AKLT state

The AKLT state is a translationally invariant matrix product state in which the same rank-3 tensor B is repeated. Here we consider a chain of length L with periodic boundary conditions. In this case, the AKLT state $|\psi\rangle$ is written as

$$
\begin{gathered}
|\psi\rangle = \sum_{\sigma_1,\cdots,\sigma_L} |\sigma_L \sigma_{L-1} \cdots \sigma_2 \sigma_1\rangle \mathrm{Tr}[B^{\sigma_L} B^{\sigma_{L-1}} \cdots B^{\sigma_2} B^{\sigma_1}], \\
B^1 = \sqrt{\frac{2}{3}} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad B^2 = \sqrt{\frac{1}{3}} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad B^3 = \sqrt{\frac{2}{3}} \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix},
\end{gathered} \tag{1}
$$

where $\sigma = 1, 2, 3$ are the indices for the $S_z = +1, 0, -1$ states at each chain site, respectively.

(a) Verify that the tensor B is both left- and right-normalized.

[Solution] B is left-normalized since

$$
\sum_\sigma (B^\sigma)^\dagger B^\sigma = \begin{pmatrix} 0 & 0 \\ 0 & 2/3 \end{pmatrix} + \begin{pmatrix} 1/3 & 0 \\ 0 & 1/3 \end{pmatrix} + \begin{pmatrix} 2/3 & 0 \\ 0 & 0 \end{pmatrix} = I, \tag{2}
$$

and right-normalized since

$$
\sum_\sigma B^\sigma (B^\sigma)^\dagger = \begin{pmatrix} 2/3 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 1/3 & 0 \\ 0 & 1/3 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 2/3 \end{pmatrix} = I. \tag{3}
$$

(b) Compute the transfer operator $T^{(\alpha,\alpha')}{}_{(\beta,\beta')} = \sum_\sigma B^{\dagger\beta'}{}_{\alpha'\sigma} B^{\alpha}{}_{\beta}{}^{\sigma} = \sum_\sigma B^{*}_{\alpha'}{}^{\beta'}{}_{\sigma} B^{\alpha}{}_{\beta}{}^{\sigma}$ without local operators. Verify that the eigenvalues of T are $(1, -1/3, -1/3, -1/3)$. Note that the arrows for the left and right legs of $B^\dagger$, indexed by α' and β', respectively, are implicitly flipped.

[Solution]

$$
\begin{aligned}
T &= (B^1)^* \otimes B^1 + (B^2)^* \otimes B^2 + (B^3)^* \otimes B^3 \\
&= \frac{2}{3} \begin{pmatrix} 0 & & & 1 \\ & 0 & & \\ & & 0 & \\ & & & 0 \end{pmatrix} + \frac{1}{3} \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & 1 \end{pmatrix} + \frac{2}{3} \begin{pmatrix} 0 & & & \\ & 0 & & \\ & & 0 & \\ 1 & & & 0 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 1 & & & 2 \\ & -1 & & \\ & & -1 & \\ 2 & & & 1 \end{pmatrix},
\end{aligned} \tag{4}
$$

where the rows of the 4×4 matrices are indexed in the order of $(\alpha, \alpha') = (1,1), (1,2), (2,1), (2,2)$, and the columns likewise. T is real symmetric, so the left and right eigenvectors are equivalent. From $\det(T - \lambda I) = (1/3 - \lambda)^2 (-1/3 - \lambda)^2 - (4/9)(-1/3 - \lambda)^2 = (\lambda - 1)(\lambda + 1/3)^3 = 0$, we identify four eigenvalues $\lambda_1 = 1$ and $\lambda_2 = \lambda_3 = \lambda_4 = -1/3$. The corresponding eigenvectors are

$$
\vec{v}_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad \vec{v}_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}, \quad \vec{v}_3 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad \vec{v}_4 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \tag{5}
$$

respectively.

(c) A transfer operator involving a local operator $\hat{O}$ acting on the physical legs of B and $B^\dagger$ is defined as

$$
[T_{\hat{O}}]^{(\alpha,\alpha')}{}_{(\beta,\beta')} = \sum_{\sigma,\sigma'} B^{\dagger\beta'}{}_{\alpha'\sigma'} [\hat{O}]^{\sigma'}{}_{\sigma} B^{\alpha}{}_{\beta}{}^{\sigma}. \tag{6}
$$

Obtain the transfer operators for $\hat{O} = \hat{S}_z$ and for $\hat{O} = \exp(\mathrm{i}\pi\hat{S}_z)$.

[Solution]

$$\begin{aligned}
T_{\hat{S}_z} &= (B^1)^* \otimes B^1 - (B^3)^* \otimes B^3 \\
&= \frac{2}{3}\begin{pmatrix} 0 & & & 1 \\ & 0 & & \\ & & 0 & \\ & & & 0 \end{pmatrix} - \frac{2}{3}\begin{pmatrix} 0 & & & \\ & 0 & & \\ & & 0 & \\ 1 & & & 0 \end{pmatrix} = \frac{2}{3}\begin{pmatrix} 0 & & & 1 \\ & 0 & & \\ & & 0 & \\ -1 & & & 0 \end{pmatrix}, \\
T_{e^{\mathrm{i}\pi\hat{S}_z}} &= -(B^1)^* \otimes B^1 + (B^2)^* \otimes B^2 - (B^3)^* \otimes B^3 \\
&= -\frac{2}{3}\begin{pmatrix} 0 & & & 1 \\ & 0 & & \\ & & 0 & \\ & & & 0 \end{pmatrix} + \frac{1}{3}\begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & 1 \end{pmatrix} - \frac{2}{3}\begin{pmatrix} 0 & & & \\ & 0 & & \\ & & 0 & \\ 1 & & & 0 \end{pmatrix} \\
&= \frac{1}{3}\begin{pmatrix} 1 & & & -2 \\ & -1 & & \\ & & -1 & \\ -2 & & & 1 \end{pmatrix}.
\end{aligned} \tag{7}$$

In the eigenbasis $\{\vec{v}_i\}$ of T, those transfer operators are given by

$$V^\dagger T_{\hat{S}_z} V = \frac{2}{3}\begin{pmatrix} 0 & -1 & & \\ 1 & 0 & & \\ & & 0 & \\ & & & 0 \end{pmatrix}, \quad V^\dagger T_{e^{\mathrm{i}\pi\hat{S}_z}} V = \mathrm{diag}([-1; 3; -1; -1])/3, \tag{8}$$

where $V = (\vec{v}_1 \ \vec{v}_2 \ \vec{v}_3 \ \vec{v}_4)$.

(d) Derive the asymptotic (i.e., $\lim_{|m-n|\to\infty} \lim_{L\to\infty}$) behaviors of

$$\begin{aligned}
\chi_{zz}(m-n) &= \langle\psi|\hat{S}_{z[m]}\hat{S}_{z[n]}|\psi\rangle, \\
\chi_{\mathrm{string}}(m-n) &= \langle\psi|\hat{S}_{z[m]} e^{\mathrm{i}\pi\hat{S}_{z[m-1]}} e^{\mathrm{i}\pi\hat{S}_{z[m-2]}} \cdots e^{\mathrm{i}\pi\hat{S}_{z[n+2]}} e^{\mathrm{i}\pi\hat{S}_{z[n+1]}} \hat{S}_{z[n]}|\psi\rangle.
\end{aligned} \tag{9}$$

Check whether you get $\chi_{zz} \sim e^{-|m-n|/\xi}$ with $\xi = 1/\log 3$ and $\chi_{\mathrm{string}} = -4/9$.

[Solution] We first compute χ_{zz}. Here $V^\dagger T_{\hat{S}_z} V$ is finite only in the $\vec{v}_1$ and $\vec{v}_2$ bases. Then we get

$$\chi_{zz}(m-n) = \frac{4}{9}\begin{pmatrix} 0 & -1 \end{pmatrix}\begin{pmatrix} 1^{|m-n|} & \\ & (-1/3)^{|m-n|} \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{-4(-1)^{|m-n|}}{9} e^{-(\log 3)|m-n|}. \tag{10}$$

Similarly, we compute χ_{string},

$$\chi_{\mathrm{string}}(m-n) = \frac{4}{9}\begin{pmatrix} 0 & -1 \end{pmatrix}\begin{pmatrix} (-1/3)^{|m-n|} & \\ & 1^{|m-n|} \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = -4/9. \tag{11}$$