Two step FVE method

A numerical modeling technique designed for error insight

2D Heat equation

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Symbol	Unit	Description
Δt	s	Time increment
Δx	m	Space increment, $\Delta x_{i+\frac{1}{2}} = x_{i+1} - x_i$
ε	$\mathrm{m\,s^{-2}}$	Multiplier of the correction term for the essential boundary condition, ζ-boundary
ε	$\mathrm{s^{-1}}$	Multiplier of the correction term for the essential boundary condition, q-boundary
ν	$\mathrm{m^2\,s^{-1}}$	Kinematic viscosity
Ω	–	Finite volume
Ψ	$\mathrm{m^2\,s^{-1}}$	Artificial smoothing coefficient
θ	–	θ-method. If $\theta = 1$ then it is a fully implicit method and if $\theta = 0$ then it is a fully explicit method.
$\boldsymbol{E}$	–	Error vector function, defined in computational space
ξ	–	Relative coordinate
ζ	m	Water level w.r.t. reference plane, positive upward
C	$\mathrm{m^{\frac{1}{2}}\,s^{-1}}$	Chézy coefficient
c_Ψ	$(.)^{-1}$	Artificial smoothing variable
c_f	–	Bed shear stress coefficient
g	$\mathrm{m\,s^{-2}}$	Gravitational constant
h	m	Total water depth
i	–	node counter
q	$\mathrm{m^2\,s^{-1}}$	The water flux in x-direction, $q = hu$
r	$\mathrm{m^2\,s^{-1}}$	The water flux in y-direction, $r = hv$
t	s	Time coordinate
u	$\mathrm{m\,s^{-1}}$	Velocity in x-direction
v	$\mathrm{m\,s^{-1}}$	Velocity in y-direction
x	m	x-coordinate
y	m	y-coordinate
z_b	m	Bed level w.r.t. reference plane, positive upward

Heat 2D

1 2D Heat equation

Consider the non-linear heat equation

$$\frac{\partial u}{\partial t} - \nabla \cdot (\mathbf{\Psi} \nabla u) = 0, \tag{1.1a}$$

with

u	Temperature, [$^\circ\mathrm{C}$].
$\mathbf{\Psi}$	2-dimensional thermal conductivity coefficient, [$\mathrm{m}^2\,\mathrm{s}^{-1}$].

$\mathbf{\Psi}$ the thermal conductivity coefficient is defined as:

$$\mathbf{\Psi} = \begin{pmatrix} \Psi_{11} & \Psi_{12} \\ \Psi_{21} & \Psi_{22} \end{pmatrix} \tag{1.2}$$

The 2D-heat equation in curvilinear (ξ, η)-coordinates reads:

$$J\frac{\partial u}{\partial t} - \nabla \cdot (\mathbf{\Psi} \nabla u) = 0, \tag{1.3}$$

Finite Volume approach

Integrating the equations over a finite volume Ω yields:

$$\int_\Omega \frac{\partial u}{\partial t}\, d\omega - \int_\Omega \nabla \cdot (\mathbf{\Psi} \nabla u)\; d\omega = 0 \tag{1.4}$$

The 2D-heat equation in curvilinear coordinates reads:

$$\int_{\Omega_{\xi\eta}} J\frac{\partial u}{\partial t}\, d\xi d\eta - \int_{\Omega_{\xi\eta}} \nabla \cdot (\mathbf{\Psi} \nabla u)\; d\xi d\eta = 0, \tag{1.5}$$

1.1 Space discretization, structured

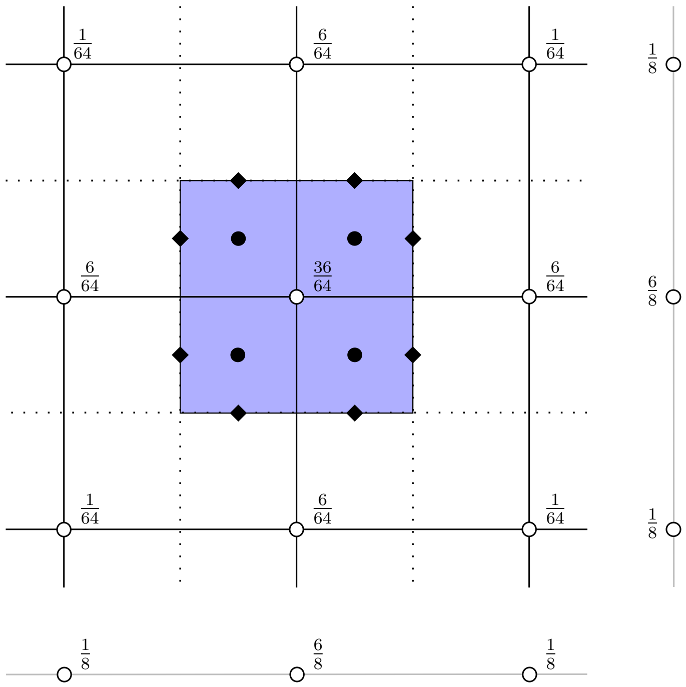

Figure 1.1: *Coefficients for the mass-matrix and the control volume in 2-dimensions in the interior area, on a structured grid. The black dots indicate the location of the quadrature points, and black diamonds the flux points.*

For the space discretization of an arbitrary function u on the quadrature point of a sub-control volume the following space interpolations are used:

$$u|_{i+\frac{1}{4},j+\frac{1}{4}} \approx \frac{1}{16}\left(9u_{i,j} + 3u_{i+1,j} + u_{i+1,j+1} + 3u_{i,j+1}\right) \tag{1.6}$$

$$u|_{i+\frac{1}{2},j+\frac{1}{4}} \approx \frac{1}{8}\left(3u_{i,j} + 3u_{i+1,j} + u_{i+1,j+1} + u_{i,j+1}\right) \tag{1.7}$$

$$u|_{i+\frac{1}{4},j+\frac{1}{2}} \approx \frac{1}{8}\left(3u_{i,j} + u_{i+1,j} + u_{i+1,j+1} + 3u_{i,j+1}\right) \tag{1.8}$$

See for the locations Figure 1.1.

1.1.1 Discretizations heat equation

The discretization of heat equation (1.4) will be presented term by term.

1.1.1.1 Time derivative

The time derivative term of the heat equation reads:

$$\int_{\Omega_{\xi\eta}} J\frac{\partial u}{\partial t}\, d\xi d\eta \tag{1.9}$$

which will be approximated by the sum of the integral over the sub-control volumes, with taking into account that the Jacobian J is constant in time. On a structured grid one control volume (cv) around a node consist of four sub-control volumes (scv_i, $i \in \{0, 1, 2, 3\}$).

$$J_{cv} \int_{cv} \frac{\partial u}{\partial t}\, d\omega = J_{scv_0} \int_{scv_0} \frac{\partial u}{\partial t}\, d\omega + J_{scv_1} \int_{scv_1} \frac{\partial u}{\partial t}\, d\omega + \\ + J_{scv_2} \int_{scv_2} \frac{\partial u}{\partial t}\, d\omega + J_{scv_3} \int_{scv_3} \frac{\partial u}{\partial t}\, d\omega \tag{1.10}$$

Fore the discretization on a curvilinear grid we get:

$$\begin{aligned} J_{cv} \int_{cv} \frac{\partial u}{\partial t}\, d\omega \approx & J_{scv_0} \Delta t_{inv} \left(u^{n+1,p+1}_{i-\frac{1}{4},j-\frac{1}{4}} - u^{n+1,n}_{i-\frac{1}{4},j-\frac{1}{4}} \right) + \\ & J_{scv_1} \Delta t_{inv} \left(u^{n+1,p+1}_{i+\frac{1}{4},j-\frac{1}{4}} - u^{n+1,n}_{i+\frac{1}{4},j-\frac{1}{4}} \right) + \\ & J_{scv_2} \Delta t_{inv} \left(u^{n+1,p+1}_{i+\frac{1}{4},j+\frac{1}{4}} - u^{n+1,n}_{i+\frac{1}{4},j+\frac{1}{4}} \right) + \\ & J_{scv_3} \Delta t_{inv} \left(u^{n+1,p+1}_{i-\frac{1}{4},j+\frac{1}{4}} - u^{n+1,n}_{i-\frac{1}{4},j+\frac{1}{4}} \right) \end{aligned} \tag{1.11}$$

with J_{scv_i} the area of the sub control volume i. For cartesian grids we have $J_{scv_i} = \frac{1}{4} \Delta x \Delta y$.

Just looking to the quadrature point of scv_2 as part of the control volume for node (i, j) the discretization reads:

$$J_{scv_2} \Delta t_{inv} \left(u^{n+1}_{i+\frac{1}{4},j+\frac{1}{4}} - u^{n+1,n}_{i+\frac{1}{4},j+\frac{1}{4}} \right) = \tag{1.12}$$

$$= J_{scv_2} \Delta t_{inv} \left[\frac{1}{16} \left(9u^{n+1}_{i,j} + 3u^{n+1}_{i+1,j} + 3u^{n+1}_{i,j+1} + u^{n+1}_{i+1,j+1} \right) + \right. \tag{1.13}$$

$$\left. - \frac{1}{16} \left(9u^{n}_{i,j} + 3u^{n}_{i+1,j} + 3u^{n}_{i,j+1} + u^{n}_{i+1,j+1} \right) \right] \tag{1.14}$$

Written in Δ-formulation it yields (using $\Delta u^{n+1,p+1} = \Delta u$):

$$J_{scv_2} \Delta t_{inv} \left(u^{n+1}_{i+\frac{1}{4},j+\frac{1}{4}} - u^{n}_{i+\frac{1}{4},j+\frac{1}{4}} \right) = \tag{1.15}$$

$$\begin{aligned} &= J_{scv_2} \Delta t_{inv} \left[\frac{1}{16} \left(9\Delta u_{i,j} + 3\Delta u_{i+1,j+1} + 3\Delta u_{i,j+1} + \Delta u_{i+1,j+1} \right) \right] + \\ &+ J_{scv_2} \Delta t_{inv} \left[\frac{1}{16} \left(9 \left(u^{n+1,p}_{i,j} - u^{n}_{i,j} \right) + 3 \left(u^{n+1,p}_{i+1,j} - u^{n}_{i+1,j} \right) + \right. \right. \\ &\qquad \left. \left. 3 \left(u^{n+1,p}_{i,j+1} - u^{n}_{i,j+1} \right) + \left(u^{n+1,p+1}_{i+1,j+1} - u^{n}_{i+1,j+1} \right) \right) \right] \end{aligned} \tag{1.16}$$

with

$$J_{scv_2} = x_\xi y_\eta - y_\xi x_\eta \tag{1.17}$$

with J_{scv_2} representing the area of scv_2, this area is computed by

$$J = \frac{1}{2}\sum_{i=0}^{3}(x_i y_{i+1} - x_{i+1} y_i), \quad \text{with } x_4 = x_0 \text{ and } y_4 = y_0 \tag{1.18}$$

1.1.1.2 Thermal conductivity

The thermal conductivity term in vector notation reads:

$$\int_\Omega \nabla \cdot (\boldsymbol{\Psi} \nabla u)\, d\omega = \oint_\Omega (\boldsymbol{\Psi}\nabla u) \cdot \boldsymbol{n}\, dl = \tag{1.19}$$

$$= \oint_\Omega \left(\left(\Psi_{11}\frac{\partial u}{\partial x} + \Psi_{12}\frac{\partial u}{\partial y} \right) n_x + \left(\Psi_{21}\frac{\partial u}{\partial x} + \Psi_{22}\frac{\partial u}{\partial y} \right) n_y \right) \|dl\| \tag{1.20}$$

with $\boldsymbol{n} = (n_x, n_y)^T$ the outward normal vector.

For the heat-equation the thermal conductivity term:

$$\frac{\partial}{\partial x}\left(\Psi_{11}\frac{\partial u}{\partial x} + \Psi_{12}\frac{\partial u}{\partial y} \right) + \frac{\partial}{\partial y}\left(\Psi_{21}\frac{\partial u}{\partial x} + \Psi_{22}\frac{\partial u}{\partial y} \right) \tag{1.21}$$

can be transformed as

step 1a

$$\frac{1}{J}\left[y_n \frac{\partial}{\partial \xi}\left(\Psi_{11}\frac{\partial u}{\partial x} + \Psi_{12}\frac{\partial u}{\partial y} \right) - y_\xi \frac{\partial}{\partial \eta}\left(\Psi_{11}\frac{\partial u}{\partial x} + \Psi_{12}\frac{\partial u}{\partial y} \right) + \right. \tag{1.22}$$

$$\left. - x_\eta \frac{\partial}{\partial \xi}\left(\Psi_{21}\frac{\partial u}{\partial x} + \Psi_{22}\frac{\partial u}{\partial y} \right) + x_\xi \frac{\partial}{\partial \eta}\left(\Psi_{21}\frac{\partial u}{\partial x} + \Psi_{22}\frac{\partial u}{\partial y} \right) \right] \tag{1.23}$$

step 1b

Assume $x_{\xi\eta} = x_{\eta\xi}$ and $y_{\xi\eta} = y_{\eta\xi}$

$$\frac{1}{J}\left[\frac{\partial}{\partial \xi}\left(y_\eta \left(\Psi_{11}\frac{\partial u}{\partial x} + \Psi_{12}\frac{\partial u}{\partial y} \right) \right) - \frac{\partial}{\partial \eta}\left(y_\xi \left(\Psi_{11}\frac{\partial u}{\partial x} + \Psi_{12}\frac{\partial u}{\partial y} \right) \right) + \right. \tag{1.24}$$

$$\left. - \frac{\partial}{\partial \xi}\left(x_\eta \left(\Psi_{21}\frac{\partial u}{\partial x} + \Psi_{22}\frac{\partial u}{\partial y} \right) \right) + \frac{\partial}{\partial \eta}\left(x_\xi \left(\Psi_{21}\frac{\partial u}{\partial x} + \Psi_{22}\frac{\partial u}{\partial y} \right) \right) \right] \tag{1.25}$$

Rearrange this equation to a divergence type in ξ- and η-direction to prepare to use Green's theorem, yields:

$$\frac{1}{J}\left[\frac{\partial}{\partial \xi}\left(y_\eta \left(\Psi_{11}\frac{\partial u}{\partial x} + \Psi_{12}\frac{\partial u}{\partial y} \right) - x_\eta \left(\Psi_{21}\frac{\partial u}{\partial x} + \Psi_{22}\frac{\partial u}{\partial y} \right) \right) + \right. \tag{1.26}$$

$$\left. + \frac{\partial}{\partial \eta}\left(-y_\xi \left(\Psi_{11}\frac{\partial u}{\partial x} + \Psi_{12}\frac{\partial u}{\partial y} \right) + x_\xi \left(\Psi_{21}\frac{\partial u}{\partial x} + \Psi_{22}\frac{\partial u}{\partial y} \right) \right) \right] \tag{1.27}$$

Applying Green's theorem, and multiplying with J (because the whole heat equation is multiplied by J)
step 2a:

$$\left(y_\eta \left(\Psi_{11} \frac{\partial u}{\partial x} + \Psi_{12} \frac{\partial u}{\partial y} \right) - x_\eta \left(\Psi_{21} \frac{\partial u}{\partial x} + \Psi_{22} \frac{\partial u}{\partial y} \right) \right) n_\xi + \tag{1.28}$$

$$+ \left(-y_\xi \left(\Psi_{11} \frac{\partial u}{\partial x} + \Psi_{12} \frac{\partial u}{\partial y} \right) + x_\xi \left(\Psi_{21} \frac{\partial u}{\partial x} + \Psi_{22} \frac{\partial u}{\partial y} \right) \right) n_\eta \tag{1.29}$$

with $\boldsymbol{n} = (n_\xi, n_\eta)^T$ the outward normal vector.

step 2b:

$$\frac{1}{J} \left[y_\eta \left(\Psi_{11} \left(y_\eta \frac{\partial u}{\partial \xi} - y_\xi \frac{\partial u}{\partial \eta} \right) + \Psi_{12} \left(-x_\eta \frac{\partial u}{\partial \xi} + x_\xi \frac{\partial u}{\partial \eta} \right) \right) + \right. \tag{1.30}$$

$$\left. - x_\eta \left(\Psi_{21} \left(y_\eta \frac{\partial u}{\partial \xi} - y_\xi \frac{\partial u}{\partial \eta} \right) + \Psi_{22} \left(-x_\eta \frac{\partial u}{\partial \xi} + x_\xi \frac{\partial u}{\partial \eta} \right) \right) \right] n_\xi + \tag{1.31}$$

$$\frac{1}{J} \left[- y_\xi \left(\Psi_{11} \left(y_\eta \frac{\partial u}{\partial \xi} - y_\xi \frac{\partial u}{\partial \eta} \right) + \Psi_{12} \left(-x_\eta \frac{\partial u}{\partial \xi} + x_\xi \frac{\partial u}{\partial \eta} \right) \right) + \right. \tag{1.32}$$

$$\left. + x_\xi \left(\Psi_{21} \left(y_\eta \frac{\partial u}{\partial \xi} - y_\xi \frac{\partial u}{\partial \eta} \right) + \Psi_{22} \left(-x_\eta \frac{\partial u}{\partial \xi} + x_\xi \frac{\partial u}{\partial \eta} \right) \right) \right] n_\eta \tag{1.33}$$

Orthotropic conductivity

When the thermal conductivity coefficient is orthotropic ($\Psi_{12} = \Psi_{21} = 0$) reduces to:

$$\frac{1}{J} \left[y_\eta \Psi_{11} \left(y_\eta \frac{\partial u}{\partial \xi} - y_\xi \frac{\partial u}{\partial \eta} \right) - x_\eta \Psi_{22} \left(-x_\eta \frac{\partial u}{\partial \xi} + x_\xi \frac{\partial u}{\partial \eta} \right) \right] n_\xi + \tag{1.34}$$

$$+ \frac{1}{J} \left[- y_\xi \Psi_{11} \left(y_\eta \frac{\partial u}{\partial \xi} - y_\xi \frac{\partial u}{\partial \eta} \right) + x_\xi \Psi_{22} \left(-x_\eta \frac{\partial u}{\partial \xi} + x_\xi \frac{\partial u}{\partial \eta} \right) \right] n_\eta \tag{1.35}$$

In case it is also orthogonal (i.e. $y_\xi = x_\eta = 0$) it reads:

$$\frac{1}{J} \left[y_\eta \Psi_{11} \left(y_\eta \frac{\partial u}{\partial \xi} \right) \right] n_\xi + \frac{1}{J} \left[x_\xi \Psi_{22} \left(x_\xi \frac{\partial u}{\partial \eta} \right) \right] n_\eta \tag{1.36}$$

Not yet documented

Define the components of $\mathbf{\Psi}$ at different locations of the volume faces where $(\mu,\nu) \in \{1,2\}$ as

$$\overline{\Psi}_{\mu\nu}\big|_{i+\frac{1}{2},j+\frac{1}{4}} = \frac{1}{8}\left(3\Psi_{\mu\nu}\big|_{i,j} + 3\Psi_{\mu\nu}\big|_{i+1,j} + \Psi_{\mu\nu}\big|_{i+1,j+1} + \Psi_{\mu\nu}\big|_{i,j+1}\right) \quad (1.37)$$

and define the partial differentials as

$$\frac{\partial u}{\partial x}\bigg|_{i+\frac{1}{2},j+\frac{1}{4}} = \frac{1}{4}\left(3\frac{u_{i+1,j} - u_{i,j}}{\Delta x_{i+\frac{1}{2},j}} + \frac{u_{i+1,j+1} - u_{i,j+1}}{\Delta x_{i+\frac{1}{2},j+1}}\right) \quad (1.38)$$

$$\frac{\partial u}{\partial y}\bigg|_{i+\frac{1}{2},j+\frac{1}{4}} = \frac{1}{2}\left(\frac{u_{i,j+1} - u_{i,j}}{\Delta y_{i,j+\frac{1}{2}}} + \frac{u_{i+1,j+1} - u_{i+1,j}}{\Delta y_{i+1,j+\frac{1}{2}}}\right) \quad (1.39)$$

2 Numerical experiment

2.1 Initial Dirac-delta function

Th analytic solution for a 2D heat equation is

$$T(t) = \frac{1}{4D\pi t} \exp\left(\frac{-x^2 - y^2}{4Dt}\right) \tag{2.1}$$

where

T	Temperature, [$^\circ\mathrm{C}$].
D	Heat conductivity, [$\mathrm{W\,m^{-1}\,K^{-1}}$].
t	Time, [s].
x	Coordinate in x-direction, [m].
y	Coordinate in y-direction, [m].

A Curvilinear coordinate transformation

Not yet documented

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