Calculus, Differential Equations, and Analysis

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Abstract

Almost every analysis related notes will be added here.

Reference books:

- Calculus:

  [1] J. Stewart, Calculus: early transcendentals, 7th ed. Belmont, CA: Brooks/Cole, Cengage Learning, 2012.

  [1] M. D. Weir, J. Hass, and G. B. Thomas, Thomas' calculus: early transcendentals, Thirteenth edition. Boston: Pearson, 2014.

- Differential Equations:

  [1] C. H. Edwards, D. E. Penney, and D. Calvis, Elementary differential equations with boundary value problems, 6. ed. Upper Saddle River, NJ: Pearson Prentice Hall, 2009.

- Analysis:

  [1] S. Abbott, Understanding analysis, 2nd edition. in Undergraduate texts in mathematics. New York: Springer, 2015.

- TBD

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Chapter 1

First Order Differential Equations

1.1 Introduction

In algebra, we typically seek the unknown *numbers* that satisfy an equation such as $x^3+7x^2-11x+41=0$. By contrast, we're challenged to find the unknown $functions y = f(x)$ for which an identity such as:

$$\frac{dy}{dx} = 2xy$$

Example. If C is a constant and

$$y(x) = Ce^{x^2}$$

then

$$\frac{dy}{dx} = 2x(Ce^{x^2}) = 2xy \tag{1}$$

Notice (1) satisfy the DE:

$$\frac{dy}{dx} = 2xy$$

Example (Newton's law of cooling)**.** Let

- T: temperature of a body
- A: temperature of surrounding medium

We have:

$$\frac{dT}{dt} = -k(T-A)$$

Example (Torricelli's law)**.** The *time rate of change* of volume V of water in a draining tank is proportional to the square root of the depth y of water in the tank:

$$\frac{dV}{dy} = -k\sqrt{y}$$

Example. For a DE:

$$\frac{dy}{dx} = y^2$$

The solution can be defined by $y(x) = 1/(C-x)$ for $x \neq C$, because:

$$\frac{dy}{dx} = \frac{1}{(C-x)^2} = y^2$$

Definition 1.1.1 (order)**.** The **order** of a DE is the order of the highest derivative that appears in it.

The most general form of an **nth-order** DE with independent variable x and unknown function or dependent variable $y = y(x)$ is:

$$F(x, y, y', y'', \cdots, y^{(n)}) = 0$$

Definition 1.1.2 (solution)**.** The continuous function $u = u(x)$ is a **solution** of the DE **on the interval** I provided that the derivatives $u', u'', \cdots, u^{(n)}$ exist on I and:

$$F(x, u, u', u'', \cdots, u^{(n)}) = 0$$

for all x in I.

We say $u = u(x)$ satisfies the DE on I.

Definition 1.1.3 (Ordinary an Partial)**.** **Ordinary** DE means that the unknown function (dependent variable) depends on only a *single* independent variable.

If the dependent variable is a function of two or more independent variables, then partial derivatives are likely to be involved; If they are, the equation is called a **partial** DE.

Example (Termal Diffusivity)**.** The temperature $u = u(x,t)$ of a long thin uniform rod at the point x at time t satisfies:

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial^2 x}$$

1.2 Solution for $dy/dx = f(x)$

If the right side of the first order DE does not involve the dependent variable y:

$$\frac{dy}{dx} = f(x) \tag{1}$$

It has a solution by integrating both sides:

$$y(x) = \int f(x)dx + C \tag{2}$$

(2) is the **general solution** to (1).

When bringing up with the initial condition, say $y(x_0) = y_0$, we can solve the constant C, and that is called **particular solution**.

Example.

$$\frac{dy}{dx} = 2x + 3, \quad y(1) = 2$$

A general solution is:

$$y(x) = \int (2x+3)dx + C = x^2 + 3x + C$$

Considering the initial condition, we have $C = -2$, so a particular solution is:

$$y(x) = x^2 + 3x - 2$$

We can also extend this to **Second-order equations**. For

$$\frac{d^2y}{dx^2} = g(x)$$

We have

$$\frac{dy}{dx} = \int g(x)dx = G(x) + C_1$$

and

$$y(x) = \int [G(x) = C_1]dx = \int G(x)dx + C_1 x + C_2$$

1.3 Solution for $dy/dx = f(x, y)$

This form cannot be easily expressed in terms of the ordinary elementary functions. We have to use graphical and numerical methods to construct approximate solutions.

1.3.1 Slope Fields and Graphical Solutions

Consider a function like:

$$y' = f(x, y)$$

At **each** point (x, y) in xy plane, we know its slope m is $m = f(x, y)$.
For a solution $y = y(x)$, each point of it (that is $(x, y(x))$) must have the correct slope.

Example ($y' = x - y$). For $y' = x - y$, let's check for different points:

- $(0, 0) = 0$
- $(0, 1) = -1$
- $(0, -1) = 1$
- $(1, 0) = 1$
- $(-1, 0) = 0$

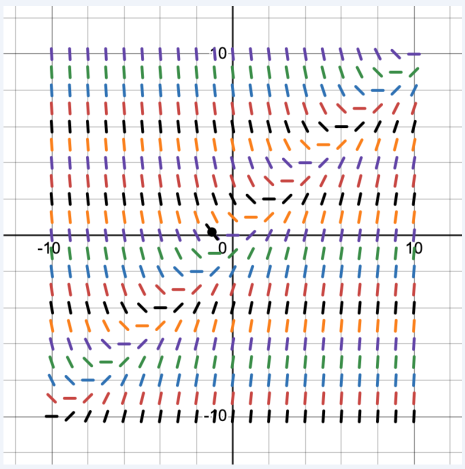

Figure 1.1: Slope fields

If we are assigned the initial condition, we can draw a curve from it.

Theorem 1.3.1 (Solution number). Suppose that both function $f(x, y)$ and its partial derivative $D_y f(x, y)$ are continuous on some rectangle R in the $xy - plane$ that contains the point (a, b) in its interior. The for some open interval I containing the point a, the initial value problem:

$$\frac{dy}{dx} = f(x, y), \quad y(a) = b$$

has one and only one solution that is defined on the interval I.

Example. This example will use the above theorem:

$$x\frac{dy}{dx} = 2y$$

Notice that to rewrite the formula into the form of the theorem, we have:

$$\frac{dy}{dx} = 2y/x$$

so we have $f(x, y) = 2y/x$, thus $\frac{\partial f}{\partial y} = 2/x$. Both functions are continuous if $x \neq 0$.

1.4 Solution for $dy/dx = g(x)h(y)$

The first-order differential equation:

$$\frac{dy}{dx} = H(x, y)$$

is called **separable** provided that $H(x, y)$ can be written as the product of a function of x and a function of y:

$$\frac{dy}{dx} = g(x)h(x) = \frac{g(x)}{f(y)}$$

where $h(y) = 1/f(y)$.

The solution will be like this:

$$\begin{aligned} f(y)dy &= g(x)dx \\ f(y)\frac{dy}{dx} &= g(x) \\ \int f(y(x))\frac{dy}{dx}dx &= \int g(x)dx + C \\ \int f(y)dy &= \int g(x)dx + C \end{aligned}$$

Example. Solve this problem:

$$\frac{dy}{dx} = 7xy, \quad y(0) = 7$$

We can separate:

$$\frac{1}{y}dy = -6xdx$$

Integrate both sides:

$$\begin{aligned} ln|y| &= -3x^2 + C \\ y &= e^{-3x^2+C} \\ y &= Ae^{-3x^2} \end{aligned}$$

Bring in the initial condition we know that $A = 7$, so $y(x) = 7e^{-3x^2}$.

Notice that we drop the absolute value because we know the initial condition, if the initial condition changes, we may want to change it to $-y$.

Example. Sometimes we can not solve y to an explicit form. Suppose we have:

$$\frac{dy}{dx} = \frac{4 - 2x}{3y^2 - 5}$$

This will be solved to:

$$y^3 - 5y = 4x - x^2 + C$$

and we can't make progress here, we call this an *implicit solution*.

Definition 1.4.1 (Sigular Solutions). It is common for a nonlinear first-order DE to have both a general solution involving an arbitrary constant C and one or several particular solutions that cannot be obtained by selecting a value for C. These exceptional solutions are frequently called **singular solutions**.

Example (singular solution). Find all solutions of the DE:

$$\frac{dy}{dx} = 6x(y-1)^{2/3}$$

Separation of the variables gives:

$$\begin{aligned}\int \frac{1}{3(y-1)^{2/3}}dy &= \int 2xdx \\ (y-1)^{1/3} &= x^2 + C \\ y(x) &= 1 + (x^2+C)^3 \qquad \text{(general solutions)}\end{aligned}$$

But the there's a singular solution $y(x) \equiv 1$ when separating the variables.

1.5 Solution for $dy/dx + P(x)y = Q(x)$ (Linear First-Order Equations)

Example (Integrating Fatcor). To solve an equation like this:

$$\frac{dy}{dx} = 2xy(y > 0)$$

we multiply both sides by the factor $1/y$ to get

$$\frac{1}{y} \cdot \frac{dy}{dx} = 2x; \quad \textit{that is}, \quad D_x(lny) = D_x(x^2)$$

For this reason, the function $\rho(y) = 1/y$ is called an **integrating factor**.

With the aid of appropriate integrating factor, we have a standard technique to solve the **linear first-order equation**:

$$\frac{dy}{dx} + P(x)y = Q(x)$$

on an interval on which the coefficient functions $P(x)$ and $Q(x)$ are continuous. We multiply both sides with integrating factor

$$\rho(x) = e^{\int P(x)dx}$$

the result is

$$e^{\int P(x)dx}\frac{dy}{dx} + P(x)e^{\int P(x)dx}y = Q(x)e^{\int P(x)dx}$$

notice that the left side is $(ye^{\int P(x)dx})'$, so we have

$$D_x[ye^{\int P(x)dx}] = Q(x)e^{\int P(x)dx}$$

finally:

$$y(x)e^{\int P(x)dx} = \int (Q(x)e^{\int P(x)dx})dx + C$$
$$y(x) = e^{-\int P(x)dx}[\int (Q(x)e^{\int P(x)dx})dx + C]$$

Example. Solve the problem:

$$\frac{dy}{dx} - y = \frac{11}{8}e^{-x/3}, \quad y(0) = -1$$

In this example, $P(x) = -1$, $Q(x) = \frac{11}{8}e^{-x/3}$, the integrating factor is $e^{\int(-1)dx} = e^{-1}$. We can get the general solution:

$$y(x) = -\frac{33}{32}e^{-x/3} + Ce^{x}$$

Bring the initial condition we can get $C = \frac{1}{32}$

About where the solution defined, we have the theorem:

Theorem 1.5.1. If the function $P(x)$ and $P(x)$ are continuous on the open interval I containing the point x_0, then the initial value problem

$$\frac{dy}{dx} + P(x)y = Q(x), \quad y(x_0) = y_0$$

has a unique solution $y(x)$ on I.

Remark. The theorem tells us that every solution is included in the general solution. Thus a linear first-order DE has no singular solutions.

Remark. The appropriate value of the constant in the general solution can be selected "automatically" by writing

$$\rho(x) = exp(\int_{x_0}^{x} P(t)dt),$$
$$y(x) = \frac{1}{\rho(x)}[y_0 + \int_{x_0}^{x} \rho(t)Q(t)dt]$$

Example (Think about the interval!)**.** Solve the initial value problem

$$x^2\frac{dy}{dx} + xy = sinx, \quad y(1) = y_0$$

The function can be transformed to

$$\frac{dy}{dx} + \frac{1}{x}y = \frac{sinx}{x}$$

with $P(x) = \frac{1}{x}$ and $Q(x) = \frac{sinx}{x}$, we have $\rho(x) = exp(\int_1^x \frac{1}{t}dt) = x$, so the particular solution can be given by:

$$y(x) = \frac{1}{x}[y_0 + \int_1^x \frac{sint}{t}dt]$$

1.6 Substitution Methods and Exact Equations

We have done equations which are separable or linear. Now we introduce **substitution methods** which sometimes transform a given DE to the form we know how to solve.

Example. The problem is in the following:

$$\frac{dy}{dx} = f(x, y), \quad v = \alpha(x, y)$$

We can get $y = \beta(x, v)$ from $v = \alpha(x, y)$ and substitute the original formula:

$$\frac{dy}{dx} = \frac{\partial\beta}{\partial x} + \frac{\partial\beta}{\partial v}\frac{dv}{dx}$$

Then we have:

$$\frac{dy}{dx} = \frac{\partial\beta}{\partial x} + \frac{\partial\beta}{\partial v}\frac{dv}{dx} = f(x, \beta(x, v))$$

Example. Solve the equation:

$$\frac{dy}{dx} = (x + y + 3)^2$$

solution. We have $v = x + y + 3$, that is

$$y = v - x - 3 \tag{1}$$

According to (1), we have:

$$\frac{dy}{dx} = \frac{dv}{dx} - 1 \tag{2}$$

Using v to substitute the original formula we have:

$$\frac{dy}{dx} = v^2 \tag{3}$$

Based on (2) and (3) we have:

$$\frac{dv}{dx} = v^2 + 1$$

Now this is a separable equation, we have:

$$\begin{aligned}
dx &= \frac{dv}{v^2 + 1} \\
x &= tan^{-1}v + C \\
x - C &= tan^{-1}v \\
v &= tan(x - C) \\
y &= tan(x - C) - x - 3
\end{aligned}$$

■

1.6.1 Homogeneous Equations

Definition 1.6.1. A **homogeneous** first-order equation is one that can be written in the form:

$$\frac{dy}{dx} = F(\frac{y}{x})$$

The substitution strategy is to use $v = \dfrac{y}{x}$.

Example. Solve the DE:

$$2xy\frac{dy}{dx} = 4x^2 + 3y^2$$

solution. Transform the DE into:

$$2\frac{dy}{dx} = 4x/y + 3y/x$$

We use $v = y/x$ to substitute the DE, we have:

$$2\frac{dy}{dx} = 4/v + 3v$$

Also, because $v = y/x$ we have:

$$\frac{dy}{dx} = v + \frac{dv}{dx}x$$

So we have:

$$v + \frac{dv}{dx}x = 2/v + 3v/2$$

(omit the rest parts) ■

1.6.2 Bernoulli Equations

Definition 1.6.2. A first-order DE of the form:

$$\frac{dy}{dx} + P(x)y = Q(x)y^n$$

is called a **Bernoulli equation**.

Notice if $n = 0$ or $n = 1$ then this is a linear DE. Otherwise we can do this substitution: $v = y^{1-n}$

Example. Solve this DE:

$$x\frac{dy}{dx} + 6y = 3xy^{4/3}$$

solution. Transform the DE:

$$\frac{dy}{dx} + (6/x)y = 3y^{4/3}$$

So we have $P(x) = 6/x$, $Q(x) = 3$, $v = y^{-1/3}$ that is $y = v^{-3}$.
Substitute and we have:

$$-3v^{-4}\frac{dv}{dx} + \frac{6v^{-3}}{x} = 3v^{-4}$$

We can transform it into a linear DE:

$$\frac{dv}{dx} + (-\frac{2}{x})v = -1$$

This is a linear first-order equations, we can multiply a factor $\rho(x) = e^{\int P(x)dx} = x^{-2}$:

$$\begin{aligned}
x^{-2}\frac{dv}{dx} - \frac{2}{x^3}v &= -x^{-2} \\
(x^{-2}v)' &= -x^{-2} \\
x^{-2}v &= \int -x^{-2} \\
x^{-2}v &= x^{-1} + C \\
v &= x + Cx^2 \\
y &= \frac{1}{(x + Cx^2)^3}
\end{aligned}$$

■

Chapter 2

Linear Equations of Higher Order

2.1 Introduction: Second-Order Linear Equations

Definition 2.1.1 (Second-Order Linear DE). A second-order differential equation in the (unknown) function $y(x)$ is one of the form:

$$G(x, y, y', y'') = 0$$

The DE is said to be **linear** provided that G is linear in the dependent variable y and its derivatives y' and y''.

Thus a linear second-order equation takes the form:

$$A(x)y'' + B(x)y' + C(x)y = F(x)$$

We don't require that $A(x), B(x), C(x)$ and $F(x)$ be linear functions of x.

Remark (Linear). This is linear:

$$e^x y'' + (cosx)y' + (1 + \sqrt{x})y = tan^{-1}x$$

These are not linear:

$$y'' = yy'$$

$$y'' + 3(y')^2 + 4y^3 = 0$$

Example (Homogeneous and Nonhomogeneous). This is nonhomogeneous:

$$x^2 y'' + 2xy' + 3y = cosx$$

Its *associated* homogeneous equation is:

$$x^2 y'' + 2xy' + 3y = 0$$

In general, the homogeneous linear equation associated with the general form is :

$$A(x)y'' + B(x)y' + C(x)y = 0$$

The importance of homogeneous linear equation is that the sum of any two solutions is again a solution, as is any constant multiple of a solution.

Theorem 2.1.1 (Superposition for Homogeneous Equations). For DE:

$$y'' + p(x)y' + q(x)y = 0$$

If we have y_1 and y_2 being two solutions for the above DE, the linear combination:

$$y = c_1 y_1 + c_2 y_2$$

is also a solution for the DE.

Theorem 2.1.2 (Existence and Uniqueness for Linear Equations)**.** Suppose that the functions p, q and f are continuous on the open interval I containing the point a. Then given any 2 numbers b_0 and b_1, the equation:

$$y'' + p(x)y' + q(x)y = f(x)$$

has a unique (that is, one and only one) solution on the entire interval I that satisfies the initial conditions:

$$y(a) = b_0, \quad y'(a) = b_1$$

Example. Verify that the functions

$$y_1(x) = e^x \quad \textit{and} \quad y_2(x) = xe^x$$

are the solutions of the differential equation

$$y'' - 2y' + y = 0$$

and then find a solution satisfying the initial conditions $y(0) = 3$, $y'(0) = 1$

solution. Verify y_1:

$$y'' - 2y' + y = e^x - 2e^x + e^x = 0$$

Verify y_2:

$$y'' - 2y' + y = (x+2)e^x - 2(x+1)e^x + xe^x = 0$$

The general solution will be:

$$y = c_1 e^x + c_2 xe^x$$

Now we can bring in the initial conditions:

$$y(0) = c_1 = 3$$
$$y'(0) = c_1 + c_2 = 1$$

Thus we have $y = 3e^x - 2xe^x$ ■

Definition 2.1.2 (Linear Independence of Two Functions)**.** Two functions defined on an open interval I are said to be **linearly independent** on I provided that neither is a constant multiple of the other.

Based on the second theorem, the homogeneous equation always has 2 linear independent solutions, therefore we have a general solution:

$$y = c_1 y_1 + c_2 y_2$$

Bring 2 initial conditions, we can have:

$$c_1 y_1 + c_2 y_2 = b_0$$
$$c_1 y_1' + c_2 y_2' = b_1$$

It can be written as:

$$\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} b_0 \\ b_1 \end{bmatrix}$$

To make it solvable, we need to have the determinant of the first matrix 0.
This is called the **Wronskian** of f and g:

Definition 2.1.3 (Wronskian). $W = \begin{vmatrix} f & g \\ f' & g' \end{vmatrix} = fg' - f'g$

Theorem 2.1.3 (Wronskian of Solutions). Suppose that y_1 and y_2 are two solutions of the homogeneous second-order linear equation:

$$y'' + p(x)y' + q(x)y = 0$$

on an open interval I on which p and q are continuous.

1. If y_1 and y_2 are linearly dependent, then $W(y_1, y_2) \equiv 0$ on I.
2. If y_1 and y_2 are linearly independent, then $W(y_1, y_2) \neq 0$ at each point of I.

Theorem 2.1.4 (General Solutions of Homogeneous Equations). Let y_1 and y_2 be 2 linearly independent solutions of the homogeneous equation

$$y'' + p(x)y' + q(x)y = 0$$

with p and q continuous on the open interval I. If Y is any solution on I, then there exist numbers c_1 and c_2 such that

$$Y(x) = c_1y_1(x) + c_2y_2(x)$$

for all x in I.

2.1.1 Linear Second-Order Equations with Constant Coefficients

For the general homogeneous equation:

$$ay'' + by' + c = 0$$

Also we observe that :

$$(e^{rx})' = re^{rx}$$

and

$$(e^{rx})'' = r^2e^{rx}$$

Any derivative of e^{rx} is a constant multiple of e^{rx}, we can substitute that in the general form equation:

$$ar^2e^{rx} + bre^{rx} + ce^{rx} = 0$$

We can get:

$$ar^2 + br + c = 0$$

This quadratic equation is called the **characteristic equation**. We care about whether the characteristic equation has 2 distinct roots.

Theorem 2.1.5 (Distict Real Roots). If the roots r_1 and r_2 are real and distinct, then:

$$y(x) = c_1e^{r_1x} + c_2e^{r_2x}$$

is a general solution of the equation.

Example. Find the general solution of

$$2y'' - 7y' + 3y = 0$$

solution. The characteristic equation is:

$$2r^2 - 7r + 3 = 0$$

We got 2 solutions $r_1 = 1/2$ and $r_2 = 3$ are real and distinct, so the general solution is:

$$y(x) = c_1 e^{x/2} + c_2 e^{3x}$$

■

Example. The DE $y'' + 2y' = 0$ has characteristic equation:

$$r^2 + 2r = r(r+2) = 0$$

with 2 distinct real roots $r_1 = 0$ and $r_2 = -2$, then we have the general solution:

$$y(x) = c_1 + c_2 e^{-2x}$$

Theorem 2.1.6 (Repeated Roots)**.** If the characteristic equation has equal (necessarily real) roots $r_1 = r_2$, then

$$y(x) = (c_1 + c_2 x)e^{r_1 x}$$

is the general solution.

Example. To solve initial value problem:

$$y'' + 2y' + y = 0$$

and $y(0) = 5$ and $y'(0) = -3$

solution. The characteristic equation is:

$$r^2 + 2r + 1 = 0$$

We have 2 same roots $r_1 = r_2 = 1$ According to the above theorem, we have:

$$\begin{aligned} y(0) &= c_1 = 5 \\ y'(0) &= -c_1 + c_2 = -3 \end{aligned}$$

which implies that $c_1 = 5$ and $c_2 = 2$. Thus the desired particular solution of the initial value problem is:

$$y(x) = 5e^{-x} + 2xe^{-x}$$

■

2.2 General Solutions of Linear Equations

Second-order linear equations generalizes in a natural way to the general n**th-order linear** differential equation of the form:

$$P_0(x)y^{(n)} + P_1(x)y^{(n-1)} + \cdots + P_{n-1}(x)y' + P_n(x)y = F(x) \tag{1}$$

There are some continuity and not equal to 0 assumptions regarding (1). By divide both sides by $P_0(x)$, we have:

$$y^{(n)} + p_1(x)y^{(n-1)} + \cdots + p_{n-1}(x)y' + p_n(x)y = f(x) \tag{2}$$

and we can get its associated homogeneous form:

$$y^{(n)} + p_1(x)y^{(n-1)} + \cdots + p_{n-1}(x)y' + p_n(x)y = 0 \tag{3}$$

Theorem 2.2.1 (Principle of Superposition for Homogeneous Equations)**.** Let $y_1, y_2, \cdots, y_n$ be n solutions of (3). The linear combination is also the solution for (3):

$$y = c_1y_1 + c_2y_2 + \cdots + c_ny_n$$

Theorem 2.2.2 (Existence and Uniqueness for Linear Equations)**.** Suppose the functions $p_1, p_2 \cdots, p_n$ and f are continuous on the open interval I containing the point a. Then given n numbers $b_0, b_1, \cdots, b_n - 1$, the nth-order linear equation (2) has a unique (that is, one and only one) solution on the entire interval I that satisfies the n initial conditions:

$$y(a) = b_0, y'(a) = b_1, \cdots, y^{(n-1)}(a) = b_{n-1}$$