2

Second Order Partial Differential Equations

"Either mathematics is too big for the human mind or the human mind is more than a machine." - Kurt Gödel (1906-1978)

2.1 Introduction

IN THIS CHAPTER WE WILL INTRODUCE several generic second order linear partial differential equations and see how such equations lead naturally to the study of boundary value problems for ordinary differential equations. These generic differential equation occur in one to three spatial dimensions and are all linear differential equations. A list is provided in Table 2.1. Here we have introduced the Laplacian operator, $\nabla^2 u = u_{xx} + u_{yy} + u_{zz}$. Depending on the types of boundary conditions imposed and on the geometry of the system (rectangular, cylindrical, spherical, etc.), one encounters many interesting boundary value problems.

Name	2 Vars	3 D
Heat Equation	$u_t = ku_{xx}$	$u_t = k\nabla^2 u$
Wave Equation	$u_{tt} = c^2 u_{xx}$	$u_{tt} = c^2 \nabla^2 u$
Laplace's Equation	$u_{xx} + u_{yy} = 0$	$\nabla^2 u = 0$
Poisson's Equation	$u_{xx} + u_{yy} = F(x,y)$	$\nabla^2 u = F(x,y,z)$
Schrödinger's Equation	$iu_t = u_{xx} + F(x,t)u$	$iu_t = \nabla^2 u + F(x,y,z,t)u$

Table 2.1: List of generic partial differential equations.

Let's look at the heat equation in one dimension. This could describe the heat conduction in a thin insulated rod of length L. It could also describe the diffusion of pollutant in a long narrow stream, or the flow of traffic down a road. In problems involving diffusion processes, one instead calls this equation the diffusion equation. [See the derivation in Section 2.2.2.]

A typical initial-boundary value problem for the heat equation would be that initially one has a temperature distribution $u(x,0) = f(x)$. Placing the bar in an ice bath and assuming the heat flow is only through the ends of the bar, one has the boundary conditions $u(0,t) = 0$ and $u(L,t) = 0$. Of course, we are dealing with Celsius temperatures and we assume there is

$$u(0,0) = 0 \qquad u(L,0) = 0$$

Figure 2.1: One dimensional heated rod of length L.

plenty of ice to keep that temperature fixed at each end for all time as seen in Figure 2.1. So, the problem one would need to solve is given as [IC = initial condition(s) and BC = boundary conditions.]

1D Heat Equation

$$
\begin{array}{llcc}
\text{PDE} & u_t = ku_{xx}, & 0 < t, & 0 \le x \le L, \\
\text{IC} & u(x,0) = f(x), & & 0 < x < L, \\
\text{BC} & u(0,t) = 0, & & t > 0, \\
& u(L,t) = 0, & & t > 0,
\end{array}
\tag{2.1}
$$

Here, k is the heat conduction constant and is determined using properties of the bar.

Another problem that will come up in later discussions is that of the vibrating string. A string of length L is stretched out horizontally with both ends fixed such as a violin string as shown in Figure 2.2. Let $u(x,t)$ be the vertical displacement of the string at position x and time t. The motion of the string is governed by the one dimensional wave equation. [See the derivation in Section 2.2.1.] The string might be plucked, giving the string an initial profile, $u(x,0) = f(x)$, and possibly each point on the string has an initial velocity $u_t(x,0) = g(x)$. The initial-boundary value problem for this problem is given below.

1D Wave Equation

$$
\begin{array}{llcc}
\text{PDE} & u_{tt} = c^2 u_{xx} & 0 < t, & 0 \le x \le L \\
\text{IC} & u(x,0) = f(x) & & 0 < x < L \\
& u_t(x,0) = g(x) & & 0 < x < L \\
\text{BC} & u(0,t) = 0 & & t > 0 \\
& u(L,t) = 0 & & t > 0
\end{array}
\tag{2.2}
$$

In this problem c is the wave speed in the string. It depends on the mass per unit length of the string, μ, and the tension, τ, placed on the string.

$$u(x,t)$$
$$u(0,0) = 0 \qquad u(L,0) = 0$$

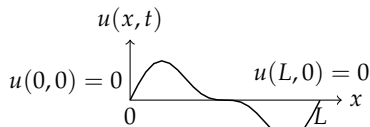

Figure 2.2: One dimensional string of length L.

There is a rich history on the study of these and other partial differential equations and much of this involves trying to solve problems in physics. Consider the one dimensional wave motion in the string. Physically, the speed of these waves depends on the tension in the string and its mass density. The frequencies we hear are then related to the string shape, or the allowed wavelengths across the string. We will be interested the harmonics, or pure sinusoidal waves, of the vibrating string and how a general wave on the string can be represented as a sum over such harmonics. This will take us into the field of spectral, or Fourier, analysis. The solution of the heat equation also involves the use of Fourier analysis. However, in this case there are no oscillations in time.

There are many applications that are studied using spectral analysis. At the root of these studies is the belief that continuous waveforms are comprised of a number of harmonics. Such ideas stretch back to the Pythagoreans study of the vibrations of strings, which led to their program of a world

of harmony. This idea was carried further by Johannes Kepler (1571-1630) in his harmony of the spheres approach to planetary orbits. In the 1700's others worked on the superposition theory for vibrating waves on a stretched spring, starting with the wave equation and leading to the superposition of right and left traveling waves. This work was carried out by people such as John Wallis (1616-1703), Brook Taylor (1685-1731) and Jean le Rond d'Alembert (1717-1783).

In 1742 d'Alembert solved the wave equation

$$c^2 \frac{\partial^2 y}{\partial x^2} - \frac{\partial^2 y}{\partial t^2} = 0,$$

where y is the string height and c is the wave speed. However, this solution led him and others, like Leonhard Euler (1707-1783) and Daniel Bernoulli (1700-1782), to investigate what "functions" could be the solutions of this equation. In fact, this led to a more rigorous approach to the study of analysis by first coming to grips with the concept of a function. For example, in 1749 Euler sought the solution for a plucked string in which case the initial condition $y(x,0) = h(x)$ has a discontinuous derivative! (We will see how this led to important questions in analysis.)

In 1753 Daniel Bernoulli viewed the solutions as a superposition of simple vibrations, or harmonics. Such superpositions amounted to looking at solutions of the form

$$y(x,t) = \sum_k a_k \sin \frac{k\pi x}{L} \cos \frac{k\pi ct}{L},$$

where the string extends over the interval $[0, L]$ with fixed ends at $x = 0$ and $x = L$.

However, the initial profile for such superpositions is given by

$$y(x,0) = \sum_k a_k \sin \frac{k\pi x}{L}.$$

It was determined that many functions could not be represented by a finite number of harmonics, even for the simply plucked string in Figure 2.4 given by an initial condition of the form

$$y(x,0) = \begin{cases} Ax, & 0 \le x \le L/2 \\ A(L-x), & L/2 \le x \le L \end{cases}$$

Thus, the solution consists generally of an infinite series of trigonometric functions.

Such series expansions were also of importance in Joseph Fourier's (1768-1830) solution of the heat equation. The use of Fourier expansions has become an important tool in the solution of linear partial differential equations, such as the wave equation and the heat equation. More generally, using a technique called the Method of Separation of Variables, allowed higher dimensional problems to be reduced to one dimensional boundary value problems. However, these studies led to very important questions, which in turn opened the doors to whole fields of analysis. Some of the problems raised were
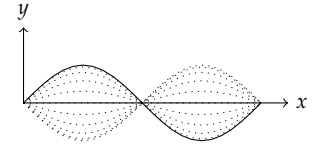

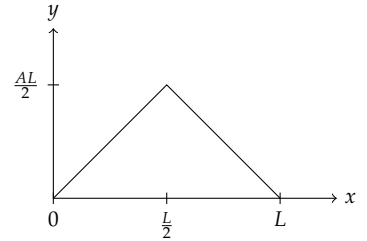Figure 2.3: Plot of the second harmonic of a vibrating string at different times.

Figure 2.4: Plot of an initial condition for a plucked string.

The one dimensional version of the heat equation is a partial differential equation for $u(x,t)$ of the form

$$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}.$$

Solutions satisfying boundary conditions $u(0,t) = 0$ and $u(L,t) = 0$, are of the form

$$u(x,t) = \sum_{n=0}^{\infty} b_n \sin \frac{n\pi x}{L} e^{-n^2\pi^2 t/L^2}.$$

In this case, setting $u(x,0) = f(x)$, one has to satisfy the condition

$$f(x) = \sum_{n=0}^{\infty} b_n \sin \frac{n\pi x}{L}.$$

This is another example leading to an infinite series of trigonometric functions.

1. What functions can be represented as the sum of trigonometric functions?

2. How can a function with discontinuous derivatives be represented by a sum of smooth functions, such as the above sums of trigonometric functions?

3. Do such infinite sums of trigonometric functions actually converge to the functions they represent?

There are many other systems for which it makes sense to interpret the solutions as sums of sinusoids of particular frequencies. For example, we can consider ocean waves. Ocean waves are affected by the gravitational pull of the moon and the sun and other numerous forces. These lead to the tides, which in turn have their own periods of motion. In an analysis of wave heights, one can separate out the tidal components by making use of Fourier analysis.

In the Section 2.4 we describe how to go about solving these equations using the method of separation of variables. We will find that in order to accommodate the initial conditions, we will need to introduce Fourier series before we can complete the problems, which will be the subject of the following chapter. However, we first derive the one-dimensional wave and heat equations.

2.2   Derivation of Generic 1D Equations

2.2.1   Derivation of Wave Equation for String

THE WAVE EQUATION FOR A ONE DIMENSIONAL STRING is derived based upon simply looking at Newton's Second Law of Motion for a piece of the string plus a few simple assumptions, such as small amplitude oscillations and constant density.

We begin with $\mathbf{F} = m\mathbf{a}$. The mass of a piece of string of length ds is $m = \rho(x)ds$. From Figure (2.5) an incremental length f the string is given by

$$\Delta s^2 = \Delta x^2 + \Delta u^2.$$

The piece of string undergoes an acceleration of $a = \frac{\partial^2 u}{\partial t^2}$.

We will assume that the main force acting on the string is that of tension. Let $T(x,t)$ be the magnitude of the tension acting on the left end of the piece of string. Then, on the right end the tension is $T(x + \Delta x, t)$. At these points the tension makes an angle to the horizontal of $\theta(x,t)$ and $\theta(x + \Delta x, t)$, respectively.

The wave equation is derived from $\mathbf{F} = m\mathbf{a}$.

Assuming that there is no horizontal acceleration, the x-component in the second law, $m\mathbf{a} = \mathbf{F}$, for the string element is given by
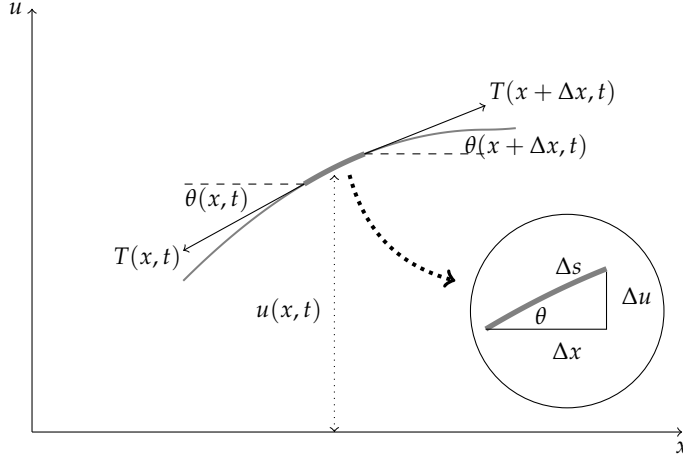
$$0 = T(x + \Delta x, t)\cos\theta(x + \Delta x, t) - T(x,t)\cos\theta(x,t).$$

The vertical component is given by

$$\rho(x)\Delta s\frac{\partial^2 u}{\partial t^2} = T(x+\Delta x,t)\sin\theta(x+\Delta x,t) - T(x,t)\sin\theta(x,t)$$

The length of the piece of string can be written in terms of Δx,

$$\Delta s = \sqrt{\Delta x^2 + \Delta u^2} = \sqrt{1 + \left(\frac{\Delta u}{\Delta x}\right)^2}\,\Delta x.$$

and the right hand sides of the component equation can be expanded about $\Delta x = 0$, to obtain

$$T(x+\Delta x,t)\cos\theta(x+\Delta x,t) - T(x,t)\cos\theta(x,t) \approx \frac{\partial(T\cos\theta)}{\partial x}(x,t)\Delta x$$

$$T(x+\Delta x,t)\sin\theta(x+\Delta x,t) - T(x,t)\sin\theta(x,t) \approx \frac{\partial(T\sin\theta)}{\partial x}(x,t)\Delta x.$$

Furthermore, we note that

$$\tan\theta = \lim_{\Delta x\to 0}\frac{\Delta u}{\Delta x} = \frac{\partial u}{\partial x}.$$

Now we can divide these component equations by Δx and let $\Delta x \to 0$. This gives the approximations

$$0 = \frac{T(x+\Delta x,t)\cos\theta(x+\Delta x,t) - T(x,t)\cos\theta(x,t)}{\Delta x}$$

$$\approx \frac{\partial(T\cos\theta)}{\partial x}(x,t)$$

$$\rho(x)\frac{\partial^2 u}{\partial t^2}\frac{\delta s}{\delta s} = \frac{T(x+\Delta x,t)\sin\theta(x+\Delta x,t) - T(x,t)\sin\theta(x,t)}{\Delta x}$$

$$\rho(x)\frac{\partial^2 u}{\partial t^2}\sqrt{1 + \left(\frac{\partial u}{\partial x}\right)^2} \approx \frac{\partial(T\sin\theta)}{\partial x}(x,t). \tag{2.3}$$

We will assume a small angle approximation, giving

$$\sin\theta \approx \tan\theta = \frac{\partial u}{\partial x},$$

$\cos \theta \approx 1$, and

$$\sqrt{1 + \left(\frac{\partial u}{\partial x}\right)^2} \approx 1.$$

Then, the horizontal component becomes

$$\frac{\partial T(x,t)}{\partial x} = 0.$$

Therefore, the magnitude of the tension $T(x,t) = T(t)$ is at most time dependent.

The vertical component equation is now

$$\rho(x)\frac{\partial^2 u}{\partial t^2} = T(t)\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = T(t)\frac{\partial^2 u}{\partial x^2}.$$

Assuming that ρ and T are constant and defining

$$c^2 = \frac{T}{\rho},$$

we obtain the one dimensional wave equation,

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}.$$

2.2.2   Derivation of 1D Heat Equation

CONSIDER A ONE DIMENSIONAL ROD of length L as shown in Figure 2.6. It is heated and allowed to sit. The heat equation is the governing equation which allows us to determine the temperature of the rod at a later time.

We begin with some simple thermodynamics. Recall that to raise the temperature of a mass m by ΔT takes thermal energy given by

$$Q = mc\Delta T,$$

assuming the mass does not go through a phase transition. Here c is the specific heat capacity of the substance. So, we will begin with the heat content of the rod as

$$Q = mcT(x,t)$$

and assume that m and c are constant.

$u(0,0) = 0 \qquad u(L,0) = 0$

$\longrightarrow x$

$0 \qquad\qquad L$

Figure 2.6: One dimensional heated rod of length L.

We will also need Fourier's law of heat transfer or heat conduction . This law simply states that heat energy flows from warmer to cooler regions and is written in terms of the heat energy flux, $\phi(x,t)$. The heat energy flux, or flux density, gives the rate of energy flow per area. Thus, the amount of heat energy flowing over the left end of the region of cross section A in time Δt is given $\phi(x,t)\Delta t A$. The units of $\phi(x,t)$ are then $J/s/m^2 = W/m^2$.

Fourier's law of heat conduction states that the flux density is proportional to the gradient of the temperature,
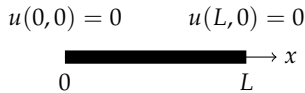
$$\phi = -K\frac{\partial T}{\partial x}.$$

Flux in
$\phi(x,t)$ ⟶

Flux out
⟶ $\phi(x+\Delta x,t)$

0   Δx L x
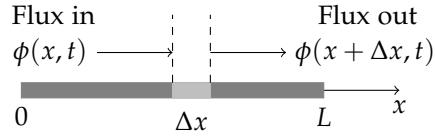
Figure 2.7: A one dimensional rod of length L. Heat can flow through increment Δx.

Here K is the thermal conductivity and the negative sign takes into account the direction of flow from higher to lower temperatures.

Now we make use of the conservation of energy. Consider a small section of the rod of width Δx as shown in Figure 2.7. The rate of change of the energy through this section is due to energy flow through the ends. Namely,

$$\text{Rate of change of heat energy} = \text{Heat in} - \text{Heat out}.$$

The energy content of the small segment of the rod is given by

$$\Delta Q = (\rho A \Delta x)cT(x,t+\Delta t) - (\rho A \Delta x)cT(x,t).$$

The flow rates across the boundaries are given by the flux.

$$(\rho A \Delta x)cT(x,t+\Delta t) - (\rho A \Delta x)cT(x,t) = [\phi(x,t) - \phi(x+\Delta x,t)]\Delta t A.$$

Dividing by Δx and Δt and letting $\Delta x, \Delta t \to 0$, we obtain

$$\frac{\partial T}{\partial t} = -\frac{1}{c\rho}\frac{\partial \phi}{\partial x}.$$

Using Fourier's law of heat conduction,

$$\frac{\partial T}{\partial t} = \frac{1}{c\rho}\frac{\partial}{\partial x}\left(K\frac{\partial T}{\partial x}\right).$$

Assuming K, c, and ρ are constant, we have the one dimensional heat equation as used in the text:

$$\frac{\partial T}{\partial t} = k\frac{\partial^2 T}{\partial x^2},$$

where $k = \frac{k}{c\rho}$.

2.3   Boundary Value Problems

YOU MIGHT HAVE ONLY SOLVED INITIAL VALUE PROBLEMS in your undergraduate differential equations class. For an initial value problem one has to solve a differential equation subject to conditions on the unknown function and its derivatives at one value of the independent variable. For example, for $x = x(t)$ we could have the initial value problem

$$x'' + x = 2, \quad x(0) = 1, \quad x'(0) = 0. \tag{2.4}$$

Typically, initial value problems involve time dependent functions and boundary value problems are spatial. So, with an initial value problem one

knows how a system evolves in terms of the differential equation and the state of the system at some fixed time. Then one seeks to determine the state of the system at a later time.

Example 2.1. *Solve the initial value problem, $x'' + 4x = \cos t$, $x(0) = 1$, $x'(0) = 0$.*

Note that the conditions are provided at one time, $t = 0$. Thus, this an initial value problem. Recall from your course on differential equations that we need to find the general solution and then apply the initial conditions. Furthermore, this is a nonhomogeneous differential equation, so the solution is a sum of a solution of the homogeneous equation and a particular solution of the nonhomogeneous equation, $x(t) = x_h(t) + x_p(t)$. [See the ordinary differential equations review in the Appendix.]

The solution of $x'' + 4x = 0$ is easily found as

$$x_h(t) = c_1 \cos 2t + c_2 \sin 2t.$$

The particular solution is found using the Method of Undetermined Coefficients. We guess a solution of the form

$$x_p(t) = A \cos t + B \sin t.$$

Differentiating twice, we have

$$x_p''(t) = -(A \cos t + B \sin t).$$

So,
$$x_p'' + 4x_p = -(A \cos t + B \sin t) + 4(A \cos t + B \sin t).$$

Comparing the right hand side of this equation with $\cos t$ in the original problem, we are led to setting $B = 0$ and $A = \frac{1}{3} \cos t$. Thus, the general solution is

$$x(t) = c_1 \cos 2t + c_2 \sin 2t + \frac{1}{3} \cos t.$$

We now apply the initial conditions to find the particular solution. The first condition, $x(0) = 1$, gives
$$1 = c_1 + \frac{1}{3}.$$

Thus, $c_1 = \frac{2}{3}$. Using this value for c_1, the second condition, $x'(0) = 0$, gives $c_2 = 0$. Therefore,
$$x(t) = \frac{1}{3}(2 \cos 2t + \cos t).$$

For boundary values problems, one knows how each point responds to its neighbors, but there are conditions that have to be satisfied at the end-points. An example would be a horizontal beam supported at the ends, like a bridge. The shape of the beam under the influence of gravity, or other forces, would lead to a differential equation and the boundary conditions at the beam ends would affect the solution of the problem. There are also a variety of other types of boundary conditions. In the case of a beam, one end could be fixed and the other end could be free to move. We will explore

the effects of different boundary conditions in our discussions and exercises. But, we will first solve a simple boundary value problem which is a slight modification of the above problem.

Example 2.2. *Solve the boundary value problem,* $x'' + x = 2$, $x(0) = 1$, $x(1) = 0$.

Note that the conditions at $t = 0$ *and* $t = 1$ *make this a boundary value problem since the conditions are given at two different points. As with initial value problems, we need to find the general solution and then apply any conditions that we may have. This is a nonhomogeneous differential equation, so the solution is a sum of a solution of the homogeneous equation and a particular solution of the nonhomogeneous equation,* $x(t) = x_h(t) + x_p(t)$. *The solution of* $x'' + x = 0$ *is easily found as*

$$x_h(t) = c_1 \cos t + c_2 \sin t.$$

The particular solution is found using the Method of Undetermined Coefficients,

$$x_p(t) = 2.$$

Thus, the general solution is

$$x(t) = 2 + c_1 \cos t + c_2 \sin t.$$

We now apply the boundary conditions and see if there are values of c_1 *and* c_2 *that yield a solution to this boundary value problem. The first condition,* $x(0) = 0$, *gives*

$$0 = 2 + c_1.$$

Thus, $c_1 = -2$. *Using this value for* c_1, *the second condition,* $x(1) = 1$, *gives*

$$0 = 2 - 2 \cos 1 + c_2 \sin 1.$$

This yields

$$c_2 = \frac{2(\cos 1 - 1)}{\sin 1}.$$

We have found that there is a solution to the boundary value problem and it is given by

$$x(t) = 2 \left(1 - \cos t \frac{(\cos 1 - 1)}{\sin 1} \sin t \right).$$

Boundary value problems arise in many physical systems, just as the initial value problems we have seen earlier. We will see in the next sections that boundary value problems for ordinary differential equations often appear in the solutions of partial differential equations. However, there is no guarantee that we will have unique solutions of our boundary value problems as we had found in the example above.

Now that we understand simple boundary value problems for ordinary differential equations, we can turn to initial-boundary value problems for partial differential equations. We will see that a common method for studying these problems is to use the method of separation of variables. In this method the problem of solving partial differential equations is to separate

the partial differential equation into several ordinary differential equations of which several are boundary value problems of the sort seen in this section.

2.4   Separation of Variables

SOLVING MANY OF THE LINEAR PARTIAL DIFFERENTIAL EQUATIONS presented in the first section can be reduced to solving ordinary differential equations. We will demonstrate this by solving the initial-boundary value problem for the heat equation as given in (2.1). We will employ a method typically used in studying linear partial differential equations, called the Method of Separation of Variables. In the next subsections we describe how this method works for the one-dimensional heat equation, one-dimensional wave equation, and the two-dimensional Laplace equation.

2.4.1   The 1D Heat Equation

WE WANT TO SOLVE THE HEAT EQUATION,

$$u_t = ku_{xx}, \quad 0 < t, \quad 0 \le x \le L.$$

subject to the boundary conditions

$$u(0,t) = 0, u(L,t) = 0, \quad t > 0,$$

and the initial condition

$$u(x,0) = f(x), \quad 0 < x < L.$$

Solution of the 1D heat equation using the method of separation of variables.

We begin by assuming that u can be written as a product of single variable functions of each independent variable,

$$u(x,t) = X(x)T(t).$$

Substituting this guess into the heat equation, we find that

$$XT' = kX''T.$$

The prime denotes differentiation with respect to the independent variable and we will suppress the independent variable in the following unless needed for emphasis.

Dividing both sides of this result by k and $u = XT$, yields

$$\frac{1}{k}\frac{T'}{T} = \frac{X''}{X}.$$

k We have separated the functions of time on one side and space on the other side. The constant k could be on either side of this expression, but we moved it to make later computations simpler.

The only way that a function of t equals a function of x is if the functions are constant functions. Therefore, we set each function equal to a constant, λ : [For example, if $Ae^{ct} = ax^2 + b$ is possible for any x or t, then this is only possible if $a = 0$, $c = 0$ and $b = A$.]

$$\underbrace{\frac{1}{k}\frac{T'}{T}}_{\text{function of } t} = \underbrace{\frac{X''}{X}}_{\text{function of } x} = \underbrace{\lambda.}_{\text{constant}}$$

This leads to two equations:

$$T' = k\lambda T, \tag{2.5}$$

$$X'' = \lambda X. \tag{2.6}$$

These are ordinary differential equations. The general solutions to these constant coefficient equations are readily found as

$$T(t) = Ae^{k\lambda t}, \tag{2.7}$$

$$X(x) = c_1 e^{\sqrt{\lambda}x} + c_2 e^{-\sqrt{\lambda}x}. \tag{2.8}$$

We need to be a little careful at this point. The aim is to force the final solutions to satisfy both the boundary conditions and initial conditions. Also, we should note that λ is arbitrary and may be positive, zero, or negative. We first look at how the boundary conditions on $u(x,t)$ lead to conditions on $X(x)$.

The first boundary condition is $u(0,t) = 0$. This implies that

$$X(0)T(t) = 0, \quad \text{for all } t.$$

The only way that this is true is if $X(0) = 0$. Similarly, $u(L,t) = 0$ for all t implies that $X(L) = 0$. So, we have to solve the boundary value problem

$$X'' - \lambda X = 0, \quad X(0) = 0 = X(L). \tag{2.9}$$

An obvious solution is $X \equiv 0$. However, this implies that $u(x,t) = 0$, which is not an interesting solution. We call such solutions, $X \equiv 0$, trivial solutions and will seek nontrivial solution for these problems.

There are three cases to consider, depending on the sign of λ.

Case I. $\underline{\lambda > 0}$

In this case we have the exponential solutions

$$X(x) = c_1 e^{\sqrt{\lambda}x} + c_2 e^{-\sqrt{\lambda}x}. \tag{2.10}$$

For $X(0) = 0$, we have

$$0 = c_1 + c_2.$$

We will take $c_2 = -c_1$. Then,

$$X(x) = c_1(e^{\sqrt{\lambda}x} - e^{-\sqrt{\lambda}x}) = 2c_1 \sinh \sqrt{\lambda}x.$$

Applying the second condition, $X(L) = 0$ yields

$$c_1 \sinh \sqrt{\lambda}L = 0.$$

This will be true only if $c_1 = 0$, since $\lambda > 0$. Thus, the only solution in this case is the trivial solution, $X(x) = 0$.

Case II. $\underline{\lambda = 0}$

For this case it is easier to set λ to zero in the differential equation. So, $X'' = 0$. Integrating twice, one finds

$$X(x) = c_1 x + c_2.$$

Setting $x = 0$, we have $c_2 = 0$, leaving $X(x) = c_1 x$. Setting $x = L$, we find $c_1 L = 0$. So, $c_1 = 0$ and we are once again left with a trivial solution.

Case III. $\underline{\lambda < 0}$

In this case is would be simpler to write $\lambda = -\mu^2$. Then the differential equation is

$$X'' + \mu^2 X = 0.$$

The general solution is

$$X(x) = c_1 \cos \mu x + c_2 \sin \mu x.$$

At $x = 0$ we get $0 = c_1$. This leaves $X(x) = c_2 \sin \mu x$.
At $x = L$, we find

$$0 = c_2 \sin \mu L.$$

So, either $c_2 = 0$ or $\sin \mu L = 0$. $c_2 = 0$ leads to a trivial solution again. But, there are cases when the sine is zero. Namely,

$$\mu L = n\pi, \quad n = 1, 2, \ldots.$$

Note that $n = 0$ is not included since this leads to a trivial solution. Also, negative values of n are redundant, since the sine function is an odd function.

In summary, we can find solutions to the boundary value problem (2.9) for particular values of λ. The solutions are

$$X_n(x) = \sin \frac{n\pi x}{L}, \quad n = 1, 2, 3, \ldots$$

for

$$\lambda_n = -\mu_n^2 = -\left(\frac{n\pi}{L}\right)^2, \quad n = 1, 2, 3, \ldots.$$

We should note that the boundary value problem in Equation (2.9) is an eigenvalue problem. We can recast the differential equation as

$$LX = \lambda X,$$

where

$$L = D^2 = \frac{d^2}{dx^2}$$

is a linear differential operator. The solutions, $X_n(x)$, are called eigenfunctions and the λ_n's are the eigenvalues. We will elaborate more on this characterization later in the next chapter.

We have found the product solutions of the heat equation (2.1) satisfying the boundary conditions. These are

$$u_n(x,t) = e^{k\lambda_n t} \sin \frac{n\pi x}{L}, \quad n = 1, 2, 3, \ldots. \tag{2.11}$$

However, these do not necessarily satisfy the initial condition $u(x,0) = f(x)$. What we do get is

$$u_n(x,0) = \sin \frac{n\pi x}{L}, \quad n = 1, 2, 3, \ldots.$$

So, if the initial condition is in one of these forms, we can pick out the right value for n and we are done.

For other initial conditions, we have to do more work. Note, since the heat equation is linear, the linear combination of the product solutions is also a solution of the heat equation. The general solution satisfying the given boundary conditions is given as

$$u(x,t) = \sum_{n=1}^{\infty} b_n e^{k\lambda_n t} \sin \frac{n\pi x}{L}. \tag{2.12}$$

The coefficients in the general solution are determined using the initial condition. Namely, setting $t = 0$ in the general solution, we have

$$f(x) = u(x,0) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}.$$

So, if we know $f(x)$, can we find the coefficients, b_n? If we can, then we will have the solution to the full initial-boundary value problem.

The expression for $f(x)$ is a Fourier sine series. We will need to digress into the study of Fourier series in order to see how one can find the Fourier series coefficients given $f(x)$. Before proceeding, we will show that this process is not uncommon by applying the Method of Separation of Variables to the wave equation in the next section.

2.4.2 *The 1D Wave Equation*

IN THIS SECTION WE WILL APPLY the Method of Separation of Variables to the one dimensional wave equation, given by

$$\frac{\partial^2 u}{\partial^2 t} = c^2 \frac{\partial^2 u}{\partial^2 x}, \quad t > 0, \quad 0 \le x \text{ł} L, \tag{2.13}$$

subject to the boundary conditions

$$u(0,t) = 0, u(L,t) = 0, \quad t > 0,$$

and the initial conditions

$$u(x,0) = f(x), u_t(x,0) = g(x), \quad 0 < x < L.$$

This problem applies to the propagation of waves on a string of length L with both ends fixed so that they do not move. $u(x,t)$ represents the vertical displacement of the string over time. The derivation of the wave equation assumes that the vertical displacement is small and the string is uniform. The constant c is the wave speed, given by

$$c = \sqrt{\frac{\tau}{\mu}},$$

where τ is the tension in the string and μ is the mass per unit length. We can understand this in terms of string instruments. The tension can be adjusted to produce different tones and the makeup of the string (nylon or steel, thick or thin) also has an effect. In some cases the mass density is changed simply by using thicker strings. Thus, the thicker strings in a piano produce lower frequency notes.

The u_{tt} term gives the acceleration of a piece of the string. The u_{xx} is the concavity of the string. Thus, for a positive concavity the string is curved upward near the point of interest. Thus, neighboring points tend to pull upward towards the equilibrium position. If the concavity is negative, it would cause a negative acceleration.

Solution of the 1D wave equation using the Method of Separation of Variables.

The solution of this problem is easily found using separation of variables. We let $u(x,t) = X(x)T(t)$. Then we find

$$XT'' = c^2 X''T,$$

which can be rewritten as

$$\frac{1}{c^2}\frac{T''}{T} = \frac{X''}{X}.$$

Again, we have separated the functions of time on one side and space on the other side. Therefore, we set each function equal to a constant, λ.

$$\underbrace{\frac{1}{c^2}\frac{T''}{T}}_{\text{function of } t} = \underbrace{\frac{X''}{X}}_{\text{function of } x} = \underbrace{\lambda.}_{\text{constant}}$$

This leads to two equations:

$$T'' = c^2 \lambda T, \tag{2.14}$$

$$X'' = \lambda X. \tag{2.15}$$

As before, we have the boundary conditions on $X(x)$:

$$X(0) = 0, \quad \text{and} \quad X(L) = 0,$$

giving the solutions, as shown in Figure 2.8,

$$X_n(x) = \sin \frac{n\pi x}{L}, \quad \lambda_n = -\left(\frac{n\pi}{L}\right)^2.$$

The main difference from the solution of the heat equation is the form of the time function. Namely, from Equation (2.14) we have to solve

$$T'' + \left(\frac{n\pi c}{L}\right)^2 T = 0. \qquad (2.16)$$

This equation takes a familiar form. We let

$$\omega_n = \frac{n\pi c}{L},$$

then we have

$$T'' + \omega_n^2 T = 0.$$

This is the differential equation for simple harmonic motion and ω_n is the angular frequency. The solutions are easily found as

$$T(t) = A_n \cos \omega_n t + B_n \sin \omega_n t. \qquad (2.17)$$

Therefore, we have found that the product solutions of the wave equation take the forms $\sin \frac{n\pi x}{L} \cos \omega_n t$ and $\sin \frac{n\pi x}{L} \sin \omega_n t$. The general solution, a superposition of all product solutions, is given by

$$u(x,t) = \sum_{n=1}^{\infty} \left[A_n \cos \frac{n\pi ct}{L} + B_n \sin \frac{n\pi ct}{L} \right] \sin \frac{n\pi x}{L}. \qquad (2.18)$$

Figure 2.8: The first three harmonics of the vibrating string.

General solution.

This solution satisfies the wave equation and the boundary conditions. We still need to satisfy the initial conditions. Note that there are two initial conditions, since the wave equation is second order in time.

First, we have $u(x,0) = f(x)$. Thus,

$$f(x) = u(x,0) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{L}. \qquad (2.19)$$

In order to obtain the condition on the initial velocity, $u_t(x,0) = g(x)$, we need to differentiate the general solution with respect to t:

$$u_t(x,t) = \sum_{n=1}^{\infty} \frac{n\pi c}{L} \left[-A_n \sin \frac{n\pi ct}{L} + B_n \cos \frac{n\pi ct}{L} \right] \sin \frac{n\pi x}{L}. \qquad (2.20)$$

Then, we have from the initial velocity

$$g(x) = u_t(x,0) = \sum_{n=1}^{\infty} \frac{n\pi c}{L} B_n \sin \frac{n\pi x}{L}. \qquad (2.21)$$

So, applying the two initial conditions, we have found that $f(x)$ and $g(x)$, are represented as Fourier sine series. In order to complete the problem we need to determine the coefficients A_n and B_n for $n = 1, 2, 3, \ldots$. Once we have these, we have the complete solution to the wave equation. We had seen similar results for the heat equation. In the next chapter we will find out how to determine these Fourier coefficients for such series of sinusoidal functions.
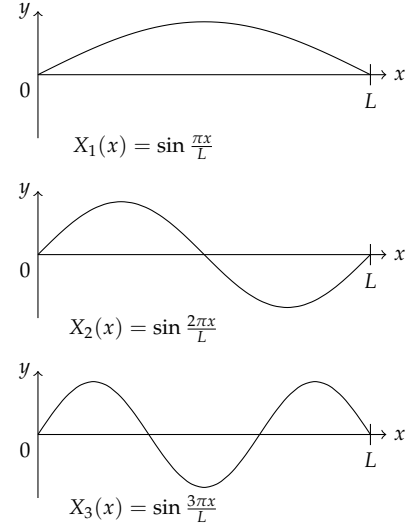
2.5 Laplace's Equation in 2D

ANOTHER GENERIC PARTIAL DIFFERENTIAL EQUATION is Laplace's equation, $\nabla^2 u = 0$. Laplace's equation arises in many applications. As an example, consider a thin rectangular plate with boundaries set at fixed temperatures. Assume that any temperature changes of the plate are governed by the heat equation, $u_t = k\nabla^2 u$, subject to these boundary conditions. However, after a long period of time the plate may reach thermal equilibrium. If the boundary temperature is zero, then the plate temperature decays to zero across the plate. However, if the boundaries are maintained at a fixed nonzero temperature, which means energy is being put into the system to maintain the boundary conditions, the internal temperature may reach a nonzero equilibrium temperature. Reaching thermal equilibrium means that asymptotically in time the solution becomes time independent. Thus, the equilibrium state is a solution of the time independent heat equation, $\nabla^2 u = 0$.

Thermodynamic equilibrium, $\nabla^2 u = 0$.

A second example comes from electrostatics. Letting $\phi(\mathbf{r})$ be the electric potential, one has for a static charge distribution, $\rho(\mathbf{r})$, that the electric field, $\mathbf{E} = \nabla\phi$, satisfies one of Maxwell's equations, $\nabla \cdot \mathbf{E} = \rho/\epsilon_0$. In regions devoid of charge, $\rho(\mathbf{r}) = 0$, the electric potential satisfies Laplace's equation, $\nabla^2\phi = 0$.

Incompressible, irrotational fluid flow, $\nabla^2\phi = 0$, for velocity $\mathbf{v} = \nabla\phi$. See more in Section 2.5.

As a final example, Laplace's equation appears in two-dimensional fluid flow. For an incompressible flow, $\nabla \cdot \mathbf{v} = 0$. If the flow is irrotational, then $\nabla \times \mathbf{v} = 0$. We can introduce a velocity potential, $\mathbf{v} = \nabla\phi$. Thus, $\nabla \times \mathbf{v}$ vanishes by a vector identity and $\nabla \cdot \mathbf{v} = 0$ implies $\nabla^2\phi = 0$. So, once again we obtain Laplace's equation.

Solutions of Laplace's equation are called *harmonic functions* and we will encounter these in Chapter 8 on complex variables and in Section 2.5 we will apply complex variable techniques to solve the two-dimensional Laplace equation. In this section we use the Method of Separation of Variables to solve simple examples of Laplace's equation in two dimensions. Three-dimensional problems will studied in Chapter 6.

Example 2.3. *Equilibrium Temperature Distribution for a Rectangular Plate*
Let's consider Laplace's equation in Cartesian coordinates,

$$u_{xx} + u_{yy} = 0, \quad 0 < x < L, \quad 0 < y < H$$

with the boundary conditions

$$u(0,y) = 0, \quad u(L,y) = 0, \quad u(x,0) = f(x), \quad u(x,H) = 0.$$

The boundary conditions are shown in Figure 6.8

As with the heat and wave equations, we can solve this problem using the method of separation of variables. Let $u(x,y) = X(x)Y(y)$. Then, Laplace's equation becomes
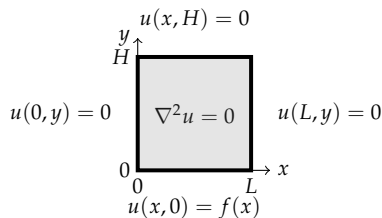
$$X''Y + XY'' = 0$$

Figure 2.9: In this figure we show the domain and boundary conditions for the example of determining the equilibrium temperature distribution for a rectangular plate.

and we can separate the x and y dependent functions and introduce a separation constant, λ,

$$\frac{X''}{X} = -\frac{Y''}{Y} = -\lambda.$$

Thus, we are led to two differential equations,

$$\begin{aligned} X'' + \lambda X &= 0, \\ Y'' - \lambda Y &= 0. \end{aligned} \qquad (2.22)$$

From the boundary condition $u(0, y) = 0, u(L, y) = 0$, we have $X(0) = 0, X(L) = 0$. So, we have the usual eigenvalue problem for $X(x)$,

$$X'' + \lambda X = 0, \quad X(0) = 0, X(L) = 0.$$

The solutions to this problem are given by

$$X_n(x) = \sin\frac{n\pi x}{L}, \quad \lambda_n = \left(\frac{n\pi}{L}\right)^2, \quad n = 1, 2, \ldots.$$

The general solution of the equation for $Y(y)$ is given by

$$Y(y) = c_1 e^{\sqrt{\lambda}y} + c_2 e^{-\sqrt{\lambda}y}.$$

The boundary condition $u(x, H) = 0$ implies $Y(H) = 0$. So, we have

$$c_1 e^{\sqrt{\lambda}H} + c_2 e^{-\sqrt{\lambda}H} = 0.$$

Thus,

$$c_2 = -c_1 e^{2\sqrt{\lambda}H}.$$

Inserting this result into the expression for $Y(y)$, we have

$$\begin{aligned} Y(y) &= c_1 e^{\sqrt{\lambda}y} - c_1 e^{2\sqrt{\lambda}H} e^{-\sqrt{\lambda}y} \\ &= c_1 e^{\sqrt{\lambda}H} \left(e^{-\sqrt{\lambda}H} e^{\sqrt{\lambda}y} - e^{\sqrt{\lambda}H} e^{-\sqrt{\lambda}y} \right) \\ &= c_1 e^{\sqrt{\lambda}H} \left(e^{-\sqrt{\lambda}(H-y)} - e^{\sqrt{\lambda}(H-y)} \right) \\ &= -2c_1 e^{\sqrt{\lambda}H} \sinh\sqrt{\lambda}(H-y). \end{aligned} \qquad (2.23)$$

Note: Having carried out this computation, we can now see that it would be better to guess this form in the future. So, for $Y(H) = 0$, one would guess a solution $Y(y) = \sinh\sqrt{\lambda}(H - y)$. For $Y(0) = 0$, one would guess a solution $Y(y) = \sinh\sqrt{\lambda}y$. Similarly, if $Y'(H) = 0$, one would guess a solution $Y(y) = \cosh\sqrt{\lambda}(H - y)$.

Since we already know the values of the eigenvalues λ_n from the eigenvalue problem for $X(x)$, we have that the y-dependence is given by

$$Y_n(y) = \sinh\frac{n\pi(H-y)}{L}.$$

So, the product solutions are given by

$$u_n(x, y) = \sin\frac{n\pi x}{L}\sinh\frac{n\pi(H-y)}{L}, \quad n = 1, 2, \ldots.$$

These solutions satisfy Laplace's equation and the three homogeneous boundary conditions and in the problem.

The remaining boundary condition, $u(x, 0) = f(x)$, still needs to be satisfied. Inserting $y = 0$ in the product solutions does not satisfy the boundary condition

unless $f(x)$ is proportional to one of the eigenfunctions $X_n(x)$. So, we first write down the general solution as a linear combination of the product solutions,

$$u(x,y) = \sum_{n=1}^{\infty} a_n \sin \frac{n\pi x}{L} \sinh \frac{n\pi (H-y)}{L}. \tag{2.24}$$

Now we apply the boundary condition, $u(x,0) = f(x)$, to find that

$$f(x) = \sum_{n=1}^{\infty} a_n \sinh \frac{n\pi H}{L} \sin \frac{n\pi x}{L}. \tag{2.25}$$

Defining $b_n = a_n \sinh \frac{n\pi H}{L}$, this becomes

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}. \tag{2.26}$$

We see that the determination of the unknown coefficients, b_n, is simply done by recognizing that this is a Fourier sine series. We now move on to the study of Fourier series and provide more complete answers in Chapter 6.

2.6   Classification of Second Order PDEs

WE HAVE STUDIED SEVERAL EXAMPLES of partial differential equations, the heat equation, the wave equation, and Laplace's equation. These equations are examples of parabolic, hyperbolic, and elliptic equations, respectively. Given a general second order linear partial differential equation, how can we tell what type it is? This is known as the classification of second order PDEs.

Let $u = u(x,y)$. Then, the general form of a linear second order partial differential equation is given by

$$a(x,y)u_{xx} + 2b(x,y)u_{xy} + c(x,y)u_{yy} + d(x,y)u_x + e(x,y)u_y + f(x,y)u = g(x,y). \tag{2.27}$$

In this section we will show that this equation can be transformed into one of three types of second order partial differential equations.

Let $x = x(\xi, \eta)$ and $y = y\xi, \eta)$ be an invertible transformation from coordinates (ξ, η) to coordinates (x,y). Furthermore, let $u(x(\xi, \eta), y(\xi, \eta)) = U(\xi, \eta)$. How does the partial differential equation (2.27) transform?

We first need to transform the derivatives of $u(x,t)$. We have

$$
\begin{aligned}
u_x &= U_\xi \xi_x + U_\eta \eta_x, \\
u_y &= U_\xi \xi_y + U_\eta \eta_y, \\
u_{xx} &= \frac{\partial}{\partial x}(U_\xi \xi_x + U_\eta \eta_x), \\
&= U_{\xi\xi}\xi_x^2 + 2U_{\xi\eta}\xi_x\eta_x + U_{\eta\eta}\eta_x^2 + U_\xi \xi_{xx} + U_\eta \eta_{xx}, \\
u_{yy} &= \frac{\partial}{\partial y}(U_\xi \xi_y + U_\eta \eta_y), \\
&= U_{\xi\xi}\xi_y^2 + 2U_{\xi\eta}\xi_y\eta_y + U_{\eta\eta}\eta_y^2 + U_\xi \xi_{yy} + U_\eta \eta_{yy},
\end{aligned}
$$

$$
\begin{aligned}
u_{xy} &= \frac{\partial}{\partial y}(U_\xi \xi_x + U_\eta \eta_x), \\
&= U_{\xi\xi}\xi_x\xi_y + U_{\xi\eta}\xi_x\eta_y + U_{\xi\eta}\xi_y\eta_x + U_{\eta\eta}\eta_x\eta_y + U_\xi\xi_{xy} + U_\eta\eta_{xy}.
\end{aligned}
$$

$$(2.28)$$

Inserting these derivatives into Equation (2.27), we have

$$
\begin{aligned}
g - fU &= au_{xx} + 2bu_{xy} + cu_{yy} + du_x + eu_y \\
&= a\left(U_{\xi\xi}\xi_x^2 + 2U_{\xi\eta}\xi_x\eta_x + U_{\eta\eta}\eta_x^2 + U_\xi\xi_{xx} + U_\eta\eta_{xx}\right) \\
&\quad + 2b\left(U_{\xi\xi}\xi_x\xi_y + U_{\xi\eta}\xi_x\eta_y + U_{\xi\eta}\xi_y\eta_x\right. \\
&\quad\quad \left. + U_{\eta\eta}\eta_x\eta_y + U_\xi\xi_{xy} + U_\eta\eta_{xy}\right) \\
&\quad + c\left(U_{\xi\xi}\xi_y^2 + 2U_{\xi\eta}\xi_y\eta_y + U_{\eta\eta}\eta_y^2 + U_\xi\xi_{yy} + U_\eta\eta_{yy}\right) \\
&\quad + d\left(U_\xi\xi_x + U_\eta\eta_x\right) \\
&\quad + e\left(U_\xi\xi_y + U_\eta\eta_y\right) \\
&= (a\xi_x^2 + 2b\xi_x\xi_y + c\xi_y^2)U_{\xi\xi} \\
&\quad + (2a\xi_x\eta_x + 2b\xi_x\eta_y + 2b\xi_y\eta_x + 2c\xi_y\eta_y)U_{\xi\eta} \\
&\quad + (a\eta_x^2 + 2b\eta_x\eta_y + c\eta_y^2)U_{\eta\eta} \\
&\quad + (a\xi_{xx} + 2b\xi_{xy} + c\xi_{yy} + d\xi_x + e\xi_y)U_\xi \\
&\quad + (a\eta_{xx} + 2b\eta_{xy} + c\eta_{yy} + d\eta_x + e\eta_y)U_\eta \\
&= AU_{\xi\xi} + 2BU_{\xi\eta} + CU_{\eta\eta} + DU_\xi + EU_\eta.
\end{aligned}
$$

$$(2.29)$$

Picking the right transformation, we can eliminate some of the second order derivative terms depending on the type of differential equation. This leads to three types: elliptic, hyperbolic, or parabolic.

For example, if transformations can be found to make $A \equiv 0$ and $C \equiv 0$, then the equation reduces to

$$U_{\xi\eta} = \text{ lower order terms.}$$

Such an equation is called hyperbolic. A generic example of a hyperbolic equation is the wave equation.

Hyperbolic case.

The conditions that $A \equiv 0$ and $C \equiv 0$ give the conditions

$$
\begin{aligned}
a\xi_x^2 + 2b\xi_x\xi_y + c\xi_y^2 &= 0. \\
a\eta_x^2 + 2b\eta_x\eta_y + c\eta_y^2 &= 0.
\end{aligned}
$$

$$(2.30)$$

We seek ξ and η satisfying these two equations, which are of the same form. Let's assume that $\xi = \xi(x, y)$ is a constant curve in the xy-plane. Furthermore, if this curve is the graph of a function, $y = y(x)$, then

$$\frac{d\xi}{dx} = \xi_x + \frac{dy}{dx}\xi_y = 0.$$

Then

$$\frac{dy}{dx} = -\frac{\xi_x}{\xi_y}.$$

Inserting this expression in $A = 0$, we have

$$
\begin{aligned}
A &= a\xi_x^2 + 2b\xi_x\xi_y + c\xi_y^2 \\
&= \xi_y^2\left(a\left(\frac{\xi_x}{\xi_y}\right)^2 + 2b\frac{\xi_x}{\xi_y} + c\right) \\
&= \xi_y^2\left(a\left(\frac{dy}{dx}\right)^2 - 2b\frac{dy}{dx} + c\right) = 0. \quad (2.31)
\end{aligned}
$$

This equation is satisfied if $y(x)$ satisfies the differential equation

$$
\frac{dy}{dx} = \frac{b \pm \sqrt{b^2 - ac}}{a}.
$$

So, for $A = 0$, we choose ξ and η to be constant on these characteristic curves.

Example 2.4. *Show that $u_{xx} - u_{yy} = 0$ is hyperbolic.*
In this case we have $a = 1 = -c$ and $b = 0$. Then,

$$
\frac{dy}{dx} = \pm 1.
$$

This gives $y(x) = \pm x + c$. So, we choose ξ and η constant on these characteristic curves. Therefore, we let $\xi = x - y$, $\eta = x + y$.

Let's see if this transformation transforms the differential equation into a canonical form. Let $u(x, y) = U(\xi, \eta)$. Then, the needed derivatives become

$$
\begin{aligned}
u_x &= U_\xi\xi_x + U_\eta\eta_x = U_\xi + U_\eta. \\
u_y &= U_\xi\xi_y + U_\eta\eta_y = -U_\xi + U_\eta. \\
u_{xx} &= \frac{\partial}{\partial x}(U_\xi + U_\eta) \\
&= U_{\xi\xi}\xi_x + U_{\xi\eta}\eta_x + U_{\eta\xi}\xi_x + U_{\eta\eta}\eta_x \\
&= U_{\xi\xi} + 2U_{\xi\eta} + U_{\eta\eta}. \\
u_{yy} &= \frac{\partial}{\partial y}(-U_\xi + U_\eta) \\
&= -U_{\xi\xi}\xi_y - U_{\xi\eta}\eta_y + U_{\eta\xi}\xi_y + U_{\eta\eta}\eta_y \\
&= U_{\xi\xi} - 2U_{\xi\eta} + U_{\eta\eta}. \quad (2.32)
\end{aligned}
$$

Inserting these derivatives into the differential equation, we have

$$
0 = u_{xx} - u_{yy} = 4U_{\xi\eta}.
$$

Thus, the transformed equation is $U_{\xi\eta} = 0$. Thus, showing it is a hyperbolic equation.

We have seen that A and C vanish for $\xi(x, y)$ and $\eta(x, y)$ constant along the characteristics

$$
\frac{dy}{dx} = \frac{b \pm \sqrt{b^2 - ac}}{a}
$$

for second order hyperbolic equations. This is possible when $b^2 - ac > 0$ since this leads to two characteristics.

In general, if we consider the second order operator

$$L[u] = a(x, y)u_{xx} + 2b(x, y)u_{xy} + c(x, y)u_{yy},$$

then this operator can be transformed to the new form

$$L'[U] = BU_{\xi\eta}$$

if $b^2 - ac > 0$. An example of a hyperbolic equation is the wave equation, $u_{tt} = u_{xx}$.

When $b^2 - ac = 0$, then there is only one characteristic solution, $\frac{dy}{dx} = \frac{b}{a}$. This is the parabolic case. But, $\frac{dy}{dx} = -\frac{\xi_x}{\xi_y}$. So,   Parabolic case.

$$\frac{b}{a} = -\frac{\xi_x}{\xi_y},$$

or

$$a\xi_x + b\xi_y = 0.$$

Also, $b^2 - ac = 0$ implies that $c = b^2/a$.

Inserting these expression into coefficient B, we have

$$
\begin{aligned}
B &= 2a\xi_x\eta_x + 2b\xi_x\eta_y + 2b\xi_y\eta_x + 2c\xi_y\eta_y \\
&= 2(a\xi_x + b\xi_y)\eta_x + 2(b\xi_x + c\xi_y)\eta_y \\
&= 2\frac{b}{a}(a\xi_x + b\xi_y)\eta_y = 0. \quad\quad\quad (2.33)
\end{aligned}
$$

Therefore, in the parabolic case, $A = 0$ and $B = 0$, and $L[u]$ transforms to

$$L'[U] = CU_{\eta\eta}$$

when $b^2 - ac = 0$. This is the canonical form for a parabolic operator. An example of a parabolic equation is the heat equation, $u_t = u_{xx}$.

Finally, when $b^2 - ac < 0$, we have the elliptic case.   In this case we   Elliptic case.
cannot force $A = 0$ or $C = 0$. However, in this case we can force $B = 0$. As we just showed, we can write

$$B = 2(a\xi_x + b\xi_y)\eta_x + 2(b\xi_x + c\xi_y)\eta_y.$$

Letting $\eta_x = 0$, we can choose ξ to satisfy $b\xi_x + c\xi_y = 0$.

$$A = a\xi_x^2 + 2b\xi_x\xi_y + c\xi_y^2 = a\xi_x^2 - c\xi_y^2 = \frac{ac - b^2}{c}\xi_x^2$$

$$C = a\eta_x^2 + 2b\eta_x\eta_y + c\eta_y^2 = c\eta_y^2$$

Furthermore, setting $\frac{ac-b^2}{c}\xi_x^2 = c\eta_y^2$, we can make $A = C$ and $L[u]$ transforms to

$$L'[U] = A[U_{\xi\xi} + U_{\eta\eta}]$$

when $b^2 - ac < 0$. This is the canonical form for an elliptic operator. An example of an elliptic equation is Laplace's equation, $u_{xx} + u_{yy} = 0$.

Classification of Second Order PDEs

The second order differential operator

$$L[u] = a(x,y)u_{xx} + 2b(x,y)u_{xy} + c(x,y)u_{yy},$$

can be transformed to one of the following forms:

- $b^2 - ac > 0$. Hyperbolic: $L[u] = B(x,y)u_{xy}$

- $b^2 - ac = 0$. Parabolic: $L[u] = C(x,y)u_{yy}$

- $b^2 - ac < 0$. Elliptic: $L[u] = A(x,y)[u_{xx} + u_{yy}]$

As a final note, the terminology used in this classification is borrowed from the general theory of quadratic equations which are the equations for translated and rotated conics. Recall that the general quadratic equation in two variable takes the form

$$ax^2 + 2bxy + cy^2 + dx + ey + f = 0. \tag{2.34}$$

One can complete the squares in x and y to obtain the new form

$$a(x-h)^2 + 2bxy + c(y-k)^2 + f' = 0.$$

So, translating points (x,y) using the transformations $x' = x - h$ and $y' = y - k$, we find the simpler form

$$ax^2 + 2bxy + cy^2 + f = 0.$$

Here we dropped all primes.

We can also introduce transformations to simplify the quadratic terms. Consider a rotation of the coordinate axes by θ,

$$\begin{aligned} x' &= x\cos\theta + y\sin\theta \\ y' &= -x\sin\theta + y\cos\theta, \end{aligned} \tag{2.35}$$

or

$$\begin{aligned} x &= x'\cos\theta - y'\sin\theta \\ y &= x'\sin\theta + y'\cos\theta. \end{aligned} \tag{2.36}$$

The resulting equation takes the form

$$Ax'^2 + 2Bx'y' + Cy'^2 + D = 0,$$

where

$$\begin{aligned} A &= a\cos^2\theta + 2b\sin\theta\cos\theta + c\sin^2\theta. \\ B &= (c-a)\sin\theta\cos\theta + b(\cos^2\theta - \sin^\theta). \\ C &= a\sin^2\theta - 2b\sin\theta\cos\theta + c\cos^2\theta. \end{aligned} \tag{2.37}$$

We can eliminate the $x'y'$ term by forcing $B = 0$. Since $\cos^2 \theta - \sin^2 \theta = \cos 2\theta$ and $\sin \theta \cos \theta = \frac{1}{2} \sin 2\theta$, we have

$$B = \frac{(c-a)}{2} \sin 2\theta + b \cos 2\theta = 0.$$

Therefore, the condition for eliminating the $x'y'$ term is

$$\cot(2\theta) = \frac{a-c}{2b}.$$

Furthermore, one can show that $b^2 - ac = B^2 - AC$. From the form $Ax'^2 + 2Bx'y' + Cy'^2 + D = 0$, the resulting quadratic equation takes one of the following forms:

- $b^2 - ac > 0$. Hyperbolic: $Ax^2 - Cy^2 + D = 0$.

- $b^2 - ac = 0$. Parabolic: $Ax^2 + By + D = 0$.

- $b^2 - ac < 0$. Elliptic: $Ax^2 + Cy^2 + D = 0$.

Thus, one can see the connection between the classification of quadratic equations and second order partial differential equations in two independent variables.

2.7 d'Alembert's Solution of the Wave Equation

A GENERAL SOLUTION OF THE ONE-DIMENSIONAL WAVE EQUATION can be found. This solution was first Jean-Baptiste le Rond d'Alembert (1717-1783) and is referred to as d'Alembert's formula. In this section we will derive d'Alembert's formula and then use it to arrive at solutions to the wave equation on infinite, semi-infinite, and finite intervals.

We consider the wave equation in the form $u_{tt} = c^2 u_{xx}$ and introduce the transformation

$$u(x,t) = U(\xi, \eta), \quad \text{where} \quad \xi = x + ct \text{ and } \eta = x - ct.$$

Note that ξ, and η are the characteristics of the wave equation.

We also need to note how derivatives transform. For example

$$\begin{aligned}
\frac{\partial u}{\partial x} &= \frac{\partial U(\xi, \eta)}{\partial x} \\
&= \frac{\partial U(\xi, \eta)}{\partial \xi} \frac{\partial \xi}{\partial x} + \frac{\partial U(\xi, \eta)}{\partial \eta} \frac{\partial \eta}{\partial x} \\
&= \frac{\partial U(\xi, \eta)}{\partial \xi} + \frac{\partial U(\xi, \eta)}{\partial \eta}.
\end{aligned} \tag{2.38}$$

Therefore, as an operator, we have

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}.$$

Similarly, one can show that

$$\frac{\partial}{\partial t} = c\frac{\partial}{\partial \xi} - c\frac{\partial}{\partial \eta}.$$

Using these results, the wave equation becomes

$$
\begin{aligned}
0 &= u_{tt} - c^2 u_{xx} \\
&= \left(\frac{\partial^2}{\partial t^2} - c^2\frac{\partial^2}{\partial x^2}\right)u \\
&= \left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)u \\
&= \left(c\frac{\partial}{\partial \xi} - c\frac{\partial}{\partial \eta} + c\frac{\partial}{\partial \xi} + c\frac{\partial}{\partial \eta}\right)\left(c\frac{\partial}{\partial \xi} - c\frac{\partial}{\partial \eta} - c\frac{\partial}{\partial \xi} - c\frac{\partial}{\partial \eta}\right)U \\
&= -4c^2\frac{\partial}{\partial \xi}\frac{\partial}{\partial \eta}U.
\end{aligned}
\tag{2.39}
$$

Therefore, the wave equation has transformed into the simpler equation,

$$U_{\eta\xi} = 0.$$

Not only is this simpler, but we see it is once again a confirmation that the wave equation is a hyperbolic equation. Of course, it is also easy to integrate. Since

$$\frac{\partial}{\partial \eta}\left(\frac{\partial U}{\partial \xi}\right) = 0,$$

$$\frac{\partial U}{\partial \xi} = \text{constant with respect to } \xi = \Gamma(\eta).$$

A further integration gives

$$U(\xi, \eta) = \int^{\eta} \Gamma(\eta')\, d\eta' + F(\xi) \equiv G(\eta) + F(\eta).$$

Therefore, we have as the general solution of the wave equation,

$$u(x, t) = F(x + ct) + G(x - ct), \tag{2.40}$$

where F and G are two arbitrary, twice differentiable functions. As t is increased, we see that $F(x + ct)$ gets horizontally shifted to the left and $G(x - ct)$ gets horizontally shifted to the right. As a result, we conclude that the solution of the wave equation can be seen as the sum of left and right traveling waves.

$u(x, t)$ = sum of left and right traveling waves.

Let's use initial conditions to solve for the unknown functions. We let

$$u(x, 0) = f(x), \quad u_t(x, 0) = g(x), \quad |x| < \infty.$$

Applying this to the general solution, we have

$$
\begin{aligned}
f(x) &= F(x) + G(x) \tag{2.41} \\
g(x) &= c[F'(x) - G'(x)]. \tag{2.42}
\end{aligned}
$$

We need to solve for $F(x)$ and $G(x)$ in terms of $f(x)$ and $g(x)$. Integrating Equation (2.42), we have

$$\frac{1}{c}\int_0^x g(s)\,ds = F(x) - G(x) - F(0) + G(0).$$

Adding this result to Equation (2.42), gives

$$F(x) = \frac{1}{2}f(x) + \frac{1}{2c}\int_0^x g(s)\,ds + \frac{1}{2}[F(0) - G(0)].$$

Subtracting from Equation (2.42), gives

$$G(x) = \frac{1}{2}f(x) - \frac{1}{2c}\int_0^x g(s)\,ds - \frac{1}{2}[F(0) - G(0)].$$

Now we can write out the solution $u(x,t) = F(x+ct) + G(x-ct)$, yielding d'Alembert's solution

d'Alembert's solution

$$u(x,t) = \frac{1}{2}[f(x+ct) + f(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,ds. \qquad (2.43)$$

When $f(x)$ and $g(x)$ are defined for all $x \in R$, the solution is well-defined. However, there are problems on more restricted domains. In the next examples we will consider the semi-infinite and finite length string problems. In each case we will need to consider the domain of dependence and the domain of influence of specific points. These concepts are shown in Figure 2.10. The domain of dependence of point P is red region. The point P depends on the values of u and u_t at points inside the domain. The domain of influence of P is the blue region. The points in the region are influenced by the values of u and u_t at P.
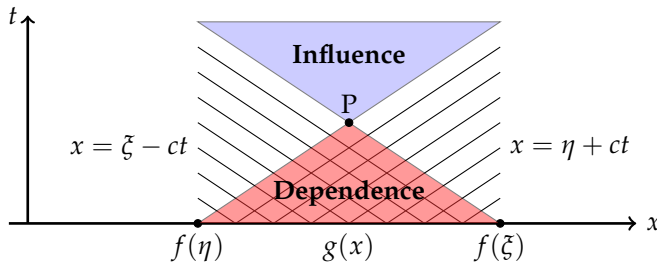
Figure 2.10: The domain of dependence of point P is red region. The point P depends on the values of u and u_t at points inside the domain. The domain of influence of P is the blue region. The points in the region are influenced by the values of u and u_t at P.

Example 2.5. *Use d'Alembert's solution to solve*

$$u_{tt} = c^2 u_{xx}, \quad u(x,0) = f(x), \quad u_t(x,0) = g(x), \quad 0 \le x < \infty.$$

The d'Alembert solution is not well-defined for this problem because $f(x-ct)$ is not defined for $x - ct < 0$ for $c, t > 0$. There are similar problems for $g(x)$. This can be seen by looking at the characteristics in the xt-plane. In Figure 2.11 there are characteristics emanating from the points marked by η_0 and ξ_0 that intersect in the domain $x > 0$. The point of intersection of the blue lines have a domain of dependence entirely in the region $x, t > 0$, however the domain of dependence of

Figure 2.11: The characteristics for the semi-infinite string.
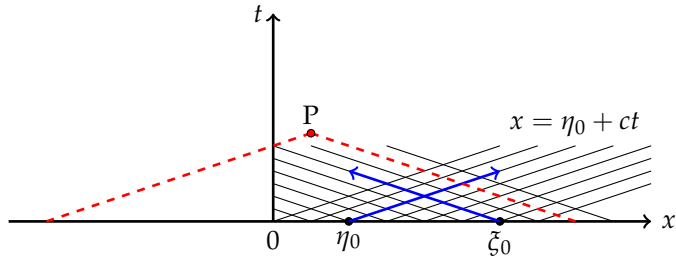
point P reaches outside this region. Only characteristics $\xi = x + ct$ reach point P, but characteristics $\eta = x - ct$ do not. But, we need $f(\eta)$ and $g(x)$ for $x < ct$ to form a solution.

This can be remedied if we specified boundary conditions at $x = 0$. For example, we will assume the end $x = 0$ is fixed,

Fixed end boundary condition

$$u(0, t) = 0, \quad t \geq 0.$$

Imagine an infinite string with one end (at $x = 0$) tied to a pole.

Since $u(x, t) = F(x + ct) + G(x - ct)$, we have

$$u(0, t) = F(ct) + G(-ct) = 0.$$

Letting $\zeta = -ct$, this gives $G(\zeta) = -F(-\zeta), \zeta \leq 0$.

Note that

$$
\begin{aligned}
G(\zeta) &= \frac{1}{2}f(\zeta) - \frac{1}{2c}\int_0^\zeta g(s)\,ds \\
-F(-\zeta) &= -\frac{1}{2}f(-\zeta) - \frac{1}{2c}\int_0^{-\zeta} g(s)\,ds \\
&= -\frac{1}{2}f(-\zeta) + \frac{1}{2c}\int_0^\zeta g(\sigma)\,d\sigma
\end{aligned}
$$

(2.44)

Comparing the expressions for $G(\zeta)$ and $-F(-\zeta)$, we see that

$$f(\zeta) = -f(-\zeta), \quad g(\zeta) = -g(-\zeta).$$

These relations imply that we can extend the functions into the region $x < 0$ if we make them odd functions, or what are called odd extensions. An example is shown in Figure 2.12.

Another type of boundary condition is if the end $x = 0$ is free,

Free end boundary condition

$$u_x(0, t) = 0, \quad t \geq 0.$$

In this case we could have an infinite string tied to a ring and that ring is allowed to slide freely up and down a pole.

One can prove that this leads to

$$f(-\zeta) = f(\zeta), \quad g(-\zeta) = g(\zeta).$$

Thus, we can use an even extension of these function to produce solutions.

Example 2.6. *Solve the initial-boundary value problem*

$$u_{tt} = c^2 u_{xx}, \quad 0 \le x < \infty, t > 0.$$

$$u(x,0) = \begin{cases} x, & 0 \le x \le 1, \\ 2 - x, & 1 \le x \le 2, \quad 0 \le x < \infty \\ 0, & x > 2, \end{cases}$$

$$u_t(x,0) = 0, \quad 0 \le x < \infty.$$

$$u(0,t) = 0, \quad t > 0. \tag{2.45}$$

This is a semi-infinite string with a fixed end. Initially it is plucked to produce a nonzero triangular profile for $0 \le x \le 2$. Since the initial velocity is zero, the general solution is found from d'Alembert's solution,

$$u(x,t) = \frac{1}{2}[f_o(x + ct) + f_o(x - ct)],$$

where $f_o(x)$ is the odd extension of $f(x) = u(x,0)$. In Figure 2.12 we show the initial condition and its odd extension. The odd extension is obtained through reflection of $f(x)$ about the origin.
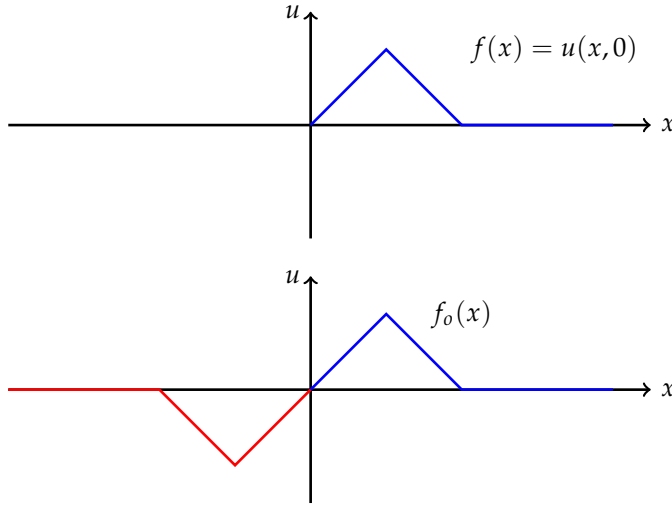
Figure 2.12: The initial condition and its odd extension. The odd extension is obtained through reflection of $f(x)$ about the origin.

The next step is to look at the horizontal shifts of $f_o(x)$. Several examples are shown in Figure 2.13.These show the left and right traveling waves.

In Figure 2.14 we show superimposed plots of $f_o(x + ct)$ and $f_o(x - ct)$ for given times. The initial profile in at the bottom. By the time $ct = 2$ the full traveling wave has emerged. The solution to the problem emerges on the right side of the figure by averaging each plot.

Example 2.7. *Use d'Alembert's solution to solve*

$$u_{tt} = c^2 u_{xx}, \quad u(x,0) = f(x), \quad u_t(x,0) = g(x), \quad 0 \le x \le \ell.$$

The general solution of the wave equation was found in the form

$$u(x,t) = F(x + ct) + G(x - ct).$$
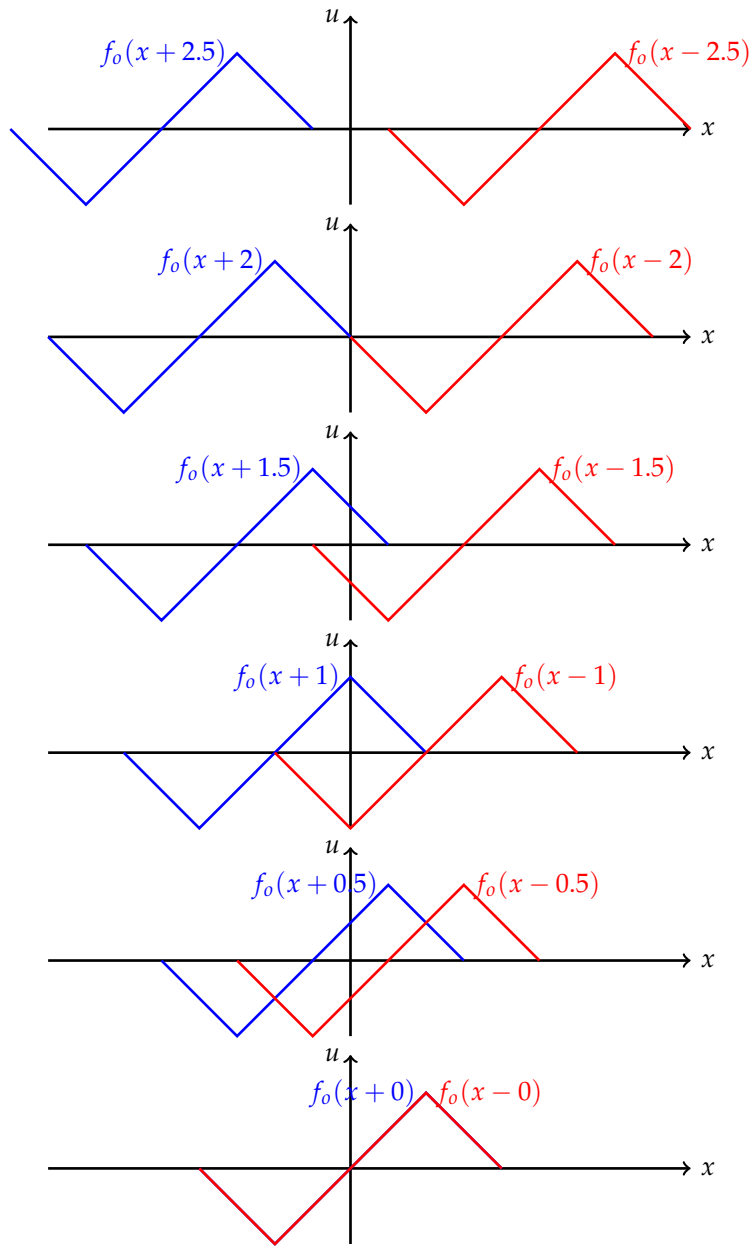
Figure 2.13: Examples of $f_o(x + ct)$ and $f_o(x - ct)$.
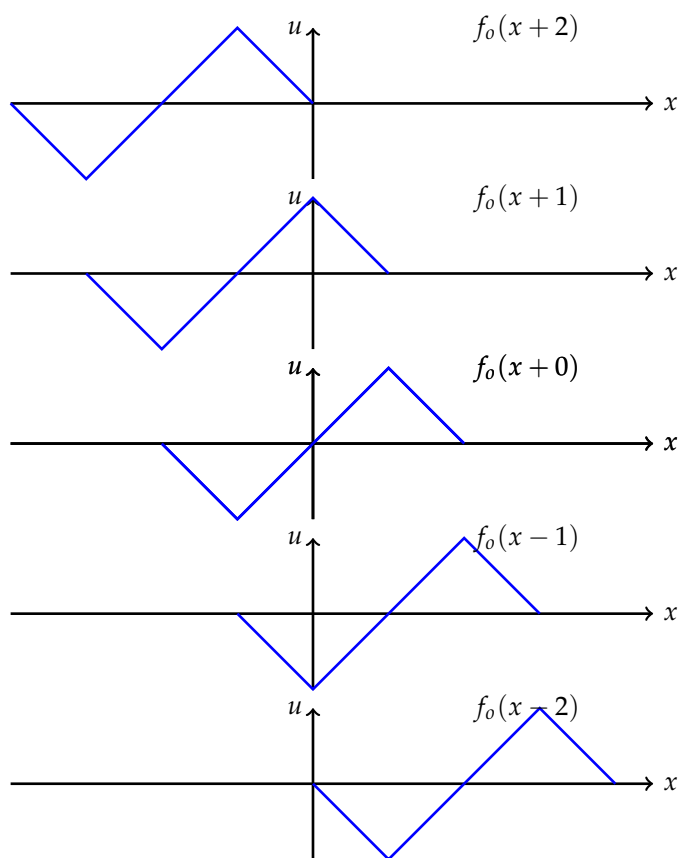
Figure 2.14: Superimposed plots of $f_o(x + ct)$ and $f_o(x - ct)$ for given times. The initial profile in at the bottom. By the time $ct = 2$ the full traveling wave has emerged.

Figure 2.15: On the left is a plot of $f(x + ct)$, $f(x - ct)$ from Figure 2.14 and the average, $u(x, t)$. On the right the solution alone is shown for $ct = 0$ at bottom to $ct = 1$ at top for the semi-infinite string problem
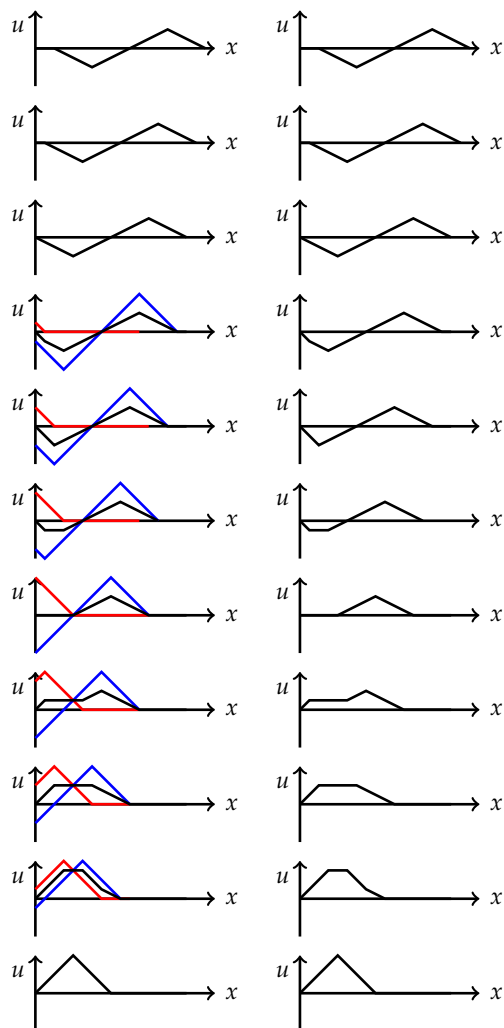
However, for this problem we can only obtain information for values of x and t such that $0 \le x + ct \le \ell$ and $0 \le x - ct \le \ell$. In Figure 2.16 the characteristics $x = \xi + ct$ and $x = \eta - ct$ for $0 \le \xi, \eta \le \ell$. The main (gray) triangle, which is the domain of dependence of the point $(\ell, 2, \ell/2c)$, is the only region in which the solution can be found based solely on the initial conditions. As with the previous problem, boundary conditions will need to be given in order to extend the domain of the solution.

In the last example we saw that a fixed boundary at $x = 0$ could be satisfied when $f(x)$ and $g(x)$ are extended as odd functions. In Figure 2.17 we indicate how the characteristics are affected by drawing in the new one as red dashed lines. This allows us to now construct solutions based on the initial conditions under the line $x = \ell - ct$ for $0 \le x \le \ell$. The new region for which we can construct solutions from the initial conditions is indicated in gray in Figure 2.17.
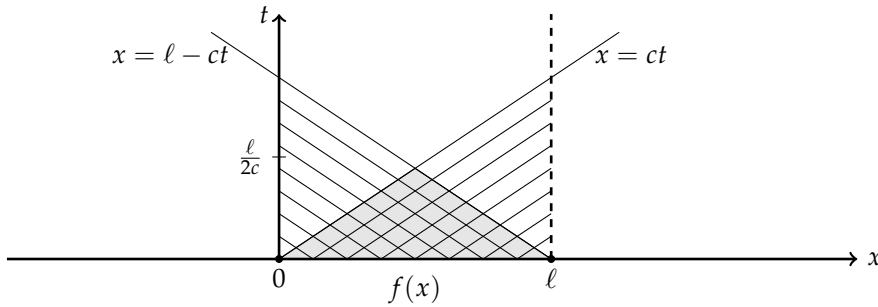
Figure 2.16: The characteristics emanating from the interval $0 \le x \le \ell$ for the finite string problem.
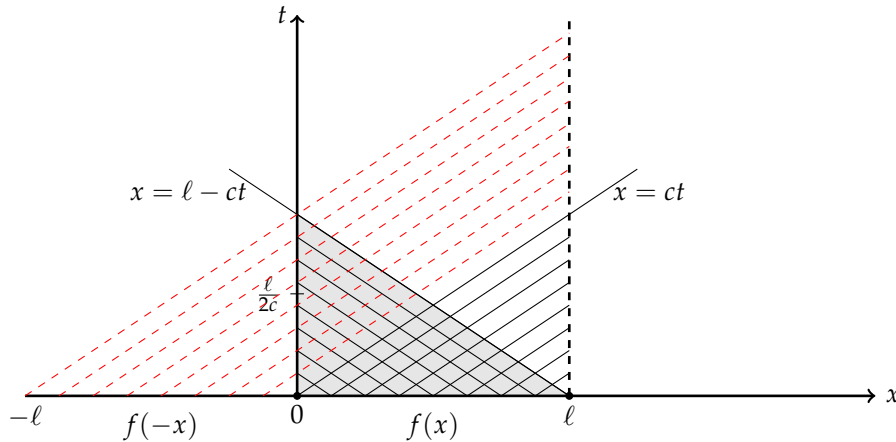

Figure 2.17: The red dashed lines are the characteristics from the interval $[-\ell, 0]$ from using the odd extension about $x = 0$.

We can add characteristics on the right by adding a boundary condition at $x = \ell$. Again, we could use fixed $u(\ell, t) = 0$, or free, $u_x(\ell, t) = 0$, boundary conditions. This allows us to now construct solutions based on the initial conditions for $\ell \le x \le 2\ell$.

Let's consider a fixed boundary condition at $x = \ell$. Then, the solution must satisfy

$$u(\ell, t) = F(\ell + ct) + G(\ell - ct) = 0.$$

To see what this means, let $\zeta = \ell + ct$. Then, this condition gives (since $ct = \zeta - \ell$)

$$F(\zeta) = -G(2\ell - \zeta), \quad \ell \le \zeta \le 2\ell.$$

Note that $G(2\ell - \zeta)$ is defined for $0 \le 2\ell - \zeta \le \ell$. Therefore, this is a well-defined extension of the domain of $F(x)$.
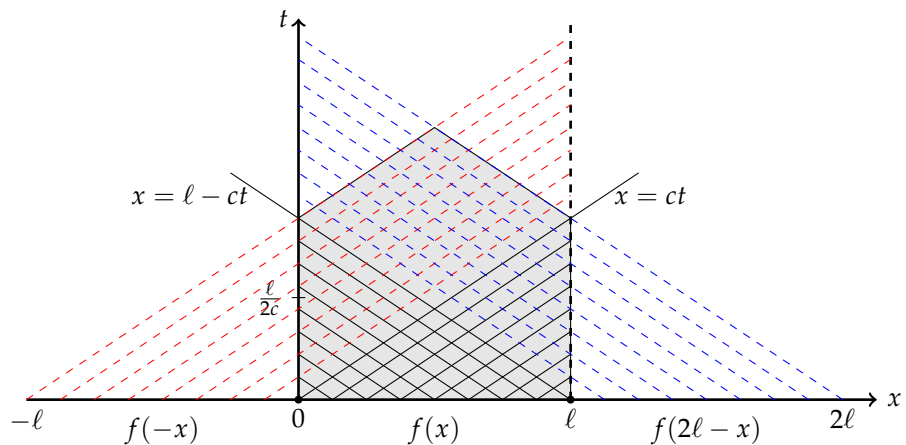
Note that

$$
\begin{aligned}
F(\zeta) &= \frac{1}{2} f(\zeta) + \frac{1}{2c} \int_0^\ell g(s)\, ds. \\
-G(2\ell - \zeta) &= -\frac{1}{2} f(2\ell - \zeta) + \frac{1}{2c} \int_0^{2\ell - \zeta} g(s)\, ds \\
&= -\frac{1}{2} f(2\ell - \zeta) - \frac{1}{2c} \int_0^\zeta g(2\ell - \sigma)\, d\sigma
\end{aligned}
$$

(2.46)

Comparing the expressions for $G(\zeta)$ and $-G(2\ell - \zeta)$, we see that

$$f(\zeta) = -f(2\ell - \zeta), \quad g(\zeta) = -g(2\ell - \zeta).$$

These relations imply that we can extend the functions into the region $x > \ell$ if we consider an odd extension of $f(x)$ and $g(x)$ about $x = \ell$.. This will give the blue dashed characteristics in Figure 2.18 and a larger gray region to construct the solution.

So far we have extended $f(x)$ and $g(x)$ to the interval $-\ell \le x \le 2\ell$ in order to determine the solution over a larger xt-domain. For example, the function $f(x)$ has been extended to

$$
f_{ext}(x) = \begin{cases}
-f(-x), & -\ell < x < 0, \\
f(x), & 0 < x < \ell, \\
-f(2\ell - x), & \ell < x < 2\ell.
\end{cases}
$$

A similar extension is needed for $g(x)$. Inserting these extended functions into d'Alembert's solution, we can determine $u(x, t)$ in the region indicated in Figure 2.18.

Even though the original region has been expanded, we have not determined how to find the solution throughout the entire strip, $[0, \ell] \times [0, \infty)$. This is accomplished by periodically repeating these extended functions with period 2ℓ. This can be shown from the two conditions

$$f(x) = -f(-x), \quad -\ell \leq x \leq 0,$$
$$f(x) = -f(2\ell - x), \quad \ell \leq x \leq 2\ell. \tag{2.47}$$

Now, consider

$$\begin{aligned} f(x + 2\ell) &= -f(2\ell - (x - 2\ell)) \\ &= -f(-x) \\ &= f(x). \end{aligned} \tag{2.48}$$

This shows that $f(x)$ is periodic with period 2ℓ. Since $g(x)$ satisfies the same conditions, then it is as well.

In Figure 2.19 we show how the characteristics are extended throughout the domain strip using the periodicity of the extended initial conditions. The characteristics from the interval endpoints zig zag throughout the domain, filling it up. In the next example we show how to construct the odd periodic extension of a specific function.
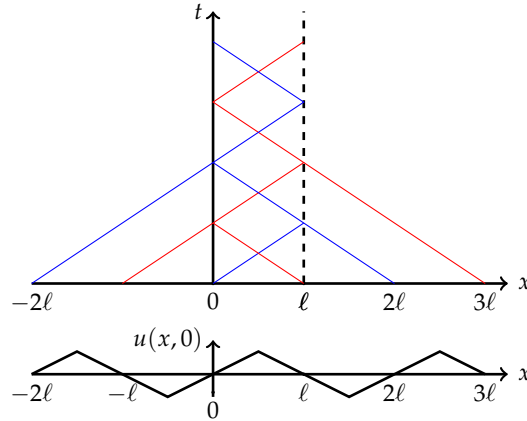

Figure 2.19: Extending the characteristics throughout the domain strip.

Example 2.8. *Construct the periodic extension of the plucked string initial profile given by*

$$f(x) = \begin{cases} x, & 0 \leq x \leq \frac{\ell}{2}, \\ \ell - x, & \frac{\ell}{2} \leq x \leq \ell, \end{cases}$$

satisfying fixed boundary conditions at $x = 0$ and $x = \ell$.

We first take the solution and add the odd extension about $x = 0$. Then we add an extension beyond $x = \ell$. This process is shown in Figure 2.20.

We can use the odd periodic function to construct solutions. In this case we use the result from the last example for obtaining the solution of the problem in which the initial velocity is zero, $u(x, t) = \frac{1}{2}[f(x + ct) + f(x - ct)]$. Translations of the odd periodic extension are shown in Figure 2.21.

Figure 2.20: Construction of odd periodic extension for (a) The initial profile, $f(x)$. (b) Make $f(x)$ an odd function on $[-\ell, \ell]$. (c) Make the odd function periodic with period 2ℓ.
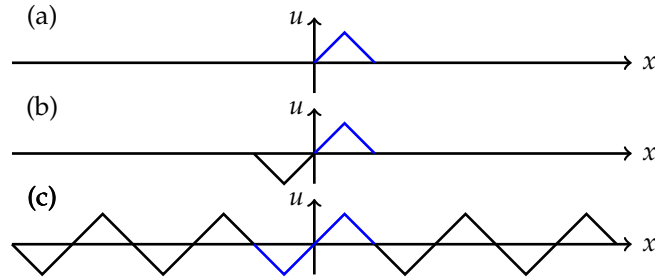
In Figure 2.22 we show superimposed plots of $f(x + ct)$ and $f(x - ct)$ for different values of ct. A box is shown inside which the physical wave can be constructed. The solution is an average of these odd periodic extensions within this box. This is displayed in Figure 2.23.

Figure 2.21: Translations of the odd periodic extension.
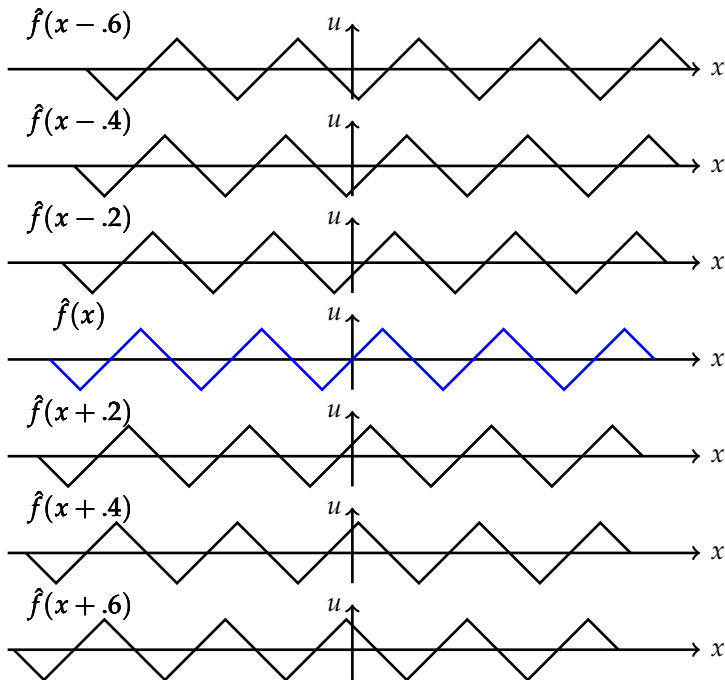
Problems

1. Solve the following initial value problems.

a. $x'' + x = 0$, $x(0) = 2$, $x'(0) = 0$.

b. $y'' + 2y' - 8y = 0$, $y(0) = 1$, $y'(0) = 2$.

c. $x^2 y'' - 2xy' - 4y = 0$, $y(1) = 1$, $y'(1) = 0$.

2. Solve the following boundary value problems directly, when possible.

a. $x'' + x = 2$, $x(0) = 0$, $x'(1) = 0$.
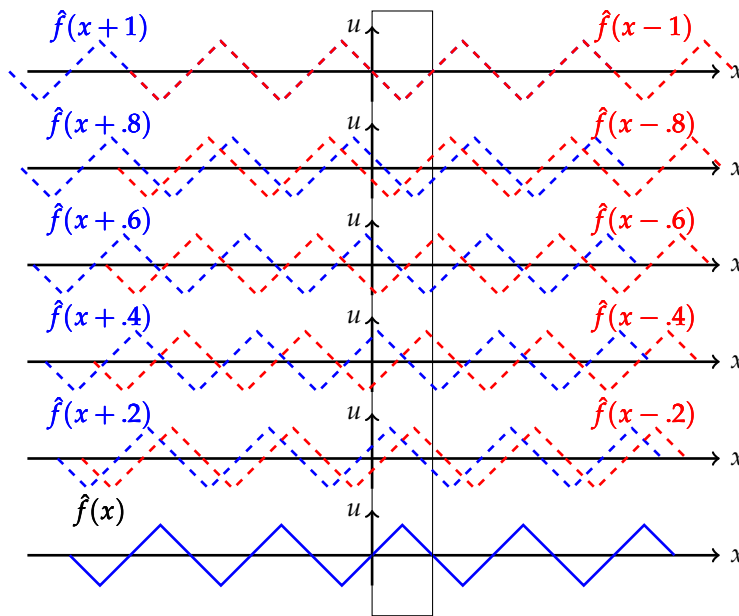
b. $y'' + 2y' - 8y = 0$, $y(0) = 1$, $y(1) = 0$.

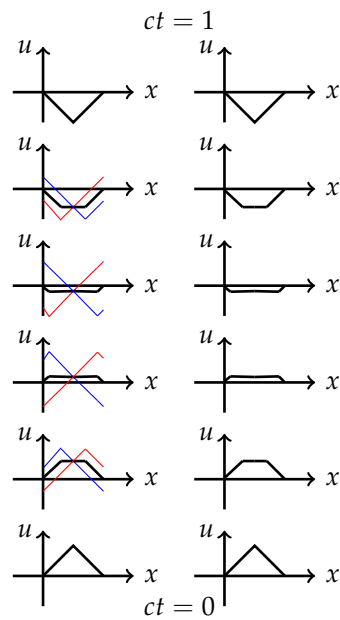Figure 2.22: Superimposed translations of the odd periodic extension.

Figure 2.23: On the left is a plot of $f(x + ct)$, $f(x - ct)$ from Figure 2.22 and the average, $u(x, t)$. On the right the solution alone is shown for $ct = 0$ to $ct = 1$.

c. $y'' + y = 0$, $y(0) = 1$, $y(\pi) = 0$.

3. Consider the boundary value problem for the deflection of a horizontal beam fixed at one end,

$$\frac{d^4 y}{dx^4} = C, \quad y(0) = 0, \quad y'(0) = 0, \quad y''(L) = 0, \quad y'''(L) = 0.$$

Solve this problem assuming that C is a constant.

4. Find the product solutions, $u(x,t) = T(t)X(x)$, to the heat equation, $u_t - u_{xx} = 0$, on $[0, \pi]$ satisfying the boundary conditions $u_x(0,t) = 0$ and $u(\pi, t) = 0$.

5. Find the product solutions, $u(x,t) = T(t)X(x)$, to the wave equation $u_{tt} = 2u_{xx}$, on $[0, 2\pi]$ satisfying the boundary conditions $u(0,t) = 0$ and $u_x(2\pi, t) = 0$.

6. Find product solutions, $u(x,t) = X(x)Y(y)$, to Laplace's equation, $u_{xx} + u_{yy} = 0$, on the unit square satisfying the boundary conditions $u(0,y) = 0$, $u(1,y) = g(y)$, $u(x,0) = 0$, and $u(x,1) = 0$.

7. Consider the following boundary value problems. Determine the eigenvalues, λ, and eigenfunctions, $y(x)$ for each problem.

    a. $y'' + \lambda y = 0$, $y(0) = 0$, $y'(1) = 0$.

    b. $y'' - \lambda y = 0$, $y(-\pi) = 0$, $y'(\pi) = 0$.

    c. $x^2 y'' + xy' + \lambda y = 0$, $y(1) = 0$, $y(2) = 0$.

    d. $(x^2 y')' + \lambda y = 0$, $y(1) = 0$, $y'(e) = 0$.

In problem d you will not get exact eigenvalues. Show that you obtain a transcendental equation for the eigenvalues in the form $\tan z = 2z$. Find the first three eigenvalues numerically.

8. Classify the following equations as either hyperbolic, parabolic, or elliptic.

    a. $u_{yy} + u_{xy} + u_{xx} = 0$.

    b. $3u_{xx} + 2u_{xy} + 5u_{yy} = 0$.

    c. $x^2 u_{xx} + 2xyu_{xy} + y^2 u_{yy} = 0$.

    d. $y^2 u_{xx} + 2xyu_{xy} + (x^2 + 4x^4)u_{yy} = 0$.

9. Use d'Alembert's solution to prove

$$f(-\zeta) = f(\zeta), \quad g(-\zeta) = g(\zeta)$$

for the semi-infinite string satisfying the free end condition $u_x(0,t) = 0$.

10. Derive a solution similar to d'Alembert's solution for the equation $u_{tt} + 2u_{xt} - 3u = 0$.

11. Construct the appropriate periodic extension of the plucked string initial profile given by

$$f(x) = \begin{cases} x, & 0 \le x \le \frac{\ell}{2}, \\ \ell - x, & \frac{\ell}{2} \le x \le \ell, \end{cases}$$

satisfying the boundary conditions at $u(0,t) = 0$ and $u_x(\ell, t) = 0$ for $t > 0$.

12. Find and sketch the solution of the problem

$$u_{tt} = u_{xx}, \quad 0 \le x \le 1, t > 0$$

$$u(x,0) = \begin{cases} 0, & 0 \le x < \frac{1}{4}, \\ 1, & \frac{1}{4} \le x \le \frac{3}{4}, \\ 0, & \frac{3}{4} < x \le 1, \end{cases}$$

$$u_t(x,0) = 0,$$

$$u(0,t) = 0, \quad t > 0,$$

$$u(1,t) = 0, \quad t > 0,$$