

LECTURE NOTES ON ALGEBRAIC LOGIC

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1. ALGEBRAS AND EQUATIONS

We begin by reviewing some fundamentals of general algebraic systems.

Definition 1.1.

- (i) A *type* is a map $\rho: \mathcal{F} \rightarrow \mathbb{N}$, where \mathcal{F} is a set of function symbols. In this case, $\rho(f)$ is said to be the *arity* of the function symbol f , for every $f \in \mathcal{F}$. Function symbols of arity zero are called *constants*.
- (ii) An *algebra* of type ρ is a pair $A = \langle A; F \rangle$ where A is a nonempty set and $F = \{f^A : f \in \mathcal{F}\}$ is a set of operations on A whose arity is determined by ρ , in the sense that each f^A has arity $\rho(f)$. The set A is called the *universe* of A .

When $\mathcal{F} = \{f_1, \dots, f_n\}$, we shall write $\langle A; f_1^A, \dots, f_n^A \rangle$ instead of $\langle A; F \rangle$. In this case, we often drop the superscripts, and write simply $\langle A; f_1, \dots, f_n \rangle$.

Classical examples of algebras are groups and rings. For instance, the type of groups ρ_G consists of a binary symbol $+$, a unary symbol $-$, and a constant symbol 0 . Then a group is an algebra $\langle G; +, -, 0 \rangle$ of type ρ_G in which $+$ is associative, 0 is a neutral element for $+$, and $-$ produces inverses.

Lattices, Heyting algebras, and modal algebras are also algebras in the above sense. For instance, the type of lattices ρ_L consists of two binary symbols \wedge and \vee and a lattice is an algebra $\langle A; \wedge, \vee \rangle$ of type ρ_L that satisfies the idempotent, commutative, associative, and absorption laws. Similarly, the type of Heyting algebras ρ_H consists of three binary operations symbols \wedge, \vee , and \rightarrow and of two constant symbols 0 and 1 . Then a Heyting algebra is an algebra $\langle A; \wedge, \vee, \rightarrow, 0, 1 \rangle$ such that $\langle A; \wedge, \vee, 0, 1 \rangle$ is a bounded lattice and, for every $a, b, c \in A$,

$$a \wedge b \leq c \iff a \leq b \rightarrow c. \quad (\text{residuation law})$$

Boolean algebras can be viewed as the Heyting algebras that satisfy the following equational version of the *excluded middle law*:

$$x \vee (x \rightarrow 0) \approx 1.$$

In this case, the complement operation $\neg x$ can be defined as $x \rightarrow 0$.

Perhaps less obviously, even algebraic structures whose operations are apparently *external* can be viewed as algebras in the sense of the above definition. For instance, modules over a ring R can be viewed as algebras whose type ρ_R extends that of groups with the unary symbols $\{\lambda_r : r \in R\}$. From this point of view, a module over R is an

algebra $\langle G; +, -, 0, \{\lambda_r : r \in R\} \rangle$ of type ρ_R such that $\langle G; +, -, 0 \rangle$ is an abelian group and, for every $r, s \in R$ and $a, c \in G$,

$$\begin{aligned}\lambda_r(a + c) &= \lambda_r(a) + \lambda_r(c) \\ \lambda_{r+s}(a) &= \lambda_r(a) + \lambda_s(a) \\ \lambda_r(\lambda_s(a)) &= \lambda_{r \cdot s}(a) \\ \lambda_1(a) &= a.\end{aligned}$$

Given a type $\rho: \mathcal{F} \rightarrow \mathbb{N}$ and a set of variables X disjoint from \mathcal{F} , the set of *terms of type ρ over X* is the least set $T_\rho(X)$ such that

- (i) $X \subseteq T_\rho(X)$;
- (ii) if $c \in \mathcal{F}$ is a constant, then $c \in T_\rho(X)$; and
- (iii) if $\varphi_1, \dots, \varphi_{\rho(f)} \in T_\rho(X)$ and $f \in \mathcal{F}$, then $f\varphi_1 \dots \varphi_{\rho(f)} \in T_\rho(X)$.

For the sake of readability, we shall often write $f(\varphi_1, \dots, \varphi_{\rho(f)})$ instead of $f\varphi_1 \dots \varphi_{\rho(f)}$. Similarly, if f is a binary operation $+$, we often write $\varphi_1 + \varphi_2$ instead of $f(\varphi_1, \varphi_2)$.

Given a term $\varphi \in T_\rho(X)$, we write $\varphi(x_1, \dots, x_n)$ to indicate that the variables occurring in φ are among x_1, \dots, x_n . Furthermore, given an algebra A of type ρ and elements $a_1, \dots, a_n \in A$, we define an element

$$\varphi^A(a_1, \dots, a_n)$$

of A , by recursion on the construction of φ , as follows:

- (i) if φ is a variable x_i , then $\varphi^A(a_1, \dots, a_n) := a_i$;
- (ii) if φ is a constant c , then c^A is the interpretation of c in A ;
- (iii) if $\varphi = f(\psi_1, \dots, \psi_m)$, then

$$\varphi^A(a_1, \dots, a_n) := f^A(\psi_1^A(a_1, \dots, a_n), \dots, \psi_m^A(a_1, \dots, a_n)).$$

An *equation of type ρ over X* is an expression of the form $\varphi \approx \psi$, where $\varphi, \psi \in T_\rho(X)$. Such an equation $\varphi \approx \psi$ is *valid* in an algebra A of type ρ , if

$$\varphi^A(a_1, \dots, a_n) = \psi^A(a_1, \dots, a_n), \text{ for every } a_1, \dots, a_n \in A,$$

in which case we say that A *satisfies* $\varphi \approx \psi$.

For instance, groups are precisely the algebras of type ρ_G that satisfy the equations

$$x + (y + z) \approx (x + y) + z \quad x + 0 \approx x \quad 0 + x \approx x \quad x + -x \approx 0 \quad -x + x \approx 0.$$

Similarly, lattices are the algebras of type ρ_L that satisfy the equations

$$\begin{array}{lll} x \wedge x \approx x & x \vee x \approx x & (\text{idempotent laws}) \\ x \wedge y \approx y \wedge x & x \vee y \approx y \vee x & (\text{commutative laws}) \\ x \wedge (y \wedge z) \approx (x \wedge y) \wedge z & x \vee (y \vee z) \approx (x \vee y) \vee z & (\text{associative laws}) \\ x \wedge (y \vee x) \approx x & x \vee (y \wedge x) \approx x. & (\text{absorption laws}) \end{array}$$

2. BASIC CONSTRUCTIONS

Algebras of the same type are called *similar* and can be compared by means of maps that preserve their structure.

Definition 2.1. Given two similar algebras A and B , a *homomorphism* from A to B is a map $f: A \rightarrow B$ such that, for every n -ary operation g of the common type and $a_1, \dots, a_n \in A$,

$$f(g^A(a_1, \dots, a_n)) = g^B(f(a_1), \dots, f(a_n)).$$

An injective homomorphism is called an *embedding* and, if there exists an embedding from A to B , we say that A *embeds* into B . Lastly, a surjective embedding is called an *isomorphism*. Accordingly, A and B are said to be *isomorphic* if there exists an isomorphism between them, in which case we write $A \cong B$.

A simple induction on the construction of terms shows that, for every pair of algebras A and B of type ρ and every term $\varphi(x_1, \dots, x_n)$ of ρ , if f is a homomorphism from A to B , then

$$f(\varphi^A(a_1, \dots, a_n)) = \varphi^B(f(a_1), \dots, f(a_n)),$$

for every $a_1, \dots, a_n \in A$. Therefore homomorphisms preserve not only basic operations, but also arbitrary terms.

In the particular case where A and B are lattices, a homomorphism from A to B is a map $f: A \rightarrow B$ such that, for every $a, c \in A$,

$$f(a \wedge^A c) = f(a) \wedge^B f(c) \quad \text{and} \quad f(a \vee^A c) = f(a) \vee^B f(c).$$

For instance, the inclusion map from the lattice $\langle \mathbb{N}; \leq \rangle$ into the lattice $\langle \mathbb{Z}; \leq \rangle$ is an injective homomorphism, that is, an embedding. Similarly, given two sets $Y \subseteq X$, the inclusion map from the powerset lattice $\langle \mathcal{P}(Y); \subseteq \rangle$ to the powerset lattice $\langle \mathcal{P}(X); \subseteq \rangle$ is also an embedding. On the other hand, if $Y \subsetneq X$, the map

$$(-) \cap Y: \mathcal{P}(X) \rightarrow \mathcal{P}(Y)$$

that sends every $Z \subseteq X$ to $Z \cap Y$ is a noninjective homomorphism from $\langle \mathcal{P}(X); \subseteq \rangle$ to $\langle \mathcal{P}(Y); \subseteq \rangle$.

Definition 2.2. Let A and B be algebras of the same type $\rho: \mathcal{F} \rightarrow \mathbb{N}$. Then A is said to be a *subalgebra* of B if $A \subseteq B$ and f^A is the restriction of f^B to A , for every $f \in \mathcal{F}$. In this case, we write $A \leq B$.

Given a class of algebras K , let

$$\mathbb{I}(K) := \{A : A \cong B \text{ for some } B \in K\}$$

$$\mathbb{S}(K) := \{A : A \leq B \text{ for some } B \in K\}.$$

When $K = \{A\}$, we write $\mathbb{I}(A)$ and $\mathbb{S}(A)$ as a shorthand for $\mathbb{I}(\{A\})$ and $\mathbb{S}(\{A\})$, respectively. The following observation is an immediate consequence of the definitions.

Proposition 2.3. Let A and B be algebras of the same type. Then $A \in \mathbb{IS}(B)$ if and only if there exists an embedding $f: A \rightarrow B$. In this case, A is isomorphic to the unique subalgebra of B with universe $f[A]$.

As we mentioned, homomorphisms can be used to compare similar algebras.

Definition 2.4. Given two similar algebras A and B , we say that A is a *homomorphic image* of B if there exists a surjective homomorphism $f: B \rightarrow A$.

Accordingly, given a class of algebras K , we set

$$\mathbb{H}(K) := \{A : A \text{ is a homomorphic image of some } B \in K\}.$$

As usual, when $K = \{A\}$, we write $\mathbb{H}(A)$ as a shorthand for $\mathbb{H}(\{A\})$.

Observe that every (not necessarily surjective) homomorphism $f: A \rightarrow B$ induces a homomorphic image of A .

Proposition 2.5. *If $f: A \rightarrow B$ is a homomorphism, then $f[A]$ is the universe of a subalgebra of B that, moreover, is a homomorphic image of A .*

Proof. Observe that $f[A]$ is nonempty, because A is. Then consider an n -ary function symbol g of the common type of A and B and $b_1, \dots, b_n \in f[A]$. Clearly, there are $a_1, \dots, a_n \in A$ such that $f(a_i) = b_i$, for every $i \leq n$. Since f is a homomorphism from A to B , we obtain

$$g^B(b_1, \dots, b_n) = g^B(f(a_1), \dots, f(a_n)) = f(g^A(a_1, \dots, a_n)) \in f[A].$$

Hence, we conclude that $f[A]$ is the universe of a subalgebra $f[A]$ of B .

Furthermore, $f: A \rightarrow f[A]$ is a homomorphism, because for every basic n -ary function symbol g of the common type and $a_1, \dots, a_n \in A$,

$$f(g^A(a_1, \dots, a_n)) = g^B(f(a_1), \dots, f(a_n)) = g^{f[A]}(f(a_1), \dots, f(a_n)),$$

where the first equality follows from the assumption that $f: A \rightarrow B$ is a homomorphism. Since the map $f: A \rightarrow f[A]$ is surjective, we conclude that $f[A] \in \mathbb{H}(A)$. \square

In view of the above result, when $f: A \rightarrow B$ is a homomorphism, we denote by $f[A]$ the unique subalgebra of B with universe $f[A]$.

For instance, let $f: \mathbb{Z} \rightarrow \mathbb{R}$ be the absolute value map, that is, the function defined by the rule

$$f(n) := \text{the absolute value of } n.$$

Observe that f is a nonsurjective homomorphism from the lattice of integers to that of reals. Furthermore, the homomorphic image $f[\langle \mathbb{Z}; \leq \rangle]$ of $\langle \mathbb{Z}; \leq \rangle$ is the lattice of natural numbers $\langle \mathbb{N}; \leq \rangle$, which, in turn, is a subalgebra of lattice of reals.

Notably, the homomorphic images of an algebra A can be “internalized” as special equivalence relations on A as follows.

Definition 2.6. A *congruence* of an algebra A is an equivalence relation θ on A such that, for every basic n -ary operation f of A and $a_1, \dots, a_n, c_1, \dots, c_n \in A$,

$$\text{if } \langle a_1, c_1 \rangle, \dots, \langle a_n, c_n \rangle \in \theta, \text{ then } \langle f^A(a_1, \dots, a_n), f^A(c_1, \dots, c_n) \rangle \in \theta. \quad (1)$$

In this case, we often write $a \equiv_\theta c$ as a shorthand for $\langle a, c \rangle \in \theta$. The poset of congruences of A ordered under the inclusion relation will be denoted by $\text{Con}(A)$.

A simple induction on the construction of terms shows that, for every congruence θ of A and every term $\varphi(x_1, \dots, x_n)$,

$$\text{if } \langle a_1, c_1 \rangle, \dots, \langle a_n, c_n \rangle \in \theta, \text{ then } \langle \varphi^A(a_1, \dots, a_n), \varphi^A(c_1, \dots, c_n) \rangle \in \theta,$$

for every $a_1, \dots, a_n \in A$. Therefore, congruences preserve not only basic operations, but also arbitrary terms. Furthermore, a simple argument shows that $\text{Con}(A)$ is an inductive closure system and, therefore, an algebraic lattice whose maximum is the total relation $A \times A$ and whose minimum is the identity relation $\text{id}_A := \{ \langle a, a \rangle : a \in A \}$.

Example 2.7 (Boolean algebras). Recall that a *filter* of a Boolean algebra A is a nonempty upset $F \subseteq A$ closed under binary meets. We denote by $\text{Fi}(A)$ the poset of filters of A ordered under the inclusion relation. It is easy to see $\text{Fi}(A)$ is an inductive closure system and, therefore, an algebraic lattice. Furthermore, the lattices $\text{Fi}(A)$ and $\text{Con}(A)$ are isomorphic via the inverse isomorphisms

$$\Omega^A(-): \text{Fi}(A) \rightarrow \text{Con}(A) \quad \text{and} \quad \tau(-): \text{Con}(A) \rightarrow \text{Fi}(A)$$

defined by the rules

$$\begin{aligned} \Omega^A(F) &:= \{ \langle a, c \rangle \in A \times A : a \rightarrow c, c \rightarrow a \in F \} \\ \tau(\theta) &:= \{ a \in A : \langle a, 1 \rangle \in \theta \}. \end{aligned}$$

Because of this, every congruence θ of a Boolean algebra A is induced by some filter F , in the sense that $\theta = \Omega^A F$. This correspondence between filters and congruences generalizes straightforwardly to all Heyting algebras. \square

Example 2.8 (Modal algebras). A *modal algebra* is an algebra $A = \langle A; \wedge, \vee, \neg, \Box, 0, 1 \rangle$ such that $\langle A; \wedge, \vee, \neg, 0, 1 \rangle$ is a Boolean algebra and \Box is a unary operation such that

$$\Box(a \wedge c) = \Box a \wedge \Box c \quad \text{and} \quad \Box 1 = 1,$$

for every $a, c \in A$. An *open filter* of a modal algebra A is a filter of the Boolean reduct of A that, moreover, is closed under the operation \Box . The poset of open filters of A ordered under the inclusion relation will be denoted by $\text{Op}(A)$. It forms an inductive closure system and, therefore, an algebraic lattice. Furthermore, the lattices $\text{Op}(A)$ and $\text{Con}(A)$ are isomorphic via the inverse isomorphisms described in Example 2.7. Because of this, every congruence of a modal algebra A has the form $\theta = \Omega^A F$, for some open filter F . \square

Example 2.9 (Groups). Similarly, it is well known that the lattice of congruences of a group is isomorphic to that of its normal subgroups. Because of this, every congruence of a group is induced by some normal subgroup. \square

As we mentioned, there is a tight correspondence between the homomorphic images and the congruences of an algebra A . On the one hand, every congruence θ of A gives rise to a homomorphic image A/θ of A . Let \mathcal{F} be the set of function symbols of A . Given $\theta \in \text{Con}(A)$ and a basic n -ary function symbol $f \in \mathcal{F}$, let $f^{A/\theta}$ be the n -ary operation on A/θ defined by the rule

$$f^{A/\theta}(a_1/\theta, \dots, a_n/\theta) := f^A(a_1, \dots, a_n)/\theta.$$

Notice that $f^{A/\theta}$ is well-defined, by condition (1). As a consequence, the structure

$$A/\theta := \langle A/\theta; \{f^{A/\theta} : f \in \mathcal{F}\} \rangle$$

is a well-defined algebra of the type as A . Furthermore, $A/\theta \in \mathbb{H}(A)$, because the map $\pi_\theta: A \rightarrow A/\theta$, defined, for every $a \in A$, as $\pi_\theta(a) := a/\theta$, is a surjective homomorphism from A to A/θ . To prove this, consider $a_1, \dots, a_n \in A$. We have

$$\begin{aligned} \pi_\theta(f^A(a_1, \dots, a_n)) &= f^A(a_1, \dots, a_n)/\theta \\ &= f^{A/\theta}(a_1/\theta, \dots, a_n/\theta) \\ &= f^{A/\theta}(\pi_\theta(a_1), \dots, \pi_\theta(a_n)), \end{aligned}$$

where the second equality follows from the definition of the operation $f^{A/\theta}$.

Corollary 2.10. *If θ is a congruence of an algebra A , then A/θ is a well-defined homomorphic image of A .*

In view of the above result, every congruence θ of an algebra A induces a homomorphic image of A , namely A/θ . The converse is also true, as we proceed to explain.

Definition 2.11. The *kernel* of a homomorphism $f: A \rightarrow B$ is the binary relation

$$\text{Ker}(f) := \{\langle a, c \rangle \in A \times A : f(a) = f(c)\}.$$

Proposition 2.12. *The kernel of a homomorphism $f: A \rightarrow B$ is a congruence of A .*

Proof. It is obvious that $\text{Ker}(f)$ is an equivalence relation on A . Therefore, to prove that $\text{Ker}(f)$ is a congruence of A , it suffices to show that it preserves the basic operations of A . Consider a basic n -ary operation g of A and $a_1, \dots, a_n, c_1, \dots, c_n \in A$ such that $\langle a_1, c_1 \rangle, \dots, \langle a_n, c_n \rangle \in \text{Ker}(f)$. By the definition of $\text{Ker}(f)$,

$$f(a_i) = f(c_i), \text{ for every } i \leq n.$$

It follows that $g^B(f(a_1), \dots, f(a_n)) = g^B(f(c_1), \dots, f(c_n))$. Since $f: A \rightarrow B$ is a homomorphism, this yields

$$f(g^A(a_1, \dots, a_n)) = g^B(f(a_1), \dots, f(a_n)) = g^B(f(c_1), \dots, f(c_n)) = f(g^A(c_1, \dots, c_n)).$$

Hence, we conclude that $\langle g^A(a_1, \dots, a_n), g^A(c_1, \dots, c_n) \rangle \in \text{Ker}(f)$, as desired. \square

The behaviour of kernels is governed by the next principle.

Fundamental Homomorphism Theorem 2.13. *If $f: A \rightarrow B$ is a homomorphism with kernel θ , then there exists a unique embedding $g: A/\theta \rightarrow B$ such that $f = g \circ \pi_\theta$.*

Proof. We begin by proving the existence of g . Let $g: A/\theta \rightarrow B$ be the map defined as $g(a/\theta) := f(a)$, for every $a \in A$. To show that g is well-defined, consider $a, c \in A$ such that $a/\theta = c/\theta$. Since $\theta = \text{Ker}(f)$, this means that $f(a) = f(c)$, as desired. Furthermore, the definition of g guarantees that $f = g \circ \pi_\theta$.

Now, observe g is injective, because, for every $a, c \in A$ such that $g(a/\theta) = g(c/\theta)$, we have $f(a) = f(c)$, that is, $\langle a, c \rangle \in \text{Ker}(f) = \theta$ and, therefore, $a/\theta = c/\theta$. Moreover, for every basic n -ary operation p of A and $a_1, \dots, a_n \in A$, we have

$$\begin{aligned} g(p^{A/\theta}(a_1/\theta, \dots, a_n/\theta)) &= g(p^A(a_1, \dots, a_n)/\theta) \\ &= f(p^A(a_1, \dots, a_n)) \\ &= p^B(f(a_1), \dots, f(a_n)) \\ &= p^B(g(a_1/\theta), \dots, g(a_n/\theta)). \end{aligned}$$

The first equality above follows from the definition of A/θ , the second and the last from the definition of g , and the third from the assumption that $f: A \rightarrow B$ is a homomorphism. Hence, we conclude that $g: A/\theta \rightarrow B$ is a homomorphism and, therefore, an embedding, as desired.

The uniqueness of g follows from the fact that, if a map g^* satisfies the condition in the statement of the theorem, then, for every $a \in A$,

$$f(a) = g^* \circ \pi_\theta(a) = g^*(a/\theta),$$

that is, g^* coincides with g . \square

Corollary 2.14. *If $f: A \rightarrow B$ is a homomorphism, then $f[A] \cong A/\text{Ker}(f)$. In particular, if f is surjective, $B \cong A/\text{Ker}(f)$.*

Proof. In the proof of the Fundamental Homomorphism Theorem we showed that the map $g: A/\text{Ker}(f) \rightarrow B$, defined by the rule $g(a/\text{Ker}(f)) := f(a)$, is an embedding of $A/\text{Ker}(f)$ into B . As g can be viewed as a surjective embedding of $A/\text{Ker}(f)$ into $f[A]$, we conclude that $f[A] \cong A/\text{Ker}(f)$. \square

At this stage, it should be clear that if θ is a congruence on an algebra A , then $\pi_\theta: A \rightarrow A/\theta$ is a surjective homomorphism whose kernel is θ . Similarly, if $f: A \rightarrow B$ is a surjective homomorphism, then $A/\text{Ker}(f) \cong B$, by Corollary 2.14. As a consequence, for every class of algebras K ,

$$\mathbb{H}(K) = \mathbb{I}\{A/\theta : A \in K \text{ and } \theta \in \text{Con}(A)\}. \quad (2)$$

Now, recall that the Cartesian product of a family of sets $\{A_i : i \in I\}$ is the set

$$\prod_{i \in I} A_i := \{f: I \rightarrow \bigcup_{i \in I} A_i : f(i) \in A_i, \text{ for all } i \in I\}.$$

In particular, if I is empty, then $\prod_{i \in I} A_i$ is the singleton containing only the empty map.

Definition 2.15. The *direct product* of a family of similar algebras $\{A_i : i \in I\}$ is the unique algebra of the common type whose universe is the Cartesian product $\prod_{i \in I} A_i$ and such that, for every basic n -ary operation symbol f and every $\vec{a}_1, \dots, \vec{a}_n \in \prod_{i \in I} A_i$,

$$f^{\prod_{i \in I} A_i}(\vec{a}_1, \dots, \vec{a}_n)(i) = f^{A_i}(\vec{a}_1(i), \dots, \vec{a}_n(i)), \text{ for every } i \in I.$$

We denote this algebra by $\prod_{i \in I} A_i$.

In this case, for every $j \in I$, the projection map on the j -th component $p_j: \prod_{i \in I} A_i \rightarrow A_j$, defined by the rule $p_j(\vec{a}) := \vec{a}(j)$, is a surjective homomorphism from $\prod_{i \in I} A_i$ to A_j .

Given a class of similar algebras K , we set

$$\mathbb{P}(K) := \{A : A \text{ is a direct product of a family } \{B_i : i \in I\} \subseteq K\}.$$

As usual, when $K = \{A\}$, we write $\mathbb{P}(A)$ as a shorthand for $\mathbb{P}(\{A\})$.

Notice that up to isomorphism, there exists a unique one-element algebra of a given type. Because of this, one-element algebras are called *trivial*. Accordingly, when the set of indexes I is empty, the direct product $\prod_{i \in I} A_i$ is the trivial algebra of the given type. It follows that $\mathbb{P}(K)$ contains always a trivial algebra, for every class of similar algebras K .

Example 2.16 (Powerset algebras). Boolean algebras of the form $\langle \mathcal{P}(X); \cap, \cup, -, \emptyset, X \rangle$ are called *powerset Boolean algebras*. Let B be the two-element Boolean algebra and observe that $\mathbb{I}\mathbb{P}(B)$ is the class of algebras isomorphic to some powerset Boolean algebra. To prove this, observe that every powerset Boolean algebra $\mathcal{P}(X)$ is isomorphic to a direct product of B via the *characteristic function* $f_X: \mathcal{P}(X) \rightarrow \prod_{x \in X} B_x$, defined by the rule

$$f(Y)(x) := \begin{cases} 1 & \text{if } x \in Y \\ 0 & \text{if } x \notin Y, \end{cases}$$

where $Y \in \mathcal{P}(X)$ and $x \in X$. By the same token, every direct product $\prod_{i \in I} B_i$ of B is isomorphic to the powerset Boolean algebra $\mathcal{P}(I)$ via the isomorphism f_I . \square

We close this section by reviewing the subdirect product construction.

Definition 2.17. A subalgebra B of a direct product $\prod_{i \in I} A_i$ is said to be a *subdirect product* of $\{A_i : i \in I\}$ if the projection map π_i is surjective, for every $i \in I$. Similarly, an embedding $f: B \rightarrow \prod_{i \in I} A_i$ is said to be *subdirect* when $f[B]$ is a subdirect product of the family $\{A_i : i \in I\}$.

Given a class of similar algebras K , we set

$$\mathbb{P}_{SD}(K) := \{A : A \text{ is a subdirect direct product of a family } \{B_i : i \in I\} \subseteq K\}.$$

As usual, when $K = \{A\}$, we write $\mathbb{P}_{SD}(A)$ as a shorthand for $\mathbb{P}_{SD}(\{A\})$. Clearly, $\mathbb{P}_{SD}(K) \subseteq \mathbb{SP}(K)$. Furthermore, $\mathbb{P}_{SD}(K)$ contains always a trivial algebra.

Example 2.18 (Distributive lattices). Let DL be the class of distributive lattices and B be the two-element distributive lattice. Birkhoff's Representation Theorem states $DL = \mathbb{IP}_{SD}(B)$. The inclusion $\mathbb{IP}_{SD}(B) \subseteq DL$ follows from the fact that DL is closed under \mathbb{I} , \mathbb{S} , and \mathbb{P} . For the other inclusion, consider a distributive lattice A and let I be the set of its prime filters. By Birkhoff's Representation Theorem, the map

$$\gamma: A \rightarrow \prod_{F \in I} B_F,$$

defined, for every $a \in A$ and $F \in I$, by the rule

$$\gamma(a)(F) := \begin{cases} 1 & \text{if } a \in F \\ 0 & \text{if } a \notin F, \end{cases}$$

is a well-defined subdirect embedding. \(\boxtimes\)

Example 2.19 (Boolean algebras). Similarly, Stone's Representation Theorem states that the class of Boolean algebras coincides with $\mathbb{IP}_{SD}(B)$, where B the two-element Boolean algebra. \(\boxtimes\)

The next result provides a general recipe to construct subdirect products.

Proposition 2.20. Let A be an algebra and $\{\theta_i : i \in I\} \subseteq \text{Con}(A)$. Then the map

$$f: A / \bigcap_{i \in I} \theta_i \rightarrow \prod_{i \in I} A / \theta_i,$$

defined, for every $a \in A$ and $j \in I$, as

$$f(a / \bigcap_{i \in I} \theta_i)(j) := a / \theta_j,$$

is a subdirect embedding.

Proof. For the sake of readability, set $B := A / \bigcap_{i \in I} \theta_i$. To prove that f is injective, consider $a, c \in A$ such that $\langle a, c \rangle \notin \bigcap_{i \in I} \theta_i$. Then there exists $j \in I$ such that $\langle a, c \rangle \notin \theta_j$ and, therefore,

$$f(a / \bigcap_{i \in I} \theta_i)(j) := a / \theta_j \neq c / \theta_j = f(c / \bigcap_{i \in I} \theta_i)(j).$$

It follows that $f(a / \bigcap_{i \in I} \theta_i) \neq f(c / \bigcap_{i \in I} \theta_i)$, as desired. Thus, f is injective. Moreover, by the definition of f , the composition $\pi_i \circ f: B \rightarrow A / \theta_i$ is surjective, for every $i \in I$.

It only remains to prove that f is a homomorphism. Consider an n -ary basic operation g and $a_1, \dots, a_n \in A$. For every $j \in I$, we have

$$\begin{aligned}
 f(g^B(a_1 / \bigcap_{i \in I} \theta_i, \dots, a_n / \bigcap_{i \in I} \theta_i))(j) &= f(g^A(a_1, \dots, a_n) / \bigcap_{i \in I} \theta_i)(j) \\
 &= g^A(a_1, \dots, a_n) / \theta_j \\
 &= g^{A/\theta_j}(a_1 / \theta_j, \dots, a_n / \theta_j) \\
 &= g^{A/\theta_j}(f(a_1 / \bigcap_{i \in I} \theta_i)(j), \dots, f(a_n / \bigcap_{i \in I} \theta_i)(j)) \\
 &= g^{\prod_{i \in I} A/\theta_i}(f(a_1 / \bigcap_{i \in I} \theta_i), \dots, f(a_n / \bigcap_{i \in I} \theta_i))(j).
 \end{aligned}$$

It follows that

$$f(g^B(a_1 / \bigcap_{i \in I} \theta_i, \dots, a_n / \bigcap_{i \in I} \theta_i)) = g^{\prod_{i \in I} A/\theta_i}(f(a_1 / \bigcap_{i \in I} \theta_i), \dots, f(a_n / \bigcap_{i \in I} \theta_i)). \quad \square$$

3. PREVARIETIES

Definition 3.1. A class of similar algebras closed \mathbb{I} , \mathbb{S} , and \mathbb{P} is said to be a *prevariety*.

Given a class of similar algebras K , there exists the least prevariety extending K , namely $\mathbb{ISP}(K)$. It is called the prevariety *generated* by K . Our aim will be to prove that prevarieties are precisely the classes of algebras axiomatized by a certain kind of infinitary formulas.

When no confusion shall arise, given a sequence \vec{a} and a set A , we write $\vec{a} \in A$ to indicate that the elements of the sequence \vec{a} belongs to A .

Definition 3.2. A *generalized quasi-equation* of type ρ is an expression Φ of the form

$$\left(\big\&_{i \in I} \varphi_i(\vec{x}) \approx \psi_i(\vec{x}) \right) \implies \varepsilon(\vec{x}) \approx \delta(\vec{x}),$$

where $\{\varphi_i \approx \psi_i : i \in I\} \cup \{\varepsilon \approx \delta\}$ is a set of equations of type ρ . Then Φ is *valid* in an algebra A of type ρ when so is its universal closure, that is, for every $\vec{a} \in A$,

$$\text{if } (\varphi_i^A(\vec{a}) = \psi_i^A(\vec{a}), \text{ for all } i \in I), \text{ then } \varepsilon^A(\vec{a}) = \delta^A(\vec{a}).$$

In this case, we often say that A *satisfies* Φ .

Notice that, in the above definition, the set of indexes I can be arbitrarily large and that the same applies to the sequence of variables \vec{x} that appear in the equations of Φ . This motivates the following.

Definition 3.3. A generalized quasi-equation is said to be

- (i) a *quasi-equation* when the index set I is finite; and
- (ii) an *equation* when the index set I is empty.

Remark 3.4. It might seem that we are using the term *equations* to refer to two distinct kinds of expressions, namely those of the form $\varepsilon \approx \delta$ and $\emptyset \implies \varepsilon \approx \delta$. This is not a problem, however, because these expressions are synonyms, in the sense that an algebra satisfies $\varepsilon \approx \delta$ if and only if it satisfies $\emptyset \implies \varepsilon \approx \delta$. Because of this, we will continue to denote equations by $\varepsilon \approx \delta$, while keeping in mind that they are special instances of generalized quasi-equations. \square

Definition 3.5. Let $\rho: \mathcal{F} \rightarrow \mathbb{N}$ be a type and X a set of variables disjoint from \mathcal{F} . The *term algebra* $T_\rho(X)$ of type ρ over X is the unique algebra of type ρ whose universe is $T_\rho(X)$ and with basic n -ary operations f defined, for every $\varphi_1, \dots, \varphi_n \in T_\rho(X)$, as

$$f^{T_\rho(X)}(\varphi_1, \dots, \varphi_n) := f(\varphi_1, \dots, \varphi_n).$$

Term algebras have the following fundamental property.

Proposition 3.6. Let A be an algebra of type ρ and X a set of variables. Every function $f: X \rightarrow A$ extends uniquely to a homomorphism $f^*: T_\rho(X) \rightarrow A$.

Proof. The unique extension f^* is defined, for every $\varphi(x_{\alpha_1}, \dots, x_{\alpha_n}) \in T_\rho(X)$, as

$$f^*(\varphi) = \varphi^A(f(x_{\alpha_1}), \dots, f(x_{\alpha_n})). \quad \square$$

Exercise 3.7.* Prove the above proposition. \square

Theorem 3.8. A class of similar algebras is a prevariety if and only if it can be axiomatized by a class of generalized quasi-equations.

Proof. The “if” part follows from the fact that the validity of generalized quasi-equations persist under the formation of isomorphic copies, subalgebras, and direct product. To prove the converse, consider a prevariety K and let Σ be the proper class of generalized quasi-equations valid in it. Let K^+ be the class of algebras in which the generalized quasi-equations in Σ are valid. Clearly, $K \subseteq K^+$. To prove the other inclusion, consider an algebra $A \in K^+$. Let also X be a set of variables for which there exists a surjective map $f: X \rightarrow A$. By Proposition 3.6, f extends to a surjective homomorphism $f^*: T_\rho(X) \rightarrow A$. Together with Corollary 2.14, this yields

$$A \cong T_\rho(X) / \text{Ker}(f^*). \quad (3)$$

Now, consider an arbitrary pair $\langle \varphi, \psi \rangle \in (T_\rho(X) \times T_\rho(X)) \setminus \text{Ker}(f^*)$. Notice that the elements of $T_\rho(X) \times T_\rho(X)$ are ordered pairs of terms and, therefore, can be viewed as equations under the identification of $\langle \varepsilon, \delta \rangle$ with $\varepsilon \approx \delta$. In this way, $\text{Ker}(f^*)$ becomes a set of equations in variables X . Bearing this in mind, consider the generalized quasi-equation

$$\Phi := \left(\& \text{Ker}(f^*) \right) \implies \varphi \approx \psi.$$

We will prove that Φ fails in A . For the sake of readability we will denote by \vec{x} the sequence of all variables in X . Observe that every element $\varepsilon \in T_\rho(X)$ is of the form $\varepsilon(\vec{x})$. Then consider the assignment $f: X \rightarrow A$. We will denote by $f(\vec{x})$ the sequence obtained by applying component-wise f to \vec{x} . For every pair $\langle \varepsilon, \delta \rangle \in \text{Ker}(f^*)$, we have

$$\varepsilon^A(f(\vec{x})) = \varepsilon^A(f^*(\vec{x})) = f^*(\varepsilon(\vec{x})) = f^*(\delta(\vec{x})) = \delta^A(f^*(\vec{x})) = \delta^A(f(\vec{x})).$$

The equalities above can be justified as follows. The first and the last holds because f^* extends f , the second and the fourth because $f^*: T_\rho(X) \rightarrow A$ is a homomorphism, and the third because $\langle \varepsilon, \delta \rangle \in \text{Ker}(f^*)$. On the other hand, since $\langle \varphi, \psi \rangle \notin \text{Ker}(f^*)$, a similar argument shows

$$\varphi^A(f(\vec{x})) \neq \psi^A(f(\vec{x})).$$

Thus, A refutes Φ , as desired.

Since $A \in K^+$, this implies that there exists some algebra $C_{\varphi,\psi} \in K$ and an assignment $g_{\varphi,\psi}: X \rightarrow C_{\varphi,\psi}$ such that

$$\varepsilon^{C_{\varphi,\psi}}(g_{\varphi,\psi}(\vec{x})) = \delta^{C_{\varphi,\psi}}(g_{\varphi,\psi}(\vec{x})), \text{ for all } \langle \varepsilon, \delta \rangle \in \text{Ker}(f^*), \text{ and } \varphi^{C_{\varphi,\psi}}(g_{\varphi,\psi}(\vec{x})) = \psi^{C_{\varphi,\psi}}(g_{\varphi,\psi}(\vec{x})).$$

Recall that $g_{\varphi,\psi}$ extends uniquely to a homomorphism $g_{\varphi,\psi}^*: T_\rho(X) \rightarrow C_{\varphi,\psi}$. Moreover, from the above display it follows

$$g_{\varphi,\psi}^*(\varepsilon) = g_{\varphi,\psi}^*(\delta), \text{ for all } \langle \varepsilon, \delta \rangle \in \text{Ker}(f^*), \text{ and } g_{\varphi,\psi}^*(\varphi) \neq g_{\varphi,\psi}^*(\psi).$$

Consequently,

$$\text{Ker}(f^*) \subseteq \text{Ker}(g_{\varphi,\psi}^*) \text{ and } \langle \varphi, \psi \rangle \notin \text{Ker}(g_{\varphi,\psi}^*).$$

It follows that

$$\text{Ker}(f^*) = \bigcap \{ \text{Ker}(g_{\varphi,\psi}^*) : \langle \varphi, \psi \rangle \in (T_\rho(X) \times T_\rho(X)) \setminus \text{Ker}(f^*) \}.$$

By Proposition 2.20, this yields

$$T_\rho(X)/\text{Ker}(f^*) \in \mathbb{IP}_{\text{SD}}(\{T_\rho(X)/\text{Ker}(g_{\varphi,\psi}^*) : \langle \varphi, \psi \rangle \in (T_\rho(X) \times T_\rho(X)) \setminus \text{Ker}(f^*)\}). \quad (4)$$

Moreover, from Corollary 2.14 and the fact that K is closed under \mathbb{I} and \mathbb{S} it follows that

$$T_\rho(X)/\text{Ker}(g_{\varphi,\psi}^*) \in \mathbb{IS}(C_{\varphi,\psi}) \subseteq K,$$

for every $\langle \varphi, \psi \rangle \in (T_\rho(X) \times T_\rho(X)) \setminus \text{Ker}(f^*)$. Consequently, (4) simplifies to

$$T_\rho(X)/\text{Ker}(f^*) \in \mathbb{IP}_{\text{SD}}(K) \subseteq K,$$

where the last inclusion follows from the fact that K is a prevariety. Together with (3), this yields $A \in \mathbb{I}(K) \subseteq K$. \square

Remark 3.9. In view of Theorem 3.8, prevarieties are classes of algebras axiomatized by classes of generalized quasi-equations. It is therefore natural to wonder whether there exists a prevariety that cannot be axiomatized by a set (as opposed to proper class) of variables. It turns out that the answer to this question depends on the set theory we live in, as the nonexistence of such a prevariety is equivalent to *Vopěnka's Principle*.

Nevertheless, prevarieties axiomatizable by a set of generalized quasi-equations admit a relatively transparent description, as we proceed to explain. Given an infinite cardinal κ and a class of algebras K , let

$$\mathbb{U}_\kappa(K) := \{A : B \in K, \text{ for all } \kappa\text{-generated } B \leq A\}.$$

Definition 3.10. Let κ be an infinite cardinal. A κ -generalized quasi-variety is a prevariety closed under \mathbb{U}_κ .

When $\kappa = \aleph_0$, we often say that K is simply a *generalized quasi-variety*. Given a class of similar algebras K , there exists the least κ -generalized quasi-variety extending K , namely $\mathbb{U}_\kappa \mathbb{ISP}(K)$. It is called the κ -generalized quasi-variety *generated* by K .

Theorem 3.11. Let κ be an infinite cardinal. A class of similar algebras is a κ -generalized quasi-variety if and only if it can be axiomatized by a set of generalized quasi-equations in which at most κ variables occur.

Proof. The “if” part follows from the fact that the validity of generalized quasi-equations in $\leq \kappa$ variables persist under the $\mathbb{I}, \mathbb{S}, \mathbb{P}, \mathbb{U}_\kappa$. To prove the converse, consider a κ -generalized quasi-variety K . Then let X be a set of variables of cardinality κ and Σ the class of generalized quasi-equations written with variables in X . Since X is a set, so is Σ . It only remains to prove that K coincides with the class K^+ of algebras satisfying the generalized quasi-equations in Σ . Clearly, $K \subseteq K^+$. To prove the other inclusion, consider an algebra $A \in K^+$. We need to prove that $A \in K$. Since K is closed under \mathbb{U}_κ , it suffices to show that all the κ -generated subalgebras of A belong to K .

Accordingly, let B be a κ -generated subalgebra of A and $Y \subseteq B$ a set of generators for B of size $\leq \kappa$. There exists a surjective map $f: X \rightarrow Y$. By Proposition 3.6, f extends to a surjective homomorphism $f^*: T_\rho(X) \rightarrow B$. Now, we repeat the argument in the proof of Theorem 3.8, obtaining $B \in K$, as desired. \square

Corollary 3.12. *A prevariety can be axiomatized by a set of generalized quasi-equations if and only if it is a κ -generalized quasi-variety, for some infinite cardinal κ .*

Exercise 3.13.* Let K be a class of similar algebras and κ an infinite cardinal. Prove that the prevariety and the κ -generalized quasi-variety generated by K are, respectively, $\mathbb{ISP}(K)$ and $\mathbb{U}_\kappa \mathbb{ISP}(K)$. \square

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