Princeton Online ./ UPENN
Money Macro Finance
Problem Set 01

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1.1

For the model with shocks to the capital stock one gets (derived in the lecture notes):

$$q = \frac{1+\kappa\bar{a}}{1+\kappa\rho} \tag{1}$$

$$i^e = \frac{\bar{a}-\rho}{1+\kappa\rho} \tag{2}$$

$$r_t = \rho + \frac{1}{\kappa}log\Big(\frac{1+\kappa\bar{a}}{1+\kappa\rho}\Big) - \delta - \frac{\sigma^2}{\eta_t} \tag{3}$$

$$n_t = \eta_t\frac{1+\kappa\bar{a}}{1+\kappa\rho}K_t \tag{4}$$

$$\frac{dK_t}{K_t} = \Big[\frac{1}{\kappa}log\Big(\frac{1+\kappa\bar{a}}{1+\kappa\rho}\Big) - \delta\Big].dt + \sigma.dZ_t \tag{5}$$

$$\frac{d\eta_t}{\eta_t} = \frac{(1-\eta_t)^2}{\eta_t^2}\sigma^2.dt + \frac{1-\eta_t}{\eta_t}\sigma.dZ_t \tag{6}$$

Now let's look at the model with technology shocks:

$$\frac{dk_t}{k_t} = \Big[\Phi\big(\frac{\bar{a}}{a_t}i_t\big) - \delta\Big].dt \tag{7}$$

$$y_t = a_t k_t \tag{8}$$

$$\frac{da_t}{a_t} = \sigma.dZ_t \tag{9}$$

To solve this model we start by postulating a diffusion process for the price of capital.

$$\frac{dq_t}{q_t} = \mu_t^q.dt + \sigma_t^q.dZ_t \tag{10}$$

And the problem of the experts is as before:

$$Max_{\{c_t^e, i_t^e, \theta_t^e\}}\ E_0 \int_0^\infty e^{-\rho t} u(c_t^e).dt \tag{11}$$

$$s.t.:\ \ \frac{dn_t}{n_t} = -\frac{c_t}{n_t}.dt + \theta_t^e.r_t.dt + (1-\theta_t^e).dr_t^K(i_t^e) \tag{12}$$

Where now:

$$dr_t^K(i_t^e) = \Big[\frac{a_t - i_t^e}{q_t} + \mu_t^q + \Phi\big(\frac{\bar{a}}{a_t}i_t\big) - \delta\Big].dt + \sigma_t^q.dZ_t \tag{13}$$

For investment we get the condition:

$$q_t^{-1} = \Phi'\big(\frac{\bar{a}}{a_t}i_t\big)\frac{\bar{a}}{a_t} \tag{14}$$

Using the functional form suggested in the lecture notes we get:

$$q_t - \frac{a_t}{\bar{a}} = \kappa i_t^e \tag{15}$$

Taking FOCs for consumption and portfolio decision we get:

$$c_t^e = \rho n_t^e \tag{16}$$

$$\frac{a_t - i_t^e}{q_t} + \Phi\big(\frac{\bar{a}}{a_t}i_t\big) - \delta + \mu_t^q - r_t = (1-\theta_t^e)(\sigma_t^q)^2 \tag{17}$$

Computing the diffusions for N_t and $q_t K_t$ and using Ito's lemma we get the diffusion for the wealth ratio of experts.

$$\frac{d\eta_t}{\eta_t} = \Big[\frac{a_t - i_t}{q_t} + (\theta_t^e\sigma_t^q)^2 - \rho\Big].dt - \theta_t^e\sigma_t^q.dZ_t \tag{18}$$

With market clearing we again have $\rho q_t = a_t - i_t^e(q_t)$. Now, however, this does not imply a constant price for capital because a_t is not constant.

Using our expression for investment:

$$\rho q_t = a_t - q_t \kappa^{-1} - \frac{a_t}{\bar{a}} \kappa^{-1} \Leftrightarrow q_t = \frac{a_t}{\bar{a}} \left[\frac{1 + \kappa \bar{a}}{1 + \kappa \rho} \right] \tag{19}$$

This tells us that $\frac{dq_t}{q_t} = \frac{da_t}{a_t} = \sigma.dZ_t$. So $\sigma_t^q = \sigma$ and $\mu_q^t = 0$.

$$i_t^e = \frac{a_t}{\bar{a}} \Big(\frac{\bar{a} - \rho}{1 + \kappa \rho} \Big) \tag{20}$$

The risk-free rate will then be:

$$r_t = \rho + \frac{1}{\kappa} log \Big(\frac{1 + \kappa \bar{a}}{1 + \kappa \rho} \Big) - \delta - \frac{1}{\eta_t} \sigma^2 \tag{21}$$

$$\frac{d\eta_t}{\eta_t} = \frac{(1 - \eta_t)^2}{\eta_t^2} \sigma^2.dt + \frac{1 - \eta_t}{\eta_t} \sigma.dZ_t \tag{22}$$

$$\frac{dK_t}{K_t} = \Big[\frac{1}{\kappa} log \Big(\frac{1 + \kappa \bar{a}}{1 + \kappa \rho} \Big) - \delta \Big].dt \tag{23}$$

The process for capital does not have the last term as before, but the process for technology does. Using Ito's lemma, the process for output would be the same as before.

1.2

If $\phi > 0$, the conclusions change a bit. In terms of the solution of the model, everything up to market clearing remains unchanged. Our equations for q_t and i_t^e are still the same.

But now this implies:

$$\frac{dq_t}{q_t} = \phi \, log \Big(\frac{\bar{a}}{a_t} \Big).dt + \sigma.dZ_t \tag{24}$$

And so we have $\mu_t^q = \phi \, log \big(\frac{\bar{a}}{a_t} \big)$ and $\sigma_t^q = \sigma$.

The process for wealth ratio of experts as well as capital are unchanged, but the risk-free rate has an additional term.

$$r_t = \rho + \frac{1}{\kappa} log \Big(\frac{1 + \kappa \bar{a}}{1 + \kappa \rho} \Big) - \delta - \frac{1}{\eta_t} \sigma^2 + \phi \, log \big(\frac{\bar{a}}{a_t} \big) \tag{25}$$

1.3

When there is a positive (negative) technology shock, this affects output directly. But because it also affects the productivity going forward, it would make it more (less) desirable to invest. So our shocks would have the initial effect on output and an increased effect of inducing investment. This second effect is not present in the model with capital shocks. The capital shock affects output through the change in capital stock, but the return on new capital one can get through investment is not changed by this.

So for the 2 models to have the same dynamics we need to shut down the investment-incentive mechanism of the technology shocks, by making them not affect the returns on future investment on capital.

2

2.1

The problem for the experts is unchanged. So we still have:

$$Max_{\{c_t^e, i_t^e, \theta_t^e\}}\, E_0 \int_0^\infty e^{-\rho t} u(c_t^e).dt \tag{26}$$

$$s.t.\ \frac{dn_t}{n_t} = -\frac{c_t}{n_t}dt + \theta_t^e r_t dt + (1-\theta_t^e) dr_t^K(i_t^e) \tag{27}$$

Additionally, the return on capital is unchanged and so is then the choice for investment, and FOCs for consumption and portfolio choice:

$$c_t^e = \rho n_t^e \tag{28}$$

$$\frac{a-i_t}{q_t} + \Phi(i_t) - \delta + \mu_t^q + \sigma\sigma_t^q - r_t = (1-\theta_t^e)(\sigma+\sigma_t^q)^2 \tag{29}$$

$$i_t = \frac{q_t - 1}{\kappa} \tag{30}$$

Now, things change a little in the market clearing part.

We still have goods market clearing:

$$C_t = (a - i_t)K_t \tag{31}$$

But now we have to seperate the consumption by the experts and households:

$$C_t = c_t^e + c_t^h = \rho N_t^e + \varrho N_t^h = \Big[\rho\eta_t + \varrho(1-\eta_t)\Big]N_t, \tag{32}$$

where η_t from now on denounces the experts wealth share.

Now using balance sheet clearing ($N_t = q_t K_t$) together with goods market clearing, and dividing through by K_t we get:

$$\Big[\rho\eta_t + \varrho(1-\eta_t)\Big]q_t = a - i_t^e(q_t) \tag{33}$$

Plugging in our expression for investment and rewriting we get:

$$q_t = \frac{1+\kappa a}{\kappa(\rho\eta_t + \varrho(1-\eta_t)) + 1} \tag{34}$$

When the two agents have the same discount factor this collapses into a constant and hence our previous result. Now it does not anymore.

Let us look at this expression as:

$$q_t = f(\eta_t) \Leftrightarrow dq_t = df(\eta_t) \tag{35}$$

We can use Ito's lemma to get the right-hand side of the equation.

$$df(\eta_t) = f'(\eta_t)d\eta_t + \frac{1}{2}f''(\eta_t)(d\eta_t)^2 = \tag{36}$$

$$df(\eta_t) = [f'(\eta_t)\mu_t^\eta \eta_t + \frac{1}{2}f''(\eta_t)(\sigma_t^\eta)^2(\eta_t)^2]dt + f''(\eta_t)\sigma_t^\eta \eta_t dZ_t \tag{37}$$

Computing these we get:

$$f'(\eta_t) = -\frac{1+\kappa a}{\Big[\kappa(\rho\eta_t + \varrho(1-\eta_t)) + 1\Big]^2}\kappa(\rho - \varrho) \tag{38}$$

$$f''(\eta_t) = \frac{1+\kappa a}{\Big[\kappa(\rho\eta_t + \varrho(1-\eta_t)) + 1\Big]^3}2\kappa^2(\rho - \varrho)^2 \tag{39}$$

So:

$$\frac{f'(\eta_t)}{q_t} = -\frac{\kappa(\rho - \varrho)}{\Big[\kappa(\rho\eta_t + \varrho(1-\eta_t)) + 1\Big]} \tag{40}$$

$$\frac{f''(\eta_t)}{2q_t} = \frac{\kappa^2(\rho - \varrho)^2}{\Big[\kappa(\rho\eta_t + \varrho(1-\eta_t)) + 1\Big]^2} \tag{41}$$

Now, to get further, we need to get an expression for the process of η_t. So let us find that and get back to the dynamics dq_t.

We first use Ito's product rule to get the dynamics of $q_t K_t$, and then using Ito's quotient rule to get the dynamics of $N_t^e / q_t K_t$. As shown in Question 1, this yields:

$$d\eta_t = \Big[\underbrace{\frac{a - i_t}{q_t} - \rho + (\theta_t^e)^2(\sigma + \sigma_t^q)^2}_{\mu_t^\eta}\Big]\eta_t dt \underbrace{- \theta_t^e(\sigma + \sigma_t^q)}_{\sigma_t^\eta}\eta_t dZ_t \tag{42}$$

We can now come back to the process of q. Simply plugging into Ito's formula yields:

$$
\frac{dq}{q} = \Bigg[\underbrace{- \frac{1+\kappa a}{\Big(\kappa(\rho\eta_t + \underline{\rho}(1-\eta_t)) + 1\Big)^2} \kappa(\rho - \underline{\rho})\mu_t^\eta \eta_t + \frac{\kappa^2(\rho-\underline{\rho})^2}{\Big(\kappa(\rho\eta_t + \underline{\rho}(1-\eta_t)) + 1\Big)^2} (\sigma_t^\eta)^2 (\eta_t)^2}_{\mu_t^q} \Bigg] dt
$$

$$
+ \underbrace{\frac{\kappa^2(\rho-\underline{\rho})^2}{\Big[\kappa(\rho\eta_t + \underline{\rho}(1-\eta_t)) + 1\Big]^2} \sigma_t^\eta \eta_t}_{\sigma_t^q} \, dZ_t \tag{43}
$$

And at least by substituting in for σ_t^η, using that $\theta_t^e = 1/\eta_t$ by capital market clearing, and rearranging(!), we get that

$$
\sigma_t^q = -\frac{(\rho - \underline{\rho})(1-\eta_t)}{1/\kappa + \rho} \sigma \tag{44}
$$

So in summary we have the following dynamics in our economy

$$
q_t = \frac{1+\kappa a}{\kappa(\rho - \underline{\rho})\eta_t + 1 + \kappa\underline{\rho}} \tag{45}
$$

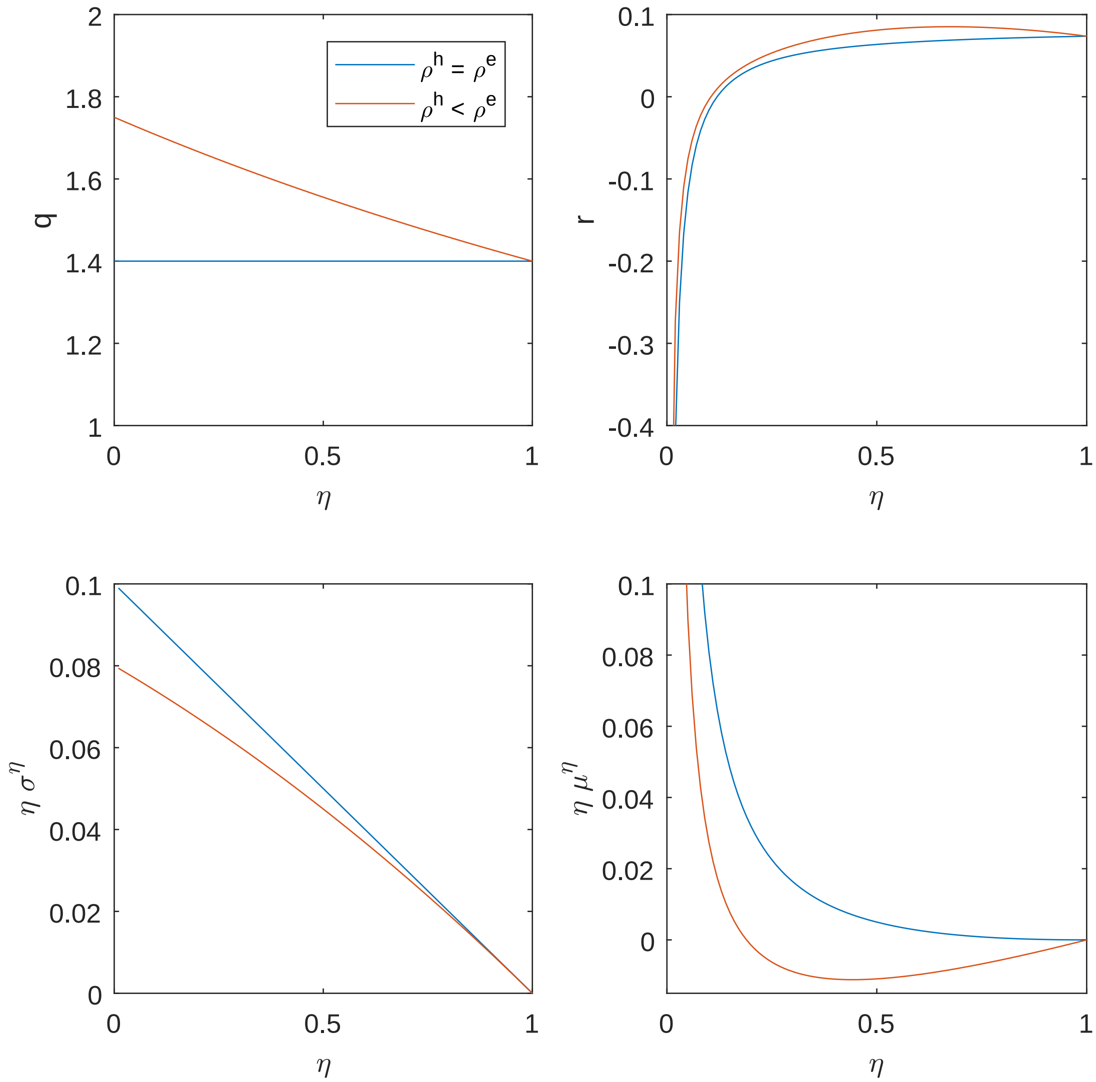
$$
i_t = \frac{a - \underline{\rho} - (\rho - \underline{\rho})\eta_t}{\kappa(\rho - \underline{\rho})\eta_t + 1 + \kappa\underline{\rho}} \tag{46}
$$

$$
\sigma_t^q = -\frac{(\rho - \underline{\rho})(1-\eta_t)}{1/\kappa + \rho} \sigma \tag{47}
$$

$$
\mu_t^\eta = -(\rho - \underline{\rho})(1-\eta_t) + (1-\eta_t^{-1})^2(\sigma + \sigma_t^q)^2 \tag{48}
$$

$$
\sigma_t^\eta = -(1-\eta_t^{-1})(\sigma + \sigma_t^q) \tag{49}
$$

2.2



The risk-free rate r, the price of capital q, the absolute drift and volatility of the state variable η (the experts wealth share), are illustrated. Both for the model where $\rho^h = \rho^e$, and the model where $\rho^h < \rho^e$, i.e. where the households are more patient than the experts ($\delta = 0$).

2.3

When $\kappa > 0$ the endogenous risk mitigates the exogenous risk. Total risk in the economy is (using our result from Question 2.1)

$$(\sigma + \sigma_t^q) = \sigma(1 - \frac{(\rho - \underline{\rho})(1 - \eta_t)}{1/\kappa + \rho}). \tag{50}$$

As the second term on the right hand side is negative and less than 1, we get that the economy helps mitigate exogenous risk, as the total risk will be less than the exogenous risk ($(\sigma + \sigma_t^p) < \sigma$).

We can see that the second term is negative since all its terms are positive, and there is a negative sign preceding it. We can see that it is less than 1, as $(\rho - \underline{\rho}) < \rho$ by construction, and $(1 - \eta_t) < 1$, making the numerator, even for the worst case where $\kappa \to \infty, 1/\kappa \to 0$, smaller than the denominator.

Intuitively, this is because as q rises, the experts take on less leverage, making endogenous risk counter-cyclical. Said differently, as the wealth share of the experts increases, the relative optimal amount of consumption increases (as the experts are more impatient), this causes an increase in the risk-free rate and the value of the asset to fall (as higher willingness to consume and less willingness to invest).

My attempt at an explanation (Joao): When a negative shock hits the capital stock, this will mechanically decrease the net worth of experts by decreasing the capital stock they own. In the model with homogeneous agents, the relative price of capital (to goods) does not react because lower capital changes both production and consumption in the same way. However, with heterogeneous agents, when the wealth share of experts goes down this affects consumption more as they are more impatient (and thus consuming a higher share of their wealth than households). Because of this, demand for consumption goods goes down and for markets to clear the relative price between consumption goods and capital goods has to go down, which is the same as saying that q_t has to go up. By going up, q_t partially mitigates the initial effect of the shock on capital, because the capital stock goes down, but the value of capital goes up, meaning that the capital loss of the experts is smaller than the initial direct effect.

2.4

The drift for the diffusion process of the wealth share gives us an idea about the way this variable will evolve over time. When the drift is positive there will be a tendency for the wealth share to increase and when it is negative there will be a tendency for it to decrease.

In the model with homogeneous agents the drift is positive for every value of the wealth share, which means that the wealth share will increase until the experts have all the wealth in the economy. With heterogenous agents, however, the story is different. For low values of the expert wealth share the drift is positive and for high values the drift is negative, so a stationary distribution should exist. This can be seen from the figure in Question 2.2. One can see that there is a wealth share, between 0 and 1, for which the drift is 0. Additionally, if we are at this wealth share and a shock pushes us towards a higher (lower) wealth share, the drift will become negative (positive), pushing us back into the previous wealth share, so this point is stable and thus we can be confident that a stationary distribution will exist.

3 This is question 3?

:

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To add matlab code
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4 This is question 4?

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$$

Plots code.

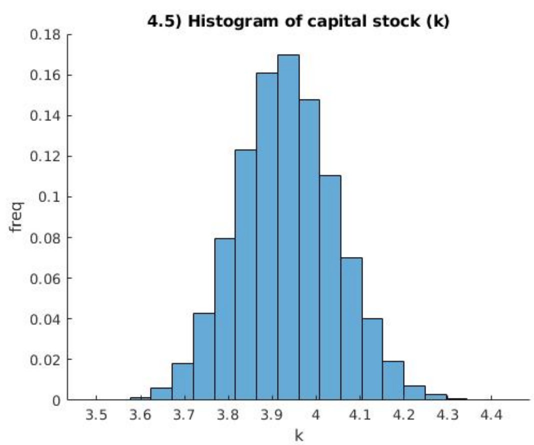

Figure 1: Distribution of capital stock with calibrated values of ρ and σ. It oscillates around the steady state.