1. Oscillations & Chaos

Outline

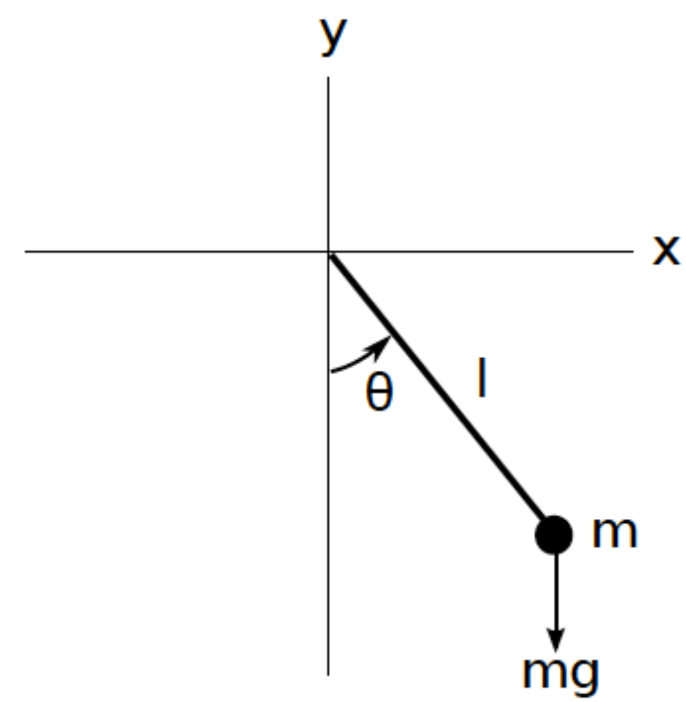
- Linear and nonlinear problems
- The pendulum and the harmonic oscillator
- Symmetries and conservation laws
- Finite difference methods
- Explanation of why some methods conserve energy and others don't: symplectic methods
- Chaotic motion
- Projects

Linear and nonlinear problems

- General analytical methods to solve differential equations rely on the superposition principle, which applies only for linear equations
- There are no general analytic methods for solving nonlinear problems
- Computer simulation works both for linear and nonlinear problems
- In simulation of a given problem issues of numerical accuracy and stability need to be controlled
- Symmetries are important: is the motion time reversal symmetric?
- Conservation laws are important: is energy conserved?

Simplest nonlinear problem: pendulum

$$\ddot{\theta} = -\frac{g}{l}\sin\theta$$



- This equation is obtained as the tangential component of Newtons law of motion:

$$F_\theta = -mg\sin\theta = ma_\theta = ml\ddot{\theta}$$

- Has no analytic solution
- Time reversal symmetry: if $\theta(t)$ is a solution then $\theta(-t)$ is also a solution
- Conservation of energy: E(t)=E(0)

$$E = \frac{m}{2}l^2\dot{\theta}^2 + mgl(1-\cos\theta)$$

$$\frac{dE}{dt} = ml^2\dot{\theta}\left(\ddot{\theta} + \frac{g}{l}\sin\theta\right) = 0$$

Harmonic oscillator

- For small oscillations the nonlinearity can be linearized: $\sin\theta \to \theta$
- Equation of the motion becomes linear:

$$\ddot{\theta} = -\frac{g}{l}\theta \Rightarrow \theta(t) = A\cos\omega t$$

- Angular frequency $\omega = \sqrt{g/l}$

- Energy $E = \frac{m}{2}l^2\dot{\theta}^2 + \frac{1}{2}mgl\theta^2 = \frac{1}{2}mglA^2$

  conserved as for the pendulum

- Prototype model for any system doing bound motion around a stable equilibrium position since

$$V(x) = V(x_0) + \underbrace{\frac{dV(x_0)}{dx}}_{=-F(x_0)=0}(x-x_0) + \frac{1}{2}\underbrace{\frac{d^2V(x_0)}{dx^2}}_{=k}(x-x_0)^2 + O(x-x_0)^3$$

Euler works poorly

- Apply Euler to the harmonic oscillator:

$$\theta(t+\Delta t) = \theta(t) + \dot{\theta}(t)\Delta t$$
$$\dot{\theta}(t+\Delta t) = \dot{\theta}(t) - \frac{g}{l}\theta(t)\Delta t$$

- Test if the Euler method conserves energy:

$$E(t+\Delta t) - E(t) =$$
$$= \left(\frac{m}{2}l^2\dot{\theta}(t+\Delta t)^2 + \frac{1}{2}mgl\theta(t+\Delta t)^2\right) - \left(\frac{m}{2}l^2\dot{\theta}(t)^2 + \frac{1}{2}mgl\theta(t)^2\right) =$$
$$= \left(\frac{m}{2}l^2\left[\dot{\theta}(t) - \frac{g}{l}\theta(t)\Delta t\right]^2 + \frac{1}{2}mgl\left[\theta(t) + \dot{\theta}(t)\Delta t\right]^2\right) - \left(\frac{m}{2}l^2\dot{\theta}(t)^2 + \frac{1}{2}mgl\theta(t)^2\right) =$$
$$= \frac{mgl}{2}\left[\dot{\theta}(t)^2 + \frac{g}{l}\theta(t)^2\right]\Delta t^2$$

- Energy conservation is violated!
- Euler fails a basic requirement of a good integrator: it should conserve energy

Euler-Cromer is much better

$$\dot{\theta}(t+\Delta t) = \dot{\theta}(t) - \frac{g}{l}\theta(t)\Delta t$$

$$\theta(t+\Delta t) = \theta(t) + \dot{\theta}(t+\Delta t)\Delta t$$

- Energy conservation test:

$$E(t+\Delta t) - E(t) =$$

$$= \left(\frac{m}{2}l^2\dot{\theta}(t+\Delta t)^2 + \frac{1}{2}mgl\theta(t+\Delta t)^2\right) - \left(\frac{m}{2}l^2\dot{\theta}(t)^2 + \frac{1}{2}mgl\theta(t)^2\right) =$$

$$= \left(\frac{m}{2}l^2[\dot{\theta}(t) - \frac{g}{l}\theta(t)\Delta t]^2 + \frac{1}{2}mgl[\theta(t) + \underbrace{\dot{\theta}(t+\Delta t)}_{=\theta(t)+\dot{\theta}(t+\Delta t)}\Delta t]^2\right) - \left(\frac{m}{2}l^2\dot{\theta}(t)^2 + \frac{1}{2}mgl\theta(t)^2\right) =$$

$$= \frac{mgl}{2}\left[\dot{\theta}(t)^2 - \frac{g}{l}\theta(t)^2\right]\Delta t^2 - 2\frac{g}{l}\dot{\theta}(t)\theta(t)\Delta t^3 + O(\Delta t)^4$$

- The Δt^2 and Δt^3 terms average to zero over a whole period
- Thus the method is conserves energy to order Δt^4
- Thus Euler-Cromer is much better than Euler

Verlet method

- Forward and backward differences

$$x(t+\Delta t) = x(t) + v(t)\Delta t + \frac{1}{2}a(t)\Delta t^2$$
$$x(t-\Delta t) = x(t) - v(t)\Delta t + \frac{1}{2}a(t)\Delta t^2$$

- Add and subtract

$$x(t+\Delta t) + x(t-\Delta t) = 2x(t) + a(t)\Delta t^2 + O(\Delta t)^4$$
$$x(t+\Delta t) - x(t-\Delta t) = 2v(t)\Delta t + O(\Delta t)^3$$

- Rearranging gives the Leapfrog method

$$x(t+\Delta t) = 2x(t) - x(t-\Delta t) + a(t)\Delta t^2 + O(\Delta t)^4$$
$$v(t) = \frac{x(t+\Delta t) - x(t-\Delta t)}{2\Delta t} + O(\Delta t)^2$$

- Disadvantage: not self starting
- The last eq gives

$$x(t-\Delta t) = x(t+\Delta t) - 2v(t)\Delta t$$

- Using this in Leapfrog gives

$$x(t+\Delta t) = x(t) + v(t)\Delta t + \frac{1}{2}a(t)\Delta t^2$$

- Next write Leapfrog one more timestep forward

$$x(t+2\Delta t) = 2x(t+\Delta t) - x(t) + a(t+\Delta t)\Delta t^2$$
$$v(t+\Delta t) = \frac{x(t+2\Delta t) - x(t)}{2\Delta t}$$

$$\Rightarrow v(t+\Delta t) = \frac{x(t+\Delta t) - x(t)}{\Delta t} + \frac{1}{2}a(t+\Delta t)\Delta t = v(t) + \frac{1}{2}[a(t+\Delta t) + a(t)]\Delta t$$

- Putting it all together gives the Verlet method:

$$\boxed{\begin{aligned} x(t+\Delta t) &= x(t) + v(t)\Delta t + \frac{1}{2}a(t)\Delta t^2 + O(\Delta t)^4 \\ v(t+\Delta t) &= v(t) + \frac{1}{2}[a(t+\Delta t) + a(t)]\Delta t + O(\Delta t)^2 \end{aligned}}$$

Both time reversal symmetric and conserves energy

Runge-Kutta

- A commonly used higher order method is obtained by averaging increments over four points in the time interval:

$$
\begin{gathered}
a_1 = a(x(t), v(t), t)\Delta t, b_1 = v(t)\Delta t \\
a_2 = a(x(t) + b_1/2, v(t) + a_1/2, t + \Delta t/2)\Delta t, b_2 = (v(t) + a_1/2)\Delta t \\
a_3 = a(x(t) + b_2/2, v(t) + a_2/2, t + \Delta t/2)\Delta t, b_3 = (v(t) + a_2/2)\Delta t \\
a_4 = a(x(t) + b_3, v(t) + a_3, t + \Delta t)\Delta t, b_4 = (v(t) + a_3)\Delta t \\
v(t + \Delta t) = v(t) + \frac{1}{6}(a_1 + 2a_2 + 2a_3 + a_4) \\
x(t + \Delta t) = x(t) + \frac{1}{6}(b_1 + 2b_2 + 2b_3 + b_4)
\end{gathered}
$$

- Here a(x(t),v(t),t) is the acceleration at position x(t), velocity v(t), and time t
  - Note that for a 1D system all quantities are scalars
  - For multiple dimensions x, v, $a_?$, $b_?$ are all vectors

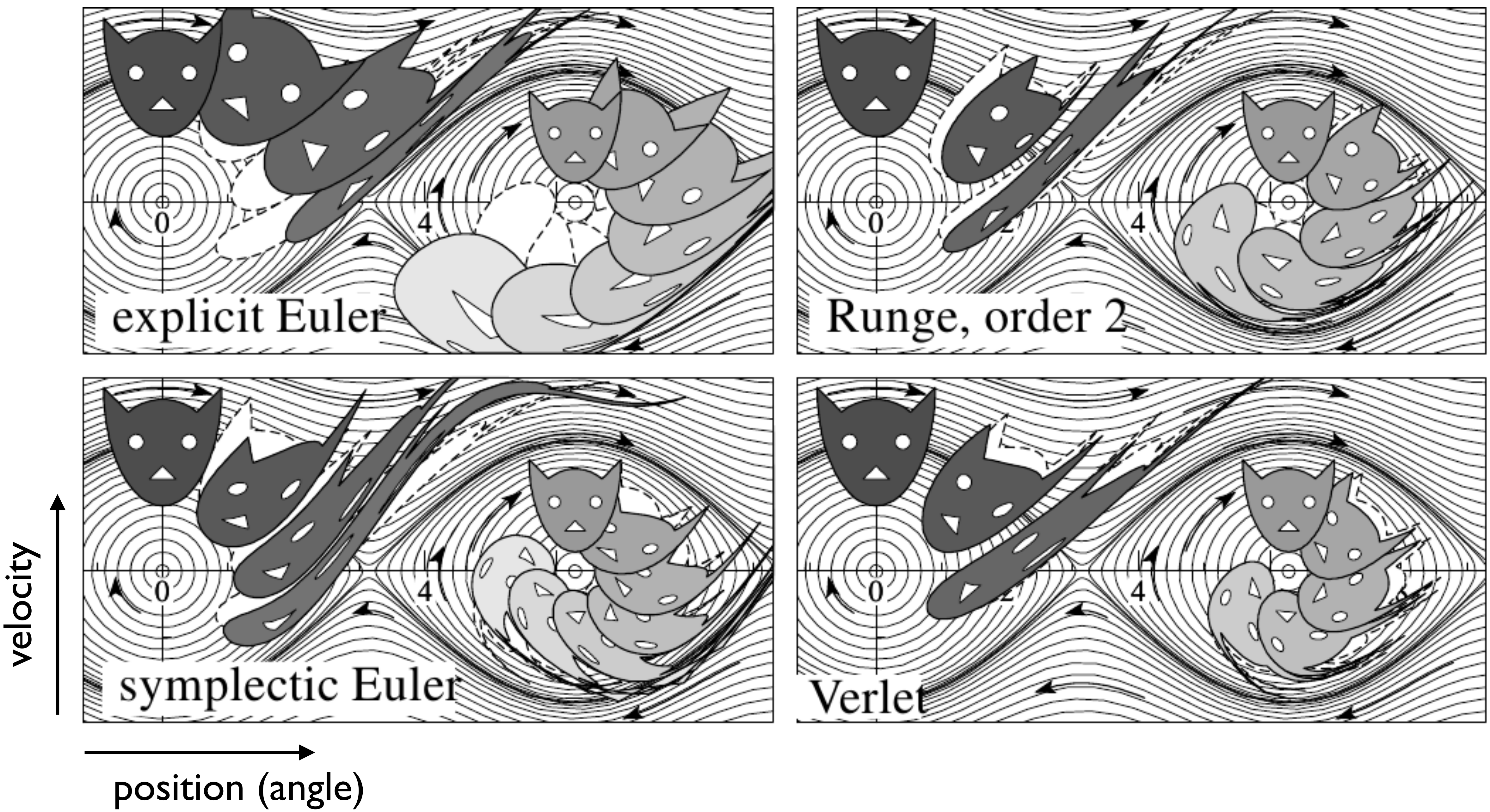
Adaptive time steps

- In problems where the acceleration is velocity dependent the above methods become less efficient
- An example of this is the double pendulum where energy exchanges between two arms to produce intervals of high speed
- In these cases adaptive time steps or predictor-corrector methods can improve performance
- Adaptive time steps means that short time steps are taken where the increments are big, and larger time steps are used when the increments are small
- The required time step can be determined on the fly in the simulation
- But causes problems with energy conservation

Why do some methods conserve energy?

- The Hamiltonian formulation of classical mechanics is based on the observation that
$$F = ma\ ,\ p = mv \Rightarrow \dot{p} = -\frac{\partial H}{\partial x}\ ,\ \dot{x} = \frac{\partial H}{\partial p}$$
where $H = \frac{p^2}{2m} + V(x)$ is called the Hamilton function and gives the total energy
- Symplectic means "area preserving", so that the area element
$dA = |dxdp|$ is unchanged under the map $(x,p) \to (x',p')$ which gives
$$dA' = \left| \left(\frac{\partial x'}{\partial x}\hat{x} + \frac{\partial p'}{\partial x}\hat{p} \right) dx \times \left(\frac{\partial x'}{\partial p}\hat{x} + \frac{\partial p'}{\partial p}\hat{p} \right) dp \right| = |J|\, dA \Rightarrow dA' = dA \text{ if the Jacobian } J = \begin{pmatrix} \frac{\partial x'}{\partial x} & \frac{\partial x'}{\partial p} \\ \frac{\partial p'}{\partial x} & \frac{\partial p'}{\partial p} \end{pmatrix} = 1$$
- This prevents the coordinates and momenta from running away, and as a consequence it is possible to show that a slightly perturbed energy is conserved
- Example: Euler-Cromer is symplectic since $J = \begin{pmatrix} 1 & 0 \\ \frac{\partial F}{\partial x}\Delta t & 1 \end{pmatrix} = 1$
- Verlet and leapfrog are also symplectic
- Euler and Runge-Kutta are not symplectic
- Higher order symplectic methods can be constructed systematically

Symplectic integrators: area preserving



integration of pendulum motion

Chaotic motion

What is chaos?

- In these lectures we study (simple) nonlinear deterministic models that can exhibit chaotic behavior

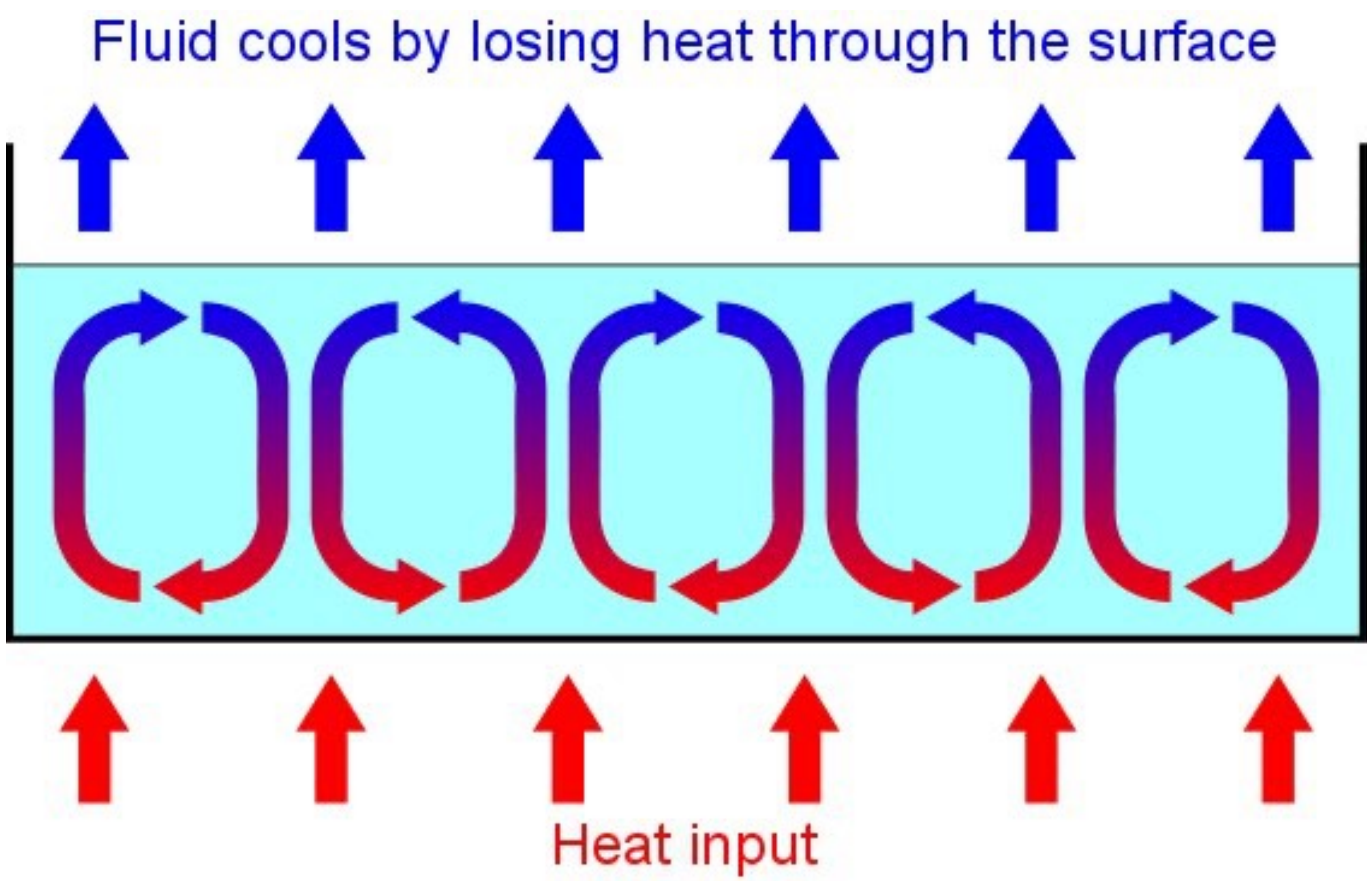
- The difference between chaos and noise (later)?

When do we encounter chaotic motion?

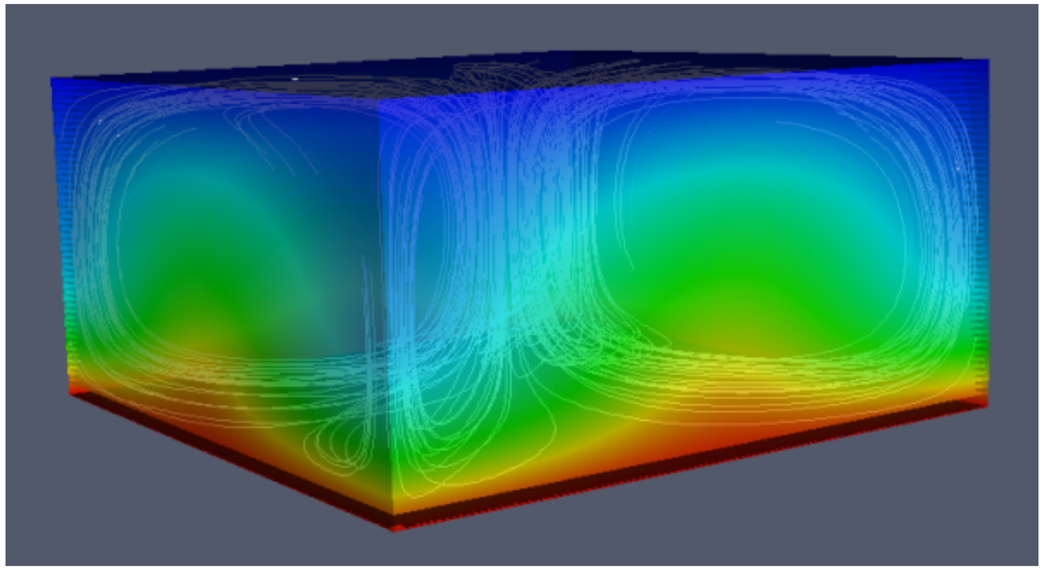
- “Top-down”:
  - You observe chaotic behavior.
    What rules govern this?
- “Bottom-up”:
  - You have a set of non-linear differential equations.
    Are the solutions chaotic or not?
- Example: butterfly effect
  in weather forecasting

Thunderbird Leadership Consulting

Rayleigh-Bénard convection



hot liquid is lighter
and wants to go up



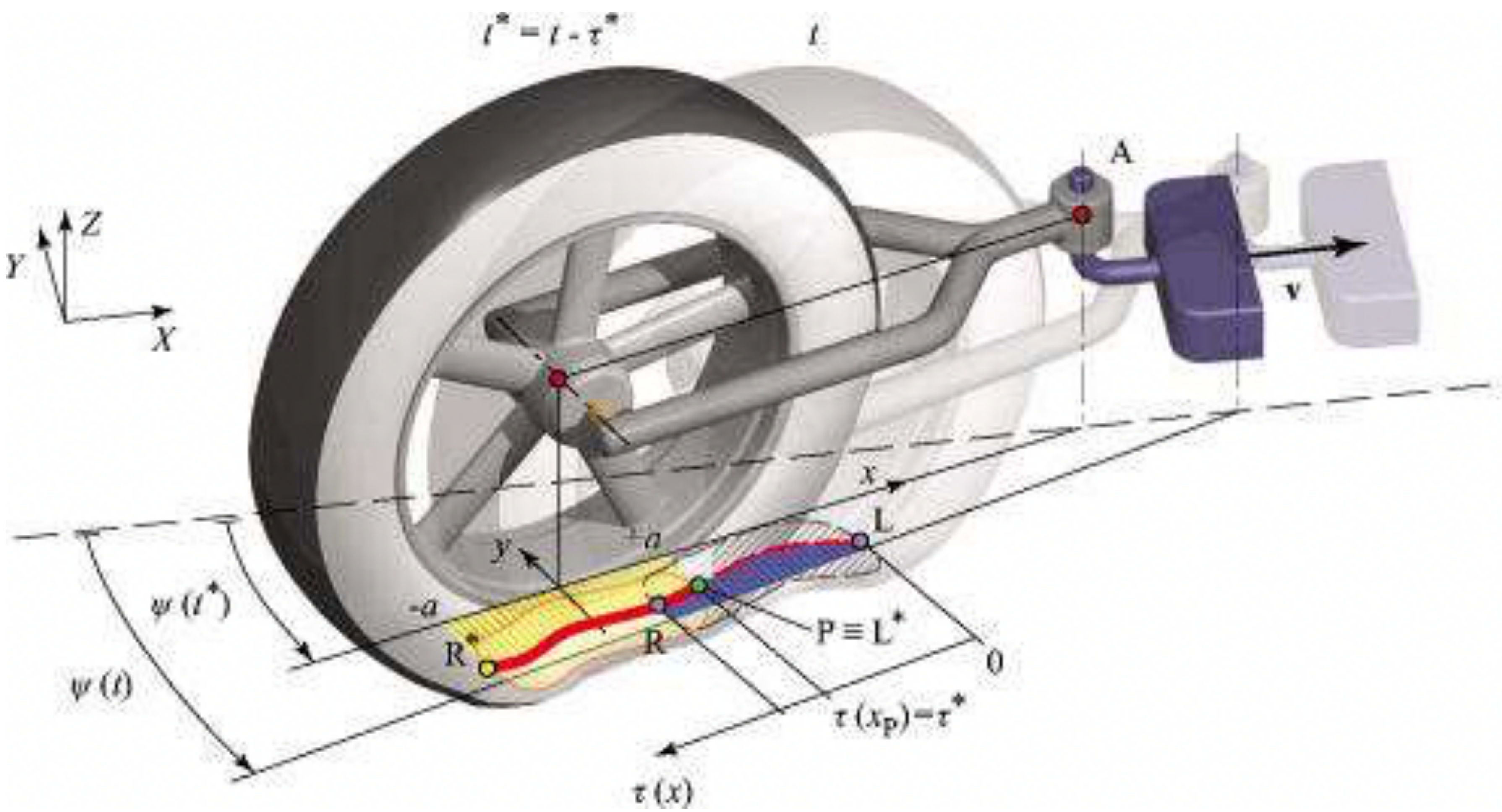
Chaos theory

- For a dynamic system to be chaotic:

1. it must be sensitive to initial conditions
2. it must be topologically mixing
3. it must have dense periodic orbits

Example: speed wobble / shimmy

- The steering wheel(s) on a vehicle, e.g. plane, shopping cart, ...



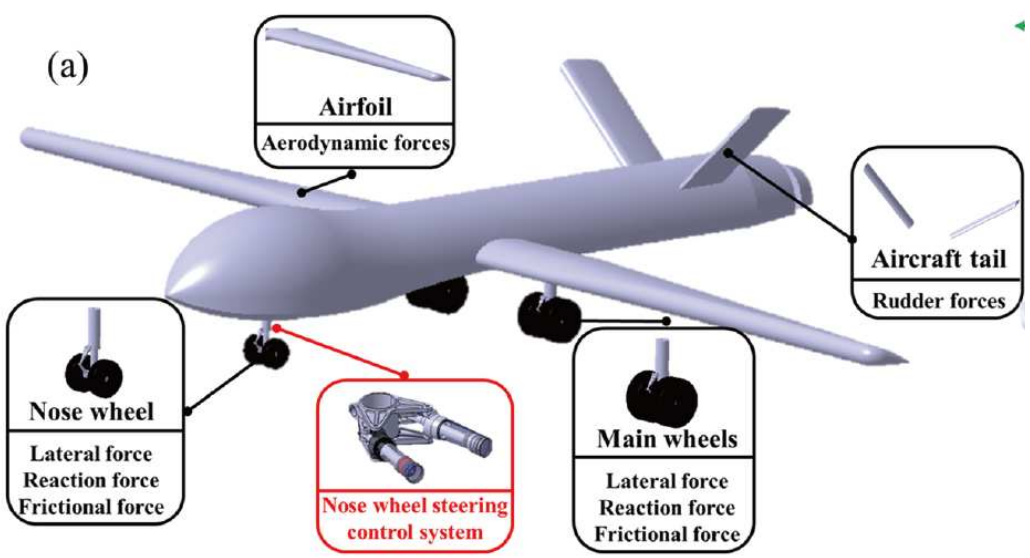
Also problems at low speeds

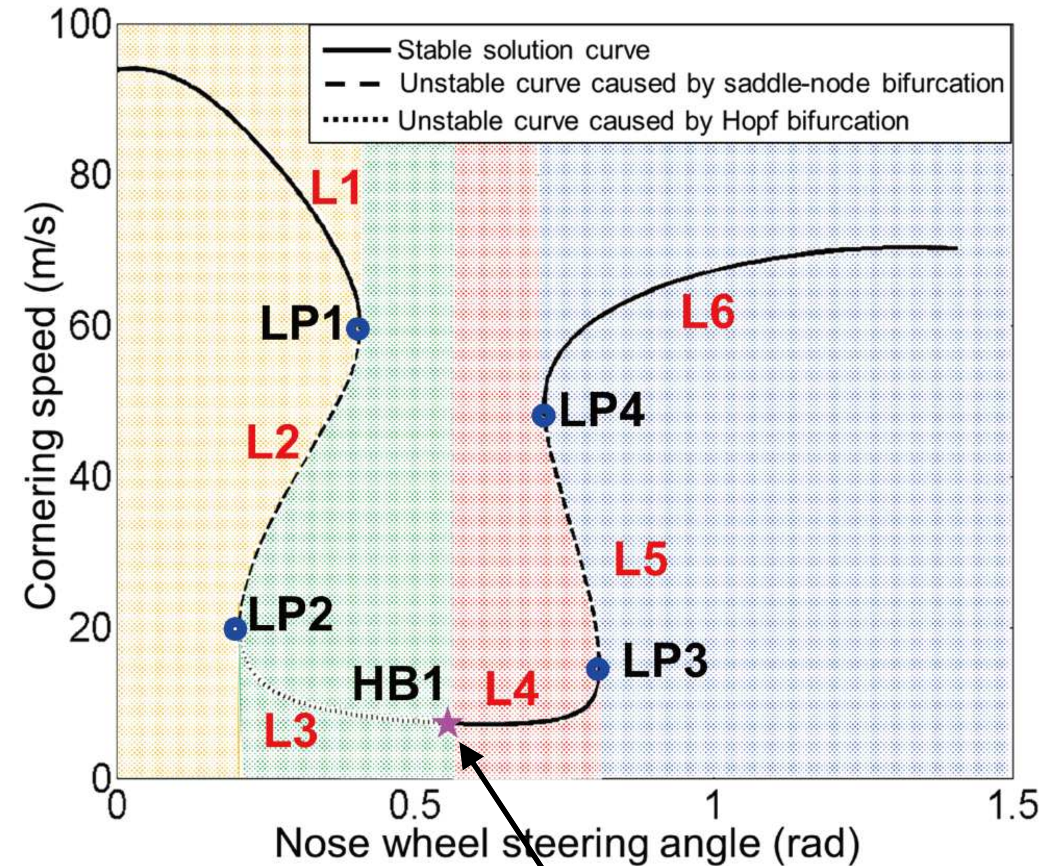
Improving aircraft nose wheel steering stability based on control continuation method

Qiaozhi YIN[1,3], Dexu KONG[4], Jiawei WANG[3], Hong NIE[1,3*] & Xiaohui WEI[2,3]

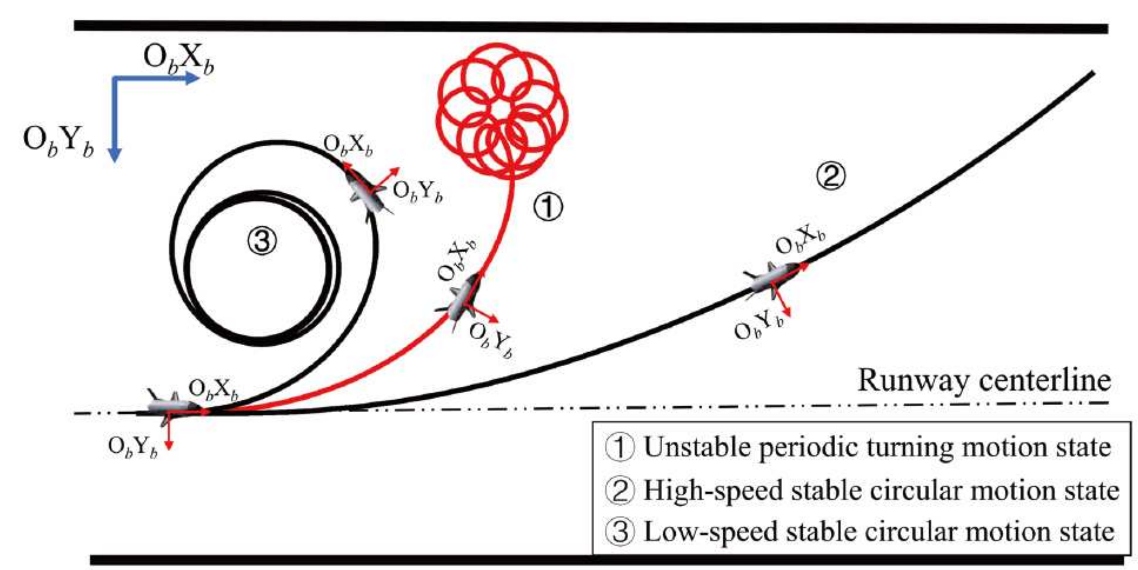
Sci. China Technol. Sci. **68**, 1320602 (2025)

https://doi.org/10.1007/s11071-023-09267-z





Hopf bifurcation



Simple model: population dynamics

$$P_{n+1} = P_n(a - b\,P_n)$$

P_n : population in year

a : coefficient for natural growth

b : coefficient for reduction by overcrowding or spread of disease

Rescaling: $P_n = (a/b)\,x_n$

Define: $r = a/4$

$$x_{n+1} = f(x) = 4\,r\,x_n(1 - x_n)$$

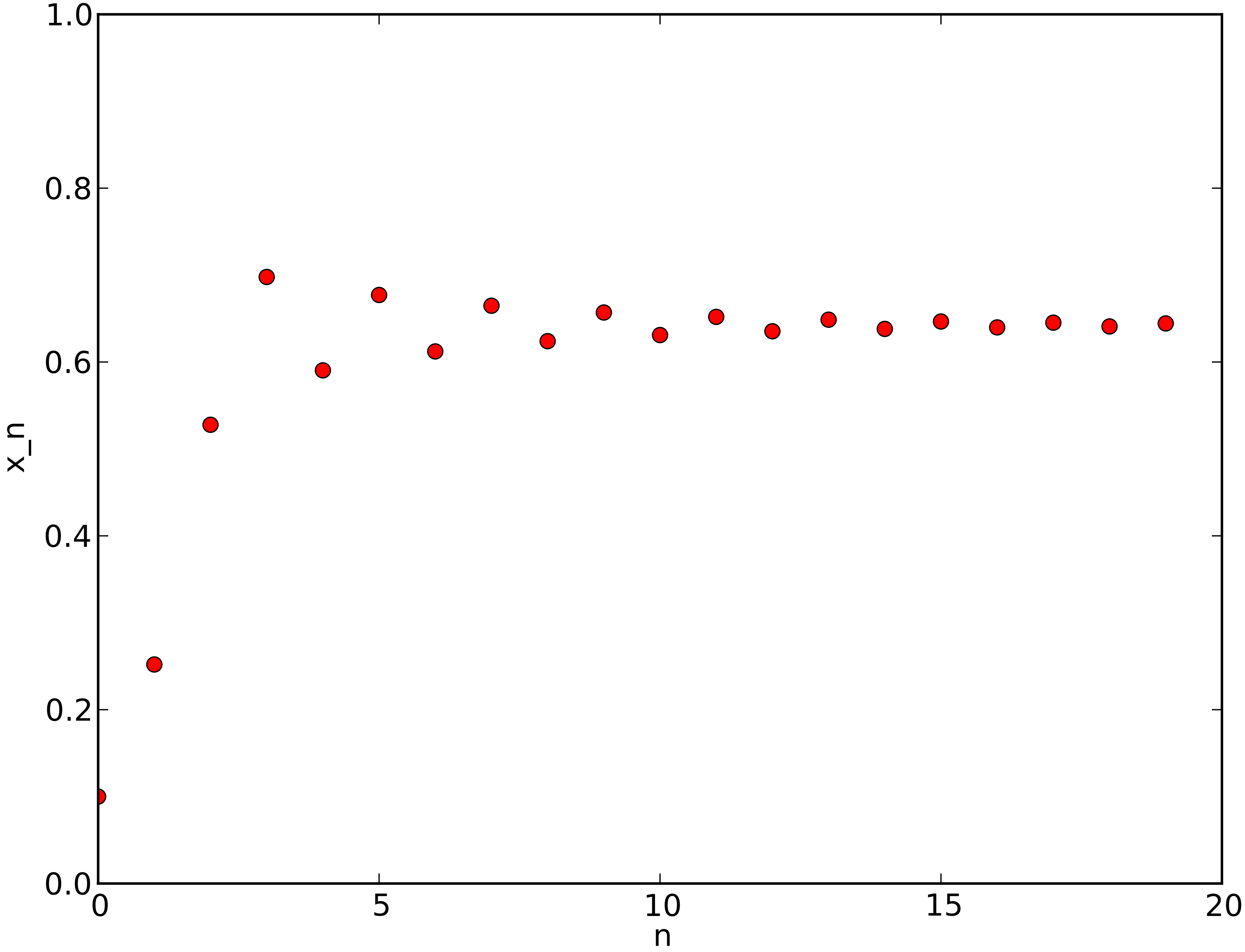
We impose $0 \leq x \leq 1\ ,\ \ 0 < r \leq 1$

Logistic map

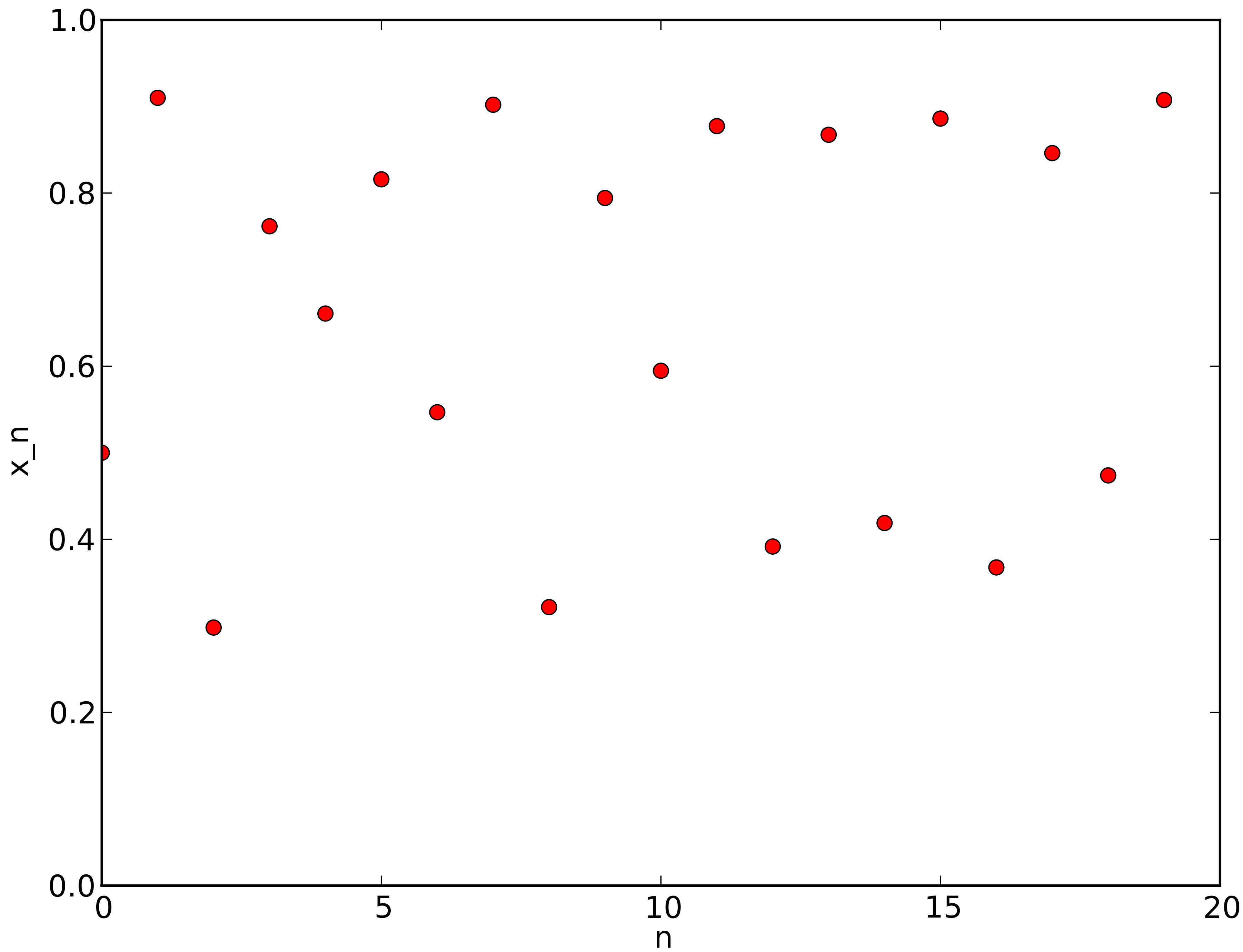
$$x_{n+1} = f(x) = 4\,r\,x_n(1 - x_n)$$

- $f(x)$ is a logistic map
- it maps any point in [0, 1] to another point in [0, 1]
- it is a deterministic prescription to find the future state given the present state of the system

$r = 0.7, x_0 = 0.1$



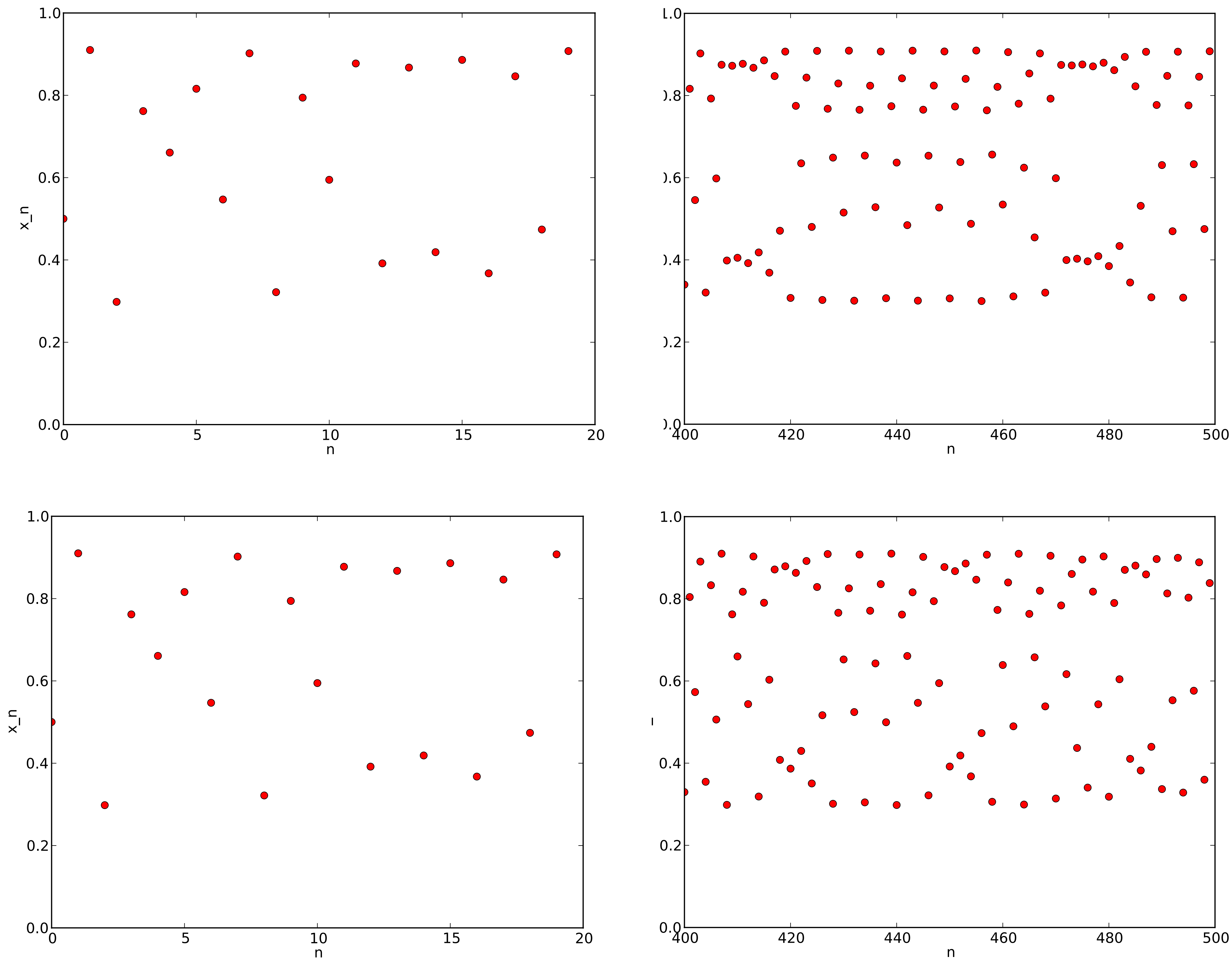
$r = 0.91, x_0 = 0.5$



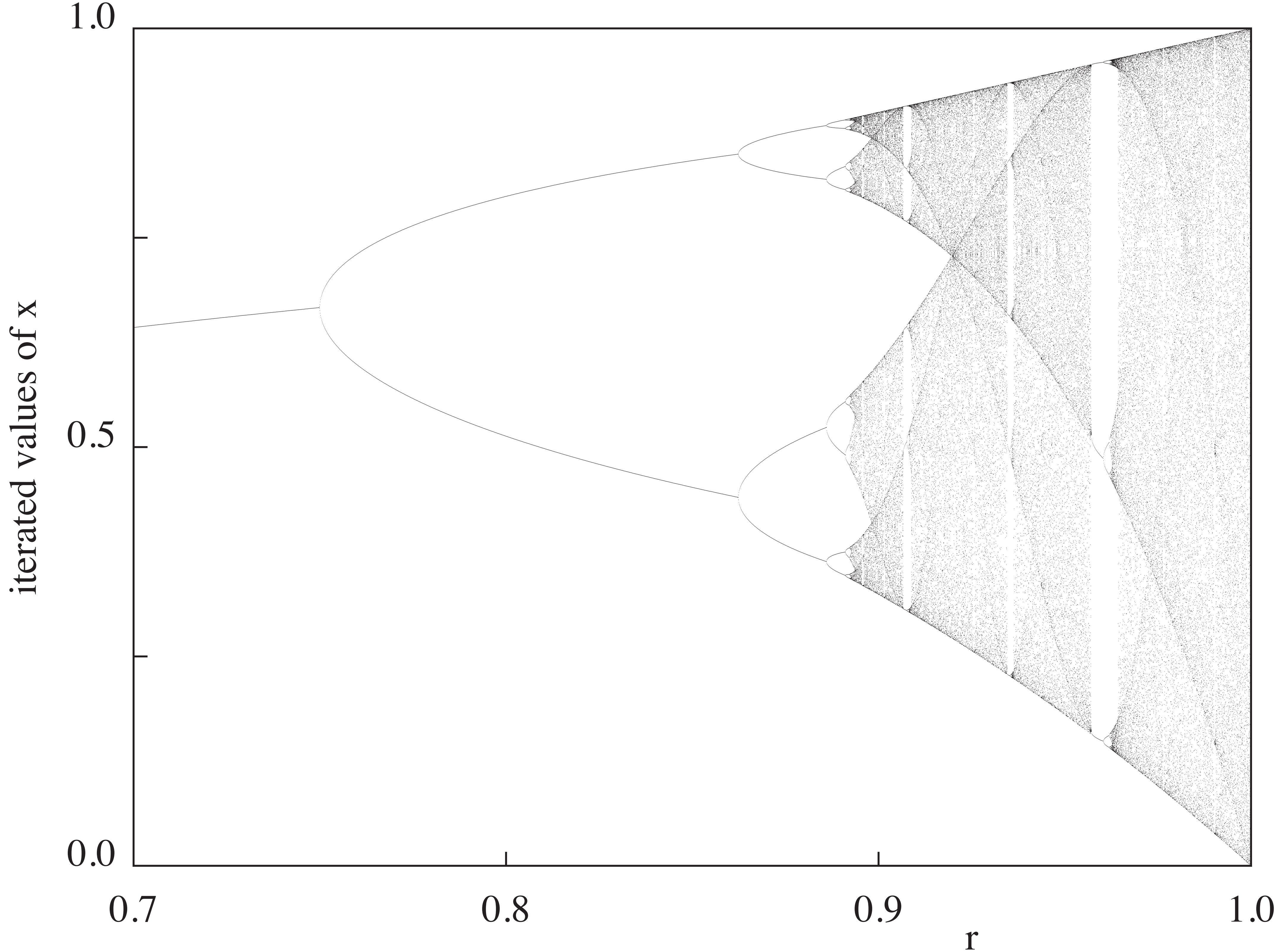
r = 0.95 and two x_0 with 0.001 diff.

x_0=0.5

x_0= 0.5001



Bifurcation diagram



Difference evolution

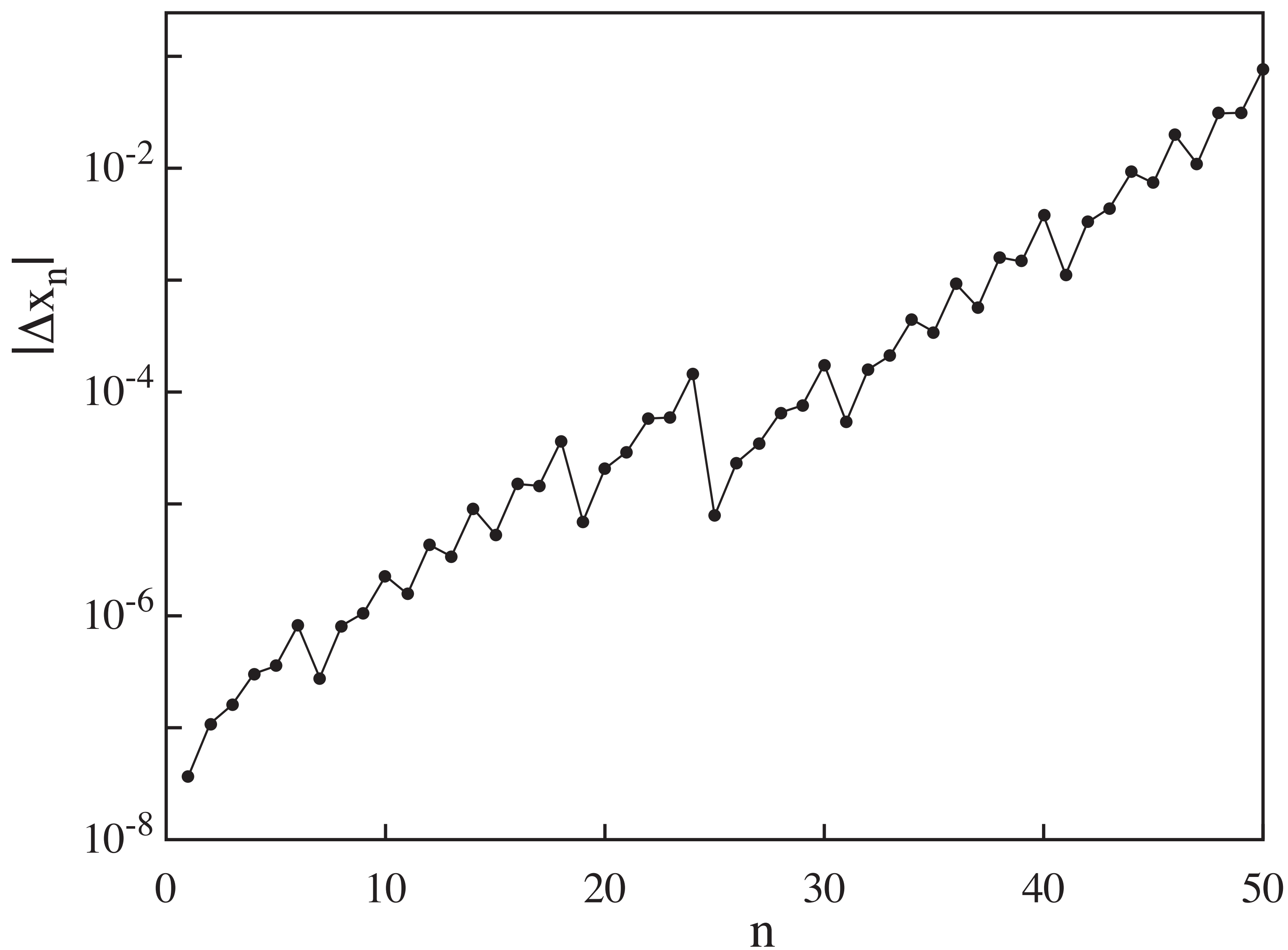


Figure 6.8: The evolution of the difference Δx_n between the trajectories of the logistic map at $r = 0.91$ for $x_0 = 0.5$ and $x_0 = 0.5001$. The separation between the two trajectories increases with n, the number of iterations, if n is not too large. (Note that $|\Delta x_1| \sim 10^{-8}$ and that the trend is not monotonic.)

Lyapunov exponent

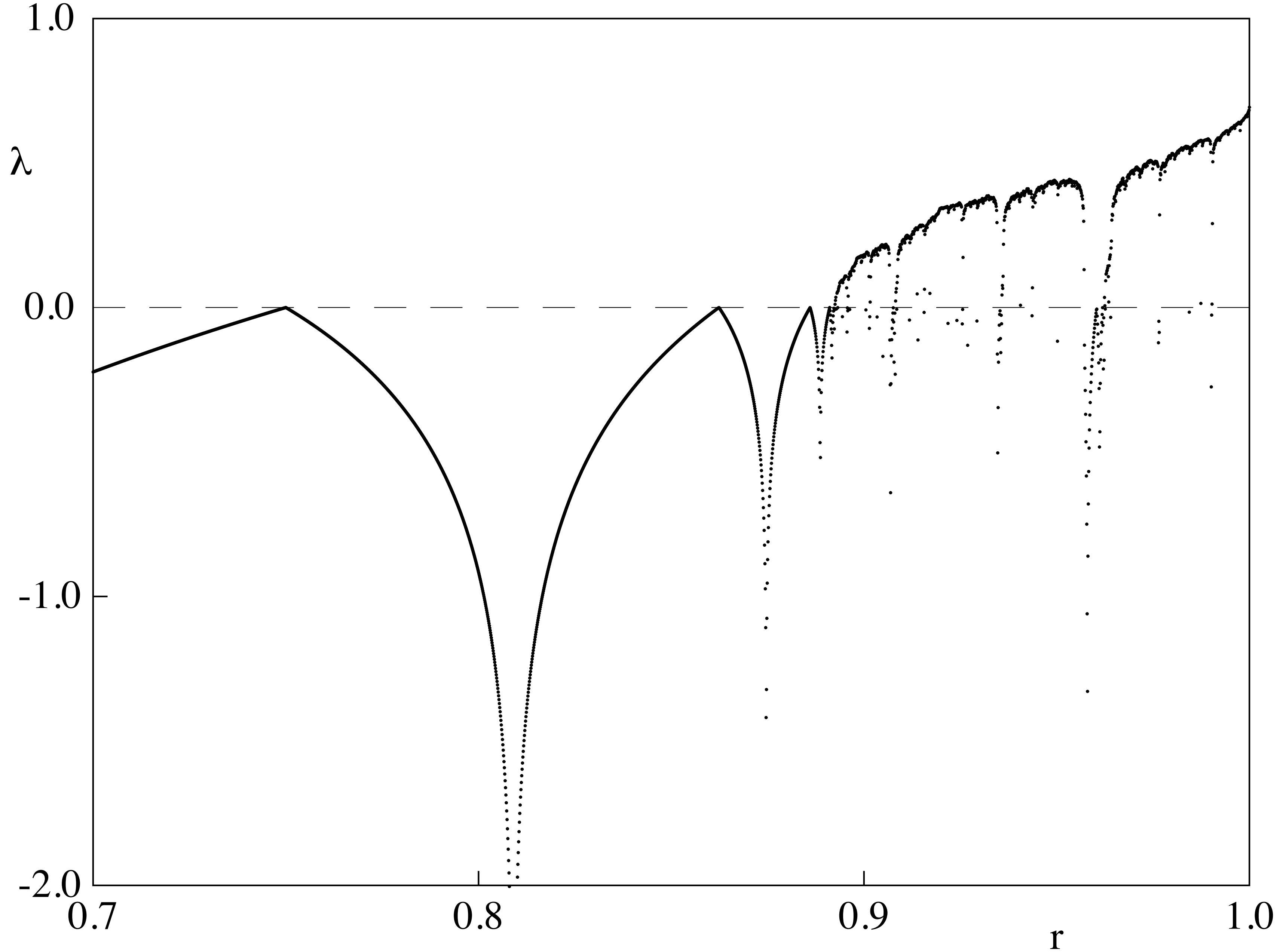
Lyapunov exponent λ : $$|\Delta x_n| = |\Delta x_0| e^{\lambda n} \tag{5}$$

After taking the logarithm: $$\lambda = \frac{1}{n} \log \left| \frac{\Delta x_n}{\Delta x_0} \right| \tag{6}$$

Rewriting: $$\lambda = \frac{1}{n} \sum_{i=0}^{i=n-1} \log \left| \frac{\Delta x_{i+1}}{\Delta x_i} \right| \tag{7}$$

$$\lambda = \lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{i=n-1} \log |f'(x_i)| \tag{8}$$

Lyapunov exponent



Hamiltonian chaos

$$H = \frac{1}{2m}(p_x^2 + p_y^2 + p_z^2) + V(x, y, z)$$

- Constants of motion
  - for time-independent systems: total energy, total momentum …
- Integrability
- More degrees of freedom than constants of motion: possibly chaotic

Double pendulum

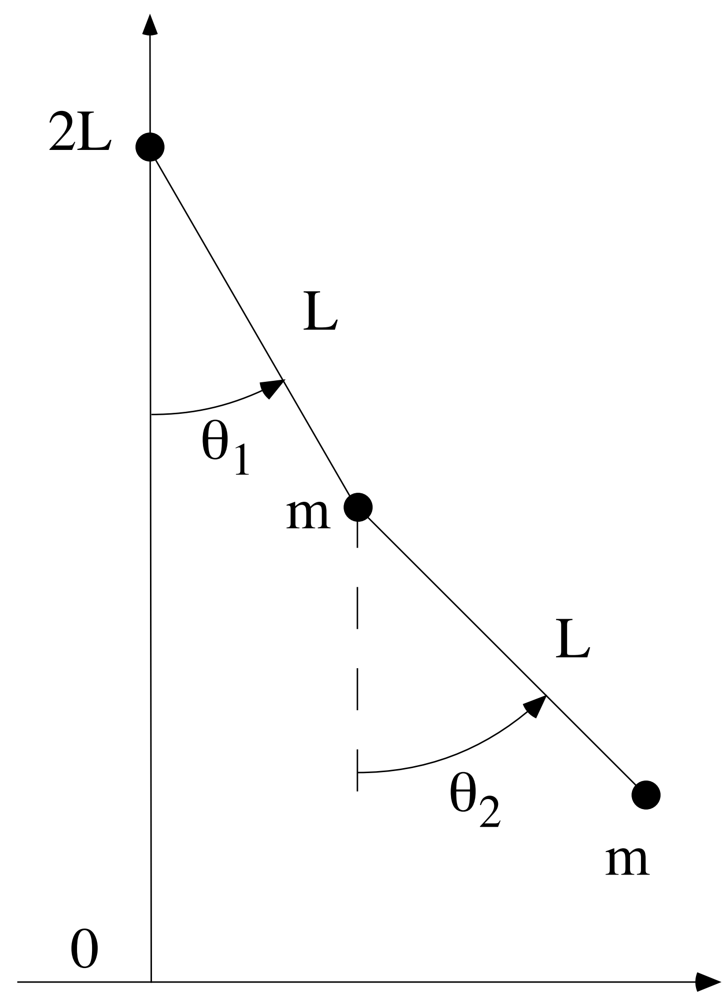


Figure 6.12: The double pendulum.

Double pendulum first recurrence map or Poincaré map

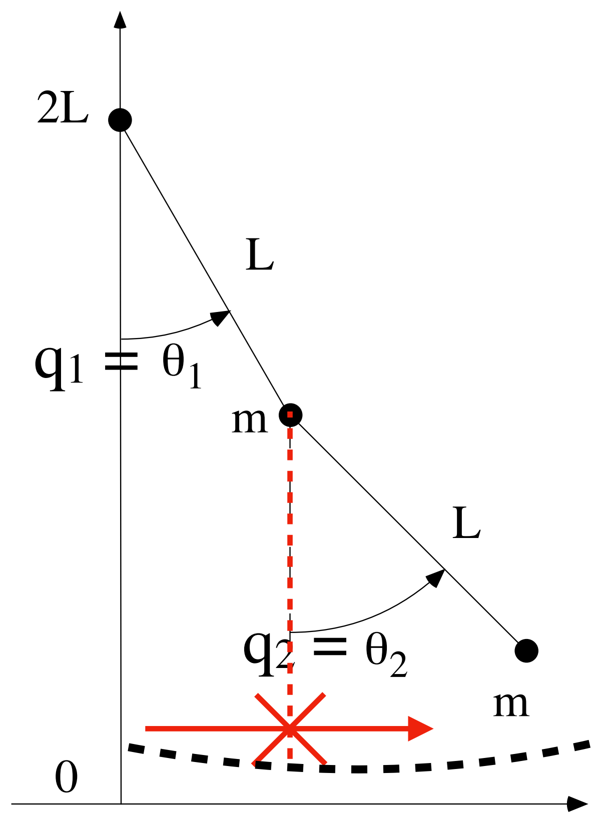


Figure 6.12: The double pendulum.

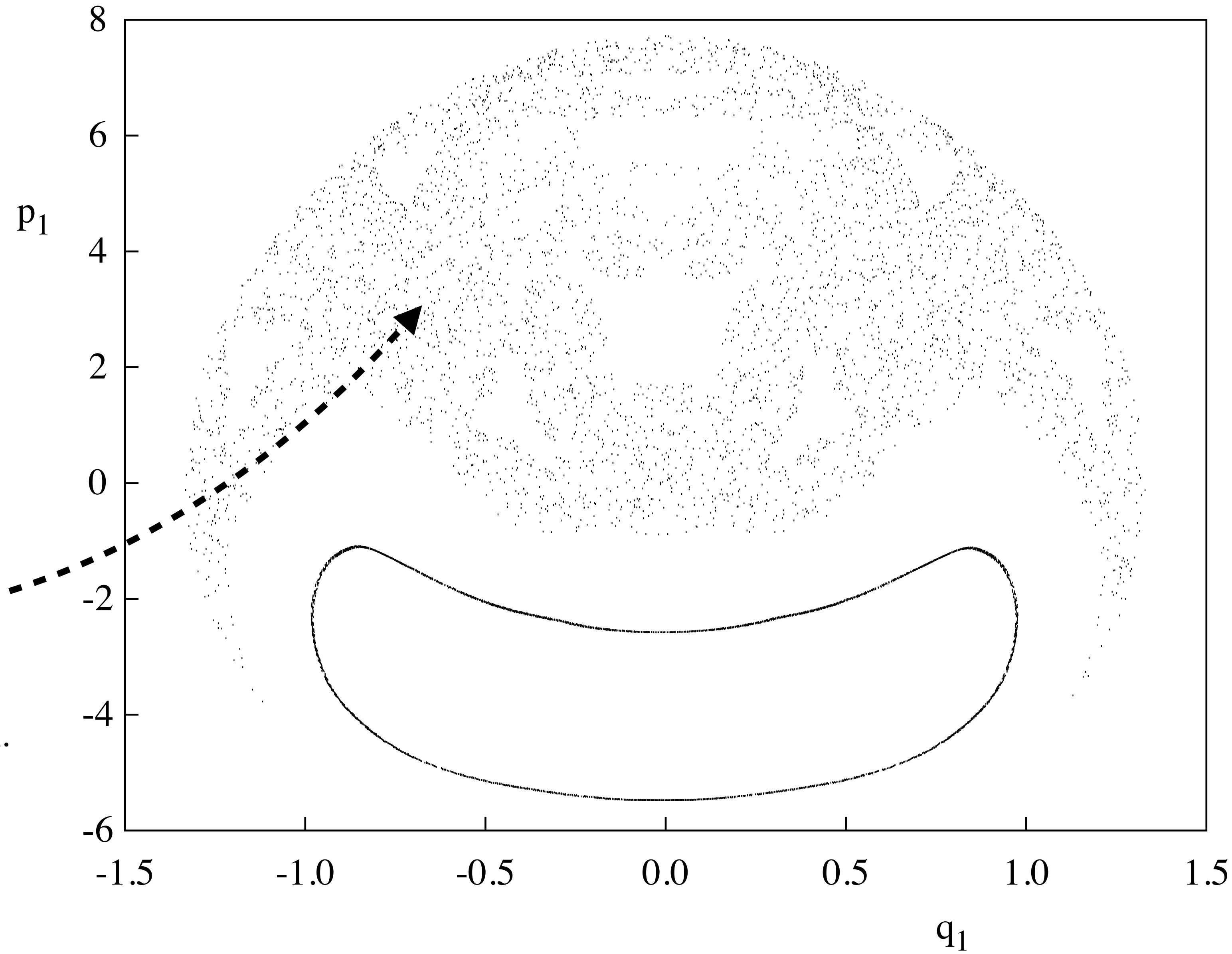


Figure 6.13: Poincaré ~~plot~~ map for the double pendulum with p_1 plotted versus q_1 for $q_2 = 0$ and ~~$p_2 > 0$.~~ $p_2 - p_1 > 0$
Two sets of initial conditions, $(q_1, q_2, p_1) = (0, 0, 0)$ and $(1.1, 0, 0)$ respectively, were used to create the plot. The initial value of the coordinate p_2 is found from (6.52) by requiring that $E = 15$.

References

- Symplectic integration is discussed, with references, in the course book, link is on the course page
- Thorough numerical analysis book:
  Geometric Numerical Integration
  Structure-Preserving Algorithms for Ordinary Differential Equations
  Authors: Ernst Hairer, Gerhard Wanner, Christian Lubich ISBN: 978-3-540-30663-4 (Print) https://link-springer-com.focus.lib.kth.se/book/10.1007/3-540-30666-8
- Simple course on galactic dynamics with different integrators:
  http://www.artcompsci.org/vol_1/v1_web/v1_web.html
- Chaos, An Introduction to Dynamical System, Kathleen T. Alligood, Tim D. Sauer, James A. Yorke, https://link-springer-com.focus.lib.kth.se/book/10.1007/978-3-642-59281-2
- Wikipedia has an informative page on chaos: http://en.wikipedia.org/wiki/Chaos_theory

Project 1.1

Study a pendulum and a harmonic oscillator using Euler-Cromer, velocity Verlet and Runge-Kutta. Assume $\sqrt{g/l} = 2\ \mathrm{s}^{-1}$ and $m = 1$ kg. Compare the different methods with each other and with the exact solution of the harmonic oscillator. Choose a good integrator and plot $\theta(t)$, $\dot{\theta}(t)$ and E. Study the dependence on the time step, try values between 0.01 and 0.2 s. Consider initial conditions $\theta(0)/\pi = 0.1$ and 0.5; $\dot{\theta}(0) = 0$.

Although Runge-Kutta has a higher order accuracy than Verlet, the former is not good for many physical simulations. Can you see why? (hint: check the integrators' behavior at sufficiently long times).

Project 1.2

Determine the period time T as a function of the initial position $\theta(0)$. Which system (harmonic osc./pendulum) has a larger period? Explain! Compare the pendulum with the perturbation series:

$$T = 2\pi\sqrt{\frac{l}{g}}\left(1 + \frac{1}{16}\theta^2(0) + \frac{11}{3072}\theta^4(0) + \frac{173}{737280}\theta^6(0) + \ldots\right)$$

Project 1.3

Study the damped harmonic oscillator equation:

$$F/m = \ddot{x} = -\omega_0^2\, x - \gamma\, \dot{x}$$

Take $\omega_0 = 2$, $\gamma = 0.5,\ 3$, $x(0) = 1$, $\dot{x}(0) = 0$. Plot $x(t)$, $v(t)$, $E(t)$. Discuss the features of these plots. Estimate the relaxation time τ, i.e. the time for the amplitude to be reduced to $1/e \approx 0.37$ of the initial amplitude (note that you should look at the "envelope", rather than at the exact value of the solution). Study the dependence of τ on γ. Find the smallest γ such that the oscillator does not pass $x = 0$. This is called the critical damping, γ_c and for $\gamma > \gamma_c$ the systems is called overdamped.

Project 1.4

Consider a damped pendulum with damping given by $-\gamma\,\dot{\theta}$. Take $\gamma = 1$, $\sqrt{g/l} = 2$, $\theta(0) = \pi/2$, $\dot{\theta}(0) = 0$. Determine the phase space portrait, i.e. plot $\dot{\theta}$ vs θ. Discuss.

Project 1.5

Similar to Problem 6.21 in the book. Double pendulum

a. Use the fourth-order Runge-Kutta algorithm (with $\Delta t = 0.003$) to simulate the double pendulum. Choose $m = 1$, $L = 1$, and $g = 9.8$. The input parameter is the total energy E. The initial values of q_1 and q_2 can be either chosen randomly within the interval $|q_i| < \pi$ or by the user. Then set the initial $p_1 = 0$, and solve for p_2 using equation (6.52) from the book:

$$H = \frac{1}{2mL^2} \frac{p_1^2 + 2p_2^2 - 2p_1 p_2 \cos(q_1 - q_2)}{1 + \sin^2(q_1 - q_2)} + mgL(3 - 2\cos q_1 - \cos q_2)$$

with $H = E$. First explore the pendulum's behavior by looking at the animation ~~by plotting the generalized coordinates and momenta as a function of time in four windows~~. Consider the energies $E = 1$, 15, and 40. Try a few initial conditions for each value of E. Visually determine whether the steady state behavior is regular or appears to be chaotic. Are there some values of E for which all the trajectories appear regular? Are there values of E for which all trajectories appear chaotic? Are there values of E for which both types of trajectories occur?

Tip: checking energy conservation is one good check for your code

b. Repeat part (a), but plot the phase space diagrams p_1 versus q_1 and p_2 versus q_2. Are these plots more useful for determining the nature of the trajectories than those drawn in part (a)?

$p_2 - p_1 > 0$

c. Draw the Poincaré plot with p_1 plotted versus q_1 only when $q_2 = 0$ and $\cancel{p_2 > 0.}$ Overlay trajectories from different initial conditions, but with the same total energy on the same plot. Duplicate the plot shown in Figure 6.13. Then produce Poincaré plots for the values of E given in part (a), with at least five different initial conditions for each energy. Describe the different types of behavior.

d. Is there a critical value of the total energy at which some chaotic trajectories first occur?

Before the next meeting

- Choose a group, link for this will be sent out soon
- Upload your report and presentation to canvas, deadline: November 5th 15:00
- Presentation meetings November 7th,
    - group 1/5: Nov 6, 8.15 - 9:00
    - group 2/6: Nov 6, 9:15 - 10:00
    - group 3/7: Nov 6, 16:15 - 17:00
    - group 4/8: Nov 6, 17:15 - 18:00
- Start in time with the assignments, it might be take more time than you initially think!
- There is an, optional, räknestuga Thursday October 30th 10:00 - 12:00 to ask question about the assignments, python, etc.
- Volunteer for kursnämnd, two free (optional) lunches!