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2DV multi-layer model, some equations

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1 Introduction

The 2DV free-surface flow equations are:

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0 \ , \tag{1}$$

$$\frac{\partial u}{\partial t} + \frac{\partial uu}{\partial x} + \frac{\partial wu}{\partial z} + \frac{1}{\rho}\frac{\partial p}{\partial x} = \frac{\partial}{\partial x}\left(2\nu\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial z}\left(\nu\frac{\partial u}{\partial z} + \nu\frac{\partial w}{\partial x}\right) \ , \tag{2}$$

$$\frac{\partial w}{\partial t} + \frac{\partial uw}{\partial x} + \frac{\partial ww}{\partial z} + \frac{1}{\rho}\frac{\partial p}{\partial z} = -g + \frac{\partial}{\partial x}\left(\nu\frac{\partial w}{\partial x} + \nu\frac{\partial u}{\partial z}\right) + \frac{\partial}{\partial z}\left(2\nu\frac{\partial w}{\partial z}\right) \ , \tag{3}$$

with bottom and free-surface boundary conditions:

$$-u\frac{\partial h}{\partial x} - w = 0 \ , \qquad z = -h \ , \tag{4}$$

$$\frac{\partial \zeta}{\partial t} + u\frac{\partial \zeta}{\partial x} - w = 0 \ , \qquad z = \zeta \ , \tag{5}$$

$$p = p_{\text{surf}} \ , \ z = \zeta \ . \tag{6}$$

In these equations, we have horizontal coordinate x, upward pointing vertical coordinate z, horizontal velocity component u, vertical velocity component w, and pressure p. Density ρ, gravitational acceleration g and atmospheric pressure at the free surface p_{surf} are taken constant. Free surface $\zeta = \zeta(x,t)$ and water depth $h = h(x)$ are both defined with respect to the horizontal plane of reference $z = 0$.

It is convenient to write the pressure as the sum of the atmospheric pressure, a hydrostatic component, and a non-hydrostatic component:

$$p = p_{\text{surf}} + \rho g(\zeta - z) + \rho q \ . \tag{7}$$

In this way, the static pressure component $p_{\text{surf}} - \rho g z$ can be eliminated from the model equations and the computations. It is advantageous to do so, because the static part can be (very) large compared to the dynamic part $\rho(g\zeta+q)$, especially for smaller waves, thereby reducing the accuracy with which the dynamic part is represented in computer memory.

With (7) substituted in the above equations, these equations become:

$$\frac{\partial u}{\partial x}+\frac{\partial w}{\partial z}=0\,, \tag{8}$$

$$\frac{\partial u}{\partial t}+\frac{\partial uu}{\partial x}+\frac{\partial wu}{\partial z}+g\frac{\partial \zeta}{\partial x}+\frac{\partial q}{\partial x}=\frac{\partial}{\partial x}\left(2\nu\frac{\partial u}{\partial x}\right)+\frac{\partial}{\partial z}\left(\nu\frac{\partial u}{\partial z}+\nu\frac{\partial w}{\partial x}\right)\,, \tag{9}$$

$$\frac{\partial w}{\partial t}+\frac{\partial uw}{\partial x}+\frac{\partial ww}{\partial z}+\frac{\partial q}{\partial z}=\frac{\partial}{\partial x}\left(\nu\frac{\partial w}{\partial x}+\nu\frac{\partial u}{\partial z}\right)+\frac{\partial}{\partial z}\left(2\nu\frac{\partial w}{\partial z}\right)\,, \tag{10}$$

with bottom and free-surface boundary conditions:

$$-u\frac{\partial h}{\partial x}-w=0\,,\, z=-h\,, \tag{11}$$

$$\frac{\partial \zeta}{\partial t}+u\frac{\partial \zeta}{\partial x}-w=0\,,\, z=\zeta\,, \tag{12}$$

$$q=0\,,\, z=\zeta\,. \tag{13}$$

It is customary in shallow-water free-surface flow models to integrate continuity equation (8) over the depth and to combine it with kinematic boundary conditions (11) and (12), to obtain:

$$\frac{\partial \zeta}{\partial t}+\int_{-h}^{\zeta} u\,dz\,. \tag{14}$$

This equation is often used instead of (12) to obtain the value of ζ. It enforces global mass conservation, even when (8) is not fully satisfied at the discrete level due to, e.g., small convergence errors. We propose to use (12) because it is easier to apply (avoids the integration of u over the depth) and because mass conservation errors can better be solved by solving (8) better, if required.

1.1 Boundary conditions

Kinematic boundary conditions (11) and (12) are conditions for normal momentum, modeling a closed boundary. Dynamic boundary condition (13) models the equilibrium of the normal stress at the free surface, neglecting the contribution of the viscous normal stress, i.e., flow pressure at the free surface equals the atmospheric pressure. Besides these conditions, we also need a condition at bottom and free surface for tangential momentum, like the irrotational flow condition if viscous effects are neglected, or a condition to model viscous effects (turbulent boundary layer at bottom, wind shear at free surface). Assuming (for the moment) no viscous effects and hence irrotational flow at bottom and free surface, the conditions for tangential momentum are:

$$\frac{\partial u}{\partial z}-\frac{\partial w}{\partial x}=0\,,\; z=-h \;\text{ and }\; z=\zeta\,. \tag{15}$$

Rotated to a local coordinate system aligned with the boundary, these conditions read:

$$\cos\theta_{\mathrm{bot}}\frac{\partial u}{\partial n}-\sin\theta_{\mathrm{bot}}\frac{\partial u}{\partial s}+\sin\theta_{\mathrm{bot}}\frac{\partial w}{\partial n}+\cos\theta_{\mathrm{bot}}\frac{\partial w}{\partial s}=0\,,\, z=-h\,, \tag{16}$$

$$\cos\theta_{\mathrm{surf}}\frac{\partial u}{\partial n}-\sin\theta_{\mathrm{surf}}\frac{\partial u}{\partial s}+\sin\theta_{\mathrm{surf}}\frac{\partial w}{\partial n}+\cos\theta_{\mathrm{surf}}\frac{\partial w}{\partial s}=0\,,\, z=\zeta\,, \tag{17}$$

with n the coordinate normal to bottom or free surface pointing outwards, with s the coordinate tangential to bottom or free surface, and with:

$$\begin{aligned} \cos\theta_{\mathrm{bot}} &= -1/\sqrt{1+(\partial h/\partial x)^2}\ ,\ \sin\theta_{\mathrm{bot}} = (\partial h/\partial x)/\sqrt{1+(\partial h/\partial x)^2}\ , \\ \cos\theta_{\mathrm{surf}} &= 1/\sqrt{1+(\partial \zeta/\partial x)^2}\ ,\ \sin\theta_{\mathrm{surf}} = (\partial \zeta/\partial x)/\sqrt{1+(\partial \zeta/\partial x)^2}\ . \end{aligned} \tag{18}$$

Neglecting the (small) local variation of θ_{bot} and θ_{surf} (or assuming a certain, irrelevant, behavior of θ_{bot} and θ_{surf} normal to bottom and free surface such that $u(\partial\cos\theta_{\mathrm{bot}}/\partial n) + w(\partial\sin\theta_{\mathrm{bot}}/\partial n) = u(\partial\sin\theta_{\mathrm{bot}}/\partial s) - w(\partial\cos\theta_{\mathrm{bot}}/\partial s)$ at $z = -h$ and $u(\partial\cos\theta_{\mathrm{surf}}/\partial n) + w(\partial\sin\theta_{\mathrm{surf}}/\partial n) = u(\partial\sin\theta_{\mathrm{surf}}/\partial s) - w(\partial\cos\theta_{\mathrm{surf}}/\partial s)$ at $z = \zeta$), (16) and (17) can be written as:

$$\frac{\partial u_s}{\partial n} - \frac{\partial u_n}{\partial s} = 0\ ,\ \ z = -h\ \text{ and }\ z = \zeta\ , \tag{19}$$

with $u_s = \cos\theta_{\mathrm{bot}} u|_{z=-h} + \sin\theta_{\mathrm{bot}} w|_{z=-h}$ and $u_n = -\sin\theta_{\mathrm{bot}} u|_{z=-h} + \cos\theta_{\mathrm{bot}} w|_{z=-h}$ at the bottom, and with $u_s = \cos\theta_{\mathrm{surf}} u|_{z=\zeta} + \sin\theta_{\mathrm{surf}} w|_{z=\zeta}$ and $u_n = -\sin\theta_{\mathrm{surf}} u|_{z=\zeta} + \cos\theta_{\mathrm{surf}} w|_{z=\zeta}$ at the free surface.

As expected, (19) is of the same form as (15): the irrotational flow condition is invariant under rotation. Using (11), which states $u_n|_{z=-h} = 0$ (impermeable bottom), the first equation of (19) can be simplified to $\partial u_s/\partial n|_{z=-h} = 0$. Such a simplification is not possible at the moving free surface; combining the second equation of (19) and (12) written as $(\cos\theta_{\mathrm{surf}})\partial\zeta/\partial t = u_n|_{z=\zeta}$ (impermeable free surface) is not useful.

In view of the numerical implementation (model equations formulated in Cartesian physical space are transformed to curvilinear computational space and then discretized), it is advisable to use (15) rather than (19) if the irrotational flow condition is to be applied. The latter would involve an additional coordinate transformation, i.e., the rotation from the global Cartesian coordinate system to the local normal coordinate system at the boundary. Notice that such a rotation cannot be avoided if the viscous shear stress along the boundaries is to be modeled (turbulent boundary layer at bottom, wind shear at free surface). For example, the condition for zero viscous shear stress at bottom and free surface reads:

$$\frac{\partial u_s}{\partial n} + \frac{\partial u_n}{\partial s} = 0\ ,\ \ z = -h\ \text{ and }\ z = \zeta\ , \tag{20}$$

and *not* $\partial u/\partial z + \partial w/\partial x = 0$, as applied in ComFLOW. That condition prescribes zero viscous shear stress in horizontal direction, *not* along the boundary.

At the bottom, condition (20) is equivalent with (19) because of $u_n|_{z=-h} = 0$. We do not have this equivalency at the free surface. Due to the moving surface viscous effects will generally be present, also if (20) is applied. Only the application of irrotational flow condition (15) ensures zero viscous stresses.

2 Coordinate transformation

The flow equations (8), (9) and (10) with boundary conditions (11), (12), (13) and (15) will be solved numerically on a structured, curvilinear grid fitted to the bottom and the free-surface boundary, cf. Figure 1. For illustration purposes, the grid shown in the figure is equidistant in x-direction and

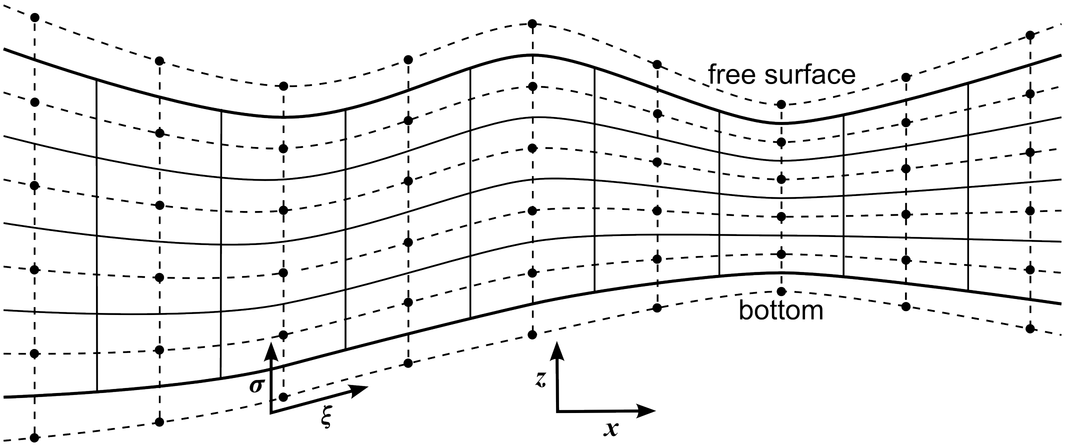

Figure 1: Schematic of applied curvilinear boundary-fitted grid. Dashed lines: grid; dots: grid points; solid lines: finite volumes; thick lines: bottom and free-surface boundary.

uniform in z-direction, but the grids that will be considered will be more general. The only restriction that will be applied is that the vertically oriented grid lines are to be strictly vertical, as in the figure.

It is useful and convenient to formulate the model equations in the computational coordinate system (τ, ξ, σ) aligned in time with the moving grid, with τ the transformed time coordinate t. The coordinate transformation gives full insight in all metric terms to be discretized, which is instrumental in making proper numerical choices and helpful in setting up the software implementation. It is possible to design a finite volume discretization with the equations formulated in the original (x, z) coordinates, but this is less transparent. It is easy to formulate accurate interpolations on the uniform Cartesian grid in computational (ξ, σ) space, but not so on the curved and stretched grid in physical (x, z) space. The uniform Cartesian grid in (ξ, σ) space also simplifies the analysis and assessment of discretization errors, including the effect of grid stretching and curvature, Borsboom (2001).

For the resulting finite volume discretization it does not matter which space is used. A finite volume scheme applied in computational space gives a discretization equivalent with a finite volume scheme applied in physical space, with the discretization of the metric terms in computational space equivalent with the size of the volumes and edges of the finite volumes in physical space.

Because of the moving free surface, the horizontal grid lines must be allowed to move. For the simulation of waves generated by the lateral motion of a vertical wave paddle the vertical grid lines must be allowed to move. The same grid motions may also be used in the future to adapt the grid dynamically to the solution. For example, to improve the simulation of short waves in deeper water, the horizontal grid lines may be concentrated near the free surface. To better resolve boundary layers, the horizontal grid lines may (also) be concentrated near the bottom. This is computationally more efficient than increasing the number of horizontal grid layers. Obviously, a high local concentration of grid lines is only possible if the applied discretization allows the use of large grid stretching. We know from 1D experiences that a compatible finite volume discretization based on piecewise linear function approximations allows stretchings up to 80%, cf. Figure 7 in Borsboom (2001). Because of the larger discretization errors on a non-uniform grid, such large stretchings will not be possible with a finite-difference-type finite volume discretization without the application of an excessive amount of artificial dissipation.

Because of the restriction (introduced to simplify things a bit, but not essential) that the vertically oriented grid lines are to be strictly vertical, we have that the horizontal physical coordinate x is invariant with the vertical computational coordinate σ. Otherwise there are no a priori spatial restrictions to the grid, but we do apply the usual restriction of grid planes defined per time level, i.e., the spatial grid planes are strictly normal to the time coordinate. In consequence, physical time t only depends on computational time τ:

$$t = t(\tau) \ , \ x = x(\tau, \xi) \ , \ z = z(\tau, \xi, \sigma) \ . \tag{21}$$

Furthermore we have, because the grid is aligned with the bottom and free-surface boundary:

$$z(\tau, \xi, \sigma_{\mathrm{bot}}) = -h(\tau, \xi) \ , \ z(\tau, \xi, \sigma_{\mathrm{surf}}) = \zeta(\tau, \xi) \ , \tag{22}$$

with σ_{bot} and σ_{surf} ($\sigma_{\mathrm{bot}} < \sigma_{\mathrm{surf}}$) the value of σ at bottom and free surface.

From coordinate transformation (21) it follows:

$$\begin{pmatrix} \frac{\partial}{\partial \tau} \\ \frac{\partial}{\partial \xi} \\ \frac{\partial}{\partial \sigma} \end{pmatrix} = \begin{pmatrix} t_\tau & x_\tau & z_\tau \\ 0 & x_\xi & z_\xi \\ 0 & 0 & z_\sigma \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial t} \\ \frac{\partial}{\partial x} \\ \frac{\partial}{\partial z} \end{pmatrix} \ , \tag{23}$$

with $t_\tau = \partial t / \partial \tau$, etc. The inverse is:

$$\begin{pmatrix} \frac{\partial}{\partial t} \\ \frac{\partial}{\partial x} \\ \frac{\partial}{\partial z} \end{pmatrix} = \frac{1}{J} \begin{pmatrix} x_\xi z_\sigma & -z_\sigma x_\tau & z_\xi x_\tau - x_\xi z_\tau \\ 0 & t_\tau z_\sigma & -t_\tau z_\xi \\ 0 & 0 & t_\tau x_\xi \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial \tau} \\ \frac{\partial}{\partial \xi} \\ \frac{\partial}{\partial \sigma} \end{pmatrix} \ , \tag{24}$$

with Jacobian $J = t_\tau x_\xi z_\sigma$, the product of time step t_τ and the finite volume surface $x_\xi z_\sigma$ if the grid size in computational space is taken unity ($\Delta\tau = \Delta\xi = \Delta\sigma = 1$).

Applying (21) and (24) to (8), (9), (10), (11), (12), (13) and (15) we obtain the flow equations formulated in computational space. Using identities like $\partial z_\tau / \partial \xi = \partial z_\xi / \partial \tau$ the equations can be recast in conservative form again. One identity needs to be mentioned separately (NB, because of the second expression in (21) we have $x_\sigma = 0$):

$$\frac{\partial (x_\xi z_\sigma)}{\partial \tau} + \frac{\partial (-z_\sigma x_\tau)}{\partial \xi} + \frac{\partial (z_\xi x_\tau - x_\xi z_\tau)}{\partial \sigma} = 0 \ . \tag{25}$$

This is the geometric conservation law. It states that a change in size of a control volume due to a grid motion (first term in left-hand side) must be balanced by the virtual fluxes through its moving faces (second and third term in left-hand side). If (25) does not hold exactly at the discrete level, then any motion of the grid may introduce spurious flow [REF to be included!!!].

Transformed continuity equation (8):

$$t_\tau \left(\frac{\partial (z_\sigma u)}{\partial \xi} + \frac{\partial (-z_\xi u + x_\xi w)}{\partial \sigma} \right) = 0 \ . \tag{26}$$

Transformed horizontal momentum equation (9):

$$
\begin{aligned}
&\frac{\partial(x_\xi z_\sigma u)}{\partial\tau} + \frac{\partial(z_\sigma(t_\tau u - x_\tau)u)}{\partial\xi} + \frac{\partial(z_\xi(x_\tau - t_\tau u)u + x_\xi(t_\tau w - z_\tau)u)}{\partial\sigma} \\
&\quad + \frac{\partial(t_\tau z_\sigma(g\zeta + q))}{\partial\xi} + \frac{\partial(-t_\tau z_\xi(g\zeta + q))}{\partial\sigma} \\
&\quad = \frac{\partial}{\partial\xi}\left(\frac{t_\tau z_\sigma 2\nu}{x_\xi z_\sigma}\left(z_\sigma\frac{\partial u}{\partial\xi} - z_\xi\frac{\partial u}{\partial\sigma}\right)\right) - \frac{\partial}{\partial\sigma}\left(\frac{t_\tau z_\xi 2\nu}{x_\xi z_\sigma}\left(z_\sigma\frac{\partial u}{\partial\xi} - z_\xi\frac{\partial u}{\partial\sigma}\right)\right) \\
&\quad + \frac{\partial}{\partial\sigma}\left(\frac{t_\tau x_\xi \nu}{x_\xi z_\sigma}\left(x_\xi\frac{\partial u}{\partial\sigma} + z_\sigma\frac{\partial w}{\partial\xi} - z_\xi\frac{\partial w}{\partial\sigma}\right)\right) .
\end{aligned}
\tag{27}
$$

Transformed vertical momentum equation (10):

$$
\begin{aligned}
&\frac{\partial(x_\xi z_\sigma w)}{\partial\tau} + \frac{\partial(z_\sigma(t_\tau u - x_\tau)w)}{\partial\xi} + \frac{\partial(z_\xi(x_\tau - t_\tau u)w + x_\xi(t_\tau w - z_\tau)w)}{\partial\sigma} + \frac{\partial(t_\tau x_\xi q)}{\partial\sigma} \\
&\quad = \frac{\partial}{\partial\xi}\left(\frac{t_\tau z_\sigma \nu}{x_\xi z_\sigma}\left(z_\sigma\frac{\partial w}{\partial\xi} - z_\xi\frac{\partial w}{\partial\sigma} + x_\xi\frac{\partial u}{\partial\sigma}\right)\right) \\
&\quad - \frac{\partial}{\partial\sigma}\left(\frac{t_\tau z_\xi \nu}{x_\xi z_\sigma}\left(z_\sigma\frac{\partial w}{\partial\xi} - z_\xi\frac{\partial w}{\partial\sigma} + x_\xi\frac{\partial u}{\partial\sigma}\right)\right) + \frac{\partial}{\partial\sigma}\left(\frac{t_\tau x_\xi 2\nu}{x_\xi z_\sigma}\left(x_\xi\frac{\partial w}{\partial\sigma}\right)\right) .
\end{aligned}
\tag{28}
$$

Transformed bottom and free-surface boundary conditions (11), (12), (13) and (15):

$$
-u\frac{\partial h}{\partial\xi} - x_\xi w = 0 \;,\; \sigma = \sigma_{\mathrm{bot}} \;, \tag{29}
$$

$$
\frac{x_\xi}{t_\tau}\frac{\partial\zeta}{\partial\tau} + (u - x_\tau/t_\tau)\frac{\partial\zeta}{\partial\xi} - x_\xi w = 0 \;,\; \sigma = \sigma_{\mathrm{surf}} \;, \tag{30}
$$

$$
q = 0 \;,\; \sigma = \sigma_{\mathrm{surf}} \;, \tag{31}
$$

$$
x_\xi\frac{\partial u}{\partial\sigma} - z_\sigma\frac{\partial w}{\partial\xi} + z_\xi\frac{\partial w}{\partial\sigma} = 0 \;,\; \sigma = \sigma_{\mathrm{bot}} \text{ and } \sigma = \sigma_{\mathrm{surf}} \;. \tag{32}
$$

Notice that the last equation can also be written in the 'conservative' form $\partial(x_\xi u + z_\xi w)/\partial\sigma - \partial(z_\sigma w)/\partial\xi = 0$, but this is not useful. For the discretization of a term like $\partial(z_\xi w)/\partial\sigma$ at a σ-location (i.e., at σ_{bot} or σ_{surf}) using piecewise linear approximations in computational space of all variables (*not* of combinations of variables!), this term has to written as the sum $z_\xi\partial w/\partial\sigma + w\partial z_\xi/\partial\sigma$ anyway, so we may just as well consider the discretization of (32) directly. Using piecewise linear approximations, both $\partial(z_\sigma w)/\partial\xi$ and $z_\sigma\partial w/\partial\xi$ can be easily discretized across a boundary finite volume (a line piece). At a corner finite volume (a point, i.e., a (ξ,σ)-location) the former has first to be written as a sum of terms, i.e., as $z_\sigma\partial w/\partial\xi + w\partial z_\sigma/\partial\xi$, so also here we find that it is easier to consider the discretization of (32) directly. NB, since x does not depend on σ, cf. (21), (the discretization of) $\partial(x_\xi u)/\partial\sigma$ equals (that of) $x_\xi\partial u/\partial\sigma$.

3 Time integration

For the discretization in time it is convenient to write the transformed flow equations (27) and (28) in compact symbolic form, collecting in a single symbol the terms that will be dealt with in the same way in the time integration.

$$\frac{1}{t_\tau}\frac{\partial(x_\xi z_\sigma u)}{\partial\tau} + \mathrm{CONVS}(u) + \mathrm{CONVT}(u) + \mathrm{PRESU} = \mathrm{VISCL}(u)\ , \tag{33}$$

$$\frac{1}{t_\tau}\frac{\partial(x_\xi z_\sigma w)}{\partial\tau} + \mathrm{CONVS}(w) + \mathrm{CONVT}(w) + \mathrm{PRESW} = \mathrm{VISCL}(w)\ , \tag{34}$$

with:

$$\mathrm{CONVS}(f) = \frac{\partial(z_\sigma u f)}{\partial\xi} + \frac{\partial(-z_\xi u + x_\xi w)f}{\partial\sigma} \tag{35}$$

$$\mathrm{CONVT}(f) = \frac{\partial(-z_\sigma x_\tau/t_\tau f)}{\partial\xi} + \frac{\partial(z_\xi x_\tau/t_\tau - x_\xi z_\tau/t_\tau)f}{\partial\sigma} \tag{36}$$

$$\mathrm{PRESU} = \frac{\partial(z_\sigma(g\zeta+q))}{\partial\xi} + \frac{\partial(-z_\xi(g\zeta+q))}{\partial\sigma} \tag{37}$$

$$\mathrm{PRESW} = \frac{\partial(x_\xi(g\zeta+q))}{\partial\sigma} \tag{38}$$

$$\mathrm{VISCL}(f) = \frac{\partial}{\partial\xi}\left(\frac{z_\sigma\nu}{x_\xi z_\sigma}\left(z_\sigma\frac{\partial f}{\partial\xi} - z_\xi\frac{\partial f}{\partial\sigma}\right)\right) - \frac{\partial}{\partial\sigma}\left(\frac{z_\xi\nu}{x_\xi z_\sigma}\left(z_\sigma\frac{\partial f}{\partial\xi} - z_\xi\frac{\partial f}{\partial\sigma}\right)\right) \tag{39}$$

$$+\frac{\partial}{\partial\sigma}\left(\frac{x_\xi\nu}{x_\xi z_\sigma}\left(x_\xi\frac{\partial f}{\partial\sigma}\right)\right)\ , \tag{40}$$

$$\mathrm{VISCU} = \frac{\partial}{\partial\xi}\left(\frac{z_\sigma\nu}{x_\xi z_\sigma}\left(z_\sigma\frac{\partial u}{\partial\xi} - z_\xi\frac{\partial u}{\partial\sigma}\right)\right) - \frac{\partial}{\partial\sigma}\left(\frac{z_\xi\nu}{x_\xi z_\sigma}\left(z_\sigma\frac{\partial u}{\partial\xi} - z_\xi\frac{\partial u}{\partial\sigma}\right)\right) \tag{41}$$

$$+\frac{\partial}{\partial\sigma}\left(\frac{x_\xi\nu}{x_\xi z_\sigma}\left(z_\sigma\frac{\partial w}{\partial\xi} - z_\xi\frac{\partial w}{\partial\sigma}\right)\right)\ . \tag{42}$$

$$\mathrm{VISCW} = \frac{\partial}{\partial\xi}\left(\frac{z_\sigma\nu}{x_\xi z_\sigma}\left(x_\xi\frac{\partial u}{\partial\sigma}\right)\right) - \frac{\partial}{\partial\sigma}\left(\frac{z_\xi\nu}{x_\xi z_\sigma}\left(x_\xi\frac{\partial u}{\partial\sigma}\right)\right) \tag{43}$$

$$+\frac{\partial}{\partial\sigma}\left(\frac{x_\xi\nu}{x_\xi z_\sigma}\left(x_\xi\frac{\partial w}{\partial\sigma}\right)\right)\ . \tag{44}$$

In the discretization in time we use $\Delta\tau = 1$, hence $t_\tau = \Delta t$. An Adams-Bashfort-Moulton predictor-corrector method will be applied.

An essential part of the numerical method is the time discretization of the geometric conservation law (25). The following discretization is satisfied exactly, where we have again used the fact that $x_\sigma = 0$:

$$\frac{1}{\Delta t}\left(\frac{\partial x^{n+1}}{\partial\xi}\frac{\partial z^{n+1}}{\partial\sigma} - \frac{\partial x^n}{\partial\xi}\frac{\partial z^n}{\partial\sigma}\right) - \frac{\partial}{\partial\xi}\left((x^{n+1}-x^n)\frac{\partial(z^n+z^{n+1})/2}{\partial\sigma}\right)$$
$$+(x^{n+1}-x^n)\frac{\partial^2(z^n+z^{n+1})/2}{\partial\xi\partial\sigma} - \frac{\partial(z^{n+1}-z^n)}{\partial\sigma}\frac{\partial(x^n+x^{n+1})/2}{\partial\xi} = 0\ . \tag{45}$$

3.1 Predictor

First the kinematic boundary condition at the free surface (30) is integrated in time:

$$\frac{\zeta^*-\zeta^n}{\Delta t} + \sum_{k=n-1}^{n}\alpha_{\mathrm{conv}}^{p,k}\left(\frac{u^k-(x^{n+1}-x^n)/\Delta t}{x_\xi^k}\frac{\partial\zeta^k}{\partial\xi} - w^k\right)_{\sigma=\sigma_{\mathrm{surf}}} = 0\ . \tag{46}$$

Notice that in this expression we should have used x^* instead of x^{n+1}, but since the grid motion in x-direction is prescribed the grid in x-direction is known, i.e., we have $x^* = x^{n+1}$.

Expression (46) is obtained by discretizing x_τ in (30) in correspondence with the discretized geometric conservation law (45). One may wonder if this is the best choice. It corresponds with using the (linear) variation of the grid in computational time τ over the current time step from t^n to t^{n+1} also at the previous time levels $t^k, k < n$, which is not correct and hence an approximation. On the other hand one may argue that, since the grid motion is to some extent arbitrary, it is allowed to fix the grid motion temporarily to the one across the current time step and to use discretization $x_\tau = x^{n+1} - x^n$ at all time levels involved in the time integration over that time step. This ignores that the solution at previous time levels is generally defined on a (slightly) different grid.

It is not well possible to do better. For example, using the (second-order accurate) finite difference approximations $x_\tau^k = (x^{k+1} - x^{k-1})/2$ (recall that we have $\Delta\tau = 1$) would limit the accuracy of the time integration of (30) to second order, regardless of the order of the applied Adams-Bashfort-Moulton method. Trying to do better becomes quite complicated and would involve grid information from many different time levels, with the risk of making the discretization rather sensitive to irregularities in the grid motion.

The previous two paragraphs are typically reasonings in physical space. A reasoning in computational space would be to realize that an Adams-Bashford-Moulton method applied to a time-dependent differential equation like (30) involves the approximation of its time integral per time step $[\tau^n, \tau^{n+1}]$, using exact expressions for the time derivatives and using approximations for all other terms that are obtained by means of extrapolations in computational time τ. This procedure leads directly to result (46).

From the solution ζ^* of (46) we obtain the prediction of the coordinate transformation in σ direction at the next time level t^{n+1}. From this transformation we obtain z_ξ^* and z_σ^* that are used in the following steps, where superscript $n + 1/2$ denotes the average between a term evaluated at t^n and at t^*.

NOTE: the predictor time step of any transport equation (if part of the model) comes here, I think. It requires the (predicted) grid at the next time level (predicted coordinate transformation at the next time level is available), the velocity at previous time levels (available), while it computes (turbulent) quantities that may be required to compute (viscosity) coefficients at the next time level in the implicit time integration of the viscosity terms in the momentum equations.

Next, the time integration of the momentum equations (33) and (34):

$$
\begin{aligned}
\frac{x_\xi^* z_\sigma^* u^* - x_\xi^n z_\sigma^n u^n}{\Delta t} &+ \sum_{k=n-1}^{n} \alpha_{\text{conv}}^{p,k} \Big(\text{CONVS}^k(u^k) + \text{CONVT}^{n+1/2}(u^k) \Big) \\
&+ \alpha_{\text{pres}}^{p,*} \text{PRESU}^* + \sum_{k=n-1}^{n} \alpha_{\text{pres}}^{p,k} \text{PRESU}^k \\
&= \theta \Big(\text{VISCL}^*(u^*) + \text{VISCU}^* \Big) \\
&+ (1-\theta) \Big(\text{VISCL}^n(u^n) + \text{VISCU}^n \Big) \,,
\end{aligned} \tag{47}
$$

$$
\begin{aligned}
\frac{x_\xi^* z_\sigma^* w^* - x_\xi^n z_\sigma^n w^n}{\Delta t} &+ \sum_{k=n-1}^{n} \alpha_{\text{conv}}^{p,k} \Big(\text{CONVS}^k(w^k) + \text{CONVT}^{n+1/2}(w^k) \Big) \\
&+ \alpha_{\text{pres}}^{p,*} \text{PRESW}^* + \sum_{k=n-1}^{n} \alpha_{\text{pres}}^{p,k} \text{PRESW}^k \\
&= \theta \Big(\text{VISCL}^*(w^*) + \text{VISCW}^* \Big) \\
&+ (1-\theta) \Big(\text{VISCL}^n(w^n) + \text{VISCW}^n \Big) \ ,
\end{aligned}
\tag{48}
$$

with $x_\xi^* = x_\xi^{n+1}$ because the grid in x-direction is prescribed and hence known. The Adams-Bashfort-Moulton parameters must satisfy $\sum_{k=n-1}^{n} \alpha_{\text{conv}}^{p,k} = \alpha_{\text{pres}}^{p,*} + \sum_{k=n-1}^{n} \alpha_{\text{pres}}^{p,k} = 1$, which is required for consistency in time.

Equations (47) and (48) are solved by combining them with continuity equation (26) at the predictor level:

$$
\frac{\partial (z_\sigma^* u^*)}{\partial \xi} + \frac{\partial (-z_\xi^* u^* + x_\xi^* w^*)}{\partial \sigma} = 0 \ , \tag{49}
$$

and with the boundary conditions (29), (31) and (32) at bottom and free surface at the predictor level:

$$
-u^* \frac{\partial h}{\partial \xi} - x_\xi^* w^* = 0 \ , \ \sigma = \sigma_{\text{bot}} \ , \tag{50}
$$

$$
q^* = 0 \ , \ \sigma = \sigma_{\text{surf}} \ , \tag{51}
$$

$$
x_\xi^* \frac{\partial u^*}{\partial \sigma} - z_\sigma^* \frac{\partial w^*}{\partial \xi} + z_\xi^* \frac{\partial w^*}{\partial \sigma} = 0 \ , \ \sigma = \sigma_{\text{bot}} \ \text{ and } \ \sigma = \sigma_{\text{surf}} \ . \tag{52}
$$

One more equation is required at bottom and free surface to allow the determination of all three unknowns per virtual grid point (u^*, w^* and q^*). As no further conditions can be imposed, the third equation to be applied at bottom and free surface is an auxiliary equation, to be obtained from the evaluation of the flow equations (47), (48) and/or (49). The most obvious (and best?) choice seems to be to apply the momentum equation normal to the boundary: evaluate (47) and (48) at bottom and at free surface, and combine both to the normal momentum equation:

.... MOMU and MOMW not at certain time level, because obtained from an integration in time

$$
-\text{MOMU}^* \frac{\partial h}{\partial \xi} - \text{MOMW}^* x_\xi^* = 0 \ , \ \sigma = \sigma_{\text{bot}} \ , \tag{53}
$$

$$
q^* = 0 \ , \ \sigma = \sigma_{\text{surf}} \ . \tag{54}
$$

3.2 Corrector

The corrector is similar to the predictor: the same steps in the same order.

$$
\begin{aligned}
\frac{\zeta^{n+1} - \zeta^n}{\Delta t} &+ \alpha_{\text{conv}}^{c,*} \left(\frac{u^* - (x^{n+1} - x^n)/\Delta t}{x_\xi^{n+1}} \frac{\partial \zeta^*}{\partial \xi} - w^* \right)_{\sigma=\sigma_{\text{surf}}} \\
&+ \sum_{k=n-1}^{n} \alpha_{\text{conv}}^{c,k} \left(\frac{u^k - (x^{n+1} - x^n)/\Delta t}{x_\xi^k} \frac{\partial \zeta^k}{\partial \xi} - w^k \right)_{\sigma=\sigma_{\text{surf}}} = 0 \ .
\end{aligned}
\tag{55}
$$

From ζ^{n+1} we obtain the coordinate transformation in σ direction at the next time level t^{n+1}, and hence z_ξ^{n+1} and z_σ^{n+1} that are used in the following steps. This time superscript $n+1/2$ denotes the average between a term evaluated at t^n and at t^{n+1}.

$$
\begin{aligned}
\frac{x_\xi^{n+1} z_\sigma^{n+1} u^{n+1} - x_\xi^n z_\sigma^n u^n}{\Delta t} &+ \alpha_{\mathrm{conv}}^{c,*}\Big(\mathrm{CONVS}^*(u^*) + \mathrm{CONVT}^{n+1/2}(u^*)\Big) \\
&+ \sum_{k=n-1}^{n} \alpha_{\mathrm{conv}}^{c,k}\Big(\mathrm{CONVS}^k(u^k) + \mathrm{CONVT}^{n+1/2}(u^k)\Big) \\
&+ \alpha_{\mathrm{pres}}^{c,n+1}\mathrm{PRESU}^{n+1} + \alpha_{\mathrm{pres}}^{c,*}\mathrm{PRESU}^* + \sum_{k=n-1}^{n} \alpha_{\mathrm{pres}}^{c,k}\mathrm{PRESU}^k \\
&= \theta\Big(\mathrm{VISCL}^{n+1}(u^{n+1}) + \mathrm{VISCU}^{n+1}\Big) \\
&+ \alpha_{\mathrm{visc}}^{c,*}\Big(\mathrm{VISCL}^{n+1}(u^*) + \mathrm{VISCU}^*\Big) \\
&+ (1-\theta-\alpha_{\mathrm{visc}}^{c,*})\Big(\mathrm{VISCL}^n(u^n) + \mathrm{VISCU}^n\Big) \ ,
\end{aligned}
\tag{56}
$$

$$
\begin{aligned}
\frac{x_\xi^{n+1} z_\sigma^{n+1} w^{n+1} - x_\xi^n z_\sigma^n w^n}{\Delta t} &+ \alpha_{\mathrm{conv}}^{c,*}\Big(\mathrm{CONVS}^*(w^*) + \mathrm{CONVT}^{n+1/2}(w^*)\Big) \\
&+ \sum_{k=n-1}^{n} \alpha_{\mathrm{conv}}^{c,k}\Big(\mathrm{CONVS}^k(w^k) + \mathrm{CONVT}^{n+1/2}(w^k)\Big) \\
&+ \alpha_{\mathrm{pres}}^{c,n+1}\mathrm{PRESW}^{n+1} + \alpha_{\mathrm{pres}}^{c,*}\mathrm{PRESW}^* + \sum_{k=n-1}^{n} \alpha_{\mathrm{pres}}^{c,k}\mathrm{PRESW}^k \\
&= \theta\Big(\mathrm{VISCL}^{n+1}(w^{n+1}) + \mathrm{VISCW}^{n+1}\Big) \\
&+ \alpha_{\mathrm{visc}}^{c,*}\Big(\mathrm{VISCL}^{n+1}(w^*) + \mathrm{VISCW}^*\Big) \\
&+ (1-\theta-\alpha_{\mathrm{visc}}^{c,*})\Big(\mathrm{VISCL}^n(w^n) + \mathrm{VISCW}^n\Big) \ ,
\end{aligned}
\tag{57}
$$

with $\alpha_{\mathrm{conv}}^{c,*} + \sum_{k=n-1}^{n} \alpha_{\mathrm{conv}}^{c,k} = \alpha_{\mathrm{pres}}^{c,*} + \sum_{k=n-1}^{n+1} \alpha_{\mathrm{pres}}^{c,k} = 1$ for consistency.

Equations (56) and (57) are solved by combining them with continuity equation (26) at the corrector level:

$$
\frac{\partial(z_\sigma^{n+1}u^{n+1})}{\partial\xi} + \frac{\partial(-z_\xi^{n+1}u^{n+1} + x_\xi^{n+1}w^{n+1})}{\partial\sigma} = 0 \ , \tag{58}
$$

and with the boundary conditions at bottom and free surface at the corrector level:

$$
-\frac{u^{n+1}}{x_\xi^{n+1}}\frac{\partial h}{\partial\xi} - w^{n+1} = 0 \ , \ \sigma = \sigma_{\mathrm{bot}} \ , \tag{59}
$$

$$
q^{n+1} = 0 \ , \ \sigma = \sigma_{\mathrm{surf}} \ , \tag{60}
$$

again together with auxiliary boundary conditions like zero shear stress at bottom and free surface.

3.3 The coefficients in predictor and corrector

Some text.

$\alpha_{\text{conv}}^{p,n-1}$	$\alpha_{\text{conv}}^{p,n}$	$\alpha_{\text{conv}}^{c,n-1}$	$\alpha_{\text{conv}}^{c,n}$	$\alpha_{\text{conv}}^{c,*}$

Table 1: Value of parameters in explicit part of Adams-Bashford-Moulton time integration (convection terms).

$\alpha_{\text{pres}}^{p,n-1}$	$\alpha_{\text{pres}}^{p,n}$	$\alpha_{\text{pres}}^{p,*}$	$\alpha_{\text{pres}}^{c,n-1}$	$\alpha_{\text{pres}}^{c,n}$	$\alpha_{\text{pres}}^{c,*}$	$\alpha_{\text{pres}}^{c,n+1}$

Table 2: Value of parameters in implicit part of Adams-Bashford-Moulton time integration (pressure terms).

	θ	$\alpha_{\text{visc}}^{c,*}$
more accurate for medium real part (around $\mathrm{CFL}_{\text{visc}} \approx 2$)	7/20	95/400
better performance for large real part (up to $\mathrm{CFL}_{\text{visc}} \approx 2$)	8/20	96/400

Table 3: Value of parameters in stiff implicit part of Adams-Bashford-Moulton time integration (viscosity terms).

4 References

Borsboom, M., 2001. “Development of a 1-D error-minimizing moving adaptive grid method.” In A. vande Wouwer, P. Saucez and W. E. Schiesser, eds., *Adaptive Method of Lines*, chap. 5, pages 139–180. CRC Press. 4