Homework 4

M1522.000900 Data Structure (2019 Fall)

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Question 1.

For a binary tree that has n leaves, the number of 2-degree nodes be $\boldsymbol{n-1}$.

In the tree that has k nodes, let the number of leaves be $L(k)$ and of 2-degree nodes be $T(k)$.
Let the proposition P for the binary tree that has k nodes, $P(k)$: $L(k) - T(k) = 1$.

Proof. $\forall k\ P(k)$

[Base case] At $k = 1$, $L(k) = 1$ and $T(k) = 0$.
$L(k) - T(k) = 1$, thus, $P(k)$ holds for $k = 1$.

[Inductive Steps] Assume that $P(k)$ holds for $k = n$.
When tree grows by 1 node, there are 3 cases by the degree.

(Case 1) The new node cannot be attached to existing node if it is 2-degree node.

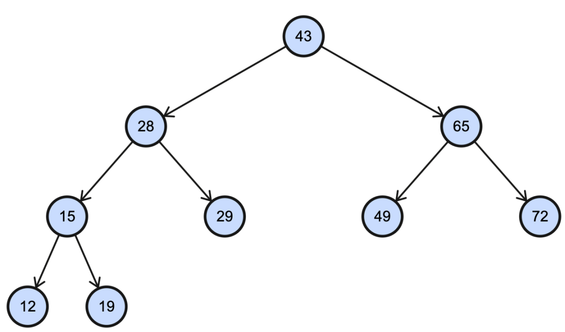
(Case 2) The new node can be attached to existing node if it is 1-degree node.
In this case, this node becomes 2-degree node by new node.
$T(n+1) = T(n) + 1\ and\ L(n+1) = L(n) + 1$
$L(n+1) - T(n+1) = L(n) - T(n) = 1$

(Case 3) The new node can be attached to existing node if it is leaf.
In this case, there is no change in $T(n+1)$ and $L(n+1)$ from $k = n$.
$L(n+1) - T(n+1) = L(n) - T(n) = 1$

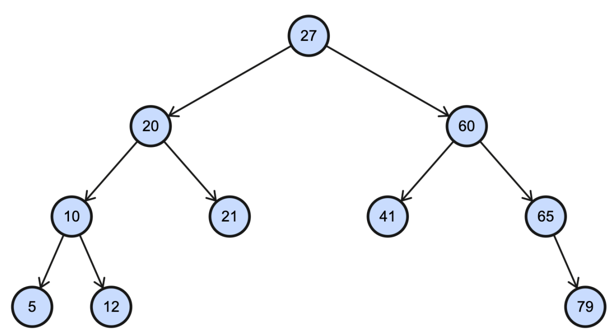
In all cases, $L(n+1) - T(n+1) = 1$.
$\therefore P(n+1)$ holds for $k = (n+1)$.

QED.

Question 2.

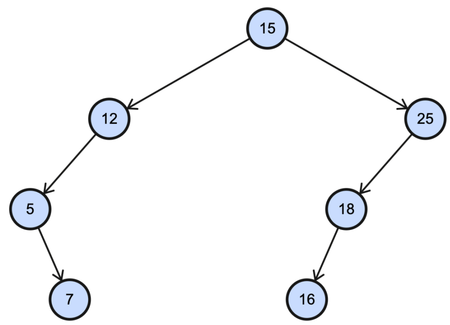


(a)



(b)

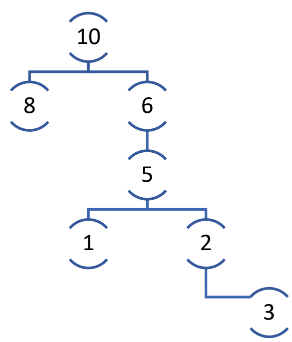
Question 3.

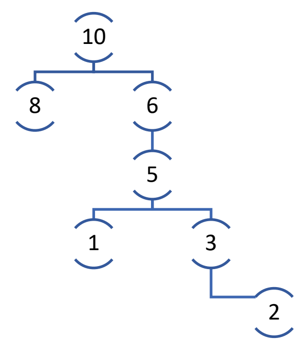
(a) 

(b) A. 15-12-5-7-25-18-16
B. 5-7-12-15-16-18-25
C. 7-5-12-16-18-25-15

Question 4.

(1) (c), (d)

(2) 



Question 5.

Let $S(h)$ be $(h-2)(2\^{}h-1)+h$.
And let a perfect binary tree of height h be T_h.

Proof. The sum of depth of each node in T_h is $S(h)$.

[Base case] In T_1, sum of depth of each node is 0.
$S(1) = -1 \cdot 0 + 1 = 1$
$\therefore\ S(h)$ holds for $h = 1$.

[Inductive Steps] Assume that $S(h)$ holds for $h = n$, then, sum of depth of each node is $S(n)$.
When T_n grows to T_{n+1}, new leaves are attached at existing leaves.
This procedure is like below,
1) T_n has 2^{n-1} leaves from its definition.
2) Each leaves in T_n became T_{n+1}'s nodes that have 2 children.
3) T_{n+1}'s leaves are 2 times of T_n's leaves.
From these, sum of depth of each node in T_{n+1} is

$$
\begin{aligned}
&= S(n) + 2^\wedge(n-1) \cdot 2 \cdot n \\
&= S(n) + 2^\wedge n \cdot n \\
&= (n-2) \cdot 2^\wedge n + 2 + 2^\wedge n \cdot n \\
&= (2n-2) \cdot 2^\wedge n + 2 \\
&= (n-1) \cdot 2 \cdot 2^\wedge n - (n-1) + (n+1) \\
&= (n-1) \cdot (2 \cdot 2^\wedge n - 1) + (n+1) \\
&= (n-1) \cdot (2^\wedge(n+1) - 1) + (n+1) \\
&= S(n+1)
\end{aligned}
$$

∴ S(h) also holds for h=(n+1).
QED.