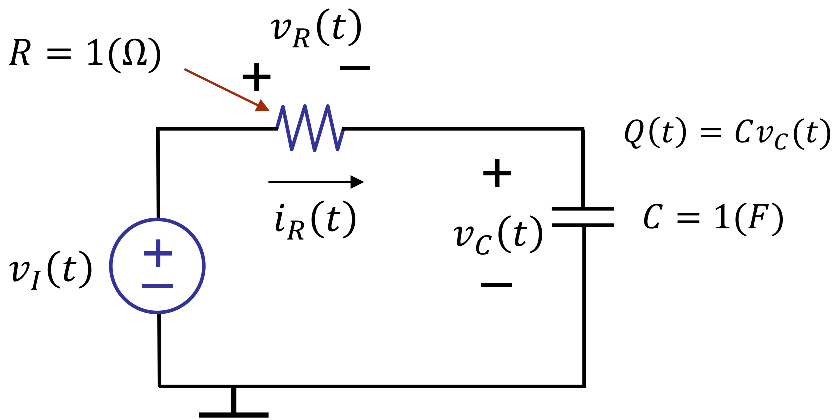
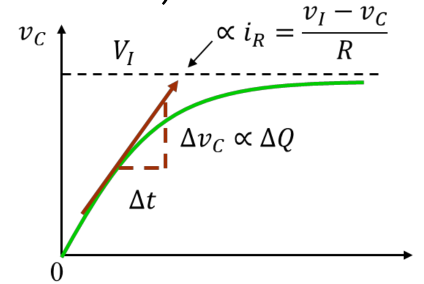
Electrical and Electronics Circuits (4190.206A 002)

- **HW #3 will be posted on ETL today**
  - The due date is Nov. 11 (Mon) 3:15pm
  - To reduce the grading burden of TA, the homework will be graded all-or-nothing style.
  - Within each problem, there are several sub-problems, but we will grade only one sub-problem within each problem, and the score of each problem will be determined by the graded problems. For example, if problem 1 is composed of 5 sub-problems, we will decide which sub-problem will be graded later, and if you solved that sub-problem correctly, you will get the full score of problem 1. In the worst case, you might have solved all other sub-problems correctly, and made a mistake only in the graded sub-problem. That is an unfortunate situation, but the score for that entire problem will become 0. Without this policy, we cannot grade so many homework efficiently. We cannot return the graded homework, so before you submit your homework, please make your own copy or scan it.

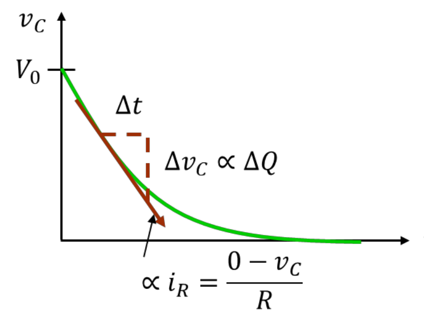
Review



Case 2)



Case 3)



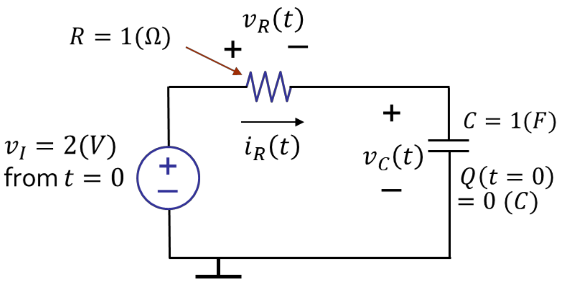
- Case 1) $Q(t=0) = 0\ (C), v_I = 0(V)$
  - No change
- Case 2) $Q(t=0) = 0\ (C),\ \ v_I(t \geq 0) = 2(V)$
  - $\Delta Q = C\Delta v_C = \left(\frac{v_I - v_C}{R}\right)\Delta t \xrightarrow{\Delta t \to 0} \quad C\frac{dv_C}{dt} = \frac{v_I - v_C}{R}$
- Case 3) $Q(t=0) = 2\ (C),\ \ v_I(t \geq 0) = 0(V)$
  - $\Delta Q = C\Delta v_C = \left(\frac{0 - v_C}{R}\right)\Delta t \xrightarrow{\Delta t \to 0} \quad C\frac{dv_C}{dt} = -\frac{v_C}{R}$

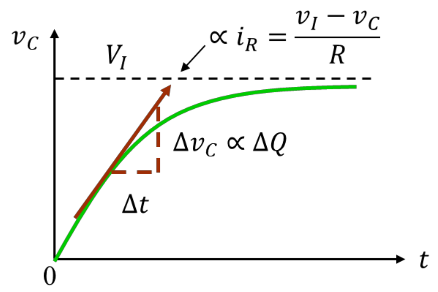
Review

- Case3) Solution for $C\frac{dv_C}{dt} = -\frac{v_C}{R}$
  - Trial solution: $\frac{1}{2^t}$, $\frac{1}{3^t}$, $\frac{1}{4^t}$, … ➔ need to decide a for $\frac{1}{a^t}$ with $a > 1$
  - By using $a = e^{\ln a}$, $\frac{1}{a^t} = \frac{1}{(e^{\ln a})^t} = \frac{1}{e^{(\ln a)t}} = e^{-(\ln a)t}$
  - Because $-(\ln a)$ is a some constant, use simple constant s
  - By trying $v_C = e^{st}$, $Cse^{st} = -\frac{e^{st}}{R}$ ➔ $s = -\frac{1}{RC}$
  - Does $v_C = e^{-t/RC}$ satisfy all the initial conditions?
    - $v_C = Ae^{-t/RC}$ also satisfy the equation
    - A is a free parameter and should be used to satisfy other conditions.

Review

- Case2) Solution for $C\frac{dv_C}{dt} = \frac{v_I - v_C}{R}$
  - When $v_I = 2(V)$, $CR\frac{dv_C}{dt} + v_C = v_I = 2$
  - Homogeneous equation
    - $CR\frac{dv_C}{dt} + v_C = 0$
    - Homogeneous solution: $v_{C,h} = Ae^{-t/RC}$
  - Non-homogeneous equation
    - $CR\frac{dv_C}{dt} + v_C = 2$
    - Particular solution
      - By guess: $v_{C,p1} = 2$
      - $v_{C,p2} = 2 + e^{-t/RC}$
      - $v_{C,p3} = 2 - e^{-t/RC}$
      - $v_{C,p4} = 2 + 5e^{-t/RC}$
      - ...
      - Particular solutions are NOT unique
      - If we know one particular solution $v_{C,p1}$, we can find infinite number of particular solutions by adding arbitrary homogeneous solution.





Review

- In this class, we are only interested in the following form of differential equations (linear ordinary differential equation)
  - $c_n \frac{d^n}{dt^n} y + c_{n-1} \frac{d^{n-1}}{dt^{n-1}} y + \cdots + c_1 \frac{d}{dt} y + c_0 y = \begin{cases} 0 \\ f(t) \end{cases}$ where $c_n, c_{n-1}, \ldots c_1, c_0$ are all real constants
  - $\boldsymbol{n}$ is called the **order** of the given differential equation

Review

- Example (homogeneous equation)
  - $\frac{d^2}{dt^2}y + 3\frac{d}{dt}y + 2y = 0$ ➔ order is 2
    - $y = e^{st}$ ➔ $(s^2 + 3s + 2)e^{st} = 0$ ➔ Solve $s^2 + 3s + 2 = 0$
    - $(s+1)(s+2) = 0$ ➔ $s = -1$ or $s = -2$
    - Homogeneous solution
      - $y = Ae^{-t} + Be^{-2t}$
  - $\frac{d^5}{dt^5}y + 2\frac{d}{dt}y + y = 0$ ➔ order is 5
    - $y = e^{st}$ ➔ $(s^5 + 2s + 1)e^{st} = 0$ ➔ Solve $s^5 + 2s + 1 = 0$
- Characteristic equation
  - Ex) $s^2 + 3s + 2 = 0$, $s^5 + 2s + 1 = 0$
  - cf. For polynomial equation, the largest exponent $\boldsymbol{n}$ is called the **degree** of polynomial

Review

- Homogeneous equation
  - $c_n \frac{d^n}{dt^n} y + c_{n-1} \frac{d^{n-1}}{dt^{n-1}} y + \cdots + c_1 \frac{d}{dt} y + c_0 y = 0$
  - $y = e^{st}$ ➔ Solve $c_n \cdot s^n + c_{n-1} \cdot s^{n-1} + \cdots + c_1 \cdot s + c_0 = 0$
  - We will have n solutions $s_1, s_2, \dots, s_{n-1}, s_n$ from the characteristic equation
  - Homogeneous solution: $y = A_1 e^{s_1 t} + A_2 e^{s_2 t} + \cdots + A_n e^{s_n t}$

  ➔ n-degree of freedom

Review

- Non-homogeneous equation
  - $\frac{d^2}{dt^2}y + 3\frac{d}{dt}y + 2y = f(t)$
  - Assume that y_h is a homogeneous solution
  - ➔ $\frac{d^2}{dt^2}y_h + 3\frac{d}{dt}y_h + 2y_h = 0$
  - Assume that y_p is a particular solution
  - ➔ $\frac{d^2}{dt^2}y_p + 3\frac{d}{dt}y_p + 2y_p = f(t)$
  - Then the total solution $y_t = y_h + y_p$ satisfies the original equation due to linear property of the given equation

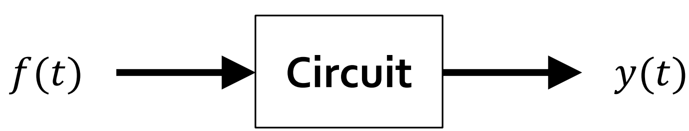
$$\frac{d^2}{dt^2}(y_h + y_p) + 3\frac{d}{dt}(y_h + y_p) + 2(y_h + y_p)$$
$$= \frac{d^2}{dt^2}y_h + 3\frac{d}{dt}y_h + 2y_h + \frac{d^2}{dt^2}y_p + 3\frac{d}{dt}y_p + 2y_p = 0 + f(t)$$

  - y_p takes care of the non-homogeneous part $f(t)$
  - y_h takes care of the initial or boundary conditions

Review

- Number of constraints
  - n -th order DE generally has n independent homogeneous solutions ➔ n degree of freedom
  - To specify **unique solution**, n-th order differential equation (DE) needs n independent constraints (e.g. initial condition or boundary conditions)

- Input and output of the system
  - $\left[c_n \frac{d^n}{dt^n} + c_{n-1} \frac{d^{n-1}}{dt^{n-1}} + \cdots + c_1 \frac{d}{dt} + c_0\right] y(t) = f(t)$
  - Non-homogeneous term $f(t)$: **input**
  - Unknown variable $y(t)$ in the homogeneous equation: **output**



Finding particular solution

- Guess the form of the particular solution
  - $\frac{d}{dt}y + 2y = 2$ 1
  - $\frac{d}{dt}y + 2y = t$ ← Ary Ao guess
  - $\frac{d}{dt}y + 2y = 3t + 1$
  - $\frac{d}{dt}y + 2y = 2t^2$
  - $\frac{d^2}{dt^2}y + 3\frac{d}{dt}y + 2y = t$
  - No dependence on the order of differential equation

Finding particular solution

- Guess the form of the particular solution
  - $\frac{d}{dt}y + 2y = e^{-t}$
  - $\frac{d}{dt}y + 2y = e^{-t} + e^{3t}$ Hint: Consider the linearity.
  - $\frac{d}{dt}y + 2y = \cos\omega t$
  - What about $\frac{d}{dt}y + 2y = e^{-t}\cos\omega t$?

Finding particular solution

- Alternative way to solve $\frac{d}{dt}y + 2y = \cos\omega t$
  - $\frac{d}{dt}y + 2y = \cos\omega t + j\sin\omega t$ ➔ consider the linearity of the solution
  - $\frac{d}{dt}y + 2y = e^{j\omega t}$ ➔ Euler relation
    - $y_p = Ae^{j\omega t}$
    - $j\omega Ae^{j\omega t} + 2Ae^{j\omega t} = e^{j\omega t}$
    - $(j\omega + 2)Ae^{j\omega t} = e^{j\omega t}$ ➔ $A = \frac{1}{2+j\omega}$
    - $A = \frac{1}{2+j\omega} \times \frac{2-j\omega}{2-j\omega} = \frac{2-j\omega}{4+\omega^2}$
    - $y_p = Ae^{j\omega t} = \frac{2-j\omega}{4+\omega^2} \times (\cos\omega t + j\sin\omega t)$
    - $y_p = \frac{1}{4+\omega^2}[2\cos\omega t + \omega\sin\omega t + j(2\sin\omega t - \omega\cos\omega t)]$
  - If the solution is a sum of a real function and an imaginary function, the real function part will satisfy $\cos\omega t$ term while the imaginary function part will satisfy $j\sin\omega t$ term
  - Solution to $\frac{d}{dt}y + 2y = \cos\omega t$ is real part of y_p

    ➔ $Re[y_p] = \frac{1}{4+\omega^2}[2\cos\omega t + \omega\sin\omega t]$

Homogeneous solution

- $\frac{d^2}{dt^2}y + 5\frac{d}{dt}y + 6y = 0$
  - Recall that solution of $s^2 + 5s + 6 = (s+2)(s+3) = 0$ characteristic equation provides e^{-2t} and e^{-3t}.

- What about $\frac{d^2}{dt^2}y + 4\frac{d}{dt}y + 4y = 0$
  - $(s+2)^2 = 0$ provides only e^{-2t}.
  - Is single homogeneous solution enough?
  - Not in general because we need two free parameters to satisfy two independent constraints.
  - Try te^{-2t}

Homogeneous solution

- What was the problem with $\frac{d^2}{dt^2}y + 4\frac{d}{dt}y + 4y = 0$?
  - $(s+2)^2 = 0$ ➔ duplicate root $s = -2$ ➔ called multiplicity of the root

- Solution for multiple identical root
  - If $s = -2$ is a root of multiplicity of 2, not only e^{-2t}, but te^{-2t} is also a solution.
  - If s =2 is a root of multiplicity of 3, e^{2t}, te^{2t}, t^2e^{2t} are solutions.
  - In general, if the multiplicity of the duplicate root s_k is m, $e^{s_kt}, te^{s_kt}, t^2e^{s_kt}, \ldots, t^{m-1}e^{s_kt}$ are the homogeneous solutions.

Proof for solution for multiple identical roots I

- Assume that $s_1, s_2, \dots s_n$ are the roots of the characteristic equation of $\left[c_n \frac{d^n}{dt^n} + c_{n-1}\frac{d^{n-1}}{dt^{n-1}} + \cdots + c_1 \frac{d}{dt} + c_0\right] y(t) = 0$

  ➔ $(s - s_1)(s - s_2)\dots(s - s_n) = 0$

  - Recall $\frac{d^2}{dt^2} y(t) = \frac{d}{dt}\left(\frac{d}{dt} y(t)\right) = \left(\frac{d}{dt}\right)^2 y(t)$
  - s corresponds to $\frac{d}{dt}$.
  - The given differential equation (DE) can be re-written as

$$\left(\frac{d}{dt} - s_1\right)\left(\frac{d}{dt} - s_2\right)\dots\left(\frac{d}{dt} - s_n\right) y(t) = 0$$

  - Note that the order of $\left(\frac{d}{dt} - s_m\right)$ factor is not important
    - Example: $\left(\frac{d}{dt} - 1\right)\left(\frac{d}{dt} - 2\right) y = \frac{d^2}{dt^2} y - 3\frac{d}{dt} y + 2y = \left(\frac{d}{dt} - 2\right)\left(\frac{d}{dt} - 1\right) y$

Proof for solution for multiple identical roots II

- $(s - s_1)(s - s_2) \ldots (s - s_n) = 0$
  - $\left(\frac{d}{dt} - s_1\right)\left(\frac{d}{dt} - s_2\right) \ldots \left(\frac{d}{dt} - s_n\right) y(t) = 0$
  - Relation between $\left(\frac{d}{dt} - s_m\right)$ factor and homo. solution $e^{s_m t}$
    - $\left(\frac{d}{dt} - s_m\right) e^{s_m t} = 0$
    - We can easily prove that $e^{s_m t}$ satisfies the given DE by shuffling the order of $\left(\frac{d}{dt} - s_k\right)$ factors.
    - $\left\{\left(\frac{d}{dt} - s_1\right) \ldots \left(\frac{d}{dt} - s_{m-1}\right)\left(\frac{d}{dt} - s_{m+1}\right) \ldots \left(\frac{d}{dt} - s_n\right)\right\}\left(\frac{d}{dt} - s_m\right) e^{s_m t} = 0$

Proof for solution for multiple identical roots III

- $\frac{d^3}{dt^3}y - 6\frac{d^2}{dt^2}y + 12\frac{d}{dt}y - 8y = \left(\frac{d}{dt} - 2\right)^3 y = 0$
  - $(s-2)^3 = 0$ ➔ duplicate root $s = 2$
  - When s =2 is a root of multiplicity of 3, e^{2t}, te^{2t}, t^2e^{2t} are solutions.
  - Example
    - $\left(\frac{d}{dt} - 2\right)e^{2t} = 2e^{2t} - 2e^{2t} = 0$
    - $\left(\frac{d}{dt} - 2\right)te^{2t} = e^{2t} + t(2e^{2t}) - 2(te^{2t}) = e^{2t}$
    - $\left(\frac{d}{dt} - 2\right)t^2e^{2t} = (2t)e^{2t} + t^2(2e^{2t}) - 2(t^2e^{2t}) = 2te^{2t}$
    - By combining the above three, $\left(\frac{d}{dt} - 2\right)^3 t^2e^{2t} = 0$
  - In general, when $l < m$, $\left(\frac{d}{dt} - s_k\right)^m t^l e^{s_k t} = 0 \Leftrightarrow (s - s_k)^m = 0$

Exercises for multiplicity of roots

- Find homogeneous solutions
  - $\left(\frac{d}{dt} - 2\right)\left(\frac{d}{dt} + 3\right) y(t) = 0$
  - $\left(\frac{d}{dt} - 1\right)^2 y(t) = 0$
  - $\left(\frac{d}{dt} + 2\right)^2 \left(\frac{d}{dt} - 2\right) y(t) = 0$
- Find particular solution
  - $\frac{d}{dt} y + 2y = e^{-2t}$

Oscillation

- Can you guess the solution for $\frac{d^2}{dt^2}y = +y$?
- Can you guess the solution for $\frac{d^2}{dt^2}y = -y$?
- When do we observe oscillation in a daily life?

Meaning of complex roots

- Recall
  - To find a particular solution for $\frac{d}{dt}y + 2y = \cos\omega t$,
  - problem was converted to $\frac{d}{dt}y + 2y = e^{j\omega t}$
  - The solution $Ae^{j\omega t}$ had complex exponent $j\omega t$ and represented a sinusoidal result

- If the root of characteristic equation has an imaginary part, the result will include oscillation

Oscillation

- Find a solution for $\frac{d^2}{dt^2}y(t) + \omega_0^2 y(t) = 0$ with the following initial conditions (assume $\omega_0 > 0$)
  - $y(0) = 1$ and $\left.\frac{dy}{dt}\right|_{t=0} = 0$
  - 2nd order differential equation requires two initial conditions
  - $s^2 = -\omega_0^2$ ➔ $s = \pm j\omega_0$ ➔ $y(t) = Ae^{j\omega_0 t} + Be^{-j\omega_0 t}$
  - $\frac{d}{dt}y(t) = Aj\omega_0 e^{j\omega_0 t} + B(-j\omega_0)e^{-j\omega_0 t}$
  - To satisfy $\left.\frac{dy}{dt}\right|_{t=0} = 0$, $Aj\omega_0 + B(-j\omega_0) = 0$ ➔ $B = A$
  - ➔ $y(t) = A\left(e^{j\omega_0 t} + e^{-j\omega_0 t}\right)$
  - To satisfy $y(0) = 1$, $2A = 1$ ➔ $y(t) = \frac{1}{2}\left(e^{j\omega_0 t} + e^{-j\omega_0 t}\right) = \cos\omega_0 t$
  - With the real-valued initial conditions, the result also should be real.

Damped oscillation

- Find a solution for $\frac{d^2}{dt^2}y + 2\frac{d}{dt}y + (1+\omega_0^2)y = 0$ (assume $\omega_0 > 0$)
  - Initial condition: $y(0) = 1$ and $\left.\frac{dy}{dt}\right|_{t=0} = -1$
  - $s^2 + 2s + (1+\omega_0^2) = 0$
  - Quadratic formula: $s = \frac{-b\pm\sqrt{b^2-4ac}}{2a} = \frac{1}{2}\left(-2 \pm \sqrt{4-4(1+\omega_0^2)}\right) = -1 \pm j\omega_0$

  ➔ $y(t) = Ae^{(-1+j\omega_0)t} + Be^{(-1-j\omega_0)t} = e^{-t}\left(Ae^{j\omega_0 t} + Be^{-j\omega_0 t}\right)$

  - To satisfy $y(0) = 1$, $A + B = 1$

  ➔ $y(t) = e^{-t}\left(Ae^{j\omega_0 t} + (1-A)e^{-j\omega_0 t}\right) = e^{-t}\left(2jA\sin\omega_0 t + e^{-j\omega_0 t}\right)$

  - $\frac{d}{dt}y(t) = e^{-t}\left\{2jA\omega_0\cos\omega_0 t - j\omega_0 e^{-j\omega_0 t}\right\} - e^{-t}\left\{2jA\sin\omega_0 t + e^{-j\omega_0 t}\right\}$
  - To satisfy $\left.\frac{dy}{dt}\right|_{t=0} = -1$, $2jA\omega_0 - j\omega_0 - 1 = -1$ ➔ $A = \frac{1}{2}$ ➔ $B = \frac{1}{2}$
  - ➔ $y(t) = e^{-t}\frac{\left(e^{j\omega_0 t} + e^{-j\omega_0 t}\right)}{2} = e^{-t}\cos\omega_0 t$
  - With the real-valued initial conditions, the result also should be real.