Example 2: Sherman-Morrison-Woodbury Formula

Theorem

***Sherman-Morrison-Woodbury Formula**: Given 4 matrices* $\mathbf{A} \in \mathbb{R}^{p\times p}, \mathbf{U} \in \mathbb{R}^{p\times q}, \mathbf{V} \in \mathbb{R}^{q\times p}, \mathbf{C} \in \mathbb{R}^{q\times q}$ *and* $\mathbf{A}, \mathbf{C}$ *are invertible. Then,*

$$(\mathbf{A} + \mathbf{UCV})^{-1} = \mathbf{A}^{-1} - \mathbf{A}^{-1}\mathbf{U}(\mathbf{C}^{-1} + \mathbf{VA}^{-1}\mathbf{U})^{-1}\mathbf{VA}^{-1} \triangleq f(\mathbf{A}, \mathbf{C}, \mathbf{U}, \mathbf{V})$$

Proof.

$$\begin{aligned}\begin{bmatrix} f(\mathbf{A}, \mathbf{C}, \mathbf{U}, \mathbf{V}) & \mathbf{O} \\ ? & \mathbf{I}_q \end{bmatrix} &= \begin{bmatrix} \mathbf{I}_p & -\mathbf{A}^{-1}\mathbf{U} \\ \mathbf{O} & \mathbf{I}_q \end{bmatrix} \begin{bmatrix} \mathbf{I}_p & \mathbf{O} \\ \mathbf{O} & (\mathbf{C}^{-1} + \mathbf{VA}^{-1}\mathbf{U})^{-1} \end{bmatrix} \begin{bmatrix} \mathbf{A}^{-1} & \mathbf{A}^{-1}\mathbf{U} \\ \mathbf{VA}^{-1} & \mathbf{C}^{-1} + \mathbf{VA}^{-1}\mathbf{U} \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{I}_p & -\mathbf{A}^{-1}\mathbf{U} \\ \mathbf{O} & \mathbf{I}_q \end{bmatrix} \begin{bmatrix} \mathbf{I}_p & \mathbf{O} \\ \mathbf{O} & (\mathbf{C}^{-1} + \mathbf{VA}^{-1}\mathbf{U})^{-1} \end{bmatrix} \begin{bmatrix} \mathbf{A}^{-1} & \mathbf{O} \\ \mathbf{VA}^{-1} & \mathbf{C}^{-1} \end{bmatrix} \begin{bmatrix} \mathbf{I}_p & \mathbf{U} \\ \mathbf{O} & \mathbf{I}_q \end{bmatrix}\end{aligned} \tag{4}$$

Multiply the four block matrices on the right to $\begin{bmatrix} \mathbf{A} + \mathbf{UCV} & \mathbf{O} \\ \mathbf{O} & \mathbf{I}_q \end{bmatrix}$ and derive an identity matrix at top-left, which implies that $(\mathbf{A} + \mathbf{UCV})f(\mathbf{A}, \mathbf{C}, \mathbf{U}, \mathbf{V}) = \mathbf{I}_p$. □