Mathematical Induction

Course Code: CSE-201 Course Title: Discrete Mathematics

Dept. of CSE
School of Engineering, Technology and Sciences

Lecture No:	**12**	**Week No:**	**6**	**Semester:**	**Autumn – 25**
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Lecture Outline

4.1 Mathematical Induction

- **Proof by Mathematical Induction**

Objectives and Outcomes

- **Objectives**: To understand matrix and matrix notation, to perform arithmetic and Boolean operations of matrices, to prove by mathematical induction.
- **Outcomes**: Students are expected to be able explain matrix and matrix notations; be able to perform arithmetic and Boolean operations of matrices; be able to prove a formula or inequality using mathematical induction.

4.1 Mathematical Induction

- Suppose that we have an infinite ladder, and we want to know whether we can reach every step on this ladder.
- We know two things –
  1) We can reach the first rung of the ladder
  2) If we can reach a particular rung of the ladder, then we can reach the next rung
- Can we conclude that we can reach every rung?

4.1 Mathematical Induction

- By (1), we know that we can reach the first rung of the ladder. Moreover, because we can reach the first rung, by (2), we can also reach the second rung; it is the next rung after the first rung.
- Applying (2) again, because we can reach the second rung, we can also reach the third rung.
- Continuing in this way, we can show that we can reach the fourth rung, the fifth rung, and so on.
  - For example, after 100 uses of (2), we know that we can reach the 101st rung

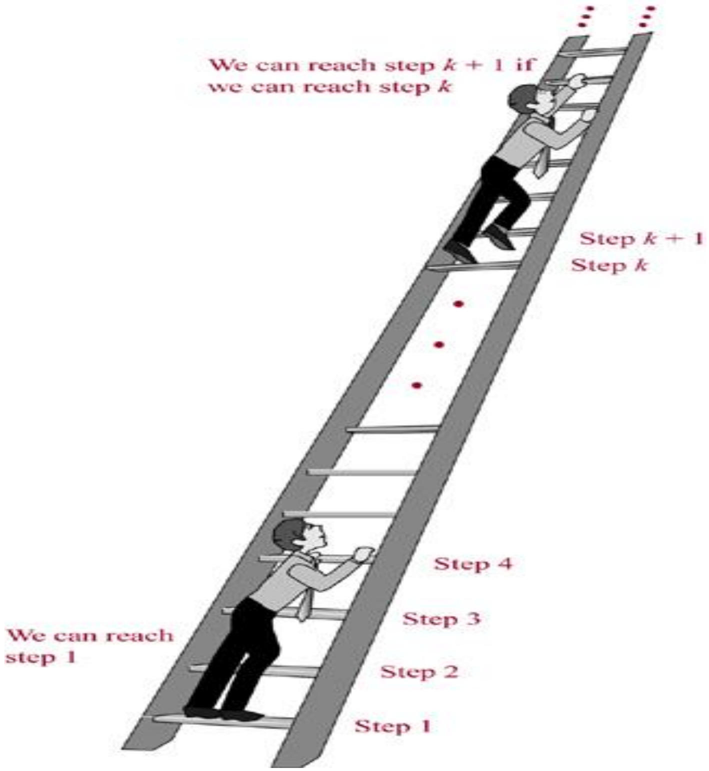
4.1 Mathematical Induction

- Can we conclude that we are able to reach every rung of this infinite ladder?
- **Answer: Yes.**

  We can verify using an important proof technique called **mathematical induction**

  We can show that $P(n)$ is true for every positive integer n, where $P(n)$ is the statement that we can reach the nth rung of the ladder.

FIGURE 1 : Climbing an Infinite Ladder



Ways to Remember How Mathematical Induction Works

E.g.

- Climbing an infinite ladder
- People telling secrets
- Infinite row of dominoes

Mathematical Induction

- Mathematical induction is an extremely important proof technique that can be used to prove assertions of this type.
- Mathematical induction is used extensively to prove results about a large variety of discrete objects.
  - For example, it is used to prove results about the complexity of algorithms, the correctness of certain types of computer programs, theorems about graphs and trees, as well as a wide range of identities and inequalities

Mathematical Induction

- Mathematical induction can be used only to prove results obtained in some other way. It is ***not*** a tool for discovering formulae or theorems.
  - The principle of mathematical induction is a useful tool for proving that a certain predicate is true for all natural numbers
  - It cannot be used to discover theorems, but only to prove them

Mathematical Induction

- **Principle of Mathematical Induction:** To prove that P(n) is true for all positive integers n, where P(n) is a propositional function, we complete two steps:
  1) **BASIS STEP**: We verify that P(1) is true.
  2) **INDUCTIVE STEP** : We show that the conditional statement P(k) $\rightarrow$ P(k+1) is true for all positive integers k
     - **Inductive hypothesis**: P(k) is true

- $[\ P(1) \wedge \forall k(P(k) \rightarrow P(k+1))] \quad \rightarrow \quad \forall n\ P(n)$

Example 1

- **Show that if n is a positive integer, then $1 + 2 + \ldots\ldots\ldots + n = n(n+1)/2$**

- **Solution**: Let $P(n)$ be the proposition that the sum of the first n positive integers is $n(n+1)/2$.

  We must do two things to prove that $P(n)$ is true for $n = 1, 2, 3, \ldots..$ Namely, we must show that P(1) is true and that the conditional statement $P(k)$ implies $P(k+1)$ is true for $k = 1, 2, 3,$

  BASIS STEP: P(1) is true, because $1 = 1(1+1)/2$

  INDUCTIVE STEP: For the inductive hypothesis, we assume that P(k) holds for an arbitrary positive integer k. That is we assume that

  $1 + 2 + \ldots..+ k = k(k+1)/2$ (INDUCTIVE HYPOTHESIS)

Example 1

- Under this assumption, it must be shown that *P(k+1)* is true, namely, that
  1 + 2 + ...+ (k+1) = (k+1)[(k+1)+1]/2 = **(k+1)(k+2)/2** is also true.

  Adding **(k+1)** to both sides of the equation in P(k), we obtain
  1 + 2 + ...+ k + (k+1) = k(k+1)/2 + (k+1)
  = [k(k+1) + 2(k+1)] / 2
  = **(k+1)(k+2)/2**

This last equation shows that P(*k*+1) is true under the assumption that P(*k*) is true. This completes the inductive step.

We have completed the basis step and inductive step, so by mathematical induction we know that P(n) is true for all positive integers n. That is, we have proven that 1 + 2 + ...+ n = n(n+1)/2 for all positive integers n.

Example 3

- Use mathematical induction to show that $1 + 2 + 2^2 + \ldots.. + 2^n = 2^{n+1} - 1$ for all **nonnegative integers *n***.
- **Solution**: Let P(n) be the proposition that $1 + 2 + 2^2 + \ldots.. + 2^n = 2^{n+1} - 1$ for the integer n.

  BASIS STEP: P(0) is true because $2^0 = 1 = 2^1 - 1$

  INDUCTIVE STEP: For the inductive hypothesis, we assume that P(k) is true.

  That is, we assume that $1 + 2 + 2^2 + \ldots.. + 2^k = 2^{k+1} - 1$ - - - - - -> (1)

  To carry out the inductive step using this assumption, we must show that when we assume that *P(k)* is true, then *P(k+1)* is also true. That is, we must show that

  $1 + 2 + 2^2 + \ldots.. + 2^{k+1} = 2^{(k+1)+1} - 1 = \mathbf{2^{k+2} - 1}$, assuming the inductive hypothesis *P(k)*.

Example 3 (Cont.)

- Under the assumption of *P(k)*, we see that

$1 + 2 + 2^2 + \ldots.. + 2^k + 2^{k+1} = (1 + 2 + 2^2 + \ldots.. + 2^k) + 2^{k+1}$

$= (2^{k+1} - 1) + 2^{k+1}$

$= 2.\ 2^{k+1} - 1$

$= \mathbf{2^{k+2} - 1}$

Because we have completed the basis step and the inductive step, by mathematical induction we know that *P(n)* is true for all nonnegative integers *n*.

That is, $1 + 2 + 2^2 + \ldots.. + 2^n = 2^{n+1} - 1$ for all nonnegative integers n.

Example 5

- **Example 5**: Use mathematical induction to prove the inequality $n < 2^n$ for all **positive integers** $\boldsymbol{n}$.
- **Solution**: Let $P(n)$ be the proposition that $n < 2^n$.

BASIS STEP: $P(1)$ is true, because $1 < 2^1 = 2$

INDUCTIVE STEP: Assume $P(k)$ is true for all positive integer k, that is, $k < 2^k$ (Inductive Hypothesis)

We need to show that if $k < 2^k$, then $k+1 < 2^{k+1}$

Now, $k + 1 < 2^k + 1$ [adding 1 to both sides of $k < 2^k$]

$\leq 2^k + 2^k = 2. 2^k = 2^{k+1}$ [note: $1 \leq 2^k$]

This shows that $P(k+1)$ is true, i.e., $k+1 < 2^{k+1}$

Because we have completed both the basis step and the inductive step, by the principle of mathematical induction we have shown that $n < 2^n$ is true for all positive integers n.

(Modified)Exercise 3

- ***Modified* Exercise 3:** Use mathematical induction to show that

  $1^2 + 2^2 + \ldots.+ n^2 = n(n+1)(2n+1) / 6$ for **all positive integers *n*.**

- **Solution:** Let P(n) be the proposition $1^2 + 2^2 + \ldots.+ n^2 = n(n+1)(2n+1)/6$

BASIS STEP: P(1) is true, because $1^2 = 1.2.3/6$

INDUCTIVE STEP: We assume that $1^2 + 2^2 + \ldots.+ k^2 = k(k+1)(2k+1)/6$ (Inductive Hypothesis)

We want to show that $1^2 + 2^2 + \ldots.+ k^2 + (k + 1)^2 = \mathbf{(k+1)(k+2)(2k+3)/6}$

Now, $1^2 + 2^2 + \ldots.+ k^2 + (k + 1)^2 = k(k+1)(2k+1)/6 + (k + 1)^2$

$= (k+1)/6\ [k(2K+1) + 6(K+1)]$

$= (k+1)/6\ [2k^2 +7k + 6]$

$= (k+1)/6\ [2k^2 + 4k + 3k+ 6]$

$= \mathbf{(k+1)(k+2)(2k+3)/6}$ So, inductive step is true.

Therefore, $1^2 + 2^2 + \ldots.+ n^2 = n(n+1)(2n+1)/6$ for all positive integers n.

Exercise 5

Exercise 5: Prove that $1^2 + 3^2 + 5^2 + \ldots + (2n+1)^2 = (n+1)(2n+1)(2n+3)/3$ whenever *n* is a **nonnegative integer**.

Solution: Let P(n) be the proposition $1^2 + 3^2 + 5^2 + \ldots + (2n+1)^2 = (n+1)(2n+1)(2n+3)/3$

BASIS STEP: P(0) is true, because $1^2 = 1.1.3/3$

INDUCTIVE STEP: We assume that $1^2 + 3^2 + 5^2 + \ldots + (2k+1)^2 = (k+1)(2k+1)(2k+3)/3$
(Inductive Hypothesis)

We want to show that $1^2 + 3^2 + 5^2 + \ldots + (2k+1)^2 + (2(k+1)+1)^2 = \mathbf{(k+2)(2k+3)(2k+5)/3}$

Now, $(1^2 + 3^2 + 5^2 + \ldots + (2k+1)^2) + (2k+3)^2 = (k+1)(2k+1)(2k+3)/3 + (2k+3)^2$

$= (2k+3)/3\ [\ (k+1)(2k+1) + 3(2k+3)\ ]$

$= (2k+3)/3\ [\ (2k^2 + 3k + 1) + (6k + 9)]$

$= (2k+3)/3\ [\ 2k^2 + 9k + 10\ ]$

$= (2k+3)/3\ [2k^2 + 4k + 5k + 10\ ] = (2k+3)/3\ [2k(k+2) + 5(k+2)]$

$= \mathbf{(k+2)(2k+3)(2k+5)/3}$

Therefore, inductive step is true

So, $1^2 + 3^2 + 5^2 + \ldots + (2n+1)^2 = (n+1)(2n+1)(2n+3)/3$, whenever *n* is a nonnegative integer

Practice @ Home

- **Relevant odd-numbered Exercises** from your text book

Books

1. *Discrete Mathematics and its applications with combinatorics and graph theory (7^{th} edition)* by Kenneth H. Rosen [Indian Adaptation by KAMALA KRITHIVASAN], published by McGraw-Hill

References

1. Discrete Mathematics, *Richard Johnsonbaugh*, Pearson education, Inc.
2. Discrete Mathematical Structures, *Bernard Kolman*, *Robert C. Busby*, *Sharon Ross,* Prentice-Hall, Inc.
3. *SCHAUM'S outlines Discrete Mathematics(2nd edition)*, by *Seymour Lipschutz*, *Marc Lipson*