Homework 1

M1522.000900 Data Structure (2019 Fall)

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Question 1.

(1) **a. Check reflective:** $a + a = 2 \cdot a \in R_1$

$\therefore R_1$ is reflective.

b. Check symmetric: if $(a, b) \in R1$, then, $a + b = 2 \cdot n (\in \mathbb{N})$.

$a + b = b + a = 2 \cdot n \in R_{1.}$

$\therefore R_1$ is symmetric.

c. Check transitive: if $(a, b) \in R1$ and $(b, c) \in R1$, then,

[*case 1*] if a is even, then b is even and c is even.
thus, $(a + c)$ is even --- (i)

[*case 2*] if a is odd, then b is odd and c is odd.
thus, $(a + c)$ is even --- (ii)

From (i)&(ii), $(a, c) \in R_1$

$\therefore$ R_1 is transitive.

$\therefore \boldsymbol{R_1}$ is an equivalence relation.

(2) **a. Check reflective:** $a + a = 2 \cdot a \notin R_2$

$\therefore R_2$ is not reflective.

$\therefore \boldsymbol{R_2}$ is *not* an equivalence relation.

(3) **a. Check reflective:** $a \times a = a2 > 0$, thus, $(a, a) \in R_3$

$\therefore R_3$ is reflective.

b. Check symmetric:

if $(a, b) \in R_3$, then, $a \times b > 0$

$a \times b = b \times a > 0$, thus, $(b, a) \in R_3$

$\therefore R_3$ is symmetric.

c. Check transitive: if $(a, b) \in R_3$ and $(b, c) \in R_3$, then,

$(a \times b) > 0$ and $(b \times c) > 0$

[*case 1*] if $a > 0$, then, $b > 0$ and $c > 0$
thus, $(a \times c) > 0$ --- (i)

[*case 2*] if $a < 0$, then, $b < 0$ and $c < 0$
thus, $(a \times c) > 0$ --- (ii)

From (i)&(ii), $(a, c) \in R_3$

$\therefore$ R_3 is transitive.

$\therefore \boldsymbol{R_3}$ is an equivalence relation.

(4) **a. Check reflective:** $a \,/\, a = 1 \in R4$

$\therefore R_4$ is reflective.

b. Check symmetric: if $(a, b) \in R_4$, then, $a/b = n(\in \mathbb{N})$

$b/a = 1/n$ is not always integer, thus, $b/a \notin R4$

$\therefore R_4$ is not symmetric

<u>**$\therefore R_4$ is *not* an equivalence relation.**</u>

(5) **a. Check reflective:** $a - a = 0 \in R_5$

$\therefore R_5$ is reflective.

b. Check symmetric: if $(a, b) \in R_5$, then, $a - b = n(\in \mathbb{N})$.

$b - a = -n \in R5$

$\therefore R_5$ is symmetric.

c. Check transitive: if $(a, b) \in R_5$ and $(b, c) \in R_5$, then,

$(a - b) = n(\in \mathbb{N})$ and $(b - c) = m\ (\in \mathbb{N})$

$(a - b) + (b - c) = (a - c) = (n + m) \in R_5$

$\therefore R_5$ is transitive.

<u>**$\therefore R_5$ is an equivalence relation.**</u>

(6) **a. Check reflective:** $|a - a| = 0 \le 2$, thus, $(a, a) \in R6$

$\therefore R_6$ is reflective.

b. Check symmetric: if $(a, b) \in R_6$, then, $|a - b| \le 2$

$|a - b| = |b - a| \le 2$, thus, $(b, a) \in R_6$

$\therefore R_6$ is symmetric.

c. Check transitive: if $(a, b) \in R_6$ and $(b, c) \in R_{6,}$ then,

$|a - b| \le 2$ and $|b - c| \le 2$

[counter case] when $a = -1, b = 1, c = 3$, then,

$|a - c| = |-1 - 3| = 4 \notin R_6$

$\therefore R_6$ is not transitive.

<u>**$\therefore R_6$ is *not* an equivalence relation.**</u>

Question 2.

(1) **a. Check antisymmetric:** if $(a, b) \in R_1$, then, $(b, a) \notin R_{1.}$

$\therefore R_1$ is antisymmetric.

b. Check transitive: if $(a, b) \in R_1$ and $(b, c) \in R_{1,}$ then,

a is grandfather of c, thus, $(a, c) \notin R_1$

$\therefore R_1$ is not transitive.

<u>**$\therefore R_1$ is *not* a partial ordering.**</u>

(2) **a. Check antisymmetric:** if $(a, b) \in R_2$, then, $(b, a) \notin R_1$.

$\therefore R_2$ is antisymmetric.

b. Check transitive: if $(a, b) \in R_2$ and $(b, c) \in R_2$, then,
a is ancestor of c, thus, $(a, c) \in R_2$

$\therefore R_2$ is transitive.

<u>$\therefore R_2$ is a partial ordering.</u>

(3) **a. Check antisymmetric:** if $(a, b) \in R_3$, then, $(b, a) \notin R_3$.

$\therefore R_3$ is antisymmetric.

b. Check transitive: if $(a, b) \in R_3$ and $(b, c) \in R_3$, then,
a is older than c, thus, $(a, c) \in R3$

$\therefore R_3$ is transitive.

<u>$\therefore R_3$ is a partial ordering.</u>

(4) **a. Check antisymmetric:** when a is sister of b, then, $(a, b) \in R_4$ and $(b, a) \in R_4$, but $a \neq b$.

$\therefore R_4$ is not antisymmetric.

<u>$\therefore R_4$ is *not* a partial ordering.</u>

(5) **a. Check antisymmetric:** $< a, b > \in R_5$ and $< b, a > \in R_5$, but, $a \neq b$.

$\therefore R_5$ is not antisymmetric.

<u>$\therefore R_5$ is *not* a partial ordering.</u>

(6) **a. Check antisymmetric:** there is no (a, b) that $< a, b > \in R_6$ and $< b, a > \in R_6$.

$\therefore R_6$ is antisymmetric

b. Check transitive: $< 2, 1 > \in R_6$ and $< 1, 3 > \in R_6$ and $< 2, 3 > \in R_6$

$\therefore R_6$ is transitive

<u>$\therefore R_6$ is a partial ordering.</u>

Question 3.

(1) To move n disks from L to R, there is process,

a. move $(n - 1)$ *disks from L to M,*
b. move n_{th} *disk from L to R,*
c. move $(n - 1)$ *disks from M to R.*

And there's no difference *from wherever to wherever* in terms of the number of movements.
So, we can define,

$$T(n) = \begin{cases} 2 \cdot T(n) + 1 & (x > 0) \\ 1 & (n = 0) \end{cases} \cdots \text{(i)}$$

$$\therefore \boldsymbol{T(10)} = 2 \cdot T(9) + 1 = \cdots = \mathbf{1023}$$

(2) **Follows *(i)* above.**

(3) Expand the equation by using (i), with adding 1 to both sides,

$$\begin{aligned} T(n) + 1 &= 2 \cdot (T(n-1) + 1) \\ T(n-1) + 1 &= 2 \cdot (T(n-2) + 1) \\ T(n-2) + 1 &= 2 \cdot (T(n-3) + 1) \\ &\cdots\cdots\cdots \\ T(2) + 1 &= 2 \cdot (T(1) + 1) \end{aligned}$$

Set $a_n = T(n) + 1$, then, a_n is geometric sequence.

$$\begin{aligned} a_n &= a_1 \cdot 2^{n-1} \\ &= (T(1) + 1) \cdot 2^{n-1} \\ &= 2^n \cdots (ii) \end{aligned}$$

by (ii), $T(n) = 2^n - 1 (n > 1)$ and equation holds when $n = 1$, therefore,

$$\therefore \boldsymbol{T(n) = 2^n - 1}$$

Question 4.

(1)

$$\begin{aligned} n = 2^i,\ & S(2^i) = 2 \cdot S(2^{i-1}) + 2^i \cdots (i) \\ n = 2^{i-1},\ & S(2^{i-1}) = 2 \cdot S(2^{i-2}) + 2^{i-1} \cdots (ii) \\ n = 2^{i-2},\ & S(2^{i-2}) = 2 \cdot S(2^{i-3}) + 2^{i-2} \cdots (iii) \\ & \cdots\cdots\cdots \\ n = 2^1,\ & S(2^1) = 2 \cdot S(2^0) + 2^1 \cdots (iv) \end{aligned}$$

From (i) to (iv), by multiply 2^kin both sides,

$$\begin{aligned} 2^0 \times S(2^i) &= 2^1 \cdot S(2^{i-1}) + 2^i \cdots (i)' \\ 2^1 \times S(2^{i-1}) &= 2^2 \cdot S(2^{i-2}) + 2^i \cdots (ii)' \\ 2^2 \times S(2^{i-2}) &= 2^3 \cdot S(2^{i-3}) + 2^i \cdots (iii)' \\ &\cdots\cdots\cdots \\ 2^{i-1} \times S(2^1) &= 2^i \cdot S(2^0) + 2^i \cdots (iv)' \end{aligned}$$

By summing up from (i)’ to (iv)’,

$$\begin{aligned} S(2^i) &= 2^i \cdot S(1) + i \cdot 2^i \\ &= i \cdot 2^i + 2^i \\ &= (i + 1) \cdot 2^i \end{aligned}$$

$2^i = n,\ S(n) = n \cdot (lg(n) + 1)\ \ (n > 1)$, and equation holds when $n = 1$, therefore,

$\therefore \boldsymbol{S(n) = n(lg(n) + 1)}$

(2) From given equation,

$S(2) = 2S(1) + 2 = 2 \cdot 1 + 2 = 4$

From closed-form solution, at $n = 2$

$S(2) = 2 \cdot (\lg(2) + 1) = 2 \cdot 2 = 4$

$\therefore$ **$\boldsymbol{S(n)}$ holds for $\boldsymbol{n = 2}$**

(3) Assuming $S(2^k) = 2^k(k + 1)$ is true.
if $n = 2^{k+1}$, then,

$$\begin{aligned} S(2^{k+1}) &= 2 \cdot S(2^k) + 2^{k+1} \\ &= 2 \cdot 2^k \cdot (k + 1) + 2^{k+1} \cdots by\ (1),\ \ when\ n = 2^k, S(n) = 2^k \cdot (k + 1) \\ &= (k + 2) \cdot 2^{k+1} \end{aligned}$$

$\therefore\ \ \boldsymbol{S(n)\ also\ holds\ for\ n = 2^{k+1}}$

Question 5.

<Basis>

$T(2) = 2 \cdot lg2 = 2$

$\therefore$ **$\boldsymbol{T(n)}$ holds for $\boldsymbol{n = 2 \cdots (i)}$**

<Inductive step>

Assuming $n = 2^k,\ T(n) = n \cdot lgn$ is true.
If $n = 2^{k+1}$, then,

$$\begin{aligned} T(2^{k+1}) &= 2 \cdot T(2^k) + 2^{k+1} \\ &= 2 \cdot k \cdot 2^k + 2^{k+1} \cdots by\ assumtion,\ \ T(2^k) = 2^k lg 2^k = k \cdot 2^k \\ &= (k + 1) \cdot 2^{k+1} \end{aligned}$$

$\therefore$ **$\boldsymbol{T(n) = n \lg n}$ also holds for $\boldsymbol{n = k + 1 \cdots (ii)}$**

<Proof>

By $(i)\&(ii)$, $\boldsymbol{T(n) = n \lg n\ \ (n \geq 2)}$