Calculus, Differential Equations, and Analysis

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Dec. 2024

Abstract

Almost every analysis related notes will be added here.
Reference books:

- Calculus:

  [1] J. Stewart, Calculus: early transcendentals, 7th ed. Belmont, CA: Brooks/Cole, Cengage Learning, 2012.

  [1] M. D. Weir, J. Hass, and G. B. Thomas, Thomas' calculus: early transcendentals, Thirteenth edition. Boston: Pearson, 2014.

- Differential Equations:

  [1] C. H. Edwards, D. E. Penney, and D. Calvis, Elementary differential equations with boundary value problems, 6. ed. Upper Saddle River, NJ: Pearson Prentice Hall, 2009.

- Analysis:

  [1] S. Abbott, Understanding analysis, 2nd edition. in Undergraduate texts in mathematics. New York: Springer, 2015.

- TBD

Contents

Chapter 1

First Order Differential Equations

1.1   Introduction

In algebra, we typically seek the unknown *numbers* that satisfy an equation such as $x^3+7x^2-11x+41 = 0$. By contrast, we're challenged to find the unknown *functions* $y = f(x)$ for which an identity such as:

$$\frac{dy}{dx} = 2xy$$

> **Example.** If C is a constant and
>
> $$y(x) = Ce^{x^2}$$
>
> then
>
> $$\frac{dy}{dx} = 2x(Ce^{x^2}) = 2xy \tag{1}$$
>
> Notice (1) satisfy the DE:
>
> $$\frac{dy}{dx} = 2xy$$

> **Example** (Newton's law of cooling)**.** Let
>
> - T: temperature of a body
>
> - A: temperature of surrounding medium
>
> We have:
>
> $$\frac{dT}{dt} = -k(T - A)$$

> **Example** (Torricelli's law)**.** The *time rate of change* of volume V of water in a draining tank is proportional to the square root of the depth y of water in the tank:
>
> $$\frac{dV}{dy} = -k\sqrt{y}$$

> **Example.** For a DE:
>
> $$\frac{dy}{dx} = y^2$$
>
> The solution can be defined by $y(x) = 1/(C - x)$ for $x \neq C$, because:
>
> $$\frac{dy}{dx} = \frac{1}{(C - x)^2} = y^2$$

> **Definition 1.1.1** (order)**.** The **order** of a DE is the order of the highest derivative that appears in it.
>
> The most general form of an **nth-order** DE with independent variable x and unknown function or dependent variable $y = y(x)$ is:
>
> $$F(x, y, y', y'', \cdots, y^{(n)}) = 0$$

> **Definition 1.1.2** (solution)**.** The continuous function $u = u(x)$ is a **solution** of the DE **on the interval** I provided that the derivatives $u', u'', \cdots, u^{(n)}$ exist on I and:
>
> $$F(x, u, u', u'', \cdots, u^{(n)}) = 0$$
>
> for all x in I.
>
> We say $u = u(x)$ satisfies the DE on I.

> **Definition 1.1.3** (Ordinary an Partial)**.** **Ordinary** DE means that the unknown function (dependent variable) depends on only a *single* independent variable.
>
> If the dependent variable is a function of two or more independent variables, then partial derivatives are likely to be involved; If they are, the equation is called a **partial** DE.
>
> > **Example** (Termal Diffusivity)**.** The temperature $u = u(x, t)$ of a long thin uniform rod at the point x at time t satisfies:
> >
> > $$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial^2 x}$$

1.2 Solution for $dy/dx = f(x)$

If the right side of the first order DE does not involve the dependent variable y:

$$\frac{dy}{dx} = f(x) \tag{1}$$

It has a solution by integrating both sides:

$$y(x) = \int f(x)dx + C \tag{2}$$

(2) is the **general solution** to (1).

When bringing up with the initial condition, say $y(x_0) = y_0$, we can solve the constant C, and that is called **particular solution**.

> **Example.**
>
> $$\frac{dy}{dx} = 2x + 3, \quad y(1) = 2$$
>
> A general solution is:
>
> $$y(x) = \int (2x + 3)dx + C = x^2 + 3x + C$$
>
> Considering the initial condition, we have $C = -2$, so a particular solution is:
>
> $$y(x) = x^2 + 3x - 2$$

We can also extend this to **Second-order equations**. For

$$\frac{d^2 y}{dx^2} = g(x)$$

We have

$$\frac{dy}{dx} = \int g(x)dx = G(x) + C_1$$

and
$$y(x) = \int [G(x) = C_1]dx = \int G(x)dx + C_1 x + C_2$$

1.3 Solution for $dy/dx = f(x, y)$

This form cannot be easily expressed in terms of the ordinary elementary functions. We have to use graphical and numerical methods to construct approximate solutions.

1.3.1 Slope Fields and Graphical Solutions

Consider a function like:
$$y' = f(x, y)$$

At **each** point (x, y) in xy plane, we know its slope m is $m = f(x, y)$.
For a solution $y = y(x)$, each point of it (that is $(x, y(x))$) must have the correct slope.

Example $(y' = x - y$). For $y' = x - y$, let's check for different points:

- $(0, 0) = 0$

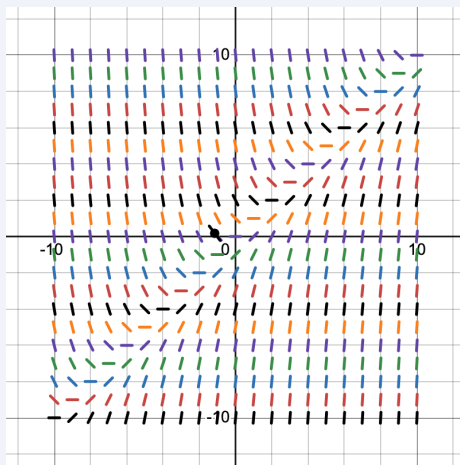- $(0, 1) = -1$

- $(0, -1) = 1$

- $(1, 0) = 1$

- $(-1, 0) = 0$

Figure 1.1: Slope fields

If we are assigned the initial condition, we can draw a curve from it.

Theorem 1.3.1 (Solution number). Suppose that both function $f(x, y)$ and its partial derivative $D_y f(x, y)$ are continuous on some rectangle R in the $xy - plane$ that contains the point (a, b) in its interior. The for some open interval I containing the point a, the initial value problem:
$$\frac{dy}{dx} = f(x, y), \quad y(a) = b$$
has one and only one solution that is defined on the interval I.

Example. This example will use the above theorem:

$$x \frac{dy}{dx} = 2y$$

Notice that to rewrite the formula into the form of the theorem, we have:

$$\frac{dy}{dx} = 2y/x$$

so we have $f(x, y) = 2y/x$, thus $\frac{\partial f}{\partial y} = 2/x$. Both functions are continuous if $x \neq 0$.

1.4 Solution for $\dfrac{dy}{dx} = g(x)h(y)$

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