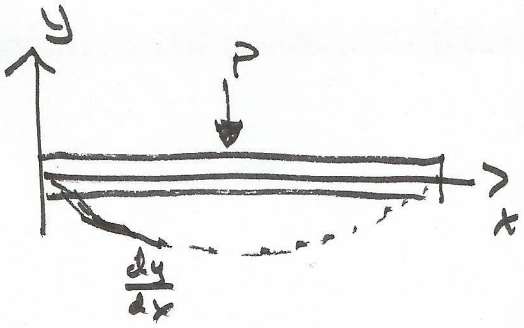
Deflection of Beams

Curvature: $\frac{1}{\rho} = \frac{M}{EI}$

$$\frac{1}{\rho} = \frac{d^2y/dx^2}{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}}$$



$\frac{dy}{dx}$ is small

$\frac{dy}{dx} \simeq 0$

$$\Downarrow$$

$$\boxed{\frac{1}{\rho} = \frac{d^2y}{dx^2}}$$

$$\boxed{\frac{M}{EI} = \frac{d^2y}{dx^2}}$$

$\Theta = \frac{dx}{dy}$ slope of the curve

$$\frac{V}{EI} = \frac{d^3y}{dx^3}$$

$$\frac{q}{EI} = \frac{d^4y}{dx^4}$$

Given $q(x)$

$$V = \int q(x)\,dx + C_v$$

$$M = \int V(x)\,dx + C_m$$

C_v and C_m obtained from static equilibrium condition

$$EI\,\theta = \int M(x)\,dx + C_1$$

$$EI\,y = \int \theta\,dx + C_2$$

C_1 & C_2 obtained from boundary conditions

Multiple Ways to Solve the eqs.

- Sectioning the beam and integrating the sections
- super position *
- moment area method
- Singularity funcs *
- numerical integration

~~Superp~~

Superposition

- Many simple beam cases have been pre-solved and the results can be algebraically combined.

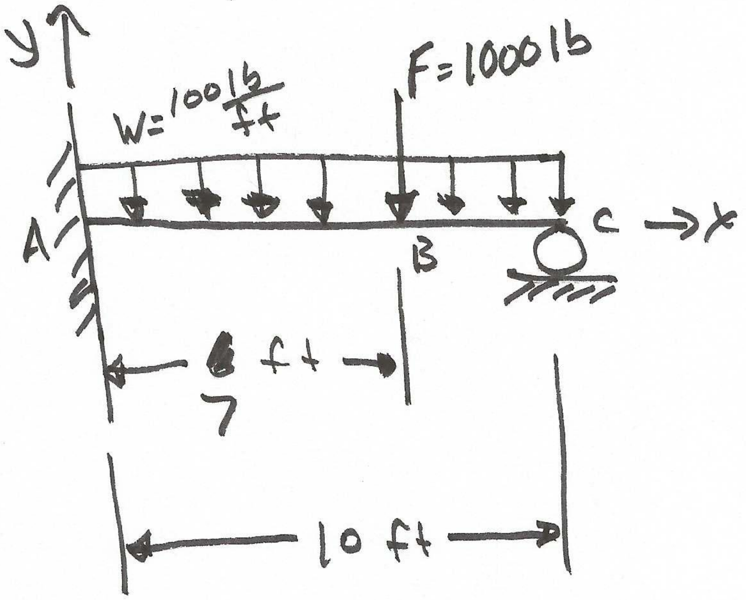
  Table A-9

- Roark's Formulas for stress and strain

SP can be used if:

- each effect must be linearly related to the load that produced it, e.g. $\delta = P \frac{L}{EA}$ &
- a load does not create a condition that affects the result of another load
- deformation from any specific load is not large enough to appreciably alter the geometric relations of the structural system

Example: Superposition



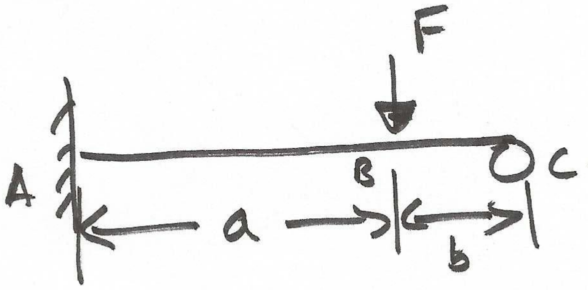
$E = 30E6$ psi

$I = 5$ in^4

~~What~~ What is y_{max}?

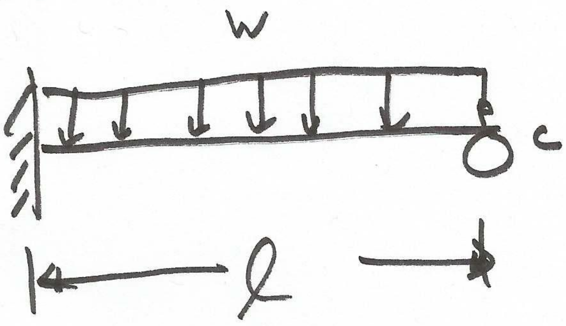
Table A-9 12

$$y_{AB} = \frac{F b x^2}{12 E I \ell^3}\left[3\ell(b^2 - \ell^2) + x(3\ell^2 - b^2)\right]$$



$$y_{BC} = y_{AB} - \frac{F(x-a)^3}{6EI}$$

Table A-9 13

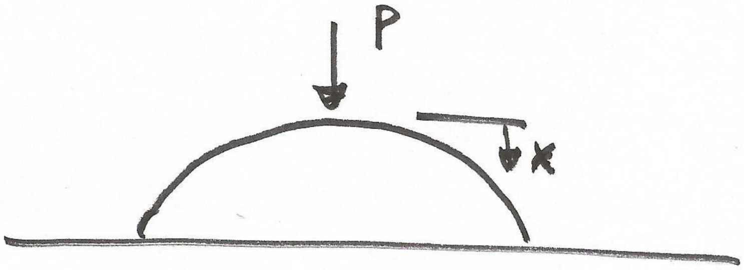


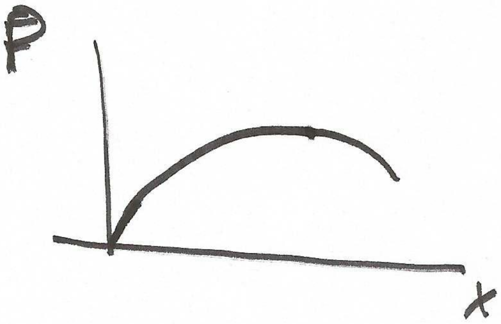
$$y = \frac{w x^2}{48EI}(\ell - x)(2x - 3\ell)$$

$$\Theta = \frac{dy}{dx} = 0 \Rightarrow \underline{\underline{y_{max}}}$$

$\cancel{y_{AB}}$

$$\left.\begin{aligned} y_{\substack{tot\\AB}} &= y_{AB} + y \\ y_{\substack{tot\\BC}} &= y_{BC} + y \end{aligned}\right\}$$ super imposed the two results





$$y_{AB} = \frac{Fbx^2}{12EIl^3}\left[3l\,(b^2-l^2) + x\,(3l^2-b^2)\right] + \frac{wx^2}{48EI}(l-x)(2x-3l)$$

$$y_{AB} = c_1\,x^4 + c_2\,x^3 + c_3\,x^2 + c_4\,x^1 + c_5$$

$$\frac{dy_{AB}}{dx} = \text{cubic polynomial} = 0$$

roots of cubic polynomial

roof , fzero , feval $\Rightarrow$ some functions in matlab

$$X = (156, \; 73'', \; -2)$$

in

$0 \leq x \leq 84''$

$\approx$

$$\boxed{y_{AB}(73'') = -0.164''}$$

Singularity Functions

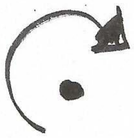
$n \geq 0$

$$\langle x-a \rangle^n = \begin{cases} 0 & x < a \\ (x-a)^n & x \geq a \end{cases}$$

$$\int \langle x-a \rangle^n \, dx = \frac{\langle x-a \rangle^{n+1}}{n+1} + C$$

$\langle x-a \rangle^{-1}$

$\langle x-a \rangle^{-2}$



$0 \quad x \neq a$

$\bullet \quad x = a$

$$EI\theta = \int M(x)\,dx$$

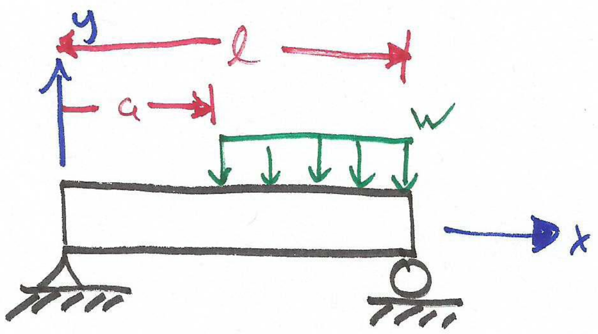
$$EIy = \int \theta(x)\,dx$$

$$EI\,\theta = \frac{R_1}{2}\langle x\rangle^2 - \frac{W}{6}\langle x-a\rangle^3 + \frac{W}{6}\cancel{\langle x-l\rangle^3}^{\,0} + \frac{R_2}{2}\cancel{\langle x-l\rangle^2}^{\,0} + C_1$$

$$\langle x\rangle^2 = x^2$$

$$M(x) = \begin{cases} f_1(x) & 0 < x \le a \\ f_2(x) & a < x \le b \\ f_3(x & b \quad c \\ f_4(x) & c \quad d \end{cases}$$

Example



$l = 10''$

$a = 4''$

$w = 100 \frac{lb}{in}$

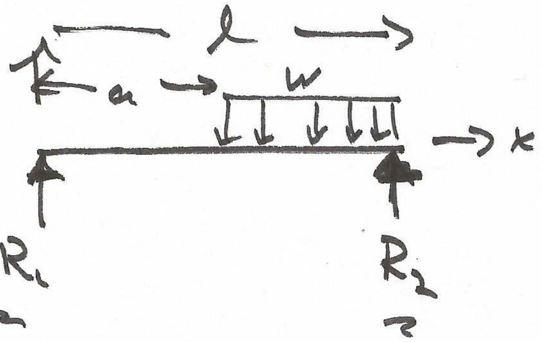
$E = 30$ Mpsi

$I = .163$ in^4

Find slope and deflection functions of x

and the maximum deflection

$$q(x) = R_1\langle x\rangle^{-1} - w\langle x-a\rangle^0 + w\langle x-l\rangle^0 + R_2\langle x-l\rangle^{-1}$$



$$V(x) = R_1\langle x\rangle^0 - w\langle x-a\rangle^1 + w\langle x-l\rangle^1 + R_2\langle x-l\rangle^0$$

$$M(x) = R_1\langle x\rangle^1 - \frac{w}{2}\langle x-a\rangle^2 + \frac{w}{2}\langle x-l\rangle^2 + R_2\langle x-l\rangle^1$$

$V=0,\ M=0$ @ $x = l^+$

$$V(l^+) = 0 = R_1 - w(l-a) + w(\cancel{l-l})^{0} + R_2$$

$$M(l^+) = 0 = R_1 l - \frac{w}{2}(l-a)^2 + \frac{w}{2}(\cancel{l-l})^2_{0} + R_2(\cancel{l-l})^1_{0}$$

$$R_1 = \frac{w}{2l}(l-a)^2$$

$\boxed{R_1 = 180\text{ lb},\ R_2 = 420\text{ lb}}$

$$\left\{ \begin{array}{l} EI\theta = \frac{R_1}{2}(x)^2 - \frac{W}{6}\langle x-a\rangle^3 + C_1 \\ EIy = \frac{R_1}{6}x^3 - \frac{W}{24}\langle x-a\rangle^4 + C_1 x + C_2 \end{array} \right.$$

$y(0)=0 \qquad y(l)=0$ <u>boundary conditions</u>

$@\ x=0:\quad 0 = C_2$

$@\ x=l:\quad 0 = \dfrac{R_1 l^3}{6} - \dfrac{W}{24}(l-a)^4 + C_1 l$

$$C_1 = -2460 \text{ lb}\cdot\text{in}^2$$

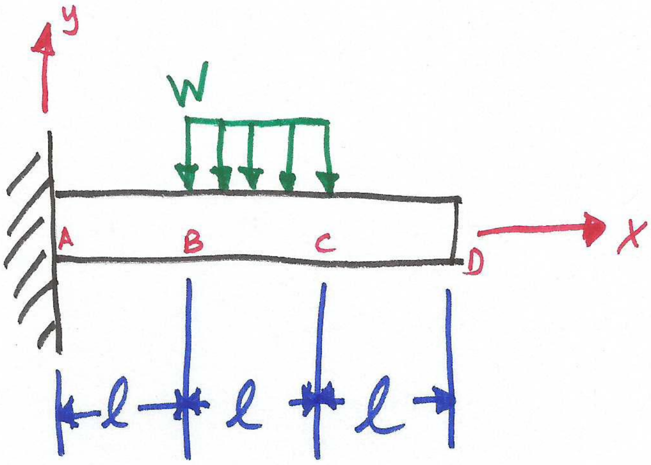
Look $a < x < l$

$$\cancel{EI}\,\theta = 0 = \frac{1}{EI}\left(90x^2 - 16\tfrac{2}{3}(x-a)^3 - 2460\right)$$

<u>Cubic polynomial</u>

$$x = (\underset{\text{bad}}{\cancel{-1.191}},\ \underline{5.264},\ \underset{\text{bad}}{\cancel{13.33}})$$

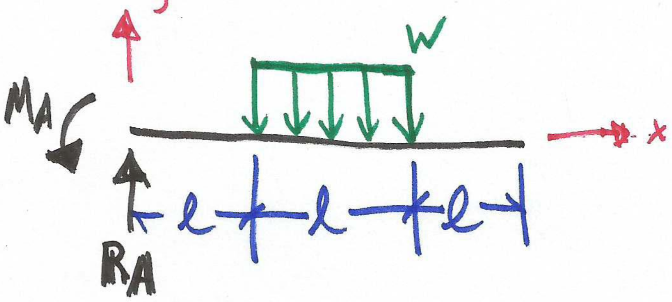
$$\boxed{y_{max} = -1.76\text{E-}3 \text{ in}}$$



Goal

Find a function for both θ and y.

FBD

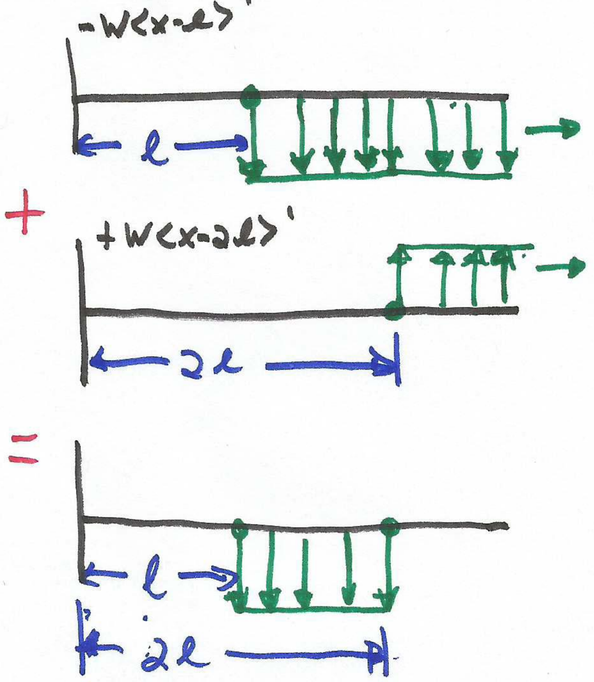


$$q(x) = R_A\langle x\rangle^{-1} - M_A\langle x\rangle^{-2}$$
$$-w\langle x-l\rangle^{0} + w\langle x-2l\rangle^{0}$$

$$V(x) = R_A\langle x\rangle^{0} - M_A\langle x\rangle^{-1}$$
$$-w\langle x-l\rangle^{1} + w\langle x-2l\rangle^{1}$$

$$M(x) = R_A\langle x\rangle^{1} - M_A\langle x\rangle^{0}$$
$$-\frac{w}{2}\langle x-l\rangle^{2} + w\langle x-2l\rangle^{2}$$

Why $-w\langle x-l\rangle^1 + w\langle x-2l\rangle^1$?



Find R_A and M_A

$V=0$ and $M=0$ @ $x=3l^+$

$$V(3l^+) = 0 = R_A - w(3l-l) + w(3l-2l)$$

$$\boxed{R_A = wl}$$

$$M(3l^+) = 0 = 3R_A l - M_A - \frac{w}{2}(3l-l)^2$$
$$+\frac{w}{2}(3l-2l)^2$$
$$0 = 3R_A l - M_A - 2wl^2$$
$$+\frac{w}{2}l^2$$

$$\boxed{M_A = \frac{3}{2}wl^2}$$

$$M(x) = R_A x - M_A - \frac{w}{2}\langle x-\ell\rangle^2 + w\langle x-2\ell\rangle^2$$

$R_A x$: $\langle x\rangle = x$ M_A: $\langle x\rangle^0 = 1$ $\Rightarrow$ for whole length of beam

$$EI\theta = \int M(x)\,dx = \frac{R_A x^2}{2} - M_A x - \frac{w}{6}\langle x-\ell\rangle^3 + \frac{w}{6}\langle x-2\ell\rangle^3 + C_1$$

$$EI y = \int \theta(x)\,dx = \frac{R_A x^3}{6} - \frac{M_A x^2}{2} - \frac{w}{24}\langle x-\ell\rangle^4 + \frac{w}{24}\langle x-2\ell\rangle^4 + C_1 x + C_2$$

$\theta(0) = 0$ $y(0) = 0$

$0 = C_1$ $0 = C_1(0) + C_2$

$C_2 = 0$

Apply boundary conditions to find C_1 and C_2.

$$\boxed{\begin{aligned} \theta &= \frac{1}{EI}\left[\frac{w\ell}{2}x^2 - \frac{3}{2}w\ell^2 x - \frac{w}{6}\langle x-\ell\rangle^3 + \frac{w}{6}\langle x-2\ell\rangle^3\right] \\ y &= \frac{1}{EI}\left[\frac{w\ell}{6}x^3 - \frac{3}{4}w\ell^2 x^2 - \frac{w}{24}\langle x-\ell\rangle^4 + \frac{w}{24}\langle x-2\ell\rangle^4\right] \end{aligned}}$$

The maximum deflection will be at the end of the cantilever so we can evaluate the singularity function @ $x = 3\ell$ to find maximum deflection.

$$y(3\ell) = \frac{1}{EI}\left[\frac{9}{2}w\ell^4 - \frac{27}{4}w\ell^4 - \frac{2}{3}w\ell^4 + \frac{w}{24}\ell^4\right]$$

$$= \boxed{-\frac{23\,w\ell^4}{8\,EI} = y_{max}}$$