MATH 3808

WINTER 2015 ASSIGNMENT 2

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1. **Hypothesis**: The observed frequencies fit with the theoretical distribution.

The probability of getting any of the outcomes is $\frac{1}{38}$. This means that over the course of 1000 round each outcome should be observed close to $e_i = \frac{1000}{38} \approx 26.31579$ times.

My results from the calculation for the chi-square distribution using Excel:

Outcome	0	00	1	2	3	4	5
frequency	26	28	32	25	27	27	12
e_i	26.31579	26.3158	26.31579	26.31579	26.31579	26.31579	26.31579
$\frac{(f_i - e_i)^2}{e_i}$	0.003789	0.10779	1.227789	0.06579	0.017789	0.017789	7.78779

Outcome	6	7	8	9	10	11	12
frequency	23	35	27	23	24	24	28
e_i	26.31579	26.31579	26.31579	26.31579	26.31579	26.31579	26.31579
$\frac{(f_i - e_i)^2}{e_i}$	0.41779	2.865789	0.017789	0.41779	0.20379	0.20379	0.107789

Outcome	13	14	15	16	17	18	19
frequency	25	38	25	21	32	35	26
e_i	26.31579	26.31579	26.31579	26.31579	26.31579	26.31579	26.31579
$\frac{(f_i - e_i)^2}{e_i}$	0.06579	5.187789	0.06579	1.07379	1.227789	2.865789	0.003789

Outcome	20	21	22	23	24	25	26
frequency	19	22	27	24	22	24	29
e_i	26.31579	26.31579	26.31579	26.31579	26.31579	26.31579	26.31579
$\frac{(f_i - e_i)^2}{e_i}$	2.03379	0.70779	0.017789	0.20379	0.70779	0.20379	0.273789

Outcome	27	28	29	30	31	32	33
frequency	25	30	20	30	24	34	27
e_i	26.31579	26.31579	26.31579	26.31579	26.31579	26.31579	26.31579
$\frac{(f_i - e_i)^2}{e_i}$	0.06579	0.515789	1.51579	0.515789	0.20379	2.243789	0.017789

***Last values on next page**

Outcome	34	35	36
frequency	29	32	19
e_i	26.31579	26.31579	26.31579
$\frac{(f_i - e_i)^2}{e_i}$	0.273789	1.227789	2.03379

After summing the calculated values, 36.716 was obtained and determined the value for $P(X \geq 26.31579) = 0.000000683$. This value is much less than 5% and thus the hypothesis should be rejected. The observed frequencies do not fit with the theoretical distribution.

2. **(a)** There are 52^4 ways to deal two ordered cards to the player and two ordered cards to the banker (ordered since repetition is allowed). Since order matters we have to figure out the probabilities of getting a Natural with an infinite deck. The differences will be that for each case e.g. $\{0,8\}$, the reverse $\{8,0\}$ will have to be found (unlike the unordered). Also for $\{4,4\}$ and $\{9,9\}$ will have to be changed to do ordered counting instead of unordered counting. The number of ways that the hand can be is also 52^2.

8-Point Natural

Type	**Probability**
$\{0,8\}$	$16 \times 4/52^2$
$\{1,7\}$	$4 \times 4/52^2$
$\{2,6\}$	$4 \times 4/52^2$
$\{3,5\}$	$4 \times 4/52^2$
$\{4,4\}$	$4 \times 4/52^2$
$\{5,3\}$	$4 \times 4/52^2$
$\{6,2\}$	$4 \times 4/52^2$
$\{7,1\}$	$4 \times 4/52^2$
$\{8,0\}$	$4 \times 16/52^2$
$\{9,9\}$	$4 \times 4/52^2$
Total	$256/52^2$

9-Point Natural

Type	**Probability**
$\{0,9\}$	$16 \times 4/52^2$
$\{1,8\}$	$4 \times 4/52^2$
$\{2,7\}$	$4 \times 4/52^2$
$\{3,6\}$	$4 \times 4/52^2$
$\{4,5\}$	$4 \times 4/52^2$
$\{5,4\}$	$4 \times 4/52^2$
$\{6,3\}$	$4 \times 4/52^2$
$\{7,2\}$	$4 \times 4/52^2$
$\{8,1\}$	$4 \times 4/52^2$
$\{9,0\}$	$4 \times 16/52^2$
Total	$256/52^2$

Since the probability of both naturals are the same and there is replacement on the cards drawn, there are $256 \times 256 = 65536$ outcomes for the four cases:

1. Player has 8, Banker has 8
2. Player has 8, Banker has 9
3. Player has 9, Banker has 8
4. Player has 9, Banker has 9

Thus the probability that both the player and the banker have a natural is:

$$\frac{4 \times 65536}{52^4} = 0.03585$$

(b) We can determine the probability of the player having a natural by adding the probability of having an 8-point natural to the probability of having a 9-point natural:

$$\frac{256}{52^2} + \frac{256}{52^2} = \frac{512}{52^2} = 0.18935$$

Since there is not a difference in how the cards are dealt to the banker, this is also the probability of the banker getting a natural.

Let A denote the event that the player has a natural

Let B denote the event that the banker has a natural

We know that $P(A) = P(B) = 0.18935$ and that $P(A \cap B) = 0.03585$ so we can simple solve the following, which will give us the probability of at least one of the player or banker has a natural:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) = 2(0.18935) - 0.03585 = 0.34285$$

3. **(a)** There are $\binom{49}{2} = 1176$ two card deals.

Straight Flush: The next two cards would be 4♥ and 5♥. There is only 1 way to get this.

Flush: The next two cards are ♥s. We have to remove the straight flush from the total.

Remaining ♥s = 10, thus $\binom{10}{2} - 1 = 44$ ways to get a flush.

Straight: The next two cards are a 4 and a 5. We have to remove the straight flush case from the total.

There is only way to get a 4 and a 5 (suit ignored). There are 4 suits, 4 ways to get each face value, thus $4 \times 4 - 1 = 15$ ways to get a straight.

Three of a Kind: The next two cards are either both an A, both a 2 or both a 3.

There are three face values to choose and each have three suits remaining, thus $3 \times \binom{3}{2} = 9$ ways to get three of a kind.

Two Pair: The next two cards are either an A and a 2, an A and a 3 or a 2 and a 3.

There are 3 face values and each have 3 suits remaining, thus $\binom{3}{2} \times \binom{3}{2} = 9$ ways to get a two pair.

High Pair: The next two cards are either both a K, Q, J or 10 or an A and a face value that is not an A, 2 or 3.

For pair of As: Three remaining As, ten face values not A, 2 or 3 and each of those ten have four remaining suits, thus $3 \times 10 \times 4 = 120$ ways to get a pair of As.

For other high pairs: Four face values (other than A) that will give a high pair and each have four remaining suits, thus $4 \times \binom{4}{2} = 24$ ways to get the other high pairs.

Add the two together to get the total ways to get a high pair: $120 + 24 = 144$.

Expected Value:

Final Hand Rank	**Number of Hands**	**Probability**	**Expected Result**
Lower Than Pair 10	954	0.81122	-0.81122
High Pair	144	0.12245	0.12245
Two Pair	9	0.00765	0.01530
Three of a Kind	9	0.00765	0.02295
Straight	15	0.01276	0.06380
Flush	44	0.03741	0.29928
Straight Flush	1	0.00085	0.17000
Expected Value	**-0.11744**		

(b) There are 48 possible card deals.

Flush: The next card is a ♥.

There are 9 remaining ♥s, thus 9 ways to get a flush.

High Pair: The next card is an A.

There are 3 remaining As, thus 3 ways to get a high pair.

Expected Value:

Final Hand Rank	**Number of Hands**	**Probability**	**Expected Result**
Lower than High Pair	36	0.75000	-0.75000
High Pair	3	0.06250	0.06250
Flush	9	0.18750	1.50000
Expected Value	**0.8125**		

4. **(a)** There are $\binom{49}{2} = 1176$ two card deals.

Straight: A three way straight, drawn cards need to be K,A or 9,K or 8,9.

Since a straight flush is not possible we do not have to account for it in the final total. Each card can be any of the suits and there are three different ways to make a straight in this case

$3 \times 4 \times 4 = 48$ ways to get a straight.

Three of a Kind: The next two cards will have to be the same face value as one of the cards currently in the hand.

Since it has to be one of the cards currently in the hand, there will be three of the face value wanted in the deck. Thus, we have $3 \times \binom{3}{2} = 9$ ways to get a three of a kind.

Two Pair: The next two cards will be two of the face values already in the hand.

Since there are three cards to choose from and each have three suits to choose from, we have $\binom{3}{2}\binom{3}{2} = 9$ ways to get a two pair.

High Pair: There are two cases for a high pair.

10, J, Q: Since the hand already has one of each of these we only need one more to get a high pair. Each has three remaining suits, thus 3×3 for the card that pairs. The other card is any card that is not one of those three, thus 10×4, which gives use $3 \times 3 \times 4 \times 10 = 360$ ways for a high pair with 10, J or Q.

K, A: Need two of one of these face values. Since there are still the four suits it is simply $2 \times \binom{4}{2} = 12$ ways to get a high pair with Ks or As.

Total: 372 ways to get a high pair with this hand

Expected Value:

Final Hand Rank	**Number of Hands**	**Probability**	**Expected Result**
Lower than High Pair	738	0.62755	-0.62755
High Pair	372	0.31633	0.31633
Two Pair	9	0.00765	0.01530
Three of a Kind	9	0.00765	0.02295
Straight	48	0.04082	0.20410
Expected Value	**-0.06887**		

With the expected value being negative, the correct play would be to take back bet #1.

(b) There are $\binom{48}{2} = 1128$ two card deals.

Straight: A three way straight, drawn cards need to be K,A or 9,K or 8,9.

Since there is one less king in play there are less ways to get one. Two of the cases of the straight now only have three suits for one card (the king).

$2 \times 3 \times 4 + 4 \times 4 = 40$ ways to get a straight.

Three of a Kind: The next two cards will have to be the same face value as one of the cards currently in the hand.

This is not affected by having less kings in play. Thus, we still have $3 \times \binom{3}{2} = 9$ ways to get a three of a kind.

Two Pair: The next two cards will be two of the face values already in the hand.

Again, not affected, we have $\binom{3}{2}\binom{3}{2} = 9$ ways to get a two pair.

High Pair: There are two cases for a high pair.

10, J, Q: Same as part (a), not affected.

K, A: Need two of one of these face values. King becomes harder to get since less in play, now there are $\binom{3}{2} + \binom{4}{2} = 9$ ways to get a high pair with Ks or As.

Total: 369 ways to get a high pair with this hand

Expected Value:

Final Hand Rank	Number of Hands	Probability	Expected Result
Lower than High Pair	701	0.62145	-0.62145
High Pair	369	0.32713	0.32713
Two Pair	9	0.00798	0.01596
Three of a Kind	9	0.00798	0.02394
Straight	40	0.03546	0.17730
Expected Value	**-0.07712**		

With the expected value being negative, the correct play would be to take back bet #1.

5. **(a)** For the hand {2♣, 2♥, 3♥, 4♥, 5♥}:

We choose the suit for the first 2 and have four choices. There will then be three other suits to select for the other four cards. This means that there are $4 \times 3 = 12$ cards in this equivalency class.

The eleven other equivalent hands are:

{2♣, 2♦, 3♦, 4♦, 5♦}, {2♣, 2♠, 3♠, 4♠, 5♠},

{2♠, 2♥, 3♥, 4♥, 5♥}, {2♠, 2♦, 3♦, 4♦, 5♦}, {2♠, 2♣, 3♣, 4♣, 5♣},

{2♦, 2♥, 3♥, 4♥, 5♥}, {2♦, 2♠, 3♠, 4♠, 5♠}, {2♦, 2♣, 3♣, 4♣, 5♣},

{2♥, 2♦, 3♦, 4♦, 5♦}, {2♥, 2♠, 3♠, 4♠, 5♠}, {2♥, 2♣, 3♣, 4♣, 5♣},

(b) Since there are five cards with two pairs, the total combination of suits is:

$$4\binom{4}{2}\binom{4}{2} = 144$$

First, let us only consider the three ways we can have suits on the 6s and 8s:

1. 6s and 8s have the same suits, e.g. {6♣, 6♥, 8♣, 8♥}
2. One 6 has the same suit as an 8, e.g. {6♣, 6♥, 8♣, 8♦}
3. They all have different suits, e.g. {6♣, 6♥, 8♦, 8♠}

This will help us when considering the 9 in the hand by considering the four potential ways a 9 can match the suits of the 6s and 8s:

1. 9 matches suit with both a 6 and an 8
    - 6s and 8s have the same suits, e.g. {6♣, 6♥, 8♣, 8♥, 9♣}, in this case there are 4 ways to choose the suits on the {6,8,9} combination, 3 ways on the {6,8} combination, so $4 \times 3 = 12$ total ways to choose the suits of this hand
    - One 6 has the same suit as an 8, e.g. {6♣, 6♥, 8♣, 8♦, 9♣}, in this case there are 4 ways to choose the suits on the {6,8,9} combination, 3 ways to choose the suit of the remaining 6 and 2 ways to choose the suit on the last 8, so $4 \times 3 \times 2 = 24$ ways to choose the suits of this hand
2. 9 matches suit with only a 6
    - One 6 has the same suit as an 8, e.g. {6♣, 6♥, 8♣, 8♦, 9♥}, in this case there are 4 ways to choose the suits on the {6,9} combination, 3 ways to choose the suit of the {6,8} combination and 2 ways to choose the suit of the remaining 8, so $4 \times 3 \times 2 = 24$ ways to choose the suits of this hand

- The 6s and 8s all have different suits, e.g. {6♣, 6♥, 8♦, 8♠, 9♣}, in this case there 4 ways to choose the suit on the {6,9}, 3 ways on the remaining 6, 2 ways to choose the 8s but since the order of the suits on the 8s does not matter we only consider it 1 way (to remove double counting), so $4 \times 3 = 12$ ways to choose the suits of this hand

3. 9 matches suit with only an 8
   - One 6 has the same suit as an 8, e.g. {6♣, 6♥, 8♣, 8♦, 9♦}, this case is the same idea as the one for 9 matching with a 6 but is a different set of hands since different face values match suit, so 24 ways to choose the suits of this hand
   - The 6s and 8s all have different suits, e.g. {6♣, 6♥, 8♦, 8♠, 9♦}, this case is the same idea as the one for 9 matching with a 6 but is a different set of hands since different face values match suit, and the order of suits on the 6s does not matter instead of the 8s (to remove double counting), so 12 ways to choose the suits of this hand
4. 9 matches suit with neither a 6 nor an 8
   - 6s and 8s have the same suits, e.g. {6♣, 6♥, 8♣, 8♥, 9♦}, in this case there are 4 ways to choose the suit of 9, 3 ways to choose the suit of the first {6,8} and 2 ways to choose the suit of the second {6,8}, but we have to divide 2 out since the order of suits does not matter on the {6,8} pairs and if we do not hands that are equivalent will be counted, we do this to remove double counting, so we have $4 \times 3 = 12$ ways to choose the suits of this hand
   - One 6 has the same suit as an 8, e.g.{6♣, 6♥, 8♣, 8♦, 9♠}, in this case there are 4 ways to choose the suit of 9, 3 ways to choose the suit of the {6,8} and 2 ways to choose the remaining suits, so 24 ways to choose the suits of this hand

Now we can check that we covered all cases by adding the number of equivalent hands that each case covers. This sums to 144, which means that all hands with these face values have been covered in one of the above equivalency cases. Thus, the following (which are the example hands) is a list of all possible non-equivalent hands:

{6♣, 6♥, 8♣, 8♥, 9♣} {6♣, 6♥, 8♣, 8♦, 9♣} {6♣, 6♥, 8♣, 8♦, 9♥} {6♣, 6♥, 8♦, 8♠, 9♣}

{6♣, 6♥, 8♣, 8♦, 9♦} {6♣, 6♥, 8♦, 8♠, 9♦} {6♣, 6♥, 8♣, 8♥, 9♦} {6♣, 6♥, 8♣, 8♦, 9♠}

6. The table for expected payoff and expected value, house edge and standard deviation further down.

Rank	Payback	Payoff	Probability	Expected Payoff
Royal Flush	30000	29999	0.00015%	0.0449985
Straight Flush	3000	2999	0.00139%	0.0416861
Four of a Kind	400	399	0.02401%	0.0957999
Full House	200	199	0.14406%	0.2866794
Flush	50	49	0.19654%	0.0963046
Straight	25	24	0.39246%	0.0941904
Three of a Kind	5	4	2.11285%	0.084514
Otherwise	0	-1	97.1285%	-0.971285
Player's EV	**-0.2271121**			

Royal Flush

There are only 4 ways to get a royal flush, so the probability is:

$$\frac{4}{\binom{52}{5}}$$

Straight Flush

There are ten possible starting points (lowest card) of a straight flush; one of them will give you a royal flush so we consider the other nine of them. There are 4 suits so the probability is:

$$\frac{9 \times 4}{\binom{52}{5}}$$

Four of a Kind

There are four of one card, each suit used, and another of the remaining twelve that can be any suit. The probability is:

$$\frac{13 \times 12\binom{4}{1}}{\binom{52}{5}}$$

Full House

There is a three of a kind and a pair in this hand. The three of a kind will have one face value of three different suits and the pair will have two different suits of one of the twelve remaining face values. The probability is:

$$\frac{13\binom{4}{3} \times 12\binom{4}{2}}{\binom{52}{5}}$$

Flush

The hand is five cards with the same suit subtract royal and straight flush cases. The probability is:

$$\frac{4\binom{13}{5} - 4 - 36}{\binom{52}{5}}$$

Straight

There are ten possible starting points (lowest card) of a straight. There are four possible suits for each card, but the hand cannot be a royal or straight flush. The probability is:

$$\frac{10 \times 4^5 - 4 - 46}{\binom{52}{5}}$$

Three of a Kind

Three card of the hand are the same face value and the other two are both different face values. The probability is:

$$\frac{13\binom{4}{3} \times \binom{12}{2}\binom{4}{1}^2}{\binom{52}{5}}$$

House Edge

The house edge for this bet is the negative of player expected value, so house edge is:

$$0.2271121 = 22.71121\%$$

Standard Deviation

Determined the variance from the players payoff and expected value (μ).

$$\begin{aligned}\sigma^2 \quad &= (29999 + 0.2271121)^2(0.0000015) + (2999 + 0.2271121)^2(0.0000139) \\ &+(399 + 0.2271121)^2(0.0002401) + (199 + 0.2271121)^2(0.0014406) \\ &+(49 + 0.2271121)^2(0.0019654) + (24 + 0.2271121)^2(0.0039246) \\ &+(4 + 0.2271121)^2(0.0211285) + (-1 + 0.2271121)^2(0.971285)\end{aligned}$$

$$= 1578.43723$$

$$\sigma \quad = 39.72955$$

7. **(a)** There are three possibilities for the dealers hand. First, the dealer's hand does not qualify. Second, the dealer's hand qualifies and beats the player's hand. Lastly, the dealer's hand qualifies but is equal in rank to the player's (this is because the player has the lowest ranked qualifying hand). There are $\binom{49}{3} = 18424$ remaining three card hands in the deck for the dealer when knowing the player's hand.

Un-qualifying Hands

Using the total possible un-qualifying hands of a full deck we have to remove all hands that contain a 2♣ or a 3♠.

Let A be the set of three card hands containing 2♣

For $|A|$ we have to choose two cards out of the remaining nine and subtract the case where the two cards are {3, 4} since that makes a straight. When we are determining the suits we have to remove the case where all the cards are ♣s.

$$|A| = \left(\binom{9}{2} - 1\right)(4^2 - 1) = 525$$

Let B be the set of three card hands containing 3♠

For $|B|$ we have to choose two cards out of the remaining nine and subtract the two cases where the two cards are {2, 4} or {4, 5} since these makes a straight. When we are determining the suits we have to remove the case where all the cards are ♠s.

$$|A| = \left(\binom{9}{2} - 2\right)(4^2 - 1) = 510$$

Now we have to find the hands that have both 2♣ and 3♠ in them. We simple have to fix those two cards and pick a third out of 7 face values, since if we pick 4 we will have a straight. In this case suit does not matter since you cannot have a flush with the two fixed cards.

$$|A \cap B| = 7 \times 4 = 28$$

Now to determine the total hands where 2♣ or 3♠ are in them:

$$|A \cup B| = |A| + |B| - |A \cap B| = 525 + 510 - 28 = 1007$$

Since we know (from Note 14) that the number of hands that are lower than a queen high is 6720, we can determine that the probability that the dealer gets an un-qualifying hand is:

$$\frac{6720 - 1007}{\binom{49}{3}} = \frac{5713}{18424} = 0.31009$$

Hands of Same Rank

We find all the hands that are of equivalent rank, so all hands of {Q, 2, 3} of the 49 remaining hands, removing flush cases. We have to choose one of the remaining suits of each of the three face values. We then have to remove the two flush cases {♥, ♦} since all of those three face values still have both of those suits in the deck.

$$\binom{3}{1}\binom{3}{1}\binom{3}{1} - 2 = 25$$

The probability of the player having tied the dealer is:

$$\frac{25}{\binom{49}{3}} = \frac{25}{18424} = 0.0013569$$

The Dealer Wins

Since we already know probability of the other events, the probability of this is simply:

$$\frac{\binom{49}{3} - 5713 - 25}{\binom{49}{3}} = \frac{12686}{18424} = 0.68856$$

Overall

Case	Payoff	Probability	Expected Return
Wins Ante, Loses Bet	1	0.31009	0.31009
Tie	0	0.0013569	0
Loses Ante and Bet	-2	0.68856	-1.37712
Expected Value		**-1.06703**	

(b) The correct way of playing this hand is to fold. This is since the expected value lower than negative one, meaning that you are more likely to lose your ante than keep it and it is extremely unlikely for you to even win the main bet. The optimal strategy is also to fold on anything lower than a (Q,6,4).