



Precalculus with Review 2: Unit 9

January 17, 2023

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**Building New Functions - See
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Part 9

Trigonometric Functions

9.1 The Unit Circle to the Function Graph

Learning Objectives

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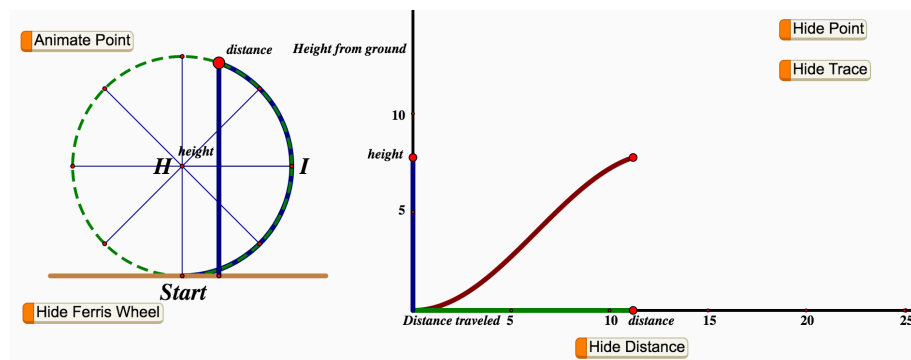
9.1.1 Traversing A Circle

Motivating Questions

- How does a point traversing a circle naturally generate a function?
- What are some important properties that characterize a function generated by a point traversing a circle?

Certain naturally occurring phenomena eventually repeat themselves, especially when the phenomenon is somehow connected to a circle. You may recall from when we first studied periodic function that we considered the case of taking a ride on a ferris wheel. We considered your height, h , above the ground and how your height changed in tandem with the distance, d , that you have traveled around the wheel. We saw snapshot of this situation, which is available as a full animation at <http://gvsu.edu/s/0Dt>.

A snapshot of the motion of a cab moving around a ferris wheel.
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Because we have two quantities changing in tandem, it is natural to wonder if it is possible to represent one as a function of the other.

Exploration In the context of the ferris wheel pictured in above, assume that the height, h , of the moving point (the cab in which you are riding), and the distance, d , that the point has traveled around the circumference of the ferris wheel are both measured in meters. Further, assume that the circumference of the ferris wheel is 150 meters. In addition, suppose that after getting in your cab at the lowest point on the wheel, you traverse the full circle several times.

- a. Recall that the circumference, C , of a circle is connected to the

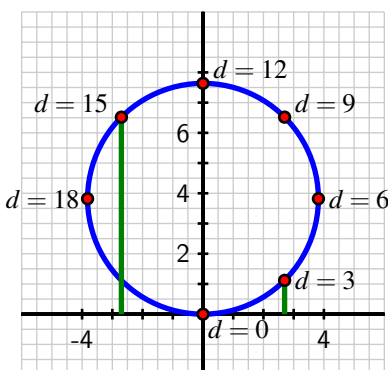
circle's radius, r , by the formula $C = 2\pi r$. What is the radius of the ferris wheel? How high is the highest point on the ferris wheel?

- How high is the cab after it has traveled $1/4$ of the circumference of the circle?
- How much distance along the circle has the cab traversed at the moment it first reaches a height of $\frac{150}{\pi} \approx 47.75$ meters?
- Can h be thought of as a function of d ? Why or why not?
- Can d be thought of as a function of h ? Why or why not?
- Why do you think the curve shown above has the shape that it does? Write several sentences to explain.

Circular Functions

The natural phenomenon of a point moving around a circle leads to interesting relationships. Let's consider a point traversing a circle of circumference 24 and examine how the point's height, h , changes as the distance traversed, d , changes.

Note particularly that each time the point traverses $\frac{1}{8}$ of the circumference of the circle, it travels a distance of $24 \cdot \frac{1}{8} = 3$ units, as seen below where each noted point lies 3 additional units along the circle beyond the preceding one.



Note that we know the exact heights of certain points. Since the circle has circumference $C = 24$, we know that $24 = 2\pi r$ and therefore $r = \frac{12}{\pi} \approx 3.82$. Hence, the point where $d = 6$ (located $1/4$ of the way along the circle) is at a height of $h = \frac{12}{\pi} \approx 3.82$. Doubling this value, the point where $d = 12$ has

height $h = \frac{24}{\pi} \approx 7.64$. Other heights, such as those that correspond to $d = 3$ and $d = 15$ (identified on the figure by the green line segments) are not obvious from the circle's radius, but can be estimated from the grid in the graph above as $h \approx 1.1$ (for $d = 3$) and $h \approx 6.5$ (for $d = 15$). Using all of these observations along with the symmetry of the circle, we can determine the other entries in the table below.

Data for height, h , as a function of distance traversed, d .

d	0	3	6	9	12	15	18	21	24
h	0	1.1	3.82	6.5	7.64	6.5	3.82	1.1	0

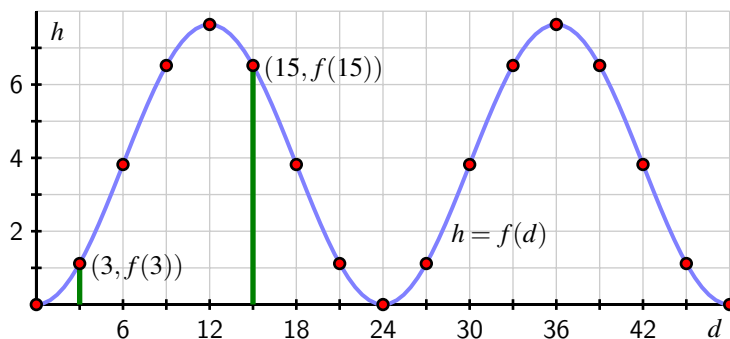
Moreover, if we now let the point continue traversing the circle, we observe that the d -values will increase accordingly, but the h -values will repeat according to the already-established pattern, resulting in the data in the table below.

Additional data for height, h , as a function of distance traversed, d .

d	24	27	30	33	36	39	42	45	48
h	0	1.1	3.82	6.5	7.64	6.5	3.82	1.1	0

It is apparent that each point on the circle corresponds to one and only one height, and thus we can view the height of a point as a function of the distance the point has traversed around the circle, say $h = f(d)$. Using the data from the two tables and connecting the points in an intuitive way, we get the graph shown below.

The height, h , of a point traversing a circle of radius 24 as a function of distance, d , traversed around the circle.

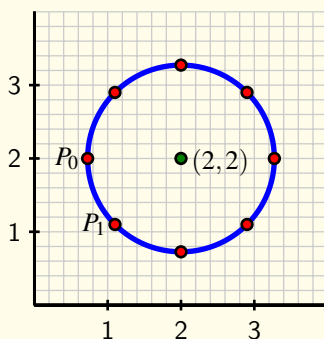


The function $h = f(d)$ we have been discussing is an example of what we will call a **circular function**. Indeed, it is apparent that if we:

- take any circle in the plane,
- choose a starting location for a point on the circle,
- let the point traverse the circle continuously,
- and track the height of the point as it traverses the circle,

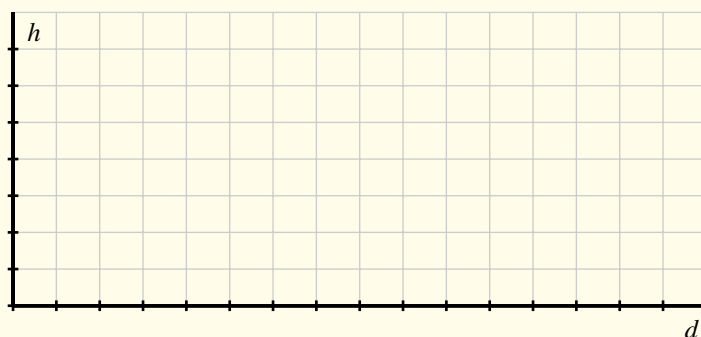
the height of the point is a function of distance traversed and the resulting graph will have the same basic shape as the curve shown in the graph above. It also turns out that if we track the location of the x -coordinate of the point on the circle, the x -coordinate is also a function of distance traversed and its curve has a similar shape to the graph of the height of the point (the y -coordinate). Both of these functions are circular functions because they are generated by motion around a circle.

Exploration Consider the circle pictured below that is centered at the point $(2, 2)$ and that has circumference 8. Assume that we track the y -coordinate (that is, the height, h) of a point that is traversing the circle counterclockwise and that it starts at P_0 as pictured.



- How far along the circle is the point P_1 from P_0 ? Why?
- Label the subsequent points in the figure P_2, P_3, \dots as we move counterclockwise around the circle. What are the exact coordinates of P_2 ? of P_4 ? Why?
- Determine the coordinates of the remaining points on the circle (exactly where possible, otherwise approximately) and hence complete the entries in the table below that track the height, h , of the point traversing the circle as a function of distance traveled, d .

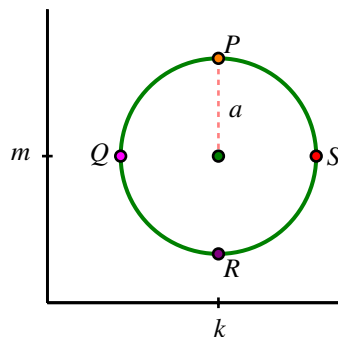
- d. By plotting the points in the table and connecting them in an intuitive way, sketch a graph of h as a function of d on the axes provided over the interval $0 \leq d \leq 16$. Clearly label the scale of your axes and the coordinates of several important points on the curve.



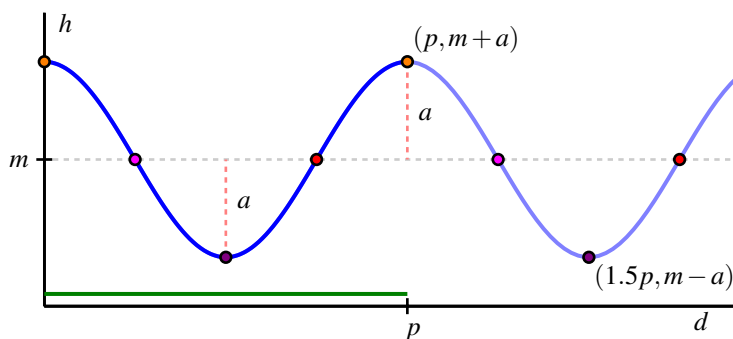
- e. What is similar about your graph in comparison to the one in we created at the beginning of this section? What is different?
- f. What will be the value of h when $d = 51$? How about when $d = 102$?

Properties of Circular Functions

Every circular function has several important features that are connected to the circle that defines the function. For the discussion that follows, we focus on circular functions that result from tracking the y -coordinate of a point traversing counterclockwise a circle of radius a centered at the point (k, m) . Further, we will denote the circumference of the circle by the letter p .



We assume that the point traversing the circle starts at P in the graph of the circle above. Its height is initially $y = m + a$, and then its height decreases to $y = m$ as we traverse to Q . Continuing, the point's height falls to $y = m - a$ at R , and then rises back to $y = m$ at S , and eventually back up to $y = m + a$ at the top of the circle. If we plot these heights continuously as a function of distance, d , traversed around the circle, we get the curve shown below.



This curve has several important features for which we introduce important terminology.

Definition The **midline** of a circular function is the horizontal line $y = m$ for which half the curve lies above the line and half the curve lies below. If the circular function results from tracking the y -coordinate of a point traversing a circle, $y = m$ corresponds to the y -coordinate of the center of the circle. In addition, the **amplitude** of a circular function is the maximum deviation of the curve from the midline. Note particularly that the value of the amplitude, a , corresponds to the radius of the circle that generates the curve.

Because we can traverse the circle in either direction and for as far as we wish, the domain of any circular function is the set of all real numbers. From our observations about the midline and amplitude, it follows that the range of a circular function with midline $y = m$ and amplitude a is the interval $[m - a, m + a]$.

This graph is an example of a periodic function. Recall the definition of a periodic function.

Definition Let f be a function whose domain and codomain are each the set of all real numbers. We say that f is **periodic** provided that there exists a real number k such that $f(x + k) = f(x)$ for every possible choice of x . The smallest value p for which $f(x + p) = f(x)$ for every choice of x is called the **period** of f .

For a circular function, the period is always the circumference of the circle that generates the curve. In the graph of the function above, we see how the curve has completed one full cycle of behavior every p units, regardless of where we start on the curve.

Circular functions arise as models for important phenomena in the world around us, such as in a *harmonic oscillator*. Consider a mass attached to a spring where the mass sits on a frictionless surface. After setting the mass in motion by stretching or compressing the spring, the mass will oscillate indefinitely back and forth, and its distance from a fixed point on the surface turns out to be given by a circular function.

The Average Rate of Change of a Circular Function

Just as there are important trends in the values of a circular function, there are also interesting patterns in the average rate of change of the function. These patterns are closely tied to the geometry of the circle.

For the next part of our discussion, we consider a circle of radius 1 centered at $(0, 0)$, and consider a point that travels a distance d counterclockwise around the circle with its starting point viewed as $(1, 0)$. We use this circle to generate the circular function $h = f(d)$ that tracks the height of the point at the moment the point has traversed d units around the circle from $(1, 0)$. Let's consider the average rate of change of f on several intervals that are connected to certain fractions of the circumference.

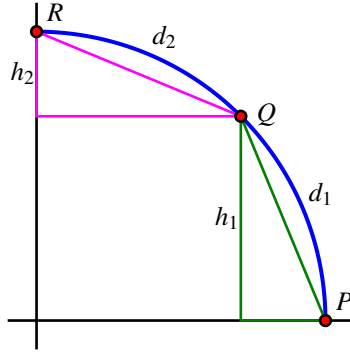
Remembering that h is a function of distance traversed along the circle, it follows that the average rate of change of h on any interval of distance between two points P and Q on the circle is given by

$$\text{AROC}_{[P,Q]} = \frac{\text{change in height}}{\text{distance along the circle}},$$

where both quantities are measured from point P to point Q .

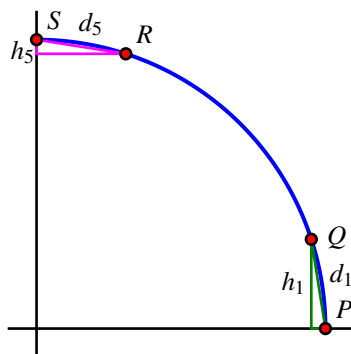
First, we consider points P , Q , and R where Q results from traversing $1/8$ of the circumference from P , and R $1/8$ of the circumference from Q . In particular, we note that the distance d_1 along the circle from P to Q is the same as the distance d_2 along the circle from Q to R , and thus $d_1 = d_2$. At the same time, it is apparent from the geometry of the circle that the change in height h_1 from P to Q is greater than the change in height h_2 from Q to R , so $h_1 > h_2$. Thus, we can say that

$$\text{AROC}_{[P,Q]} = \frac{h_1}{d_1} > \frac{h_2}{d_2} = \text{AROC}_{[Q,R]}.$$



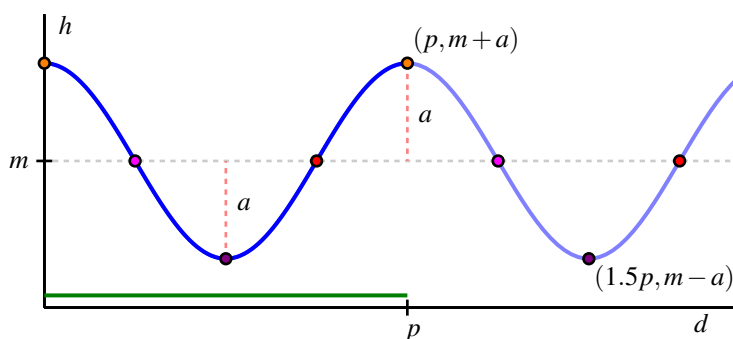
The differences in certain average rates of change appear to become more extreme if we consider shorter arcs along the circle. Next we consider traveling $1/20$ of the circumference along the circle. In the graph below, points P and Q lie $1/20$ of the circumference apart, as do R and S , so here $d_1 = d_5$. In this situation, it is the case that $h_1 > h_5$ for the same reasons as above, but we can say even more. From the green triangle, we see that $h_1 \approx d_1$ (while $h_1 < d_1$), so that $\text{AROC}_{[P,Q]} = \frac{h_1}{d_1} \approx 1$. At the same time, in the magenta triangle in the figure we see that h_5 is very small, especially in comparison to d_5 , and thus $\text{AROC}_{[R,S]} = \frac{h_5}{d_5} \approx 0$. Hence, in this graph,

$$\text{AROC}_{[P,Q]} \approx 1 \text{ and } \text{AROC}_{[R,S]} \approx 0.$$



This information tells us that a circular function appears to change most rapidly for points near its midline and to change least rapidly for points near its highest and lowest values.

We can study the average rate of change not only on the circle itself, but also on a generic circular function graph, and thus make conclusions about where the function is increasing, decreasing, concave up, and concave down.



Summary

- When a point traverses a circle, a corresponding function can be generated by tracking the height of the point as it moves around the circle, where height is viewed as a function of distance traveled around the circle. We call such a function a *circular function*.
- Circular functions have several standard features. The function has a *midline* that is the line for which half the points on the curve lie above the line and half the points on the curve lie below. A circular function's *amplitude* is the maximum deviation of the

function value from the midline; the amplitude corresponds to the radius of the circle that generates the function. Circular functions also repeat themselves, and we call the smallest value of p for which $f(x + p) = f(x)$ for all x the period of the function. The period of a circular function corresponds to the circumference of the circle that generates the function.

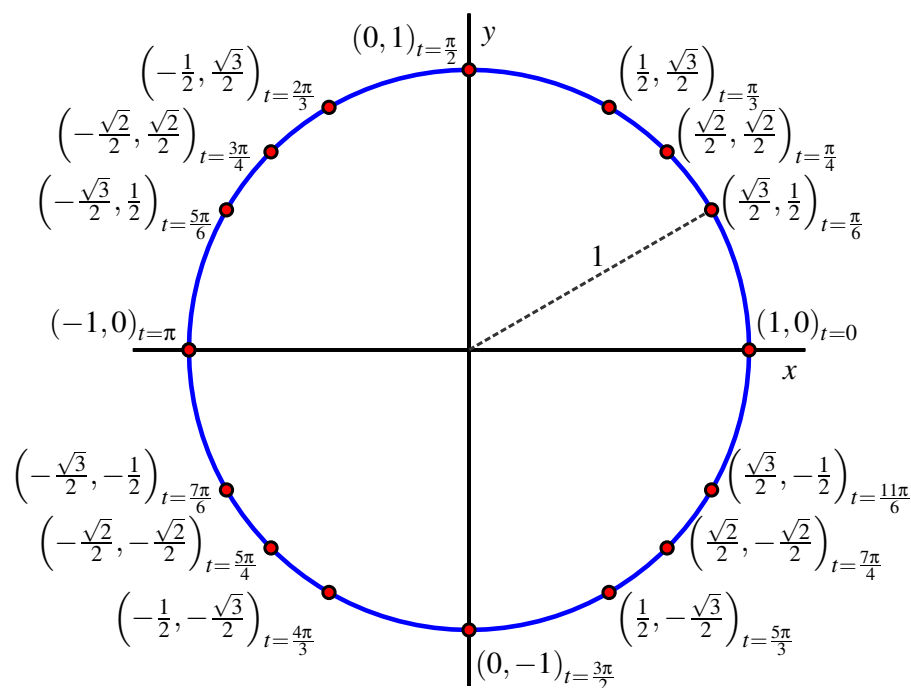
- Non-constant linear functions are either always increasing or always decreasing; quadratic functions are either always concave up or always concave down. Circular functions are sometimes increasing and sometimes decreasing, plus sometimes concave up and sometimes concave down. These behaviors are closely tied to the geometry of the circle.

9.1.2 The Sine and Cosine Functions

Motivating Questions

- What are the sine and cosine functions and how do they arise from a point traversing the unit circle?
- What important properties do the sine and cosine functions share?

In the last section, we saw how tracking the height of a point that is traversing a circle generates a periodic function. Previously, we also identified a collection of special points on the unit circle.

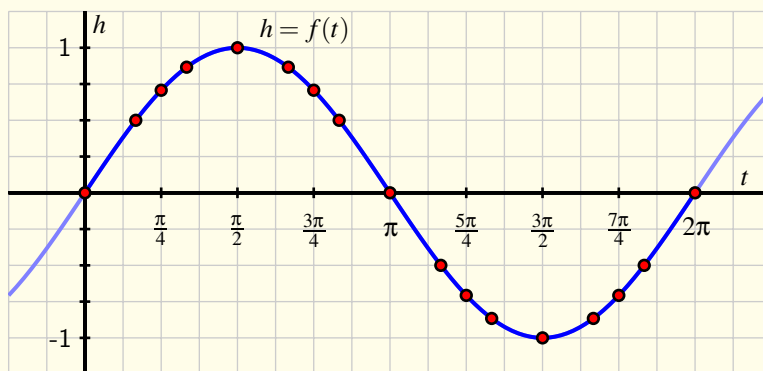


You can also use the *Desmos* file:

Desmos link: <https://www.desmos.com/calculator/jgddn7tzxg>

Exploration

If we consider the unit circle, start at $t = 0$, and traverse the circle counterclockwise, we may view the height, h , of the traversing point as a function of the angle, t , in radians. From there, we can plot the resulting (t, h) ordered pairs and connect them to generate the circular function pictured below.



- What is the exact value of $h\left(\frac{\pi}{4}\right)$? of $h\left(\frac{\pi}{3}\right)$?
- Complete the following table with the exact values of h that correspond to the stated inputs.

t	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
h									

t	π	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{7\pi}{4}$	$\frac{11\pi}{6}$	2π
h									

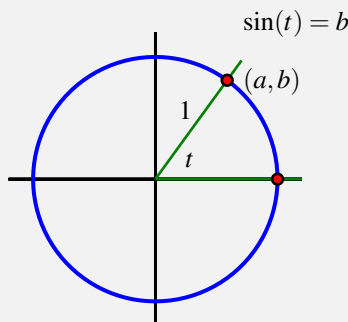
- What is the exact value of $h\left(\frac{11\pi}{4}\right)$? of $h\left(\frac{14\pi}{3}\right)$?
- Give four different values of t for which $h(t) = -\frac{\sqrt{3}}{2}$.

The Definition of the Sine Function

The circular function that tracks the height of a point on the unit circle traversing counterclockwise from $(1, 0)$ as a function of the corresponding central angle

(in radians) is one of the most important functions in mathematics. As such, we give the function a name: the **sine** function.

Definition



Given a central angle in the unit circle that measures t radians and that intersects the circle at both $(1, 0)$ and (a, b) , we define the **sine of t** , denoted $\sin(t)$, by the rule

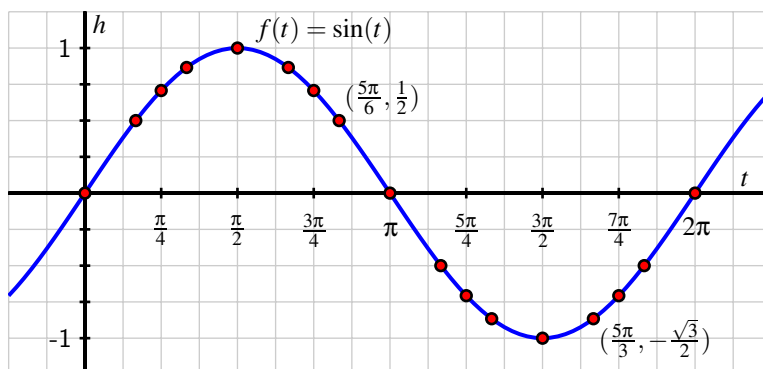
$$\sin(t) = b.$$

Because of the correspondence between radian angle measure and distance traversed on the unit circle, we can also think of $\sin(t)$ as identifying the y -coordinate of the point after it has traveled t units counterclockwise along the circle from $(1, 0)$. Note particularly that we can consider the sine of negative inputs: for instance, $\sin(-\frac{\pi}{2}) = -1$.

Based on our earlier work with the unit circle, we know many different exact values of the sine function, and summarize these in the table below:

t	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
h	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
t	π	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{7\pi}{4}$	$\frac{11\pi}{6}$	2π
h	0	$-\frac{1}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{3}}{2}$	-1	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{1}{2}$	0

Moreover, if we now plot these points in the usual way, we get the familiar circular wave function that comes from tracking the height of a point traversing a circle. We often call this graph the **sine wave**.



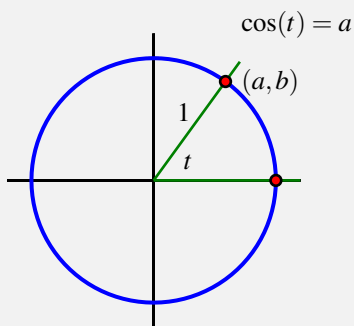
At <https://www.desmos.com/calculator/f9foqx24ct> you can explore and investigate a helpful Desmos animation that shows how this motion around the circle generates the sine graph.

Desmos link: <https://www.desmos.com/calculator/f9foqx24ct>

The Definition of the Cosine Function

Given any central angle of radian measure t in the unit circle with one side passing through the point $(1, 0)$, the angle generates a unique point (a, b) that lies on the circle. Just as we can view the y -coordinate as a function of t , the x -coordinate is likewise a function of t . We therefore make the following definition.

Definition

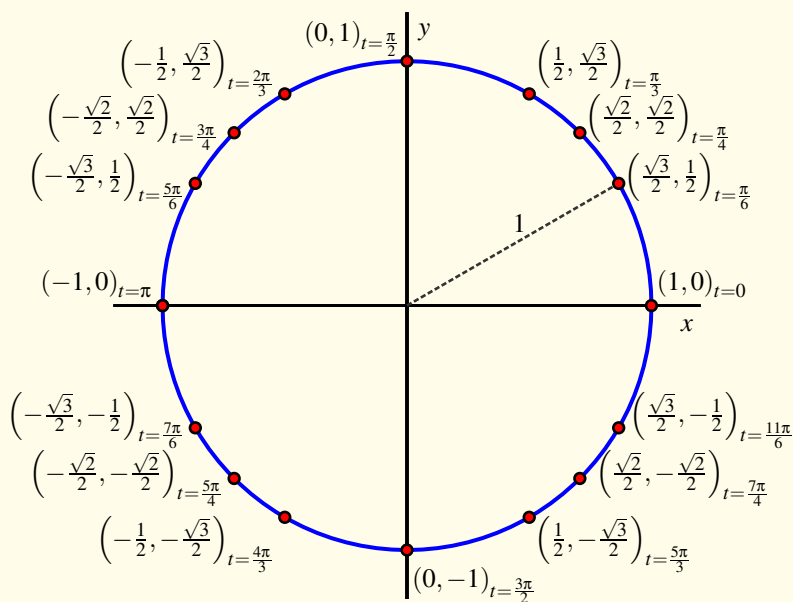


Given a central angle in the unit circle that measures t radians and that intersects the circle at both $(1, 0)$ and (a, b) , we define the **cosine of t** , denoted $\cos(t)$, by the rule

$$\cos(t) = a.$$

Again because of the correspondence between the radian measure of an angle and arc length along the unit circle, we can view the value of $\cos(t)$ as tracking the x -coordinate of a point traversing the unit circle clockwise a distance of t units along the circle from $(1, 0)$. We now use the data and information we have developed about the unit circle to build a table of values of $\cos(t)$ as well as a graph of the curve it generates.

Exploration Let $k = g(t)$ be the function that tracks the x -coordinate of a point traversing the unit circle counterclockwise from $(1, 0)$. That is, $g(t) = \cos(t)$. Use the information we know about the unit circle to respond to the following questions.



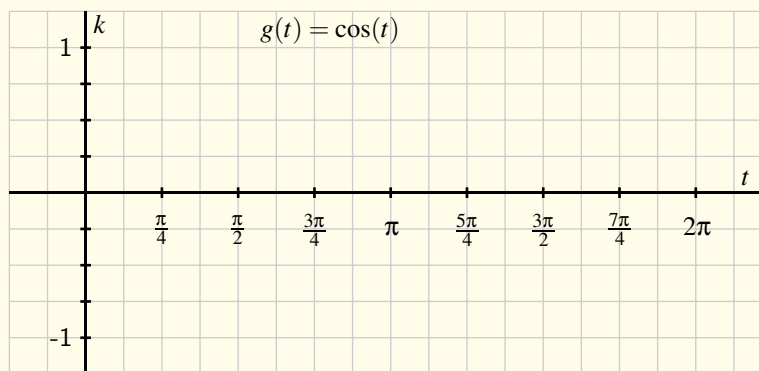
- What is the exact value of $\cos(\frac{\pi}{6})$? of $\cos(\frac{5\pi}{6})$? $\cos(-\frac{\pi}{3})$?
- Complete the following table with the exact values of k that cor-

respond to the stated inputs.

t	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
h									

t	π	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{7\pi}{4}$	$\frac{11\pi}{6}$	2π
h									

- c. On the axes provided, sketch an accurate graph of $k = \cos(t)$. Label the exact location of several key points on the curve.



- d. What is the exact value of $\cos(\frac{11\pi}{4})$? of $\cos(\frac{14\pi}{3})$?
- e. Give four different values of t for which $\cos(t) = -\frac{\sqrt{3}}{2}$.
- f. How is the graph of $k = \cos(t)$ different from the graph of $h = \sin(t)$? How are the graphs similar?

As we work with the sine and cosine functions, it's always helpful to remember their definitions in terms of the unit circle and the motion of a point traversing the circle. At <https://www.desmos.com/calculator/9s1ms0onlyf> you can explore and investigate a helpful Desmos animation that shows how this motion around the circle generates the cosine graph.

Desmos link: <https://www.desmos.com/calculator/9s1ms0onlyf>

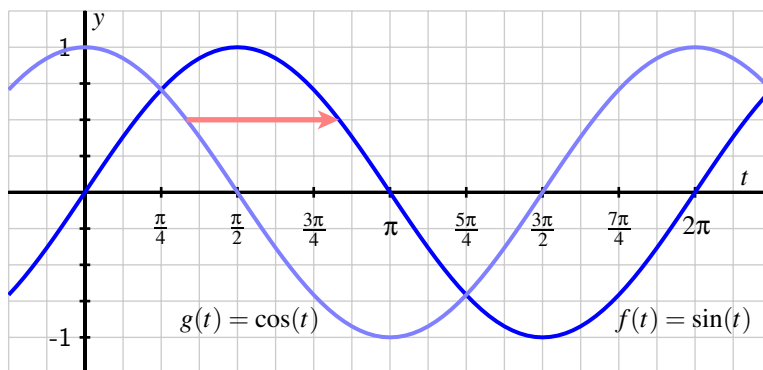
Properties of Sine and Cosine

Because the sine function results from tracking the y -coordinate of a point traversing the unit circle and the cosine function from the x -coordinate, the two functions have several shared properties of circular functions.

For both $f(t) = \sin(t)$ and $g(t) = \cos(t)$,

- the domain of the function is all real numbers;
- the range of the function is $[-1, 1]$;
- the midline of the function is $y = 0$;
- the amplitude of the function is $a = 1$;
- the period of the function is $p = 2\pi$.

It is also insightful to juxtapose the sine and cosine functions' graphs on the same coordinate axes. When we do, as seen in Figure ??, we see that the curves can be viewed as horizontal translations of one another.



In particular, since the sine graph can be viewed as the cosine graph shifted $\frac{\pi}{2}$ units to the right, it follows that for any value of t ,

$$\sin(t) = \cos\left(t - \frac{\pi}{2}\right).$$

Similarly, since the cosine graph can be viewed as the sine graph shifted left,

$$\cos(t) = \sin\left(t + \frac{\pi}{2}\right).$$

Because each of the two preceding equations hold for every value of t , they are often referred to as *identities*.

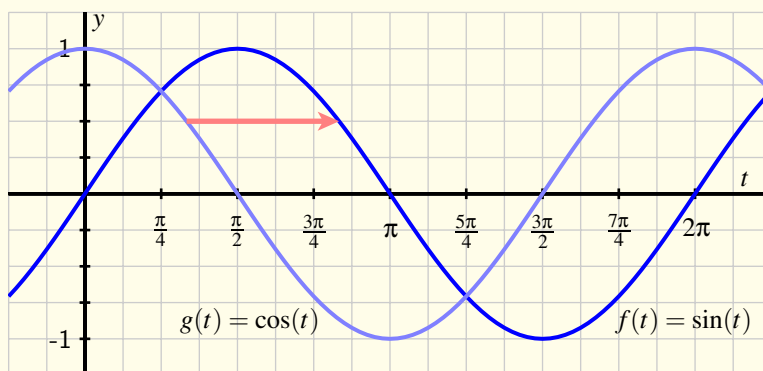
In light of the definitions of the sine and cosine functions, we can now view any point (x, y) on the unit circle as being of the form $(\cos(t), \sin(t))$, where t is the measure of the angle whose vertices are $(1, 0)$, $(0, 0)$, and (x, y) . Note particularly that since $x^2 + y^2 = 1$, it is also true that $\cos^2(t) + \sin^2(t) = 1$. We call this fact the Fundamental Trigonometric Identity.

For any real number t ,

$$\cos^2(t) + \sin^2(t) = 1.$$

There are additional trends and patterns in the two functions' graphs that we explore further in the following activity.

Exploration Use the figure below to assist in answering the following questions.



- Give an example of the largest interval you can find on which $f(t) = \sin(t)$ is decreasing.
- Give an example of the largest interval you can find on which $f(t) = \sin(t)$ is decreasing and concave down.
- Give an example of the largest interval you can find on which $g(t) = \cos(t)$ is increasing.
- Give an example of the largest interval you can find on which $g(t) = \sin(t)$ is increasing and concave up.
- Without doing any computation, on which interval is the average rate of change of $g(t) = \cos(t)$ greater: $[\pi, \pi + 0.1]$ or $\left[\frac{3\pi}{2}, \frac{3\pi}{2} + 0.1\right]$? Why?

- f. In general, how would you characterize the locations on the sine and cosine graphs where the functions are increasing or decreasing most rapidly?
- g. For which quadrants of the x - y plane is $\cos(t)$ negative for an angle in that quadrant?

Using Computing Technology

We have established that we know the exact value of $\sin(t)$ and $\cos(t)$ for any of the t -values labeled on the unit circle, as well as for any such $t \pm 2j\pi$, where j is a whole number, due to the periodicity of the functions. But what if we want to know $\sin(1.35)$ or $\cos(\frac{\pi}{5})$ or values for other inputs not in the table?

Any standard computing device a scientific calculator, *Desmos*, *Geogebra*, *WolframAlpha*, etc. has the ability to evaluate the sine and cosine functions at any input we desire. Because the input is viewed as an angle, each computing device has the option to consider the angle in radians or degrees. *It is always essential that you are sure which type of input your device is expecting.* Our computational device of choice is *Desmos*. In *Desmos*, you can change the input type between radians and degrees by clicking the wrench icon in the upper right and choosing the desired units. Radians is the default, and radians is what we will primarily use in both this class and calculus.

It takes substantial and sophisticated mathematics to enable a computational device to evaluate the sine and cosine functions at any value we want; the algorithms involve an idea from calculus known as an infinite series. While your computational device is powerful, it's both helpful and important to understand the meaning of these values on the unit circle and to remember the special points for which we know the outputs of the sine and cosine functions exactly.

Exploration

Answer the following questions exactly wherever possible. If you estimate a value, do so to at least 5 decimal places of accuracy.

- a. The x coordinate of the point on the unit circle that lies in the third quadrant and whose y -coordinate is $y = -\frac{3}{4}$.
- b. The y -coordinate of the point on the unit circle generated by a central angle in standard position that measures $t = 2$ radians.
- c. The x -coordinate of the point on the unit circle generated by a central angle in standard position that measures $t = -3.05$ radians.
- d. The value of $\cos(t)$ where t is an angle in Quadrant II that satisfies

$$\sin(t) = \frac{1}{2}.$$

- e. The value of $\sin(t)$ where t is an angle in Quadrant III for which $\cos(t) = -0.7$.
- f. The average rate of change of $f(t) = \sin(t)$ on the intervals $[0.1, 0.2]$ and $[0.8, 0.9]$.
- g. The average rate of change of $g(t) = \cos(t)$ on the intervals $[0.1, 0.2]$ and $[0.8, 0.9]$.

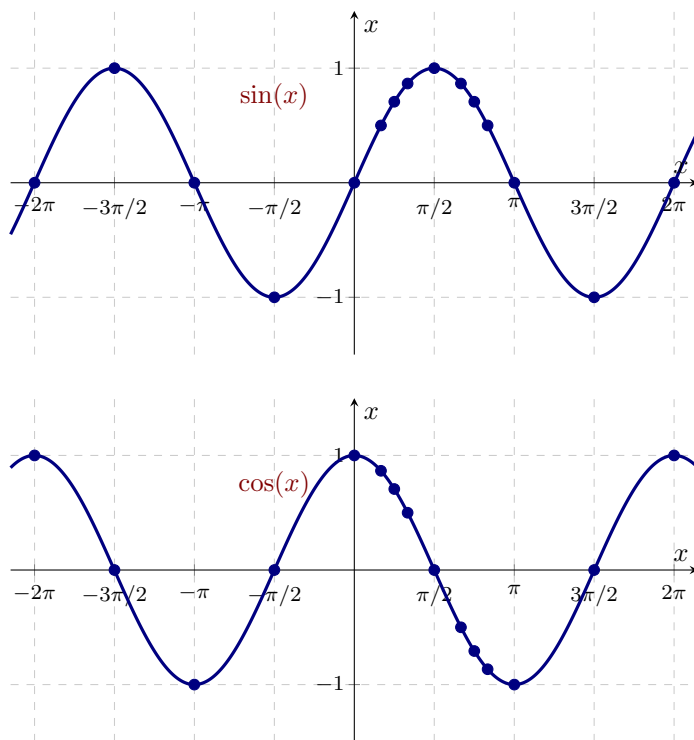
Summary

- The sine and cosine functions result from tracking the y - and x -coordinates of a point traversing the unit circle counterclockwise from $(1, 0)$. The value of $\sin(t)$ is the y -coordinate of a point that has traversed t units along the circle from $(1, 0)$ (or equivalently that corresponds to an angle of t radians), while the value of $\cos(t)$ is the x -coordinate of the same point.
- The sine and cosine functions are both periodic functions that share the same domain (the set of all real numbers), range (the interval $[-1, 1]$), midline ($y = 0$), amplitude ($a = 1$), and period ($P = 2\pi$). In addition, the sine function is horizontal shift of the cosine function by $\frac{\pi}{2}$ units to the right, so $\sin(t) = \cos(t - \frac{\pi}{2})$ for any value of t .
- If t corresponds to one of the special angles that we know on the unit circle, we can compute the values of $\sin(t)$ and $\cos(t)$ exactly. For other values of t , we can use a computational device to estimate the value of either function at a given input; when we do so, we must take care to know whether we are computing in terms of radians or degrees.

9.1.3 Creating a New Function: Tangent

Introduction

We are now going to determine the graph of the tangent function by analyzing what we now know about the sine and cosine functions. As a reminder, here is a graph of those functions with some important points marked. Specifically the points at all multiples of $\frac{\pi}{2}$ have been marked, as well as at the standard points $x = \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{2\pi}{3}, \frac{3\pi}{4},$ and $\frac{5\pi}{6}$.

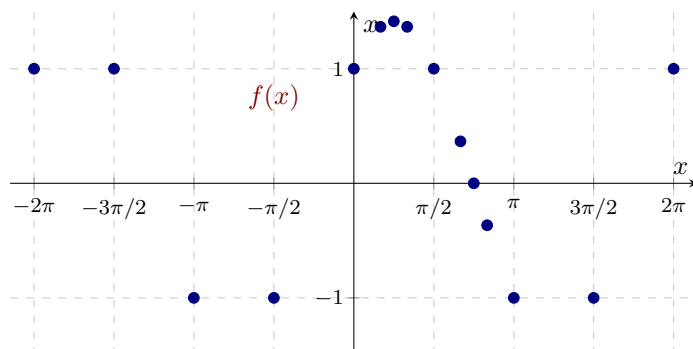


Graph of $f(x) = \sin(x) + \cos(x)$

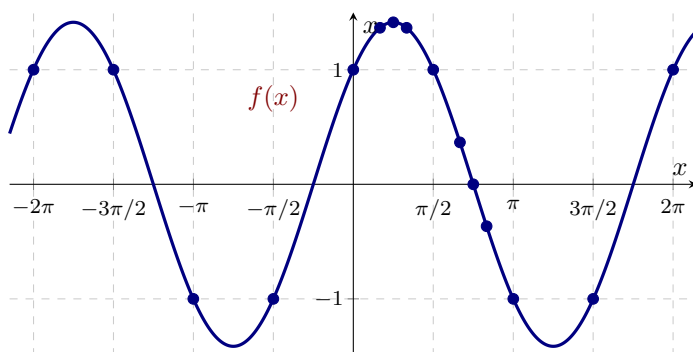
Before considering the function $\tan(x) = \frac{\sin(x)}{\cos(x)}$, let's consider the (possibly) more straightforward $f(x) = \sin(x) + \cos(x)$. Let's practice with this function creating a graph of a sum of two known functions.

What is the value of $f(0)$? We know $f(0) = \sin(0) + \cos(0) = 0 + 1 = 1$.

We can easily calculate the values of f at all of the important points marked in the graphs above. Let us plot the points of f corresponding to them.



Those extra points plotted between $x = 0$ and $x = \pi$ show us the behavior of this function f . Notice that between $x = 0$ and $x = \frac{\pi}{2}$, the graph increases to a peak, then decreases in a very sinusoidal manner. If we continue this with standard values and “connect the dots”, we end up with the following graph.



We’ve ended up with another periodic function that looks like a stretched and shifted version of sine or cosine.

Now, let’s try again using division instead of addition.

Determining the Graph of Tangent

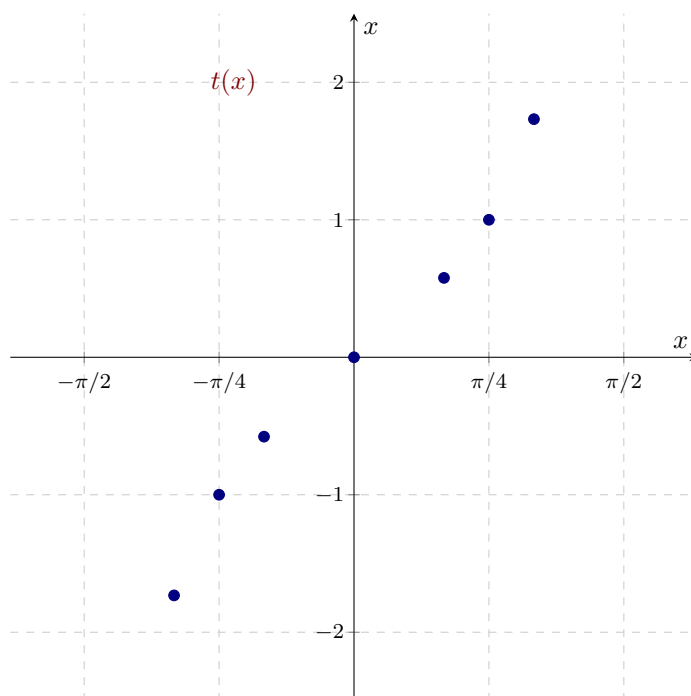
Recall that $\tan(x) = \frac{\sin(x)}{\cos(x)}$.

Notice that this function is undefined at all x -values with $\cos(x) = 0$. That means the function $\tan(x)$ is not defined for $x = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \dots$

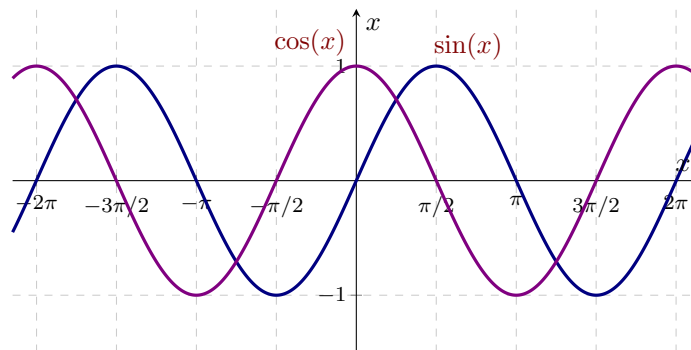
We can calculate values of $\tan(x)$ for other inputs. $\tan(0) = \frac{\sin(0)}{\cos(0)} = \frac{0}{1} = 0$. The following table lists some of the other values arising from this division. Notice that in this table we have chosen to rationalize the denominators of the fractions that have appeared. That is, we have written $\frac{1}{\sqrt{2}}$ as $\frac{\sqrt{2}}{2}$, by multiplying the fraction by 1 written in the form $\frac{\sqrt{2}}{\sqrt{2}}$. Similarly $\frac{1}{\sqrt{3}}$ is written as $\frac{\sqrt{3}}{3}$.

x	$\sin(x)$	$\cos(x)$	$t(x) = \frac{\sin(x)}{\cos(x)}$
$-\frac{\pi}{3}$	$-\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$-\sqrt{3}$
$-\frac{\pi}{4}$	$-\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	-1
$-\frac{\pi}{6}$	$-\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{3}$
0	0	1	0
$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1
$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$

If we plot these points, we find the following graph.



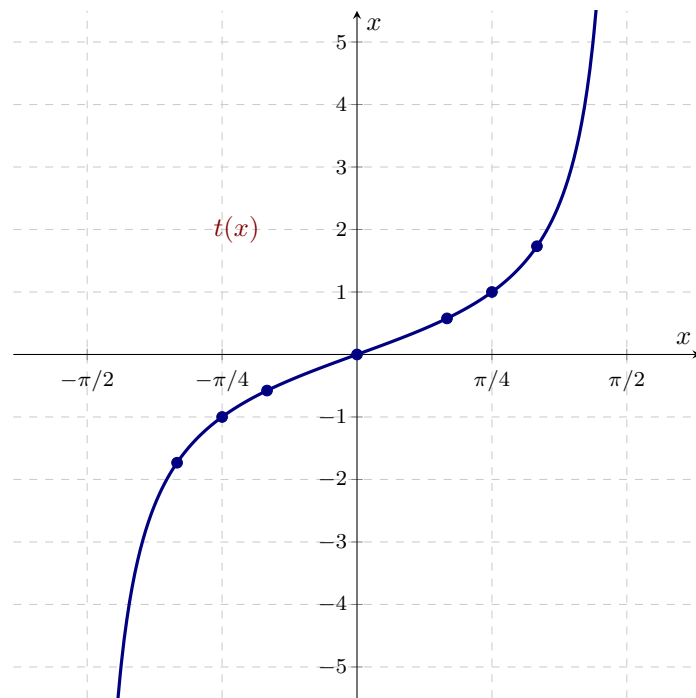
Let's think about what happens if x is a number really close to $\frac{\pi}{2}$, but just a little bit smaller than $\frac{\pi}{2}$. Notice from the graphs that the value of $\sin(x)$ will be a positive number that is really close to 1 and the value of $\cos(x)$ will be really close to 0 but still positive.



What happens if we take 1 and divide it by a small positive number? Let's look at a table of values to see.

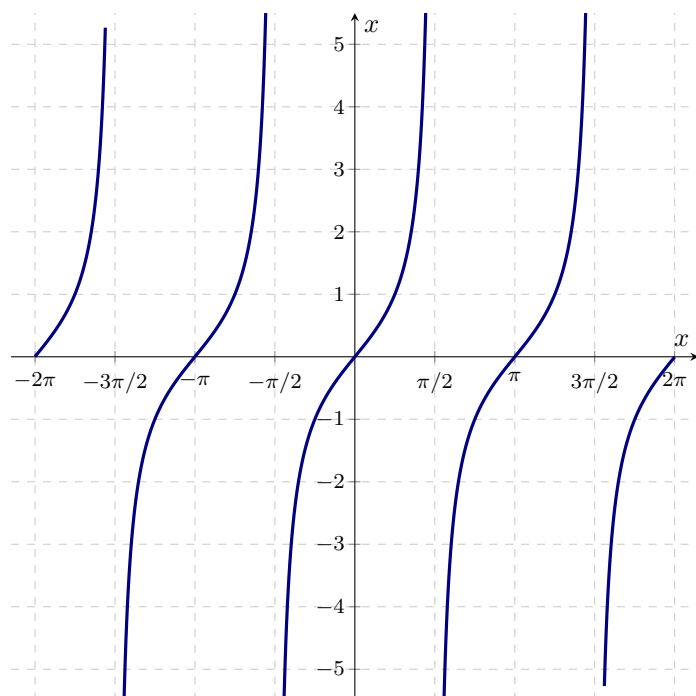
z	$\frac{1}{z}$
1	1
$\frac{1}{2}$	2
$\frac{1}{3}$	3
$\frac{1}{10}$	10
$\frac{1}{100}$	100
$\frac{1}{1000}$	1000

Notice that as the numbers $z = 1, 1/2, 1/3, \dots$ got smaller and smaller, the values of $\frac{1}{z} = 1, 2, 3, \dots$ got larger and larger? That is the same thing we are noticing in the graph of the function t we are building above. For values of x really close to $\pi/2$, but still less than $\pi/2$, the value of $t(x)$ is basically 1 divided by a very small positive number. This table of values tells us that the smaller that denominator gets, the larger the fraction becomes. Adding this behavior to the graph of t gives the following.



By repeating similar calculations for other standard inputs, we arrive at the following graph.

Creating a New Function: Tangent



As you can see from the graph, $\tan(x)$ is an odd, periodic function with period π (not 2π like sine and cosine).

9.1.4 Graphs of Secant, Cosecant, and Cotangent

Motivating Questions

- What do the graphs of secant, cosecant, and cotangent look like?
- What are some important properties of these graphs? Where do they have asymptotes?

Like the tangent function, the secant, cosecant, and cotangent functions are defined in terms of the sine and cosine functions, so we can determine the exact values of these functions at each of the special points on the unit circle. In addition, we can use our understanding of the unit circle and the properties of the sine and cosine functions to determine key properties of these other trigonometric functions.

The Secant Function

We begin by investigating the secant function. Using the fact that $\sec(t) = \frac{1}{\cos(t)}$, we note that anywhere $\cos(t) = 0$, the value of $\sec(t)$ is undefined. We record such instances in the following table by writing “ u ”. At all other points, the value of the secant function is simply the reciprocal of the cosine function’s value. Since $|\cos(t)| \leq 1$ for all t , it follows that $|\sec(t)| \geq 1$ for all t (for which the secant’s value is defined).

Values of Secant in Quadrant I

t	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$\cos(t)$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
$\sec(t)$	1	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2	u

Values of Secant in Quadrant II

t	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
$\cos(t)$	0	$-\frac{1}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{3}}{2}$	-1
$\sec(t)$	u	-2	$-\sqrt{2}$	$-\frac{2}{\sqrt{3}}$	-1

Values of Secant in Quadrant III

t	π	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$
$\cos(t)$	-1	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{1}{2}$	0
$\sec(t)$	-1	$-\frac{2}{\sqrt{3}}$	$-\sqrt{2}$	-2	u

Values of Secant in Quadrant IV

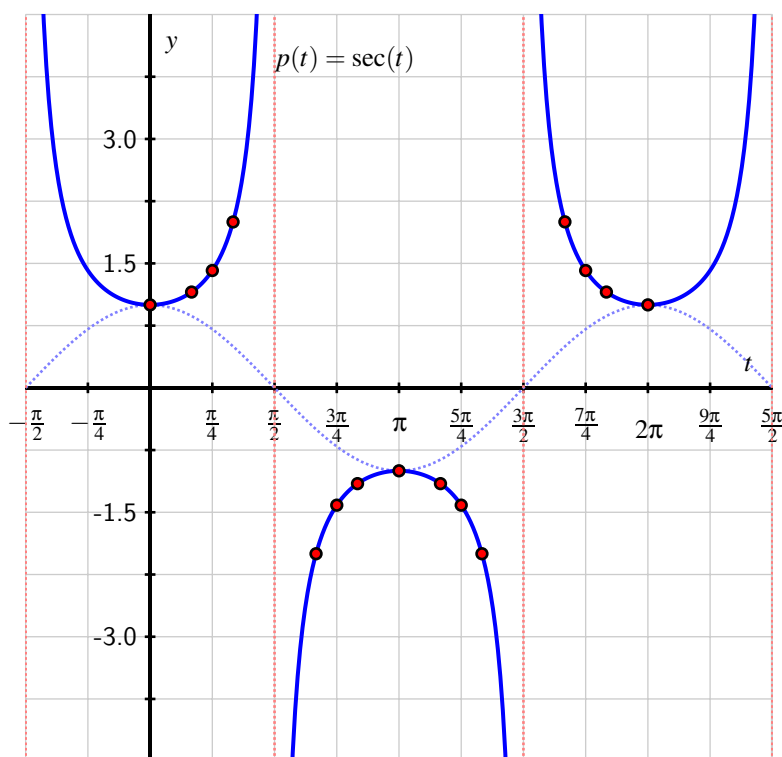
t	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{7\pi}{4}$	$\frac{11\pi}{6}$	2π
$\cos(t)$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
$\sec(t)$	u	2	$\sqrt{2}$	$\frac{2}{\sqrt{3}}$	1

These tables help us identify trends in the secant function. The sign of $\sec(t)$ matches the sign of $\cos(t)$ and thus is positive in Quadrant I, negative in Quadrant II, negative in Quadrant III, and positive in Quadrant IV.

In addition, we observe that as t -values in the first quadrant get closer to $\frac{\pi}{2}$, $\cos(t)$ gets closer to 0 (while being always positive). Since the numerator of the secant function is always 1, having its denominator approach 0 means that $\sec(t)$ increases without bound as t approaches $\frac{\pi}{2}$ from the left side. Once t

is slightly greater than $\frac{\pi}{2}$ in Quadrant II, the value of $\cos(t)$ is negative (and close to zero). This makes the value of $\sec(t)$ decrease without bound (negative and getting further away from 0) for t approaching $\frac{\pi}{2}$ from the right side, and results in $p(t) = \sec(t)$ having a vertical asymptote at $t = \frac{\pi}{2}$. The periodicity and sign behavior of $\cos(t)$ mean this asymptotic behavior of the secant function will repeat.

Plotting the data in the table along with the expected asymptotes and connecting the points intuitively, we see the graph of the secant function.



We see from both the table and the graph that the secant function has period $P = 2\pi$. We summarize our recent work as follows.

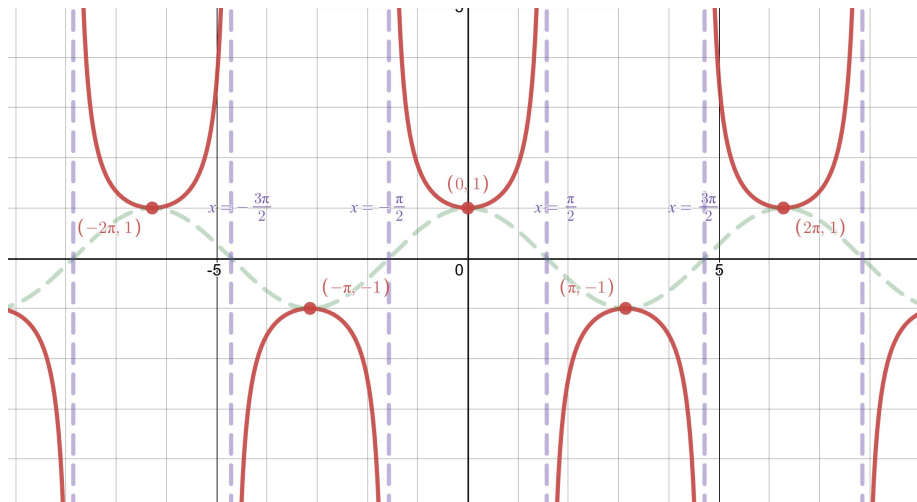
Properties of the secant function.

For the function $p(t) = \sec(t)$,

- its domain is the set of all real numbers except $t = \frac{\pi}{2} \pm k\pi$ where k is any whole number;

- its range is the set of all real numbers y such that $|y| \geq 1$;
- its period is $P = 2\pi$.

We can see the secant function in Desmos as well.



Try playing with the secant graph yourself.

Desmos link: <https://www.desmos.com/calculator/gmsanlrjza>

The Cosecant Function

Graphing the cosecant function is extremely similar to graphing the secant function, except we use $\csc(t) = \frac{1}{\sin(t)}$ so we are flipping over the values of sine instead of the values of cosine. Since the sine and cosine graphs look very similar except they are shifted by $\frac{\pi}{2}$, the secant and cosecant graphs will also look very similar but be shifted from one another in the exact same way.

Let's create the table of famous values for cosecant.

Values of Cosecant in Quadrant I

t	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$\sin(t)$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
$\csc(t)$	u	2	$\sqrt{2}$	$\frac{2}{\sqrt{3}}$	1

Values of Cosecant in Quadrant II

t	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
$\sin(t)$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
$\csc(t)$	1	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2	u

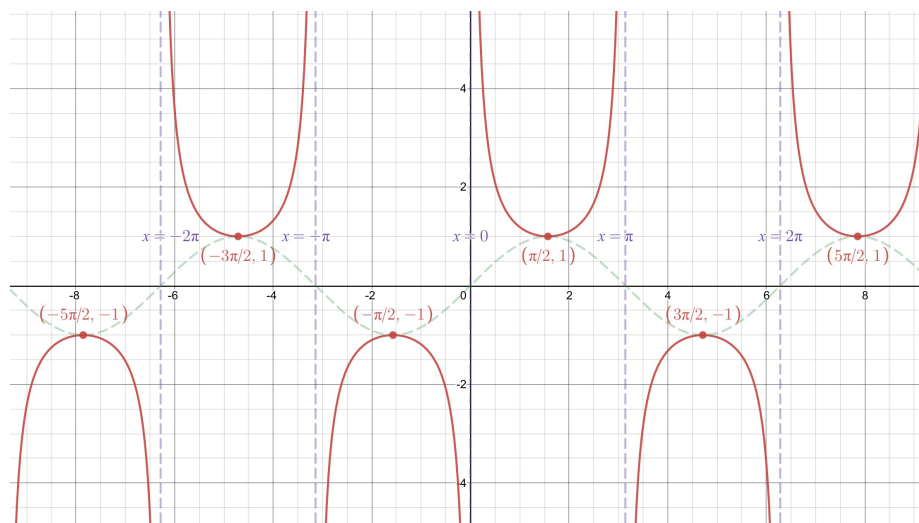
Values of Cosecant in Quadrant III

t	π	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$
$\sin(t)$	0	$-\frac{1}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{3}}{2}$	-1
$\csc(t)$	u	-2	$-\sqrt{2}$	$-\frac{2}{\sqrt{3}}$	-1

Values of Cosecant in Quadrant IV

t	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{7\pi}{4}$	$\frac{11\pi}{6}$	2π
$\sin(t)$	-1	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{1}{2}$	0
$\csc(t)$	-1	$-\frac{2}{\sqrt{3}}$	$-\sqrt{2}$	-2	u

Plotting the data in the table along with the expected asymptotes and connecting the points intuitively, we see the graph of the cosecant function.



We see from both the table and the graph that the cosecant function has period $P = 2\pi$. We summarize our recent work as follows.

Properties of the cosecant function.

For the function $p(t) = \csc(t)$,

- its domain is the set of all real numbers except $t = \pm k\pi$ where k is any whole number;
- its range is the set of all real numbers y such that $|y| \geq 1$;
- its period is $P = 2\pi$.

Try playing with the cosecant graph yourself.

Desmos link: <https://www.desmos.com/calculator/norjxi7z4r>

The Cotangent Function

Graphing the cotangent function is similar to graphing the secant and cosecant functions, except we use $\cot(t) = \frac{1}{\tan(t)}$ so we are flipping over the values of tangent. Since the tangent graph has a period of π , the graph of cotangent will also have a period of π . Therefore, we only need to calculate tables of values for the first two quadrants.

Let's create the table of famous values for cotangent.

Values of Cotangent in Quadrant I

t	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$\tan(t)$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	u
$\cot(t)$	u	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$	0

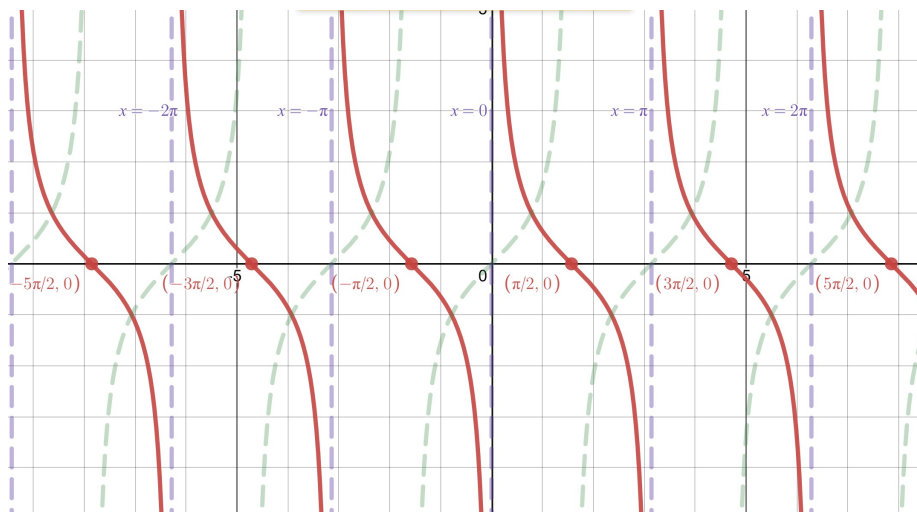
Notice that something unusual happened here. Even though tangent is undefined at $t = \frac{\pi}{2}$, we have that $\cot(\frac{\pi}{2}) = 0$. This is because $\cot(t) = \frac{\cos(t)}{\sin(t)}$ and so we can define $\cot(\frac{\pi}{2}) = \frac{\cos(\frac{\pi}{2})}{\sin(\frac{\pi}{2})} = \frac{0}{1} = 0$.

Values of Cotangent in Quadrant II

t	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
$\tan(t)$	u	$-\sqrt{3}$	-1	$-\frac{1}{\sqrt{3}}$	0
$\cot(t)$	0	$-\frac{1}{\sqrt{3}}$	-1	$-\sqrt{3}$	u

Plotting the data in the table along with the expected asymptotes and connecting the points intuitively, we see the graph of the cotangent function.

Graphs of Secant, Cosecant, and Cotangent



We summarize our recent work as follows.

Properties of the cotangent function.

For the function $p(t) = \csc(t)$,

- its domain is the set of all real numbers except $t = \pm k \frac{\pi}{2}$ where k is any whole number;
- its range is the set of all real numbers;
- its period is $P = \pi$.

Try playing with the cotangent graph yourself.

Desmos link: <https://www.desmos.com/calculator/yhmzxaq7ep>

9.2 Trig Functions

Learning Objectives

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9.2.1 Trig Functions as Functions

Motivating Questions

- How do trigonometric functions interact with other functions?
- How do we find zeros of trigonometric functions?
- How does average rate of change look like with trigonometric functions?

Trig Function Compositions

Trigonometric functions can be composed with any of the types of functions that we have already seen. Just as with other function compositions, we need to be mindful of the domains and ranges of our functions.

Example 1. Let's consider the following functions: $f(x) = \sin(x)$ and $g(x) = 3x^2$.

Find the function below and state its domain and range

- (a) $f(g(x))$
- (b) $g(f(x))$
- (c) $f(f(x))$

Explanation First let's find the domain and range of $f(x)$ and $g(x)$. The domain for both $f(x)$ and $g(x)$ is $(-\infty, \infty)$. The range for $f(x)$ is $[-1, 1]$ and the range for $g(x)$ is $[0, \infty)$. Now we can look at the compositions.

- (a) $f(g(x)) = \sin(3x^2)$
The domain is $(-\infty, \infty)$ and the range is $[-1, 1]$.
- (b) $g(f(x)) = 3\sin(x)^2$
The domain is $(-\infty, \infty)$. The range is $[0, 3]$
- (c) $f(f(x)) = \sin(\sin(x))$
The domain is $(-\infty, \infty)$. The range is $[-\sin(1), \sin(1)]$.

Finding Zeros of Trigonometric Functions.

Example 2. $f(x) = \sin^2(x) - 1$

Find the zeros of f for $0 \leq x \leq 2\pi$

Explanation First we need to set f to equal 0

$$\sin^2(x) - 1 = 0$$

Now we can recognize that we have a difference of squares, so we have the following:

$$\sin^2(x) - 1 = (\sin(x) + 1)(\sin(x) - 1)$$

Now we can set each part equal to zero. So we have

$$\sin(x) + 1 = 0 \text{ and } \sin(x) - 1 = 0$$

After simplifying them a bit we have

$$\sin(x) = -1 \text{ and } \sin(x) = 1$$

Because these are famous values of $\sin(x)$, we can find values for x without using inverse trigonometry.

$$x = -\frac{\pi}{2}, \frac{\pi}{2}$$

Example 3. $f(x) = \sin^2(x) + \frac{5}{2}\sin(x) - \frac{3}{2}$

Find the zeros of f for $0 \leq x \leq 2\pi$

Explanation First we need to set f to equal 0

$$\sin^2(x) + \frac{5}{2}\sin(x) - \frac{3}{2} = 0$$

Thankfully f can factor nicely, so we have

$$\sin^2(x) + \frac{5}{2}\sin(x) - \frac{3}{2} = (\sin(x) - \frac{1}{2})(\sin(x) + 3)$$

Now we set each part equal to zero.

$$\sin(x) - \frac{1}{2} = 0 \text{ and } \sin(x) + 3 = 0$$

After simplifying them a bit we have

$$\sin(x) = \frac{1}{2} \text{ and } \sin(x) = -3$$

-3 is outside of the range of $\sin(x)$ so we can disregard that term. Our only concern is the famous value where $\sin(x)$ is $\frac{1}{2}$. This happens to occur in two places for $0 \leq x \leq 2\pi$, so our answer is $x = \frac{\pi}{6}, \frac{5\pi}{6}$

Average Rate of Change with Trigonometry

We can still find average rate of change with trigonometric functions, but because they are periodic there can be some interesting results.

Example 4. Let $f(x) = \sin(x)$

$$(a) \text{ AROC}_{[\frac{\pi}{6}, \frac{3\pi}{4}]}$$

$$(b) \text{ AROC}_{[\frac{\pi}{3}, \frac{2\pi}{3}]}$$

Explanation

$$(a) \text{ AROC}_{[\frac{\pi}{6}, \frac{3\pi}{4}]} = \frac{\sin(\frac{3\pi}{4}) - \sin(\frac{\pi}{6})}{\frac{3\pi}{4} - \frac{\pi}{6}}$$

First we substitute our trig values

$$\frac{\sin(\frac{3\pi}{4}) - \sin(\frac{\pi}{6})}{\frac{3\pi}{4} - \frac{\pi}{6}} = \frac{\frac{\sqrt{2}}{2} - \frac{1}{2}}{\frac{3\pi}{4} - \frac{\pi}{6}}$$

Now we simplify our fractions

$$\frac{\frac{\sqrt{2}}{2} - \frac{1}{2}}{\frac{3\pi}{4} - \frac{\pi}{6}} = \frac{\frac{\sqrt{2}-1}{2}}{\frac{9\pi}{12} - \frac{2\pi}{12}} = \frac{\frac{\sqrt{2}-1}{2}}{\frac{7\pi}{12}}$$

$$\frac{\frac{\sqrt{2}-1}{2}}{\frac{7\pi}{12}} = \frac{6(\sqrt{2}-1)}{7\pi}$$

$$\text{AROC}_{[\frac{\pi}{6}, \frac{3\pi}{4}]} = \frac{6\sqrt{2}-6}{7\pi}$$

$$(b) \text{ AROC}_{[\frac{\pi}{3}, \frac{2\pi}{3}]} = \frac{\sin(\frac{2\pi}{3}) - \sin(\frac{\pi}{3})}{\frac{2\pi}{3} - \frac{\pi}{3}}$$

First we substitute our trig values

$$\frac{\sin(\frac{2\pi}{3}) - \sin(\frac{\pi}{3})}{\frac{2\pi}{3} - \frac{\pi}{3}} = \frac{\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2}}{\frac{2\pi}{3} - \frac{\pi}{3}}$$

Now we simplify

$$\frac{\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2}}{\frac{2\pi}{3} - \frac{\pi}{3}} = \frac{0}{\frac{\pi}{3}}$$

We can see that we have 0 in the numerator, which means that our average rate of change will be 0. This is because we are working with a periodic function and picked values in similar positions. Even though we had positive and negative rates of change at some point between $\frac{\pi}{3}$ and $\frac{2\pi}{3}$, we ended up with the same y value, so *on average* there was no change.

9.2.2 Trig Functions Transformations

Motivating Questions

- How do the three standard transformations (vertical translation, horizontal translation, and vertical scaling) affect the midline, amplitude, range, and period of sine and cosine curves?
- What algebraic transformation results in horizontal stretching or scaling of a function?
- How can we determine a formula involving sine or cosine that models any circular periodic function for which the midline, amplitude, period, and an anchor point are known?

Recall our previous work with transformations, where we studied how the graph of the function g defined by $g(x) = af(x - b) + c$ is related to the graph of f , where a , b , and c are real numbers with $a \neq 0$. Because such transformations can shift and stretch a function, we are interested in understanding how we can use transformations of the sine and cosine functions to fit formulas to circular functions.

Example 5. Let $f(t) = \cos(t)$. First, answer all of the questions below without using Desmos; then use Desmos to confirm your conjectures. For each prompt, describe the graphs of g and h as transformations of f and, in addition, state the amplitude, midline, and period of both g and h .

- $g(t) = 3 \cos(t)$ and $h(t) = -\frac{1}{4} \cos(t)$
- $g(t) = \cos(t - \pi)$ and $h(t) = \cos\left(t + \frac{\pi}{2}\right)$
- $g(t) = \cos(t) + 4$ and $h(t) = \cos(t) - 2$
- $g(t) = 3 \cos(t - \pi) + 4$ and $h(t) = -\frac{1}{4} \cos\left(t + \frac{\pi}{2}\right) - 2$

Explanation

- $g(t)$ Has a vertical stretch by 3 and an amplitude of 3. $h(t)$ has a reflection across the x -axis with a vertical shrink of $\frac{1}{4}$ and an amplitude of $\frac{1}{4}$. $g(t)$ and $h(t)$ both have a midline at $y = 0$ and a period of 2π .
- $g(t)$ Has a horizontal shift to the right of π . $h(t)$ has a horizontal shift to the left of $\frac{\pi}{2}$. $g(t)$ and $h(t)$ both have a midline at $y = 0$, a period of 2π , and an amplitude of 1.

- c. $g(t)$ Has a vertical shift up of 4 and a midline at $y = 4$. $h(t)$ has a vertical shift down of 2 and a midline of $y = -2$. $g(t)$ and $h(t)$ both have a period of 2π and an amplitude of 1.
- d. $g(t)$ Has a vertical stretch by 3, a horizontal shift to the right of π , and a vertical shift up of 4. $g(t)$ has an amplitude of 3 and a midline at $y = 4$. $h(t)$ has a reflection across the x -axis with a vertical shrink of $\frac{1}{4}$, a horizontal shift to the left of $\frac{\pi}{2}$, and a vertical shift down of 2. $h(t)$ has an amplitude of $\frac{1}{4}$ and a midline of $y = -2$. $g(t)$ and $h(t)$ both have a period of 2π .

Shifts and vertical stretches of the sine and cosine functions

We know that the standard functions $f(t) = \sin(t)$ and $g(t) = \cos(t)$ are circular functions that each have midline $y = 0$, amplitude $a = 1$, period $p = 2\pi$, and range $[-1, 1]$. This suggests the following general principles.

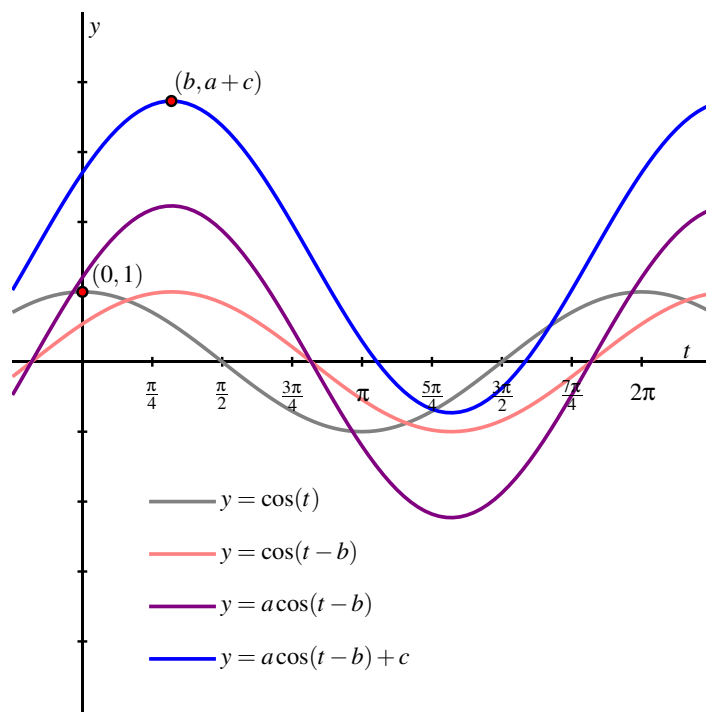
Transformations of sine and cosine with shifts and vertical stretches.

Given real numbers a , c , and d with $a \neq 0$, the functions

$$f(t) = a \cos(t - c) + d \text{ and } g(t) = a \sin(t - c) + d$$

each represent a horizontal shift by c units to the right, followed by a vertical stretch by a units, followed by a vertical shift of d units, applied to the parent function ($\cos(t)$ or $\sin(t)$, respectively). The resulting circular functions have midline $y = d$, amplitude a , range $[d - a, d + a]$, and period $p = 2\pi$. In addition, the point $(c, a + d)$ lies on the graph of f and the point (c, d) lies on the graph of g .

In the figure below, we see how the overall transformation $k(t) = a \cos(t - b) + c$ comes from executing a sequence of simpler ones. The original parent function $y = \cos(t)$ (in dark gray) is first shifted b units right to generate the light red graph of $y = \cos(t - b)$. In turn, that graph is then scaled vertically by a to generate the purple graph of $y = a \cos(t - b)$. Finally, the purple graph is shifted c units vertically to result in the final graph of $y = a \cos(t - b) + c$ in blue.



It is often useful to follow one particular point through a sequence of transformations. In the above figure, we see the red point that is located at $(0, 1)$ on the original function $y = \cos(t)$, as well as the point $(c, a + d)$ that is the corresponding point on $k(t) = a \cos(t - c) + d$ under the overall transformation. Note that the point $(b, a + c)$ results from the input, $t = c$, that makes the argument of the cosine function zero: $k(c) = a \cos(c - c) + d = a \cos(0) + d$.

While the sine and cosine functions extend infinitely in either direction, it's natural to think of the point $(0, 1)$ as the “starting point” of the cosine function, and similarly the point $(0, 0)$ as the starting point of the sine function. We will refer to the corresponding points $(c, a + d)$ and (c, d) on $k(t) = a \cos(t - c) + d$ and $h(t) = a \sin(t - c) + d$ as anchor points. Anchor points, along with other information about a circular function's amplitude, midline, and period help us to determine a formula for a function that fits a given situation.

Exploration Consider a spring-mass system where the weight resting on a frictionless table. We let $s(t)$ denote the distance from the wall (where the spring is attached) to the weight at time t in seconds and know that the weight oscillates periodically with a minimum value of $s(t) = 2$ feet and a maximum value of $s(t) = 7$ feet with a period of 2π . We also know that $s(0) = 4.5$ and $s\left(\frac{\pi}{2}\right) = 2$.

Determine a formula for $s(t)$ in the form $s(t) = a \cos(t - b) + c$ or $s(t) = a \sin(t - c) + d$. Is it possible to find two different formulas that work? For any formula you find, identify the anchor point.

Subsection 2.4.2 Horizontal scaling

There is one more very important transformation of a function that we've not yet explored. Given a function $y = f(x)$, we want to understand the related function $g(x) = f(bx)$, where b is a positive real number. The sine and cosine functions are ideal functions with which to explore these effects; moreover, this transformation is crucial for being able to use the sine and cosine functions to model phenomena that oscillate at different frequencies.

By using a graphing utility such as Desmos, we can explore the effect of the transformation $g(t) = f(bt)$, where $f(t) = \sin(t)$.

By experimenting with various values or a slider, we gain an intuitive sense for how the value of b affects the graph of $h(t) = f(bt)$ in comparison to the graph of $f(t)$. When $b = 2$, we see that the graph of h is oscillating twice as fast as the graph of f since $h(t) = f(2t)$ completes two full cycles over an interval in which f completes one full cycle. In contrast, when $b = \frac{1}{2}$, the graph of h oscillates half as fast as the graph of f , as $h(t) = f(\frac{1}{2}t)$ completes only half of one cycle over an interval where $f(t)$ completes a full one.

We can also understand this from the perspective of function composition. To evaluate $h(t) = f(2t)$, at a given value of t , we first multiply the input t by a factor of 2, and then evaluate the function f at the result. An important observation is that

$$h\left(\frac{1}{2}t\right) = f\left(2 \cdot \frac{1}{2}t\right) = f(t).$$

This tells us that the point $(\frac{1}{2}t, f(t))$ lies on the graph of h since an input of $\frac{1}{2}t$ in h results in the value $f(t)$. At the same time, the point $(t, f(t))$ lies on the graph of f . Thus we see that the correlation between points on the graphs of f and h (where $h(t) = f(2t)$) is

$$(t, f(t)) \rightarrow \left(\frac{1}{2}t, f(t)\right).$$

We can therefore think of the transformation $h(t) = f(2t)$ as achieving the output values of f twice as fast as the original function $f(t)$ does. Analogously, the transformation $h(t) = f(\frac{1}{2}t)$ will achieve the output values of f only half as quickly as the original function.

Given a function $y = f(t)$ and a real number $b > 0$, the transformed function $y = h(t) = f(bt)$ is a *horizontal stretch* of the graph of f . Every point $(t, f(t))$ on the graph of f gets stretched horizontally to the corresponding point $(\frac{1}{b}t, f(t))$ on the graph of h . If $0 < b < 1$, the graph of v is a stretch of f away from the y -axis by a factor of $\frac{1}{b}$; if $b > 1$, the graph of h is a compression of f toward the y -axis by a factor of $\frac{1}{b}$. The only point on the graph of f that is unchanged by the transformation is $(0, f(0))$.

Transformations of sine and cosine with any shift or stretch.

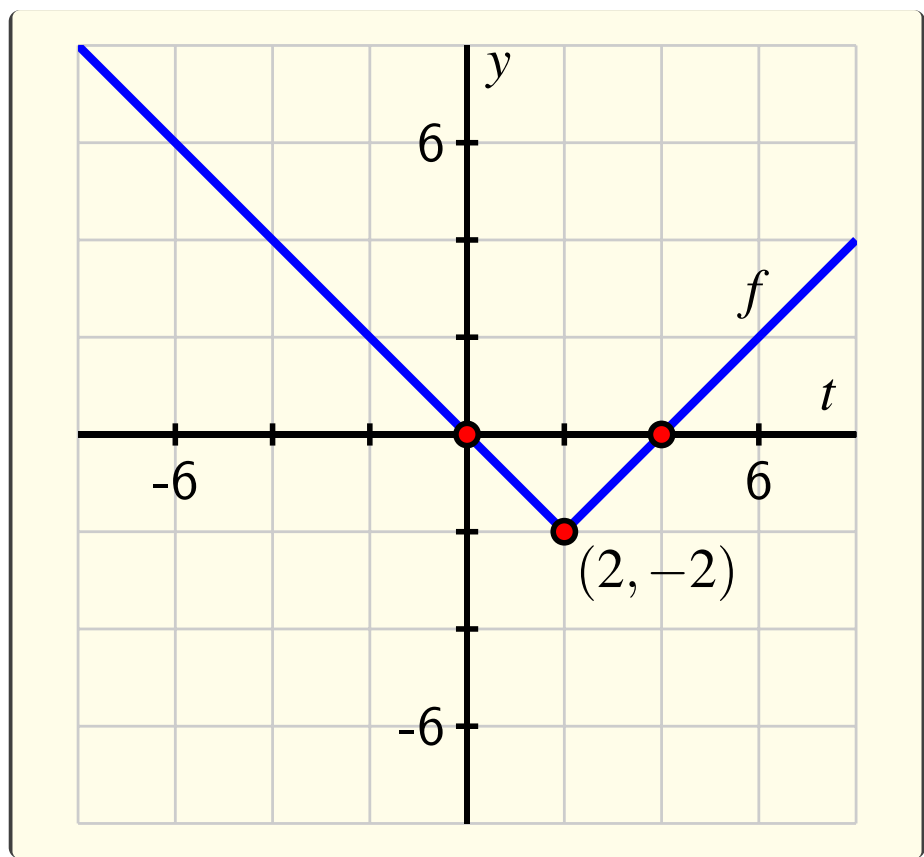
Given real numbers a , b , c , and d with $a \neq 0$ and $b > 0$, the functions

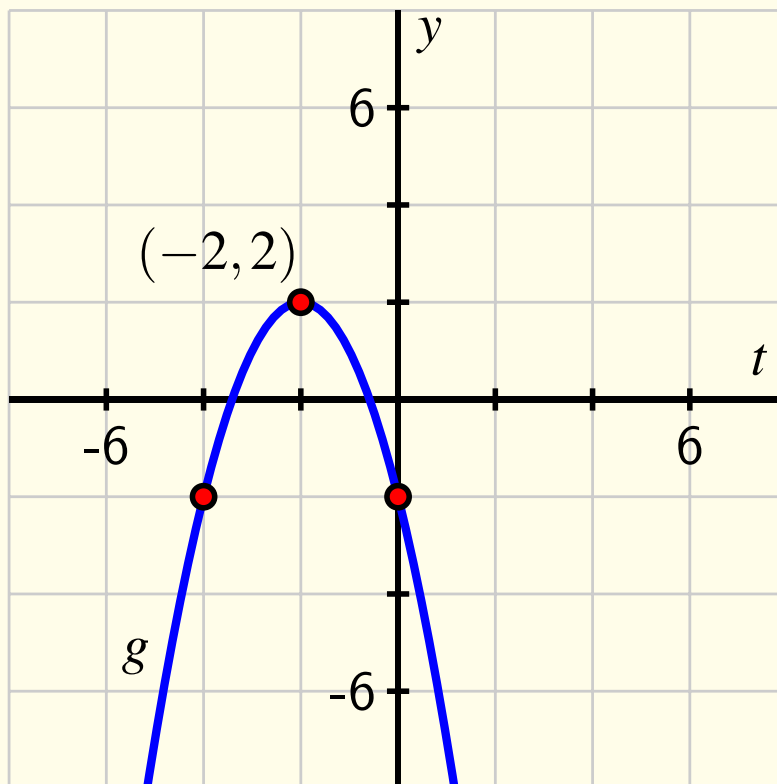
$$f(t) = a \cos(b(t - c)) + d \text{ or } g(t) = a \sin(b(t - c)) + d.$$

each represent a horizontal shift by c units to the right, followed by a vertical stretch by a units, followed by a vertical shift of d units, applied to the parent function $(\cos(t)$ or $\sin(t)$, respectively). They also contain a horizontal scaling of f by a factor of $\frac{1}{b}$. The resulting circular functions have midline $y = d$, amplitude a , range $[d - a, d + a]$, and period $p = 2\pi$. In addition, the point $(c, a + d)$ lies on the graph of f and the point (c, d) lies on the graph of g .

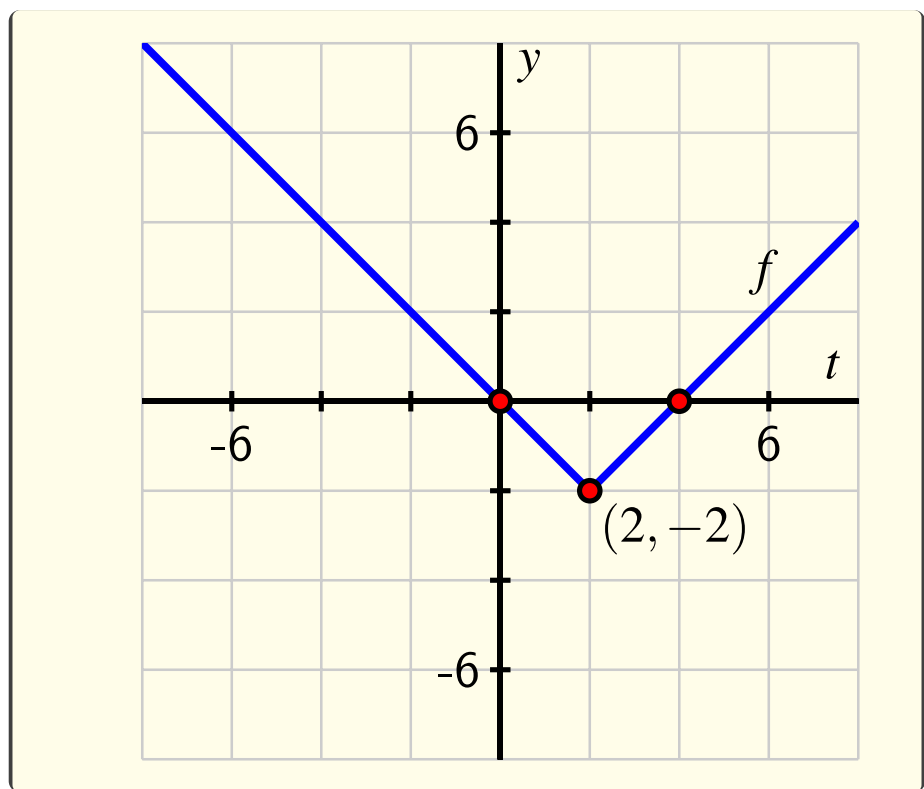
While we will soon focus on horizontal stretches of the sine and cosine functions for the remainder of this section, it's important to note that horizontal scaling follows the same principles for any function we choose.

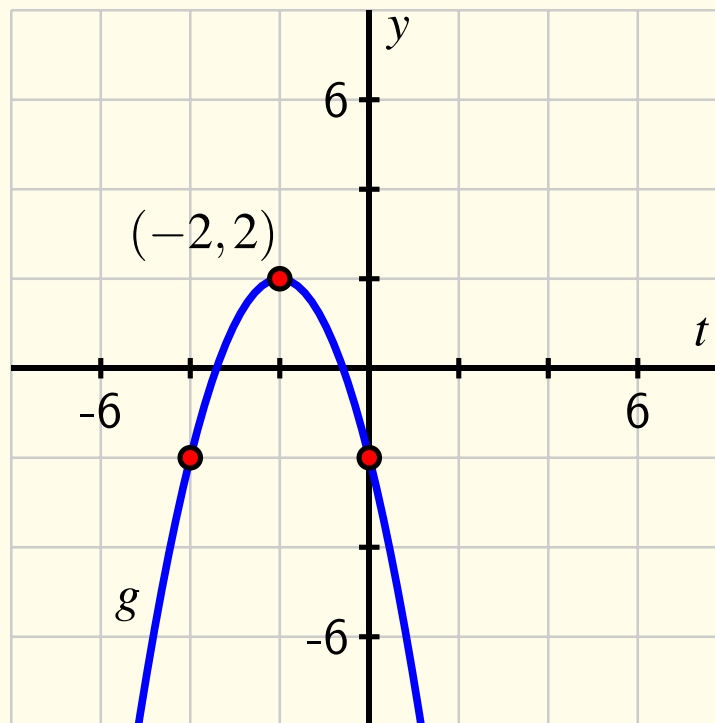
Exploration Consider the functions f and g given in the following figures.





- a. On the same axes as the plot of $y = f(t)$, sketch the following graphs: $y = h(t) = f(\frac{1}{3}t)$ and $y = j(t) = r = f(4t)$. Be sure to label several points on each of f , h , and k with arrows to indicate their correspondence. In addition, write one sentence to explain the overall transformations that have resulted in h and j from f .
- b. On the same axes as the plot of $y = g(t)$, sketch the following graphs: $y = k(t) = g(2t)$ and $y = m(t) = g(\frac{1}{2}t)$. Be sure to label several points on each of g , k , and m with arrows to indicate their correspondence. In addition, write one sentence to explain the overall transformations that have resulted in k and m from g .
- c. On the additional copies of the two figures below, sketch the graphs of the following transformed functions: $y = r(t) = 2f(\frac{1}{2}t)$ and $y = s(t) = \frac{1}{2}g(2t)$. As above, be sure to label several points on each graph and indicate their correspondence to points on the original parent function.





- d. Describe in words how the function $y = r(t) = 2f(\frac{1}{2}t)$ is the result of two elementary transformations of $y = f(t)$. Does the order in which these transformations occur matter? Why or why not?

Circular functions with different periods

Because the circumference of the unit circle is 2π , the sine and cosine functions each have period 2π . Of course, as we think about using transformations of the sine and cosine functions to model different phenomena, it is apparent that we will need to generate functions with different periods than 2π . For instance, if a ferris wheel makes one revolution every 5 minutes, we'd want the period of the function that models the height of one car as a function of time to be $P = 5$. Horizontal scaling of functions enables us to generate circular functions with any period we desire.

We begin by considering two basic examples. First, let $f(t) = \sin(t)$ and $g(t) = f(2t) = \sin(2t)$. We know from our most recent work that this transformation results in a horizontal compression of the graph of $\sin(t)$ by a factor of $\frac{1}{2}$ toward

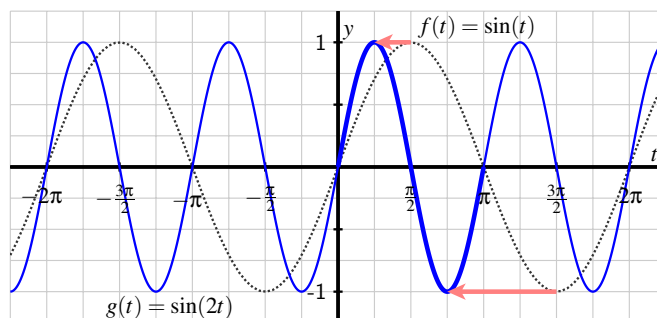


Figure 1: A plot of the parent function, $f(t) = \sin(t)$ (dashed, in gray), and the transformed function $g(t) = f(2t) = \sin(2t)$ (in blue).

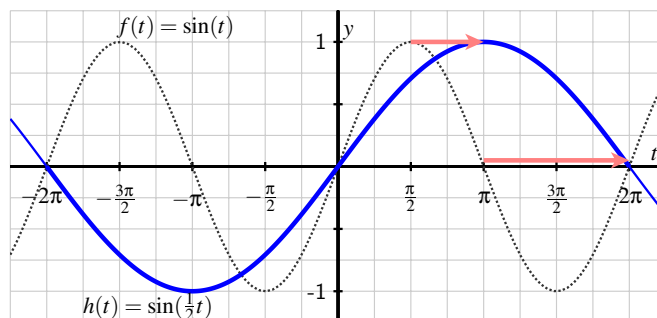


Figure 2: A plot of the parent function, $f(t) = \sin(t)$ (dashed, in gray), and the transformed function $h(t) = f(\frac{1}{2}t) = \sin(\frac{1}{2}t)$ (in blue).

the y -axis. If we plot the two functions on the same axes, it becomes apparent how this transformation affects the period of f .

From the graph, we see that $g(t) = \sin(2t)$ oscillates twice as frequently as $f(t) = \sin(t)$, and that g completes a full cycle on the interval $[0, \pi]$, which is half the length of the period of f . Thus, the “2” in $f(2t)$ causes the period of f to be $\frac{1}{2}$ as long; specifically, the period of g is $P = \frac{1}{2}(2\pi) = \pi$.

On the other hand, if we let $h(t) = f(\frac{1}{2}t) = \sin(\frac{1}{2}t)$, the transformed graph h is stretched away from the y -axis by a factor of 2. This has the effect of doubling the period of f , so that the period of h is $P = 2 \cdot 2\pi = 4\pi$, as seen in the previous figure.

Our observations generalize for any positive constant $b > 0$. In the case where

$b = 2$, we saw that the period of $g(t) = \sin(2t)$ is $P = \frac{1}{2} \cdot 2\pi$, whereas in the case where $b = \frac{1}{2}$, the period of $h(t) = \sin(\frac{1}{2}t)$ is $P = 2 \cdot 2\pi = \frac{1}{\frac{1}{2}} \cdot 2\pi$. Identical reasoning holds if we are instead working with the cosine function. In general, we can say the following.

For any constant $b > 0$, the period of the functions $\sin(bt)$ and $\cos(bt)$ is

$$P = \frac{2\pi}{b}.$$

Thus, if we know the b -value from the given function, we can deduce the period. If instead we know the desired period, we can determine b by the rule $b = \frac{2\pi}{P}$.

Example 6. *Determine the exact period, amplitude, and midline of each of the following functions. In addition, state the range of each function and any horizontal shift that has been introduced to the graph. Make your conclusions without consulting Desmos, and then use the program to check your work.*

- a. $p(x) = \sin(10x) + 2$
- b. $q(x) = -3\cos(0.25x) - 4$
- c. $r(x) = 2\sin\left(\frac{\pi}{4}x\right) + 5$
- d. $w(x) = 2\cos\left(\frac{\pi}{2}(x - 3)\right) + 5$
- e. $u(x) = -0.25\sin(3x - 6) + 5$

Explanation

- a. $p(x)$ has a period of $\frac{\pi}{5}$, an amplitude of 1 and a midline of $y = 2$.
The range of $p(x)$ is $[1, 3]$.
- b. $q(x)$ has a period of 8π , an amplitude of 3 and a midline of $y = -4$.
The range of $q(x)$ is $[-7, -1]$.
- c. $r(x)$ has a period of 8, an amplitude of 2 and a midline of $y = 5$.
The range of $r(x)$ is $[3, 7]$.
- d. $w(x)$ has a period of 4, an amplitude of 2 and a midline of $y = 5$.
The range of $w(x)$ is $[3, 7]$.
There is a horizontal shift of 3 to the right.
- e. $u(x)$ has a period of $\frac{2\pi}{3}$, an amplitude of 0.25 and a midline of $y = 5$.
The range of $u(x)$ is $[4.75, 5.25]$.
There is a horizontal shift of 2 to the right.

Exploration Consider a spring-mass system where the weight is hanging from the ceiling in such a way that the following is known: we let $d(t)$ denote the distance from the ceiling to the weight at time t in seconds and know that the weight oscillates periodically with a minimum value of $d(t) = 1.5$ and a maximum value of $d(t) = 4$, with a period of 3, and you know $d(0.5) = 2.75$ and $d(1.25) = 4$.

State the midline, amplitude, range, and an anchor point for the function, and hence determine a formula for $d(t)$ in the form $a \cos(k(t + b)) + c$ or $a \sin(k(t + b)) + c$. Show your work and thinking, and use *Desmos* appropriately to check that your formula generates the desired behavior.

Summary

- Given real numbers a , c , and d with $a \neq 0$, the functions

$$f(t) = a \cos(b(t - c)) + d \text{ and } g(t) = a \sin(b(t - c)) + d$$

each represent a horizontal shift by c units to the right, followed by a vertical stretch by a units, followed by a vertical shift of d units, applied to the parent function ($\cos(t)$ or $\sin(t)$, respectively). The resulting circular functions have midline $y = d$, amplitude a , range $[d - a, d + a]$, and period $p = 2\pi$. In addition, the anchor point $(c, a + d)$ lies on the graph of f and the anchor point (c, d) lies on the graph of h .

- Given a function f and a constant $b > 0$, the algebraic transformation $h(t) = f(bt)$ results in horizontal scaling of f by a factor of $\frac{1}{b}$. In particular, when $b > 1$, the graph of f is compressed toward the y -axis by a factor of $\frac{1}{b}$ to create the graph of h , while when $0 < b < 1$, the graph of f is stretched away from the y -axis by a factor of $\frac{1}{b}$ to create the graph of h .
- Given any circular periodic function for which the midline, amplitude, period, and an anchor point are known, we can find a corresponding formula for the function of the form

$$f(t) = a \cos(b(t - c)) + d \text{ or } g(t) = a \sin(b(t - c)) + d.$$

Each of these functions has period midline $y = d$, amplitude a , and period $P = \frac{2\pi}{b}$. The point $(c, a + d)$ lies on f and the point (c, d) lies on g .

9.3 Some Applications of Trigonometry

Learning Objectives

- Solving Trigonometric Equations
 - Solving elementary trigonometric equations.
 - Solving trigonometric equations using identities.
 - Finding solutions on restricted domains.
- Applications of Trigonometric Functions
 - Applications involving trigonometric functions

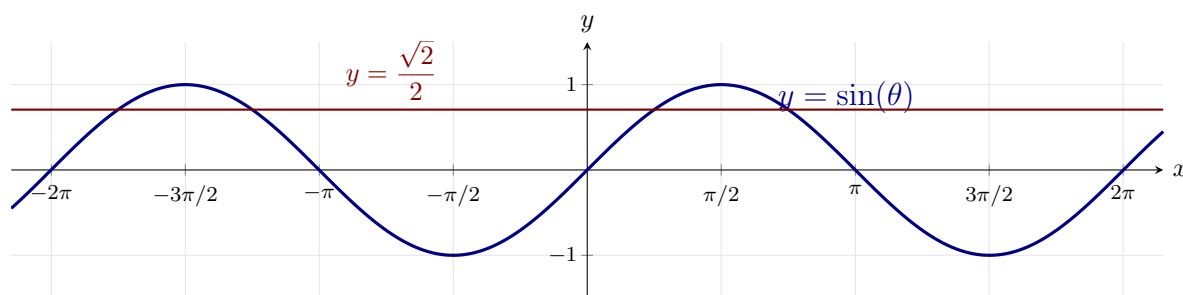
9.3.1 Solving Trigonometric Equations

Introduction

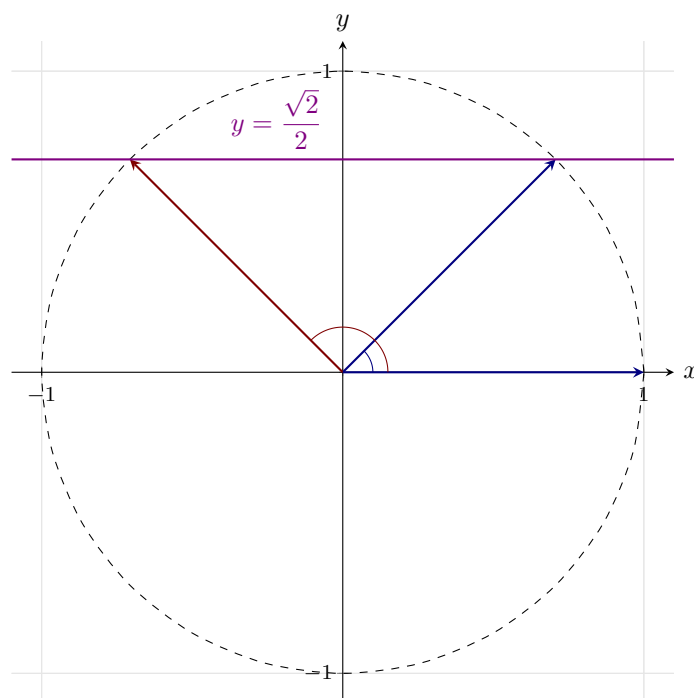
Frequently we are in the situation of having to determine precisely which angles satisfy a particular equation. Something like $\sin(\theta) = \frac{\sqrt{2}}{2}$. We know that

$\sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$, meaning that $\theta = \frac{\pi}{4}$ is a solution of this equation, but is that the only solution or *are there more?*

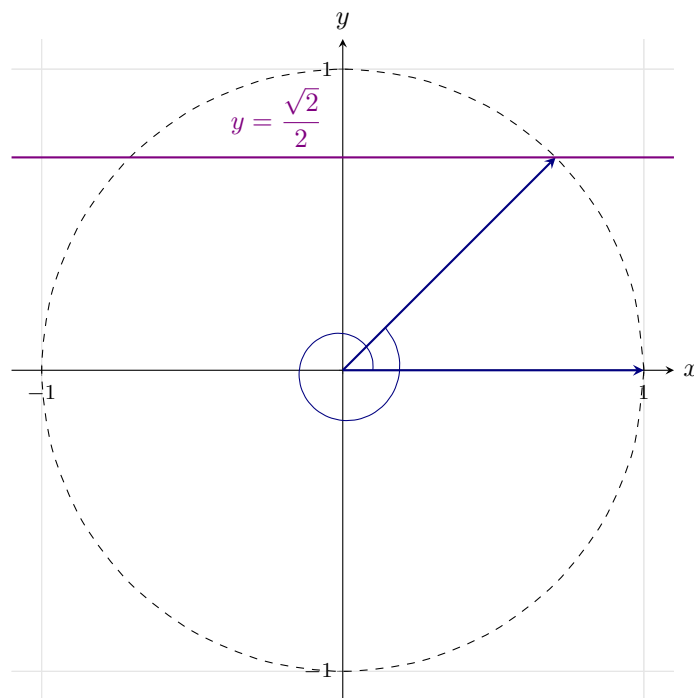
Let's look at the graph of the sine function.



Notice that the graph of $\sin(\theta)$ and the graph of the constant function $y = \frac{\sqrt{2}}{2}$ intersect many times, not just once. In fact, since sine is a periodic function, these graphs intersect infinitely many times. Each of these intersections represents a single solution of the equation $\sin(\theta) = \frac{\sqrt{2}}{2}$. We need a process to identify and write down each of these solutions. Let's start by looking at the unit circle. Remember that sine values correspond to the y -coordinate of points on the unit circle. This equation is asking us to find all the points on the unit circle with a y -coordinate of $\frac{\sqrt{2}}{2}$.



You see that there are two locations on the unit circle with y -coordinate equal to $\frac{\sqrt{2}}{2}$, one in the first quadrant and another in the second. As we mentioned earlier, the first quadrant angle is $\theta = \frac{\pi}{4}$. The angle in the second quadrant has reference angle $\frac{\pi}{4}$, which means the angle is $\frac{3\pi}{4}$. Those are the only two points on the circle with that y -coordinate, but remember that there are many other angles which are coterminal with those. For instance:



The only solutions are the angles $\frac{\pi}{4}$, $\frac{3\pi}{4}$, and *all the angles coterminal with them*. Since the sine function has period 2π , that means any other solution has to be an integer multiple of 2π away from one of these first two solutions. Putting that together, our solutions are:

$$\theta = \frac{\pi}{4} + 2\pi k, \frac{3\pi}{4} + 2\pi k, k \text{ any integer.}$$

The steps we've followed are summarized in the following.

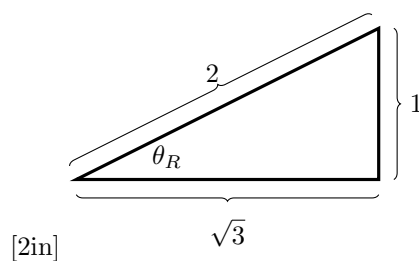
To solve a trigonometric equation:

- (a) Find the reference angle of the solutions. Typically the standard values will help identify this.
- (b) Find all solutions on a single period of the function. Use the graph, the unit circle, and the reference angle to identify these.
- (c) Find all solutions. Use the period of the function to find all requested solutions.

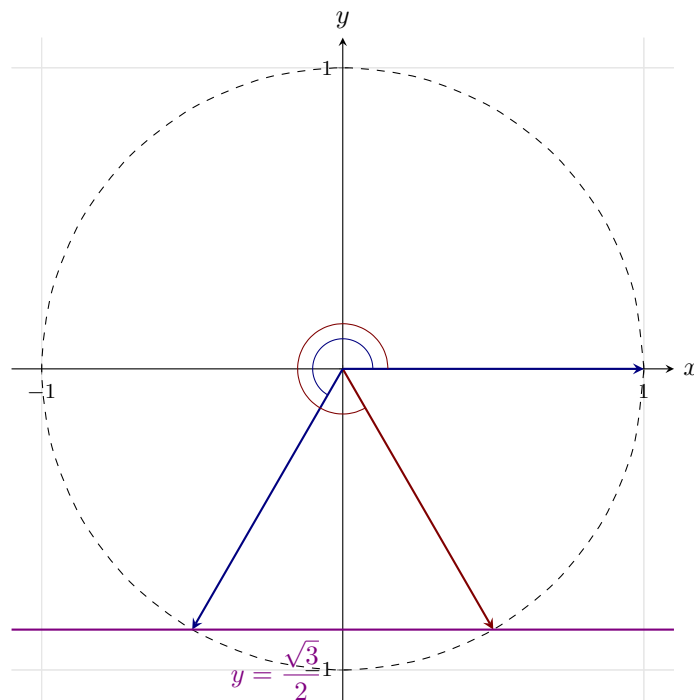
Example 7. Solve the equation:

$$\sin(\theta) = -\frac{1}{2}.$$

Explanation We'll start by finding the reference angle, θ_R , the acute angle between the terminal side of θ and the x -axis. The reference angle satisfies $\sin(\theta_R) = \frac{1}{2}$ and the negative sign will be used to indicate the quadrant of the angle.



From the picture we see $\theta_R = \frac{\pi}{6}$. Let's look at the unit circle.



In one period $[0, 2\pi)$, there are two angles that have reference angle $\frac{\pi}{6}$ and have negative sine value. One is in quadrant 3, and one in quadrant 4. That means the solutions in the interval $[0, 2\pi)$ are $\frac{5\pi}{6}$ and $\frac{11\pi}{6}$.

To find all solutions, we have to add all multiples of 2π to these. The solutions

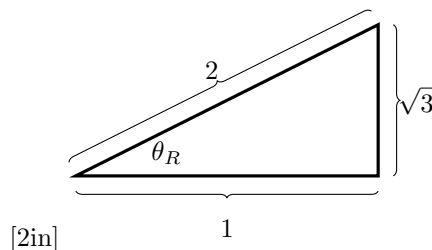
are then

$$\theta = \frac{5\pi}{6} + 2\pi k, \frac{11\pi}{6} + 2\pi k, k \text{ any integer.}$$

Example 8. Solve the equation:

$$\tan(\theta) = -\sqrt{3}.$$

Explanation We'll start by finding the reference angle, θ_R , the acute angle between the terminal side of θ and the x -axis. The reference angle satisfies $\tan(\theta_R) = \sqrt{3}$ and the negative sign will be used to indicate the quadrant of the angle. Since tangent is opposite over adjacent, we have the following triangle.



From the picture we see $\theta_R = \frac{\pi}{3}$. Let's look at the unit circle.

The tangent function has one period $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. In the interval $\left(-\frac{\pi}{2}, 0\right)$ (which is Quadrant IV), tangent is negative while in $\left(0, \frac{\pi}{2}\right)$ (which is Quadrant I), tangent is positive. For $\tan(\theta)$ to be negative in this interval, we need θ to be in $\left(-\frac{\pi}{2}, 0\right)$. The only angle in that interval with reference angle $\frac{\pi}{3}$ is $\theta = -\frac{\pi}{3}$. This is the only solution on this period.

Remember that the tangent function has period π , unlike sine and cosine which have period 2π . On the period $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, tangent is one-to-one, so there is exactly one angle which gives the desired output value. Sine and cosine are not one-to-one across a full period.

To find all solutions, we have to add all multiples of π to this. The solutions are then

$$\theta = \frac{2\pi}{3} + \pi k, k \text{ any integer.}$$

Let's try one a bit more complicated.

Example 9. Solve the equation:

$$\cos(\theta)(\cos(\theta) + 1) = \sin^2(\theta).$$

Explanation We'll start by simplifying a bit.

$$\begin{aligned}\cos(\theta) (\cos(\theta) + 1) &= \sin^2(\theta) \\ \cos^2(\theta) + \cos(\theta) &= \sin^2(\theta) \\ \cos^2(\theta) - \sin^2(\theta) + \cos(\theta) &= 0 \\ \cos^2(\theta) - (1 - \cos^2 \theta) + \cos(\theta) &= 0 \\ 2 \cos^2(\theta) + \cos(\theta) - 1 &= 0.\end{aligned}$$

Notice that this equation is quadratic in $\cos(\theta)$. We can factor it like we try to do to solve any other quadratic equation:

$$(\cos(\theta) + 1) (2 \cos(\theta) - 1) = 0.$$

On the interval $[0, 2\pi)$, $\cos(\theta) = -1$ has only one solution, $\theta = \pi$. For $\cos(\theta) = \frac{1}{2}$, we see that the reference angle $\theta_R = \frac{\pi}{3}$. Since cosine is positive in Quadrants I and IV, we find solutions $\theta = \frac{\pi}{3}$ and $\frac{5\pi}{3}$.

All solutions are:

$$\theta = \pi + 2\pi k, \frac{\pi}{3} + 2\pi k, \frac{5\pi}{3} + 2\pi k, \text{ } k \text{ any integer.}$$

9.3.2 Applications of Trigonometry

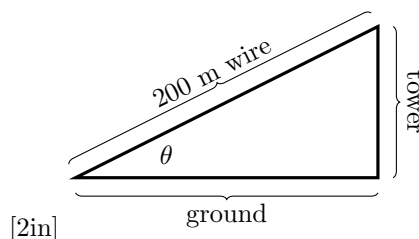
Applications of Trigonometry

In the previous sections, you have been learning about trigonometric functions in the abstract. In this section, we wish to apply them.

Example 10. *A wire 200 meters long is attached to the top of a tower. When pulled taut, it makes a 60° angle with the ground. How tall is the tower? How far away from the base of the tower does the wire hit the ground?*

Explanation

In application problems, we are often given data about angles measured in degrees. It is up to us to translate this into radians for our calculations. Let's call the angle we're given as θ , so $\theta = \frac{\pi}{3}$ radians. Let's draw a diagram of this situation.



Based on this image, the height of the tower is the opposite the angle we know, and the distance along the ground is adjacent. That means the tower height is related to the angle by $\sin(\theta) = \frac{\text{tower}}{200}$ and the distance across the ground is given by $\cos(\theta) = \frac{\text{ground}}{200}$.

Since $\sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$, the tower height can be computed by:

$$\begin{aligned}\sin\left(\frac{\pi}{3}\right) &= \frac{\text{tower}}{200} \\ \text{tower} &= 200 \sin\left(\frac{\pi}{3}\right) \\ &= 200 \left(\frac{\sqrt{3}}{2}\right) \\ &= 100\sqrt{3}.\end{aligned}$$

The tower has a height of $100\sqrt{3}$ m, which is approximately 173.21 m.

The EXACT VALUE of the height is $100\sqrt{3}$ m. Saying that this is approximately 173.21 m is provided to give us an indication of scale. Our actual answer is the exact value, not this approximation.

We'll follow a similar calculation to find the distance from the base of the tower to the wire along the ground.

$$\begin{aligned}\cos\left(\frac{\pi}{3}\right) &= \frac{\text{ground}}{200} \\ \text{ground} &= 200 \cos\left(\frac{\pi}{3}\right) \\ &= 200 \left(\frac{1}{2}\right) \\ &= 100.\end{aligned}$$

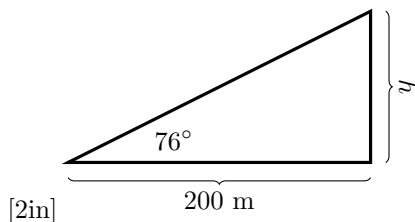
The wire hits the ground 100 m from the base of the tower.

Example 11. *A camera is setup 200 m from the base of a building, pointed at the top of the building. If the angle-of-elevation is measured as 76° , find the height of the building.*

Explanation

The angle-of-elevation means the angle between the camera's line-of-sight and horizontal. Since the camera is setup 200 meters from the building, this gives us a right triangle where we know the base angle and the length of the adjacent side.

If we call the height of the building as h , then we have a triangle

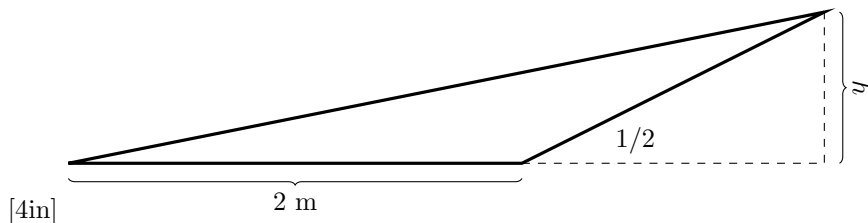


That means $\tan(76^\circ) = \frac{h}{200}$, so $h = 200 \tan(76^\circ)$. The exact height is $200 \tan(76^\circ)$ meters. This is approximately 802.16 meters.

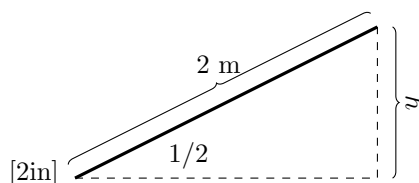
Example 12. *A 4 meter long piece of wire is going to be bent at its midpoint. The right side of the wire is bent up through an angle of $\frac{1}{2}$. The two ends of the wire are joined by a piece of string, creating an obtuse triangle. What is the area of the resulting triangle?*

Explanation

That the wire is bent at its midpoint, means the resulting triangle will have two sides of length 2 m. Call the height of the triangle h .



Let us focus on the dotted triangle on the right side



Notice that the height of this right triangle is the same as the height of the obtuse triangle above. Since we know the hypotenuse and base angle of this right triangle, we can find the height (the opposite side) using sine.

$$\sin\left(\frac{1}{2}\right) = \frac{h}{2}$$

$$2 \sin\left(\frac{1}{2}\right) = h.$$

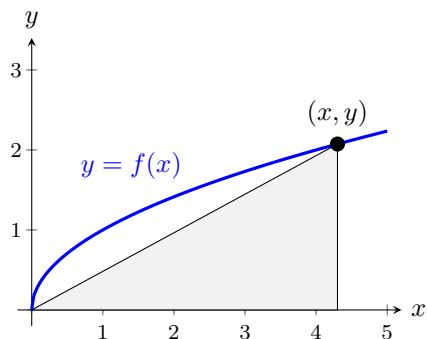
Be careful here! That angle $\frac{1}{2}$ is not in degrees, it's in radians. (You can tell, because there is no “degrees symbol”.) If you are going to approximate this value, make sure you are using radians.

Now the area of the whole triangle is:

$$\begin{aligned} A &= \frac{1}{2}bh \\ &= \frac{1}{2}(2) \left(2 \sin\left(\frac{1}{2}\right) \right) \\ &= 2 \sin\left(\frac{1}{2}\right). \end{aligned}$$

The exact value of the area is $2 \sin\left(\frac{1}{2}\right) m^2$. Using a calculator, this is approximately $0.959 m^2$. (Remember that m^2 is the abbreviation for “square meters”.)

Example 13. A right triangle is constructed by taking a point (x, y) on the graph of the function $f(x) = \sqrt{x}$, drawing a line vertically downward to the x -axis, then connecting both of those points to the origin as in the picture below.



For one particular point (x, y) , the acute angle between the hypotenuse of the triangle and the positive x -axis is found to measure $\frac{\pi}{6}$ radians. Find the coordinates of the point (x, y) .

Explanation

Since the hypotenuse of the right triangle runs from the origin, which has coordinates $(0, 0)$, to the point (x, y) , the horizontal side of the triangle has length x and vertical side has length y . We know that the value of tangent is given by the ratio of the opposite side length divided by the adjacent side length. Using the given angle of $\frac{\pi}{6}$, this means $\tan\left(\frac{\pi}{6}\right) = \frac{1}{\sqrt{3}} = \frac{y}{x}$. That is, $x = y\sqrt{3}$

We also know that the point lies on the graph of $f(x) = \sqrt{x}$, which means $y = \sqrt{x}$.

This gives us a nonlinear system of two equations:

$$\begin{cases} x = y\sqrt{3} \\ y = \sqrt{x} \end{cases}$$

Squaring the bottom equation yields $y^2 = x$. When substituting the top equation into the bottom equation, we arrive at:

$$\begin{aligned} y^2 &= y\sqrt{3} \\ y^2 - y\sqrt{3} &= 0 \\ y(y - \sqrt{3}) &= 0 \end{aligned}$$

so either $y = 0$ or $y = \sqrt{3}$.

The value $y = 0$ corresponds to the point $(0, 0)$ on the graph, which does not yield any angle. This is an extraneous solution, which is discarded.

Substituting the value $y = \sqrt{3}$ into $x = y\sqrt{3}$ gives $x = (\sqrt{3})\sqrt{3} = 3$. The point is $(3, \sqrt{3})$.

Part 10

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