ECONOMICS 714

Computational Economics

Fall 2019 - Homework #2

Juan Castellanos Silván

January 10, 2020

1 Firms' dynamic problem

Consider the following problem:

$$J\left(states_{it}\right)=\max_{controls_{it}}\left\{d_{it}+\mathbb{E}_t\frac{1}{1+r_t^i}J\left(states_{it+1}\right)\right\} \tag{1}$$

subject to

$$d_{it}=s_{it}-w_t\left(l_{1,it}+l_{2,it}\right)+b_{it+1}-\left(1+r_t\right)b_{it}-0.02\left(b_{it+1}-b_{it}\right)^2-0.01\left(b_{it}-0.2\right)^2 \tag{2}$$

$$s_{it}+x_{it}=\left(\theta_1\left(k_{1,it}^T\right)^{\frac{\lambda_1-1}{\lambda_1}}+\left(1-\theta_1\right)\left(k_{1,it}^I\right)^{\frac{\lambda_1-1}{\lambda_1}}\right)^{\frac{\lambda_1}{\lambda_1-1}\alpha_1}\left(e^{z_{1,it}}l_{1,it}\right)^{\gamma_1} \tag{3}$$

$$k_{it+1}^T=\left(1-\delta_T\right)k_{it}^T+x_{it} \tag{4}$$

$$k_{it+1}^I=\left(\theta_2\left(k_{2,it}^T\right)^{\frac{\lambda_2-1}{\lambda_2}}+\left(1-\theta_2\right)\left(k_{2,it}^I\right)^{\frac{\lambda_2-1}{\lambda_2}}\right)^{\frac{\lambda_2}{\lambda_2-1}\alpha_2}\left(e^{z_{2,it}}l_{2,it}\right)^{\gamma_2}+\left(1-\delta_I\right)k_{it}^I \tag{5}$$

$$k_{it}^T=k_{1,it}^T+k_{2,it}^T \tag{6}$$

$$k_{it}^I=k_{1,it}^I+k_{2,it}^I \tag{7}$$

where the individual states of the firm are: $states_{it}=\left(k_{it}^T,k_{it}^I,b_{it},z_{1,it},z_{2,it}\right)$ and the controls are : $controls_{it}=\left(k_{1,it}^T,k_{1,it}^I,k_{it+1}^T,l_{1,it},l_{2,it},b_{it+1}\right)$.

Notice that we can simplify the above problem if we rearrange the constraints. Plug (3) & (4) into (2), and (6) & (7) into (5). Thus, we obtain

$$\begin{split}d_{it}=\left(\theta_1\left(k_{1,it}^T\right)^{\frac{\lambda_1-1}{\lambda_1}}+\left(1-\theta_1\right)\left(k_{1,it}^I\right)^{\frac{\lambda_1-1}{\lambda_1}}\right)^{\frac{\lambda_1}{\lambda_1-1}\alpha_1}\left(e^{z_{1,it}}l_{1,it}\right)^{\gamma_1}-k_{it+1}^T+\left(1-\delta_T\right)k_{it}^T+\\-w_t\left(l_{1,it}+l_{2,it}\right)+b_{it+1}-\left(1+r_t\right)b_{it}-0.02\left(b_{it+1}-b_{it}\right)^2-0.01\left(b_{it}-0.2\right)^2\end{split} \tag{8}$$

$$k_{it+1}^I=\left(\theta_2\left(k_{it}^T-k_{1,it}^T\right)^{\frac{\lambda_2-1}{\lambda_2}}+\left(1-\theta_2\right)\left(k_{it}^I-k_{1,it}^I\right)^{\frac{\lambda_2-1}{\lambda_2}}\right)^{\frac{\lambda_2}{\lambda_2-1}\alpha_2}\left(e^{z_{2,it}}l_{2,it}\right)^{\gamma_2}+\left(1-\delta_I\right)k_{it}^I \tag{9}$$

Consequently the firm's dynamic problem reduces to (1) subject to (8) and (9). It is important to mention that the law of motion of intangible capital – eq. (9) – is relevant for the states tomorrow, while the dividends equation – eq. (8) – determines the instantaneous utility of the firm. Thus, the optimality conditions of the firm are:

$$\begin{aligned}
(k^T_{1,it}): \quad & \alpha_1\theta_1\left(k^T_{1,it}\right)^{-\frac{1}{\lambda_1}}\left(\theta_1\left(k^T_{1,it}\right)^{\frac{\lambda_1-1}{\lambda_1}}+(1-\theta_1)\left(k^I_{1,it}\right)^{\frac{\lambda_1-1}{\lambda_1}}\right)^{\frac{\lambda_1}{\lambda_1-1}\alpha_1-1}\left(e^{z_{1,it}}l_{1,it}\right)^{\gamma_1}+ \\
& +\mathbb{E}_t\left[\frac{1}{1+r^i_t}\frac{\partial k^I_{it+1}}{\partial k^T_{1,it}}\,J_{k^I_{it+1}}(states_{it+1})\right]=0
\end{aligned} \tag{10}$$

$$\begin{aligned}
(k^I_{1,it}): \quad & \alpha_1(1-\theta_1)\left(k^I_{1,it}\right)^{-\frac{1}{\lambda_1}}\left(\theta_1\left(k^T_{1,it}\right)^{\frac{\lambda_1-1}{\lambda_1}}+(1-\theta_1)\left(k^I_{1,it}\right)^{\frac{\lambda_1-1}{\lambda_1}}\right)^{\frac{\lambda_1}{\lambda_1-1}\alpha_1-1}\left(e^{z_{1,it}}l_{1,it}\right)^{\gamma_1}+ \\
& +\mathbb{E}_t\left[\frac{1}{1+r^i_t}\frac{\partial k^I_{it+1}}{\partial k^I_{1,it}}\,J_{k^I_{it+1}}(states_{it+1})\right]=0
\end{aligned} \tag{11}$$

$$(k^T_{it+1}): \quad 1=\mathbb{E}_t\left[\frac{1}{1+r^i_t}J_{k^T_{it+1}}(states_{it+1})\right] \tag{12}$$

$$(b_{it+1}): \quad 1=0.04\,(b_{it+1}-b_{it})+\mathbb{E}_t\left[\frac{1}{1+r^i_t}J_{b_{it+1}}(states_{it+1})\right] \tag{13}$$

$$(l_{1,it}): \quad w_t=\gamma_1(e^{z_{1,it}})^{\gamma_1}(l_{1,it})^{\gamma_1-1}\left(\theta_1\left(k^T_{1,it}\right)^{\frac{\lambda_1-1}{\lambda_1}}+(1-\theta_1)\left(k^I_{1,it}\right)^{\frac{\lambda_1-1}{\lambda_1}}\right)^{\frac{\lambda_1}{\lambda_1-1}\alpha_1} \tag{14}$$

$$(l_{2,it}): \quad w_t=\mathbb{E}_t\left[\frac{1}{1+r^i_t}\frac{\partial k^I_{it+1}}{\partial l^T_{2,it}}\,J_{k^I_{it+1}}(states_{it+1})\right] \tag{15}$$

Applying the Envelope Theorem, we have that

$$\begin{aligned}
J_{k^I_{it}}(states_{it}) &= \alpha_1(1-\theta_1)\left(k^I_{1,it}\right)^{-\frac{1}{\lambda_1}}\left(\theta_1\left(k^T_{1,it}\right)^{\frac{\lambda_1-1}{\lambda_1}}+(1-\theta_1)\left(k^I_{1,it}\right)^{\frac{\lambda_1-1}{\lambda_1}}\right)^{\frac{\lambda_1}{\lambda_1-1}\alpha_1-1}\left(e^{z_{1,it}}l_{1,it}\right)^{\gamma_1} \\
J_{k^T_{it}}(states_{it}) &= \alpha_1\theta_1\left(k^T_{1,it}\right)^{-\frac{1}{\lambda_1}}\left(\theta_1\left(k^T_{1,it}\right)^{\frac{\lambda_1-1}{\lambda_1}}+(1-\theta_1)\left(k^I_{1,it}\right)^{\frac{\lambda_1-1}{\lambda_1}}\right)^{\frac{\lambda_1}{\lambda_1-1}\alpha_1-1}\left(e^{z_{1,it}}l_{1,it}\right)^{\gamma_1}+(1-\delta_T) \\
J_{b_{it}}(states_{it}) &= (1+r_t)-0.04\,(b_{it+1}-b_{it})+0.02\,(b_{it}-0.2)
\end{aligned}$$

The partial derivatives that result from applying the chain rule are given by

$$\begin{aligned}
\frac{\partial k^I_{it+1}}{\partial k^T_{1,it}} &= -\theta_2\alpha_2(k^T_{it}-k^T_{1,it})^{-\frac{1}{\lambda_2}}\left(\theta_2\left(k^T_{it}-k^T_{1,it}\right)^{\frac{\lambda_2-1}{\lambda_2}}+(1-\theta_2)\left(k^I_{it}-k^I_{1,it}\right)^{\frac{\lambda_2-1}{\lambda_2}}\right)^{\frac{\lambda_2}{\lambda_2-1}\alpha_2-1}\left(e^{z_{2,it}}l_{2,it}\right)^{\gamma_2} \\
\frac{\partial k^I_{it+1}}{\partial k^I_{1,it}} &= -(1-\theta_2)\alpha_2(k^I_{it}-k^I_{1,it})^{-\frac{1}{\lambda_2}}\left(\theta_2\left(k^T_{it}-k^T_{1,it}\right)^{\frac{\lambda_2-1}{\lambda_2}}+(1-\theta_2)\left(k^I_{it}-k^I_{1,it}\right)^{\frac{\lambda_2-1}{\lambda_2}}\right)^{\frac{\lambda_2}{\lambda_2-1}\alpha_2-1}\left(e^{z_{2,it}}l_{2,it}\right)^{\gamma_2}+ \\
& \quad +(1-\delta_I) \\
\frac{\partial k^I_{it+1}}{\partial l_{2,it}} &= \gamma_2\left(e^{z_{2,it}}\right)^{\gamma_2}(l_{2,it})^{\gamma_2-1}\left(\theta_2\left(k^T_{it}-k^T_{1,it}\right)^{\frac{\lambda_2-1}{\lambda_2}}+(1-\theta_2)\left(k^I_{it}-k^I_{1,it}\right)^{\frac{\lambda_2-1}{\lambda_2}}\right)^{\frac{\lambda_2}{\lambda_2-1}\alpha_2}
\end{aligned}$$

Consequently, the equilibrium conditions are given by:

$$
\begin{aligned}
(k^T_{1,it}):\quad & \alpha_1\theta_1\left(k^T_{1,it}\right)^{-\frac{1}{\lambda_1}}\left(\theta_1\left(k^T_{1,it}\right)^{\frac{\lambda_1-1}{\lambda_1}}+(1-\theta_1)\left(k^I_{1,it}\right)^{\frac{\lambda_1-1}{\lambda_1}}\right)^{\frac{\lambda_1}{\lambda_1-1}\alpha_1-1}(e^{z_{1,it}}l_{1,it})^{\gamma_1}=\\
&=\mathbb{E}_t\left[\frac{1}{1+r^i_t}\theta_2\alpha_2(k^T_{it}-k^T_{1,it})^{-\frac{1}{\lambda_2}}\left(\theta_2\left(k^T_{it}-k^T_{1,it}\right)^{\frac{\lambda_2-1}{\lambda_2}}+(1-\theta_2)\left(k^I_{it}-k^I_{1,it}\right)^{\frac{\lambda_2-1}{\lambda_2}}\right)^{\frac{\lambda_2}{\lambda_2-1}\alpha_2-1}(e^{z_{2,it}}l_{2,it})^{\gamma_2}\right.\\
&\left.\alpha_1(1-\theta_1)\left(k^I_{1,it+1}\right)^{-\frac{1}{\lambda_1}}\left(\theta_1\left(k^T_{1,it+1}\right)^{\frac{\lambda_1-1}{\lambda_1}}+(1-\theta_1)\left(k^I_{1,it+1}\right)^{\frac{\lambda_1-1}{\lambda_1}}\right)^{\frac{\lambda_1}{\lambda_1-1}\alpha_1-1}(e^{z_{1,it+1}}l_{1,it+1})^{\gamma_1}\right]
\end{aligned}
\tag{16}
$$

$$
\begin{aligned}
(k^I_{1,it}):\quad & \alpha_1(1-\theta_1)\left(k^I_{1,it}\right)^{-\frac{1}{\lambda_1}}\left(\theta_1\left(k^T_{1,it}\right)^{\frac{\lambda_1-1}{\lambda_1}}+(1-\theta_1)\left(k^I_{1,it}\right)^{\frac{\lambda_1-1}{\lambda_1}}\right)^{\frac{\lambda_1}{\lambda_1-1}\alpha_1-1}(e^{z_{1,it}}l_{1,it})^{\gamma_1}=\\
&=\mathbb{E}_t\left[\frac{1}{1+r^i_t}\left\{(1-\theta_2)\alpha_2(k^I_{it}-k^I_{1,it})^{-\frac{1}{\lambda_2}}\left(\theta_2\left(k^T_{it}-k^T_{1,it}\right)^{\frac{\lambda_2-1}{\lambda_2}}+(1-\theta_2)\left(k^I_{it}-k^I_{1,it}\right)^{\frac{\lambda_2-1}{\lambda_2}}\right)^{\frac{\lambda_2}{\lambda_2-1}\alpha_2-1}(e^{z_{2,it}}l_{2,it})^{\gamma_2}\right.\right.\\
&\left.\left.+(1-\delta_I)\right\}\alpha_1(1-\theta_1)\left(k^I_{1,it+1}\right)^{-\frac{1}{\lambda_1}}\left(\theta_1\left(k^T_{1,it+1}\right)^{\frac{\lambda_1-1}{\lambda_1}}+(1-\theta_1)\left(k^I_{1,it+1}\right)^{\frac{\lambda_1-1}{\lambda_1}}\right)^{\frac{\lambda_1}{\lambda_1-1}\alpha_1-1}(e^{z_{1,it+1}}l_{1,it+1})^{\gamma_1}\right]
\end{aligned}
\tag{17}
$$

$$
\begin{aligned}
(k^T_{it+1}):\quad & 1=\mathbb{E}_t\left[\frac{1}{1+r^i_t}\alpha_1\theta_1\left(k^T_{1,it+1}\right)^{-\frac{1}{\lambda_1}}\left(\theta_1\left(k^T_{1,it+1}\right)^{\frac{\lambda_1-1}{\lambda_1}}+(1-\theta_1)\left(k^I_{1,it+1}\right)^{\frac{\lambda_1-1}{\lambda_1}}\right)^{\frac{\lambda_1}{\lambda_1-1}\alpha_1-1}\right.\\
&\left.(e^{z_{1,it+1}}l_{1,it+1})^{\gamma_1}+(1-\delta_T)\right]
\end{aligned}
\tag{18}
$$

$$
(b_{it+1}):\quad 1=0.04\,(b_{it+1}-b_{it})+\mathbb{E}_t\left[\frac{1}{1+r^i_t}(1+r_{t+1})-0.04\,(b_{it+2}-b_{it+1})+0.02\,(b_{it+1}-0.2)\right]
\tag{19}
$$

$$
(l_{1,it}):\quad w_t=\gamma_1(e^{z_{1,it}})^{\gamma_1}(l_{1,it})^{\gamma_1-1}\left(\theta_1\left(k^T_{1,it}\right)^{\frac{\lambda_1-1}{\lambda_1}}+(1-\theta_1)\left(k^I_{1,it}\right)^{\frac{\lambda_1-1}{\lambda_1}}\right)^{\frac{\lambda_1}{\lambda_1-1}\alpha_1}
\tag{20}
$$

$$
\begin{aligned}
(l_{2,it}):\quad & w_t=\mathbb{E}_t\left[\frac{1}{1+r^i_t}\gamma_2\,(e^{z_{2,it}})^{\gamma_2}\,(l_{2,it})^{\gamma_2-1}\left(\theta_2\left(k^T_{it}-k^T_{1,it}\right)^{\frac{\lambda_2-1}{\lambda_2}}+(1-\theta_2)\left(k^I_{it}-k^I_{1,it}\right)^{\frac{\lambda_2-1}{\lambda_2}}\right)^{\frac{\lambda_2}{\lambda_2-1}\alpha_2}\right.\\
&\left.\alpha_1(1-\theta_1)\left(k^I_{1,it+1}\right)^{-\frac{1}{\lambda_1}}\left(\theta_1\left(k^T_{1,it+1}\right)^{\frac{\lambda_1-1}{\lambda_1}}+(1-\theta_1)\left(k^I_{1,it+1}\right)^{\frac{\lambda_1-1}{\lambda_1}}\right)^{\frac{\lambda_1}{\lambda_1-1}\alpha_1-1}(e^{z_{1,it+1}}l_{1,it+1})^{\gamma_1}\right]
\end{aligned}
\tag{21}
$$

In short, equations (16) - (21) characterize the optimal behavior of the firm.

1.1 Smolyak's algorithm

In this section, we will use Chebyshev polynomials and the Smolyak interpolation algorithm to solve the firm's optimization problem presented above. In particular, the Smolyak method fits within the class of projection methods.

The main idea of the algorithm is to find a grid of points $\mathbb{G}(q,d) \in [-1,1]$ and an approximating function $\widehat{f}(x|b,q,d) : [-1,-1]^d \to \mathbb{R}$ such that at the points $x_i \in \mathbb{G}(q,d)$, the unknown function $f(.)$ and $\widehat{f}(.|b,q,d)$ are equal, while at the points $x_i \notin \mathbb{G}(q,d)$, $d(.|b,q,d)$ is close to the unknown function $f(.)$.

For this particular application, we have three endogenous state variables, $d = 3$, and we will select $q = d + \mu = 5$, where $\mu = 2$ is a measure of the precision of the approximation. Thus, the grid of points is given by

$$\begin{aligned}
\mathbb{G}(5,3) &= \bigcup_{3 \leq |\mathbf{i}| \leq 5} \left(\mathcal{G}^{i_1} \times \mathcal{G}^{i_2} \times \mathcal{G}^{i_3}\right) \\
&= \bigcup_{(i_1,i_2,i_3)\mathbb{Z}^3_{++}:\, i_1+i_2+i_3=5} \left(\mathcal{G}^{m(i_1)} \times \mathcal{G}^{m(i_2)} \times \mathcal{G}^{m(i_3)}\right) \\
&= \left(\mathcal{G}^{m(1)} \times \mathcal{G}^{m(1)} \times \mathcal{G}^{m(3)}\right) \cup \left(\mathcal{G}^{m(1)} \times \mathcal{G}^{m(2)} \times \mathcal{G}^{m(2)}\right) \cup \left(\mathcal{G}^{m(1)} \times \mathcal{G}^{m(3)} \times \mathcal{G}^{m(1)}\right) \cup \\
&\quad \left(\mathcal{G}^{m(2)} \times \mathcal{G}^{m(1)} \times \mathcal{G}^{m(2)}\right) \cup \left(\mathcal{G}^{m(2)} \times \mathcal{G}^{m(2)} \times \mathcal{G}^{m(1)}\right) \cup \left(\mathcal{G}^{m(3)} \times \mathcal{G}^{m(1)} \times \mathcal{G}^{m(1)}\right)
\end{aligned}$$

As a result, the number of points in the grid is equal to 25 since $1 + 4n + 4\frac{n(n-1)}{2} = 25$. These points are represented in the scatter plot below.

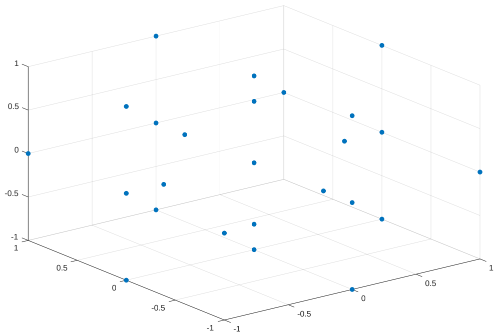

Figure 1: Smolyak grid points

The next step consist of construction a Smolyak polynomial function. Let $\widehat{f}^{d,\mu}$ denote this function in dimension d with approximation level μ. In our particular application, the Smolyak interpolant is a linear combination of tensor-product operators $p^{|i|}$ given by

$$\begin{aligned}
\widehat{f}^{3,2}(x_1, x_2, x_3; b) &= \sum_{3 \leq |i| \leq 5} (-1)^{5-|i|} \begin{pmatrix} 2 \\ 5-|i| \end{pmatrix} p^{|i|} \\
&= \sum_{3 \leq |i| \leq 5} (-1)^{5-|i|} \left(\frac{2}{(5-|i|)!(2-(5-|i|))!}\right) p^{|i|} \\
&= \frac{1}{3} p^{|3|} - 2p^{|4|} + 2p^{|5|} \\
&= \frac{1}{3} p^{1,1,1} - 2\left(p^{2,1,1} + p^{1,2,1} + p^{1,1,2}\right) + 2\left(p^{3,1,1} + \dots + p^{1,2,2}\right) \\
&= \frac{1}{3} b_{111} - 2\left(b_{111} + b_{211}\Psi_2(x_1) + b_{311}\Psi_3(x_1) + \dots\right) + \\
&\qquad + 2(b_{111} + b_{211}\Psi_2(x_1) + b_{311}\Psi_3(x_1) + b_{411}\Psi_4(x_1) + b_{511}\Psi_5(x_1) + \dots)
\end{aligned}$$

Notice that the criterion $3 \leq |i| \leq 5$ is satisfied 10 times, 1 time for $|i| = 3$, 3 times for $|i| = 4$, and 6 times for $|i| = 5$. Also notice that the number of basis functions in the polynomial is equal to the number of points in the grid, i.e. we have 25 basis functions.

The closed-form expression for Smolyak interpolation coefficients in $\widehat{f}^{3,2}$ when the multidimensional grid points and basis functions are constructed using unidimensional Chebyshev polynomials and their extrema is given by:

$$b_{\ell_1 \dots \ell_d} = \frac{2^d}{\left(m\left(i_1\right)-1\right)\cdots\left(m\left(i_d\right)-1\right)} \cdot \frac{1}{c_{\ell_1}\cdots c_{\ell_d}} \times \sum_{j_1=1}^{m(i_1)} \cdots \sum_{j_d=1}^{m(i_d)} \frac{\Psi_{\ell_1}\left(\zeta_{j_1}\right)\cdots\Psi_{\ell_d}\left(\zeta_{j_d}\right)\cdot f\left(\zeta_{j_1},\dots,\zeta_{j_d}\right)}{c_{j_1}\cdots c_{j_d}}$$

For example, for b_{211}, we get

$$\begin{aligned}
b_{211} &= \frac{2^3}{3-1} \cdot \frac{1}{c_2} \sum_{j_1=1}^{3} \frac{\Psi_2(\zeta_{j_1}) f(\zeta_{j_1},0,0)}{c_{j_1} \cdot 1 \cdot 1} \\
&= 2\left[\frac{\Psi_2(\zeta_1) f(\zeta_1,0,0)}{c_1} + \frac{\Psi_2(\zeta_2) f(\zeta_2,0,0)}{c_2} + \frac{\Psi_2(\zeta_1) f(\zeta_2,0,0)}{c_3}\right] \\
&= -f(-1,0,0) \;+\; f(1,0,0)
\end{aligned}$$

However, this formula does not apply for the constant b_{111}, which can be retrieved exploiting the fact that $\widehat{f}^{3,2}(0,0,0;b) = f(0,0,0)$.

1.2 A third order perturbation