

Project Based Calculus II Project

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May 18, 2018

1 CALCULUS AND BASEBALL

In this project we explore three of the many applications of calculus to baseball. The physical interactions of the game, especially the collision of ball and bat are quite complex and their models are discussed in detail in a book by Robert Adair, *The Physics of Baseball*.

1. It may surprise you to learn that the collision of baseball and bat lasts only about a thousandth of a second. Here we calculate the average force on the bat during this collision by first computing the change in the ball's momentum.

The *momentum* p of an object is the product of its *mass* m and its *velocity* v , that is, $p = mv$. Suppose an object, moving along a straight line, is acted on by a force $F = f(t)$ that is continuous function of time.

- (a) Show that the change in momentum over a time interval $[t_0, t_1]$ is equal to the integral of F from t_0 to t_1 ; that is, show that

$$p(t_1) - p(t_0) = \int_{t_0}^{t_1} F(t) dt$$

$$\begin{aligned} F &= m \frac{dv}{dt} \\ \int_{t_0}^{t_1} F(t) dt &= \int_{t_0}^{t_1} m \frac{dv}{dt} dt \\ \int_{t_0}^{t_1} m \frac{dv}{dt} dt &= m \int_{t_0}^{t_1} dv \\ m \int_{t_0}^{t_1} dv &= [mv]_{t_0}^{t_1} \end{aligned}$$

$$[mv]_{t_0}^{t_1} = mv_1 - mv_0$$

$$mv_1 - mv_0 = p(t_1) - p(t_0)$$

$$p(t_1) - p(t_0) = p(t_1) - p(t_0)$$

- (b) A pitcher throws a 90-mi/h fastball to a batter, who hits a line drive directly back to the pitcher. The ball is in contact with the bat for 0.001 s and leaves the bat with velocity 110 mi/h. A baseball weighs 5oz and, in US Customary unites, its mass is measured in slugs: $m = w/g$ where $g = 32 \text{ ft/s}^2$.

- (i) Find the change in the ball's momentum.

$$v_1 = 110 \text{ mi/h} = \frac{110(5280)}{3600} \text{ ft/s} = 161.333 \text{ ft/s}$$

$$v_0 = -90 \text{ mi/h} = -132 \text{ ft/s}$$

Baseball's mass

$$m = \frac{5}{512}$$

$$p(t_1) - p(t_0) = mv_1 - mv_0$$

$$mv_1 - mv_0 = \frac{5}{512} [161.333 - (-132)]$$

$$= 2.86 \text{ slug-ft/s}$$

- (ii) Find the average force on the bat.

$$\int_0^{0.001} F(t) \, dt$$

$$p(0.001) - p(0) = 2.86$$

So the average on the interval $[0, 0.001]$

$$\frac{1}{0.001} \int_0^{0.001} F(t) \, dt$$

$$\int_0^{0.001} F(t) \, dt = \frac{1}{0.001} (2.86)$$

$$= 2860 \text{ lb.}$$

2. In this problem we calculate the work required for a pitcher to throw a 90-mi/h fastball by first considering kinetic energy.

The *kinetic energy* K of an object of mass m and velocity v is give by $k = \frac{1}{2}mv^2$.

Suppose an object of mass m , moving in a straight line, is acted on by a force $F = F(s)$ that depends on its position s . According to Newton's Second Law:

$$F(s) = ma = m \frac{dv}{dt}$$

where a and v denote the acceleration and velocity of the object.

- (a) Show that the work done in moving the object from a position s_0 to a position s_1 is equal to the change in the object's kinetic energy; that is show that

$$W = \int_{s_0}^{s_1} F(s) \, ds = \frac{1}{2}mv_1^2 - \frac{1}{2}mv_0^2$$

$$W = \int_{s_0}^{s_1} F(s) \, ds$$

$$F(s) = m v \frac{dv}{ds}$$

By substitution. . .

$$W = \int_{s_0}^{s_1} F(s) \, ds = \int_{s_0}^{s_1} m v \, dv$$

$$= \int_{s_0}^{s_1} m v \, dv = \left[\frac{1}{2} m v^2 \right]_{v_0}^{v_1}$$

$$\left[\frac{1}{2} m v^2 \right]_{v_0}^{v_1} = \frac{1}{2} m v_1^2 - \frac{1}{2} m v_0^2$$

- (b) How many foot-pounds of work does it take to throw a basketball at a speed of 90 mi/h?

Given Information . . .

$$90 \text{ mi/h} = 132 \text{ ft/s}$$

$$v_0 = 0$$

then . . .

$$v_1 = 132 \text{ ft/s}$$

$$m = \frac{5}{512}$$

$$W = \frac{1}{2} m v_1^2 - \frac{1}{2} m v_0^2$$

$$= \frac{1}{2} * \frac{5}{512} * (132)^2$$

$$= 85 \text{ ft/lb}$$

3. (a) An outfielder fields a baseball 280 ft away from home plate and throws it directly to the catcher with an initial velocity of 100 ft/s. Assume that the velocity $v(t)$ of the ball after t seconds satisfies the differential equation $\frac{dv}{dt} = -\frac{1}{10} v$ because of air resistance. How long does it take for the ball to reach home plate?

Here we have a differential equation of the form

$$\frac{dv}{dt} = v(0)e^{kt}$$

In this case,

$$k = -\frac{1}{10} \text{ and } v(0) = 100 \text{ ft/s, so } v(t) = 100e^{(-\frac{t}{10})}$$

We are interested in the time t that the ball takes to travel 280 ft, so we find the distance function

$$s(t) = \int_0^t v(x) dx$$

$$= \int_0^t 100e^{-\frac{x}{10}} dx$$

$$= 100[-10e^{-\frac{x}{10}}]_0^t$$

$$= -1000e^{-\frac{t}{10}}$$

Now we set $s(t) = 280$ and solve for t :

$$280 = 1000(1 - e^{-\frac{t}{10}})$$

$$= 1 - e^{-\frac{t}{10}} \quad v = \frac{7}{25}$$

$$-\frac{1}{10} t = \ln(1 - \frac{7}{25})$$

$$t = 3.285 \text{ seconds}$$

- (b) The manager of the team wonders whether the ball will reach home plate sooner if it is delayed by an infielder. The shortstop can position himself directly between the outfielder and home plate, catch the ball thrown by the outfielder, turn, and throw the ball to the catcher with an initial velocity of 105 ft/s. The manager clocks the relay time of the shortstop at half a second. How far from home plate should the shortstop position himself to minimize the total time for the ball to reach the plate? Should the manager encourage a direct throw or a delayed throw? What if the shortstop can throw at 115 ft/s.

Let x be the distance of the shortstop from homeplate. We calculate the time for the ball to reach homeplate as a function of x , then differentiate with respect to x to find the value of x which corresponds to the minimum time. The total time that it takes the ball to reach home is the sum of the times of the two throws, plus the relay time $\frac{1}{2}$

seconds. The distance from the fielder to the shortstop is $280 - x$, so to find the time t_1 taken by the first throw, we solve the equation

$$\begin{aligned} s_1(t_1) &= 280 - x \\ 1 - e^{-\frac{t_1}{10}} &= \frac{280-x}{1000} \\ t_1 &= -10 \ln \frac{720+x}{1000}. \end{aligned}$$

We find the time t_2 taken by the second throw if the shortstop throws with velocity w , since we see that this velocity varies in the rest of the problem.

We use $v = we^{-\frac{t}{10}} = 1 - \frac{x}{10w}$ and isolate t_2 in the equation $s(t_2) = 10w(1 - e^{-\frac{t_2}{10}}) = x$

$$\begin{aligned} e^{-\frac{t_2}{10}} &= 1 - \frac{x}{10w} \\ t_2 &= -10 \ln \frac{10w-x}{10w} \end{aligned}$$

so the total time is

$$t_w(x) = \frac{1}{2} - 10 \left[\ln \frac{720+x}{1000} + \ln \frac{10w-x}{10w} \right].$$

To find the minimum, we differentiate

$$\frac{dt_w}{dx} = -10 \left[\frac{1}{720+x} - \frac{1}{10w-x} \right]$$

which changes from negative to positive when $720 + x = 10w - x$.

By the First Derivative Test, t_w has a minimum at this distance from the shortstop to home plate. So if the shortstop throws at $w = 105$ from a point $x = 5(105) - 260 = 165$ ft from home plate, the minimum time is

$$t_{105}(165) = \frac{1}{2} - 10 \left(\ln \frac{720+165}{1000} + \ln \frac{1050-165}{1050} \right) = 3.431 \text{ seconds.}$$

This is longer than the time taken in part (a), so in this case, the manager should encourage a direct throw.

If $w = 115$ ft/s, then $x = 215$ ft from home, and the minimum time is $t_{115} = \frac{1}{2}$

$$10 \left(\ln \frac{720+215}{1000} + \ln \frac{1150-215}{1150} \right) = 3.242 \text{ seconds.}$$

This is less than the time taken in part (a), so in this case, the manager should encourage a relayed throw.

- (c) For what throwing velocity of the shortstop does a relayed throw take the same time as a direct throw?

In general, the minimum time is

$$\begin{aligned} t_w(5w - 260) &= \frac{1}{2} - 10 \left[\ln \frac{260+5w}{1000} + \ln \frac{260+5w}{10w} \right] \\ &= \frac{1}{2} - 10 \ln \frac{(w+72)^2}{400w} \end{aligned}$$

We want to find out when this is about 3.285 seconds, the same time as the direct throw. We can estimate that this is the case for $w = 112.8$ ft.s. So if the shortstop can throw the ball with this velocity, then a relayed throw takes the same time as a direct throw.