Management Science (BUS306)

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Section 0. Course?@Overview

What is management science?

From
https://en.wikipedia.org/wiki/Management_science

Management science (MS), is the broad interdisciplinary study of problem solving and decision making in human organizations, with strong links to economics, business, engineering, and other sciences.

What is management science?

It uses various scientific research-based principles, strategies, and analytical methods including mathematical modeling, statistics and numerical algorithms to improve an organization's ability to enact rational and meaningful management decisions by arriving at optimal or near optimal solutions to complex decision problems.

What is management science?

In short, management sciences help businesses to achieve goals using various scientific methods.

What is management science?

The field was initially an outgrowth of applied mathematics, where early challenges were problems relating to the optimization of systems which could be modeled linearly, i.e., determining the optima (maximum value of profit, assembly line performance, crop yield, bandwidth, etc. or minimum of loss, risk, costs, etc.) of some objective function.

What is management science?

Today, management science encompasses any organizational activity for which the problem can be structured as a functional system so as to obtain a solution set with identifiable characteristics.

What is management science?: Overview

Management science is concerned with a number of different areas of study:

What is management science?: Overview

One is developing and applying models and concepts that may prove useful in helping to illuminate management issues and solve managerial problems.

What is management science?: Overview

The models used can often be represented mathematically, but sometimes computer-based, visual or verbal representations are used as well or instead.

What is management science?: Overview

Management science research can be done on three levels:

What is management science?: Overview

The fundamental level lies in three mathematical disciplines: probability, optimization, and dynamical systems theory.

What is management science?: Overview

The modeling level is about building models, analyzing them mathematically, gathering and analyzing data, implementing models on computers, solving them, experimenting with them–all this is part of management science research on the modeling level.

What is management science?: Overview

This level is mainly instrumental, and driven mainly by statistics and econometrics.

What is management science?: Overview

The application level, just as in any other engineering and economics disciplines, strives to make a practical impact and be a driver for change in the real world.

What is management science?: Overview

The management scientist's mandate is to use rational, systematic, science-based techniques to inform and improve decisions of all kinds.

What is management science?: Overview

The techniques of management science are not restricted to business applications but may be applied to military, medical, public administration, charitable groups, political groups or community groups.

What is management science?: History

Its origins can be traced to operations research, which made its debut during World War II when the Allied forces recruited scientists of various disciplines to assist with military operations.

What is management science?: History

The application of these models within the corporate sector became known as management science.

In 1967 Stafford Beer characterized the field of management science as “the business use of operations research.”

What is management science?: Applications

Applications of management science are abundant in industry as airlines, manufacturing companies, service organizations, military branches, and in government.

What is management science?: Applications

The range of problems and issues to which management science has contributed insights and solutions is vast.

What is management science?: Applications

It includes:

• scheduling airlines, both planes and crew,

• deciding the appropriate place to site new facilities such as a warehouse or factory,

• managing the flow of water from reservoirs,

• identifying possible future development paths for parts of the telecommunications industry,

• establishing the information needs and appropriate systems to supply them within the health service, and

• identifying and understanding the strategies adopted by companies for their information systems

Prerequisites

This subject has no formal prerequisites.

Every time some basic knowledge is necessary for this course, I will explain the knowledge from scratch.

Course Structure

BUS306 is basically structured as two longer modules.

- Linear Programming Theory and Applications
- Integer Programming Theory and Applications

If I have time, I will cover some other interesting topic(s) such as dynamic programming.

Textbook

There is no required textbook for the subject.

Students can refer to the following book, out of print but available free online, for some of the material.

Bradley, S. P., A. C. Hax, and T. L. Magnanti. "*Applied Mathematical Programming*." Addison-Wesley, 1977. ISBN-13: 978-0201004649.

Textbook

One other book I highly recommend not only for its Chapter 8 Linear Programming and Game Theory, but also for the preceding chapters is the following:

Strang, Gilbert. "*Linear Algebra and Its Applications, 4th Edition*." Academic Press, 2006. ISBN-13: 978-0030105678.

Textbook for Software

Please download the following free book from https://www.r-bloggers.com/modeling-and-solving-linear-programming-with-r-free-book/

Sallan, J. M., Lordan, O., Fernandez, V. "*Modeling and Solving Linear Programming with R*" OmniaScience (Omnia Publisher SL) 2015. ISBN-13: 978-8494422935.

Textbook for Software

"*Modeling and Solving Linear Programming with R*" is a freely available book about solving linear programming problems and exercises with R.

Textbook for Software

R is a free software environment for statistical computing and graphics. It compiles and runs on a wide variety of UNIX platforms, Windows and MacOS.

To download R, please choose your preferred CRAN mirror.

Textbook for Software

This book provides a brief introduction to linear programming, an introduction of solving linear programming problems with R and a set of exercises.

For each exercise a possible solution through linear programming is introduced together with the code to solve it in R and its numerical solution.

Optimization

Optimization is everywhere.

The prevailing social norm: If you are not optimizing your objective whatever it is, then you are required to explain why not.

Optimization

- At personal level,
  - finding the best school, the best university to you,
  - finding the best subjects, or the best classes,[1]
  - finding the best career choices to you,[2]
  - figuring out best use of our time,
  - given a budget constraint, trying to figure out a mix of products that gives you the highest utility or the best value.[3]

---

[1] trying to get highest GPA by taking only soft and easy courses, or trying to get highest GPA by taking courses that enhance your horizon, even if those courses are harder?

[2] perhaps conditionally and sequentially.

[3] as learned in Microeconomics 101, I suppose.

Optimization

- In corporate environment,
  - minimizing cost of acquiring a set of needed supplies,
  - figuring out the most efficient way of using these supplies in production,
  - minimizing production costs,
  - maximizing impact of advertising,[4]
  - minimizing cost of getting product to customers,
  - maximizing the value of a company to its shareholders.

[4] As measured in what?

Optimization: Decision variables

• Decision variables: the elements that are under the control of the decision maker.

For instance,

- The level of investments in a portfolio,
- The subjects/classes a student take in a term,
- The work schedules of each employee.

Optimization: Objective function

• A (single) objective function: a function of the decision variables that needs to be optimized.

For instance,

- maximize expected return of a portfolio,
- make this term as carefree as possible,
- minimize the labor cost.

Optimization: Constraints

- Constraints: restrictions placed on the decision variables.

Business or legal constraints are, for instance,

- there is at most 20% cap in stocks in the portfolio,[5]
- students must have taken a prerequisite of a subject before taking the subject,
- no employees can work more than 5 consecutive days, and at most 8 hours a day.

[5] For risk aversion, I suppose.

Optimization: Constraints

- Constraints: restrictions placed on the decision variables.

Physical constraints are, for instance,

- none of the investment instruments have a negative investment in a portfolio,
- students cannot take courses not offered in that term,
- no employees can work a negative amount of time.

FORMULATION OF SOME EXAMPLES

In order to provide a preliminary understanding of the types of problems to which mathematical programming can be applied, and to illustrate the kind of rationale that should be used in formulating linear-programming problems, we will present in this section a highly simplified examples and their corresponding linear programming formulations.

A simple example from Sallan, Lordan, and Fernandez Page 13-14

A small business manufactures and sells two products, named product 1 and product 2.

To manufacture these products, the business needs labor and capital, the latter in the form of a machine.

A simple example from Sallan, Lordan, and Fernandez Page 13-14

To manufacture each tonne of product 1 requires 30 working hours from its employees. Similary, each tonne of product 2 requires 20 working hours.

A simple example from Sallan, Lordan, and Fernandez Page 13-14

The business can ask from its employees a maximum of 2,700 combined working hours for the period considered.[6]

[6] For instance, 2,700 combined working hours is stipulated in the labor contract and the employee union sues the business or goes on strike whenever the business tries to ask working hours in excess of 2,700 hours.

A simple example from Sallan, Lordan, and Fernandez Page 13-14

Each tonne of product 1 and product 2 needs 5 and 10 machine hours respectively.[7]

[7] Funny, but these products can be manufactured using the same machine. Here it is also implicitly assumed that the business has only one machine of this kind.

A simple example from Sallan, Lordan, and Fernandez Page 13-14

There are 850 machine hours available for the period considered.[8]

---

[8] After that, for instance, the machine needs the regularly scheduled servicing.

A simple example from Sallan, Lordan, and Fernandez Page 13-14

Each tonne of product 1 yields 20 million euros of profit, while product 2 yields 60 million euros for each tonne sold.

A simple example from Sallan, Lordan, and Fernandez Page 13-14

For technical reasons, the firm must produce a minimum of 95 tonnes in total between both products.[9]

[9] There are many possible reasons for this kind of constraint. For instance, the business may wishes to keep its economy of scale intact, not to disappoint its preferred and thus important customers, to maintain its periodical production plan, to satisfy the demand from the stock market if it is listed.

A simple example from Sallan, Lordan, and Fernandez Page 13-14

We need to know how many tonnes of product 1 and 2 must be produced to maximize total profit.

A simple example from Sallan, Lordan, and Fernandez Page 13-14

First, we need to define the decision variables.

- P_1 number of tonnes produced and sold of product 1
- P_2 number of tonnes produced and sold of product 2

A simple example from Sallan, Lordan, and Fernandez Page 13-14

The *cost coefficients*[10] of these variables are 20 and 60, respectively.

[10] Unfortunate usage, but this word is commonly used. In this case, a profit coefficients would be more appropriate. The unit is in million euros.

A simple example from Sallan, Lordan, and Fernandez Page 13-14

The objective function $z = z(P_1, P_2)$ is defined multiplying each variable by its corresponding cost coefficient as

$$z = 20 \cdot P_1 + 60 \cdot P_2.$$

A simple example from Sallan, Lordan, and Fernandez Page 13-14

The constraints of this problem are as follows:

A simple example from Sallan, Lordan, and Fernandez Page 13-14

(Working Hours Constraint)

A constraint WH that the total amount of working hours employed in product 1 (consuming 30 working hours per tonne) and in product 2 (consuming 20 working hours per tonne) is less than or equal to 2,700 hours, or

$$30 \cdot P_1 + 20 \cdot P_2 \leq 2,700.$$

A simple example from Sallan, Lordan, and Fernandez Page 13-14

(Machine Hours Constraint)

A constraint MH that the total amount of machine hours used for product 1 (consuming 5 machine hours) and for product 2 (consuming 10 machine hours) is less than or equal to 850 hours, or

$$5 \cdot P_1 + 10 \cdot P_2 \leq 850.$$

A simple example from Sallan, Lordan, and Fernandez Page 13-14

(Production Minimum Constraint)

A constraint PM that the total units produced and sold are greater than or equal to 95 tonnes, or

$$P_1 + P_2 \geq 95.$$

A simple example from Sallan, Lordan, and Fernandez Page 13-14

(Non-negativity Constraint)

A constraint NN that the decision variables for this problem, the amounts of production for P_1 and P_2, are non-negative, or

$$\begin{aligned} P_1 &\geq 0, \\ P_2 &\geq 0. \end{aligned}$$

Solving a simple example from Sallan, Lordan, and Fernandez Page 13-14 in R

We put all this together, then the problem is summarized as:

$$\max z = 20 \cdot P_1 + 60 \cdot P_2,$$

such that

$$\begin{aligned} 30 \cdot P_1 + 20 \cdot P_2 &\leq 2,700, \\ 5 \cdot P_1 + 10 \cdot P_2 &\leq 850, \\ P_1 + P_2 &\geq 95, \\ P_1 &\geq 0, \\ P_2 &\geq 0. \end{aligned}$$

Solving a simple example from Sallan, Lordan, and Fernandez Page 13-14 in R

There are several solvers available for solving linear programming models.

Solving a simple example from Sallan, Lordan, and Fernandez Page 13-14 in R

As a preparation for that, we practice writing the problem of this kind in standard form

Solving a simple example from Sallan, Lordan, and Fernandez Page 13-14 in R

with an objective function of the n decision variables x_j, $j = 1, \ldots, n$ with the corresponding cost coefficients c_j

Solving a simple example from Sallan, Lordan, and Fernandez Page 13-14 in R

and a set of m constraints, in which a linear combination of the decision variables x_j affected by the corresponding coefficients a_{ij} has to be less than or equal to its right hand side value b_i, $i = 1, \ldots, m$.[11]

[11] Constraints with signs greater than or equal to, equalities, or their mix are also possible.

Solving a simple example from Sallan, Lordan, and Fernandez Page 13-14 in R

A problem like ours can be written in standard form as

$$\max z = c_1 x_1 + \cdots + c_n x_n$$

subject to

$$\begin{aligned} a_{11}x_1 + \cdots + a_{1n}x_n &\leq b_1, \\ a_{21}x_1 + \cdots + a_{2n}x_n &\leq b_2, \\ &\vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n &\leq b_m, \\ x_i &\geq 0, \text{ for } i = 1, \ldots, n. \end{aligned}$$

Solving a simple example from Sallan, Lordan, and Fernandez Page 13-14 in R

For the objective function, we can rewrite

$$\max z \quad = \quad c_1 x_1 + \cdots + c_n x_n$$

as

$$\max z \quad = \quad [c_1, \ldots, c_n] \cdot \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \boldsymbol{c}^T \boldsymbol{x},$$

where

$$\boldsymbol{c} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}, \qquad \boldsymbol{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}.$$

Solving a simple example from Sallan, Lordan, and Fernandez Page 13-14 in R

As for the m constraints, we can rewrite

$$\begin{aligned} a_{11}x_1 + \cdots + a_{1n}x_n &\leq b_1, \\ &\vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n &\leq b_m, \end{aligned}$$

as

$$\boldsymbol{A}\boldsymbol{x} = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ & \vdots & \\ a_{m1} & \ddots & a_{mn} \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \leq \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix} = \boldsymbol{b},$$

where

$$\boldsymbol{A} = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ & \vdots & \\ a_{m1} & \ddots & a_{mn} \end{bmatrix}, \quad \boldsymbol{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}, \quad \boldsymbol{b} = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}.$$

Solving a simple example from Sallan, Lordan, and Fernandez Page 13-14 in R

As for n non-negativity constraints, we can rewrite

$$x_i \quad \geq \quad 0, \;\; \text{for } i = 1, \ldots, n.$$

as

$$\boldsymbol{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \geq \mathbf{0}$$

where

$$\boldsymbol{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}, \qquad \mathbf{0} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}.$$

Solving a simple example from Sallan, Lordan, and Fernandez Page 13-14 in R

In other words, we can express the production problem in matrix form as

$$\max z = \boldsymbol{c}^T \boldsymbol{x}$$

subject to

$$\begin{aligned} \boldsymbol{A}\boldsymbol{x} &\leq \boldsymbol{b}, \\ \boldsymbol{x} &\geq \boldsymbol{0}. \end{aligned}$$

Solving a simple example from Sallan, Lordan, and Fernandez Page 13-14 in R

Using the same matrix syntax, we can write the standard form of minimum of a linear program as:

$$\min z = \boldsymbol{c}^T \boldsymbol{x}$$

subject to

$$\begin{aligned} \boldsymbol{A}\boldsymbol{x} &\geq \boldsymbol{b}, \\ \boldsymbol{x} &\geq \boldsymbol{0}. \end{aligned}$$

Solving a simple example from Sallan, Lordan, and Fernandez Page 13-14 in R

Note that non-equality constraints can be transformed by changing the signs of all terms of the constraint:

$$a_{i1}x_1 + \cdots + a_{in}x_n \geq b_i$$

is equal to

$$-a_{i1}x_1 - \cdots - a_{in}x_n \leq -b_i.$$

Solving a simple example from Sallan, Lordan, and Fernandez Page 13-14 in R

Remember that the problem at hand is summarized as:

$$\max z = 20 \cdot P_1 + 60 \cdot P_2,$$

such that

$$\begin{aligned} 30 \cdot P_1 + 20 \cdot P_2 &\leq 2,700, \\ 5 \cdot P_1 + 10 \cdot P_2 &\leq 850, \\ P_1 + P_2 &\geq 95, \\ P_1 &\geq 0, \\ P_2 &\geq 0. \end{aligned}$$

Solving a simple example from Sallan, Lordan, and Fernandez Page 13-14 in R

We can rewrite this problem somewhat as:

$$\max z = 20 \cdot P_1 + 60 \cdot P_2,$$

such that

$$\begin{aligned} 30 \cdot P_1 + 20 \cdot P_2 &\leq 2,700, \\ 5 \cdot P_1 + 10 \cdot P_2 &\leq 850, \\ -P_1 - P_2 &\leq -95, \\ P_1 &\geq 0, \\ P_2 &\geq 0. \end{aligned}$$

Solving a simple example from Sallan, Lordan, and Fernandez Page 13-14 in R

Let us write this problem in standard form.

Solving a simple example from Sallan, Lordan, and Fernandez Page 13-14 in R

As for the objective function,

$$\max z \quad = \quad 20 \cdot P_1 + 60 \cdot P_2,$$

we can rewrite as

$$\max z = \boldsymbol{c}^T \boldsymbol{x}$$

where

$$\boldsymbol{c} = \begin{bmatrix} 20 \\ 60 \end{bmatrix}, \qquad \boldsymbol{x} = \begin{bmatrix} P_1 \\ P_2 \end{bmatrix}.$$

Solving a simple example from Sallan, Lordan, and Fernandez Page 13-14 in R

As for the $m = 3$ constraints,

$$\begin{aligned} 30 \cdot P_1 + 20 \cdot P_2 &\leq 2,700, \\ 5 \cdot P_1 + 10 \cdot P_2 &\leq 850, \\ P_1 + P_2 &\geq 95, \end{aligned}$$

we have

$$\boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b},$$

where

$$\boldsymbol{A} = \begin{bmatrix} 30 & 20 \\ 5 & 10 \\ -1 & -1 \end{bmatrix}, \quad \boldsymbol{x} = \begin{bmatrix} P_1 \\ P_2 \end{bmatrix}, \quad \boldsymbol{b} = \begin{bmatrix} 2700 \\ 850 \\ -95 \end{bmatrix}.$$

Solving a simple example from Sallan, Lordan, and Fernandez Page 13-14 in R

There are several solvers available for solving linear programming models.

Solving a simple example from Sallan, Lordan, and Fernandez Page 13-14 in R

A list can be found in http://bit.ly/1zkJpVw

Solving a simple example from Sallan, Lordan, and Fernandez Page 13-14 in R

The following packages can be of interest for R users:

- **lp_solve** is implemented through the **lpSolve** and **lpSolveAPI** packages
- GLPK is implemented through the **Rglpk** package
- SYMPHONY is implemented through **Rsymphony**

Solving a simple example from Sallan, Lordan, and Fernandez Page 13-14 in R

All solver are implemented as R functions, and parameters can be passed to these functions as R matrices and vectors.

Solving a simple example from Sallan, Lordan, and Fernandez Page 13-14 in R

Most R packages solving LP implement solvers as functions, whose input variables are:

- A character variable indicating if we have a maximization or minimization problem,
- Vectors with cost coefficients $\boldsymbol{c}$,
- A set of constraints with a matrix $\boldsymbol{A}$ and right hand side values $\boldsymbol{b}$,
- A character vector with the constraint signs, either "$\leq$"or "$\geq$".

Solving a simple example from Sallan, Lordan, and Fernandez Page 13-14 in R

Let us start solving this problem with **lpSolve** package.

Solving a simple example from Sallan, Lordan, and Fernandez Page 13-14 in R

The default behavior of R is to try to run in the language you run Windows in.

The following instruction is to run R in a language environment other than your run Windows in. I assume that many of you may not need to know.

Solving a simple example from Sallan, Lordan, and Fernandez Page 13-14 in R

1. Open command prompt from Windows System Tool. In many cases, your directory pointer is pointing towards

   `c:/Users/Your User Name>`

2. Change the directory to

   `c:/Program Files/R/R-3.3.2/bin/x64>`

3. Run Rgui using language setting as English:

   `c:/Program Files/R/R-3.3.2/bin/x64>Rgui language=en`

Solving a simple example from Sallan, Lordan, and Fernandez Page 13-14 in R

First we need to download and install **lpSolve** package from CRAN repositories. Character vector of the name of package whose current versions should be downloaded from the repositories must be specified as an argument in

```
> install.packages("lpSolve")
```

Solving a simple example from Sallan, Lordan, and Fernandez Page 13-14 in R

Then in your screen, you get the following messages.

```
Installing package into ?eC:/Users/Your User Name/
Documents/R/win-library/3.3?f (as ?elib?f is unspecif
```

Solving a simple example from Sallan, Lordan, and Fernandez Page 13-14 in R

Then you are asked to select a CRAN mirror for use in this session by

```
--- Please select a CRAN mirror for use in this
session ---
```

Solving a simple example from Sallan, Lordan, and Fernandez Page 13-14 in R

After three lines of messages regarding where it is downloaded and what is downloaded along with its size, you get the following messages.

```
package ?elpSolve?f successfully unpacked and MD5 sum
check
```

Solving a simple example from Sallan, Lordan, and Fernandez Page 13-14 in R

followed by

```
The downloaded binary packages are in
C:/Users/Your User Name/AppData/Local/Temp/
Rtmp6RQdMt/downloaded_packages
```

Solving a simple example from Sallan, Lordan, and Fernandez Page 13-14 in R

Messages you get after executing `install.packages` tell you which directory (called a library) the package files are installed in, and it tells you whether the package was installed successfully.

Solving a simple example from Sallan, Lordan, and Fernandez Page 13-14 in R

If you install many packages, you end up with a collection of many packages.

If R loaded all of them at the beginning of each session, that would take a lot of memory and time.

Solving a simple example from Sallan, Lordan, and Fernandez Page 13-14 in R

In general, before you can use a package, you have to load it into R memory by using the library() function.

Solving a simple example from Sallan, Lordan, and Fernandez Page 13-14 in R

So that is what we do here as well by

```
> library(lpSolve)
```

Solving a simple example from Sallan, Lordan, and Fernandez Page 13-14 in R

Usage of `lp` can be found in many places such as https://cran.r-project.org/web/packages/lpSolve/lpSolve.pdf

Solving a simple example from Sallan, Lordan, and Fernandez Page 13-14 in R

`lp` has the following arguments:

```
lp (direction="min", objective.in, const.mat,
const.dir, const.rhs, transpose.constraints=TRUE,
compute.sens=0, int.vec, presolve=0, binary.vec,
all.int=FALSE, all.bin=FALSE, scale = 196,
dense.const, num.bin.solns=1, use.rw=FALSE)
```

Solving a simple example from Sallan, Lordan, and Fernandez Page 13-14 in R

Of which `direction`, `objective.in` which is our $\boldsymbol{c}$, `const.mat` which is our $\boldsymbol{A}$, `const.rhs` which is our $\boldsymbol{b}$ and a character vector with the constraint signs `const.dir` are absolutely necessary.

Solving a simple example from Sallan, Lordan, and Fernandez Page 13-14 in R

Since our problem requires `direction` to be `''max''`, we need to define the remaining parameters $\boldsymbol{c}$, $\boldsymbol{A}$, and $\boldsymbol{b}$ as well as a character vector with the constraint signs `const.dir`.

Solving a simple example from Sallan, Lordan, and Fernandez Page 13-14 in R

Let us start with $\boldsymbol{c}$ as

```
> coefficients.of.obj.fun.c<-c(20,60)
> coefficients.of.obj.fun.c
[1] 20 60
```

Solving a simple example from Sallan, Lordan, and Fernandez Page 13-14 in R

Then we define $\boldsymbol{A}$ as

```
> constraint.coefficients.matrix.A<-
+matrix(c(30,20,5,10,-1,-1),ncol=2, byrow=TRUE)
> constraint.coefficients.matrix.A
[,1] [,2]
[1,] 30 20
[2,] 5 10
[3,] -1 -1
```

Solving a simple example from Sallan, Lordan, and Fernandez Page 13-14 in R

Next we define $\boldsymbol{b}$ as

```
> constraint.right.hand.side.values.b<-
+c(2700,850,-95)
> constraint.right.hand.side.values.b
[1] 2700 850 -95
```

Solving a simple example from Sallan, Lordan, and Fernandez Page 13-14 in R

Finally a character vector with the constraint signs as

```
> constraint.direction<-c("<=","<=","<=")
> constraint.direction
[1] "<=" "<=" "<="
```

Solving a simple example from Sallan, Lordan, and Fernandez Page 13-14 in R

We are ready to solve the problem of subsection 2.2.2 as

```
> solution.for.problem.2.2.2<-
+lp("max",coefficients.of.obj.fun.c,
+ constraint.coefficients.matrix.A,
+ constraint.direction,
+ constraint.right.hand.side.values.b)
```

Solving a simple example from Sallan, Lordan, and Fernandez Page 13-14 in R

The optimal value of the objective function can be obtained by

```
> solution.for.problem.2.2.2
Success: the objective function is 4900
```

Solving a simple example from Sallan, Lordan, and Fernandez Page 13-14 in R

The optimal decision variable values can be obtained by

```
> solution.for.problem.2.2.2$solution
[1] 20 75
```

Solving a simple example from Sallan, Lordan, and Fernandez Page 13-14 in R

Remember that the original problem at hand is summarized as:

$$\max z = 20 \cdot P_1 + 60 \cdot P_2,$$

such that

$$\begin{aligned} 30 \cdot P_1 + 20 \cdot P_2 &\leq 2,700, \\ 5 \cdot P_1 + 10 \cdot P_2 &\leq 850, \\ P_1 + P_2 &\geq 95, \\ P_1 &\geq 0, \\ P_2 &\geq 0. \end{aligned}$$

Short title

Instructor: Yuichiro Kanazawa, Ph.D.

Course?@Overvi

Visualization

The Simplex Method

Duality

Consulting

Solving a simple example from Sallan, Lordan, and Fernandez Page 13-14 in R

If we were to use the original problem alternatively, we can define $\boldsymbol{A}$ as

```
> original.constraint.coefficients.matrix.A<-
+matrix(c(30,20,5,10,1,1),ncol=2, byrow=TRUE)
> original.constraint.coefficients.matrix.A
[,1] [,2]
[1,] 30 20
[2,] 5 10
[3,] 1 1
```

Solving a simple example from Sallan, Lordan, and Fernandez Page 13-14 in R

Then we define $\boldsymbol{b}$ as

```
> original.constraint.right.hand.side.values.b<-
+c(2700,850,95)
> original.constraint.right.hand.side.values.b
[1] 2700 850 95
```

Solving a simple example from Sallan, Lordan, and Fernandez Page 13-14 in R

Finally a character vector with the constraint signs as

```
> original.constraint.direction<-c("<=","<=",">=")
> original.constraint.direction
[1] "<=" "<=" ">="
```

Solving a simple example from Sallan, Lordan, and Fernandez Page 13-14 in R

We are now ready to solve the problem of subsection 2.2.2 alternatively as

```
> alternative.solution.for.problem.2.2.2<-

+lp("max",coefficients.of.obj.fun.c,

+ original.constraint.coefficients.matrix.A,

+ original.constraint.direction,

+ original.constraint.right.hand.side.values.b)
```

Solving a simple example from Sallan, Lordan, and Fernandez Page 13-14 in R

The optimal value of the objective function can be obtained by

```
> alternative.solution.for.problem.2.2.2
Success: the objective function is 4900
```

Solving a simple example from Sallan, Lordan, and Fernandez Page 13-14 in R

The optimal decision variable values can be obtained by

```
> alternative.solution.for.problem.2.2.2$solution
[1] 20 75
```

Short title

Instructor: Yuichiro Kanazawa, Ph.D.

Course?@Overvi

Visualization

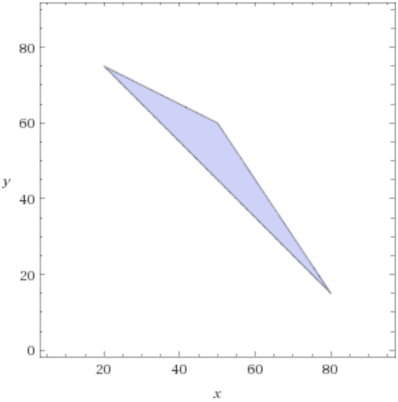
The Simplex Method

Duality

Consulting

Solving a simple example from Sallan, Lordan, and Fernandez Page 13-14 in R

It turns out the three constraints form the following simplex.



Solving a simple example from Sallan, Lordan, and Fernandez Page 13-14 in R

This can be produced by googling wolfram alpha widget inequality plotter for linear programming or directly http-ing

http://www.wolframalpha.com/widgets/view.jsp?id=7fa77b668578a893653c674b2be3865c.

Solving a simple example from Sallan, Lordan, and Fernandez Page 13-14 in R

Once you are there, you type in

```
30x+20y <= 2700
5x+10y <=850
x+y >=95
x>=0
y>=0
```

in appropriate boxes and click submit.

Portfolio Selection from Bradley, Hax and Magnanti Page 8-10 in R

A portfolio manager in charge of a bank portfolio has $10 million to invest.

The securities available for purchase, as well as their respective quality ratings, maturities, and yields, are shown in Table 110.

Bond name	*Bond type*	*Quality scales Moody's*	*Quality scales Bank's*	*Years to maturity*	*Yield to maturity*	*After-tax yield*
A	Municipal	Aa	2	9	4.3	4.3
B	Agency	Aa	2	15	5.4	2.7
C	Government	Aaa	1	4	5.0	2.5
D	Government	Aaa	1	3	4.4	2.2
E	Municipal	Ba	5	2	4.5	4.5

Table: The securities available for purchase

Portfolio Selection from Bradley, Hax and Magnanti Page 8-10 in R

The bank places the following policy limitations on the portfolio manager's actions:

1. A portfolio manager in charge of a bank portfolio has \$10 million to invest.

2. Government and agency bonds must total at least \$4 million.

3. The average quality of the portfolio cannot exceed 1.4 on the bank's quality scale. (Note that a low number on this scale means a high-quality bond.)

4. The average years to maturity of the portfolio must not exceed 5 years.

Portfolio Selection from Bradley, Hax and Magnanti Page 8-10 in R

Problem 1. Assuming that the objective of the portfolio manager is to maximize after-tax earnings and that the tax rate is 50 percent, what bonds should he purchase?

Portfolio Selection from Bradley, Hax and Magnanti Page 8-10 in R

Problem 2. If it became possible to borrow up to \$1 million at 5.5 percent before taxes, how should his selection be changed?

Portfolio Selection from Bradley, Hax and Magnanti Page 8-10 in R

The decision variables[12] for this problem are simply the dollar amount of each security to be purchased:

$$
\begin{aligned}
x_A &= \text{Amount to be invested in bond A; in millions of dollars,}\\
x_B &= \text{Amount to be invested in bond B; in millions of dollars,}\\
x_C &= \text{Amount to be invested in bond C; in millions of dollars,}\\
x_D &= \text{Amount to be invested in bond D; in millions of dollars,}\\
x_E &= \text{Amount to be invested in bond E; in millions of dollars.}
\end{aligned}
$$

[12] The elements that are under the control of the decision maker.

Portfolio Selection problem 1 from Bradley, Hax and Magnanti Page 8-10 in R

We must now determine the form of the objective function[13].

[13] A function of the decision variables and is to be maximized or minimized depending on the problem.

Portfolio Selection problem 1 from Bradley, Hax and Magnanti Page 8-10 in R

Assuming that all securities are purchased at par (face value) and held to maturity and that the income on municipal bonds is tax-exempt, the after-tax earnings are given by:

$$z = 0.043 \cdot x_A + 0.027 \cdot x_B + 0.025 \cdot x_C + 0.022 \cdot x_D + 0.045 \cdot x_E.$$

Portfolio Selection problem 1 from Bradley, Hax and Magnanti Page 8-10 in R

Now let us consider each of the constraints[14] of the problem.

[14] Restrictions on the decision variables

Portfolio Selection from Bradley, Hax and Magnanti Page 8-10 in R

0. Although the following is not explicitly stated, we ought to assume that:

$$\begin{aligned} x_A &\geq 0, \\ x_B &\geq 0, \\ x_C &\geq 0, \\ x_D &\geq 0, \\ x_E &\geq 0. \end{aligned}$$

Portfolio Selection problem 1 from Bradley, Hax and Magnanti Page 8-10 in R

1. The portfolio manager has only a total of ten million dollars to invest, and therefore:

$$x_A + x_B + x_C + x_D + x_E \leq 10.$$

Portfolio Selection problem 1 from Bradley, Hax and Magnanti Page 8-10 in R

2. Further, of this amount at least \$4 million must be invested in government and agency bonds. Hence,

$$x_B + x_C + x_D \geq 4.$$

Portfolio Selection problem 1 from Bradley, Hax and Magnanti Page 8-10 in R

3. The average quality of the portfolio, which is given by the ratio of the total quality to the total value of the portfolio, must not exceed 1.4:

$$\frac{2 \cdot x_A + 2 \cdot x_B + x_C + x_D + 5 \cdot x_E}{x_A + x_B + x_C + x_D + x_E} \leq 1.4.$$

Portfolio Selection problem 1 from Bradley, Hax and Magnanti Page 8-10 in R

Note that the inequality is less-than-or-equal-to, since a low number on the bank's quality scale means a high-quality bond.

Portfolio Selection problem 1 from Bradley, Hax and Magnanti Page 8-10 in R

By clearing the denominator and re-arranging terms, we find that this inequality is clearly equivalent to the linear constraint:

$$0.6 \cdot x_A + 0.6 \cdot x_B - 0.4 \cdot x_C - 0.4 \cdot x_D + 3.6 \cdot x_E \leq 0.$$

Portfolio Selection problem 1 from Bradley, Hax and Magnanti Page 8-10 in R

4. The constraint on the average maturity of the portfolio is a similar ratio. The average maturity must not exceed five years:

$$\frac{9 \cdot x_A + 15 \cdot x_B + 4 \cdot x_C + 3 \cdot x_D + 2 \cdot x_E}{x_A + x_B + x_C + x_D + x_E} \leq 5,$$

Portfolio Selection problem 1 from Bradley, Hax and Magnanti Page 8-10 in R

which is equivalent to the linear constraint:

$$4 \cdot x_A + 10 \cdot x_B - x_C - 2 \cdot x_D - 3 \cdot x_E \leq 0.$$

Portfolio Selection problem 1 from Bradley, Hax and Magnanti Page 8-10 in R

Note that the two ratio constraints are, in fact, *nonlinear* constraints, which would require sophisticated computational procedures if included in this form.

Portfolio Selection problem 1 from Bradley, Hax and Magnanti Page 8-10 in R

However, simply multiplying both sides of each ratio constraint by its denominator (which must be nonnegative since it is the sum of nonnegative variables) transforms this nonlinear constraint into a simple linear constraint.

Portfolio Selection problem 1 from Bradley, Hax and Magnanti Page 8-10 in R

We can summarize our formulation as follows:

$$\begin{aligned} \max z &= 0.043 \cdot x_A + 0.027 \cdot x_B + 0.025 \cdot x_C \\ &\quad +0.022 \cdot x_D + 0.045 \cdot x_E, \end{aligned}$$

such that

$$\begin{aligned} x_A + x_B + x_C + x_D + x_E &\leq 10, \\ x_B + x_C + x_D &\geq 4, \\ 0.6 \cdot x_A + 0.6 \cdot x_B - 0.4 \cdot x_C - 0.4 \cdot x_D + 3.6 \cdot x_E &\leq 0, \\ 4 \cdot x_A + 10 \cdot x_B - x_C - 2 \cdot x_D - 3 \cdot x_E &\leq 0, \\ x_A \geq 0, \quad x_B \geq 0, \quad x_C \geq 0, \quad x_D \geq 0, \quad x_E &\geq 0. \end{aligned}$$

Portfolio Selection problem 1 from Bradley, Hax and Magnanti Page 8-10 in R

Or as follows:

$$\begin{aligned}\max z &= 0.043 \cdot x_A + 0.027 \cdot x_B + 0.025 \cdot x_C \\ &\quad +0.022 \cdot x_D + 0.045 \cdot x_E,\end{aligned}$$

such that

$$\begin{aligned} x_A + x_B + x_C + x_D + x_E &\leq 10, \\ -x_B - x_C - x_D &\leq -4, \\ 0.6 \cdot x_A + 0.6 \cdot x_B - 0.4 \cdot x_C - 0.4 \cdot x_D + 3.6 \cdot x_E &\leq 0, \\ 4 \cdot x_A + 10 \cdot x_B - x_C - 2 \cdot x_D - 3 \cdot x_E &\leq 0, \\ x_A \geq 0, \quad x_B \geq 0, \quad x_C \geq 0, \quad x_D \geq 0, \quad x_E &\geq 0. \end{aligned}$$

Solving a Portfolio Selection problem 1 from Bradley, Hax and Magnanti Page 8-10 in R

Let us write this problem in standard form.

Solving a Portfolio Selection problem 1 from Bradley, Hax and Magnanti Page 8-10 in R

As for the objective function,

$$\begin{aligned}\max z &= 0.043 \cdot x_A + 0.027 \cdot x_B + 0.025 \cdot x_C \\ &\quad +0.022 \cdot x_D + 0.045 \cdot x_E,\end{aligned}$$

we can rewrite as

$$\max z = \boldsymbol{c}^T \boldsymbol{x}$$

where

$$\boldsymbol{c} = \begin{bmatrix} 0.043 \\ 0.027 \\ 0.025 \\ 0.022 \\ 0.045 \end{bmatrix}, \qquad \boldsymbol{x} = \begin{bmatrix} x_A \\ x_B \\ x_C \\ x_D \\ x_E \end{bmatrix}.$$

Solving a Portfolio Selection problem 1 from Bradley, Hax and Magnanti Page 8-10 in R

As for the $m = 4$ constraints,

$$\begin{array}{rcl} x_A + x_B + x_C + x_D + x_E & \leq & 10, \\ -x_B - x_C - x_D & \leq & -4, \\ 0.6 \cdot x_A + 0.6 \cdot x_B - 0.4 \cdot x_C - 0.4 \cdot x_D + 3.6 \cdot x_E & \leq & 0, \\ 4 \cdot x_A + 10 \cdot x_B - x_C - 2 \cdot x_D - 3 \cdot x_E & \leq & 0, \end{array}$$

we have

$$\boldsymbol{A}\boldsymbol{x} \quad \leq \quad \boldsymbol{b},$$

where

$$\boldsymbol{A} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & -1 & -1 & -1 & 0 \\ 0.6 & 0.6 & -0.4 & -0.4 & 3.6 \\ 4 & 10 & -1 & -2 & -3 \end{bmatrix}, \boldsymbol{x} = \begin{bmatrix} x_A \\ x_B \\ x_C \\ x_D \\ x_E \end{bmatrix}, \boldsymbol{b} = \begin{bmatrix} 10 \\ -4 \\ 0 \\ 0 \end{bmatrix}.$$

Solving a Portfolio Selection problem 1 from Bradley, Hax and Magnanti Page 8-10 in R

Since our problem requires `direction` to be `‘‘max’’`, we need to define the remaining parameters $\boldsymbol{c}$, $\boldsymbol{A}$, and $\boldsymbol{b}$ as well as a character vector with the constraint signs `const.dir`.

Solving a Portfolio Selection problem 1 from Bradley, Hax and Magnanti Page 8-10 in R

Let us start with c as

```
> portfolio.obj.fun.c<-c(0.043,0.027,0.025,
+0.022,0.045)
> portfolio.obj.fun.c
[1] 0.043 0.027 0.025 0.022 0.045
```

Solving a Portfolio Selection problem 1 from Bradley, Hax and Magnanti Page 8-10 in R

Then we define $\boldsymbol{A}$ as

```
> portfolio.const.mat.A<-
+matrix(c(1,1,1,1,1,0,-1,-1,-1,0,
+0.6,0.6,-0.4,-0.4,3.6,4,10,-1,-2,-3),
+ncol=5, byrow=TRUE)
```

Solving a Portfolio Selection problem 1 from Bradley, Hax and Magnanti Page 8-10 in R

```
> portfolio.const.mat.A
[,1] [,2] [,3] [,4] [,5]
[1,] 1.0 1.0 1.0 1.0 1.0
[2,] 0.0 -1.0 -1.0 -1.0 0.0
[3,] 0.6 0.6 -0.4 -0.4 3.6
[4,] 4.0 10.0 -1.0 -2.0 -3.0
```

Solving a Portfolio Selection problem 1 from Bradley, Hax and Magnanti Page 8-10 in R

Next we define $\boldsymbol{b}$ as

```
> portfolio.const.rhs.b<-c(10,-4,0,0)

> portfolio.const.rhs.b

[1] 10 -4 0 0
```

Solving a Portfolio Selection problem 1 from Bradley, Hax and Magnanti Page 8-10 in R

Finally a character vector with the constraint signs as

```
> portfolio.const.dir<-c("<=","<=","<=","<=")
> portfolio.const.dir
[1] "<=" "<=" "<=" "<="
```

Solving a Portfolio Selection problem 1 from Bradley, Hax and Magnanti Page 8-10 in R

We are ready to solve the portfolio selection problem as

```
> portfolio.solution<-
+lp("max",portfolio.obj.fun.c,
+ portfolio.const.mat.A,
+ portfolio.const.dir,
+ portfolio.const.rhs.b)
```

Solving a Portfolio Selection problem 1 from Bradley, Hax and Magnanti Page 8-10 in R

The optimal value of the objective function can be obtained by

```
> portfolio.solution
Success: the objective function is 0.2983636
```

Solving a Portfolio Selection problem 1 from Bradley, Hax and Magnanti Page 8-10 in R

The optimal decision variable values can be obtained by

```
> portfolio.solution$solution

[1] 2.1818182 0.0000000 7.3636364 0.0000000 0.4545455
```

Solving a Portfolio Selection problem 1 from Bradley, Hax and Magnanti Page 8-10 in R

Alternatively, we can define $\boldsymbol{A}$ as

```
> original.portfolio.const.mat.A<-
+matrix(c(1,1,1,1,1,0,1,1,1,0,
+0.6,0.6,-0.4,-0.4,3.6,4,10,-1,-2,-3),
+ncol=5, byrow=TRUE)
> original.portfolio.const.mat.A
[,1] [,2] [,3] [,4] [,5]
[1,] 1.0 1.0 1.0 1.0 1.0
[2,] 0.0 1.0 1.0 1.0 0.0
[3,] 0.6 0.6 -0.4 -0.4 3.6
[4,] 4.0 10.0 -1.0 -2.0 -3.0
```

Solving a Portfolio Selection problem 1 from Bradley, Hax and Magnanti Page 8-10 in R

Then we define $\boldsymbol{b}$ as

```
> original.portfolio.const.rhs.b<-
+c(10,4,0,0)
> original.portfolio.const.rhs.b
[1] 10 4 0 0
```

Solving a Portfolio Selection problem 1 from Bradley, Hax and Magnanti Page 8-10 in R

Finally a character vector with the constraint signs as

```
> original.portfolio.const.dir<-
+c("<=",">=","<=","<=")
> original.portfolio.const.dir
[1] "<=" ">=" "<=" "<="
```

Solving a Portfolio Selection problem 1 from Bradley, Hax and Magnanti Page 8-10 in R

We are ready to solve the portfolio selection problem as

```
> original.portfolio.solution<-
+lp("max",portfolio.obj.fun.c,
+ original.portfolio.const.mat.A,
+ original.portfolio.const.dir,
+ original.portfolio.const.rhs.b)
```

Solving a Portfolio Selection problem 1 from Bradley, Hax and Magnanti Page 8-10 in R

The optimal value of the objective function can be obtained by

```
> original.portfolio.solution
Success: the objective function is 0.2983636
```

Solving a Portfolio Selection problem 1 from Bradley, Hax and Magnanti Page 8-10 in R

The optimal decision variable values can be obtained by

```
> original.portfolio.solution$solution
[1] 2.1818182 0.0000000 7.3636364 0.0000000 0.4545455
```

Portfolio Selection

Either way, we obtain the maximal after-tax earnings with the tax rate of 50 percent is 0.2983636, with 2.1818182 million dollars of Bond A, 7.3636364 million dollars of Bond C and 0.4545455 million dollars of Bond E.

In the book, the optimal after-tax earnings would be 0.294, with 3.36 million dollars of Bond A, 6.48 million dollars of Bond D and 0.16 million dollars of Bond E.

Portfolio Selection problem 1 from Bradley, Hax and Magnanti Page 8-10 in R

Which is correct?

Portfolio Selection problem 2 from Bradley, Hax and Magnanti Page 8-10 in R

Now consider the additional possibility of being able to borrow up to \$1 million at 5.5 percent before taxes.

Essentially, we can increase our cash supply above ten million by borrowing at an after-tax rate of 2.75 percent.

Portfolio Selection problem 2 from Bradley, Hax and Magnanti Page 8-10 in R

We can define a new decision variable as follows:

$$y \quad = \quad \text{amount borrowed in millions of dollars.}$$

Portfolio Selection problem 2 from Bradley, Hax and Magnanti Page 8-10 in R

There is an upper bound on the amount of funds that can be borrowed, and hence

$$y \leq 1.$$

Portfolio Selection problem 2 from Bradley, Hax and Magnanti Page 8-10 in R

The cash constraint is then modified to reflect that the total amount purchased must be less than or equal to the cash that can be made available including borrowing:

$$x_A + x_B + x_C + x_D + x_E \leq 10 + y.$$

or

$$x_A + x_B + x_C + x_D + x_E - y \leq 10.$$

Portfolio Selection problem 2 from Bradley, Hax and Magnanti Page 8-10 in R

Now, since the borrowed money costs 2.75 percent after taxes, the new after-tax earnings are:

$$\begin{aligned} z &= 0.043x_A + 0.027x_B + 0.025x_C \\ &\qquad +0.022x_D + 0.045x_E - 0.0275y. \end{aligned}$$

Portfolio Selection problem 2 from Bradley, Hax and Magnanti Page 8-10 in R

We can summarize our formulation as follows:

$$\begin{aligned} \max z &= 0.043 \cdot x_A + 0.027 \cdot x_B + 0.025 \cdot x_C \\ &\quad +0.022 \cdot x_D + 0.045 \cdot x_E - 0.0275y, \end{aligned}$$

such that

$$\begin{aligned} x_A + x_B + x_C + x_D + x_E - y &\leq 10, \\ y &\leq 1, \\ x_B + x_C + x_D &\geq 4, \\ 0.6 \cdot x_A + 0.6 \cdot x_B - 0.4 \cdot x_C - 0.4 \cdot x_D + 3.6 \cdot x_E &\leq 0, \\ 4 \cdot x_A + 10 \cdot x_B - x_C - 2 \cdot x_D - 3 \cdot x_E &\leq 0, \\ x_A \geq 0, \quad x_B \geq 0, \quad x_C \geq 0, \quad x_D \geq 0, \quad x_E &\geq 0. \end{aligned}$$

Portfolio Selection problem 2 from Bradley, Hax and Magnanti Page 8-10 in R

Or as follows:

$$\begin{aligned}\max z &= 0.043 \cdot x_A + 0.027 \cdot x_B + 0.025 \cdot x_C \\ &\qquad +0.022 \cdot x_D + 0.045 \cdot x_E - 0.0275y,\end{aligned}$$

such that

$$\begin{aligned} x_A + x_B + x_C + x_D + x_E - y &\leq 10, \\ y &\leq 1, \\ -x_B - x_C - x_D &\leq -4, \\ 0.6 \cdot x_A + 0.6 \cdot x_B - 0.4 \cdot x_C - 0.4 \cdot x_D + 3.6 \cdot x_E &\leq 0, \\ 4 \cdot x_A + 10 \cdot x_B - x_C - 2 \cdot x_D - 3 \cdot x_E &\leq 0, \\ x_A \geq 0, \quad x_B \geq 0, \quad x_C \geq 0, \quad x_D \geq 0, \quad x_E &\geq 0. \end{aligned}$$

Solving a Portfolio Selection problem 2 from Bradley, Hax and Magnanti Page 8-10 in R

Let us write this problem in standard form.

Solving a Portfolio Selection problem 2 from Bradley, Hax and Magnanti Page 8-10 in R

As for the objective function,

$$\begin{aligned}\max z &= 0.043 \cdot x_A + 0.027 \cdot x_B + 0.025 \cdot x_C \\ &\qquad +0.022 \cdot x_D + 0.045 \cdot x_E - 0.0275y,\end{aligned}$$

we can rewrite as

$$\max z = \boldsymbol{c}^T \boldsymbol{x}$$

where

$$\boldsymbol{c} = \begin{bmatrix} 0.043 \\ 0.027 \\ 0.025 \\ 0.022 \\ 0.045 \\ -0.0275 \end{bmatrix}, \qquad \boldsymbol{x} = \begin{bmatrix} x_A \\ x_B \\ x_C \\ x_D \\ x_E \\ y \end{bmatrix}.$$

Solving a Portfolio Selection problem 2 from Bradley, Hax and Magnanti Page 8-10 in R

As for the $m = 5$ constraints,

$$\begin{aligned} x_A + x_B + x_C + x_D + x_E &\leq 10, \\ y &\leq 1, \\ -x_B - x_C - x_D &\leq -4, \\ 0.6 \cdot x_A + 0.6 \cdot x_B - 0.4 \cdot x_C - 0.4 \cdot x_D + 3.6 \cdot x_E &\leq 0, \\ 4 \cdot x_A + 10 \cdot x_B - x_C - 2 \cdot x_D - 3 \cdot x_E &\leq 0, \end{aligned}$$

we have

$$\boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b},$$

where

$$\boldsymbol{A} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & -1 & -1 & -1 & 0 & 0 \\ 0.6 & 0.6 & -0.4 & -0.4 & 3.6 & 0 \\ 4 & 10 & -1 & -2 & -3 & 0 \end{bmatrix}, \boldsymbol{x} = \begin{bmatrix} x_A \\ x_B \\ x_C \\ x_D \\ x_E \\ y \end{bmatrix}, \boldsymbol{b} = \begin{bmatrix} 10 \\ 1 \\ -4 \\ 0 \\ 0 \end{bmatrix}.$$

Solving a Portfolio Selection problem 1 from Bradley, Hax and Magnanti Page 8-10 in R

Since our problem requires `direction` to be `''max''`, we need to define the remaining parameters $\boldsymbol{c}$, $\boldsymbol{A}$, and $\boldsymbol{b}$ as well as a character vector with the constraint signs `const.dir`.

Solving a Portfolio Selection problem 2 from Bradley, Hax and Magnanti Page 8-10 in R

Let us start with $\boldsymbol{c}$ as

```
> portfolio2.obj.fun.c<-c(0.043,0.027,0.025,
+0.022,0.045,-0.0275)
> portfolio2.obj.fun.c
[1] 0.043 0.027 0.025 0.022 0.045 -0.0275
```

Solving a Portfolio Selection problem 2 from Bradley, Hax and Magnanti Page 8-10 in R

Then we define $\boldsymbol{A}$ as

```
> portfolio2.const.mat.A<-
+matrix(c(1,1,1,1,1,-1,0,0,0,0,0,1,0,-1,-1,-1,0,0,
+0.6,0.6,-0.4,-0.4,3.6,0,4,10,-1,-2,-3,0),
+ncol=6, byrow=TRUE)
```

Solving a Portfolio Selection problem 2 from Bradley, Hax and Magnanti Page 8-10 in R

```
> portfolio.const.mat.A
[,1] [,2] [,3] [,4] [,5] [,6]
[1,] 1.0 1.0 1.0 1.0 1.0 -1
[2,] 0.0 0.0 0.0 0.0 0.0 1
[3,] 0.0 -1.0 -1.0 -1.0 0.0 0
[4,] 0.6 0.6 -0.4 -0.4 3.6 0
[5,] 4.0 10.0 -1.0 -2.0 -3.0 0
```

Solving a Portfolio Selection problem 2 from Bradley, Hax and Magnanti Page 8-10 in R

Next we define $\boldsymbol{b}$ as

```
> portfolio2.const.rhs.b<-c(10,1,-4,0,0)

> portfolio2.const.rhs.b

[1] 10 1 -4 0 0
```

Solving a Portfolio Selection problem 2 from Bradley, Hax and Magnanti Page 8-10 in R

Finally a character vector with the constraint signs as

```
> portfolio2.const.dir<-c("<=","<=","<=","<=","<=")

> portfolio2.const.dir

[1] "<=" "<=" "<=" "<=" "<="
```

Solving a Portfolio Selection problem 2 from Bradley, Hax and Magnanti Page 8-10 in R

We are ready to solve the portfolio selection problem as

```
> portfolio2.solution<-
+lp("max",portfolio2.obj.fun.c,
+ portfolio2.const.mat.A,
+ portfolio2.const.dir,
+ portfolio2.const.rhs.b)
```

Solving a Portfolio Selection problem 2 from Bradley, Hax and Magnanti Page 8-10 in R

The optimal value of the objective function can be obtained by

```
> portfolio2.solution
Success: the objective function is 0.3007
```

Solving a Portfolio Selection problem 2 from Bradley, Hax and Magnanti Page 8-10 in R

The optimal decision variable values can be obtained by

```
> portfolio2.solution$solution
[1] 2.4 0.0 8.1 0.0 0.5 1.0
```

Solving a Portfolio Selection problem 2 from Bradley, Hax and Magnanti Page 8-10 in R

Alternatively, we can define $\boldsymbol{A}$ as

```
> original.portfolio2.const.mat.A<-
+matrix(c(1,1,1,1,1,-1,0,0,0,0,0,1,0,1,1,1,0,0,
+0.6,0.6,-0.4,-0.4,3.6,0,4,10,-1,-2,-3,0),
+ncol=6, byrow=TRUE)
```

Solving a Portfolio Selection problem 2 from Bradley, Hax and Magnanti Page 8-10 in R

```
> original.portfolio2.const.mat.A
[,1] [,2] [,3] [,4] [,5] [,6]
[1,] 1.0 1.0 1.0 1.0 1.0 -1
[2,] 0.0 0.0 0.0 0.0 0.0 1
[3,] 0.0 1.0 1.0 1.0 0.0 0
[4,] 0.6 0.6 -0.4 -0.4 3.6 0
[5,] 4.0 10.0 -1.0 -2.0 -3.0 0
```

Solving a Portfolio Selection problem 2 from Bradley, Hax and Magnanti Page 8-10 in R

Then we define $\boldsymbol{b}$ as

```
> original.portfolio2.const.rhs.b<-
+c(10,1,4,0,0)
> original.portfolio2.const.rhs.b
[1] 10 1 4 0 0
```

Solving a Portfolio Selection problem 2 from Bradley, Hax and Magnanti Page 8-10 in R

Finally a character vector with the constraint signs as

```
> original.portfolio2.const.dir<-
+c("<=","<=",">=","<=","<=")
> original.portfolio2.const.dir
[1] "<=" "<=" ">=" "<=" "<="
```

Solving a Portfolio Selection problem 2 from Bradley, Hax and Magnanti Page 8-10 in R

We are ready to solve the portfolio selection problem as

```
> original.portfolio2.solution<-
+lp("max",portfolio2.obj.fun.c,
+ original.portfolio2.const.mat.A,
+ original.portfolio2.const.dir,
+ original.portfolio2.const.rhs.b)
```

Solving a Portfolio Selection problem 2 from Bradley, Hax and Magnanti Page 8-10 in R

The optimal value of the objective function can be obtained by

```
> original.portfolio2.solution
Success: the objective function is 0.3007
```

Solving a Portfolio Selection problem 2 from Bradley, Hax and Magnanti Page 8-10 in R

The optimal decision variable values can be obtained by

```
> original.portfolio2.solution$solution
[1] 2.4 0.0 8.1 0.0 0.5 1.0
```

Portfolio Selection

Either way, we obtain the maximal after-tax earnings with the tax rate of 50 percent is 0.3007, with 2.4 million dollars of Bond A, 8.1 million dollars of Bond C, 0.5 million dollars of Bond E, and 1 million dollars of borrowing.

In the book, the optimal after-tax earnings would be 0.296, with 3.7 million dollars of Bond A, 7.13 million dollars of Bond D, 0.18 million dollars of Bond E, and 1 million dollars of borrowing.

Portfolio Selection problem 1 from Bradley, Hax and Magnanti Page 8-10 in R

Which is correct?

Visualizing linear programming in 2 dimensions

Suppose we have the following linear programming problem

Maximize

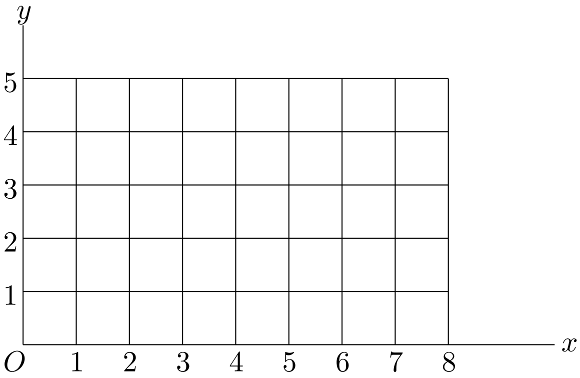
$$z = 3x + 5y$$

subject to

$$\begin{aligned} 2x + 3y &\leq 10, && (2.1) \\ x + 2y &\leq 6, && (2.2) \\ x + y &\leq 5, && (2.3) \\ x &\leq 4, && (2.4) \\ y &\leq 3, && (2.5) \\ x, y &\geq 0. && (2.6) \end{aligned}$$

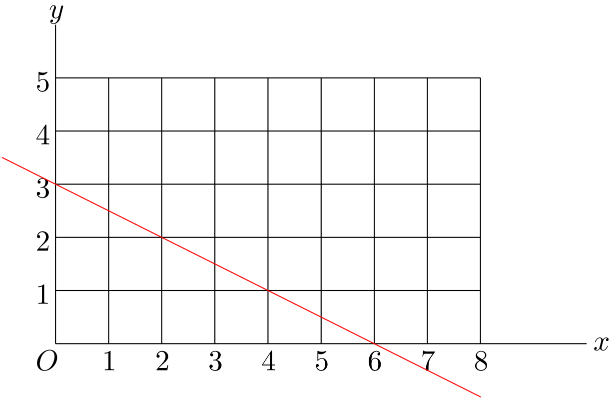
Visualizing linear programming in 2 dimensions

Let us use the following coordinated grid to visualize the previous linear programming problem.



Visualizing linear programming in 2 dimensions

For a constraint $x + 2y \leq 6$ in (2.2), notice that a single linear inequality determines a unique half-plane.

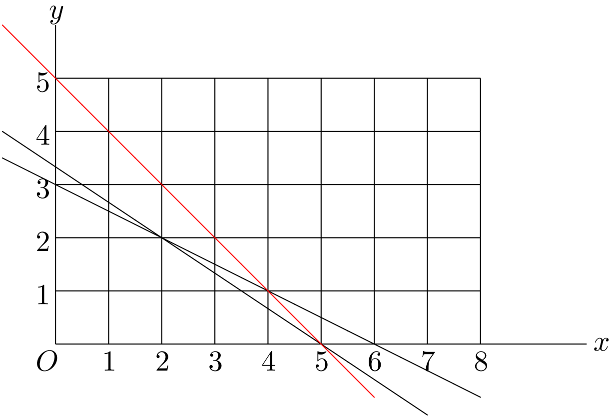


Visualizing linear programming in 2 dimensions

If we add another constraint $2x + 3y \leq 10$ in (2.1), then

Visualizing linear programming in 2 dimensions

If we add one more constraint $x + y \leq 5$ in (2.3), then



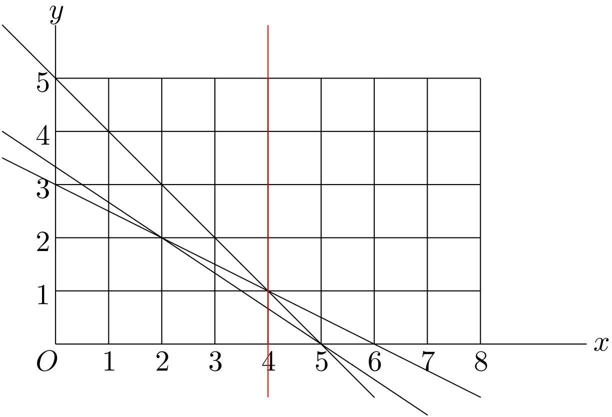
Visualizing linear programming in 2 dimensions

A constraint is called redundant if deleting the constraint does not increase the size of the feasible region.

A constraint $x + y \leq 5$ in (2.3) is redundant.

Visualizing linear programming in 2 dimensions

If we add one more constraint $x \leq 4$ in (2.4), then



Visualizing linear programming in 2 dimensions

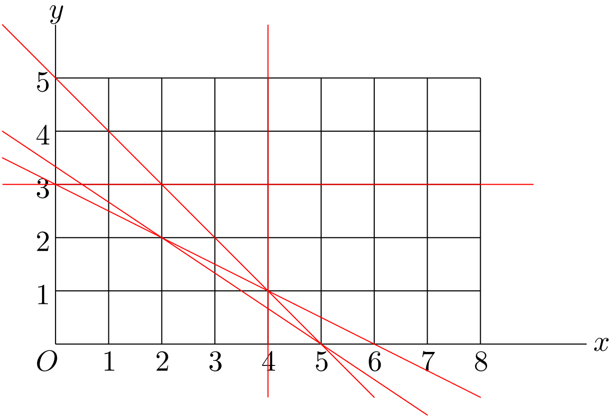
If we add one more constraint $y \leq 3$ in (2.5), then

Visualizing linear programming in 2 dimensions

A constraint $y \leq 3$ in (2.5) is redundant.

Visualizing linear programming in 2 dimensions

With an obvious constraint $x \geq 0$ and $y \geq 0$ in (2.6), we have now graphed the feasible region.



Visualizing linear programming in 2 dimensions

That is, we have the following feasible region.

Visualizing linear programming in 2 dimensions

How many constraints are redundant?

1. One
2. Two
3. More than two

Convex Sets

A set S is convex if for every two points in the set, the line segment joining the points is also in the set; that is,

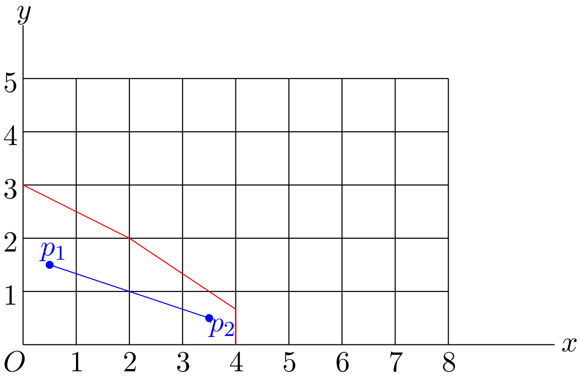
If $p_1, p_2 \in S$, then so is

$$(1-\lambda)\,p_1 + \lambda p_2$$

for $0 \leq \lambda \leq 1$.

Convex Sets

In the following picture, you see any point on the line segment connecting p_1 and p_2 belongs to the feasible region.



Convex Sets

Theorem

The feasible region of a linear program is convex.

More on Convexity

From https://en.wikipedia.org/wiki/Convex_set

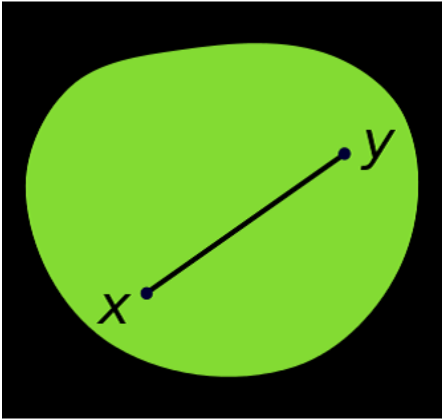
In a Euclidean space (or, more generally in an affine space), a convex set is a region such that, for every pair of points within the region, every point on the straight line segment that joins the pair of points is also within the region.

More on Convexity

Below is an illustration of a convex set which looks somewhat like a deformed circle. The (black) line segment joining points x and y lies completely within the (green) set.

Since this is true for any points x and y within the set that we might choose, the set is convex.

More on Convexity

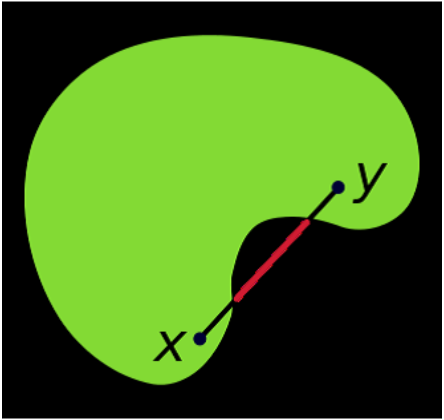


More on Convexity

Below is an illustration of a non-convex set.

Since the red part of the (black and red) line-segment joining the points x and y lies outside of the (green) set, the set is non-convex.

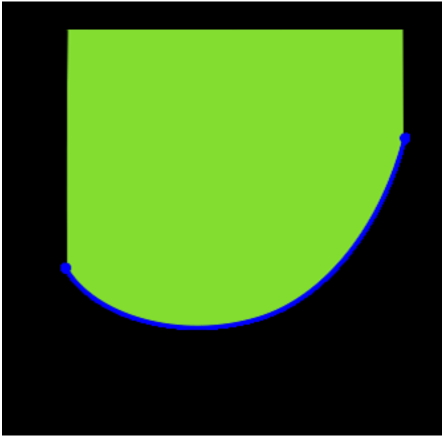
More on Convexity



More on Convexity

A function below is convex if and only if its epigraph, the region (in green) above its graph (in blue), is a convex set.

More on Convexity



Corner Points or vertex

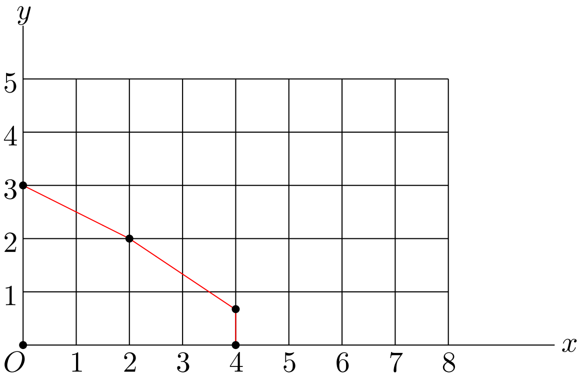
A corner point (also called an extreme point) of the feasible region is a point that is not the midpoint of two other points of the feasible region.

Corner Points or vertex

Where are the corner points of this feasible region?

Corner Points or vertex

The corner points of this feasible region are below.

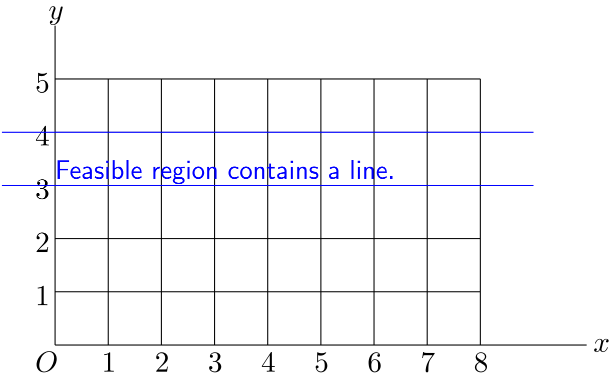


Corner Points or vertex

A feasible LP region has a corner point so long as it does not contain a line.

No corner point

No corner point.



Facts about Corner Points or vertex

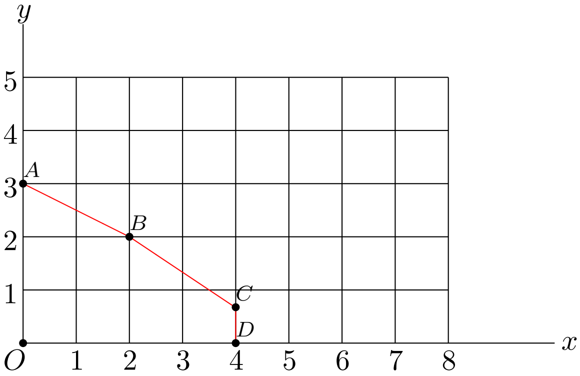
If every variable is non-negative, and if the feasible region is non-empty, then there is a corner point.

Facts about Corner Points or vertex

In two dimensions, a corner point is at the intersection of two equality constraints.

Facts about Corner Points or vertex

For instance, we have five corner points below.



Facts about Corner Points or vertex

Corner point $A = (0, 3)$ is at the intersection of two equality constraints $x = 0$ and $x + 2y = 6$.

Similarly, corner point $B = (2, 2)$ is at the intersection of two equality constraints $x + 2y = 6$ and $2x + 3y = 10$.

Corner point $C = (4, 2/3)$ is at the intersection of two equality constraints $2x + 3y = 10$ and $x = 4$.

Corner point $D = (4, 0)$ is at the intersection of two equality constraints $x = 4$ and $y = 0$.

Corner point $0 = (0, 0)$ is at the intersection of two equality constraints $x = 0$ and $y = 0$.

Facts about Corner Points or vertex

Take, for instance, corner point $B = (2, 2)$ for our problem.

We can rephrase the statement that corner point $B = (2, 2)$ is at the intersection of two equality constraints $x + 2y = 6$ and $2x + 3y = 10$ as that corner point $B = (2, 2)$ is the solution of the following system of linear equations:

$$\begin{aligned} x + 2y &= 6, \\ 2x + 3y &= 10. \end{aligned}$$

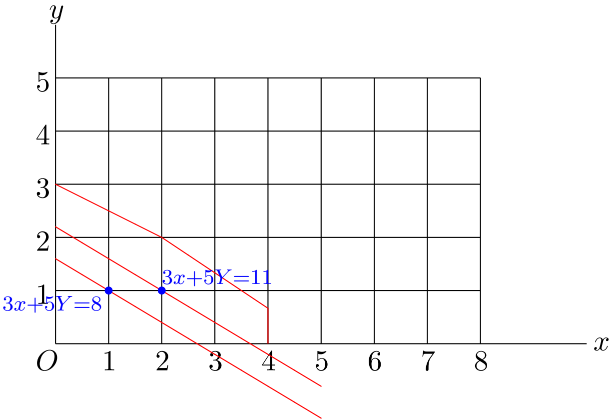
Visualizing linear programming in 2 dimensions

The geometrical method for optimizing the objective function $3x + 5y$.

Graph points such that $3x + 5y = z$ for various values of z.

Visualizing linear programming in 2 dimensions

Choose z maximal.



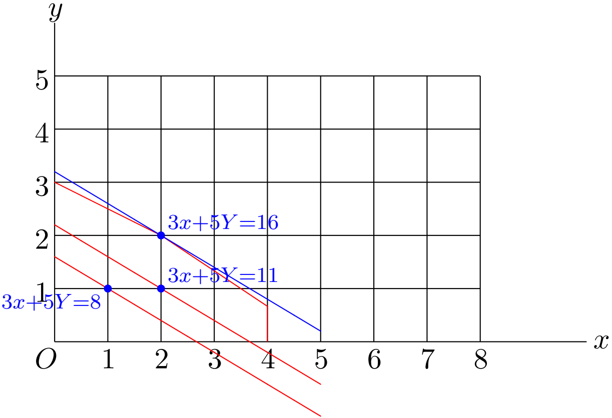
Visualizing linear programming in 2 dimensions

Find the maximum value z such that there is a feasible solution with $3x + 5y = z$.

Move the line with profit p parallel as much as possible.

Visualizing linear programming in 2 dimensions

The optimal solution occurs at a corner point or vertex of the feasible region..



Optimality at Corner Points or vertex

This turns out to be a general property of linear programming:

Theorem

If a problem has an optimal solution, there is always a corner point or vertex of the feasible region that is optimal.

Towards the simplex method

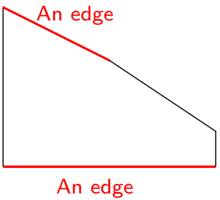
The *simplex method* for finding an optimal solution to a general linear program exploits this property by starting at a vertex and moving from vertex to vertex, improving the value of the objective function with each move.

Towards the simplex algorithm

- More geometrical notions -edges and rays
- Then ... the simplex algorithm

Edges of the feasible region

In two dimensions, an edge of the feasible region is one of the line segments making up the boundary of the feasible region.



Edges of the feasible region

In three dimensions, an edge of the feasible region is one of the line segments making up the framework of a polyhedron[15].

[15] From https://en.wikipedia.org/wiki/Polyhedro. In elementary geometry, a polyhedron (plural polyhedra or polyhedrons) is a solid in three dimensions with flat polygonal faces, straight edges and sharp corners or vertices.

Edges of the feasible region

The edges are where the faces intersect each other. A face is a flat region of the feasible region.

Fig.: Regular Tetrahedron[16]

[16] From https://en.wikipedia.org/wiki/Polyhedro.

Edges of the feasible region

In three dimensions, the "edges" are the intersections of two constraints.

The corner points are the the intersection of three constraints.

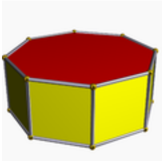


Fig.: Octagonal prism[17]

[17] From https://en.wikipedia.org/wiki/Polyhedro.

Short title

Instructor: Yuichiro Kanazawa, Ph.D.

Course?@Overview

Visualization

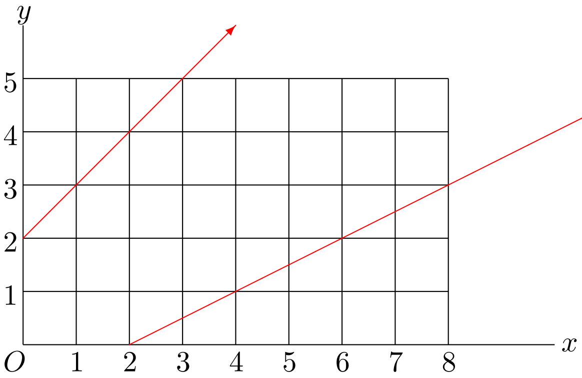
The Simplex Method

Duality

Consulting

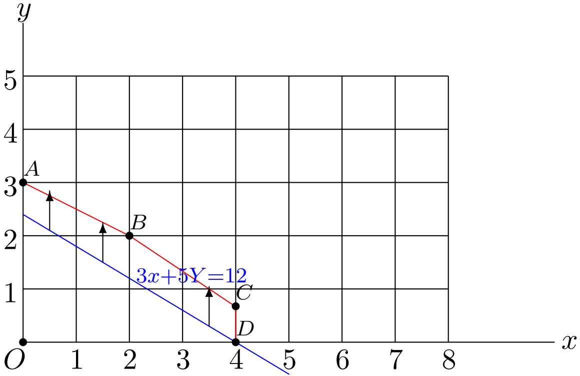
Extreme Rays

An extreme ray is like an edge, but it starts at a corner point and goes on infinitely. Two extreme rays in the picture below.



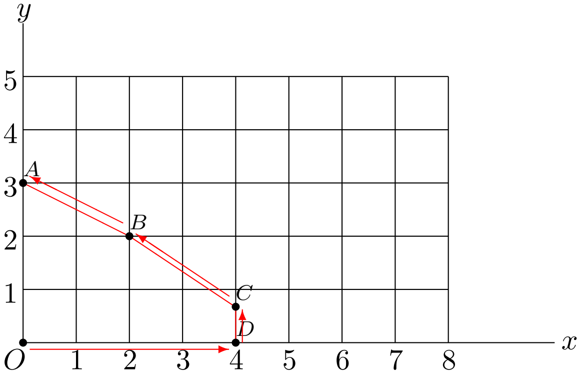
The Simplex Method

Start at any feasible corner point. For instance, start with corner point D. Starting from O is possible, but rather absurd.



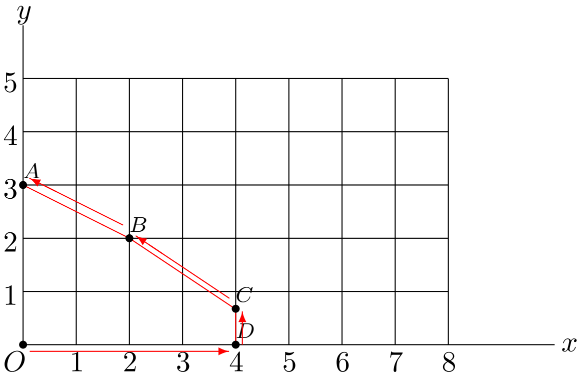
The Simplex Method

Find an edge (or extreme ray) in which the objective value is continually improving. For instance, find an edge from D to C to B to A.



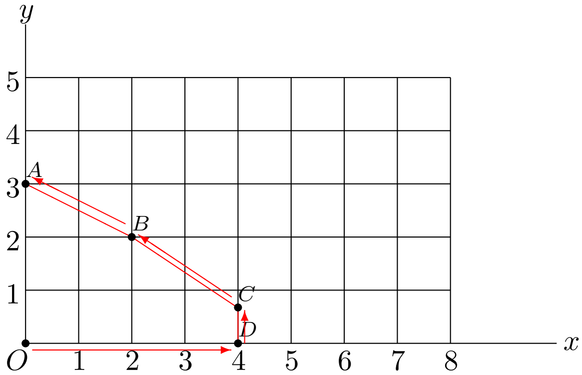
The Simplex Method

Go to the next corner point. If there is no such corner point, stop. The objective function is unbounded.



The Simplex Method

Continue until no adjacent corner point has a better objective value.



The Simplex Method

In two dimensions it is pretty easy to find a corner point to start at, especially if the LP is already graphed.

But with larger LPs, it is surprisingly tricky.

Facts about Corner Points or vertex

This slide already appeared, but I am quoting again.

Take, for instance, corner point $B = (2, 2)$ for our problem.

We can rephrase the statement that corner point $B = (2, 2)$ is at the intersection of two equality constraints $x + 2y = 6$ and $2x + 3y = 10$ as that corner point $B = (2, 2)$ is the solution of the following system of linear equations:

$$\begin{aligned} x + 2y &= 6, \\ 2x + 3y &= 10. \end{aligned}$$

System of linear equations: A Review

From https://en.wikipedia.org/wiki/System_of_linear_equations

In mathematics, a **system of linear equations** (or **linear system**) is a collection of two or more linear equations involving the same set of variables.

System of linear equations: A Review

For example,

$$\begin{aligned} 3x + 2y - z &= 1 \\ 2x - 2y + 4z &= -2 \\ -x + \tfrac{1}{2}y - z &= 0 \end{aligned}$$

is a system of three equations in the three variables x, y, z.

System of linear equations: A Review

A **solution** to a linear system is an assignment of values to the variables such that all the equations are simultaneously satisfied.

System of linear equations: A Review

A solution to the system above is given by

$$\begin{aligned} x &= 1 \\ y &= -2 \\ z &= -2 \end{aligned}$$

since it makes all three equations valid.

System of linear equations: A Review

A linear system in three variables determines a collection of planes. The intersection point is the solution.

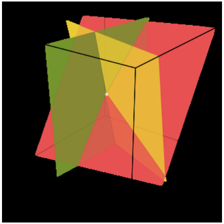


Fig.: From https://en.wikipedia.org/wiki/System_of_linear_equations.

System of linear equations: A Review

In mathematics, the theory of linear systems is the basis and a fundamental part of linear algebra, a subject which is used in most parts of modern mathematics.

System of linear equations: A Review

Computational algorithms for finding the solutions are an important part of numerical linear algebra, and play a prominent role in engineering, physics, chemistry, computer science, and economics.

System of linear equations: A Review

Elementary example

The simplest kind of linear system involves two equations and two variables:

$$\begin{aligned} 2x + 3y &= \ \ 6 \\ 4x + 9y &= 15. \end{aligned}$$

System of linear equations: A Review

One method for solving such a system is as follows.

First, solve the top equation for x in terms of y:

$$x = 3 - \frac{3}{2}y.$$

System of linear equations: A Review

Second, substitute this expression for x into the bottom equation:

$$4\left(3 - \frac{3}{2}y\right) + 9y = 12 + 3y = 15.$$

This results in a single equation involving only the variable y. Solving gives $y = 1$.

Third, substituting this back into the equation for x yields $x = 3/2$.

System of linear equations: A Review

This method of eliminating one variable (in this case x first by expressing x in terms of y), solving for the other variable (in this case y), and substituting back for the first variable (in this case x) generalizes to systems with additional variables.

It is called Gaussian elimination. We will soon review Gaussian elimination when we come back to the simplex method.

System of linear equations: A Review

General form

A general system of m linear equations with n unknowns can be written as

$$\begin{array}{ccccccc} a_{11}x_1 & + & a_{12}x_2 & +\cdots+ & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & +\cdots+ & a_{2n}x_n & = & b_2 \\ \vdots & & \vdots & & \vdots & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & +\cdots+ & a_{mn}x_n & = & b_m. \end{array}$$

Here $x_1, x_2, \ldots, x_n$ are the unknowns, $a_{11}, a_{12}, \ldots, a_{mn}$ are the coefficients of the system, and $b_1, b_2, \ldots, b_m$ are the constant terms.

System of linear equations: A Review

Vector equation

One extremely helpful view is that each unknown is a weight for a column vector in a linear combination.

$$x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots + x_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

System of linear equations: A Review

This allows all the language and theory of vector spaces (or more generally, modules) to be brought to bear.

System of linear equations: A Review

For example, the collection of all possible linear combinations of the vectors on the left-hand side is called their span, and the equations have a solution just when the right-hand vector is within that span.

System of linear equations: A Review

If every vector within that span has exactly one expression as a linear combination of the given left-hand vectors, then any solution is unique.

System of linear equations: A Review

Matrix equation

The vector equation is equivalent to a matrix equation of the form

$$A\mathbf{x} = \mathbf{b}$$

where A is an $m \times n$ matrix, x is a column vector with n entries, and b is a column vector with m entries.

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

System of linear equations: A Review

The number of vectors in a basis for the span is now expressed as the rank of the matrix.

System of linear equations: A Review

Solution set

A solution of a linear system is an assignment of values to the variables $x_1, x_2, \ldots, x_n$ such that each of the equations is satisfied.

The set of all possible solutions is called the solution set.

System of linear equations: A Review

A linear system may behave in any one of three possible ways:

1. The system has *infinitely many solutions*.
2. The system has a single *unique solution*.
3. The system has *no solution*.

System of linear equations: A Review

Geometric interpretation

For a system involving two variables (x and y), each linear equation determines a line on the xy-plane.

System of linear equations: A Review

Because a solution to a linear system must satisfy all of the equations, the solution set is the intersection of these lines, and is hence either a line, a single point, or the empty set.

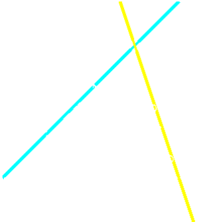


Fig.: From https://en.wikipedia.org/wiki/System_of_linear_equations.

System of linear equations: A Review

For three variables, each linear equation determines a plane in three-dimensional space, and the solution set is the intersection of these planes.

System of linear equations: A Review

Thus the solution set may be a plane, a line, a single point, or the empty set.

System of linear equations: A Review

For example, as three parallel planes do not have a common point, the solution set of their equations is empty; the solution set of the equations of three planes intersecting at a point is single point; if three planes pass through two points, their equations have at least two common solutions; in fact the solution set is infinite and consists in all the line passing through these points.

System of linear equations: A Review

For n variables, each linear equation determines a hyperplane in n-dimensional space.

The solution set is the intersection of these hyperplanes.

System of linear equations: A Review

General behavior

In general, the behavior of a linear system is determined by the relationship between the number of equations and the number of unknowns:

System of linear equations: A Review

Usually, a system with fewer equations than unknowns has infinitely many solutions, but it may have no solution.

Such a system is known as an underdetermined system.

System of linear equations: A Review

One equation in the case of two variables, we typically have

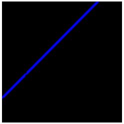


Fig.: From https://en.wikipedia.org/wiki/System_of_linear_equations.

The first system has infinitely many solutions, namely all of the points on the blue line.

System of linear equations: A Review

The solution set for two equations in three variables is usually a line.

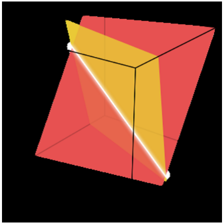


Fig.: From https://en.wikipedia.org/wiki/System_of_linear_equations.

System of linear equations: A Review

Usually, a system with the same number of equations and unknowns has a single unique solution.

System of linear equations: A Review

Two equations in the case of two variables, we typically have

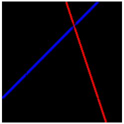


Fig.: From https://en.wikipedia.org/wiki/System_of_linear_equations.

The second system has a single unique solution, namely the intersection of the two lines.

System of linear equations: A Review

Usually, a system with more equations than unknowns has no solution.

Such a system is also known as an overdetermined system.

System of linear equations: A Review

Three equations in the case of two variables, we typically have

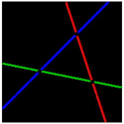


Fig.: From
https://en.wikipedia.org/wiki/System_of_linear_equations.

The third system has no solutions, since the three lines share no common point.

System of linear equations: A Review

Keep in mind that the pictures above show only the most common case.

System of linear equations: A Review

It is possible for a system of two equations and two unknowns to have no solution (if the two lines are parallel), or for a system of three equations and two unknowns to be solvable (if the three lines intersect at a single point).

System of linear equations: A Review

In general, a system of linear equations may behave differently from expected if the equations are linearly dependent, or if two or more of the equations are inconsistent.

System of linear equations: A Review

Linear Independence

The equations of a linear system are linearly independent if none of the equations can be derived algebraically from the others.

System of linear equations: A Review

When the equations are linearly independent, each equation contains new information about the variables, and removing any of the equations increases the size of the solution set.

System of linear equations: A Review

For example, the equations

$$3x + 2y = 6 \quad \text{and} \quad 6x + 4y = 12$$

are not independent—they are the same equation when scaled by a factor of two, and they would produce identical graphs.

This is an example of equivalence in a system of linear equations.

System of linear equations: A Review

For a more complicated example, the equations

$$\begin{aligned} x - 2y &= -1 \\ 3x + 5y &= 8 \\ 4x + 3y &= 7 \end{aligned}$$

are not independent, because the third equation is the sum of the other two.

System of linear equations: A Review

Indeed, any one of these equations can be derived from the other two, and any one of the equations can be removed without affecting the solution set.

The graphs of these equations are three lines that intersect at a single point $(1, 1)$ as seen in the next slide.

System of linear equations: A Review

The equations $x - 2y = -1$, $3x + 5y = 8$, and $4x + 3y = 7$ are linearly dependent.

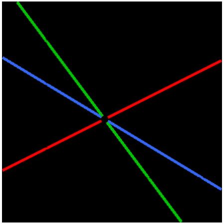


Fig.: From
`https://en.wikipedia.org/wiki/System_of_linear_equations`.

System of linear equations: A Review

Consistency

A linear system is inconsistent if it has no solution, and otherwise it is said to be consistent.

When the system is inconsistent, it is possible to derive a contradiction from the equations, that may always be rewritten as the statement $0 = 1$.

System of linear equations: A Review

For example, the equations

$$3x + 2y = 6 \quad \text{and} \quad 3x + 2y = 12$$

are inconsistent. In fact, by subtracting the first equation from the second one and multiplying both sides of the result by $1/6$, we get $0 = 1$.

System of linear equations: A Review

The graphs of these equations on the xy-plane are a pair of parallel lines.

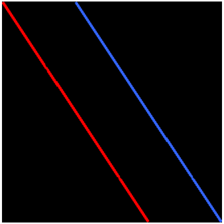


Fig.: From
https://en.wikipedia.org/wiki/System_of_linear_equations.

System of linear equations: A Review

It is possible for three linear equations to be inconsistent, even though any two of them are consistent together. For example, the equations

$$\begin{aligned} x + y &= 1 \\ 2x + y &= 1 \\ 3x + 2y &= 3 \end{aligned}$$

are inconsistent.

Adding the first two equations together gives $3x + 2y = 2$, which can be subtracted from the third equation to yield $0 = 1$.

System of linear equations: A Review

Note that any two of these equations have a common solution.

The same phenomenon can occur for any number of equations.

System of linear equations: A Review

In general, inconsistencies occur if the left-hand sides of the equations in a system are linearly dependent, and the constant terms do not satisfy the dependence relation.

System of linear equations: A Review

A system of equations whose left-hand sides are linearly independent is always consistent.

System of linear equations: A Review

Solving a linear system

System of linear equations: A Review

Describing the solution

When the solution set is finite, it is reduced to a single element.

In this case, the unique solution is described by a sequence of equations whose left-hand sides are the names of the unknowns and right-hand sides are the corresponding values, for example $(x = 3,\ y = -2,\ z = 6)$.

System of linear equations: A Review

It can be difficult to describe a set with infinite solutions.

Typically, some of the variables are designated as free (or independent, or as parameters), meaning that they are allowed to take any value, while the remaining variables are dependent on the values of the free variables.

System of linear equations: A Review

For example, consider the following system:

$$\begin{aligned} x + 3y - 2z &= 5 \\ 3x + 5y + 6z &= 7 \end{aligned}$$

The solution set to this system can be described by the following equations:

$$x = -7z - 1 \quad \text{and} \quad y = 3z + 2.$$

Here z is the free variable, while x and y are dependent on z.

System of linear equations: A Review

Any point in the solution set can be obtained by first choosing a value for z, and then computing the corresponding values for x and y.

System of linear equations: A Review

Each free variable gives the solution space one degree of freedom, the number of which is equal to the dimension of the solution set.

System of linear equations: A Review

For example, the solution set for the above equation is a line, since a point in the solution set can be chosen by specifying the value of the parameter z.

An infinite solution of higher order may describe a plane, or higher-dimensional set.

System of linear equations: A Review

Different choices for the free variables may lead to different descriptions of the same solution set.

System of linear equations: A Review

For example, the solution to the above equations can alternatively be described as follows:

$$y = -\frac{3}{7}x + \frac{11}{7} \quad \text{and} \quad z = -\frac{1}{7}x - \frac{1}{7}.$$

Here x is the free variable, and y and z are dependent.

System of linear equations: A Review

Elimination of variables

The simplest method for solving a system of linear equations is to repeatedly eliminate variables.

System of linear equations: A Review

This method can be described as follows:

1. In the first equation, solve for one of the variables in terms of the others.
2. Substitute this expression into the remaining equations. This yields a system of equations with one fewer equation and one fewer unknown.
3. Continue until you have reduced the system to a single linear equation.
4. Solve this equation, and then back-substitute until the entire solution is found.

System of linear equations: A Review

For example, consider the following system:

$$\begin{aligned} x + 3y - 2z &= 5 \\ 3x + 5y + 6z &= 7 \\ 2x + 4y + 3z &= 8 \end{aligned}$$

System of linear equations: A Review

Solving the first equation for x gives $x = 5 + 2z - 3y$, and plugging this into the second and third equation yields

$$\begin{aligned} -4y + 12z &= -8 \\ -2y + \ \ 7z &= -2 \end{aligned}$$

System of linear equations: A Review

Solving the first of these equations for y yields $y = 2 + 3z$, and plugging this into the second equation yields $z = 2$.

System of linear equations: A Review

We now have:

$$
\begin{aligned}
x &= 5 + 2z - 3y \\
y &= 2 + 3z \\
z &= 2
\end{aligned}
$$

System of linear equations: A Review

Substituting $z = 2$ into the second equation gives $y = 8$, and substituting $z = 2$ and $y = 8$ into the first equation yields $x = -15$.

Therefore, the solution set is the single point $(x, y, z) = (-15, 8, 2)$.

System of linear equations: A Review

Elimination of variables

In linear algebra, Gaussian elimination (also known as row reduction) is an algorithm for solving systems of linear equations.

System of linear equations: A Review

It is understood as a sequence of operations performed on the corresponding matrix of coefficients.

System of linear equations: A Review

The method is named after Carl Friedrich Gauss (1777–1855), although it was known to Chinese mathematicians as early as 179 AD.

System of linear equations: A Review

To perform row reduction on a matrix, one uses a sequence of elementary row operations to modify the matrix until the lower left-hand corner of the matrix is filled with zeros, as much as possible.

System of linear equations: A Review

There are three types of elementary row operations:

1. Swapping two rows,
2. Multiplying a row by a non-zero number (scalar),
3. Adding a multiple of one row to another row.

System of linear equations: A Review

Because these operations are reversible, the augmented matrix produced always represents a linear system that is equivalent to the original.

System of linear equations: A Review

For example, let us reconsider the system when we repeatedly eliminate variables:

$$\begin{aligned} x + 3y - 2z &= 5 \\ 3x + 5y + 6z &= 7 \\ 2x + 4y + 3z &= 8 \end{aligned}$$

System of linear equations: A Review

We represent the linear system as an augmented matrix:

$$\left[\begin{array}{ccc|c} 1 & 3 & -2 & 5 \\ 3 & 5 & 6 & 7 \\ 2 & 4 & 3 & 8 \end{array}\right].$$

System of linear equations: A Review

Please follow how the three types of elementary row operations are conducted.

$$\left[\begin{array}{rrr|r} 1 & 3 & -2 & 5 \\ 3 & 5 & 6 & 7 \\ 2 & 4 & 3 & 8 \end{array}\right] \Longleftrightarrow \left[\begin{array}{rrr|r} 1 & 3 & -2 & 5 \\ 0 & -4 & 12 & -8 \\ 2 & 4 & 3 & 8 \end{array}\right].$$

System of linear equations: A Review

$$\left[\begin{array}{rrr|r} 1 & 3 & -2 & 5 \\ 0 & -4 & 12 & -8 \\ 2 & 4 & 3 & 8 \end{array}\right] \Longleftrightarrow \left[\begin{array}{rrr|r} 1 & 3 & -2 & 5 \\ 0 & -4 & 12 & -8 \\ 0 & -2 & 7 & -2 \end{array}\right].$$

System of linear equations: A Review

$$\left[\begin{array}{rrr|r} 1 & 3 & -2 & 5 \\ 0 & -4 & 12 & -8 \\ 0 & -2 & 7 & -2 \end{array}\right] \Longleftrightarrow \left[\begin{array}{rrr|r} 1 & 3 & -2 & 5 \\ 0 & 1 & -3 & 2 \\ 0 & -2 & 7 & -2 \end{array}\right].$$

System of linear equations: A Review

$$\left[\begin{array}{rrr|r} 1 & 3 & -2 & 5 \\ 0 & 1 & -3 & 2 \\ 0 & -2 & 7 & -2 \end{array}\right] \Longleftrightarrow \left[\begin{array}{rrr|r} 1 & 3 & -2 & 5 \\ 0 & 1 & -3 & 2 \\ 0 & 0 & 1 & 2 \end{array}\right].$$

Notice that the augmented matrix above right corresponds to

$$\begin{aligned} x &= 5 + 2z - 3y, \\ y &= 2 + 3z, \\ z &= 2. \end{aligned}$$

System of linear equations: A Review

$$\left[\begin{array}{ccc|c} 1 & 3 & -2 & 5 \\ 0 & 1 & -3 & 2 \\ 0 & 0 & 1 & 2 \end{array}\right] \Longleftrightarrow \left[\begin{array}{ccc|c} 1 & 3 & -2 & 5 \\ 0 & 1 & 0 & 8 \\ 0 & 0 & 1 & 2 \end{array}\right].$$

System of linear equations: A Review

$$\left[\begin{array}{ccc|c} 1 & 3 & -2 & 5 \\ 0 & 1 & 0 & 8 \\ 0 & 0 & 1 & 2 \end{array}\right] \Longleftrightarrow \left[\begin{array}{ccc|c} 1 & 3 & 0 & 9 \\ 0 & 1 & 0 & 8 \\ 0 & 0 & 1 & 2 \end{array}\right].$$

System of linear equations: A Review

$$\left[\begin{array}{ccc|c} 1 & 3 & 0 & 9 \\ 0 & 1 & 0 & 8 \\ 0 & 0 & 1 & 2 \end{array}\right] \Longleftrightarrow \left[\begin{array}{ccc|r} 1 & 0 & 0 & -15 \\ 0 & 1 & 0 & 8 \\ 0 & 0 & 1 & 2 \end{array}\right].$$

System of linear equations: A Review

The last matrix is in reduced row echelon form to be explained below, and represents the system $x = -15, y = 8, z = 2$.

System of linear equations: A Review

In R, we can solve this system of linear equations as follows.

First. we define the coefficient matrix A on the left-hand side as

```
> A<-matrix(c(1,3,-2,3,5,6,2,4,3),ncol=3,byrow=TRUE)

> A

[,1] [,2] [,3]
[1,] 1 3 -2
[2,] 3 5 6
[3,] 2 4 3
```

System of linear equations: A Review

Then we define the right hand side B as vector as

```
> b<-c(5,7,8)
> b
[1] 5 7 8
```

System of linear equations: A Review

Finally we solve the system of linear equations as

```
> solve(A,b)
```

to obtain

```
[1] -15 8 2
```

System of linear equations: A Review

We can also solve this system of linear equations by first obtaining an inverse matrix of A and premultiply the obtained inverse matrix of A by b as follows.

Inverse matrix of A is obtained by

```
> Inv.A<-solve(A)
> Inv.A
[,1] [,2] [,3]
[1,] 2.25 4.25 -7
[2,] -0.75 -1.75 3
[3,] -0.50 -0.50 1
> Inv.A%*%b
[,1]
[1,] -15
[2,] 8
[3,] 2
```

System of linear equations: A Review

This inverse matrix has the following characteristics:

```
> Inv.A%*%A

[,1] [,2] [,3]
[1,] 1 -3.552714e-15 0.000000e+00
[2,] 0 1.000000e+00 1.776357e-15
[3,] 0 0.000000e+00 1.000000e+00
```

System of linear equations: A Review

Elimination of variables

A comparison with the example in the previous section on the algebraic elimination of variables shows that these two methods are in fact the same; the difference lies in how the computations are written down.

System of linear equations: A Review

For each row in a matrix, if the row does not consist of only zeros, then the left-most non-zero entry is called the leading coefficient (or pivot) of that row.

System of linear equations: A Review

If two leading coefficients are in the same column, then a row operation of third type (see above) could be used to make one of those coefficients zero.

System of linear equations: A Review

Then by using the row swapping operation, one can always order the rows so that for every non-zero row, the leading coefficient is to the right of the leading coefficient of the row above.

System of linear equations: A Review

If this is the case, then matrix is said to be in row echelon form. So the lower left part of the matrix contains only zeros, and all of the zero rows are below the non-zero rows.

System of linear equations: A Review

The word “echelon”is used here because one can roughly think of the rows being ranked by their size, with the largest being at the top and the smallest being at the bottom.

System of linear equations: A Review

For example, the following matrix is in row echelon form, and its leading coefficients are shown in red.

$$\begin{bmatrix} 0 & \mathbf{2} & 1 & -1 \\ 0 & 0 & \mathbf{3} & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

It is in echelon form because the zero row is at the bottom, and the leading coefficient of the second row (in the third column), is to the right of the leading coefficient of the first row (in the second column).

System of linear equations: A Review

A matrix is said to be in reduced row echelon form if furthermore all of the leading coefficients are equal to 1 (which can be achieved by using the elementary row operation of the second type), and in every column containing a leading coefficient, all of the other entries in that column are zero (which can be achieved by using elementary row operations of the third type).

System of linear equations: A Review

This is an example of a matrix in reduced row echelon form:

$$\begin{bmatrix} \mathbf{1} & 0 & a_1 & 0 & b_1 \\ 0 & \mathbf{1} & a_2 & 0 & b_2 \\ 0 & 0 & 0 & \mathbf{1} & b_3 \end{bmatrix}$$

The Simplex Method

The simplex method consists of repeating two steps:

Step 0. Getting LPs into the canonical form.

Step 1. Find a corner(vertex) by solving simultaneous equations.

Step 2. Find the value of the objective function at the corner(vertex).

Step 3. Repeat **Step 1** and **Step 2** until the value of the objective function does not improve.

The Simplex Method: Canonical Form

Getting LPs into the correct form called canonical form for the simplex method by:

- changing inequality constraints (other than non-negativity constraints) to equality constraints
- putting the objective function

The Simplex Method: Canonical Form

Consider a linear program in which all variables are non-negative.

$$
\begin{aligned}
\max z = 20x_1 + 30x_2 & \\
\text{such that} & \\
x_1 + 2x_2 &\leq 800 \\
3x_1 + 4x_2 &\leq 1800 \\
3x_1 + x_2 &\leq 1500 \\
x_1, x_2 &\geq 0.
\end{aligned}
$$

The Simplex Method: Canonical Form

Its feasible region is plotted below:

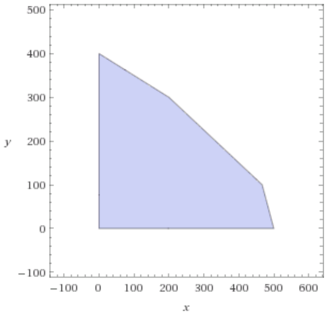


Fig.: Drawn by Wolfram Alpha Widgets Inequality Plotter for Linear Programming.

The Simplex Method: Canonical Form

How can we convert inequality constraints into equality constraints? Take, for example, the following constraint

$$x_1 + 2x_2 \quad \leq \quad 800.$$

The Simplex Method: Canonical Form

Notice that we can convert a "$\leq$"constraint into a "$=$"constraint by adding a slack variable, constrained to be ≥ 0.

For instance, if we let $s_1 = 800 - x_1 - 2x_2$, then the inequality above implies that $s_1 \geq 0$ and that

$$x_1 + 2x_2 + s_1 = 800.$$

The Simplex Method: Canonical Form

Similar operation will make the original problem into the canonical form:

$$\max z = 20x_1 + 30x_2$$

such that

$$\begin{aligned} x_1 + 2x_2 + s_1 &= 800 \\ 3x_1 + 4x_2 + s_2 &= 1800 \\ 3x_1 + x_2 + s_3 &= 1500 \\ x_1, x_2, s_1, s_2, s_3 &\geq 0. \end{aligned}$$

The Simplex Method: LP tableau

We use the following template.

basic var.	z	x_1	x_2	s_1	s_2	s_3	RHS	Increment

The Simplex Method: LP tableau

We create an linear programming tableau or LP tableau from the canonical form.

basic var.	z	x_1	x_2	s_1	s_2	s_3	RHS	Increment
	1	-20	-30	0	0	0	0	
	0	1	2	1	0	0	800	
	0	3	4	0	1	0	1800	
	0	3	1	0	0	1	1500	

The Simplex Method: LP tableau

In the LP tableau, the basic variables are the variables corresponding to the identity matrix.

The LP tableau has four equations, so its identity matrix is

$$I = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The Simplex Method: LP tableau

We determine which variables out of $z, x_1, x_2, s_1, s_2, s_3$ are the four basic variables corresponding to the LP tableau.

Remember to include z because its coefficient in the first row is always 1 and its coefficients in the other rows are always 0.

The Simplex Method: LP tableau

Since the basic variables are the variables corresponding to the identity matrix, they are z, s_1, s_2, s_3 for this example.

The nonbasic variables are the remaining variables. For this example, they are x_1, x_2.

The Simplex Method: LP tableau

We write down the basic variables in the leftmost column of the LP tableau.

basic var.	z	x_1	x_2	s_1	s_2	s_3	RHS	Increment
$\mathbf{z}$	1	-20	-30	0	0	0	0	
$\mathbf{s_1}$	0	1	2	1	0	0	800	
$\mathbf{s_2}$	0	3	4	0	1	0	1800	
$\mathbf{s_3}$	0	3	1	0	0	1	1500	

The Simplex Method: LP tableau in canonical form

An LP tableau is in canonical form if all of the following are true.

1. All decision variables other than the objective function z take non-negative value
2. All (other) constraints are equality constraints
3. The RHS is non-negative (except for objective function row, which is always 0)
4. For each row i, there is a column equal to unit vector 1_i whose i-th row is 1 and the remaining rows are 0.

The Simplex Method: LP tableau in canonical form

The LP tableau below is in canonical form:

basic var.	z	x_1	x_2	s_1	s_2	s_3	RHS	Increment
$\mathbf{z}$	1	-20	-30	0	0	0	0	
$\mathbf{s_1}$	0	1	2	1	0	0	800	
$\mathbf{s_2}$	0	3	4	0	1	0	1800	
$\mathbf{s_3}$	0	3	1	0	0	1	1500	

The Simplex Method: basic feasible solutions

For the LP tableau above, the basic feasible solution (henceforth, bfs) is the unique solution obtained by setting the non-basic variables to 0.

That is,

$z = 0, x_1 = 0, x_2 = 0, s_1 = 800, s_2 = 1800, s_3 = 1500.$

The Simplex Method: basic feasible solutions

Since we see that this basic feasible solution $(x_1, x_2) = (0, 0)$ corresponds to one of the corner points in the Wolfram Alpha Widgets Picture, what our operation does so far is that we are able to find out at least one corner point to start with, though obviously the value of the objective function at this corner point is lousy one.

The Simplex Method: Optimality conditions for a maximization problem

Theorem (Optimality Condition)

A basic feasible solution is optimal if every coefficient in the z-row is non-negative.

The Simplex Method: Optimality conditions for a maximization problem

Since coefficients for x_1 and x_2 are respectively -20 and -30, our first attempt at $z = 0$, $x_1 = 0$, $x_2 = 0$, $s_1 = 800$, $s_2 = 1800$, $s_3 = 1500$ is not optimal, according to the theorem.

basic var.	z	x_1	x_2	s_1	s_2	s_3	RHS	Increment
z	1	-20	-30	0	0	0	0	
s_1	0	1	2	1	0	0	800	
s_2	0	3	4	0	1	0	1800	
s_3	0	3	1	0	0	1	1500	

The Simplex Method: Optimality conditions for a maximization problem

What this theorem says is actually not that difficult to understand.

For the LP tableau above, z-row $z - 20x_1 - 30x_2 = 0$ is really

$$z = 20x_1 + 30x_2.$$

Thus by increasing either x_1 or x_2 from $x_1 = 0, x_2 = 0$ will increase the objective function z.

The Simplex Method: How to obtain a better solution if the bfs is not optimal

Since the number of basic feasible variables must be equal to the number of equations to solve the LP tableau, we need to determine which new variable to enter into the LP tableau as a new basic feasible solution (entering variable) and which variable of the previous basic feasible solution to exit (exiting variable) from the LP tableau.

The Simplex Method: How to obtain a better solution if the bfs is not optimal

For the LP tableau below, we have two candidates x_1 and x_2 for an entering variable. Since

$$z = 20x_1 + 30x_2.$$

increasing x_1 by 1 will increase z by 20, while increasing x_2 by 1 will increase z by 30.

The Simplex Method: How to obtain a better solution if the bfs is not optimal

If there is more than one column so that the entry in the objective row is negative, then (in order to find the maximum) the choice of which one to add to the set of basic variables is somewhat arbitrary and several entering variable choice rules such as Devex algorithm have been developed.

From
https://en.wikipedia.org/wiki/Simplex_algorithm

The Simplex Method: How to obtain a better solution if the bfs is not optimal

We choose x_2 over x_1 as the first entering variable because one unit increase in x_2 is larger.

Notice that this implies that you are moving from $(x_1, x_2) = (0, 0)$ towards along y-axis to reach the corner $(x_1, x_2) = (0, 400)$ in the Wolfram Alpha Widgets Picture.

The Simplex Method: How to obtain a better solution if the bfs is not optimal

In other words, moving x_2 first while keeping $x_1 = 0$ is tantamount to hoping that going to hunt corners in the Wolfram Alpha Widgets Picture this way clockwise from $(x_1, x_2) = (0, 0)$ to $(x_1, x_2) = (0, 400)$ will reach the optimal corner faster.

This is just a hope.

The Simplex Method: How to obtain a better solution if the bfs is not optimal

You need to find a rule that increasing x_2 while maintaining $x_1 = 0$ will not overshoot and go out of the feasible region.

basic var.	z	x_1	x_2	s_1	s_2	s_3	RHS	Increment
$\mathbf{z}$	1	-20	-30	0	0	0	0	
$\mathbf{s_1}$	0	1	2	1	0	0	800	
$\mathbf{s_2}$	0	3	4	0	1	0	1800	
$\mathbf{s_3}$	0	3	1	0	0	1	1500	

The Simplex Method: How to obtain a better solution if the bfs is not optimal

Notice that choosing x_2 as the entering variable means we increase x_2 while maintaining $x_1 = 0$.

Two observations are in order here:

- Decreasing x_2 would not be an option;
- Increasing x_2 too big will go out of the feasible region.

The Simplex Method: How to obtain a better solution if the bfs is not optimal

We have three inequality constraints:

$$
\begin{aligned}
x_1 + 2x_2 \leq 800 &\iff x_1 + 2x_2 + s_1 = 800, \\
3x_1 + 4x_2 \leq 1800 &\iff 3x_1 + 4x_2 + s_2 = 1800, \\
3x_1 + x_2 \leq 1500 &\iff 3x_1 + x_2 + s_3 = 1500, \\
x_1, x_2 \geq 0 &\iff x_1, x_2, s_1, s_2, s_3 \geq 0.
\end{aligned}
$$

The Simplex Method: How to obtain a better solution if the bfs is not optimal

Since x_1 is kept at 0, we realize that each of the following inequalities tries to tell you that we are able to increase x_2 by $800/2 = 400$ or $1800/4 = 450$, or $1500/1 = 1500$ while satisfying the corresponding constraint.

$$
\begin{aligned}
x_1 + 2x_2 \le 800 &\iff x_1 + 2x_2 + s_1 = 800,\\
3x_1 + 4x_2 \le 1800 &\iff 3x_1 + 4x_2 + s_2 = 1800,\\
3x_1 + x_2 \le 1500 &\iff 3x_1 + x_2 + s_3 = 1500,\\
x_1, x_2 \ge 0 &\iff x_1, x_2, s_1, s_2, s_3 \ge 0.
\end{aligned}
$$

The Simplex Method: How to obtain a better solution if the bfs is not optimal

However, in order to satisfy all three constraints, we can only increase the minimum of the three 400, 450, and 1500. This rule is called the minimum ratio rule.

$$
\begin{aligned}
x_1 + 2x_2 \leq 800 &\iff x_1 + 2x_2 + s_1 = 800, \\
3x_1 + 4x_2 \leq 1800 &\iff 3x_1 + 4x_2 + s_2 = 1800, \\
3x_1 + x_2 \leq 1500 &\iff 3x_1 + x_2 + s_3 = 1500, \\
x_1, x_2 \geq 0 &\iff x_1, x_2, s_1, s_2, s_3 \geq 0.
\end{aligned}
$$

The Simplex Method: How to obtain a better solution if the bfs is not optimal

The Minimum Ratio Rule

Operationally, we calculate the ratio of RHS to the column for the first entering variable x_2 for the row 2 to 4 as follows:

basic var.	z	x_1	x_2	s_1	s_2	s_3	RHS	Increment
z	1	-20	-30	0	0	0	0	
$\mathbf{s_1}$	0	1	2	1	0	0	800	$800/2 = 400$
$\mathbf{s_2}$	0	3	4	0	1	0	1800	$1800/4 = 450$
$\mathbf{s_3}$	0	3	1	0	0	1	1500	$1500/1 = 1500$

The Simplex Method: How to obtain a better solution if the bfs is not optimal

For the LP tableau above, we have three candidates s_1, s_2, and s_3 for an exiting variable.

Since s_1 row generates the smallest increment of the three rows s_1, s_2, and s_3, this implies that s_1 will be the first exiting variable.

The Simplex Method: How to obtain a better solution if the bfs is not optimal

We need to make the x_2 column to be $(0, 1, 0, 0)$ in the following LP tableau with three types of elementary row operations with s_1 row x_2 column element 2 as a pivot:

1. Swapping two rows,
2. Multiplying a row by a non-zero number (scalar),
3. Adding a multiple of one row to another row.

basic var.	z	x_1	x_2	s_1	s_2	s_3	RHS	Increment
$\mathbf{z}$	1	-20	-30	0	0	0	0	
$\mathbf{s_1}$	0	1	$\mathbf{2}$	1	0	0	800	$800/2 = 400$
$\mathbf{s_2}$	0	3	4	0	1	0	1800	$1800/4 = 450$
$\mathbf{s_3}$	0	3	1	0	0	1	1500	$1500/1 = 1500$

The Simplex Method: How to obtain a better solution if the bfs is not optimal

(1) Divide s_1 row by 2 to obtain a pivot row.

basic var.	z	x_1	x_2	s_1	s_2	s_3	RHS	Increment
$\mathbf{z}$	1	-20	-30	0	0	0	0	
$\mathbf{x_2}$	**0**	**0.5**	**1**	**0.5**	**0**	**0**	**400**	
$\mathbf{s_2}$	0	3	4	0	1	0	1800	
$\mathbf{s_3}$	0	3	1	0	0	1	1500	

The Simplex Method: How to obtain a better solution if the bfs is not optimal

(2) Subtract the pivot row $\times 4$ from s_2 row to obtain

basic var.	z	x_1	x_2	s_1	s_2	s_3	RHS	Increment
z	1	−20	−30	0	0	0	0	
$\mathbf{x_2}$	0	0.5	**1**	0.5	0	0	400	
$\mathbf{s_2}$	**0**	**1**	**0**	**−2**	**1**	**0**	**200**	
$\mathbf{s_3}$	0	3	1	0	0	1	1500	

The Simplex Method: How to obtain a better solution if the bfs is not optimal

(3) Subtract the pivot row from s_3 row to obtain

basic var.	z	x_1	x_2	s_1	s_2	s_3	RHS	Increment
$\mathbf{z}$	1	-20	-30	0	0	0	0	
$\mathbf{x_2}$	0	0.5	$\mathbf{1}$	0.5	0	0	400	
$\mathbf{s_2}$	0	1	0	-2	1	0	200	
$\mathbf{s_3}$	$\mathbf{0}$	$\mathbf{2.5}$	$\mathbf{0}$	$\mathbf{-0.5}$	$\mathbf{0}$	$\mathbf{1}$	$\mathbf{1100}$	

The Simplex Method: How to obtain a better solution if the bfs is not optimal

(4) Subtract the pivot row $\times -30$ from z row to obtain

basic var.	z	x_1	x_2	s_1	s_2	s_3	RHS	Increment
$\mathbf{z}$	$\mathbf{1}$	$\mathbf{-5}$	$\mathbf{0}$	$\mathbf{15}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{12000}$	
$\mathbf{x_2}$	0	0.5	$\mathbf{1}$	0.5	0	0	400	
$\mathbf{s_2}$	0	1	0	-2	1	0	200	
$\mathbf{s_3}$	0	2.5	0	-0.5	0	1	1100	

The Simplex Method: How to obtain a better solution if the bfs is not optimal

For the LP tableau above, the basic feasible solution (bfs) is the unique solution obtained by setting the non-basic variables to 0.

That is,

$z = 12000, x_1 = 0, x_2 = 400, s_1 = 0, s_2 = 200, s_3 = 1100.$

The Simplex Method: How to obtain a better solution if the bfs is not optimal

Remember the optimality condition:

Theorem (Optimality Condition)

A basic feasible solution is optimal if every coefficient in the z-row is non-negative.

The Simplex Method: How to obtain a better solution if the bfs is not optimal

Since z row x_1 column element in the LP tableau below has -5, increasing x_1 by 1 increases the objective function by 5.

basic var.	z	x_1	x_2	s_1	s_2	s_3	RHS	Increment
$\mathbf{z}$	$\mathbf{1}$	$\mathbf{-5}$	$\mathbf{0}$	$\mathbf{15}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{12000}$	
$\mathbf{x_2}$	0	0.5	$\mathbf{1}$	0.5	0	0	400	
$\mathbf{s_2}$	0	1	0	-2	1	0	200	
$\mathbf{s_3}$	0	2.5	0	-0.5	0	1	1100	

The Simplex Method: How to obtain a better solution if the bfs is not optimal

That means that there is still a room for improvement.

The Simplex Method: How to obtain a better solution if the bfs is not optimal

The second entering variable must be x_1 because that variable is the only one to manupulate.

Increasing x_1 now implies that we are moving from a corner at $(x_1, x_2) = (0, 400)$ to another corner to its right in the Wolfram Alpha Widgets Picture.

The Simplex Method: How to obtain a better solution if the bfs is not optimal

In order to employ the minimum ratio rule, we calculate the ratio of RHS to the column for the second entering variable x_2 for the row 2 to 4 as follows:

basic var.	z	x_1	x_2	s_1	s_2	s_3	RHS	Increment
$\mathbf{z}$	1	-5	0	15	0	0	12000	
$\mathbf{x_2}$	0	0.5	1	0.5	0	0	400	$400/0.5 = 800$
$\mathbf{s_2}$	0	1	0	-2	1	0	200	$200/1 = 200$
$\mathbf{s_3}$	0	2.5	0	-0.5	0	1	1100	$1100/2.5 = 440$

The Simplex Method: How to obtain a better solution if the bfs is not optimal

For the LP tableau above, we have two candidates s_2 and s_3 for an exiting variable.

Since s_2 row generates the smallest increment of the three rows s_2 and s_3, this implies that s_2 will be the second exiting variable.

The Simplex Method: How to obtain a better solution if the bfs is not optimal

We need to make the x_1 column to be $(0, 0, 1, 0)$ in the following LP tableau with three types of elementary row operations with s_2 row x_1 column element as a pivot:

1. Swapping two rows,
2. Multiplying a row by a non-zero number (scalar),
3. Adding a multiple of one row to another row.

basic var.	z	x_1	x_2	s_1	s_2	s_3	RHS	Increment
z	1	-5	0	15	0	0	12000	
$\mathbf{x_2}$	0	0.5	1	0.5	0	0	400	
$\mathbf{s_2}$	0	**1**	0	-2	1	0	200	
$\mathbf{s_3}$	0	2.5	0	-0.5	0	1	1100	

The Simplex Method: How to obtain a better solution if the bfs is not optimal

(1) Divide s_2 row by 1 (in theory, but if you do this, then you have a serious problem) to obtain a pivot row.

basic var.	z	x_1	x_2	s_1	s_2	s_3	RHS	Increment
$\mathbf{z}$	1	-5	0	15	0	0	12000	
$\mathbf{x_2}$	0	0.5	1	0.5	0	0	400	
$\mathbf{x_1}$	$\mathbf{0}$	$\mathbf{1}$	$\mathbf{0}$	$\mathbf{-2}$	$\mathbf{1}$	$\mathbf{0}$	$\mathbf{200}$	
$\mathbf{s_3}$	0	2.5	0	-0.5	0	1	1100	

The Simplex Method: How to obtain a better solution if the bfs is not optimal

(2) Subtract the pivot row $\times 2.5$ from s_3 row to obtain

basic var.	z	x_1	x_2	s_1	s_2	s_3	RHS	Increment
$\mathbf{z}$	1	-5	0	15	0	0	12000	
$\mathbf{x_2}$	0	0.5	1	0.5	0	0	400	
$\mathbf{x_1}$	0	$\mathbf{1}$	0	-2	1	0	200	
$\mathbf{s_3}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{4.5}$	$\mathbf{-2.5}$	$\mathbf{1}$	$\mathbf{600}$	

The Simplex Method: How to obtain a better solution if the bfs is not optimal

(3) Subtract the pivot row $\times 0.5$ from x_2 row to obtain

basic var.	z	x_1	x_2	s_1	s_2	s_3	RHS	Increment
$\mathbf{z}$	1	-5	0	15	0	0	12000	
$\mathbf{x_2}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{1}$	$\mathbf{1.5}$	$\mathbf{-0.5}$	$\mathbf{0}$	$\mathbf{300}$	
$\mathbf{x_1}$	0	$\mathbf{1}$	0	-2	1	0	200	
$\mathbf{s_3}$	0	0	0	4.5	-2.5	1	600	

The Simplex Method: How to obtain a better solution if the bfs is not optimal

(4) Subtract the pivot row $\times -5$ from z row to obtain

basic var.	z	x_1	x_2	s_1	s_2	s_3	RHS	Increment
$\mathbf{z}$	$\mathbf{1}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{5}$	$\mathbf{5}$	$\mathbf{0}$	$\mathbf{13000}$	
$\mathbf{x_2}$	0	0	1	1.5	-0.5	0	300	
$\mathbf{x_1}$	0	$\mathbf{1}$	0	-2	1	0	200	
$\mathbf{s_3}$	0	0	0	4.5	-2.5	1	600	

The Simplex Method: How to obtain a better solution if the bfs is not optimal

For the LP tableau above, the basic feasible solution (bfs) is the unique solution obtained by setting the non-basic variables to 0.

That is,

$z = 13000, x_1 = 200, x_2 = 300, s_1 = 0, s_2 = 0, s_3 = 600.$

The Simplex Method: How to obtain a better solution if the bfs is not optimal

Again remember the optimality condition:

Theorem (Optimality Condition)

A basic feasible solution is optimal if every coefficient in the z-row is non-negative.

The Simplex Method: How to obtain a better solution if the bfs is not optimal

Since z row x_1 column element in the LP tableau below does not have any negative value, increasing x_1 nor x_2 by 1 will no longer increases the objective function according to the optimality condition theorem.

The Simplex Method: How to obtain a better solution if the bfs is not optimal

Theorem aside, let us look at the original problem in canonical form as:

$$\max z = 20x_1 + 30x_2$$

such that

$$\begin{array}{rcl} x_1 + 2x_2 + s_1 & = & 800 \\ 3x_1 + 4x_2 + s_2 & = & 1800 \\ 3x_1 + x_2 + s_3 & = & 1500 \\ x_1, x_2, s_1, s_2, s_3 & \geq & 0. \end{array}$$

The Simplex Method: How to obtain a better solution if the bfs is not optimal

Substituting $z = 13000$, $x_1 = 200$, $x_2 = 300$, $s_1 = 0$, $s_2 = 0$, $s_3 = 600$ for the original problem in canonical form above makes:

$$\max 13000 = 20 \cdot 200 + 30 \cdot 300$$

$$\begin{array}{rcl} & \text{such that} & \\ 200 + 2 \cdot 300 + 0 & = & 800 \\ 3 \cdot 200 + 4 \cdot 300 + 0 & = & 1800 \\ 3 \cdot 200 + 300 + 600 & = & 1500 \\ x_1 = 200, x_2 = 300, s_1 = 0, s_2 = 0, s_3 = 600 & \geq & 0. \end{array}$$

The Simplex Method: How to obtain a better solution if the bfs is not optimal

We know that the third equality in canonical form is not utilized to obtain this optimal because $s_3 = 600$ and is not zero.

The Simplex Method: How to obtain a better solution if the bfs is not optimal

This shows that the intersection between the first two equality constraints in canonical form constitute the optimal.

The Simplex Method: How to obtain a better solution if the bfs is not optimal

In other words, the operation that makes s_1 as the first exiting variable and forces it to be 0, and then makes s_2 as the second exiting variable and forces it to be 0 gives the solution to the following:

$$\begin{aligned} x_1 + 2x_2 \le 800 \quad &\Longleftrightarrow \quad x_1 + 2x_2 + s_1 = 800, \\ 3x_1 + 4x_2 \le 1800 \quad &\Longleftrightarrow \quad 3x_1 + 4x_2 + s_2 = 1800, \\ x_1, x_2 \ge 0 \quad &\Longleftrightarrow \quad x_1, x_2, s_1 = 0, s_2 = 0 \ge 0. \end{aligned}$$

And this operation finds the next corner point $(x_1, x_2) = (200, 300)$.

The Simplex Method: How to obtain a better solution if the bfs is not optimal

For this problem, we took three steps to find the optimal solution:

Step 1. Find $(x_1, x_2) = (0, 0)$;

Step 2. Find $(x_1, x_2) = (0, 400)$;

Step 3. Find $(x_1, x_2) = (200, 300)$;

The Simplex Method: in R

```
> install.packages("lpSolve")

> library(lpSolve)
```

The Simplex Method: in R

```
> simplex1.obj.fun.c<-c(20,30)
> simplex1.obj.fun.c
[1] 20 30
```

The Simplex Method: in R

```
> simplex1.const.mat.A<-matrix(c(1,2,3,4,3,1),
+ ncol=2,byrow=TRUE)
> simplex1.const.mat.A
[,1] [,2]
[1,] 1 2
[2,] 3 4
[3,] 3 1
```

The Simplex Method: in R

```
> simplex1.const.rhs.b<-c(800,1800,1500)
> simplex1.const.rhs.b
[1] 800 1800 1500
> simplex1.const.dir<-c("<=","<=","<=")
> simplex1.const.dir
[1] "<=" "<=" "<="
```

The Simplex Method: in R

```
> simplex1<-lp("max",simplex1.obj.fun.c,
+ simplex1.const.mat.A,
+ simplex1.const.dir,
+ simplex1.const.rhs.b)
> simplex1
Success: the objective function is 13000
> simplex1$solution
[1] 200 300
```

Practice Problem 1.

Consider a linear program in which all variables are non-negative.

$$\max z = 3x_1 + 4x_2$$

$$\begin{array}{rcl} \text{such that} & & \\ x_1 + 2x_2 & \leq & 6 \\ x_1 + x_2 & \leq & 4 \\ x_1, x_2 & \geq & 0. \end{array}$$

Practice Problem 1.

We use the following template.

basic var.	z	x_1	x_2	s_1	s_2	RHS	Increment

Practice Problem 2.

$$\max z = x_1 + 4x_2 + 6x_3$$

such that

$$\begin{aligned} x_1 + x_2 + x_3 &\leq 2 \\ x_2 + x_3 &\leq 1 \\ x_1, x_2, x_3 &\geq 0. \end{aligned}$$

Practice Problem 2.

We use the following template.

basic var.	z	x_1	x_2	x_3	s_1	s_2	RHS	Increment

The Simplex Method: in R

```
> install.packages("lpSolve")

> library(lpSolve)
```

The Simplex Method: in R

```
> practice1.obj.fun.c<-c(3,4)

> practice1.obj.fun.c

[1] 3 4
```

The Simplex Method: in R

```
> practice1.const.mat.A<-matrix(c(1,2,1,1),
+ ncol=2,byrow=TRUE)
> practice1.const.mat.A
[,1] [,2]
[1,] 1 2
[2,] 1 1
```

The Simplex Method: in R

```
> practice1.const.rhs.b<-c(6,4)

> practice1.const.rhs.b

[1] 6 4

> practice1.const.dir<-c("<=","<=")

> practice1.const.dir

[1] "<=" "<="
```

The Simplex Method: in R

```
> practice1<-lp("max",practice1.obj.fun.c,
+ practice1.const.mat.A,
+ practice1.const.dir,
+ practice1.const.rhs.b)
> practice1
Success: the objective function is 14
> practice1$solution
[1] 2 2
```

The Simplex Method: in R

```
> practice2.obj.fun.c<-c(1,4,6)

> practice2.obj.fun.c

[1] 1 4 6
```

The Simplex Method: in R

```
> practice2.const.mat.A<-matrix(c(1,1,1,0,1,1),
+ ncol=2,byrow=TRUE)
> practice2.const.mat.A
[,1] [,2] [,3]
[1,] 1 1 1
[2,] 0 1 1
```

The Simplex Method: in R

```
> practice2.const.rhs.b<-c(2,1)

> practice2.const.rhs.b

[1] 2 1

> practice2.const.dir<-c("<=","<=")

> practice2.const.dir

[1] "<=" "<="
```

The Simplex Method: in R

```
> practice2<-lp("max",practice2.obj.fun.c,
+ practice2.const.mat.A,
+ practice2.const.dir,
+ practice2.const.rhs.b)
> practice2
Success: the objective function is 7
> practice2$solution
[1] 1 0 1
```

Duality

Given any linear program, there is another related linear program called the dual.

In this section, we will develop an understanding of the dual linear program.

Duality

This understanding translates to important insights about many optimization problems and algorithms in business and economics.

Duality

We begin in the next section by exploring the main concepts of duality through the simple graphical example of building cars and trucks.

Duality

Let us consider a simplified model of an automobile manufacturer that produces cars and trucks.

Duality

Manufacturing is organized into four departments: sheet metal stamping, engine assembly, automobile assembly, and truck assembly.

The capacity of each department is limited.

Duality: Producing Cars and Trucks

The following table provides the percentages of each department's monthly capacity that would be consumed by constructing a thousand cars or a thousand trucks:

Department	Automobile	Truck
metal stamping	4%	2.86%
engine assembly	3%	6%
automobile assembly	4.44%	0%
truck assembly	0%	6.67%

Table: Manufacturing Capacity of an Automobile Manufacturer

Duality: Producing Cars and Trucks

The accounting department estimates a profit of \$3,000 per car produced and \$2,500 per truck produced.

Duality: Producing Cars and Trucks

If the company decides only to produce cars, it could produce 22,500 ($\approx 1,000 \times (100\%/4.44\%$[18]$)$) of them, generating a total profit of $67.5 ($\$3,000$ per car $\times\ 22,500$) million.

[18] Since automobile assembly is the most limiting for manufacturing an automobile in terms of capacity.

Duality: Producing Cars and Trucks

On the other hand, if it only produces trucks, it can produce 15,000 ($\approx 1,000 \times (100\%/6.67$[19]$\%)$) of them, with a total profit of \$37.5 ($\$2,500$ per car $\times\ 15,000$) million.

[19] Since truck assembly is the most limiting for manufacturing a truck in terms of capacity.

Duality: Producing Cars and Trucks

So should the company only produce cars?

Duality: Producing Cars and Trucks

Let us formulate a linear program that will lead us to the optimal solution.

Duality: Producing Cars and Trucks

Define decision variables x_1 and x_2 to be the number in thousands of cars and trucks, respectively, to produce each month.

Duality: Producing Cars and Trucks

Together, they can be thought of as a vector $\boldsymbol{x} = (x_1, x_2)^T \in \Re^2$.

Duality: Producing Cars and Trucks

These quantities have to be positive, so we introduce a constraint $\boldsymbol{x}^T \geq \mathbf{0}$ where $\mathbf{0} = (0, 0)^T$.

Duality: Producing Cars and Trucks

Several additional constraints arise from capacity limitations.

Duality: Producing Cars and Trucks

The car assembly and truck assembly departments limit production according to

$$\begin{aligned} 4.44x_1 &\leq 100, \quad \text{and} \\ 6.67x_2 &\leq 100. \end{aligned}$$

Duality: Producing Cars and Trucks

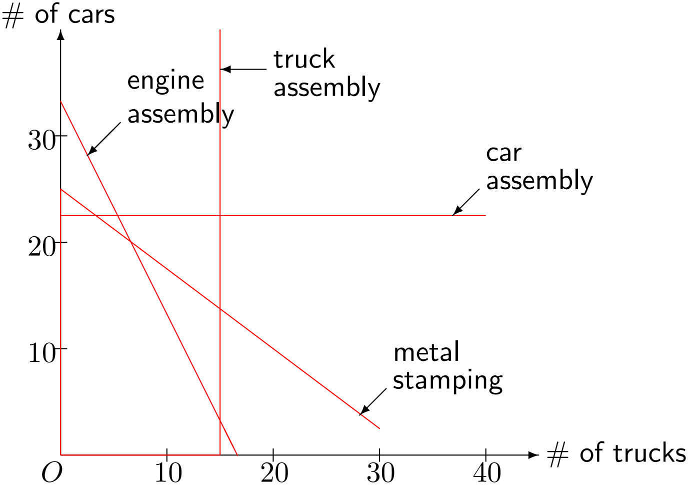
The metal stamping and engine assembly activities also introduce constraints:

$$\begin{aligned} 4x_1 + 2.86x_2 &\leq 100, \quad \text{and} \\ 3x_1 + 6x_2 &\leq 100. \end{aligned}$$

Duality: Producing Cars and Trucks

The set of vectors $\boldsymbol{x} = (x_1, x_2)^T \in \Re^2$ that satisfy these constraints is illustrated in the figure below.

Sensitivity Analysis: Producing Cars and Trucks



Duality: Producing Cars and Trucks

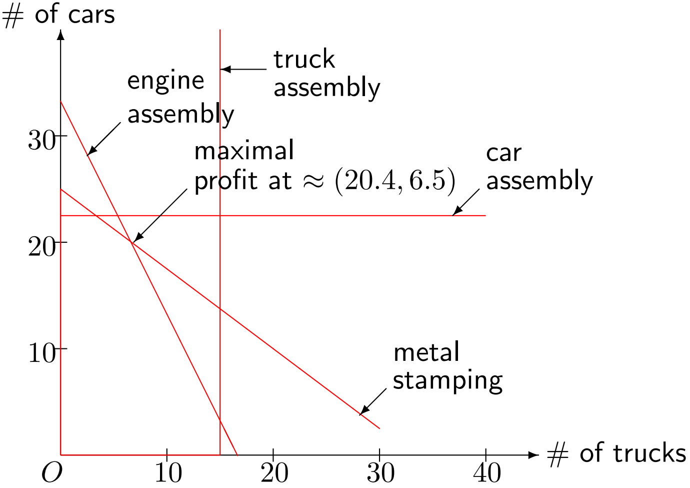
The anticipated profit in thousands of dollars associated with production quantities x_1 and x_2 is

$$Z = 3x_1 + 2.5x_2.$$

Duality: Producing Cars and Trucks

The diagram below also identifies the feasible solution that maximizes profit, which is given approximately by $x_1 = 20.4$ and $x_2 = 6.5$.

Sensitivity Analysis: Producing Cars and Trucks



Duality: Producing Cars and Trucks

The optimal profit is over 77.3 million per month, which exceeds by about 10 million the profit associated with producing only cars.

Duality: Producing Cars and Trucks

Note that this solution involves making use of the entire capacity available for metal stamping and engine assembly, but does not maximize use of capacity to assemble either cars or trucks.

Duality: Producing Cars and Trucks

The linear programming program is given by

$$
\begin{array}{lrl}
\text{maximize} & Z = 3x_1 + 2.5x_2 & \text{(profit in thousands of dollars)} \\
\text{subject to} & 4.44x_1 \leq 100 & \text{(car assembly capacity)} \\
 & 6.67x_2 \leq 100 & \text{(truck assembly capacity)} \\
 & 4x_1 + 2.86x_2 \leq 100 & \text{(metal stamping capacity)} \\
 & 3x_1 + 6x_2 \leq 100 & \text{(engine assembly capacity)} \\
 & x_1 \geq 0, x_2 \geq 0 & \text{(nonnegative production)}
\end{array}
$$

Duality: Producing Cars and Trucks

Written in matrix notation, the linear program becomes

$$\begin{array}{rrcl} \text{maximize} & z & = & \boldsymbol{c}^T \boldsymbol{x} \\ \text{subject to} & \boldsymbol{A}\boldsymbol{x} & \leq & \boldsymbol{b}, \\ & \boldsymbol{x} & \geq & 0 \end{array}$$

where

$$\boldsymbol{c} = \begin{bmatrix} 3 \\ 2.5 \end{bmatrix}, \qquad \boldsymbol{A} = \begin{bmatrix} 4.44 & 0 \\ 0 & 6.67 \\ 4 & 2.86 \\ 3 & 6 \end{bmatrix}, \quad \boldsymbol{b} = \begin{bmatrix} 100 \\ 100 \\ 100 \\ 100 \end{bmatrix}.$$

Duality: Producing Cars and Trucks

The optimal solution of our problem is a basic feasible solution.

Duality: Producing Cars and Trucks

Since there are two decision variables, each basic feasible solution is characterized by a set of two linearly independent binding constraints.[20]

[20] An inequality constraint is said to be binding if it holds with equality at the optimal point.

Duality: Producing Cars and Trucks

At the optimal solution, the two binding constraints are those associated with metal stamping and engine assembly capacity.

Duality: Producing Cars and Trucks

Hence, the optimal solution is the unique solution to a pair of linear equations:

$$
\begin{aligned}
4x_1 + 2.86x_2 &= 100 \quad \text{(metal stamping capacity is binding)} \\
3x_1 + 6x_2 &= 100 \quad \text{(engine assembly capacity is binding).}
\end{aligned}
$$

Duality: Producing Cars and Trucks

In matrix form, these equations can be written as $\bar{\boldsymbol{A}}\boldsymbol{x}^{\mathsf{opt}} = \bar{\boldsymbol{b}}$, where

$$\bar{\boldsymbol{A}} = \begin{bmatrix} 4 & 2.86 \\ 3 & 6 \end{bmatrix}, \quad \bar{\boldsymbol{b}} = \begin{bmatrix} 100 \\ 100 \end{bmatrix}.$$

It is critically important to realize that a solution to linear programming problem comes down to a solution to a system of binding linear constraints.

System of linear equations: A Review continued

From
https://en.wikipedia.org/wiki/Rank_(linear_algebra)
In linear algebra, the rank of a matrix A is the dimension of the vector space generated (or spanned) by its columns.

The rank is commonly denoted rank(A).

System of linear equations: A Review continued

The column rank of A is the dimension of the column space of A, while the row rank of A is the dimension of the row space of A.

System of linear equations: A Review continued

Put it differently, the column rank of A is the number of linearly independent columns of A, while the row rank of A is the number of linearly independent rows of A.

System of linear equations: A Review continued

Theorem (A fundamental result in linear algebra)

The column rank and the row rank are always equal.

This number (i.e., the number of linearly independent rows or columns) is simply called the rank of A.

System of linear equations: A Review continued

The matrix

$$\begin{bmatrix} 1 & 0 & 1 \\ -2 & -3 & 1 \\ 3 & 3 & 0 \end{bmatrix}$$

has rank 2: the first two rows are linearly independent, so the rank is at least 2, but all three rows are linearly dependent (the third is equal to the second subtracted from the first) so the rank must be less than 3.

System of linear equations: A Review continued

The matrix

$$A = \begin{bmatrix} 1 & 1 & 0 & 2 \\ -1 & -1 & 0 & -2 \end{bmatrix}$$

has rank 1: there are nonzero columns, so the rank is positive, but any pair of columns is linearly dependent.

System of linear equations: A Review continued

Similarly, the transpose

$$A^T = \begin{bmatrix} 1 & -1 \\ 1 & -1 \\ 0 & 0 \\ 2 & -2 \end{bmatrix}$$

of A has rank 1.

System of linear equations: A Review continued

Since the column vectors of A are the row vectors of the transpose of A, the statement that the column rank of a matrix equals its row rank is equivalent to the statement that the rank of a matrix is equal to the rank of its transpose, i.e., $\text{rank}(A) = \text{rank}(A^T)$.

System of linear equations: A Review continued

Computing the rank of a matrix

A common approach to finding the rank of a matrix is to reduce it to a simpler form, generally row echelon form, by elementary row operations.

System of linear equations: A Review continued

Once in row echelon form, the rank equals the number of pivots (or basic columns) and also the number of non-zero rows.

System of linear equations: A Review continued

For example, the matrix A given by

$$A = \begin{bmatrix} 1 & 2 & 1 \\ -2 & -3 & 1 \\ 3 & 5 & 0 \end{bmatrix}$$

can be put in reduced row-echelon form by using the following elementary row operations:

System of linear equations: A Review continued

$$\begin{bmatrix} 1 & 2 & 1 \\ -2 & -3 & 1 \\ 3 & 5 & 0 \end{bmatrix} R_2 \Longleftrightarrow 2r_1 + r_2 \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 3 & 5 & 0 \end{bmatrix}$$

System of linear equations: A Review continued

$$\begin{bmatrix}1 & 2 & 1\\0 & 1 & 3\\3 & 5 & 0\end{bmatrix} R_3 \Longleftrightarrow -3r_1 + r_3 \begin{bmatrix}1 & 2 & 1\\0 & 1 & 3\\0 & -1 & -3\end{bmatrix}$$

System of linear equations: A Review continued

$$\begin{bmatrix}1 & 2 & 1\\0 & 1 & 3\\0 & -1 & -3\end{bmatrix} R_3 \Longleftrightarrow r_2 + r_3 \begin{bmatrix}1 & 2 & 1\\0 & 1 & 3\\0 & 0 & 0\end{bmatrix}$$

System of linear equations: A Review continued

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix} R_1 \Longleftrightarrow -2r_2 + r_1 \begin{bmatrix} 1 & 0 & -5 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

The final matrix (in row echelon form) has two non-zero rows and thus the rank of matrix A is 2.

System of linear equations: A Review continued

In linear algebra, an n-by-n square matrix A is called invertible (also nonsingular or nondegenerate) if there exists an n-by-n square matrix B such that

$$\mathbf{AB} = \mathbf{BA} = \mathbf{I}_n$$

where I_n denotes the n-by-n identity matrix with ones on the main diagonal and zeros elsewhere and the multiplication used is ordinary matrix multiplication.

System of linear equations: A Review continued

If this is the case, then the matrix B is uniquely determined by A and is called the inverse of A, denoted by A^{-1}.

System of linear equations: A Review continued

Theorem (The invertible matrix theorem)

Let A be a square n by n matrix. The following statements are equivalent:
(1) A is invertible;
(2) A has full rank; that is, $\mathrm{rank}(A)=n$;
(3) The columns of A are linearly independent;
(4) The columns of A form a basis of $\Re^n$;
(5) The transpose A^T is an invertible matrix (hence rows of A are linearly independent, span $\Re^n$, and form a basis of $\Re^n$).

System of linear equations: A Review continued

Theorem (Other properties)

The following properties hold for an invertible matrix A:

(1) $(A^{-1})^{-1} = A$;

(2) $(kA)^{-1} = k^{-1}A^{-1}$ for nonzero scalar k;

(3) $(A^T)^{-1} = (A^{-1})^T$;

(4) For any invertible n-by-n matrices A and B, $(AB)^{-1} = B^{-1}A^{-1}$.

Duality: Producing Cars and Trucks

Since the number of linearly independent columns or rows is 2, the 2 by 2 matrix $\bar{\boldsymbol{A}}$ has full rank or rank $(\bar{\boldsymbol{A}})$=2.

$$\bar{\boldsymbol{A}} = \left[\begin{array}{cc} 4 & 2.86 \\ 3 & 6 \end{array} \right].$$

Duality: Producing Cars and Trucks

This is obvious enough, but if you wish in general to find out the rank of a matrix using R, you need to install package "Matrix" and read the package into memory.

You can find the details of the package "Matrix" at `https://cran.r-project.org/web/packages/Matrix/Matrix.pdf`.

Duality: Producing Cars and Trucks

In R, we can obtain the rank of matrix $\bar{A}$ as follows:

```
> install.packages("Matrix")

> library(Matrix)
```

Duality: Producing Cars and Trucks

We define $\bar{A}$ as

```
> Abar<-matrix(c(4,2.86,3,6),
+ ncol=2,byrow=TRUE)
```

```
> Abar

[,1] [,2]
[1,] 4 2.86
[2,] 3 6.00
```

Duality: Producing Cars and Trucks

Then the rank of the matrix $\bar{A}$ is 2 as shown the first output:

```
> rankMatrix(Abar)

[1] 2
attr(,"method")
[1] "tolNorm2"
attr(,"useGrad")
[1] FALSE
attr(,"tol")
[1] 4.440892e-16
```

where `method` is a character string specifying the computational `method` for the rank, `tol` is nonnegative number specifying a (relative, "scalefree") `tol`erance for testing of "practically zero" with specific meaning depending on method.

Duality: Producing Cars and Trucks

Since $\bar{\boldsymbol{A}}$ has full rank or rank $(\bar{\boldsymbol{A}})$=2, it has an inverse $\bar{\boldsymbol{A}}^{-1}$.
Inversion of 2 by 2 matrices can be done as follows:

$$\mathbf{A}^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

Therefore

$$\begin{aligned} \bar{\boldsymbol{A}}^{-1} &= \begin{bmatrix} 4 & 2.86 \\ 3 & 6 \end{bmatrix}^{-1} = \frac{1}{4 \cdot 6 - 2.86 \cdot 3} \begin{bmatrix} 6 & -2.86 \\ -3 & 4 \end{bmatrix} \\ &\approx \begin{bmatrix} 0.389 & -0.185 \\ -0.195 & 0.259 \end{bmatrix}. \end{aligned}$$

Duality: Producing Cars and Trucks

In R, we can obtain the inverse $\bar{\boldsymbol{A}}^{-1}$ of $\bar{\boldsymbol{A}}$ as follows:

```
> Inv.Abar<-solve(Abar)
```

```
> Inv.Abar

[,1] [,2]
[1,] 0.3891051 -0.1854734
[2,] -0.1945525 0.2594034
```

Duality: Producing Cars and Trucks

The optimal solution of the linear program is given by $\boldsymbol{x}^{\text{opt}} = \bar{\boldsymbol{A}}^{-1}\bar{\boldsymbol{b}}$, and therefore, the optimal profit is $c^T\boldsymbol{x}^{\text{opt}} = c^T\bar{\boldsymbol{A}}^{-1}\bar{\boldsymbol{b}} \approx 77.3$.

Duality: Producing Cars and Trucks

For this, we first define $\bar{\boldsymbol{b}}$ as

```
> bbar<-matrix(c(100,100),ncol=1,byrow=TRUE)

> bbar

[,1]
[1,] 100
[2,] 100
```

Duality: Producing Cars and Trucks

We then define c as

```
> c<-matrix(c(3,2.5),ncol=1,byrow=TRUE)

> c

[,1]
[1,] 3.0
[2,] 2.5
```

Duality: Producing Cars and Trucks

Then we obtain $c^T \bar{\boldsymbol{A}}^{-1} \bar{\boldsymbol{b}} \approx 77.3$ as

```
> t(c)%*%Inv.Abar%*%bbar

[,1]
[1,] 77.3022
```

Duality: Producing Cars and Trucks

This solution turns out to be equivalent to linear programming solution to the original problem as shown below:

```
> install.packages("lpSolve")

> library("lpSolve")
```

Duality: Producing Cars and Trucks

```
> dir<-c("<=","<=","<=","<=")
> dir
[1] "<=" "<=" "<=" "<="
```

Duality: Producing Cars and Trucks

```
> A<-matrix(c(4.44,0,0,6.67,4,2.86,3,6),
+ ncol=2,byrow=TRUE)

> A

[,1] [,2]
[1,] 4.44 0.00
[2,] 0.00 6.67
[3,] 4.00 2.86
[4,] 3.00 6.00
```

Duality: Producing Cars and Trucks

```
> b<-matrix(c(100,100,100,100),
+ ncol=1,byrow=TRUE)

> b

     [,1]
[1,]  100
[2,]  100
[3,]  100
[4,]  100
```

Duality: Producing Cars and Trucks

```
> lp("max",c,A,dir,b)
Success: the objective function is 77.3022
> lp("max",c,A,dir,b)$solution
[1] 20.363165 6.485084
```

The Dual Linear Program

Remember that the percentages of each department's monthly capacity that would be consumed by building a thousand cars or a thousand trucks:

Department	Automobile	Truck
metal stamping	4%	2.86%
engine assembly	3%	6%
automobile assembly	4.44%	0%
truck assembly	0%	6.67%

Table: Manufacturing Capacity of an Automobile Manufacturer

Also the marketing department estimates a profit of \$3,000 per car produced and \$2,500 per truck produced.

Duality: Producing Cars and Trucks

Let the maximal profit in our original linear program is $z(\mathbf{0})$, then the original linear programming problem is

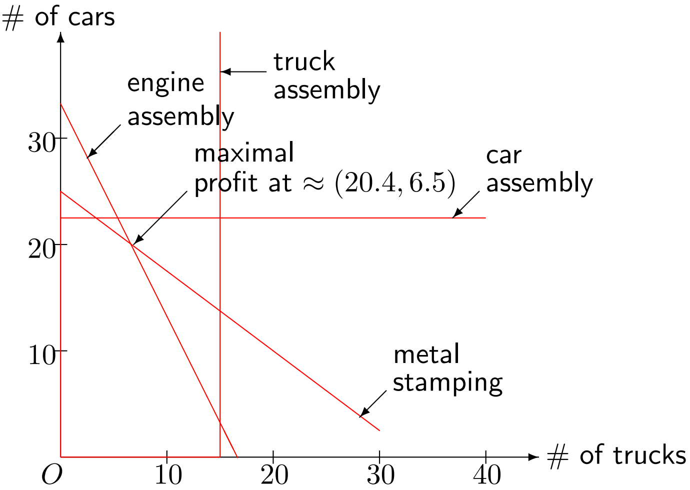
$$\begin{array}{lrcl} \text{maximize} & z(\mathbf{0}) & = & \boldsymbol{c}^T\boldsymbol{x} \\ \text{subject to} & \boldsymbol{A}\boldsymbol{x} & \leq & \boldsymbol{b}, \\ & \boldsymbol{x} & \geq & \mathbf{0} \end{array}$$

where

$$\boldsymbol{c} = \begin{bmatrix} 3 \\ 2.5 \end{bmatrix}, \qquad \boldsymbol{A} = \begin{bmatrix} 4.44 & 0 \\ 0 & 6.67 \\ 4 & 2.86 \\ 3 & 6 \end{bmatrix}, \quad \boldsymbol{b} = \begin{bmatrix} 100 \\ 100 \\ 100 \\ 100 \end{bmatrix}.$$

In a graph, we have the following:

Sensitivity Analysis: Producing Cars and Trucks



Sensitivity Analysis: Producing Cars and Trucks

Since the metal stamping and engine assembly capacity constraints are binding at the optimal solution to the linear program, or:

$$
\begin{aligned}
4x_1 + 2.86x_2 &= 100 \quad \text{(metal stamping capacity is binding)} \\
3x_1 + 6x_2 &= 100 \quad \text{(engine assembly capacity is binding).}
\end{aligned}
$$

Duality: Producing Cars and Trucks

In matrix form, these equations can be written as $\bar{\boldsymbol{A}}\boldsymbol{x}^{\text{opt}} = \bar{\boldsymbol{b}}$, where

$$\bar{\boldsymbol{A}} = \begin{bmatrix} 4 & 2.86 \\ 3 & 6 \end{bmatrix}, \quad \bar{\boldsymbol{b}} = \begin{bmatrix} 100 \\ 100 \end{bmatrix}.$$

Since $\bar{\boldsymbol{A}}$ is invertible with $\bar{\boldsymbol{A}}^{-1}$ as its inverse, we have

$$\boldsymbol{x}^{\text{opt}} = \bar{\boldsymbol{A}}^{-1}\bar{\boldsymbol{b}},$$

and the maximal profit for the original linear programming problem is

$$z(\mathbf{0}) \quad = \quad \boldsymbol{c}^T\boldsymbol{x}^{\text{opt}} = \boldsymbol{c}^T\bar{\boldsymbol{A}}^{-1}\bar{\boldsymbol{b}}.$$

Sensitivity Analysis: Producing Cars and Trucks

Suppose we wish to increase profit by expanding manufacturing capacities.

In such a situation, it is useful to think of profit as a function of a vector $\mathbf{\Delta} \in \Re^4$ of changes to capacity.

Sensitivity Analysis: Producing Cars and Trucks

We denote this profit by $z(\boldsymbol{\Delta})$, defined to be the maximal objective value associated with the linear programming problem

$$\begin{array}{rrcl} \text{maximize} & z(\boldsymbol{\Delta}) & = & \boldsymbol{c}^T\boldsymbol{x} \\ \text{subject to} & \boldsymbol{A}\boldsymbol{x} & \leq & \boldsymbol{b}+\boldsymbol{\Delta}, \\ & \boldsymbol{x} & \geq & \mathbf{0} \end{array}$$

where

$$\boldsymbol{c} = \begin{bmatrix} 3 \\ 2.5 \end{bmatrix}, \qquad \boldsymbol{A} = \begin{bmatrix} 4.44 & 0 \\ 0 & 6.67 \\ 4 & 2.86 \\ 3 & 6 \end{bmatrix}, \quad \boldsymbol{b} = \begin{bmatrix} 100+\Delta \\ 100+\Delta \\ 100+\Delta \\ 100+\Delta \end{bmatrix}.$$

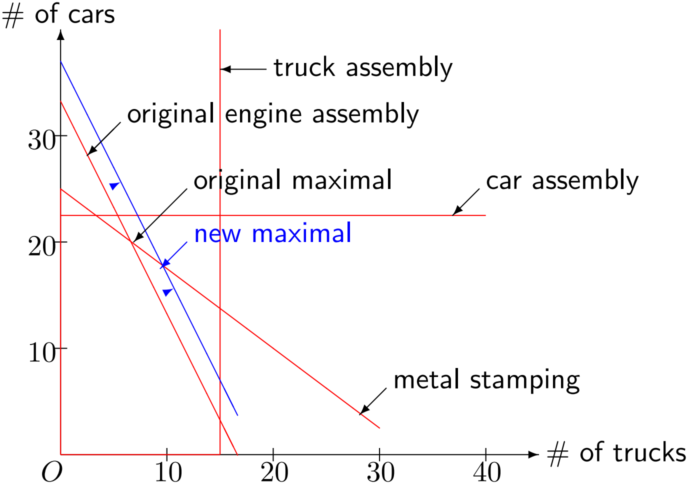
Sensitivity Analysis: Producing Cars and Trucks

We will examine how incremental changes $z(\mathbf{\Delta})$ in capacities influence the optimal profit.

The study of such changes is called sensitivity analysis.

We have a graph below where the engine assembly capacity is increased by a small amount.

Sensitivity Analysis: Producing Cars and Trucks



Sensitivity Analysis: Producing Cars and Trucks

Knowing that the metal stamping and engine assembly capacity constraints are binding, this new optimal solution must be given by

$$\boldsymbol{x}^{\text{new opt}} = \bar{\boldsymbol{A}}^{-1}\left(\bar{\boldsymbol{b}} + \bar{\boldsymbol{\Delta}}\right),$$

where

$$\bar{\boldsymbol{\Delta}} = \begin{bmatrix} \Delta_3 \\ \Delta_4 \end{bmatrix}.$$

The new optimal profit must be

$$z\left(\boldsymbol{\Delta}\right) = \boldsymbol{c}^T \boldsymbol{x}^{\text{new opt}} = \boldsymbol{c}^T \bar{\boldsymbol{A}}^{-1}\left(\bar{\boldsymbol{b}} + \bar{\boldsymbol{\Delta}}\right)$$

Sensitivity Analysis: Producing Cars and Trucks

Furthermore, the difference in profit is

$$z\left(\boldsymbol{\Delta}\right) - z(\mathbf{0}) = \boldsymbol{c}^T \bar{\boldsymbol{A}}^{-1} \bar{\boldsymbol{\Delta}}.$$

Sensitivity Analysis: Producing Cars and Trucks

This matrix equation provides a way to gauge the impact of changes in capacities on optimal profit in the event that the set of binding constraints does not change.

Sensitivity Analysis: Producing Cars and Trucks

It turns out that this also gives us the information required to conduct sensitivity analysis.

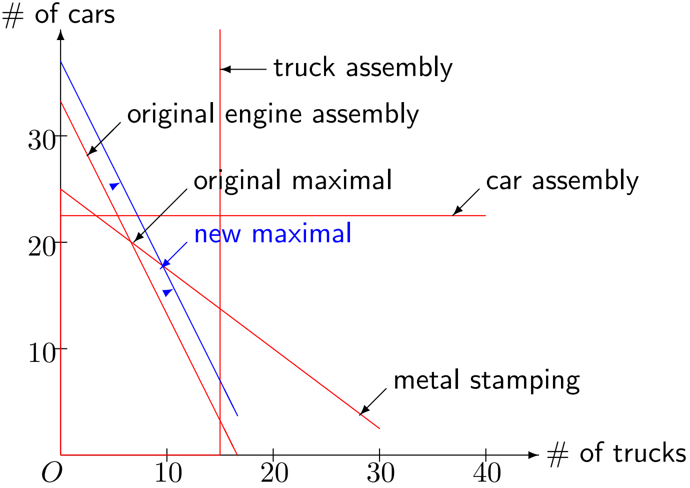
Sensitivity Analysis: Producing Cars and Trucks

This is because small changes in capacities will not change which constraints are binding.

Sensitivity Analysis: Producing Cars and Trucks

To understand why, consider the case where the engine assembly capacity is increased by a small amount as shown below.

Sensitivity Analysis: Producing Cars and Trucks



Sensitivity Analysis: Producing Cars and Trucks

Clearly, the new optimal solution is still at the intersection where metal stamping and engine assembly capacity constraints are binding.

Sensitivity Analysis: Producing Cars and Trucks

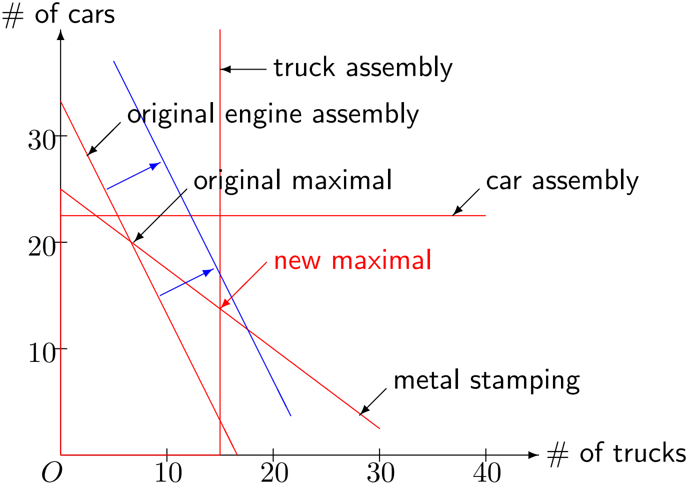
Similarly, one can easily see that incremental changes in any of the other capacity constraints will not change the fact that metal stamping and engine assembly capacity constraints are binding.

Sensitivity Analysis: Producing Cars and Trucks

This observation does not hold when we consider large changes.

As illustrated in Figure below, sufficiently large changes can result in a different set of binding constraints.

Sensitivity Analysis: Producing Cars and Trucks



Sensitivity Analysis: Producing Cars and Trucks

The figure shows how after a large increase in engine assembly capacity, the associated constraint is no longer binding.

Instead, the truck assembly capacity constraint becomes binding.

Sensitivity Analysis: Producing Cars and Trucks

The *sensitivity* y_i of profit to quantity of the i-th resource is the rate at which $z(\mathbf{\Delta})$ increases as Δ_i. i-th element in $\mathbf{\Delta}$ increases by one unit, starting from $\Delta_i = 0$.

Sensitivity Analysis: Producing Cars and Trucks

It is clear that small changes in **nonbinding capacities** do not influence profit at all.

Hence, sensitivities y_1 and y_2 of profit to quantity of the car assembly capacity and truck assembly capacity are $y_1 = y_2 = 0$.

Sensitivity Analysis: Producing Cars and Trucks

From the preceding discussion, we have

$$z(\boldsymbol{\Delta}) - z(\mathbf{0}) = \boldsymbol{c}^T \bar{\boldsymbol{A}}^{-1} \bar{\boldsymbol{\Delta}}.$$

where

$$\boldsymbol{c} = \begin{bmatrix} 3 \\ 2.5 \end{bmatrix}, \quad \bar{\boldsymbol{A}} = \begin{bmatrix} 4 & 2.86 \\ 3 & 6 \end{bmatrix}$$

Sensitivity Analysis: Producing Cars and Trucks

Therefore

$$
\begin{aligned}
\bar{\boldsymbol{y}} = [y_3 \quad y_4] &= \boldsymbol{c}^T \bar{\boldsymbol{A}}^{-1} \\
&= [3 \quad 2.5] \begin{bmatrix} 4 & 2.86 \\ 3 & 6 \end{bmatrix}^{-1} \\
&\approx [3 \quad 2.5] \begin{bmatrix} 0.389 & -0.185 \\ -0.195 & 0.259 \end{bmatrix} \\
&\approx [0.681 \quad 0.092].
\end{aligned}
$$

Sensitivity Analysis: Producing Cars and Trucks

In other words, the sensitivity is about \$0.681 million per percentage of metal stamping capacity and \$0.092 million per percentage of engine assembly capacity.

Sensitivity Analysis: Producing Cars and Trucks

If a 1% increase in metal stamping capacity requires the same investment as a 1% increase in engine assembly, we should invest in metal stamping.

Shadow Prices and Valuation of the Firm: Producing Cars and Trucks

The sensitivities of profit to resource quantities are commonly called shadow prices.

Each i-th resource has a shadow price y_i.

Shadow Prices and Valuation of the Firm: Producing Cars and Trucks

The shadow price associated with a particular constraint tells you **how much the optimal value of the objective would increase per unit increase in the amount of resources available**.

Shadow Prices and Valuation of the Firm: Producing Cars and Trucks

Interpretation 1

The shadow price associated with a resource tells you how much **more profit** you would get by increasing the amount of that resource by **one unit**.

Shadow Prices and Valuation of the Firm: Producing Cars and Trucks

Interpretation 2

The shadow price associated with a resource represents the maximal price at which we should be willing to buy **additional unit** of that resource.

Shadow Prices and Valuation of the Firm: Producing Cars and Trucks

Interpretation 3

The shadow price associated with a resource represents the minimal price at which we should be willing to sell **one unit** of that resource.

Shadow Prices and Valuation of the Firm: Producing Cars and Trucks

A shadow price might therefore be thought of as the value per unit of a resource.

These three interpretations are important, but not surprising.

Shadow Prices and Valuation of the Firm: Producing Cars and Trucks

In our example of building cars and trucks, shadow prices for car and truck assembly capacity are zero.

Shadow Prices and Valuation of the Firm: Producing Cars and Trucks

Shadow prices of engine assembly and metal stamping capacity, on the other hand, are \$0.092 and \$0.681 million per percent.

Shadow Prices and Valuation of the Firm: Producing Cars and Trucks

Based on the discussion in the previous section, if the metal stamping and engine assembly capacity constraints remain binding when resource quantities are set at $\boldsymbol{b} + \Delta$, the optimal profit is given by

$$z(\boldsymbol{\Delta}) = z(\boldsymbol{0}) + \boldsymbol{y}^{\boldsymbol{T}} \boldsymbol{\Delta},$$

where

$$\boldsymbol{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0.681 \\ 0.092 \end{bmatrix}.$$

Shadow Prices and Valuation of the Firm: Producing Cars and Trucks

I stated that a shadow price might therefore be thought of as the value per unit of a resource.

Shadow Prices and Valuation of the Firm: Producing Cars and Trucks

This fact can be restated in the following way:

If we compute the value of our entire stock of resources based on shadow prices, we get our optimal profit.

Shadow Prices and Valuation of the Firm: Producing Cars and Trucks

For instance, in our example of building cars and trucks, we have

$$0.092 \times 100 + 0.681 \times 100 = 77.3.$$

Shadow Prices and Valuation of the Firm: Producing Cars and Trucks

This is not just a coincidence but reflects a fundamental property of shadow prices.

This is a truly surprising and an important result.

Shadow Prices and Valuation of the Firm: Producing Cars and Trucks

From the discussion above we know that as long as the metal stamping and engine assembly constraints are binding,

$$z(\boldsymbol{\Delta}) = z(\mathbf{0}) + \boldsymbol{y}^{\boldsymbol{T}} \boldsymbol{\Delta}.$$

Shadow Prices and Valuation of the Firm: Producing Cars and Trucks

If we let $\boldsymbol{\Delta} = -\boldsymbol{b}$ in the linear program

$$\begin{array}{rrcl} \text{maximize} & z & = & \boldsymbol{c}^T \boldsymbol{x} \\ \text{subject to} & \boldsymbol{A}\boldsymbol{x} & \leq & \boldsymbol{b} + \boldsymbol{\Delta}, \\ & \boldsymbol{x} & \geq & \mathbf{0}, \end{array}$$

then the resulting linear program has 0 capacity at each plant, so the optimal solution is 0, with associated profit of 0.

Shadow Prices and Valuation of the Firm: Producing Cars and Trucks

Moreover, both the metal stamping and engine assembly constraints are still binding.

Shadow Prices and Valuation of the Firm: Producing Cars and Trucks

This means that

$$\mathbf{0} = z\left(-\boldsymbol{b}\right) = z\left(\mathbf{0}\right) + \boldsymbol{y}^{\boldsymbol{T}}\left(-\boldsymbol{b}\right).$$

Shadow Prices and Valuation of the Firm: Producing Cars and Trucks

Rearranging this gives that

$$z(\mathbf{0}) = \boldsymbol{y}^{\boldsymbol{T}}\boldsymbol{b}.$$

where

$$\boldsymbol{y} = \begin{bmatrix} 0 \\ 0 \\ 0.681 \\ 0.092 \end{bmatrix}, \qquad \boldsymbol{b} = \begin{bmatrix} 100 \\ 100 \\ 100 \\ 100 \end{bmatrix}.$$

Shadow Prices and Valuation of the Firm: Producing Cars and Trucks

This is a remarkable fundamental result:

the net value of our current resources, valued at their shadow prices, is equal to the maximal profit that we can obtain through operation of the firm, i.e., **the value of the firm**.

Shadow Prices and Valuation of the Firm: Producing Cars and Trucks

In other words, the current resources valued at their shadow prices is the value at and above which the firm can be acquired.

Shadow Prices and Valuation of the Firm: Producing Cars and Trucks

If an acquisition proposal for this firm is priced below this value, it would be more profitable for the current stock holders of the company to keep the company and have them operate as is.

Shadow Prices and Valuation of the Firm: Producing Cars and Trucks

Interpretation 4

To the extent a firm's operation is closely approximated by a linear program problem, **the resulting shadow prices associated with its resources determines the value of the firm.**

The Dual Linear Program

Shadow prices solve another linear program, called the *dual*.

The Dual Linear Program

The original linear program of interest—in this case, the one involving decisions on quantities of cars and trucks to build in order to maximize profit—is called the *primal*.

The Dual Linear Program

We now formulate the dual.

The Dual Linear Program

Consider a situation where we are managing the firm but do not know linear programming.

The Dual Linear Program

Therefore, we do not know exactly what the optimal decisions or optimal profit are.

The Dual Linear Program

Company X approaches us and expresses a desire to purchase all the capacity at our four factories, that is, metal stamping, engine assembly, automobile assembly, and truck assembly factories.

The Dual Linear Program

We enter into a negotiation over the prices

$$\boldsymbol{y} = \begin{bmatrix} y_1 & \text{(for car assembly factory)} \\ y_2 & \text{(for truck assembly factory)} \\ y_3 & \text{(for metal stamping factory)} \\ y_4 & \text{(for engine assembly factory)} \end{bmatrix} \in \Re^4$$

that we should charge per percentage of capacity at each of our four factories. Note that each element in $\boldsymbol{y}$ is in million dollars per 1%.

The Dual Linear Program

Thus company X pays

$$z = 100y_1 + 100y_2 + 100y_3 + 100y_4.$$

Since each element in $\boldsymbol{y}$ is in million dollars per 1%, multiplying one hundred percent and adding all the prices give the total price of our 100% manufacturing capacities in terms of 1,000 dollars.

Obviously company X wishes to minimize this amount.

The Dual Linear Program

To make this deal to be of any interest to us, the prices must be nonnegative:

$$\boldsymbol{y} \geq \mathbf{0}.$$

The Dual Linear Program

Remember that the percentages of each department's monthly capacity that would be consumed by building a thousand cars or a thousand trucks:

Department	Automobile	Truck
metal stamping	4%	2.86%
engine assembly	3%	6%
automobile assembly	4.44%	0%
truck assembly	0%	6.67%

Table: Manufacturing Capacity of an Automobile Manufacturer

Also the marketing department estimates a profit of \$3,000 per car produced and \$2,500 per truck produced.

The Dual Linear Program

We argue that **selling a bundle of capacity that could be used to produce 1,000 automobiles must be at least as profitable as producing those 1,000 automobile**.

The Dual Linear Program

When you manufacture 1,000 cars, you earn 3,000 dollars/car $\times$ 1,000 cars, which equals 3 million dollars.

The Dual Linear Program

When you sell to company X a 1,000 car manufacturing capacity, you will earn 4.44%$\times y_1$ million dollars/%, 4%$\times y_3$ million dollars/%, and 3%$\times y_4$ million dollars/% combined.

Hence in million dollars, we have the following car manufacturing constraint:

$$4.44y_1 + 4y_3 + 3y_4 \geq 3.$$

The Dual Linear Program

We also argue that **selling a bundle of capacity that could be used to produce 1,000 trucks is at least as profitable as producing those 1,000 trucks**.

The Dual Linear Program

When you manufacture 1,000 trucks, you earn 2,500 dollars/car $\times$ 1,000 cars, which equals 2.5 million dollars.

The Dual Linear Program

When you sell to company X a 1,000 truck manufacturing capacity, you will earn 6.67%$\times y_2$ million dollars/%, 2.86%$\times y_3$ million dollars/%, and 6%$\times y_4$ million dollars/% combined.

Hence in million dollars, we also have the following truck manufacturing constraint:

$$6.67y_2 + 2.86y_3 + 6y_4 \geq 2.5.$$

The Dual Linear Program

Given our requirements, Company X solves a linear program to determine prices that minimize the amount it would have to pay to purchase all of our capacity:

$$
\begin{aligned}
\text{minimize} \quad & 100y_1 + 100y_2 + 100y_3 + 100y_4 \\
\text{subject to} \quad & 4.44y_1 + 4y_3 + 3y_4 \geq 3, \quad \text{(car production)} \\
& 6.67y_2 + 2.86y_3 + 6y_4 \geq 2.5, \quad \text{(truck production)} \\
& y_1, y_2, y_3, y_4 \geq 0. \quad \text{(nonnegative prices)}
\end{aligned}
$$

Duality: Producing Cars and Trucks

In matrix notation, we have

$$\text{minimize} \quad \begin{bmatrix} 100 \\ 100 \\ 100 \\ 100 \end{bmatrix}^T \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix}$$

$$\text{subject to} \quad \begin{bmatrix} 4.44 & 0 & 4 & 3 \\ 0 & 6.67 & 2.86 & 6 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} \geq \begin{bmatrix} 3 \\ 2.5 \end{bmatrix},$$

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} \geq \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Duality: Producing Cars and Trucks

Note that the original linear program is

$$\begin{array}{rrcl} \text{maximize} & z & = & \boldsymbol{c}^T \boldsymbol{x} \\ \text{subject to} & \boldsymbol{A}\boldsymbol{x} & \leq & \boldsymbol{b}, \\ & \boldsymbol{x} & \geq & \boldsymbol{0} \end{array}$$

where

$$\boldsymbol{c} = \left[\begin{array}{c} 3 \\ 2.5 \end{array} \right], \qquad \boldsymbol{A} = \left[\begin{array}{cc} 4.44 & 0 \\ 0 & 6.67 \\ 4 & 2.86 \\ 3 & 6 \end{array} \right], \quad \boldsymbol{b} = \left[\begin{array}{c} 100 \\ 100 \\ 100 \\ 100 \end{array} \right].$$

Duality: Producing Cars and Trucks

Therefore this new dual problem can be written as

$$\begin{array}{rrcl} & \text{minimize} & & \boldsymbol{b}^T\boldsymbol{y} \\ \text{subject to} & \boldsymbol{A}^T\boldsymbol{y} & \geq & \boldsymbol{c}, \\ & \boldsymbol{y} & \geq & \boldsymbol{0}, \end{array}$$

where

$$\boldsymbol{c} = \begin{bmatrix} 3 \\ 2.5 \end{bmatrix}, \qquad \boldsymbol{A} = \begin{bmatrix} 4.44 & 0 \\ 0 & 6.67 \\ 4 & 2.86 \\ 3 & 6 \end{bmatrix}, \quad \boldsymbol{b} = \begin{bmatrix} 100 \\ 100 \\ 100 \\ 100 \end{bmatrix}.$$

Duality: Producing Cars and Trucks

We now obtain the optimal solution to this dual linear programming problem as shown below:

```
> install.packages("lpSolve")

> library("lpSolve")
```

Duality: Producing Cars and Trucks

```
> dual.dir<-c(">=",">=")
> dual.dir
[1] ">=" ">="
```

Duality: Producing Cars and Trucks

If you saved the previous R workspace, you do not need to define $\boldsymbol{A}$ as below:

```
> A<-matrix(c(4.44,0,0,6.67,4,2.86,3,6),
+ ncol=2,byrow=TRUE)

> A

[,1] [,2]
[1,] 4.44 0.00
[2,] 0.00 6.67
[3,] 4.00 2.86
[4,] 3.00 6.00
```

Duality: Producing Cars and Trucks

If you saved the previous R workspace, you do not need to define $\boldsymbol{b}$ as below:

```
> b<-matrix(c(100,100,100,100),
+ ncol=1,byrow=TRUE)

> b

     [,1]
[1,]  100
[2,]  100
[3,]  100
[4,]  100
```

Duality: Producing Cars and Trucks

If you saved the previous R workspace, you do not need to define c as below:

```
> c<-matrix(c(3,2.5),ncol=1,byrow=TRUE)

> c

[,1]
[1,] 3.0
[2,] 2.5
```

Duality: Producing Cars and Trucks

```
> lp("min",t(b),t(A),dual.dir,c)

Success: the objective function is 77.3022

> lp("min",t(b),t(A),dual.dir,c)$solution

[1] 0.0000000 0.0000000 0.6809339 0.0920882
```

The Dual Linear Program

The optimal solution to this linear program is

$$\boldsymbol{y} = \begin{bmatrix} 0 \\ 0 \\ 0.092 \\ 0.681 \end{bmatrix},$$

and the minimal value of the objective function is 77.3.

The Dual Linear Program

Remarkably, we have recovered the shadow prices and the optimal profit!

The Dual Linear Program

It is not a coincidence that

• **the minimal cost in the dual equals the optimal profit in the primal**

and that

• **the optimal solution of the dual is the vector of shadow prices for each one of the resouces of the primal**.

The Dual Linear Program

These are fundamental relations between the *primal* and the *dual*.

The Dual Linear Program

We offer an **intuitive** or **heuristic** explanation now.

The Dual Linear Program

The constraints ensure that we receive at least as much money from selling all the manufacturing capacity as we would from manufacturing.

The Dual Linear Program

Therefore, it seems clear that the minimal cost in the dual is at least as large as the maximal profit in the primal.

This fact is known as **weak duality**.

The Dual Linear Program

Another result, referred to as **strong duality**, asserts that the minimal cost in the dual equals the maximal profit in the primal.

The Dual Linear Program

This is not obvious.

The Dual Linear Program

It is motivated to some extent, though, by the fact that Company X is trying to get the best deal it can.

The Dual Linear Program

It is natural to think that if Company X negotiates effectively, it should be able to acquire all our resources for an amount of money equal that we would obtain as profit from manufacturing.

This would imply strong duality.

The Dual Linear Program

Why, now, should an optimal solution to the dual provide shadow prices?

The Dual Linear Program

To see this, consider changing the resource quantities by a small amount $\mathbf{\Delta} \in \Re^4$.

The Dual Linear Program

Then, the dual based on the primal on the left becomes as shown below on the right

$$\begin{array}{llcll} \text{maximize} & \boldsymbol{c}^T\boldsymbol{x} & & \text{minimize} & (\boldsymbol{b}+\boldsymbol{\Delta})^T\boldsymbol{y} \\ \text{subject to} & \boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b}+\boldsymbol{\Delta} & \text{and} & \text{subject to} & \boldsymbol{A}^T\boldsymbol{y} \geq \boldsymbol{c} \\ & \boldsymbol{x} \geq \boldsymbol{0} & & & \boldsymbol{y} \geq \boldsymbol{0}. \end{array}$$

The Dual Linear Program

From our previous derivation, the maximal profit in the primal and the minimal cost in the dual are both equal to $z(\boldsymbol{\Delta})$.

The Dual Linear Program

Suppose that the optimal solution to the dual is unique—as is the case in our example of building cars and trucks.

The Dual Linear Program

Then, for sufficiently small $\boldsymbol{\Delta}$, the optimal solutions to the dual with and without the increment $\boldsymbol{\Delta}$ stay as $z\left(\boldsymbol{\Delta}\right) = \left(\boldsymbol{b} + \boldsymbol{\Delta}\right)^T \boldsymbol{y}^{\text{opt}}$ and $z\left(\mathbf{0}\right) = \boldsymbol{b}^T \boldsymbol{y}^{\text{opt}}$.

The Dual Linear Program

Therefore the optimal profit in the dual should change by

$$z(\mathbf{\Delta}) - z(\mathbf{0}) = (\boldsymbol{b} + \mathbf{\Delta})^T \boldsymbol{y}^{\text{opt}} - \boldsymbol{b}^T \boldsymbol{y}^{\text{opt}} = \mathbf{\Delta}^T \boldsymbol{y}^{\text{opt}}.$$

The Dual Linear Program

In our example, the profit increment corresponding to 1% increase on the all the constraints in for the dual is expressed as

$$\begin{aligned}
\boldsymbol{\Delta}^T \boldsymbol{y}^{\text{opt}} &= [1 \quad 1 \quad 1 \quad 1] \begin{bmatrix} 0 \\ 0 \\ 0.092 \\ 0.681 \end{bmatrix} \\
&= 1 \times 0.092 + 1 \times 0.681 \\
&= 0.773.
\end{aligned}$$

Duality: Producing Cars and Trucks

The linear programming program is given by

$$
\begin{array}{lrl}
\text{maximize} & Z = 3x_1 + 2.5x_2 & \text{(profit in thousands of dollars)} \\
\text{subject to} & 4.44x_1 \leq 100 & \text{(car assembly capacity)} \\
 & 6.67x_2 \leq 100 & \text{(truck assembly capacity)} \\
 & 4x_1 + 2.86x_2 \leq 100 & \text{(metal stamping capacity)} \\
 & 3x_1 + 6x_2 \leq 100 & \text{(engine assembly capacity)} \\
 & x_1 \geq 0, x_2 \geq 0 & \text{(nonnegative production)}
\end{array}
$$

The Dual Linear Program: Another View

Let us estimate the mazimized (optimal) profit from the four constraints (inequalities) we have.

The Dual Linear Program: Another View

Let us define the coefficients of magnification or shrinkage for those inequalities be y_1, y_2, y_3, and y_4 respectively.

This means for instance, if $y_3 = 2$, then $4x_1 + 2.86x_2 \leq 100$ is magnified by the factor of 2 to obtain $8x_1 + 5.72x_2 \leq 200$.

For other coefficients, we apply the same rule.

The Dual Linear Program: Another View

In order to estimate the maximum of $Z = 3x_1 + 2.5x_2$, we magnify the first constraint by the factor of y_1, the second by y_2, the third by y_3, and the fourth by y_4, where $y_1, y_2, y_3, y_4 \geq 0$.

The Dual Linear Program: Another View

In other words, we create the following four inequalities:

$$
\begin{aligned}
4.44x_1 \times y_1 &\leq 100\,(\text{car assembly capacity}) \times y_1 \\
6.67x_2 \times y_2 &\leq 100\,(\text{truck assembly capacity}) \times y_2 \\
(4x_1 + 2.86x_2) \times y_3 &\leq 100\,(\text{metal stamping capacity}) \times y_3 \\
(3x_1 + 6x_2) \times y_4 &\leq 100\,(\text{engine assembly capacity}) \times y_4
\end{aligned}
$$
[21]

[21] In order to maintain the direction of these inequalities, we need to force $y_1, y_2, y_3, y_4 \geq 0$

The Dual Linear Program: Another View

We then add them up and summarize the left hand side with respect to x_1 and x_2 to manufacture the following inequality.

$$\begin{aligned} & 4.44x_1 \times y_1 + 6.67x_2 \times y_2 \\ & \qquad + (4x_1 + 2.86x_2) \times y_3 + (3x_1 + 6x_2) \times y_4 \\ = \quad & (4.44y_1 + 4y_3 + 3y_4)\, x_1 + (6.67y_2 + 2.86y_3 + 6y_4)\, x_2 \\ \leq \quad & 100y_1 + 100y_2 + 100y_3 + 100y_4. \end{aligned}$$

The Dual Linear Program: Another View

Remember we start out this business because we wish to have a good estimate of the maximum of the objective function of the original linear programming problem:

$$z = 3x_1 + 2.5x_2.$$

A linear combination of the all the four constraints will generate the following inequality

$$\begin{aligned} & (4.44y_1 + 4y_3 + 3y_4)\, x_1 + (6.67y_2 + 2.86y_3 + 6y_4)\, x_2 \\ \leq \quad & 100y_1 + 100y_2 + 100y_3 + 100y_4. \end{aligned}$$

The Dual Linear Program: Another View

To estimate how big $z = 3x_1 + 2.5x_2$ can be given the four constraints now reduces to estimate the minimal of the right hand side $100y_1 + 100y_2 + 100y_3 + 100y_4$ of the previous inequality while maintaining $4.44y_1 + 4y_3 + 3y_4 \geq 3$ and $6.67y_2 + 2.86y_3 + 6y_4 \geq 2.5$.

The Dual Linear Program: Another View

In other words, we are trying to solve the following problem:

$$\begin{array}{rl} \text{minimize} & 100y_1 + 100y_2 + 100y_3 + 100y_4 \\ \text{subject to} & 4.44y_1 + 4y_3 + 3y_4 \geq 3 \\ & 6.67y_2 + 2.86y_3 + 6y_4 \geq 2.5 \\ & y_1, y_2, y_3, y_4 \geq 0, \end{array}$$

which is a dual to the original linear programming problem.

The Dual Linear Program: Another View

We can regard the dual to the primal linear programming problem as a mathematical effort to accurately estimate the maximal value of the objective function.

The Dual Linear Program: Another View

In this sense, it is natural for us to expect the result to obtain the best estimate via dual to coincide with the solution to the primal linear programming problem.

The Dual Linear Program: Another View

This gives you a flavor of another view on how the dual can be formulated from the primal linear programming problem.

The Dual Linear Program: Another View

However in the previous slide we had the following:

To estimate how big $z = 3x_1 + 2.5x_2$ can be given the four constraints now reduces to estimate the minimal of the right hand side $100y_1 + 100y_2 + 100y_3 + 100y_4$ of the previous inequality while maintaining $4.44y_1 + 4y_3 + 3y_4 \geq 3$ and $6.67y_2 + 2.86y_3 + 6y_4 \geq 2.5$.

We need to deal with the last part somewhat more carefully.

The Dual Linear Program: Another View

For this we need to rewrite the origianl linear programming program slightly as

$$\begin{array}{rrl} \text{maximize} & Z = 3x_1 + 2.5x_2 & \text{(profit in thousands of dollars)} \\ \text{subject to} & 4.44x_1 \leq 100 & \text{(car assembly capacity)} \\ & 6.67x_2 \leq 100 & \text{(truck assembly capacity)} \\ & 4x_1 + 2.86x_2 \leq 100 & \text{(metal stamping capacity)} \\ & 3x_1 + 6x_2 \leq 100 & \text{(engine assembly capacity)} \\ & -x_1 \leq 0 & \text{(nonnegative production)} \\ & -x_2 \leq 0 & \text{(nonnegative production)} \end{array}$$

The Dual Linear Program: Another View

We want to derive a surrogate constraint for the objective function of the form

$$c_1x_1 + c_2x_2 \leq Z_u$$

by using the 4 capacity constraints and 2 nonnegative production constraints above.

The Dual Linear Program: Another View

The rules for doing this are:

1. We can multiply any inequality by a nonnegative scalar.
2. We can add any set of inequalities together.

The Dual Linear Program: Another View

If we can obtain an inequality whose left-hand-side coefficients exactly match those of the objective function, then the right-hand-side constant is an upper bound on the optimal value of z.

The Dual Linear Program: Another View

Let $y_1, \ldots, y_4$ be the multipliers of the first four rows of the capacity constraints and let $u_1, \ldots, u_2$ be the multipliers of the last two rows corresponding to the nonnegativity constraints.

The Dual Linear Program: Another View

Multiplying each inequality by the appropriate multiplier and adding the inequalities gives the surrogate constraint:

$$\begin{aligned} & 4.44x_1 \times y_1 + 6.67x_2 \times y_2 \\ & \qquad + (4x_1 + 2.86x_2) \times y_3 + (3x_1 + 6x_2) \times y_4 \\ & \qquad + u_1 \times (-x_1) + u_2 \times (-x_2) \\ = \quad & (4.44y_1 + 4y_3 + 3y_4 - u_1)\, x_1 \\ & \qquad + (6.67y_2 + 2.86y_3 + 6y_4 - u_2)\, x_2 \\ \leq \quad & 100y_1 + 100y_2 + 100y_3 + 100y_4. \end{aligned}$$

The Dual Linear Program: Another View

This is an objective function upper bound constraint if

(B1) The y_i's, $i = 1, \ldots, 4$ and u_j's, $j = 1, \ldots, 2$ are nonnegative,

(B2) The coefficients $(4.44y_1 + 4y_3 + 3y_4 - u_1)$ and $(6.67y_2 + 2.86y_3 + 6y_4 - u_2)$ of the surrogate constraint for x_1 and x_2 respectively correspond to the coefficients 3 and 2.5 of the objective function.

The Dual Linear Program: Another View

The y_i's, $i = 1, \ldots, 4$ and u_j's, $j = 1, \ldots, 2$ satisfy (B1) and (B2) if and only if

$$\begin{aligned} 4.44y_1 + 4y_3 + 3y_4 - u_1 &= 3, \\ 6.67y_2 + 2.86y_3 + 6y_4 - u_2 &= 2.5. \end{aligned}$$

The Dual Linear Program: Another View

Equivalently if the y_i's, $i = 1, \ldots, 4$, satisfy

$$
\begin{array}{rcl}
4.44y_1 + 4y_3 + 3y_4 = 3 + u_1 & \geq & 3, \\
6.67y_2 + 2.86y_3 + 6y_4 = 2.5 + u_2 & \geq & 2.5 \\
y_1, y_2, y_3, y_4 & \geq & 0.
\end{array}
$$

and we wish to choose y_1, y_2, y_3, y_4 so as to minimize

$$100y_1 + 100y_2 + 100y_3 + 100y_4.$$

The Dual Linear Program: Another View

Therefore we are ready to show in general:

Theorem (Lemma)

Solving the dual is equivalent to finding the set of multipliers that gives us a surrogate objective constraint having the least upper bound for the objective function z of the primal.

The Dual Linear Program: Another View

(**Proof**) The dual to a canonical maximization linear programming problem

$$\begin{array}{rrcl} \text{maximize} & z & = & c_1x_1 + \cdots + c_nx_n \\ \text{subject to} & & & a_{11}x_1 + \cdots + a_{1n}x_n \le b_1 \\ & & & \vdots \qquad\qquad\quad \vdots \\ & & & a_{m1}x_1 + \cdots + a_{mn}x_n \le b_m \\ & & & x_1 \ge 0, \ldots, x_n \ge 0 \end{array}$$

is the canonical minimization linear programming problem

$$\begin{array}{rrcl} \text{minimize} & w & = & b_1y_1 + \cdots + b_my_m \\ \text{subject to} & & & a_{11}y_1 + \cdots + a_{m1}y_m \ge c_1 \\ & & & \vdots \qquad\qquad\quad \vdots \\ & & & a_{1n}y_1 + \cdots + a_{mn}y_m \ge c_n \\ & & & y_1 \ge 0, \ldots, y_n \ge 0. \end{array}$$

The Dual Linear Program: Another View

Let $y_1, \ldots, y_m$ be the multipliers of the first m rows of the capacity constraints and let $u_1, \ldots, u_n$ be the multipliers of the last n rows corresponding to the nonnegativity constraints.

The Dual Linear Program: Another View

Multiplying each inequality by the appropriate multiplier and adding the inequalities gives the surrogate inequality

$$\left(\sum_{i=1}^{m} y_i a_{i1} - u_1\right) x_1 + \cdots + \left(\sum_{i=1}^{m} y_i a_{in} - u_n\right) x_1 \leq \sum_{i=1}^{m} y_i b_i.$$

The Dual Linear Program: Another View

Notice that this is an objective function upper bound constraint if

(B1) The y_i's, $i = 1, \ldots, m$ and u_j's, $j = 1, \ldots, n$ are nonnegative,

(B2) The coefficients $c_j = \sum_{i=1}^{m} y_i a_{ij} - u_j$ for $j = 1, \ldots, n$.

The Dual Linear Program: Another View

If we use the matrix form of the maximization linear programming problem:

$$\begin{aligned} \text{maxmize} \quad z \quad = \quad & \boldsymbol{c}^T \boldsymbol{x} \\ & \boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b} \\ & -\boldsymbol{x} \leq \boldsymbol{0} \end{aligned}$$

and let $\boldsymbol{y} = (y_1, \ldots, y_m)^T$ be the column vector or $m \times 1$ matrix of row multipliers and $\boldsymbol{u} = (u_1, \ldots, u_n)$ be the row vector or $1 \times n$ of nonnegativity constraint multipliers, then the implied equality constraint will be

$$\boldsymbol{y}^T \boldsymbol{A}\boldsymbol{x} - \boldsymbol{u}\boldsymbol{x} = \left(\boldsymbol{y}^T \boldsymbol{A} - \boldsymbol{u}\right) \boldsymbol{x} \leq \boldsymbol{y}^T \boldsymbol{b}.$$

The Dual Linear Program: Another View

In order that (B1) and (B2) be satisfied we need $\boldsymbol{y} \geq 0$, $\boldsymbol{u} \geq 0$, and $\boldsymbol{c}^T = \boldsymbol{y}^T \boldsymbol{A} - \boldsymbol{u}$, or equivalently

$$\begin{aligned} \boldsymbol{y}^T \boldsymbol{A} &\geq \boldsymbol{c}^T, \\ \boldsymbol{y} &\geq \boldsymbol{0}. \end{aligned}$$

or

$$\begin{aligned} \boldsymbol{A}^T \boldsymbol{y} &\geq \boldsymbol{c}, \\ \boldsymbol{y} &\geq \boldsymbol{0}. \end{aligned}$$

The Dual Linear Program: Another View

Further, to get the least upper bound on z we need to minimize the right-hand-side term $\boldsymbol{y}^T\boldsymbol{b}$.

Knowing $\boldsymbol{y}^T\boldsymbol{b} = \boldsymbol{b}^T\boldsymbol{y}$, we put this together gives the following linear programming problem

$$\begin{array}{lll} \text{minimize} \quad w & = & \boldsymbol{b}^T\boldsymbol{y} \\ & & \boldsymbol{A}^T\boldsymbol{y} \leq \boldsymbol{c} \\ & & \boldsymbol{y} \leq \boldsymbol{0}, \end{array}$$

which is the matrix form of the dual problem.

The Dual Linear Program: Another View

Dual linear programming problem can be derived from an effort to recover the least upper bound of the **primal** objective function via a surrogate constraint manufactured from a set of the **primal** constraints.

The Dual Linear Program: Another View

Theorem (Lemma)

Let $\boldsymbol{x} = (\bar{x}_1, \ldots, \bar{x}_n)$ be a feasible solution to the primal linear programming problem with objective function value $\bar{z}$, and let $\bar{y} = (\bar{y}_1, \ldots, \bar{y}_m)$ be a feasible solution to the dual with objective function value $\bar{w}$. Then $\bar{z} \leq \bar{w}$, and if $\bar{z} = \bar{w}$ then $\bar{x}$ and $\bar{y}$ areis equivalent to finding the set of multipliers that gives us a surrogate objective constraint having the least upper bound for the objective function z of the primal.

Consulting

This section tries to simulate a possible consulting session with your client in a manufacturing or logistics, but it can be applied to a service sector as well.

Consulting

Of course, we understand that every client is different and you have to establish your style as consultant.

Consulting

Nevertheless this section gives you a general idea regarding **what you "need" to do** and **how you "phrase" what you need to say**.

Consulting

Suppose that your client is the owner of a small manufacturing firm that is technologically able to manufacture only products 1 and 2.

Your client wants to maximize profit from this operation.

Consulting

Since you are an entry-level consultant in management science[22], you emphasize **economic fundamentals** in the study of product costing, decision making, and evaluation in organizations.

[22] Frankly, in linear programming only so far.

Consulting

You client knows the existence of economies of scale, the cost advantages that enterprises obtain due to size, output, or scale of operation, with cost per unit of output generally decreasing with increasing scale due to the fact that:

- fixed costs are spread out over more units of output;
- operational efficiency is also greater with increasing scale, leading to lower variable cost as well.

Consulting

You client also knows:

Economies of scale apply to a variety of organizational and business situations and at various levels, such as a business or manufacturing unit, plant or an entire enterprise.

Consulting

For example:

• A large manufacturing facility would be expected to have a lower cost per unit of output than a smaller facility, all other factors being equal.

• A company with many facilities should have a cost advantage over a competitor with fewer.

Consulting

Your client provides you with his production standards:

1. The manufacturing process requires that product 2 pass through Machine A and Machine B.
2. Product 1, however, requires no work on Machine B.

Consulting

The other production standards are as follows:

1. One unit of product 1 earns him 38 dollars, while the same unit of product 2 earns him 33 dollars.

2. The following cost will be incurred:

Material cost	5 dollars (per pound)	(all variable)
Labor cost	13 dollars (per hour)	(all variable)
Overhead	50 dollars	(all fixed)

The Dual Linear Program: Practice

3. The usage and machine hours available will be as follows:

Usage	**Product 1**	**Product 2**
Direct materials (in pounds)	1	2
Direct labor (in labor hours)	2	1
Machine A (in machine hours)	1	1
Machine B (in machine hours)	0	1

	Machine A	**Machine B**
Machine hours available:	150	50

Consulting

You may wish to confirm your understanding of these production standards by saying, for instance,

"I understand that each unit of product 1 requires one pound (5 dollars worth) of direct material, two hours of labor (26 dollars worth), and one machine hour from machine A. Is this correct?"

Consulting

"Similarly, each unit of product 2 requires two pounds (10 dollars worth) of direct material, one hour of labor (13 dollars worth), and one machine hour each from both machines A and B. Am I right?"

Consulting

Having understood the setting, your inclination as a consultant would be to formulate the optimal product mix problem as a linear programming problem.

Consulting

It is likely that your client would not be so happy for linear programming formulation yet, because he knows economy of scale is involved.

Consulting

You may wish, for instance, to craft a three-pronged defense: Attack the universality of economy of scale; Limit your analysis both locally and short-run.

Consulting

Defense 1. Economies of scale often have limits, such as passing the optimum design point where costs per additional unit begin to increase.

Consulting

For example, you go on to explain:

"Common limits include exceeding the nearby raw material supply, such as wood in the lumber, pulp and paper industry;"

"A common limit for low cost per unit weight commodities is saturating the regional market, thus having to ship product uneconomical distances;"

"Other limits include using energy less efficiently or having a higher defect rate,"

Consulting

followed up by the graphics like the one in https://en.wikipedia.org/wiki/Economies_of_scale#Physical_and_engineering_basis

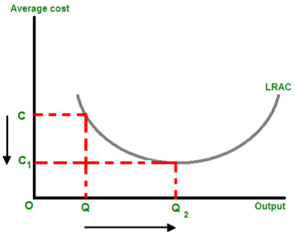


Fig.: As quantity of production increases from Q to Q2, the average cost of each unit decreases from C to C1. LRAC is the long run average cost.

Consulting

Defense 2. Linearity is only an approximation. it is valid only locally

Consulting

You may even wish to admit ludicrousness of this assumption by saying:

> *If costs and revenues were globally linear in units sold, and selling price exceeded unit variable costs, an infinite amount of profit could be made by selling an infinite amount of product, which is against both economic theory and reality.*

So linearity is an assumption that is useful only in local analysis.

Consulting

Defense 3. We are crafting short-run analysis.

Short-run analysis assumes one or more inputs are fixed.

Consulting

You may wish to say:

> *Only by combining a local linear approximation with a short-run perspective for costs*[23] *and revenue, we can sensibly speak of an* **optimal product mix**.

[23] In economics, this kind of formulation in production is called a constant-returns-to-scale technology.

Consulting

These defenses allow the consultant, you, to formulate the optimal product mix problem as a linear programming problem.

Consulting

Let q_1 and q_2 denote the quantities of product 1 and 2, respectively.

Let z_1 and z_2 denote the quantities of materials and labor.

Consulting

Later you would rather eliminate z_1 and z_2 from the picture, but at this stage, easy to understand formulation for your client would perhaps be ideal.

Consulting

You formulate the linear programming problem as below:

$$
\begin{array}{rcl}
\text{Maximize} z & = & 38q_1 + 33q_2 - 5z_1 - 13z_2 - 50 \\
\text{subject to} & & q_1 + q_2 \leq 150 \\
& & q_2 \leq 50 \\
& & q_1 + 2q_2 - z_1 = 0 \\
& & 2q_1 + q_2 - z_2 = 0 \\
& & q_1, q_2, z_1, z_2 \geq 0
\end{array}
$$

Consulting

For instance, you explain to your client: "The objective function is revenue minus variable costs and the fixed overhead."

Consulting

You also comment:

"The fixed overhead is not affected by the levels of q_1, q_2, z_1, and z_2."

"Therefore the amount of fixed costs cannot affect the optimal amounts of the outputs and inputs."

Consulting

You explain to your client:

"The constraints tell us that the total amount of input used (demanded) must not exceed the total supply of the resource."

Consulting

You need to emphasize:

"The first two constraints differ from the next two, because the supply of machine hours is regarded as fixed **in the short run**, while the supply of materials and labor is not **even in the short run**[24].

[24] You are able to say this because in your defense, you already claimed to have employed short run perspective in your analysis.

Consulting

You need to emphasize:

"Please note that the third and fourth constraints are **tight**."

Consulting

"In other words, the third constraint tells us it is not a good idea to acquire more or less materials (z_1) than will be used in production ($q_1 + 2q_2$)."

"Likewise, the fourth constraint tells us it is not a good idea to acquire more or less labor (z_2) than will be used in production ($2q_1 + q_2$)."

Consulting

"The third constraint also tells us that materials (z_1) can follow with the production level $(q_1 + 2q_2)$ even in the short-run."

"Likewise, the fourth constraint tells us that labor (z_2) can follow with the production level $(2q_1 + q_2)$ even in the short-run."

Consulting

You substitute $z_1 = q_1 + 2q_2$ and $z_2 = 2q_1 + q_2$ for the objective function and you obtain the following reframed linear programming problem:

$$
\begin{aligned}
\text{Maximize} z &= 7q_1 + 10q_2 - 50 \\
\text{subject to} \quad & q_1 + q_2 \leq 150 \\
& q_2 \leq 50 \\
& q_1, q_2 \geq 0
\end{aligned}
$$

Consulting

Before proceeding to solve this linear programming problem, you need to explain what you just did.

This is a simple calculation, but the calculation like this must always be followed up by an explanation or two about the intention as to what you are trying to accomplish.

Consulting

For instance, you need to say something like:

"Please observe that the unit contribution margins have appeared as coefficients in the new objective function."

Consulting

"For product 1 unit revenue of 38 dollars with the corresponding expenses of 5 dollars in material and $13 \times 2 = 26$ dollars in labor with the resulting margin of 7 dollars.

"This 7 dollar margin has appeared as coefficient of q_1."

Consulting

"Similarly for product 2 unit revenue of 33 dollars with the corresponding expenses of $5 \times 2 = 10$ dollars in material and 13 dollars in labor with the resulting margin of 10 dollars."

"This 10 dollar margin has appeared as coefficient of q_2."

Consulting

What you just did is you found out unit contribution margins by allocating variable costs to each unit using linear programming.

A property of optimality (that the materials and labor constraints are tight) has “told” us to allocate the costs, and how to do it.

Consulting

With graph or with `lpSolve`, you find that the optimal profit is 1,150USD with $q_1 = 100$ and $q_2 = 50$.

In class, you are done. In consulting, you need to give meaning to what you found.

Consulting

For instance, you may wish to say something like:

"Even though product 1 has the lower contribution margin, we make more product 1 than 2 to maximize profit."

"This can be attributed to the presence and form of the second constraint—product 1 uses no Machine B hours."

Consulting

You may even say something like:

"In general, the incremental profit per unit of scarce resource is what matters. The trick is to figure out *which* are the scarce resources."

Consulting

At this stage, you may wish to summarize what you have just did by say something like:

"So far we have learned that when choosing optimal product mix we are interested in the marginal profit associated with each product[25] and its demand on resources.[26]"

[25] Of 7 and 10 dollars for product 1 and 2 in the objective function.

[26] As expressed in the constraints.

Consulting

"We used a linear programming to tell us how to best utilize the capacity that is assumed fixed."

Consulting

You are now transitioning to a different topic, an important topic of shadow price.

Consulting

In order to attract interest of your client, you may for instance start by saying:

"In managerial settings, we[27] often find it a useful simplification to assume some resources could be changed in the short run like the supply of materials, but others could not, for instance, the supply of machine hours."

[27] This "we" is we, the management science consultant.

Consulting

You steer your conversation to an idea of shadow prices, but you need to prime your client so that he is ready.

"A natural issue that arises in this context is when and where capacity should be expanded."

Consulting

Instead of introducing duality now and tax your client who is a good business person but who is not particularly interested in linear programming, you may wish to use an intuitive argument :

"Suppose you have the option of expanding capacity by renovating Machine A."

Consulting

"What is the maximum amount you should be willing to pay to expand capacity by one Machine A hour?"

Consulting

"Of course, it is the additional profit obtained by using this extra capacity optimally."

Consulting

"Imagine that you were handed one hour of capacity of Machine A."

"How would you be able to optimally exploit the situation?"

Consulting

A little bit of devil's advocate would be good:

"Since both products use exactly one Machine A hour, we might feel that we should make one more of the product with the higher unit contribution margin."

Consulting

"Under this premise we would like to make one more unit of product 2, since the unit contribution margins of products 1 and 2 are \$7 and \$10, respectively."

Consulting

"If this were feasible, profit would go up by $10."

Consulting

"Remember the Machine B constraint is also tight, since the optimal solution (before expansion) is $q_1 = 100$, $q_2 = 50$."

Consulting

"While we would like to make one more unit of product 2, this is infeasible; making additional units of product 2 would violate the Machine B constraint."

Consulting

"Instead, we optimally use one more hour of Machine A capacity to make one more unit of product 1, and profit increases by $7."

Consulting

"This is the benefit from adding one Machine A hour, and we would choose to purchase this additional hour if and only if its cost were less than \$7."

Consulting

"Assume, instead, we add one more hour of Machine B time. That is, Machine A capacity is 150, but Machine B capacity is now 51."

Consulting

"Since only product 2 uses Machine B, and this constraint is tight, we would like to produce one more unit of product 2 (it uses exactly one hour)."

Consulting

"If we could produce one more unit of product 2 without affecting production of product 1, profit would go up by $10."

Consulting

"But remember that the Machine A constraint is also tight."

Consulting

"In fact, since each unit of product 2 uses one hour of Machine A time, there will be one less hour available for making product 1."

Consulting

"Further, since each unit of product 1 also uses one hour of Machine A time, increasing production of product 2 by one unit means we must decrease production of product 1 by one unit."

Consulting

"Therefore, the net benefit from expand Machine B capacity by one hour is $1 \times \$10 - 1 \times \$7 = \$3$."

Consulting

"The maximum your company is willing to pay for one more hour of Machine B capacity is thus $3, since that would be the increase in profit if the extra hour were used optimally."

Consulting

"We have used our intuition and the solution to the linear programming problem to find the benefit obtained of adding one hour of capacity of Machine A and product 2."

Consulting

"We have also used our intuition and the solution to the linear programming problem to find the benefit obtained of adding one hour of capacity of Machine B and the increasing production of product 2, while decreasing production of product 1."

Consulting

Now you need to introduce a topic of shadow prices as the solution to the dual program.

Consulting

Recall that the marginal benefit of adding a unit of capacity to a right-hand side of a constraint is referred to as the shadow price of that resource.

Consulting

It is time to introduce the word shadow price.

"We have calculated the net benefits from expanding Machine A and B capacity by one hour and those net benefits are called **shadow prices of Machine A and B** respectively."

Consulting

But the word shadow price sounds technical, and a kind of weird. So you had better to start from there.

Consulting

"Everyone understands "price." It is a money you exchange (pay) for a resource, good, or service **through a market**."

Consulting

"A shadow price, on the contrary, for a resource, good, or service is not based on actual market exchange, but is **mathematically derived**[28] from **indirect data** obtained from related markets[29]."

[28] In our case, via linear programming.

[29] In our case, via a cost and production schedule and a set of short-run and per unit revenue, all of which are necessary components for exchanging those through a market.

Consulting

"The other important points about **shadow prices** are that it is calculated **without consideration for the total cost** and for a particular resource **at the margin**."

Consulting

Now you shoulder the responsibility of having to explain these two phrases **without consideration for the total cost** and **at the margin**.

So,

Consulting

"When you consider a long trip in your car, the cost of gas, the expressway fees, lodging, and food will be budgeted."

"But you are unlikely to include the wear on the tires or the cost of the money you might have borrowed to purchase the car."

Consulting

"When you consider a relatively small day trip to a park by your car, the cost of gas may be factored."

"But you are even less likely to include the wear on the tires or the cost of the money you might have borrowed to purchase the car."

Consulting

"In a nutshell, there will be a cost incurred other than the cost of gas for this small day trip, but you disregard them."

Consulting

"In many cases, this kind of thinking is not only acceptable but very useful and that is why we have calculated the shadow prices.

Consulting

"This is why we say **shadow price** is calculated **without consideration for the total cost** and **at the margin**."

Consulting

"Increasing one machine hour either for Machine A or B:

• may require another lot of that material with considerable expenses; or

• may tire workforce so much so that many of the employees may decide to go on strike or quit this manufacturing facility altogether."

Consulting

“However, we do not consider these potential threshold effects when we calculate **shadow price**.”

“This is why we say **shadow price** is calculated **without consideration for the total cost** and **at the margin**.”

Consulting

Hopefully, these explanations at least mitigate resistance or uncomfortableness on the part of your client and make him to accept the concept of “shadow” price, at least halfheartedly.

Consulting

"So far we employ the linear programming to provide useful information regarding capacity expansion decisions.

Consulting

"Please note that the marginal benefit of adding a unit of capacity to a right-hand side of a constraint we have been talking about is referred to as the **shadow price of that resource**."

Consulting

"In our work we obtained the shadow prices for the Machine A and B at 7 and 3 dollars per hour, respectively."

Consulting

"These 7 and 3 dollars per hour respectively for Machine A and B are the additional profits you could have gotten, had you added one machine hour to Machine A or B."

Consulting

This line of explanation is correct, but they are **quite insufficient to convey the deep meaning of shadow prices**.

Consulting

The real danger is if you present shadow prices as marginal benefit of adding a unit of capacity **only**, then you will be stuck when you introduce its dual program.

Consulting

In all likelihood, your client will insist that maximization, rather than minimization, of the objective function that consists of shadow prices because on the surface, shadow prices are benefit after all and because he fails to appreciate the real implications of shadow-prices-as-benefits explanation.

Consulting

You may wish to say something like:

"But you can also interpret these **shadow prices** as sort of sacrifices as well."

Consulting

“For instance, shadow price of 7 dollars per hour for Machine A is a proxy (meaning not through-a-market) value of one machine hour for Machine A, defined by what your client **should be willing to buy** or, in other words, **must give up to gain** one extra hour for Machine A.”

Consulting

"Your client **must give up to gain** one extra hour for Machine A because machine hour for Machine A is scarce (in the short run) resource relative to needs of manufacturing product 1 and 2."

Consulting

You may also wish to say something like:

"This scarcity is clearly manifested when we explain shadow prices informally."

Consulting

"Note that the use of machine hour for Machine A in manufacturing product 1 prevents their use of manufacturing product 2, and vice versa."

Consulting

"Similarly, the use of machine hour for Machine B in manufacturing product 2 prevents their use of manufacturing product 1, even though the use of machine hour for Machine B in manufacturing product 1 does not prevent their use of manufacturing product 2 because product 1 does not use Machine B."

Consulting

"You see **the use of resources in one way prevents their use in other ways** when resources are scarce relative to needs."

Consulting

The concept of opportunity cost is in order:

"Opportunity cost refers to a benefit that a person could have received, but gave up, to take another course of action."

Consulting

"Shadow price of 7 dollars per hour for Machine A is what your client **must give up to gain** one extra hour for Machine A, this shadow price of 7 dollars per hour for Machine A is an **opportunity cost** to gain one extra hour for Machine A."

Consulting

"This way of looking at/interpreting these **shadow price** as **opportunity cost of investing in one machine hour** leads us to frame our short-run, profit-maximization problem somewhat differently."

Consulting

"Rather than thinking in terms of maximizing profit using the scarce resources efficiently, we might be able to think in terms of minimizing the opportunity cost of scarce and fixed (in the short-run) resources."

Consulting

"The opportunity-cost-minimizing problem, it turns out, is also a linear program, called the **dual program**."

Consulting

"The **dual program** can be obtained directly (and even mechanically) from the original profit-maximizing program, which we shall call the **primal program**."

Consulting

"Below are written the primal and dual programs for our problem. We drop fixed overhead from the primal because the fixed costs are not affected by the levels of q_1, q_2."

Consulting

Primal

$$\begin{array}{lrcll} \text{Maximize} & z & = & 7q_1 + 10q_2 & \\ \text{subject to} & & & q_1 + q_2 \leq 150 & (w_1) \\ & & & q_2 \leq 50 & (w_2) \\ & & & q_1, q_2 \geq 0 & \end{array}$$

Dual

$$\begin{array}{lrcll} \text{Mimimize} & w & = & 150w_1 + 50w_2 & \\ \text{subject to} & & & w_1 \qquad \geq 7 & (q_1) \\ & & & w_1 + w_2 \geq 10 & (q_2) \\ & & & w_1, w_2 \geq 0 & \end{array}$$

Consulting

You may ask yourself:

"What intuition can we use to guide us in writing down the dual program?"

Consulting

"We already know that the shadow prices of Machine A and B to be 7 and 3 dollars respectively. "

Consulting

"For the first two constraints $w_1 \geq 7$ and $w_1 + w_2 \geq 10$, we tentatively hypothesize that these w_1 and w_2 are the shadow prices of Machine A and B, because if they were, then both of these constraints would be satisfied at $w_1 = 7$ and $w_2 = 3$."

Consulting

"Now we look at the objective function."

Consulting

"We know the coefficients 150 and 50 are respectively the machine hours available for Machine A and B."

Consulting

"If w_1 and w_2 were respectively the shadow prices of Machine A and B,

the objective function would be the (shadow) price w_1 per hour of Machine A times the amount of Machine A hours

plus

the (shadow) price w_2 per hour of Machine B times the amount of Machine B hours."

Consulting

"The (shadow) price w_1 per hour of Machine A times the amount of Machine A hours is the (mathematically-derived, that is, shadow) price of Machine A."

Consulting

Similarly, "the (shadow) price w_2 per hour of Machine B times the amount of Machine B hours is the (mathematically-derived, that is, shadow) price of Machine B."

Consulting

"If you added these two, then the resulting combination of the prices would be the (mathematically-derived, that is, shadow) price of Machines A and B."

Consulting

"Note that in the short-run, we assume that machine hours of Machine A and B cannot be increased, so these two resources are **constraining** or **tight**."

Consulting

"Note also that in the short-run, we assume that the other two resources—material and labor—can be supplied according to the level of production."

"Therefore material and labor constraints are not **constraining** or **tight**."

Consulting

"So if w_1 and w_2 were respectively the shadow prices of Machine A and B, the objective function would be the total (mathematically-derived, that is, shadow) price of all the resources, tight or otherwise."

Consulting

Previously, we stated:

"Shadow price of 7 dollars per hour for Machine A is what your client **must give up to gain** one extra hour for Machine A, this shadow price of 7 dollars per hour for Machine A is an **opportunity cost** to gain one extra hour for Machine A."

Consulting

"The same can be said for shadow price of 3 dollar per hour for Machine B"

Consulting

Therefore

"We can interpret the objective function as the **total opportunity cost in dollars of all of the resources "** if w_1 and w_2 were the shadow prices of Machine A and B respectively."

Consulting

"Since this $w = 150w_1 + 50w_2$ is the total opportunity cost, of course you wish to minimize w."

Consulting

"We are hoping the meaning of the dual comes to you gradually."

Consulting

"Since we just find out what the dual objective seems to mean, let us revisit these two constraints (q_1) and (q_2) again."

Consulting

"Take the second constraint in the dual program: $w_1 + w_2 \geq 10$."

Consulting

"The left-hand side is the opportunity cost of producing one unit of product 2."

Consulting

"This is because the left-hand side is the amount (one hour) of Machine A used to make one unit of product 2 times the price w_1 per hour of Machine A plus the amount (one hour) of Machine B used to make one unit of product 2 times the price w_2 per hour of Machine B. "

Consulting

"The right-hand side is the per-unit profit or benefit obtained from one unit of product 2."

Consulting

"The second constraint in the dual program says that the cost of a unit of product 2 is greater than or equal to the benefit."

Consulting

"Now you see why it is aptly labeled as (q_2): The constraint of $w_1 + w_2 \geq 10$ is about product 2."

Consulting

"Similarly, the first constraint says that the cost of a unit of product 1 is greater than or equal to the benefit."

Consulting

"Let us remind ourselves how a firm producing a single product or service determines its production level."

Consulting

"Under the usual assumptions of increasing marginal cost and decreasing marginal revenue, a firm producing a single product or service would produce up to the level where marginal revenue is equal to marginal cost. "

Consulting

"This is because if the firm would produce less than this level, it would lose profit it could have gotten."

Consulting

"This is also because if the firm would produce more than this level, it would reduce the profit it could have gotten."

Consulting

"We are applying the same logic to a multi-product firm here."

Consulting

"So finally it looks like this dual program's solution produce as its solution the **shadow prices**."

Consulting

"In light of this economic interpretation, you may wish to ask why the constraints are "greater than or equal to "and not "equal to.""

Consulting

"The answer is that we do not *require* that a multi-product firm manufacture every product that is technologically feasible to the firm."

Consulting

"If under the given technology, the cost of the resources to produce a product is found to be larger than its benefit, the product is simply not produced."

Consulting

"When that happens, the product is simply too expensive to make. There are better uses of the resources. You know that this is an acceptable course of action for any enterprises."

Consulting

"A vivid example is silk diapers."

Consulting

"It is technologically feasible to make diapers out of silk, but there are other, better uses of silk."

Consulting

"I earlier suggested that the dual program could be obtained mechanically."

Consulting

"Inspection of the two programs reveals the following."

Consulting

"We form the first constraint in the dual program by reading down the column of the primal program corresponding to the q_1 variable."

Consulting

"The coefficient on q_1 in the (w_1) constraint of the primal is 1, and the coefficient on q_1 in the (w_2) constraint is zero, leading to $1w_1 + 0w_2$. "

Consulting

"The right-hand side of the (q_1) dual constraint is the coefficient on q_1 in the primal objective function, 7. "

Consulting

"We form the second dual constraint by reading down the column of the primal program corresponding to the q_2 variable. "

Consulting

"The coefficient on q_2 in the (w_1) constraint of the primal is 1, and the coefficient on q_2 in the (w_2) constraint is one, leading to $1w_1 + 1w_2$. "

Consulting

"The right-hand side of the (q_2) dual constraint is the coefficient on q_2 in the primal objective function, 10. "

Consulting

"The above dual program can be solved graphically, because there are only two dual variables. of course you can use `lpSolve` as well."

Consulting

"Thus, the solution is a familiar one: the dual variables (shadow prices) are 7 and 3."

Consulting

"Now it is almost certain that this dual program's solution produces as its solution the **shadow prices**."

Consulting

"By the way, this mechanical formation of the dual also reveals that **dual of the dual is the primal** property of linear programming problem."

Consulting

"It may be a bit intriguing that the dual program's objective function when evaluated at its optimal solution is equal to the primal program's objective function when evaluated at its optimal solution."

Consulting

"Note that in the above the primal program objective function without the fixed overhead at the optimal solution was $7(100) + 10(50) = 1,200$."

Consulting

"Now let's try substituting the optimal solution to the dual program into the dual objective function:
$150(7) + 50(3) = 1,200.$"

Consulting

"The value of the primal objective function at the optimal solution to the primal program and the value of the dual objective function at the optimal solution to the dual objective function are equal."

Consulting

“Is this a coincidence, or is this a property of linear programming that holds in general?”

Consulting

"Remarkably, the answer is the latter."

Consulting

Suppose $\boldsymbol{q} = (q_1, \ldots, q_n)$ is a feasible solution to the primal program with m constraints, and $\boldsymbol{w} = (w_1, \ldots, w_m)$ is a feasible solution to the corresponding dual program. Then if the primal objective function evaluated at $\boldsymbol{q}$ is equal to the dual objective function evaluated at $\boldsymbol{w}$, then both problems are optimized by $\boldsymbol{q}$ and $\boldsymbol{w}$.

Consulting

"Another important point is the result also "goes the other way.""

Consulting

"That is, if the primal and dual objective functions are not equal at a particular feasible solution, then we know both of those solutions are not optimal."

Consulting

"The Observation is not difficult to prove. We prove it for a special case: two constraints and two variables."

Consulting

The primal program is:

$$\begin{array}{rcl} \text{Maximize} \quad z & = & c_1 q_1 + c_2 q_2 \\ \text{subject to} & & A q_1 + B q_2 \leq b_1 \\ & & C q_1 + D q_2 \leq b_2 \\ & & q_1, q_2 \geq 0. \end{array}$$

The dual program is:

$$\begin{array}{rcl} \text{Mimimize} \quad w & = & b_1 w_1 + b_2 w_2 \\ \text{subject to} & & A w_1 + C w_2 \geq c_1 \\ & & B w_1 + D w_2 \geq c_2 \\ & & w_1, w_2 \geq 0. \end{array}$$

Consulting

First, using dual feasibility, multiply the first dual constraint by $q_1 \geq 0$ (which will not change the direction of the inequality).

Multiply the second dual constraint by $q_2 \geq 0$ as well.

Consulting

This produces:

$$\begin{aligned} q_1\left(Aw_1 + Cw_2\right) &\geq q_1 c_1, \\ q_2\left(Bw_1 + Dw_2\right) &\geq q_2 c_2. \end{aligned}$$

Consulting

These constraints together imply:

$$c_1 q_1 + c_2 q_2 \leq (Aw_1 + Cw_2)\, q_1 + (Bw_1 + Dw_2)\, q_2.$$

Consulting

Rearrange the right-hand side of this inequality:

$$\begin{aligned}(Aw_1 + Cw_2)\, q_1 + (Bw_1 + Dw_2)\, q_2 \\ = (Aq_1 + Bq_2)\, w_1 + (Cq_1 + Dq_2)\, w_2.\end{aligned}$$

Consulting

Therefore

$$c_1 q_1 + c_2 q_2 \leq (Aq_1 + Bq_2)\, w_1 + (Cq_1 + Dq_2)\, w_2.$$

Consulting

Now, by the constraints in the primal program:

$$
\begin{aligned}
c_1 q_1 + c_2 q_2 &\leq (Aq_1 + Bq_2)\, w_1 + (Cq_1 + Dq_2)\, w_2 \\
&\leq b_1 w_1 + b_2 w_2
\end{aligned}
$$

Consulting

But this states that the primal objective function can never be larger than the dual objective function, if the proposed solutions are feasible in their respective programs.

Consulting

If $c_1q_1 + c_2q_2 = b_1w_1 + b_2w_2$, we know we have made the primal objective as large as it can possibly be–it is maximized!

Consulting

In parallel fashion, if $c_1q_1 + c_2q_2 = b_1w_1 + b_2w_2$, we know we have made the dual objective function as small as it can be—it is minimized!

Consulting

Thus, if feasible dual and primal solutions lead to the same objective function values, then both solutions are optimal.