree 9 that satisfies $p(0) = -2$ and p has the graph shown in Figure **??**. Assume that all of the zeros of p are shown in the figure.

c. A polynomial q of degree 8 with 3 distinct real zeros (possibly of different multiplicities) such that q has the sign chart in Figure **??** and satisfies $q(0) = -10$.





d. A polynomial q of degree 9 with 3 distinct real zeros (possibly of different multiplicities) such that q satisfies the sign chart in Figure **??** and satisfies $q(0) = -10$.

e. A polynomial p of degree 11 that satisfies $p(0) = -2$ and p has the graph shown in Figure **??**. Assume that all of the zeros of p are shown in the figure.

Summary

- From a polynomial function's algebraic structure, we can deduce several key traits of the function.
- If the function is in standard form, say $p(x) = a_0 + a_1x + a_2x^2 + \cdots + a_{n-1}x^{n-1} + a_nx^n$, we know that its degree is n and that when we zoom out, p looks like a_nx^n and thus has the same long-range behavior as a_nx^n. Thus, p is chair-shaped if n is odd and U-shaped if n is even. Whether $\lim_{n\to\infty} p(x)$ is $+\infty$ or $-\infty$ depends on the sign of a_n.
- If the function is in factored form, say $p(x) = a_n(x - r_1)(x - r_2)\cdots(x - r_n)$ (where the r_i's are possibly not distinct and possibly complex), we can quickly determine both the degree of the polynomial (n) and the locations of its zeros, as well as their multiplicities.
- A sign chart is a visual way to identify all of the locations where a function is zero along with the sign of the function on the various intervals the zeros create. A sign chart gives us an overall sense of the graph of the function, but without concerning ourselves with any specific values of the function besides the zeros.
- When a polynomial p has a repeated factor such as $(x - 5)(x - 5)(x - 5) = (x - 5)^3$, we way that $x = 5$ is a zero of multiplicity 3. At the point $x = 5$ where p will cross the x-axis, up close it will look like a cubic polynomial and thus be chair-shaped. In general,

if $(x - r)^n$ is a factor of a polynomial p so that $x = r$ is a zero of multiplicity n, the polynomial will behave near $x = r$ like a polynomial of degree n.

6.3 Zeros of Famous Functions

Learning Objectives

- Zeros of Rational Functions
- Zeros of Functions with Radicals
- Zeros of Exponential Functions
  - Recall that $\ln(x)$ is the inverse of e^x
  - Defining the logarithm of a general base, by reflection of the graph.
  - Problems you can do without rules of logarithms.

6.3.1 Zeros of Rational Functions

Introduction

Suppose Julia is taking her family on a boat trip 12 miles down the river and back. The river flows at a speed of 2 miles per hour and she wants to drive the boat at a constant speed, v miles per hour downstream and back upstream. Due to the current of the river, the actual speed of travel is $v+2$ miles per hour going downstream, and $v-2$ miles per hour going upstream. If Julia plans to spend 8 hours for the whole trip, how fast should she drive the boat?

The time it takes Julia to drive the boat downstream is $\frac{12}{v+2}$ hours and upstream is $\frac{12}{v-2}$ hours. The function to model the whole trip's time is

$$t(v) = \frac{12}{v-2} + \frac{12}{v+2}$$

where t stands for time in hours. The trip will take 8 hours, so we want $t(v)$ to equal 8, and we have:

$$\frac{12}{v-2} + \frac{12}{v+2} = 8.$$

To solve this equation algebraically, we would start by subtracting 8 from both sides to obtain:

$$\frac{12}{v-2} + \frac{12}{v+2} - 8 = 0.$$

This has taken our equation involving rational functions, and converted it into the problem of determining the zeros of a single rational function. Namely, we are really just finding the zeros of $s(v) = \frac{12}{v-2} + \frac{12}{v+2} - 8$. (Notice that the function was changed by subtracting the 8, so we had to use a new name for it.)

In the same way, whenever we are asked to find the solution of a rational equation, it is equivalent to finding the zeros of a rational function instead.

Zeros of Rational Functions

Example 70. *Let us finish the calculation started in the Introduction. Find the zeros of* $s(v) = \frac{12}{v-2} + \frac{12}{v+2} - 8$.

Explanation

We will begin by combining the left-hand side into a single fraction. Notice the fractions that appear have a common denominator of $(v-2)(v+2) = v^2 - 4$.

$$\begin{aligned}
s(v) &= \frac{12}{v-2} + \frac{12}{v+2} - 8 \\
&= \frac{12}{v-2} \cdot \left(\frac{v+2}{v+2}\right) + \frac{12}{v+2} \cdot \left(\frac{v-2}{v-2}\right) - 8\left(\frac{(v+2)(v-2)}{(v+2)(v-2)}\right) \\
&= \frac{12(v+2)}{(v-2)(v+2)} + \frac{12(v-2)}{(v+2)(v-2)} - \frac{8(v^2-4)}{(v+2)(v-2)} \\
&= \frac{12v+24}{(v+2)(v-2)} + \frac{12v-24}{(v+2)(v-2)} - \frac{8v^2-32}{(v+2)(v-2)} \\
&= \frac{(12v+24)+(12v-24)-(8v^2-32)}{(v+2)(v-2)} \\
&= \frac{-8v^2+(12v+12v)+(24-24+32)}{(v+2)(v-2)} \\
&= \frac{-8v^2+24v+32}{(v+2)(v-2)}.
\end{aligned}$$

That means $s(v) = 0$ is equivalent to the equation $\dfrac{-8v^2+24v+32}{(v+2)(v-2)} = 0$.

Since a fraction is zero if and only if the numerator is zero (and the denominator is nonzero), we need to look at $-8v^2 + 24v + 32 = 0$. We'll start by factoring, since we see a common factor of 8 in the coefficients. Actually, let's factor out -8 to clean up the sign of the leading term: $-8v^2 + 24v + 32 = -8(v^2 - 3v - 4)$. The quadratic factor $v^2 - 3v - 4$ can be factored to $(v-4)(v+1)$. That means:

$$\begin{aligned}
\frac{-8v^2+24v+32}{(v+2)(v-2)} &= 0 \\
-8v^2+24v+32 &= 0 \\
-8(v^2-3v-4) &= 0 \\
-8(v-4)(v+1) &= 0.
\end{aligned}$$

Setting each factor equal to 0 we see that either $-8 = 0$ (which is impossible), $v - 4 = 0$ (which gives a possible solution of $v = 4$), and $v + 1 = 0$ (which gives a possible solution of $v = -1$.

There are two POSSIBLE solutions, $v = -1$ and $v = 4$. The process of solving a rational equation like this can sometimes introduce extraneous solution. That is, a number that appears to be a solution, but doesn't actually satisfy the original equation.

Let's plug both of these possibilities back into our original formula for $s(v)$ to verify that they are actually solutions.

$$\begin{aligned} s(-1) &= \frac{12}{(-1)-2} + \frac{12}{(-1)+2} - 8 \\ &= \frac{12}{-3} + \frac{12}{1} - 8 \\ &= -4 + 12 - 8 = 0 \end{aligned}$$

$$\begin{aligned} s(4) &= \frac{12}{(4)-2} + \frac{12}{(4)+2} - 8 \\ &= \frac{12}{2} + \frac{12}{6} - 8 \\ &= 6 + 2 - 8 = 0. \end{aligned}$$

That means both $v = -1$ and $v = 4$ are solutions to the rational equation $\frac{12}{v-2} + \frac{12}{v+2} - 8 = 0$.

Let's remember where this example came from. In the example, v represented a speed, so it cannot be negative. The only solution is $v = 4$ miles per hour.

Example 71. *Let f be the function given by $f(x) = \frac{1}{x-4} + x - \frac{x-3}{x-4}$. Find the zeros of the rational function f.*

Explanation

We are being asked to solve the equation

$$\frac{1}{x-4} + x - \frac{x-3}{x-4} = 0$$

Notice that the only denominators appearing in the fractions are $x - 4$, so the common denominator is $x - 4$. We start by combining these terms into a single fraction with that denominator.

$$\begin{aligned}
f(x) &= \frac{1}{x-4} + x - \frac{x-3}{x-4} \\
&= \frac{1}{x-4} + x \cdot \left(\frac{x-4}{x-4}\right) - \frac{x-3}{x-4} \\
&= \frac{1}{x-4} + \frac{x(x-4)}{x-4} - \frac{x-3}{x-4} \\
&= \frac{1}{x-4} + \frac{x^2-4x}{x-4} - \frac{x-3}{x-4} \\
&= \frac{(1)+(x^2-4x)-(x-3)}{x-4} \\
&= \frac{x^2+(-4x-x)+(1+3)}{x-4} \\
&= \frac{x^2-5x+4}{x-4}
\end{aligned}$$

Setting the numerator equal to zero and factoring gives the following.

$$\begin{aligned}
f(x) &= 0 \\
\frac{x^2-5x+4}{x-4} &= 0 \\
x^2-5x+4 &= 0 \\
(x-4)(x-1) &= 0.
\end{aligned}$$

By setting each of these factors equal to 0 we see that either $x-4=0$ (which gives a possible solution of $x=4$) and $x-1=0$ (which gives a possible solution of $x=1$). The two possible solutions are $x=4$ and $x=1$. Let's check them.

$$\begin{aligned}
f(1) &= \frac{1}{(1)-4} + (1) - \frac{(1)-3}{(1)-4} \\
&= \frac{1}{-3} + 1 - \frac{-2}{-3} \\
&= -\frac{1}{3} + 1 - \frac{2}{3} = 0.
\end{aligned}$$

However, $x=4$ is not in the domain of f, since it makes the denominators of the first and third terms zero. That is, $x=4$ is an extraneous solution.

The only solution is $x=1$.

Example 72. *Let g be the function given by* $g(p) = \dfrac{3}{p-2} + \dfrac{5}{p+2} - \dfrac{12}{p^2-4}$. *Find the zeros of the rational function g.*

Explanation

Since $p^2 - 4 = (p+2)(p-2)$, the least common denominator between these three fractions is $(p+2)(p-2) = p^2 - 4$. As before, we start by combining into a single fraction with that denominator.

$$\begin{aligned}
g(p) &= \frac{3}{p-2} + \frac{5}{p+2} - \frac{12}{p^2-4} \\
&= \frac{3}{p-2} \cdot \left(\frac{p+2}{p+2}\right) + \frac{5}{p+2} \cdot \left(\frac{p-2}{p-2}\right) - \frac{12}{p^2-4} \\
&= \frac{3(p+2)}{(p-2)(p+2)} + \frac{5(p-2)}{(p+2)(p-2)} - \frac{12}{p^2-4} \\
&= \frac{3p+6}{(p+2)(p-2)} + \frac{5p-10}{(p+2)(p-2)} - \frac{12}{(p+2)(p-2)} \\
&= \frac{(3p+6)+(5p-10)-(12)}{(p+2)(p-2)} \\
&= \frac{(3p+5p)+(6-10-12)}{(p+2)(p-2)} \\
&= \frac{8p-16}{(p+2)(p-2)}
\end{aligned}$$

Setting the numerator equal to zero gives the following.

$$\begin{aligned}
8p - 16 &= 0 \\
8p &= 16 \\
p &= \frac{16}{8} = 2.
\end{aligned}$$

There is one possible solution, at $p = 2$.

However, $p = 2$ is not in the domain of g, since it makes the denominators of the first and third terms zero. That is, $p = 2$ is an extraneous solution.

The function g does not have any zeros.

Let's look at this last example a bit more. We combined the three terms of $g(p)$

into a single fraction, but that fraction was not in its reduced form.

$$\begin{aligned}
g(p) &= \frac{3}{p-2} + \frac{5}{p+2} - \frac{12}{p^2-4} \\
&= \frac{8p-16}{(p+2)(p-2)} \\
&= \frac{8(p-2)}{(p+2)(p-2)} \\
&= \frac{8\cancel{(p-2)}}{(p+2)\cancel{(p-2)}} \\
&= \frac{8}{(p+2)} \quad \text{, for } p \neq 2.
\end{aligned}$$

Why was $p = -2$ not a zero of the function? Because it was also a zero of the denominator. (Notice the common factor of $p - 2$ in both the numerator and denominator.) Rewriting the fraction in lowest terms, we see that the numerator is never zero, since it's a constant 8.

From this example, it may seem that reducing the rational function to lowest terms will always help you bypass the extraneous solutions. That is not the case.

Example 73. *Let r be the function given by $r(t) = \dfrac{(t+3)(t^2-2t+1)}{t^2-1}$. Find the zeros of r.*

Explanation

Since we are already given the formula for r as a single fraction, let us simplify.

$$\begin{aligned}
r(t) &= \frac{(t+3)(t^2-2t+1)}{t^2-1} \\
&= \frac{(t+3)(t-1)^2}{(t+1)(t-1)} \\
&= \frac{(t+3)(t-1)(t-1)}{(t+1)(t-1)} \\
&= \frac{(t+3)(t-1)\cancel{(t-1)}}{(t+1)\cancel{(t-1)}} \\
&= \frac{(t+3)(t-1)}{(t+1)}
\end{aligned}$$

The fraction will be zero when the numerator is zero. Setting each of the factors of the numerator equal to zero gives $t + 3 = 0$ (which gives a possible solution of $t = -3$), and $t - 1 = 0$ (which gives a possible solution of $t = 1$).

When we cancelled out the common factor of $t-1$ from the numerator and the denominator, we changed the function without mentioning it. This new fraction $\dfrac{(t+3)(t-1)}{(t+1)}$ has $t = 1$ in its domain, but it is not in the domain of r. To be thorough, after that cancellation we should have written

$$r(t) = \frac{(t+3)(t-1)}{(t+1)}\text{ , for } t \neq 1$$

to indicate that we're still using the original domain of r.

The function r has a single zero, at $x = -3$.

6.3.2 Zeros of Functions with Radicals

Introduction

In the previous section we found zeros for rational functions and ran into the problem of extraneous solutions. Radical functions are another instance where we have to check for those.

Earlier we discussed that domains of the polynomial functions given by the form x^n, for n a positive, whole number. If n is odd the range is $(-\infty, \infty)$. That means for every number b, there is a number a with $a^n = b$. In this case, there is exactly one number a with this property, which is denoted by $a = \sqrt[n]{b}$. This means $\sqrt[n]{b}$ exists, when n is odd, for all real numbers b.
If n is even the range of x^n is $[0, \infty)$. This means that for every nonnegative number b, there is a number a with $a^n = b$. In this case, there are exactly two numbers a with this property (unless $b = 0$). Those numbers have the same absolule value, but one is positive and the other is negative. The positive one is denoted by $\sqrt[n]{b}$. This means $\sqrt[n]{b}$ exists, when n is even, only for nonnegative real numbers b and the result $\sqrt[n]{b}$ is nonnegative.

Definition The **principal n^{th} root function** is given by $\sqrt[n]{x}$, for n a positive, whole number called the **index of the radical**. The value of $\sqrt[n]{x}$ has the property that $\left(\sqrt[n]{x}\right)^n = x$.

For n odd, the domain of $\sqrt[n]{x}$ is $(-\infty, \infty)$ and the range is $(-\infty, \infty)$. For n even, the domain of $\sqrt[n]{x}$ is $[0, \infty)$ and the range is $[0, \infty)$.

Zeros of Rational Functions

Example 74. *Let g be the function given by $g(x) = \sqrt{1 - x^2}$. Find the zeros of g.*

Explanation

$$
\begin{aligned}
g(x) &= 0 \\
\sqrt{1-x^2} &= 0 \\
\left(\sqrt{1-x^2}\right)^2 &= (0)^2 \\
1-x^2 &= 0 \\
(1+x)(1-x) &= 0
\end{aligned}
$$

From these factors, either $x=-1$ or $x=1$. Checking these gives:

$$
\begin{aligned}
g(1) &= \sqrt{1-(1)^2} = 0 \\
g(-1) &= \sqrt{1-(-1)^2} = 0
\end{aligned}
$$

Both of these check out, so the zeroes are $x=\pm 1$.

Example 75. *Let f be the function given by $f(x)=\sqrt{3x+7}-x-1$. Find the zeros of the function f.*

Explanation

To solve the equation $\sqrt{3x+7}-x-1=0$, we'll isolate the radical and square both sides.

$$
\begin{aligned}
\sqrt{3x+7}-x-1 &= 0 \\
\sqrt{3x+7} &= x+1 \\
\left(\sqrt{3x+7}\right)^2 &= (x+1)^2 \\
3x+7 &= x^2+2x+1 \\
x^2-x-6 &= 0 \\
(x+2)(x-3) &= 0
\end{aligned}
$$

Setting these factors each equal to zero gives us the possible solutions $x=-2$ and $x=3$. Let's check them.

$$
\begin{aligned}
f(3) &= \sqrt{3(3)+7}-(3)-1 \\
&= \sqrt{16}-4 = 0
\end{aligned}
$$

$$
\begin{aligned}
f(-2) &= \sqrt{3(-2)+7}-(-2)-1 \\
&= \sqrt{1}+1 = 2
\end{aligned}
$$

Notice that $f(-2)$ is not zero, so $x=-2$ is an extraneous solution.

The only zero is $x=3$.

Example 76. *Let s be the function given by $s(t)=t-\sqrt[3]{t^3+3t^2-6t-8}+1$. Find the zeros of the function s.*

Explanation

As before, we'll isolate the radical. However, the radical here is a cube root so we have to raise each side to the third power, instead of squaring them.

$$\begin{aligned} s(t) &= 0 \\ t-\sqrt[3]{t^3+3t^2-6t-8}+1 &= 0 \\ t+1 &= \sqrt[3]{t^3+3t^2-6t-8} \\ (t+1)^3 &= \left(\sqrt[3]{t^3+3t^2-6t-8}\right)^3 \\ t^3+3t^2+3t+1 &= t^3+3t^2-6t-8 \\ 9t &= -9 \\ t &= -1 \end{aligned}$$

The only possible zero is $t=-1$. Let's check it.

$$\begin{aligned} s(-1) &= (-1)-\sqrt[3]{(-1)^3+3(-1)^2-6(-1)-8}+1 \\ &= -\sqrt[3]{-1+3+6-8} = 0. \end{aligned}$$

The function s has a single zero, at $t=-1$.

Example 77. *Let f be the function given by $f(x)=3+\sqrt{4-x}-\sqrt{2x+1}$. Find the zeros of f.*

Explanation

In this case, there are multiple radicals and we can't isolate them both simultaneously. Instead, we'll isolate just one of them first.

$$\begin{aligned}
f(x) &= 0 \\
3 + \sqrt{4-x} - \sqrt{2x+1} &= 0 \\
3 + \sqrt{4-x} &= \sqrt{2x+1} \\
\left(3 + \sqrt{4-x}\right)^2 &= \left(\sqrt{2x+1}\right)^2 \\
9 + 6\sqrt{4-x} + (4-x) &= 2x+1 \\
6\sqrt{4-x} &= 3x - 12 \\
3\left(2\sqrt{4-x}\right) &= 3(x-4) \\
2\sqrt{4-x} &= x - 4 \\
\left(2\sqrt{4-x}\right)^2 &= (x-4)^2 \\
4(4-x) &= x^2 - 8x + 16 \\
16 - 4x &= x^2 - 8x + 16 \\
x^2 - 4x &= 0 \\
x(x-4) &= 0
\end{aligned}$$

Setting each of these factors equal to zero gives the two possible zeros $x = 0$ and $x = 4$. Let's check them.

$$\begin{aligned}
f(0) &= 3 + \sqrt{4-(0)} - \sqrt{2(0)+1} \\
&= 3 + \sqrt{4} - \sqrt{1} = 4
\end{aligned}$$

$$\begin{aligned}
f(4) &= 3 + \sqrt{4-(4)} - \sqrt{2(4)+1} \\
&= 3 + \sqrt{0} - \sqrt{9} = 0.
\end{aligned}$$

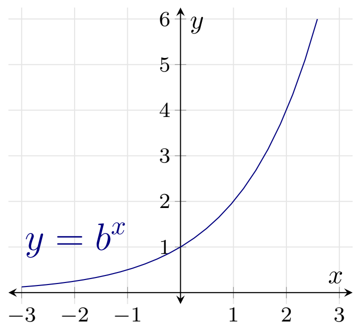
The possible zero at $x = 0$ is an extraneous solution. The only zero of f is $x = 4$.

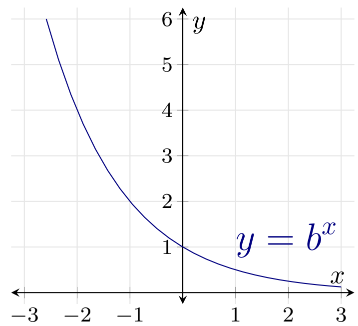
6.3.3 Zeros of Exponential Functions

Introduction

Zeros of Exponential Functions

As we saw previously, there are two varieties of elementary exponential functions: Increasing and Decreasing. The exponential function f given by $f(x) = b^x$ is increasing if $b > 1$ and decreasing if $0 < b < 1$. Graphically, the two situations resemble the following.





These functions have domain $(-\infty, \infty)$ and range $(0, \infty)$. Notice that 0 is not in the range. That means the exponential function $f(x) = b^x$ has no zeros. The translated exponential functions, however, $g(x) = b^x + c$ will have a zero if c is negative.

> Remember that the natural logarithm, $\ln(x)$, is the inverse of the exponential function e^x.

That means the composition $\ln(e^x) = x$ for all values of x. If we isolate the exponential on one side of our equation, we can use the logarithm to "undo" it.

Example 78. *Let f be the function given by $f(x) = 4e^x - 5$. Find the zeros of f.*

Explanation

$$
\begin{aligned}
f(x) &= 0 \\
4e^x - 5 &= 0 \\
4e^x &= 5 \\
e^x &= \frac{5}{4} \\
\ln(e^x) &= \ln\left(\frac{5}{4}\right) \\
x &= \ln\left(\frac{5}{4}\right)
\end{aligned}
$$

This function has only a single zero, at $x = \ln\left(\frac{5}{4}\right)$.

The key to finding the zero in this example was being able to use the inverse function of e^x to bring down that variable. By examining the graphs of the exponentials above, you will notice that they pass the horizontal line test. That is, the exponential function $f(x) = b^x$ is a one-to-one function for any $b > 0$, $b \neq 1$. This means each of those exponential functions has an inverse, not just the base e exponential. These inverses are called logarithms.

> **Definition** For a constant $b > 0$, $b \neq 1$, the **logarithm** with base b, $\log_b(x)$, is the inverse of the exponential function b^x. The domain of $\log_b(x)$ is $(0, \infty)$ and the range of $\log_b(x)$ is $(-\infty, \infty)$.

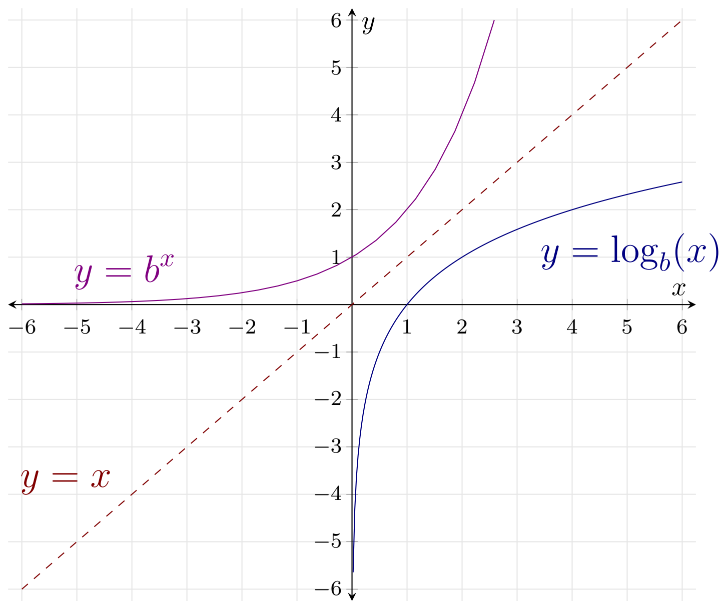
Remember that if f and f^{-1} are inverse functions, the domain of f is the range of f^{-1}, and the range of f is the domain of f^{-1}.

That the functions given by $\log_b(x)$ and b^x are inverses means:

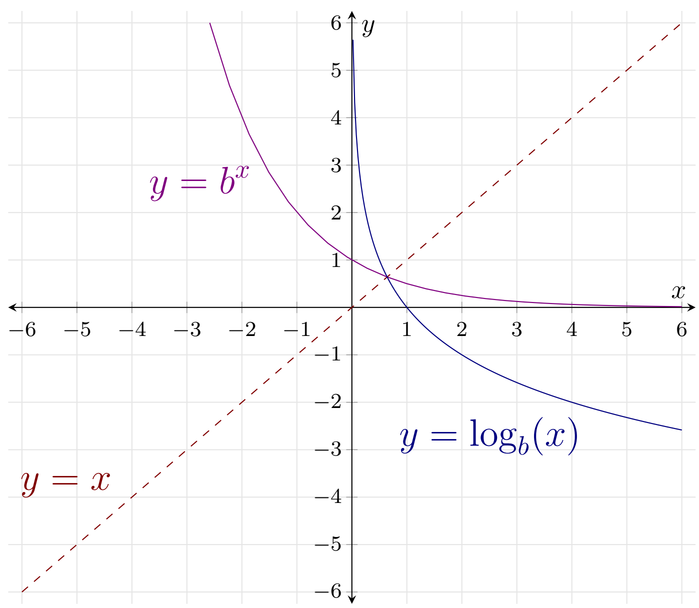
(a) $\log_b(b^x) = x$ for all x in $(-\infty, \infty)$

(b) $b^{\log_b(x)} = x$ for all x in $(0, \infty)$

The graphs of the exponentials b^x allow us to find the graphs of the corresponding logarithms by reflecting across the line $y = x$. For $b > 1$ we have this graph.



For $0 < b < 1$ we have this graph.



Here is a link to exponential functions and logarithms plotted on the same graph in Desmos. Move the slider for the base value of b and see how the two graphs respond. Desmos link: `https://www.desmos.com/calculator/q0aivjmasd`.

Example 79. *Let g be the function given by $g(x) = 2 \cdot 6^x - 5$. Find the zeros* *of the function g.*

Explanation

Be careful with the order of operations here. Remember that $2 \cdot 6^x$ is not the same as 12^x.

$$\begin{aligned} g(x) &= 0 \\ 2 \cdot 6^x - 5 &= 0 \\ 2 \cdot 6^x &= 5 \\ 6^x &= \frac{5}{2} \\ \log_6(6^x) &= \log_6\left(\frac{5}{2}\right) \\ x &= \log_6\left(\frac{5}{2}\right) \end{aligned}$$

The function g has a zero at $x = \log_6\left(\frac{5}{2}\right)$.

Example 80. *Let h be the function given by $h(t) = \left(\frac{1}{2}\right)^t + 3$. Find the zeros of h.*

Explanation

$$\begin{aligned} h(t) &= 0 \\ \left(\frac{1}{2}\right)^t + 3 &= 0 \\ \left(\frac{1}{2}\right)^t &= -3 \end{aligned}$$

Our next step would be to take the logarithm, base $\frac{1}{2}$, of both sides to isolate the variable t, but that would mean taking the logarithm of -3. The domain of $\log_{1/2}(t)$ is $(0, \infty)$, so the logarithm of -3 does not exist. Said another way, the function $\left(\frac{1}{2}\right)^t$ has range $(0, \infty)$, so there is no value of t for which $\left(\frac{1}{2}\right)^t$ is -3.

This function has no zeros.

Notice that 0 is in the range of the logarithms. The fact that $b^0 = 1$ for all $b \neq 0$, means that for each logarithm, $\log_b(1) = 0$. Each logarithm $\log_b(x)$ has a zero at $x = 1$. If the function is modified, we can use the fact that $b^{\log_b(x)} = x$ for all x in $(-\infty, \infty)$ to find the zeros.

Example 81. *Let f be the function given by $f(x) = 3\log_5(x) + 7$. Find the zeros of f.*

Explanation

$$\begin{aligned}
f(x) &= 0\\
3\log_5(x) + 7 &= 0\\
3\log_5(x) &= -7\\
\log_5(x) &= -\frac{7}{3}\\
5^{\log_5(x)} &= 5^{-\frac{7}{3}}\\
x &= 5^{-\frac{7}{3}}
\end{aligned}$$

The function f has a zero at $x = 5^{-\frac{7}{3}}$.

Example 82. *Let k be the function given by $k(t) = \dfrac{2t\log_5(t)}{3e^t + 1}$. Find the zeros of f.*

Explanation We know that a fraction is zero precisely when the numerator is zero.

$$\begin{aligned}
k(t) &= 0\\
2t\log_5(t) &= 0
\end{aligned}$$

Setting these factors equal to zero we find either $2t = 0$, giving us the possible zero at $t = 0$, or $\log_5(t) = 0$, giving us the possible zero at $t = 1$. Let us check them.

$$\begin{aligned}
k(1) &= \frac{2(1)\log_5(1)}{3e^1 + 1}\\
&= \frac{2(0)}{3e + 1} = 0
\end{aligned}$$

However, $t = 0$ is not in the domain of k, since the $\log_5(t)$ factor would be undefined.

The function k has a zero at $x = 1$.

Part 7

Manipulating Functions

7.1 Function Transformations

Learning Objectives

- Vertical and Horizontal Shifts
  - How to shift a function vertically
  - How to shift a function horizontally
  - Combining shifts and properties of quadratics (vertex, completing the square)
- Stretching Functions
  - Vertical stretch
  - Horizontal stretch
- Reflections of Functions
  - Reflections across the x-axis, the y-axis, the origin, $y = x$
  - Connect reflections to inverses, even, and odd functions

7.1.1 Vertical and Horizontal Shifts

Motivating Questions

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7.1.2 Stretching Functions

Motivating Questions

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7.1.3 Reflections of Functions

Motivating Questions

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7.2 Systems of Equations

Learning Objectives

- From Systems to Solutions
  - What is a system of equations?
  - What is a solution to a system?
  - Solving systems via graphs
- Solving Systems Algebraically
  - Eliminating Variables
  - Substitution Method
- Applications of Systems of Equations
  - Word problems
  - Mixture Problems

7.2.1 From Systems to Solutions

Motivating Questions

- What is a system of equations?
- What is a solution to a system?
- How can we solve systems of equations using graphs?

Introduction

We have already seen many techniques for solving equations. Until now, however, we have only solved equations of the form $f(x) = 0$ for the variable x. In this section, we will consider equations with more than one variable and discuss how to solve them.

Consider a peculiar grocery store where the prices of all the items for sale are not listed, and you only find out the total cost of your purchase. Say you buy 6 mangos and 3 bananas, and your total cost is 9 dollars. Assume all the mangos cost the same amount and all the bananas cost the same amount. Without making any more purchases, is it possible find out how much a mango and a banana cost on their own?

Let's create an equation to describe this situation. Let x be a variable representing the cost of a mango, and let y be a variable representing the cost of a banana. Then, the equation

$$6x + 3y = 9$$

represents that buying 6 mangos at a cost of x dollars and 3 bananas at a cost of y dollars yields a total cost of 9 dollars.

You might have noticed that plugging $x = 1$ and $y = 1$ into the equation gives us a true statement, so you might conclude that mangos and bananas both cost 1 dollar. However, notice that plugging $x = 1.20$ and $y = 0.60$ into the equation also gives us a true statement, so it's also possible that mangos cost \$1.20 and bananas cost \$0.60. Even more worrying is that $x = 0$ and $y = 3$ also gives us a solution to the equation: is this store peculiar enough to be giving away mangos for free and charging \$3 per banana?

Examining the equation we set up can give us more insight. Let's rearrange the equation to solve for y in terms of x:

$$\begin{aligned} 6x + 3y &= 9 \\ 3y &= 9 - 6x \\ y &= 3 - 2x. \end{aligned}$$

Now it becomes clearer what's going on. Whatever x is, we can find a value of y that satisfies our original equation. No matter the cost of a single mango, there's a way to price the bananas so that our equation is true! This means it's impossible to find the price of a single mango or a single banana with the information you've been given. We need more data!

Systems of linear equations

In order to collect more information, you go back to the store and buy 2 mangos and 2 banana for a total cost of \$3.20. This can be modeled by the equation

$$2x + 2y = 3.2.$$

Keep in mind that this x and y are the same x and y from before, so in order to find the cost of a mango and a banana, we must find x and y that satisfy both equations

$$\begin{cases} 6x + 3y = 9 \\ 2x + 2y = 3.2 \end{cases}$$

at the same time. This coupling of two (or more) linear equations is called a *system of linear equations*.

Definition A **linear equation of two variables** is an equation of the form

$$a_1x + a_2y = c,$$

where a_1, a_2, and c are real numbers and at least one of a_1 and a_2 is nonzero.
A **system of linear equations of two variables** is a collection of two or more linear equations of two variables.
We say a **solution** to a system of linear equations of two variables is a point (x, y) satisfying all equations in the system.

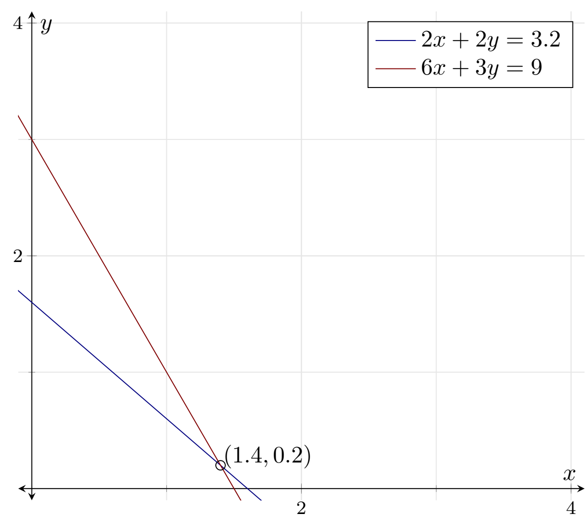
The key to identifying linear equations is to note that the variables involved are to the first power and that the coefficients of the variables are numbers. Some examples of equations which are non-linear are $x^2 + y = 1$, $xy = 5$ and $e^{2x} + \ln(y) = 1$. Note that we can still have systems of non-linear equations, but they can be much more difficult to solve.

Finding solutions graphically

Let's return to our example from earlier and try to find a solution to the system of linear equation:

$$\begin{cases} 6x + 3y = 9 \\ 2x + 2y = 3.2 \end{cases}.$$

We want to find x and y satisfying both equations in the system. If x and y satisfy $6x + 3y = 9$, then the point (x, y) lies on the graph of $6x + 3y = 9$. Similarly, if x and y satisfy $2x+2y = 3.2$, then the point (x, y) lies on the graph of $2x + 2y = 3.2$. Therefore, to find any solutions, we can look at the graphs of $6x + 3y = 9$ and $2x + 2y = 3.2$, and see if there are any points that lie at the intersection of the two graphs:



By inspecting the graph, we see that these two lines intersect only at $(1.4, 0.2)$, so the only solution to the system is $x = 1.4$ and $y = 0.2$.

In context, this means that mangos cost \$1.40 each and bananas cost \$0.20 each. Note that in order to have exactly one solution to our system of linear equations in two variables, we needed the system to have two equations.

Note that not every system of linear equations will have one solution. If the graphs of the two equations are parallel, they will never intersect, so there won't be any solutions. Additionally, if the two equations are represented by the same graph, there will be infinitely many intersection points, and therefore, infinitely many solutions.

Next, we will see some methods for solving systems of equations algebraically.

7.3 End of Semester Project

Learning Objectives

- Creating a Font
  - Definition of Function
  - Function Transformations
  - Domains and ranges, especially restricted domains
  - Interperting graphs

7.3.1 Creating a Font

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Part 8

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