

Two step FVE method

A numerical modeling technique designed for error insight

2D Heat equation

J. Mooiman

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Authors

Jan Mooiman

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- 1.1 Definition of the grid to solve the 2D-shallow water equations in the interior area

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List of Symbols

Symbol	Unit	Description
Δt	s	Time increment
Δx	m	Space increment, $\Delta x_{i+\frac{1}{2}} = x_{i+1} - x_i$
ε	m s^{-2}	Multiplier of the correction term for the essential boundary condition, ζ -boundary
ε	s^{-1}	Multiplier of the correction term for the essential boundary condition, q -boundary
ν	$\text{m}^2 \text{s}^{-1}$	Kinematic viscosity
Ω	—	Finite volume
Ψ	$\text{m}^2 \text{s}^{-1}$	Artificial smoothing coefficient
θ	—	θ -method. If $\theta = 1$ then it is a fully implicit method and if $\theta = 0$ then it is a fully explicit method.
E	—	Error vector function, defined in computational space
ξ	—	Relative coordinate
ζ	m	Water level w.r.t. reference plane, positive upward
C	$\text{m}^{\frac{1}{2}} \text{s}^{-1}$	Chézy coefficient
c_Ψ	$(.)^{-1}$	Artificial smoothing variable
c_f	—	Bed shear stress coefficient
g	m s^{-2}	Gravitational constant
h	m	Total water depth
i	—	node counter
q	$\text{m}^2 \text{s}^{-1}$	The water flux in x -direction, $q = hu$
r	$\text{m}^2 \text{s}^{-1}$	The water flux in y -direction, $r = hv$
t	s	Time coordinate
u	m s^{-1}	Velocity in x -direction
v	m s^{-1}	Velocity in y -direction
x	m	x -coordinate
y	m	y -coordinate
z_b	m	Bed level w.r.t. reference plane, positive upward

Heat 2D

1 2D Heat equation

Consider the non-linear heat equation

$$\frac{\partial u}{\partial t} - \nabla \cdot (\Psi \nabla u) = 0, \quad (1.1a)$$

with

- u Temperature, [°C].
 Ψ 2-dimensional thermal diffusivity, [$\text{m}^2 \text{s}^{-1}$].

Ψ the thermal diffusivity is defined as:

$$\Psi = \begin{pmatrix} \Psi_{11} & \Psi_{12} \\ \Psi_{21} & \Psi_{22} \end{pmatrix} \quad (1.2)$$

The 2D-heat equation in curvilinear coordinates read:

$$J \frac{\partial u}{\partial t} - \nabla \cdot (\Psi \nabla u) = 0, \quad (1.3)$$

Finite Volume approach

Integrating the equations over a finite volume Ω yields:

$$\int_{\Omega} \frac{\partial u}{\partial t} d\omega - \int_{\Omega} \nabla \cdot (\Psi \nabla u) d\omega = 0 \quad (1.4)$$

The 2D-heat equation in curvilinear coordinates read:

$$\int_{\Omega_{\xi\eta}} J \frac{\partial u}{\partial t} d\xi d\eta - \int_{\Omega_{\xi\eta}} \nabla \cdot (\Psi \nabla u) d\xi d\eta = 0, \quad (1.5)$$

1.1 Space discretization, structured

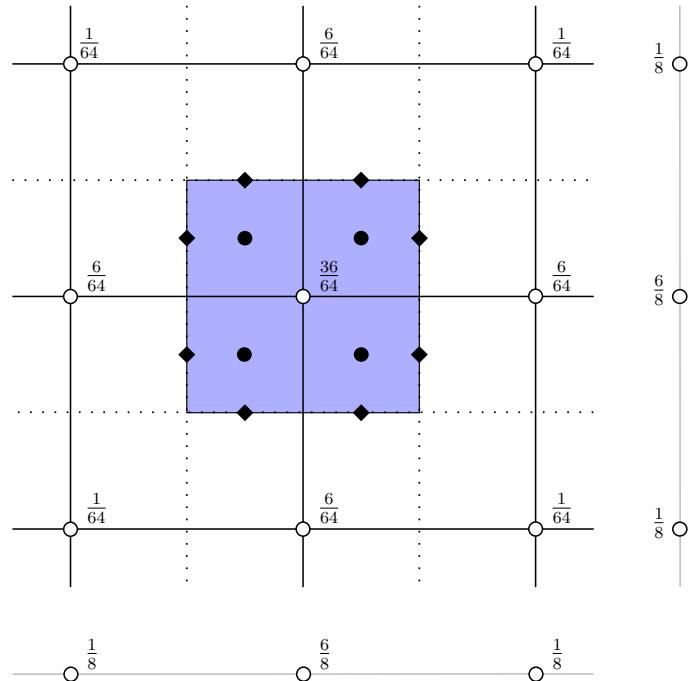


Figure 1.1: Coefficients for the mass-matrix and the control volume in 2-dimensions in the interior area, on a structured grid. The black dots indicate the location of the quadrature points, and black diamonds the flux points.

For the space discretization of an arbitrary function u on the quadrature point of a sub-control volume the following space interpolations are used:

$$u|_{i+\frac{1}{4},j+\frac{1}{4}} \approx \frac{1}{16} (9u_{i,j} + 3u_{i+1,j} + u_{i+1,j+1} + 3u_{i,j+1}) \quad (1.6)$$

$$u|_{i+\frac{1}{2},j+\frac{1}{4}} \approx \frac{1}{8} (3u_{i,j} + 3u_{i+1,j} + u_{i+1,j+1} + u_{i,j+1}) \quad (1.7)$$

$$u|_{i+\frac{1}{4},j+\frac{1}{2}} \approx \frac{1}{8} (3u_{i,j} + u_{i+1,j} + u_{i+1,j+1} + 3u_{i,j+1}) \quad (1.8)$$

See for the locations [Figure 1.1](#).

1.1.1 Discretizations heat equation

The discretization of heat [equation \(1.4\)](#) will be presented term by term.

1.1.1.1 Time derivative

The time derivative term of the heat equation reads:

$$\int_{\Omega_{\xi\eta}} J \frac{\partial u}{\partial t} d\xi d\eta \quad (1.9)$$

which will be approximated by the sum of the integral over the sub-control volumes, with taking into account that the Jacobian J is constant in time. On a structured grid one control volume (cv) around a node consist of four sub-control volumes (scv_i , $i \in \{0, 1, 2, 3\}$).

$$\begin{aligned} J_{cv} \int_{cv} \frac{\partial u}{\partial t} d\omega &= J_{scv_0} \int_{scv_0} \frac{\partial u}{\partial t} d\omega + J_{scv_1} \int_{scv_1} \frac{\partial u}{\partial t} d\omega + \\ &+ J_{scv_2} \int_{scv_2} \frac{\partial u}{\partial t} d\omega + J_{scv_3} \int_{scv_3} \frac{\partial u}{\partial t} d\omega \end{aligned} \quad (1.10)$$

For the discretization on a curvilinear grid we get:

$$\begin{aligned} J_{cv} \int_{cv} \frac{\partial u}{\partial t} d\omega &\approx J_{scv_0} \Delta t_{inv} \left(u_{i-\frac{1}{4}, j-\frac{1}{4}}^{n+1, p+1} - u_{i-\frac{1}{4}, j-\frac{1}{4}}^{n+1, n} \right) + \\ &J_{scv_1} \Delta t_{inv} \left(u_{i+\frac{1}{4}, j-\frac{1}{4}}^{n+1, p+1} - u_{i+\frac{1}{4}, j-\frac{1}{4}}^{n+1, n} \right) + \\ &J_{scv_2} \Delta t_{inv} \left(u_{i+\frac{1}{4}, j+\frac{1}{4}}^{n+1, p+1} - u_{i+\frac{1}{4}, j+\frac{1}{4}}^{n+1, n} \right) + \\ &J_{scv_3} \Delta t_{inv} \left(u_{i-\frac{1}{4}, j+\frac{1}{4}}^{n+1, p+1} - u_{i-\frac{1}{4}, j+\frac{1}{4}}^{n+1, n} \right) \end{aligned} \quad (1.11)$$

with J_{scv_i} the area of the sub control volume i . For cartesian grids we have $J_{scv_i} = \frac{1}{4} \Delta x \Delta y$.

Just looking to the quadrature point of scv_2 as part of the control volume for node (i, j) the discretization reads:

$$J_{scv_2} \Delta t_{inv} \left(u_{i+\frac{1}{4}, j+\frac{1}{4}}^{n+1} - u_{i+\frac{1}{4}, j+\frac{1}{4}}^n \right) = \quad (1.12)$$

$$= J_{scv_2} \Delta t_{inv} \left[\frac{1}{16} (9u_{i,j}^{n+1} + 3u_{i+1,j}^{n+1} + 3u_{i,j+1}^{n+1} + u_{i+1,j+1}^{n+1}) + \right. \quad (1.13)$$

$$\left. - \frac{1}{16} (9u_{i,j}^n + 3u_{i+1,j}^n + 3u_{i,j+1}^n + u_{i+1,j+1}^n) \right] \quad (1.14)$$

Written in Δ -formulation it yields (using $\Delta u^{n+1, p+1} = \Delta u$):

$$J_{scv_2} \Delta t_{inv} \left(u_{i+\frac{1}{4}, j+\frac{1}{4}}^{n+1} - u_{i+\frac{1}{4}, j+\frac{1}{4}}^n \right) = \quad (1.15)$$

$$= J_{scv_2} \Delta t_{inv} \left[\frac{1}{16} (9\Delta u_{i,j} + 3\Delta u_{i+1,j+1} + 3\Delta u_{i,j+1} + \Delta u_{i+1,j+1}) \right] +$$

$$+ J_{scv_2} \Delta t_{inv} \left[\frac{1}{16} \left(9(u_{i,j}^{n+1,p} - u_{i,j}^n) + 3(u_{i+1,j}^{n+1,p} - u_{i+1,j}^n) + 3(u_{i,j+1}^{n+1,p} - u_{i,j+1}^n) + (u_{i+1,j+1}^{n+1,p} - u_{i+1,j+1}^n) \right) \right] \quad (1.16)$$

with

$$J_{scv_2} = x_\xi y_\eta - y_\xi x_\eta \quad (1.17)$$

with J_{scv_2} representing the area of scv_2 , this area is computed by

$$J = \frac{1}{2} \sum_{i=0}^3 (x_i y_{i+1} - x_{i+1} y_i), \quad \text{with } x_4 = x_0 \text{ and } y_4 = y_0 \quad (1.18)$$

1.1.1.2 Viscosity

The viscosity term in vector notation reads:

$$\int_{\Omega} \nabla \cdot (\Psi \nabla u) d\omega = \oint_{\Omega} (\Psi \nabla u) \cdot \mathbf{n} dl = \quad (1.19)$$

$$= \oint_{\Omega} \left(\left(\Psi_{11} \frac{\partial u}{\partial x} + \Psi_{12} \frac{\partial u}{\partial y} \right) n_x + \left(\Psi_{21} \frac{\partial u}{\partial x} + \Psi_{22} \frac{\partial u}{\partial y} \right) n_y \right) \|dl\| \quad (1.20)$$

with $\mathbf{n} = (n_x, n_y)^T$ the outward normal vector.

For the heat-equation the diffusity term can be transformed as:

$$\frac{\partial}{\partial x} \left(\Psi_{11} \frac{\partial u}{\partial x} + \Psi_{12} \frac{\partial u}{\partial y} \right) + \frac{\partial}{\partial y} \left(\Psi_{21} \frac{\partial u}{\partial x} + \Psi_{22} \frac{\partial u}{\partial y} \right) \quad (1.21)$$

step 1

$$\frac{1}{J} \left[y_n \frac{\partial}{\partial \xi} \left(\Psi_{11} \frac{\partial u}{\partial x} + \Psi_{12} \frac{\partial u}{\partial y} \right) - y_\xi \frac{\partial}{\partial \eta} \left(\Psi_{11} \frac{\partial u}{\partial x} + \Psi_{12} \frac{\partial u}{\partial y} \right) \right] + \quad (1.22)$$

$$+ \frac{1}{J} \left[-x_\eta \frac{\partial}{\partial \xi} \left(\Psi_{21} \frac{\partial u}{\partial x} + \Psi_{22} \frac{\partial u}{\partial y} \right) + x_\xi \frac{\partial}{\partial \eta} \left(\Psi_{21} \frac{\partial u}{\partial x} + \Psi_{22} \frac{\partial u}{\partial y} \right) \right] \quad (1.23)$$

step 2

$$\frac{1}{J} \left[y_\eta \frac{\partial}{\partial \xi} \left(\Psi_{11} \left(y_\eta \frac{\partial u}{\partial \xi} - y_\xi \frac{\partial u}{\partial \eta} \right) + \Psi_{12} \left(-x_\eta \frac{\partial u}{\partial \xi} + x_\xi \frac{\partial u}{\partial \eta} \right) \right) + \right. \quad (1.24)$$

$$\left. - y_\xi \frac{\partial}{\partial \eta} \left(\Psi_{11} \left(y_\eta \frac{\partial u}{\partial \xi} - y_\xi \frac{\partial u}{\partial \eta} \right) + \Psi_{12} \left(-x_\eta \frac{\partial u}{\partial \xi} + x_\xi \frac{\partial u}{\partial \eta} \right) \right) \right] + \quad (1.25)$$

$$+ \frac{1}{J} \left[-x_\eta \frac{\partial}{\partial \xi} \left(\Psi_{21} \left(y_\eta \frac{\partial u}{\partial \xi} - y_\xi \frac{\partial u}{\partial \eta} \right) + \Psi_{22} \left(-x_\eta \frac{\partial u}{\partial \xi} + x_\xi \frac{\partial u}{\partial \eta} \right) \right) + \right. \quad (1.26)$$

$$\left. + x_\xi \frac{\partial}{\partial \eta} \left(\Psi_{21} \left(y_\eta \frac{\partial u}{\partial \xi} - y_\xi \frac{\partial u}{\partial \eta} \right) + \Psi_{22} \left(-x_\eta \frac{\partial u}{\partial \xi} + x_\xi \frac{\partial u}{\partial \eta} \right) \right) \right] \quad (1.27)$$

$$\frac{1}{J} \left[y_\eta y_{\xi\eta} \Psi_{11} \frac{\partial u}{\partial \xi} + y_\eta y_\eta \frac{\partial}{\partial \xi} \left(\Psi_{11} \frac{\partial u}{\partial \xi} \right) - y_\eta y_{\xi\xi} \Psi_{11} \frac{\partial u}{\partial \eta} - y_\eta y_\xi \frac{\partial}{\partial \xi} \left(\Psi_{11} \frac{\partial u}{\partial \eta} \right) + \right. \quad (1.28)$$

$$\left. - y_\eta x_{\xi\eta} \Psi_{12} \frac{\partial u}{\partial \xi} - y_\eta x_\eta \frac{\partial}{\partial \xi} \left(\Psi_{12} \frac{\partial u}{\partial \xi} \right) + y_\eta x_{\xi\xi} \Psi_{12} \frac{\partial u}{\partial \eta} + y_\eta x_\xi \frac{\partial}{\partial \xi} \left(\Psi_{12} \frac{\partial u}{\partial \eta} \right) + \right. \quad (1.29)$$

$$\left. - y_\xi y_{\eta\eta} \Psi_{11} \frac{\partial u}{\partial \xi} - y_\xi y_\eta \frac{\partial}{\partial \eta} \left(\Psi_{11} \frac{\partial u}{\partial \xi} \right) + y_\xi y_{\eta\xi} \Psi_{11} \frac{\partial u}{\partial \eta} + y_\xi y_\xi \frac{\partial}{\partial \eta} \left(\Psi_{11} \frac{\partial u}{\partial \eta} \right) + \right. \quad (1.30)$$

$$\left. + y_\xi x_{\eta\eta} \Psi_{12} \frac{\partial u}{\partial \xi} + y_\xi x_\eta \frac{\partial}{\partial \eta} \left(\Psi_{12} \frac{\partial u}{\partial \xi} \right) - y_\xi x_{\eta\xi} \Psi_{12} \frac{\partial u}{\partial \eta} - y_\xi x_\xi \frac{\partial}{\partial \eta} \left(\Psi_{12} \frac{\partial u}{\partial \eta} \right) \right] + \quad (1.31)$$

$$\frac{1}{J} \left[-x_\eta y_{\xi\eta} \Psi_{21} \frac{\partial u}{\partial \xi} - x_\eta y_\eta \frac{\partial}{\partial \xi} \left(\Psi_{21} \frac{\partial u}{\partial \xi} \right) + x_\eta y_{\xi\xi} \Psi_{21} \frac{\partial u}{\partial \eta} + x_\eta y_\xi \frac{\partial}{\partial \xi} \left(\Psi_{21} \frac{\partial u}{\partial \eta} \right) + \right. \quad (1.32)$$

$$\left. + x_\eta x_{\xi\eta} \Psi_{22} \frac{\partial u}{\partial \xi} + x_\eta x_\eta \frac{\partial}{\partial \xi} \left(\Psi_{22} \frac{\partial u}{\partial \xi} \right) - x_\eta x_{\xi\xi} \Psi_{22} \frac{\partial u}{\partial \eta} - x_\eta x_\xi \frac{\partial}{\partial \eta} \left(\Psi_{22} \frac{\partial u}{\partial \eta} \right) + \right. \quad (1.33)$$

$$\left. + x_\xi y_{\eta\eta} \Psi_{21} \frac{\partial u}{\partial \xi} + x_\xi y_\eta \frac{\partial}{\partial \eta} \left(\Psi_{21} \frac{\partial u}{\partial \xi} \right) - x_\xi y_{\eta\xi} \Psi_{21} \frac{\partial u}{\partial \eta} - x_\xi y_\xi \frac{\partial}{\partial \eta} \left(\Psi_{21} \frac{\partial u}{\partial \eta} \right) + \right. \quad (1.34)$$

$$\left. - x_\xi x_{\eta\eta} \Psi_{22} \frac{\partial u}{\partial \xi} - x_\xi x_\eta \frac{\partial}{\partial \eta} \left(\Psi_{22} \frac{\partial u}{\partial \xi} \right) + x_\xi x_{\eta\xi} \Psi_{22} \frac{\partial u}{\partial \eta} + x_\xi x_\xi \frac{\partial}{\partial \eta} \left(\Psi_{22} \frac{\partial u}{\partial \eta} \right) \right] \quad (1.35)$$

Integration of the term for the eight quadrature points qp , for each face of the control volume scv we get:

$$\left(\Psi_{11} \frac{\partial u}{\partial x} + \Psi_{12} \frac{\partial u}{\partial y} \right) n_x \|dl\| + \left(\Psi_{21} \frac{\partial u}{\partial x} + \Psi_{22} \frac{\partial u}{\partial y} \right) n_y \|dl\| \quad (1.36)$$

If the viscosity is isotropic ($\Psi_{12} = \Psi_{21} = 0$) these equations reduce to:

$$\left(\Psi_{11} \frac{\partial u}{\partial x} \right) n_x \|dl\| + \left(\Psi_{22} \frac{\partial u}{\partial y} \right) n_y \|dl\| \quad (1.37)$$

Define the components of Ψ at different locations with $(\mu, \nu) \in \{1, 2\}$ as

$$\bar{\Psi}_{\mu\nu} \Big|_{i+\frac{1}{2}, j+\frac{1}{4}} = \frac{1}{8} \left(3\Psi_{\mu\nu} \Big|_{i,j} + 3\Psi_{\mu\nu} \Big|_{i+1,j} + \Psi_{\mu\nu} \Big|_{i+1,j+1} + \Psi_{\mu\nu} \Big|_{i,j+1} \right) \quad (1.38)$$

and define the partial differentials as

$$\frac{\partial u}{\partial x} \Big|_{i+\frac{1}{2}, j+\frac{1}{4}} = \frac{1}{4} \left(3 \frac{u_{i+1,j} - u_{i,j}}{\Delta x_{i+\frac{1}{2},j}} + \frac{u_{i+1,j+1} - u_{i,j+1}}{\Delta x_{i+\frac{1}{2},j+1}} \right) \quad (1.39)$$

$$\frac{\partial u}{\partial y} \Big|_{i+\frac{1}{2}, j+\frac{1}{4}} = \frac{1}{2} \left(\frac{u_{i,j+1} - u_{i,j}}{\Delta y_{i,j+\frac{1}{2}}} + \frac{u_{i+1,j+1} - u_{i+1,j}}{\Delta y_{i+1,j+\frac{1}{2}}} \right) \quad (1.40)$$

In case the viscosity is isotropic it reads:

$$\Psi = \begin{pmatrix} \Psi & 0 \\ 0 & \Psi \end{pmatrix} \quad (1.41)$$

(isotropic, $\Psi_{11} = \Psi_{22} = \Psi$ and $\Psi_{12} = \Psi_{21} = 0$). and then the discretization is straight forward.

A Curvilinear coordinate transformation

A.0.1 Viscosity term

The x and y -momentum equation diffusion terms are treated here as well due to their complexity. They can be transformed as:

$$\frac{\partial}{\partial x} \left(2\nu h \frac{\partial(q/h)}{\partial x} \right) = \quad (\text{A.1})$$

$$= \frac{1}{J} \left[y_\eta \frac{\partial}{\partial \xi} \left(\frac{2\nu h}{J} \left(y_\eta \frac{\partial(q/h)}{\partial \xi} - y_\xi \frac{\partial(q/h)}{\partial \eta} \right) \right) - y_\xi \frac{\partial}{\partial \eta} \left(\frac{2\nu h}{J} \left(y_\eta \frac{\partial(q/h)}{\partial \xi} - y_\xi \frac{\partial(q/h)}{\partial \eta} \right) \right) \right] = \quad (\text{A.2})$$

$$= \frac{1}{J} \left[y_\eta \left(\frac{\partial}{\partial \xi} \left(\frac{2\nu h}{J} \right) \left(y_\eta \frac{\partial(q/h)}{\partial \xi} - y_\xi \frac{\partial(q/h)}{\partial \eta} \right) + \frac{2\nu h}{J} \frac{\partial}{\partial \xi} \left(y_\eta \frac{\partial(q/h)}{\partial \xi} - y_\xi \frac{\partial(q/h)}{\partial \eta} \right) \right) + - y_\xi \left(\frac{\partial}{\partial \eta} \left(\frac{2\nu h}{J} \right) \left(y_\eta \frac{\partial(q/h)}{\partial \xi} - y_\xi \frac{\partial(q/h)}{\partial \eta} \right) + \frac{2\nu h}{J} \frac{\partial}{\partial \eta} \left(y_\eta \frac{\partial(q/h)}{\partial \xi} - y_\xi \frac{\partial(q/h)}{\partial \eta} \right) \right) \right] = \quad (\text{A.3})$$

$$= \frac{2}{J} \left[y_\eta \left(\frac{\partial}{\partial \xi} \left(\frac{\nu h}{J} \right) \left(y_\eta \frac{\partial(q/h)}{\partial \xi} - y_\xi \frac{\partial(q/h)}{\partial \eta} \right) + \frac{\nu h}{J} \frac{\partial}{\partial \xi} \left(y_\eta \frac{\partial(q/h)}{\partial \xi} - y_\xi \frac{\partial(q/h)}{\partial \eta} \right) \right) + - y_\xi \left(\frac{\partial}{\partial \eta} \left(\frac{\nu h}{J} \right) \left(y_\eta \frac{\partial(q/h)}{\partial \xi} - y_\xi \frac{\partial(q/h)}{\partial \eta} \right) + \frac{\nu h}{J} \frac{\partial}{\partial \eta} \left(y_\eta \frac{\partial(q/h)}{\partial \xi} - y_\xi \frac{\partial(q/h)}{\partial \eta} \right) \right) \right] \quad (\text{A.4})$$

$$- \frac{\partial}{\partial x} \left(2\nu h \frac{\partial(q/h)}{\partial x} \right) - \frac{\partial}{\partial y} \left(\nu h \frac{\partial(r/h)}{\partial x} + \nu h \frac{\partial(q/h)}{\partial y} \right) = \quad (\text{A.5})$$

Email: jan.mooiman@outlook.com
[GitHub Mooiman](#)