Calculus, Differential Equations, and Analysis

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Abstract

Almost every analysis related notes will be added here.

Reference books:

- Calculus:

  [1] J. Stewart, Calculus: early transcendentals, 7th ed. Belmont, CA: Brooks/Cole, Cengage Learning, 2012.

  [1] M. D. Weir, J. Hass, and G. B. Thomas, Thomas' calculus: early transcendentals, Thirteenth edition. Boston: Pearson, 2014.

- Differential Equations:

  [1] C. H. Edwards, D. E. Penney, and D. Calvis, Elementary differential equations with boundary value problems, 6. ed. Upper Saddle River, NJ: Pearson Prentice Hall, 2009.

- Analysis:

  [1] S. Abbott, Understanding analysis, 2nd edition. in Undergraduate texts in mathematics. New York: Springer, 2015.

- TBD

Contents

Chapter 1

First Order Differential Equations

1.1 Introduction

In algebra, we typically seek the unknown *numbers* that satisfy an equation such as $x^3+7x^2-11x+41 = 0$. By contrast, we're challenged to find the unknown $functions y = f(x)$ for which an identity such as:

$$\frac{dy}{dx} = 2xy$$

Example. If C is a constant and

$$y(x) = Ce^{x^2}$$

then

$$\frac{dy}{dx} = 2x(Ce^{x^2}) = 2xy \tag{1}$$

Notice (1) satisfy the DE:

$$\frac{dy}{dx} = 2xy$$

Example (Newton's law of cooling)**.** Let

- T: temperature of a body
- A: temperature of surrounding medium

We have:

$$\frac{dT}{dt} = -k(T - A)$$

Example (Torricelli's law)**.** The *time rate of change* of volume V of water in a draining tank is proportional to the square root of the depth y of water in the tank:

$$\frac{dV}{dy} = -k\sqrt{y}$$

Example. For a DE:

$$\frac{dy}{dx} = y^2$$

The solution can be defined by $y(x) = 1/(C - x)$ for $x \neq C$, because:

$$\frac{dy}{dx} = \frac{1}{(C-x)^2} = y^2$$

Definition 1.1.1 (order). The **order** of a DE is the order of the highest derivative that appears in it.

The most general form of an **nth-order** DE with independent variable x and unknown function or dependent variable $y = y(x)$ is:

$$F(x, y, y', y'', \cdots, y^{(n)}) = 0$$

Definition 1.1.2 (solution). The continuous function $u = u(x)$ is a **solution** of the DE **on the interval** I provided that the derivatives $u', u'', \cdots, u^{(n)}$ exist on I and:

$$F(x, u, u', u'', \cdots, u^{(n)}) = 0$$

for all x in I.

We say $u = u(x)$ satisfies the DE on I.

Definition 1.1.3 (Ordinary an Partial). **Ordinary** DE means that the unknown function (dependent variable) depends on only a *single* independent variable.

If the dependent variable is a function of two or more independent variables, then partial derivatives are likely to be involved; If they are, the equation is called a **partial** DE.

Example (Termal Diffusivity). The temperature $u = u(x, t)$ of a long thin uniform rod at the point x at time t satisfies:

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial^2 x}$$

1.2 Solution for $dy/dx = f(x)$

If the right side of the first order DE does not involve the dependent variable y:

$$\frac{dy}{dx} = f(x) \tag{1}$$

It has a solution by integrating both sides:

$$y(x) = \int f(x)dx + C \tag{2}$$

(2) is the **general solution** to (1).

When bringing up with the initial condition, say $y(x_0) = y_0$, we can solve the constant C, and that is called **particular solution**.

Example.

$$\frac{dy}{dx} = 2x + 3, \quad y(1) = 2$$

A general solution is:

$$y(x) = \int (2x + 3)dx + C = x^2 + 3x + C$$

Considering the initial condition, we have $C = -2$, so a particular solution is:

$$y(x) = x^2 + 3x - 2$$

We can also extend this to **Second-order equations**. For

$$\frac{d^2y}{dx^2} = g(x)$$

We have

$$\frac{dy}{dx} = \int g(x)dx = G(x) + C_1$$

and

$$y(x) = \int [G(x) = C_1]dx = \int G(x)dx + C_1x + C_2$$

1.3 Solution for $dy/dx = f(x, y)$

This form cannot be easily expressed in terms of the ordinary elementary functions. We have to use graphical and numerical methods to construct approximate solutions.

1.3.1 Slope Fields and Graphical Solutions

Consider a function like:

$$y' = f(x, y)$$

At **each** point (x, y) in xy plane, we know its slope m is $m = f(x, y)$.
For a solution $y = y(x)$, each point of it (that is $(x, y(x))$) must have the correct slope.

Example ($y' = x - y$). For $y' = x - y$, let's check for different points:

- $(0, 0) = 0$
- $(0, 1) = -1$
- $(0, -1) = 1$
- $(1, 0) = 1$
- $(-1, 0) = 0$

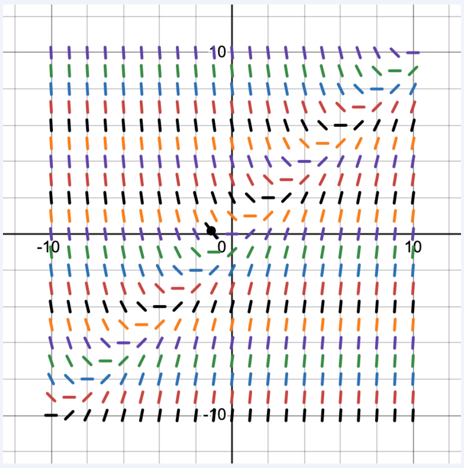

Figure 1.1: Slope fields

If we are assigned the initial condition, we can draw a curve from it.

Theorem 1.3.1 (Solution number). Suppose that both function $f(x, y)$ and its partial derivative $D_y f(x, y)$ are continuous on some rectangle R in the $xy - plane$ that contains the point (a, b) in its interior. The for some open interval I containing the point a, the initial value problem:

$$\frac{dy}{dx} = f(x, y), \quad y(a) = b$$

has one and only one solution that is defined on the interval I.

Example. This example will use the above theorem:

$$x\frac{dy}{dx} = 2y$$

Notice that to rewrite the formula into the form of the theorem, we have:

$$\frac{dy}{dx} = 2y/x$$

so we have $f(x, y) = 2y/x$, thus $\frac{\partial f}{\partial y} = 2/x$. Both functions are continuous if $x \neq 0$.

1.4 Solution for $dy/dx = g(x)h(y)$

The first-order differential equation:

$$\frac{dy}{dx} = H(x, y)$$

is called **separable** provided that $H(x, y)$ can be written as the product of a function of x and a function of y:

$$\frac{dy}{dx} = g(x)h(x) = \frac{g(x)}{f(y)}$$

where $h(y) = 1/f(y)$.

The solution will be like this:

$$\begin{aligned} f(y)dy &= g(x)dx \\ f(y)\frac{dy}{dx} &= g(x) \\ \int f(y(x))\frac{dy}{dx}dx &= \int g(x)dx + C \\ \int f(y)dy &= \int g(x)dx + C \end{aligned}$$

Example. Solve this problem:

$$\frac{dy}{dx} = 7xy, \quad y(0) = 7$$

We can separate:

$$\frac{1}{y}dy = -6xdx$$

Integrate both sides:

$$\begin{aligned} ln|y| &= -3x^2 + C \\ y &= e^{-3x^2+C} \\ y &= Ae^{-3x^2} \end{aligned}$$

Bring in the initial condition we know that $A = 7$, so $y(x) = 7e^{-3x^2}$.

Notice that we drop the absolute value because we know the initial condition, if the initial condition changes, we may want to change it to $-y$.

Example. Sometimes we can not solve y to an explicit form. Suppose we have:

$$\frac{dy}{dx} = \frac{4 - 2x}{3y^2 - 5}$$

This will be solved to:

$$y^3 - 5y = 4x - x^2 + C$$

and we can't make progress here, we call this an *implicit solution*.

Definition 1.4.1 (Sigular Solutions)**.** It is common for a nonlinear first-order DE to have both a general solution involving an arbitrary constant C and one or several particular solutions that cannot be obtained by selecting a value for C. These exceptional solutions are frequently called **singular solutions**.

Example (singular solution)**.** Find all solutions of the DE:

$$\frac{dy}{dx} = 6x(y-1)^{2/3}$$

Separation of the variables gives:

$$\begin{aligned} \int \frac{1}{3(y-1)^{2/3}} dy &= \int 2x dx \\ (y-1)^{1/3} &= x^2 + C \\ y(x) &= 1 + (x^2 + C)^3 \end{aligned} \qquad \text{(general solutions)}$$

But the there's a singular solution $y(x) \equiv 1$ when separating the variables.

1.5 Linear First-Order Equations

Example (Integrating Fatcor)**.** To solve an equation like this:

$$\frac{dy}{dx} = 2xy(y > 0)$$

we multiply both sides by the factor $1/y$ to get

$$\frac{1}{y} \cdot \frac{dy}{dx} = 2x; \quad \textit{that is}, \quad D_x(lny) = D_x(x^2)$$

For this reason, the function $\rho(y) = 1/y$ is called an **integrating factor**.

With the aid of appropriate integrating factor, we have a standard technique to solve the **linear first-order equation**:

$$\frac{dy}{dx} + P(x)y = Q(x)$$

on an interval on which the coefficient functions $P(x)$ and $Q(x)$ are continuous. We multiply both sides with integrating factor

$$\rho(x) = e^{\int P(x)dx}$$

the result is

$$e^{\int P(x)dx}\frac{dy}{dx} + P(x)e^{\int P(x)dx}y = Q(x)e^{\int P(x)dx}$$

notice that the left side is $(ye^{\int P(x)dx})'$, so we have

$$D_x[ye^{\int P(x)dx}] = Q(x)e^{\int P(x)dx}$$

finally:

$$\begin{aligned} y(x)e^{\int P(x)dx} &= \int (Q(x)e^{\int P(x)dx})dx + C \\ y(x) &= e^{-\int P(x)dx}[\int (Q(x)e^{\int P(x)dx})dx + C] \end{aligned}$$

Example. Solve the problem:

$$\frac{dy}{dx} - y = \frac{11}{8}e^{-x/3}, \quad y(0) = -1$$

In this example, $P(x) = -1$, $Q(x) = \frac{11}{8}e^{-x/3}$, the integrating factor is $e^{\int(-1)dx} = e^{-1}$. We can get the general solution:

$$y(x) = -\frac{33}{32}e^{-x/3} + Ce^{x}$$

Bring the initial condition we can get $C = \frac{1}{32}$

About where the solution defined, we have the theorem:

Theorem 1.5.1. If the function $P(x)$ and $P(x)$ are continuous on the open interval I containing the point x_0, then the initial value problem

$$\frac{dy}{dx} + P(x)y = Q(x), \quad y(x_0) = y_0$$

has a unique solution $y(x)$ on I.

Remark. The theorem tells us that every solution is included in the general solution. Thus a linear first-order DE has no singular solutions.

Remark. The appropriate value of the constant in the general solution can be selected "automatically" by writing

$$\rho(x) = exp(\int_{x_0}^{x} P(t)dt),$$
$$y(x) = \frac{1}{\rho(x)}[y_0 + \int_{x_0}^{x} \rho(t)Q(t)dt]$$

Example (Think about the interval!)**.** Solve the initial value problem

$$x^2\frac{dy}{dx} + xy = sinx, \quad y(1) = y_0$$

The function can be transformed to

$$\frac{dy}{dx} + \frac{1}{x}y = \frac{sinx}{x}$$

with $P(x) = \frac{1}{x}$ and $Q(x) = \frac{sinx}{x}$, we have $\rho(x) = exp(\int_1^x \frac{1}{t}dt) = x$, so the particular solution can be given by:

$$y(x) = \frac{1}{x}[y_0 + \int_1^x \frac{sint}{t}dt]$$

1.6 Substitution Methods and Exact Equations