Asymptotic Equivalence of Wald, Likelihood Ratio, and Score Tests

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Introduction

Wald test, Likelihood ratio (LR) test and Score test are three classical tests that compare a restricted model (the null model) against an unrestricted model (the alternative model). In this essay, simulations are conducted with Poisson models to illustrate the theory that if the null hypothesis is true, (1) the distribution of the three test statistics follow chi-square distribution asymptotically, (2) the differences among the three tests vanish as $n \to \infty$. Finally, we will set the null hypothesis to be false and check the results of the three test statistics.

Description of Simulation Setup

The simulation starts with generating data from the Poisson model with a $n \times 2$ matrix $\boldsymbol{X}$ of normally distributed values and a 2×1 vector of regression coefficients $\boldsymbol{\beta} = (\beta_1, \beta_2)'$.

$$y_i | x_i \sim Poisson\left(exp^{x_i' \beta}\right), \qquad i = 1, \cdots, n \tag{1}$$

Here we focus on the hypothesis of the parameter β_2. First, when the null hypothesis is true (H_0: $\beta_2 = 0$), we will (1) simulate the three tests under a large sample size to see the results of the three tests, including the distributions of test statistics and test results, (2) simulate the three tests under different sample sizes to see the differences of the test results. Second, when the null hypothesis is false (H_0: $\beta_2 = 0.5$), we will simulate the three tests under a large sample size to see the results of the three tests. To maximize the loglikelihood, Newton-Raphson algorithm is used.

(a) Simulation results when the null is true

Asymptotic equivalence of three tests

To see the asymptotic distribution of the test statistics for the three tests, we simulated 1,000 datasets for each test statistic under H_0: $\beta_2 = 0$ and plotted the histogram of the three test statistics. As shown in Figure 1, all the three test statistics are quite close to Chi-square distribution with 1 degrees of freedom, which is in line with the theory that the three tests are asymptotically Chi-square distributed when H_0 is true.

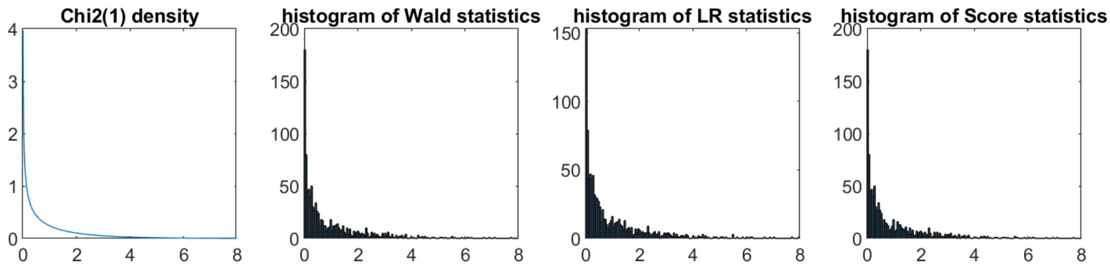

Figure 1. Histogram of different test statistics under true hypothesis

To have a better understanding of how the three tests perform on a single dataset, we simulated a 1000×2 matrix $\boldsymbol{X}$ to investigate the result of the three tests for testing H_0: $\beta_2 = 0$. As shown in Table

1, the test statistics of the three tests are quite close, and the p-values of three tests all suggest that we should not reject $H_0: \beta_2 = 0$ at 5% significance level.

	Wald test	LR test	Score test
Test statistics	0.6783	0.6772	0.6783
P-value	0.4102	0.4105	0.4102

Table 1. Simulation results under true hypothesis

Simulation under different sample sizes

To get the simulation results under different sample sizes, we set the sizes of dataset from 5 to 200 by 5. As shown in Figure 2, there are some differences when the sample size is small but the error vanishes (three lines coincidence) when sample size is large enough, which is consistent with the theory that the differences among the three tests vanish as $n \to \infty$.

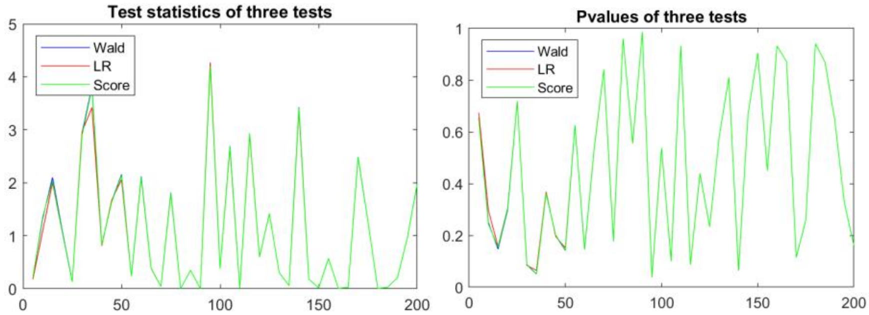

Figure 2. Simulation with different sample sizes

(b) Simulation results when the null is false

We simulated 1,000 datasets for each test statistic under $H_0: \beta_2 = 0.5$. As shown in Figure 2, the three test statistics are no longer Chi-square distributed.

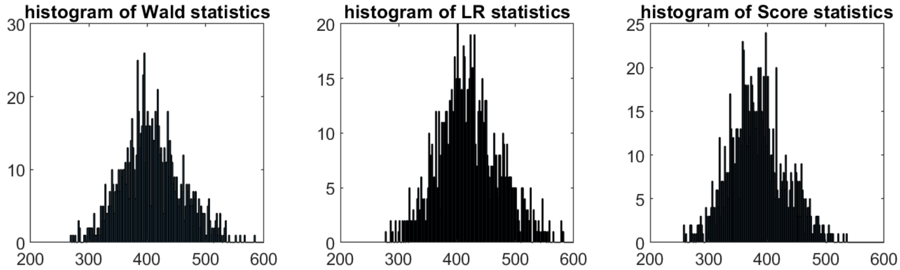

Figure 3. Histogram of test statistics under false hypothesis

If we simulate the three tests on a single dataset, we will find that the values of the test statistics are considerably large, and the p-values of the three tests are very close to zero, which suggest we should reject $H_0: \beta_2 = 0.5$ (Table 2).

	Wald test	LR test	Score test
Test statistics	460.7480	461.2101	419.1153
P-value	0.0000	0.0000	0.0000

Table 2. Simulation results under false hypothesis