A numerical modeling technique designed for error insight

With application to 1D shallow-water flow simulations (rivers, channels, pipes, ...)

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~~CASA Colloquium, March 18, 2020, Eindhoven University of Technology~~
H2I meeting, March 29 & April 12, 2021, Deltares
NSS meeting, April 22, 2021, Deltares

Contents

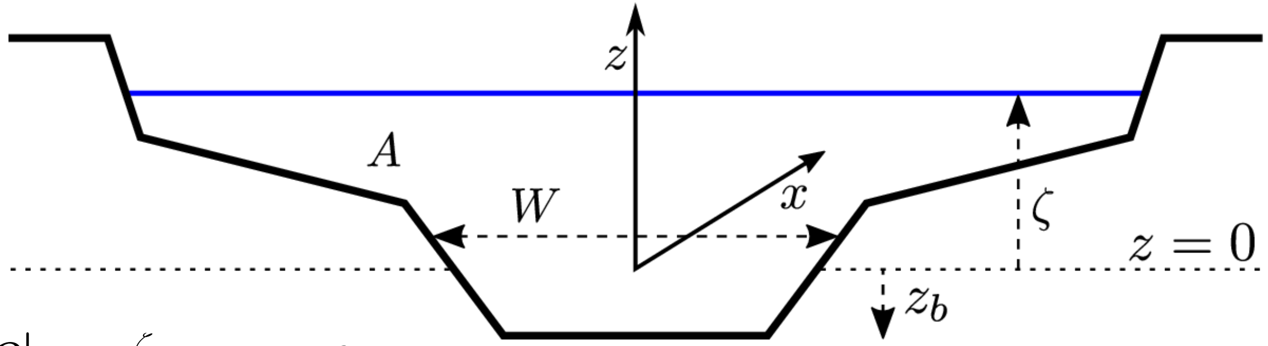
- Examples of 1D shallow-water flow applications
- The 1D shallow-water flow model equations
- Computational model: the components
- Numerical implementation: properties
- Two-step numerical modeling approach
- Finite volume method designed for error analysis
- Test: flow over a weir (with grid adaptation)
- Application: Libyan man-made river
- Concluding remarks
- What's next

(this presentation and other info in \Bulletin\borsb_m\Adaptive-grid_SUPSUB)

1D shallow-water flow applications

1D shallow-water equations

$$\frac{\partial A}{\partial t}+\frac{\partial Q}{\partial x}=0$$



$$\frac{\partial Q}{\partial t}+\frac{\partial Q^2/A}{\partial x}+gA\frac{\partial \zeta}{\partial x}+g\frac{Q|Q|}{C^2RA}=g\int_{z_b}^{\zeta}(\zeta-z)\frac{\partial W}{\partial x}dz$$

- $W=W(x,z)$: channel width ($W=0$ below bottom $z=z_b$)
- $\zeta=\zeta(x,t)$: free surface
- $A=\int_{z_b}^{\zeta}Wdz=A(x,\zeta)$: wetted cross-sectional area (→ effective depth $h=A(x,\zeta)/W(x,\zeta)$)
- $R=A/P$: hydraulic radius (± average depth), with P wetted perimeter
- $Q=Q(x,t)$: discharge (→ average velocity $u=Q/A$)

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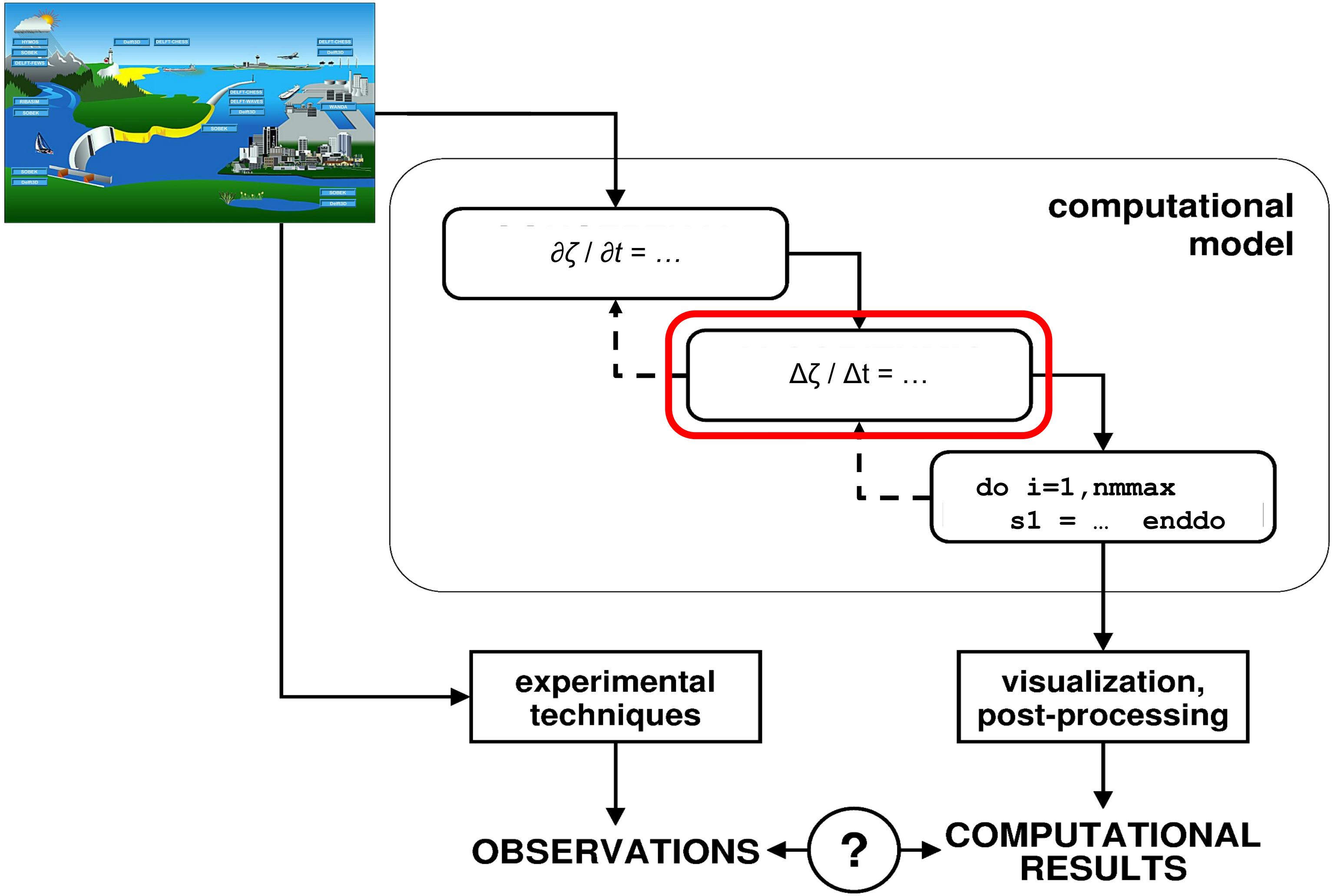
Computational model: the components



this presentation is about the numerical implementation

(plus user interface, grid generator, Kalman filter, …)

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Computational model: the numerical implementation

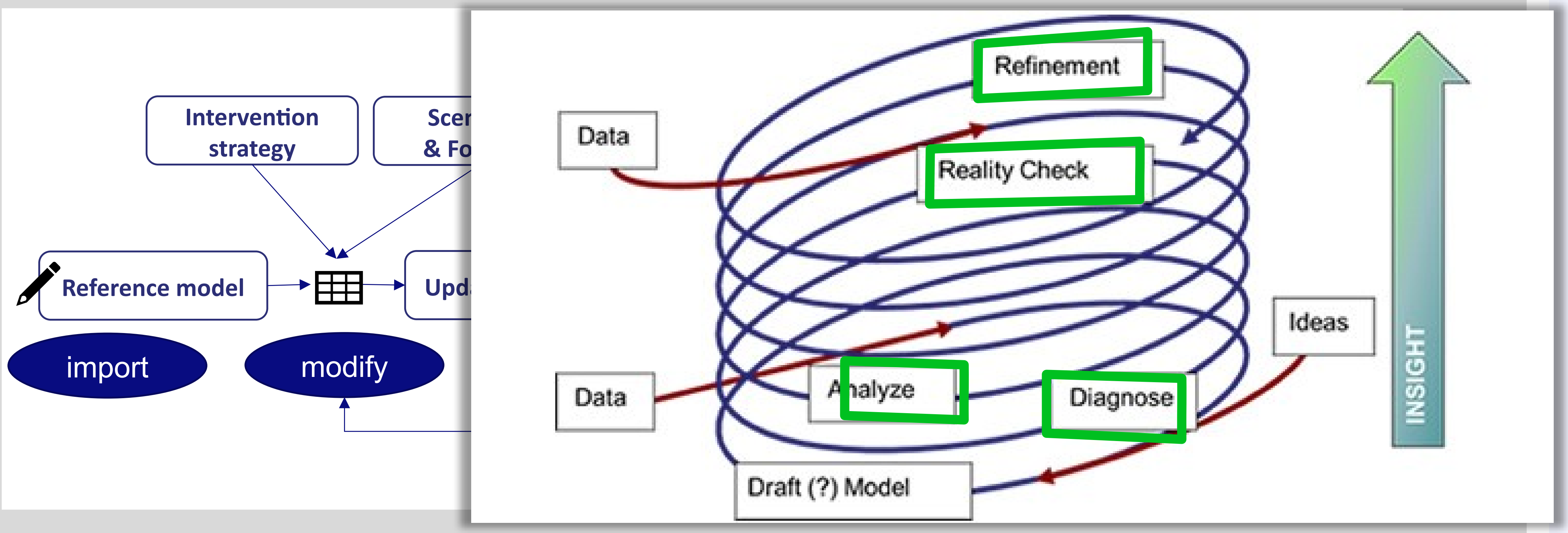
Properties that we are looking for:

- accuracy
- robustness
- efficiency/parallelizability
- flexibility/general applicability
- user friendliness, ease of use (e.g., virtually **no** options)
- relevance/fit for purpose
- reusability, extendibility, modularity (esp. for developers and maintainers)
- costs and risks (short-/long-term, in development/maintenance/applicability, ...)
  - ease of implementation
  - maintainability
  - ease of analysis
- reliability (insight in modeling errors), traceability (what to do to improve results)

Computational Framework

provides process support for model analyst

From presentation Peter Gijsbers
BlueEarth Engine (Computational Framework)
February 11, 2021



The basic idea

1. "*… introduce (artificial) dissipative terms ... to give the shocks* (MB: and other non-smooth features) *a thickness ... somewhat larger than the spacing of the points of the network.*" (regularization à la LES)

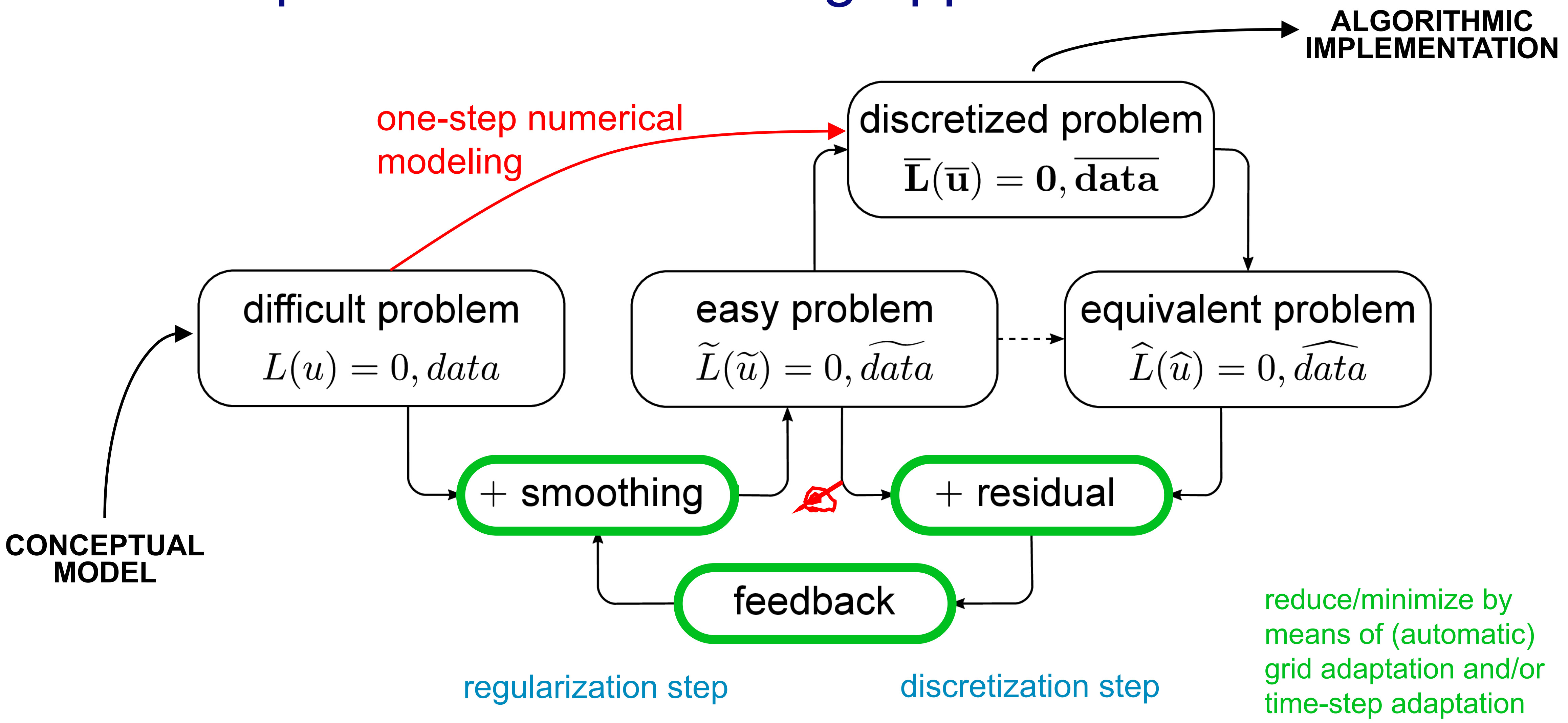
2. "*Then the difference equations may be used for the* ***entire*** *calculation, just as though there were no shocks* (MB: and other non-smooth features) *at all.*" (unified numerical implementation possible)

3. "*The quantitative influence of these terms can be made as small as one wishes by choice of a sufficiently fine mesh* (MB: and a sufficiently small time step) *…*" (basis for (automatic) grid adaptation and time-step adaptation)

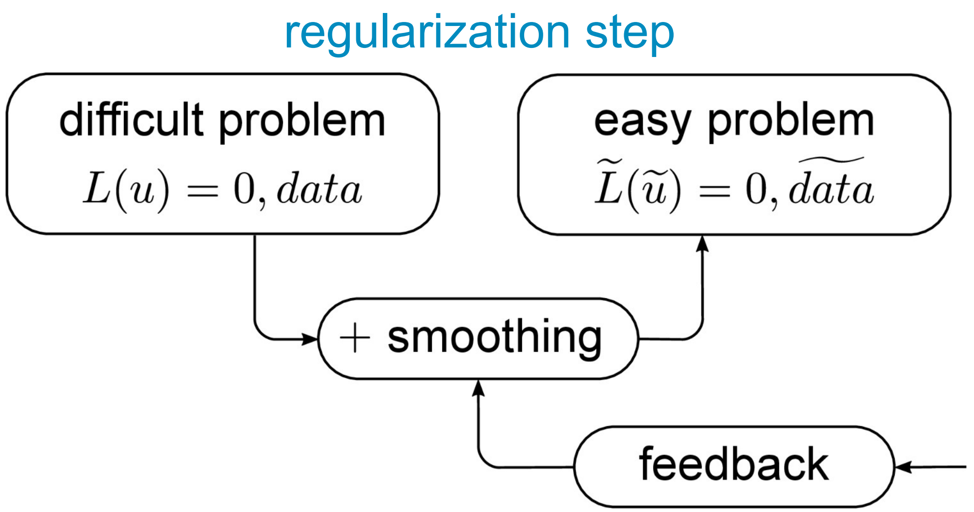
4. "*… components whose wave-lengths are of order* Δx (MB: and whose time scales are of order Δt) *are always falsified somewhat.*" (dynamic subgrid effects can NOT be simulated!)

(Von Neumann & Richtmyer, J. Appl. Phys. 1950)

Two-step numerical modeling approach



Difficult Easy 1D shallow-water equations



$$\frac{\partial \tilde{A}}{\partial t} + \frac{\partial \tilde{Q}}{\partial x} = 0$$

$$\frac{\partial \tilde{Q}}{\partial t} + \frac{\partial \tilde{Q}^2/\tilde{A}}{\partial x} + g\tilde{A}\frac{\partial \tilde{\zeta}}{\partial x} + g\frac{\tilde{Q}|\tilde{Q}|}{C^2 R\tilde{A}} = 0 \frac{\partial}{\partial x}\left(\tilde{A}\tilde{\nu}_{art}\frac{\partial \tilde{Q}/\tilde{A}}{\partial x}\right)$$

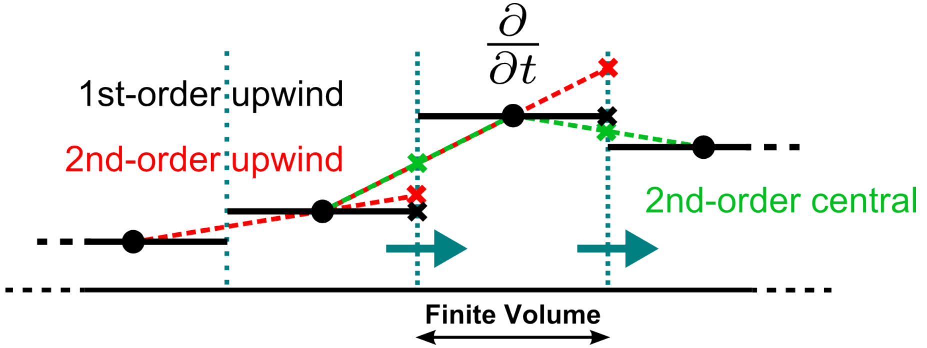
cont.eq. and mom.eq. combined to energy equation:

$$\frac{\partial}{\partial t}\left(\tilde{A}\frac{\tilde{u}^2}{2} + g\int_{z_b}^{\zeta} z\tilde{W}dz\right) + \frac{\partial}{\partial x}\left(\tilde{Q}\left(\frac{\tilde{u}^2}{2} + g\tilde{\zeta}\right)\right) = \frac{\partial}{\partial x}\left(\tilde{Q}\tilde{\nu}_{art}\frac{\partial \tilde{u}}{\partial x}\right) - \tilde{A}\tilde{\nu}_{art}\left(\frac{\partial \tilde{u}}{\partial x}\right)^2 - g\frac{\tilde{u}^2|\tilde{Q}|}{C^2 R}$$

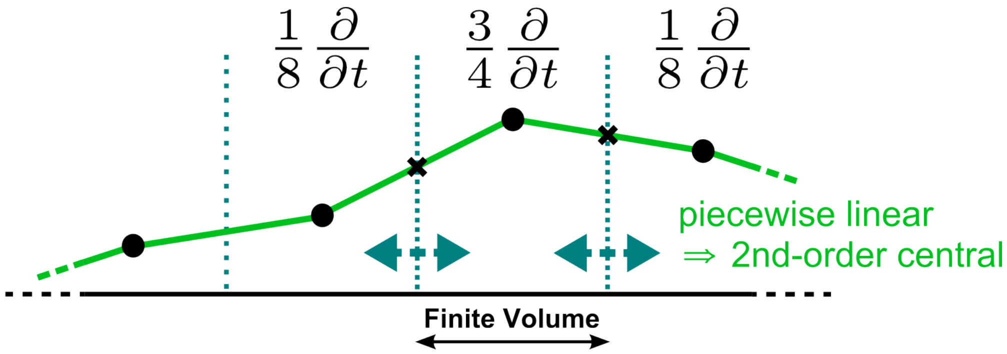
→ take $\tilde{\nu}_{art} \propto \Delta x\left(\left|err\left(\sqrt{\hat{u}^2/2}\right)\right| + \left|err\left(\sqrt{g\hat{\zeta}}\right)\right|\right)$

Finite volume-type collocated discretizations

- finite difference-like
  → ± piecewise constant $\overline{A}, \overline{Q}, \overline{\zeta}, \overline{W}, \ldots$

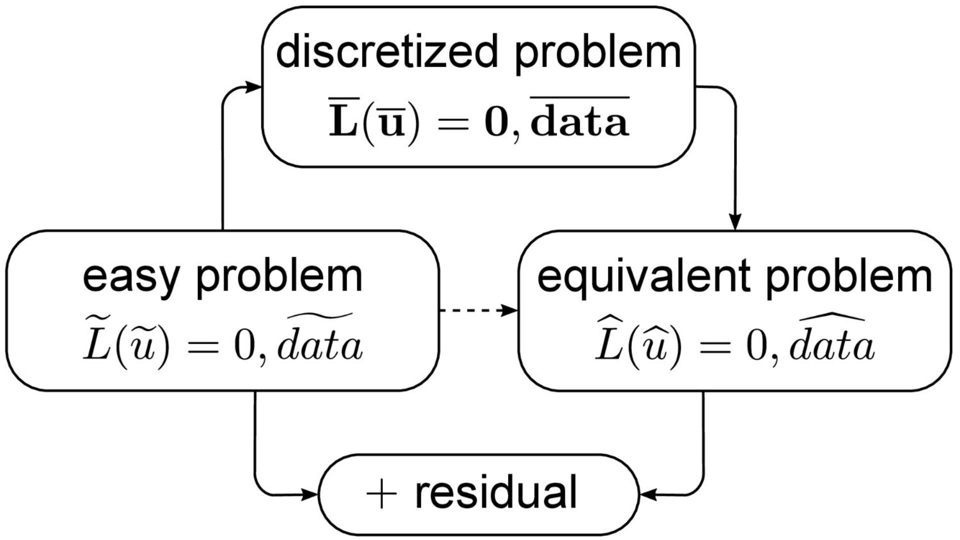


- finite element-like
  piecewise linear $\overline{A}, \overline{Q}, \overline{\zeta}, \overline{W}, \ldots$



Error analysis

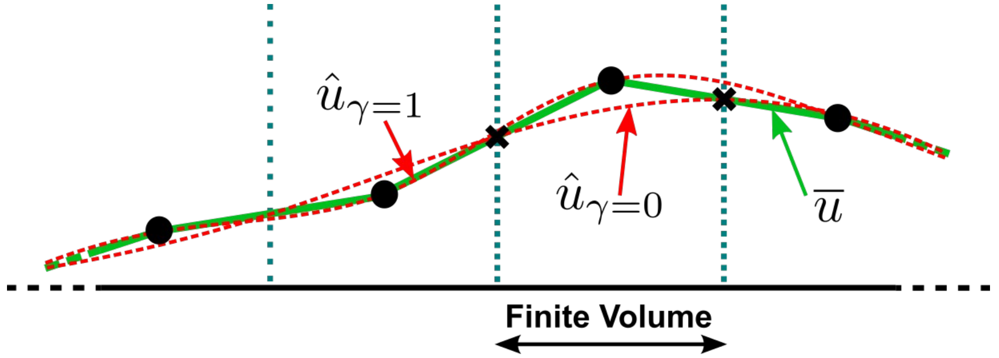
discretization step



Discretization error (residual) $R(\hat{u}) = \hat{L}(\hat{u}) - \tilde{L}(\hat{u})$

We have $$0 = \overline{\mathbf{L}}_i(\overline{\mathbf{u}}) = \int_{\xi_{i-1/2}}^{\xi_{i+1/2}} \tilde{L}(\overline{u})\, d\xi = \int_{\overline{x}_{i-1/2}}^{\overline{x}_{i+1/2}} \tilde{L}(\overline{u})\, dx = \int_{\overline{x}_{i-1/2}}^{\overline{x}_{i+1/2}} \hat{L}(\hat{u})\, dx$$

Hence $$\int_{\overline{x}_{i-1/2}}^{\overline{x}_{i+1/2}} R(\hat{u})\, dx = \overline{\mathbf{L}}_i(\overline{\mathbf{u}}) - \int_{\overline{x}_{i-1/2}}^{\overline{x}_{i+1/2}} \tilde{L}(\hat{u})\, dx$$



(Borsboom, Ch.5 in *Adaptive Methods of Lines*, 2001)

Equivalent 1D shallow-water-like equations

$$\frac{\partial}{\partial t}\left(\tilde{A}+\gamma_0 D_x(\hat{A})\right)+\frac{\partial}{\partial x}\left(\tilde{Q}+\gamma_1 D_x(\hat{Q})\right)=0$$

$$\frac{\partial}{\partial t}\left(\tilde{Q}+\gamma_0 D_x(\hat{Q})\right)+\frac{\partial}{\partial x}\left(\frac{\left(\tilde{Q}+\gamma_1 D_x(\hat{Q})\right)^2}{\tilde{A}+\gamma_1 D_x(\hat{A})}\right)+g\left(\left(\tilde{A}+\gamma_0 D_x(\hat{A})\right)\frac{\partial\tilde{\zeta}+\gamma_1 D_x(\hat{\zeta})}{\partial x}+\frac{\partial\hat{A}}{\partial x}\frac{D_x(\hat{\zeta})}{24}\right)$$

$$+g\frac{\left(\tilde{Q}+\gamma_0 D_x(\hat{Q})\right)\left|\tilde{Q}+\gamma_0 D_x(\hat{Q})\right|}{C^2\left(\tilde{R}+\gamma_0 D_x(\hat{R})\right)\left(\tilde{A}+\gamma_0 D_x(\hat{A})\right)}=\frac{\partial}{\partial x}\left(\left(\tilde{v}_{art}+\gamma_1 D_x(\hat{v}_{art})\right)\frac{\partial\tilde{Q}+\gamma_2 D_x(\hat{Q})}{\partial x}\right)$$

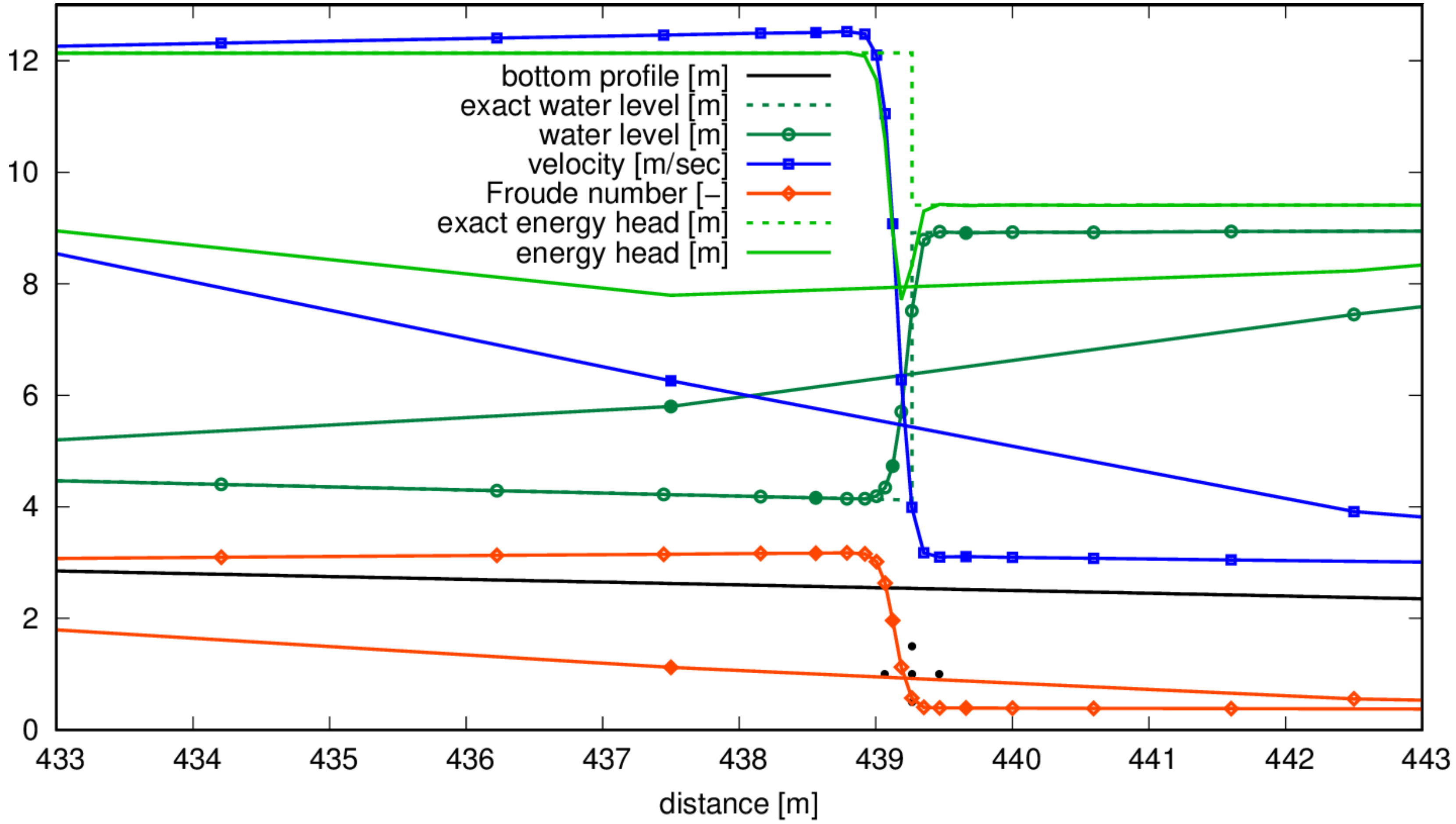
$$-\frac{\partial}{\partial x}\left(\frac{\left(\tilde{v}_{art}+\gamma_1 D_x(\hat{v}_{art})\right)\left(\tilde{Q}+\gamma_1 D_x(\hat{Q})\right)}{\tilde{A}+\gamma_1 D_x(\hat{A})}\frac{\partial\tilde{A}+\gamma_2 D_x(\hat{A})}{\partial x}\right)$$

with $D_x=\Delta\xi^2\left(\partial^2/\partial\xi^2-x_{\xi\xi}/x_\xi\,\partial/\partial\xi\right)+O\left(\Delta\xi^4\right)=x_\xi{}^2\partial^2/\partial x^2+O\left(\Delta\xi^4\right)\approx\Delta x^2\partial^2/\partial x^2$,

and with $\gamma_0=(\gamma-1/3)/8$, $\gamma_1=\gamma/8$, $\gamma_2=(\gamma-2/3)/8$.

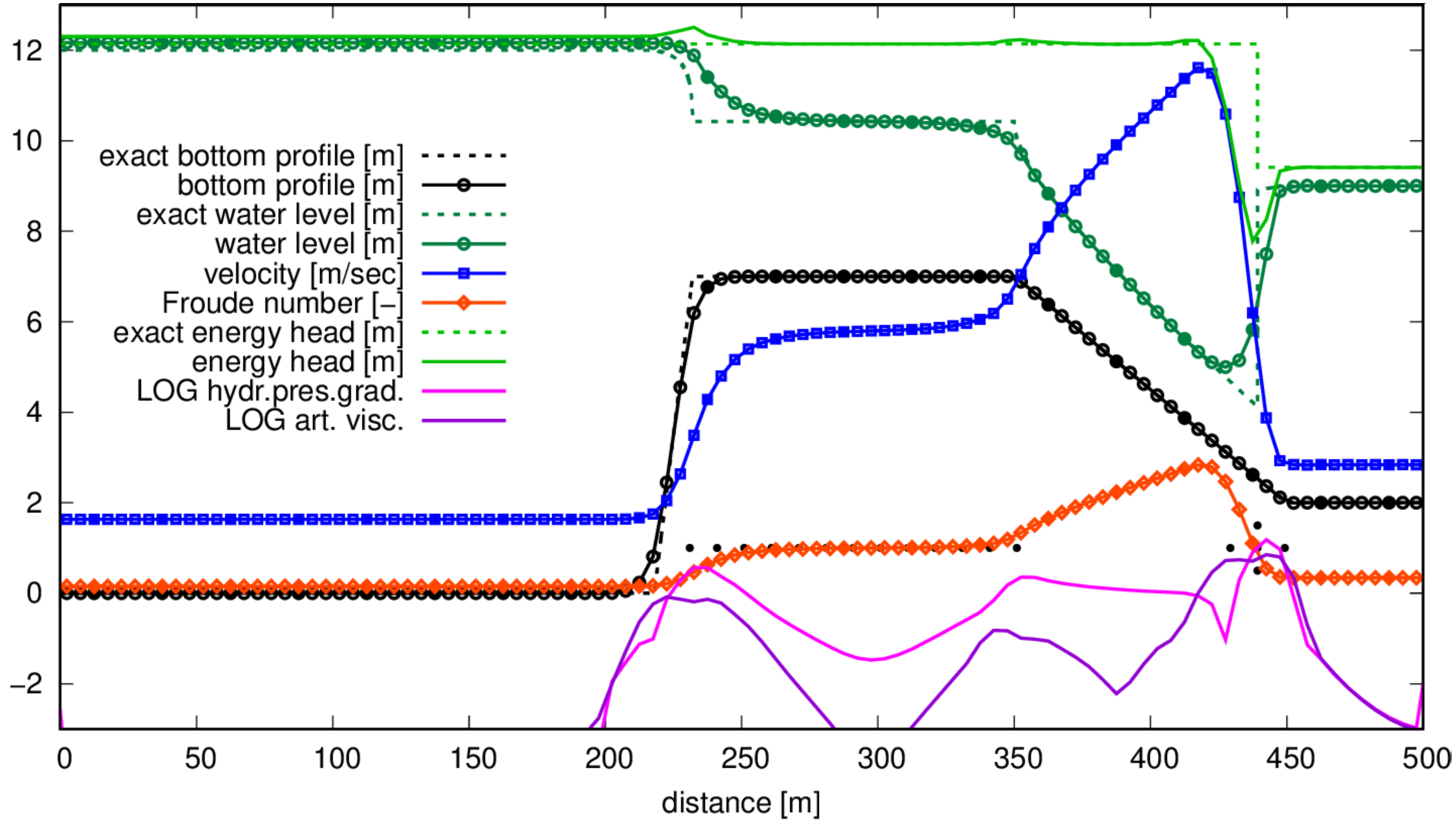
(Borsboom, Ch.5 in *Adaptive Methods of Lines*, 2001)

Sub-/supercritical steady flow over a weir



100 finite volumes, uniform and adaptive grid, solution around jump

Sub-/supercritical steady flow over a weir



100 finite volumes, smoothed geometry

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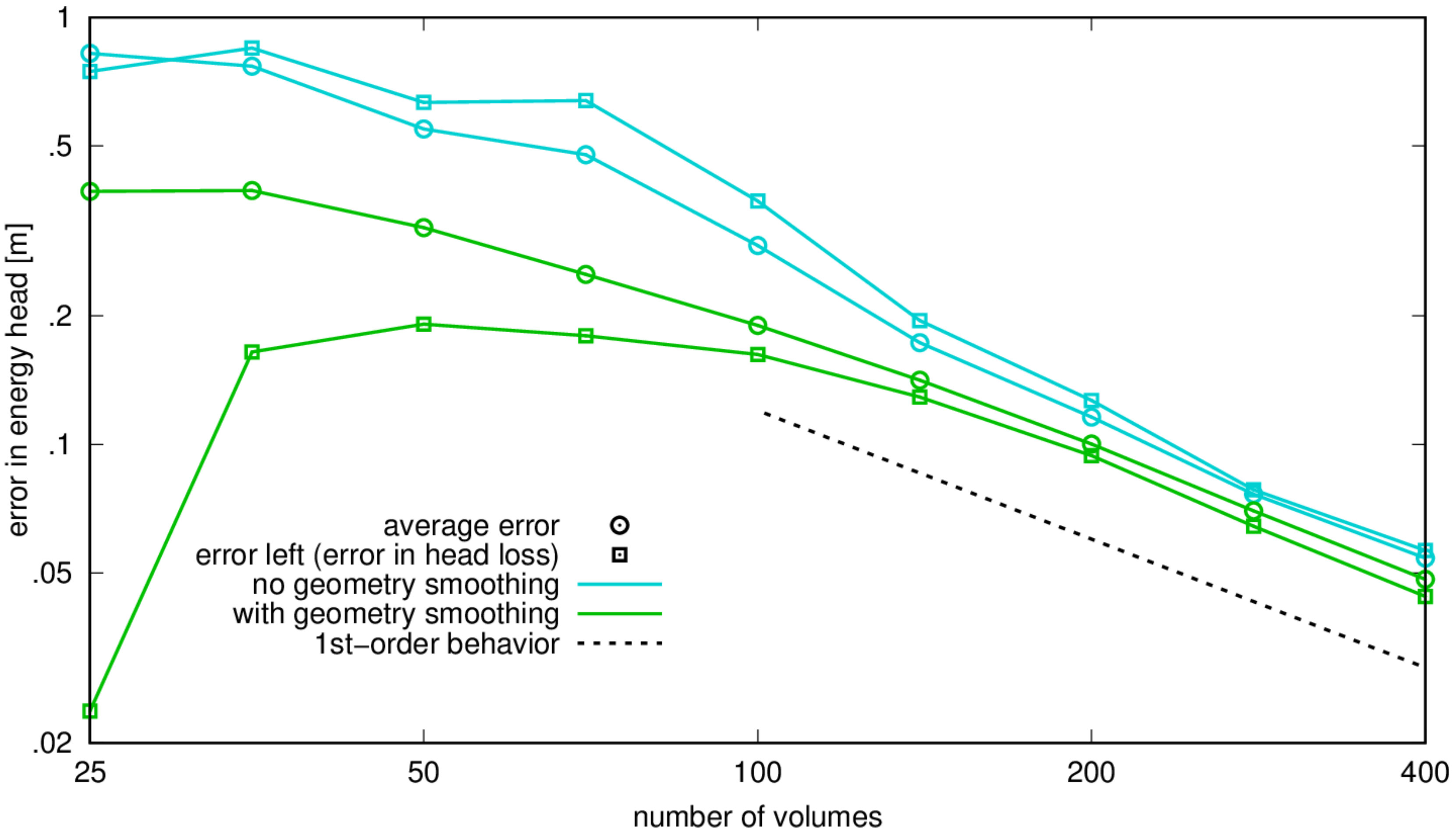
Sub-/supercritical steady flow over a weir



100 finite volumes, error on non-smooth and smoothed geometry

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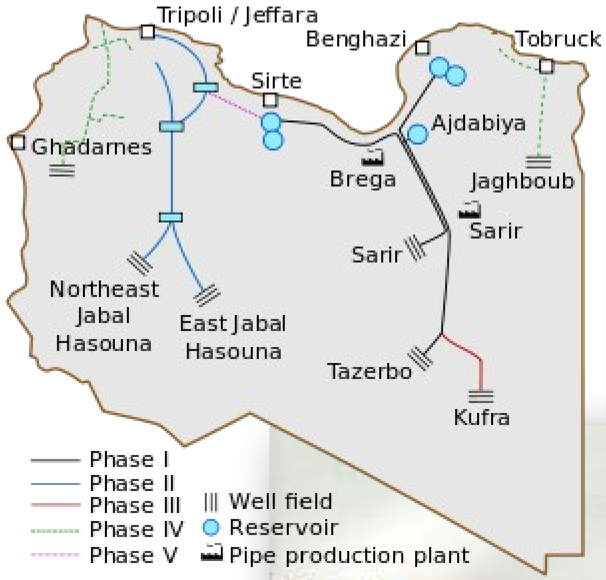
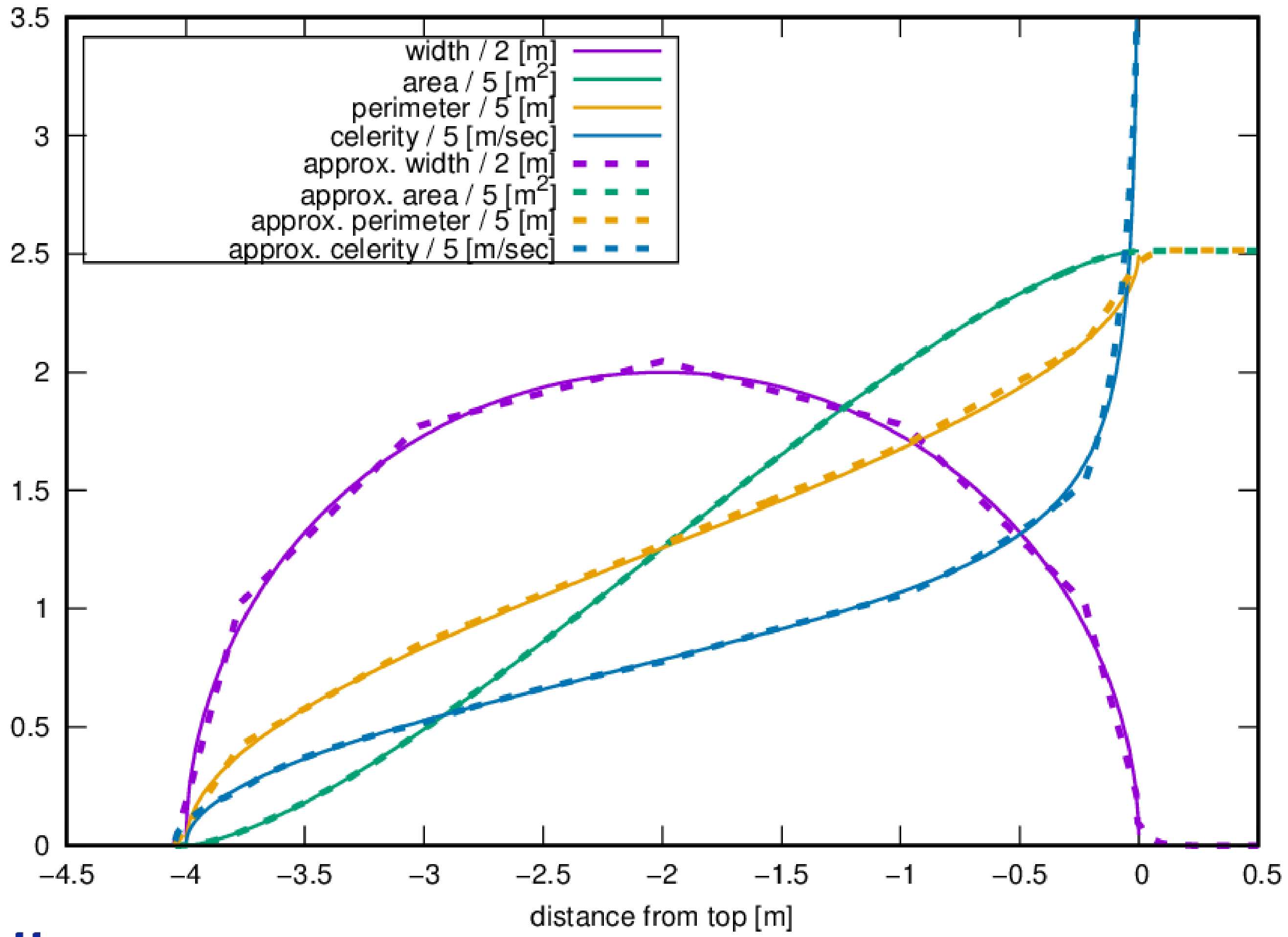
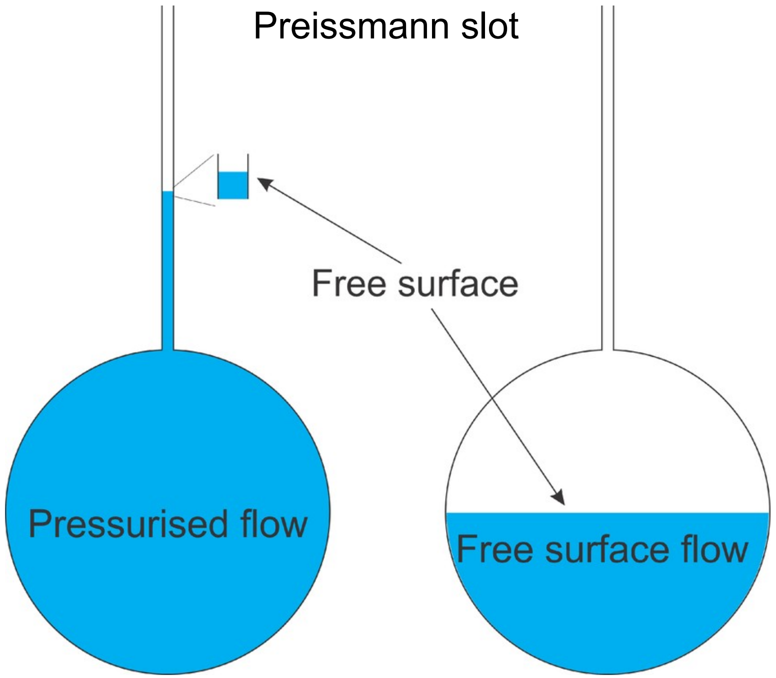
The great man-made river



The great man-made river

4m diameter pipes through Libyan desert
circular cross section approximated by dodecagon with added Preissmann slot

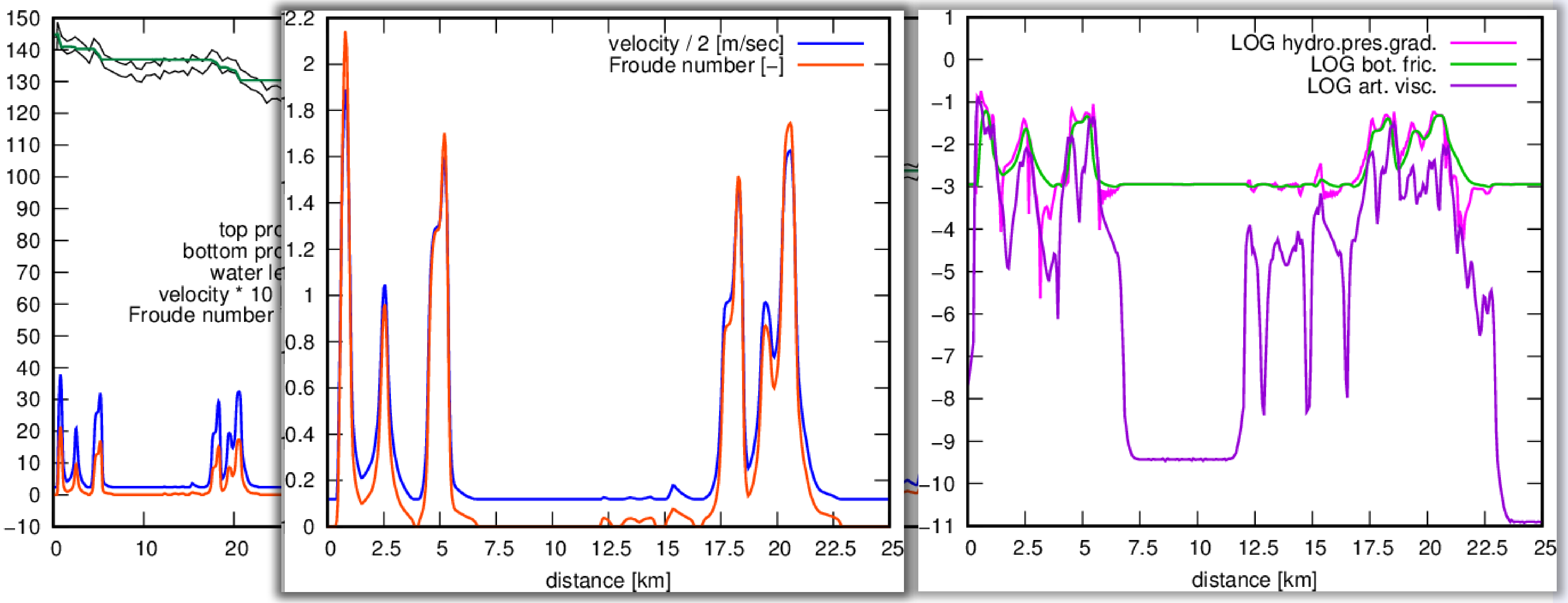




From presentation Michiel Tukker
Free surface flow in Wanda
April 10, 2020

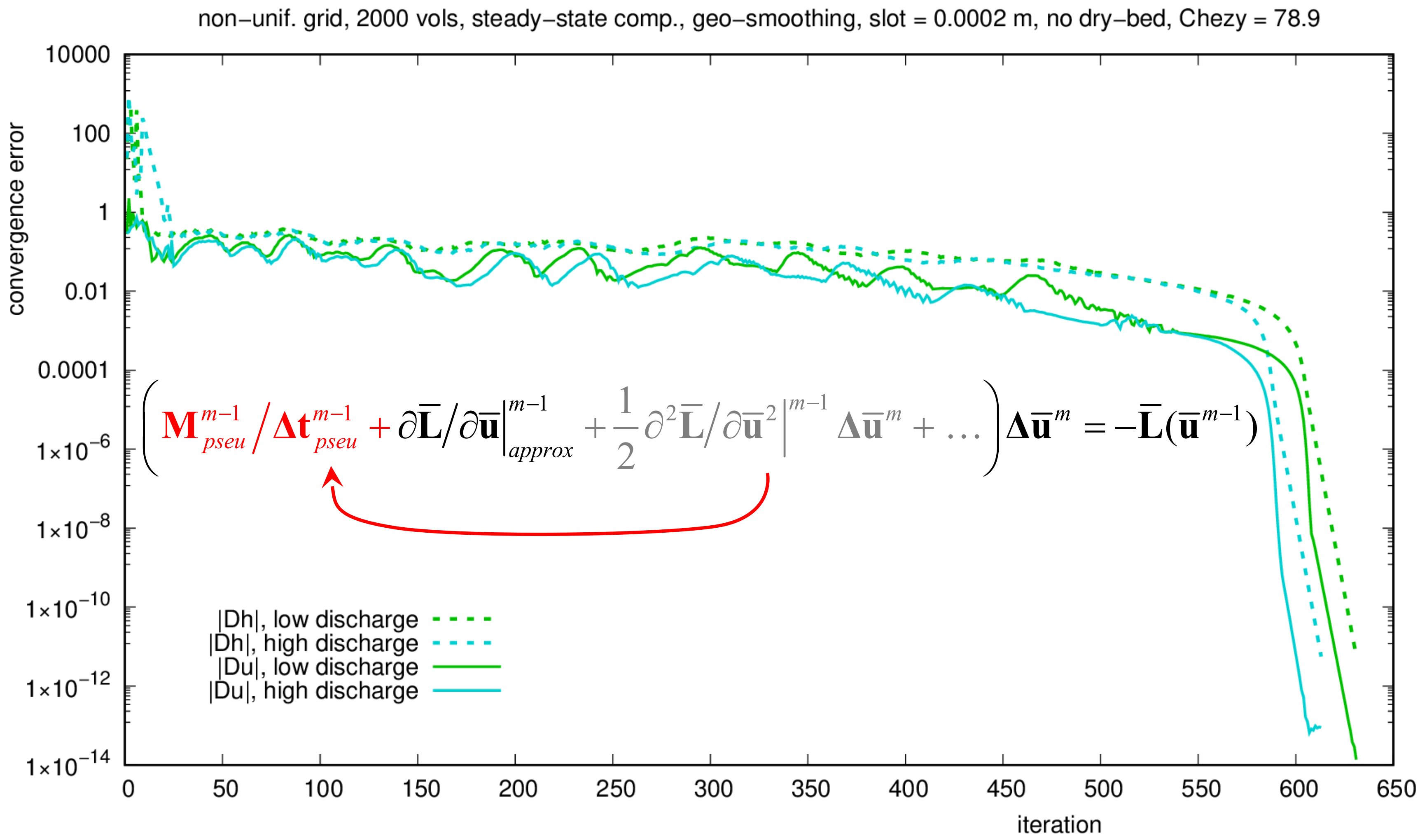
Steady-state great man-made river

381.2km long 4m diameter pipe through Libyan desert



2000 finite volumes: 1815 of 54.96m in first 100km, 165 of 1648.8m in last 270km, 20 in between

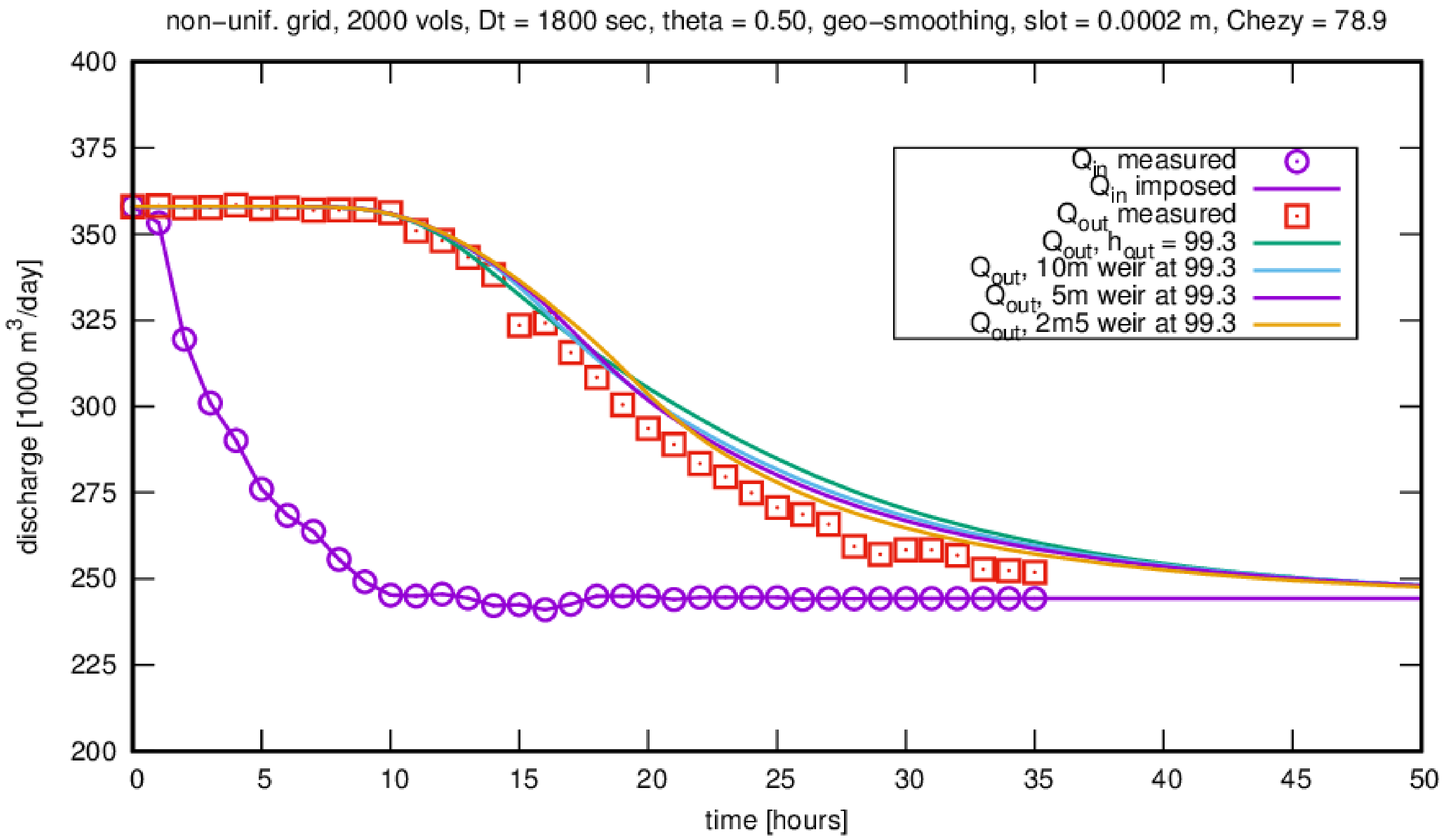
Steady-state great man-made river



2000 finite volumes, convergence history

Unsteady great man-made river

Comparison with measurements, high discharge



$$\max\left(|u|\Delta t/\Delta x\right)\approx 125,\ \ \max\left(\sqrt{gh}\Delta t/\Delta x\right)\approx 26{,}000$$

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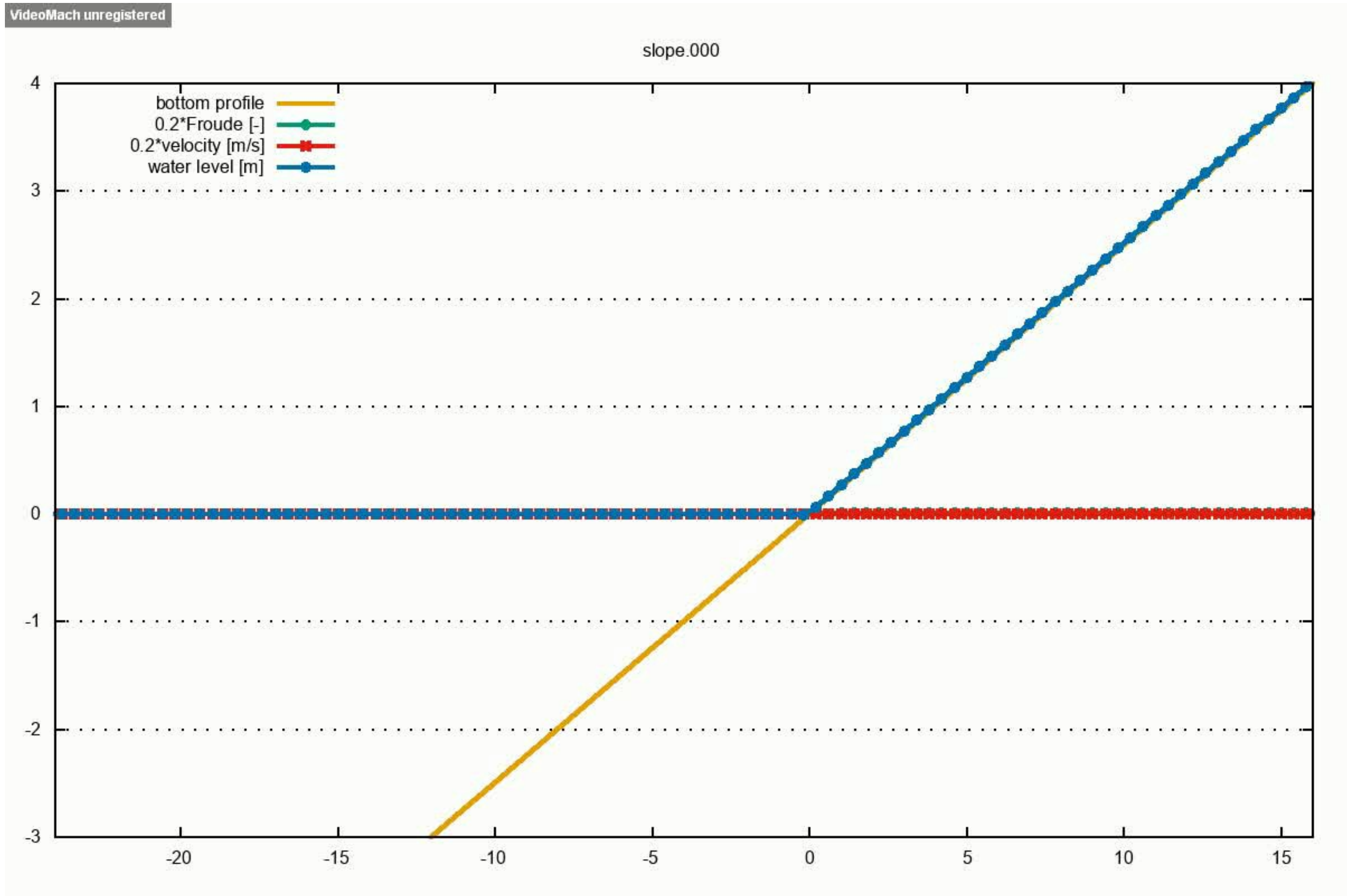
Wave run-up/run-down

$$domain\ length = 40,\ \ \Delta x = 0.4,\ \ period = 10,\ \ \Delta t = 0.1,$$
$$Chezy = 100,\ \ \theta = 0.6,$$
$$\max\left(|u|\Delta t/\Delta x\right) \approx 2,$$
$$\max\left(\sqrt{gh}\Delta t/\Delta x\right) \approx 2$$



9–17 iterations per time step
(12.3 on average)
→ better initial solutions required to speed up convergence per time step

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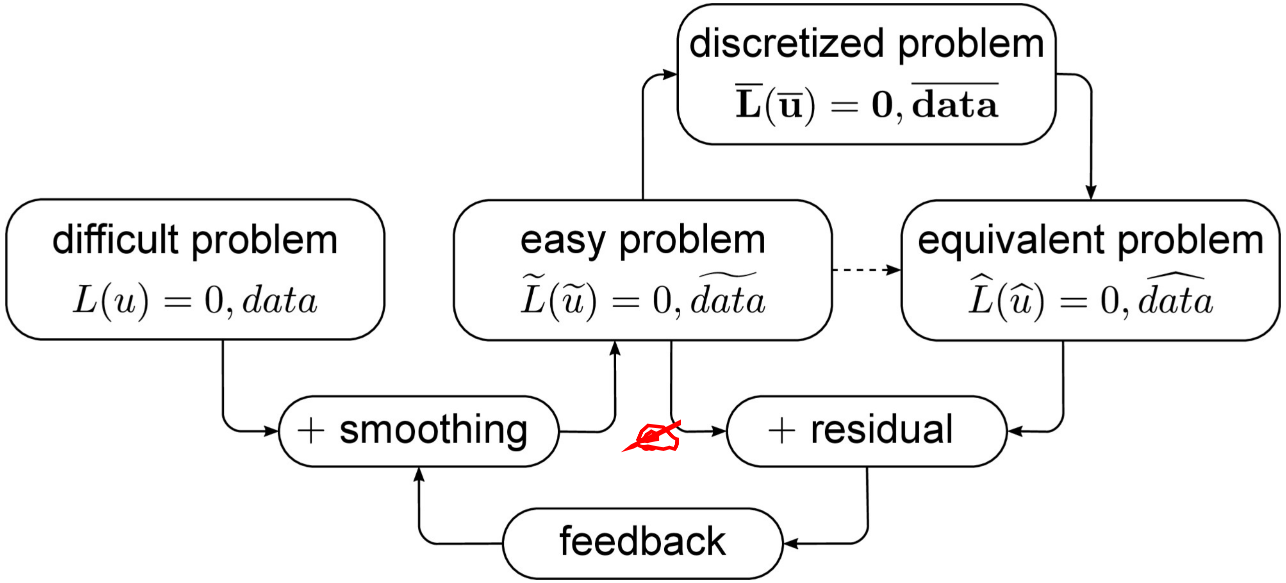
Summarizing …

Properties of space discretization:

- Smoothing errors dominant, discretization errors negligible
  → numerical errors can be interpreted physically!
  → numerical errors can be compared to physical errors!
- Pretty accurate
- Continuous and smooth
  - very fast solvers
  - robust model
- (block-)Structured grids required
  - flexibility by means of DD and IBM

Time integration:

- To be investigated



Motto

SUPPRESS THE WIGGLES, BUT SUCH THAT THEY KEEP TELLING YOU SOMETHING!

(variation on Gresho & Lee, "*Don't suppress the wiggles—They're telling you something!*", Comput. Fluids 1981)

What’s next

SUPSUB next version:

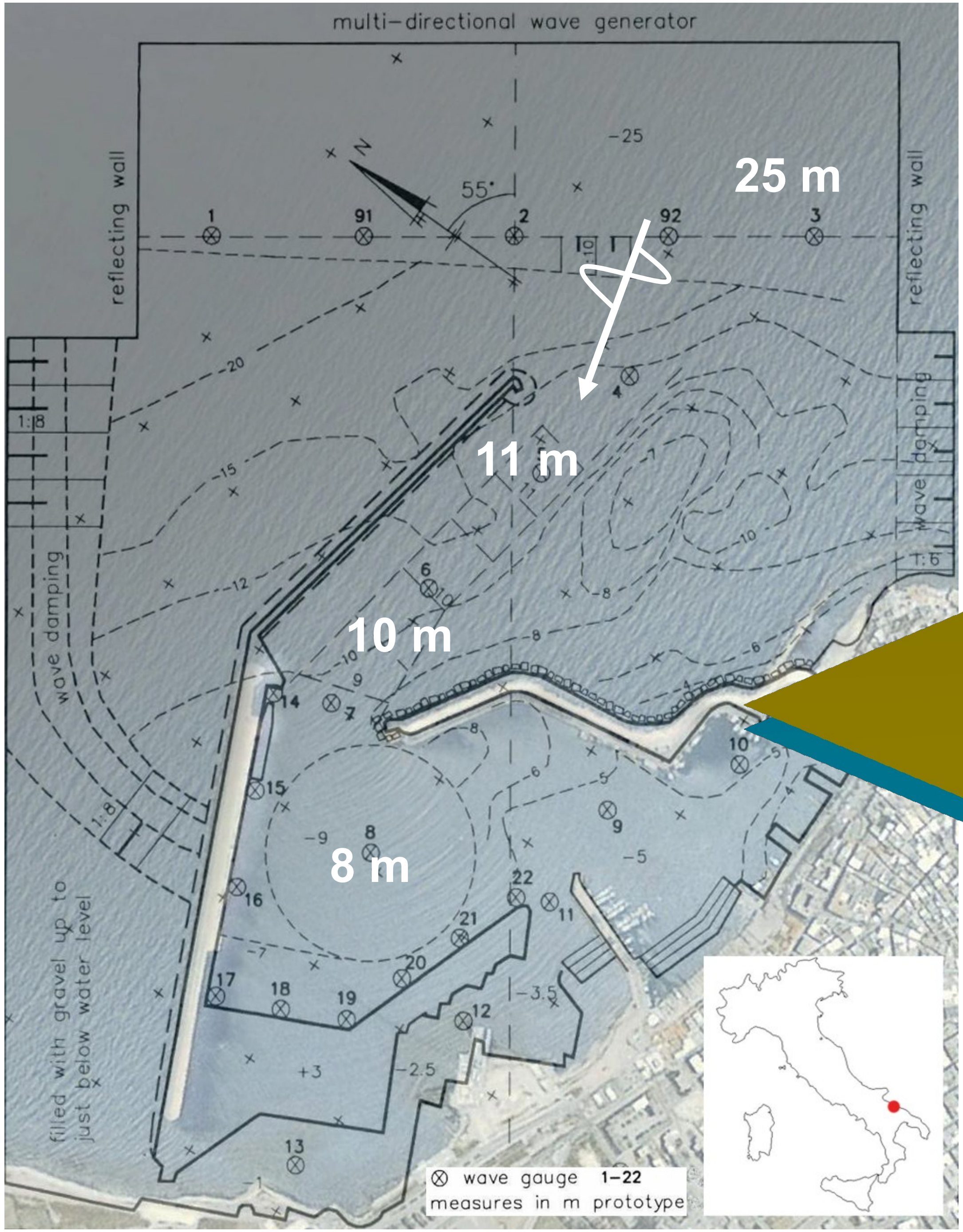
- embed as single-pipe component in WANDA?
- better initialization per time step (unsteady applications)
  → increase of robustness and efficiency
- better pseudo time-step scaling (based on residual and *full* linearization error)
  → increase of robustness and efficiency
- proper geometry smoothing (boundary/initial condition smoothing p.m.)
  → some increase of accuracy, robustness, efficiency
- moving error-minimizing grid adaptation (p.m.)
  → large increase of accuracy, efficiency, complexity

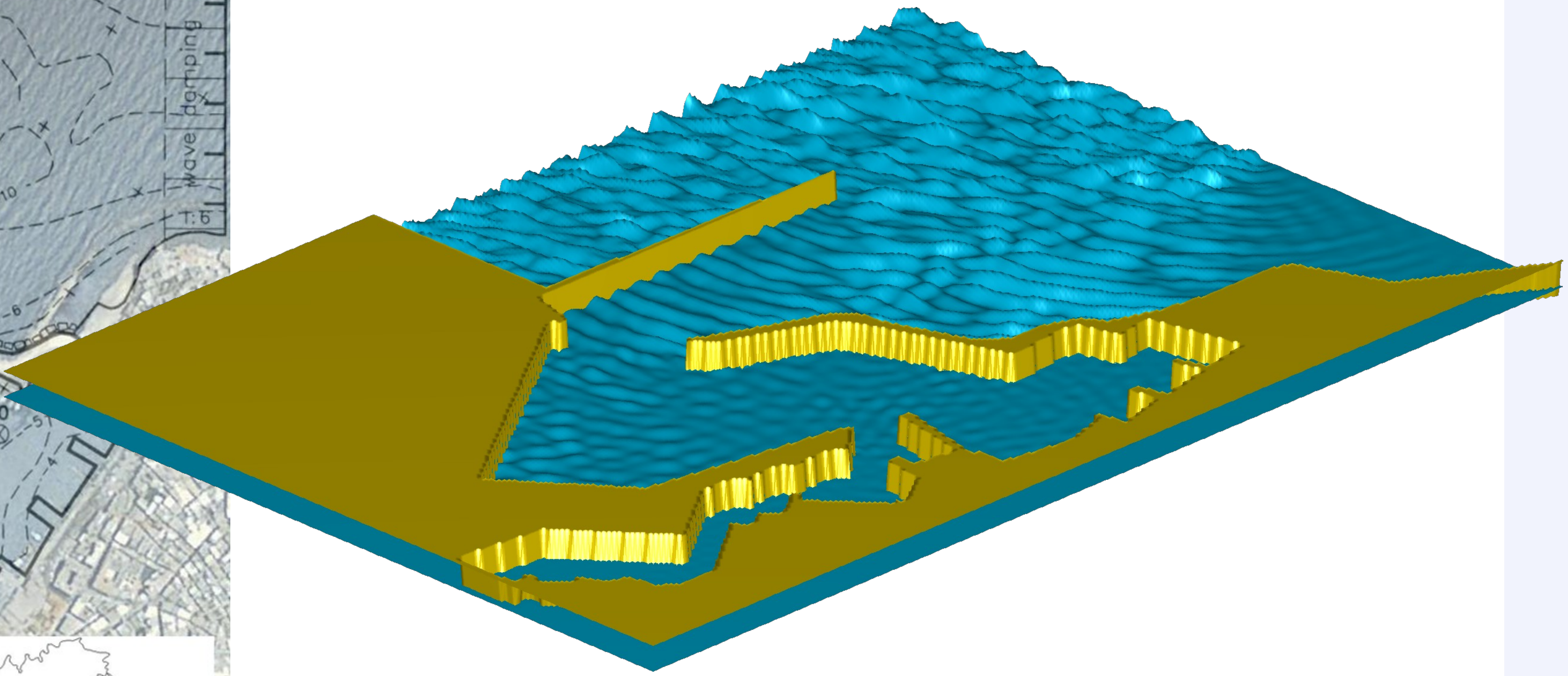
Wave modeling (Boussinesq-type, multi-layer, GABCs):

- extension to 2DV, IMEX time integration?
- 2DH/3D with DD and IBM for modeling flexibility (p.m.)

Remarks, discussion, questions?

A wave modeling example

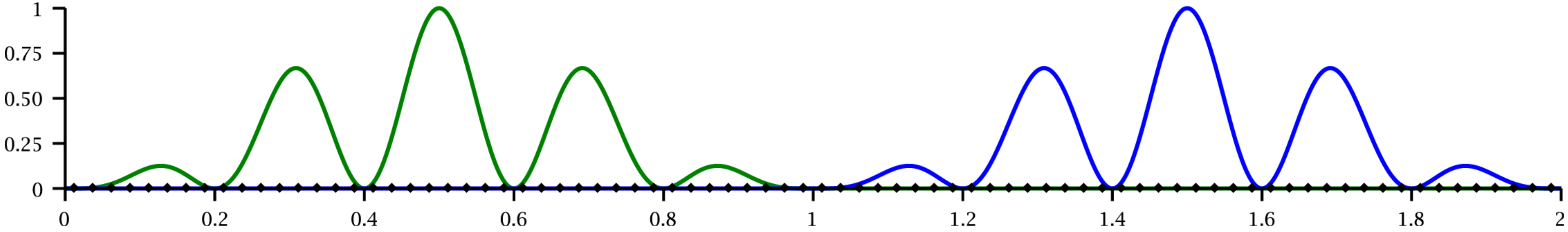




Port of Molfetta, Italy

Performance of several space discretizations

Transport of wavy signal

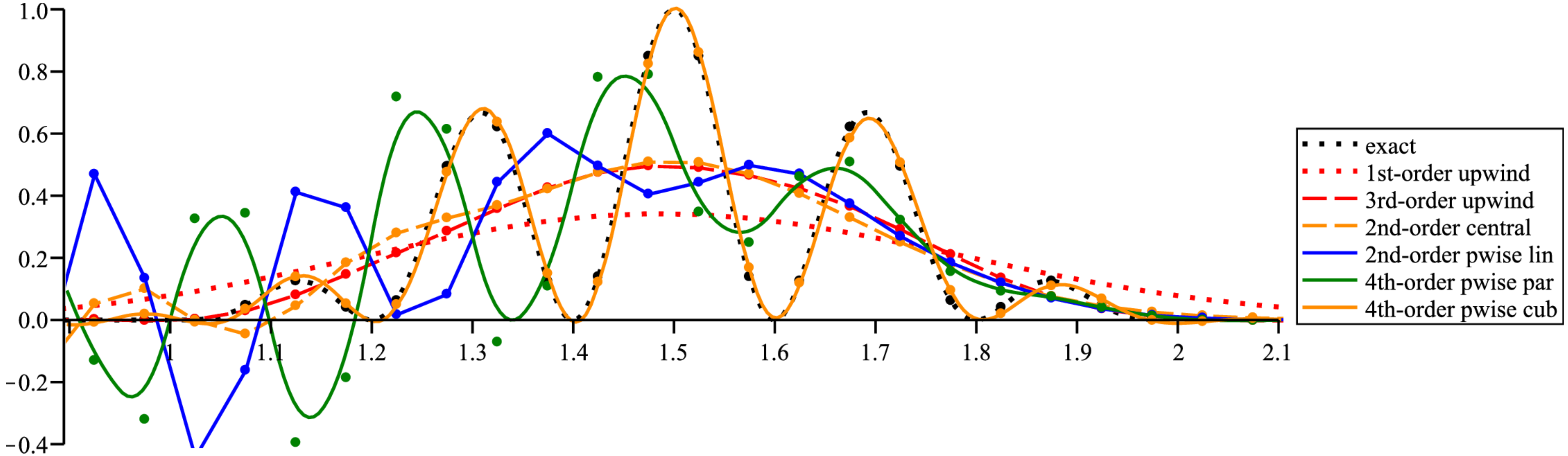


initial solution

final solution

Performance of several space discretizations

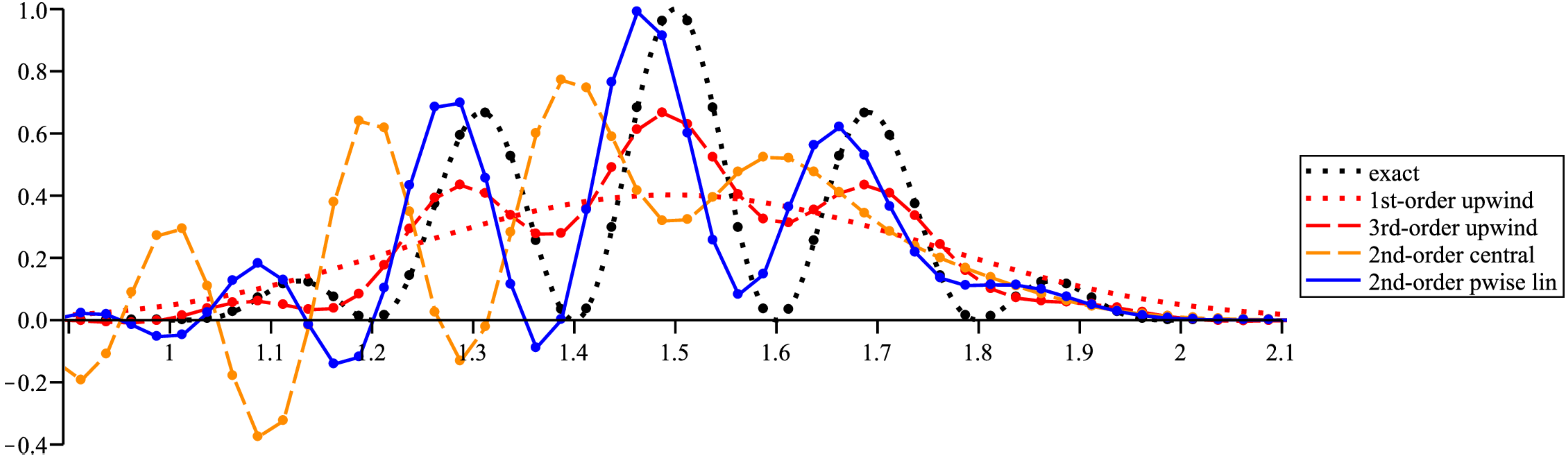
Transport of wavy signal



20 finite volumes (Δx=0.05)

Performance of several space discretizations

Transport of wavy signal



160 finite volumes ($\Delta x = 0.00625$)

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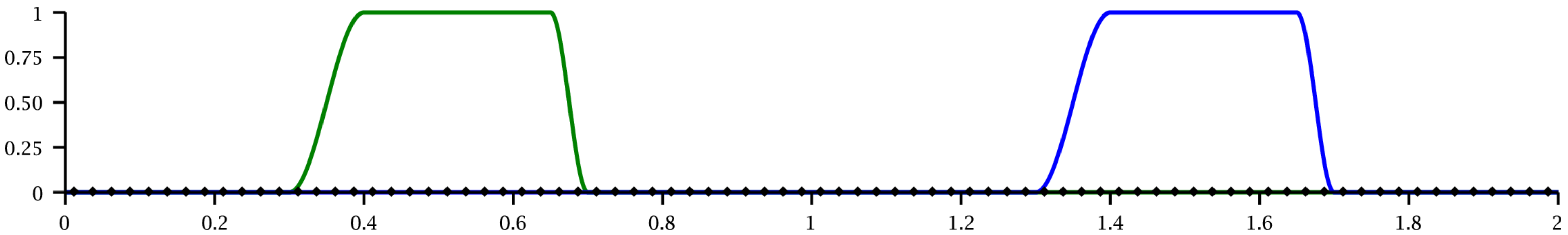
Performance of several space discretizations

Transport of block with cosine slopes

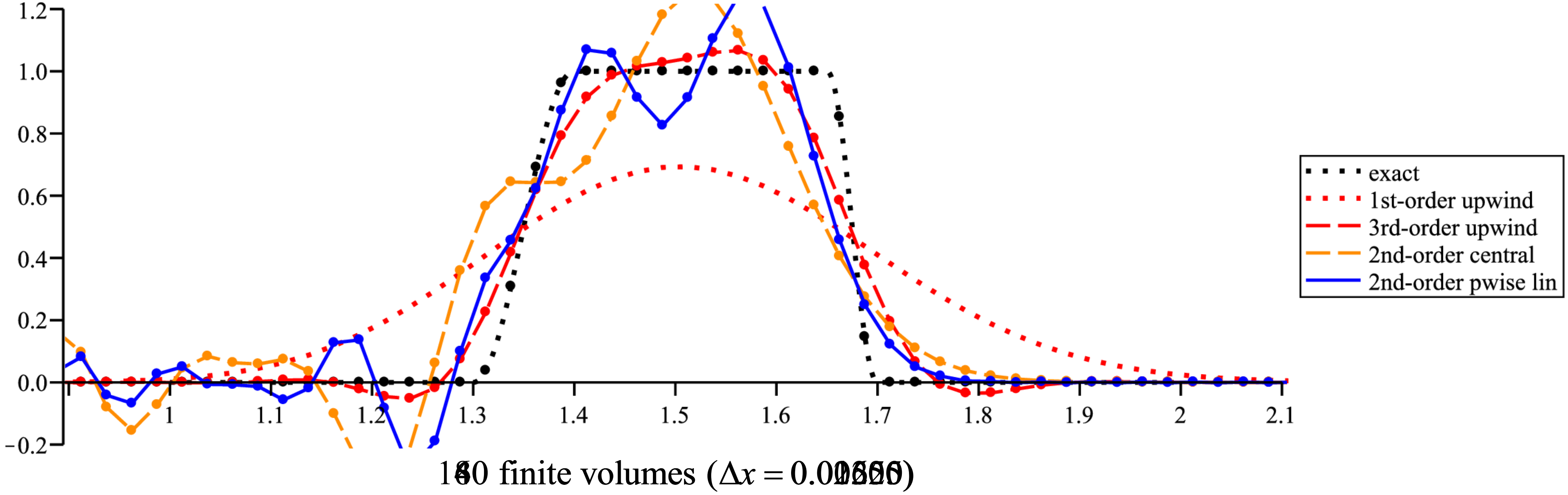


initial solution

final solution

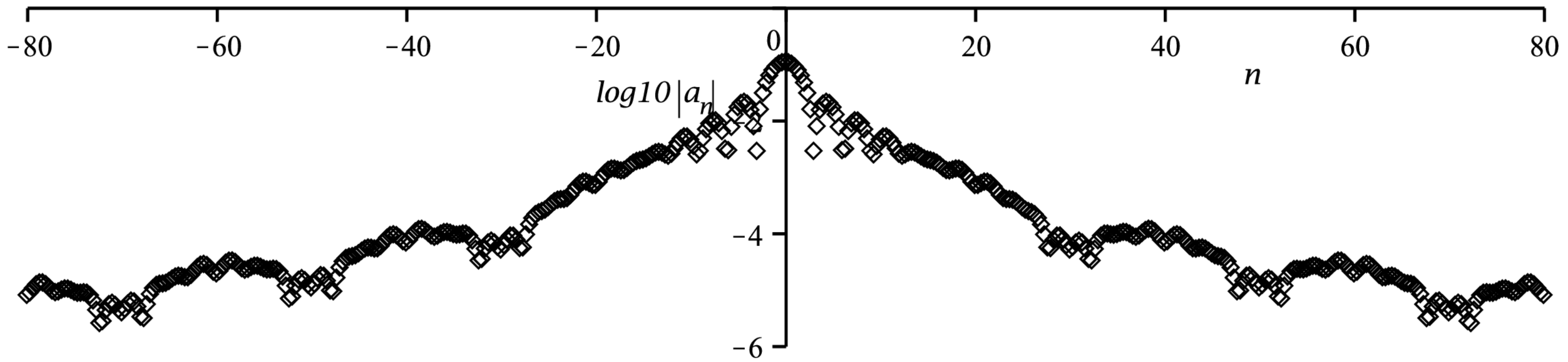
Performance of several space discretizations

Transport of block with cosine slopes

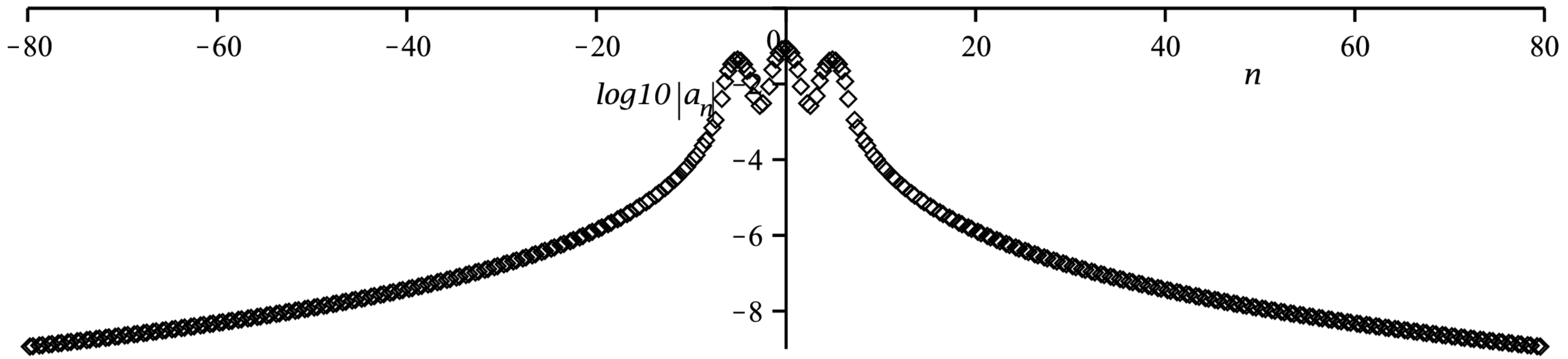


Fourier-mode decompositions

Block with cosine slopes



Wavy signal



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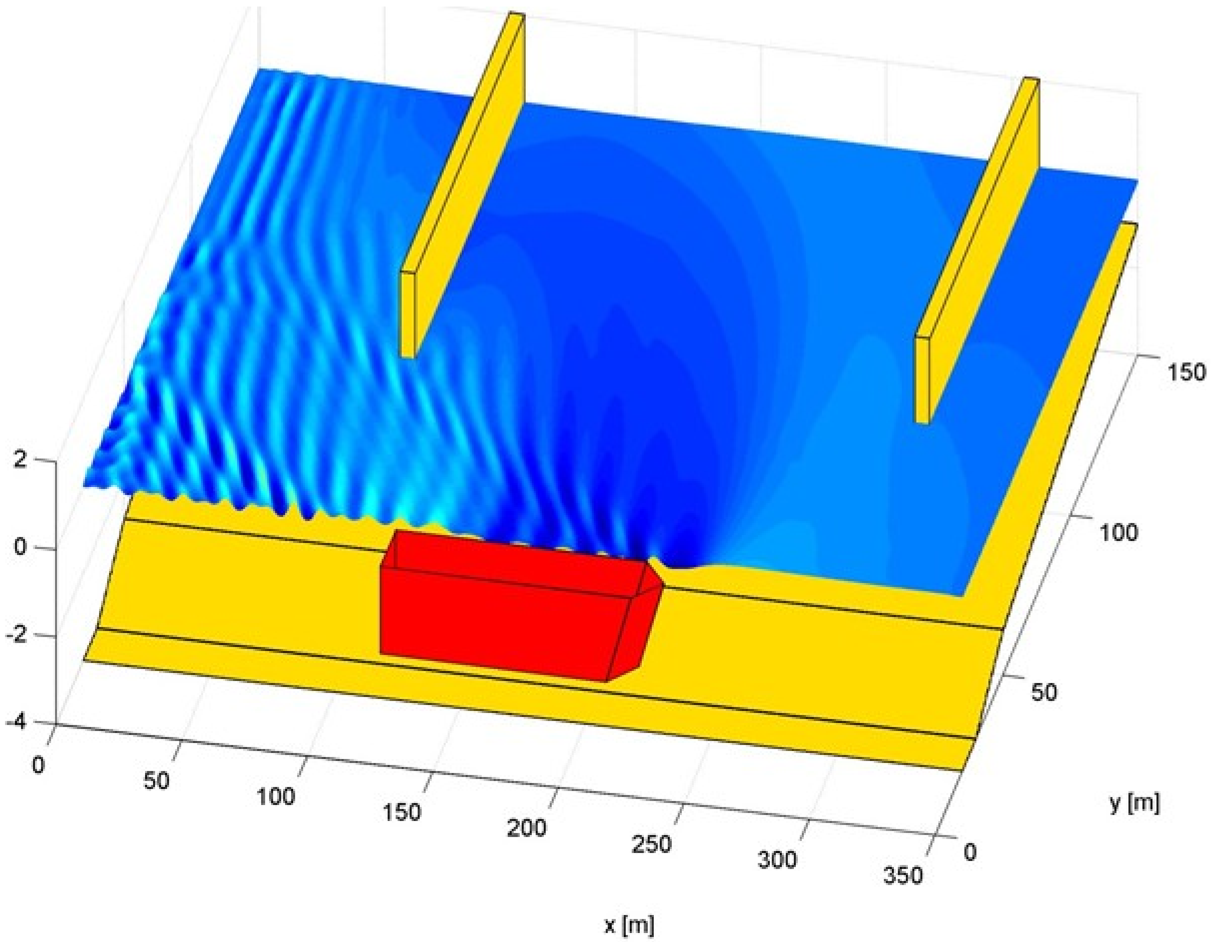
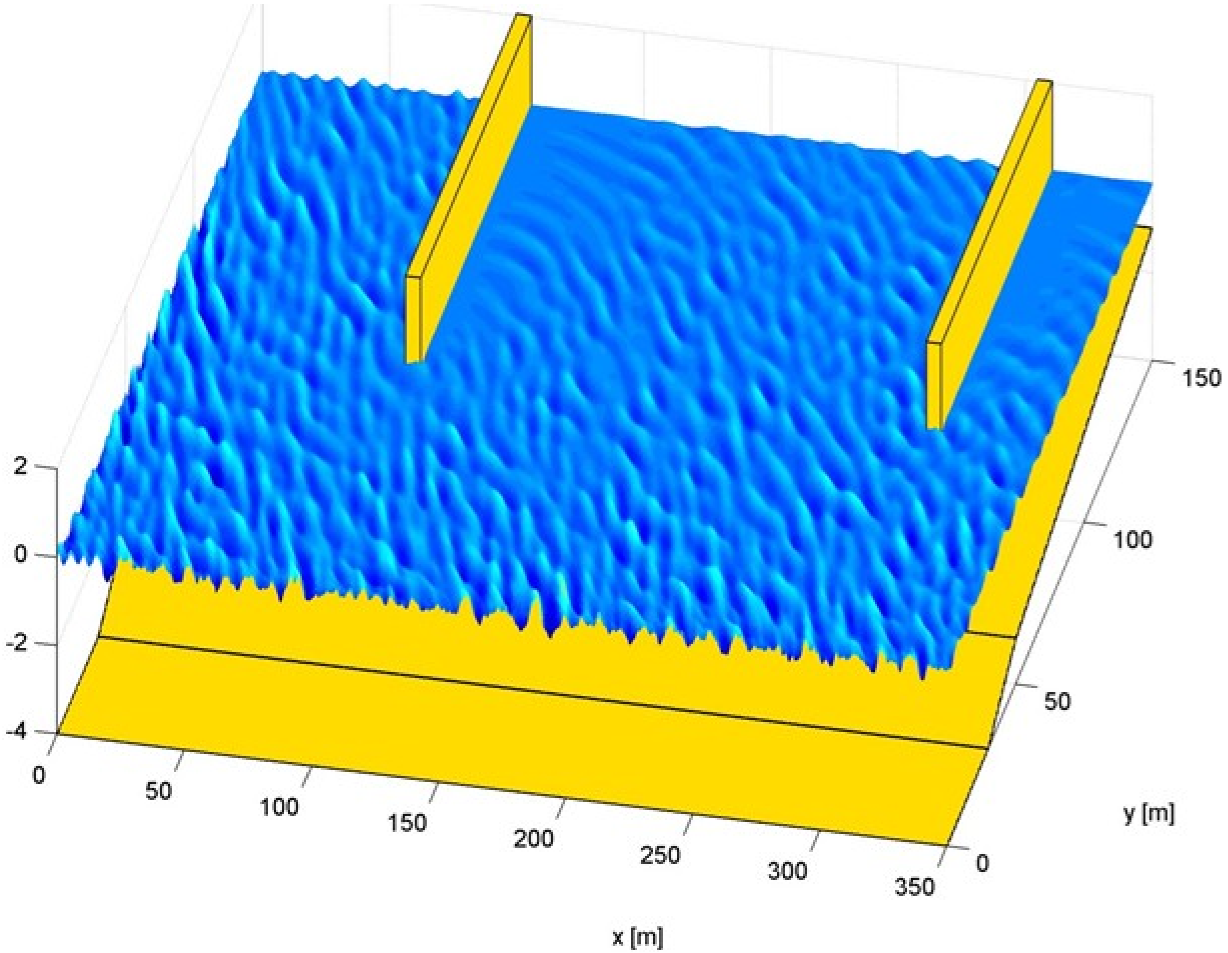
Some more wave modeling examples



Ship-induced waves in a river section



Wind waves in a river section

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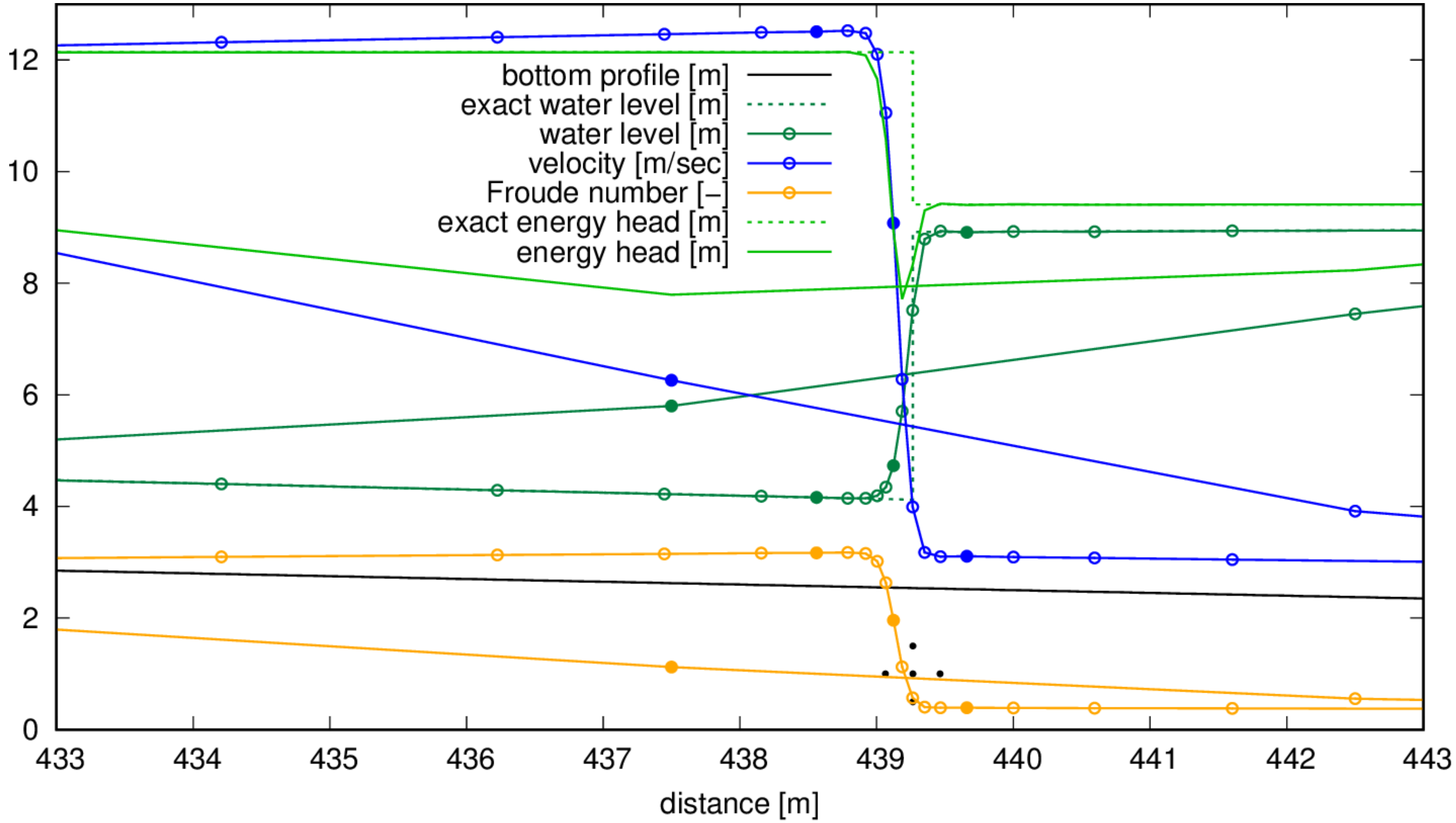
Sub-/supercritical steady flow over a weir



100 finite volumes, uniform and adaptive grid, solution around jump