Note: Your TA probably will not cover all the problems. This is totally fine, the discussion worksheets are deliberately made long so they can serve as a resource you can use to practice, reinforce, and build upon concepts discussed in lecture, readings, and the homework.

1 Asymptotics and Limits

If we would like to prove asymptotic relations instead of just using them, we can use limits.

Asymptotic Limit Rules: If $f(n), g(n) \geq 0$:

- If $\lim_{n\to\infty} \frac{f(n)}{g(n)} < \infty$, then $f(n) = \mathcal{O}(g(n))$.
- If $\lim_{n\to\infty} \frac{f(n)}{g(n)} = c$, for some $c > 0$, then $f(n) = \Theta(g(n))$.
- If $\lim_{n\to\infty} \frac{f(n)}{g(n)} > 0$, then $f(n) = \Omega(g(n))$.

Note that these are all sufficient (and not necessary) conditions involving limits, and are not true definitions of $\mathcal{O}$, Θ, and Ω. We highly recommend checking on your own that these statements are correct!)

(a) Prove that $n^3 = \mathcal{O}(n^4)$.

$$\lim_{n\to\infty} \frac{n^3}{n^4} = \lim_{n\to\infty} \frac{1}{n} = 0$$

(b) Find an $f(n), g(n) \geq 0$ such that $f(n) = \mathcal{O}(g(n))$, yet $\lim_{n\to\infty} \frac{f(n)}{g(n)} \neq 0$.

$f(n) = n$
$g(n) = n$

(c) Prove that for any $c > 0$, we have $\log n = \mathcal{O}(n^c)$.

Hint: Use L'Hôpital's rule: If $\lim_{n\to\infty} f(n) = \lim_{n\to\infty} g(n) = \infty$, then $\lim_{n\to\infty} \frac{f(n)}{g(n)} = \lim_{n\to\infty} \frac{f'(n)}{g'(n)}$ (if the RHS exists)

$$\lim_{n\to\infty} \frac{\log n}{n^c} = \lim_{n\to\infty} \frac{\frac{1}{n}}{c \cdot n^{c-1}} = \lim_{n\to\infty} \frac{1}{c \cdot n^c} = 0$$

(d) Find an $f(n), g(n) \geq 0$ such that $f(n) = \mathcal{O}(g(n))$, yet $\lim_{n\to\infty} \frac{f(n)}{g(n)}$ does not exist. In this case, you would be unable to use limits to prove $f(n) = \mathcal{O}(g(n))$.

Hint: think about oscillating functions!

$f(n) = n \cdot \sin(n)$
$g(n) = n$

2 Asymptotic Complexity Comparisons

(a) Order the following functions so that for all i, j, if f_i comes before f_j in the order then $f_i = O(f_j)$. Do not justify your answers.

- $f_1(n) = 3^n$
- ✓ $f_2(n) = n^{\frac{1}{3}}$
- ✓ $f_3(n) = 12$
- ✓ $f_4(n) = 2^{\log_2 n}$
- ✓ $f_5(n) = \sqrt{n}$
- ✓ $f_6(n) = 2^n$
- ✓ $f_7(n) = \log_2 n$
- ✓ $f_8(n) = 2^{\sqrt{n}}$
- ✓ $f_9(n) = n^3$

As an answer you may just write the functions as a list, e.g. $f_8, f_9, f_1, \ldots$

$3 \leq 7 \leq 2 \leq 5 \leq 4 \leq 9 \leq 8 \leq 6 \leq 1$

(b) In each of the following, indicate whether $f = O(g)$, $f = \Omega(g)$, or both (in which case $f = \Theta(g)$). ***Briefly*** justify each of your answers. Recall that in terms of asymptotic growth rate, constant < logarithmic < polynomial < exponential.

	$f(n)$	$g(n)$
(i)	$\log_3 n$	$\log_4(n)$
(ii)	$n \log(n^4)$	$n^2 \log(n^3)$
(iii)	$\sqrt{n}$	$(\log n)^3$
(iv)	$n + \log n$	$n + (\log n)^2$

(i) $\frac{f(n)}{g(n)} = \frac{\log_3(n)}{\log_4(n)} = \frac{\ln 4}{\ln 3}$ $f = \Theta(g)$

(ii) $= \frac{n \log(n^4)}{n^2 \log(n^3)} = \frac{\log(n^4)}{\log(n^{3n})} = \frac{4}{3n}$ $f = O(g)$

(iii) $= \frac{\sqrt{n}}{(\log n)^3}$ $f = \Omega(g)$

(iv) $= \frac{n + \log n}{n + (\log n)^2} = \frac{1 + \frac{\log n}{n}}{1 + \frac{(\log n)^2}{n}}$ $f = \Theta(g)$

3 Recurrence Relations

Solve the following recurrence relations, assuming base cases $T(0) = T(1) = 1$:

(a) $T(n) = 2 \cdot T(n/2) + O(n)$

(b) $T(n) = T(n-1) + n$

(c) $T(n) = 3 \cdot T(n-2) + 5$

$\boxed{3b}$

$T(n) - T(n-1) = n$

$(T(n) - T(n-1)) + (T(n-1) - T(n-2)) + \cdots + (T(1) - T(0))$

$= n + (n-1) + \cdots + 1$

$T(n) - T(0) = \frac{n}{2}(1+n)$

$T(n) = \frac{1}{2}n^2 + \frac{1}{2}n + 1$

$\boxed{3c}$

$$
\begin{aligned}
T(2k) &= 3T(2(k-1)) + 5 \\
&= 3(3T(2(k-2)) + 5) + 5 \\
&= 3^2 T(2(k-2)) + (3+1)\cdot 5 \\
&= 3^3 T(2(k-3)) + (3^2 + 3^1 + 3^0)\cdot 5 \\
&= 3^k T(0) + (3^{k-1} + 3^{k-2} + \cdots + 3^0)\cdot 5 \\
&= 3^k \cdot 1 + \frac{3^0(1-3^k)}{1-3}\cdot 5 \\
&= \frac{7}{2}\cdot 3^k - \frac{5}{2}
\end{aligned}
$$

$$
\begin{aligned}
T(2k+1) &= 3T(2k-1) + 5 \\
&= 3^k T(1) + (3^{k-1} + 3^{k-2} + \cdots + 3^0)\cdot 5 \\
&= 3^k \cdot 1 + \frac{3^0(1-3^k)}{1-3}\cdot 5 \\
&= \frac{7}{2}\cdot 3^k - \frac{5}{2}
\end{aligned}
$$

(d) $T(n) = 2 \cdot T(n/2) + O(n \log n)$

(e) $T(n) = 3T(n^{1/3}) + O(\log n)$

(f) $T(n) = T(n-1) + T(n-2)$

