Two step FVE method

A numerical modeling technique designed for error insight

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Authors

Jan Mooiman

Partners

Members of the team (dd. 22 May 2024)

Adri Mourits

Bea Saggiorato

Frank Platzek

Jan Mooiman

Mart Borsboom

Sven Westerbeek

Thea Vuik

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Symbol	Unit	Description
Δt	s	Time increment
Δx	m	Space increment, $\Delta x_{i+\frac{1}{2}} = x_{i+1} - x_i$
ε_ζ	$\mathrm{m\,s^{-2}}$	Multiplier of the correction term for the essential boundary condition, ζ-boundary
ε_q	$\mathrm{s^{-1}}$	Multiplier of the correction term for the essential boundary condition, q-boundary
μ	m	Centre of the Gaussian hump
ν	$\mathrm{m^2\,s^{-1}}$	Kinematic viscosity
Ω	$-$	Finite volume
Ψ	$\mathrm{m^2\,s^{-1}}$	Artificial smoothing coefficient
σ	m	Spreading of the Gaussian hump
θ	$-$	θ-method. If $\theta = 1$ then it is a fully implicit method and if $\theta = 0$ then it is a fully explicit method.
$\boldsymbol{E}$	$-$	Error vector function, defined in computational space
ξ	$-$	Relative coordinate
ζ	m	Water level w.r.t. reference plane, positive upward
C	$\mathrm{m^{\frac{1}{2}}\,s^{-1}}$	Chézy coefficient
c_Ψ	$(.)^{-1}$	Artificial smoothing variable
c_f	$-$	Bed shear stress coefficient
g	$\mathrm{m\,s^{-2}}$	Gravitational constant
h	m	Total water depth
i	$-$	node counter
q	$\mathrm{m^2\,s^{-1}}$	The water flux in x-direction, $q = hu$
r	$\mathrm{m^2\,s^{-1}}$	The water flux in y-direction, $r = hv$
t	s	Time coordinate
u	$\mathrm{m\,s^{-1}}$	Velocity in x-direction
v	$\mathrm{m\,s^{-1}}$	Velocity in y-direction
x	m	x-coordinate
y	m	y-coordinate
z_b	m	Bed level w.r.t. reference plane, positive upward

1 2D Shallow water equations

Consider the non-linear wave equation

$$\frac{\partial h}{\partial t} + \nabla \cdot \boldsymbol{q} = 0, \tag{1.1a}$$

$$\frac{\partial \boldsymbol{q}}{\partial t} + \nabla \cdot \left(\frac{\boldsymbol{q}\boldsymbol{q}^T}{h} \right) + gh\nabla\zeta + \\ + c_f \left(\frac{\boldsymbol{q}\,|\boldsymbol{q}|}{h^2} \right) - \nabla \cdot \left(\nu h \left(\nabla \boldsymbol{q}/h + \nabla \boldsymbol{q}^T/h \right) \right) = 0, \tag{1.1b}$$

$$\zeta = h + z_b, \tag{1.1c}$$

with

ζ	Water level w.r.t. reference plane ($\zeta = h + z_b$), [m].
h	Water depth ($h = \zeta - z_b$), [m].
z_b	Bed level w.r.t. reference plane, [m].
$\boldsymbol{q}$	Flow, defined as $\boldsymbol{q} = (q, r)^T = (hu, hv)^T$, $[\mathrm{m^2\,s^{-1}}]$.
$\boldsymbol{u}$	Velocity vector, defined as $\boldsymbol{u} = (u, v)^T$, $[\mathrm{m\,s^{-1}}]$.
c_f	Bed shear stress coefficient, [−].
	Chézy: $c_f = g/C^2$
C	Chézy coefficient, $[\mathrm{m^{\frac{1}{2}}\,s^{-1}}]$.
g	Acceleration due to gravity, $[\mathrm{m\,s^{-2}}]$.
ν	Kinematic viscosity, $[\mathrm{m^2\,s^{-1}}]$.

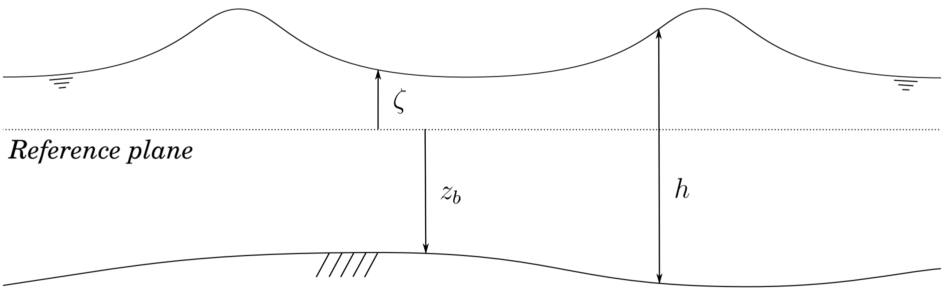

Figure 1.1: *Definition sketch of water level (ζ), bed level (z_b) and total water depth (h)*

Finite Volume approach

Integrating the equations over a finite volume Ω yields:

$$\int_\Omega \frac{\partial h}{\partial t}\, d\omega + \int_\Omega \nabla \cdot \boldsymbol{q}\, d\omega = 0, \tag{1.2a}$$

$$\int_\Omega \frac{\partial \boldsymbol{q}}{\partial t}\, d\omega + \int_\Omega \nabla \cdot \left(\frac{\boldsymbol{q}\boldsymbol{q}^T}{h}\right) d\omega + \int_\Omega gh\nabla\zeta\, d\omega + \\ + \int_\Omega c_f \left(\frac{\boldsymbol{q}\,|\boldsymbol{q}|}{h^2}\right) d\omega - \int_\Omega \nabla \cdot \left(\nu h \left(\nabla \boldsymbol{q}/h + \nabla \boldsymbol{q}^T/h\right)\right) d\omega = 0, \tag{1.2b}$$

$$\int_\Omega \zeta\, d\omega = \int_\Omega h\, d\omega + \int_\Omega z_b\, d\omega, \tag{1.2c}$$

The 'linear' shallow water equations in curvilinear coordinates read:

$$J\frac{\partial h}{\partial t} + \frac{\partial}{\partial \xi}\left(y_\eta\, q - x_\eta\, r\right) + \frac{\partial}{\partial \eta}\left(-y_\xi\, q + x_\xi\, r\right) = 0, \tag{1.3}$$

$$J\frac{\partial q}{\partial t} + gh\left(y_\eta \frac{\partial \zeta}{\partial \xi} - y_\xi \frac{\partial \zeta}{\partial \eta}\right) = 0, \qquad \textit{q-momentum eq.} \tag{1.4}$$

$$J\frac{\partial r}{\partial t} + gh\left(-x_\eta \frac{\partial \zeta}{\partial \xi} + x_\xi \frac{\partial \zeta}{\partial \eta}\right) = 0, \qquad \textit{r-momentum eq.} \tag{1.5}$$

1.1 Space discretization, structured

For the space discretizations of an arbitrary function u on the quadrature point of a sub-control volume the following space interpolations are used, $u \in \{h, q, r\}$:

$$u|_{i+\frac{1}{4},j+\frac{1}{4}} \approx \frac{1}{16}\left(9u_{i,j} + 3u_{i+1,j} + u_{i+1,j+1} + 3u_{i,j+1}\right) \tag{1.6}$$

$$u|_{i+\frac{1}{2},j+\frac{1}{4}} \approx \frac{1}{8}\left(3u_{i,j} + 3u_{i+1,j} + u_{i+1,j+1} + u_{i,j+1}\right) \tag{1.7}$$

$$u|_{i+\frac{1}{4},j+\frac{1}{2}} \approx \frac{1}{8}\left(3u_{i,j} + u_{i+1,j+1} + u_{i+1,j} + 3u_{i,j+1}\right) \tag{1.8}$$

See for the locations Figure 1.2.

1.1.1 Discretizations continuity equation

The discretization of continuity equation (1.3) will be presented term by term.

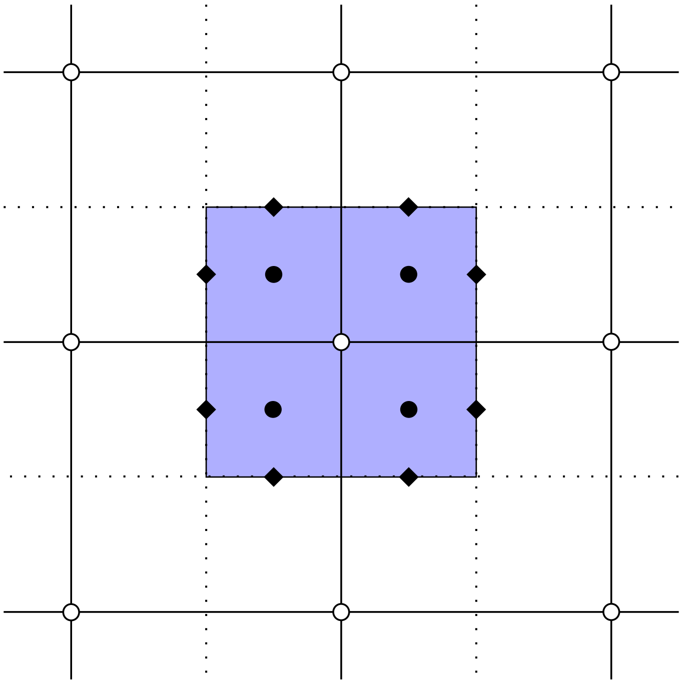

Figure 1.2: *Definition of the grid to solve the 2D-shallow water equations in the interior area. The black dots indicate the location of the quadrature points, and black diamonds the flux points.*

1.1.1.1 Time derivative

The discretization of the time derivative term of the continuity equation reads:

$$J \int_{\Omega} \frac{\partial h}{\partial t}\, d\omega \tag{1.9}$$

which will be approximated by the sum of the integral over the sub-control volumes. On a structured grid one control volume (cv) around a node consist of four sub-control volumes (scv_i, $i \in \{0, 1, 2, 3\}$).

$$\begin{aligned} J_{cv} \int_{cv} \frac{\partial h}{\partial t}\, d\omega = J_{scv_0} \int_{scv_0} \frac{\partial h}{\partial t}\, d\omega + J_{scv_1} \int_{scv_1} \frac{\partial h}{\partial t}\, d\omega + \\ + J_{scv_2} \int_{scv_2} \frac{\partial h}{\partial t}\, d\omega + J_{scv_3} \int_{scv_3} \frac{\partial h}{\partial t}\, d\omega \end{aligned} \tag{1.10}$$

For a curvilinear grid we get:

$$\begin{aligned} J_{cv} \int_{cv} \frac{\partial h}{\partial t}\, d\omega \approx & J_{scv_0} \Delta t_{inv} \left(h^{n+1,p+1}_{i-\frac{1}{4},j-\frac{1}{4}} - h^{n+1,n}_{i-\frac{1}{4},j-\frac{1}{4}} \right) + \\ & J_{scv_1} \Delta t_{inv} \left(h^{n+1,p+1}_{i+\frac{1}{4},j-\frac{1}{4}} - h^{n+1,n}_{i+\frac{1}{4},j-\frac{1}{4}} \right) + \\ & J_{scv_2} \Delta t_{inv} \left(h^{n+1,p+1}_{i+\frac{1}{4},j+\frac{1}{4}} - h^{n+1,n}_{i+\frac{1}{4},j+\frac{1}{4}} \right) + \\ & J_{scv_3} \Delta t_{inv} \left(h^{n+1,p+1}_{i-\frac{1}{4},j+\frac{1}{4}} - h^{n+1,n}_{i-\frac{1}{4},j+\frac{1}{4}} \right) \end{aligned} \tag{1.11}$$

with J_{scv_i} the area of the sub control volume i. For cartesian coordinates we have $J_{scv_i} = \frac{1}{4}\Delta x \Delta y$.

Just looking to the quadrature point of scv_2 as part of the control volume for node (i, j) the discretization reads:

$$J_{scv_2} \Delta t_{inv} \left(h^{n+1}_{i+\frac{1}{4},j+\frac{1}{4}} - h^{n+1,n}_{i+\frac{1}{4},j+\frac{1}{4}} \right) = \tag{1.12}$$

$$= J_{scv_2} \Delta t_{inv} \left[\frac{1}{16} \left(9h^{n+1}_{i,j} + 3h^{n+1}_{i+1,j} + 3h^{n+1}_{i,j+1} + h^{n+1}_{i+1,j+1} \right) + \right. \tag{1.13}$$

$$\left. - \frac{1}{16} \left(9h^{n}_{i,j} + 3h^{n}_{i+1,j} + 3h^{n}_{i,j+1} + h^{n}_{i+1,j+1} \right) \right] \tag{1.14}$$

Written in Δ-formulation it reads:

$$\begin{aligned} &J_{scv_2} \Delta t_{inv} \left(h^{n+1}_{i+\frac{1}{4},j+\frac{1}{4}} - h^{n}_{i+\frac{1}{4},j+\frac{1}{4}} \right) = \\ &= J_{scv_2} \Delta t_{inv} \left[\frac{1}{16} \left(9\Delta h^{n+1,p+1}_{i,j} + 3\Delta q^{n+1,p+1}_{i+1,j+1} + 3\Delta q^{n+1,p+1}_{i,j+1} + \Delta h^{n+1,p+1}_{i+1,j+1} \right) \right] + \\ &+ J_{scv_2} \Delta t_{inv} \left[\frac{1}{16} \left(9h^{n+1,p}_{i,j} + 3h^{n+1,p}_{i+1,j} + 3h^{n+1,p}_{i,j+1} + h^{n+1,p+1}_{i+1,j+1} \right) + \right. \\ &\quad \left. - \frac{1}{16} \left(9h^{n}_{i,j} + 3h^{n}_{i+1,j} + 3h^{n}_{i,j+1} + h^{n}_{i+1,j+1} \right) \right] \end{aligned} \tag{1.15}$$

with

$$J_{scv_2} = x_\xi y_\eta - y_\xi x_\eta \tag{1.16}$$

where x_ξ, y_η, y_ξ and x_η at the quadrature point qp are

$$x_{\xi_{qp}} = \frac{1}{4}(3(x_{i+1,j} - x_{i,j}) + 1(x_{i+1,j+1} - x_{i,j+1})) \tag{1.17}$$

$$y_{\eta_{qp}} = \frac{1}{4}(3(y_{i,j+1} - y_{i,j}) + 1(y_{i+1,j+1} - y_{i+1,j})) \tag{1.18}$$

$$y_{\xi_{qp}} = \frac{1}{4}(3(y_{i+1,j} - y_{i,j}) + 1(y_{i+1,j+1} - y_{i,j+1})) \tag{1.19}$$

$$x_{\eta_{qp}} = \frac{1}{4}(3(x_{i,j+1} - x_{i,j}) + 1(x_{i+1,j+1} - x_{i+1,j})) \tag{1.20}$$

1.1.1.2 Mass flux

The discretization of the mass flux term of the continuity equation

$$\oint_{\partial\Omega_{\xi\eta}} \left((y_\eta q - x_\eta r)\, n_\xi + (-y_\xi q + x_\xi r)\, n_\eta \right) dl \tag{1.21}$$

will be given is this section (equation (A.25), multiplied by J). Where $\boldsymbol{n} = (n_\xi, n_\eta)^T$ is the outward normal vector.

The linearization in time reads:

$$\begin{aligned} \oint_{\partial\Omega_{\xi\eta}} \Big[&\left(y_\eta \left(q^{n+\theta,p} + \theta\Delta q \right) - x_\eta \left(r^{n+\theta,p} + \theta\Delta r \right) \right) n_\xi \\ &+ \left(-y_\xi \left(r^{n+\theta,p} + \theta\Delta r \right) + x_\xi \left(r^{n+\theta,p} + \theta\Delta r \right) \right) n_\eta \Big] dl \end{aligned} \tag{1.22}$$

This terms need to be computed for each of the control volume faces, taken into account the outward normal. Eight faces on a structured grid. For the quadrature point $(i+\frac{1}{2}, j+\frac{1}{4})$ where $\boldsymbol{n}=(1,0)^T$ and $\|dl\|=\frac{1}{2}$, reads:

$$\left(y_\eta\left(q^{n+\theta,p}+\theta\Delta q\right)-x_\eta\left(r^{n+\theta,p}+\theta\Delta r\right)\right)n_\xi\,\|dl\|\approx \tag{1.23}$$

$$\approx\left[\frac{1}{2}y_\eta\left(q^{n+\theta,p}+\theta\Delta q\right)-\frac{1}{2}x_\eta\left(r^{n+\theta,p}+\theta\Delta r\right)\right]n_\xi \tag{1.24}$$

with

$$y_\eta\big|_{i+\frac{1}{2},j+\frac{1}{4}}=\frac{1}{2}\left(y_{i,j+1}-y_{i,j}\right)+\frac{1}{2}\left(y_{i+1,j+1}-y_{i+1,j}\right) \tag{1.25}$$

$$x_\eta\big|_{i+\frac{1}{2},j+\frac{1}{4}}=\frac{1}{4}\left(3\left(x_{i+1,j}-x_{i,j}\right)+\left(x_{i+1,j+1}-x_{i,j+1}\right)\right) \tag{1.26}$$

and for terms related to q we get (similar for r):

$$q\big|^{n+\theta,p}_{i+\frac{1}{2},j+\frac{1}{4}}=\frac{1}{8}\left(3q^{n+\theta,p}_{i,j}+3q^{n+\theta,p}_{i+1,j}+1q^{n+\theta,p}_{i,j+1}+1q^{n+\theta,p}_{i+1,j+1}\right) \tag{1.27}$$

$$\theta\Delta q\big|_{i+\frac{1}{2},j+\frac{1}{4}}=\theta\frac{1}{8}\left(3\Delta q_{i,j}+3\Delta q_{i+1,j}+1\Delta q_{i,j+1}+1\Delta q_{i+1,j+1}\right) \tag{1.28}$$

Equation (1.27) is part of the right hand side and equation (1.28) is a part of the matrix coefficients.

1.1.2 Discretizations momentum equations

The discretization of momentum equation (1.2b) will be presented term by term.

1.1.2.1 Time derivative

The discretization of the time derivative term of the momentum equation is only shown for the q-momentum equation, the time derivative for the r-momentum equation is similar. The time derivative for the q-momentum equation reads:

$$J\int_\Omega\frac{\partial q}{\partial t}\,d\omega \tag{1.29}$$

which will be approximated by the sum of the integral over the sub-control volumes. On a structured grid one control volume (cv) around a node consist of four sub-control volumes (scv_i, $i\in\{0,1,2,3\}$). For a curvilinear grid we get:

$$\begin{aligned}\int_{cv}\frac{\partial q}{\partial t}\,d\omega\approx & J_{scv_0}\Delta t_{inv}\left(q^{n+1}_{i-\frac{1}{4},j-\frac{1}{4}}-q^{n}_{i-\frac{1}{4},j-\frac{1}{4}}\right)+\\ & J_{scv_1}\Delta t_{inv}\left(q^{n+1}_{i+\frac{1}{4},j-\frac{1}{4}}-q^{n}_{i+\frac{1}{4},j-\frac{1}{4}}\right)+\\ & J_{scv_3}\Delta t_{inv}\left(q^{n+1}_{i+\frac{1}{4},j+\frac{1}{4}}-q^{n}_{i+\frac{1}{4},j+\frac{1}{4}}\right)+\\ & J_{scv_3}\Delta t_{inv}\left(q^{n+1}_{i-\frac{1}{4},j+\frac{1}{4}}-q^{n}_{i-\frac{1}{4},j+\frac{1}{4}}\right)\end{aligned} \tag{1.30}$$

with J_{scv_i} the area of the sub control volume i, for cartesian coordinates we have $J_{scv_i} = \frac{1}{4}\Delta x \Delta y$.

Just looking to the quadrature point of scv_2 as part of the control volume for node (i, j) the discretization reads:

$$J_{scv_2}\Delta t_{inv}\left(q^{n+1}_{i+\frac{1}{4},j+\frac{1}{4}} - q^{n+1,n}_{i+\frac{1}{4},j+\frac{1}{4}}\right) = \tag{1.31}$$

$$= J_{scv_2}\Delta t_{inv}\left[\frac{1}{16}\left(9q^{n+1}_{i,j} + 3q^{n+1}_{i+1,j} + 3q^{n+1}_{i,j+1} + q^{n+1}_{i+1,j+1}\right) + \right. \tag{1.32}$$

$$\left. - \frac{1}{16}\left(9q^{n}_{i,j} + 3q^{n}_{i+1,j} + 3q^{n}_{i,j+1} + q^{n}_{i+1,j+1}\right)\right] \tag{1.33}$$

Written in Δ-formulation it reads:

$$\begin{aligned}
&J_{scv_2}\Delta t_{inv}\left(q^{n+1}_{i+\frac{1}{4},j+\frac{1}{4}} - q^{n}_{i+\frac{1}{4},j+\frac{1}{4}}\right) = \\
&= J_{scv_2}\Delta t_{inv}\left[\frac{1}{16}\left(9\Delta q^{n+1,p+1}_{i,j} + 3\Delta q^{n+1,p+1}_{i+1,j+1} + 3\Delta q^{n+1,p+1}_{i,j+1} + \Delta q^{n+1,p+1}_{i+1,j+1}\right)\right] + \\
&+ J_{scv_2}\Delta t_{inv}\left[\frac{1}{16}\left(9q^{n+1,p}_{i,j} + 3q^{n+1,p}_{i+1,j} + 3q^{n+1,p}_{i,j+1} + q^{n+1,p+1}_{i+1,j+1}\right) + \right. \\
&\quad \left. - \frac{1}{16}\left(9q^{n}_{i,j} + 3q^{n}_{i+1,j} + 3q^{n}_{i,j+1} + q^{n}_{i+1,j+1}\right)\right]
\end{aligned} \tag{1.34}$$

with

$$J_{scv_2} = x_\xi y_\eta - y_\xi x_\eta \tag{1.35}$$

where x_ξ, y_η, y_ξ and x_η at the quadrature point qp are

$$x_{\xi_{qp}} = \frac{1}{4}(3(x_{i+1,j} - x_{i,j}) + 1(x_{i+1,j+1} - x_{i,j+1})) \tag{1.36}$$

$$y_{\eta_{qp}} = \frac{1}{4}(3(y_{i,j+1} - y_{i,j}) + 1(y_{i+1,j+1} - y_{i+1,j})) \tag{1.37}$$

$$y_{\xi_{qp}} = \frac{1}{4}(3(y_{i+1,j} - y_{i,j}) + 1(y_{i+1,j+1} - y_{i,j+1})) \tag{1.38}$$

$$x_{\eta_{qp}} = \frac{1}{4}(3(x_{i,j+1} - x_{i,j}) + 1(x_{i+1,j+1} - x_{i+1,j})) \tag{1.39}$$

1.1.2.2 Pressure term

In this section we use that the pressure term is dependent on ζ. The discretization of the pressure term

$$gh\left(y_\eta\frac{\partial\zeta}{\partial\xi} - y_\xi\frac{\partial\zeta}{\partial\eta}\right), \qquad q\textit{-momentum eq.} \tag{1.40}$$

$$gh\left(-x_\eta\frac{\partial\zeta}{\partial\xi} + x_\xi\frac{\partial\zeta}{\partial\eta}\right). \qquad r\textit{-momentum eq.} \tag{1.41}$$

will be given in this section (multiplied by J). The integral over a control volume will be the sum of integrals over the sub control volumes. On a structured grid one control volume (cv) around a node consist of four sub-control volumes (scv_i, $i \in \{0,1,2,3\}$). Thus

$$gh\nabla\zeta\big|_{sc} = gh\nabla\zeta\big|_{scv_0} + gh\nabla\zeta\big|_{scv_1} + gh\nabla\zeta\big|_{scv_2} + gh\nabla\zeta\big|_{scv_3} \tag{1.42}$$

q-momentum

Considering one sub-control volume and only for the x-direction (similar for the y-direction) then the linearization in time reads:

$$gh\left(y_\eta \frac{\partial\zeta}{\partial\xi} - y_\xi \frac{\partial\zeta}{\partial\eta}\right) \approx \tag{1.43}$$

$$\approx g\left(h_{qp}^{n+\theta,p} + \theta\Delta h_{qp}\right) \times \tag{1.44}$$

$$\times\left(y_{\eta_{qp}} \frac{\partial}{\partial\xi}\left(\zeta_{qp}^{n+\theta,p} + \theta\Delta\zeta_{qp}\right) - y_{\xi_{qp}} \frac{\partial}{\partial\eta}\left(\zeta_{qp}^{n+\theta,p} + \theta\Delta\zeta_{qp}\right)\right) \tag{1.45}$$

with qp the location of the quadrature point in the sub-control volume. Assume that the higher order terms are negligible then the discretization for each of the 4 sub-control volumes reads:

$$gh_{qp}^{n+\theta,p}\left(y_{\eta_{qp}} \frac{\partial}{\partial\xi}\left(\zeta_{qp}^{n+\theta,p}\right) - y_{\xi_{qp}} \frac{\partial}{\partial\eta}\left(\zeta_{qp}^{n+\theta,p}\right)\right) + \tag{1.46}$$

$$+\, gh_{qp}^{n+\theta,p}\left(y_{\eta_{qp}} \frac{\partial}{\partial\xi}\left(\theta\Delta\zeta_{qp}\right) - y_{\xi_{qp}} \frac{\partial}{\partial\eta}\left(\theta\Delta\zeta_{qp}\right)\right) + \tag{1.47}$$

$$+\, g\left(y_{\eta_{qp}} \frac{\partial}{\partial\xi}\left(\zeta_{qp}^{n+\theta,p}\right) - y_{\xi_{qp}} \frac{\partial}{\partial\eta}\left(\zeta_{qp}^{n+\theta,p}\right)\right)\theta\Delta h_{qp} \tag{1.48}$$

Just looking to the quadrature point of scv_2 as part of the control volume for node (i,j) the discretization reads (term by term):

$$h_{i+\frac{1}{4},j+\frac{1}{4}}^{n+\theta,p} \approx \frac{1}{16}\left(9h_{i,j}^{n+\theta,p} + 3h_{i+1,j}^{n+\theta,p} + h_{i+1,j+1}^{n+\theta,p} + 3h_{i,j+1}^{n+\theta,p}\right) \tag{1.49}$$

$$\frac{\partial\zeta_{i+\frac{1}{4},j+\frac{1}{4}}^{n+\theta,p}}{\partial\xi} \approx \left[3\left(\zeta_{i+1,j} - \zeta_{i,j}\right) + \left(\zeta_{i+1,j+1} - \zeta_{i,j+1}\right)\right] \tag{1.50}$$

$$\frac{\partial\zeta_{i+\frac{1}{4},j+\frac{1}{4}}^{n+\theta,p}}{\partial\eta} \approx \left[3\left(\zeta_{i,j+1} - \zeta_{i,j}\right) + \left(\zeta_{i+1,j+1} - \zeta_{i+1,j}\right)\right] \tag{1.51}$$

$$\frac{\partial\Delta\zeta_{i+\frac{1}{4},j+\frac{1}{4}}^{n+1,p+1}}{\partial\xi} \approx \frac{1}{4}\left[3\left(\Delta\zeta_{i+1,j} - \Delta\zeta_{i,j}\right) + \left(\Delta\zeta_{i+1,j+1} - \Delta\zeta_{i,j+1}\right)\right] \tag{1.52}$$

$$\frac{\partial\Delta\zeta_{i+\frac{1}{4},j+\frac{1}{4}}^{n+1,p+1}}{\partial\eta} \approx \frac{1}{4}\left[3\left(\Delta\zeta_{i,j+1} - \Delta\zeta_{i,j}\right) + \left(\Delta\zeta_{i+1,j+1} - \Delta\zeta_{i+1,j}\right)\right] \tag{1.53}$$

$$\Delta h_{i+\frac{1}{4},j+\frac{1}{4}}^{n+1,p+1} \approx \frac{1}{16}\left(9\Delta h_{i,j} + 3\Delta h_{i+1,j} + \Delta h_{i+1,j+1} + 3\Delta h_{i,j+1}\right) \tag{1.54}$$

r-momentum

For the r-momentum equation:

$$gh\left(-x_\eta\frac{\partial\zeta}{\partial\xi}+x_\xi\frac{\partial\zeta}{\partial\eta}\right)\approx \tag{1.55}$$

$$\approx g\left(h_{qp}^{n+\theta,p}+\theta\Delta h_{qp}\right)\times \tag{1.56}$$

$$\times\left(-x_{\eta_{qp}}\frac{\partial}{\partial\xi}\left(\zeta_{qp}^{n+\theta,p}+\theta\Delta\zeta_{qp}\right)+x_{\xi_{qp}}\frac{\partial}{\partial\eta}\left(\zeta_{qp}^{n+\theta,p}+\theta\Delta\zeta_{qp}\right)\right) \tag{1.57}$$

with qp the location of the quadrature point in the sub-control volume. The discretization for each of the 4 sub-control volumes reads:

$$gh_{qp}^{n+\theta,p}\left(-x_{\eta_{qp}}\frac{\partial}{\partial\xi}\left(\zeta_{qp}^{n+\theta,p}\right)+x_{\xi_{qp}}\frac{\partial}{\partial\eta}\left(\zeta_{qp}^{n+\theta,p}\right)\right)+ \tag{1.58}$$

$$+gh_{qp}^{n+\theta,p}\left(-x_{\eta_{qp}}\frac{\partial}{\partial\xi}\left(\theta\Delta\zeta_{qp}\right)+x_{\xi_{qp}}\frac{\partial}{\partial\eta}\left(\theta\Delta\zeta_{qp}\right)\right)+ \tag{1.59}$$

$$+g\left(-x_{\eta_{qp}}\frac{\partial}{\partial\xi}\left(\zeta_{qp}^{n+\theta,p}\right)+x_{\xi_{qp}}\frac{\partial}{\partial\eta}\left(\zeta_{qp}^{n+\theta,p}\right)\right)\theta\Delta h_{qp} \tag{1.60}$$

1.1.2.3 Convection

The discretization of the convection term

$$\oint_{\partial\Omega_{\xi\eta}}\left[\left(y_\eta q-x_\eta r\right)\frac{q}{h}n_\xi+\left(-y_\xi q+x_\xi r\right)\frac{q}{h}n_\eta\right]dl,\qquad q\text{-momentum eq.} \tag{1.61}$$

$$\oint_{\partial\Omega_{\xi\eta}}\left[\left(y_\eta q-x_\eta r\right)\frac{r}{h}n_\xi+\left(-y_\xi q+x_\xi r\right)\frac{r}{h}n_\eta\right]dl,\qquad r\text{-momentum eq.} \tag{1.62}$$

will be given in this section (equation (A.31), multiplied with J). Where $\boldsymbol{n}=(n_\xi,n_\eta)^T$ is the outward normal vector.

Considering one sub-control volume and only for the x-direction (similar for the y-direction) then the linearization in time reads:

$$\oint_{\partial\Omega_{\xi\eta}}\left[\left(y_\eta\left(q^{n+\theta,p}+\theta\Delta q\right)-x_\eta\left(r^{n+\theta,p}+\theta\Delta r\right)\right)\frac{q^{n+\theta,p}+\theta\Delta q}{h^{n+\theta,p}+\theta\Delta h}n_\xi+\right.$$
$$\left.+\left(-y_\xi\left(q^{n+\theta,p}+\theta\Delta q\right)+x_\xi\left(r^{n+\theta,p}+\theta\Delta r\right)\right)\frac{q^{n+\theta,p}+\theta\Delta q}{h^{n+\theta,p}+\theta\Delta h}n_\eta\right]dl \tag{1.63}$$

These terms need to be computed for each of the control volume faces, taken into account the outward normal. Eight faces on a structured grid.

The linearization in time for the q-momentum equation reads (see **??**):

$$\frac{1}{2}\left(y_\eta q^{n+\theta,p} - x_\eta r^{n+\theta,p}\right)\frac{q^{n+\theta,p}}{h^{n+\theta,p}}n_\xi + \frac{1}{2}\left(-y_\xi q^{n+\theta,p} + x_\xi r^{n+\theta,p}\right)\frac{q^{n+\theta,p}}{h^{n+\theta,p}}n_\eta + \tag{1.64}$$

$$+\frac{1}{2}\theta\Delta h\left[\left(\frac{-y_\eta (q^{n+\theta,p})^2 + x_\eta q^{n+\theta,p} r^{n+\theta,p}}{(h^{n+\theta,p})^2}\right)n_\xi + \left(\frac{-x_\xi q^{n+\theta,p} r^{n+\theta,p} + y_\xi (r^{n+\theta,p})^2}{(h^{n+\theta,p})^2}\right)n_\eta\right] + \tag{1.65}$$

$$+\frac{1}{2}\theta\Delta q\left[\left(\frac{2y_\eta q^{n+\theta,p} - x_\eta r^{n+\theta,p}}{h^{n+\theta,p}}\right)n_\xi + \left(\frac{-2y_\xi q^{n+\theta,p} + x_\xi r^{n+\theta,p}}{h^{n+\theta,p}}\right)n_\eta\right] + \tag{1.66}$$

$$+\frac{1}{2}\theta\Delta r\left[\left(\frac{-x_\eta q^{n+\theta,p}}{h^{n+\theta,p}}\right)n_\xi + \left(\frac{x_\xi q^{n+\theta,p}}{h^{n+\theta,p}}\right)n_\eta\right] \tag{1.67}$$

and for the r-momentum equation:

$$\oint_{\partial\Omega_{\xi\eta}}\left[\left(y_\eta\left(q^{n+\theta,p} + \theta\Delta q\right) - x_\eta\left(r^{n+\theta,p} + \theta\Delta r\right)\right)\frac{r^{n+\theta,p} + \theta\Delta r}{h^{n+\theta,p} + \theta\Delta h}n_\xi + \right.$$
$$\left. + \left(-y_\xi\left(q^{n+\theta,p} + \theta\Delta q\right) + x_\xi\left(r^{n+\theta,p} + \theta\Delta r\right)\right)\frac{r^{n+\theta,p} + \theta\Delta r}{h^{n+\theta,p} + \theta\Delta h}n_\eta\right] dl \tag{1.68}$$

The linearization in time for the r-momentum equation reads (see **??**):

$$\frac{1}{2}\left(y_\eta q^{n+\theta,p} - x_\eta r^{n+\theta,p}\right)\frac{r^{n+\theta,p}}{h^{n+\theta,p}}n_\xi + \frac{1}{2}\left(-y_\xi q^{n+\theta,p} + x_\xi r^{n+\theta,p}\right)\frac{r^{n+\theta,p}}{h^{n+\theta,p}}n_\eta + \tag{1.69}$$

$$+\frac{1}{2}\theta\Delta h\left[\left(\frac{-y_\eta q^{n+\theta,p} r^{n+\theta,p} + x_\eta (r^{n+\theta,p})^2}{(h^{n+\theta,p})^2}\right)n_\xi + \left(\frac{y_\xi q^{n+\theta,p} r^{n+\theta,p} - x_\xi (r^{n+\theta,p})^2}{(h^{n+\theta,p})^2}\right)n_\eta\right] + \tag{1.70}$$

$$+\frac{1}{2}\theta\Delta q\left[\left(\frac{y_\eta r^{n+\theta,p}}{h^{n+\theta,p}}\right)n_\xi + \left(\frac{-y_\xi r^{n+\theta,p}}{h^{n+\theta,p}}\right)n_\eta\right] \tag{1.71}$$

$$+\frac{1}{2}\theta\Delta r\left[\left(\frac{y_\eta q^{n+\theta,p} - 2x_\eta r^{n+\theta,p}}{h^{n+\theta,p}}\right)n_\xi + \left(\frac{-y_\xi q^{n+\theta,p} + 2x_\xi r^{n+\theta,p}}{h^{n+\theta,p}}\right)n_\eta\right] + \tag{1.72}$$

These terms need to be computed for each of the control volume faces, taken into account the outward normal. Eight faces on a cartesian/curvilinear grid.

1.1.2.4 Bed shear stress

The bed shear stress term in vector notation reads:

$$\int_\Omega c_f \left(\frac{\boldsymbol{q}\,|\boldsymbol{q}|}{h^2} \right) \, d\omega. \tag{1.73}$$

The components for the two momentum equations read:

$$F_q = \int_\Omega c_f \left(\frac{q\,|\boldsymbol{q}|}{h^2} \right) \, d\omega, \qquad \textit{q-momentum eq.} \tag{1.74}$$

$$F_r = \int_\Omega c_f \left(\frac{r\,|\boldsymbol{q}|}{h^2} \right) \, d\omega, \qquad \textit{r-momentum eq.} \tag{1.75}$$

and $|\boldsymbol{q}| = \sqrt{q^2 + r^2}$. To avoid the discontinuity in the first derivative around zero, the absolute function will be approximated by (the Newton iteration process needs C^1–continue functions)

$$|\boldsymbol{q}| \approx |\widetilde{\boldsymbol{q}}| = \left((q^2 + r^2)^2 + \varepsilon^4 \right)^{\frac{1}{4}}. \tag{1.76}$$

The bed shear stress then reads:

$$F_q \approx \Delta x \Delta y \, c_f \left(\frac{q\,|\widetilde{\boldsymbol{q}}|}{h^2} \right), \qquad \textit{q-momentum eq.} \tag{1.77}$$

$$F_r \approx \Delta x \Delta y \, c_f \left(\frac{r\,|\widetilde{\boldsymbol{q}}|}{h^2} \right), \qquad \textit{r-momentum eq.} \tag{1.78}$$

The Jacobian for the bed shear stress $F(h, q, r)$ reads:

$$\begin{pmatrix} \frac{\partial F_q}{\partial h} & \frac{\partial F_q}{\partial q} & \frac{\partial F_q}{\partial r} \\ \frac{\partial F_r}{\partial h} & \frac{\partial F_r}{\partial q} & \frac{\partial F_r}{\partial r} \end{pmatrix} = \tag{1.79}$$

$$= \begin{pmatrix} -2c_f \frac{q|\widetilde{\boldsymbol{q}}|}{h^3} & c_f \frac{|\widetilde{\boldsymbol{q}}|}{h^2} + c_f \frac{q}{h^2} \frac{q(q^2+r^2)}{|\widetilde{\boldsymbol{q}}|^3} & c_f \frac{q}{h^2} \frac{r(q^2+r^2)}{|\widetilde{\boldsymbol{q}}|^3} \\ -2c_f \frac{r|\widetilde{\boldsymbol{q}}|}{h^3} & c_f \frac{r}{h^2} \frac{q(q^2+r^2)}{|\widetilde{\boldsymbol{q}}|^3} & c_f \frac{|\widetilde{\boldsymbol{q}}|}{h^2} + c_f \frac{r}{h^2} \frac{r(q^2+r^2)}{|\widetilde{\boldsymbol{q}}|^3} \end{pmatrix} \tag{1.80}$$

All these coefficients of the Jacobian should be evaluated at the quadrature point of the sub-control volumes and evaluated at time level $(n + \theta, p)$. The same applies also for the right hand side and evaluated at time level $(n + \theta, p)$.

The linearization in time for the q-momentum equation reads (see **??**):

$$c_f\left(\frac{q_{qp}^{n+\theta,p}\left|\widetilde{\boldsymbol{q}^{n+\theta,p}}\right|}{(h_{qp}^{n+\theta,p})^2}\right)-\left(2c_f\frac{q_{qp}^{n+\theta,p}\left|\widetilde{\boldsymbol{q}_{qp}^{n+\theta,p}}\right|}{(h_{qp}^{n+\theta,p})^3}\right)\theta\Delta h_{qp}+ \tag{1.81}$$

$$+\left(c_f\frac{\left|\widetilde{\boldsymbol{q}_{qp}^{n+\theta,p}}\right|}{(h_{qp}^{n+\theta,p})^2}+c_f\frac{q_{qp}^{n+\theta,p}}{(h_{qp}^{n+\theta,p})^2}\frac{q_{qp}^{n+\theta,p}\left((q_{qp}^{n+\theta,p})^2+(r_{qp}^{n+\theta,p})^2\right)}{\left|\widetilde{\boldsymbol{q}_{qp}^{n+\theta,p}}\right|^3}\right)\theta\Delta q_{qp}+ \tag{1.82}$$

$$+\left(c_f\frac{q_{qp}^{n+\theta,p}}{(h_{qp}^{n+\theta,p})^2}\frac{r_{qp}^{n+\theta,p}\left((q_{qp}^{n+\theta,p})^2+(r_{qp}^{n+\theta,p})^2\right)}{\left|\widetilde{\boldsymbol{q}_{qp}^{n+\theta,p}}\right|^3}\right)\theta\Delta r_{qp} \tag{1.83}$$

and for the r-momentum equation:

$$c_f\left(\frac{r_{qp}^{n+\theta,p}\left|\widetilde{\boldsymbol{q}^{n+\theta,p}}\right|}{(h_{qp}^{n+\theta,p})^2}\right)-\left(2c_f\frac{r_{qp}^{n+\theta,p}\left|\widetilde{\boldsymbol{q}_{qp}^{n+\theta,p}}\right|}{(h_{qp}^{n+\theta,p})^3}\right)\theta\Delta h_{qp}+ \tag{1.84}$$

$$+\left(c_f\frac{r_{qp}^{n+\theta,p}}{(h_{qp}^{n+\theta,p})^2}\frac{q_{qp}^{n+\theta,p}\left((q_{qp}^{n+\theta,p})^2+(r_{qp}^{n+\theta,p})^2\right)}{\left|\widetilde{\boldsymbol{q}_{qp}^{n+\theta,p}}\right|^3}\right)\theta\Delta q_{qp}+ \tag{1.85}$$

$$+\left(c_f\frac{\left|\widetilde{\boldsymbol{q}_{qp}^{n+\theta,p}}\right|}{(h_{qp}^{n+\theta,p})^2}+c_f\frac{r_{qp}^{n+\theta,p}}{(h_{qp}^{n+\theta,p})^2}\frac{r_{qp}^{n+\theta,p}\left((q_{qp}^{n+\theta,p})^2+(r_{qp}^{n+\theta,p})^2\right)}{\left|\widetilde{\boldsymbol{q}_{qp}^{n+\theta,p}}\right|^3}\right)\theta\Delta r_{qp} \tag{1.86}$$

with

$$\left|\widetilde{\boldsymbol{q}_{qp}^{n+\theta,p}}\right|=\left((q_{qp}^2+r_{qp}^2)^2+\varepsilon^4\right)^{\frac{1}{4}}. \tag{1.87}$$

These terms need to be computed for each quadrature points (qp) of the sub-control volumes.

1.1.2.5 Viscosity

The viscosity term in vector notation reads:

$$\int_\Omega \nabla\cdot\left(\nu h\left(\nabla(\boldsymbol{q}/h)+\nabla(\boldsymbol{q}^T/h)\right)\right)\,d\omega=\oint_\Omega\left(\nu h\left(\nabla(\boldsymbol{q}/h)+\nabla(\boldsymbol{q}^T/h)\right)\right)\cdot\boldsymbol{n}\,dl \tag{1.88}$$

with $\boldsymbol{n} = (n_x, n_y)^T$ the outward normal vector. Written in components ($\boldsymbol{q} = (q, r)^T$):

$$\oint_\Omega \left(\nu h \left(\nabla(\boldsymbol{q}/h) + \nabla(\boldsymbol{q}^T/h)\right)\right) \boldsymbol{\cdot} \boldsymbol{n}\, dl = \tag{1.89}$$

$$= \oint_\Omega \nu h \left(\begin{pmatrix} \frac{\partial(q/h)}{\partial x} & \frac{\partial(q/h)}{\partial y} \\ \frac{\partial(r/h)}{\partial x} & \frac{\partial(r/h)}{\partial y} \end{pmatrix} + \begin{pmatrix} \frac{\partial(q/h)}{\partial x} & \frac{\partial(r/h)}{\partial x} \\ \frac{\partial(q/h)}{\partial y} & \frac{\partial(r/h)}{\partial y} \end{pmatrix} \right) \boldsymbol{\cdot} \boldsymbol{n}\, dl \tag{1.90}$$

The components for the two momentum equations read:

$$F_q = \oint_\Omega \nu h \left[2\frac{\partial(q/h)}{\partial x} n_x + \left(\frac{\partial(q/h)}{\partial y} + \frac{\partial(r/h)}{\partial x} \right) n_y \right] dl \qquad \textit{q-momentum eq.} \tag{1.91}$$

$$F_r = \oint_\Omega \nu h \left[\left(\frac{\partial(r/h)}{\partial x} + \frac{\partial(q/h)}{\partial y} \right) n_x + 2\frac{\partial(r/h)}{\partial y} n_y \right] dl \qquad \textit{r-momentum eq.} \tag{1.92}$$

Written in linear terms for the derivative which read:

$$\nu h \frac{\partial(q/h)}{\partial x} = \nu \left(\frac{\partial q}{\partial x} - \frac{q}{h}\frac{\partial h}{\partial x} \right), \quad \nu h \frac{\partial(q/h)}{\partial y} = \nu \left(\frac{\partial q}{\partial y} - \frac{q}{h}\frac{\partial h}{\partial y} \right), \tag{1.93}$$

$$\nu h \frac{\partial(r/h)}{\partial x} = \nu \left(\frac{\partial r}{\partial x} - \frac{r}{h}\frac{\partial h}{\partial x} \right), \quad \nu h \frac{\partial(r/h)}{\partial y} = \nu \left(\frac{\partial r}{\partial y} - \frac{r}{h}\frac{\partial h}{\partial y} \right) \tag{1.94}$$

then the equations for the momentum equations read:

$$F_q = \oint_\Omega \left[2\nu \left(\frac{\partial q}{\partial x} - \frac{q}{h}\frac{\partial h}{\partial x} \right) n_x + \left(\nu \left(\frac{\partial q}{\partial y} - \frac{q}{h}\frac{\partial h}{\partial y} \right) + \nu \left(\frac{\partial r}{\partial x} - \frac{r}{h}\frac{\partial h}{\partial x} \right) \right) n_y \right] dl \tag{1.95}$$

$$F_r = \oint_\Omega \left[\left(\nu \left(\frac{\partial r}{\partial x} - \frac{r}{h}\frac{\partial h}{\partial x} \right) + \nu \left(\frac{\partial q}{\partial y} - \frac{q}{h}\frac{\partial h}{\partial y} \right) \right) n_x + 2\nu \left(\frac{\partial r}{\partial y} - \frac{r}{h}\frac{\partial h}{\partial y} \right) n_y \right] dl \tag{1.96}$$

These equations need to be discretized on the quadrature points qp at the control volume faces. The discretization is first given for expression $\mathcal{A}$ (using result from **??** for the quotient):

$$\nu_{qp} \frac{\partial q_{qp}^{n+\theta,p+1}}{\partial x} - \nu_{qp} \frac{q^{n+\theta,p+1}}{h_{qp}^{n+\theta,p+1}} \frac{\partial h_{qp}^{n+\theta,p+1}}{\partial x} \approx \tag{1.97}$$

$$\begin{aligned} &\approx \nu_{qp} \frac{\partial q_{qp}^{n+\theta,p}}{\partial x} + \nu_{qp} \frac{\partial}{\partial x}(\theta \Delta q_{qp}) + \\ &- \nu_{qp} \left(\frac{q_{qp}^{n+\theta,p}}{h_{qp}^{n+\theta,p}} - \frac{q_{qp}^{n+\theta,p}}{(h_{qp}^{n+\theta,p})^2} \theta \Delta h_{qp} + \frac{1}{h_{qp}^{n+\theta,p}} \theta \Delta q_{qp} \right) \left(\frac{\partial h_{qp}^{n+\theta,p}}{\partial x} + \frac{\partial}{\partial x}(\theta \Delta h_{qp}) \right) \approx \end{aligned} \tag{1.98}$$

$$\approx \underbrace{\nu_{qp}\left(\frac{\partial q_{qp}^{n+\theta,p}}{\partial x} - \frac{q_{qp}^{n+\theta,p}}{h_{qp}^{n+\theta,p}}\frac{\partial h_{qp}^{n+\theta,p}}{\partial x}\right)}_{\textit{to right hand side}} + \tag{1.99}$$

$$+ \underbrace{\nu_{qp}\frac{q_{qp}^{n+\theta,p}}{(h_{qp}^{n+\theta,p})^2}\frac{\partial h_{qp}^{n+\theta,p}}{\partial x}}_{\mathcal{A}_1}\theta\Delta h_{qp} + \underbrace{\left(-\nu_{qp}\frac{1}{h_{qp}^{n+\theta,p}}\frac{\partial h_{qp}^{n+\theta,p}}{\partial x}\right)}_{\mathcal{B}_1}\theta\Delta q_{qp} \tag{1.100}$$

$$+ \underbrace{\nu_{qp}}_{\mathcal{C}_1}\frac{\partial}{\partial x}(\theta\Delta q_{qp}) + \underbrace{\left(-\nu_{qp}\frac{q_{qp}^{n+\theta,p}}{h_{qp}^{n+\theta,p}}\right)}_{\mathcal{D}_1}\frac{\partial}{\partial x}(\theta\Delta h_{qp}) \tag{1.101}$$

the terms $O(\Delta h\frac{\partial\Delta h}{\partial x}, \Delta q\frac{\partial\Delta h}{\partial x})$ are assumed to negligible. In a similar way for the other three expressions:

$$\nu_{qp}\frac{\partial q_{qp}^{n+\theta,p+1}}{\partial y} - \nu_{qp}\frac{q^{n+\theta,p+1}}{h_{qp}^{n+\theta,p+1}}\frac{\partial h_{qp}^{n+\theta,p+1}}{\partial y} \approx \tag{1.102}$$

$$\approx \underbrace{\nu_{qp}\left(\frac{\partial q_{qp}^{n+\theta,p}}{\partial y} - \frac{q_{qp}^{n+\theta,p}}{h_{qp}^{n+\theta,p}}\frac{\partial h_{qp}^{n+\theta,p}}{\partial y}\right)}_{\textit{to right hand side}} + \tag{1.103}$$

$$+ \underbrace{\nu_{qp}\frac{q_{qp}^{n+\theta,p}}{(h_{qp}^{n+\theta,p})^2}\frac{\partial h_{qp}^{n+\theta,p}}{\partial y}}_{\mathcal{A}_2}\theta\Delta h_{qp} + \underbrace{\left(-\nu_{qp}\frac{1}{h_{qp}^{n+\theta,p}}\frac{\partial h_{qp}^{n+\theta,p}}{\partial y}\right)}_{\mathcal{B}_2}\theta\Delta q_{qp} \tag{1.104}$$

$$+ \underbrace{\nu_{qp}}_{\mathcal{C}_2}\frac{\partial}{\partial y}(\theta\Delta q_{qp}) + \underbrace{\left(-\nu_{qp}\frac{q_{qp}^{n+\theta,p}}{h_{qp}^{n+\theta,p}}\right)}_{\mathcal{D}_2}\frac{\partial}{\partial y}(\theta\Delta h_{qp}) \tag{1.105}$$

and

$$\nu_{qp}\frac{\partial r_{qp}^{n+\theta,p+1}}{\partial x} - \nu_{qp}\frac{r^{n+\theta,p+1}}{h_{qp}^{n+\theta,p+1}}\frac{\partial h_{qp}^{n+\theta,p+1}}{\partial x} \approx \tag{1.106}$$

$$\approx \underbrace{\nu_{qp}\left(\frac{\partial r_{qp}^{n+\theta,p}}{\partial x} - \frac{r_{qp}^{n+\theta,p}}{h_{qp}^{n+\theta,p}}\frac{\partial h_{qp}^{n+\theta,p}}{\partial x}\right)}_{\textit{to right hand side}} + \tag{1.107}$$

$$+ \underbrace{\nu_{qp}\frac{r_{qp}^{n+\theta,p}}{(h_{qp}^{n+\theta,p})^2}\frac{\partial h_{qp}^{n+\theta,p}}{\partial x}}_{\mathcal{A}_3}\theta\Delta h_{qp} + \underbrace{\left(-\nu_{qp}\frac{1}{h_{qp}^{n+\theta,p}}\frac{\partial h_{qp}^{n+\theta,p}}{\partial x}\right)}_{\mathcal{B}_3}\theta\Delta r_{qp} \tag{1.108}$$

$$+ \underbrace{\nu_{qp}}_{\mathcal{C}_3}\frac{\partial}{\partial x}(\theta\Delta r_{qp}) + \underbrace{\left(-\nu_{qp}\frac{r_{qp}^{n+\theta,p}}{h_{qp}^{n+\theta,p}}\right)}_{\mathcal{D}_3}\frac{\partial}{\partial x}(\theta\Delta h_{qp}) \tag{1.109}$$

and

$$\nu_{qp}\frac{\partial r_{qp}^{n+\theta,p+1}}{\partial y} - \nu_{qp}\frac{r^{n+\theta,p+1}}{h_{qp}^{n+\theta,p+1}}\frac{\partial h_{qp}^{n+\theta,p+1}}{\partial y} \approx \tag{1.110}$$

$$\approx \underbrace{\nu_{qp}\left(\frac{\partial r_{qp}^{n+\theta,p}}{\partial y} - \frac{r_{qp}^{n+\theta,p}}{h_{qp}^{n+\theta,p}}\frac{\partial h_{qp}^{n+\theta,p}}{\partial y}\right)}_{\textit{to right hand side}} + \tag{1.111}$$

$$+ \underbrace{\nu_{qp}\frac{r_{qp}^{n+\theta,p}}{(h_{qp}^{n+\theta,p})^2}\frac{\partial h_{qp}^{n+\theta,p}}{\partial y}}_{\mathcal{A}_4}\theta\Delta h_{qp} + \underbrace{\left(-\nu_{qp}\frac{1}{h_{qp}^{n+\theta,p}}\frac{\partial h_{qp}^{n+\theta,p}}{\partial y}\right)}_{\mathcal{B}_4}\theta\Delta r_{qp} \tag{1.112}$$

$$+ \underbrace{\nu_{qp}}_{\mathcal{C}_4}\frac{\partial}{\partial y}(\theta\Delta r_{qp}) + \underbrace{\left(-\nu_{qp}\frac{r_{qp}^{n+\theta,p}}{h_{qp}^{n+\theta,p}}\right)}_{\mathcal{D}_4}\frac{\partial}{\partial y}(\theta\Delta h_{qp}) \tag{1.113}$$

This terms need to be computed for each of the control volume faces, taken into account the outward normal. Eight faces on a cartesian/curvilinear grid.

1.1.3 Discretization at boundary

For the 2D non-linear wave equations (equation (1.2)) at each boundary boundary conditions need to be prescribed, the number of boundary conditions depends on the flow direction on the boundary. Considering a hyperbolic system, if the flow is flowing into the domain two boundary conditions need to prescribed and when the flow is flowing out the domain just one boundary need to prescribed. This is according the characteristic theory of 2D hyperbolic systems (Daubert and Graffe, 1967). The ingoing information is called the **essential** boundary condition (Dirichlet or Neumann condition). And a boundary condition to handle the outgoing wave is called the **natural** boundary condition. So for inflow there are **two essential** and **one natural** boundary condition and for outflow there is **one essential** boundary condition and **two natural** boundary conditions.

The natural boundary condition is always located at the boundary of a control volume and the essential (the genuine) boundary condition is always located inside the last control volume of the grid, as indicated in Figure 1.3.

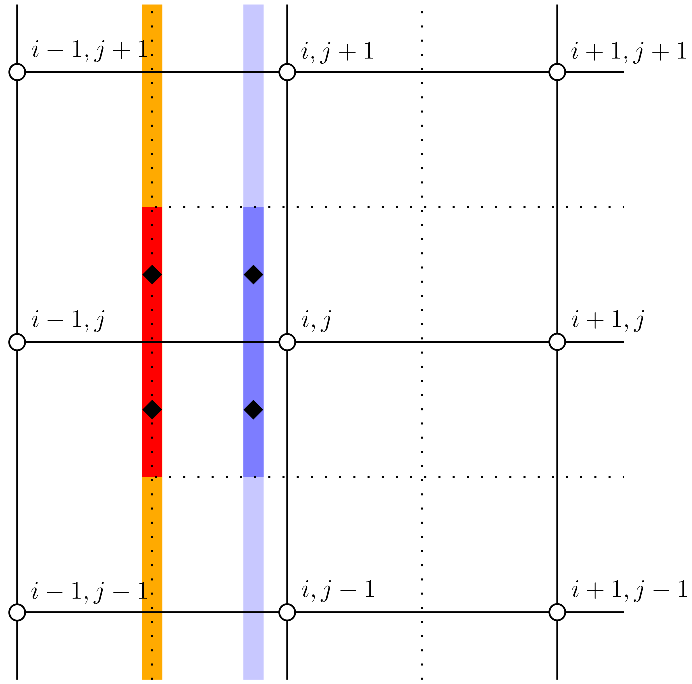

Figure 1.3: *Dotted lines indicate the border of the control volumes. The natural boundary condition is located at the boundary of a control volume, the orange line and the essential boundary condition is located inside the last control volume, the cyan-colored line. The black diamonds are the location of the quadrature points at the boundary.*

The boundary conditions in this section are presented for the left/west boundary. First the **essential** boundary conditions are discussed and after that the **natural** boundary condition. A similar derivation can be given for right/east boundary.

1.1.3.1 Essential boundary condition

The **essential** boundary condition is to be assumed somewhere in the first control volume, ($x_{i_{bc}}$ with $i_{bc} \in [i - \frac{1}{2}, i + \frac{1}{2}]$). For simplicity the boundary condition is chosen to be on node $i = 1$ (location x_1).

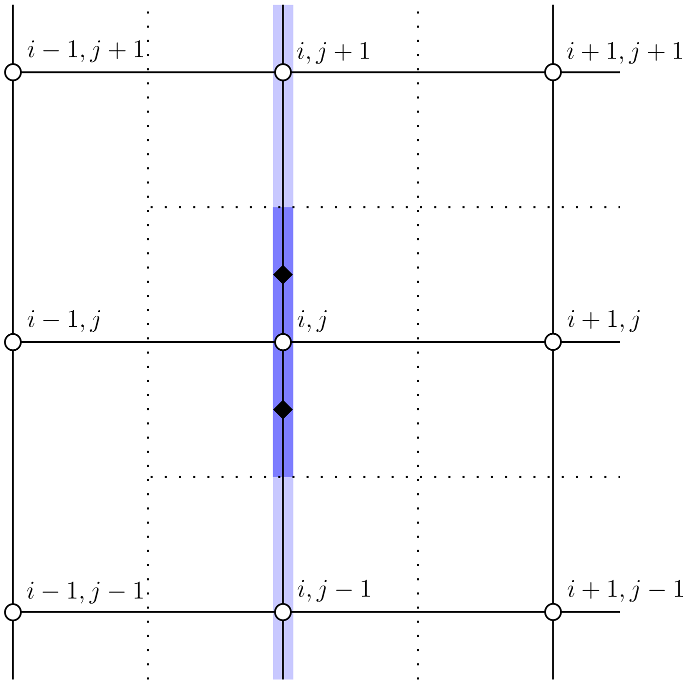

Figure 1.4: *Essential boundary condition, dotted lines indicate the border of the control volumes.*

The **essential** boundary condition for the left/west boundary at x_1 reads, describing the ingoing wave (indicated with h^+, q^+, r^+) with as less as possible disturbing the outgoing wave (**??**):

$$\left(\sqrt{gh} - \frac{q}{h}\right)\frac{\partial h^+}{\partial t} + \frac{\partial q^+}{\partial t} = F(t) \tag{1.114}$$

$$\left(\sqrt{gh} + \frac{q}{h}\right)\frac{\partial h^+}{\partial t} - \frac{\partial q^+}{\partial t} = 0 \tag{1.115}$$

Equation (1.115) means that the ingoing wave does not disturb the outgoing wave. And we assume normal incoming waves, which means that $r^+ = 0$.

The **essential** boundary condition for the right/east boundary at $x_{I+\frac{1}{2}}$ reads, describing the ingoing wave (indicated with h^-, q^-, r^-) with as less as possible disturbing the outgoing wave (**??**):

$$\left(\sqrt{gh} + \frac{q}{h}\right)\frac{\partial h^-}{\partial t} - \frac{\partial q^-}{\partial t} = G(t) \tag{1.116}$$

$$\left(\sqrt{gh} - \frac{q}{h}\right)\frac{\partial h^-}{\partial t} + \frac{\partial q^-}{\partial t} = 0 \tag{1.117}$$

Equation (1.117) means that the ingoing wave does not disturb the outgoing wave.

Given water level at left/west boundary

Adding the equations ((1.114) + (1.115)) yields

$$2\sqrt{gh}\frac{\partial h}{\partial t} = F(t) \tag{1.118}$$

So the essential boundary condition for incoming signal (if $\partial z_b/\partial t = 0$) reads

$$\boxed{\left(\sqrt{gh} - \frac{q}{h}\right)\frac{\partial h^+}{\partial t} + \frac{\partial q^+}{\partial t} = 2\sqrt{gh}\frac{\partial \zeta_{given}}{\partial t} + \varepsilon(\zeta_{given} - \zeta)} \tag{1.119}$$

a correction term is added, to prevent drifting away of the solution (an integration constant is missing). The variable ε has dimension $[\mathrm{m\,s^{-2}}]$.

The discretization of boundary equation (1.127) at $x = i + \frac{1}{2}$ reads (when $\partial z_b/\partial t = 0$):

$$\left(\sqrt{gh^{n+\theta,p+1}} - \frac{q^{n+\theta,p+1}}{h^{n+\theta,p+1}}\right)\frac{\partial h^+}{\partial t} + \frac{\partial q^+}{\partial t} =$$
$$= 2\sqrt{gh^{n+\theta,p+1}}\frac{\partial \zeta_{given}}{\partial t} + \varepsilon\left((\zeta_{given} - z_b) - h^{n+1,p}\right) \tag{1.120}$$

Given water flux at left/west boundary

Subtracting the equations ((1.114) − (1.115)), yields:

$$-2\frac{q}{h}\frac{\partial h}{\partial t} + 2\frac{\partial q}{\partial t} = F(h, q, t) \tag{1.121}$$

So the essential boundary condition for incoming signal reads

$$\left(\sqrt{gh} - \frac{q}{h}\right)\frac{\partial h^+}{\partial t} + \frac{\partial q^+}{\partial t} = -2\frac{q}{h}\frac{\partial h}{\partial t} + 2\frac{\partial q}{\partial t} \tag{1.122}$$

Using equation (1.115) (ingoing information does not disturb outgoing information)

$$\left(\sqrt{gh} + \frac{q}{h}\right)\frac{\partial h^+}{\partial t} - \frac{\partial q^+}{\partial t} = 0 \Rightarrow \frac{\partial h^+}{\partial t} = \frac{1}{\sqrt{gh} + \frac{q}{h}}\frac{\partial q^+}{\partial t} \tag{1.123}$$

substituting equation (1.123) into the right hand side of equation (1.122), yields

$$\boxed{\left(\sqrt{gh} - \frac{q}{h}\right)\frac{\partial h^+}{\partial t} + \frac{\partial q^+}{\partial t} = 2\left(\frac{\sqrt{gh}}{\sqrt{gh} + \frac{q}{h}}\right)\frac{\partial q_{given}}{\partial t} + \varepsilon(q_{given} - q)} \tag{1.124}$$

a correction term is added, to prevent drifting away of the solution (an integration constant is missing). The variable ε has dimension $[\mathrm{s^{-1}}]$. The discretization of boundary equation (1.124) at $x = i + \frac{1}{2}$ reads

$$\left(\sqrt{gh^{n+\theta,p+1}} - \frac{q^{n+\theta,p+1}}{h^{n+\theta,p+1}}\right)\frac{\partial h}{\partial t} + \frac{\partial q}{\partial t} =$$
$$= 2\left(\frac{\sqrt{gh^{n+\theta,p+1}}}{\sqrt{gh^{n+\theta,p+1}} + \frac{q^{n+\theta,p+1}}{h^{n+\theta,p+1}}}\right)\frac{\partial q_{given}}{\partial t} + \varepsilon\left(q_{given} - q^{n+1,p}\right) \tag{1.125}$$

Given water level at left/west boundary

Adding the equations ((**??**) + (**??**)) yields

$$2\sqrt{gh}\frac{\partial h}{\partial t} = F(t) \tag{1.126}$$

So the essential boundary condition for incoming signal (if $\partial z_b/\partial t = 0$) reads

$$\boxed{\left(\sqrt{gh} - \frac{q}{h}\right)\frac{\partial h^+}{\partial t} + \frac{\partial q^+}{\partial t} = 2\sqrt{gh}\frac{\partial \zeta_{given}}{\partial t} + \varepsilon(\zeta_{given} - \zeta)} \tag{1.127}$$

a correction term is added, to prevent drifting away of the solution (an integration constant is missing). The variable ε has dimension $[\mathrm{m\,s^{-2}}]$.

The discretization of boundary equation (1.127) at $x = i + \frac{1}{2}$ reads

$$\begin{aligned}&\left(\sqrt{gh^{n+\theta,p+1}} - \frac{q^{n+\theta,p+1}}{h^{n+\theta,p+1}}\right)\frac{\partial h^+}{\partial t} + \frac{\partial q^+}{\partial t} = \\ &= 2\sqrt{gh^{n+\theta,p+1}}\frac{\partial \zeta_{given}}{\partial t} + \varepsilon\left((\zeta_{given} - z_b) - h^{n+1,p}\right)\end{aligned} \tag{1.128}$$

1.1.3.2 Natural boundary condition

The **natural** boundary condition for the left/west boundary, describing the undisturbed outgoing wave, reads (**??**):

$$-\left(\sqrt{gh} + \frac{q}{h}\right)\underbrace{\left(\frac{\partial h}{\partial t} + \frac{\partial q}{\partial x} + \dots\right)}_{\text{continuity eq.}} + \underbrace{\left(\frac{\partial q}{\partial t} + gh\frac{\partial \zeta}{\partial x} + \dots\right)}_{\text{momentum eq.}} = 0 \tag{1.129}$$

where q is normal to this boundary.

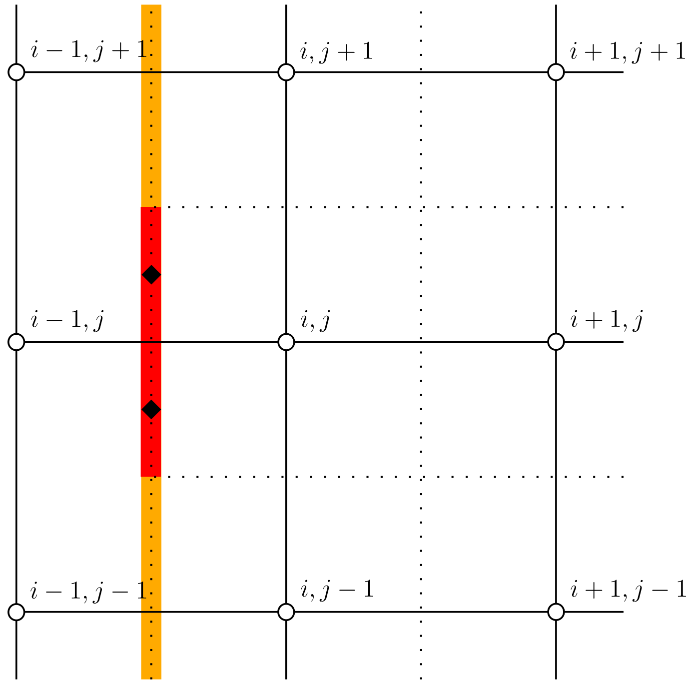

Figure 1.5: *Natural boundary condition ($i = 1$), dotted lines indicate the border of the control volumes. The diamonds indicate the quadrature points.*

Time derivative, continuity equation

At the left/west boundary ($x_{i-\frac{1}{2},j-\frac{1}{4}}$ with $i = 1$) the time discretization of the continuity equation for the **natural** boundary condition, describing the outgoing wave, reads:

$$\int_\Gamma \frac{\partial h}{\partial t}\, ds \bigg|_{i-\frac{1}{2},j-\frac{1}{4}} \approx \frac{\frac{1}{2}\Delta y_{i-\frac{1}{2},j-\frac{1}{2}}}{\Delta t} \left(h^{n+1}_{i-\frac{1}{2},j-\frac{1}{4}} - h^{n}_{i-\frac{1}{2},j-\frac{1}{4}} \right) \tag{1.130}$$

with $\Delta y_{i-\frac{1}{2},j-\frac{1}{2}} = y_{i-\frac{1}{2},j} - y_{i-\frac{1}{2},j-1}$.

A diffusion like term is added to the boundary equation to damp the reflection of spurious waves. A coefficient α_{bnd} is placed before that extra term, the optimal value of this coefficient is taken from the analysis in Borsboom (2009).

$$\begin{aligned} \frac{\frac{1}{2}\Delta y_{i-\frac{1}{2},j-\frac{1}{2}}}{\Delta t} \Bigg[\frac{1}{2} \left(h^{n+1,p+1}_{i-1,j-\frac{1}{4}} + h^{n+1,p+1}_{i,j-\frac{1}{4}} \right) + \frac{\alpha_{bnd}}{2} \left(h^{n+1,p+1}_{i-1,j-\frac{1}{4}} - 2h^{n+1,p+1}_{i,j-\frac{1}{4}} + h^{n+1,p+1}_{i+1,j-\frac{1}{4}} \right) + \\ - \left(\frac{1}{2} \left(h^{n}_{i-1,j-\frac{1}{4}} + h^{n}_{i,j-\frac{1}{4}} \right) + \frac{\alpha_{bnd}}{2} \left(h^{n}_{i-1,j-\frac{1}{4}} - 2h^{n}_{i,j-\frac{1}{4}} + h^{n}_{i+1,j-\frac{1}{4}} \right) \right) \Bigg] \end{aligned} \tag{1.131}$$

After rearranging the equation to the Δ-formulation, the implicit and the explicit part reads:

$$\frac{\frac{1}{2}\Delta y_{i-\frac{1}{2},j-\frac{1}{2}}}{\Delta t}\left(\frac{1}{2}\left(\Delta h^{n+1,p+1}_{i-1,j-\frac{1}{4}}+\Delta h^{n+1,p+1}_{i,j-\frac{1}{4}}\right)+\right.$$
$$+\left.\frac{\alpha_{bnd}}{2}\left(\Delta h^{n+1,p+1}_{i-1,j-\frac{1}{4}}-2\Delta h^{n+1,p+1}_{i,j-\frac{1}{4}}+\Delta h^{n+1,p+1}_{i+1,j-\frac{1}{4}}\right)\right)+$$
$$+\frac{\frac{1}{2}\Delta y_{i-\frac{1}{2},j-\frac{1}{2}}}{\Delta t}\left\{\frac{1}{2}\left(h^{n+1,p}_{i-1,j-\frac{1}{4}}+h^{n+1,p}_{i,j-\frac{1}{4}}\right)+\frac{\alpha_{bnd}}{2}\left(h^{n+1,p}_{i-1,j-\frac{1}{4}}-2h^{n+1,p}_{i,j-\frac{1}{4}}+h^{n+1,p}_{i+1,j-\frac{1}{4}}\right)+\right.$$
$$\left.-\left(\frac{1}{2}\left(h^{n}_{i-1,j-\frac{1}{4}}+h^{n}_{i,j-\frac{1}{4}}\right)+\frac{\alpha_{bnd}}{2}\left(h^{n}_{i-1,j-\frac{1}{4}}-2h^{n}_{i,j-\frac{1}{4}}+h^{n}_{i+1,j-\frac{1}{4}}\right)\right)\right\} \tag{1.132}$$

Mass flux, continuity equation

At the left/west boundary ($x_{i-\frac{1}{2},j-\frac{1}{4}}$ with $i=1$) the discretization of the mass flux for the **natural** boundary condition, describing the outgoing wave (assuming $\partial r/\partial y=0$), reads:

$$\int_\Gamma \frac{\partial q}{\partial x}\,ds \approx \frac{\frac{1}{2}\Delta y_{i-\frac{1}{2},j-\frac{1}{2}}}{\Delta x}\left(q^{n+\theta,p+1}_{i}-q^{n+\theta,p+1}_{i-1}\right) \tag{1.133}$$

which will be approximated by

$$\frac{\frac{1}{2}\Delta y_{i-\frac{1}{2},j-\frac{1}{2}}}{\Delta x}\left(\left(q^{n+\theta,p}_{i,j-\frac{1}{4}}+\theta\Delta q^{n+1,p+1}_{i,j-\frac{1}{4}}\right)-\left(q^{n+\theta,p+1}_{i-1,j-\frac{1}{4}}+\theta\Delta q^{n+1,p+1}_{i-1,j-\frac{1}{4}}\right)\right) \tag{1.134}$$

$$\Leftrightarrow \tag{1.135}$$

$$\frac{\frac{1}{2}\Delta y_{i-\frac{1}{2},j-\frac{1}{2}}}{\Delta x}\theta\left(\Delta q^{n+1,p+1}_{i,j-\frac{1}{4}}-\Delta q^{n+1,p+1}_{i-1,j-\frac{1}{4}}\right)+\frac{1}{\Delta x}\left\{q^{n+\theta,p}_{i,j-\frac{1}{4}}-q^{n+\theta,p+1}_{i-1,j-\frac{1}{4}}\right\} \tag{1.136}$$

Time derivative, momentum equation

At the left/west boundary ($x_{i-\frac{1}{2},j-\frac{1}{4}}$ with $i=1$) the time discretization of the momentum equation for the **natural** boundary condition, describing the outgoing wave, reads:

$$\int_\Gamma \frac{\partial q}{\partial t}\,ds\bigg|_{i-\frac{1}{2},j-\frac{1}{4}} \approx \frac{\frac{1}{2}\Delta y_{i-\frac{1}{2},j-\frac{1}{2}}}{\Delta t}\left(q^{n+1}_{i-\frac{1}{2},j-\frac{1}{4}}-q^{n}_{i-\frac{1}{2},j-\frac{1}{4}}\right) \tag{1.137}$$

A diffusion like term is added to the boundary equation to damp the reflection of spurious waves. A coefficient α_{bnd} is placed before that extra term, the optimal value of this coefficient is taken from the analysis in Borsboom (2009).

$$\frac{\frac{1}{2}\Delta y_{i-\frac{1}{2},j-\frac{1}{2}}}{\Delta t}\left[\frac{1}{2}\left(q^{n+1,p+1}_{i-1,j-\frac{1}{4}}+q^{n+1,p+1}_{i,j-\frac{1}{4}}\right)+\frac{\alpha_{bnd}}{2}\left(q^{n+1,p+1}_{i-1,j-\frac{1}{4}}-2q^{n+1,p+1}_{i,j-\frac{1}{4}}+q^{n+1,p+1}_{i+1,j-\frac{1}{4}}\right)+\right.$$
$$\left.-\left(\frac{1}{2}\left(q^{n}_{i-1,j-\frac{1}{4}}+q^{n}_{i,j-\frac{1}{4}}\right)+\frac{\alpha_{bnd}}{2}\left(q^{n}_{i-1,j-\frac{1}{4}}-2q^{n}_{i,j-\frac{1}{4}}+q^{n}_{i+1,j-\frac{1}{4}}\right)\right)\right] \tag{1.138}$$

After rearranging the equation to the Δ-formulation, the implicit and the explicit part reads:

$$
\begin{aligned}
&\frac{\frac{1}{2}\Delta y_{i-\frac{1}{2},j-\frac{1}{2}}}{\Delta t}\left(\frac{1}{2}\left(\Delta q_{i-1,j-\frac{1}{4}}^{n+1,p+1}+\Delta q_{i,j-\frac{1}{4}}^{n+1,p+1}\right)+\right.\\
&\quad\left.+\frac{\alpha_{bnd}}{2}\left(\Delta q_{i-1,j-\frac{1}{4}}^{n+1,p+1}-2\Delta q_{i,j-\frac{1}{4}}^{n+1,p+1}+\Delta q_{i+1,j-\frac{1}{4}}^{n+1,p+1}\right)\right)+\\
&\quad+\frac{\frac{1}{2}\Delta y_{i-\frac{1}{2},j-\frac{1}{2}}}{\Delta t}\left\{\frac{1}{2}\left(q_{i-1,j-\frac{1}{4}}^{n+1,p}+q_{i,j-\frac{1}{4}}^{n+1,p}\right)+\frac{\alpha_{bnd}}{2}\left(q_{i-1,j-\frac{1}{4}}^{n+1,p}-2q_{i,j-\frac{1}{4}}^{n+1,p}+q_{i+1,j-\frac{1}{4}}^{n+1,p}\right)+\right.\\
&\quad\left.-\frac{1}{2}\left(q_{i-1,j-\frac{1}{4}}^{n}+q_{i,j-\frac{1}{4}}^{n}\right)-\frac{\alpha_{bnd}}{2}\left(q_{i-1,j-\frac{1}{4}}^{n}-2q_{i,j-\frac{1}{4}}^{n}+q_{i+1,j-\frac{1}{4}}^{n}\right)\right\}
\end{aligned}
\tag{1.139}
$$

Pressure term, momentum equation

At the left/west boundary ($x_{i-\frac{1}{2},j-\frac{1}{4}}$ with $i=1$) the discretization of the pressure term for the **natural** boundary condition, describing the outgoing wave, reads:

$$
\int_\Gamma\left(gh\frac{\partial\zeta}{\partial x}\right)\,ds\bigg|_{i-\frac{1}{2},j-\frac{1}{4}}\approx\frac{1}{2}\Delta y_{i-\frac{1}{2},j-\frac{1}{2}}\;gh_{i-\frac{1}{2},j-\frac{1}{4}}^{n+\theta,p+1}\frac{\partial}{\partial x}\left(\zeta_{i-\frac{1}{2},j-\frac{1}{4}}^{n+\theta,p+1}\right)
\tag{1.140}
$$

In a formulation of the shallow-water equations, where the equation for the free-surface level ζ reduces to $\zeta=h+z_b$ (excluding drying and flooding), the equations can be simplified, because $\Delta\zeta=\Delta h$ (when z_b is not time dependent). In this case, the contributions to the $\Delta\zeta$-equations need to be incorporated in the Δh-equations. The pressure term will then be approximated by

$$
\begin{aligned}
&\frac{\frac{1}{2}\Delta y_{i-\frac{1}{2},j-\frac{1}{2}}}{\Delta x}gh_{i-\frac{1}{2},j-\frac{1}{4}}^{n+\theta,p}\left(\zeta_{i,j-\frac{1}{4}}^{n+\theta,p}-\zeta_{i-1,j-\frac{1}{4}}^{n+\theta,p}\right)+\\
&\quad+\frac{\frac{1}{2}\Delta y_{i-\frac{1}{2},j-\frac{1}{2}}}{\Delta x}g\left(\zeta_{i,j-\frac{1}{4}}^{n+\theta,p}-\zeta_{i-1,j-\frac{1}{4}}^{n+\theta,p}\right)\theta\Delta h_{i-\frac{1}{2},j-\frac{1}{4}}^{n+1,p+1}+\\
&\quad+\frac{\frac{1}{2}\Delta y_{i-\frac{1}{2},j-\frac{1}{2}}}{\Delta x}gh_{i-\frac{1}{2},j-\frac{1}{4}}^{n+\theta,p}\theta\left(\Delta\zeta_{i,j-\frac{1}{4}}^{n+1,p+1}-\Delta\zeta_{i-1,j-\frac{1}{4}}^{n+1,p+1}\right)
\end{aligned}
\tag{1.141}
$$

After rearranging the equation into an implicit and an explicit part it reads:

$$
\frac{\frac{1}{2}\Delta y_{i-\frac{1}{2},j-\frac{1}{2}}}{\Delta x}g\left(\zeta_{i,j-\frac{1}{4}}^{n+\theta,p}-\zeta_{i-1,j-\frac{1}{4}}^{n+\theta,p}\right)\theta\Delta h_{i-\frac{1}{2},j-\frac{1}{4}}^{n+1,p+1}+
\tag{1.142}
$$

$$
\begin{aligned}
&+\frac{\frac{1}{2}\Delta y_{i-\frac{1}{2},j-\frac{1}{2}}}{\Delta x}gh_{i-\frac{1}{2},j-\frac{1}{4}}^{n+\theta,p}\theta\left(\Delta\zeta_{i}^{n+1,p+1}-\Delta\zeta_{i-1,j-\frac{1}{4}}^{n+1,p+1}\right)+\\
&+\left\{\frac{\frac{1}{2}\Delta y_{i-\frac{1}{2},j-\frac{1}{2}}}{\Delta x}gh_{i-\frac{1}{2}}^{n+\theta,p}\left(\zeta_{i,j-\frac{1}{4}}^{n+\theta,p}-\zeta_{i-1,j-\frac{1}{4}}^{n+\theta,p}\right)\right\}
\end{aligned}
\tag{1.143}
$$

Convection, momentum equation

The finite volume flux through the boundary line element ds yields

$$\int_\Gamma \left(\nabla \cdot \frac{\boldsymbol{q}\boldsymbol{q}^T}{h}\right) ds = \int_\Gamma \begin{pmatrix} \frac{\partial qq/h}{\partial x} + \frac{\partial rq/h}{\partial y} \\ \frac{\partial qr/h}{\partial x} + \frac{\partial rr/h}{\partial y} \end{pmatrix} \cdot \boldsymbol{n}\, ds = \tag{1.144}$$

$$= \left[\left(\frac{\partial qq/h}{\partial x} + \frac{\partial rq/h}{\partial y} \right) n_x + \left(\frac{\partial qr/h}{\partial x} + \frac{\partial rr/h}{\partial y} \right) n_y \right] \|ds\| \tag{1.145}$$

with $\boldsymbol{n} = (n_x, n_y)^T$ the outward normal vector.

Written in linear terms for the derivative they read:

$$\frac{\partial (qq/h)}{\partial x} = \frac{2q}{h}\frac{\partial q}{\partial x} - \frac{q^2}{h^2}\frac{\partial h}{\partial x} \tag{1.146}$$

$$\frac{\partial (rq/h)}{\partial y} = \frac{q}{h}\frac{\partial r}{\partial y} + \frac{r}{h}\frac{\partial q}{\partial y} - \frac{rq}{h^2}\frac{\partial h}{\partial y} \tag{1.147}$$

$$\frac{\partial (qr/h)}{\partial x} = \frac{r}{h}\frac{\partial q}{\partial x} + \frac{q}{h}\frac{\partial r}{\partial x} - \frac{qr}{h^2}\frac{\partial h}{\partial x} \tag{1.148}$$

$$\frac{\partial (rr/h)}{\partial y} = \frac{2r}{h}\frac{\partial q}{\partial y} - \frac{r^2}{h^2}\frac{\partial h}{\partial y} \tag{1.149}$$

Consider on a cartesian grid the boundary at the west/left side of the domain, ranging over the interval $\left[\boldsymbol{x}_{i-\frac{1}{2},j-\frac{1}{2}}, \boldsymbol{x}_{i-\frac{1}{2},j+\frac{1}{2}}\right]$. In case we have normal flow at the boundary ($r = 0$) this boundary term reads:

$$\int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} \left(\frac{\partial qq/h}{\partial x}\right) ds \approx \frac{1}{2}\Delta y_{i-\frac{1}{2},j-\frac{1}{2}} \left.\frac{\partial qq/h}{\partial x}\right|_{i-\frac{1}{2},j-\frac{1}{4}} + \frac{1}{2}\Delta y_{i-\frac{1}{2},j+\frac{1}{2}} \left.\frac{\partial qq/h}{\partial x}\right|_{i-\frac{1}{2},j+\frac{1}{4}} \tag{1.150}$$

which will be approximated at the quatrature point qp by

$$\left.\left(\frac{2q}{h}\frac{\partial q}{\partial x} - \frac{q^2}{h^2}\frac{\partial h}{\partial x}\right)\right|_{qp} \approx \tag{1.151}$$

$$\approx \underbrace{\frac{2q_{qp}^{n+\theta,p}}{h_{qp}^{n+\theta,p}}\frac{\partial}{\partial x}\left(q_{qp}^{n+\theta,p}\right) - \frac{(q_{qp}^{n+\theta,p})^2}{(h_{qp}^{n+\theta,p})^2}\frac{\partial}{\partial x}\left(h_{qp}^{n+\theta,p}\right)}_{\text{to right hand side}} +$$

$$+ \underbrace{\left(-\frac{2q_{qp}^{n+\theta,p}}{(h_{qp}^{n+\theta,p})^2}\frac{\partial}{\partial x}\left(q_{qp}^{n+\theta,p}\right) + \frac{2(q_{qp}^{n+\theta,p})^2}{(h_{qp}^{n+\theta,p})^3}\frac{\partial}{\partial x}\left(h_{qp}^{n+\theta,p}\right)\right)}_{\mathcal{A}} \theta\Delta h_{qp}^{n+1,p+1} +$$

$$+ \underbrace{\left(\frac{2}{h_{qp}^{n+\theta,p}}\frac{\partial}{\partial x}\left(q_{qp}^{n+\theta,p}\right) - \frac{2q_{qp}^{n+\theta,p}}{(h_{qp}^{n+\theta,p})^2}\frac{\partial}{\partial x}\left(h_{qp}^{n+\theta,p}\right)\right)}_{\mathcal{B}} \theta\Delta q_{qp}^{n+1,p+1} +$$

$$+\underbrace{\left(-\frac{(q_{qp}^{n+\theta,p})^2}{(h_{qp}^{n+\theta,p})^2}\right)}_{\mathcal{C}}\theta\frac{\partial}{\partial x}\left(\Delta h_{qp}^{n+1,p+1}\right)+\underbrace{\frac{2q_{qp}^{n+\theta,p}}{h_{qp}^{n+\theta,p}}}_{\mathcal{D}}\theta\frac{\partial}{\partial x}\left(\Delta q_{qp}^{n+1,p+1}\right)$$

(1.152)

where $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$ multiplied by θ are the coefficients in the matrix of the Δ-formulation.

Bed shear stress, momentum equation

The bed shear stress term at the boundary reads:

$$\int_\Gamma c_f\left(\frac{\boldsymbol{q}\,|\boldsymbol{q}|}{h^2}\right)ds. \tag{1.153}$$

The components for the two momentum equations read:

$$F_q=\int_\Gamma c_f\left(\frac{q\,|\boldsymbol{q}|}{h^2}\right)ds, \qquad \textit{q-momentum eq.} \tag{1.154}$$

$$F_r=\int_\Gamma c_f\left(\frac{r\,|\boldsymbol{q}|}{h^2}\right)ds, \qquad \textit{r-momentum eq.} \tag{1.155}$$

Not yet documented

1.1.4 Discretization at corner

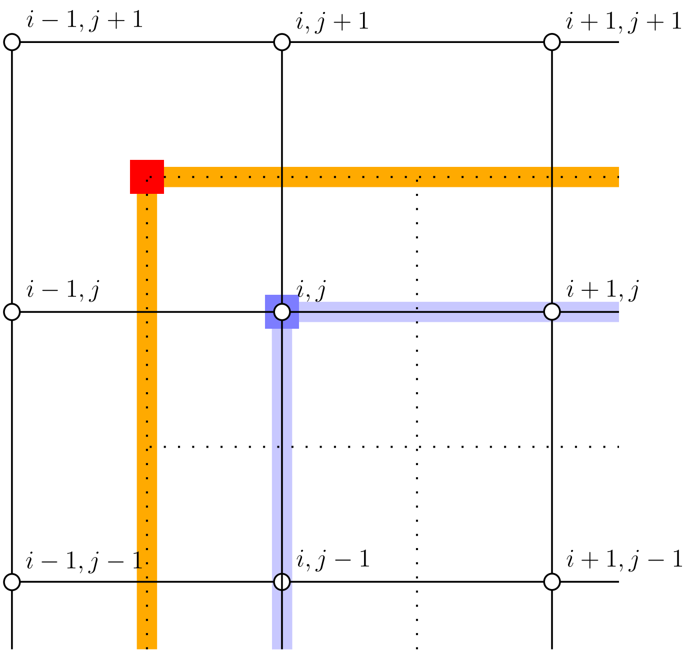

Figure 1.6: *Coefficients for the mass-matrix in 2-dimensions on a structured grid at a corner. No line integrals are performed in the corner.*

1.1.4.1 Weakly reflective boundary conditions

Consider the following weakly reflective boundary conditions.

For inflow:

$$q_{i+\frac{1}{2}} + \sqrt{gh_{i+\frac{1}{2}}} = \sqrt{gh^{\infty}_{i+\frac{1}{2}}}, \quad \text{inflow} \tag{1.156}$$

$$r_{i+\frac{1}{2}} = 0, \quad \text{inflow} \tag{1.157}$$

For outflow:

$$\left.\frac{\partial r}{\partial y}\right|_{i+\frac{1}{2}} = 0, \quad \text{outflow} \tag{1.158}$$

$$q_{i+\frac{1}{2}} - \sqrt{gh_{i+\frac{1}{2}}} = 0, \quad \text{outflow} \tag{1.159}$$

References

Borsboom, M. (2009). *MapleSoft file: "transpeq-analysisdiscretizationinsidedomain&@boundaries.mw"*.

Daubert, A. and O. Graffe (1967). “Quelques aspects des écoulements presque horizontaux à deux dimensions en plan et non permanents, application aux estuaires”. In: *La Houille Blanche - Revue internationale de l’eau* 8, pp. 847–860. DOI: 10.1051/lhb/1967059. URL: https://doi.org/10.1051/lhb/1967059.

A Curvilinear coordinate transformation

A curvilinear coordinate transformation is employed to enable the calculation of non-rectangular bodies of water. Two grids are introduced: a curvilinear xy-grid, following the curvature of the shallow water body and a computational/numerical $\xi\eta$-grid, created by mapping the curvilinear xy-grid to an orthogonal coordinate system via a coordinate transformation. The global Cartesian xy-coordinate system is used for reference of the numerical grid. An example of this is sketched in Figure A.1.

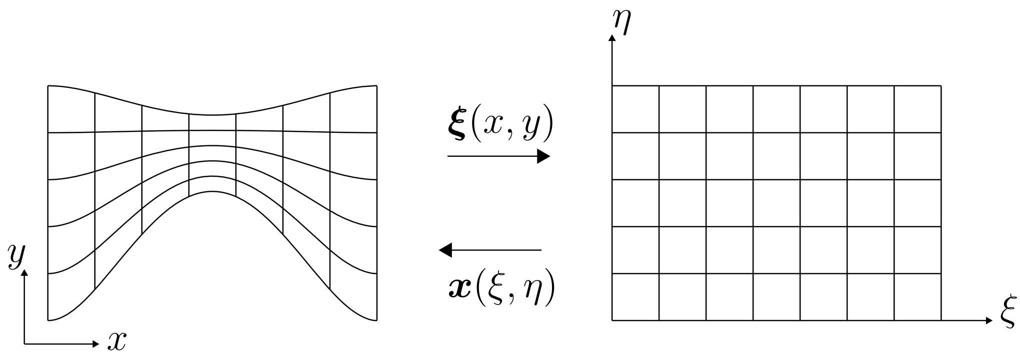

Figure A.1: *Mesh mapping from the physical Cartesian xy-grid to the numerical $\xi\eta$-grid and vice versa.*

Following the chain rule and assuming time-independent grids, we can express the differential operators in the global coordinate system as functions of the differential operators in the body-fitted curvilinear grid as follows:

$$\frac{\partial}{\partial x} = \frac{\partial \xi}{\partial x}\frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial x}\frac{\partial}{\partial \eta}, \tag{A.1}$$

$$\frac{\partial}{\partial y} = \frac{\partial \xi}{\partial y}\frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial y}\frac{\partial}{\partial \eta}, \tag{A.2}$$

in matrix-vector notation these equations reads:

$$\begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{pmatrix} = \begin{pmatrix} \frac{\partial \xi}{\partial x} & \frac{\partial \eta}{\partial x} \\ \frac{\partial \xi}{\partial y} & \frac{\partial \eta}{\partial y} \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial \xi} \\ \frac{\partial}{\partial \eta} \end{pmatrix}. \tag{A.3}$$

where the transformation coefficients $\frac{\partial \xi}{\partial x}$, $\frac{\partial \eta}{\partial x}$, $\frac{\partial \xi}{\partial y}$, $\frac{\partial \eta}{\partial y}$ are unknown and but can be determined by the grid definitions.

The goal is to express the transformation coefficients of equation (A.3) in terms of its inverted derivatives (grid definitions). The inverse transformation reads:

$$\begin{pmatrix} \frac{\partial}{\partial \xi} \\ \frac{\partial}{\partial \eta} \end{pmatrix} = \begin{pmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{pmatrix}. \tag{A.4}$$

We introduce a shorthand notation for the transformation coefficients with a subscript, e.g. $\xi_x = \frac{\partial \xi}{\partial x}$ and calculate the inverse of the right-hand side to give:

$$\begin{pmatrix} x_\xi & y_\xi \\ x_\eta & y_\eta \end{pmatrix}^{-1} = \frac{1}{x_\xi y_\eta - y_\xi x_\eta} \begin{pmatrix} y_\eta & -y_\xi \\ -x_\eta & x_\xi \end{pmatrix} = \frac{1}{J} \begin{pmatrix} y_\eta & -y_\xi \\ -x_\eta & x_\xi \end{pmatrix} \tag{A.5}$$

with the Jacobian determinant defined as:

$$J = \begin{vmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \eta} \\ \frac{\partial y}{\partial \xi} & \frac{\partial y}{\partial \eta} \end{vmatrix} = x_\xi y_\eta - y_\xi x_\eta. \tag{A.6}$$

leading to

$$\xi_x = \frac{1}{J} y_\eta, \quad \eta_x = -\frac{1}{J} y_\xi, \quad \xi_y = -\frac{1}{J} x_\eta, \quad \eta_y = \frac{1}{J} x_\xi. \tag{A.7}$$

where $J = x_\xi y_\eta - x_\eta y_\xi$ is the determinant of the coordinate transformation matrix.

Hence, we can use equation (A.7) to transform the partial derivatives (equation (A.3)) leading to:

$$\frac{\partial}{\partial x} = \xi_x \frac{\partial}{\partial \xi} + \eta_x \frac{\partial}{\partial \eta} = \frac{1}{J} \left(y_\eta \frac{\partial}{\partial \xi} - y_\xi \frac{\partial}{\partial \eta} \right) \tag{A.8}$$

$$\frac{\partial}{\partial y} = \xi_y \frac{\partial}{\partial \xi} + \eta_y \frac{\partial}{\partial \eta} = \frac{1}{J} \left(-x_\eta \frac{\partial}{\partial \xi} + x_\xi \frac{\partial}{\partial \eta} \right) \tag{A.9}$$

Second derivatives

Second derivatives are instead transformed by applying this operator twice. This yields:

$$\frac{\partial^2}{\partial x^2} = \frac{1}{J} \Big[y_\eta \frac{\partial}{\partial \xi} \Big(\frac{1}{J} \Big(y_\eta \frac{\partial}{\partial \xi} - y_\xi \frac{\partial}{\partial \eta} \Big) \Big) - y_\xi \frac{\partial}{\partial \eta} \Big(\frac{1}{J} \Big(y_\eta \frac{\partial}{\partial \xi} - y_\xi \frac{\partial}{\partial \eta} \Big) \Big) \Big], \tag{A.10}$$

$$\frac{\partial^2}{\partial y^2} = \frac{1}{J} \Big[-x_\eta \frac{\partial}{\partial \xi} \Big(\frac{1}{J} \Big(-x_\eta \frac{\partial}{\partial \xi} + x_\xi \frac{\partial}{\partial \eta} \Big) \Big) + x_\xi \frac{\partial}{\partial \eta} \Big(\frac{1}{J} \Big(-x_\eta \frac{\partial}{\partial \xi} + x_\xi \frac{\partial}{\partial \eta} \Big) \Big) \Big], \tag{A.11}$$

which can be rewritten to:

$$\begin{aligned} \frac{\partial^2}{\partial x^2} &= \frac{y_\eta}{J^2} y_\eta \frac{\partial^2}{\partial \xi^2} + y_\xi \frac{y_\xi}{J^2} \frac{\partial^2}{\partial \eta^2} - 2 \frac{y_\eta}{J^2} y_\xi \frac{\partial^2}{\partial \xi \partial \eta} + \\ &+ \left(\frac{y_\xi y_\eta J_\eta}{J^3} - \frac{y_\eta^2 J_\xi}{J^3} + \frac{y_\eta y_{\xi\eta}}{J^2} - \frac{y_{\eta\eta} y_\xi}{J^2} \right) \frac{\partial}{\partial \xi} + \\ &+ \left(\frac{y_\eta y_\xi J_\xi}{J^3} - \frac{y_\xi^2 J_\eta}{J^3} + \frac{y_{\xi\eta} y_\xi}{J^2} - \frac{y_{\xi\xi} y_\eta}{J^2} \right) \frac{\partial}{\partial \eta} \end{aligned} \tag{A.12}$$

$$
\begin{aligned}
\frac{\partial^2}{\partial y^2} = x_\eta \frac{x_\eta}{J^2}\frac{\partial^2}{\partial \xi^2} + x_\xi \frac{x_\xi}{J^2}\frac{\partial^2}{\partial \eta^2} - 2x_\xi \frac{x_\eta}{J^2}\frac{\partial^2}{\partial \xi \partial \eta} + \\
+ \left(\frac{x_\xi x_\eta J_\eta}{J^3} - \frac{x_\eta^2 J_\xi}{J^3} + \frac{x_{\xi\eta} x_\eta}{J^2} - \frac{x_{\eta\eta} x_\xi}{J^2}\right)\frac{\partial}{\partial \xi} + \\
+ \left(\frac{x_\eta x_\xi J_\xi}{J^3} - \frac{x_\xi^2 J_\eta}{J^3} + \frac{x_{\xi\eta} x_\xi}{J^2} - \frac{x_{\xi\xi} x_\eta}{J^2}\right)\frac{\partial}{\partial \eta},
\end{aligned}
\tag{A.13}
$$

after repeated application of the chain rule, which should equal the application of the chain rule directly:

$$
\frac{\partial^2}{\partial x^2} = \xi_x^2 \frac{\partial^2}{\partial \xi^2} + \eta_x^2 \frac{\partial^2}{\partial \eta^2} + 2\eta_x \xi_x \frac{\partial^2}{\partial \xi \partial \eta} + \xi_{xx}\frac{\partial}{\partial \xi} + \eta_{xx}\frac{\partial}{\partial \eta}, \tag{A.14}
$$

$$
\frac{\partial^2}{\partial y^2} = \xi_y^2 \frac{\partial^2}{\partial \xi^2} + \eta_y^2 \frac{\partial^2}{\partial \eta^2} + 2\eta_y \xi_y \frac{\partial^2}{\partial \xi \partial \eta} + \xi_{yy}\frac{\partial}{\partial \xi} + \eta_{yy}\frac{\partial}{\partial \eta}. \tag{A.15}
$$

from which we can find the following relations:

$$
\xi_{xx} = \frac{y_\xi y_\eta J_\eta}{J^3} - \frac{y_\eta^2 J_\xi}{J^3} + \frac{y_\eta y_{\xi\eta}}{J^2} - \frac{y_{\eta\eta} y_\xi}{J^2} \tag{A.16}
$$

$$
\eta_{xx} = \frac{y_\eta y_\xi J_\xi}{J^3} - \frac{y_\xi^2 J_\eta}{J^3} + \frac{y_{\xi\eta} y_\xi}{J^2} - \frac{y_{\xi\xi} y_\eta}{J^2} \tag{A.17}
$$

$$
\xi_{yy} = \frac{x_\xi x_\eta J_\eta}{J^3} - \frac{x_\eta^2 J_\xi}{J^3} + \frac{x_{\xi\eta} x_\eta}{J^2} - \frac{x_{\eta\eta} x_\xi}{J^2} \tag{A.18}
$$

$$
\eta_{yy} = \frac{x_\eta x_\xi J_\xi}{J^3} - \frac{x_\xi^2 J_\eta}{J^3} + \frac{x_{\xi\eta} x_\xi}{J^2} - \frac{x_{\xi\xi} x_\eta}{J^2} \tag{A.19}
$$

A.1 Transforming the terms

For completeness, we show the transformation of various terms here since some particular choices will be made to ease the implementation. For example, transforming the spatial derivatives in the continuity equation in a conservative form

$$
\frac{\partial q}{\partial x} + \frac{\partial r}{\partial y} = \frac{1}{J}\left(y_\eta \frac{\partial q}{\partial \xi} - y_\xi \frac{\partial q}{\partial \eta} + \left(-x_\eta \frac{\partial r}{\partial \xi} + x_\xi \frac{\partial r}{\partial \eta} \right) \right) \tag{A.20}
$$

and adding $(y_{\eta\xi} - y_{\xi\eta})q$ and $(x_{\eta\xi} - x_{\xi\eta})r$, yields

$$
\frac{\partial q}{\partial x} + \frac{\partial r}{\partial y} = \frac{1}{J}\left(y_\eta \frac{\partial q}{\partial \xi} - y_\xi \frac{\partial q}{\partial \eta} + y_{\eta\xi} q - y_{\xi\eta} q \right. \tag{A.21}
$$

$$
\left. - x_\eta \frac{\partial r}{\partial \xi} + x_\xi \frac{\partial r}{\partial \eta} + x_{\eta\xi} r - x_{\xi\eta} r \right) \tag{A.22}
$$

$$
= \frac{1}{J}\left(\frac{\partial}{\partial \xi}\left(y_\eta q - x_\eta r \right) + \frac{\partial}{\partial \eta}\left(-y_\xi q + x_\xi r \right) \right), \tag{A.23}
$$

Taking the finite volume approach, yields

$$\frac{1}{J}\int_{\Omega_{\xi\eta}}\begin{pmatrix}\frac{\partial}{\partial\xi}\\ \frac{\partial}{\partial\eta}\end{pmatrix}\cdot\begin{pmatrix}y_\eta q - x_\eta r\\ -y_\xi q + x_\xi r\end{pmatrix}\,dl = \frac{1}{J}\oint_{\partial\Omega_{\xi\eta}}\begin{pmatrix}y_\eta q - x_\eta r\\ -y_\xi q + x_\xi r\end{pmatrix}\cdot\begin{pmatrix}n_\xi\\ n_\eta\end{pmatrix}\,dl = \tag{A.24}$$

$$\frac{1}{J}\oint_{\partial\Omega_{\xi\eta}}\left(\left(y_\eta q - x_\eta r\right)n_\xi + \left(-y_\xi q + x_\xi r\right)n_\eta\right)\,dl \tag{A.25}$$

A.1.1 Convection term

The convection term in the x-momentum equation can be transformed as:

$$\frac{\partial(q^2/h)}{\partial x} + \frac{\partial(qr/h)}{\partial y} = \tag{A.26}$$

$$= \frac{1}{J}\left(y_\eta\frac{\partial(q^2/h)}{\partial\xi} - y_\xi\frac{\partial(q^2/h)}{\partial\eta} - x_\eta\frac{\partial(qr/h)}{\partial\xi} + x_\xi\frac{\partial(qr/h)}{\partial\eta}\right) = \tag{A.27}$$

$$\begin{aligned}= \frac{1}{J}\Bigg(& y_\eta\frac{\partial(q^2/h)}{\partial\xi} - y_\xi\frac{\partial(q^2/h)}{\partial\eta} + \frac{q^2}{h}y_{\xi\eta} - \frac{q^2}{h}y_{\eta\xi} + \\ & - x_\eta\frac{\partial(qr/h)}{\partial\xi} + x_\xi\frac{\partial(qr/h)}{\partial\eta} - \frac{qr}{h}x_{\xi\eta} + \frac{qr}{h}x_{\eta\xi}\Bigg) =\end{aligned} \tag{A.28}$$

$$= \frac{1}{J}\frac{\partial}{\partial\xi}\left(y_\eta\frac{q^2}{h} - x_\eta\frac{qr}{h}\right) + \frac{1}{J}\frac{\partial}{\partial\eta}\left(-y_\xi\frac{q^2}{h} + x_\xi\frac{qr}{h}\right). \tag{A.29}$$

Taking the finite volume approach and applying Green's theorem, yields

$$\frac{1}{J}\oint_{\partial\Omega}\left(\left(y_\eta\frac{q^2}{h} - x_\eta\frac{qr}{h}\right)n_\xi + \left(-y_\xi\frac{q^2}{h} + x_\xi\frac{qr}{h}\right)n_\eta\right)\,dl. \tag{A.30}$$

which is equal to (and look to the similarity with equation (A.25))

$$\frac{1}{J}\oint_{\partial\Omega}\left[\left(y_\eta q - x_\eta r\right)\frac{q}{h}n_\xi + \left(-y_\xi q + x_\xi r\right)\frac{q}{h}n_\eta\right]\,dl. \tag{A.31}$$

A similar transformation is valid for the convection term in the y-momentum equation, which reads

$$\frac{\partial(rq/h)}{\partial x} + \frac{\partial(r^2/h)}{\partial y} = \tag{A.32}$$

$$= \frac{1}{J}\left(y_\eta\frac{\partial(rq/h)}{\partial\xi} - y_\xi\frac{\partial(rq/h)}{\partial\eta} - x_\eta\frac{\partial(r^2/h)}{\partial\xi} + x_\xi\frac{\partial(r^2/h)}{\partial\eta}\right) = \tag{A.33}$$

$$\begin{aligned}= \frac{1}{J}\Bigg(& y_\eta\frac{\partial(rq/h)}{\partial\xi} - y_\xi\frac{\partial(rq/h)}{\partial\eta} + \frac{rq}{h}y_{\xi\eta} - \frac{rq}{h}y_{\eta\xi} + \\ & - x_\eta\frac{\partial(r^2/h)}{\partial\xi} + x_\xi\frac{\partial(r^2/h)}{\partial\eta} - \frac{r^2}{h}x_{\xi\eta} + \frac{r^2}{h}x_{\eta\xi}\Bigg) =\end{aligned} \tag{A.34}$$

$$= \frac{1}{J}\frac{\partial}{\partial \xi}\left(y_\eta \frac{rq}{h} - x_\eta \frac{r^2}{h}\right) + \frac{1}{J}\frac{\partial}{\partial \eta}\left(-y_\xi \frac{rq}{h} + x_\xi \frac{r^2}{h}\right). \tag{A.35}$$

Taking the finite volume approach and applying Green's theorem, yields

$$\frac{1}{J}\oint_{\partial\Omega}\left(\left(y_\eta \frac{rq}{h} - x_\eta \frac{r^2}{h}\right) n_\xi + \left(-y_\xi \frac{rq}{h} + x_\xi \frac{r^2}{h}\right) n_\eta\right) dl \tag{A.36}$$

which is equal to (and look to the similarity with equation (A.25))

$$\frac{1}{J}\oint_{\partial\Omega}\left[(y_\eta q - x_\eta r)\frac{r}{h} n_\xi + (-y_\xi q + x_\xi r)\frac{r}{h} n_\eta\right] dl \tag{A.37}$$

A.1.2 Viscosity term

The x and y-momentum equation diffusion terms are treated here as well due to their complexity. They can be transformed as:

$$\frac{\partial}{\partial x}\left(2\nu h \frac{\partial (q/h)}{\partial x}\right) = \tag{A.38}$$

$$= \frac{1}{J}\left[y_\eta \frac{\partial}{\partial \xi}\left(\frac{2\nu h}{J}\left(y_\eta \frac{\partial (q/h)}{\partial \xi} - y_\xi \frac{\partial (q/h)}{\partial \eta}\right)\right) - y_\xi \frac{\partial}{\partial \eta}\left(\frac{2\nu h}{J}\left(y_\eta \frac{\partial (q/h)}{\partial \xi} - y_\xi \frac{\partial (q/h)}{\partial \eta}\right)\right)\right] = \tag{A.39}$$

$$\begin{aligned} = \frac{1}{J}\Bigg[& y_\eta \frac{\partial}{\partial \xi}\left(\frac{2\nu h}{J}\right)\left(y_\eta \frac{\partial (q/h)}{\partial \xi} - y_\xi \frac{\partial (q/h)}{\partial \eta}\right) + \frac{2\nu h}{J}\frac{\partial}{\partial \xi}\left(y_\eta \frac{\partial (q/h)}{\partial \xi} - y_\xi \frac{\partial (q/h)}{\partial \eta}\right) + \\ & - y_\xi \left(\frac{\partial}{\partial \eta}\left(\frac{2\nu h}{J}\right)\left(y_\eta \frac{\partial (q/h)}{\partial \xi} - y_\xi \frac{\partial (q/h)}{\partial \eta}\right) + \frac{2\nu h}{J}\frac{\partial}{\partial \eta}\left(y_\eta \frac{\partial (q/h)}{\partial \xi} - y_\xi \frac{\partial (q/h)}{\partial \eta}\right)\right)\Bigg] = \end{aligned} \tag{A.40}$$

$$\begin{aligned} = \frac{2}{J}\Bigg[& y_\eta \frac{\partial}{\partial \xi}\left(\frac{\nu h}{J}\right)\left(y_\eta \frac{\partial (q/h)}{\partial \xi} - y_\xi \frac{\partial (q/h)}{\partial \eta}\right) + \frac{\nu h}{J}\frac{\partial}{\partial \xi}\left(y_\eta \frac{\partial (q/h)}{\partial \xi} - y_\xi \frac{\partial (q/h)}{\partial \eta}\right) + \\ & - y_\xi \left(\frac{\partial}{\partial \eta}\left(\frac{\nu h}{J}\right)\left(y_\eta \frac{\partial (q/h)}{\partial \xi} - y_\xi \frac{\partial (q/h)}{\partial \eta}\right) + \frac{\nu h}{J}\frac{\partial}{\partial \eta}\left(y_\eta \frac{\partial (q/h)}{\partial \xi} - y_\xi \frac{\partial (q/h)}{\partial \eta}\right)\right)\Bigg] \end{aligned} \tag{A.41}$$

for the x-momentum equation the viscous terms can be transformed as:

$$-\frac{\partial}{\partial x}\left(2\nu h \frac{\partial (q/h)}{\partial x}\right) - \frac{\partial}{\partial y}\left(\nu h \frac{\partial (r/h)}{\partial x} + \nu h \frac{\partial (q/h)}{\partial y}\right) = \tag{A.42}$$

$$\begin{aligned} = & -\frac{y_\eta}{J}\frac{\partial}{\partial \xi}\left(2\nu h \frac{\partial (q/h)}{\partial x}\right) + \frac{y_\xi}{J}\frac{\partial}{\partial \eta}\left(2\nu h \frac{\partial (q/h)}{\partial x}\right) + \\ & + \frac{x_\eta}{J}\frac{\partial}{\partial \xi}\left(\nu h \frac{\partial (r/h)}{\partial x} + \nu h \frac{\partial (q/h)}{\partial y}\right) - \frac{x_\xi}{J}\frac{\partial}{\partial \eta}\left(\nu h \frac{\partial (r/h)}{\partial x} + \nu h \frac{\partial (q/h)}{\partial y}\right) \end{aligned} \tag{A.43}$$

$$
\begin{aligned}
=&-\frac{1}{J}\frac{\partial}{\partial\xi}\left[2\nu y_\eta h\frac{\partial(q/h)}{\partial x}-\nu h x_\eta\frac{\partial(r/h)}{\partial x}+\nu h x_\eta\frac{\partial(q/h)}{\partial y}\right]+\\
&+\frac{1}{J}\frac{\partial}{\partial\eta}\left[2\nu y_\xi h\frac{\partial(q/h)}{\partial x}-\nu h x_\xi\frac{\partial(r/h)}{\partial x}+\nu h x_\xi\frac{\partial(q/h)}{\partial y}\right]
\end{aligned}
\tag{A.44}
$$

$$
\begin{aligned}
=&-\frac{1}{J}\frac{\partial}{\partial\xi}\left[\frac{\nu h}{J}\left(2y_\eta y_\eta\frac{\partial(q/h)}{\partial\xi}-2y_\eta y_\xi\frac{\partial(q/h)}{\partial\eta}-x_\eta y_\eta\frac{\partial(r/h)}{\partial\xi}+\right.\right.\\
&\left.\left.+x_\eta y_\xi\frac{\partial(r/h)}{\partial\eta}-x_\eta^2\frac{\partial(q/h)}{\partial\xi}+x_\eta x_\xi\frac{\partial(q/h)}{\partial\eta}\right]+\right.\\
&+\frac{1}{J}\frac{\partial}{\partial\eta}\left[\frac{\nu h}{J}\left(2y_\xi y_\eta\frac{\partial(q/h)}{\partial\xi}-2y_\xi y_\xi\frac{\partial(q/h)}{\partial\eta}-x_\xi y_\eta\frac{\partial(r/h)}{\partial\xi}+x_\xi y_\xi\frac{\partial(r/h)}{\partial\eta}+\right.\right.\\
&\left.\left.-x_\xi x_\eta\frac{\partial(q/h)}{\partial\xi}+x_\xi^2\frac{\partial(q/h)}{\partial\eta}\right]\right.
\end{aligned}
\tag{A.45}
$$

$$
\begin{aligned}
=&-\frac{1}{J}\frac{\partial}{\partial\xi}\left[\frac{\nu h}{J}\left(-x_\eta\frac{\partial}{\partial\xi}\left(x_\eta\frac{q}{h}+y_\eta\frac{r}{h}\right)+x_\eta\frac{\partial}{\partial\eta}\left(x_\xi\frac{q}{h}+y_\xi\frac{r}{h}\right)+\right.\right.\\
&\left.\left.+2y_\eta\frac{\partial(y_\eta\, q/h)}{\partial\xi}-2y_\eta\frac{\partial(y_\xi\, q/h)}{\partial\eta}\right]\right.\\
&+\frac{1}{J}\frac{\partial}{\partial\eta}\left[\frac{\nu h}{J}\left(-x_\xi\frac{\partial}{\partial\xi}\left(x_\eta\frac{q}{h}+y_\eta\frac{r}{h}\right)+x_\xi\frac{\partial}{\partial\eta}\left(x_\xi\frac{q}{h}+y_\xi\frac{r}{h}\right)+\right.\right.\\
&\left.\left.+2y_\xi\frac{\partial(y_\eta\, q/h)}{\partial\xi}-2y_\xi\frac{\partial(y_\xi\, q/h)}{\partial\eta}\right]\right.
\end{aligned}
\tag{A.46}
$$

for the y-momentum equation the viscous terms can be transformed as:

$$
-\frac{\partial}{\partial x}\left(\nu h\frac{\partial(r/h)}{\partial x}+\nu h\frac{\partial(q/h)}{\partial y}\right)-\frac{\partial}{\partial y}\left(2\nu h\frac{\partial(r/h)}{\partial y}\right)=
\tag{A.47}
$$

$$
\begin{aligned}
=&-\frac{y_\eta}{J}\frac{\partial}{\partial\xi}\left(\nu h\frac{\partial(r/h)}{\partial x}+\nu h\frac{\partial(q/h)}{\partial y}\right)+\frac{y_\xi}{J}\frac{\partial}{\partial\eta}\left(\nu h\frac{\partial(r/h)}{\partial x}+\nu h\frac{\partial(q/h)}{\partial y}\right)+\\
&+\frac{x_\eta}{J}\frac{\partial}{\partial\xi}\left(2\nu h\frac{\partial(r/h)}{\partial y}\right)-\frac{x_\xi}{J}\frac{\partial}{\partial\eta}\left(2\nu h\frac{\partial(r/h)}{\partial y}\right)=
\end{aligned}
\tag{A.48}
$$

$$
\begin{aligned}
=&-\frac{1}{J}\frac{\partial}{\partial\xi}\left[\nu h y_\eta\frac{\partial(r/h)}{\partial x}+\nu h y_\eta\frac{\partial(q/h)}{\partial y}-2\nu h x_\eta\frac{\partial(r/h)}{\partial y}\right]+\\
&+\frac{1}{J}\frac{\partial}{\partial\eta}\left[\nu h y_\xi\frac{\partial(r/h)}{\partial x}+\nu h y_\xi\frac{\partial(q/h)}{\partial y}-2\nu h x_\xi\frac{\partial(r/h)}{\partial y}\right]=
\end{aligned}
\tag{A.49}
$$

$$
\begin{aligned}
=&-\frac{1}{J}\frac{\partial}{\partial\xi}\left[\frac{\nu h y_\eta y_\eta}{J}\frac{\partial(r/h)}{\partial\xi}-\frac{\nu h y_\eta y_\xi}{J}\frac{\partial(r/h)}{\partial\eta}+\frac{\nu h y_\eta x_\eta}{J}\frac{\partial(q/h)}{\partial\xi}-\frac{\nu h y_\eta x_\xi}{J}\frac{\partial(q/h)}{\partial\eta}+\right.\\
&\left.\qquad+\frac{2\nu h x_\eta^2}{J}\frac{\partial(r/h)}{\partial\xi}-\frac{2\nu h x_\xi x_\eta}{J}\frac{\partial(r/h)}{\partial\eta}\right]+\\
&+\frac{1}{J}\frac{\partial}{\partial\eta}\left[\frac{\nu h y_\xi y_\eta}{J}\frac{\partial(r/h)}{\partial\xi}-\frac{\nu h y_\xi y_\xi}{J}\frac{\partial(r/h)}{\partial\eta}+\frac{\nu h y_\xi x_\eta}{J}\frac{\partial(q/h)}{\partial\xi}-\frac{\nu h y_\xi x_\xi}{J}\frac{\partial(q/h)}{\partial\eta}+\right.\\
&\left.\qquad+\frac{2\nu h x_\eta x_\xi}{J}\frac{\partial(r/h)}{\partial\xi}-\frac{2\nu h x_\xi^2}{J}\frac{\partial(r/h)}{\partial\eta}\right]=
\end{aligned}
\tag{A.50}
$$

$$
\begin{aligned}
=&-\frac{1}{J}\frac{\partial}{\partial\xi}\left[\frac{\nu h}{J}\left(y_\eta^2\frac{\partial(r/h)}{\partial\xi}-y_\eta y_\xi\frac{\partial(r/h)}{\partial\eta}+y_\eta x_\eta\frac{\partial(q/h)}{\partial\xi}-y_\eta x_\xi\frac{\partial(q/h)}{\partial\eta}+\right.\right.\\
&\left.\left.\qquad+2x_\eta^2\frac{\partial(r/h)}{\partial\xi}-2x_\xi x_\eta\frac{\partial(r/h)}{\partial\eta}\right)\right]+\\
&+\frac{1}{J}\frac{\partial}{\partial\eta}\left[\frac{\nu h}{J}\left(y_\xi y_\eta\frac{\partial(r/h)}{\partial\xi}-y_\xi^2\frac{\partial(r/h)}{\partial\eta}+y_\xi x_\eta\frac{\partial(q/h)}{\partial\xi}-y_\xi x_\xi\frac{\partial(q/h)}{\partial\eta}+\right.\right.\\
&\left.\left.\qquad+2x_\eta x_\xi\frac{\partial(r/h)}{\partial\xi}-2x_\xi^2\frac{\partial(r/h)}{\partial\eta}\right)\right]=
\end{aligned}
\tag{A.51}
$$

$$
\begin{aligned}
=&-\frac{1}{J}\frac{\partial}{\partial\xi}\left[\frac{\nu h}{J}\left(y_\eta\frac{\partial}{\partial\xi}\left(y_\eta\frac{r}{h}+x_\eta\frac{q}{h}\right)-y_\eta\frac{\partial}{\partial\eta}\left(y_\xi\frac{r}{h}+x_\xi\frac{q}{h}\right)+\right.\right.\\
&\left.\left.\qquad+2x_\eta\frac{\partial(x_\eta\, r/h)}{\partial\xi}-2x_\eta\frac{\partial(x_\xi\, r/h)}{\partial\eta}\right)\right]\\
&+\frac{1}{J}\frac{\partial}{\partial\eta}\left[\frac{\nu h}{J}\left(y_\xi\frac{\partial}{\partial\xi}\left(y_\eta\frac{r}{h}+x_\eta\frac{q}{h}\right)+\right.\right.\\
&\left.\left.\qquad-y_\xi\frac{\partial}{\partial\eta}\left(y_\xi\frac{r}{h}+x_\xi\frac{q}{h}\right)+2x_\xi\frac{\partial(x_\eta\, r/h)}{\partial\xi}-2x_\xi\frac{\partial(x_\xi\, r/h)}{\partial\eta}\right)\right].
\end{aligned}
\tag{A.52}
$$

A.2 Transformed 2D Shallow Water Equations

Given the above, the 2D Shallow water equations can be transformed to generalized coordinates, resulting in:

$$
J\frac{\partial h}{\partial t}+\frac{\partial}{\partial\xi}\left(y_\eta\, q-x_\eta\, r\right)+\frac{\partial}{\partial\eta}\left(-y_\xi\, q+x_\xi\, r\right)=0, \tag{A.53}
$$

$$
\begin{aligned}
&J\frac{\partial q}{\partial t}+\frac{\partial}{\partial \xi}\left(y_\eta\,\frac{q^2}{h}-x_\eta\,\frac{qr}{h}\right)+\frac{\partial}{\partial \eta}\left(-y_\xi\,\frac{q^2}{h}+x_\xi\,\frac{qr}{h}\right)+gh\left(y_\eta\,\frac{\partial \zeta}{\partial \xi}-y_\xi\,\frac{\partial \zeta}{\partial \eta}\right)+\\
&\quad+Jc_f\frac{q\,|\mathbf{q}|}{h^2}+\frac{\partial}{\partial \xi}\left[\frac{\nu h}{J}\left(x_\eta\frac{\partial}{\partial \xi}\left(x_\eta\frac{q}{h}+y_\eta\frac{r}{h}\right)-x_\eta\frac{\partial}{\partial \eta}\left(x_\xi\frac{q}{h}+y_\xi\frac{r}{h}\right)+\right.\right.\\
&\quad\left.\left.-\,2y_\eta\frac{\partial(y_\eta\,q/h)}{\partial \xi}+2y_\eta\frac{\partial(y_\xi\,q/h)}{\partial \eta}\right)\right]+\frac{\partial}{\partial \eta}\left[\frac{\nu h}{J}\left(-\,x_\xi\frac{\partial}{\partial \xi}\left(x_\eta\frac{q}{h}+y_\eta\frac{r}{h}\right)+\right.\right.\\
&\quad\left.\left.+\,x_\xi\frac{\partial}{\partial \eta}\left(x_\xi\frac{q}{h}+y_\xi\frac{r}{h}\right)+2y_\xi\frac{\partial(y_\eta\,q/h)}{\partial \xi}-2y_\xi\frac{\partial(y_\xi\,q/h)}{\partial \eta}\right)\right]=0,
\end{aligned}
\tag{A.54}
$$

$$
\begin{aligned}
&J\frac{\partial r}{\partial t}+\frac{\partial}{\partial \xi}\left(y_\eta\,\frac{qr}{h}-x_\eta\,\frac{r^2}{h}\right)+\frac{\partial}{\partial \eta}\left(-y_\xi\,\frac{qr}{h}+x_\xi\,\frac{r^2}{h}\right)+gh\left(-x_\eta\,\frac{\partial \zeta}{\partial \xi}+x_\xi\,\frac{\partial \zeta}{\partial \eta}\right)+\\
&\quad+Jc_f\frac{r\,|\mathbf{q}|}{h^2}+\frac{\partial}{\partial \xi}\left[\frac{\nu h}{J}\left(-\,y_\eta\frac{\partial}{\partial \xi}\left(x_\eta\frac{q}{h}+y_\eta\frac{r}{h}\right)+\right.\right.\\
&\quad\left.\left.+\,y_\eta\frac{\partial}{\partial \eta}\left(x_\xi\frac{q}{h}+y_\xi\frac{r}{h}\right)-2x_\eta\frac{\partial(x_\eta\,r/h)}{\partial \xi}+2x_\eta\frac{\partial(x_\xi\,r/h)}{\partial \eta}\right)\right]+\\
&\quad+\frac{\partial}{\partial \eta}\left[\frac{\nu h}{J}\left(y_\xi\frac{\partial}{\partial \xi}\left(x_\eta\frac{q}{h}+y_\eta\frac{r}{h}\right)+\right.\right.\\
&\quad\left.\left.-\,y_\xi\frac{\partial}{\partial \eta}\left(x_\xi\frac{q}{h}+y_\xi\frac{r}{h}\right)+2x_\xi\frac{\partial(x_\eta\,r/h)}{\partial \xi}-2x_\xi\frac{\partial(x_\xi\,r/h)}{\partial \eta}\right)\right]=0.
\end{aligned}
\tag{A.55}
$$

with transformation coefficients presented using the subscript notation. However, the above equations are in differential form. We will write the equations in an integral form:

$$
\int\frac{\partial \mathbf{U}}{\partial t}dV+\int\nabla\cdot\mathbf{F}dV-\int\mathbf{S}dV=0 \tag{A.56}
$$

where the ∇-operator for use in the $\xi\eta$-grid, now reads:

$$
\nabla=\begin{pmatrix}\frac{\partial}{\partial\xi}\\ \frac{\partial}{\partial\eta}\end{pmatrix}, \tag{A.57}
$$

However, this definition is not necessary yet as we will be applying Gauss' theorem (for internal points) to equation (A.56), which yields:

$$
\int\frac{\partial \mathbf{U}}{\partial t}dV+\oint\mathbf{F}\cdot\mathbf{n}dS-\int\mathbf{S}dV=0, \tag{A.58}
$$

where $\mathbf{F}$ consists of continuity, convective and viscous contributions which are split in ξ and η directions:

$$
\mathbf{F}_{\texttt{cont},\xi}=\begin{pmatrix}y_\eta\,q-x_\eta\,r\\0\\0\end{pmatrix},\quad \mathbf{F}_{\texttt{cont},\eta}=\begin{pmatrix}-y_\xi\,q+x_\xi\,r\\0\\0\end{pmatrix} \tag{A.59}
$$

$$
\mathbf{F}_{\texttt{conv},\xi} = \begin{pmatrix} 0 \\ y_\eta \frac{q^2}{h} - x_\eta \frac{qr}{h} \\ y_\eta \frac{qr}{h} - x_\eta \frac{r^2}{h} \end{pmatrix}, \quad \mathbf{F}_{\texttt{conv},\eta} = \begin{pmatrix} 0 \\ -y_\xi \frac{q^2}{h} + x_\xi \frac{qr}{h} \\ -y_\xi \frac{qr}{h} + x_\xi \frac{r^2}{h} \end{pmatrix} \tag{A.60}
$$

$$
\mathbf{F}_{\texttt{visc},\xi} = \begin{pmatrix} 0 \\ \frac{\nu h}{J}\left[x_\eta \frac{\partial}{\partial \xi}\left(x_\eta \frac{q}{h} + y_\eta \frac{r}{h}\right) - x_\eta \frac{\partial}{\partial \eta}\left(x_\xi \frac{q}{h} + y_\xi \frac{r}{h}\right) - 2y_\eta \frac{\partial (y_\eta\ q/h)}{\partial \xi} + 2y_\eta \frac{\partial (y_\xi\ q/h)}{\partial \eta}\right] \\ \frac{\nu h}{J}\left[- y_\eta \frac{\partial}{\partial \xi}\left(x_\eta \frac{q}{h} + y_\eta \frac{r}{h}\right) + y_\eta \frac{\partial}{\partial \eta}\left(x_\xi \frac{q}{h} + y_\xi \frac{r}{h}\right) - 2x_\eta \frac{\partial (x_\eta\ r/h)}{\partial \xi} + 2x_\eta \frac{\partial (x_\xi\ r/h)}{\partial \eta}\right] \end{pmatrix}. \tag{A.61}
$$

$$
\mathbf{F}_{\texttt{visc},\eta} = \begin{pmatrix} 0 \\ \frac{\nu h}{J}\left[- x_\xi \frac{\partial}{\partial \xi}\left(x_\eta \frac{q}{h} + y_\eta \frac{r}{h}\right) + x_\xi \frac{\partial}{\partial \eta}\left(x_\xi \frac{q}{h} + y_\xi \frac{r}{h}\right) + 2y_\xi \frac{\partial (y_\eta\ q/h)}{\partial \xi} - 2y_\xi \frac{\partial (y_\xi\ q/h)}{\partial \eta}\right] \\ \frac{\nu h}{J}\left[y_\xi \frac{\partial}{\partial \xi}\left(x_\eta \frac{q}{h} + y_\eta \frac{r}{h}\right) - y_\xi \frac{\partial}{\partial \eta}\left(x_\xi \frac{q}{h} + y_\xi \frac{r}{h}\right) + 2x_\xi \frac{\partial (x_\eta\ r/h)}{\partial \xi} - 2x_\xi \frac{\partial (x_\xi\ r/h)}{\partial \eta}\right] \end{pmatrix} \tag{A.62}
$$

Applying the chain rule to the derivatives of combined functions yields:

$$
\mathbf{F}_{\texttt{visc},\xi} = \begin{pmatrix} 0 \\
\frac{\nu h}{J}\Bigg[\underbrace{-\frac{x_\eta q}{h^2}\frac{\partial (x_\eta\ h)}{\partial \xi} + \frac{x_\eta}{h}\frac{\partial (x_\eta\ q)}{\partial \xi} - \frac{x_\eta r}{h^2}\frac{\partial (y_\eta\ h)}{\partial \xi} + \frac{x_\eta}{h}\frac{\partial (y_\eta\ r)}{\partial \xi}}_{x_\eta \frac{\partial}{\partial \xi}\left(x_\eta \frac{q}{h} + y_\eta \frac{r}{h}\right)} \\
\underbrace{-\frac{x_\eta}{h}\frac{\partial (x_\xi\ q)}{\partial \eta} + \frac{x_\eta q}{h^2}\frac{\partial (x_\xi\ h)}{\partial \eta} - \frac{x_\eta}{h}\frac{\partial (y_\xi\ r)}{\partial \eta} + \frac{x_\eta r}{h^2}\frac{\partial (y_\xi\ h)}{\partial \eta}}_{-x_\eta \frac{\partial}{\partial \eta}\left(x_\xi \frac{q}{h} + y_\xi \frac{r}{h}\right)} \\
\underbrace{-\frac{2y_\eta}{h}\frac{\partial (y_\eta\ q)}{\partial \xi} + \frac{2y_\eta q}{h^2}\frac{\partial (y_\eta\ h)}{\partial \xi} + \frac{2y_\eta}{h}\frac{\partial (y_\xi\ q)}{\partial \eta} - \frac{2y_\eta q}{h^2}\frac{\partial (y_\xi\ h)}{\partial \eta}}_{-2y_\eta \frac{\partial (x_\eta\ q/h)}{\partial \xi} + y x_\eta \frac{\partial (x_\xi\ q/h)}{\partial \eta}}\Bigg] \\
\frac{\nu h}{J}\Bigg[\underbrace{-\frac{y_\eta}{h}\frac{\partial (x_\eta\ q)}{\partial \xi} + \frac{y_\eta\ q}{h^2}\frac{\partial (x_\eta\ h)}{\partial \xi} - \frac{y_\eta}{h}\frac{\partial (y_\eta\ r)}{\partial \xi} + \frac{y_\eta\ r}{h^2}\frac{\partial (y_\eta\ h)}{\partial \xi}}_{-y_\eta \frac{\partial}{\partial \xi}\left(x_\eta \frac{q}{h} + y_\eta \frac{r}{h}\right)} \\
\underbrace{+\frac{y_\eta}{h}\frac{\partial (x_\xi\ q)}{\partial \eta} - \frac{y_\eta q}{h^2}\frac{\partial (x_\xi\ h)}{\partial \eta} + \frac{y_\eta}{h}\frac{\partial (y_\xi\ r)}{\partial \eta} - \frac{y_\eta r}{h^2}\frac{\partial (y_\xi\ h)}{\partial \eta}}_{y_\eta \frac{\partial}{\partial \eta}\left(x_\xi \frac{q}{h} + y_\xi \frac{r}{h}\right)} \\
\underbrace{-\frac{2x_\eta}{h}\frac{\partial (x_\eta\ r)}{\partial \xi} + \frac{2x_\eta r}{h^2}\frac{\partial (x_\eta\ h)}{\partial \xi} + \frac{2x_\eta}{h}\frac{\partial (x_\xi\ r)}{\partial \eta} - \frac{2x_\eta r}{h^2}\frac{\partial (x_\xi\ h)}{\partial \eta}}_{-2x_\eta \frac{\partial (x_\eta\ r/h)}{\partial \xi} + 2x_\eta \frac{\partial (x_\xi\ r/h)}{\partial \eta}}\Bigg]
\end{pmatrix} \tag{A.63}
$$

$$
\mathbf{F}_{\mathtt{visc},\eta} = \begin{pmatrix} 0 \\
\frac{\nu h}{J}\Bigg[\underbrace{-\frac{x_\xi}{h}\frac{\partial(x_\eta\, q)}{\partial\xi} + \frac{x_\xi\, q}{h^2}\frac{\partial(x_\eta\, h)}{\partial\xi} - \frac{x_\xi}{h}\frac{\partial(y_\eta\, r)}{\partial\xi} + \frac{x_\xi\, r}{h^2}\frac{\partial(y_\eta\, h)}{\partial\xi}}_{-x_\xi\frac{\partial}{\partial\xi}\left(x_\eta\frac{q}{h}+y_\eta\frac{r}{h}\right)} \\
\underbrace{+\frac{x_\xi}{h}\frac{\partial(x_\xi\, q)}{\partial\eta} - \frac{x_\xi\, q}{h^2}\frac{\partial(x_\xi\, h)}{\partial\eta} + \frac{y_\xi}{h}\frac{\partial(x_\xi\, r)}{\partial\eta} - \frac{y_\xi\, r}{h^2}\frac{\partial(x_\xi\, h)}{\partial\eta}}_{x_\xi\frac{\partial}{\partial\eta}\left(x_\xi\frac{q}{h}+y_\xi\frac{r}{h}\right)} \\
\underbrace{+\frac{2y_\xi}{h}\frac{\partial(y_\eta\, q)}{\partial\xi} - \frac{2y_\xi\, q}{h^2}\frac{\partial(y_\eta\, h)}{\partial\xi} - \frac{2y_\xi}{h}\frac{\partial(y_\xi\, q)}{\partial\eta} + \frac{2y_\xi\, q}{h^2}\frac{\partial(y_\xi\, h)}{\partial\eta}}_{2y_\xi\frac{\partial(y_\eta\, q/h)}{\partial\xi}-2y_\xi\frac{\partial(y_\xi\, q/h)}{\partial\eta}}\Bigg] \\
\frac{\nu h}{J}\Bigg[\underbrace{\frac{y_\xi}{h}\frac{\partial(x_\eta\, q)}{\partial\xi} - \frac{y_\xi\, q}{h^2}\frac{\partial(x_\eta\, h)}{\partial\xi} + \frac{y_\xi}{h}\frac{\partial(y_\eta\, r)}{\partial\xi} + \frac{y_\xi\, r}{h^2}\frac{\partial(y_\eta\, h)}{\partial\xi}}_{y_\xi\frac{\partial}{\partial\xi}\left(x_\eta\frac{q}{h}+y_\eta\frac{r}{h}\right)} \\
\underbrace{-\frac{y_\xi}{h}\frac{\partial(x_\xi\, q)}{\partial\eta} + \frac{y_\xi q}{h^2}\frac{\partial(x_\xi\, h)}{\partial\eta} - \frac{y_\xi}{h}\frac{\partial(y_\xi\, r)}{\partial\eta} - \frac{y_\xi\, r}{h^2}\frac{\partial(y_\xi\, h)}{\partial\eta}}_{-y_\xi\frac{\partial}{\partial\eta}\left(x_\xi\frac{q}{h}+y_\xi\frac{r}{h}\right)} \\
\underbrace{+\frac{2x_\xi}{h}\frac{\partial(x_\eta\, r)}{\partial\xi} - \frac{2x_\xi\, r}{h^2}\frac{\partial(x_\eta\, h)}{\partial\xi} - \frac{2x_\xi}{h}\frac{\partial(x_\xi\, r)}{\partial\eta} + \frac{2x_\xi\, r}{h^2}\frac{\partial(x_\xi\, h)}{\partial\eta}}_{2x_\xi\frac{\partial(x_\eta\, r/h)}{\partial\xi}-2x_\xi\frac{\partial(x_\xi\, r/h)}{\partial\eta}}\Bigg]
\end{pmatrix} \tag{A.64}
$$

The source terms are split again into pressure gradient and bed shear stress-based sources, $\mathbf{S} = \mathbf{S}_{\mathrm{pg}} + \mathbf{S}_{\mathrm{bs}}$. Note that the expression for the bed shear stress matches that presented before in **??** due to the absence of spatial derivatives.

$$
\mathbf{S}_{\mathrm{pg}} = \begin{pmatrix} 0 \\ gh\left(y_\eta\,\frac{\partial\zeta}{\partial\xi} - y_\xi\,\frac{\partial\zeta}{\partial\eta}\right) \\ gh\left(-x_\eta\,\frac{\partial\zeta}{\partial\xi} + x_\xi\,\frac{\partial\zeta}{\partial\eta}\right) \end{pmatrix}, \quad \mathbf{S}_{\mathrm{bs}} = \begin{pmatrix} 0 \\ -Jc_f\frac{q|\mathbf{q}|}{h^2} \\ -Jc_f\frac{r|\mathbf{q}|}{h^2} \end{pmatrix}, \tag{A.65}
$$

A.3 Transformed Jacobians

We will derive the Jacobians in the numerical grid from the above definitions of the flux and source terms.

A.3.1 Continuity Flux Jacobians

In ξ, the continuity flux and its Jacobian read:

$$\mathbf{F}_{\texttt{cont},\xi} = \begin{pmatrix} y_\eta\, q - x_\eta\, r \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{J}^{CT,\xi} = \begin{pmatrix} 0 & y_\eta & -x_\eta \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{A.66}$$

The continuity flux in η and its Jacobian read:

$$\mathbf{F}_{\texttt{cont},\eta} = \begin{pmatrix} -y_\xi\, q + x_\xi\, r \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{J}^{CT,\eta} = \begin{pmatrix} 0 & -y_\xi & x_\xi \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{A.67}$$

A.3.2 Convective Flux Jacobians

In ξ, the convective flux and its Jacobian read:

$$\mathbf{F}_{\texttt{conv},\xi} = \begin{pmatrix} 0 \\ y_\eta \frac{q^2}{h} - x_\eta \frac{qr}{h} \\ y_\eta \frac{qr}{h} - x_\eta \frac{r^2}{h} \end{pmatrix}, \quad \mathbf{J}^{CF,\xi} = \begin{pmatrix} 0 & 0 & 0 \\ -y_\eta \frac{q^2}{h^2} + x_\eta \frac{qr}{h^2} & y_\eta \frac{2q}{h} - x_\eta \frac{r}{h} & -x_\eta \frac{q}{h} \\ -y_\eta \frac{qr}{h^2} + x_\eta \frac{r^2}{h^2} & y_\eta \frac{q}{h} & y_\eta \frac{q}{h} - x_\eta \frac{2r}{h} \end{pmatrix}. \tag{A.68}$$

The convective flux in η and its Jacobian read:

$$\mathbf{F}_{\texttt{conv},\eta} = \begin{pmatrix} 0 \\ y_\eta \frac{q^2}{h} - x_\eta \frac{qr}{h} \\ y_\eta \frac{qr}{h} - x_\eta \frac{r^2}{h} \end{pmatrix}, \quad \mathbf{J}^{CF,\eta} = \begin{pmatrix} 0 & 0 & 0 \\ -y_\eta \frac{q^2}{h^2} + x_\eta \frac{qr}{h^2} & y_\eta \frac{2q}{h} - x_\eta \frac{r}{h} & -x_\eta \frac{q}{h} \\ -y_\eta \frac{qr}{h^2} + x_\eta \frac{r^2}{h^2} & y_\eta \frac{r}{h} & y_\eta \frac{q}{h} - x_\eta \frac{2r}{h} \end{pmatrix}. \tag{A.69}$$

A.3.3 Viscous Flux Jacobians

The viscous flux terms are not repeated here for brevity. The jacobians of $\mathbf{F}_{\texttt{conv},\xi}$ and $\mathbf{F}_{\texttt{conv},\eta}$ are extensive and split up in their respective columns, for example:

$$\mathbf{J}^{\mathrm{F},\xi} = \begin{pmatrix} \dfrac{\partial \mathbf{J}^{\mathrm{F},\xi}}{\partial h} & \dfrac{\partial \mathbf{J}^{\mathrm{F},\xi}}{\partial q} & \dfrac{\partial \mathbf{J}^{\mathrm{F},\xi}}{\partial r} \end{pmatrix} \tag{A.70}$$

where the column vectors are:

$$\frac{\partial \mathbf{J}^{\mathrm{VF},\xi}}{\partial h} = \begin{pmatrix} 0 \\[2ex]
\begin{aligned}
&\frac{\nu}{J}\Bigg[-\frac{x_\eta q}{h^2}\frac{\partial(x_\eta\ h)}{\partial\xi} + \frac{x_\eta}{h}\frac{\partial(x_\eta\ q)}{\partial\xi} - \frac{x_\eta r}{h^2}\frac{\partial(y_\eta\ h)}{\partial\xi} + \frac{x_\eta}{h}\frac{\partial(y_\eta\ r)}{\partial\xi} \\
&\quad -\frac{x_\eta}{h}\frac{\partial(x_\xi\ q)}{\partial\eta} + \frac{x_\eta q}{h^2}\frac{\partial(x_\xi\ h)}{\partial\eta} - \frac{x_\eta}{h}\frac{\partial(y_\xi\ r)}{\partial\eta} + \frac{x_\eta r}{h^2}\frac{\partial(y_\xi\ h)}{\partial\eta} \\
&\quad -\frac{2y_\eta}{h}\frac{\partial(y_\eta\ q)}{\partial\xi} + \frac{2y_\eta q}{h^2}\frac{\partial(y_\eta\ h)}{\partial\xi} + \frac{2y_\eta}{h}\frac{\partial(y_\xi\ q)}{\partial\eta} - \frac{2y_\eta q}{h^2}\frac{\partial(y_\xi\ h)}{\partial\eta}\Bigg] \\
&+\frac{\nu h}{J}\Bigg[\frac{x_\eta q}{2h^3}\frac{\partial(x_\eta\ h)}{\partial\xi} - \frac{x_\eta q}{h^2}\frac{\partial(x_\eta\ \cdot)}{\partial\xi} - \frac{x_\eta}{h^2}\frac{\partial(x_\eta\ q)}{\partial\xi} + \frac{x_\eta r}{2h^3}\frac{\partial(y_\eta\ h)}{\partial\xi} \\
&\quad -\frac{x_\eta r}{h^2}\frac{\partial(y_\eta\ \cdot)}{\partial\xi} - \frac{x_\eta}{h^2}\frac{\partial(y_\eta\ r)}{\partial\xi} + \frac{x_\eta}{h^2}\frac{\partial(x_\xi\ q)}{\partial\eta} - \frac{x_\eta q}{2h^3}\frac{\partial(x_\xi\ h)}{\partial\eta} \\
&\quad +\frac{x_\eta q}{h^2}\frac{\partial(x_\xi\ \cdot)}{\partial\eta} + \frac{x_\eta}{h^2}\frac{\partial(y_\xi\ r)}{\partial\eta} - \frac{x_\eta r}{2h^3}\frac{\partial(y_\xi\ h)}{\partial\eta} + \frac{2y_\eta}{h^2}\frac{\partial(y_\eta\ q)}{\partial\xi} \\
&\quad -\frac{2y_\eta q}{2h^3}\frac{\partial(y_\eta\ h)}{\partial\xi} + \frac{2y_\eta q}{h^2}\frac{\partial(y_\eta\ \cdot)}{\partial\xi} - \frac{2y_\eta}{h^2}\frac{\partial(y_\xi\ q)}{\partial\eta} + \frac{2y_\eta q}{2h^3}\frac{\partial(y_\xi\ h)}{\partial\eta} - \frac{2y_\eta q}{h^2}\frac{\partial(y_\xi\ \cdot)}{\partial\eta}\Bigg]
\end{aligned} \\[2ex]
\begin{aligned}
&\frac{\nu}{J}\Bigg[-\frac{y_\eta}{h}\frac{\partial(x_\eta\ q)}{\partial\xi} + \frac{y_\eta\ q}{h^2}\frac{\partial(x_\eta\ h)}{\partial\xi} - \frac{y_\eta}{h}\frac{\partial(y_\eta\ r)}{\partial\xi} + \frac{y_\eta\ r}{h^2}\frac{\partial(y_\eta\ h)}{\partial\xi} \\
&\quad +\frac{y_\eta}{h}\frac{\partial(x_\xi\ q)}{\partial\eta} - \frac{y_\eta q}{h^2}\frac{\partial(x_\xi\ h)}{\partial\eta} + \frac{y_\eta}{h}\frac{\partial(y_\xi\ r)}{\partial\eta} - \frac{y_\eta r}{h^2}\frac{\partial(y_\xi\ h)}{\partial\eta} \\
&\quad -\frac{2x_\eta}{h}\frac{\partial(x_\eta\ r)}{\partial\xi} + \frac{2x_\eta r}{h^2}\frac{\partial(x_\eta\ h)}{\partial\xi} + \frac{2x_\eta}{h}\frac{\partial(x_\xi\ r)}{\partial\eta} - \frac{2x_\eta r}{h^2}\frac{\partial(x_\xi\ h)}{\partial\eta}\Bigg] \\
&+\frac{\nu h}{J}\Bigg[\frac{y_\eta}{h^2}\frac{\partial(x_\eta\ q)}{\partial\xi} - \frac{y_\eta\ q}{2h^3}\frac{\partial(x_\eta\ h)}{\partial\xi} + \frac{y_\eta\ q}{h^2}\frac{\partial(x_\eta\ \cdot)}{\partial\xi} + \frac{y_\eta}{h^2}\frac{\partial(y_\eta\ r)}{\partial\xi} \\
&\quad -\frac{y_\eta\ r}{2h^3}\frac{\partial(y_\eta\ h)}{\partial\xi} + \frac{y_\eta\ r}{h^2}\frac{\partial(y_\eta\ \cdot)}{\partial\xi} - \frac{y_\eta}{h^2}\frac{\partial(x_\xi\ q)}{\partial\eta} + \frac{y_\eta q}{2h^3}\frac{\partial(x_\xi\ h)}{\partial\eta} \\
&\quad -\frac{y_\eta q}{h^2}\frac{\partial(x_\xi\ \cdot)}{\partial\eta} - \frac{y_\eta}{h^2}\frac{\partial(y_\xi\ r)}{\partial\eta} + \frac{y_\eta r}{2h^3}\frac{\partial(y_\xi\ h)}{\partial\eta} - \frac{y_\eta r}{h^2}\frac{\partial(y_\xi\ \cdot)}{\partial\eta} \\
&\quad +\frac{2x_\eta}{h^2}\frac{\partial(x_\eta\ r)}{\partial\xi} - \frac{2x_\eta r}{2h^3}\frac{\partial(x_\eta\ h)}{\partial\xi} + \frac{2x_\eta r}{h^2}\frac{\partial(x_\eta\ \cdot)}{\partial\xi} - \frac{2x_\eta}{h^2}\frac{\partial(x_\xi\ r)}{\partial\eta} \\
&\quad +\frac{2x_\eta r}{2h^3}\frac{\partial(x_\xi\ h)}{\partial\eta} - \frac{2x_\eta r}{h^2}\frac{\partial(x_\xi\ \cdot)}{\partial\eta}\Bigg]
\end{aligned}
\end{pmatrix} \tag{A.71}$$

$$\frac{\partial \mathbf{J}^{\mathrm{VF},\xi}}{\partial q} = \begin{pmatrix} 0 \\[2ex]
\begin{aligned}
&\frac{\nu h}{J}\Bigg[-\frac{x_\eta}{h^2}\frac{\partial(x_\eta\ h)}{\partial\xi} + \frac{x_\eta}{h}\frac{\partial(x_\eta\ \cdot)}{\partial\xi} - \frac{x_\eta}{h}\frac{\partial(x_\xi\ \cdot)}{\partial\eta} + \frac{x_\eta}{h^2}\frac{\partial(x_\xi\ h)}{\partial\eta} \\
&\quad -\frac{2y_\eta}{h}\frac{\partial(y_\eta\ \cdot)}{\partial\xi} + \frac{2y_\eta}{h^2}\frac{\partial(y_\eta\ h)}{\partial\xi} + \frac{2y_\eta}{h}\frac{\partial(y_\xi\ )}{\partial\eta} - \frac{2y_\eta}{h^2}\frac{\partial(y_\xi\ h)}{\partial\eta}\Bigg]
\end{aligned} \\[2ex]
\frac{\nu h}{J}\Bigg[-\frac{y_\eta}{h}\frac{\partial(x_\eta\ \cdot)}{\partial\xi} + \frac{y_\eta}{h^2}\frac{\partial(x_\eta\ h)}{\partial\xi} + \frac{y_\eta}{h}\frac{\partial(x_\xi\ \cdot)}{\partial\eta} - \frac{y_\eta}{h^2}\frac{\partial(x_\xi\ h)}{\partial\eta}\Bigg]
\end{pmatrix}, \tag{A.72}$$

and

$$
\frac{\partial \mathbf{J}^{\mathrm{VF},\xi}}{\partial r} = \left(\begin{array}{c}
0 \\[2em]
\frac{\nu h}{J}\Bigg[-\frac{x_\eta}{h^2}\frac{\partial(y_\eta\ h)}{\partial \xi} + \frac{x_\eta}{h}\frac{\partial(y_\eta\ \cdot)}{\partial \xi} \\
-\frac{x_\eta}{h}\frac{\partial(y_\xi\ \cdot)}{\partial \eta} + \frac{x_\eta}{h^2}\frac{\partial(y_\xi\ h)}{\partial \eta}\Bigg] \\
\frac{\nu h}{J}\Bigg[-\frac{y_\eta}{h}\frac{\partial(y_\eta\ \cdot)}{\partial \xi} + \frac{y_\eta}{h^2}\frac{\partial(y_\eta\ h)}{\partial \xi} \\
+\frac{y_\eta}{h}\frac{\partial(y_\xi)}{\partial \eta} - \frac{y_\eta}{h^2}\frac{\partial(y_\xi\ h)}{\partial \eta} \\
-\frac{2x_\eta}{h}\frac{\partial(x_\eta\ \cdot)}{\partial \xi} + \frac{2x_\eta}{h^2}\frac{\partial(x_\eta\ h)}{\partial \xi} + \frac{2x_\eta}{h}\frac{\partial(x_\xi\ \cdot)}{\partial \eta} - \frac{2x_\eta}{h^2}\frac{\partial(x_\xi\ h)}{\partial \eta}\Bigg]
\end{array} \right), \tag{A.73}
$$

and in the η-direction:

$$
\frac{\partial \mathbf{J}^{\mathrm{VF},\eta}}{\partial h} = \left(\begin{array}{c}
0 \\[2em]
\frac{\nu}{J}\Bigg[-\frac{x_\xi}{h}\frac{\partial(x_\eta\ q)}{\partial \xi} + \frac{x_\xi\ q}{h^2}\frac{\partial(x_\eta\ h)}{\partial \xi} - \frac{x_\xi}{h}\frac{\partial(y_\eta\ r)}{\partial \xi} + \frac{x_\xi\ r}{h^2}\frac{\partial(y_\eta\ h)}{\partial \xi} \\
+\frac{x_\xi}{h}\frac{\partial(x_\xi\ q)}{\partial \eta} - \frac{x_\xi\ q}{h^2}\frac{\partial(x_\xi\ h)}{\partial \eta} + \frac{y_\xi}{h}\frac{\partial(x_\xi\ r)}{\partial \eta} - \frac{y_\xi\ r}{h^2}\frac{\partial(x_\xi\ h)}{\partial \eta} \\
+\frac{2y_\xi}{h}\frac{\partial(y_\eta\ q)}{\partial \xi} - \frac{2y_\xi\ q}{h^2}\frac{\partial(y_\eta\ h)}{\partial \xi} - \frac{2y_\xi}{h}\frac{\partial(y_\xi\ q)}{\partial \eta} + \frac{2y_\xi\ q}{h^2}\frac{\partial(y_\xi\ h)}{\partial \eta}\Bigg] \\
+\frac{\nu h}{J}\Bigg[+\frac{x_\xi}{h^2}\frac{\partial(x_\eta\ q)}{\partial \xi} - \frac{x_\xi\ q}{2h^3}\frac{\partial(x_\eta\ h)}{\partial \xi} + \frac{x_\xi\ q}{h^2}\frac{\partial(x_\eta\ \cdot)}{\partial \xi} + \frac{x_\xi}{h^2}\frac{\partial(y_\eta\ r)}{\partial \xi} \\
-\frac{x_\xi\ r}{2h^3}\frac{\partial(y_\eta\ h)}{\partial \xi} + \frac{x_\xi\ r}{h^2}\frac{\partial(y_\eta\ \cdot)}{\partial \xi} - \frac{x_\xi}{h^2}\frac{\partial(x_\xi\ q)}{\partial \eta} + \frac{x_\xi\ q}{2h^3}\frac{\partial(x_\xi\ h)}{\partial \eta} \\
-\frac{x_\xi\ q}{h^2}\frac{\partial(x_\xi\ \cdot)}{\partial \eta} - \frac{y_\xi}{h^2}\frac{\partial(x_\xi\ r)}{\partial \eta} + \frac{y_\xi\ r}{2h^3}\frac{\partial(x_\xi\ h)}{\partial \eta} - \frac{y_\xi\ r}{h^2}\frac{\partial(x_\xi\ \cdot)}{\partial \eta} \\
-\frac{2y_\xi}{h^2}\frac{\partial(y_\eta\ q)}{\partial \xi} + \frac{2y_\xi\ q}{2h^3}\frac{\partial(y_\eta\ h)}{\partial \xi} - \frac{2y_\xi\ q}{h^2}\frac{\partial(y_\eta\ \cdot)}{\partial \xi} + \frac{2y_\xi}{h^2}\frac{\partial(y_\xi\ q)}{\partial \eta} \\
-\frac{2y_\xi\ q}{2h^3}\frac{\partial(y_\xi\ h)}{\partial \eta} + \frac{2y_\xi\ q}{h^2}\frac{\partial(y_\xi\ \cdot)}{\partial \eta}\Bigg] \\[2em]
\frac{\nu}{J}\Bigg[\frac{y_\xi}{h}\frac{\partial(x_\eta\ q)}{\partial \xi} - \frac{y_\xi\ q}{h^2}\frac{\partial(x_\eta\ h)}{\partial \xi} + \frac{y_\xi}{h}\frac{\partial(y_\eta\ r)}{\partial \xi} + \frac{y_\xi\ r}{h^2}\frac{\partial(y_\eta\ h)}{\partial \xi} \\
-\frac{y_\xi}{h}\frac{\partial(x_\xi\ q)}{\partial \eta} + \frac{y_\xi q}{h^2}\frac{\partial(x_\xi\ h)}{\partial \eta} - \frac{y_\xi}{h}\frac{\partial(y_\xi\ r)}{\partial \eta} - \frac{y_\xi\ r}{h^2}\frac{\partial(y_\xi\ h)}{\partial \eta} \\
+\frac{2x_\xi}{h}\frac{\partial(x_\eta\ r)}{\partial \xi} - \frac{2x_\xi\ r}{h^2}\frac{\partial(x_\eta\ h)}{\partial \xi} - \frac{2x_\xi}{h}\frac{\partial(x_\xi\ r)}{\partial \eta} + \frac{2x_\xi\ r}{h^2}\frac{\partial(x_\xi\ h)}{\partial \eta}\Bigg] \\
+\frac{\nu h}{J}\Bigg[-\frac{y_\xi}{h^2}\frac{\partial(x_\eta\ q)}{\partial \xi} + \frac{y_\xi\ q}{2h^3}\frac{\partial(x_\eta\ h)}{\partial \xi} - \frac{y_\xi\ q}{h^2}\frac{\partial(x_\eta\ \cdot)}{\partial \xi} - \frac{y_\xi}{h^2}\frac{\partial(y_\eta\ r)}{\partial \xi} \\
-\frac{y_\xi\ r}{2h^3}\frac{\partial(y_\eta\ h)}{\partial \xi} + \frac{y_\xi\ r}{h^2}\frac{\partial(y_\eta\ \cdot)}{\partial \xi} + \frac{y_\xi}{h^2}\frac{\partial(x_\xi\ q)}{\partial \eta} - \frac{y_\xi q}{2h^3}\frac{\partial(x_\xi\ h)}{\partial \eta} \\
+\frac{y_\xi q}{h^2}\frac{\partial(x_\xi\ \cdot)}{\partial \eta} + \frac{y_\xi}{h^2}\frac{\partial(y_\xi\ r)}{\partial \eta} + \frac{y_\xi\ r}{2h^3}\frac{\partial(y_\xi\ h)}{\partial \eta} - \frac{y_\xi\ r}{h^2}\frac{\partial(y_\xi\ \cdot)}{\partial \eta} \\
-\frac{2x_\xi}{h^2}\frac{\partial(x_\eta\ r)}{\partial \xi} + \frac{2x_\xi\ r}{2h^3}\frac{\partial(x_\eta\ h)}{\partial \xi} - \frac{2x_\xi\ r}{h^2}\frac{\partial(x_\eta\ \cdot)}{\partial \xi} + \frac{2x_\xi}{h^2}\frac{\partial(x_\xi\ r)}{\partial \eta} \\
-\frac{2x_\xi\ r}{2h^3}\frac{\partial(x_\xi\ h)}{\partial \eta} + \frac{2x_\xi\ r}{h^2}\frac{\partial(x_\xi\ \cdot)}{\partial \eta}\Bigg]
\end{array} \right), \tag{A.74}
$$

$$
\frac{\partial \mathbf{J}^{\mathrm{VF},\eta}}{\partial q} = \begin{pmatrix} 0 \\ \frac{\nu h}{J}\left[-\frac{x_\xi}{h}\frac{\partial(x_\eta\ \cdot)}{\partial\xi} + \frac{x_\xi}{h^2}\frac{\partial(x_\eta\ h)}{\partial\xi} + \frac{x_\xi}{h}\frac{\partial(x_\xi\ \cdot)}{\partial\eta} - \frac{x_\xi}{h^2}\frac{\partial(x_\xi\ h)}{\partial\eta} \right. \\ \left. +\frac{2y_\xi}{h}\frac{\partial(y_\eta\ \cdot)}{\partial\xi} - \frac{2y_\xi}{h^2}\frac{\partial(y_\eta\ h)}{\partial\xi} - \frac{2y_\xi}{h}\frac{\partial(y_\xi\ \cdot)}{\partial\eta} + \frac{2y_\xi}{h^2}\frac{\partial(y_\xi\ h)}{\partial\eta} \right] \\ +\frac{\nu h}{J}\left[\frac{y_\xi}{h}\frac{\partial(x_\eta\ \cdot)}{\partial\xi} - \frac{y_\xi}{h^2}\frac{\partial(x_\eta\ h)}{\partial\xi} - \frac{y_\xi}{h}\frac{\partial(x_\xi\ \cdot)}{\partial\eta} + \frac{y_\xi}{h^2}\frac{\partial(x_\xi\ h)}{\partial\eta} \right] \end{pmatrix}, \tag{A.75}
$$

and

$$
\frac{\partial \mathbf{J}^{\mathrm{VF},\eta}}{\partial r} = \begin{pmatrix} 0 \\ \frac{\nu h}{J}\left[-\frac{x_\xi}{h}\frac{\partial(y_\eta\ \cdot)}{\partial\xi} + \frac{x_\xi}{h^2}\frac{\partial(y_\eta\ h)}{\partial\xi} + \frac{y_\xi}{h}\frac{\partial(x_\xi\ \cdot)}{\partial\eta} - \frac{y_\xi}{h^2}\frac{\partial(x_\xi\ h)}{\partial\eta} \right] \\ \frac{\nu h}{J}\left[+\frac{y_\xi}{h}\frac{\partial(y_\eta\ \cdot)}{\partial\xi} + \frac{y_\xi}{h^2}\frac{\partial(y_\eta\ h)}{\partial\xi} - \frac{y_\xi}{h}\frac{\partial(y_\xi\ \cdot)}{\partial\eta} - \frac{y_\xi}{h^2}\frac{\partial(y_\xi\ h)}{\partial\eta} \right. \\ \left. +\frac{2x_\xi}{h}\frac{\partial(x_\eta\ \cdot)}{\partial\xi} - \frac{2x_\xi}{h^2}\frac{\partial(x_\eta\ h)}{\partial\xi} - \frac{2x_\xi}{h}\frac{\partial(x_\xi\ \cdot)}{\partial\eta} + \frac{2x_\xi}{h^2}\frac{\partial(x_\xi\ h)}{\partial\eta} \right] \end{pmatrix}, \tag{A.76}
$$

Email: jan.mooiman@outlook.com
GitHub Mooiman