Graph Theory

Ch.1: Introduction to Graphs

Prepared By

Dr. Ibrahim Attiya

Course Objectives

- The understanding of *fundamental definitions and properties of graphs*.
- The ability to *read and write rigorous mathematical proofs involving graphs*.
- The ability to apply *effective methods and algorithms for solving typical graph problems in practice*.
- Recognition of the *numerous applications of graph theory* in computer science and engineering.

Student Assessment Criteria

✓ Assignments	10 Points
✓ Mid-term Exam	10 Points
✓ Practical Exam	10 Points
✓ Oral Exam	10 Points
✓ Final Exam	60 Points

Outline

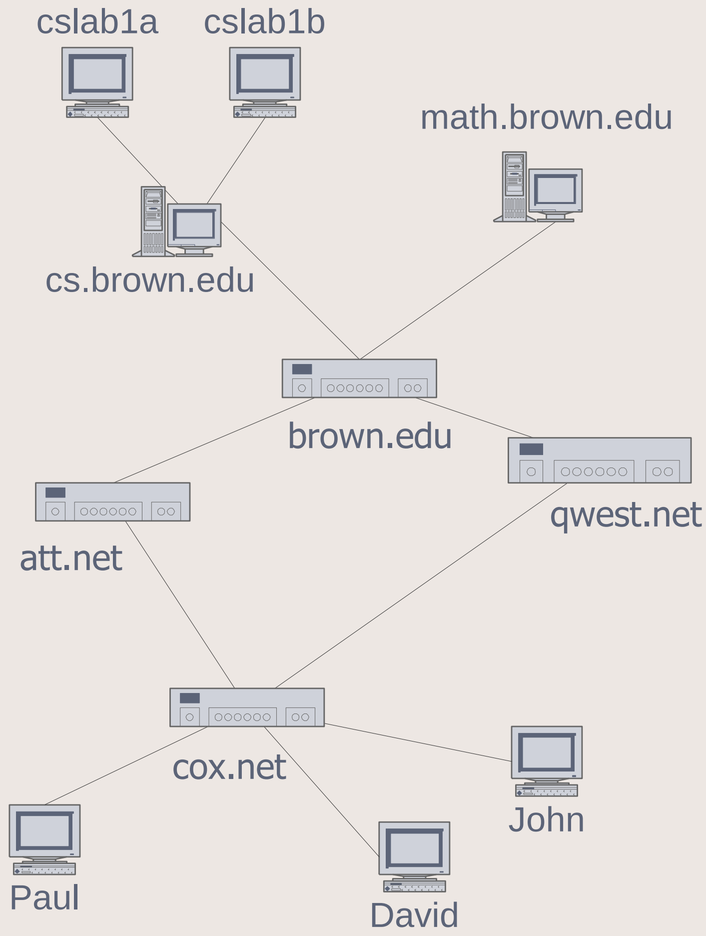
- ❑ Why Graph Theory?
- ❑ Some Applications of Graphs
- ❑ Examples of Graphs
- ❑ History of Graph Theory
- ❑ Definitions of Graphs
- ❑ Incidence and Degree
- ❑ Isomorphism
- ❑ Complete Graph
- ❑ Bipartite Graph
- ❑ Directed Graph or Digraph

Why Graph Theory ?

- ✓ Graphs used to model pair wise relations between objects.
- ✓ Generally a network can be represented by a graph.
- ✓ Many practical problems can be easily represented in terms of graph theory.
- ✓ Graph theory has diverse applications in the areas of computer science, biology, chemistry, physics, and engineering.

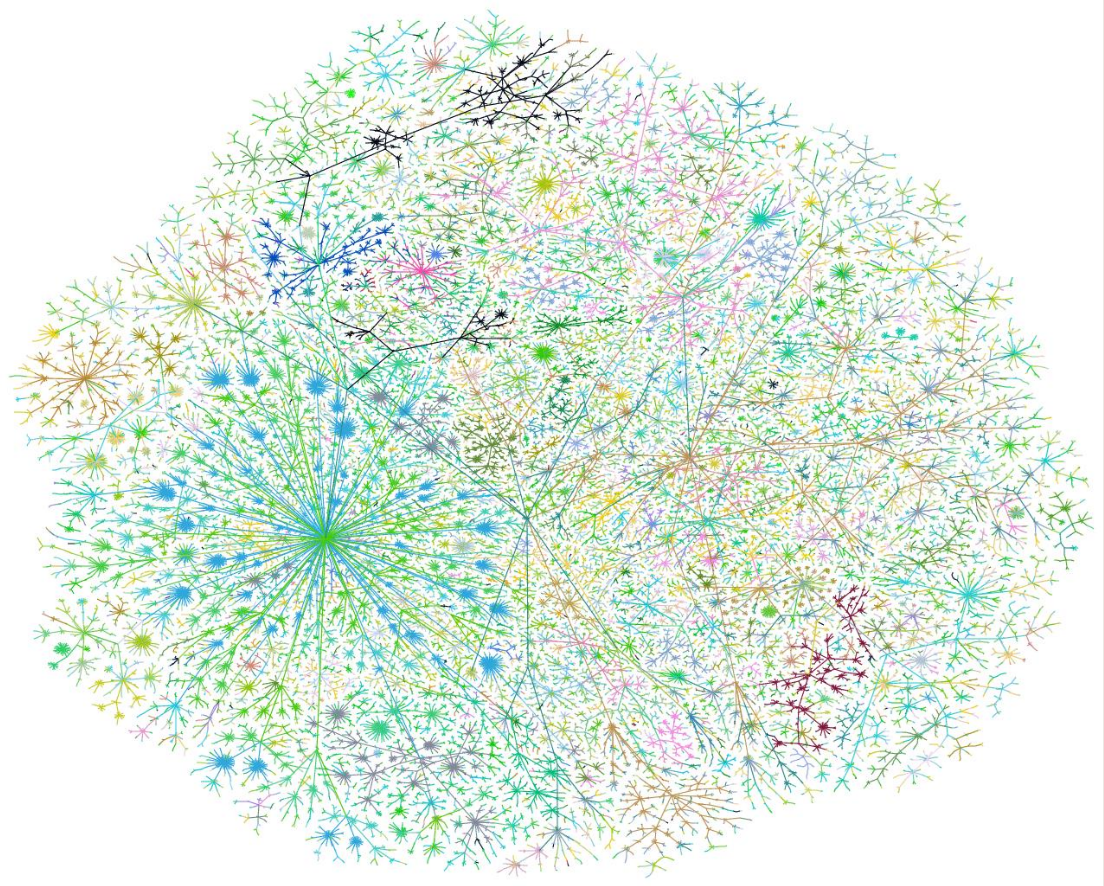
Applications

- Electronic circuits
  - Printed circuit board
  - Integrated circuit
- Transportation networks
  - Highway network
  - Flight network
- Computer networks
  - Local area network
  - Internet
  - Web
- Databases
  - Entity-relationship diagram



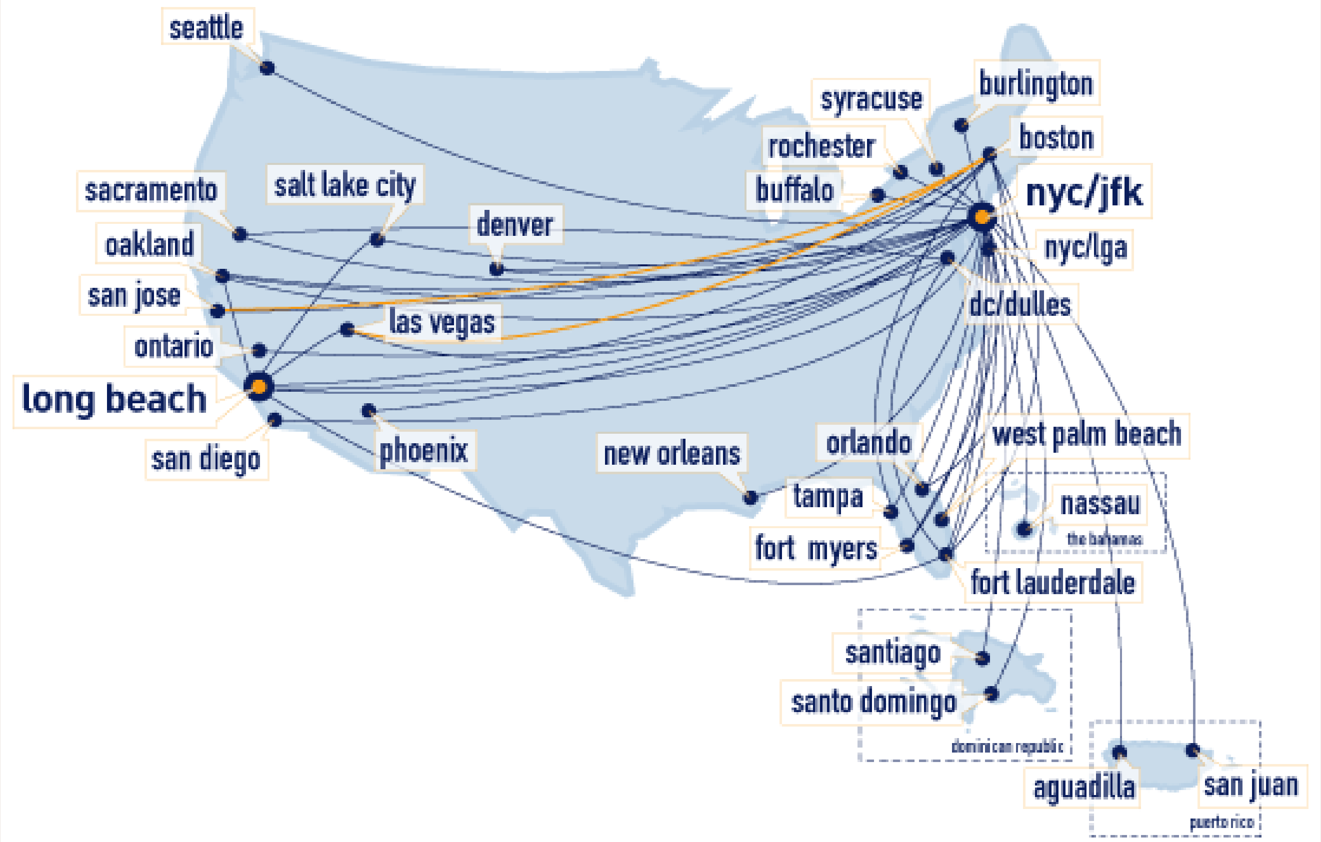
Examples of Graphs

The Internet (circa 1999)



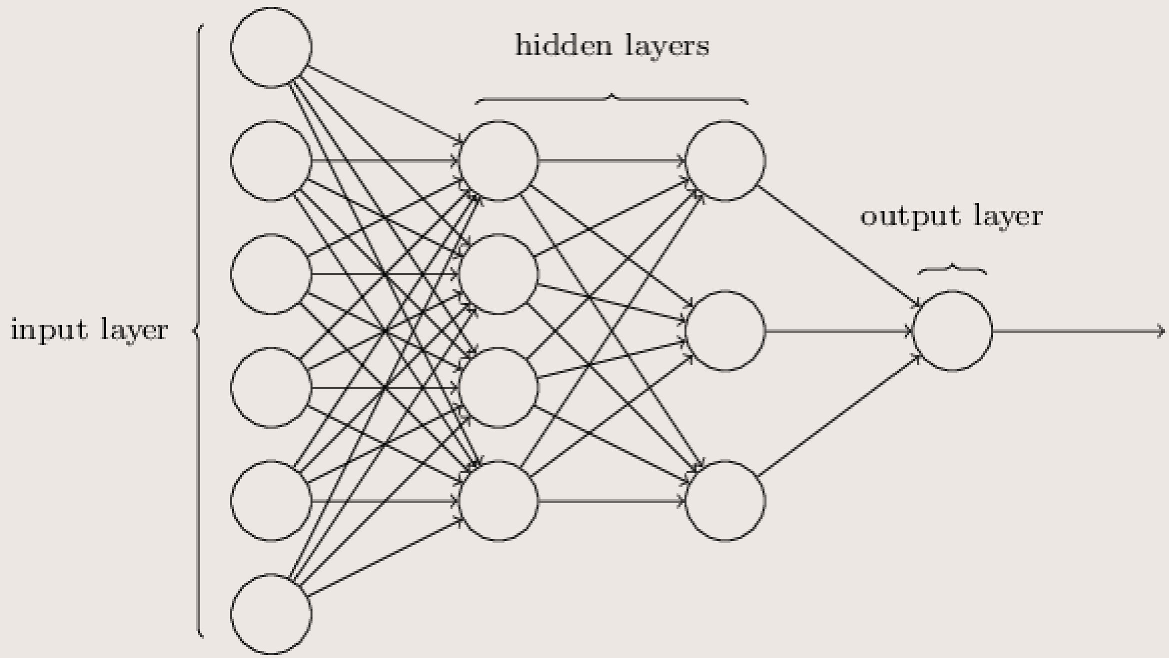
Examples of Graphs

Flight networks (Jet Blue, for example)



Examples of Graphs

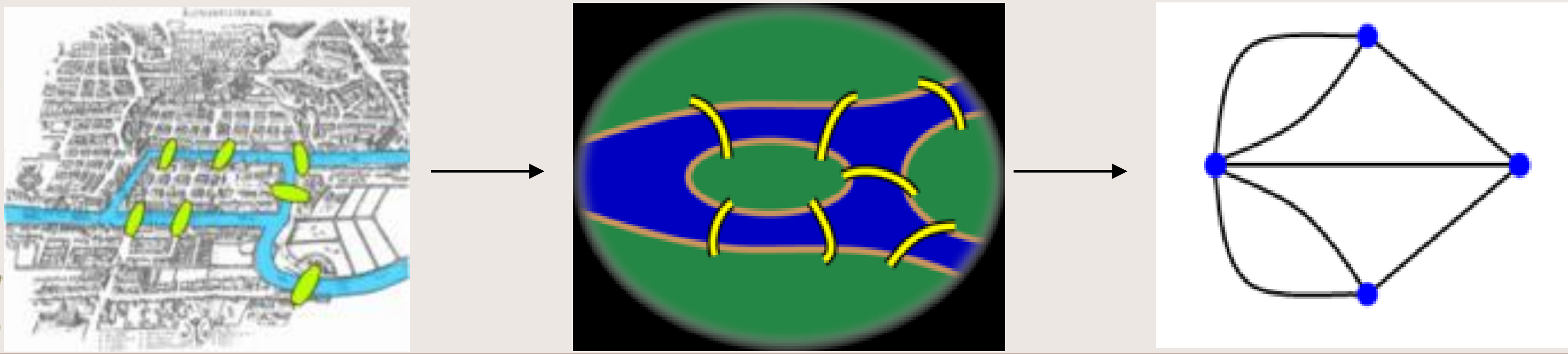
Neural networks



Graph Theory - History

Leonhard Euler's paper on "*Seven Bridges of Königsberg*",

published in 1736.

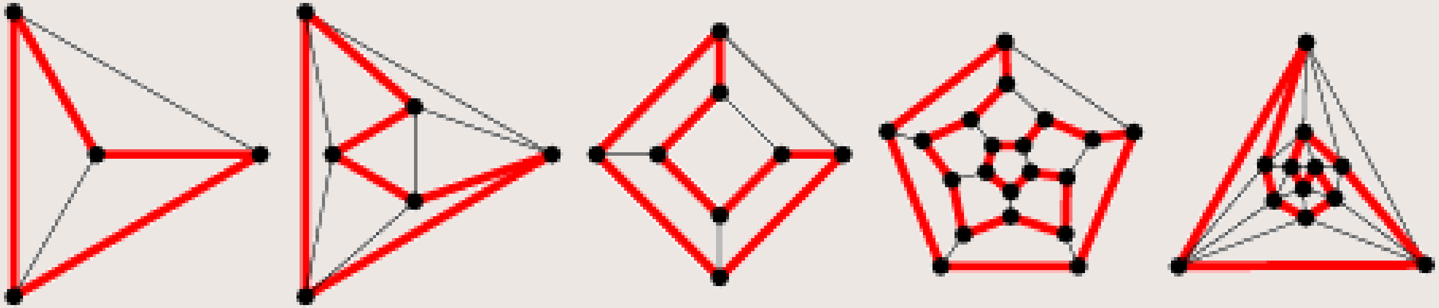


Graph Theory - History

Cycles in Polyhedra

Thomas P. Kirkman

William R. Hamilton

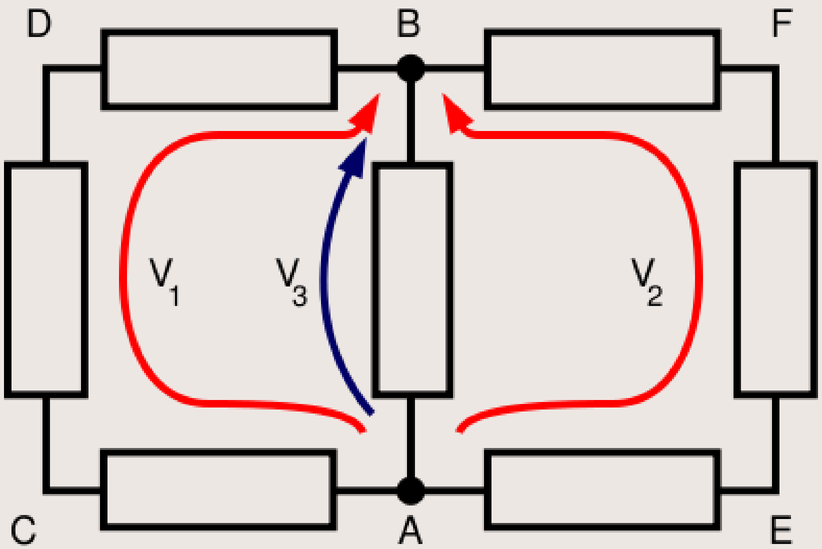


Hamiltonian cycles in Platonic graphs

Graph Theory - History

Trees in Electric Circuits

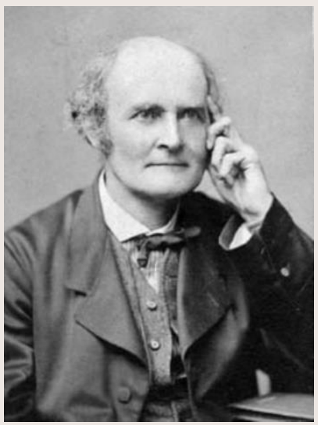
Gustav Kirchhoff



$$V_1 = V_2 = V_3$$

Graph Theory - History

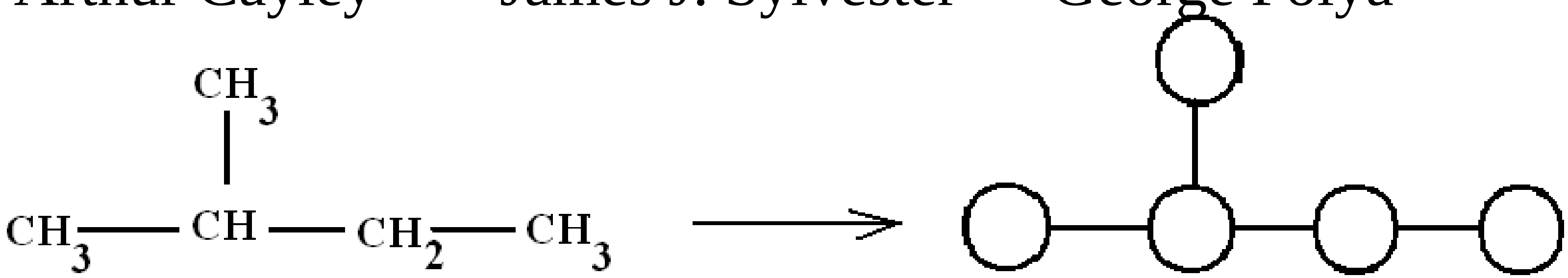
Enumeration of Chemical Isomers



Arthur Cayley

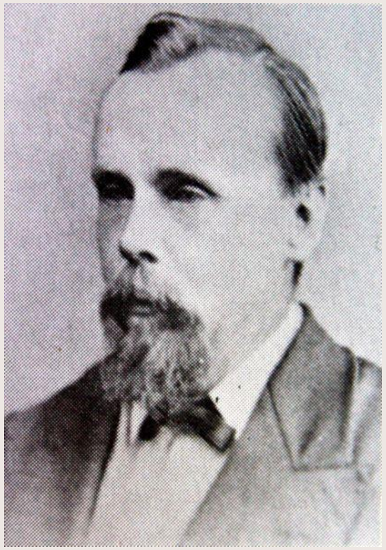
James J. Sylvester

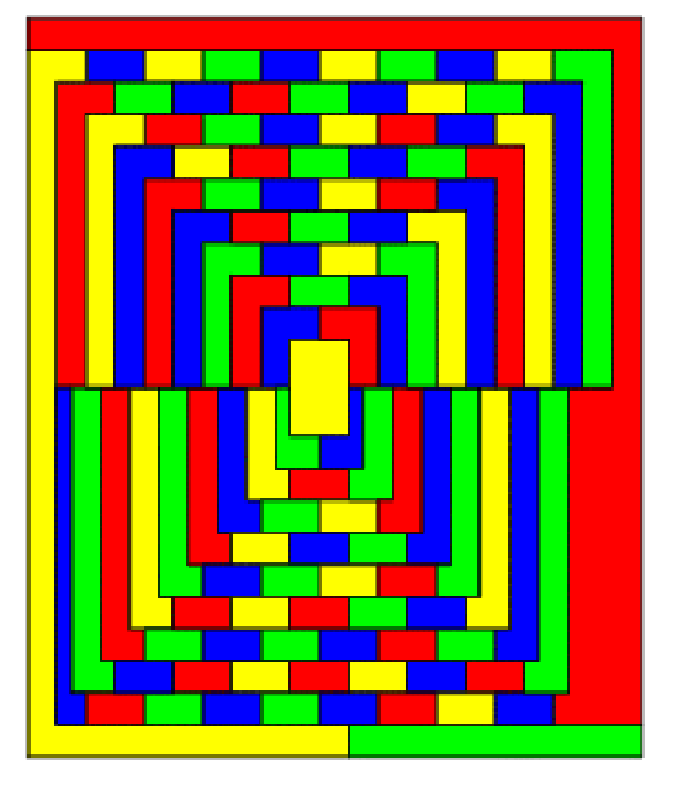
George Polya



Graph Theory - History

Four Colors of Maps





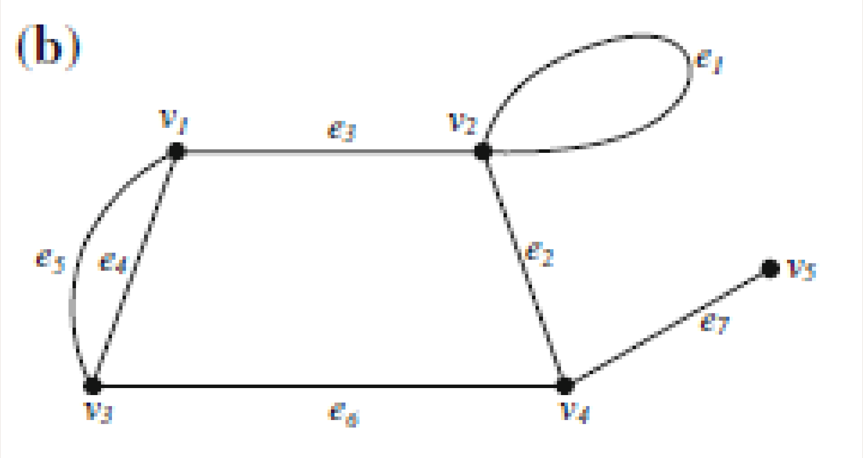
Francis Guthrie Auguste DeMorgan

Definitions of Graphs

- A **graph** $G = (V, E)$ consists of two finite sets. V; the vertex set of the graph, which is a non-empty set of elements called ***vertices*** and E; the edge set of the graph, which is a possibly empty set of elements called ***edges***, such that each edge e in E is assigned as an unordered pair of vertices $= (u, v)$, called the end vertices of e.
- **Order and size**: We define $|V| = n$ to be the ***order*** of G and $|E| = m$ to be the ***size*** of G.

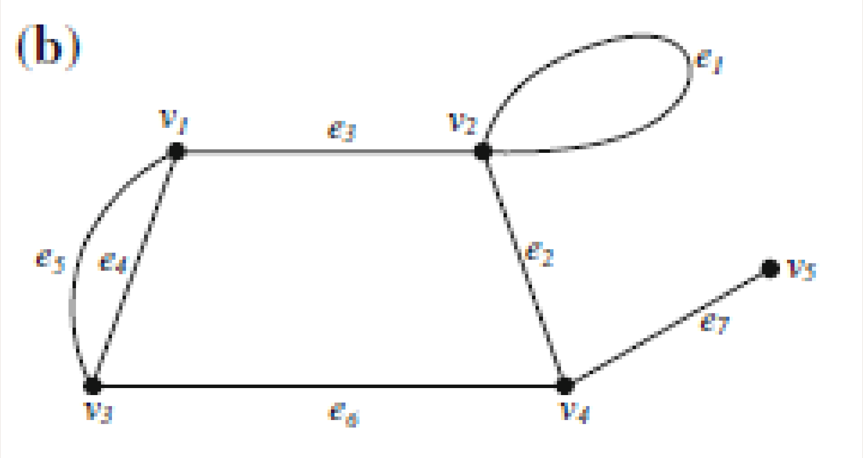
Definitions of Graphs

- The definition of a graph allows the possibility of the edge e having identical end vertices. Such an edge having the same vertex as both of its end vertices is called a ***<u>self-loop</u>*** (or simply a ***<u>loop</u>***).
- Edge e_1 in the following fig. is a self-loop.



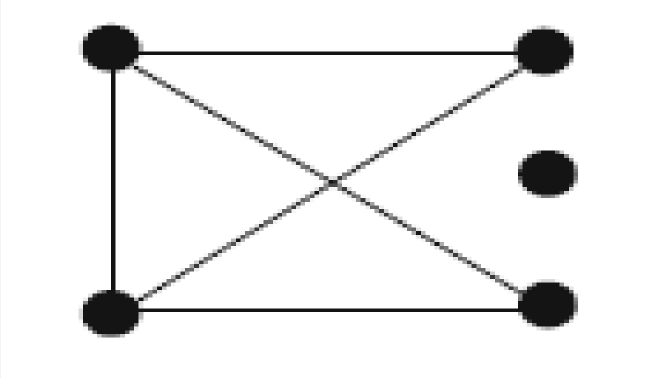
Definitions of Graphs

- Also, note that the definition of graph allows that more than one edge is associated with a given pair of vertices.
- Edges e_4 and e_5 in this figure are referred to as ***parallel edges***.



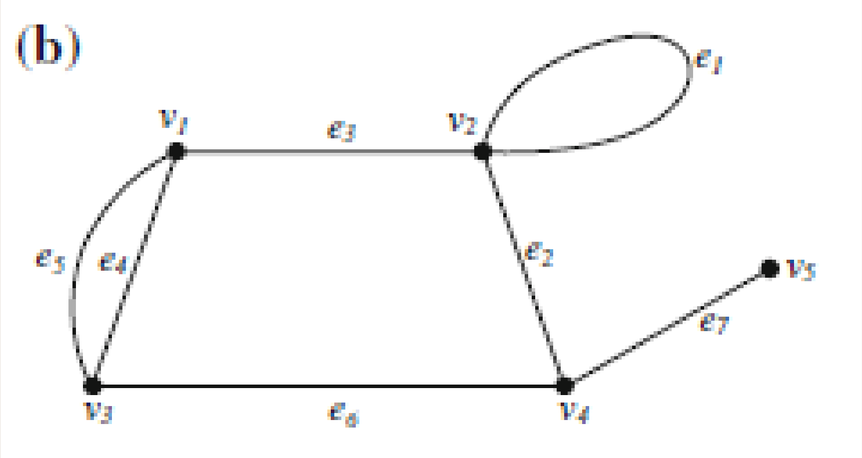
Definitions of Graphs

- A graph, that has neither self-loops nor parallel edges, is called a ***simple graph***.
- An example of a simple graph is given in the following figure.



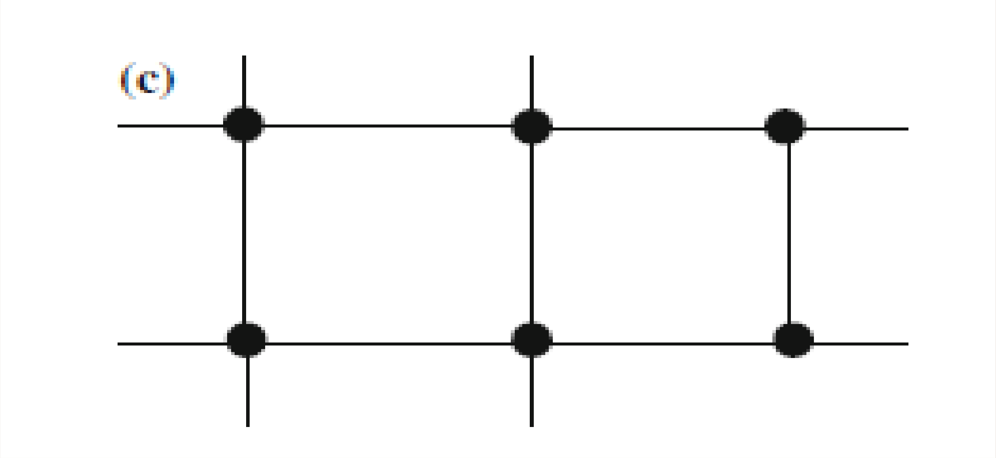
Definitions of Graphs

- ***Multigraph***: A multigraph G is an ordered pair $G = (V, E)$ with V a set of vertices or nodes and E a multiset of unordered pairs of vertices called edges.
- An example of a multigraph is given in this figure.



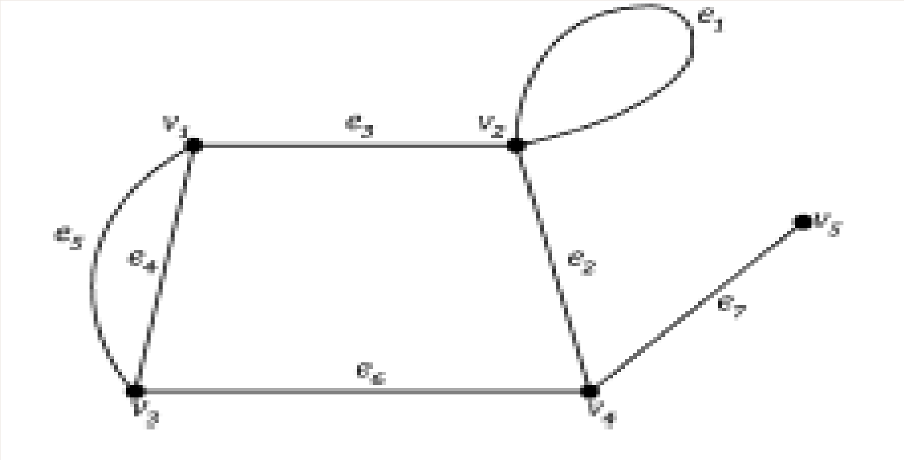
Definitions of Graphs

- **Finite** and **Infinite graph**: A graph with a finite number of vertices as well as finite number of edges is called a ***finite graph*** ; otherwise it is an ***infinite graph*** as shown in the following figure.



Incidence and Degree

- When a vertex v_i is an end vertex of some edge e_j, v_i, and e_j are said to ***be incident*** with each other.
- A graph with five vertices and seven edges is shown in the following figure. Edges e_2, e_6, and e_7 are incident with vertex v_4.



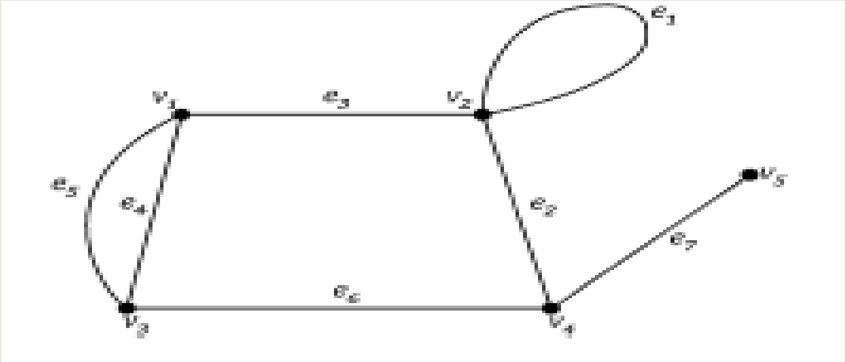
Incidence and Degree

- ***Adjacent***: Two nonparallel edges are said to be adjacent if they are incident on a common vertex. For example, e_2 and e_7 are adjacent.
- Similarly, two vertices are said to be ***adjacent*** if they are the end vertices of the same edge.
- In the previous figure, v_4 and v_5 are adjacent, but v_1 and v_4 are not.

Incidence and Degree

- ***Degree***: Let v be a vertex of the graph G. The degree $d(v)$ of v is the number of edges of G incident with v, counting each self-loop **twice**.
- For example, in the given figure, $d(v_1) = 3 = d(v_3) = d(v_4)$, $d(v_2) = 4$ *and* $d(v_5) = 1$.

$d(v_1) + d(v_3) + d(v_4) + \ldots\ldots + d(v_5) = 14$ = twice the number of edges.



Incidence and Degree

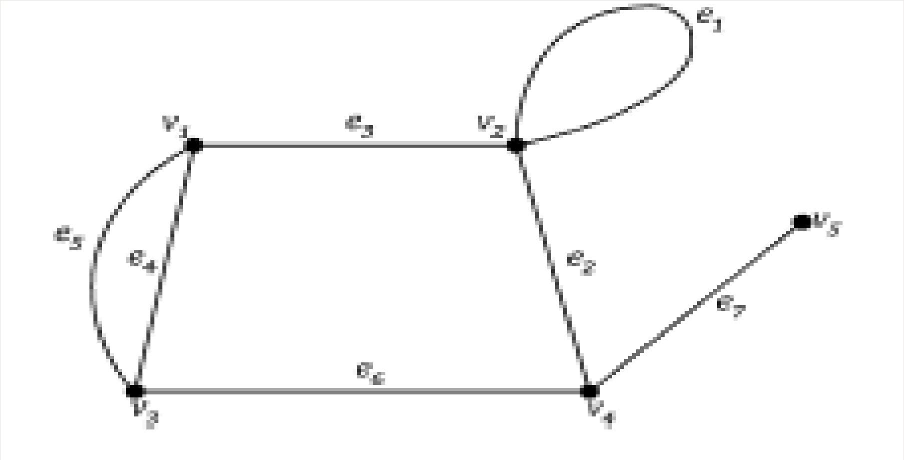
- **Theorem 1.1** *For any graph G with e edges and n vertices* $v_1, v_2, v_3\ldots\ldots v_n$

$$\sum_{i=1}^{n} d(v_i) = 2e$$

- *Proof* Each edge, since it has two end vertices, contributes precisely two to the sum of the degrees of all vertices in G. When the degrees of the vertices are summed each edge is counted twice.

Incidence and Degree

- **Odd and even vertices**: A vertex of a graph is called odd or even depending on whether its degree is odd or even.
- In the graph of the below figure, there is an even number of odd vertices.



Incidence and Degree

- **Theorem 1.2** (Handshaking lemma) *In any graph G, there is an even number of odd vertices.*

Proof If we consider the vertices with odd and even degrees separately, the equation

$\sum_{i=1}^{n} d(v_i) = 2e$ can be expressed as equation

$$\sum_{i=1}^{n} d(v_i) = \sum_{\text{even}} d(v_j) + \sum_{\text{odd}} d(v_k)$$

Let W be the set of odd vertices of G, and let U be the set of even vertices of G. Then for each $u \in U$, $d(u)$ is even and so $\sum_{u \in U} d(u)$, being a sum of even numbers, is even.

However,

$$\sum_{u \in U} d(u) + \sum_{w \in W} d(w) = \sum_{v \in V} d(v) = 2e, \text{ by Theorem 1.1}$$

Thus,

$$\sum_{w \in W} d(w) = 2e - \sum_{u \in U} d(u), \text{ is even. (being the difference of two even numbers)}$$

As all the terms in $\sum_{w \in W} d(w)$ are odd and their sum is even, there must be an even number of odd vertices. □

Incidence and Degree

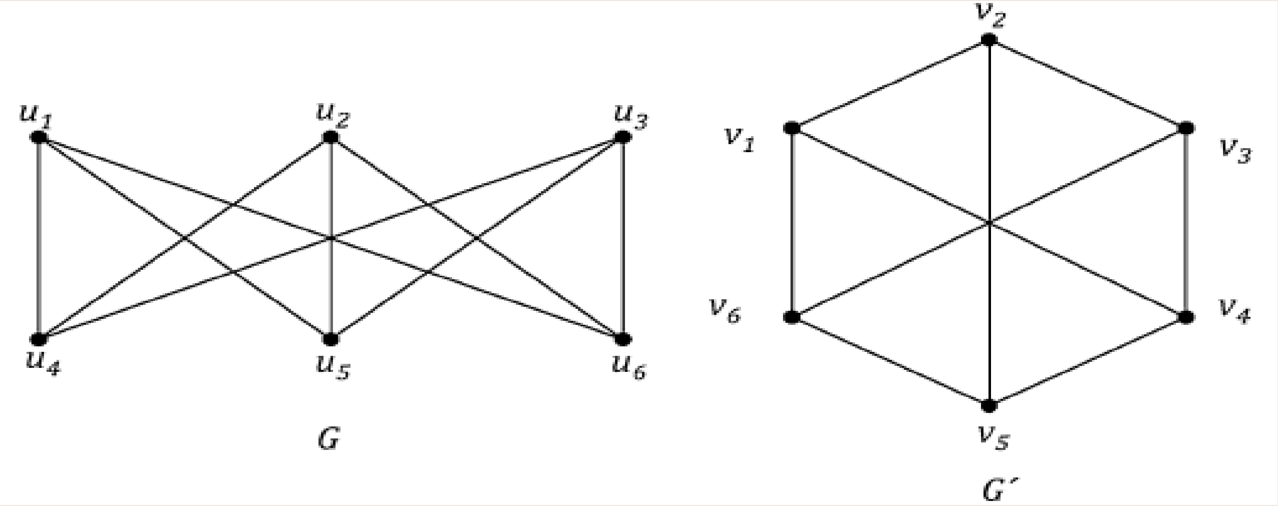
- ***Isolated vertex***: A vertex having no incident edge is called an isolated vertex..
- ***Pendant vertex***: A vertex of degree one is called a pendant vertex.
- ***Null graph***: If $E = Ø$, in a graph $G = (V, E)$, then such a graph without any edges is called a null graph.

Isomorphism

- A graph $G_1 = (V_1, E_1)$ is said to be isomorphic to the graph $G_2 = (V_2, E_2)$ if there is a one-to-one correspondence between the vertex sets V_1 and V_2 and a one-to-one correspondence between the edge sets E_1 and E_2 in such a way that if e_1 is an edge with end vertices u_1 and v_1 in G_1 then the corresponding edge e_2 in G_2 has its end vertices u_2 and v_2 in G_2 which corresponds to u_1 and v_1 , respectively. Such a pair of correspondence is called a graph ***isomorphism***.

Isomorphism

- Example 1.1 Show that the following two graphs in Fig. 1.6 are isomorphic.



Solution:

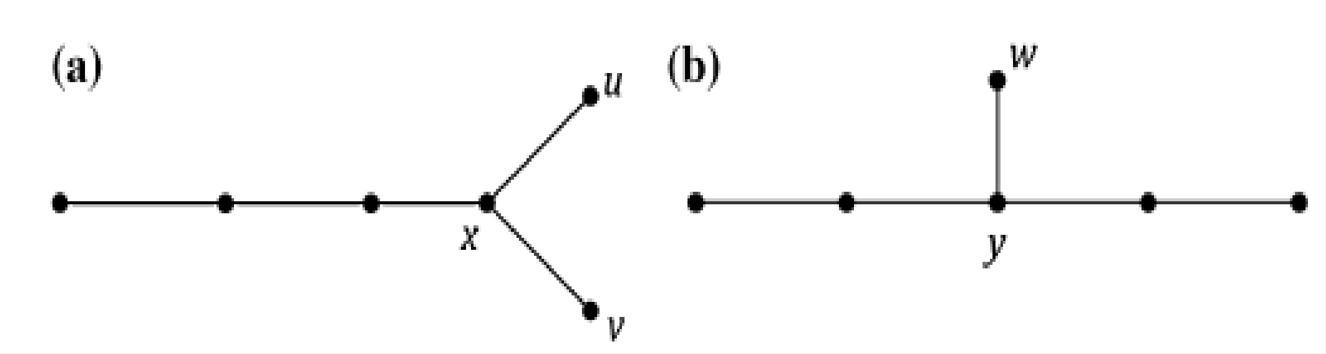
We see that both the graphs G and G' have equal number of vertices and edges. The vertex corresponds are given below:

$$u_1 \leftrightarrow v_1, u_2 \leftrightarrow v_3, u_3 \leftrightarrow v_5, u_4 \leftrightarrow v_2, u_5 \leftrightarrow v_4, u_6 \leftrightarrow v_6 \text{ or } u_5 \leftrightarrow v_6, u_6 \leftrightarrow v_4.$$

Hence, the two graphs are isomorphic.

Isomorphism

- Example 1.2 Check whether the graphs in Fig. 1.7 are isomorphic.



Solution:

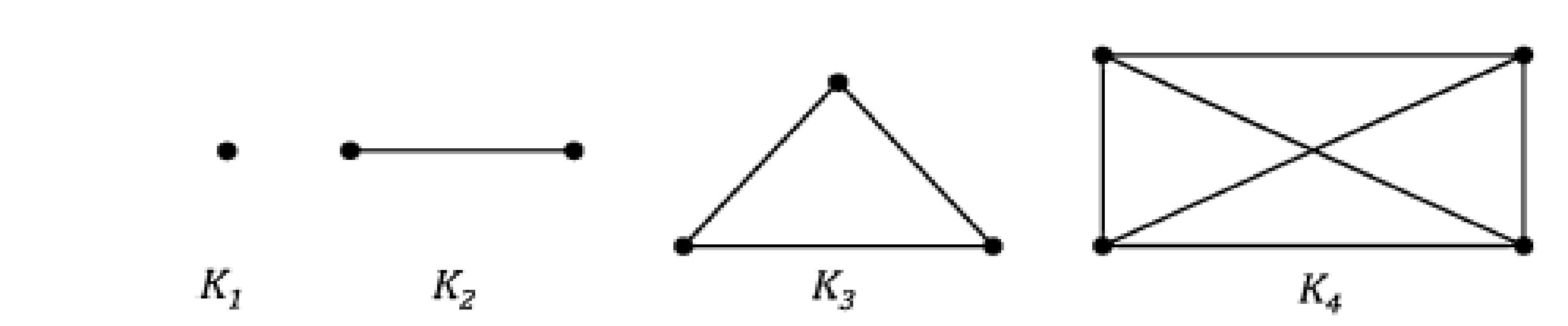
The graphs in Fig. 1.7a and b are not isomorphic. If the graph 1.7a were to be isomorphic to the one in 1.7b, vertex *x* must correspond to *y*; because there are no other vertices of degree three. Now in 1.7b, there is only one pendant vertex *w* adjacent to y; while in 1.7a there are two pendant vertices *u* and *v* adjacent to *x*.

Complete Graph

- A ***complete graph*** is a simple graph in which each pair of distinct vertices is joined by an edge. In other words, a simple graph in which there exists an edge between every pair of vertices is called a complete graph.
- It follows that the graph has $n\,(n\text{-}1)/2$ edges (since there are n - 1 edges incident with each of the n vertices, so a total of $n(n\text{-}1)$, but divide by 2 since $(v_j,v_i) = (v_i,v_j)$)

Complete Graph

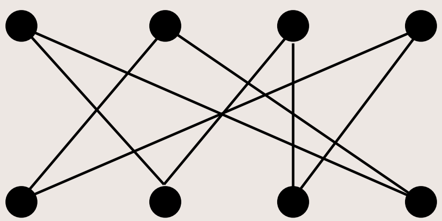
- **Corollary** *The maximum number of edges in a simple graph with n vertices is n (n-*1*)/*2*. Given any two complete graphs with the same number of vertices, n, then they are* ***isomorphism***.
- The complete graph of n vertices is denoted by K_n.

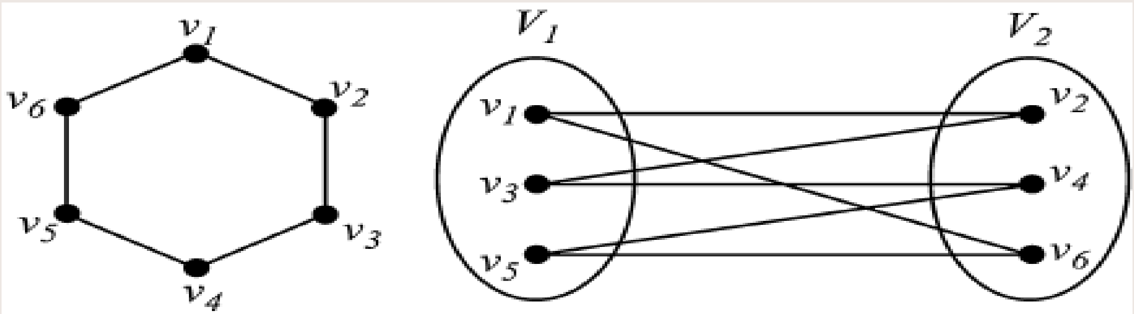


- K_1, K_2, K_3, K_4 are examples of complete graphs.

Bipartite Graph

- **Definition.** Let G be a graph. If the vertex set V of G can be partitioned into two non-empty subsets X and Y (i.e., $X \cup Y = V$ and $X \cap Y = \emptyset$) in such a way that, each edge of G has one end in X and other end in Y, then G is called bipartite. The partition $V = X \cup Y$ is called a bipartition of G.

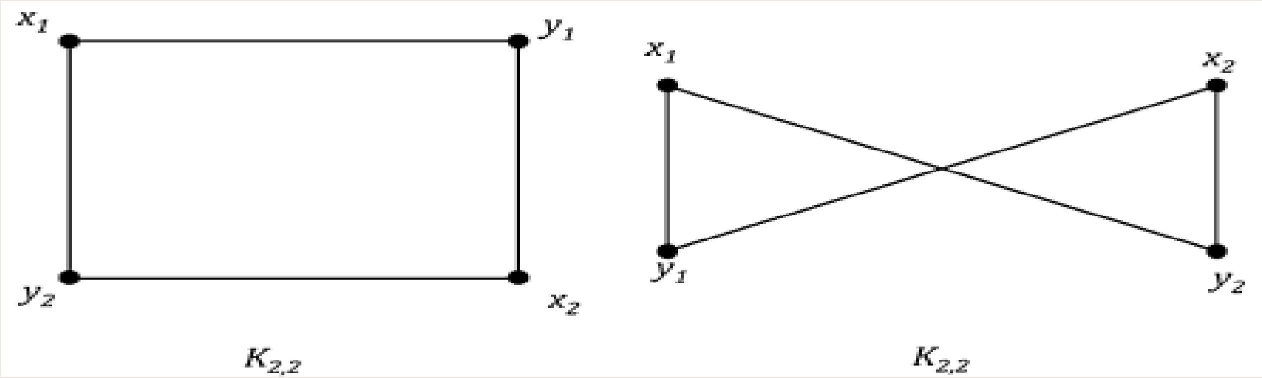




The above figures cite examples of Bipartite graphs.

Complete Bipartite Graph

- **Definition.** A ***complete Bipartite graph*** is a simple bipartite graph G, with bipartition $V = X \cup Y$ in which every vertex in X is adjacent to every vertex of Y. If X has m vertices and Y has n vertices, such a graph is denoted by $K_{m,n}$.



$K_{2,2}$ is an example of a complete Bipartite graph.

Directed Graph or Digraph

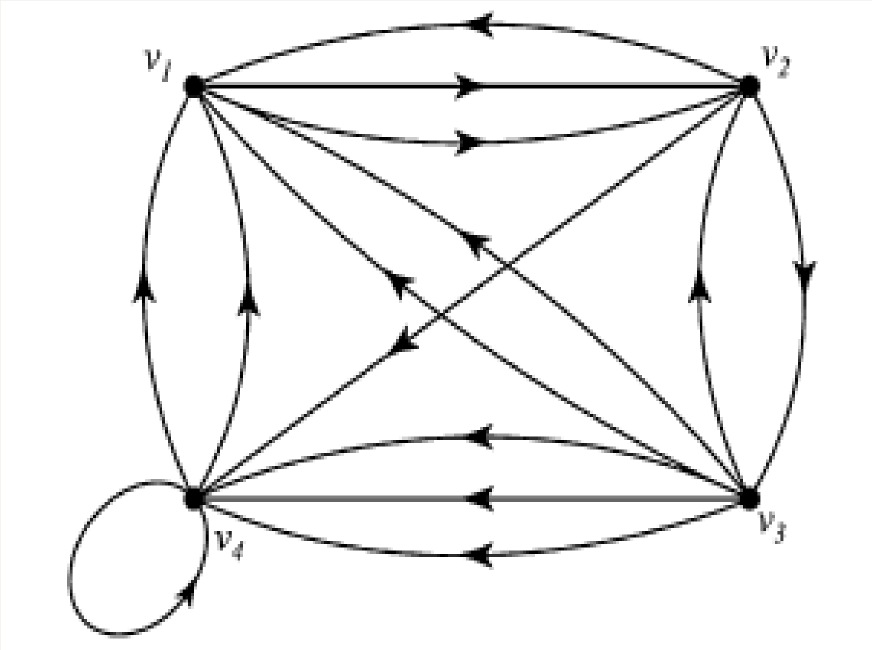
- A ***digraph*** (or a ***directed graph***) $G = (V_G, E_G)$ consists of the two sets:

1. A vertex set V_G; nonempty set, whose elements are called vertices or nodes.
2. An edge set or arc set E_G; possibly empty set, whose elements are called directed edges or arcs, such that each directed edge in EG is assigned an order pair of vertices (u, v), i.e., $E_G \subseteq V_G \times V_G$.

Directed Graph or Digraph

- For $u, v \in V_G$; an arc or a directed edge $e = (u, v) \in V_G$ is denoted by uv and implies that e is directed from u to v.

- The given figure shows a directed graph or digraph.



Directed Graph or Digraph

- **In-degree** and **Out-degree**: The in-degree and the out-degree of a vertex are defined as follows:

1. In a digraph G, the number of edges incident out of a vertex v is called the out-degree of v. It is denoted by $\text{degree}^{+}(v)$ or $d^{+}(v)$.
2. In a digraph G, the number of edges incident into a vertex v is called the in-degree of v: It is denoted by $\text{degree}^{-}(v)$ or $d^{-}(v)$.

- The total degree (or simply degree) of v is $d^{+}(v) = \text{degree}^{+}(v) + \text{degree}^{-}(v)$.

Directed Graph or Digraph

- The total degree (or simply degree) of v is $d^+(v) = \text{degree}^+(v) + \text{degree}^-(v)$.

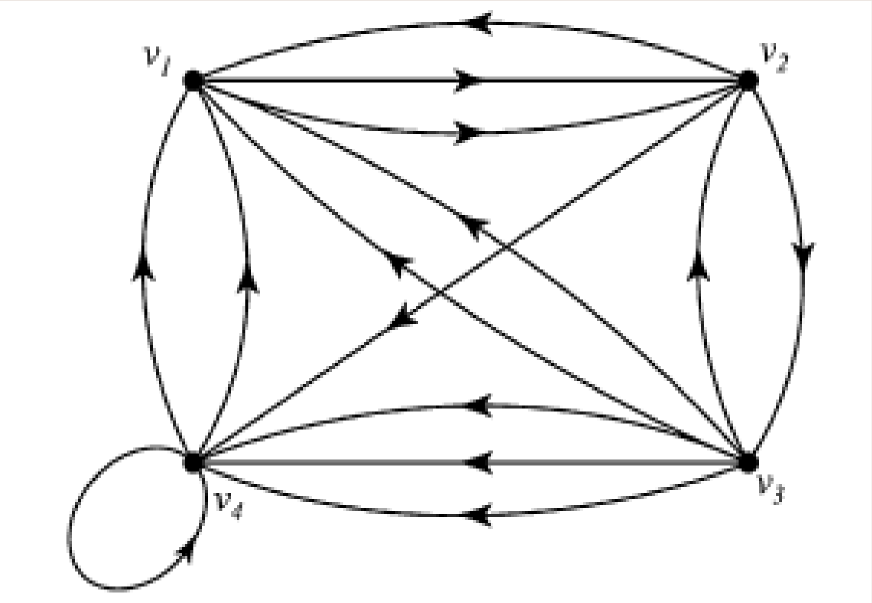
In this case, we have the following Handshaking Lemma.

- **Lemma 1.1** Let G be a digraph. Then

$$\sum_{v \in G} \text{degree}^+(v) = |E_G| = \sum_{v \in G} \text{degree}^-(v)$$

Directed Graph or Digraph

- *Example 1.3* Find the in-degree and out-degree of each vertex of the following directed graph. Also, verify that the sum of the in-degrees (or the out-degrees) equals the number of edges.



Directed Graph or Digraph

- *Solution of the previous example:*

$$\text{degree}^+(v_1) = 2 \quad \text{degree}^-(v_1) = 5$$

$$\text{degree}^+(v_2) = 3 \quad \text{degree}^-(v_2) = 3$$

$$\text{degree}^+(v_3) = 6 \quad \text{degree}^-(v_3) = 1$$

$$\text{degree}^+(v_4) = 3 \quad \text{degree}^-(v_4) = 5$$

Here, we see that

$$\sum_{v \in G} \text{degree}^+(v) = \sum_{v \in G} \text{degree}^-(v) = 14 = \text{the number of edges of } G.$$

Any Questions?