Electrical and Electronics Circuits (4190.206A 002)

- **HW #3 is posted on ETL**
  - The due date is Nov. 11 (Mon) 3:15pm
- Exams schedule
  - 2^{nd} mid-term exam:
    - 11/18(Mon) – my 1^{st} priority
    - 11/13(Wed)
  - Final exam: 12/11 (Wed) 2:00pm~3:15pm

- Do you need additional review class before the exam?

- Self-attendance check

Review

- $\boldsymbol{n}$-th **order** linear ordinary differential equation
  - $c_n \frac{d^n}{dt^n} y + c_{n-1} \frac{d^{n-1}}{dt^{n-1}} y + \cdots + c_1 \frac{d}{dt} y + c_0 y = \begin{cases} 0 \\ f(t) \end{cases}$ where $c_n, c_{n-1}, \ldots c_1, c_0$ are all real constants $=1$ of them

    $\frac{dy}{dt} y \Leftarrow$ not linear.

    <It's not in our observe>
  - Linearity of equation ➔ sum of particular solution and homogeneous solution

    still survive, not unique
- ② Particular solution
  - Try functions that look similar to $f(t)$ with enough degree of freedom i.e., $At^2 + Bt + C = t^2$

    procedure
- ① Homogeneous solution
  - Homogeneous equation ➔ $y = e^{st}$ ➔ (turn into) characteristic equation ➔ $s_1, s_2, \ldots, s_{n-1}, s_n$ roots ➔ $A_1 e^{s_1 t}, A_2 e^{s_2 t}, \ldots, A_n e^{s_n t}$ $\Leftarrow$ algebric equation.
  - ③ Problem with the identical roots ➔ $t^m e^{s_k t}$
- Use $\boldsymbol{n}$-degree of freedom $A_1, A_2, \ldots, A_n$ to $\boxed{\text{satisfy } \boldsymbol{n} \text{ constraints}}$

freedom 자유도.

Homogeneous solution

- $\frac{d^2}{dt^2}y + 5\frac{d}{dt}y + 6y = 0$
  - Recall that solution of $s^2 + 5s + 6 = (s+2)(s+3) = 0$ characteristic equation provides e^{-2t} and e^{-3t}.

    linear combination of those two

- What about $\frac{d^2}{dt^2}y + 4\frac{d}{dt}y + 4y = 0$

  unfortunately
  - $(s+2)^2 = 0$ provides only e^{-2t}.
  - Is single homogeneous solution enough?
  - Not in general because we need two free parameters to satisfy two independent constraints.
  - Try te^{-2t}

Homogeneous solution

- What was the problem with $\frac{d^2}{dt^2}y + 4\frac{d}{dt}y + 4y = 0$?
  - $(s+2)^2 = 0$ ➔ duplicate root $s = -2$ ➔ called multiplicity of the root

- Solution for multiple identical root
  - If $s = -2$ is a root of multiplicity of 2, not only e^{-2t}, but te^{-2t} is also a solution.

    $(s-2)^3$
  - If s =2 is a root of multiplicity of 3, e^{2t}, te^{2t}, t^2e^{2t} are solutions.
  - In general, if the multiplicity of the duplicate root s_k is m, $e^{s_k t}, te^{s_k t}, t^2 e^{s_k t}, \ldots, t^{m-1} e^{s_k t}$ are the homogeneous solutions.

Proof for solution for multiple identical roots I

- $\left[c_n \frac{d^n}{dt^n} + c_{n-1} \frac{d^{n-1}}{dt^{n-1}} + \cdots + c_1 \frac{d}{dt} + c_0\right] y(t) = 0$

  ➔ $y_h(t) = e^{st}$ ➔ $[c_n s^n + c_{n-1} s^{n-1} + \cdots + c_1 s + c_0] e^{st} = 0$

- Assume that $s_1, s_2, \ldots s_n$ are the roots of the characteristic equation

  ➔ $(s - s_1)(s - s_2) \ldots (s - s_n) = 0$

  =

  - Recall $\frac{d^2}{dt^2} y(t) = \frac{d}{dt}\left(\frac{d}{dt} y(t)\right) = \left(\frac{d}{dt}\right)^2 y(t)$
  - s corresponds to $\frac{d}{dt}$.
  - The given differential equation (DE) can be re-written as

$$\left(\frac{d}{dt} - s_1\right)\left(\frac{d}{dt} - s_2\right) \ldots \left(\frac{d}{dt} - s_n\right) y(t) = 0$$

simple replace

  - Note that the order of $\left(\frac{d}{dt} - s_m\right)$ factor is not important (cf. but the factor is important)
    - Example: $\left(\frac{d}{dt} - 1\right)\left(\frac{d}{dt} - 2\right) y = \frac{d^2}{dt^2} y - 3\frac{d}{dt} y + 2y = \left(\frac{d}{dt} - 2\right)\left(\frac{d}{dt} - 1\right) y$

Proof for solution for multiple identical roots II

- $(s-s_1)(s-s_2)\ldots(s-s_n)=0$
  - $\left(\frac{d}{dt}-s_1\right)\left(\frac{d}{dt}-s_2\right)\ldots\left(\frac{d}{dt}-s_n\right)y(t)=0$
  - Relation between $\left(\frac{d}{dt}-s_m\right)$ factor and homo. solution $e^{s_m t}$
    - $\left(\frac{d}{dt}-s_m\right)e^{s_m t}=0$
    - We can easily prove that $e^{s_m t}$ satisfies the given DE by shuffling the order of $\left(\frac{d}{dt}-s_k\right)$ factors.
    - $\left\{\left(\frac{d}{dt}-s_1\right)\ldots\left(\frac{d}{dt}-s_{m-1}\right)\left(\frac{d}{dt}-s_{m+1}\right)\ldots\left(\frac{d}{dt}-s_n\right)\right\}\left(\frac{d}{dt}-s_m\right)e^{s_m t}=0$

ex) $\left(\frac{d}{dt}-2\right)y=0 \rightarrow \left(\frac{d}{dt}-2\right)t\,e^{2t}=t^0 e^{2t}$

$\hookrightarrow \underbrace{\left(\frac{d}{dt}-2\right)\left(\frac{d}{dt}-2\right)}_{\text{multiple Identical roots.}} t e^{2t} \Rightarrow 0$

$m \quad m-1$

Proof for solution for multiple identical roots III

- $\frac{d^3}{dt^3}y - 6\frac{d^2}{dt^2}y + 12\frac{d}{dt}y - 8y = \left(\frac{d}{dt} - 2\right)^3 y = 0$
  - $(s-2)^3 = 0$ ➔ duplicate root $s = 2$
  - When s =2 is a root of multiplicity of 3, e^{2t}, te^{2t}, t^2e^{2t} are solutions.
  - Example
    - $\left(\frac{d}{dt} - 2\right)e^{2t} = 2e^{2t} - 2e^{2t} = 0$ ① 곱미분 ② 거치는 방법 → multiply "t"
    - $\left(\frac{d}{dt} - 2\right)te^{2t} = e^{2t} + t(2e^{2t}) - 2(te^{2t}) = e^{2t}$
    - $\left(\frac{d}{dt} - 2\right)t^2e^{2t} = (2t)e^{2t} + t^2(2e^{2t}) - 2(t^2e^{2t}) = 2te^{2t}$
    - By combining the above three, $\left(\frac{d}{dt} - 2\right)^3 t^2e^{2t} = 0$
  - In general, when $l < m$, $\left(\frac{d}{dt} - s_k\right)^m t^l e^{s_k t} = 0 \Leftrightarrow (s - s_k)^m = 0$

Exercises for multiplicity of roots

- Find homogeneous solutions
  - $\left(\frac{d}{dt}-2\right)\left(\frac{d}{dt}+3\right)y(t)=0$ ⇒ $Ae^{2t}+Be^{-3t}$
  - $\left(\frac{d}{dt}-1\right)^2 y(t)=0$ ⇒ $Ae^{t}+Bte^{t}$
  - $\left(\frac{d}{dt}+2\right)^2\left(\frac{d}{dt}-2\right)y(t)=0$ $Ae^{-2t}+Bte^{-2t}+Ce^{2t}$

easy

결론

$y=Ate^{-2t}$ 를 put in 해.

- Find particular solution
  - $\frac{d}{dt}y+2y=e^{-2t}$

put this $y=Ae^{-2t}$ to the homo. part. ⇒ $\frac{dy}{dt}+2y=-2Ae^{-2t}+2Ae^{-2t}=0$

- we want $0e^{-2t}$ but became
- particular solution overwrapped by h.

But this don't.

Q. what's the problem?

$s=\pm 1$ $\quad Ae^{t}+Be^{-t}$

$s=\pm j$ $\quad Ae^{jt}+Be^{-jt} \Rightarrow (A+B)\cos t + (A-B)j\sin t$.

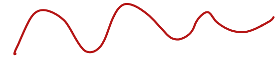
$\Downarrow$ $\cos t + j\sin t$ $\quad$ $\Downarrow$ $\cos t - j\sin t$

Oscillation

- Can you guess the solution for $\frac{d^2}{dt^2}y = +y$?

$s^2=1 \rightarrow s=\pm 1 \rightarrow Ae^{t}+Be^{-t}$

sin, cos,



- Can you guess the solution for $\frac{d^2}{dt^2}y = -y$?

$s^2=-1 \rightarrow s=\pm j \rightarrow Ae^{jt}+Be^{-jt}$

- When do we observe oscillation in a daily life?

Meaning of complex roots

- Recall
  - To find a particular solution for $\frac{d}{dt}y + 2y = \cos \omega t$,
  - problem was converted to $\frac{d}{dt}y + 2y = e^{j\omega t}$
  - The solution $Ae^{j\omega t}$ had complex exponent $j\omega t$ and represented a sinusoidal result

- If the root of characteristic equation has an imaginary part, the result will include oscillation

Oscillation

- Find a solution for $\frac{d^2}{dt^2}y(t) + \omega_0^2 y(t) = 0$ with the following initial conditions (assume $\omega_0 > 0$)
  - $y(0) = 1$ and $\left.\frac{dy}{dt}\right|_{t=0} = 0$
  - 2nd order differential equation requires two initial conditions
  - $s^2 = -\omega_0^2 \Rightarrow s = \pm j\omega_0 \Rightarrow y(t) = Ae^{j\omega_0 t} + Be^{-j\omega_0 t}$
  - $\frac{d}{dt}y(t) = Aj\omega_0 e^{j\omega_0 t} + B(-j\omega_0)e^{-j\omega_0 t}$
  - To satisfy $\left.\frac{dy}{dt}\right|_{t=0} = 0$, $Aj\omega_0 + B(-j\omega_0) = 0 \Rightarrow B = A$
  - $\Rightarrow y(t) = A\left(e^{j\omega_0 t} + e^{-j\omega_0 t}\right)$
  - To satisfy $y(0) = 1$, $2A = 1 \Rightarrow y(t) = \frac{1}{2}\left(\underbrace{e^{j\omega_0 t} + e^{-j\omega_0 t}}_{2\cos w_0 t \text{ (by euler)}}\right) = \cos\omega_0 t$
  - With the real-valued initial conditions, the result also should be real.

※ After solving, imaginary part will disappear!
⇔ If there exist, you should did some mistake.

(Tn) HW. verify? Just plugin.

Damped oscillation

- Find a solution for $\frac{d^2}{dt^2}y + 2\frac{d}{dt}y + (1+\omega_0^2)y = 0$ (assume $\omega_0 > 0$)
  - Initial condition: $y(0) = 1$ and $\left.\frac{dy}{dt}\right|_{t=0} = -1$
  - $s^2 + 2s + (1+\omega_0^2) = 0$
  - Quadratic formula: $s = \frac{-b\pm\sqrt{b^2-4ac}}{2a} = \frac{1}{2}\left(-2 \pm \sqrt{4-4(1+\omega_0^2)}\right) = -1 \pm j\omega_0$

  $\Rightarrow y(t) = Ae^{(-1+j\omega_0)t} + Be^{(-1-j\omega_0)t} = e^{-t}\left(Ae^{j\omega_0 t} + Be^{-j\omega_0 t}\right)$

  - To satisfy $y(0) = 1$, $A + B = 1$

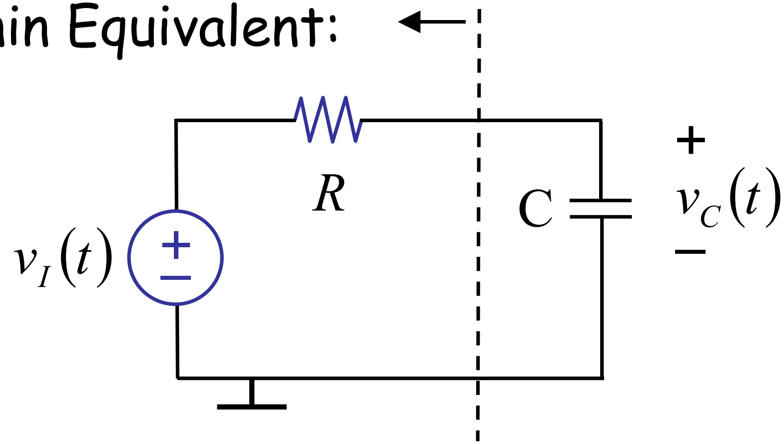
  $\Rightarrow y(t) = e^{-t}\left(Ae^{j\omega_0 t} + (1-A)e^{-j\omega_0 t}\right) = e^{-t}\left(2jA\sin\omega_0 t + e^{-j\omega_0 t}\right)$

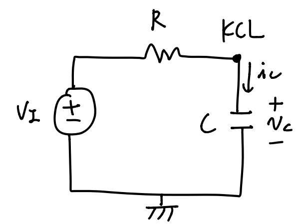
  - $\frac{d}{dt}y(t) = e^{-t}\left\{2jA\omega_0\cos\omega_0 t - j\omega_0 e^{-j\omega_0 t}\right\} - e^{-t}\left\{2jA\sin\omega_0 t + e^{-j\omega_0 t}\right\}$
  - To satisfy $\left.\frac{dy}{dt}\right|_{t=0} = -1$, $2jA\omega_0 - j\omega_0 - 1 = -1 \Rightarrow A = \frac{1}{2} \Rightarrow B = \frac{1}{2}$
  - $\Rightarrow y(t) = e^{-t}\frac{\left(e^{j\omega_0 t} + e^{-j\omega_0 t}\right)}{2} = e^{-t}\cos\omega_0 t$
  - With the real-valued initial conditions, the result also should be real.
- Plot of (exponential) x (sinusoidal)
- If non-homogeneous part $f(t)$ is (exponential) x (sinusoidal), use Euler relation.

Initial condition.

Analysis of an RC circuit (Section 10.1)

Thévenin Equivalent:





$(t=0)$

$v_c(t=0) = v_0$

- $Q = Cv_c \Rightarrow Q_0 = Cv_0$
- KCL, $\frac{v_c - v_I}{R} + i_c = 0$
- $i_c = C \cdot \frac{dv_c}{dt}$

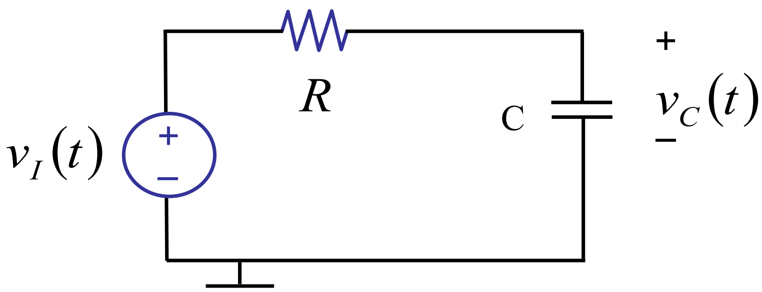
Node method:

$$\frac{v_C - v_I}{R} + C\frac{dv_C}{dt} = 0$$

$$RC\frac{dv_C}{dt} + v_C = v_I \quad \begin{cases} t \geq t_0 \\ v_C(t_0) \text{ given} \end{cases}$$

RC time constant

Example Analysis of an RC circuit



$$v_I(t) = V_I$$

$$v_C(0) = V_0 \quad \text{given}$$

$$RC\,\frac{dv_C}{dt} + v_C = V_I \qquad \text{———} \quad \otimes$$

Example Analysis of an RC circuit

⊗ RC time constant?

$$v_I(t) = V_I$$

$$v_C(0) = V_0 \quad \text{given}$$

(homogeneous)

$$RC \cdot \frac{dv_C}{dt} + v_C = 0.$$

$$RC\frac{dv_C}{dt} + v_C = V_I \quad \text{———} \quad \otimes$$

$$\boxed{\begin{array}{ccccc} v_C(t) & = & v_{CH}(t) & + & v_{CP}(t) \\ \text{total} & & \text{homogeneous} & & \text{particular} \end{array}}$$

Method of homogeneous and particular solutions:

1. Find the homogeneous solution.
2. Find the particular solution.
3. The total solution is the sum of the particular and homogeneous solutions.
4. Use the initial conditions to solve for the remaining constants.

Homogeneous Solution

$$RC\frac{dv_{CH}}{dt}+v_{CH}=0 \quad \text{—— (y)}$$

v_{CH}: solution to the homogeneous (y) equation (set drive to zero)

$v_{CH}=Ae^{st}$ assume solution of this form. A, s ?

$$RC\frac{dAe^{st}}{dt}+Ae^{st}=0 \qquad RCAs\cancel{e^{st}}+A\cancel{e^{st}}=0$$

Discard trivial $A=0$ solution,

$RCs+1=0$ Characteristic equation

$$\Rightarrow \quad s=-\frac{1}{RC}$$

$$\boxed{v_{CH}=Ae^{\frac{-t}{RC}}}$$

RC called time constant τ

Particular Solution

$$RC\,\frac{dv_{CP}}{dt} + v_{CP} = V_I$$

$$\boxed{v_{CP} = V_I}$$ works

$$RC\,\underbrace{\cancel{\frac{dV_I}{dt}}}_{0} + V_I = V_I$$

In general, use trial and error.

v_{CP} : any solution that satisfies the original equation ⊗

Total Solution

$$v_C = v_{CP} + v_{CH}$$

$$v_C = V_I + A\,e^{\frac{-t}{RC}}$$

Find remaining unknown from initial conditions:

Given, $v_C = V_0$ at $\mathrm{t} = 0$

so, $V_0 = V_I + A$

or $A = V_0 - V_I$

thus $$\boxed{v_C = V_I + \left(V_0 - V_I\right)\,e^{\frac{-t}{RC}}}$$

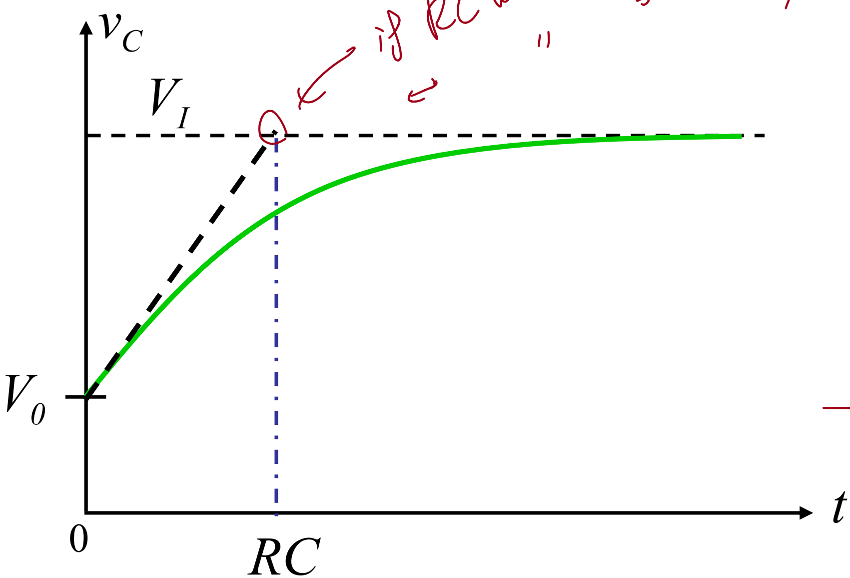
also $$i_C = C\frac{dv_C}{dt} = -\frac{\left(V_0 - V_I\right)}{R}e^{\frac{-t}{RC}}$$

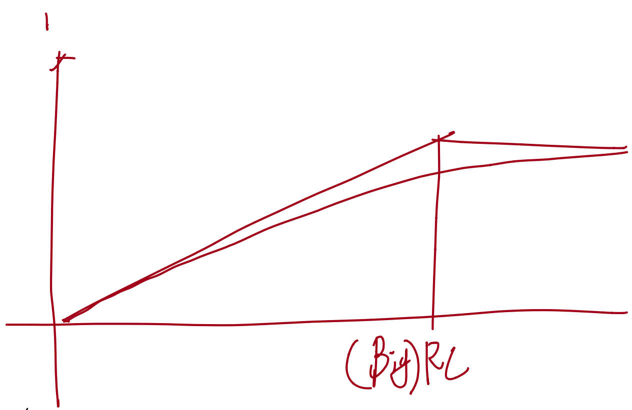
Plot of Total Solution

$$v_C = V_I + (V_0 - V_I)\, e^{\frac{-t}{RC}}$$

if RC was very large ⇒ (slow) (How fast it reaches to assymtotic (final) value) ⇒ circuit will respond.

" short ⇒ sharp





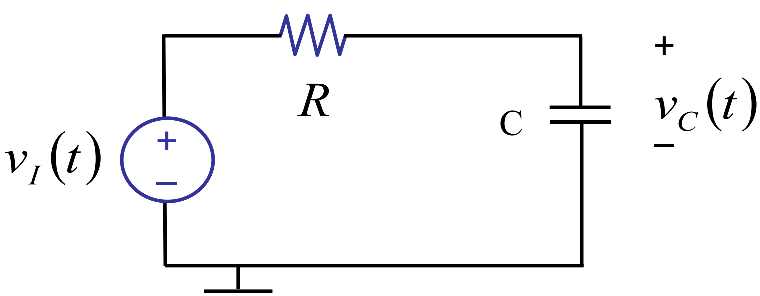
Cf. RL circuit & RC Circuit: mirror image of each other.

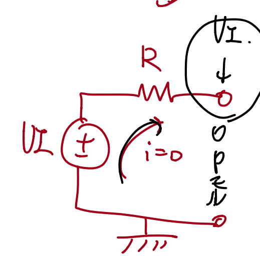
Intuitive Analysis

This should turn into pure ratio.
(Do not imply any of measured value (V, A, Ω, ...)).

- Response to step input
- $v_C = V_I + (V_0 - V_I)e^{-t/RC} = \boxed{V_I\left(1 - e^{-t/RC}\right) + V_0 e^{-t/RC}}$
- At $t = 0$, $1 - e^{-\frac{t}{RC}} = 0$ and $e^{-t/RC} = 1$
- At $t = \infty$, $1 - e^{-\frac{t}{RC}} = 1$ and $e^{-t/RC} = 0$
- Especially at $t = \infty$, treat capacitor as open circuit

간단하게 생각.

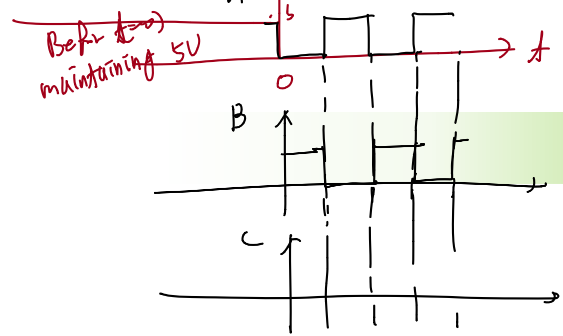


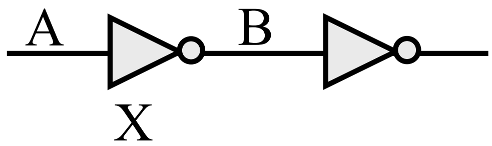


So, Final value is V_I.

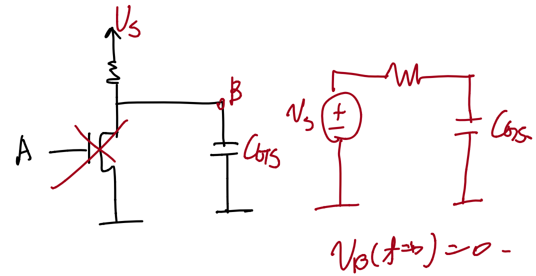
A

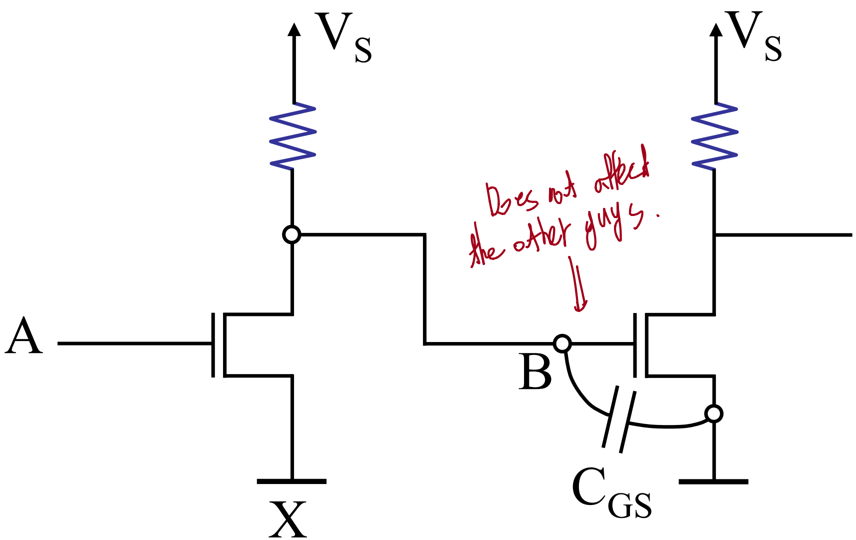
Double Inverter Circuit

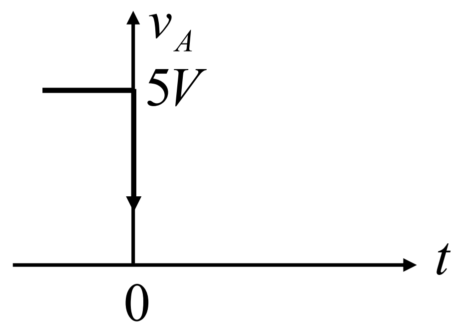




First, rising delay t_r at B



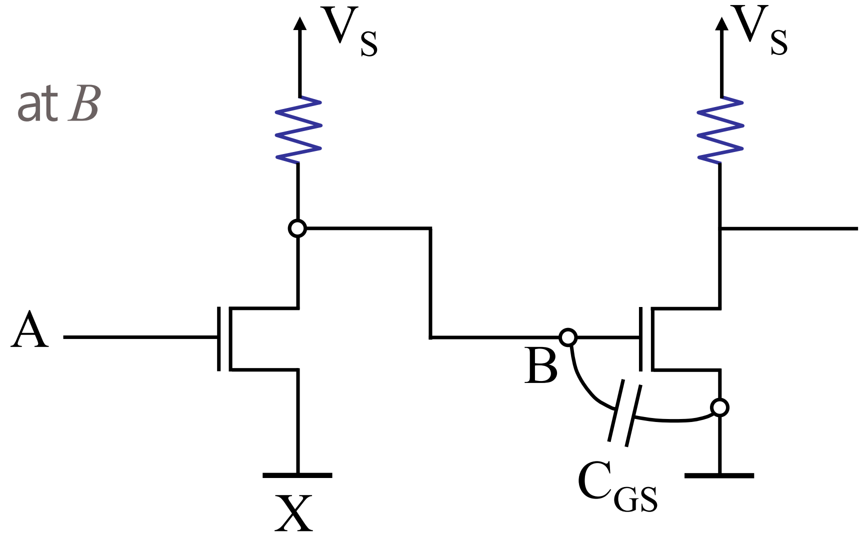


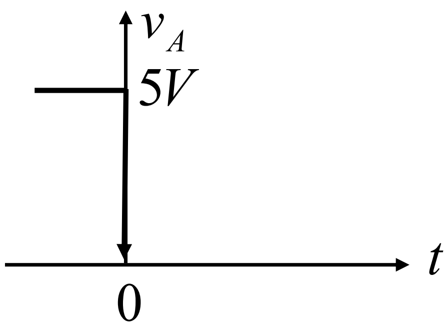


1 → 0 at A

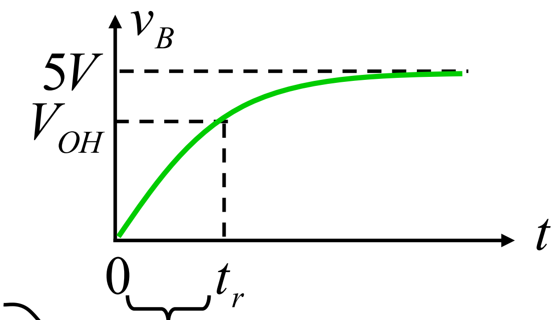
Double Inverter Circuit

Rising Delay t_r at B



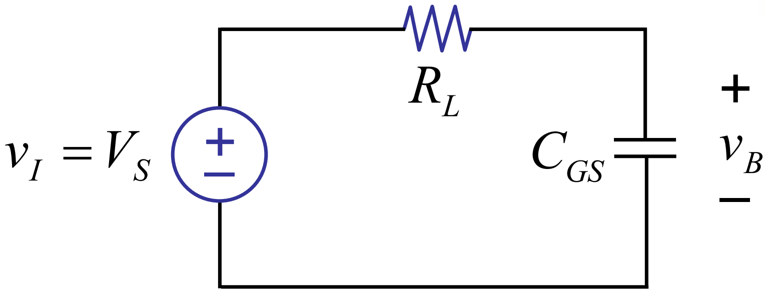


1 → 0 at A



rising delay of X

Equivalent circuit for 0➔1 at B



$$v_I = V_S$$
$$v_B(0) = 0$$
for $t \geq 0$

$$v_B = V_S + (0 - V_S)\, e^{\frac{-t}{R_L C_{GS}}}$$

Now, we need to find t for which $v_B = V_{OH}$.

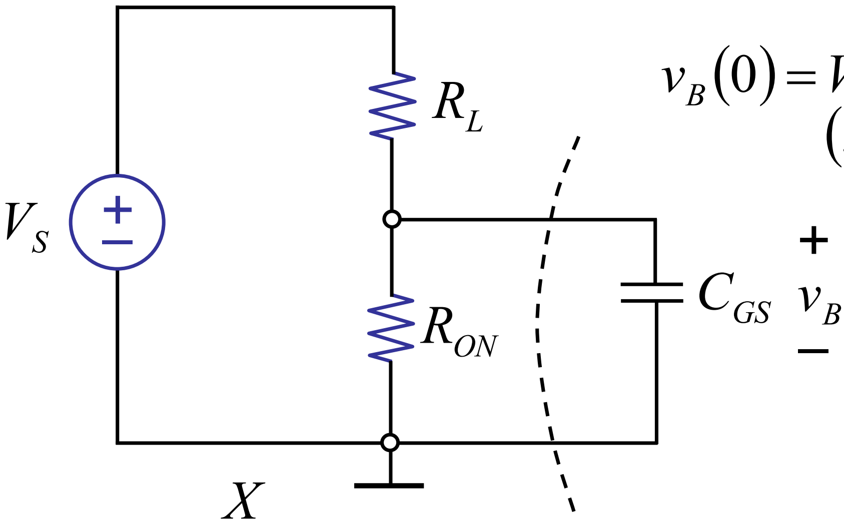
$$v_{OH} = V_S - V_S\, e^{\frac{-t}{R_L C_{GS}}}$$

Find t_r :
$$V_S\, e^{\frac{-t_r}{R_L C_{GS}}} = V_S - V_{OH}$$

$$t_r = -R_L C_{GS} \ln \frac{V_S - V_{OH}}{V_S}$$

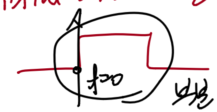
Falling Delay

Equivalent circuit for 1 → 0 at B



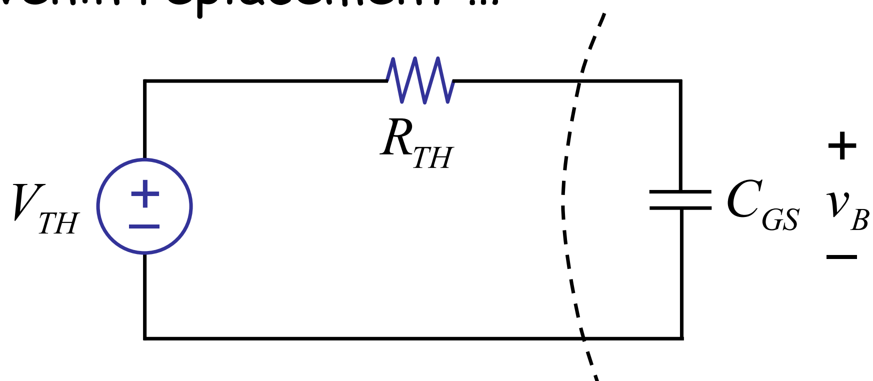
Initial condition을 바꾸다 $(0 \to V_S)$ (5)

$$v_B(0) = V_S$$
$$(5V)$$



보는 관점.

Thévenin replacement …



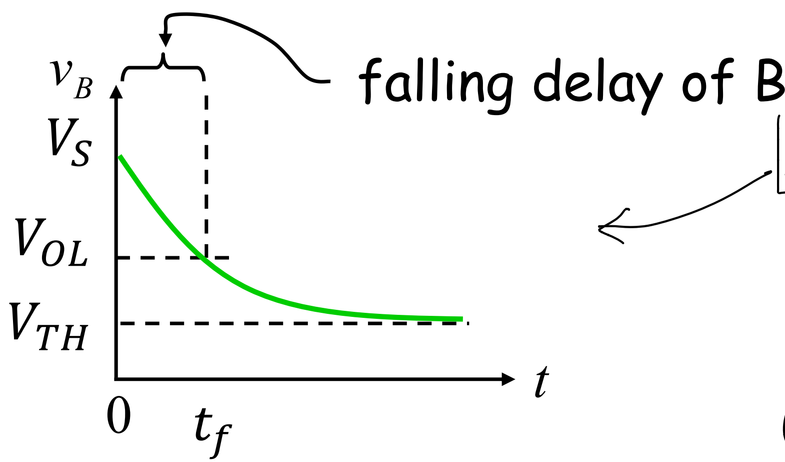
Falling Delay

$$v_B = V_{TH} + (V_S - V_{TH})\, e^{\frac{-t}{R_{TH} C_{GS}}}$$

Falling decay t_f is the t for which v_B falls to V_{OL}

$$V_{OL} = V_{TH} + (V_S - V_{TH})\, e^{\frac{-t_f}{R_{TH} C_{GS}}}$$

or
$$t_f = -R_{TH} C_{GS} \ln \frac{V_{OL} - V_{TH}}{V_S - V_{TH}}$$



$\boxed{\text{Intuitive way ;}}$

Input이 constant일 때,
경험적으로 initial value와 final value를
알아 계산할 수 있다. (differentiation
(개형을 미리 알고). 없이).