

Factored Form

We explore different methods for solving quadratic functions.

We have previously looked at different forms of quadratic functions. We've looked at standard form and vertex form, where characteristics like y -intercept and vertex can be found easily by looking at the function. Another useful way to look at quadratic functions is to have them written out as a product of linear factors. This can help us to quickly determine the x -intercepts of a quadratic function and to get a good idea of the position and shape of the graph. Not all quadratics can be written in factored form, so we will begin by addressing those.

Remark Irreducible quadratic factors are quadratic factors that when set equal to zero only have complex roots. As a result they cannot be reduced into factors containing only real numbers, hence the name irreducible.

As seen in the graphs below, the graphs of the functions do not cross the x -axis, so they do not have x -intercepts. The first graph, $y = x^2 + x + 1$ is entirely above the x -axis and the second graph, $y = -x^2 + x - 1$ is entirely below the x -axis. Since neither of them cross the x -axis, they have no x -intercepts and are irreducible.

Intercept Form

Intercept Form of a Quadratic Function

A quadratic function whose graph has x intercepts at the points $(r, 0)$ and $(s, 0)$ can be written as:

$$f(x) = a(x - r)(x - s)$$

Factoring from Standard Form when $a = 1$

When $a = 1$, our factored form is the same as our Intercept form.

Example 1. Factor the following quadratic into a product of linear factors:

$$x^2 + 3x + 2$$

Learning outcomes:
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Explanation For us to begin factoring this quadratic, we have to look at the b and c terms. We are looking for 2 numbers that multiply to 2 (or c) and add up to 3 (or b). By going through the factors of 2 we can see that the only numbers that satisfy these conditions are 2 and 1.

$$2 + 1 = 3$$

$$2 \cdot 1 = 2$$

This means that we can factor the quadratic the following way:

$$x^2 + 3x + 2 = (x + 2)(x + 1)$$

The quadratic is now written as a product of linear factors and because $a = 1$, these are also our x -intercepts for our function.

Factoring from Standard Form when $a > 1$

Example 2. Factor the following quadratic into a product of linear components:

$$f(x) = 3x^2 - 4x - 7$$

Explanation First we will start by pulling out a 3 from every term.

$$3(x^2 - \frac{4}{3}x - \frac{7}{3})$$

We now have to find factors that add up to $\frac{4}{3}$ and multiply to $-\frac{7}{3}$. In this particular case, we can see that the difference between the numerators (7&4) is 3, and since $\frac{3}{3} = 1$ our job will be a little easier. This leads us to the following factors:

$$\frac{3}{3} + \frac{-7}{3} = -\frac{4}{3}$$

$$\frac{3}{3} \cdot \frac{-7}{3} = -\frac{7}{3}$$

So, our factored form is as follows:

$$f(x) = 3(x + \frac{3}{3})(x - \frac{7}{3})$$

We now have the equation written as a product of linear components, and if we want to find the x intercepts we can set each of the components equal to 0.

$$x + \frac{3}{3} = 0$$

$$x = -\frac{3}{3}$$

$$x = -1$$

$$x - \frac{7}{3} = 0$$

$$x = \frac{7}{3}$$

So our solutions or x intercepts are $x = -1$ and $x = \frac{7}{3}$

Quadratic Formula

When there appears to be no easy way to factor a quadratic, our best option is to use the Quadratic Formula. Let's try the previous example with the Quadratic Formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

when $ax^2 + bx + c = 0$

Example 3. Find the solutions to the following quadratic equation:

$$3x^2 - 4x - 7 = 0$$

Explanation

$$\begin{aligned}
 x &= \frac{-(-4) \pm \sqrt{(-4)^2 - 4(3)(-7)}}{2(3)} \\
 &= \frac{4 \pm \sqrt{16 - (-84)}}{6} \\
 &= \frac{4 \pm \sqrt{100}}{6} \\
 &= \frac{4 \pm 10}{6} \\
 x &= \frac{4 + 10}{6} & x &= \frac{4 - 10}{6} \\
 x &= \frac{14}{6} & x &= \frac{-6}{6} \\
 x &= \frac{7}{3} & x &= -1
 \end{aligned}$$

We still get $x = -1$ and $x = \frac{7}{3}$ as our solutions. This can be a very useful tool especially with more complicated quadratic equations.

Factoring when missing a term

We've talked about factoring $ax^2 + bx + c$ when all terms are present, but what do we do when one of the terms is missing? Since there are three terms we have three different cases to address.

The first case also happens to be the easiest to solve. How do you solve a quadratic that is missing the ax^2 term? This is a bit of a trick question, because without an x^2 term, we are no longer dealing with a quadratic.

Example 4. *Solve the following equation*

$$2x - 9 = 0$$

Explanation We can see here that we are only dealing with a linear terms and there are no quadratic (x^2) terms. This means we do not have to factor and we can solve for x directly.

$$\begin{aligned} 2x - 9 &= 0 \\ 2x &= 9 \\ x &= \frac{9}{2} \end{aligned}$$

The second case is when the middle bx term is missing.

Example 5. *Factor the quadratic $f(x) = x^2 - 9$ into linear components.*

Explanation This quadratic is a special case called “difference of squares.” There is no “middle” term and the remaining two terms are both perfect squares, so we can use a shortcut when factoring.

Difference of Squares

When a, b are non zero.

$$a^2 - b^2 = (a + b)(a - b)$$

In our case, we can see that x^2 is a perfect square and 9 is also a perfect square because $9 = 3^2$. This means that our original quadratic will be factored like this:

$$x^2 - 9 = (x + 3)(x - 3)$$

We can also think of it in the same way as factoring other quadratics. Since there is no middle term, we can look at factors of -9 that add up to 0. 3 and -3 add up to 0 and multiply out to -9 . The difference of squares is just a useful pattern that helps to speed up our factoring process.

The last case is when there is no constant or c term.

Example 6. *Factor the quadratic $f(x) = x^2 + 2x$ into linear components.*

Explanation Since there is a common x factor in both terms we can pull out that factor and we are left with a product of linear components.

$$\begin{aligned} f(x) &= x^2 + 2x \\ &= x \cdot x + 2x \\ &= x(x + 2) \end{aligned}$$

Again, our factoring is already simplified. We do not have go through the whole process of factoring. If we have a quadratic function with only the ax^2 and bx term, then we will always be able to pull out at least an x term when factoring.