# Calculus, Differential Equations, and Analysis

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**Abstract**

Almost every analysis related notes will be added here.

Reference books:

- Calculus:

  [1] J. Stewart, Calculus: early transcendentals, 7th ed. Belmont, CA: Brooks/Cole, Cengage Learning, 2012.

  [1] M. D. Weir, J. Hass, and G. B. Thomas, Thomas' calculus: early transcendentals, Thirteenth edition. Boston: Pearson, 2014.

- Differential Equations:

  [1] C. H. Edwards, D. E. Penney, and D. Calvis, Elementary differential equations with boundary value problems, 6. ed. Upper Saddle River, NJ: Pearson Prentice Hall, 2009.

- Analysis:

  [1] S. Abbott, Understanding analysis, 2nd edition. in Undergraduate texts in mathematics. New York: Springer, 2015.

- TBD

# Contents

# Chapter 1

# First Order Differential Equations

## 1.1   Introduction

In algebra, we typically seek the unknown *numbers* that satisfy an equation such as $x^3+7x^2-11x+41 = 0$. By contrast, we're challenged to find the unknown $functions\, y = f(x)$ for which an identity such as:

$$\frac{dy}{dx} = 2xy$$

**Example.** If $C$ is a constant and

$$y(x) = Ce^{x^2}$$

then

$$\frac{dy}{dx} = 2x(Ce^{x^2}) = 2xy \tag{1}$$

Notice (1) satisfy the DE:

$$\frac{dy}{dx} = 2xy$$

**Example** (Newton's law of cooling)**.** Let

- $T$: temperature of a body

- $A$: temperature of surrounding medium

We have:

$$\frac{dT}{dt} = -k(T - A)$$

**Example** (Torricelli's law)**.** The *time rate of change* of volume $V$ of water in a draining tank is proportional to the square root of the depth $y$ of water in the tank:

$$\frac{dV}{dy} = -k\sqrt{y}$$

**Example.** For a DE:

$$\frac{dy}{dx} = y^2$$

The solution can be defined by $y(x) = 1/(C - x)$ for $x \neq C$, because:

$$\frac{dy}{dx} = \frac{1}{(C - x)^2} = y^2$$

> **Definition 1.1.1** (order). The **order** of a DE is the order of the highest derivative that appears in it.
>
> The most general form of an **nth-order** DE with independent variable $x$ and unknown function or dependent variable $y = y(x)$ is:
>
> $$F(x, y, y', y'', \cdots, y^{(n)}) = 0$$

> **Definition 1.1.2** (solution). The continuous function $u = u(x)$ is a **solution** of the DE **on the interval** $I$ provided that the derivatives $u', u'', \cdots, u^{(n)}$ exist on $I$ and:
>
> $$F(x, u, u', u'', \cdots, u^{(n)}) = 0$$
>
> for all $x$ in $I$.
>
> We say $u = u(x)$ satisfies the DE on $I$.

> **Definition 1.1.3** (Ordinary an Partial). **Ordinary** DE means that the unknown function (dependent variable) depends on only a *single* independent variable.
>
> If the dependent variable is a function of two or more independent variables, then partial derivatives are likely to be involved; If they are, the equation is called a **partial** DE.
>
> > **Example** (Termal Diffusivity). The temperature $u = u(x, t)$ of a long thin uniform rod at the point $x$ at time $t$ satisfies:
> >
> > $$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial^2 x}$$

## 1.2 Solution for $dy/dx = f(x)$

If the right side of the first order DE does not involve the dependent variable $y$:

$$\frac{dy}{dx} = f(x) \tag{1}$$

It has a solution by integrating both sides:

$$y(x) = \int f(x)dx + C \tag{2}$$

(2) is the **general solution** to (1).

When bringing up with the initial condition, say $y(x_0) = y_0$, we can solve the constant $C$, and that is called **particular solution**.

> **Example.**
>
> $$\frac{dy}{dx} = 2x + 3, \quad y(1) = 2$$
>
> A general solution is:
>
> $$y(x) = \int (2x + 3)dx + C = x^2 + 3x + C$$
>
> Considering the initial condition, we have $C = -2$, so a particular solution is:
>
> $$y(x) = x^2 + 3x - 2$$

We can also extend this to **Second-order equations**. For

$$\frac{d^2 y}{dx^2} = g(x)$$

We have

$$\frac{dy}{dx} = \int g(x)dx = G(x) + C_1$$

and
$$y(x) = \int [G(x) = C_1] dx = \int G(x) dx + C_1 x + C_2$$

## 1.3  Solution for $dy/dx = f(x, y)$

This form cannot be easily expressed in terms of the ordinary elementary functions. We have to use graphical and numerical methods to construct approximate solutions.

### 1.3.1  Slope Fields and Graphical Solutions

Consider a function like:
$$y' = f(x, y)$$

At **each** point $(x, y)$ in $xy$ plane, we know its slope $m$ is $m = f(x, y)$.

For a solution $y = y(x)$, each point of it (that is $(x, y(x))$) must have the correct slope.

**Example** $(y' = x - y$ ). For $y' = x - y$, let's check for different points:

- $(0, 0) = 0$

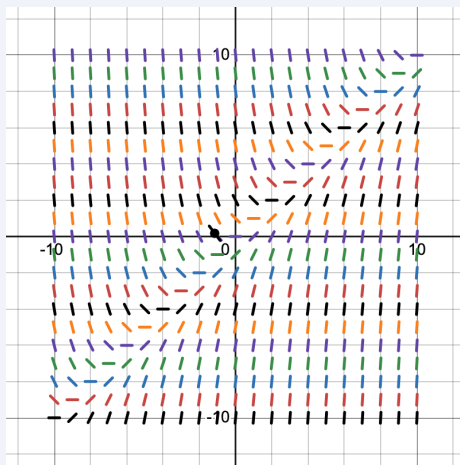- $(0, 1) = -1$

- $(0, -1) = 1$

- $(1, 0) = 1$

- $(-1, 0) = 0$



Figure 1.1: Slope fields

If we are assigned the initial condition, we can draw a curve from it.

**Theorem 1.3.1** (Solution number). Suppose that both function $f(x, y)$ and its partial derivative $D_y f(x, y)$ are continuous on some rectangle $R$ in the $xy - plane$ that contains the point $(a, b)$ in its interior. The for some open interval $I$ containing the point $a$, the initial value problem:

$$\frac{dy}{dx} = f(x, y), \quad y(a) = b$$

has one and only one solution that is defined on the interval $I$.

**Example.** This example will use the above theorem:

$$x\frac{dy}{dx} = 2y$$

Notice that to rewrite the formula into the form of the theorem, we have:

$$\frac{dy}{dx} = 2y/x$$

so we have $f(x, y) = 2y/x$, thus $\frac{\partial f}{\partial y} = 2/x$. Both functions are continuous if $x \neq 0$.

## 1.4 Solution for $dy/dx = g(x)h(y)$

The first-order differential equation:

$$\frac{dy}{dx} = H(x, y)$$

is called **separable** provided that $H(x, y)$ can be written as the product of a function of $x$ and a function of $y$:

$$\frac{dy}{dx} = g(x)h(x) = \frac{g(x)}{f(y)}$$

where $h(y) = 1/f(y)$.

The solution will be like this:

$$f(y)dy = g(x)dx$$
$$f(y)\frac{dy}{dx} = g(x)$$
$$\int f(y(x))\frac{dy}{dx}dx = \int g(x)dx + C$$
$$\int f(y)dy = \int g(x)dx + C$$

**Example.** Solve this problem:

$$\frac{dy}{dx} = 7xy, \quad y(0) = 7$$

We can separate:

$$\frac{1}{y}dy = -6xdx$$

Integrate both sides:

$$ln|y| = -3x^2 + C$$
$$y = e^{-3x^2+C}$$
$$y = Ae^{-3x^2}$$

Bring in the initial condition we know that $A = 7$, so $y(x) = 7e^{-3x^2}$.

Notice that we drop the absolute value because we know the initial condition, if the initial condition changes, we may want to change it to $-y$.

**Example.** Sometimes we can not solve $y$ to an explicit form. Suppose we have:

$$\frac{dy}{dx} = \frac{4 - 2x}{3y^2 - 5}$$

This will be solved to:
$$y^3 - 5y = 4x - x^2 + C$$
and we can't make progress here, we call this an *implicit solution*.

**Definition 1.4.1** (Sigular Solutions). It is common for a nonlinear first-order DE to have both a general solution involving an arbitrary constant $C$ and one or several particular solutions that cannot be obtained by selecting a value for $C$. These exceptional solutions are frequently called **singular solutions**.

**Example** (singular solution). Find all solutions of the DE:
$$\frac{dy}{dx} = 6x(y-1)^{2/3}$$
Separation of the variables gives:
$$\int \frac{1}{3(y-1)^{2/3}} dy = \int 2x dx$$
$$(y-1)^{1/3} = x^2 + C$$
$$y(x) = 1 + (x^2 + C)^3 \qquad \text{(general solutions)}$$
But the there's a singular solution $y(x) \equiv 1$ when separating the variables.

## 1.5   Linear First-Order Equations

**Example** (Integrating Fatcor). To solve an equation like this:
$$\frac{dy}{dx} = 2xy(y > 0)$$
we multiply both sides by the factor $1/y$ to get
$$\frac{1}{y} \cdot \frac{dy}{dx} = 2x; \quad \textit{that is,} \quad D_x(lny) = D_x(x^2)$$
For this reason, the function $\rho(y) = 1/y$ is called an **integrating factor**.

With the aid of appropriate integrating factor, we have a standard technique to solve the **linear first-order equation**:
$$\frac{dy}{dx} + P(x)y = Q(x)$$
on an interval on which the coefficient functions $P(x)$ and $Q(x)$ are continuous. We multiply both sides with integrating factor
$$\rho(x) = e^{\int P(x)dx}$$
the result is
$$e^{\int P(x)dx} \frac{dy}{dx} + P(x)e^{\int P(x)dx}y = Q(x)e^{\int P(x)dx}$$
notice that the left side is $(ye^{\int P(x)dx})'$, so we have
$$D_x[ye^{\int P(x)dx}] = Q(x)e^{\int P(x)dx}$$
finally:
$$y(x)e^{\int P(x)dx} = \int (Q(x)e^{\int P(x)dx})dx + C$$
$$y(x) = e^{-\int P(x)dx}[\int (Q(x)e^{\int P(x)dx})dx + C]$$

**Example.** Solve the problem:

$$\frac{dy}{dx} - y = \frac{11}{8}e^{-x/3}, \quad y(0) = -1$$

In this example, $P(x) = -1$, $Q(x) = \frac{11}{8}e^{-x/3}$, the integrating factor is $e^{\int(-1)dx} = e^{-1}$.
We can get the general solution:

$$y(x) = -\frac{33}{32}e^{-x/3} + Ce^x$$

Bring the initial condition we can get $C = \frac{1}{32}$

About where the solution defined, we have the theorem:

**Theorem 1.5.1.** If the function $P(x)$ and $P(x)$ are continuous on the open interval $I$ containing the point $x_0$, then the initial value problem

$$\frac{dy}{dx} + P(x)y = Q(x), \quad y(x_0) = y_0$$

has a unique solution $y(x)$ on $I$.

**Remark.** The theorem tells us that every solution is included in the general solution. Thus a linear first-order DE has no singular solutions.

**Remark.** The appropriate value of the constant in the general solution can be selected "automatically" by writing

$$\rho(x) = exp(\int_{x_0}^x P(t)dt),$$

$$y(x) = \frac{1}{rho(x)}[y_0 + \int_{x_0}^x \rho(t)Q(t)dt]$$

**Example** (Think about the interval!)**.** Solve the initial value problem

$$x^2\frac{dy}{dx} + xy = sinx, \quad y(1) = y_0$$

The function can be transformed to

$$\frac{dy}{dx} + \frac{1}{x}y = \frac{sinx}{x}$$

with $P(x) = \frac{1}{x}$ and $Q(x) = \frac{sinx}{x}$, we have $\rho(x) = exp(\int_1^x \frac{1}{t}dt) = x$, so the particular solution can be given by:

$$y(x) = \frac{1}{x}[y_0 + \int_1^x \frac{sint}{t}dt]$$

## 1.6   Substitution Methods and Exact Equations