Homework II - Econ 714

A firm i at period t is endowed with two technologies. The first technology produces business output y_{it} according to:

$$y_{it} = \left(\theta_1 \left(k_{1,it}^T \right)^{\frac{\lambda_1 - 1}{\lambda_1}} + (1 - \theta_1) \left(k_{1,it}^I \right)^{\frac{\lambda_1 - 1}{\lambda_1}} \right)^{\frac{\lambda_1}{\lambda_1 - 1}\alpha_1} \left(e^{z_{1,it}} l_{1,it} \right)^{\gamma_1}$$

where $k_{1,it}^T$ is tangible capital, $k_{1,it}^I$ is intangible capital, $l_{1,it}$ is labor, and $z_{1,it}$ is an idiosyncratic productivity shock. The productivity shock $z_{1,it}$ follows a Markov Chain with support and transition matrix:

$$\begin{pmatrix} 0.95 \\ 1.05 \end{pmatrix}, \begin{pmatrix} 0.95 & 0.05 \\ 0.05 & 0.95 \end{pmatrix}.$$

The firm can use the business output y_{it} for investment, x_{it}, in tangible capital, k_{it+1}^T, that will come on-line at the start of the next period or for final sales, s_{it}:

$$y_{it} = s_{it} + x_{it}.$$

Note that we are not constraining final sales or investment to be positive: the firm can buy business output from other firms if it desires to invest in tangible capital more than it produced or disinvest from existing capital.

Given a level of investment x_{it}, tangible capital evolves as:

$$k_{it+1}^T = (1 - \delta_T) k_{it}^T + x_{it}$$

where δ_T is the depreciation rate of tangible capital.

The second technology produces firm-specific intangible capital k_{it+1}^I that will come on-line at the start of the next period according to:

$$k_{it+1}^I = \left(\theta_2 \left(k_{2,it}^T \right)^{\frac{\lambda_2 - 1}{\lambda_2}} + (1 - \theta_2) \left(k_{2,it}^I \right)^{\frac{\lambda_2 - 1}{\lambda_2}} \right)^{\frac{\lambda_2}{\lambda_2 - 1}\alpha_2} \left(e^{z_{2,it}} l_{2,it} \right)^{\gamma_2} + (1 - \delta_I) k_{it}^I$$

where δ_I is the depreciation rate of intangible capital and $z_{2,it}$ is an intangible investment-specific productivity shock that follows a Markov Chain with support and transition matrix:

$$\begin{pmatrix} 0.90 \\ 1.10 \end{pmatrix}, \begin{pmatrix} 0.90 & 0.10 \\ 0.10 & 0.90 \end{pmatrix}$$

The total amount of tangible and intangible capital used by the firm must satisfy:

$$\begin{aligned} k_{it}^T &= k_{1,it}^T + k_{2,it}^T. \\ k_{it}^I &= k_{1,it}^I + k_{2,it}^I. \end{aligned}$$

Finally, the firm pays off as dividends:

$$d_{it} = s_{it} - w_t \left(l_{1,it} + l_{2,it}\right) + b_{it+1} - \left(1 + r_t\right) b_{it} - 0.02 \left(b_{it+1} - b_{it}\right)^2 - 0.01 \left(b_{it} - 0.2\right)^2$$

Thus, the individual states of the firm are: $states_{it} = \left(k_{it}^T, k_{it}^I, b_{it+1}, z_{1,it}, z_{2,it}\right)$ and the controls: $controls_{it} = \left(k_{1,it}^T, k_{1,it}^I, k_{it+1}^T, l_{1,it}, l_{2,it}, b_{it+1}\right)$. Note the problem of the firm has a block-recursive structure. Once the firms decide $k_{1,it}^T$, $k_{1,it}^I$, and k_{it+1}^T, the choices of $l_{1,it}$ and $l_{2,it}$ come directly from the optimality conditions derived from the production functions and the wage.

Given a discount factor for outside investors r_t^i, the value function of the firm is then given by:

$$J(states_{it}) = \max_{controls_{it}} \left\{ d_{it} + \mathbb{E}_t \frac{1}{1 + r_t^i} J(states_{it+1}) \right\}$$

s.t.

$$d_{it} = s_{it} - w_t \left(l_{1,it} + l_{2,it}\right) + b_{it+1} - \left(1 + r_t\right) b_{it} - 0.02 \left(b_{it+1} - b_{it}\right)^2 - 0.01 \left(b_{it} - 0.2\right)^2$$

$$s_{it} + x_{it} = \left(\theta_1 \left(k_{1,it}^T\right)^{\frac{\lambda_1 - 1}{\lambda_1}} + \left(1 - \theta_1\right) \left(A_t k_{1,it}^I\right)^{\frac{\lambda_1 - 1}{\lambda_1}}\right)^{\frac{\lambda_1}{\lambda_1 - 1}\alpha_1} \left(e^{z_{1,it}} l_{1,it}\right)^{\gamma_1}$$

$$k_{it+1}^T = \left(1 - \delta_T\right) k_{it}^T + x_{it}$$

$$k_{it+1}^I = A_t \left(\theta_2 \left(k_{2,it}^T\right)^{\frac{\lambda_2 - 1}{\lambda_2}} + \left(1 - \theta_2\right) \left(k_{2,it}^I\right)^{\frac{\lambda_2 - 1}{\lambda_2}}\right)^{\frac{\lambda_2}{\lambda_2 - 1}\alpha_2} \left(e^{z_{2,it}} l_{2,it}\right)^{\gamma_2} + \left(1 - \delta_I\right) k_{it}^I$$

$$k_{it}^T = k_{1,it}^T + k_{2,it}^T$$

$$k_{it}^I = k_{1,it}^I + k_{2,it}^I$$

$z_{1,it}$ and $z_{2,it}$ follow their Markov Chains

The calibration of the model is:

λ_1	1.1	α_1	0.3	δ_T	0.10
λ_2	0.9	α_2	0.4	δ_I	0.15
θ_1	0.5	γ_1	0.6	r_t^i	0.025
θ_2	0.4	γ_2	0.5	r_t	0.03

1. Question I

Compute the model above using Chebyshev polynomials. Use Smolyak interpolation with $\mathbb{G}(5,3)$ for the three continuous state variables as described in my handbook chapter. Report the value function and decision rules and simulate the behavior of the firm for 1,000 periods.

2. Question II

Compute the model above using a third perturbation. Again, report the value function and decision rules and simulate the behavior of the firm for 1,000 periods. You can substitute the Markov chain process for AR(1) processes:

$$z_{1,it} = 0.95 z_{1,it-1} + 0.05 \varepsilon_{1,it},\ \varepsilon_{1,it} \sim \mathcal{N}(0,1)$$
$$z_{2,it} = 0.9 z_{1,it-1} + 0.1 \varepsilon_{1,it},\ \varepsilon_{2,it} \sim \mathcal{N}(0,1)\,.$$