Two step FVE method

A numerical modeling technique designed for error insight

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2025-10-08 10:13:52

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Symbol	Unit	Description
Δt	s	Time increment
Δx	m	Space increment, $\Delta x_{i+\frac{1}{2}} = x_{i+1} - x_i$
ε	$\mathrm{m\,s^{-2}}$	Multiplier of the correction term for the essential boundary condition, ζ-boundary
ε	$\mathrm{s^{-1}}$	Multiplier of the correction term for the essential boundary condition, q-boundary
ν	$\mathrm{m^2\,s^{-1}}$	Kinematic viscosity
Ω	$-$	Finite volume
Ψ	$\mathrm{m^2\,s^{-1}}$	Artificial smoothing coefficient
θ	$-$	θ-method. If $\theta = 1$ then it is a fully implicit method and if $\theta = 0$ then it is a fully explicit method.
$\boldsymbol{E}$	$-$	Error vector function, defined in computational space
ξ	$-$	Relative coordinate
ζ	m	Water level w.r.t. reference plane, positive upward
C	$\mathrm{m^{\frac{1}{2}}\,s^{-1}}$	Chézy coefficient
c_Ψ	$(.)^{-1}$	Artificial smoothing variable
c_f	$-$	Bed shear stress coefficient
g	$\mathrm{m\,s^{-2}}$	Gravitational constant
h	m	Total water depth
i	$-$	node counter
q	$\mathrm{m^2\,s^{-1}}$	The water flux in x-direction, $q = hu$
r	$\mathrm{m^2\,s^{-1}}$	The water flux in y-direction, $r = hv$
t	s	Time coordinate
u	$\mathrm{m\,s^{-1}}$	Velocity in x-direction
v	$\mathrm{m\,s^{-1}}$	Velocity in y-direction
x	m	x-coordinate
y	m	y-coordinate
z_b	m	Bed level w.r.t. reference plane, positive upward

Heat 2D

1 Heat 2D equation

Consider the non-linear heat equation

$$\frac{\partial u}{\partial t} - \nabla \cdot (\boldsymbol{\Psi} \nabla u) = 0, \tag{1.1a}$$

with

u	Temperature, $[{}^\circ\mathrm{C}]$.
$\boldsymbol{\Psi}$	2-dimensional thermal diffusivity, $[\mathrm{m}^2\,\mathrm{s}^{-1}]$.

$\boldsymbol{\Psi}$ the thermal diffusivity is defined as:

$$\boldsymbol{\Psi} = \begin{pmatrix} \Psi_{11} & \Psi_{12} \\ \Psi_{21} & \Psi_{22} \end{pmatrix} \tag{1.2}$$

Finite Volume approach

Integrating the equations over a finite volume Ω yields:

$$\int_\Omega \frac{\partial h}{\partial t}\, d\omega - \int_\Omega \nabla \cdot (\boldsymbol{\Psi} \nabla u)\; d\omega = 0 \tag{1.3}$$

with

1.1 Space discretization, structured

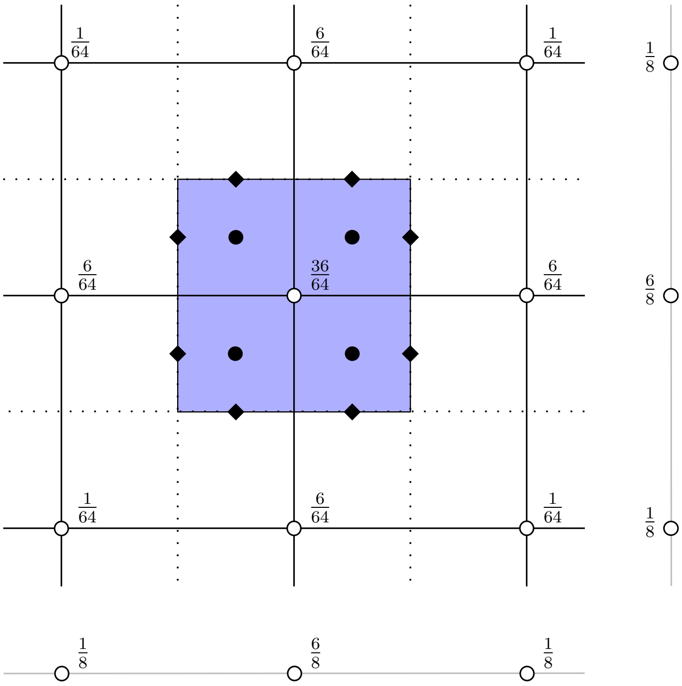

Figure 1.1: *Coefficients for the mass-matrix and the control volume in 2-dimensions in the interior area, on a structured grid. The black dots indicate the location of the quadrature points, and black diamonds the flux points.*

For the space discretization of an arbitrary function u on the quadrature point of a sub-control volume the following space interpolations are used:

$$u|_{i+\frac{1}{4},j+\frac{1}{4}} \approx \frac{1}{16}\left(9u_{i,j} + 3u_{i+1,j} + u_{i+1,j+1} + 3u_{i,j+1}\right) \tag{1.4}$$

$$u|_{i+\frac{1}{2},j+\frac{1}{4}} \approx \frac{1}{8}\left(3u_{i,j} + 3u_{i+1,j} + u_{i+1,j+1} + u_{i,j+1}\right) \tag{1.5}$$

$$u|_{i+\frac{1}{4},j+\frac{1}{2}} \approx \frac{1}{8}\left(3u_{i,j} + u_{i+1,j} + u_{i+1,j+1} + 3u_{i,j+1}\right) \tag{1.6}$$

See for the locations Figure 1.1.

1.1.1 Discretizations heat equation

The discretization of heat equation (1.3) will be presented term by term.

1.1.1.1 Time derivative

The discretization of the time derivative term of the continuity equation reads:

$$\int_{\Omega} \frac{\partial u}{\partial t}\, d\omega \tag{1.7}$$

which will be approximated by the sum of the integral over the sub-control volumes. On a structured grid one control volume (cv) around a node consist of four sub-control volumes (scv_i, $i \in \{0,1,2,3\}$).

$$\int_{cv} \frac{\partial u}{\partial t}\, d\omega = \int_{scv_0} \frac{\partial u}{\partial t}\, d\omega + \int_{scv_1} \frac{\partial u}{\partial t}\, d\omega + \int_{scv_2} \frac{\partial u}{\partial t}\, d\omega + \int_{scv_3} \frac{\partial u}{\partial t}\, d\omega \quad (1.8)$$

For a cartesian grid we get:

$$\begin{aligned}
\int_{cv} \frac{\partial h}{\partial t}\, d\omega \approx & \frac{1}{4}\Delta x \Delta y \Delta t_{inv} \left(h^{n+1,p+1}_{i-\frac{1}{4},j-\frac{1}{4}} - h^{n+1,n}_{i-\frac{1}{4},j-\frac{1}{4}} \right) + \\
& \frac{1}{4}\Delta x \Delta y \Delta t_{inv} \left(h^{n+1,p+1}_{i+\frac{1}{4},j-\frac{1}{4}} - h^{n+1,n}_{i+\frac{1}{4},j-\frac{1}{4}} \right) + \\
& \frac{1}{4}\Delta x \Delta y \Delta t_{inv} \left(h^{n+1,p+1}_{i+\frac{1}{4},j+\frac{1}{4}} - h^{n+1,n}_{i+\frac{1}{4},j+\frac{1}{4}} \right) + \\
& \frac{1}{4}\Delta x \Delta y \Delta t_{inv} \left(h^{n+1,p+1}_{i-\frac{1}{4},j+\frac{1}{4}} - h^{n+1,n}_{i-\frac{1}{4},j+\frac{1}{4}} \right)
\end{aligned} \quad (1.9)$$

Just looking to the quadrature point of scv_0 as part of the control volume for node (i,j) the discretization reads:

$$\frac{1}{4}\Delta x \Delta y \Delta t_{inv} \left(h^{n+1}_{i-\frac{1}{4},j-\frac{1}{4}} - h^{n+1,n}_{i-\frac{1}{4},j-\frac{1}{4}} \right) = \quad (1.10)$$

$$= \frac{1}{4}\Delta x \Delta y \Delta t_{inv} \left[\frac{1}{16}\left(9h^{n+1}_{i,j} + 3h^{n+1}_{i-1,j} + 3h^{n+1}_{i,j-1} + h^{n+1}_{i-1,j-1} \right) + \right. \quad (1.11)$$

$$\left. - \frac{1}{16}\left(9h^{n}_{i,j} + 3h^{n}_{i-1,j} + 3h^{n}_{i,j-1} + h^{n}_{i-1,j-1} \right) \right] \quad (1.12)$$

Written in Δ-formulation it reads:

$$\begin{aligned}
& \frac{1}{4}\Delta x \Delta y \Delta t_{inv} \left(h^{n+1}_{i-\frac{1}{4},j-\frac{1}{4}} - h^{n}_{i-\frac{1}{4},j-\frac{1}{4}} \right) = \\
& = \frac{1}{4}\Delta x \Delta y \Delta t_{inv} \left[\frac{1}{16}\left(9\Delta h^{n+1,p+1}_{i,j} + 3\Delta q^{n+1,p+1}_{i-1,j} + 3\Delta q^{n+1,p+1}_{i,j-1} + \Delta h^{n+1,p+1}_{i-1,j-1} \right) \right] + \\
& + \frac{1}{4}\Delta x \Delta y \Delta t_{inv} \left[\frac{1}{16}\left(9h^{n+1,p}_{i,j} + 3h^{n+1,p}_{i-1,j} + 3h^{n+1,p}_{i,j-1} + h^{n+1,p+1}_{i-1,j-1} \right) + \right. \\
& \left. - \frac{1}{16}\left(9h^{n}_{i,j} + 3h^{n}_{i-1,j} + 3h^{n}_{i,j-1} + h^{n}_{i-1,j-1} \right) \right]
\end{aligned} \quad (1.13)$$

1.1.1.2 Viscosity

The viscosity term in vector notation reads:

$$\int_{\Omega} \nabla \cdot (\boldsymbol{\Psi} \nabla u)\, d\omega = \oint_{\Omega} (\boldsymbol{\Psi} \nabla u) \cdot \boldsymbol{n}\, dl \approx \quad (1.14)$$

$$\oint_{\Omega} \left(\left(\Psi_{11} \frac{\partial u}{\partial x} + \Psi_{12} \frac{\partial u}{\partial y} \right) n_x + \left(\Psi_{21} \frac{\partial u}{\partial x} + \Psi_{22} \frac{\partial u}{\partial y} \right) n_y \right) \|dl\| \quad (1.15)$$

with $\boldsymbol{n} = (n_x, n_y)^T$ the outward normal vector.

Integration of the term for the eight quadrature points qp, for each face of the control volume scv we get:

$$\left(\Psi_{11} \frac{\partial u}{\partial x} + \Psi_{12} \frac{\partial u}{\partial y} \right) n_x \|dl\| + \left(\Psi_{21} \frac{\partial u}{\partial x} + \Psi_{22} \frac{\partial u}{\partial y} \right) n_y \|dl\| \quad (1.16)$$

If the viscosity is isotropic ($\Psi_{12} = \Psi_{21} = 0$) these equations reduce to:

$$\left(\Psi_{11} \frac{\partial u}{\partial x}\right) n_x \left\| dl \right\| + \left(\Psi_{22} \frac{\partial u}{\partial y}\right) n_y \left\| dl \right\| \tag{1.17}$$

Define the components of $\mathbf{\Psi}$ at different locations with $(\mu, \nu) \in \{1, 2\}$ as

$$\overline{\Psi}_{\mu\nu}\Big|_{i+\frac{1}{2},j+\frac{1}{4}} = \frac{1}{8}\left(3\Psi_{\mu\nu}\big|_{i,j} + 3\Psi_{\mu\nu}\big|_{i+1,j} + \Psi_{\mu\nu}\big|_{i+1,j+1} + \Psi_{\mu\nu}\big|_{i,j+1}\right) \tag{1.18}$$

and define the partial differentials as

$$\left.\frac{\partial u}{\partial x}\right|_{i+\frac{1}{2},j+\frac{1}{4}} = \frac{1}{4}\left(3\frac{u_{i+1,j} - u_{i,j}}{\Delta x_{i+\frac{1}{2},j}} + \frac{u_{i+1,j+1} - u_{i,j+1}}{\Delta x_{i+\frac{1}{2},j+1}}\right) \tag{1.19}$$

$$\left.\frac{\partial u}{\partial y}\right|_{i+\frac{1}{2},j+\frac{1}{4}} = \frac{1}{2}\left(\frac{u_{i,j+1} - u_{i,j}}{\Delta y_{i,j+\frac{1}{2}}} + \frac{u_{i+1,j+1} - u_{i+1,j}}{\Delta y_{i+1,j+\frac{1}{2}}}\right) \tag{1.20}$$

In case the viscosity is isotropic it reads:

$$\mathbf{\Psi} = \begin{pmatrix} \Psi & 0 \\ 0 & \Psi \end{pmatrix} \tag{1.21}$$

(isotropic, $\Psi_{11} = \Psi_{22} = \Psi$ and $\Psi_{12} = \Psi_{21} = 0$). and then the discretization is straight forward.

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