6B-1

EMRT = estimated mean response time

confidence interval $[p, q]$ $EMRT_{S2} - EMRT_{S1}$

$p, q > 0$ system 1 is better with probability $(1-\alpha)$

$p, q < 0$ system 2

$p > 0, q < 0$ not different

common random numbers can reduce the variance if the behaviour of the two system is positively correlated.

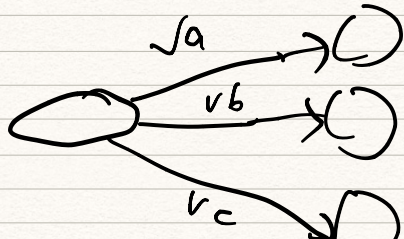
---

7A (1)

$X_{TA}$ = Throughput of travel site

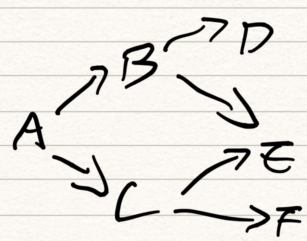
$X_a$ = Throughput of airline web service

$$X_a \geq V_a \times X_{TA}$$



$$X_{TA} \leq \min\left\{\frac{X_a}{V_a}, \frac{X_b}{V_b}, \frac{X_c}{V_c}\right\}$$

bottleneck



$$X_A \leq \min\left\{\frac{X_B}{V_{AB}}, \frac{X_C}{V_{AC}}, \cancel{\frac{X_D}{V_{AB}V_{BD}}}, \frac{X_F}{V_{AC}V_{CF}}, \frac{X_E}{V_{AB}V_{BE} + V_{AC}V_{CE}}\right\}$$

$\frac{V_D}{V_{BD}} \geq X_B$ $\quad X_B \leq V_D \cdot V_{BD}$ **7A (2)**

| | | | | | |
|---|---|---|---|---|---|
| $\frac{1}{4}$ | 0.2 | 0.2 | $\rightarrow$ | 0.2 | |
| $\frac{1}{4}$ | 0.3 | 0.3 | $\rightarrow$ | 0.3 | $\rightarrow$ 0.275 |
| $\frac{1}{4}$ | 0.2 | 0.3 | $\rightarrow$ | 0.3 | |
| $\frac{1}{4}$ | 0.3 | 0.2 | $\rightarrow$ | 0.3 | |

fork-join system: parallel invocation, all services must complete at the joining point before the next service can start

server 1 to $N-1$: mean service time $= S$

server $N$: $g \times S$ $\quad T(g) =$ Response time of this system

$T(1) = (1 + \frac{1}{2} + \frac{1}{3} + \cdots \frac{1}{N})S = H_N$ $\quad H_1 = 1$ ...... $H_2 = 1 + \frac{1}{2}$ ...

$H_N = N$-th harmonic number

when $T(g)$ for $g > 1$ use Markov chain

states $(i, j, k)$ $\quad i\,(0 \sim N-1)$ fast web $\quad k\,(0, \cdots N)$ yet to complete

$j\,(0,1)$ slow web

$$T(g) = \frac{S}{(N-1+\frac{1}{g})P(N)}$$

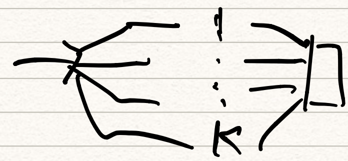
$$P(N) = \left[1 + \sum_{i=1}^{N-2} F(i) + gF(1) + V\right]^{-1}$$

$$V = \frac{1}{g} \sum_{j=1}^{N-1} \frac{1}{j} \sum_{i=j}^{N-1} F(i) \qquad F(i) = \prod_{j=1}^{N-i-1} \frac{N-j}{N-j-1+\frac{1}{g}}$$

when $g = 1$ $\quad T(g) = H_N S$

when there are $n-1$ jobs in the QN, 7A (3)
the average number of jobs in the subsystem
is $Z$, when there're $n$ jobs in the system.
the average time each job requires is $S$ (exp)
$\mu = \frac{1}{S}$, waiting time $= S \times Z$ service time $= S \times H_k$

response time $= S \times (H_k + Z)$




for $k=1$ $H_1 = 1$

$I$ = Number of subsystems in the QN MVA
$S_i$ = Avg service time of a station in subsystem $i$
$k_i$ = parallel stations in subsystem $i$
$R_i(n)$ = response time at subsystem $i$ when there're $n$ jobs in the QN
$\bar{n}_i(n)$ = Avg number of jobs at subsystem $i$ when there are $n$ jobs in the QN
$V_i$ = Visit ratio of subsystem $i$

$$R_i(n) = S_i \times (H_{k_i} + \bar{n}_i(n-1))$$

$$X_0(n) = \frac{n}{R_0(n)} = \frac{n}{\sum_{i=1}^{I} V_i R_i(n)} = \frac{n}{\sum_{i=1}^{I} D_i (H_{k_i} + \bar{n}_i(n-1))}$$

$$\bar{n}_i(n) = V_i \times X_0(n) \times R_i(n) = D_i X_0(n) (H_{k_i} + \bar{n}_i(n-1))$$

$n=0$ $\bar{n}_1(0) = \bar{n}_2(0) = \bar{n}_3(0) = 0$

$n=1$ ......

7B (1)

| | Resource 1 | Resource 2 | Resource 3 |
|---|---|---|---|
| speed | 1000 | 2000 | 3000 |
| cost | 0.1 | 0.25 | 0.6 |

requires : $10^7$ $T \le 4800$ cost $\le 1500$

$x_1, x_2, x_3$ (fraction)

$T_1 = \dfrac{10^7 \times x_1}{1000}$ $T_2 = \dfrac{10^7 \times x_2}{2000}$ $T_3 = \dfrac{10^7 \times x_3}{3000}$

$C_1 = T_1 \times 0.1$ $C_2 = T_2 \times 0.25$ $C_3 = T_3 \times 0.6$

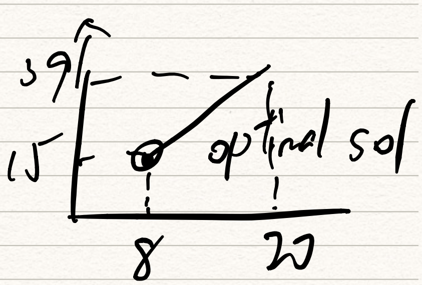
min $T$ $T \ge T_1$, $T \ge T_2$, $T \ge T_3$

subject to $T \le 4800$, $C_1 + C_2 + C_3 \le 1500$

$x_1 + x_2 + x_3 = 1$, $x_1, x_2, x_3 \ge 0$

$\min\limits_{x} \; 2x - 1$

subject to $x \le 20$, $x \ge 8$



Given: 08A

$n$ resources

Resource $i$ offers speed of $p_i$ million cycles/sec with cost $c_i$ dollars/sec

set up cost for using Resource $i$ is $S_i$ dollars

customer require $N$ million cycles

completion time <= Tmax

cost $\leq$ Cmax

$$y_i \begin{cases} 1 & \text{if Resource } i \text{ is chosen} \\ 0 & \end{cases}$$

$x_i$ = fraction of the job allocated to Resource $i$

$T$ = Completion time, $\max \frac{N x_i}{p_i}$

$$C = \sum_{i=1}^{n} \left( \frac{c_i N x_i}{p_i} + S_i y_i \right) \quad x_i \leq y_i$$

(y=1 x=0 will not be chosen)

The problem formulation is $\min T$

subject to

$$T \geqslant \frac{N x_i}{p_i} \quad i = 1 \cdots n$$

$$T \leq T_{max}$$

$$\sum_{i=1}^{n} \left( \frac{c_i N x_i}{p_i} + S_i y_i \right) \leq C_{max}$$

$$x_i \in y_i \quad i = 1 \cdots n$$

$$\sum_{i=1}^{n} x_i = 1$$

$$x_i \geqslant 0 \quad y_i \in \{0, 1\} \quad i = 1 \cdots n$$

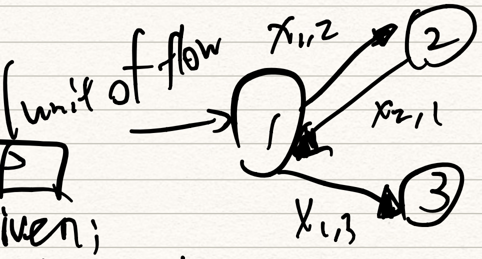
Network flow problems are important of integer programming. 08B (1)

Network is represented as a directed graph

$G = (N, E)$

$N$ = the set of nodes $N = \{1, 2, 3, 4, 5, 6\}$

$E$ = the set of directed edges $E = \{(1,2), (1,3) \cdots\}$




$1 + x_{2,1} = x_{1,2} + x_{1,3}$

$\lambda = 4,1$ source node destination node check $\boxed{1}$

$\boxed{SSP}$

Given;

A directed graph $G = (N, E)$

A flow of size 1 enter at node s (source) and leaves at node d (destination)

It costs $c_{i,j}$ for using directed edge $\langle i, j \rangle$

Aim: the total cost is minimized

the entire flow must use only one path

$$x_{i,j} = \begin{cases} 1 & \text{used edge} \\ 0 \end{cases}$$

$$\sum_{j:\langle 1,j\rangle \in E} x_{1,j} - \sum_{j:\langle j,1\rangle \in E} x_{j,1} = 1 \quad \text{(source)}$$

$$\sum_{j:\langle 6,j\rangle \in E} x_{6,j} - \sum_{j:\langle j,6\rangle \in E} x_{j,6} = -1 \quad \text{(destination)}$$

$$\sum_{j:\langle 2,j\rangle \in E} x_{2,j} - \sum_{j:\langle j,2\rangle \in E} x_{j,2} = 0 \quad \text{(inter)}$$

Shortest path problem can be formulated as 0&B (2)

$$\min \sum_{\langle i,j\rangle \in E} c_{i,j} x_{i,j}$$

Subject to

$$\sum_{j:\langle i,j\rangle \in E} x_{i,j} - \sum_{j:\langle j,i\rangle \in E} x_{j,i} = \begin{cases} 1 & \text{if } i = s \\ 0 & \text{if } i \in N - \{s,d\} \\ -1 & \text{if } i = d \end{cases}$$

$$x_{i,j} \in \{0,1\} \text{ for all } \langle i,j\rangle \in E$$

## Non-unit flow and link capacity

Given:

A directed graph $(N,E)$

* A flow of size $f$ with source node $s$ and destination node $d$

It costs $c_{ij}$ (per unit flow) for the flow to use directed edge $(i,j)$

* The capacity of the directed edge $(i,j)$ is $b_{ij}$

Aim: The total cost is minimised

the entire flow must use only one path

* The flow on any directed edge does not exceed it's capacity

$$x_{i,j} = \begin{cases} 1 & \text{if directed edge } (i,j) \text{ is used} \\ 0 & \end{cases}$$

The amount of flow on directed edge $(i,j)$ will be $f x_{ij}$

The problem formulation is 08B (3)

$$\min \sum_{(i,j)\in E} c_{ij} x_{ij}$$

subject to

$$\sum_{j:(i,j)\in E} x_{ij} - \sum_{j:(j,i)\in E} x_{ji} = \begin{cases} 1 & \text{if } i=s \\ 0 & \text{if } i\in N-\{s,d\} \\ -1 & \text{if } i=d \end{cases}$$

$$f x_{ij} \le b_{ij} \text{ for all } (i,j)\in E \quad ***$$

$$x_{ij} \in \{0,1\} \text{ for all } (i,j)\in E$$

*** this constraint ensures that only links with sufficient capacity maybe chosen to carry the flow

Solution: Eliminate edges with insufficient capacity then Dijkstra

$\boxed{\text{Traffic engineering problem}}$

Given:

A directed graph $(N,E)$

* m flows (index by $k=1,2..m$)
* Flow k has size $f_k$, source node $s_k$. destination $d_k$
* It costs $c_{ij}$ for a unit of flow to use directed edge $(i,j)$

The capacity of directed edge is $b_{ij}$

Aim: total cost is minimised The entire flow use one path

* total flow on a directed edge doesn't exceed capacity.

The problem formulation is 08B

$$\min \sum_{(i,j)\in E} \sum_{k=1}^{m} c_{ij} f_k x_{ijk} \qquad (4)$$

subject to

$$\sum_{j:(i,j)\in E} x_{ijk} - \sum_{j:(j,i)\in E} x_{jik} = \begin{cases} 1 & \text{if } i = s_k \\ 0 & \text{if } i \in N - \{s_k, d_k\} \text{ *} \\ -1 & \text{if } i = d_k \end{cases}$$

$$\sum_{k=1}^{m} f_k x_{ijk} \le b_{ij} \text{ for all } (i,j)\in E \text{ **}$$

$$x_{ijk} \in \{0,1\} \text{ for all } (i,j)\in E, k=1,\dots,m$$

* One set of flow balance constraint per flow. Enforces

Enforces flow $k$ is from $s_k$ to $d_k$

** Total flow on a link does not exceed it's capacity

$\boxed{\text{The coverage problem}}$ 09A (1)

Given:

A number of potential access point locations $L_1, L_2 \cdots L_p$

A number of stations $S_1, S_2 \cdots S_n$

Binary constant $\delta_{ij}$ where

$$\delta_{ij} = \begin{cases} 1 & \text{if station } S_i \text{ is covered by an AP at } L_j \\ 0 & \text{otherwise} \end{cases}$$

Aim: Find the minimum number of access points let all stations are covered.

Decision variables: 09A

$$X_j = \begin{cases} 1 & \text{if an AP is to be installed at } L_j \\ 0 & \text{otherwise} \end{cases}$$ (2)

Integer programming formulation:

$$\min \sum_{j=1}^{P} X_j$$

subject to
$$\sum_{j=1}^{P} \delta_{ij} X_j \geq 1 \quad \forall i = 1 \ldots, n$$
$$X_j \in \{0, 1\}$$

09B
(1)

$\boxed{\text{restricted range of values}}$

$$x = \sum_{i=1}^{m} a_i y_i \qquad \sum_{i=1}^{m} y_i = 1 \qquad y_i \in \{0, 1\}$$

$\boxed{\text{Either-or}}$

one of the constraints must be satisfied

$$\sum_{i=1}^{n} a_{1,i} X_i \geq b_1 \qquad \sum_{i=1}^{n} a_{2,i} X_i \geq b_2$$

then $$\sum_{i=1}^{n} a_{1,i} X_i \geq b_1 P \qquad \sum_{i=1}^{n} a_{2,i} X_i \geq b_2 (1-P)$$
$$P \in \{0, 1\}$$

$\boxed{\text{if-then}}$ if $f(x_1 \cdot\cdot x_2 \cdot\cdot x_n) > 0$, then $g(x_1, x_2 \cdot\cdot x_n) \geq 0$

$\Downarrow$ $f(x_1, x_2 \cdots x_n) \leq M_1 (1-P)$

$-g(x_1, x_2 \cdots x_n) \leq M_2 P$

$\boxed{\text{piecewise linear functions}}$ 09B (2)

5 dollar/sec for the first 5000 sec

2 — — — — — — — the next 15000 sec

1 — — — thereafter

$0 \le t \le 5000 \quad c = 5t$

$5000 \le t \le 20000 \quad c = 25000 + 2(t - 5000)$

$t \geq 20000 \quad c = 55000 + (t - 20000)$

$0 \le ty_1 \le 5000y_1$

$5000y_2 \le ty_2 \le 20000y_2$

$20000y_3 \le ty_3$

$\text{cost} = y_1(5t) + y_2(2t + 15000) + y_3(t + 35000)$

$y_1 + y_2 + y_3 = 1$ non-linear constraints

Define $t_i = ty_i$ for $i = 1, 2, 3$

$\text{cost} = 5t_1 + 2t_2 + 15000y_2 + t_3 + 35000y_3$