Noise dynamically suppresses chaos in neural networks

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Noise is ubiquitous in neural systems due to intrinsic stochasticity or external drive. For deterministic dynamics, neural networks of randomly coupled units display a transition to chaos at a critical coupling strength. Here, we investigate the effect of additive white noise on the transition. We develop the dynamical mean-field theory yielding the statistics of the activity and the maximum Lyapunov exponent. A closed form expression determines the transition from the regular to the chaotic regime. Noise suppresses chaos by a dynamic mechanism, shifting the transition to significantly larger coupling strengths than predicted by local stability analysis. The decay time of the autocorrelation function does not diverge at the transition, but peaks slightly above the critical coupling strength.

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Collective phenomena that emerge in systems with many interacting degrees of freedom present a central quest in physics. In nature and technology, these systems operate in the presence of noise or time-dependent drive, rendering their dynamics stochastic or nonautonomous. Large random networks of neuron-like units are a prominent example, which exhibit chaotic dynamics [1–3]. Their information processing capabilities have been a focus in neuroscience [4] and in machine learning [5]. In particular, the capacity is optimal close to the onset of chaos [6–8].

In the absence of noise or inputs, networks of randomly coupled rate neurons display a transition from a fixed point to deterministic chaos at a critical coupling strength [1]. The transition is well understood by dynamical mean-field theory, originally developed for spin glasses [9]. In particular, the onset of chaos is equivalent to the emergence of a decaying autocorrelation function, the dynamical order parameter of the transition. The decay of the autocorrelation function has been used in several subsequent studies as a criterion for chaotic dynamics [10–12]. It was further demonstrated that the transition happens precisely when the fixed point becomes linearly unstable. This identifies the spectral radius of the random connectivity matrix [13, 14] as the parameter controlling the transition. In stochastic systems, a decaying autocorrelation function does not indicate chaotic dynamics. The stochasticity causes perpetual decorrelation also for regular dynamics. Thus the transition to chaos, if existent at all, must be qualitatively different to the deterministic case.

It is controversially discussed, whether the deterministic picture explains a transition to chaotic dynamics in spiking neural networks [15], since the realization noise of the spiking dynamics is missing. It was shown that the spiking noise smooths the transition [16] and that the correlation time in spiking simulations does not peak at the point where the transition is predicted by the spectral radius of the Jacobian [17, 18]. However, the fundamental mechanism by which noise affects the maximum Lyapunov exponent and the location of the transition are not understood.

The effect of noise on the transition to chaos has so far only been studied for a time-discrete system [19], whose order parameter is a single number, the variance of the mean-field. Therefore these results can not be transferred to time-continuous systems whose order parameter, in the deterministic case, is a function of time-lag.

To focus on the generic influence of noise on the transition to chaos, we subject the seminal model by Sompolinsky et al. [1] to white noise and develop the dynamical mean-field theory for the resulting stochastic system. The autocorrelation function can be understood as the motion of a classical particle in a potential and we find that the noise amounts to the initial kinetic energy. We show that the loss of local stability is only a necessary condition for the transition from regular to chaotic dynamics. We derive an exact condition showing that noise suppresses the transition significantly stronger than expected from its effect on local stability. This is explained by a dynamic effect of the noise: it sharpens the autocorrelation function decreasing the maximum Lyapunov exponent. The Lyapunov exponent becomes positive precisely at the point when the curvature of the autocorrelation function at time-lag zero becomes negative. This is equivalent to a parity between the strength of the incoming fluctuations and the variance of a single unit. Furthermore the decay time of the autocorrelation function does not diverge at the transition. Its maximum is reduced by noise and occurs slightly above the critical coupling strength.

We study the continous-time dynamics of a random network of N neurons, whose states $x_i(t) \in \mathbb{R}$, $i = 1, \ldots, N$, evolve according to the system of stochastic differential equations

$$\frac{dx_i}{dt} = -x_i + \sum_{j=1}^{N} J_{ij}\phi(x_j) + \xi_i(t)\,. \tag{1}$$

The J_{ij} are independent and identically Gaussian distributed random coupling weights with zero mean and variance g^2/N, where the intensive gain parameter g controls the recurrent coupling strength or, equivalently, the weight heterogeneity of the network. The $\xi_i(t)$ are independent Gaussian white noise processes with autocorrelation function $\langle \xi_i(t)\xi_i(s)\rangle = 2\sigma^2\delta(t-s)$. We choose the sigmoidal transfer function $\phi(x) = \tanh(x)$, such that without noise, i.e., $\sigma = 0$, the model agrees with the one studied in [1].

The dynamical system (1) contains two sources of randomness: quenched disorder due to the random coupling weights and temporal fluctuations by the noise. A particular realization of the random couplings J_{ij} defines a fixed network configuration and its dynamical properties usually vary between different realizations. For large network size N, however, certain quantities are self-averaging meaning that their values for a typical realization can be obtained by an average over network configurations [20]. An important example is the population-averaged autocorrelation function.

We study the dynamical mean-field theory, which, in the limit of $N \to \infty$, describes the statistical properties of the system under the joint distribution of disorder, noise and possibly random initial conditions. The theory can be derived via a heuristic "local chaos" assumption [21] or using a generating functional formulation [9, 22], while a rigorous proof employs large deviation techniques [23]. The general idea is that for large network size N the local recurrent input $\sum_{j=1}^{N} J_{ij}\phi(x_j)$ in (1) approaches a Gaussian process $\eta(t)$ with self-consistently determined statistics. As a result we obtain the dynamical mean-field equation characterizing the statistics of a typical neuron:

$$\frac{dx}{dt} = -x + \eta(t) + \xi(t)\,. \tag{2}$$

Here, $\xi(t)$ is a Gaussian white noise process as in (1), independent of $\eta(t)$. Because the random couplings J_{ij} have zero mean, the Gaussian process $\eta(t)$ is centered, $\langle \eta(t)\rangle = 0$, and thus fully specified by its autocorrelation function

$$\langle \eta(t)\eta(s)\rangle = g^2\langle \phi(x(t))\phi(x(s))\rangle\,. \tag{3}$$

The expectation on the right-hand side is taken with respect to the distribution of $x(t)$.

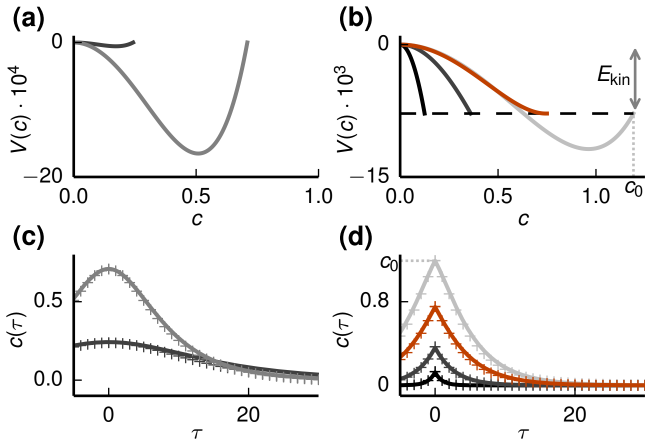

Figure 1: **Solution of the mean-field theory.** Upper row: Self-consistent potential (7) for noise-free case $\sigma = 0$ **(a)** and noisy case $\sigma = \sqrt{0.125}$ **(b)**. Coupling strength g encoded in grayscale from black to light gray with $g \in [0, 1.2, 1.48, 1.7]$ and in red for $g = 1.48$ indicating the critical coupling (12) in the noisy case. Dashed line shows the initial kinetic energy $E_{\mathrm{kin}} = \sigma^4/2$. Lower row: Corresponding autocorrelation function (solid line) in comparison to simulations (crosses) for noise free **(c)** and noisy case **(d)**. The largest c-values of the curves in panel **(a)**,**(b)** mark c_0 and thus correspond to the values at zero in **(c)**,**(d)**, as indicated for $g = 1.7$ (gray dotted lines). Network size in simulations is $N = 10000$.

Our goal is to determine the mean-field autocorrelation function $\langle x(t)x(s)\rangle$, which also describes the population-averaged autocorrelation function due to self-averaging. Assuming that $x(t)$ is a stationary process, $c(\tau) := \langle x(t+\tau)x(t)\rangle$ obeys the differential equation

$$\ddot{c} = \frac{d^2c}{d\tau^2} = c - g^2 f_\phi(c, c_0) - 2\sigma^2\delta(\tau) \tag{4}$$

with $c_0 = c(0) = \langle x(t)^2\rangle$. The Dirac δ inhomogeneity originates from the white noise autocorrelation function and is absent in [1]. In (4) we write $\langle \phi(x(t+\tau))\phi(x(t))\rangle = f_\phi(c(\tau), c_0)$, introducing the notation

$$f_u(c(\tau), c_0) = \iint u\left(\sqrt{c_0 - \tfrac{c(\tau)^2}{c_0}}\,z_1 + \tfrac{c(\tau)}{\sqrt{c_0}}\,z_2\right) u\left(\sqrt{c_0}\,z_2\right)\, Dz_1 Dz_2 \tag{5}$$

with the Gaussian integration measure $Dz = \exp(-z^2/2)/\sqrt{2\pi}\,dz$ and an arbitrary function $u(x)$. This representation holds since $x(t)$ is itself a Gaussian process. Note that (5) reduces to one-dimensional integrals for $f_u(c_0, c_0) = \langle u(x)^2\rangle$ and $f_u(0, c_0) = \langle u(x)\rangle^2$, where x is Gaussian distributed with zero mean and variance c_0.

We reformulate (4) as a one-dimensional motion of a classical particle in a potential, i.e.,

$$\ddot{c} = -V'(c) - 2\sigma^2\delta(\tau)\,, \tag{6}$$

where we define

$$V(c) = V(c; c_0) = -\frac{1}{2}c^2 + g^2 f_\Phi(c, c_0) - g^2 f_\Phi(0, c_0)\,. \tag{7}$$

with $\Phi(x) = \int_0^x \phi(y)\,dy$ and $\partial/\partial c\, f_\Phi(c, c_0) = f_\phi(c, c_0)$ following from Price's theorem [24]. The potential (7) depends on the initial value c_0, which has to be determined self-consistently. We obtain c_0 from classical energy conservation $\frac{1}{2}\dot{c}^2 + V(c) = \text{constant}$. Considering $\tau \geq 0$ and the symmetry of $c(\tau)$, the noise term in (6) amounts to an initial velocity $\dot{c}(0+) = -\sigma^2$ and thus to the kinetic energy $\frac{1}{2}\dot{c}^2(0+) = \sigma^4/2$. Considering the shape of the potential (Figure 1a,b) and that $|c(\tau)| \leq c_0$, the solution $c(\tau)$ and its first derivative must approach zero as $\tau \to \infty$. Thus we obtain the self-consistency condition for c_0 as

$$\frac{1}{2}\sigma^4 + V(c_0; c_0) = V(0; c_0) = 0. \tag{8}$$

For the noiseless case Figure 1a,c shows the resulting potential and the corresponding self-consistent autocorrelation function $c(\tau)$, whose decay is equivalent to chaotic dynamics [1]. Approaching the transition from above, $g \to g_c = 1$, the decay time diverges due to the transition from chaos to a fixed point. This picture breaks down in the stochastic system (Figure 1 b,d), where $c(\tau)$ is always decaying and has a kink at zero. The analytical predictions are in excellent agreement with the population averaged autocorrelation function obtained from numerical simulations showing that the self-averaging property is fulfilled. In the following we will derive a condition for the transition from regular to chaotic dynamics in the noisy case.

The maximum Lyapunov exponent quantifies how sensitive the dynamics depends on the initial conditions [25]. It measures the asymptotic growth rate of infinitesimal perturbations. For stochastic dynamics the stability of the solution for a fixed realization of the noise is also characterized by the maximum Lyapunov exponent [26]: If it is negative, trajectories with different initial conditions converge to the same time-dependent solution, i.e., the dynamics is stable. If it is positive, the distance between two initially arbitrary close trajectories grows exponentially in time, i.e., the dynamics exhibits sensitive dependence on the initial conditions and is hence chaotic.

The linear stability of the dynamical system (1) is analyzed via the variational equation

$$\frac{d}{dt} y_i(t) = -y_i(t) + \sum_{j=1}^{N} J_{ij}\phi'(x_j(t))\, y_j(t)\,, \tag{9}$$

$i = 1, \ldots, N$, describing the temporal evolution of an infinitesimal deviation $y_i(t)$ about a reference trajectory $x_i(t)$. The maximum Lyapunov exponent can then be

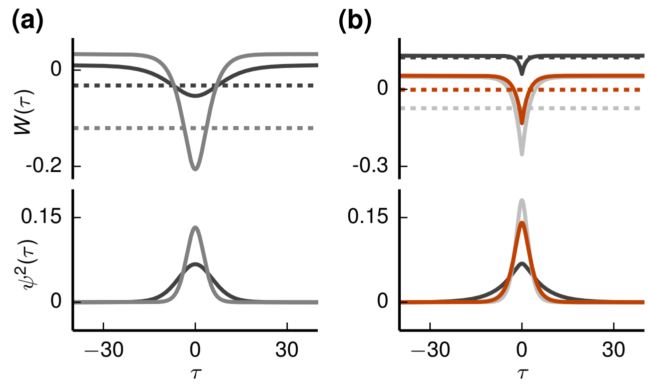

Figure 2: **Solution of Schrödinger equation.** Upper part of vertical axis: Quantum potential W (solid line) and ground state energy E_0 (dashed line) for noise free case **(a)** and noisy case **(b)**. Lower part of vertical axis: Corresponding ground state wave functions. Parameters and color code same as in Figure 1 (noisy case for $g = 0$ left out here).

defined as

$$\lambda_{\max} = \lim_{t\to\infty} \frac{1}{2t} \ln \sum_{i=1}^{N} y_i^2(t)\,, \tag{10}$$

if a generic initial perturbation $y_i(0)$ with $\sum_{i=1}^{N} y_i^2(0) = 1$ is chosen.

Typically, Lyapunov exponents cannot be calculated analytically. However, for large network size N one can obtain a mean-field description of the variational equation (9) and introduce a dynamical mean-field variable $y(t)$. Self-averaging suggests that the squared Euclidean norm appearing in the definition of the maximum Lyapunov exponent (10) can be approximated as $\sum_{i=1}^{N} y_i^2(t) \approx N\,K(t,t)$. Here, $K(t,s) = \langle y(t)y(s)\rangle$ is the mean-field autocorrelation function, which obeys the linear partial differential equation $(\partial_t + 1)(\partial_s + 1)\,K(t,s) = g^2 f_{\phi'}(c(t-s), c_0)\,K(t,s)$. A separation ansatz in the coordinates $\tau = t - s$ and $T = t + s$ then yields an eigenvalue problem in the form of a time-independent Schrödinger equation [1, 16]

$$\left[-\partial_\tau^2 - V''(c(\tau))\right]\psi(\tau) = E\,\psi(\tau), \tag{11}$$

where now τ plays the role of a spatial coordinate. The ground state energy E_0 of (11) determines the asymptotic growth rate of $K(t,t)$ as $t \to \infty$ and, hence, the maximum Lyapunov exponent via $\lambda_{\max} = -1 + \sqrt{1 - E_0}$. Therefore, the dynamics is predicted to become chaotic if $E_0 < 0$. The negative second derivative of the classical potential $V(c)$ evaluated along the self-consistent autocorrelation function $c(\tau)$ yields the quantum potential $W(\tau) = -V''(c(\tau)) = 1 - g^2 f_{\phi'}(c(\tau), c_0)$. It is shown together with the ground state energy and wave function in Figure 2. The latter are obtained as solutions of a finite difference discretization of (11).

For the noiseless case Sompolinsky et al. [1] showed that a decaying autocorrelation function corresponds to a positive maximum Lyapunov exponent. This follows from the observation that for $g > 1$ the derivative of the self-consistent autocorrelation function $\dot{c}(\tau)$ solves the Schrödinger equation with $E = 0$. Because $\dot{c}(\tau)$ is an eigenfunction with a single node, there exists a ground state with energy $E_0 < 0$. Hence, the dynamics is chaotic and the maximum Lyapunov exponent crosses zero at $g = 1$ (Figure 3a).

In the presence of noise the maximum Lyapunov exponent becomes positive at a critical coupling strength $g_c > 1$; depending on the noise intensity the transition is shifted to larger values (Figure 3a). The mean-field prediction $\lambda_{\max} = -1 + \sqrt{1 - E_0}$ shows excellent agreement with the maximum Lyapunov exponents obtained in simulations using a standard algorithm [25]. Since the ground state energy E_0 must be larger than the minimum $W(0) = 1 - g^2\langle[\phi'(x)]^2\rangle$ of the quantum potential, an upper bound for $\lambda_{\max}$ is provided by $-1+g\sqrt{\langle[\phi'(x)]^2\rangle}$ leading to a necessary condition for a positive maximum Lyapunov exponent. Interestingly, $g\sqrt{\langle[\phi'(x)]^2\rangle}$ is also the spectral radius of the Jacobian matrix (shifted by one) in (9) as estimated by random matrix theory [13, 14]. Therefore, the dynamics are expected to become locally unstable if this spectral radius exceeds unity. However, close to the critical coupling g_c, the maximum Lyapunov exponent is clearly smaller than the upper bound, which is a good approximation only for small g (Figure 3a, inset). This confirms the condition being only necessary; the actual transition occurs at substantially larger coupling strengths (Figure 3b).

We now set out to derive the exact transition curve, i.e., the curve (g_c, σ_c) in parameter space at which the system becomes chaotic. To this end we determine a ground state with vanishing energy $E_0 = 0$. As in the noiseless case, $\dot{c}(\tau)$ solves (11) for $E = 0$, except at $\tau = 0$ where it exhibits a jump caused by the noise. However, due to linearity $|\dot{c}(\tau)|$ would be a continuous and symmetric solution with zero nodes. Therefore, if its derivative is continuous as well, requiring $\ddot{c}(0+) = 0$, it constitutes the searched for ground state. This is in contrast to the noiseless case, where $\dot{c}(\tau)$ corresponds to the first excited state. Consequently, with (4) we find the transition condition

$$g_c^2\langle\phi^2(x)\rangle - c_0 = 0\,, \tag{12}$$

in which c_0 is determined by the self-consistency condition (8) resulting in the transition curve (g_c, σ_c) (Figure 3b).

At the transition the classical self-consistent potential has a horizontal tangent at c_0, while in the chaotic regime a minimum emerges (Figure 1b). This implies that the curvature $\ddot{c}(0+)$ of the autocorrelation function at zero changes sign from positive to negative (Figure 1d). Furthermore, according to (12) the system becomes chaotic precisely when the variance $c_0 = \langle x^2\rangle$ of a typical single unit equals the variance $g^2\langle\phi^2(x)\rangle$ of the recurrent input. Close to the transition a standard perturbative approach shows that $\lambda_{\max}$ is proportional to $g^2\langle\phi^2(x)\rangle - c_0$, indicating a self-stabilizing effect: Since both terms grow with increasing coupling g, the growth of their difference is attenuated, qualitatively explaining the reduction of slope of the maximum Lyapunov exponent in the vicinity of the transition (Figure 3a).

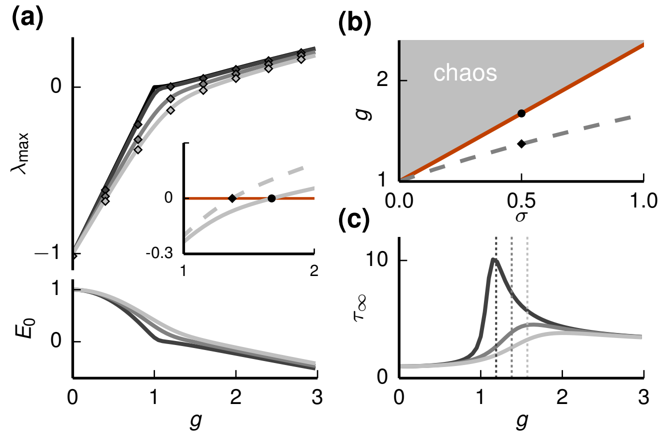

Figure 3: **Transition to chaos. (a)** Upper part of vertical axis: Maximal Lyapunov exponent $\lambda_{\max}$ as a function of the coupling strength g for different noise levels encoded in grayscale from black to light gray with $\sigma \in [0, \sqrt{0.0125}, \sqrt{0.125}, \sqrt{0.25}]$. Numerical solution (solid line) and simulation (diamonds). Network size in simulations is $N = 5000$. Comparison to upper bound for maximum Lyapunov exponent $-1+g\sqrt{\langle[\phi'(x)]^2\rangle}$ (dashed) for $\sigma = \sqrt{0.25}$ in inset. Zero crossings marked with dot and diamond. Lower part of vertical axis: Ground state energy E_0 as function of g. **(b)** Phase diagram with transition curve (solid red curve) obtained from (12) and necessary condition $g^2\langle{\phi'}^2(x)\rangle > 1$ (gray dashed curve). Dot and diamond correspond to zero crossings in inset in (a). **(c)** Asymptotic decay time constant of autocorrelation function. Vertical dashed lines mark the transition to chaos. Color code as in (a).

The condition (12) predicts the transition to happen at significantly larger coupling strengths compared to the necessary condition $g\sqrt{\langle[\phi'(x)]^2\rangle} > 1$ (Figure 3b). This is surprising as the necessary condition also implies locally unstable dynamics, as noted above. This difference is explained in an illustrative manner, considering that the effect of noise is twofold in the continuous-time system. First, because $\phi'(x)$ is maximal at the origin, noise reduces the averaged squared slope in $g^2\langle[\phi'(x)]^2\rangle$, thereby stabilizing the dynamics. We interpret this as an essentially static effect since this can be fully attributed to the increase of the instantaneous variance c_0 due to the noise. However, in addition noise reduces the decay time of the autocorrelation function (Figure 1c,d), which sharpens the quantum potential (Figure 2). This shifts the ground-state energy to larger values resulting in a further decreased Lyapunov exponent. Thus we identify a second stabilization mechanism, which depends on the shape of the autocorrelation function and thus constitutes a dynamical effect.

Analytical results for the transition to chaos in stochastic networks so far only exist for time-discrete dynamics [19]. Their mean-field theory corresponds to static solutions of the time-continuous system [1, 21]. The absence of temporal correlations greatly simplifies the analysis, reducing the order parameter to a single real number, the variance of the mean-field. The necessary condition in the time-continuous stochastic system found here is therefore also sufficient for the transition to chaos in those systems [19, their eq. 13].

Finally, we consider the effect of the noise on the asymptotic decay time $\tau_\infty^{-1} = \sqrt{1 - g\langle\phi'(x)\rangle^2}$ of the autocorrelation function (Figure 3d). For low noise the decay constant shows a sharp peak at the transition, reflecting the diverging time scale in the noise-free case. For larger noise amplitudes this peak is strongly suppressed. Remarkably, the decay time does not have its maximum at the transition, but rather peaks at larger coupling values. This observations suggest that the absence of a diverging time scale in spiking neural networks [17] is due to the spiking noise.

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[1] H. Sompolinsky, A. Crisanti, and H. J. Sommers, Phys. Rev. Lett. **61**, 259 (1988).
[2] C. van Vreeswijk and H. Sompolinsky, Science **274**, 1724 (1996).
[3] M. Monteforte and F. Wolf, Phys. Rev. Lett. **105**, 268104 (2010).
[4] W. Maass, T. Natschläger, and H. Markram, Neural Comput. **14**, 2531 (2002).
[5] H. Jaeger and H. Haas, Science **304**, 78 (2004).
[6] R. Legenstein and W. Maass, Neural Networks **20**, 323 (2007).
[7] D. Sussillo and L. F. Abbott, Neuron **63**, 544 (2009).
[8] T. Toyoizumi and L. Abbott, Phys. Rev. E **84**, 051908 (2011).
[9] H. Sompolinsky and A. Zippelius, Phys. Rev. B **25**, 6860 (1982).
[10] K. Rajan, L. Abbott, and H. Sompolinsky, Phys. Rev. E **82**, 011903 (2010).
[11] J. Aljadeff, M. Stern, and T. Sharpee, Phys. Rev. Lett. **114**, 088101 (2015).
[12] O. Harish and D. Hansel, PLoS Comput Biol **11**, e1004266 (2015).
[13] H. Sommers, A. Crisanti, H. Sompolinsky, and Y. Stein, Phys. Rev. Lett. **60**, 1895 (1988).
[14] K. Rajan and L. Abbott, Phys. Rev. Lett. **97**, 188104 (2006).
[15] S. Ostojic, Nat. Neurosci. **17**, 594 (2014).
[16] J. Kadmon and H. Sompolinsky, Phys. Rev. X **5**, 041030 (2015).
[17] R. Engelken, F. Farkhooi, D. Hansel, C. van Vreeswijk, and F. Wolf, bioRxiv p. 017798 (2015).
[18] S. Ostojic, bioRxiv p. 020354 (2015).
[19] L. Molgedey, J. Schuchhardt, and H. Schuster, Phys. Rev. E **69**, 3717 (1992).
[20] K. Fischer and J. Hertz, *Spin glasses* (Cambridge University Press, 1991).
[21] S.-I. Amari, Systems, Man and Cybernetics, IEEE Transactions on pp. 643–657 (1972).
[22] A. Crisanti and H. Sompolinsky, Phys. Rev. A **36**, 4922 (1987).
[23] T. Cabana and J. Touboul, J. statist. Phys. **153**, 211 (2013).
[24] A. Papoulis, *Probability, Random Variables, and Stochastic Processes* (McGraw-Hill, Boston, Massachusetts, 1991), 3rd ed.
[25] J.-P. Eckmann and D. Ruelle, Reviews of modern physics **57**, 617 (1985).
[26] G. Lajoie, K. K. Lin, and E. Shea-Brown, Phys. Rev. E **87**, 052901 (2013).