Linear Algebra

Change of basis:

Let $S=\{u_1, u_2, \dots, u_n\}$
and $S'=\{v_1, v_2, \dots, v_n\}$
be the set of two basis of vector space V.

old → new $S \to S' \Rightarrow P$
new → old $S' \to S \Rightarrow Q$

For old every vector of S' should be written in linear combination of S

$v_1 = a_{11}u_1 + a_{12}u_2 + \dots + a_{1n}u_n$

$v_2 = a_{21}u_1 + a_{22}u_2 + \dots + a_{2n}u_n$

• **Example:**

$S=\{u_1, u_2\}=\{(1,2),(3,5)\}$
and
$S'=\{v_1, v_2\}=\{(1,-1),(1,-2)\}$

Find change of basis from old to new

$S \to S'$

$v_1 = x u_1 + y u_2$ —(i)

$v_2 = x u_1 + y u_2$ —(ii)

For equ (i)

$(1,-1) = x(1,2) + y(3,5)$

$(1,-1) = (x, 2x) + (3y, 5y)$

$(1,-1) = (x+3y, 2x+5y)$

$x + 3y = +1$ —(a)

$2x + 5y = -1$ —(b)

from equ (a) $\boxed{x = 1-3y}$ put in equ (b)

$2(1-3y) + 5y = -1$

$2 - 6y + 5y = -1$

$-y = -1-2$

$\boxed{y = 3}$ Put in equ (a)

$x + 3y = 1$

$x + 3(3) = 1$

$x + 9 = 1$

$\boxed{x = -8}$

For equ (ii)

$(1,-2) = x(1,2) + y(3,5)$

$(1,-2) = (x+3y, 2x+5y)$

$x + 3y = 1$ —— (a)

$2x + 5y = -2$ —— (b)

$\boxed{x = -11}$ $\boxed{y = 4}$

Put values in equ (i) and (ii)

$v_1 = -8u_1 + 3u_2$

$v_2 = -11u_1 + 4u_2$

$$P = \begin{bmatrix} -8 & 3 \\ -11 & 4 \end{bmatrix}^t$$

$$\boxed{P = \begin{bmatrix} -8 & -11 \\ 3 & 4 \end{bmatrix}}$$

Now for new to old

$S' \to S$

You can use equations and solve similarly

$u_1 = xv_1 + yv_2$ —— (i)

$u_2 = xv_1 + yv_2$ —— (ii)

or you can simbly find it

by taking inverse of P

$$\boxed{Q = P^{-1}}$$

• **Similarity:**

There will be similarity b/w two matrices A and B if $P(S \to S')$ and $Q(S' \to S)$

$$\boxed{B = P^{-1} A P}$$

• **Theorems:**

Let P be the change of basis from old to new of V. S and S' are basis of V. Let $\vec{v} \in V$ then

$$P[v]_{S'} = [v]_S$$

To prove the theorem Let's continue previous example with $v = (1,2)$

$$P = \begin{bmatrix} -8 & -11 \\ 3 & 4 \end{bmatrix}$$

for $[V]_{S'}$ we represent V as linear combination of S'

$$V = xV_1 + yV_2 \quad \text{—(i)}$$

$$(1,2) = x(1,-1) + y(1,-2)$$

$$(1,2) = (x+y, -x-2y)$$

$$x+y=1$$

$$-x-2y=2$$

$$\boxed{y=1-x}$$

$$-x-2(1-x)=2$$

$$-x-2+2x=2$$

$$\boxed{x=4}$$

$$x+y=1$$

$$4+y=1$$

$$\boxed{y=-3}$$

put in equ (i)

$$V = 4V_1 - 3V_2$$

$$[V]_{S'} = \begin{bmatrix} 4 & -3 \end{bmatrix}^t = \boxed{\begin{bmatrix} 4 \\ -3 \end{bmatrix}}$$

Similarly you will find $[V]_S$ and prove by taking products.

- **Theorem for operators**

$$[T]_{S'} = P^{-1}[T]_S P$$

$T(V_1) = xV_1 + yV_2$
$T(V_2) = xV_1 + yV_2$

- **Eignvalues, Eigen vectors and Diagonalization:**

Linear System of the form

$$Ax = \lambda x$$

- **Example:**

$x_1 + 3x_2 = \lambda x_1$
$4x_1 + 2x_2 = \lambda x_2$

- **Sol:**

Write in matrix form

$$\begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \lambda \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$A \quad x = \lambda x$$

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \lambda \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - \begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

For subtraction order must be same so to equalize order we introduce identity matrix of 2x2

$$\lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - \begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

By multiplying $(\lambda I - A)x = 0$

$$\begin{bmatrix} \lambda - 1 & -3 \\ -4 & \lambda - 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$(\lambda I - A) \quad x = 0$$

- **Find Eigen Value:**

Put determinant $(\lambda I - A) = 0$

$$|\lambda I - A| = \begin{vmatrix} \lambda - 1 & -3 \\ -4 & \lambda - 2 \end{vmatrix} = 0$$

$(\lambda-1)(\lambda-2)-(-3)(-4)=0$

$\lambda^2-2\lambda-\lambda+2-12=0$

$\lambda^2-3\lambda-10=0$

$\lambda^2-5\lambda+2\lambda-10=0$

$\lambda(\lambda-5)+2(\lambda-5)=0$

$\lambda+2=0$ $\quad$ $\lambda-5=0$

$\boxed{\lambda=-2}$ $\quad$ $\boxed{\lambda=5}$

So Eigen values are:

$\boxed{\lambda_1=5}$ $\quad$ $\boxed{\lambda_2=-2}$

• <u>Find Eigen vector:</u>

Put

$(\lambda I-A)x=0$ — (A)

For $\lambda_1=5$

$$\lambda_1 I-A=\begin{bmatrix}\lambda_1-1 & -3\\ -4 & \lambda_1-2\end{bmatrix}$$

$$\lambda_1 I-A=\begin{bmatrix}5-1 & -3\\ -4 & 5-2\end{bmatrix}$$

$$(\lambda_1 I-A)=\begin{bmatrix}4 & -3\\ -4 & 3\end{bmatrix}$$

Put in equation (A)

$$\begin{bmatrix} 4 & -3 \\ -4 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$4x_1 - 3x_2 = 0$ — (i)

$-4x_1 + 3x_2 = 0$

$$V_1 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

∴ As the values of x_1 and x_2 are 0.

So we take $\boxed{x_2 = t}$ Put in equ (i)

$4x_1 - 3\cancel{x_2}^{t} = 0$

$\boxed{x_1 = \frac{3}{4} t}$

$$\boxed{V_1 = \begin{bmatrix} \frac{3}{4} t \\ t \end{bmatrix}}$$

(Eigen vector1)

For $\lambda_2 = -2$

$$(\lambda_2 I - A) = \begin{bmatrix} -2-1 & -3 \\ -4 & -2-2 \end{bmatrix}$$

$$(\lambda_2 I - A) = \begin{bmatrix} -3 & -3 \\ -4 & -4 \end{bmatrix}$$

Put in equ (A)

$$\begin{bmatrix} -3 & -3 \\ -4 & -4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$-3x_1 - 3x_2 = 0$

$-4x_1 - 4x_2 = 0$

Similarly

$x_2 = t$

$x_1 = -t$

$$\boxed{V_2 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -t \\ t \end{bmatrix}}$$

(Eigen vector2)

- **Diagonalization :-**

$$P = [v_1 \ v_2] = \begin{bmatrix} 3/4t & -t \\ t & t \end{bmatrix}$$

$$D = P^{-1}AP \text{ —— (A)}$$

and For diagonalization

$$D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

Where λ_1 and λ_2 are Eigen values.

If we are taking with t then we take t as 1.

Put $t = 1$

$$\boxed{P = \begin{bmatrix} 3/4 & -1 \\ 1 & 1 \end{bmatrix}}$$

Now you can easily find D by putting P in equ (A)