# Physics 514 – Percolation

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# Chapter 1

# Percolation



## 1.1 Introduction

While the Monte Carlo and molecular dynamics methods can, in principle, be used to study any physical system, difficulties appear close to phase transitions. In our investigation of phase transitions we will start with percolation, a very simple and purely geometric topic.

Although there is no dynamics of any kind involved, percolation nevertheless exhibits complex behavior and a phase transition. This simple model will allow us to introduce many concepts that we will need later for the simulation of dynamic systems. These concepts include

- Phase transitions
- Scaling
- Finite size effects and finite size scaling (FSS)
- Monte Carlo simulations
- Analytic and numeric renormalization group methods

Percolation problems appear in a wide range of physical contexts

- *Oil fields*: The 'Swiss cheese model' can be used as a simplified model for the storage and flow of liquids in porous media. The porous medium is modeled by spherical cavities distributed randomly in the surrounding medium. If the density of cavities gets larger they start to overlap and form large clusters of cavities. Important questions include:
  - For oil drilling we want to know how the amount of oil stored varies with the size of the oil field
  - fluids can only flow through the medium if there is a cluster connecting opposite sides. Such a cluster is called a 'percolating' cluster. What is the density of cavities is needed to create a percolating cluster?
  - How does the speed of the fluid flow through the medium depend on the density?
- Forest fires: we model forest fires by assuming that a tree neighboring a burning tree catches fire with probability $p$. Fire fighters will want to know:
  - How much of the forest will burn?
  - Will the fire spread throughout the whole forest?
- Spread of diseases: The spread of diseases can be modeled in a simplified way similar to the forest fire model. Now the important question is: what part of the population will fall ill? Will the disease be an epidemic, spreading throughout the whole country (or the whole world)?

- Conductance of wire meshes:
  - How many links can be cut in a wire mesh so that it is still conducting?
  - What is the resistivity as a function of the ratio of cut links?
- Vulnerability of the internet: The internet was designed in 1964 to make computer networks reliable even in the case of attacks in war time. The most urgent question is:
  - What portion of the internet is still connected if a fraction $p$ of switches fails?
- Gelation of liquids: Gelation of liquids can be modeled by allowing a molecule to form a bond with a neighboring molecule with probability $p$. Again the interesting questions are
  - What is the average size of a molecule cluster as a function of $p$.
  - What is the critical concentration at which the largest molecule cluster percolates and the liquid solidifies?
- baking of cookies: This is an example given in Gould and Tobochnik[1] Distribute several drops of dough randomly on a cookie tray. When baking the dough will spread and the cookies that are close will bake together. If the number of drops is too large we will obtain a percolating cookie, spanning the baking tray from one edge to the other.



[1] Introduction to Computer Simulation Methods, Harvey Gould, Jan Tobochnik, Wolfgang Christian, Addison-Wesley (2007), ISBN 9780805377583

you can find a more extensive discussion of the problem and its applications in 'Introduction To Percolation Theory' by Dietrich Gould and Ammon Aharony[2].

## 1.2 Site percolation on a square lattice

The simplest of all percolation models is the site percolation model on a two-dimensional square lattice. Each square shall be occupied with probability $p$. Occupied squares that share edges form a cluster.

For large $p$ we obtain a percolating cluster, i.e. a cluster that spans the lattice from one edge to the other. In the infinite lattice limit there is a sharp transition at a critical density $p_c$. For $p < p_c$ there is *never* a percolating cluster, and for $p > p_c$ there is *always* a percolating cluster. We will ask the following questions:



- What is the critical concentration $p_c$
- How do the number of clusters and the average cluster size $S$ depend on $p$?
- What is the probability $P$ that a site is on the percolating cluster?
- What is the resistivity/conductance of the percolating cluster?
- How does finite lattice size $L$ change the results?
- How do the results depend on the lattice structure and on the specific percolation model used?

[2]Introduction To Percolation Theory, Dietrich Stauffer, Ammon Aharony, CRC Press, (1994), ISBN 9780748402533

The answer to the last question will be that close to the critical concentration $p_c$ the properties depend only on the dimensionality $d$ and on the type (continuum or lattice), but the do not depend on the specific lattice structure or percolation model. Thus our results obtained for the percolation transition on the square lattice will be 'universal' in the sense that they will apply also to all other two-dimensional percolation models, like the forest fire model, the spread of diseases, and the problems encountered when baking cookies.



## 1.3 Exact solutions

### 1.3.1 One dimension

In one dimension the problem can be solved exactly. We will use this case to introduce some of the quantities of interest.

The one-dimensional system is just a chain, and any 'hole' in this chain will break up the percolating cluster. Thus the critical percolation probability is $p_c = 1$.

The probability that a site is the left edge of a cluster of *finite* size $s$ is simply

$$n_s = (1-p)^2 p^s, \tag{1.1}$$

as the cluster consists of $s$ occupied sites neighbored by two empty ones. The probability that a random site is anywhere on an $s$-site cluster is $sn_s$. The sum over all cluster sizes $s$ leads to a sum rule which is not only valid in one dimension: the probability that a site is on

a cluster of any size is just $p$:

$$P + \sum_s s n_s = p \tag{1.2}$$

Here $P$ is the probability that a site is on the *infinite* percolating cluster (there is only one). In one dimensions, $P = 1$ for $p = p_c = 1$ and $P = 0$ otherwise.

The average cluster size is

$$S = \frac{\sum_s s^2 n_s}{\sum_s s n_s} = \frac{(1+p)}{1-p} \sim (p_c - p)^{-1} \tag{1.3}$$

At this phase transition, the average cluster size diverges.

The pair correlation function $g(r)$ is the probability that for an occupied site a site at distance $r$ belongs to the same cluster. In one dimensions it is

$$g(r) = p^{|r|} = \exp(-|r|/\xi), \tag{1.4}$$

where

$$\xi = -\frac{1}{\log p} \sim (p_c - p)^{-1} \tag{1.5}$$

is the *correlation length*. Again, there is a sum rule for the pair correlation function:

$$\sum_{r=-\infty}^{\infty} g(r) = S \tag{1.6}$$



### 1.3.2 Bethe lattice – Infinite dimensions

Another exact solution is available in infinite dimensions on the Bethe lattice. The Bethe lattice or Cayley tree in which every site is connected to three or more neighbors. The Bethe lattice has no loops, meaning that it is not

possible to follow along different bonds to get back to the same place.

This lattice is infinite dimensional: in $d$ dimensions the volume scales $\sim L^d$ and the surface $\sim L^{d-1}$, thus surface $\propto$ volume$^{1-1/d}$. The Bethe lattice with $R$ generations of sites has a boundary of $z(1-z)^{R-1}$ sites and a volume of $1 + z[(z-1)^R - 1]/(z-2)$ sites. For large $R$ we have surface $\sim$ volume $\times(z-2)/(z-1)$ and thus $d = \infty$ (no $d$-dependence of ratio).

As the Bethe lattice has no loops, everything can be calculated exactly. Let us follow a branch in a cluster. At each site it is connected to $z$ sites and branches out to $z-1$ new branches. If each site is occupied with probability $p$ then at each site the cluster branches into $p(z-1)$ occupied branches. If $p(z-1) < 1$ the number of branches decreases and the cluster will be finite. Otherwise the number of branches increases and the cluster will be infinite. Thus

$$p_c = \frac{1}{z-1} \tag{1.7}$$



$P$, the probability of being on a percolating cluster, can be calculated exactly for $z = 3$ (see Aharony and Stauffer). We define by $Q$ the probability that a cluster starting at the root of one of the branches does *not* connect to infinity. This is the cause if either the root of the branch is empty or it is occupied but neither of the branches connected to it extends to infinity. Thus for $z = 3$:

$$Q = (1-p) + pQ^2 \tag{1.8}$$

This equation has two solutions: $Q = 1$ and $Q =$

$(1-p)/p$. The probability that a site is *not* connected on the percolating cluster is then $1-p+pQ^3$, as it is either empty or occupied but not connected to infinity by any of the $z=3$ branches connected to the site. This gives $P=0$, corresponding to $p<p_c$ and, for $p>p_c$:

$$P = 1-(1-p+pQ^3) = p\left[1-\left(\frac{1-p}{p}\right)^3\right] \sim (p-p_c), \tag{1.9}$$

for larger connectivity $z>3$ similar calculations still yield the same power law.

A similar argument can be found for the mean cluster size for $p<p_c$. Let us call the size of a cluster on a branch $T$. This is 0 if the root of the branch is empty and $1+(z-1)T$ if it is occupied. Thus $T=p(1+(z-1)T)$, with the solution $T=p/(p+1-pz)$. The size of the total cluster is the root plus the three branches:

$$S = 1+zT = \frac{1+p}{1-(z-1)p} \sim (p-p_c)^{-1} \tag{1.10}$$

The probability that a site is at an edge of a cluster scales as

$$\frac{n_s(p)}{n_s(p_c)} \sim \exp(-cs), \tag{1.11}$$

with $c \sim (p-p_c)^2$

## 1.4 Scaling

We have seen that in both exactly solvable cases the interesting quantities have power law singularities at $p_c$.

The generalization of this to arbitrary lattices is the 'scaling ansatz'. The Scaling ansatz is just that: an ansatz, not a mathematical theorem. However it can be motivated from the fractal behavior of the percolating cluster, from renormalization group arguments and from the good agreement of this ansatz with numerical results.

### 1.4.1 The scaling ansatz

We generalize the scaling for the *average cluster size* to:

$$S \sim |p - p_c|^{-\gamma}, \tag{1.12}$$

In the example above, $\gamma$ was 1, but will allow more general coefficients. On the 'percolating' site of the phase transition, where $p > p_c$, we exclude the percolating cluster that would give an infinite contribution.

The *correlation length* $\xi$ is defined as

$$\xi^2 = \frac{\sum_r r^2 g(r)}{\sum_r g(r)}, \tag{1.13}$$

which is equivalent to the exponential decay constant of the correlation function in the previous example. The sum over all distances can be split into sums over all clusters, adding the contribution of each cluster:

$$\xi^2 = \frac{2\sum_{cluster} R^2_{cluster} n(cluster)^2}{\sum_{cluster} n(cluster)^2}, \tag{1.14}$$

where $R_{cluster}$ is the average cluster radius,

$$R^2_{cluster} = \frac{1}{2n(cluster)^2} \sum_{ij \in cluster} |r_i - r_j|^2 \tag{1.15}$$

$n(cluster)$ the size of the cluster and $r_i$ the location of the $i-th$ site.

For the correlation $\xi$ we define an exponent $\nu$:

$$\xi \sim |p - p_c|^{-\nu} \tag{1.16}$$

The pair correlation function $g(r)$ for $p \neq p_c$ decays exponentially with the correlation length $\xi$. At the critical concentration $p = p_c$ however the correlation length $\xi$ diverges and we assume a power law:

$$g(r) \sim r^{-d-2+\eta} \tag{1.17}$$

For the probability $P$ that a site is on the percolating cluster we make the ansatz

$$P \sim (p - p_c)^{\beta}. \tag{1.18}$$

Finally we define an exponent for the cluster density $M_0 = \sum_s n_s$ which scales as

$$M_0 \sim |p - p_c|^{2-\alpha} \tag{1.19}$$

This is a total of five exponents: $\alpha, \beta, \gamma, \eta$, and $\nu$. There are *scaling relations* between those exponents, for details see *e.g.* the book by Aharony and Stauffer:

$$2 - \alpha = 2\beta + \gamma \tag{1.20}$$

This last relation is the so-called *scaling law*.



### 1.4.2 Fractals

For further discussion we need to introduce the concept of fractal dimensions. As you can see by looking at pictures of percolation, the percolating cluster at criticality is a complex object that is self-similar on all length scales.

This self-similarity follows naturally from the divergence of the dominant length scale $\xi$ at the critical point and is reflected in the power-law behavior of all properties at the critical point. Power laws are the only scale-invariant functions:

$$f(r/l) \sim f(r) \quad \text{as} \quad r^{\zeta} \sim (r/l)^{\zeta} \tag{1.21}$$

Thus self-similarity and fractal behavior are intimately related to the scaling ansatz. Whether you use the scaling ansatz to motivate fractal behavior or use apparent fractal behavior to motivate a scaling ansatz is a matter of taste.

Self-similar objects like the percolating cluster at criticality are called fractals, since their dimension $D$ defined by the relationship of volume to linear dimension is a non-integral fraction:

$$V(R) \sim R^{D} \tag{1.22}$$

This is in contrast to simple objects like lines, squares, or cubes, which have integer dimension.



Applying these ideas to clusters we make the ansatz

$$R_s \sim s^{1/D} \tag{1.23}$$

for the average radius of a cluster of $s$ sites. This allows us to evaluate Eq. 1.14 as

$$\xi^2 = \frac{2\sum_s R_s^2 s^2 n_s}{\sum_s s^2 n_s}, \tag{1.24}$$

since $R_s$ is the mean distance between two sites in a cluster, a site is connected to $s$ other sites, and $sn_s$ is the probability of the site belonging to an $s$-site cluster.

with a bit of algebra (again, see the book) one can find

$$D = (\beta + \gamma)/\nu = d - \beta/\nu \tag{1.25}$$

$$d - 2 + \eta = 2\beta/\nu \tag{1.26}$$

A third law is the so-called 'hyper scaling' law:

$$d\nu = \gamma + 2\beta = 2 - \alpha. \tag{1.27}$$

It can be obtained by combining the previous scaling relations. Scaling relations involving dimensions are usually called 'hyper scaling' laws. While the other scaling laws hold for all dimensions, the last one will eventually break down for large dimensions; it is clearly not valid for the infinite-dimensional Bethe lattice. It holds up to the 'upper critical dimension' $d_u = 6$ in the case of percolation. For $d \geq d_u$ the exponents are always those of the infinite-d Bethe lattice.



## 1.5 Renormalization group

The renormalization group method is intricately linked to the self similarity of the percolating cluster and to the scaling ansatz. It also provides a motivation for the scaling ansatz.

The idea is to ignore unimportant microscopic details and to concentrate on the important physics on large scales. We do that by replacing a $b \times b$ square of our square lattice by a single square. We choose it to be filled if a percolating cluster exists on the $b \times b$ lattice and empty otherwise. This process is iterated until we are left with just one single square which is either filled or empty.

### 1.5.1 The square lattice

As the simplest example let us choose $b = 2$. On a $2 \times 2$ square there are two vertically spanning clusters with 2 occupied sites, four spanning clusters with three occupied sites, and one spanning cluster with four occupied sites. The total probability $R(p)$ for a vertically spanning cluster on a $2 \times 2$ cluster is thus

$$R(p) = 2p^2(1-p)^2 + 4p^3(1-p) + p^4 \tag{1.28}$$

The RG transformation

$$p \leftarrow R(p) = 2p^2(1-p)^2 + 4p^3(1-p) + p^4 \tag{1.29}$$

has two trivial fixed points $p = 0$ and $p = 1$ as well as one non-trivial fixed point $p_2^* = (\sqrt{5}-1)/2 \approx 0.6180$. This is surprisingly close to the correct result $p_c = 0.5927$.

To obtain better accuracy one needs to work with larger cell sizes $b$. In the limit $b \to \infty$ we have $p_b \to p_c$ and the renormalization group calculation becomes exact.

It is important to note that the choice of percolation criterion is ambiguous. Here we have chosen to define a percolating cluster as one that spans the lattice vertically. We could also the one which spans the cluster horizontally, with the same results. The other choices, while giving the same results in the limit $b \to \infty$ have larger finite size corrections. If we define a cluster as percolating if it spans horizontally or vertically, the renormalization group transform is

$$R(p) = 4p^2(1-p)^2 + 4p^3(1-p) + p^4 \tag{1.30}$$

with a fixed point at $3-\sqrt{5})/2 \approx 0.382$. If on the other hand we define a percolating cluster as spanning both horizontally and vertically, the renormalization group transform is

$$R(p) = 4p(1-p)^3 + p^4, \tag{1.31}$$

with a fixed point at $\sqrt{13}-1)/6 \approx 0.768$. Both these estimates are much farther from $p_c$. Our first choice thus turns out to have been the best.

The RG method provides us not only with an estimate for $p_c$, but also with estimates for the exponents. In one step we transform:

$$p' = R(p) \tag{1.32}$$

$$\xi' = \xi/b \tag{1.33}$$

At the same time the scaling law is valid and we obtain

$$(p'-p_b^*)^{-\nu} = \xi' = \frac{\xi}{b} = \frac{1}{b}(p-p_b^*)^{-\nu}. \tag{1.34}$$

By expanding $R(p)$ in a Taylor series around $p_b^*$ we obtain

$$\nu = \frac{\log b}{\log \left.\frac{dR}{dp}\right|_{p_b^*}} \tag{1.35}$$

For $b=2$ this gives the estimate $\nu \approx 1.635$, compared to the exact value of $\nu = 4/3$.



### 1.5.2 The triangular lattice

A better estimate is obtained on the triangular lattice. There we replace three sites by one ($b^2 = 3$), thus obtaining the transformation

$$p \leftarrow R(p) = 3p^2(1-p) + p^3 \tag{1.36}$$

with fixed points at 0, 1/2, and 1. In this case surprisingly the value $p^*_{\sqrt{3}} = p_c$ is exact.

Also the estimate for $\nu$ is much better: 1.355 It will be necessary to go to larger values of $b$ if we want to improve the accuracy of our results. Up to $b = 6$ we might be able to determine the RG equations exactly. For larger $b$ we have to use numerical methods, which we will treat later.

## 1.6 Mote Carlo Simulation



Monte Carlo simulations are the simplest method to investigate percolation. We will focus on three types of questions that can be answered by Monte Carlo simulations:

- What is the p-dependence of an arbitrary quantity $X$.
- What is the critical probability $p_c$?
- What are the values of the universal critical exponents?

$X$ can be a quantity like $\xi, S, P$, or any other interesting observable. For now we work only with finite lattices with linear dimension $L < \infty$. The problem of finite size scaling to the infinite system will be discussed later.

The expectation value of the quantity $X$ can be calculated exactly for small lattices by a sum over all possible configurations on a system with $N = L^d$ lattice sites:

$$\langle X \rangle = \sum_{n=0}^{N} \sum_{c \in \mathcal{C}_{N,n}} p^n (1-p)^{N-n} X(c), \tag{1.37}$$

where $\mathcal{C}_{N,n}$ is the set of configurations of $n$ occupied sites in a lattice with $N$ sites, and $X(c)$ is the value of the quantity $X$ measured in the configuration $c$. It is obvious that this sum can be performed exactly only for small lattice sizes up to $N \approx 30$ sizes, as the number of terms increases exponentially $\sim 2^N$.

### 1.6.1 Monte Carlo estimates

Larger lattice sizes with $N$ up to $10^6$ can no longer be done exactly, but Monte Carlo summation can provide estimates of this average to any desired accuracy. The average over the complete set of configurations is replaced by a random sample of $M \approx 10^6 \ldots 10^{12}$ configurations $c_i$ drawn randomly with the correct probabilities $p^n(1-p)^{N-n}$ for a configuration with $n$ occupied sites.

To create configurations with the correct probabilities we draw a pseudo-random number $u$, uniformly distributed in $[0, 1[$for each site $J$. We set the site occupied if $u < p$ and empty otherwise. This corresponds to importance sampling, where each configuration is created with the correct probability.



### 1.6.2 Cluster labeling

Next we identify clusters using, for example, the Hoshen -Kopelman cluster labeling algorithm, which works in the following way:

- Allocate an array of size $N$ to store the cluster label for each site
- Loop through all sites $i$ in the lattice. For occupied

sites check if a cluster label has already been assigned to any occupied neighboring sites.

  - If no neighboring site is occupied or has a label assigned, assign a new cluster label to this site.
  - If one neighboring site is occupied and has a cluster label assigned, assign the same label to this site.
  - If more than one neighboring sites are occupied and all have the same label assigned, assign this same label to the current site.
  - If more than one neighboring site is occupied and the sites have different labels assigned, we have a problem. The current site connects to two cluster parts which until now were disconnected and labeled as different clusters. We take the smallest of label numbers as the proper label and assign this to the current site. For the other, larger, labels we would need to relabel all wrongly labeled sites.



- We use the following trick to avoid relabeling all wrongly labeled sites: for each label we keep a list of proper labels, initialized originally to the label itself. To obtain the cluster label of a site we first obtain the label number assigned to the site. Next we check its proper label. If the two agree we are done. Otherwise we replace the label stored by the proper label and repeat this process until the label agrees with its proper label. This way we avoid relabeling all wrongly labeled sites.

Having thus created a configuration and identified the existing clusters we can measure the value of any quantity of interest. Care must be taken that in the averages over clusters for $S$ and other quantities any existing percolating cluster is always excluded, since its contribution is infinite in the thermodynamic limit $L \to \infty$.

### 1.6.3 Finite size effects

The Monte Carlo results obtained for $P$ as a function of different system sizes show rapid convergence for $p \ll p_c$ and $p \gg p_c$. However, for $p \sim p_c$ the convergence as a function of system size is very slow, and the development of the power law singularity $P \propto (p - p_c)^\beta$ in the infinite system is only really visible for large system sizes.

This means that we have no good way to directly estimate $p_c$ and the exponents. The problem is caused the fact that the correlation length $\xi$, which diverges as $|p - p_c|^{-\nu}$, gets to be larger than the lattice size $L$ near criticality and we would need lattices $L \gg \xi \to \infty$, which is impossible.



### 1.6.4 Finite size scaling

The new idea is to extend scaling so that it also contains the system size $L$ as a parameter. The motivation is simple. A finite system size $L$ introduces a cutoff for all length scales. This implies that $\xi$ cannot grow larger than $L$ on the finite system. This, in turn, means that a finite system size $L$ has the same effect as a finite distance from the critical point, related by $L \sim \xi \propto (p - p_c)^{-\nu}$.

Thus, for example, at the critical point

$$P(L) \sim (p - p_c)^{\beta} \sim L^{-\beta/\nu} \tag{1.38}$$

More formally we can make the following scaling ansatz: close to criticality the only important length scale is $\xi$. The effects of finite system sizes are thus determined only by the ration $L/\xi$. For a quantity $X$, which diverges as $(p - p_c)^{-\chi}$ for $L \gg \xi$ we make the ansatz

$$X(L, p) = (p - p_c)^{-\chi} \tilde{\mathcal{F}}_1(L/\xi) = (p - p_c)^{-\chi} \tilde{\mathcal{F}}_1((p - p_c)L^{1/\nu}) \tag{1.39}$$

or, equivalently

$$X(L, \xi) = \xi^{\chi/\nu} \mathcal{F}_2(L/\xi) \propto \begin{cases} \xi^{\chi/\nu}, & \text{for } L \gg \xi. \\ L^{\chi/\nu}, & \text{for } \xi \gg L. \end{cases} \tag{1.40}$$

Applying this scaling ansatz to the size of the percolating cluster $L^d P$ we immediately derive the second expression for the fractal dimension $D$ in Eq. 1.25:

$$L^D = L^d P(L, \xi) = L^d \xi^{-\beta/\nu} f(L/\xi) \propto L^{d-\beta/\nu} \tag{1.41}$$

here we have chosen $L = const \times \xi$, allowing us to replace $\xi$ by $L$, and giving a constant value for the scaling function $f(L/\xi)$.

Thus we see that finite size scaling allows us to determine the ratio of exponents like $\beta/\nu$ by calculating the $L$-dependence of $P$ at $p = p_c$. One problem still remains: how can we determine $p_c$ on a finite lattice? The answer is again finite size scaling. Consider the probability $\Pi(p)$ for the existence of a percolating cluster. In the infinite system it is:

$$\Pi(p) = \theta(p - p_c) \tag{1.42}$$

(with $\theta$ the Heaviside step function). In a finite system the step function is smeared out. We can make the usual finite size scaling ansatz, using equation 1.39:

$$\Pi(p, L) = \Phi((p - p_c)L^{1/\nu}) \tag{1.43}$$

The derivative $d\Pi/dp$ gives the probability for first finding a percolating cluster at concentrations in the interval $[p, p + dp[$. The probability $p$ at which a pwercoluting cluster appears can easily be measured in a Monte Carlo simulation. Its average

$$p_{av} = \int p \frac{d\Pi}{dp} dp \tag{1.44}$$

is slightly different from the exact value $p_c$ on any finite lattice, but converges like

$$p_{av} - p_c \propto L^{-1/\nu} \tag{1.45}$$

as can be seen by integrating the scaling ansatz 1.43. Similarly the variance

$$\Delta^2 = \int (\tilde{p} - p_{av})^2 \frac{d\Pi}{dp} dp \tag{1.46}$$

decreases as

$$\Delta \propto L^{-1/\nu} \tag{1.47}$$

With this information we can obtain $\nu$ and $p_c$ from the finite size scaling of the average $p$ at which a percolating cluster appears on a finite lattice. The algorithm is as follows:

1. Draw a pseudo-random number $u_j$, uniformly distributed in $[0, 1[$ for each site $j$.

2. We use a binary search to determine the probability $\tilde{p}$ where a percolating cluster appears for this realization of the $u_j$s. We start by checking $\tilde{p} = 0.5$, and then depending on whether a percolating cluster exists try $\tilde{p} = 0.25$ or $\tilde{p} = 0.75$, and so on. This gives us one sample for $\tilde{p}$.

3. Get a new set of $u_j$s and repeat, to get a good average of $\Delta$ and $p_{av}$.

4. Finally we fit the results obtained for several lattice sizes $L$ to obtain $p_c$ and $\nu$ as fitting parameters.

5. Once we have good estimates for $p_c$ we can use simulations at $p_c$ as a function of $L$ to get further exponents. By measuring $P(p_c)$ as a function of $L$ and fitting we measure $\beta/\nu$. Similarly, by fitting $S(p_c)$ as a function of $L$ we estimate $\gamma/\nu$.

The scaling laws which relate the exponents can be used as a check on our simulations and as a tool to increase the accuracy of our estimates.