SVM

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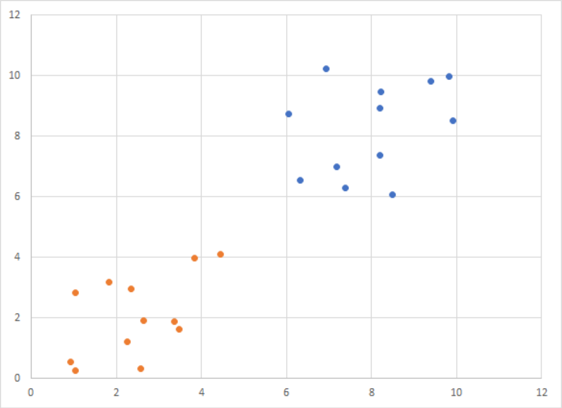
Recap and Today

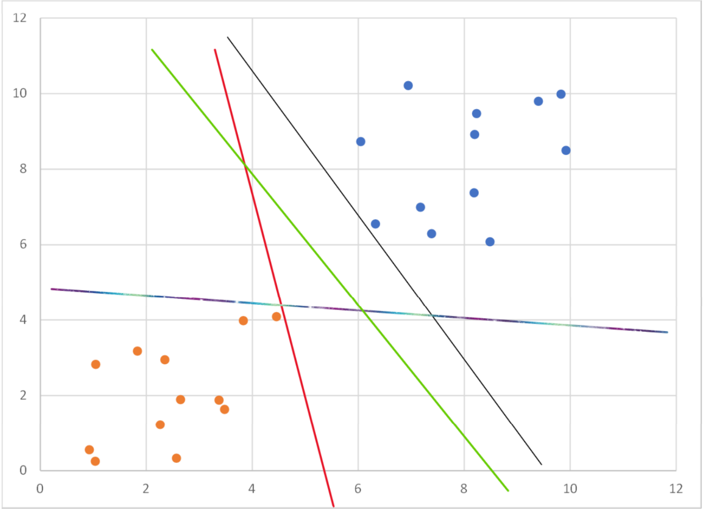
- SVM
- More of SVM

Support Vector Machine - Introduction

- Supervised Learning Algorithm for Classification
- Given N training points in which n_1 are of type A, n_2 are of type B, draw the **best** line(plane)
- To begin with, assume that the training points are linearly separable



Which is the best line?



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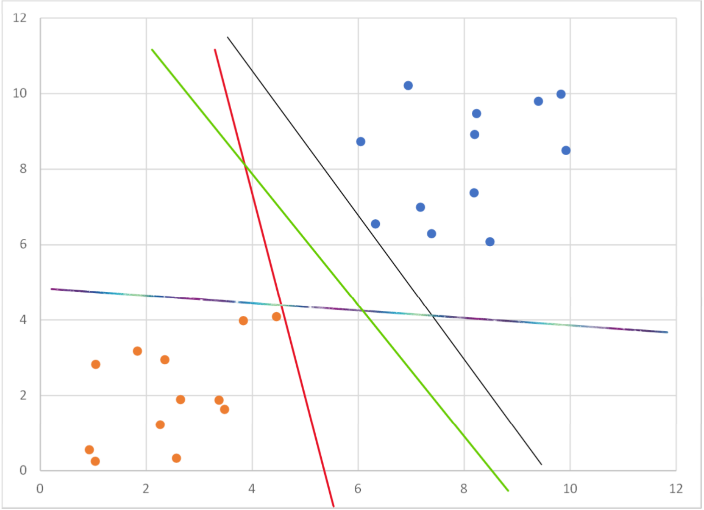

What makes us think it is the green line? Can we make the ideas a bit more precise?

Notations

Let the data-set be denoted as -

S.No	X_1	X_2	Y (+1 or -1)
1	x_{11}	x_{12}	+1
2	x_{21}	x_{22}	-1
3	x_{31}	x_{32}	-1
. . .	. . .	. . .	. . .
N	x_{N1}	x_{N2}	+1

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Let the equation of the line be $w_1 x_1 + w_2 x_2 + b = 0$
We need to determine w_1, w_2 and b

Obtaining w_1, w_2 and b

- Arbitratily choose w_1, w_2 and b such that
  $w_1 x_{i1} + w_2 x_{i2} + b > 0$ whenever $y_i = +1$
  $w_1 x_{i1} + w_2 x_{i2} + b < 0$ whenever $y_i = -1$
- Simply put, for all points, $y_i(w_1 x_{i1} + w_2 x_{i2} + b) > 0$
- Criteria - Consider a line. Find the distance of the line from all the training examples (or points). Look at the minimum of all these distances.
  We are interested in the line for which this minimum distance is as large as possible.
- $$\max_{(w_1,w_2,b)} \left(\min_{i=\{1,2,\ldots,N\}} \frac{|w_1 x_{i1} + w_2 x_{i2} + b|}{\sqrt{w_1^2 + w_2^2}} \right)$$
- The **max** in the equation in **maximize** and **min** in the equation is **minimum**

Optimization Problem

The optimization problem thus becomes -

$$\max_{(w_1,w_2,b)} \left(\min_{i=\{1,2,....,N\}} \frac{|w_1 x_{i1} + w_2 x_{i2} + b|}{\sqrt{w_1^2 + w_2^2}} \right)$$

The **max** in the equation in **maximize** and **min** in the equation is **minimum**

subject to the following **N** constraints - $y_i(w_1 x_{i1} + w_2 x_{i2} + b) > 0$

Optimization Problem

The optimization problem thus becomes -

$$\max_{(w_1,w_2,b)} \left(\frac{1}{\sqrt{w_1^2 + w_2^2}} \left[\min_{i=\{1,2,....,N\}} \left(|w_1 x_{i1} + w_2 x_{i2} + b| \right) \right] \right)$$

subject to the following **N** constraints -

$$y_i(w_1 x_{i1} + w_2 x_{i2} + b) > 0$$

Optimization Problem

The optimization problem thus becomes -

$$\max_{(w_1,w_2,b)} \left(\frac{1}{\sqrt{w_1^2 + w_2^2}} \left[\min_{i=\{1,2,....,N\}} (|w_1 x_{i1} + w_2 x_{i2} + b|) \right] \right)$$

subject to the following **N** constraints -

$$y_i(w_1 x_{i1} + w_2 x_{i2} + b) > 0$$

As scaling all w_1, w_2 and b by the same factor (non-zero) doesn't change the line (or hyperplane), we will choose w_1, w_2 and b such that -

$\min_{i=\{1,2,....,N\}}(|w_1 x_{i1} + w_2 x_{i2} + b|) = 1$
The **min** in the above equation is **minimum**

Optimization Problem

The optimization problem thus becomes -

$$\max_{(w_1,w_2,b)} \left(\frac{1}{\sqrt{w_1^2 + w_2^2}} \right)$$

subject to the following **2N** constraints -

$$y_i(w_1 x_{i1} + w_2 x_{i2} + b) > 0$$

$$\min_{i=\{1,2,....,N\}}(|w_1 x_{i1} + w_2 x_{i2} + b|) = 1$$

Optimization Problem

The optimization problem thus becomes -

$$\min_{(w_1,w_2,b)} \left(\frac{w_1^2 + w_2^2}{2} \right)$$

subject to the following **2N** constraints -

$$y_i(w_1 x_{i1} + w_2 x_{i2} + b) > 0$$

$$\min_{i=\{1,2,....,N\}}(|w_1 x_{i1} + w_2 x_{i2} + b|) = 1$$

Optimization Problem

The optimization problem thus becomes -

$$\min_{(w_1,w_2,b)} \left(\frac{w_1^2 + w_2^2}{2}\right)$$

subject to the following **2N** constraints -

$$y_i(w_1x_{i1} + w_2x_{i2} + b) > 0$$

$$\min_{i=\{1,2,....,N\}}(|w_1x_{i1} + w_2x_{i2} + b|) = 1$$

$\min_{i=\{1,2,....,N\}}(|w_1x_{i1} + w_2x_{i2} + b|) = 1$ implies -
$|w_1x_{i1} + w_2x_{i2} + b| >= 1 \qquad \forall i = \{1, 2, ..., N\}$ or

$|y_i(w_1x_{i1} + w_2x_{i2} + b)| >= 1 \qquad \forall i = \{1, 2, ..., N\}$ or
$y_i(w_1x_{i1} + w_2x_{i2} + b) >= 1 \qquad \forall i = \{1, 2, ..., N\}$ The implies condition is not both ways, but still it can be replaced in this problem because ??

Optimization Problem

The implies condition can be replaced in this problem because ??
After some algebra, the optimization problem becomes -

$$\min_{(w_1,w_2,b)} \left(\frac{w_1^2 + w_2^2}{2} \right)$$

subject to the following **N** constraints -

$$y_i(w_1 x_{i1} + w_2 x_{i2} + b) >= 1 \qquad \forall i = \{1, 2, ..., N\}$$

Why the name support vector?

Consider the following two optimizaion problems -

$$\min_{(w_1,w_2,b)} \left(\frac{w_1^2 + w_2^2}{2}\right)$$

w_1^*, w_2^*, b^*

subject to the following **N** constraints -

$$y_i(w_1x_{i1} + w_2x_{i2} + b) >= 1 \qquad \forall i = \{1, 2, ..., N\}$$

$$\min_{(w_1,w_2,b)} \left(\frac{w_1^2 + w_2^2}{2}\right) - \sum_{i=1}^{N} \alpha_i(y_i(w_1x_{i1} + w_2x_{i2} + b) - 1)$$

subject to no constraints, only the fact that all α_i's are either zero or positive

Which of these two optimization problems has a lower value?

Why the name support vector?

$$\min_{(w_1,w_2,b)} \left(\frac{w_1^2 + w_2^2}{2}\right)$$

subject to the following **N** constraints -

$$y_i(w_1x_{i1} + w_2x_{i2} + b) >= 1 \qquad \forall i = \{1, 2, ..., N\}$$

$$\min_{(w_1,w_2,b)} \left(\frac{w_1^2 + w_2^2}{2}\right) - \sum_{i=1}^{N} \alpha_i(y_i(w_1x_{i1} + w_2x_{i2} + b) - 1)$$

subject to no constraints, only the fact that all α_i's are either zero or positive

The one in the red box has a lower value. Now, let us keep playing with putting different values of α_i's.

Thank you for your attention