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# A numerical modeling technique designed for error insight

## With application to 1D shallow-water flow simulations (rivers, channels, pipes, ...)

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# Contents

- Examples of 1D shallow-water flow applications
- The 1D shallow-water flow model equations
- Computational model: the components
- Numerical implementation: properties
- Two-step numerical modeling approach
- Finite volume method designed for error analysis
- Test: flow over a weir (including grid adaptation)
- Application: Libyan man-made river
- Concluding remarks

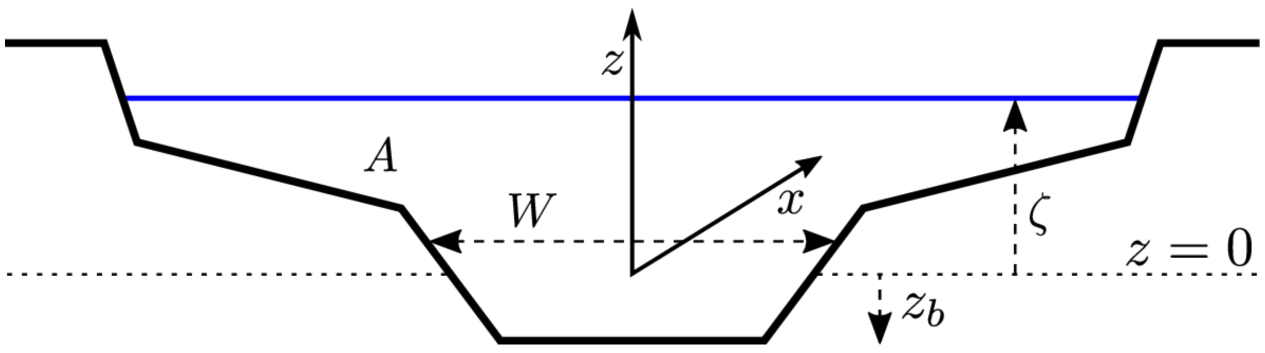
(this presentation and other info available)

# 1D shallow-water flow applications

# 1D shallow-water equations

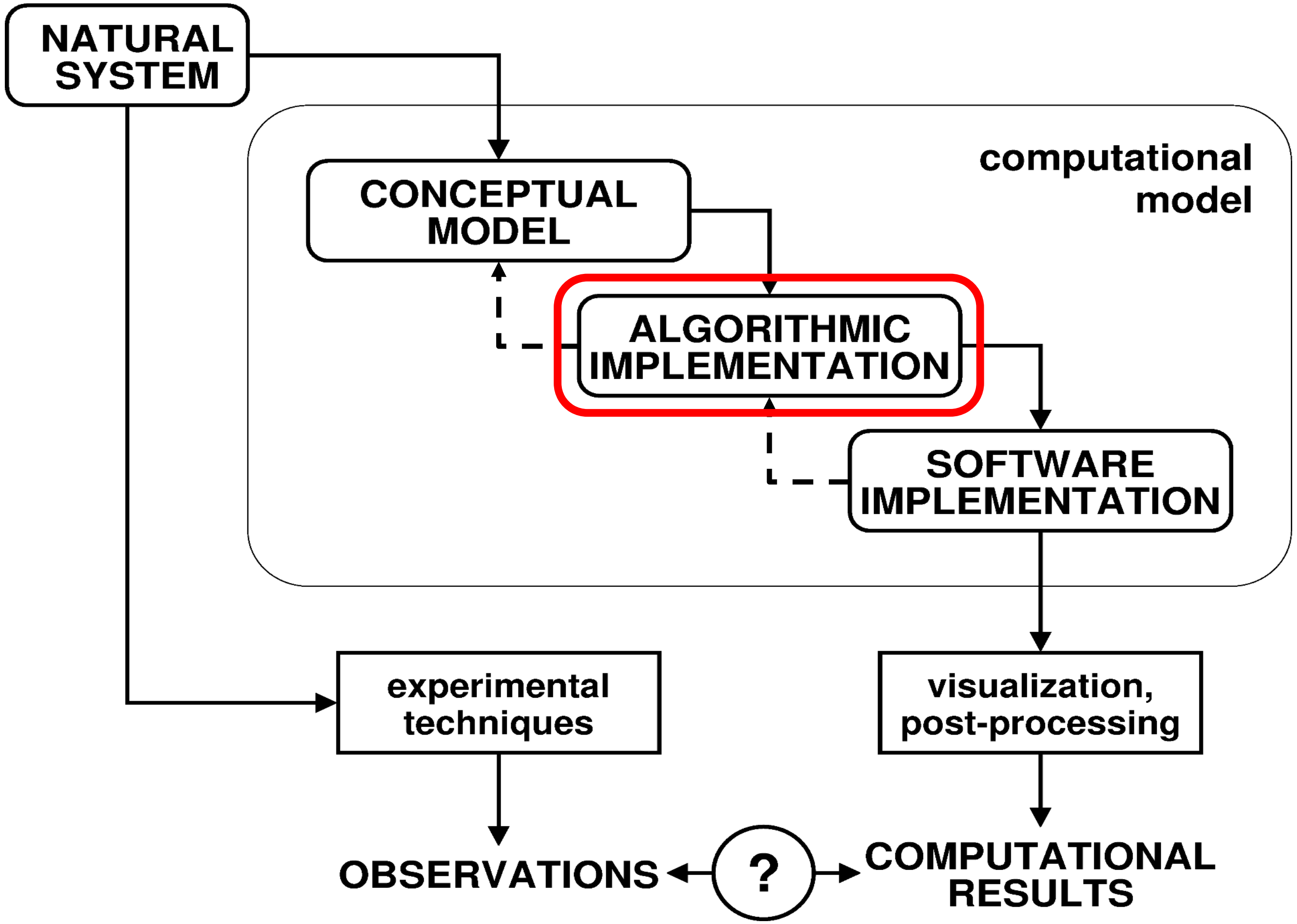
$$\frac{\partial A}{\partial t} + \frac{\partial Q}{\partial x} = 0$$

$$\frac{\partial Q}{\partial t} + \frac{\partial Q^2/A}{\partial x} + gA\frac{\partial \zeta}{\partial x} + g\frac{Q|Q|}{C^2 RA} = 0$$




- $W = W(x,z)$ : channel width ($W = 0$ below bottom $z = z_b$)
- $\zeta = \zeta(x,t)$ : free surface
- $A = \int_{z_b}^{\zeta} W dz = A(x,\zeta)$ : wetted cross-sectional area (→ effective depth $h = A(x,\zeta)/W(x,\zeta)$)
- $R = A/P$: hydraulic radius (± average depth), with $P$ wetted perimeter
- $Q = Q(x,t)$ : discharge (→ average velocity $u = Q/A$ )

# Computational model: the components




this presentation is about the numerical implementation

(plus user interface, grid generator, Kalman filter, …)

# Computational model: the requirements

| Computational models should be (or not) sufficiently … | which is determined by the model components: |
| --- | --- |
| • robust | Phys, Num |
| • maintainable/modular | Phys, Num, Soft |
| • realizable/analyzable/verifiable | Phys, Num, Soft |
| • flexible/generally applicable | Phys, Num, Soft |
| • extendable/improvable/reusable | Phys, Num, Soft |
| • efficient/parallelizable | Phys, Num, Soft |
| • easy to use/user friendly (e.g., virtually **no** options) | Phys, Num, Soft |
| • quickly available | Phys, Num, Soft |
| • relevant/fit for purpose/useful | Phys, Num |
| • accurate | Phys, Num |
| • reliable (insight in modeling errors) | Phys, Num |
| • traceable (what to do to improve results) | Phys, Num |
| • simple/understandable/teachable | Phys, Num, Soft |

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"*Too often, failure to agree with experiment is mistakenly ascribed to the turbulence model rather than inadequate numerics.*" (Johnson *et al.*, Math. Model. Nat. Phenom. 2011)

# The basic idea

1. "*… introduce (artificial) dissipative terms ... to give the shocks* (MB: and other non-smooth features) *a thickness ... somewhat larger than the spacing of the points of the network.*" (regularization à la LES)

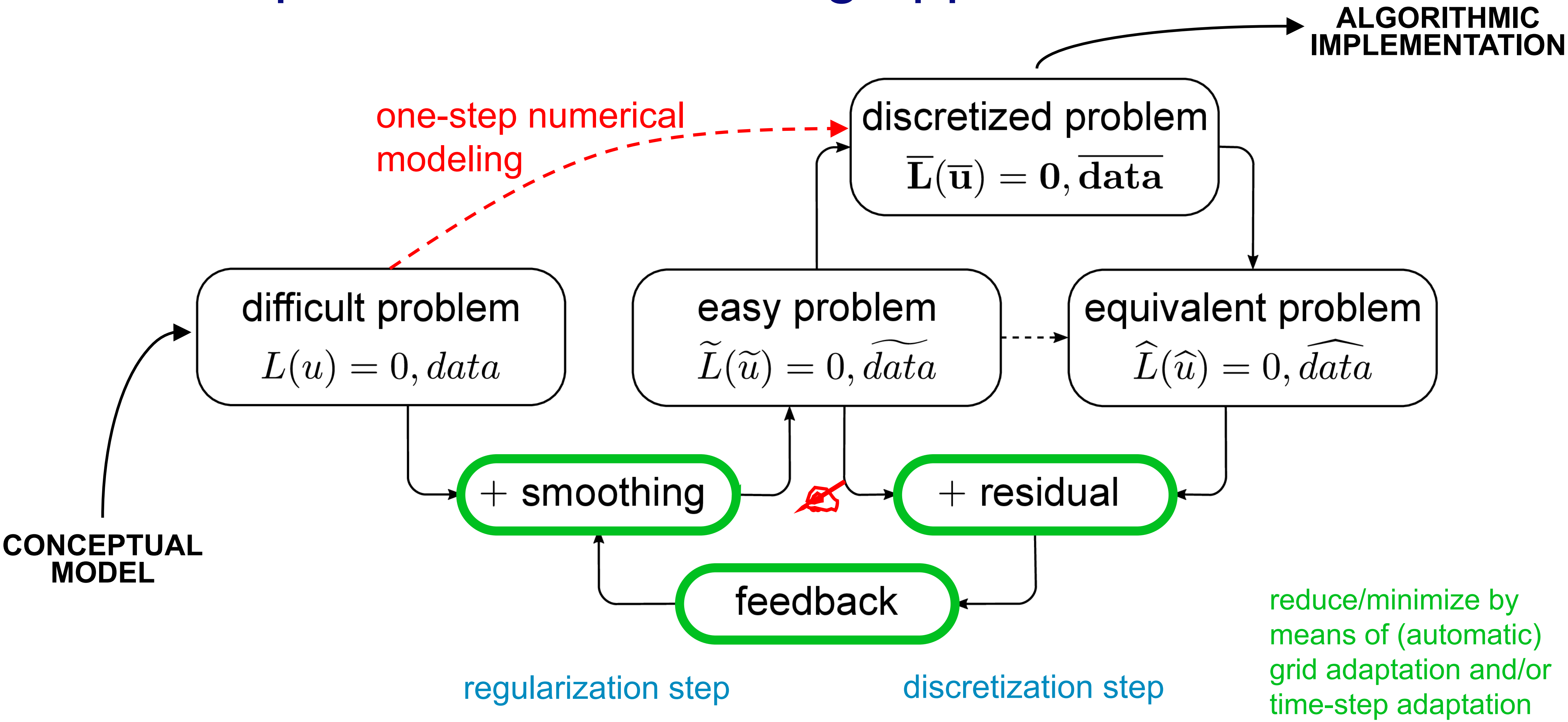
2. "*Then the difference equations may be used for the* ***entire*** *calculation, just as though there were no shocks* (MB: and other non-smooth features) *at all.*" (unified numerical implementation possible)

3. "*The quantitative influence of these terms can be made as small as one wishes by choice of a sufficiently fine mesh* (MB: and a sufficiently small time step) *…*" (basis for (automatic) grid adaptation and time-step adaptation)

4. "*… components whose wave-lengths are of order* $\Delta x$ (MB: and whose time scales are of order $\Delta t$) *are always falsified somewhat.*" (dynamic subgrid effects can NOT be simulated!)

(Von Neumann & Richtmyer, J. Appl. Phys. 1950)

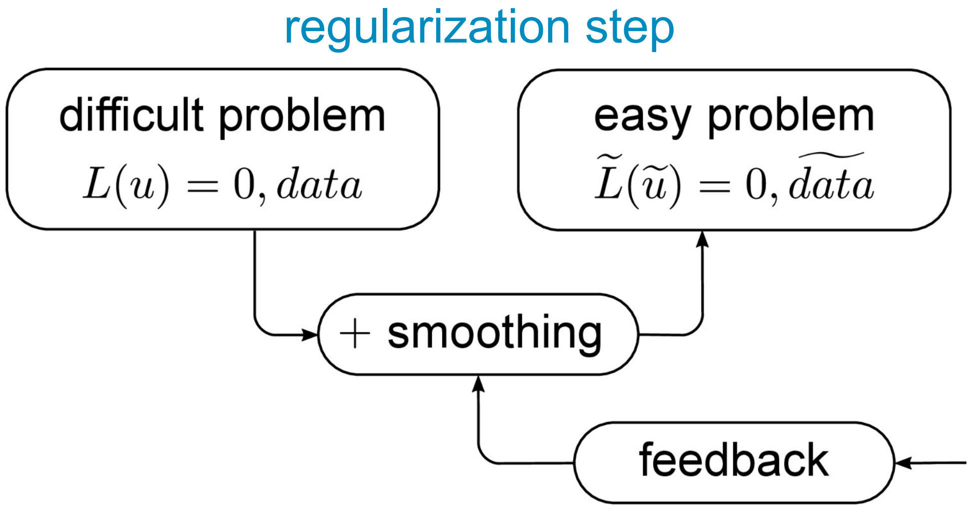
# Two-step numerical modeling approach



# Difficult Easy 1D shallow-water equations

$$\frac{\partial \tilde{A}}{\partial t} + \frac{\partial \tilde{Q}}{\partial x} = 0$$

$$\frac{\partial \tilde{Q}}{\partial t} + \frac{\partial \tilde{Q}^2/\tilde{A}}{\partial x} + g\tilde{A}\frac{\partial \tilde{\zeta}}{\partial x} + g\frac{\tilde{Q}|\tilde{Q}|}{C^2 R\tilde{A}} = 0\frac{\partial}{\partial x}\left(\tilde{A}\tilde{v}_{art}\frac{\partial \tilde{Q}/\tilde{A}}{\partial x}\right)$$



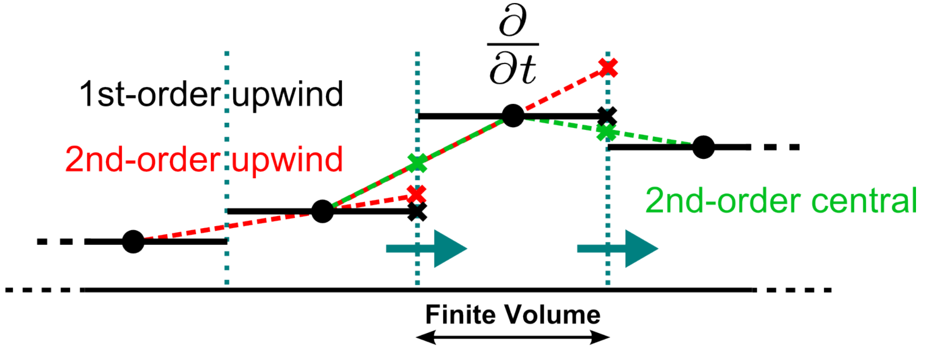

cont.eq. and mom.eq. combined to energy equation (with $\tilde{u} = \tilde{Q}/\tilde{A}$):

$$\frac{\partial}{\partial t}\left(\tilde{A}\frac{\tilde{u}^2}{2} + g\int_{z_b}^{\zeta} z\tilde{W}dz\right) + \frac{\partial}{\partial x}\left(\tilde{Q}\left(\frac{\tilde{u}^2}{2} + g\tilde{\zeta}\right)\right) = \frac{\partial}{\partial x}\left(\tilde{Q}\tilde{v}_{art}\frac{\partial \tilde{u}}{\partial x}\right) - \tilde{A}\tilde{v}_{art}\left(\frac{\partial \tilde{u}}{\partial x}\right)^2 - g\frac{\tilde{u}^2|\tilde{Q}|}{C^2 R}$$

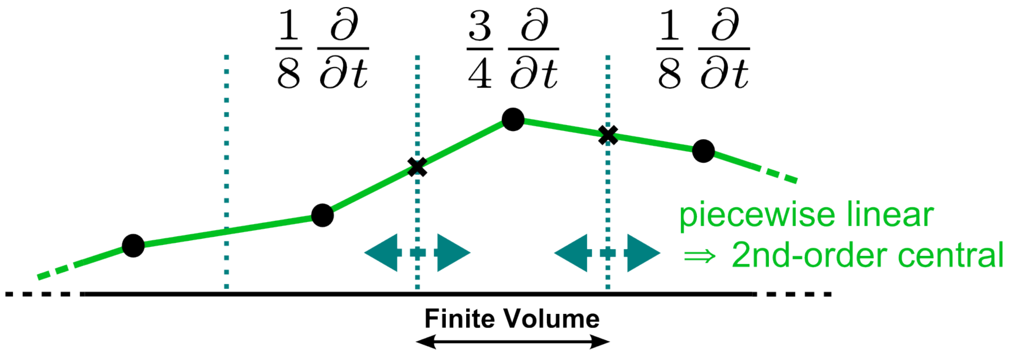
→ take $\tilde{v}_{art} \propto \Delta x\left(\left|err\left(\sqrt{\hat{u}^2/2}\right)\right| + \left|err\left(\sqrt{g\hat{\zeta}}\right)\right|\right)$

# Finite volume-type collocated discretizations

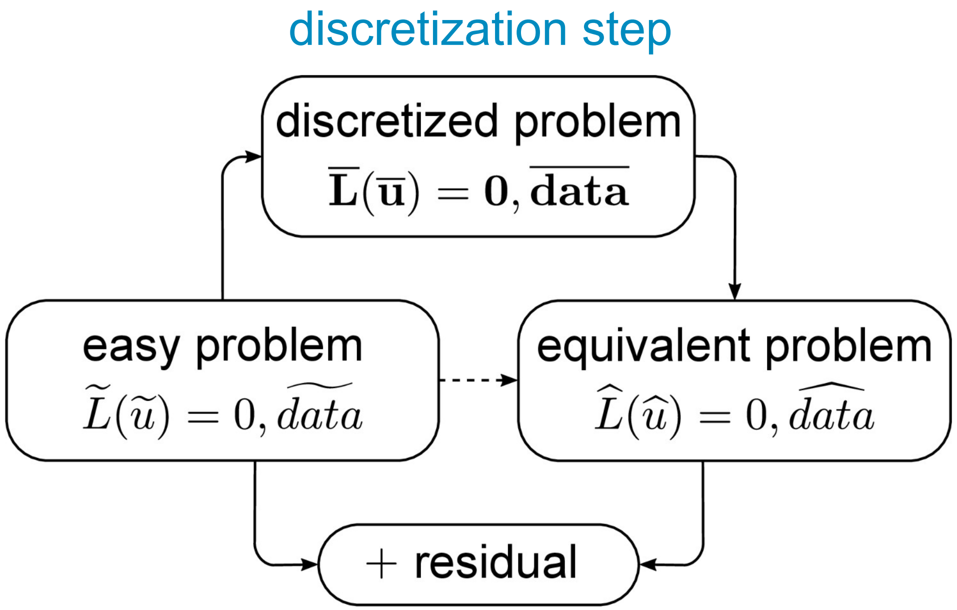
- finite difference-like
  → ± piecewise constant $\bar{A}, \bar{Q}, \bar{\zeta}, \bar{W}, \ldots$




- finite element-like
  piecewise linear $\bar{A}, \bar{Q}, \bar{\zeta}, \bar{W}, \ldots$



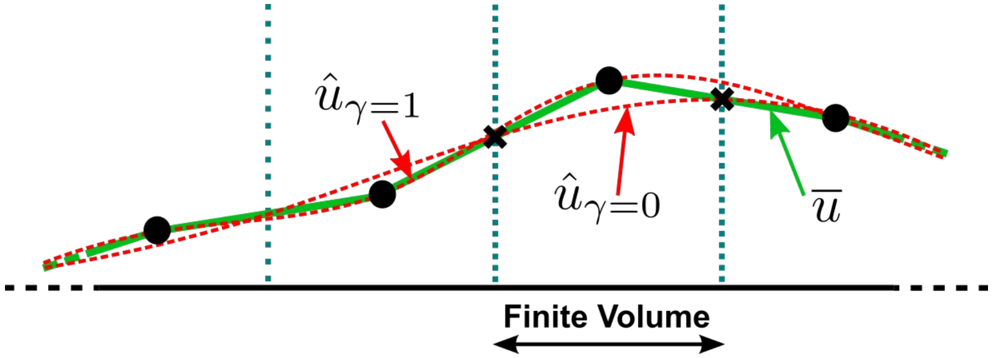
# Error analysis




Discretization error (residual) $R(\hat{u}) = \hat{L}(\hat{u}) - \tilde{L}(\hat{u})$

We have $$0 = \overline{\mathbf{L}}_i(\overline{\mathbf{u}}) = \int_{\xi_{i-1/2}}^{\xi_{i+1/2}} \tilde{L}(\overline{u})\, d\xi = \int_{\overline{x}_{i-1/2}}^{\overline{x}_{i+1/2}} \tilde{L}(\overline{u})\, dx = \int_{\overline{x}_{i-1/2}}^{\overline{x}_{i+1/2}} \hat{L}(\hat{u})\, dx$$

Hence $$\int_{\overline{x}_{i-1/2}}^{\overline{x}_{i+1/2}} R(\hat{u})\, dx = \overline{\mathbf{L}}_i(\overline{\mathbf{u}}) - \int_{\overline{x}_{i-1/2}}^{\overline{x}_{i+1/2}} \tilde{L}(\hat{u})\, dx$$




(Borsboom, Ch.5 in *Adaptive Methods of Lines*, 2001)

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# Equivalent 1D shallow-water-like equations

$$\frac{\partial}{\partial t}\left(\tilde{A}+\gamma_0 D_x(\hat{A})\right)+\frac{\partial}{\partial x}\left(\tilde{Q}+\gamma_1 D_x(\hat{Q})\right)=0$$

$$\frac{\partial}{\partial t}\left(\tilde{Q}+\gamma_0 D_x(\hat{Q})\right)+\frac{\partial}{\partial x}\left(\frac{\left(\tilde{Q}+\gamma_1 D_x(\hat{Q})\right)^2}{\tilde{A}+\gamma_1 D_x(\hat{A})}\right)+g\left(\left(\tilde{A}+\gamma_0 D_x(\hat{A})\right)\frac{\partial \tilde{\zeta}+\gamma_1 D_x(\hat{\zeta})}{\partial x}+\frac{\partial \hat{A}}{\partial x}\frac{D_x(\hat{\zeta})}{24}\right)$$

$$+g\frac{\left(\tilde{Q}+\gamma_0 D_x(\hat{Q})\right)\left|\tilde{Q}+\gamma_0 D_x(\hat{Q})\right|}{C^2\left(\tilde{R}+\gamma_0 D_x(\hat{R})\right)\left(\tilde{A}+\gamma_0 D_x(\hat{A})\right)}=\frac{\partial}{\partial x}\left(\left(\tilde{v}_{art}+\gamma_1 D_x(\hat{v}_{art})\right)\frac{\partial \tilde{Q}+\gamma_2 D_x(\hat{Q})}{\partial x}\right)$$

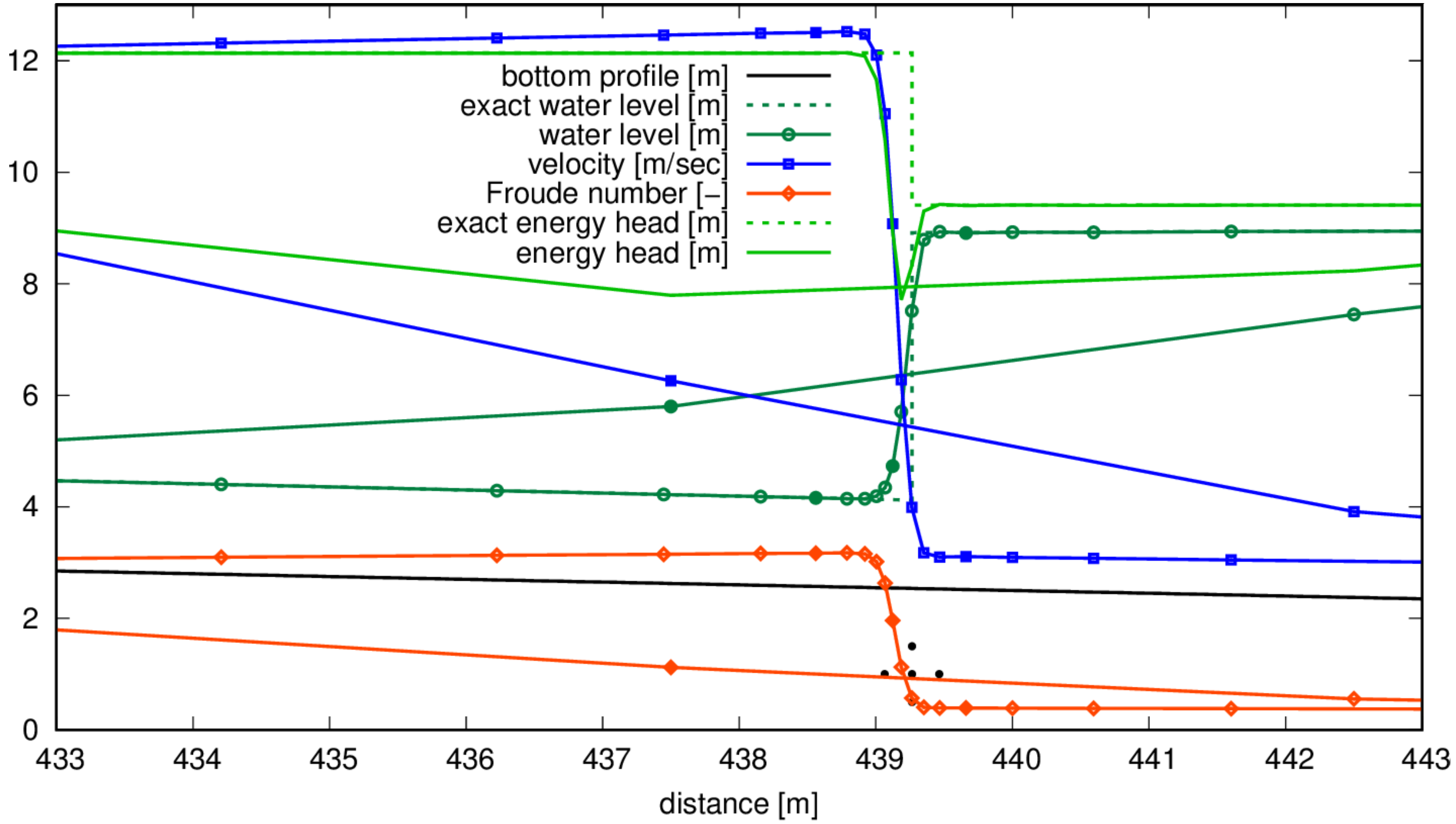
$$-\frac{\partial}{\partial x}\left(\frac{\left(\hat{\tilde{v}}_{art}+\gamma_1 D_x(\hat{v}_{art})\right)\left(\hat{\tilde{Q}}+\gamma_1 D_x(\hat{Q})\right)}{\hat{\tilde{A}}+\gamma_1 D_x(\hat{A})}\frac{\partial \hat{\tilde{A}}+\gamma_2 D_x(\hat{A})}{\partial x}\right)$$

with $D_x=\Delta\xi^2\left(\partial^2/\partial\xi^2-x_{\xi\xi}/x_\xi\,\partial/\partial\xi\right)+O\left(\Delta\xi^4\right)=x_\xi{}^2\partial^2/\partial x^2+O\left(\Delta\xi^4\right)\approx\Delta x^2\partial^2/\partial x^2$,

and with $\gamma_0=(\gamma-1/3)/8$, $\gamma_1=\gamma/8$, $\gamma_2=(\gamma-2/3)/8$.

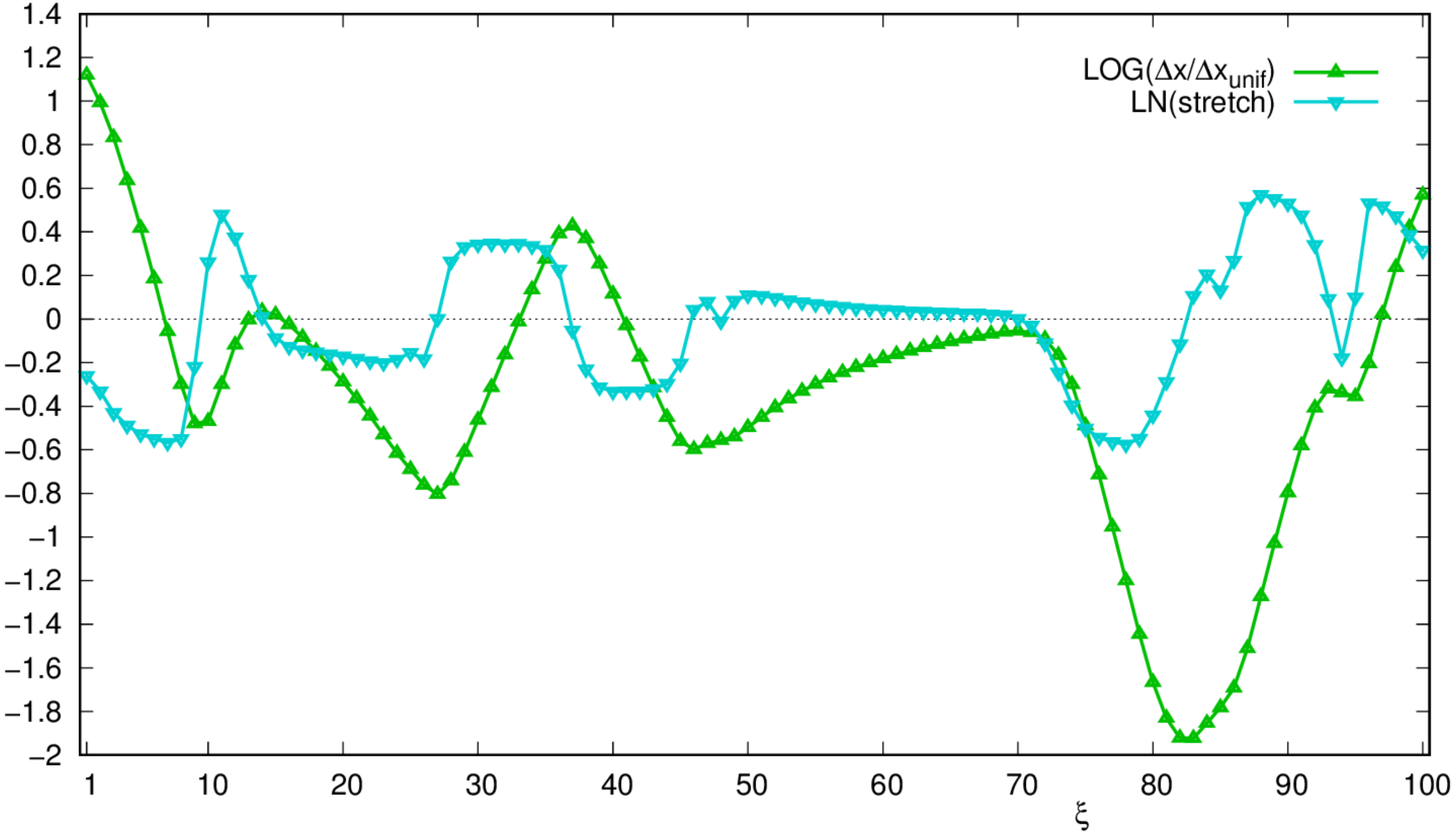
(Borsboom, Ch.5 in *Adaptive Methods of Lines*, 2001)

# Sub-/supercritical steady flow over a weir



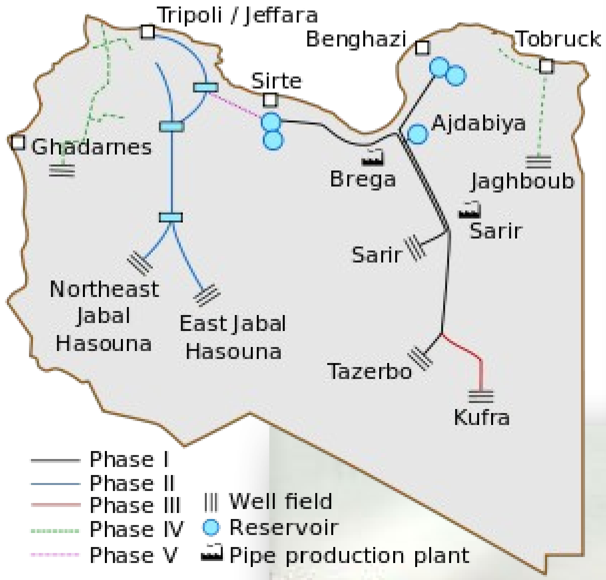

100 finite volumes, uniform and adaptive grid, solution around jump

# Sub-/supercritical steady flow over a weir




100 finite volumes, adaptive grid, grid size and grid stretching

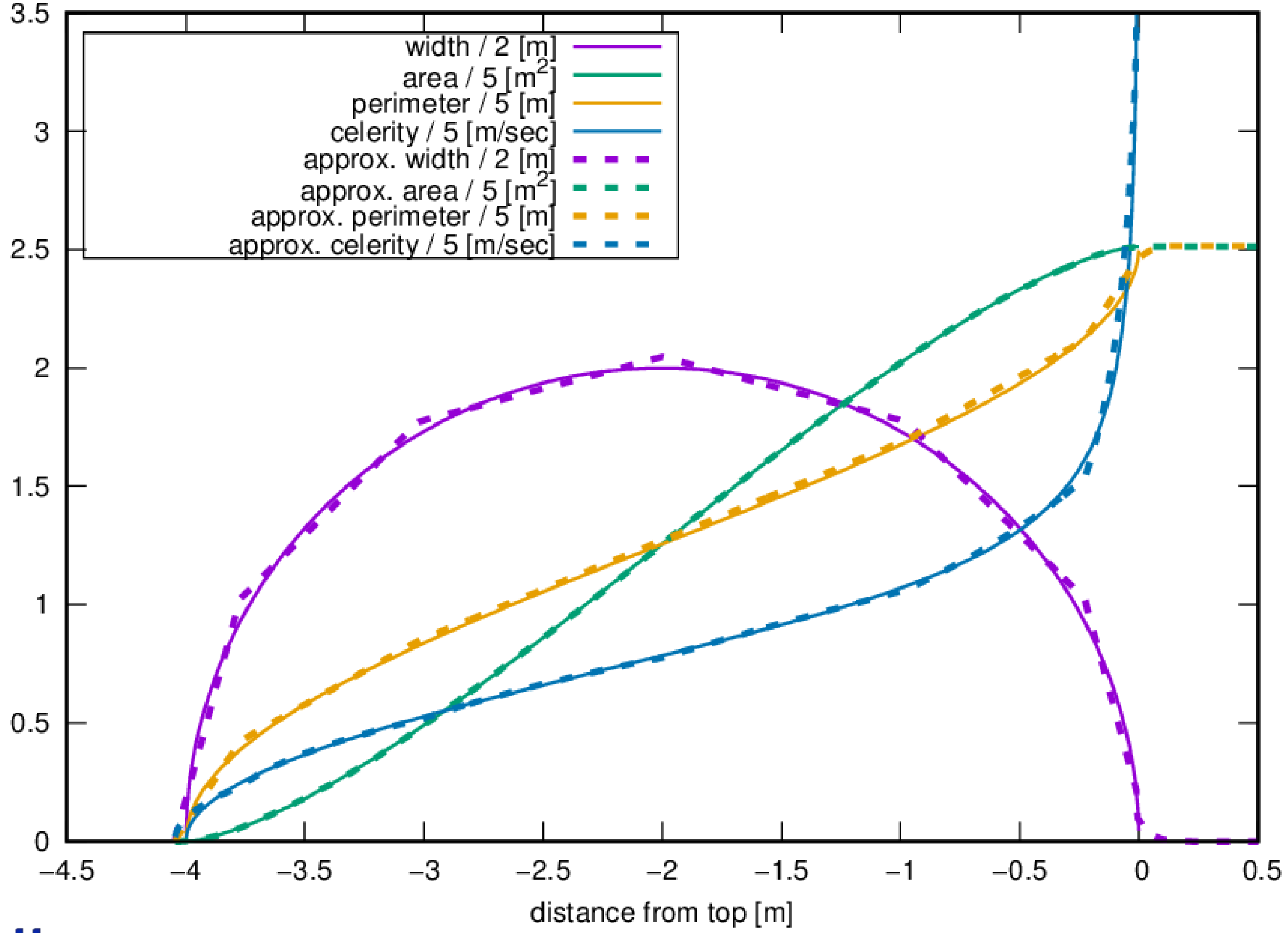
# The great man-made river

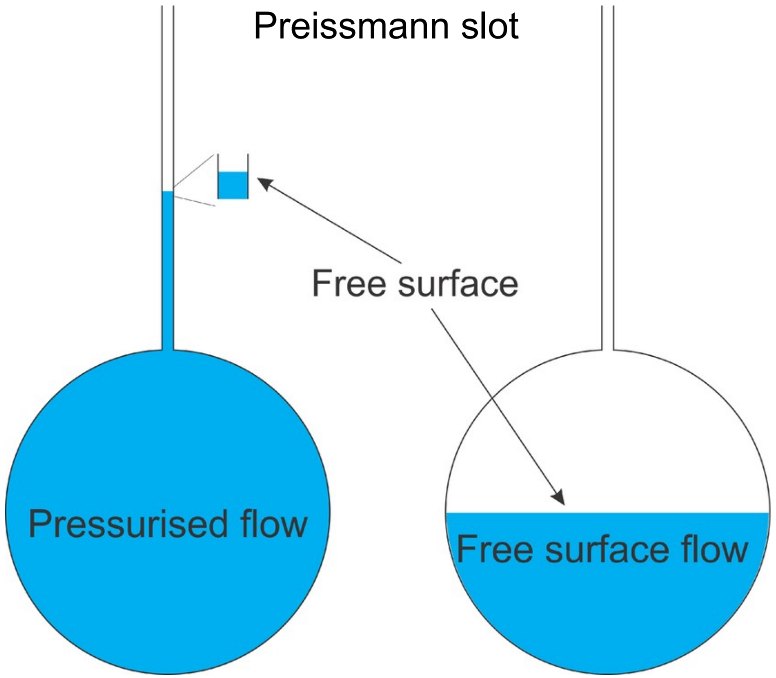




# The great man-made river

4m diameter pipes through Libyan desert
circular cross section approximated by dodecagon with added Preissmann slot




From presentation Michiel Tukker
*Free surface flow in Wanda*
April 10, 2020

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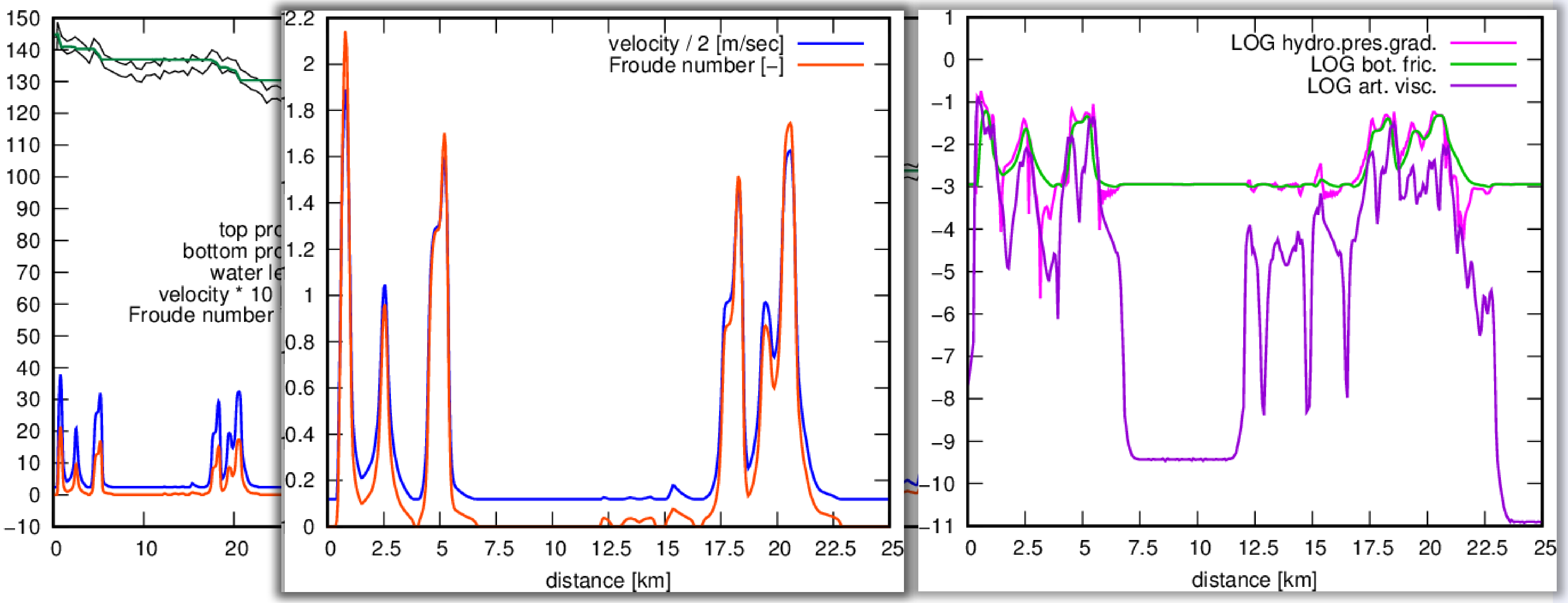
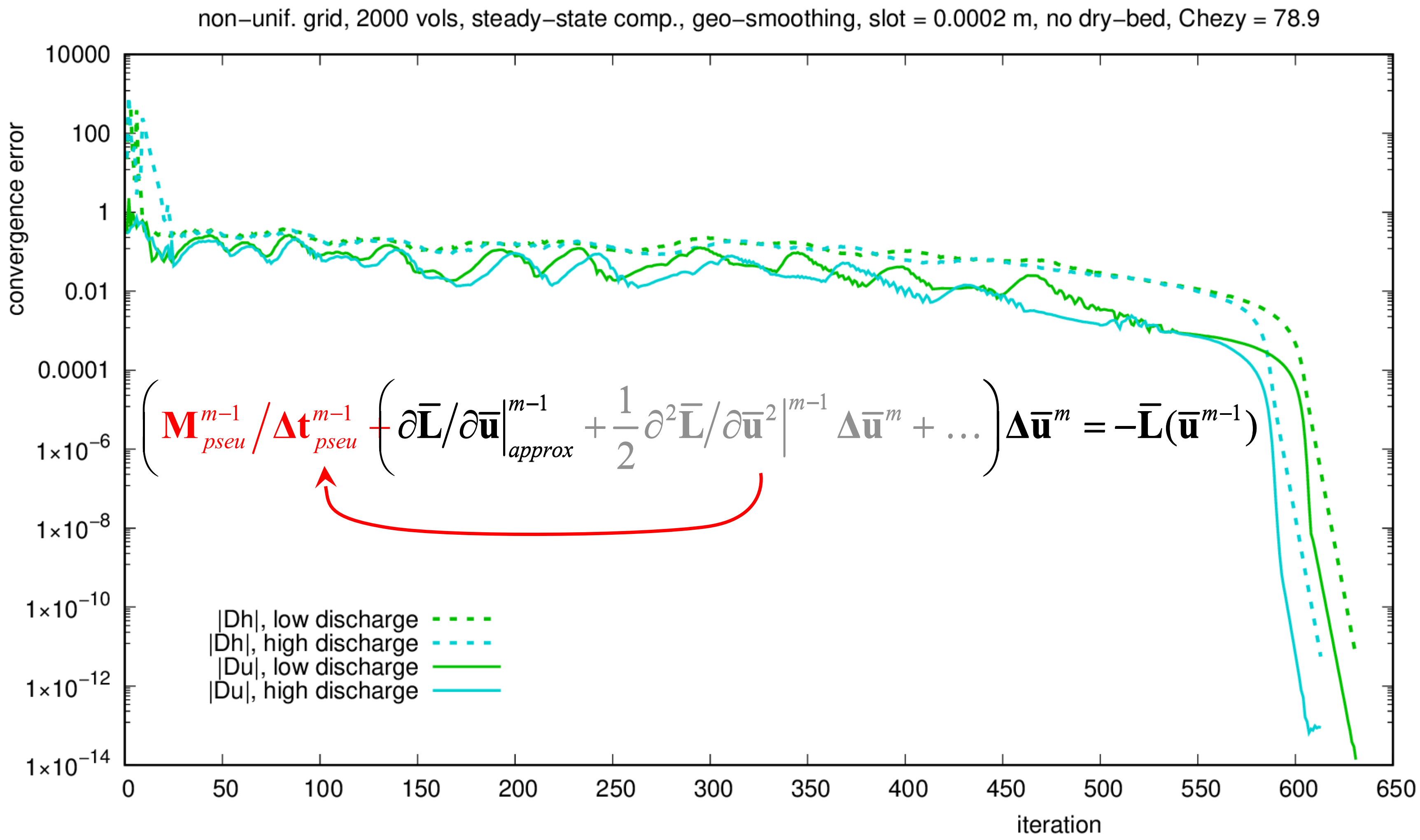
# Steady-state great man-made river

381.2km long 4m diameter pipe through Libyan desert




2000 finite volumes: 1815 of 54.96m in first 100km, 165 of 1648.8m in last 270km, 20 in between

# Steady-state great man-made river

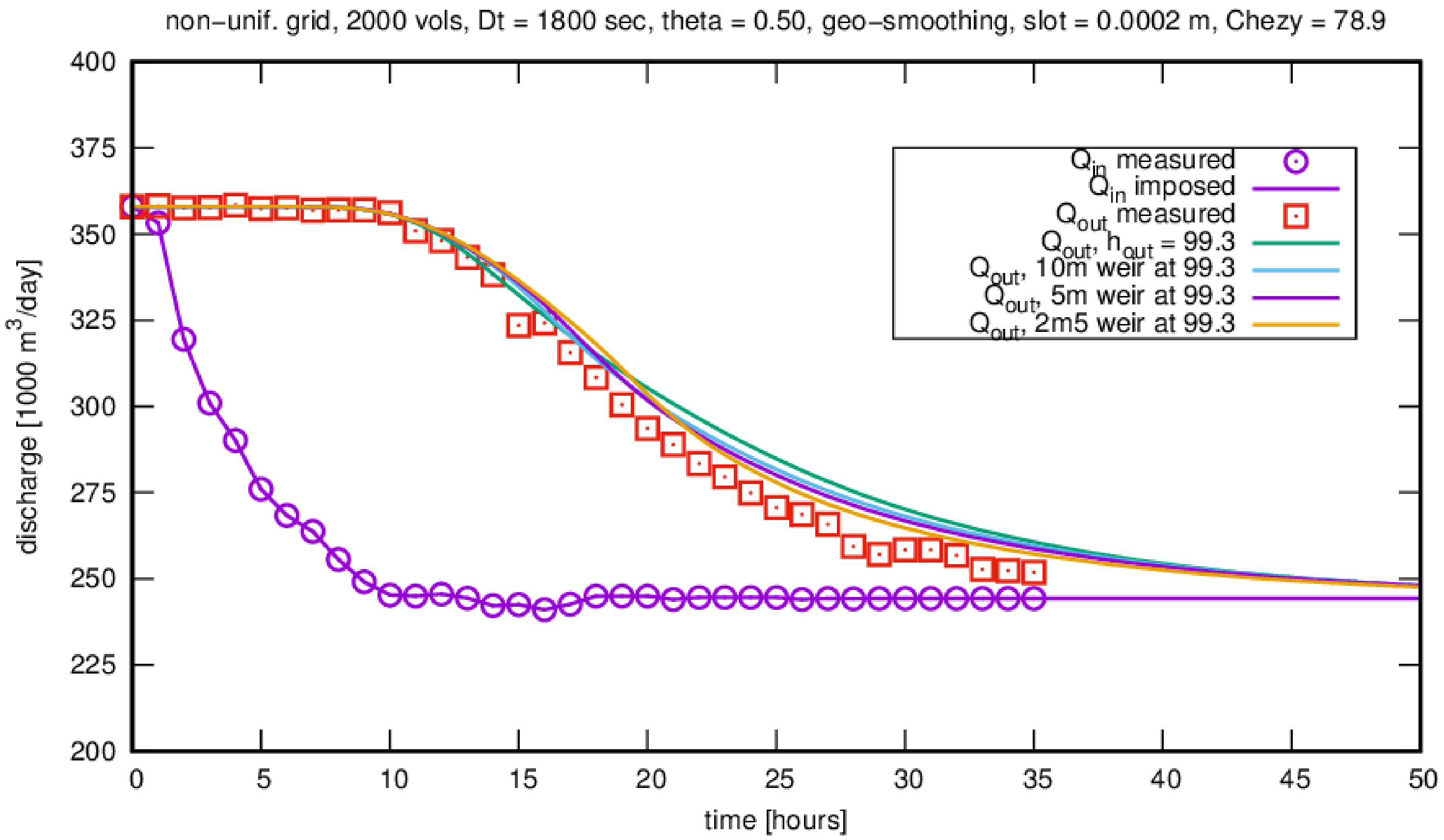



$$\left( \mathbf{M}_{pseu}^{m-1} \big/ \Delta \mathbf{t}_{pseu}^{m-1} + \left( \partial \overline{\mathbf{L}} / \partial \overline{\mathbf{u}} \Big|_{approx}^{m-1} + \frac{1}{2} \partial^2 \overline{\mathbf{L}} / \partial \overline{\mathbf{u}}^2 \Big|^{m-1} \Delta \overline{\mathbf{u}}^m + \ldots \right) \right) \Delta \overline{\mathbf{u}}^m = -\overline{\mathbf{L}}(\overline{\mathbf{u}}^{m-1})$$

2000 finite volumes, convergence history

# Unsteady great man-made river

Comparison with measurements, high discharge




$$\max\left(|u|\Delta t/\Delta x\right) \approx 125, \;\; \max\left(\sqrt{gh}\Delta t/\Delta x\right) \approx 26{,}000$$

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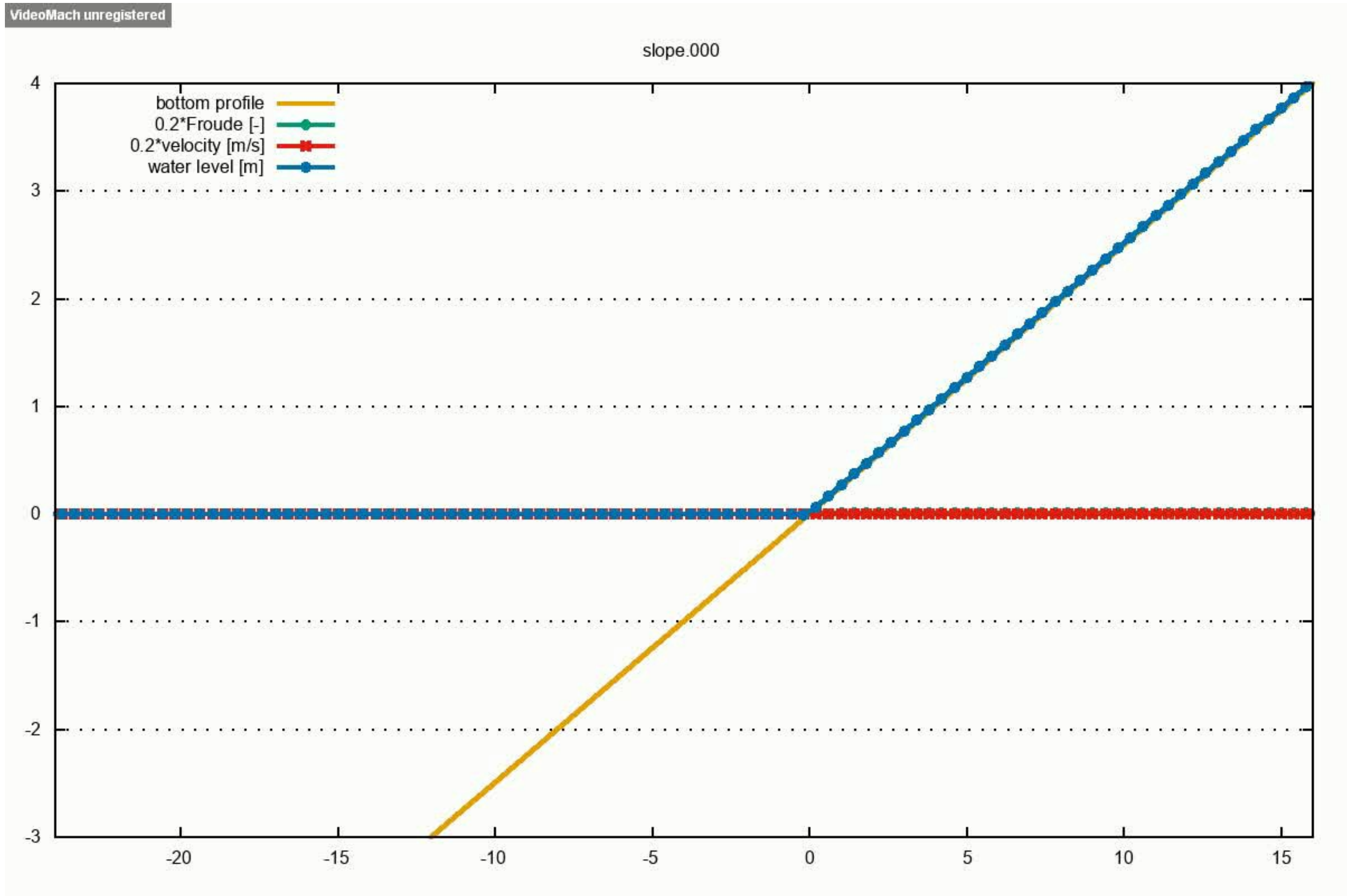
# Wave run-up/run-down

$$domain\ length = 40,\ \ \Delta x = 0.4,\ \ period = 10,\ \ \Delta t = 0.1,$$

$$Chezy = 100,\ \ \theta = 0.6,$$

$$\max\left(|u|\,\Delta t/\Delta x\right) \approx 2,$$

$$\max\left(\sqrt{gh}\,\Delta t/\Delta x\right) \approx 2$$




9–17 iterations per time step

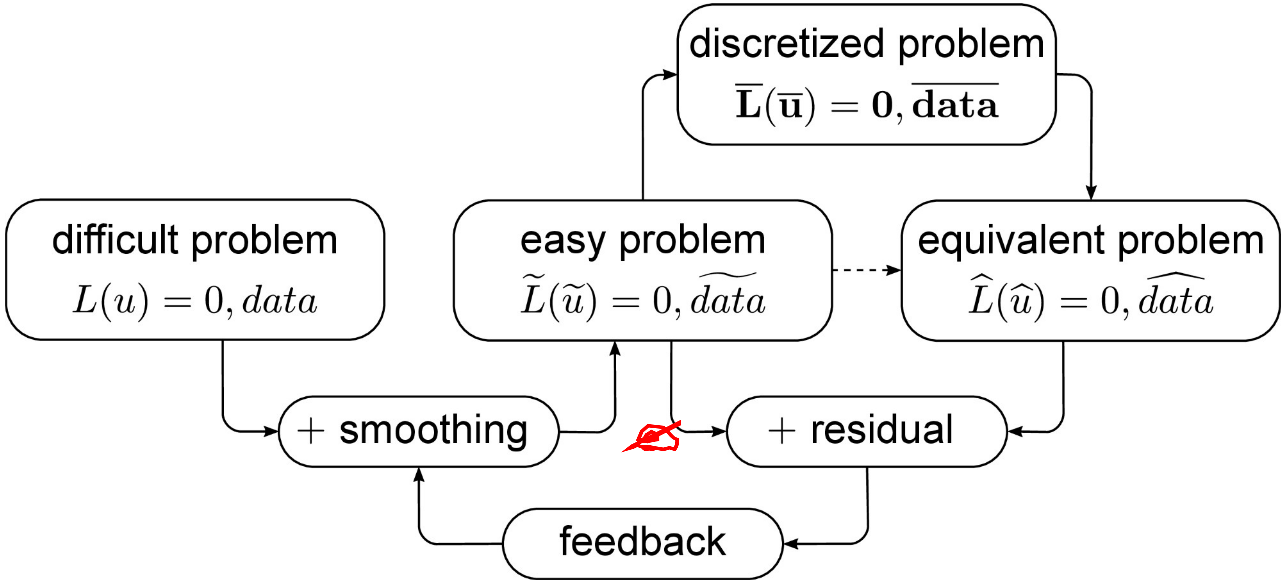
(12.3 on average)

→ better initial solutions required to speed up convergence per time step

# Summarizing …

Properties of space discretization:

- Smoothing errors dominant, discretization errors negligible
  → numerical errors can be interpreted physically!
  → numerical errors can be compared to physical errors!
- Pretty accurate
- Continuous and smooth
  - very fast solvers
  - robust model
- (block-)Structured grids required
  - flexibility by means of DD and IBM




Time integration:

- To be investigated

# Motto

**SUPPRESS THE WIGGLES, BUT SUCH THAT THEY KEEP TELLING YOU SOMETHING!**

(variation on Gresho & Lee, "*Don't suppress the wiggles—They're telling you something!*", Comput. Fluids 1981)

# Remarks, discussion, questions?

# 1D shallow-water equations

Constant channel width $W$: $A = W\left(\zeta - z_b\right) = Wh$

$$\frac{\partial h}{\partial t} + \frac{\partial q}{\partial x} = 0$$

$$\frac{\partial q}{\partial t} + \frac{\partial\, q^2/h}{\partial x} + g\frac{\partial h^2/2}{\partial x} + g\frac{P q|q|}{W C^2 h^2} = -gh\frac{\partial z_b}{\partial x}$$

- $h = h(x,t) = \zeta(x,t) - z_b(x)$ : depth
- $q = q(x,t)$: depth-integrated velocity (→ average velocity $u = q/h$ )

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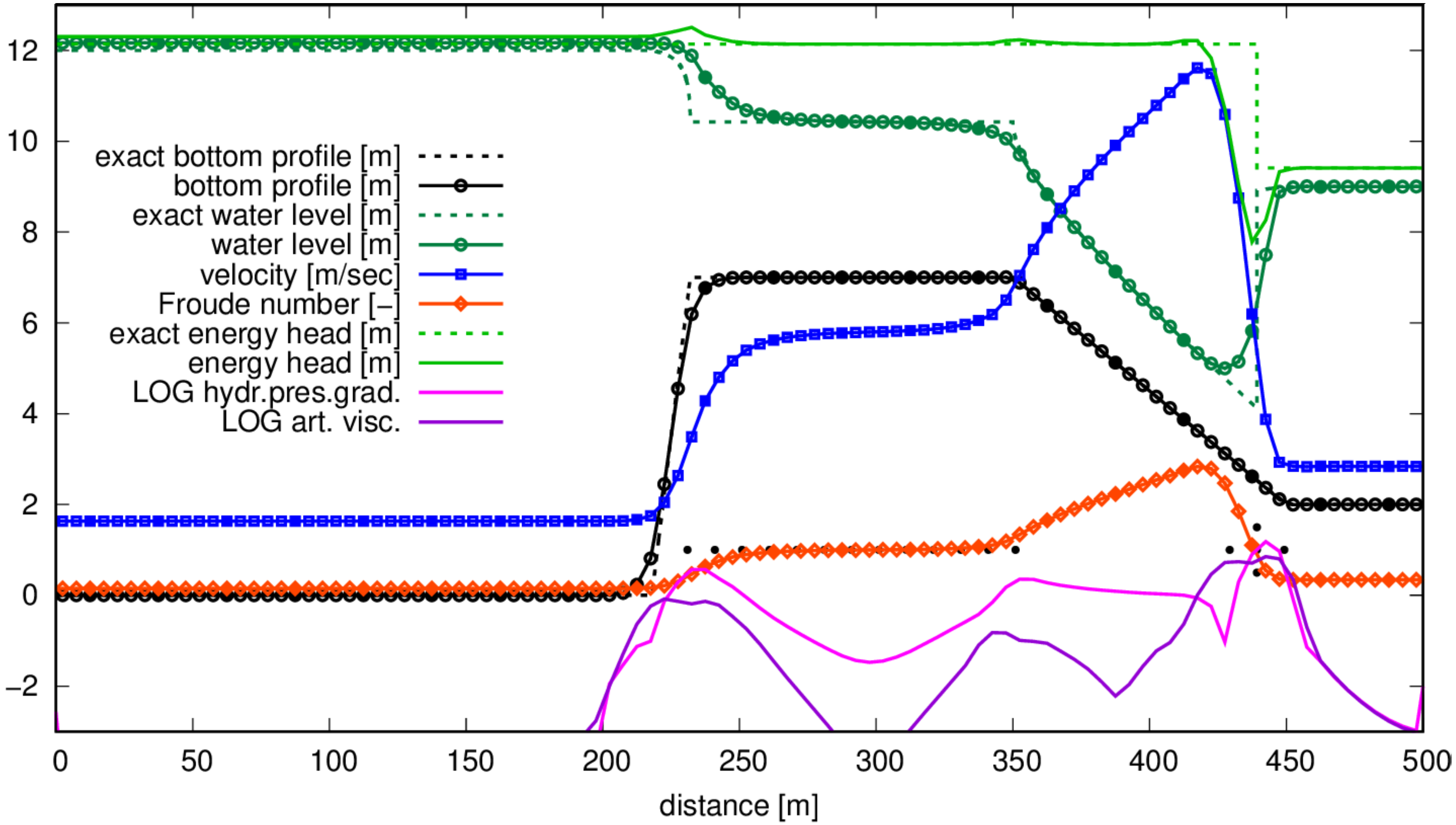
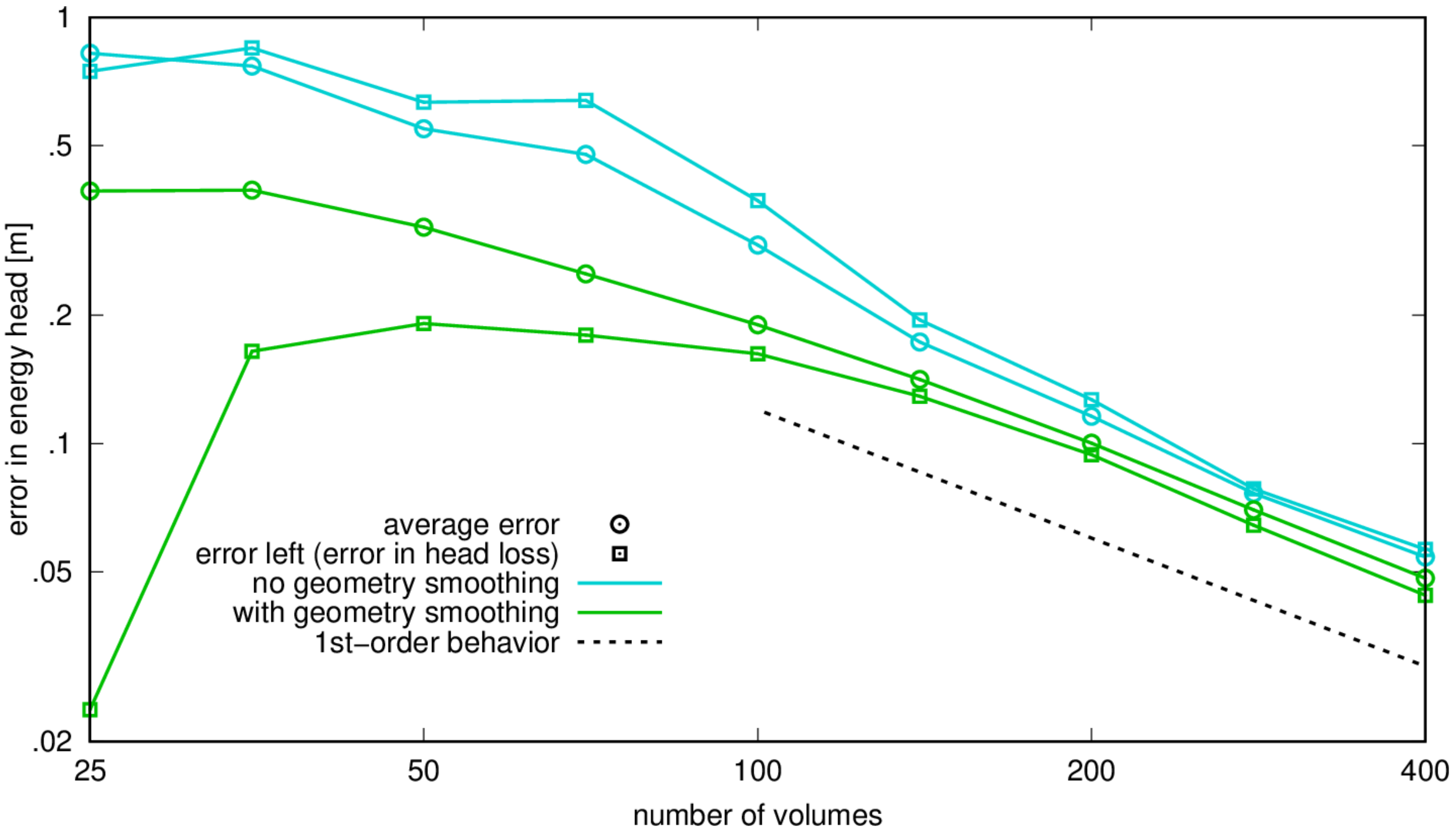
# Sub-/supercritical steady flow over a weir




100 finite volumes, smoothed geometry

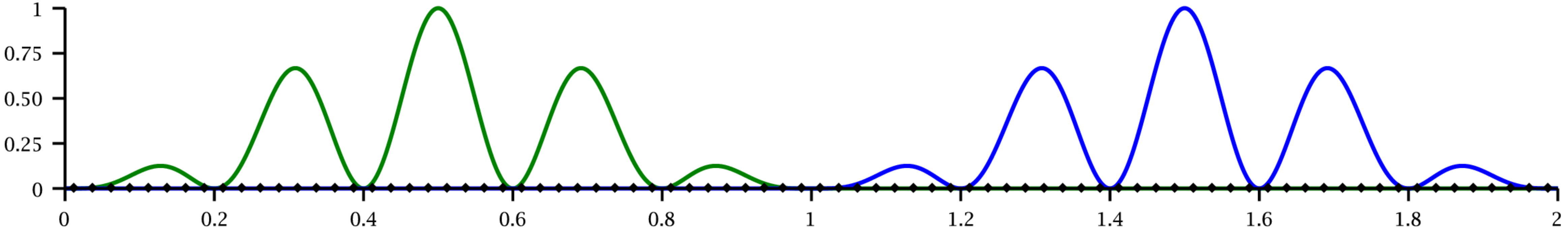
# Sub-/supercritical steady flow over a weir




100 finite volumes, error on non-smooth and smoothed geometry

# Performance of several space discretizations

Transport of wavy signal




initial solution

final solution

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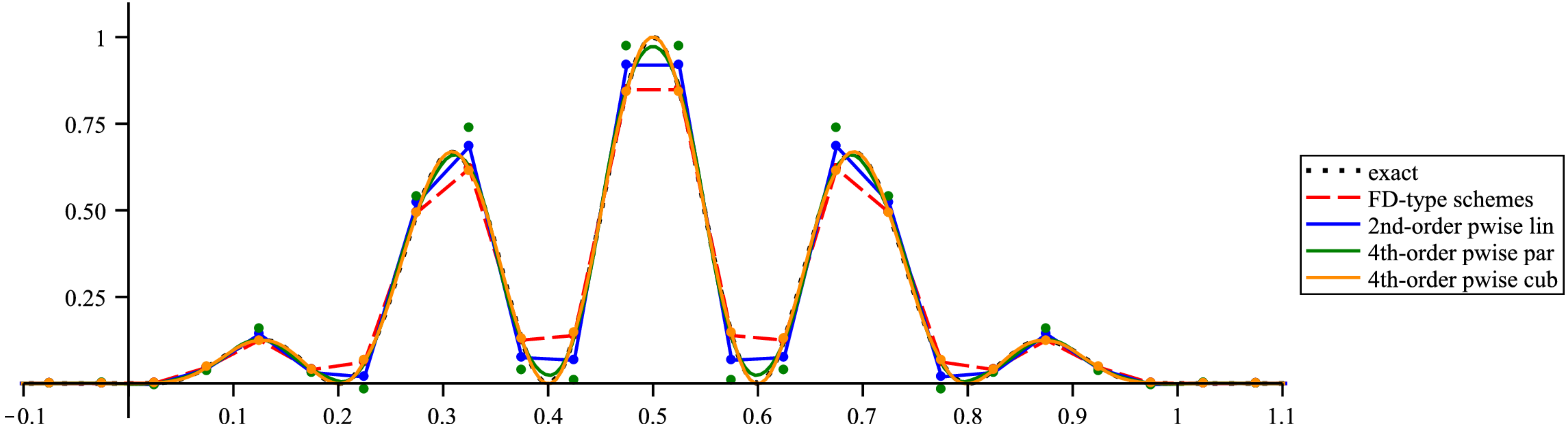
# Performance of several space discretizations

Transport of wavy signal, initial solution




20 finite volumes ($\Delta x$=0.05)

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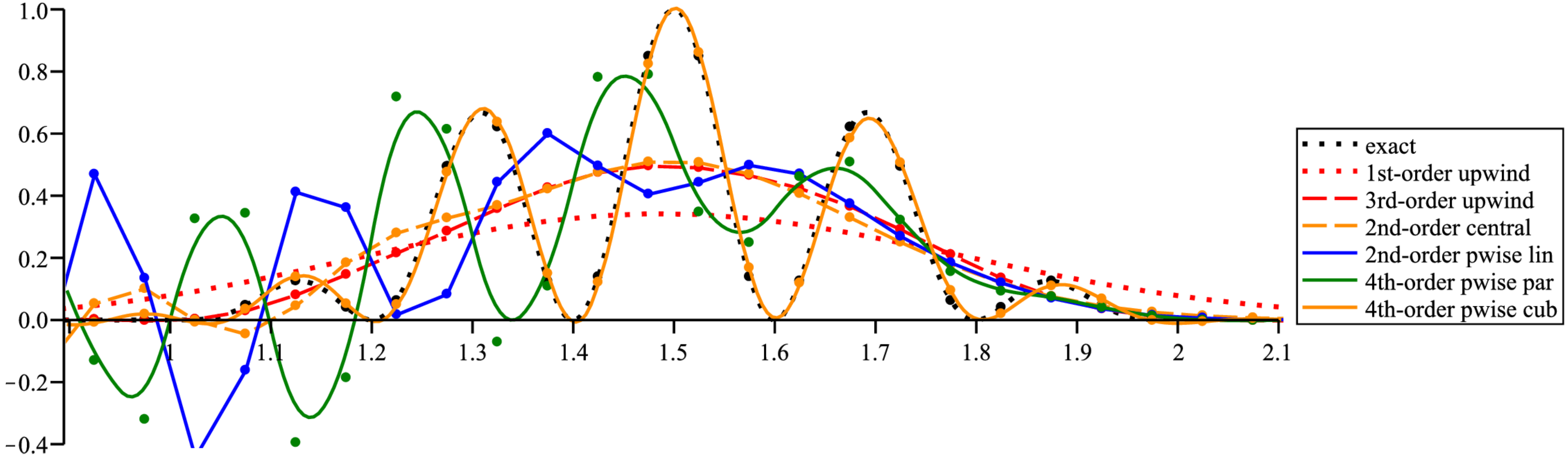
# Performance of several space discretizations

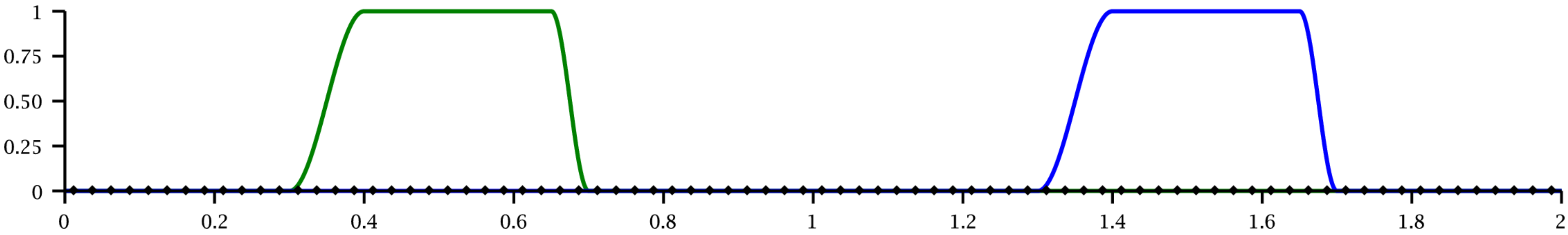
Transport of wavy signal, final solution




20 finite volumes ($\Delta x$=0.05)

# Performance of several space discretizations

Transport of block with cosine slopes

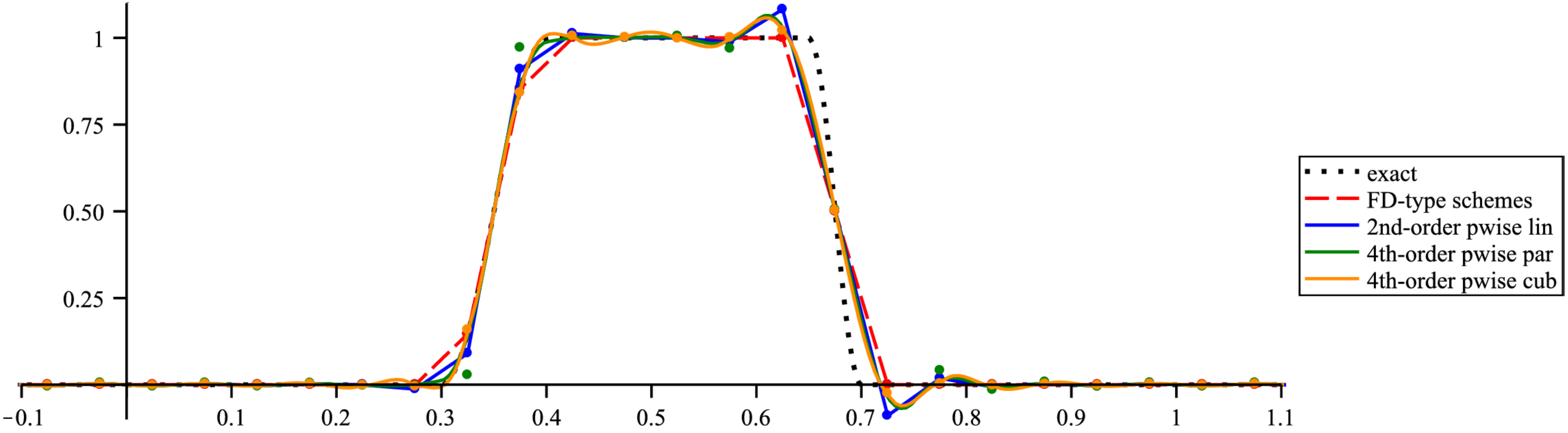



initial solution

final solution

# Performance of several space discretizations

Transport of block with cosine slopes, initial solution




20 finite volumes ($\Delta x$=0.05)

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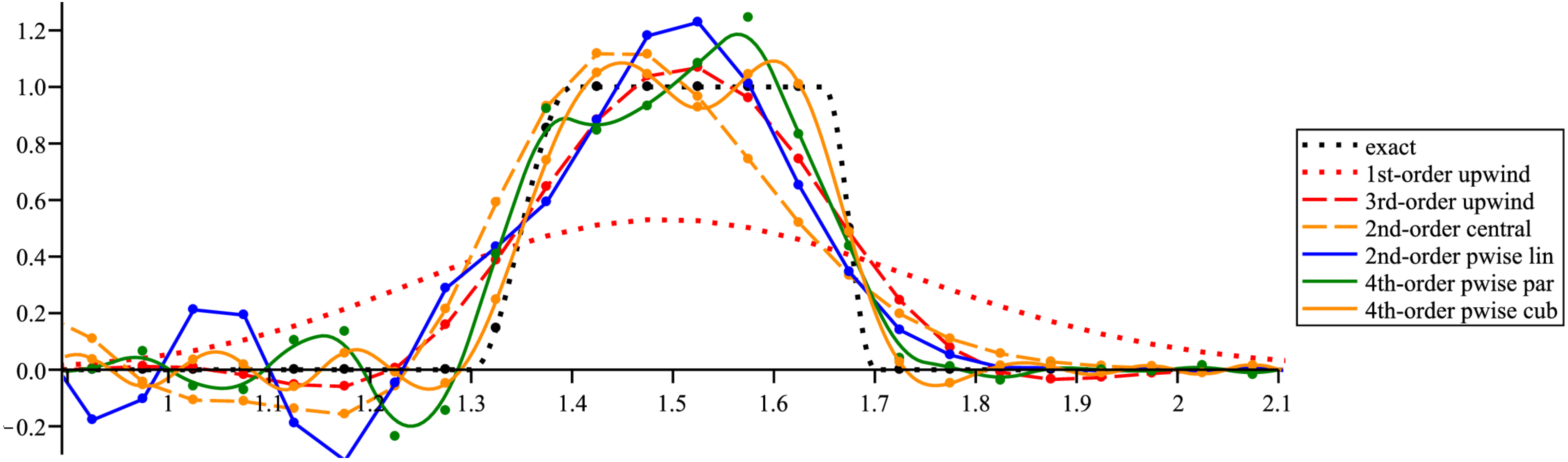
# Performance of several space discretizations

Transport of block with cosine slopes, final solution




20 finite volumes ($\Delta x$=0.05)

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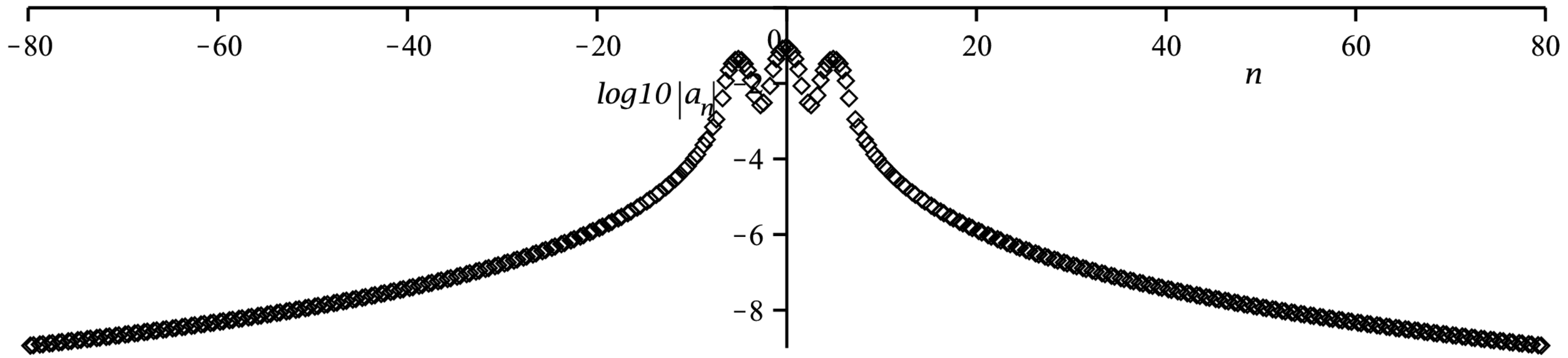
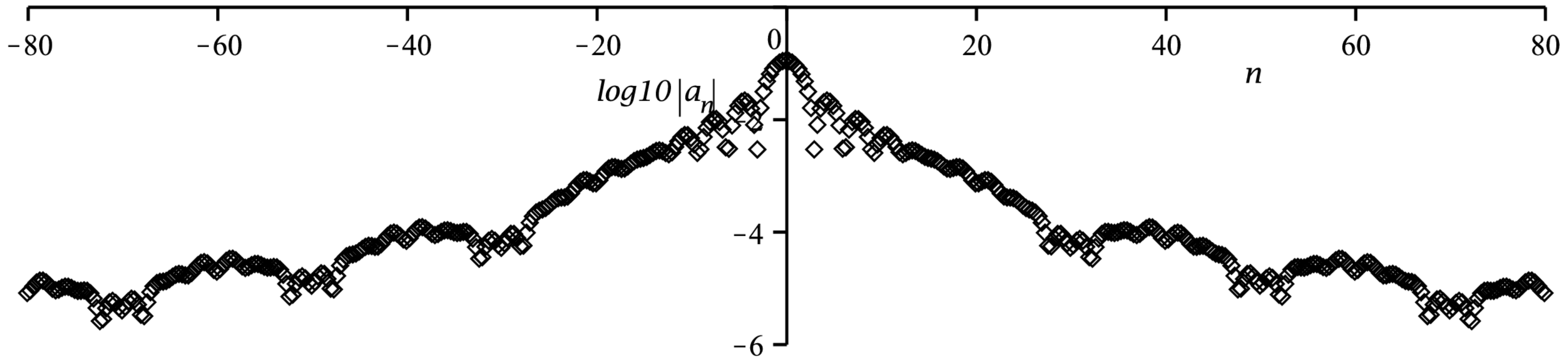
# Fourier-mode decompositions

Wavy signal




Block with cosine slopes



# Solution method

- Fully implicit time integration (for steady state set $1/\Delta t = 0$ and $\theta = 1$ )

$$0 = -\bar{\mathbf{L}}(\bar{\mathbf{u}}^{n-1},\bar{\mathbf{u}}^{n}) = -\bar{\mathbf{M}}\frac{\bar{\mathbf{u}}^{n} - \bar{\mathbf{u}}^{n-1}}{\Delta t^{n}} - \bar{\mathbf{f}}\left(\theta\bar{\mathbf{u}}^{n} + (1-\theta)\bar{\mathbf{u}}^{n-1}\right)$$

- Newton linearization

$$\left(\bar{\mathbf{M}}/\Delta t^{n} + \theta \left.\frac{\partial \bar{\mathbf{f}}}{\partial \bar{\mathbf{u}}^{n}}\right|^{m-1}\right)\Delta\bar{\mathbf{u}}^{n,m} = -\bar{\mathbf{M}}\frac{\bar{\mathbf{u}}^{n,m-1} - \bar{\mathbf{u}}^{n-1}}{\Delta t^{n}} - \bar{\mathbf{f}}\left(\theta\bar{\mathbf{u}}^{n,m-1} + \left(1-\theta\right)\bar{\mathbf{u}}^{n-1}\right)$$

- Stabilization/regularization

$$\left(\mathbf{M}^{n,m-1}_{pseu}/\mathbf{\Delta t}^{n,m-1}_{pseu} + \left.\frac{\partial \bar{\mathbf{L}}}{\partial \bar{\mathbf{u}}^{n}}\right|^{m-1}\right)\Delta\bar{\mathbf{u}}^{n,m} = -\bar{\mathbf{L}}(\bar{\mathbf{u}}^{n-1/2,m-1})$$

# What's next

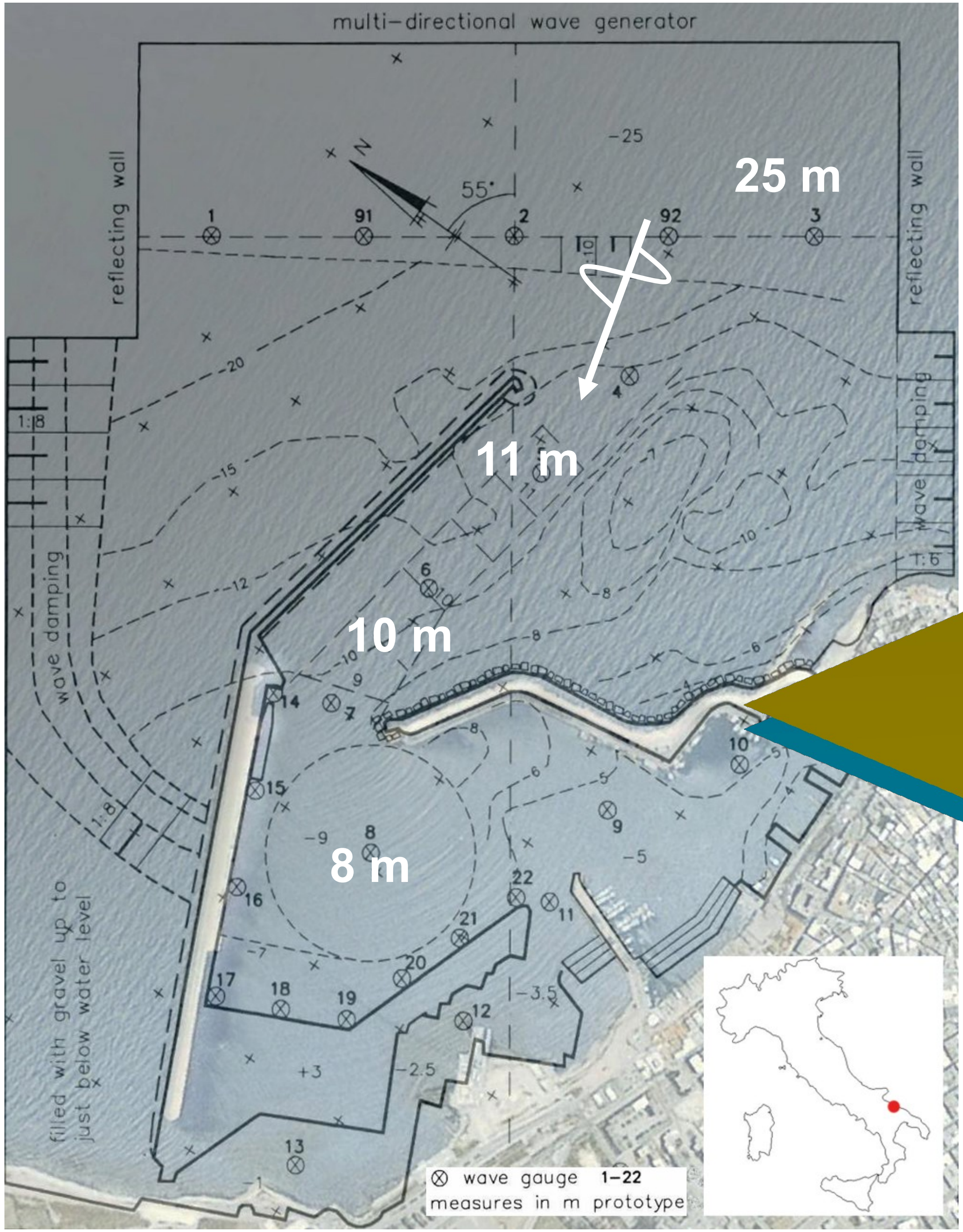
SUPSUB next version:

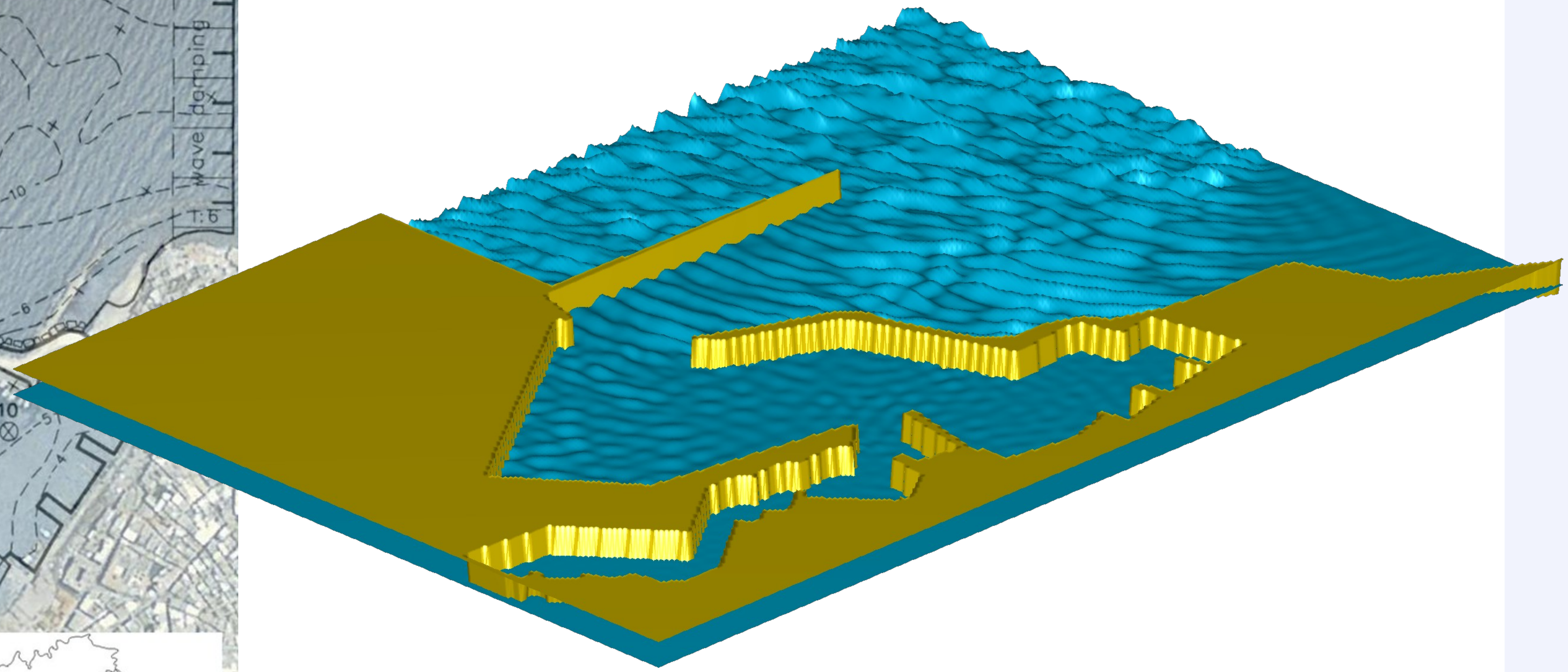
- embed as single-pipe component in WANDA?
- better initialization per time step (unsteady applications)
  → increase of robustness and efficiency
- better pseudo time-step scaling (based on residual and *full* linearization error)
  → increase of robustness and efficiency
- proper geometry smoothing (boundary/initial condition smoothing p.m.)
  → some increase of accuracy, robustness, efficiency
- moving error-minimizing grid adaptation (p.m.)
  → large increase of accuracy, efficiency, complexity

Wave modeling (Boussinesq-type, multi-layer, GABCs):

- extension to 2DV, IMEX time integration?
- 2DH/3D with DD and IBM for modeling flexibility (p.m.)

# A wave modeling example



Port of Molfetta, Italy