Theory Of Computation

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Abstract

The note is taken by studying 18.404J/6.5400 *Theory of Computation* course by professor Michael Sipser of MIT. The course material can be downloaded in MIT OpenCourseWare. Meanwhile, most contents in this note will also be derived from his book *Introduction to the Theory of Computation, third edition*.

The course is divided into 2 parts, computational theory and complexity theory. Computational theory is developed during 1930s - 1950s. It concerns about what is computable. This note will be focused on the first part.

Example. Program verification, mathematical truth

Example (Models of Computation)**.** Finite automata, Turing machines, $\cdots$

Contents

Chapter 1

Introduction, Finite Automata, Regular Expressions

The theory of computation begins with a question: What is a computer. The real computer is too complicated to understand, to start with, we use an idealized computer called **computational model**.

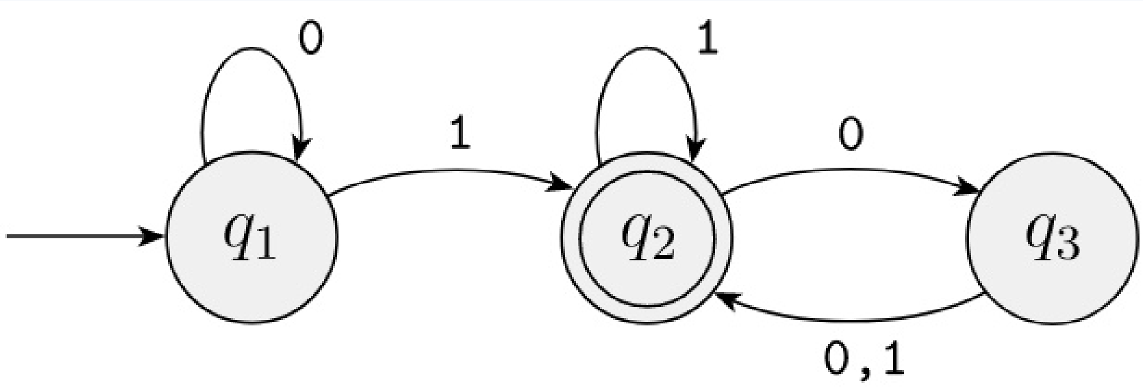
The simplest model among them is **finite state machine** or **finite automaton**.

1.1 Finite Automata

Finite automata are good models for computers with an extremely limited amount of memory.

Finite automata and their probabilistic counterpart **Markov chains** are useful tools when we're attempting to recognize patterns in data. Markov chains have even been used to model and predict price changes in financial markets.

Example (Finite Automata Example). Here's an example of finite automata:



- The figure is called **state diagram** of M_1.
- Three **states**: q_1, q_2 and q_3.
- **Start state**: q_1.
- **Accept state**: q_2.
- The arrows going from one state to another are called **transitions**.

When the automaton receives an input string such as `1101`, it processes that string and produces an output. The output is either **accept** or **reject**:

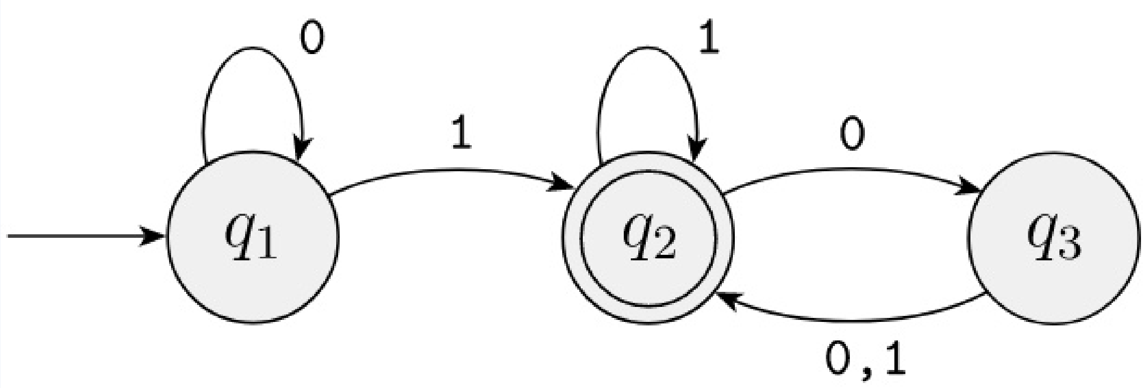
1. Start in state q_1
2. Read `1`, follow transition from q_1 to q_2

3. Read `1`, follow transition from q_2 to q_2
4. Read `0`, follow transition from q_2 to q_3
5. Read `1`, follow transition from q_3 to q_2
6. *Accept* because M_1 is in an accept state q_2 at the end of the input

Definition 1.1.1 (Formal Definition of A Finite Automaton). A **finite automaton** is a 5-tuple $(Q, \Sigma, \sigma, q_0, F)$, where

1. Q is a finite set called **state**
2. Σ is a finite set called the **alphabet**
3. $\sigma : Q \times \Sigma \Rightarrow Q$ is the **transition function**
4. $q_0 \in Q$ is the **start state**
5. $F \subseteq Q$ is the **set of accept state**

Example (Revisit Finite Automata Example). Let's revisit the finite automata example M_1 and see from the formal definition perspective:



We can describe M_1 formally by writing $M_1 = (Q, \Sigma, \sigma, q_1, F)$, where

1. $Q = \{q_1, q_2, q_3\}$
2. $\Sigma = \{0, 1\}$
3. σ is described as

	0	1
q_1	q_1	q_2
q_2	q_3	q_2
q_3	q_2	q_2

4. q_1 is the start state
5. $F = \{q_2\}$

If A is the set of all strings that machine M accepts, we say that A is the **language of machine M** and write $L(M) = A$. We say that **M recognizes A** or that **M accepts A**. Here because *accept* has different meaning, we use *recognize* for the language.

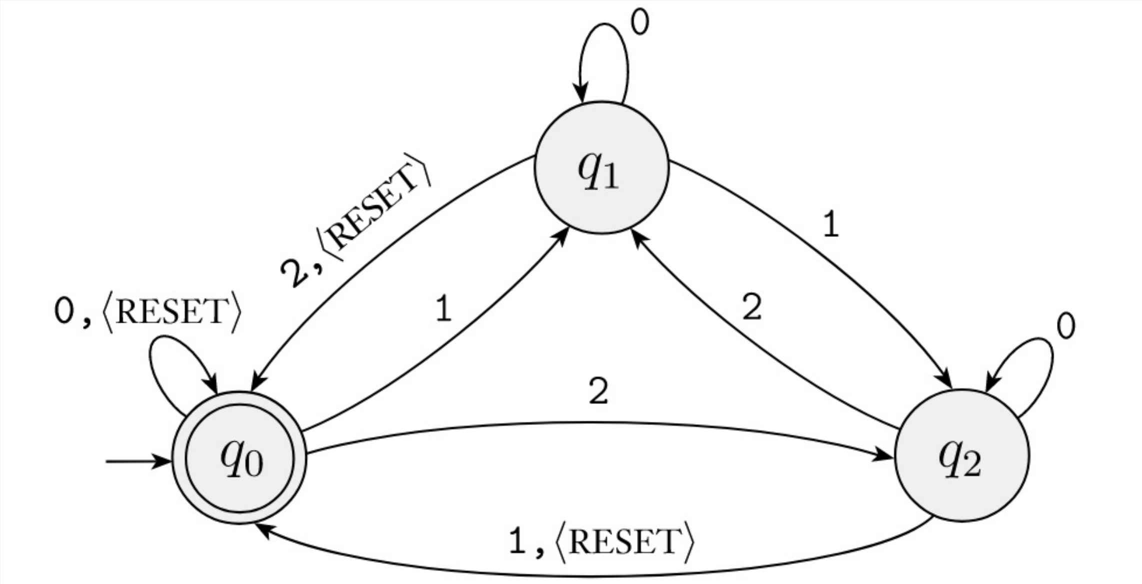
Remark. A machine may accept several strings, but it always recognizes only one language. If the machine accepts no strings, it still recognizes one language – namely, the empty language $\emptyset$.

Example (Revisit Finite Automata Example: Language). In our example, the language set A can be represented as:

A = $\{\omega|\omega$ contains at least one `1` and an even number of `0`s follow the last `1`$\}$.

Then $L(M_1) = A$, or equivalently, M_1 recognizes A.

Example. When describing such a machine:



The alphabet $\Sigma = \{1, 2, 3, < RESET >\}$, we treat $< RESET >$ as a single symbol.

The machine keeps a running count of the sum of the numerical input symbols it reads, modulo 3. Every time it receives $< RESET >$ symbol, it resets the count to 0. It accepts if the sum is 0 modulo 3.

1.2 Formal Definition of Computation

Let $M = (Q, \Sigma, \sigma, q_0, F)$ be a FA and let $\omega = \omega_1\omega_2 \cdots \omega_n$ be a string where each ω_i is a member of alphabet Σ. Then M **accepts** ω if a sequence of state $r_0, r_1, \cdots, r_n$ in Q exists with three conditions:

1. $r_0 = q_0$ (machine starts at initial state)
2. $\sigma(r_i, \omega_{i+1}) = r_{i+1}$ (machine goes form state to state following the transition function)
3. $r_n \in F$ (machine accepts its input if it ends up in an accept state)

We say that M recognizes language A if $A = \{\omega|M accepts \omega\}$

Note. A is the language, ω is the accepted string. A is the set of all instances of ω.

We say a machine "accepts" a string, and a machine "recognizes" a language.

Definition 1.2.1 (Regular Language). A language is called a **regular language** if some finite automaton recognizes it.

Example. Let B = $\{ \ \omega \mid \omega$ has even number of 1s $\}$

B is a regular language.

Example. Let C = $\{ \ \omega \mid \omega$ has equal numbers of 0s and 1s $\}$

C is <u>not</u> a regular language.

1.3 Regular Expressions

1.3.1 Regular Operations

Definition 1.3.1. Let A and B be languages, we define the regular operations **union** , **concatenation** , and **start** as follows:

- Union: $A \cup B = \{x | x \in A || x \in B\}$
- Concatenation: $A \circ B = \{xy | x \in A \& y \in B\}$
- Star: $A^* = \{x_1 x_2 \cdots x_k | k \geq 0 \& x_i \in A\}$

Notice that ϵ(empty language) always belongs to $A*$.

Example. Σ^*1 is the language end with 1

Remark. Show finite automata equivalent to regular expressions.

1.3.2 Closure Properties

Theorem 1.3.1. The class of regular language is closed under the union operation.
In other words, if A_1 and A_2 are regular languages, so is $A_1 \cup A_2$.

Proof. Let $M_1 = \{Q_1, \Sigma, \sigma_1, q_1, F_1\}$ recognize A_1.
Let $M_2 = \{Q_2, \Sigma, \sigma_2, q_2, F_2\}$ recognize A_2. (assuming in the same alphabet to make the proof simple)
Construct $M = (Q, \Sigma, \sigma, q_0, F)$ recognizing $A_1 \cup A_2$.
M should accept input w if either M_1 or M_2 accepts w.
Component of M:

- $Q = Q_1 \times Q_2$
- $q_0 = (q_1, q_2)$
- $\sigma((q, r), a) = (\sigma_1(q, a), \sigma(r, a))$
- $F = (F_1 \times Q_2) \cup (Q_1 \times F2)$
- not $F = \cancel{F_1 \times F2}$ (this gives intersection!)

■

Example (What is close?). Positive integers close under addition but not close under subtraction.

Theorem 1.3.2. The class of regular language is closed under the concatenation operation.
In other words, if A_1 and A_2 are regular languages then so is $A_1 \circ A_2$.