Applications of Singular Value Decomposition in Image Analysis

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Abstract

This report explores the use of Singular Value Decomposition (SVD) in image analysis and compression. We review theoretical foundations of SVD and its applications in low-rank approximation of images. Both grayscale and color images are considered, and the methodology to perform SVD-based compression in Python is described. Finally, the report outlines the experimental approach and explains how image metrics such as Peak Signal-to-Noise Ratio (PSNR) and compression ratio can be computed.

1 Introduction

The Singular Value Decomposition (SVD) is a fundamental tool in linear algebra with wide applications in numerical analysis, data compression, and image processing. Classical references include [1, 5, 4]. The SVD decomposes a matrix into orthogonal bases for its row and column spaces and a diagonal matrix of singular values, capturing intrinsic structural information about the data.

2 Literature Review

The truncated SVD has been extensively studied as a method for dimensionality reduction and regularization [2]. Principal Component Analysis (PCA), a related technique, also leverages eigen-decomposition for data compression [3]. Recent studies in image analysis utilize SVD to compress images while preserving perceptual quality [6, 7]. Modern image compression pipelines often combine SVD with advanced color-space and filtering techniques.

3 Theoretical Background

3.1 Singular Value Decomposition

Given a matrix $A \in \mathbb{R}^{m\times n}$, its singular value decomposition is

$$A = U\Sigma V^T,$$

where $U \in \mathbb{R}^{m\times m}$ and $V \in \mathbb{R}^{n\times n}$ are orthogonal matrices, and $\Sigma \in \mathbb{R}^{m\times n}$ is a diagonal matrix containing singular values $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0$ with $r = \mathrm{rank}(A)$.

3.2 Low-Rank Approximation

The rank-k approximation A_k of A can be expressed in the more common and intuitive form:

$$A_k = \sum_{i=1}^{k} \sigma_i u_i v_i^T,$$

where u_i and v_i are the i-th left and right singular vectors, respectively, and σ_i is the i-th singular value.

This approximation minimizes the Frobenius norm of the error:

$$\|A - A_k\|_F = \min_{\text{rank}(B)=k} \|A - B\|_F.$$

3.3 Application to Image Analysis

Images can be represented as matrices with pixel intensities. For grayscale images, each pixel corresponds to a single value; for color images, each channel (Red, Green, Blue) is represented as a separate matrix. Applying SVD to these matrices allows compression by storing only the most significant singular values and vectors, which often capture most of the visual content.

4 Experimental Setup

We perform the following steps in Python for both grayscale and color images:

1. Load the original image.
2. For grayscale, convert the image to a 2D matrix.
3. For color images, handle each RGB channel separately.
4. Compute SVD and perform rank-k approximations for selected values of k.
5. Compute metrics: Peak Signal-to-Noise Ratio (PSNR) and compression ratio (CR).
6. Save compressed images and record metrics for analysis.

The experimental section is modular and allows extension to other image datasets. All code and results are maintained in separate folders to keep the report clean.

5 Conclusion and Perspectives

This report demonstrates the use of SVD for image compression and low-rank approximation. The methodology is applicable to grayscale and color images. Future work could involve integrating SVD-based compression with other dimensionality reduction or deep learning methods, as well as extending the analysis to video sequences and large-scale image datasets.

References

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