Solutions: Asignment 1: ODE and PDEs

African institute for Mathematical Sciences, Senegal
Academic year: 2023-2024

Exercise 1

1-) *Determine the basic properties of the following equation and solve the the equation :* $y' = y^2(1+t^2), \qquad y(0) = 1.$

2-) *Let us consider the following differential equation :*

$$t^2y'' - t(t+2)y' + (t+2)y = 0. \tag{1}$$

(a) *Show that les functions* $y_1(t) = t$ *and* $y_2(t) = te^t$ *are two solutions linearly independant of* (1).

(b) *Show that the function* $f(t) = Ay_1(t) + By_2(t)$ *is a solution of* (1), *where* A *and* B *are constants.*

Solution 1

1-) *The basic properties of the following equation* $y' = y^2(1+t^2)$, $y(0) = 1$.

- *linear differential equation*
- *First order, degree 1*
- *variables coefficients*
- *homogeneous*
- *initial value*

Solving the equation : By separable of variables, we have

$$\frac{dy}{y^2} = (1+t^2)dt \Longleftrightarrow -\frac{1}{y} = t + \frac{1}{3}t^3 + C.$$

Since $y(0) = 1$, *we have* $C = -1$. *Then*

$$y(t) = \frac{3}{3 - t^3 - 3t}.$$

2-)

(a) *Let us show that les functions* $y_1(t) = t$ *and* $y_2(t) = te^t$ *are two solutions linearly independant of* (1). *By substitution, it is easy to show that of* y_1 *and* y_2 *respectively are solutions to the equation* (1). *Moreover,* y_1 *and* y_2 *are linearly independant.*

(b) *Let us show that the function $f(t) = Ay_1(t) + By_2(t)$ is a solution of (1), where A and B are constants. The substitution of f in the equation (1) leads us to the result.*

Exercise 2 *Find the general solution of*

1-) $3\partial_x u - 2\partial_y u + u = x$.

2-) $y\partial_x u - x\partial_y u = 0$.

Solution 2 *Let us find the general solution of the following equations :*

1-) $3\partial_x u - 2\partial_y u + u = x$.

- *Method 1 : Using the characteric method, we have* $\frac{dx}{3} = -\frac{dy}{2} = \frac{du}{x-u}$.

$$\frac{dx}{3} = -\frac{dy}{2} \Longleftrightarrow -2x - 3y = C_1.$$

$$\frac{dx}{3} = \frac{du}{x-u} \Longleftrightarrow \frac{du}{dx} + \frac{1}{3}u = \frac{1}{3}x \text{ (1D-ODE)}.$$

The integration factor is

$$\mu(x) = e^{\int \frac{1}{3}dx} = e^{\frac{x}{3}}.$$

Then, by integration by parts, we get

$$\mu(x)u(x,y) = \int \frac{x}{3}e^{\frac{x}{3}} = xe^{\frac{x}{3}} - 3e^{\frac{x}{3}} + C_2(C_1).$$

Therefore, the general solution is

$$u(x,y) = x - 3 + C_2(-2x-3y)e^{-\frac{x}{3}}.$$

- *Method 2 : Using the transformation $u(x,y) = v(w,z)$, where* $\begin{cases} w = -2x - 3y \\ z = y, \end{cases}$ *we have $-2\partial_z v + v = -\frac{1}{2}(w+3z)$. Then, the integration factor is*

$$\mu(z) = e^{-\int \frac{1}{2}dz} = e^{-\frac{z}{2}}.$$

Thus, by integration by parts, we get

$$\mu(z)v(w,z) = \frac{1}{4}\int (w+3z)e^{-\frac{z}{2}}dz = \frac{1}{2}(w+3z+6)e^{-\frac{z}{2}} + C_2(w).$$

Then,

$$v(w,z) = \frac{1}{2}(w+3z+6) + C_2(w)e^{\frac{z}{2}}.$$

Therefore, the general solution is

$$u(x,y) = v(w,z) = x - 3 + C_2(-2x-3y)e^{\frac{y}{2}}.$$

2-) $y\partial_x u - x\partial_y u = 0.$

By characteristic method, we have

$$\frac{dy}{y} = -\frac{dy}{x} = \frac{du}{0} \Longleftrightarrow \left\{ \begin{array}{l} x^2 + y^2 = C_1 \\ u(x,y) = C_2(C_1). \end{array} \right.$$

Then, the general solution is

$$u(x,y) = C_2(x^2 + y^2).$$

Exercise 3 *Write the conservation law form* $\partial_t u + \partial_x \phi = 0$*, by finding the flux function of*

1-) $\partial_t u + c\partial_x u = 0.$

2-) $\partial_t u + u^2\partial_x u + \partial_{xxx} u = 0.$

Solution 3 *The conservation law :*

1-) We have $\partial_t u + \partial_x \phi = 0$*, with* $\phi(u) = cu$*.*

2-) We have $\partial_t u + \partial_x \phi = 0$*, with* $\phi(u) = \frac{1}{3}u^3 + \partial_{xx} u$*.*

Exercise 4 *Solve the following*

1-) $\partial_t u + 4\partial_x u = 0,\ u(x,0) = \frac{1}{1+x^2}.$

2-) $y\partial_x u + x\partial_y u = 0,\ u(0,y) = e^{-y^2}.$

Solution 4 *Let us solve the following equation :*

1-) $\partial_t u + 4\partial_x u = 0,\ u(x,0) = \frac{1}{1+x^2}.$

- *By characteristics method, we have*

$$dt = \frac{dx}{4} = \frac{du}{0} \Longleftrightarrow \left\{ \begin{array}{l} \frac{du}{dt} = 0 \\ \frac{dx}{dt} = 4. \end{array} \right.$$

Then, $u(x,t) = u(x_0, 0) = \frac{1}{1+x_0^2}$ *along* $x(t) = 4t + x_0$*.*

2-) $y\partial_x u + x\partial_y u = 0,\ u(0,y) = e^{-y^2}.$

- *By characteristics method, we have*

$$\frac{dx}{y} = \frac{dy}{x} = \frac{du}{0} \Longleftrightarrow \left\{ \begin{array}{l} xdx = ydy \\ du = 0. \end{array} \right. \Longleftrightarrow \left\{ \begin{array}{l} \frac{x^2-y^2}{2} = C_1 \\ u(x,y) = C_2(C_1). \end{array} \right.$$

Inserting the initial condition, we have $C_1 = -\frac{y^2}{2}$ *and* $u(x,y) = C_2(-\frac{y^2}{2}) = e^{-y^2}$*.*

Exercise 5 *Let us consider the following two equations with initial condition* $u(x,0) = \frac{1}{1+x^2}$.

i) $\partial_t u + u\partial_x u = 0,\ |x| < \infty,\ t > 0.$

ii) $\partial_t u + u^2\partial_x u = 0,\ |x| < \infty,\ t > 0.$

1-) Find the plot of characteristics.

2-) Analytically, determine the breaking time.

3-) Plot the solution $u(x,t)$ *at times before and after breaking time.*

Solution 5 *Let us consider the following two equations with initial condition* $u(x,0) = \frac{1}{1+x^2}$.

i) $\partial_t u + u\partial_x u = 0,\ |x| < \infty,\ t > 0.$

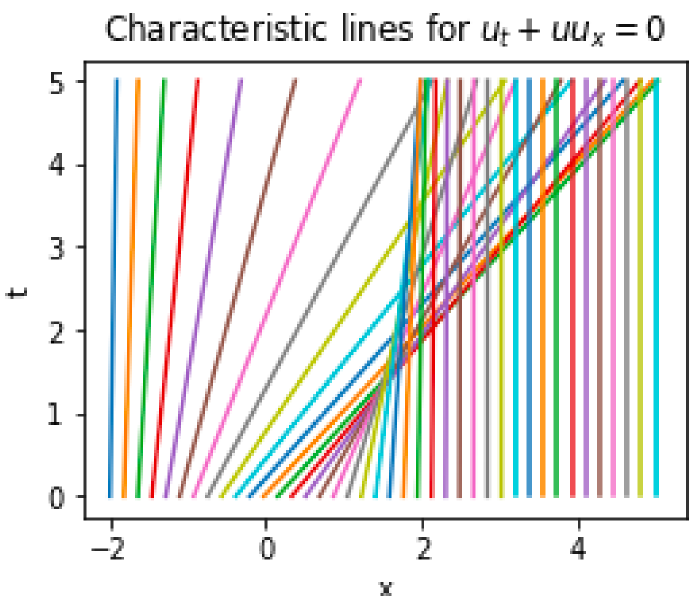
1-) The plot of characteristics.

- *By characteristics method, we have*

$$\begin{cases} \frac{du}{dt} = 0 \\ \frac{dx}{dt} = u. \end{cases}$$

Then, u *is constant along* $\frac{dx}{dt} = u$.
Since u *is constant, this equation can be integrated to yield* $x(t) = u(x_0,0)t + x_0$. *Inserting the initial condition,* $x = \left(\frac{1}{1+x_0^2}\right)t + x_0$. *Therefore, the solution is* $u(x,t) = \frac{1}{1+x_0^2}$ *along* $x = \left(\frac{1}{1+x_0^2}\right)t + x_0$.

- *Plot :*



2-) Let us determine the breaking time.
Since $u_0 = u(x,0) = \frac{1}{1+x^2}$, *then, we have*

$$F(\xi) = \frac{1}{1+\xi^2} \Longleftrightarrow F'(\xi) = -\frac{2\xi}{(1+\xi^2)^2}.$$

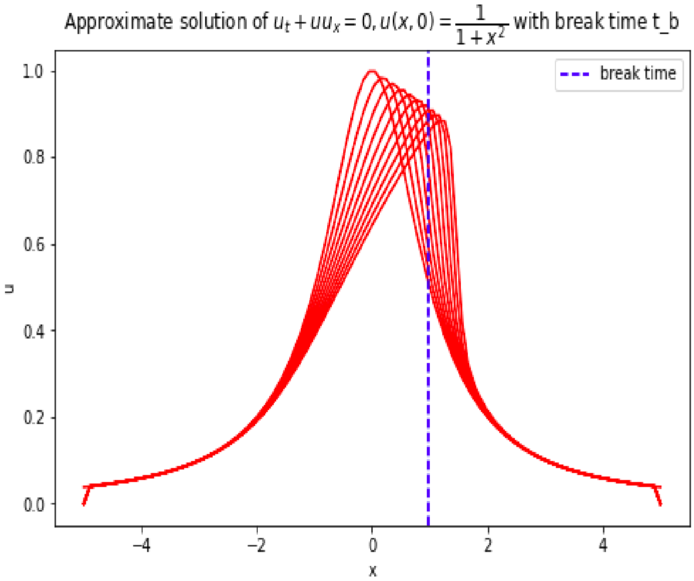
This gives $t = -\frac{1}{F'(\xi)} = \frac{(1+\xi^2)^2}{2\xi}$. *To find the minimum time (break time), we set the derivative equal to zero and for* ξ.
We have

$$\frac{(1+\xi^2)(3\xi^2-1)}{2\xi^2} = \frac{(1+\xi^2)(\sqrt{3}\xi+1)(\sqrt{3}\xi-1)}{2\xi^2} = 0.$$

Thus, the minimum occurs for $\xi = \frac{\sqrt{3}}{3}$. *Therefore, the break time is*

$$t_b = t\left(\frac{\sqrt{3}}{3}\right) = \frac{8\sqrt{3}}{9}.$$

3-) Plot of the solution $u(x,t)$ *at times before and after breaking time.*



ii) $\partial_t u + u^2 \partial_x u = 0$, $|x| < \infty$, $t > 0$.

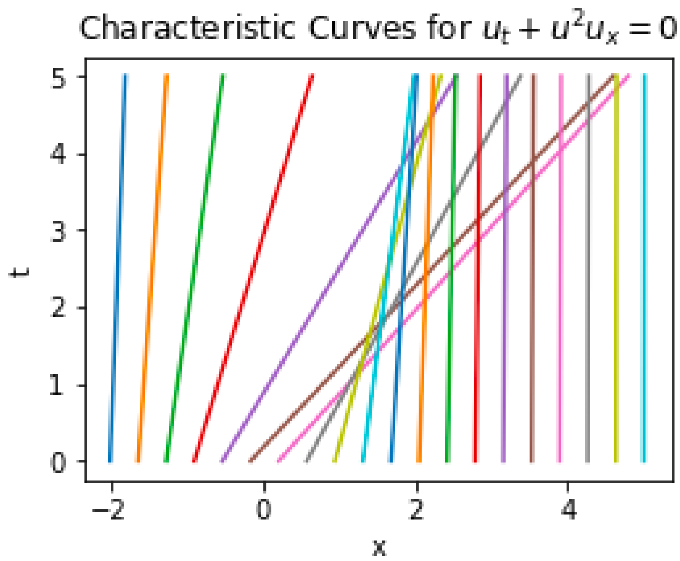
1-) The plot of characteristics.

- *By characteristics method, we have*

$$\begin{cases} \frac{du}{dt} = 0 \\ \frac{dx}{dt} = u^2. \end{cases}$$

Since u *is constant, this equation can be integrated to yield* $x(t) = u(x_0,0)^2 t + x_0$. *Inserting the initial condition,* $x = \frac{1}{(1+x_0^2)^2} t + x_0$. *Therefore, the solution is* $u(x,t) = \frac{1}{(1+x_0^2)^2}$ *along* $x = \frac{1}{(1+x_0^2)^2} t + x_0$.

- *Plot :*



2-) Let us determine the breaking time. Since $u_0 = u(x,0) = \frac{1}{1+x^2}$, *then, we have*

$$F(\xi) = \frac{1}{(1+\xi^2)^2} \iff F'(\xi) = -\frac{4\xi}{(1+\xi^2)^3}.$$

This gives $t = -\frac{1}{F'(\xi)} = \frac{(1+\xi^2)^3}{4\xi}$. *To find the minimum time (break time), we set the derivative equal to zero and for* ξ.
We have

$$\frac{(1+\xi^2)^2(5\xi^2-1)}{4\xi^2} = \frac{2(\sqrt{5}\xi-1)(\sqrt{5}\xi+1)}{(4\xi^2} = 0.$$

Thus, the minimum occurs for $\xi = \frac{\sqrt{5}}{5}$. *Therefore, the break time is*

$$t_b = t\left(\frac{\sqrt{5}}{5}\right) = \frac{54\sqrt{5}}{125}.$$

3-) Plot of the solution $u(x,t)$ *at times before and after breaking time.*