LECTURE NOTES ON THE ALGEBRA OF LOGIC

T. MORASCHINI

1. ALGEBRAS AND EQUATIONS

We begin by reviewing some fundamentals of general algebraic systems. For more information, the reader may consult [4, 12].

Definition 1.1.

(i) A *type* is a map $\rho\colon \mathcal{F} \to \mathbb{N}$, where $\mathcal{F}$ is a set of function symbols. In this case, $\rho(f)$ is said to be the *arity* of the function symbol f, for every $f \in \mathcal{F}$. Function symbols of arity zero are called *constants*.

(ii) An *algebra* of type ρ is a pair $\boldsymbol{A} = \langle A; F \rangle$ where A is a nonempty set and $F = \{f^{\boldsymbol{A}} : f \in \mathcal{F}\}$ is a set of operations on A whose arity is determined by ρ, in the sense that each $f^{\boldsymbol{A}}$ has arity $\rho(f)$. The set A is called the *universe* of $\boldsymbol{A}$.

When $\mathcal{F} = \{f_1, \dots, f_n\}$, we shall write $\langle A; f_1^{\boldsymbol{A}}, \dots, f_n^{\boldsymbol{A}} \rangle$ instead of $\langle A; F \rangle$. In this case, we often drop the superscripts, and write simply $\langle A; f_1, \dots, f_n \rangle$.

Classical examples of algebras are groups and rings. For instance, the type of groups ρ_G consists of a binary symbol $+$, a unary symbol $-$ and a constant symbol 0. Then a group is an algebra $\langle G; +, -, 0 \rangle$ of type ρ_G in which $+$ is associative, 0 is a neutral element for $+$ and $-$ produces inverses.

Lattices, Heyting algebras and modal algebras are also algebras in the above sense. For instance, the type of lattices ρ_L consists of two binary symbols $\wedge$ and $\vee$ and a lattice is an algebra $\langle A; \wedge, \vee \rangle$ of type ρ_L that satisfies the idempotent, commutative, associative and absorption laws. Similarly, the type of Heyting algebras ρ_H consists of three binary operations symbols $\wedge, \vee$ and $\to$ and of two constant symbols 0 and 1. Then a Heyting algebra is an algebra $\langle A; \wedge, \vee, \to, 0, 1 \rangle$ such that $\langle A; \wedge, \vee, 0, 1 \rangle$ is a bounded lattice and, for every $a, b, c \in A$,

$$a \wedge b \leqslant c \iff a \leqslant b \to c. \qquad (\textit{residuation law})$$

Boolean algebras can be viewed as the Heyting algebras that satisfy the following equational version of the *excluded middle law*:

$$x \vee (x \to 0) \approx 1.$$

In this case, the complement operation $\neg x$ can be defined as $x \to 0$.

Perhaps less obviously, even algebraic structures whose operations are apparently *external* can be viewed as algebras in the sense of the above definition. For instance, modules over a ring $\boldsymbol{R}$ can be viewed as algebras whose type ρ_R extends that of groups with the unary symbols $\{\lambda_r : r \in R\}$. From this point of view, a module over $\boldsymbol{R}$ is an

Date: June 13, 2021.

algebra $\langle G; +, -, 0, \{\lambda_r : r \in R\}\rangle$ of type ρ_R such that $\langle G; +, -, 0\rangle$ is an abelian group and, for every $r, s \in R$ and $a, c \in G$,

$$\begin{aligned}\lambda_r(a + c) &= \lambda_r(a) + \lambda_r(c)\\ \lambda_{r+s}(a) &= \lambda_r(a) + \lambda_s(a)\\ \lambda_r(\lambda_s(a)) &= \lambda_{r\cdot s}(a)\\ \lambda_1(a) &= a.\end{aligned}$$

Given a type $\rho\colon \mathcal{F} \to \mathbb{N}$ and a set of variables X disjoint from $\mathcal{F}$, the set of *terms of type ρ over X* is the least set $T_\rho(X)$ such that

(i) $X \subseteq T_\rho(X)$;
(ii) if $c \in \mathcal{F}$ is a constant, then $c \in T_\rho(X)$; and
(iii) if $\varphi_1, \dots, \varphi_{\rho(f)} \in T_\rho(X)$ and $f \in \mathcal{F}$, then $f\varphi_1 \dots \varphi_{\rho(f)} \in T_\rho(X)$.

For the sake of readability, we shall often write $f(\varphi_1, \dots, \varphi_{\rho(f)})$ instead of $f\varphi_1 \dots \varphi_{\rho(f)}$. Similarly, if f is a binary operation $+$, we often write $\varphi_1 + \varphi_2$ instead of $f(\varphi_1, \varphi_2)$.

Definition 1.2. Let $\rho\colon \mathcal{F} \to \mathbb{N}$ be a type and X a set of variables disjoint from $\mathcal{F}$. The *term algebra* $\boldsymbol{T}_\rho(X)$ of type ρ over X is the unique algebra of type ρ whose universe is $T_\rho(X)$ and with basic n-ary operations f defined, for every $\varphi_1, \dots, \varphi_n \in T_\rho(X)$, as

$$f^{\boldsymbol{T}_\rho(X)}(\varphi_1, \dots, \varphi_n) := f(\varphi_1, \dots, \varphi_n).$$

When no confusion might arise, we drop the subscript and write $T(X)$ instead of $T_\rho(X)$. Term algebras have the following fundamental property.

Proposition 1.3. *Let $\boldsymbol{A}$ be an algebra of type ρ and X a set of variables. Every function $f\colon X \to A$ extends uniquely to a homomorphism $f^*\colon \boldsymbol{T}_\rho(X) \to \boldsymbol{A}$.*

Proof. The unique extension f^* is defined, for every $\varphi(x_{\alpha_1}, \dots x_{\alpha_n}) \in T_\rho(X)$, as

$$f^*(\varphi) = \varphi^{\boldsymbol{A}}(f(x_{\alpha_1}), \dots, f(x_{\alpha_n})). \qquad \boxtimes$$

Exercise 1.4. Prove the above proposition. $\boxtimes$

Corollary 1.5. *If $f\colon \boldsymbol{T}(Var) \to \boldsymbol{A}$ and $g\colon \boldsymbol{B} \to \boldsymbol{A}$ are homomorphisms and g is surjective, there exists a homomorphism $h\colon \boldsymbol{T}(Var) \to \boldsymbol{B}$ such that $f = g \circ h$.*

Proof. As $g\colon \boldsymbol{B} \to \boldsymbol{A}$ is surjective, for every $n \in \mathbb{N}$ there exists some $b_n \in B$ such that $f(x_n) = g(b_n)$. By Proposition 1.3, there exists a homomorphism $h\colon \boldsymbol{T}(Var) \to \boldsymbol{B}$ such that $h(x_n) = b_n$, for all $n \in \mathbb{N}$. As a consequence,

$$g \circ h(x_n) = g(b_n) = f(x_n), \text{ for all } n \in \mathbb{N}.$$

Since Var is a set of generators for $\boldsymbol{T}(Var)$ and both $g \circ h$ and f are homomorphisms, we conclude that $f = g \circ h$. $\boxtimes$

Given a term $\varphi \in T_\rho(X)$, we write $\varphi(x_1, \dots, x_n)$ to indicate that the variables occurring in φ are among $x_1, \dots, x_n$. Furthermore, given an algebra $\boldsymbol{A}$ of type ρ and elements $a_1, \dots, a_n \in A$, we define an element

$$\varphi^{\boldsymbol{A}}(a_1, \dots, a_n)$$

of A, by recursion on the construction of φ, as follows:

(i) if φ is a variable x_i, then $\varphi^{\boldsymbol{A}}(a_1, \dots, a_n) := a_i$;

(ii) if φ is a constant c, then $c^{\boldsymbol{A}}$ is the interpretation of c in $\boldsymbol{A}$;
(iii) if $\varphi = f(\psi_1, \dots, \psi_m)$, then

$$\varphi^{\boldsymbol{A}}(a_1, \dots, a_n) \coloneqq f^{\boldsymbol{A}}(\psi_1^{\boldsymbol{A}}(a_1, \dots, a_n), \dots, \psi_m^{\boldsymbol{A}}(a_1, \dots, a_n)).$$

An *equation of type ρ over X* is an expression of the form $\varphi \approx \psi$, where $\varphi, \psi \in T_\rho(X)$. We denote by $E_\rho(X)$ the set of equations of type ρ over X. Such an equation $\varphi \approx \psi$ is *valid* in an algebra $\boldsymbol{A}$ of type ρ, if

$$\varphi^{\boldsymbol{A}}(a_1, \dots, a_n) = \psi^{\boldsymbol{A}}(a_1, \dots, a_n), \text{ for every } a_1, \dots, a_n \in A,$$

in which case we say that $\boldsymbol{A}$ *satisfies* $\varphi \approx \psi$.

For instance, groups are precisely the algebras of type ρ_G that satisfy the equations

$$x + (y + z) \approx (x + y) + z \quad x + 0 \approx x \quad 0 + x \approx x \quad x + -x \approx 0 \quad -x + x \approx 0.$$

Similarly, lattices are the algebras of type ρ_L that satisfy the equations

$$\begin{array}{ccr} x \wedge x \approx x & x \vee x \approx x & (\textit{idempotent laws}) \\ x \wedge y \approx y \wedge x & x \vee y \approx y \vee x & (\textit{commutative laws}) \\ x \wedge (y \wedge z) \approx (x \wedge y) \wedge z & x \vee (y \vee z) \approx (x \vee y) \vee z & (\textit{associative laws}) \\ x \wedge (y \vee x) \approx x & x \vee (y \wedge x) \approx x. & (\textit{absorption laws}) \end{array}$$

From now on, we will work with a fixed denumerable set of variables

$$Var = \{x_n : n \in \mathbb{N}\}.$$

Accordingly, when we write $x, y, z \dots$ for variables, it should be understood that these are variables in Var.

2. Basic constructions

Algebras of the same type are called *similar* and can be compared by means of maps that preserve their structure.

Definition 2.1. Given two similar algebras $\boldsymbol{A}$ and $\boldsymbol{B}$, a *homomorphism* from $\boldsymbol{A}$ to $\boldsymbol{B}$ is a map $f \colon A \to B$ such that, for every n-ary operation g of the common type and $a_1, \dots, a_n \in A$,

$$f(g^{\boldsymbol{A}}(a_1, \dots, a_n)) = g^{\boldsymbol{B}}(f(a_1), \dots, f(a_n)).$$

An injective homomorphism is called an *embedding* and, if there exists an embedding from $\boldsymbol{A}$ to $\boldsymbol{B}$, we say that $\boldsymbol{A}$ *embeds* into $\boldsymbol{B}$. Lastly, a surjective embedding is called an *isomorphism*. Accordingly, $\boldsymbol{A}$ and $\boldsymbol{B}$ are said to be *isomorphic* if there exists an isomorphism between them, in which case we write $\boldsymbol{A} \cong \boldsymbol{B}$.

A simple induction on the construction of terms shows that, for every pair of algebras $\boldsymbol{A}$ and $\boldsymbol{B}$ of type ρ and every term $\varphi(x_1, \dots, x_n)$ of ρ, if f is a homomorphism from $\boldsymbol{A}$ to $\boldsymbol{B}$, then

$$f(\varphi^{\boldsymbol{A}}(a_1, \dots, a_n)) = \varphi^{\boldsymbol{B}}(f(a_1), \dots, f(a_n)),$$

for every $a_1, \dots, a_n \in A$. Therefore homomorphisms preserve not only basic operations, but also arbitrary terms.

In the particular case where $\boldsymbol{A}$ and $\boldsymbol{B}$ are lattices, a homomorphism from $\boldsymbol{A}$ to $\boldsymbol{B}$ is a map $f \colon A \to B$ such that, for every $a, c \in A$,

$$f(a \wedge^{\boldsymbol{A}} c) = f(a) \wedge^{\boldsymbol{B}} f(c) \;\text{ and }\; f(a \vee^{\boldsymbol{A}} c) = f(a) \vee^{\boldsymbol{B}} f(c).$$

For instance, the inclusion map from the lattice $\langle \mathbb{N}; \leqslant \rangle$ into the lattice $\langle \mathbb{Z}; \leqslant \rangle$ is an injective homomorphism, that is, an embedding. Similarly, given two sets $Y \subseteq X$, the inclusion map from the powerset lattice $\langle \mathcal{P}(Y); \subseteq \rangle$ to the powerset lattice $\langle \mathcal{P}(X); \subseteq \rangle$ is also an embedding. On the other hand, if $Y \subsetneq X$, the map

$$(-) \cap Y \colon \mathcal{P}(X) \to \mathcal{P}(Y)$$

that sends every $Z \subseteq X$ to $Z \cap Y$ is a noninjective homomorphism from $\langle \mathcal{P}(X); \subseteq \rangle$ to $\langle \mathcal{P}(Y); \subseteq \rangle$.

Definition 2.2. Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be algebras of the same type $\rho \colon \mathcal{F} \to \mathbb{N}$. Then $\boldsymbol{A}$ is said to be a *subalgebra* of $\boldsymbol{B}$ if $A \subseteq B$ and $f^{\boldsymbol{A}}$ is the restriction of $f^{\boldsymbol{B}}$ to A, for every $f \in \mathcal{F}$. In this case, we write $\boldsymbol{A} \leqslant \boldsymbol{B}$.

Given a class of algebras K, let

$$\mathbb{I}(\mathsf{K}) := \{\boldsymbol{A} : \boldsymbol{A} \cong \boldsymbol{B} \text{ for some } \boldsymbol{B} \in \mathsf{K}\}$$
$$\mathbb{S}(\mathsf{K}) := \{\boldsymbol{A} : \boldsymbol{A} \leqslant \boldsymbol{B} \text{ for some } \boldsymbol{B} \in \mathsf{K}\}.$$

When $\mathsf{K} = \{\boldsymbol{A}\}$, we write $\mathbb{I}(\boldsymbol{A})$ and $\mathbb{S}(\boldsymbol{A})$ as a shorthand for $\mathbb{I}(\{\boldsymbol{A}\})$ and $\mathbb{S}(\{\boldsymbol{A}\})$, respectively. The following observation is an immediate consequence of the definitions.

Proposition 2.3. *Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be algebras of the same type. Then $\boldsymbol{A} \in \mathbb{IS}(\boldsymbol{B})$ if and only if there exists an embedding $f \colon \boldsymbol{A} \to \boldsymbol{B}$. In this case, $\boldsymbol{A}$ is isomorphic to the unique subalgebra of $\boldsymbol{B}$ with universe $f[A]$.*

As we mentioned, homomorphisms can be used to compare similar algebras.

Definition 2.4. Given two similar algebras $\boldsymbol{A}$ and $\boldsymbol{B}$, we say that $\boldsymbol{A}$ is a *homomorphic image* of $\boldsymbol{B}$ if there exists a surjective homomorphism $f \colon \boldsymbol{B} \to \boldsymbol{A}$.

Accordingly, given a class of algebras K, we set

$$\mathbb{H}(\mathsf{K}) := \{\boldsymbol{A} : \boldsymbol{A} \text{ is a homomorphic image of some } \boldsymbol{B} \in \mathsf{K}\}.$$

As usual, when $\mathsf{K} = \{\boldsymbol{A}\}$, we write $\mathbb{H}(\boldsymbol{A})$ as a shorthand for $\mathbb{H}(\{\boldsymbol{A}\})$.

Observe that every (not necessarily surjective) homomorphism $f \colon \boldsymbol{A} \to \boldsymbol{B}$ induces a homomorphic image of $\boldsymbol{A}$.

Proposition 2.5. *If $f \colon \boldsymbol{A} \to \boldsymbol{B}$ is a homomorphism, then $f[A]$ is the universe of a subalgebra of $\boldsymbol{B}$ that, moreover, is a homomorphic image of $\boldsymbol{A}$.*

Proof. Observe that $f[A]$ is nonempty, because A is. Then consider an n-ary function symbol g of the common type of $\boldsymbol{A}$ and $\boldsymbol{B}$ and $b_1, \dots, b_n \in f[A]$. Clearly, there are $a_1, \dots, a_n \in A$ such that $f(a_i) = b_i$, for every $i \leqslant n$. Since f is a homomorphism from $\boldsymbol{A}$ to $\boldsymbol{B}$, we obtain

$$g^{\boldsymbol{B}}(b_1, \dots, b_n) = g^{\boldsymbol{B}}(f(a_1), \dots, g(a_n)) = f(g^{\boldsymbol{A}}(a_1, \dots, a_n)) \in f[A].$$

Hence, we conclude that $f[A]$ is the universe of a subalgebra $f[\boldsymbol{A}]$ of $\boldsymbol{B}$.

Furthermore, $f \colon \boldsymbol{A} \to f[\boldsymbol{A}]$ is a homomorphism, because for every basic n-ary function symbol g of the common type and $a_1, \dots, a_n \in A$,

$$f(g^{\boldsymbol{A}}(a_1, \dots, a_n)) = g^{\boldsymbol{B}}(f(a_1), \dots, f(a_n)) = g^{f[\boldsymbol{A}]}(f(a_1), \dots, f(a_n)),$$

where the first equality follows from the assumption that $f \colon \boldsymbol{A} \to \boldsymbol{B}$ is a homomorphism. Since the map $f \colon A \to f[A]$ is surjective, we conclude that $f[\boldsymbol{A}] \in \mathbb{H}(\boldsymbol{A})$. ⊠

In view of the above result, when $f\colon \boldsymbol{A} \to \boldsymbol{B}$ is a homomorphism, we denote by $f[\boldsymbol{A}]$ the unique subalgebra of $\boldsymbol{B}$ with universe $f[A]$.

For instance, let $f\colon \mathbb{Z} \to \mathbb{R}$ be the absolute value map, that is, the function defined by the rule

$$f(n) \coloneqq \text{the absolute value of } n.$$

Observe that f is a nonsurjective homomorphism from the lattice of integers to that of reals. Furthermore, the homomorphic image $f[\langle \mathbb{Z}; \leqslant \rangle]$ of $\langle \mathbb{Z}; \leqslant \rangle$ is the lattice of natural numbers $\langle \mathbb{N}; \leqslant \rangle$, which, in turn, is a subalgebra of lattice of reals.

Notably, the homomorphic images of an algebra $\boldsymbol{A}$ can be "internalized" as special equivalence relations on $\boldsymbol{A}$ as follows.

Definition 2.6. A *congruence* of an algebra $\boldsymbol{A}$ is an equivalence relation θ on A such that, for every basic n-ary opearation f of $\boldsymbol{A}$ and $a_1, \dots, a_n, c_1, \dots, c_n \in A$,

$$\text{if } \langle a_1, c_1 \rangle, \dots, \langle a_n, c_n \rangle \in \theta, \text{ then } \langle f^{\boldsymbol{A}}(a_1, \dots, a_n), f^{\boldsymbol{A}}(c_1, \dots, c_n) \rangle \in \theta. \tag{1}$$

In this case, we often write $a \equiv_\theta c$ as a shorthand for $\langle a, c \rangle \in \theta$. The poset of congruences of $\boldsymbol{A}$ ordered under the inclusion relation will be denoted by $\mathsf{Con}(\boldsymbol{A})$.

A simple induction on the construction of terms shows that, for every congruence θ of $\boldsymbol{A}$ and every term $\varphi(x_1, \dots, x_n)$,

$$\text{if } \langle a_1, c_1 \rangle, \dots, \langle a_n, c_n \rangle \in \theta, \text{ then } \langle \varphi^{\boldsymbol{A}}(a_1, \dots, a_n), \varphi^{\boldsymbol{A}}(c_1, \dots, c_n) \rangle \in \theta,$$

for every $a_1, \dots, a_n \in A$. Therefore, congruences preserve not only basic operations, but also arbitrary terms. Furthermore, a simple argument shows that $\mathsf{Con}(\boldsymbol{A})$ is a complete (indeed algebraic) lattice whose maximum is the total relation $A \times A$ and whose minimum is the identity relation $\mathrm{id}_A \coloneqq \{\langle a, a \rangle : a \in A\}$.

Example 2.7 (Heyting algebras)**.** Recall that a *filter* of a Heyting algebra $\boldsymbol{A}$ is a nonempty upset $F \subseteq A$ closed under binary meets. We denote by $\mathsf{Fi}(\boldsymbol{A})$ the poset of filters of $\boldsymbol{A}$ ordered under the inclusion relation. It is easy to see $\mathsf{Fi}(\boldsymbol{A})$ is a complete lattice. Furthermore, the lattices $\mathsf{Fi}(\boldsymbol{A})$ and $\mathsf{Con}(\boldsymbol{A})$ are isomorphic via the inverse isomorphisms

$$\boldsymbol{\Omega}^{\boldsymbol{A}}(-)\colon \mathsf{Fi}(\boldsymbol{A}) \to \mathsf{Con}(\boldsymbol{A}) \quad \text{and} \quad \boldsymbol{\tau}^{\boldsymbol{A}}(-)\colon \mathsf{Con}(\boldsymbol{A}) \to \mathsf{Fi}(\boldsymbol{A})$$

defined by the rules

$$\begin{aligned} \boldsymbol{\Omega}^{\boldsymbol{A}}(F) &\coloneqq \{\langle a, c \rangle \in A \times A : a \to c, c \to a \in F\} \\ \boldsymbol{\tau}^{\boldsymbol{A}}(\theta) &\coloneqq \{a \in A : \langle a, 1 \rangle \in \theta\}. \end{aligned}$$

Because of this, every congruence θ of a Heyting algebra $\boldsymbol{A}$ is induced by some filter F, in the sense that $\theta = \boldsymbol{\Omega}^{\boldsymbol{A}} F$. ⊠

Example 2.8 (Modal algebras)**.** A *modal algebra* is an algebra $\boldsymbol{A} = \langle A; \wedge, \vee, \neg, \Box, 0, 1 \rangle$ such that $\langle A; \wedge, \vee, \neg, 0, 1 \rangle$ is a Boolean algebra and $\Box$ is a unary operation such that

$$\Box(a \wedge c) = \Box a \wedge \Box c \quad \text{and} \quad \Box 1 = 1,$$

for every $a, c \in A$. An *open filter* of a modal algebra $\boldsymbol{A}$ is a filter of the Boolean reduct of $\boldsymbol{A}$ that, moreover, is closed under the operation $\Box$. The poset of open filters of $\boldsymbol{A}$ ordered under the inclusion relation will be denoted by $\mathsf{Op}(\boldsymbol{A})$. It forms a complete lattice. Furthermore, the lattices $\mathsf{Op}(\boldsymbol{A})$ and $\mathsf{Con}(\boldsymbol{A})$ are isomorphic via the inverse isomorphisms described in Example 2.7. Because of this, every congruence of a modal algebra $\boldsymbol{A}$ has the form $\theta = \boldsymbol{\Omega}^{\boldsymbol{A}} F$, for some open filter F. ⊠

Example 2.9 (Groups)**.** Similarly, it is well known that the lattice of congruences of a group is isomorphic to that of its normal subgroups. Because of this, every congruence of a group is induced by some normal subgroup. ⊠

As we mentioned, there is a tight correspondence between the homomorphic images and the congruences of an algebra $\boldsymbol{A}$. On the one hand, every congruence θ of $\boldsymbol{A}$ gives rise to a homomorphic image $\boldsymbol{A}/\theta$ of $\boldsymbol{A}$. Let $\mathcal{F}$ be the set of function symbols of $\boldsymbol{A}$. Given $\theta \in \mathsf{Con}(\boldsymbol{A})$ and a basic n-ary function symbol $f \in \mathcal{F}$, let $f^{\boldsymbol{A}/\theta}$ be the n-ary operation on A/θ defined by the rule

$$f^{\boldsymbol{A}/\theta}(a_1/\theta, \dots, a_n/\theta) \coloneqq f^{\boldsymbol{A}}(a_1, \dots, a_n)/\theta.$$

Notice that $f^{\boldsymbol{A}/\theta}$ is well-defined, by condition (1). As a consequence, the structure

$$\boldsymbol{A}/\theta \coloneqq \langle A/\theta; \{f^{\boldsymbol{A}/\theta} : f \in \mathcal{F}\}\rangle$$

is a well-defined algebra of the type as $\boldsymbol{A}$. Furthermore, $\boldsymbol{A}/\theta \in \mathbb{H}(\boldsymbol{A})$, because the map $\pi_\theta \colon A \to A/\theta$, defined, for every $a \in A$, as $\pi_\theta(a) \coloneqq a/\theta$, is a surjective homomorphism from $\boldsymbol{A}$ to $\boldsymbol{A}/\theta$. To prove this, consider $a_1, \dots, a_n \in A$. We have

$$\begin{aligned} \pi_\theta(f^{\boldsymbol{A}}(a_1, \dots, a_n)) &= f^{\boldsymbol{A}}(a_1, \dots, a_n)/\theta \\ &= f^{\boldsymbol{A}/\theta}(a_1/\theta, \dots, a_n/\theta) \\ &= f^{\boldsymbol{A}/\theta}(\pi_\theta(a_1), \dots, \pi_\theta(a_n)), \end{aligned}$$

where the second equality follows from the definition of the operation $f^{\boldsymbol{A}/\theta}$.

Corollary 2.10. *If θ is a congruence of an algebra $\boldsymbol{A}$, then $\boldsymbol{A}/\theta$ is a well-defined homomorphic image of $\boldsymbol{A}$.*

In view of the above result, every congruence θ of an algebra $\boldsymbol{A}$ induces a homomorphic image of $\boldsymbol{A}$, namely $\boldsymbol{A}/\theta$. The converse is also true, as we proceed to explain.

Definition 2.11. The *kernel* of a homomorphism $f \colon \boldsymbol{A} \to \boldsymbol{B}$ is the binary relation

$$\mathsf{Ker}(f) \coloneqq \{\langle a, c\rangle \in A \times A : f(a) = f(c)\}.$$

Proposition 2.12. *The kernel of a homomorphism $f \colon \boldsymbol{A} \to \boldsymbol{B}$ is a congruence of $\boldsymbol{A}$.*

Proof. It is obvious that $\mathsf{Ker}(f)$ is an equivalence relation on A. Therefore, to prove that $\mathsf{Ker}(f)$ is a congruence of $\boldsymbol{A}$, it suffices to show that it preserves the basic operations of $\boldsymbol{A}$. Consider a basic n-ary operation g of $\boldsymbol{A}$ and $a_1, \dots, a_n, c_1, \dots, c_n \in A$ such that $\langle a_1, c_1\rangle, \dots, \langle a_n, c_n\rangle \in \mathsf{Ker}(f)$. By the definition of $\mathsf{Ker}(f)$,

$$f(a_i) = f(c_i), \text{ for every } i \leqslant n.$$

It follows that $g^{\boldsymbol{B}}(f(a_1), \dots, f(a_n)) = g^{\boldsymbol{B}}(f(c_1), \dots, f(c_n))$. Since $f \colon \boldsymbol{A} \to \boldsymbol{B}$ is a homomorphism, this yields

$$f(g^{\boldsymbol{A}}(a_1, \dots, a_n)) = g^{\boldsymbol{B}}(f(a_1), \dots, f(a_n)) = g^{\boldsymbol{B}}(f(c_1), \dots, f(c_n)) = f(g^{\boldsymbol{A}}(c_1, \dots, c_n)).$$

Hence, we conclude that $\langle g^{\boldsymbol{A}}(a_1, \dots, a_n), g^{\boldsymbol{A}}(c_1, \dots, c_n)\rangle \in \mathsf{Ker}(f)$, as desired. ⊠

The behaviour of kernels is governed by the next principle.

Fundamental Homomorphism Theorem 2.13. *If $f \colon \boldsymbol{A} \to \boldsymbol{B}$ is a homomorphism with kernel θ, then there exists a unique embedding $g \colon \boldsymbol{A}/\theta \to \boldsymbol{B}$ such that $f = g \circ \pi_\theta$.*

Proof. We begin by proving the existence of g. Let $g\colon A/\theta \to B$ be the map defined as $g(a/\theta) := f(a)$, for every $a \in A$. To show that g is well-defined, consider $a, c \in A$ such that $a/\theta = c/\theta$. Since $\theta = \mathsf{Ker}(f)$, this means that $f(a) = f(c)$, as desired. Furthermore, the definition of g guarantees that $f = g \circ \pi_\theta$.

Now, observe g is injective, because, for every $a, c \in A$ such that $g(a/\theta) = g(c/\theta)$, we have $f(a) = f(c)$, that is, $\langle a, c\rangle \in \mathsf{Ker}(f) = \theta$ and, therefore, $a/\theta = c/\theta$. Moreover, for every basic n-ary operation p of $\boldsymbol{A}$ and $a_1, \dots, a_n \in A$, we have

$$\begin{aligned} g(p^{\boldsymbol{A}/\theta}(a_1/\theta, \dots, a_n/\theta)) &= g(p^{\boldsymbol{A}}(a_1, \dots, a_n)/\theta)\\ &= f(p^{\boldsymbol{A}}(a_1, \dots, a_n))\\ &= p^{\boldsymbol{B}}(f(a_1), \dots, f(a_n))\\ &= p^{\boldsymbol{B}}(g(a_1/\theta), \dots, g(a_n/\theta)). \end{aligned}$$

The first equality above follows from the definition of $\boldsymbol{A}/\theta$, the second and the last from the definition of g and the third from the assumption that $f\colon \boldsymbol{A} \to \boldsymbol{B}$ is a homomorphism. Hence, we conclude that $g\colon \boldsymbol{A}/\theta \to \boldsymbol{B}$ is a homomorphism and, therefore, an embedding, as desired.

The uniqueness of g follows from the fact that, if a map g^* satisfies the condition in the statement of the theorem, then, for every $a \in A$,

$$f(a) = g^* \circ \pi_\theta(a) = g^*(a/\theta),$$

that is, g^* coincides with g. ⊠

Corollary 2.14. *If $f\colon \boldsymbol{A} \to \boldsymbol{B}$ is a homomorphism, then $f[\boldsymbol{A}] \cong \boldsymbol{A}/\mathsf{Ker}(f)$. In particular, if f is surjective, $\boldsymbol{B} \cong \boldsymbol{A}/\mathsf{Ker}(f)$.*

Proof. In the proof of the Fundamental Homomorphism Theorem we showed that the map $g\colon A/\mathsf{Ker}(f) \to B$, defined by the rule $g(a/\mathsf{Ker}(f)) := f(a)$, is an embedding of $\boldsymbol{A}/\mathsf{Ker}(f)$ into $\boldsymbol{B}$. As g can be viewed as a surjective embedding of $\boldsymbol{A}/\mathsf{Ker}(f)$ into $f[\boldsymbol{A}]$, we conclude that $f[\boldsymbol{A}] \cong \boldsymbol{A}/\mathsf{Ker}(f)$. ⊠

At this stage, it should be clear that if θ is a congruence on an algebra $\boldsymbol{A}$, then $\pi_\theta\colon \boldsymbol{A} \to \boldsymbol{A}/\theta$ is a surjective homomorphism whose kernel is θ. Similarly, if $f\colon \boldsymbol{A} \to \boldsymbol{B}$ is a surjective homomorphism, then $\boldsymbol{A}/\mathsf{Ker}(f) \cong \boldsymbol{B}$, by Corollary 2.14. As a consequence, for every class of algebras K,

$$\mathbb{H}(\mathsf{K}) = \mathbb{I}\{\boldsymbol{A}/\theta : \boldsymbol{A} \in \mathsf{K} \text{ and } \theta \in \mathsf{Con}(\boldsymbol{A})\}. \tag{2}$$

Now, recall that the Cartesian product of a family of sets $\{A_i : i \in I\}$ is the set

$$\prod_{i\in I} A_i := \{f\colon I \to \bigcup_{i \in I} A_i \ :\ f(i) \in A_i, \text{for all } i \in I\}.$$

In particular, if I is empty, then $\prod_{i\in I} A_i$ is the singleton containing only the empty map.

Definition 2.15. The *direct product* of a family of similar algebras $\{\boldsymbol{A}_i : i \in I\}$ is the unique algebra of the common type whose universe is the Cartesian product $\prod_{i\in I} A_i$ and such that, for every basic n-ary operation symbol f and every $\vec{a}_1, \dots, \vec{a}_n \in \prod_{i\in I} A_i$,

$$f^{\prod_{i\in I} \boldsymbol{A}_i}(\vec{a}_1, \dots, \vec{a}_n)(i) = f^{\boldsymbol{A}_i}(\vec{a}_1(i), \dots, \vec{a}_n(i)), \text{ for every } i \in I.$$

We denote this algebra by $\prod_{i\in I} \boldsymbol{A}_i$.

In this case, for every $j \in I$, the projection map on the j-th component $p_j \colon \prod_{i \in I} A_i \to A_j$, defined by the rule $p_j(\vec{a}) := \vec{a}(j)$, is a surjective homomorphism from $\prod_{i \in I} \boldsymbol{A}_i$ to $\boldsymbol{A}_j$.

Given a class of similar algebras K, we set

$$\mathbb{P}(\mathsf{K}) := \{\boldsymbol{A} : \boldsymbol{A} \text{ is a direct product of a family } \{\boldsymbol{B}_i : i \in I\} \subseteq \mathsf{K}\}.$$

As usual, when $\mathsf{K} = \{\boldsymbol{A}\}$, we write $\mathbb{P}(\boldsymbol{A})$ as a shorthand for $\mathbb{P}(\{\boldsymbol{A}\})$.

Notice that up to isomorphism, there exists a unique one-element algebra of a given type. Because of this, one-element algebras are called *trivial*. Accordingly, when the set of indexes I is empty, the direct product $\prod_{i \in I} \boldsymbol{A}_i$ is the trivial algebra of the given type. It follows that $\mathbb{P}(\mathsf{K})$ contains always a trivial algebra, for every class of similar algebras K.

Example 2.16 (Powerset algebras)**.** Boolean algebras of the form $\langle \mathcal{P}(X); \cap, \cup, -, \emptyset, X \rangle$ are called *powerset Boolean algebras*. Let $\boldsymbol{B}$ be the two-element Boolean algebra and observe that $\mathbb{IP}(\boldsymbol{B})$ is the class of algebras isomorphic to some powerset Boolean algebra. To prove this, observe that every powerset Boolean algebra $\mathcal{P}(X)$ is isomorphic to a direct product of $\boldsymbol{B}$ via the *characteristic function* $f_X \colon \mathcal{P}(X) \to \prod_{x \in X} \boldsymbol{B}_x$, defined by the rule

$$f(Y)(x) := \begin{cases} 1 & \text{if } x \in Y \\ 0 & \text{if } x \notin Y, \end{cases}$$

where $Y \in \mathcal{P}(X)$ and $x \in X$. By the same token, every direct product $\prod_{i \in I} \boldsymbol{B}_i$ of $\boldsymbol{B}$ is isomorphic to the powerset Boolean algebra $\mathcal{P}(I)$ via the isomorphism f_I. $\boxtimes$

We close this section by reviewing the subdirect product construction.

Definition 2.17. A subalgebra $\boldsymbol{B}$ of a direct product $\prod_{i \in I} \boldsymbol{A}_i$ is said to be a *subdirect product* of $\{\boldsymbol{A}_i : i \in I\}$ if the projection map p_i is surjective, for every $i \in I$. Similarly, an embedding $f \colon \boldsymbol{B} \to \prod_{i \in I} \boldsymbol{A}_i$ is said to be *subdirect* when $f[\boldsymbol{B}]$ is a subdirect product of the family $\{\boldsymbol{A}_i : i \in I\}$.

Given a class of similar algebras K, we set

$$\mathbb{P}_{\mathrm{SD}}(\mathsf{K}) := \{\boldsymbol{A} : \boldsymbol{A} \text{ is a subdirect direct product of a family } \{\boldsymbol{B}_i : i \in I\} \subseteq \mathsf{K}\}.$$

As usual, when $\mathsf{K} = \{\boldsymbol{A}\}$, we write $\mathbb{P}_{\mathrm{SD}}(\boldsymbol{A})$ as a shorthand for $\mathbb{P}_{\mathrm{SD}}(\{\boldsymbol{A}\})$. Clearly, $\mathbb{P}_{\mathrm{SD}}(\mathsf{K}) \subseteq \mathbb{SP}(\mathsf{K})$. Furthermore, $\mathbb{P}_{\mathrm{SD}}(\mathsf{K})$ contains always a trivial algebra.

Example 2.18 (Distributive lattices)**.** Let DL be the class of distributive lattices and $\boldsymbol{B}$ be the two-element distributive lattice. Birkhoff's Representation Theorem states that $\mathsf{DL} = \mathbb{IP}_{\mathrm{SD}}(\boldsymbol{B})$. The inclusion $\mathbb{IP}_{\mathrm{SD}}(\boldsymbol{B}) \subseteq \mathsf{DL}$ follows from the fact that DL is closed under $\mathbb{I}, \mathbb{S}$ and $\mathbb{P}$. For the other inclusion, consider a distributive lattice $\boldsymbol{A}$ and let I be the set of its prime filters. By Birkhoff's Representation Theorem, the map

$$\gamma \colon \boldsymbol{A} \to \prod_{F \in I} \boldsymbol{B}_F,$$

defined, for every $a \in A$ and $F \in I$, by the rule

$$\gamma(a)(F) := \begin{cases} 1 & \text{if } a \in F \\ 0 & \text{if } a \notin F, \end{cases}$$

is a well-defined subdirect embedding. $\boxtimes$

Example 2.19 (Boolean algebras)**.** Similarly, Stone's Representation Theorem states that the class of Boolean algebras coincides with $\mathbb{IP}_{\mathrm{SD}}(\boldsymbol{B})$, where $\boldsymbol{B}$ the two-element Boolean algebra. $\boxtimes$

The next result provides a general recipe to construct subdirect products.

Proposition 2.20. *Let $\boldsymbol{A}$ be an algebra and $\{\theta_i : i \in I\} \subseteq \mathsf{Con}(\boldsymbol{A})$. Then the map*

$$f\colon \boldsymbol{A}/\bigcap_{i\in I}\theta_i \to \prod_{i\in I}\boldsymbol{A}/\theta_i,$$

defined, for every $a \in A$ and $j \in I$, as

$$f(a/\bigcap_{i\in I}\theta_i)(j) \coloneqq a/\theta_j,$$

is a subdirect embedding.

Proof. For the sake of readability, set $\boldsymbol{B} \coloneqq \boldsymbol{A}/\bigcap_{i\in I}\theta_i$. To prove that f is injective, consider $a, c \in A$ such that $\langle a, c\rangle \notin \bigcap_{i\in I}\theta_i$. Then there exists $j \in I$ such that $\langle a, c\rangle \notin \theta_j$ and, therefore,

$$f(a/\bigcap_{i\in I}\theta_i)(j) \coloneqq a/\theta_j \neq c/\theta_j = f(c/\bigcap_{i\in I}\theta_i)(j).$$

It follows that $f(a/\bigcap_{i\in I}\theta_i) \neq f(c/\bigcap_{i\in I}\theta_i)$. Thus, f is injective. Moreover, by the definition of f, the composition $p_i \circ f\colon \boldsymbol{B} \to \boldsymbol{A}/\theta_i$ is surjective, for every $i \in I$.

It only remains to prove that f is a homomorphism. Consider an n-ary basic operation g and $a_1, \dots, a_n \in A$. For every $j \in I$, we have

$$\begin{aligned}
f(g^{\boldsymbol{B}}(a_1/\bigcap_{i\in I}\theta_i, \dots, a_n/\bigcap_{i\in I}\theta_i))(j) &= f(g^{\boldsymbol{A}}(a_1, \dots, a_n)/\bigcap_{i\in I}\theta_i)(j)\\
&= g^{\boldsymbol{A}}(a_1, \dots, a_n)/\theta_j\\
&= g^{\boldsymbol{A}/\theta_j}(a_1/\theta_j, \dots, a_n/\theta_j)\\
&= g^{\boldsymbol{A}/\theta_j}(f(a_1/\bigcap_{i\in I}\theta_i)(j), \dots, f(a_n/\bigcap_{i\in I}\theta_i)(j))\\
&= g^{\prod_{i\in I}\boldsymbol{A}/\theta_i}(f(a_1/\bigcap_{i\in I}\theta_i), \dots, f(a_n/\bigcap_{i\in I}\theta_i))(j).
\end{aligned}$$

It follows that

$$f(g^{\boldsymbol{B}}(a_1/\bigcap_{i\in I}\theta_i, \dots, a_n/\bigcap_{i\in I}\theta_i)) = g^{\prod_{i\in I}\boldsymbol{A}/\theta_i}(f(a_1/\bigcap_{i\in I}\theta_i), \dots, f(a_n/\bigcap_{i\in I}\theta_i)). \qquad \boxtimes$$

3. Propositional logics and equational completeness theorems

For general information on propositional logics we refer the reader to [17, 22, 23, 24, 45]. Recall that a *closure operator* on a set A is a map $C\colon \mathcal{P}(A) \to \mathcal{P}(A)$ such that, for every $X \subseteq Y \subseteq A$,

$$X \subseteq C(X) = C(C(X)) \text{ and } C(X) \subseteq C(Y).$$

Given a closure operator C on A, a subset $X \subseteq A$ is said to be *closed* if $X = C(X)$. A *closure system* on A is a family $\mathcal{C} \subseteq \mathcal{P}(A)$ that contains A and such that $\bigcap \mathcal{F}$, for every nonempty $\mathcal{F} \subseteq \mathcal{C}$. Closure operators and systems on A are two faces of the same coin. More precisely, if the family of closed sets of a closure operator on A is a closure system on A. On the other hand, if $\mathcal{C}$ is a closure system on A, then the map $C\colon \mathcal{P}(A) \to \mathcal{P}(A)$, defined by the rule

$$C(X) \coloneqq \bigcap\{Y \in \mathcal{C} : X \subseteq Y\},$$

is a closure operator on A. These transformations between closure operators and systems on A are one inverse to the other.

Exercise 3.1. Prove that these transformations are well-defined and one inverse to the other. ⊠

Another way of presenting closure operators or systems is by means of the following concept.

Definition 3.2. A *consequence relation* on a set A is a binary relation $\vdash \subseteq \mathcal{P}(A) \times A$ such that, for every $X \cup Y \cup \{a\} \subseteq A$,

(i) if $a \in X$, then $X \vdash a$; and
(ii) if $X \vdash y$ for all $y \in Y$ and $Y \vdash a$, then $X \vdash a$.

Furthermore, $\vdash$ is said to be *finitary* when, for every $X \cup \{a\} \subseteq A$,

$$\text{if } X \vdash a \text{, there exists a finite } Y \subseteq X \text{ such that } Y \vdash a.$$

Remark 3.3. The relation $X \vdash a$ should be read, intuitively, as "X proves a" or "a follows from X". In this reading, the demand expressed by condition (i) is rather natural, while (ii) is an abstract of the Cut rule. ⊠

Formally speaking, a consequence relation on a set A is a binary relation $\vdash \subseteq \mathcal{P}(A) \times A$. However, to simplify the notation, we will often write $a_1, \dots, a_n \vdash c$ as a shorthand for $\{a_1, \dots, a_n\} \vdash c$. Similarly, we will use $X, a \vdash c$ as a shorthand for $X \cup \{a\} \vdash c$. Lastly, for every set of formulas $X \cup Y \cup \{a, c\}$, we write

(i) $X \vdash Y$, when $X \vdash y$ for every $y \in Y$;
(ii) $a \dashv\vdash c$, when $a \vdash c$ and $c \vdash a$; and
(iii) $X \dashv\vdash Y$, when $X \vdash Y$ and $Y \vdash X$.

Definition 3.4. Let $\vdash$ be a consequence relation on a set A. A *theory* of $\vdash$ is a subset $X \subseteq A$ such that, for every $a \in A$, if $X \vdash a$, then $a \in X$. The set of theories of A will be denoted by $\mathcal{T}h(\vdash)$.

It is easy to see that $\mathcal{T}h(\vdash)$ is a closure system on A. Moreover, given a closure operator C on A, the following is a consequence relation on A:

$$\{\langle X, a\rangle \in \mathcal{P}(A) \times A : a \in C(X)\}.$$

Together with the correspondence between closure systems and operators, these transformations induce a one-to-one correspondence between consequence relations, closure operators and closure systems on A.

Exercise 3.5. Prove these facts. ⊠

In the context of logic, the term algebra $\boldsymbol{T}_\rho(Var)$ is often called the *algebra of formulas* (of type ρ) and its elements are referred to as *formulas*. An *endomorphism* of an algebra $\boldsymbol{A}$ is a homomorphism whose domain and codomain is $\boldsymbol{A}$. Endomorphisms of the algebra of formulas play a fundamental role in logic.

Definition 3.6. A *substitution* of type ρ is an endomorphism σ of $\boldsymbol{T}_\rho(Var)$.

When the type ρ is clear from the context, we will simply say that σ is a substitution.

In view of Proposition 1.3 and of the fact that Var is a set of generators for $\boldsymbol{T}_\rho(Var)$, every function $\sigma\colon Var \to T_\rho(Var)$ can be uniquely extended to a substitution σ^+ of type ρ, namely the function defined by the rule

$$\varphi(x_1, \dots, x_n) \longmapsto \varphi(\sigma(x_1), \dots, \sigma(x_n)).$$

Because of this, substitutions of type ρ can be presented by exhibiting functions $\sigma\colon Var \to T_\rho(Var)$.

Definition 3.7. A *logic* of type ρ is a consequence relation $\vdash$ on the set of formulas $T_\rho(Var)$ that, moreover, is *substitution invariant* in the sense that for every substitution σ of type ρ and every set of formulas $\Gamma \cup \{\varphi\} \subseteq T_\rho(Var)$,

$$\text{if } \Gamma \vdash \varphi, \text{ then } \sigma[\Gamma] \vdash \sigma(\varphi).$$

Remark 3.8. As mentioned above, $\Gamma \vdash \varphi$ should be read as "Γ proves φ" or "φ follows from Γ". The requirement that $\vdash$ is substitution invariant, instead, is intended to capture the idea that logical inferences are true only in virtue of their form (as opposed to their content). ⊠

Example 3.9 (Hilbert calculi)**.** We work within a fixed, but arbitrary, type ρ. A *rule* is an expression of the form $\Gamma \rhd \varphi$, where $\Gamma \cup \{\varphi\} \subseteq T_\rho(Var)$. In this case, Γ is said to be the set of *premises* of the rule and φ the *conclusion*. When $\Gamma = \emptyset$, the rule $\Gamma \rhd \varphi$ is sometimes called an *axiom*. A *Hilbert calculus* is a set of rules.

Every Hilbert calculus H induces a logic, as we proceed to explain. Consider a set of formulas $\Gamma \cup \{\varphi\} \subseteq T_\rho(Var)$. A *proof of φ from Γ in* H is a well-ordered sequence $\langle \psi_\alpha : \alpha \leqslant \gamma \rangle$ of formulas $\psi_\alpha \in T_\rho(Var)$ whose last element ψ_γ is φ and such that, for every $\alpha \leqslant \gamma$, either $\psi_\alpha \in \Gamma$ or there exist a substitution σ and a rule $\Delta \rhd \delta$ in H such that the formulas in $\sigma[\Delta]$ occur in the initial segment $\langle \psi_\beta : \beta < \alpha \rangle$ and $\psi_\alpha = \sigma(\delta)$.

The logic $\vdash_{\mathsf{H}}$ induced by H is defined, for every $\Gamma \cup \{\varphi\} \subseteq T_\rho(Var)$, as

$$\Gamma \vdash_{\mathsf{H}} \varphi \Longleftrightarrow \text{there exists a proof of } \varphi \text{ from } \Gamma \text{ in } \mathsf{H}.$$

As expected, $\vdash_{\mathsf{H}}$ is a logic in the sense of Definition 3.7. Furthermore, it is the least logic $\vdash$ such that $\Gamma \vdash \varphi$, for every rule $\Gamma \rhd \varphi$ in H.

A logic $\vdash$ is said to be *axiomatized* by a Hilbert calculus H when it coincides with $\vdash_{\mathsf{H}}$. Notice that every logic $\vdash$ is vacuously axiomatized by the Hilbert calculus

$$\{\Gamma \rhd \varphi : \Gamma \vdash \varphi\}.$$

Because of this, axiomatizations in terms of Hilbert calculi H acquire special interest when H is finite or, at least, recursive. ⊠

When no confusion shall arise, given a sequence $\vec{a}$ and a set A, we write $\vec{a} \in A$ to indicate that the elements of the sequence $\vec{a}$ belong to A. The following concept is instrumental to exhibit further examples of logics.

Definition 3.10. Let K be a class of similar algebras We define a binary relation $\vDash_{\mathsf{K}} \subseteq \mathcal{P}(E_\rho(Var)) \times E_\rho(Var)$ as follows:

$$\begin{aligned}\Theta \vDash_{\mathsf{K}} \varepsilon \approx \delta \Longleftrightarrow{} & \text{for every } \boldsymbol{A} \in \mathsf{K} \text{ and every } \vec{a} \in A,\\ & \quad \text{if } \varphi^{\boldsymbol{A}}(\vec{a}) = \psi^{\boldsymbol{A}}(\vec{a}) \text{ for all } \varphi \approx \psi \in \Theta, \text{ then } \varepsilon^{\boldsymbol{A}}(\vec{a}) = \delta^{\boldsymbol{A}}(\vec{a}).\end{aligned}$$

The relation $\vDash_{\mathsf{K}}$ is known as the *equational consequence relative to* K.

Example 3.11 (Equationally defined logics)**.** We work within a fixed, but arbitrary, type ρ. Given a set of equations $\boldsymbol{\tau}(x)$ in a single variable x and a set of formulas $\Gamma \cup \{\varphi\} \subseteq T(Var)$, we abbreviate

$$\{\varepsilon(\varphi) \approx \delta(\varphi) \colon \varepsilon \approx \delta \in \boldsymbol{\tau}\} \text{ as } \boldsymbol{\tau}(\varphi), \text{ and } \bigcup_{\gamma \in \Gamma} \boldsymbol{\tau}(\gamma) \text{ as } \boldsymbol{\tau}[\Gamma].$$

Given a class of algebras K and a set of equations $\boldsymbol{\tau}(x)$, we define a logic $\vdash_{\mathsf{K},\tau}$ as follows: for every $\Gamma \cup \{\varphi\} \subseteq T(Var)$,

$$\Gamma \vdash_{\mathsf{K},\boldsymbol{\tau}} \varphi \Longleftrightarrow \boldsymbol{\tau}[\Gamma] \vDash_{\mathsf{K}} \boldsymbol{\tau}(\varphi).$$

It is easy to prove that $\vdash_{\mathsf{K},\tau}$ is indeed a logic in the sense of Definition 3.7. Notice that, in this case, $\vdash$ is related to K by a *completeness theorem* witnessed by the set of equations $\boldsymbol{\tau}(x)$ that allows to translate formulas into equations and, therefore, to interpret $\vdash_{\mathsf{K},\tau}$ into $\vDash_{\mathsf{K}}$.

For instance, the completeness theorem of classical propositional logic **CPC** with respect to the class of Boolean algebras BA states precisely that **CPC** coincides with $\vdash_{\mathsf{BA},\boldsymbol{\tau}}$ where $\boldsymbol{\tau} = \{x \approx 1\}$. Similarly, the completeness theorem of intuitionistic propositional logic **IPC** with respect to the class of Heyting algebras HA states precisely that **IPC** coincides with $\vdash_{\mathsf{HA},\boldsymbol{\tau}}$ where $\boldsymbol{\tau} = \{x \approx 1\}$. Because of this, **CPC** and **IPC** can be defined as follows: for every set of formulas $\Gamma \cup \{\varphi\}$ of the appropriate type,

$$\Gamma \vdash_{\mathbf{CPC}} \varphi \Longleftrightarrow \boldsymbol{\tau}[\Gamma] \vDash_{\mathsf{BA}} \boldsymbol{\tau}(\varphi)$$
$$\Gamma \vdash_{\mathbf{IPC}} \varphi \Longleftrightarrow \boldsymbol{\tau}[\Gamma] \vDash_{\mathsf{HA}} \boldsymbol{\tau}(\varphi),$$

where $\boldsymbol{\tau} = \{x \approx 1\}$. ⊠

The relation between logic and algebra is often explained in terms of the existence of equational completeness theorems. The following definition makes this concept precise. As we will see, however, equational completeness theorems alone are not sufficient to account for the relation between logic and algebra.

Definition 3.12. A logic $\vdash$ is said to admit an *equational completeness theorem* if there are a set of equations $\boldsymbol{\tau}(x)$ and a class K of algebras such that for all $\Gamma \cup \{\varphi\} \subseteq T(Var)$,

$$\Gamma \vdash \varphi \Longleftrightarrow \boldsymbol{\tau}[\Gamma] \vDash_{\mathsf{K}} \boldsymbol{\tau}(\varphi).$$

In this case, $\vdash$ coincides with $\vdash_{\mathsf{K},\boldsymbol{\tau}}$ and K is said to be a $\boldsymbol{\tau}$*-algebraic semantics* (or simply an *algebraic semantics*) for $\vdash$.

This notion was introduced in [7] and studied in depth in [10, 40, 43]. For instance, the classes of Boolean and Heyting algebras are, respectively, $\boldsymbol{\tau}$-algebraic semantics for **CPC** and **IPC** where $\boldsymbol{\tau} = \{x \approx 1\}$.

Another familiar example of equational completeness theorem arises from the field of modal logic. Let Fr be the class of all Kripke frames. We can associate two distinct logics with Fr, see for instance [31, 32]. The *global consequence* $\mathbf{K}_g$ of the modal system $\mathbf{K}$ is the logic defined, for every set of modal formulas $\Gamma \cup \{\varphi\}$, as follows:

$$\begin{aligned}\Gamma \vdash_{\mathbf{K}_g} \varphi \Longleftrightarrow\ & \text{for every } \langle W, R\rangle \in \mathsf{Fr} \text{ and evaluation } v \text{ in } \langle W, R\rangle, \\ & \text{if } w, v \Vdash \Gamma \text{ for all } w \in W, \text{ then } w, v \Vdash \varphi \text{ for all } w \in W.\end{aligned}$$

On the other hand, the *local consequence* $\mathbf{K}_\ell$ of the modal system $\mathbf{K}$ is defined, for every set of modal formulas $\Gamma \cup \{\varphi\}$, as follows:

$$\begin{aligned}\Gamma \vdash_{\mathbf{K}_\ell} \varphi \Longleftrightarrow & \text{ for every } \langle W, R\rangle \in \mathsf{Fr}, w \in W, \text{ and evaluation } v \text{ in } \langle W, R\rangle,\\ & \text{ if } w, v \Vdash \Gamma, \text{ then } w, v \Vdash \varphi.\end{aligned}$$

It is easy to see that $\mathbf{K}_g$ and $\mathbf{K}_\ell$ are logics. Moreover, they are distinct, because

$$x \vdash_{\mathbf{K}_g} \Box x \text{ and } x \nvdash_{\mathbf{K}_\ell} \Box x. \tag{3}$$

Exercise 3.13. Prove that $\mathbf{K}_g$ and $\mathbf{K}_\ell$ are logics. Notice also that the modal system $\mathbf{K}$ is not a logic itself, because it is not a consequence relation. Indeed, there are two ways to turn $\mathbf{K}$ into a logic, namely, $\mathbf{K}_g$ and $\mathbf{K}_\ell$. ⊠

Exercise 3.14. Prove that $\mathbf{K}_g$ and $\mathbf{K}_\ell$ have the same *theorems*, i.e., formulas provable from the empty set. Prove also that the set of their theorems is the modal system $\mathbf{K}$. This indicates that, even in the modal setting, logics should not be identified with their sets of theorems. ⊠

The global consequence $\mathbf{K}_g$ is related to the class MA of modal algebras by the following equational completeness theorem.

Theorem 3.15. *For every set $\Gamma \cup \{\varphi\}$ of modal formulas,*

$$\Gamma \vdash_{\mathbf{K}_g} \varphi \Longleftrightarrow \boldsymbol{\tau}[\Gamma] \vDash_{\mathsf{MA}} \boldsymbol{\tau}(\varphi),$$

where $\boldsymbol{\tau} = \{x \approx 1\}$. Consequently, the class of modal algebras is a $\boldsymbol{\tau}$-algebraic semantics for $\mathbf{K}_g$.

In order to prove it, recall that a filter on a Boolean algebra $\boldsymbol{A}$ is said to be *proper* when it differs from A. Moreover, a proper filter U of $\boldsymbol{A}$ is said to be a *ultrafilter* of $\boldsymbol{A}$ if it is maximal among the proper filters of $\boldsymbol{A}$ or, equivalently, if

$$a \in U \text{ or } \neg a \in U, \text{ for every } a \in A.$$

While the following result holds in ZFC, it cannot be proved in ZF (although it is strictly weaker then the axiom of choice).

Ultrafilter Lemma 3.16. *Every proper filter on a Boolean algebra can be extended to a ultrafilter.*

We are now ready to prove Theorem 3.15.

Proof sketch. It suffices to prove that

$$\Gamma \nvdash_{\mathbf{K}_g} \varphi \Longleftrightarrow \boldsymbol{\tau}[\Gamma] \nvDash_{\mathsf{MA}} \boldsymbol{\tau}(\varphi).$$

Suppose first that $\Gamma \nvdash_{\mathbf{K}_g} \varphi$. Then there are a Kripke frame $\langle W, R\rangle$, an evaluation v in it and a world u such that

$$w, v \Vdash \Gamma \text{ for all } w \in W \text{ and } u, v \nVdash \varphi. \tag{4}$$

Then consider the complex algebra of $\langle W, R\rangle$, that is, the structure

$$\boldsymbol{A} := \langle \mathcal{P}(W); \cap, \cup, -, \Box, \emptyset, W\rangle,$$

where $-$ is set theoretic complement and, for every $V \subseteq W$,

$$\Box V := \{w \in W : \text{if } \langle w, t\rangle \in R, \text{ then } t \in V\}.$$

It is easy to prove that $\boldsymbol{A}$ is a modal algebra. Then consider the unique homomorphism $f\colon \boldsymbol{T}(Var) \to \boldsymbol{A}$ such that

$$f(x) = \{w \in W : w, v \Vdash x\},$$

for every $x \in Var$. A simple induction of the construction of terms shows that, for every formula ψ,

$$f(\psi) = \{w \in W : w, v \Vdash \psi\}.$$

Together with (4), this yields

$$f[\Gamma] \subseteq \{W\} \text{ and } f(\varphi) \neq W.$$

Hence, we conclude that $\boldsymbol{\tau}[\Gamma] \nvDash_{\mathsf{MA}} \boldsymbol{\tau}(\varphi)$.

To prove the converse, suppose that $\boldsymbol{\tau}[\Gamma] \nvDash_{\mathsf{MA}} \boldsymbol{\tau}(\varphi)$. Then there are a modal algebra $\boldsymbol{A}$ and a homomorphism $f\colon \boldsymbol{T}(Var) \to \boldsymbol{A}$ such that

$$f[\Gamma] \subseteq \{1\} \text{ and } f(\varphi) \neq 1.$$

Then consider the Kripke frame dual to $\boldsymbol{A}$, that is, the structure $\langle W, R\rangle$, where W is the set of ultrafilters of $\boldsymbol{A}$ and R the binary relation on W defined as follows:

$$R := \{\langle U, V\rangle \in W \times W : \{a \in A : \Box a \in U\} \subseteq V\}.$$

Let then $v\colon Var \to \mathcal{P}(W)$ be the evaluation in $\langle W, R\rangle$ defined by the rule

$$v(x) := \{U \in W : f(x) \in U\}.$$

An easy induction on the construction of terms shows that, for every formula ψ,

$$\{U \in W : f(\psi) \in U\} = \{U \in W : U, v \Vdash \psi\}. \tag{5}$$

Now, since $f(\varphi) \neq 1$, the Ultrafilter Lemma guarantees the existence of an ultrafilter F such that $f(\varphi) \notin F$. Furthermore, as every ultrafilter contain 1, from $f[\Gamma] \subseteq \{1\}$ it follows that $f[\Gamma] \subseteq U$, for all $U \in W$. In short,

$$f[\Gamma] \subseteq U \text{ for all } U \in W \text{ and } f(\varphi) \notin F.$$

Together with (5), this yields

$$U, v \Vdash \Gamma \text{ for all } U \in W \text{ and } F, v \nVdash \varphi.$$

Hence, we conclude that $\Gamma \nvdash_{\mathbf{K}_g} \varphi$. ⊠

At this stage, it is tempting to conjecture that the relation between logic and algebra can be explained in terms of equational completeness theorems only. As we anticipated, however, this is not the case. For instance, the relation between **CPC** and BA cannot be explained in terms of completeness theorems only, because the class of Heyting algebras HA is also an algebraic semantics for **CPC**. To explain why, it is convenient to recall the following classical result relating **CPC** and **IPC** [27].

Givenko's Theorem 3.17. *For every set of formulas $\Gamma \cup \{\varphi\}$,*

$$\Gamma \vdash_{\mathbf{CPC}} \varphi \iff \{\neg\neg\gamma \colon \gamma \in \Gamma\} \vdash_{\mathbf{IPC}} \neg\neg\varphi.$$

As a consequence, we obtain the desired result.

Corollary 3.18. *The class of Heyting algebras is an algebraic semantics for* **CPC**.

Proof. For every set of formulas $\Gamma \cup \{\varphi\}$, we have

$$\Gamma \vdash_{\mathbf{CPC}} \varphi \iff \{\neg\neg\gamma \colon \gamma \in \Gamma\} \vdash_{\mathbf{IPC}} \neg\neg\varphi$$
$$\iff \{\neg\neg\gamma \approx 1 \colon \gamma \in \Gamma\} \vDash_{\mathsf{HA}} \neg\neg\varphi \approx 1.$$

The first equivalent above is Glivenko's Theorem, while the second is a consequence of the completeness theorem of **IPC** with respect to HA. As a consequence, the class of Heyting algebras is a $\boldsymbol{\tau}$-algebraic semantics for **IPC**, where $\boldsymbol{\tau} = \{\neg\neg x \approx 1\}$. ⊠

This means that the univocal relation between **CPC** and the class of Boolean algebras cannot be explained in terms of the existence of completeness theorems only. As we shall see, this relation arises from a deeper phenomenon, known as *algebraizability* [7].

Exercise 3.19. One may wonder whether the fact that **CPC** has many distinct algebraic semantics cannot be amended by restricting our attention to $\boldsymbol{\tau}$-algebraic semantics where $\boldsymbol{\tau} = \{x \approx 1\}$. This is not the case, as this exercise asks you to check. Let $\boldsymbol{A}$ be the three-element algebra $\langle \{0, 1, a\}; \wedge, \vee, \neg, 0, 1\rangle$ where $\langle A; \wedge, \vee\rangle$ is the lattice with order $0 < a < 1$ and $\neg \colon A \to A$ is the map described by the rule

$$\neg 0 = \neg a = 1 \text{ and } \neg 1 = 0.$$

Clearly, $\boldsymbol{A}$ is not a Boolean algebra (as there is no three-element Boolean algebra). Prove that $\{\boldsymbol{A}\}$ is $\boldsymbol{\tau}$-algebraic semantics for **CPC** where $\boldsymbol{\tau} = \{x \approx 1\}$. Hint: use the fact that the two-element Boolean algebra is a homomorphic image of $\boldsymbol{A}$. ⊠

Indeed the existence of equational completeness theorems between a logic and a class of algebras turns out to be a very weak relation, as shown in [10, 40]. For instance, while many interesting logics lack a natural equational completeness theorem, they still admit a nonstandard one. This is the case of $\mathbf{K}_\ell$, as we proceed to explain.

A logic $\vdash$ is said to be *protoalgebraic* if there exists a set $\Delta(x, y)$ of formulas such that $\emptyset \vdash \Delta(x, x)$ and $x, \Delta(x, y) \vdash y$. Notice that all logics $\vdash$ with a binary connective $\to$ such that $\emptyset \vdash x \to x$ and $x, x \to y \vdash y$ are protoalgebraic, as witnessed by the set $\Delta \coloneqq \{x \to y\}$. Furthermore, a logic $\vdash$ is said to be *nontrivial* if $x \nvdash y$.

Theorem 3.20 (M. [40, Thm. 9.3])**.** *A nontrivial protoalgebraic logic. $\vdash$ has an algebraic semantics if and only it there are two distinct formulas φ and ψ that are logically equivalent in the sense that*

$$\delta(\varphi, \vec{z}) \dashv\vdash \delta(\psi, \vec{z}), \textit{ for all } \delta(x, \vec{z}) \in T(Var).$$

As a consequence, we obtain the following.

Corollary 3.21. *The logic $\mathbf{K}_\ell$ has an algebraic semantics.*

Proof. Clearly, $\mathbf{K}_\ell$ is nontrivial and protoalgebraic. Furthermore, the formulas x and $x \wedge x$ are distinct, but logical equivalent in $\mathbf{K}_\ell$. Therefore, $\mathbf{K}_\ell$ has an algebraic semantics in view of Theorem 3.20. ⊠

On the other hand, $\mathbf{K}_\ell$ lacks any natural equational completeness theorem.

Theorem 3.22 (M. [40, Cor. 9.7])**.** *No class of modal algebras is an algebraic semantics for $\mathbf{K}_\ell$.*

Proof. We begin by proving that, for all $\varphi, \psi \in T(Var)$,

$$\mathsf{MA} \vDash \varphi \approx \psi \iff \varphi \dashv\vdash_{\mathbf{K}_\ell} \psi. \tag{6}$$

To this end, observe that

$$\begin{aligned}\mathsf{MA} \vDash \varphi \approx \psi &\iff \mathsf{MA} \vDash \varphi \leftrightarrow \psi \approx 1\\ &\iff \emptyset \vdash_{\mathbf{K}_g} \varphi \leftrightarrow \psi\\ &\iff \emptyset \vdash_{\mathbf{K}_\ell} \varphi \leftrightarrow \psi\\ &\iff \varphi \dashv\vdash_{\mathbf{K}_\ell} \psi.\end{aligned}$$

The above equivalence are justified as follows. The first is an easy property of Boolean algebras, the second is a consequence of Theorem 3.15, the third holds because $\mathbf{K}_g$ and $\mathbf{K}_\ell$ have the same theorems (see Exercise 3.14) and the last one because $\mathbf{K}_\ell$ has a standard deduction theorem.

Now, suppose, with a view to contradiction, that $\mathbf{K}_\ell$ has a $\boldsymbol{\tau}$-algebraic semantics $\mathsf{K} \subseteq \mathsf{MA}$. This implies that there exists an equation $\varepsilon \approx \delta \in \boldsymbol{\tau}$ such that $\mathsf{MA} \nvDash \varepsilon \approx \delta$. Thus, in view of the above display, we can assume, by symmetry, that $\varepsilon \nvdash_{\mathbf{K}_\ell} \delta$. This means that there are a Kripke frame $\mathbb{X} = \langle X, R \rangle$, an element $w \in X$ and a valuation v in $\mathbb{X}$ such that $w, v \Vdash \varepsilon$ and $w, v \nVdash \delta$.

Let $\mathbb{X}^+ = \langle X^+; R^+ \rangle$ be the Kripke frame obtained by adding a new point w^+ to $\mathbb{X}$ and defining the relation R^+ as follows:

$$\langle p, q \rangle \in R^+ \iff p = w^+ \text{ or } \langle p, q \rangle \in R.$$

Let also v^+ be the unique evaluation in $\mathbb{X}^+$ such that for every $y \in Var$ and $q \in \mathbb{X}^+$:

$$q, v^+ \Vdash y \iff \text{ either } (q \in X \text{ and } q, v \Vdash y) \text{ or } q = w^+.$$

From the definition of $\mathbb{X}^+$ and v^+ it follows that

$$q, v^+ \Vdash \varphi \iff q, v \Vdash \varphi$$

for all $\varphi \in T(Var)$ and $q \in \mathbb{X}$. Consequently, as $w, v \Vdash \varepsilon$ and $w, v \nVdash \delta$,

$$w^+, v^+ \Vdash x \text{ and } w^+, v^+ \nVdash \Box(\varepsilon \to \delta).$$

This implies

$$x \nvdash_{\mathbf{K}_\ell} \Box(\varepsilon \to \delta).$$

On the other hand, clearly $\emptyset \vdash_{\mathbf{K}_\ell} \Box(\delta \to \delta)$. Consequently,

$$x, \Box(\delta \to \delta) \nvdash_{\mathbf{K}_\ell} \Box(\varepsilon \to \delta). \tag{7}$$

Now, observe that, for every $\varphi, \psi \in T(Var)$,

$$\varepsilon(x) \approx \delta(x), \varphi(\Box(\delta \to \delta)) \approx \psi(\Box(\delta \to \delta)) \vDash_{\mathsf{K}} \varphi(\Box(\varepsilon \to \delta)) \approx \psi(\Box(\varepsilon \to \delta)).$$

Since $\varepsilon \approx \delta \in \boldsymbol{\tau}(x)$, this implies

$$\boldsymbol{\tau}(x), \boldsymbol{\tau}(\Box(\delta \to \delta)) \vDash_{\mathsf{K}} \boldsymbol{\tau}(\Box(\varepsilon \to \delta)).$$

Since K is a $\boldsymbol{\tau}$-algebraic semantics for $\mathbf{K}_\ell$, this yields $x, \Box(\delta \to \delta) \vdash_{\mathbf{K}_\ell} \Box(\varepsilon \to \delta)$, a contradiction with (7). ⊠

While, in view of Theorem 3.22, most logics have an algebraic semantics, examples of logics lacking any algebraic semantics are known since [6].

Exercise 3.23. Prove that $\mathbf{K}_\ell$ has a standard deduction theorem, i.e., that for every set of formulas $\Gamma \cup \{\psi, \varphi\}$,

$$\Gamma, \psi \vdash_{\mathbf{K}_\ell} \varphi \iff \Gamma \vdash_{\mathbf{K}_\ell} \psi \to \varphi.$$

Prove that this is not the case for $\mathbf{K}_g$. ⊠

Exercise 3.24. Prove that no class of distributive lattices is an algebraic semantics for the $\langle \wedge, \vee \rangle$-fragment $\mathbf{CPC}_{\wedge\vee}$ of $\mathbf{CPC}$. Hint: use the fact that every equation in a single variable holds in the class of distributive lattices.

Furthermore, prove that $\mathbf{CPC}_{\wedge\vee}$ has a nonstandard algebraic semantics. To this end, consider the three-element algebra $\boldsymbol{A} = \langle \{0^+, 0^-, 1\}; \wedge, \vee \rangle$ whose binary commutative operations are defined by the following tables

$\wedge$	0^-	0^+	1
0^-	0^+	0^+	0^+
0^+		0^-	0^+
1			1

$\vee$	0^-	0^+	1
0^-	0^+	0^+	1
0^+		0^-	1
1			1

and prove that $\{\boldsymbol{A}\}$ is a $\boldsymbol{\tau}$-algebraic semantics for $\boldsymbol{\tau} = \{x \approx x \wedge x\}$. Conclude that $\mathbf{CPC}_{\wedge\vee}$ is another example of logic that admits a nonstandard algebraic semantics, but lacks a standard one. $\boxtimes$

4. Ultraproducts and universal classes

In order to understand the relation between logic and algebra, we need to take a short detour in universal algebra and the theory of quasi-varieties. We begin by reviewing a product-like construction known as *ultraproduct* [2, 14]. First, recall that ultrafilters on powerset Boolean algebras $\mathcal{P}(X)$ are also called *ultrafilters on* X. Then let $\{\boldsymbol{A}_i : i \in I\}$ be a family of similar algebras. The *equalizer* $[\![\vec{a} = \vec{c}]\!]$ of a pair of elements $\vec{a}, \vec{c} \in \prod_{i \in I} A_i$ is the set of indexes on which the sequences $\vec{a}$ and $\vec{c}$ agree, that is,

$$[\![\vec{a} = \vec{c}]\!] := \{i \in I : \vec{a}(i) = \vec{c}(i)\}.$$

Moreover, given an ultrafilter U on the index set I, let θ_U be the binary relation on the Cartesian product $\prod_{i \in I} A_i$ defined as

$$\theta_U := \{\langle \vec{a}, \vec{c} \rangle : [\![\vec{a} = \vec{c}]\!] \in U\}.$$

Proposition 4.1. *If $\{\boldsymbol{A}_i : i \in I\}$ is a family of similar algebras and U an ultrafilter on I, then θ_U is a congruence of $\prod_{i \in I} \boldsymbol{A}_i$.*

Proof. We begin by proving that θ_U is an equivalence relation on $\prod_{i \in I} A_i$. To this end, consider $\vec{a}, \vec{b}, \vec{c} \in \prod_{i \in I} A_i$. We have

$$[\![\vec{a} = \vec{a}]\!] = \{i \in I : \vec{a}(i) = \vec{a}(i)\} = I.$$

Observe that $I \in U$, since U is a nonempty upset of $\mathcal{P}(I)$. Together with the above display, this yields $[\![\vec{a} = \vec{a}]\!] \in U$ and, therefore, $\langle \vec{a}, \vec{a} \rangle \in \theta_U$. It follows that θ_U is reflexive. To prove that it is symmetric, suppose that $\langle \vec{a}, \vec{c} \rangle \in \theta_U$. Then $[\![\vec{a} = \vec{c}]\!] \in U$. Since $[\![\vec{a} = \vec{c}]\!] = [\![\vec{c} = \vec{a}]\!]$, this implies $[\![\vec{c} = \vec{a}]\!] \in U$ and, therefore, $\langle \vec{c}, \vec{a} \rangle \in \theta_U$. Lastly, to prove that θ_U is transitive, suppose that $\langle \vec{a}, \vec{b} \rangle, \langle \vec{b}, \vec{c} \rangle \in \theta_U$, that is, $[\![\vec{a} = \vec{b}]\!], [\![\vec{b} = \vec{c}]\!] \in U$. Since U is closed under binary meets,

$$[\![\vec{a} = \vec{b}]\!] \cap [\![\vec{b} = \vec{c}]\!] \in U$$

Clearly, $[\![\vec{a} = \vec{b}]\!] \cap [\![\vec{b} = \vec{c}]\!] \subseteq [\![\vec{a} = \vec{c}]\!]$. Since U is an upset of $\mathcal{P}(I)$, we obtain that $[\![\vec{a} = \vec{c}]\!] \in U$, whence $\langle \vec{a}, \vec{c} \rangle \in \theta_U$. We conclude that θ_U is an equivalence relation.

To prove that θ_U is a congruence, it only remains to show that it preserves the basic operations. Accordingly, let f be a basic n-ary operation and $\vec{a}_1, \dots, \vec{a}_n, \vec{c}_1, \dots, \vec{c}_n \in \prod_{i \in I} A_i$ such that

$$\langle \vec{a}_1, \vec{c}_1 \rangle, \dots, \langle \vec{a}_n, \vec{c}_n \rangle \in \theta_U.$$

By definition of θ_U, this amounts to $[\![\vec{a}_1 = \vec{c}_1]\!], \dots, [\![\vec{a}_n = \vec{c}_n]\!] \in U$. Since U is a filter, it is closed under finite meets, whence

$$[\![\vec{a}_1 = \vec{c}_1]\!] \cap \dots \cap [\![\vec{a}_n = \vec{c}_n]\!] \in U. \tag{8}$$

We will show that

$$[\![\vec{a}_1 = \vec{c}_1]\!] \cap \dots \cap [\![\vec{a}_n = \vec{c}_n]\!] \subseteq [\![f^{\prod_{i \in I} \boldsymbol{A}_i}(\vec{a}_1, \dots, \vec{a}_n) = f^{\prod_{i \in I} \boldsymbol{A}_i}(\vec{c}_1, \dots, \vec{c}_n)]\!]. \tag{9}$$

To this end, consider $j \in [\![\vec{a}_1 = \vec{c}_1]\!] \cap \dots \cap [\![\vec{a}_n = \vec{c}_n]\!]$. We have

$$\vec{a}_1(j) = \vec{c}_1(j), \dots, \vec{a}_n(j) = \vec{c}_n(j).$$

Consequently,

$$\begin{aligned} f^{\prod_{i \in I} \boldsymbol{A}_i}(\vec{a}_1, \dots, \vec{a})(j) &= f^{\boldsymbol{A}_j}(\vec{a}_1(j), \dots, \vec{a}_n(j)) \\ &= f^{\boldsymbol{A}_j}(\vec{c}_1(j), \dots, \vec{c}_n(j)) \\ &= f^{\prod_{i \in I} \boldsymbol{A}_i}(\vec{c}_1, \dots, \vec{c})(j), \end{aligned}$$

that is, $j \in [\![f^{\prod_{i \in I} \boldsymbol{A}_i}(\vec{a}_1, \dots, \vec{a}_n) = f^{\prod_{i \in I} \boldsymbol{A}_i}(\vec{c}_1, \dots, \vec{c}_n)]\!]$. This establishes (9). Since U is an upset of $\mathcal{P}(I)$, from (8) and (9) it follows

$$[\![f^{\prod_{i \in I} \boldsymbol{A}_i}(\vec{a}_1, \dots, \vec{a}_n) = f^{\prod_{i \in I} \boldsymbol{A}_i}(\vec{c}_1, \dots, \vec{c}_n)]\!] \in U.$$

Hence, we conclude that $\langle f^{\prod_{i \in I} \boldsymbol{A}_i}(\vec{a}_1, \dots, \vec{a}_n), f^{\prod_{i \in I} \boldsymbol{A}_i}(\vec{c}_1, \dots, \vec{c}_n) \rangle \in \theta_U$, as desired. ⊠

In view of the above result, we can make the following definition.

Definition 4.2. An *ultraproduct* of a family of similar algebras $\{\boldsymbol{A}_i : i \in I\}$ is an algebra of the form $\prod_{i \in I} \boldsymbol{A}_i / \theta_U$, for some ultrafilter U on I.

Given a class of similar algebras K, we set

$$\mathbb{P}_{\text{U}}(\mathsf{K}) \coloneqq \{\boldsymbol{A} : \boldsymbol{A} \text{ is an ultraproduct of a family } \{\boldsymbol{B}_i : i \in I\} \subseteq \mathsf{K}\}.$$

Notice that $\mathbb{P}_{\text{U}}(\mathsf{K}) \subseteq \mathbb{HP}(\mathsf{K})$. Furthermore, as usual, when $\mathsf{K} = \{\boldsymbol{A}\}$, we write $\mathbb{P}_{\text{U}}(\boldsymbol{A})$ as a shorthand for $\mathbb{P}_{\text{U}}(\{\boldsymbol{A}\})$.

Exercise 4.3. Prove that if U is not free (that is, it is principal), then $\prod_{i \in I} \boldsymbol{A}_i / \theta_U$ is isomorphic to some $\boldsymbol{A}_i$. Conclude that if I is finite, then $\prod_{i \in I} \boldsymbol{A}_i / \theta_U$ belongs to $\mathbb{I}\{\boldsymbol{A}_i : i \in I\}$. Because of this, interesting ultraproducts arise from free ultrafilters only. ⊠

Exercise 4.4. Prove that K is a finite set of finite algebras, $\mathbb{P}_{\text{U}}(\mathsf{K}) \subseteq \mathbb{I}(\mathsf{K})$. ⊠

The importance of ultraproducts is largely due to the following result [12, Thm. V.2.9].

Łoś' Theorem 4.5. *Let $\{\boldsymbol{A}_i : i \in I\}$ be a family of similar algebras, U an ultrafilter on I and $\phi(x_1, \dots, x_n)$ a first order formula. For every $\vec{a}_1, \dots, \vec{a}_n \in \prod_{i \in I} A_i$,*

$$\prod_{i \in I} \boldsymbol{A}_i / \theta_U \vDash \phi(\vec{a}_1/\theta_U, \dots, \vec{a}_n/\theta_U) \iff \{i \in I : \boldsymbol{A}_i \vDash \phi(\vec{a}_1(i), \dots, \vec{a}_n(i))\} \in U.$$

Corollary 4.6. *Let $\{\boldsymbol{A}_i : i \in I\}$ be a family of similar algebras, U an ultrafilter on I and ϕ a sentence. If ϕ is valid in all the $\boldsymbol{A}_i$, then it is valid in $\prod_{i \in I} \boldsymbol{A}_i / \theta_U$.*

In view Łos' Theorem, ultraproducts are instrumental to construct nonstandard models of first order theories. For instance, let $\mathbb{N} = \langle \mathbb{N}; s, +, \cdot, 0\rangle$ be the standard model of Peano Arithmetic. If U is an ultrafilter on $\mathbb{N}$, the ultraproduct $\prod_{n\in\mathbb{N}} \mathbb{N}_n / U$ is *elementarily equivalent* to $\mathbb{N}$, that is, it satisfies the same sentences as $\mathbb{N}$. On the other hand, it is not hard to see that if U is free, $\prod_{n\in\mathbb{N}} \mathbb{N}_n / U$ is uncountable and, therefore, contains many "infinite" (or nonstandard) natural numbers.

For the present purpose, however, we will not need the full strength of Łoś' Theorem and, therefore, we shall omit its proof. Instead, we shall focus on a particular embedding theorem for ultraproducts that depends on the following notion.

Definition 4.7. A *local subgraph* $\mathbb{X}$ of an algebra $\boldsymbol{A}$ is a finite subset $X \subseteq A$ endowed with the restriction of finitely many basic operations of $\boldsymbol{A}$ to X.

In this case, $\mathbb{X}$ is a finite *partial* algebra of finite type (even when the type of $\boldsymbol{A}$ is infinite).

Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be similar algebras and $\mathbb{X}$ a local subgraph of $\boldsymbol{A}$. A map $f \colon X \to B$ is said to be an *embedding* of $\mathbb{X}$ into $\boldsymbol{B}$ if it is injective and, for every basic n-ary operation g of the type of $\mathbb{X}$ and $a_1, \dots, a_n \in X$ such that $g^{\boldsymbol{A}}(a_1, \dots, a_n) \in X$,

$$f(g^{\boldsymbol{A}}(a_1, \dots, a_n)) = g^{\boldsymbol{B}}(f(a_1), \dots, f(a_n)).$$

Theorem 4.8. *Let $\mathsf{K} \cup \{\boldsymbol{A}\}$ be a class of similar algebras. If every local subgraph of $\boldsymbol{A}$ can be embedded into some member of K, then $\boldsymbol{A} \in \mathbb{ISP}_{\mathrm{U}}(\mathsf{K})$.*

Proof. Let I be the set of local subgraphs of $\boldsymbol{A}$. By assumption, for every $\mathbb{X} \in I$ there are an algebra $\boldsymbol{B}_{\mathbb{X}} \in \mathsf{K}$ and an embedding $h_{\mathbb{X}} \colon \mathbb{X} \to \boldsymbol{B}_{\mathbb{X}}$. We define a partial order $\sqsubseteq$ on I as follows:

$$\mathbb{X} \sqsubseteq \mathbb{Y} \Longleftrightarrow X \subseteq Y \text{ and the type of } \mathbb{Y} \text{ extends that of } \mathbb{X}.$$

Then, for every $\mathbb{X} \in I$, define

$$J_{\mathbb{X}} := \{\mathbb{Y} \in I \colon \mathbb{X} \sqsubseteq \mathbb{Y}\}.$$

Moreover, let $\mathcal{F}$ be the filter of $\mathcal{P}(I)$ generated by $\{J_{\mathbb{X}} \colon \mathbb{X} \in I\}$. Recall that

$$\mathcal{F} = \{Y \subseteq I : J_{\mathbb{X}_1} \cap \dots \cap J_{\mathbb{X}_n} \subseteq Y, \text{ for some } \mathbb{X}_1, \dots, \mathbb{X}_n \in I\}.$$

We will prove that $\mathcal{F}$ is proper. To this end, consider $\mathbb{X}_1, \dots, \mathbb{X}_n \in I$. Then let $\mathbb{Y}$ be the local subgraph of $\boldsymbol{A}$ with universe $Y := X_1 \cup \dots \cup X_n$ and whose type in the union of the types of the various $\mathbb{X}_i$. Then

$$\mathbb{X}_i \sqsubseteq \mathbb{Y}, \text{ for every } i \leqslant n,$$

that is, $\mathbb{Y} \in J_{\mathbb{X}_1} \cap \dots \cap J_{\mathbb{X}_n}$. It follows that $\emptyset \notin \mathcal{F}$ and, therefore, that $\mathcal{F}$ is proper. As $\mathcal{F}$ is a proper filter, by the Ultrafilter Lemma, it can be extended to an ultrafilter U on I.

Now, consider a map

$$f \colon A \to \prod_{\mathbb{X}\in I} B_{\mathbb{X}}$$

such that $f(a)(\mathbb{X}) = h_{\mathbb{X}}(a)$, for every $a \in A$ and $\mathbb{X} \in I$ such that $a \in \mathbb{X}$. Moreover, let

$$f^* \colon A \to \prod_{\mathbb{X}\in I} B_{\mathbb{X}} / \theta_U$$

be the map defined by the rule

$$f^*(a) := f(a)/\theta_U.$$

We will show f^* is an embedding of $\boldsymbol{A}$ into $\prod_{\mathbb{X}\in I} \boldsymbol{B}_{\mathbb{X}} / \theta_U$.

In order to prove that f^* is injective, consider a pair of distinct elements $a, c \in A$. Consider a local subgraph $\mathbb{Y}$ of $\boldsymbol{A}$ containing a and c. We will show that

$$J_{\mathbb{Y}} \subseteq \{\mathbb{X} \in I : f(a)(\mathbb{X}) \neq f(c)(\mathbb{X})\} \tag{10}$$

Consider $\mathbb{X} \in J_{\mathbb{Y}}$. Then $\mathbb{Y} \sqsubseteq \mathbb{X}$ and, therefore, $a, c \in Y \subseteq X$. Since $a, c \in X$, we have

$$f(a)(\mathbb{X}) = h_{\mathbb{X}}(a) \text{ and } f(c)(\mathbb{X}) = h_{\mathbb{X}}(c).$$

Furthermore, $h_{\mathbb{X}}(a) \neq h_{\mathbb{X}}(c)$, because $h_{\mathbb{X}}$ is injective and $a \neq c$. This yields $f(a)(\mathbb{X}) \neq f(c)(\mathbb{X})$, establishing (10).

Recall that the definition of U guarantees that $J_{\mathbb{Y}} \in \mathcal{F} \subseteq U$. Therefore, since U is an upset of $\mathcal{P}(I)$, we can apply (10) obtaining

$$I \smallsetminus [\![f(a) = f(c)]\!] = \{\mathbb{X} \in I : f(a)(\mathbb{X}) \neq f(c)(\mathbb{X})\} \in U.$$

Since U is a proper filter, this implies

$$[\![f(a) = f(c)]\!] \notin U$$

and, therefore,

$$f^*(a) = f(a)/\theta_U \neq f(c)/\theta_U = f^*(c).$$

Hence, we conclude that f^* is injective.

To prove that it is a homomorphism, consider a basic n-ary operation g and $a_1, \dots, a_n \in A$. Then consider a local subgraph $\mathbb{Y}$ of $\boldsymbol{A}$ whose universe contains $a_1, \dots, a_n, g^{\boldsymbol{A}}(a_1, \dots, a_n)$ and whose type contains g. We will prove that

$$J_{\mathbb{Y}} \subseteq [\![f(g^{\boldsymbol{A}}(a_1, \dots, a_n)) = g^{\prod_{\mathbb{X} \in I} \boldsymbol{B}_{\mathbb{X}}}(f(a_1), \dots, f(a_n))]\!]. \tag{11}$$

Consider $\mathbb{V} \in J_{\mathbb{Y}}$. Since $\mathbb{Y} \sqsubseteq \mathbb{V}$, the type of $\mathbb{V}$ contains g and $a_1, \dots, a_n, g^{\boldsymbol{A}}(a_1, \dots, a_n) \in V$. Since $a_1, \dots, a_n, g^{\boldsymbol{A}}(a_1, \dots, a_n) \in V$, we have

$$\begin{aligned} f(a_1)(\mathbb{V}) &= h_{\mathbb{V}}(a_1) \\ &\vdots \\ f(a_n)(\mathbb{V}) &= h_{\mathbb{V}}(a_n) \\ f(g^{\boldsymbol{A}}(a_1, \dots, a_n))(\mathbb{V}) &= h_{\mathbb{V}}(g^{\boldsymbol{A}}(a_1, \dots, a_n)). \end{aligned}$$

Furthermore, as the type of $\mathbb{V}$ contains g,

$$h_{\mathbb{V}}(g^{\boldsymbol{A}}(a_1, \dots, a_n)) = g^{\boldsymbol{B}_{\mathbb{V}}}(h_{\mathbb{V}}(a_1), \dots, h_{\mathbb{V}}(a_n)).$$

From the above displays it follows

$$f(g^{\boldsymbol{A}}(a_1, \dots, a_n))(\mathbb{V}) = g^{\boldsymbol{B}_{\mathbb{V}}}(f(a_1)(\mathbb{V}), \dots, f(a_n)(\mathbb{V})) = g^{\prod_{\mathbb{X} \in I} \boldsymbol{B}_{\mathbb{X}}}(f(a_1), \dots, f(a_n))(\mathbb{V}),$$

that is, $\mathbb{V} \in [\![f(g^{\boldsymbol{A}}(a_1, \dots, a_n)) = g^{\prod_{\mathbb{X} \in I} \boldsymbol{B}_{\mathbb{X}}}(f(a_1), \dots, f(a_n))]\!]$. This establishes (11). Lastly, as $J_{\mathbb{Y}} \in U$ and U is an upset of $\mathcal{P}(I)$, condition (11) implies

$$[\![f(g^{\boldsymbol{A}}(a_1, \dots, a_n)) = g^{\prod_{\mathbb{X} \in I} \boldsymbol{B}_{\mathbb{X}}}(f(a_1), \dots, f(a_n))]\!] \in U,$$

and, therefore,

$$\begin{aligned} f^*(g^{\boldsymbol{A}}(a_1, \dots, a_n)) &= f(g^{\boldsymbol{A}}(a_1, \dots, a_n))/\theta_U \\ &= g^{\prod_{\mathbb{X}\in I} \boldsymbol{B}_{\mathbb{X}}}(f(a_1), \dots, f(a_n))/\theta_U \\ &= g^{\prod_{\mathbb{X}\in I} \boldsymbol{B}_{\mathbb{X}}/\theta_U}(f(a_1)/\theta_U, \dots, f(a_n)/\theta_U) \\ &= g^{\prod_{\mathbb{X}\in I} \boldsymbol{B}_{\mathbb{X}}/\theta_U}(f^*(a_1), \dots, f^*(a_n)). \end{aligned}$$

Hence, we conclude that f^* is a homomorphism and, therefore, an embedding of $\boldsymbol{A}$ into $\prod_{\mathbb{Y}\in I} \boldsymbol{B}_{\mathbb{Y}}/\theta_U$. As a consequence,

$$\boldsymbol{A} \in \mathbb{ISP}_{\mathrm{U}}(\{\boldsymbol{B}_{\mathbb{X}} : \mathbb{X} \in I\}) \subseteq \mathbb{ISP}_{\mathrm{U}}(\mathsf{K}). \qquad \boxtimes$$

Corollary 4.9. *Every algebra embeds into an ultraproduct of its finitely generated subalgebras.*

Exercise 4.10. Consider the consequence relation $\vdash$ on the set of sentences of a given algebraic language defined as follows:

$$\begin{aligned} \Gamma \vdash \varphi \Longleftrightarrow\ & \text{for every algebra } \boldsymbol{A} \text{ of the appropriate type,} \\ & \text{if } \boldsymbol{A} \vDash \Gamma, \text{ then } \boldsymbol{A} \vDash \varphi. \end{aligned}$$

Notice that, in view of the Completeness Theorem, $\vdash$ is classical first order logic for algebraic languages. This exercise asks you to prove the Compactness Theorem for $\vdash$ using the ultraproduct construction (the same proof works for relational languages too). To this end, consider a set of sentences $\Gamma \cup \{\varphi\}$ such that $\Delta \nvdash \varphi$, for every finite $\Delta \subseteq \Gamma$. Then we can associate with every finite $\Delta \subseteq \Gamma$ an algebra $\boldsymbol{A}_\Delta$ such that

$$\boldsymbol{A}_\Delta \vDash \Delta \ \text{ and } \ \boldsymbol{A} \nvDash \varphi.$$

Prove that there exists an ultraproduct $\boldsymbol{B}$ of the family $\{\boldsymbol{A}_\Delta : \Delta \text{ is a finite subset of } \Gamma\}$ such that $\boldsymbol{B} \vDash \Gamma$ and $\boldsymbol{B} \nvDash \varphi$. Then conclude that $\Gamma \nvdash \varphi$, as desired. Hint: use Łoś' Theorem. $\boxtimes$

Theorem 4.8 is related to the following first order formulas.

Definition 4.11. A sentence is said to be *universal* if it is of the form $\forall x_1, \dots, x_n \varphi$ for some quantifier free formula φ. Accordingly, a class of similar algebras is said to be *universal* if it can be axiomatized by a set of universal sentences.

The following concept is instrumental for describing universal classes.

Definition 4.12. Let $\mathbb{X}$ be a local subgraph of an algebra $\boldsymbol{A}$ and assume that the universe and the type of $\mathbb{X}$ are, respectively, $\{a_1, \dots, a_n\}$ and $f_1, \dots, f_t$.

(i) The *positive atomic diagram* of $\mathbb{X}$ is the set of equations

$$\mathcal{D}^+(\mathbb{X}) := \{f_i(x_{m_1}, \dots, x_{m_k}) \approx x_j : m_1, \dots m_k, j \leqslant n \text{ and } i \leqslant t \text{ and } f_i^{\boldsymbol{A}}(a_{m_1}, \dots, a_{m_k}) = a_j\}.$$

(ii) The *negative atomic diagram* of $\mathbb{X}$ is the set of negated equations

$$\mathcal{D}^-(\mathbb{X}) := \{x_m \not\approx x_k : m, k \leqslant n \text{ and } a_m \neq a_k\}.$$

Theorem 4.13 (Łoś)**.** *A class of similar algebras is universal if and only if it is closed under $\mathbb{I}, \mathbb{S}$ and $\mathbb{P}_{\mathrm{U}}$.*

Proof. The implication from left to right follows from the easy observation that the validity of universal sentences is preserved by $\mathbb{I}, \mathbb{S}$ and $\mathbb{P}_{\mathrm{U}}$. To prove the converse, consider a class K of similar algebras closed under $\mathbb{I}, \mathbb{S}$ and $\mathbb{P}_{\mathrm{U}}$. Moreover, let $\mathrm{Th}_{\forall}(\mathsf{K})$ be the set of universal sentences valid in K and let $\mathsf{K}_{\forall}$ be the class of algebras satisfying $\mathrm{Th}_{\forall}(\mathsf{K})$. In order to conclude the proof, it suffices to show that $\mathsf{K} = \mathsf{K}_{\forall}$.

The inclusion $\mathsf{K} \subseteq \mathsf{K}_{\forall}$ follows immediately from the fact that $\mathsf{K} \vDash \mathrm{Th}_{\forall}(\mathsf{K})$. To prove the other inclusion, consider $\boldsymbol{A} \in \mathsf{K}_{\forall}$. We will show that every local subgraph of $\boldsymbol{A}$ can be embedded into an element of K. To this end, consider a local subgraph $\mathbb{X}$ of $\boldsymbol{A}$ with universe $\{a_1, \dots, a_n\}$ and take the sentence

$$\Phi := \exists x_1, \dots, x_n \Big(\bigwith \mathcal{D}^{+}(\mathbb{X}) \,\&\, \bigwith \mathcal{D}^{-}(\mathbb{X}) \Big).$$

Now, suppose, with a view to contradiction, that $\mathsf{K} \vDash \neg\Phi$. Since $\neg\Phi$ is equivalent to a universal sentence and $\boldsymbol{A} \vDash \mathrm{Th}_{\forall}(\mathsf{K})$, we obtain $\boldsymbol{A} \vDash \neg\Phi$. But this is false, as witnessed by the assignment $x_1 \longmapsto a_1, \dots, x_n \longmapsto a_n$ in $\boldsymbol{A}$. Thus, we conclude that $\mathsf{K} \nvDash \neg\Phi$. Consequently, there exists an element $\boldsymbol{B} \in \mathsf{K}$ such that $\boldsymbol{B} \vDash \Phi$. Let then $b_1, \dots, b_n$ be the elements that witness the validity of the existential part of Φ in $\boldsymbol{B}$. The map $f \colon X \to B$ defined by the rule $a_i \longmapsto b_i$ is an embedding of $\mathbb{X}$ into $\boldsymbol{B}$, as desired.

Since every local subgraph of $\boldsymbol{A}$ can be embedded into a member of K, we can apply Theorem 4.8 obtaining that $\boldsymbol{A} \in \mathbb{ISP}_{\mathrm{U}}(\mathsf{K})$. As, by assumption, K is closed under $\mathbb{I}, \mathbb{S}$ and $\mathbb{P}_{\mathrm{U}}$, this yields $\boldsymbol{A} \in \mathsf{K}$. Hence, we conclude that $\mathsf{K}_{\forall} \subseteq \mathsf{K}$. ⊠

Given a class of similar algebras K, the least universal class extending K exists and will be denoted by $\mathbb{U}(\mathsf{K})$ and called the universal class *generated* by K.

Corollary 4.14. *If K is a class of similar algebras, $\mathbb{U}(\mathsf{K}) = \mathbb{ISP}_{\mathrm{U}}(\mathsf{K})$.*

Proof. From Theorem 4.13 and the fact that $\mathbb{U}(\mathsf{K})$ is a universal class it follows that it is closed under $\mathbb{I}, \mathbb{S}$ and $\mathbb{P}_{\mathrm{U}}$, whence $\mathbb{ISP}_{\mathrm{U}}(\mathsf{K}) \subseteq \mathbb{U}(\mathsf{K})$. To prove the reverse inclusion, observe that $\mathbb{U}(\mathsf{K})$ is the class of all algebras satisfying all the universal sentences valid in K. In the proof of Theorem 4.13, we showed that this guarantees the inclusion $\mathbb{U}(\mathsf{K}) \subseteq \mathbb{ISP}_{\mathrm{U}}(\mathsf{K})$. ⊠

Corollary 4.15. *Let $\mathsf{K} \cup \{\boldsymbol{A}\}$ be a class of similar algebras. If $\boldsymbol{A} \in \mathbb{U}(\mathsf{K})$, then every local subgraph of $\boldsymbol{A}$ embeds into some member of K.*

Proof. This is established in the second part of the proof of Theorem 4.13. ⊠

The following provides an algebraic path to the strong finite model property in logic.

Definition 4.16. A class of similar algebras K is said to have the *finite embeddability property* (FEP, for short) if every local subgraph of a member of K can be embedded into a finite member of K.

Given a class of algebras K, we denote by $\mathsf{K}^{<\omega}$ the class of its finite members.

Proposition 4.17. *A universal class K has the FEP if and only if $\mathsf{K} = \mathbb{U}(\mathsf{K}^{<\omega})$.*

Proof. Suppose first that K has the FEP. Then

$$\mathsf{K} \subseteq \mathbb{ISP}_{\mathrm{U}}(\mathsf{K}^{<\omega}) = \mathbb{U}(\mathsf{K}^{<\omega}).$$

The first inclusion in the above display follows from Theorem 4.8 and the second from Corollary 4.14. Furthermore, since K is a universal class, $\mathbb{U}(\mathsf{K}^{<\omega}) \subseteq \mathsf{K}$. Therefore, we conclude that $\mathsf{K} = \mathbb{U}(\mathsf{K}^{<\omega})$. To prove the converse, suppose that $\mathsf{K} = \mathbb{U}(\mathsf{K}^{<\omega})$. By Corollary

4.15, every local subgraph of a member of K can be embedded into some element of $\mathsf{K}^{<\omega}$, that is, K has the FEP. ⊠

The *universal theory* $\mathrm{Th}_\forall(\mathsf{K})$ of a class of algebras K is the set of universal sentences valid in K. $\mathrm{Th}_\forall(\mathsf{K})$ is said to be *decidable* when so is the problem of determining whether a universal sentence is valid in K. Furthermore, we say that a class of algebras is *finitely axiomatizable* if it can be axiomatized by finitely many sentences. The following result can be traced back at least to [35].

Proposition 4.18. *Let K be a finitely axiomatizable class of algebras. If K has the FEP, then $\mathrm{Th}_\forall(\mathsf{K})$ is decidable.*

Proof. In order to prove that $\mathrm{Th}_\forall(\mathsf{K})$ is decidable it suffices to show that

(i) the problem of determining whether a universal sentence belongs to $\mathrm{Th}_\forall(\mathsf{K})$ is semidecidable; and
(ii) the problem of determining whether a universal sentence does not belong to $\mathrm{Th}_\forall(\mathsf{K})$ is semidecidable.

Condition (i) holds, because K is finitely axiomatizable. Therefore, it only remains to prove (ii). To this end, let Σ be a finite set of axioms for K and $\{f_1, \dots, f_n\}$ the function symbols that appear in Σ. Given a universal sentence Φ, we enumerate the finite models $\boldsymbol{A}_1, \boldsymbol{A}_2 \dots$ of Σ in the type $\{f_1, \dots, f_n, g_1, \dots, g_m\}$, where $g_1, \dots, g_m$ are the function symbols that occur in Φ. This can be done mechanically, because Σ is finite. Our algorithm tests if Φ fails in some $\boldsymbol{A}_n$. If this is the case, it stops and answers that Φ does not belong to $\mathrm{Th}_\forall(\mathsf{K})$, otherwise it runs forever.

In order to establish (ii), it suffices to prove that the algorithm stops if and only if $\Phi \notin \mathrm{Th}_\forall(\mathsf{K})$. First, it if stops, then Φ fails in some $\boldsymbol{A}_n$. Let then $\boldsymbol{B}$ be an algebra of the type of K obtained by expanding $\boldsymbol{A}_n$ with an arbitrary interpretation of the missing function symbols. From $\boldsymbol{A}_n \nvDash \Phi$ it follows $\boldsymbol{B} \nvDash \Phi$. Moreover, since $\boldsymbol{A}_n \vDash \Sigma$, we obtain $\boldsymbol{B} \vDash \Sigma$ and, therefore, $\boldsymbol{B} \in \mathsf{K}$ (as Σ axiomatizes K). Hence, we conclude that $\Phi \notin \mathrm{Th}_\forall(\mathsf{K})$.

To prove the converse, consider a universal sentence $\forall \vec{x} \varphi$ that fails in K. Then there exist $\boldsymbol{B} \in \mathsf{K}$ and $b_1, \dots, b_n \in B$ such that $\boldsymbol{B} \nvDash \varphi(b_1, \dots, b_n)$. Let $\mathbb{X}$ be the local subgraph of $\boldsymbol{B}$ whose universe is $\{b_1, \dots, b_n\}$ and whose type consists of the function symbols occurring in φ. Since K has the FEP, there is an embedding $f \colon \mathbb{X} \to \boldsymbol{C}$, for some $\boldsymbol{C} \in \mathsf{K}^{<\omega}$. It follows that $\boldsymbol{C} \nvDash \varphi(f(a_1), \dots, f(a_n))$, whence $\forall \vec{x} \varphi$. Let $\boldsymbol{C}^-$ be the reduct of $\boldsymbol{C}$ is the language $\mathscr{L}$ consisting of the function symbols occurring in $\Sigma \cup \{\varphi\}$. Clearly, $\boldsymbol{C}^- \nvDash \varphi(f(a_1), \dots, f(a_n))$ and, therefore, $\forall \vec{x} \varphi$ fails in $\boldsymbol{C}^-$. As $\boldsymbol{C}^-$ is a finite model of Σ in the language $\mathscr{L}$, there must be some $n \in \mathbb{N}$ such that $\boldsymbol{C}^- \cong \boldsymbol{A}_n$. It follows that $\forall \vec{x} \varphi$ fails in $\boldsymbol{A}_n$, whence the algorithm stops, as desired. ⊠

Example 4.19 (Lattices)**.** We will prove that the class Latt of all lattices has the FEP. For consider a lattice $\boldsymbol{A}$ and let $\mathbb{X}$ be one of its local subgraphs. Let also

$$B := \{a_1 \wedge^{\boldsymbol{A}} \cdots \wedge^{\boldsymbol{A}} a_n : a_1, \dots, a_n \in X \text{ and } n \in \mathbb{N}\}.$$

Since the operation $\wedge^{\boldsymbol{A}}$ is idempotent, commutative and associative, the set B is finite. furthermore, B can be viewed as a subposet of $\boldsymbol{A}$. Let $\boldsymbol{B}^+$ be the poset obtained extending $\langle B; \leqslant \rangle$ with a new top element. Since $\boldsymbol{B}^+$ is a finite meet-semilattice with maximum, it is also a lattice. Furthermore, $\mathbb{X}$ embeds into $\boldsymbol{B}^+$. Hence, Latt has the FEP. By Propositions 4.18 and 4.17, $\mathrm{Th}_\forall(\mathsf{Latt})$ is decidable [48] and $\mathsf{Latt} = \mathbb{U}(\mathsf{Latt}^{<\omega})$. On the other hand, the first order theory of distributive lattices (and, therefore, of any nontrivial equational class of lattices) is undecidable [29]. ⊠

Example 4.20 (Heyting algebras)**.** We will prove that the class HA of Heyting algebras has the FEP. For consider a Heyting algebra $\boldsymbol{A}$ and let $\mathbb{X}$ be one of its local subgraphs. Then let $\boldsymbol{B}$ be the bounded sublattice of $\boldsymbol{A}$ generated by $\mathbb{X}$. Notice that $\boldsymbol{B}$ is finite, because it is a finitely generated bounded distributive lattice. Since $\boldsymbol{B}$ is a finite distributive lattice, it can be viewed as a finite Heyting algebra $\boldsymbol{B}^{+}$. Furthermore, it is easy to see that the identity map is an embedding of $\mathbb{X}$ into $\boldsymbol{B}^{+}$. Hence, we conclude that HA has the FEP. By Propositions 4.18 and 4.17, $\mathrm{Th}_{\forall}(\mathsf{HA})$ is decidable and $\mathsf{HA} = \mathbb{U}(\mathsf{HA}^{<\omega})$ [36]. On the other hand, the first order theory of nontrivial equational class of Heyting algebras other than that of Boolean algebras is known to be undecidable [11].

Notably, the fact that $\mathrm{Th}_{\forall}(\mathsf{HA})$ is decidable implies that **IPC** is also decidable, in the sense that, given a finite set of formulas $\{\gamma_1, \dots, \gamma_n, \varphi\}$, we can decide whether $\gamma_1, \dots, \gamma_n \vdash_{\mathbf{IPC}} \varphi$. This is because, in view of the fact that HA is a $\{x \approx 1\}$-algebraic semantics for **IPC**, we have

$$\begin{aligned}\gamma_1, \dots, \gamma_n \vdash_{\mathbf{IPC}} \varphi &\Longleftrightarrow \gamma_1 \approx 1, \dots, \gamma_n \approx 1 \vDash_{\mathsf{HA}} \varphi \approx 1 \\ &\Longleftrightarrow \mathsf{HA} \vDash \forall \vec{x}\, ((\gamma_1 \approx 1 \,\&\, \dots \,\&\, \gamma_n \approx 1) \Longrightarrow \varphi \approx 1).\end{aligned}$$

As $\forall \vec{x}\, ((\gamma_1 \approx 1 \,\&\, \dots \,\&\, \gamma_n \approx 1) \Longrightarrow \varphi \approx 1)$ is a universal sentence, we can decide whether it holds in HA or not, because the universal theory of HA is decidable. ⊠

Exercise 4.21. The following proof the class MA of modal algebras has the FEP originates in [35]. Let $\mathbb{X}$ be a local subgraph of a model algebra $\boldsymbol{A}$. Moreover, let $\boldsymbol{B}$ be the Boolean subalgebra of $\boldsymbol{A}$ generated by $X \cup \{\Box 0\}$. We consider the algebra algebra $\boldsymbol{B}^{+}$ obtained by endowing $\boldsymbol{B}$ with a unary operation $\Box$ defined as follows:

$$\Box^{\boldsymbol{B}^{+}} b := \bigvee^{\boldsymbol{B}} \{a \in B : \Box^{\boldsymbol{A}} a \in B \text{ and } a \leqslant b\}.$$

Prove that $\boldsymbol{B}^{+}$ is a modal algebra and that $\mathbb{X}$ embeds into $\boldsymbol{B}^{+}$. Then conclude that $\mathrm{Th}_{\forall}(\mathsf{MA})$ is decidable and $\mathsf{MA} = \mathbb{U}(\mathsf{MA}^{<\omega})$. Use these facts to infer that the logic $\mathbf{K}_g$ is decidable. ⊠

5. Quasi-varieties

For a detailed presentation of the theory of quasi-varieties, we refer the reader to [28, 33].

Definition 5.1. A class of similar algebras closed under $\mathbb{I}, \mathbb{S}, \mathbb{P}$ and $\mathbb{P}_{\mathrm{U}}$ is said to be a *quasi-variety*.

Examples of quasi-varieties include the classes of Boolean, Heyting and modal algebras, as well as the class of (bounded) distributive lattices and groups. Our aim will be to prove that quasi-varieties are precisely the classes of algebras axiomatized by the following kind of first order formulas.

Definition 5.2. A *quasi-equation* of type ρ is an expression Φ of the form

$$(\varphi_1 \approx \psi_1 \,\&\, \dots \,\&\, \varphi_n \approx \psi_n) \Longrightarrow \varepsilon \approx \delta,$$

where $\{\varphi_1 \approx \psi_1, \dots, \varphi_n \approx \psi_n, \varepsilon \approx \delta\}$ is a set of equations of type ρ. Then Φ is *valid* in an algebra $\boldsymbol{A}$ of type ρ when so is its universal closure $\forall \vec{x}\, \Phi$, that is, for every $\vec{a} \in A$,

$$\text{if } \varphi_1^{\boldsymbol{A}}(\vec{a}) = \psi_1^{\boldsymbol{A}}(\vec{a}), \dots, \varphi_n^{\boldsymbol{A}}(\vec{a}) = \psi_n^{\boldsymbol{A}}(\vec{a}), \text{ then } \varepsilon^{\boldsymbol{A}}(\vec{a}) = \delta^{\boldsymbol{A}}(\vec{a}).$$

In this case, we often say that $\boldsymbol{A}$ *satisfies* Φ and write $\boldsymbol{A} \vDash \Phi$. A quasi-equation is said to be an *equation* when its antecedent is empty.

Notably, for every class K of algebras and equations $\varphi_1 \approx \psi_1, \dots, \varphi_n \approx \psi_n, \varepsilon \approx \delta$,

$$\mathsf{K} \vDash (\mathop{\&}_{i \leqslant n} \varphi_i \approx \psi_i) \Longrightarrow \varepsilon \approx \delta \text{ iff } \{\varphi_1 \approx \psi_1, \dots, \varphi_n \approx \psi_n\} \vDash_{\mathsf{K}} \varepsilon \approx \delta.$$

Remark 5.3. The reader might have noticed that expressions of the form $\varepsilon \approx \delta$ and $\emptyset \Longrightarrow \varepsilon \approx \delta$ are both called *equations*. This is not a problem, because they are synonyms, in the sense that an algebra satisfies $\varepsilon \approx \delta$ if and only if it satisfies $\emptyset \Longrightarrow \varepsilon \approx \delta$. Because of this, we will continue to denote equations by $\varepsilon \approx \delta$, while keeping in mind that they can be viewed as quasi-equations whose antecedent is empty. ⊠

The aim of this section is to prove the following classical result.

Maltsev's Theorem 5.4. *A class of similar algebras is a quasi-variety if and only if it can be axiomatized by a set of quasi-equations.*

Proof. The "only if" part follows from the fact that the validity of quasi-equations is preserved by the class operators $\mathbb{I}, \mathbb{S}, \mathbb{P}$ and $\mathbb{P}_{\mathrm{U}}$. To prove the converse, consider a quasi-variety K and let Σ the set of quasi-equations valid in K. Let also K^+ be the class of algebras axiomatized by Σ. Our aim is to prove that $\mathsf{K} = \mathsf{K}^+$.

The inclusion $\mathsf{K} \subseteq \mathsf{K}^+$ is straightforward. To prove the other one, consider an algebra $\boldsymbol{A} \in \mathsf{K}^+$. In view of Theorem 4.8, in order to show that $\boldsymbol{A} \in \mathsf{K}$, it suffices to prove that every local subgraph of $\boldsymbol{A}$ embeds in some members of K. This is because, in this case, $\boldsymbol{A} \in \mathbb{ISP}_{\mathrm{U}}(\mathsf{K}) \subseteq \mathsf{K}$. Since K is closed under $\mathbb{I}, \mathbb{S}$ and $\mathbb{P}_{\mathrm{U}}$, this implies $\boldsymbol{A} \in \mathsf{K}$, as desired.

Then consider a local subgraph $\mathbb{X}$ of $\boldsymbol{A}$ with universe $\{a_1, \dots, a_n\}$. Observe that both $\mathcal{D}^+(\mathbb{X})$ and $\mathcal{D}^-(\mathbb{X})$ are finite sets. Then take an enumeration

$$\mathcal{D}^-(\mathbb{X}) = \{\varepsilon_1 \not\approx \delta_1, \dots, \varepsilon_t \not\approx \delta_t\}.$$

Moreover, for each $i \leqslant t$, consider the quasi-equation

$$\Phi_i := \Big(\mathop{\&} \mathcal{D}^+(\mathbb{X})\Big) \Longrightarrow \varepsilon_i \approx \delta_i.$$

As witnessed by the natural assignment

$$x_1 \longmapsto a_1, \dots, x_n \longmapsto a_n,$$

the quasi-equations $\Phi_1, \dots, \Phi_t$ fail in $\boldsymbol{A}$. Since $\boldsymbol{A}$ satisfies all the quasi-equations valid in K, this implies that each Φ_i fails in some $\boldsymbol{B}_i \in \mathsf{K}$ under an assignment

$$x_1 \longmapsto b_1^i, \dots, x_n \longmapsto b_n^i. \tag{12}$$

Now, consider the map $h \colon X \to (B_1 \times \cdots \times B_t)$, defined by the rule

$$a_1 \longmapsto \langle b_1^1, \dots, b_1^t \rangle, \dots, a_n \longmapsto \langle b_n^1, \dots, b_n^t \rangle.$$

We will prove that h is an embedding of $\mathbb{X}$ into $\boldsymbol{B}_1 \times \cdots \times \boldsymbol{B}_t$. To prove that h is injective, consider two distinct elements $a_p, a_q \in X$. Then the formula $x_p \not\approx x_q$ belongs to $\mathcal{D}^-(\mathbb{X})$. Consequently, there exists $i \leqslant t$ such that

$$\Phi_i = \Big(\mathop{\&} \mathcal{D}^+(\mathbb{X})\Big) \Longrightarrow x_p \approx x_q.$$

Since Φ_i fails in $\boldsymbol{B}_i$ under the assignment in (12), we obtain $b^i_p \neq b^i_q$. As a consequence,

$$h(a_p)(i) = b^i_p \neq b^i_q = h(a_q)(i)$$

and, therefore, $h(a_p) \neq h(a_q)$. Hence, h is injective. To prove that it preserves the partial operations, consider a basic m-ary operation f in the type of $\mathbb{X}$ and $a_{k_1}, \dots, a_{k_m} \in X$ such that $f^{\boldsymbol{A}}(a_{k_1}, \dots, a_{k_m}) \in X$. Then there exists some $p \leqslant n$ such that $a_p = f^{\boldsymbol{A}}(a_{k_1}, \dots, a_{k_m})$. Moreover, the equation

$$f(x_{k_1}, \dots, x_{k_m}) \approx x_p$$

belongs to $\mathcal{D}^+(\mathbb{X})$. As each quasi-equation Φ_i fails under the assignment in (12), the same assignment satisfies the antecedent of Φ_i, namely $\mathcal{D}^+(\mathbb{X})$. It follows that

$$f^{\boldsymbol{B}_i}(b^i_{k_1}, \dots, b^i_{k_m}) = b^i_p, \text{ for each } i \leqslant t.$$

As a consequence, for every $i \leqslant t$,

$$\begin{aligned}
h(f^{\boldsymbol{A}}(a_{k_1}, \dots, a_{k_m}))(i) &= h(a_p)(i) \\
&= b^i_p \\
&= f^{\boldsymbol{B}_i}(b^i_{k_1}, \dots, b^i_{k_m}) \\
&= f^{\boldsymbol{B}_i}(h(a_{k_1})(i), \dots, h(a_{k_m})(i)) \\
&= f^{\boldsymbol{B}_1 \times \cdots \times \boldsymbol{B}_t}(h(a_{k_1}), \dots, h(a_{k_m}))(i).
\end{aligned}$$

Thus, $h(f^{\boldsymbol{A}}(a_{k_1}, \dots, a_{k_m})) = f^{\boldsymbol{B}_1 \times \cdots \times \boldsymbol{B}_t}(h(a_{k_1}), \dots, h(a_{k_m}))$. We conclude that $h \colon \mathbb{X} \to (\boldsymbol{B}_1 \times \cdots \times \boldsymbol{B}_t)$ is an embedding. Since $\boldsymbol{B}_1, \dots, \boldsymbol{B}_t \in \mathsf{K}$ and K is closed under $\mathbb{P}$, the direct product $\boldsymbol{B}_1 \times \cdots \times \boldsymbol{B}_t$ belongs to K. Hence, $\mathbb{X}$ embeds into some member of K, as desired. ⊠

Exercise 5.5. In view of Łoś' Theorem quasi-equations persist in ultraproducts. If you are not familiar with the proof of Łoś' Theorem, offer a direct proof of this fact. ⊠

Given a class of similar algebras K, the least quasi-variety extending K exists and will be denoted by $\mathbb{Q}(\mathsf{K})$ and called the quasi-variety *generated* by K.

Corollary 5.6. *Let* K *be a class of similar algebras. Then* $\mathbb{Q}(\mathsf{K}) = \mathbb{ISPP}_{\text{U}}(\mathsf{K})$. *If in addition* K *is a finite set of finite algebras,* $\mathbb{Q}(\mathsf{K}) = \mathbb{ISP}(\mathsf{K})$.

Proof. The inclusion $\mathbb{ISPP}_{\text{U}}(\mathsf{K}) \subseteq \mathbb{Q}(\mathsf{K})$ is straightforward. To prove the other one, consider $\boldsymbol{A} \in \mathbb{Q}(\mathsf{K})$. By Maltsev's Theorem, $\mathbb{Q}(\mathsf{K})$ is the class of all algebras satisfying the quasi-equations valid in K. The proof of the hard part of Maltsev's Theorem show that $\boldsymbol{A} \in \mathbb{ISP}_{\text{U}}\mathbb{P}(\mathsf{K})$. Therefore, it only remains to show that $\mathbb{P}_{\text{U}}\mathbb{P}(\mathsf{K}) \subseteq \mathbb{ISPP}_{\text{U}}(\mathsf{K})$, which is left as an exercise. This shows that $\mathbb{Q}(\mathsf{K}) = \mathbb{ISPP}_{\text{U}}(\mathsf{K})$.

To prove the second part of the statement, suppose that K is a finite set of finite algebras. This guarantees that $\mathbb{P}_{\text{U}}(\mathsf{K}) \subseteq \mathbb{I}(\mathsf{K})$ (see Exercise 4.4). As a consequence, $\mathbb{Q}(\mathsf{K}) = \mathbb{ISPP}_{\text{U}}(\mathsf{K}) = \mathbb{ISP}(\mathsf{K})$, as desired. ⊠

Exercise 5.7. Prove that if K is a class of similar algebras, then $\mathbb{P}_{\text{U}}\mathbb{P}(\mathsf{K}) \subseteq \mathbb{ISPP}_{\text{U}}(\mathsf{K})$. ⊠

Example 5.8 (Quasi-varieties)**.** In view of Examples 4.19 and 4.20, we know that

$$\mathsf{Latt} = \mathbb{ISP}_{\text{U}}(\mathsf{Latt}^{<\omega}) \text{ and } \mathsf{HA} = \mathbb{ISP}_{\text{U}}(\mathsf{HA}^{<\omega}).$$

This implies that $\mathsf{Latt} \subseteq \mathbb{Q}(\mathsf{Latt}^{<\omega})$ and $\mathsf{HA} \subseteq \mathbb{Q}(\mathsf{HA}^{<\omega})$. Since both Latt and HA are closed under $\mathbb{I}, \mathbb{S}, \mathbb{P}$ and $\mathbb{P}_{\mathrm{U}}$, this yields

$$\mathsf{Latt} = \mathbb{Q}(\mathsf{Latt}^{<\omega}) \text{ and } \mathsf{HA} = \mathbb{Q}(\mathsf{HA}^{<\omega}).$$

Let DL be the class of distributive lattices and $\boldsymbol{B}$ the two-element distributive lattice. In view of Example 2.18, we have $\mathsf{DL} = \mathbb{IP}_{\mathrm{SD}}(\boldsymbol{B})$. Clearly, $\mathbb{IP}_{\mathrm{SD}}(\boldsymbol{B}) \subseteq \mathbb{ISP}(\boldsymbol{B}) \subseteq \mathbb{Q}(\boldsymbol{B})$. On the other hand, since DL is closed under $\mathbb{I}, \mathbb{S}, \mathbb{P}$ and $\mathbb{P}_{\mathrm{U}}$, we obtain $\mathbb{Q}(\boldsymbol{B}) \subseteq \mathsf{DL}$. Hence, $\mathsf{DL} = \mathbb{Q}(\boldsymbol{B})$. On the other hand, $\mathbb{U}(\boldsymbol{B}) = \mathbb{ISP}_{\mathrm{U}}(\boldsymbol{B}) = \mathbb{I}(\boldsymbol{B})$. Similarly, the class of Boolean algebras is the quasi-variety generated by the two-element Boolean algebra (see 2.19, if necessary). ⊠

Exercise 5.9. Prove that there is no finite Heyting algebra $\boldsymbol{A}$ such that $\mathsf{HA} = \mathbb{Q}(\boldsymbol{A})$, cf. with $\mathsf{HA} = \mathbb{Q}(\mathsf{HA}^{<\omega})$. Prove that, however, there exists an infinite Heyting algebra $\boldsymbol{A}$ such that $\mathsf{HA} = \mathbb{Q}(\boldsymbol{A})$ [30, 50], see also [19, 42]. ⊠

The next observation will be needed later on.

Theorem 5.10. *If K is a quasi-variety, the consequence relation $\vDash_{\mathsf{K}}$ is finitary.*

Proof. In view of Maltsev's Theorem, K can be axiomatized by a set Σ of quasi-equations. Formally speaking, this means that K is the class of models of the set of sentences

$$\Sigma_{\forall} := \{\forall \vec{x} \varphi : \varphi(\vec{x}) \in \Sigma\}.$$

Let then $\Theta \cup \{\varphi \approx \psi\} \subseteq E(Var)$ be such that $\Theta \vDash_{\mathsf{K}} \varphi \approx \psi$. Moreover, let $\{a_n : n \in \mathbb{N}\}$ be a set of new constants. Given a formula $\varphi(x_1, \dots, x_n)$, we write

$$\varphi(\vec{a}) \text{ as a shorthand for } \varphi(a_1, \dots, a_n).$$

Since $\Sigma_{\forall}$ axiomatizes K, the fact that $\Theta \vDash_{\mathsf{K}} \varphi \approx \psi$ is equivalent to the demand that

$$\Sigma_{\forall} \cup \{\varepsilon(\vec{a}) \approx \delta(\vec{a}) : \varepsilon \approx \delta \in \Theta\} \vdash \varphi(\vec{a}) \approx \psi(\vec{a}),$$

where $\vdash$ is the derivability symbol of classical first order logic. Consequently, by the Compactness Theorem, there exists a finite $\Delta \subseteq \Theta$ such that

$$\Sigma_{\forall} \cup \{\varepsilon(\vec{a}) \approx \delta(\vec{a}) : \varepsilon \approx \delta \in \Delta\} \vdash \varphi(\vec{a}) \approx \psi(\vec{a}).$$

Since $\Sigma_{\forall}$ axiomatizes K, this means that $\Delta \vDash_{\mathsf{K}} \varphi \approx \psi$, as desired. ⊠

Observe that, for every class K of similar algebras, $\mathbb{I}(\mathsf{K}) \cup \mathbb{P}_{\mathrm{U}}(\mathsf{K}) \subseteq \mathbb{HP}(\mathsf{K})$. Because of this, the following classes of algebras [4, 12] are also quasi-varieties.

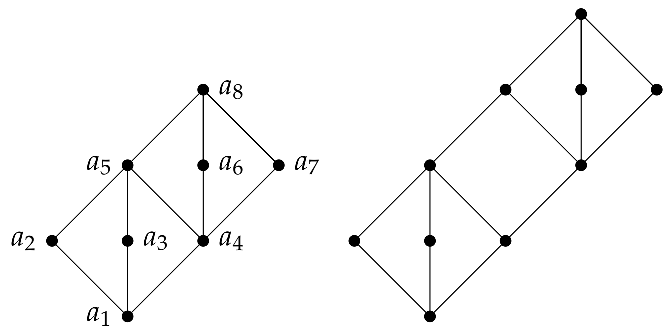
Definition 5.11. A class of similar algebras closed under $\mathbb{H}, \mathbb{S}$ and $\mathbb{P}$ is said to be a *variety*.

Notably, varieties coincide with equational classes of algebras [4, Thm. 4.41].

Birkhoff's Theorem 5.12. *A class of similar algebras is a variety if and only if it can be axiomatized by a set of equations.*

As a consequence, examples of varieties include the classes of (distributive) lattices, Heyting, Boolean and modal algebras. While every variety is a quasi-variety, the converse is not true in general, as we proceed to explain.

Example 5.13 (Lattices)**.** Consider the lattices $\boldsymbol{A}$ and $\boldsymbol{B}$ depicted below. We will show that the quasi-variety $\mathbb{Q}(\boldsymbol{B})$ is not closed under $\mathbb{H}$ and, therefore, is not a variety.



$\boldsymbol{A}$ $\boldsymbol{B}$

To this end, let $\mathcal{D}^+(\boldsymbol{A})$ be the positive atomic diagram of $\boldsymbol{A}$ written with the variables $x_1, \dots, x_8$ corresponding to the elements $a_1, \dots, a_8$ and consider the quasi-equation

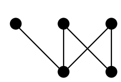
$$\Phi = \mathop{\&} \mathcal{D}^+(\boldsymbol{A}) \Longrightarrow x_1 \approx x_8.$$

Notice that $\boldsymbol{B}$ validates Φ. To prove this, consider an assignment $f \colon \{x_1, \dots, x_8\} \to B$ that validates $\mathcal{D}^+(\boldsymbol{A})$ in $\boldsymbol{B}$. Using the definition of $\mathcal{D}^+(\boldsymbol{A})$, it is easy to see that the map $h \colon A \to B$ that sends a_i to $f(a_i)$ is a homomorphism from $\boldsymbol{A}$ to $\boldsymbol{B}$. Since $\mathsf{Con}(\boldsymbol{A}) = \{\mathrm{id}_A, A \times A\}$, the kernel $\mathsf{Ker}(h)$ is either id_A or $A \times A$. Notice that there is no embedding of $\boldsymbol{A}$ into $\boldsymbol{B}$. Therefore, $\mathsf{Ker}(h)$ cannot be the identity relation. It follows that $\mathsf{Ker}(h) = A \times A$. In particular, $\langle a_1, a_8 \rangle \in \mathsf{Ker}(h)$ and, therefore, $f(x_1) = h(a_1) = f(a_8) = f(x_8)$. Hence, we conclude that $\boldsymbol{B} \vDash \Phi$, as desired. Moreover, Φ fails in $\boldsymbol{A}$, as witnessed by the assignment

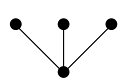
$$x_1 \longmapsto a_1, \dots, x_8 \longmapsto a_8.$$

In brief, Φ holds in $\boldsymbol{B}$ but fails in $\boldsymbol{A}$. By Maltsev's Theorem, we conclude that $\boldsymbol{A}$ does not belong to the quasi-variety $\mathbb{Q}(\boldsymbol{B})$ generated by $\boldsymbol{B}$. On the other hand, $\boldsymbol{A}$ is a homomorphic image of $\boldsymbol{B}$ (obtained by glueing two pairs of elements of $\boldsymbol{B}$). Thus, the quasi-variety $\mathbb{Q}(\boldsymbol{B})$ is not closed under $\mathbb{H}$. ⊠

Example 5.14 (Heyting algebras)**.** We assume the reader is familiar with *Esakia duality* for Heyting algebras [20, 21]. Let $\boldsymbol{A}$ be the finite Heyting algebra whose dual Esakia space is the following poset $\boldsymbol{A}_*$ endowed with the discrete topology.



Or aim is to show that $\mathbb{Q}(\boldsymbol{A})$ is not closed under $\mathbb{H}$ and, therefore, fails to be a variety. To this end, consider the finite Heyting algebra $\boldsymbol{B}$ whose dual Esakia space is the the rooted poset $\boldsymbol{B}_*$ (endowed with the discrete topology) depicted below.

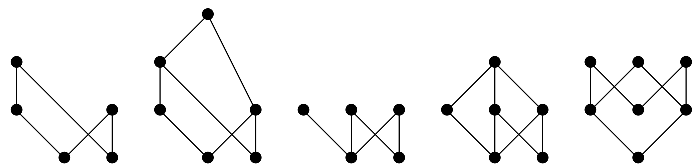


Notice that, as $\boldsymbol{B}_*$ is rooted and finite, the algebra $\boldsymbol{B}$ is subdirectly irreducible. Moreover, observe that, as $\boldsymbol{B}_*$ is an upset of $\boldsymbol{A}_*$, by Esakia duality we obtain $\boldsymbol{B} \in \mathbb{H}(\boldsymbol{A})$. Therefore, it suffices to show that $\boldsymbol{B} \notin \mathbb{Q}(\boldsymbol{A})$. Suppose the contrary, with a view to contradiction. By Corollary 5.6,

$$\boldsymbol{B} \in \mathbb{ISP}(\boldsymbol{A}) \subseteq \mathbb{IP}_{\text{SD}}\mathbb{S}(\boldsymbol{A}).$$

As $\boldsymbol{B}$ is subdirectly irreducible, we conclude that $\boldsymbol{B} \in \mathbb{IS}(\boldsymbol{A})$. By Esakia duality, this means that $\boldsymbol{B}_*$ is a p-morphic image of $\boldsymbol{A}_*$. But a quick inspection of the posets $\boldsymbol{A}_*$ and $\boldsymbol{B}_*$ shows that this is impossible, a contradiction. Hence, we conclude that $\mathbb{Q}(\boldsymbol{A})$ does not contain $\boldsymbol{B}$ and, therefore, is not closed under $\mathbb{H}$. ⊠

Exercise 5.15. Let $\boldsymbol{A}_1, \dots, \boldsymbol{A}_5$ be the Heyting algebras whose Esakia duals are the five posets depicted below.



A variety K is said to be *primitive* whenever every quasi-variety $\mathsf{M} \subseteq \mathsf{K}$ is a variety. Prove that if a variety K of Heyting algebras is primitive, then it omits $\boldsymbol{A}_1, \dots, \boldsymbol{A}_5$.

Notably, the converse is also true: a variety K of Heyting algebras is primitive if and only if it omits $\boldsymbol{A}_1, \dots, \boldsymbol{A}_5$ [15, 16], see also [5]. A similar description of primitive varieties of K4-algebras has been obtain in [46]. These results admit a logical interpretation in terms of structural completeness [3, 44, 47]. ⊠

If K is a quasi-variety and θ a congruence of some $\boldsymbol{A} \in \mathsf{K}$, the algebra $\boldsymbol{A}/\theta$ need not belong to K. This makes the following concept attractive.

Definition 5.16. Let $\mathsf{K} \cup \{\boldsymbol{A}\}$ be a class of similar algebras. A congruence $\theta \in \mathsf{Con}(\boldsymbol{A})$ is said to be a K-*congruence* of $\boldsymbol{A}$ if $\boldsymbol{A}/\theta \in \mathsf{K}$. We denote the poset of K-congruences of $\boldsymbol{A}$, ordered under the inclusion relation, by $\mathsf{Con}_{\mathsf{K}}(\boldsymbol{A})$.

Proposition 5.17. *If K is a quasi-variety, then $\mathsf{Con}_{\mathsf{K}}(\boldsymbol{A})$ is a complete lattice in which meets are intersections, for every algebra $\boldsymbol{A}$ of the type of K.*

Proof. Then let $\boldsymbol{A}$ be an algebra of the type of K. Since K is a quasi-variety, it is closed under $\mathbb{P}_{\text{SD}}$ and it contains a trivial algebra (the subdirect product of the empty family). Therefore, as K is closed under $\mathbb{I}$ by assumption, it contains all trivial algebras and, in particular, $\boldsymbol{A}/(A \times A)$. Thus, $A \times A \in \mathsf{Con}_{\mathsf{K}}(\boldsymbol{A})$. Then consider a nonempty family $\{\theta_i : i \in I\} \subseteq \mathsf{Con}_{\mathsf{K}}(\boldsymbol{A})$. By Proposition 2.20,

$$\boldsymbol{A}/\bigcap_{i \in I}\theta_i \in \mathbb{IP}_{\text{SD}}(\{\boldsymbol{A}/\theta_i : i \in I\}).$$

Observe that $\{\boldsymbol{A}/\theta_i : i \in I\} \subseteq \mathsf{K}$, since the various θ_i are K-congruences of $\boldsymbol{A}$. Together with the above display and the assumption that K is closed under $\mathbb{I}$ and $\mathbb{P}_{\text{SD}}$, this yields $\boldsymbol{A}/\bigcap_{i \in I}\theta_i \in \mathsf{K}$, whence $\bigcap_{i \in I}\theta_i \in \mathsf{Con}_{\mathsf{K}}(\boldsymbol{A})$. It follows that $\mathsf{Con}_{\mathsf{K}}(\boldsymbol{A})$ has a minimum, namely $A \times A$ and infima of nonempty families. Therefore, arbitrary infima exist in $\mathsf{Con}_{\mathsf{K}}(\boldsymbol{A})$ and, therefore, $\mathsf{Con}_{\mathsf{K}}(\boldsymbol{A})$ is a complete lattice. ⊠

Corollary 5.18. *If K is a quasi-variety and $\boldsymbol{A}$ an algebra of the same type, the map*

$$\mathrm{Cg}^{\boldsymbol{A}}_{\mathsf{K}} \colon \mathcal{P}(A \times A) \to \mathcal{P}(A \times A)$$

that associates the least K*-congruence of* $\boldsymbol{A}$ *containing a subset* X *with a subset* $X \subseteq A \times A$ *is a closure operator on* $A \times A$*.*

Remark 5.19. The proof of Proposition 5.17 depends only on the fact that K is closed under $\mathbb{I}, \mathbb{S}$ and $\mathbb{P}$. Classes of similar algebras closed under $\mathbb{I}, \mathbb{S}$ and $\mathbb{P}$ are called *prevarieties*. Notably, they coincide with the classes of algebras that can be axiomatized by *proper classes* of infinitary quasi-equations. The demand that proper classes can be replaced by sets in the axiomatization of prevarieties is equivalent to an independent set theoretical principle known as *Vopěnka Principle* [28, Prop. 2.3.18], see also [1]. ⊠

6. Algebraizable logics

We are now ready to introduce a robust theory of algebraization that will account for the relation between logic and algebra [7], see also [8, 17, 22, 23, 24]. To this end, for every set of formulas $\Delta(x, y)$ and set of equations $\Theta \cup \{\varepsilon \approx \delta\}$, we shall abbreviate

$$\{\varphi(\varepsilon, \delta) \colon \varphi(x, y) \in \Delta\} \text{ as } \Delta(\varepsilon, \delta), \text{ and } \bigcup_{\varphi \approx \psi \in \Theta} \Delta(\varphi, \psi) \text{ as } \Delta[\Theta].$$

Definition 6.1 (Blok & Pigozzi)**.** A finitary logic $\vdash$ is said to be *algebraizable* if there are a finite set of equations $\boldsymbol{\tau}(x)$, a finite set of formulas $\Delta(x, y)$, and a quasi-variety K such that

$$\Gamma \vdash \varphi \Longleftrightarrow \boldsymbol{\tau}[\Gamma] \vDash_{\mathsf{K}} \boldsymbol{\tau}(\varphi) \tag{Alg1}$$

$$\Theta \vDash_{\mathsf{K}} \varepsilon \approx \delta \Longleftrightarrow \Delta[\Theta] \vdash \Delta(\varepsilon, \delta) \tag{Alg2}$$

$$\varphi \dashv\vdash \Delta[\boldsymbol{\tau}(\varphi)] \tag{Alg3}$$

$$\varepsilon \approx \delta \;=\!\mid\vDash_{\mathsf{K}} \boldsymbol{\tau}[\Delta(\varepsilon, \delta)] \tag{Alg4}$$

for every set of formulas $\Gamma \cup \{\varphi\}$ and every set of equations $\Theta \cup \{\varepsilon \approx \delta\}$. In this case, K is said to be an *equivalent algebraic semantics* for $\vdash$. In addition, we say that $\boldsymbol{\tau}, \Delta$ and K *witness* the algebraizability of the logic $\vdash$.

Condition (Alg1) expresses the demand that K is a $\boldsymbol{\tau}$-algebraic semantics for $\vdash$, namely, that $\vdash$ can be interpreted into $\vDash_{\mathsf{K}}$ by means of the set of equations $\boldsymbol{\tau}(x)$ that allows to translate a set of formulas Γ into a set of equations $\boldsymbol{\tau}[\Gamma]$. Condition (Alg2) states that this interpretation can be *reversed*, in the sense that $\vDash_{\mathsf{K}}$ can also be interpreted into $\vdash$ by means of the set of formulas $\Delta(x, y)$ that allows to translate sets of equations Θ into sets of formulas $\Delta[\Theta]$. Lastly, conditions (Alg3) and (Alg4) guarantee that these two interpretations are *inverses of each other* up to provability equivalence.

Remark 6.2. While the problem of determining whether logics presented by Hilbert calculi are algebraizable is undecidable [37], the same problem becomes decidable, although complete for EXPTIME [39], for logics presented by finite sets of finite logical matrices in the sense of [49]. ⊠

The definition of an algebraizable logic can be made more concise as follows.

Proposition 6.3. *The following conditions are equivalent for a finitary logic* $\vdash$*:*

(i) $\vdash$ *is algebraizable;*
(ii) *There are a set of equations* $\boldsymbol{\tau}(x)$*, a set of formulas* $\Delta(x, y)$ *and a quasi-variety* K *that satisfy* (Alg1) *and*

$$x \approx y \;=\!\mid\vDash_{\mathsf{K}} \boldsymbol{\tau}[\Delta(x, y)]; \tag{Alg4*}$$

(iii) *There are a set of equations $\boldsymbol{\tau}(x)$, a set of formulas $\Delta(x,y)$ and a quasi-variety K that satisfy* (Alg2) *and*
$$x \dashv\vdash \Delta[\boldsymbol{\tau}(x)]. \tag{Alg3*}$$

In this case, $\boldsymbol{\tau}$, Δ, and K witness the algebraizability of $\vdash$.

Proof. The implication (i)⇒(ii) is straightforward. To prove (ii)⇒(iii), observe that (Alg4*) implies that, for every $\varphi \approx \psi \in \boldsymbol{\tau}$,
$$\varphi \approx \psi \mathrel{=\!\!|\!\!\models_{\mathsf{K}}} \boldsymbol{\tau}[\Delta(\varphi,\psi)].$$
Since $\boldsymbol{\tau}[\Delta[\boldsymbol{\tau}(x)]] = \bigcup\{\boldsymbol{\tau}[\Delta(\varphi,\psi)] : \varphi \approx \psi \in \boldsymbol{\tau}\}$, this yields
$$\boldsymbol{\tau}(x) \mathrel{=\!\!|\!\!\models_{\mathsf{K}}} \boldsymbol{\tau}[\Delta[\boldsymbol{\tau}(x)]].$$
By (Alg1), we conclude that $x \dashv\vdash \Delta[\boldsymbol{\tau}(x)]$. Then consider a set of equations $\Theta \cup \{\varepsilon \approx \delta\}$. We have
$$\Theta \vDash_{\mathsf{K}} \varepsilon \approx \delta \iff \boldsymbol{\tau}[\Delta[\Theta]] \vDash_{\mathsf{K}} \boldsymbol{\tau}[\Delta(\varepsilon,\delta)] \iff \Delta[\Theta] \vdash \Delta(\varepsilon,\delta),$$
where the first equivalence follows from (Alg4*) and the second from (Alg1).

(iii)⇒(i): Since $\vdash$ is finitary and $\Delta[\boldsymbol{\tau}(x)] \vdash x$, there exists a finite $\boldsymbol{\tau}' \subseteq \boldsymbol{\tau}$ such that $\Delta[\boldsymbol{\tau}(x)] \vdash x$. Together with the assumption that $x \vdash \Delta[\boldsymbol{\tau}(x)]$, this yields $x \dashv\vdash \Delta[\boldsymbol{\tau}'(x)]$. Therefore, the pair $\boldsymbol{\tau}'$ and Δ satisfy conditions (Alg2) and (Alg3*). Furthermore, a proof similar to the one of the implication (ii)⇒(iii) establishes (Alg1) and (Alg4*) for $\boldsymbol{\tau}'$ and Δ.

Now, by (Alg4*), we have $x \approx y \mathrel{=\!\!|\!\!\models_{\mathsf{K}}} \boldsymbol{\tau}'[\Delta(x,y)]$. Since, by Theorem 5.10, the relation $\vDash_{\mathsf{K}}$ is finitary, there exists a finite $\Delta' \subseteq \Delta$ such that $x \approx y \mathrel{=\!\!|\!\!\models_{\mathsf{K}}} \boldsymbol{\tau}'[\Delta'(x,y)]$. Therefore, the pair $\boldsymbol{\tau}'$ and Δ' satisfy conditions (Alg1) and (Alg4*). Furthermore, a proof similar to the one of the implication (ii)⇒(iii) establishes (Alg2) and (Alg3*) for $\boldsymbol{\tau}'$ and Δ'. As a consequence, the pair of finite sets $\boldsymbol{\tau}'$ and Δ' satisfy (Alg1), (Alg2), (Alg3*) and (Alg4*). Since conditions (Alg3*) and (Alg4*) imply (Alg3) and (Alg4), respectively, we conclude that $\vdash$ is algebraizable and that its algebraizability is witnessed by $\boldsymbol{\tau}$, Δ, and K. ⊠

Example 6.4 (Algebraizable logics)**.** Consider the following sets of equations and formulas, respectively,
$$\boldsymbol{\tau}(x) := \{x \approx 1\} \text{ and } \Delta(x,y) := \{x \to y, y \to x\}.$$
We shall prove that **IPC** is algebraizable and that its algebraizability is witnessed by $\boldsymbol{\tau}, \Delta$ and the variety HA of Heyting algebras. First, recall that HA is a $\boldsymbol{\tau}$-algebraic semantics for **IPC**. Moreover, observe that
$$x \approx y \mathrel{=\!\!|\!\!\models_{\mathsf{HA}}} \{x \to y \approx 1, y \to x \approx 1\}. \tag{13}$$
To prove this, consider a Heyting algebra $\boldsymbol{A}$ and $a, c \in A$. Using the residuation law and the fact that 1 is the maximum of $\boldsymbol{A}$, we obtain
$$\begin{aligned} a = c &\iff (a \leqslant c \text{ and } c \leqslant a) \\ &\iff (1 \wedge a \leqslant \text{ and } 1 \wedge c \leqslant a) \\ &\iff (1 \leqslant a \to c \text{ and } 1 \leqslant c \to a) \\ &\iff a \to c = 1 \text{ and } c \to a = 1. \end{aligned}$$

This establishes (13), which is precisely (Alg4*). Hence, **IPC** satisfies (Alg1) and (Alg4*) with respect to $\boldsymbol{\tau}, \Delta$ and HA. By Proposition 6.3, we conclude that **IPC** is algebraizable and that its algebraizability is witnessed by $\boldsymbol{\tau}, \Delta$ and HA.

A similar argument shows that $\mathbf{K}_g$ is algebraizable and that its algebraizability is witnessed by $\boldsymbol{\tau}, \Delta$ and the class of modal algebras MA. Similarly, **CPC** is algebraizable and its algebraizability is witnessed by $\boldsymbol{\tau}, \Delta$ and the class of Boolean algebras BA. ⊠

Example 6.5 (A nonalgebraizable logic)**.** On the other hand, not every logic is algebraizable. As an exemplification, we will prove that $\mathbf{K}_\ell$ is not algebraizable.

Suppose the contrary, with a view to contradition. Then there are a set of equations $\boldsymbol{\tau}(x)$, a set of formulas $\Delta(x,y)$ and a quasi-variety K witnessing the algebraizbaility of $\mathbf{K}_\ell$.

We shall see that K is a class of modal algebras. As the class MA of modal algebras is a variety, it suffices to show that every equation valid in MA is also valid in K. To this end, let $\varepsilon \approx \delta$ be an equation valid in MA. From (Alg1) and (Alg4) it follows

$$\emptyset \vdash_{\mathbf{K}_\ell} \Delta(x,x) \iff \emptyset \vDash_{\mathsf{K}} \boldsymbol{\tau}[\Delta(x,x)] \iff \emptyset \vDash_{\mathsf{K}} x \approx x.$$

Since $\emptyset \vDash_{\mathsf{K}} x \approx x$, we conclude that $\emptyset \vdash_{\mathbf{K}_\ell} \Delta(x,x)$. By substitution invariance, this yields

$$\emptyset \vdash_{\mathbf{K}_\ell} \Delta(\varepsilon,\varepsilon). \tag{14}$$

Since $\varepsilon \approx \delta$ is valid in MA, for every $\varphi(x,y) \in \Delta$, we have

$$\mathsf{MA} \vDash \varphi(\varepsilon,\varepsilon) \approx \varphi(\varepsilon,\delta).$$

Together with (6), this yields $\Delta(\varepsilon,\varepsilon) \dashv\vdash_{\mathbf{K}_\ell} \Delta(\varepsilon,\delta)$. Since, by (14), $\emptyset \vdash_{\mathbf{K}_\ell} \Delta(\varepsilon,\varepsilon)$, this implies $\emptyset \vdash_{\mathbf{K}_\ell} \Delta(\varepsilon,\delta)$. By (Alg1), we obtain $\emptyset \vDash_{\mathsf{K}} \boldsymbol{\tau}[\Delta(\varepsilon,\delta)]$. Using (Alg4), we conclude that K satisfies $\varepsilon \approx \delta$ and, therefore, that K is a class of modal algebras.

Since $\mathbf{K}_\ell$ is related to K by (Alg4), this implies that $\mathbf{K}_\ell$ has an algebraic semantics, namely K, that is a class of modal algebras. But this contradicts Theorem 3.22. ⊠

As we mentioned, the theory of algebraizable logics allows to associate with a logic a unique *distinguished* algebraic semantics.

Theorem 6.6. *If $\boldsymbol{\tau}_1, \Delta_1, \mathsf{K}_1$ and $\boldsymbol{\tau}_2, \Delta_2, \mathsf{K}_2$ witness the algebraizability of a logic $\vdash$, then*

$$\mathsf{K}_1 = \mathsf{K}_2 \qquad \boldsymbol{\tau}_1(x) \mathrel{=\!\mid\mid\!\!\vDash_{\mathsf{K}_1}} \boldsymbol{\tau}_2(x) \qquad \Delta_1(x,y) \dashv\vdash \Delta_2(x,y).$$

Proof. We begin by proving $\Delta_1(x,y) \dashv\vdash \Delta_2(x,y)$. By symmetry, it suffices to prove $\Delta_1(x,y) \vdash \Delta_2(x,y)$. To this end, consider $\varphi \in \Delta_2$. Since $\boldsymbol{\tau}_1, \Delta_1, \mathsf{K}_1$ witness the algebraizability of $\vdash$, from (Alg4) and (Alg1) it follows

$$\boldsymbol{\tau}_1(x), x \approx y \vDash_{\mathsf{K}_1} \boldsymbol{\tau}_1(y) \iff \boldsymbol{\tau}_1(x), \boldsymbol{\tau}_1[\Delta_1(x,y)] \vDash_{\mathsf{K}_1} \boldsymbol{\tau}_1(y) \iff x, \Delta_1(x,y) \vdash y$$

and

$$x \approx y \vDash_{\mathsf{K}_1} \varphi(x,x) \approx \varphi(x,y) \iff \Delta_1(x,y) \vdash \Delta_1(\varphi(x,x), \varphi(x,y)).$$

Since $\boldsymbol{\tau}_1(x), x \approx y \vDash_{\mathsf{K}_1} \boldsymbol{\tau}_1(y)$ and $x \approx y \vDash_{\mathsf{K}_1} \varphi(x,x) \approx \varphi(x,y)$, we obtain that

$$x, \Delta_1(x,y) \vdash y \;\text{ and }\; \Delta_1(x,y) \vdash \Delta_1(\varphi(x,x), \varphi(x,y)).$$

Moreover, by substitution invariance, we get $\varphi(x,x), \Delta_1(\varphi(x,x), \varphi(x,y)) \vdash \varphi(x,y)$. Together with the above display, we conclude that

$$\varphi(x,x), \Delta_1(x,y) \vdash \varphi(x,y).$$

Now, as explained in Example 6.5, the fact that $\boldsymbol{\tau}_2, \Delta_2, \mathsf{K}_2$ witness the algebraizability of $\vdash$ guarantees that $\emptyset \vdash \Delta_2(x,x)$. As a consequence, $\emptyset \vdash \varphi(x,x)$. Together with the above display, this yields $\Delta_1(x,y) \vdash \varphi(x,y)$ and, therefore, $\Delta_1(x,y) \vdash \Delta_2(x,y)$. Hence, we conclude that $\Delta_1(x,y) \dashv\vdash \Delta_2(x,y)$, as desired.

Then we turn to prove that $\mathsf{K}_1 = \mathsf{K}_2$. Since K_1 and K_2 are quasi-varieties, it suffices to show that they satisfy the same quasi-equations with variables in Var. To prove this, consider a finite set of equation $\Theta \cup \{\varepsilon \approx \delta\} \subseteq E(Var)$. We have

$$\begin{aligned}
\mathsf{K}_1 \vDash \text{\&}\,\Theta \Longrightarrow \varepsilon \approx \delta &\iff \Theta \vDash_{\mathsf{K}_1} \varepsilon \approx \delta \\
&\iff \Delta_1[\Theta] \vdash \Delta_1(\varepsilon, \delta) \\
&\iff \Delta_2[\Theta] \vdash \Delta_2(\varepsilon, \delta) \\
&\iff \Theta \vDash_{\mathsf{K}_2} \varepsilon \approx \delta \\
&\iff \mathsf{K}_2 \vDash \text{\&}\,\Theta \Longrightarrow \varepsilon \approx \delta.
\end{aligned}$$

The above equivalence can be justified as follows. The first and the last are straightforward, the second from (Alg2) and the assumption that $\boldsymbol{\tau}_1, \Delta_1, \mathsf{K}_1$ witness the algebraizability of $\vdash$, the third from $\Delta_1(x, y) \dashv\vdash \Delta_2(x, y)$ and the fourth from the assumption that $\boldsymbol{\tau}_2, \Delta_2, \mathsf{K}_2$ witness the algebraizability of $\vdash$. Hence, we conclude that $\mathsf{K}_1 = \mathsf{K}_2$.

It only remains to prove that $\boldsymbol{\tau}_1(x) \mathrel{=\!\mid\!\vDash_{\mathsf{K}_1}} \boldsymbol{\tau}_2(x)$. To prove this, observe that

$$\begin{aligned}
\boldsymbol{\tau}_1(x) \mathrel{=\!\mid\!\vDash_{\mathsf{K}_1}} \boldsymbol{\tau}_2(x) &\iff \Delta_1[\boldsymbol{\tau}_1(x)] \dashv\vdash \Delta_1[\boldsymbol{\tau}_2(x)] \\
&\iff \Delta_1[\boldsymbol{\tau}_1(x)] \dashv\vdash \Delta_2[\boldsymbol{\tau}_2(x)] \\
&\iff x \dashv\vdash x.
\end{aligned}$$

The above equivalence can be justified as follows. The first follows from (Alg2) and the assumption that $\boldsymbol{\tau}_1, \Delta_1, \mathsf{K}_1$ witness the algebraizability of $\vdash$, the third from $\Delta_1(x, y) \dashv\vdash \Delta_2(x, y)$ and the last from (Alg3) and the fact that both $\boldsymbol{\tau}_1, \Delta_1, \mathsf{K}_1$ and $\boldsymbol{\tau}_2, \Delta_2, \mathsf{K}_2$ witness the algebraizability of $\vdash$. Since $x \dashv\vdash x$, we conclude that $\boldsymbol{\tau}_1(x) \mathrel{=\!\mid\!\vDash_{\mathsf{K}_1}} \boldsymbol{\tau}_2(x)$. ⊠

Corollary 6.7. *Every algebraizable logic has a unique equivalent algebraic semantics.*

Proof. Immediate from Theorem 6.6. ⊠

Example 6.8 (Equivalent algebraic semantics)**.** In view of Example 6.5,

(i) The unique equivalent algebraic semantics of **CPC** is BA;
(ii) The unique equivalent algebraic semantics of **IPC** is HA;
(iii) The unique equivalent algebraic semantics of $\mathbf{K}_g$ is MA.

In particular, while **CPC** has various algebraic semantics (for instance, BA and HA), the class of Boolean algebras is its unique equivalent algebraic semantics. ⊠

At this stage, it is natural to wonder whether every quasi-variety is the equivalent algebraic semantics of some algebraizable logic. While this is not the case in general [25] (as explained in the next exercise), still every quasi-variety is *categorically equivalent* to the equivalent algebraic semantics of an algebraizable logic [41], see also [9]. As a consequence, the categorical properties of every every quasi-variety can be studied though the eyes of an algebraizable logic. Notably, category equivalences and adjunctions between quasi-varieties have been described in detail in [18, 34, 38], see also [26].

Exercise 6.9. A class of algebras K that satisfies $f(x, \dots, x) \approx x$ for each of its basic operations f is said to be *idempotent*. For instance, all classes of lattices are idempotent. Prove that a nontrivial idempotent quasi-variety cannot be the equivalent algebraic semantics of any algebraizable logic.

To this end, you might wish to use the following strategy: suppose, with a view to contradiction, that there exists a nontrivial idempotent quasi-variety K that, moreover, is the equivalent algebraic semantics of some algebraizable logic $\vdash$. First show that $\emptyset \vdash x$. By substitution invariance, derive $\emptyset \vdash \Delta(x,y)$. Use this fact to infer that K is trivial and, therefore, to obtain a contradiction. ⊠

Definition 6.10. Given two logics $\vdash$ and $\vdash^*$ of the same type. Then $\vdash^*$ is said to be

(i) an *extension* of $\vdash$ if, $\Gamma \vdash^* \varphi$, for every $\Gamma \cup \{\varphi\} \subseteq T(Var)$ such that $\Gamma \vdash \varphi$; and
(ii) an *axiomatic extension* of $\vdash$ if there exists a set of formulas Σ such that $\sigma[\Sigma] \subseteq \Sigma$ for every substitution σ and, for every $\Gamma \cup \{\varphi\} \subseteq T(Var)$,

$$\Gamma \vdash^* \varphi \Longleftrightarrow \Gamma, \Sigma \vdash \varphi.$$

In this case, we say that Σ *axiomatizes* $\vdash^*$ *relative to* $\vdash$.

Axiomatic extensions of **IPC** have been called *superintuitionistic logics* in the literature.

Remark 6.11. Notice that when a logic $\vdash^*$ is an axiomatic extension of a logic $\vdash$, axiomatized relative to $\vdash$ by Σ, it can be axiomatized by extending any Hilbert axiomatization for $\vdash$ with the set of axioms $\{\emptyset \rhd \varphi : \varphi \in \Sigma\}$. ⊠

Definition 6.12. Let K and M be quasi-varieties of the same type. Then M is said to be

(i) a *subquasi-variety* of K if $\mathsf{M} \subseteq \mathsf{K}$; and
(ii) a *relative subvariety* of K if it is axiomatized relative to K by a set of equations.

Notice that the relative subvarieties of a variety are precisely its subvarieties.

Given a finitary logic $\vdash$, the posets $\mathsf{fEx}(\vdash)$ and $\mathsf{aEx}(\vdash)$ of finitary and axiomatic extensions of $\vdash$, respectively, ordered under the inclusion relation form two complete lattices. Similarly, given a quasi-variety K, the posets $\mathsf{sQ}(\mathsf{K})$ and $\mathsf{sV}(\mathsf{K})$ of subquasi-varieties and relative subvarieties of K, respectively, ordered under the inclusion relation form two complete lattices. In view of the next result, axiomatic extensions of an algebraizable logic can be studied through the lenses of the relative subvarieties of its equivalent algebraic semantics, which, in turn, are amenable to the methods of model theory, universal algebra and duality theory. This approach has proved very fruitful in the study of superintuitionistic and normal modal logics [13, 31].

Theorem 6.13. *Let $\vdash$ be an algebraizable logic and K its equivalent algebraic semantics. The lattices $\mathsf{fEx}(\vdash)$ and $\mathsf{sQ}(\mathsf{K})$ are dually isomorphic under the map that sends a finitary extension of $\vdash$ to its equivalent algebraic semantics. This dual isomorphism restricts to one between $\mathsf{aEx}(\vdash)$ and $\mathsf{sV}(\mathsf{K})$.*

Remark 6.14. Notice that the above result implicitly states that algebraizability persists under the formation of finitary extensions. Moreover, it generalizes the well-known fact that the lattices of superintuitionistic logics and of normal modal logics are dually isomorphic to those of varieties of Heyting and modal algebras, respectively. ⊠

7. Deductive filters

Deductive closed sets of formulas have been called theories. This notion can be extended as follows.

Definition 7.1. A subset F of the universe of an algebra $\boldsymbol{A}$ is said to be a *deductive filter* of a logic $\vdash$ on $\boldsymbol{A}$ when, for every $\Gamma \cup \{\varphi\} \subseteq T(Var)$,

$$\text{if } \Gamma \vdash \varphi\text{, then for every homomorphism } f\colon \boldsymbol{T}(Var) \to \boldsymbol{A},$$
$$\text{if } f[\Gamma] \subseteq F\text{, then } f(\varphi) \in F.$$

We denote by $\mathsf{Fi}_{\vdash}(\boldsymbol{A})$ the poset of deductive filters of $\vdash$ on $\boldsymbol{A}$ ordered under inclusion.

The following observations is a direct consequence of the definition of a deductive filter.

Proposition 7.2. *If $\vdash$ is a logic and $\boldsymbol{A}$ an algebra, then $\mathsf{Fi}_{\vdash}(\boldsymbol{A})$ is a complete lattice in which meets are intersections.*

Recall that the set of theories of a logic $\vdash$ is denoted by $\mathcal{T}h(\vdash)$. When convenient, we regard $\mathcal{T}h(\vdash)$ as a lattice ordered under inclusion.

Proposition 7.3. *If $\vdash$ is a logic, the lattice $\mathcal{T}h(\vdash)$ of theories of $\vdash$ coincides with the lattice $\mathsf{Fi}_{\vdash}(\boldsymbol{T}(Var))$ of filters of $\vdash$ on the formula algebra $\boldsymbol{T}(Var)$.*

Proof. Consider a set of formulas Γ. Suppose first that Γ is a theory of $\vdash$. Then consider a set of formulas $\Sigma \cup \{\varphi\}$ such that $\Sigma \vdash \varphi$ and a homomorphism $\sigma\colon \boldsymbol{T}(Var) \to \boldsymbol{T}(Var)$ such that $\sigma[\Sigma] \subseteq \Gamma$. Notice that σ is a substitution. Consequently, as $\vdash$ is substitution invariant, $\sigma[\Sigma] \vdash \sigma(\varphi)$. Now, since $\sigma[\Sigma] \subseteq \Gamma$, we have $\Gamma \vdash \sigma[\Sigma]$. Together with $\sigma[\Sigma] \vdash \sigma(\varphi)$ and the Cut principle, this implies $\Gamma \vdash \sigma(\varphi)$. Since Γ is a theory of $\vdash$, this yields $\sigma(\varphi) \in \Gamma$. Hence, we conclude that Γ is a filter of $\vdash$ on $\boldsymbol{T}(Var)$.

Conversely, suppose that Γ is a filter of $\vdash$ on $\boldsymbol{T}(Var)$. Then consider a formula φ such that $\Gamma \vdash \varphi$ and let $id\colon \boldsymbol{T}(Var) \to \boldsymbol{T}(Var)$ be the identity homomorphism. Since $id[\Gamma] = \Gamma$ and $\Gamma \vdash \varphi$, the assumption that Γ is a filter of $\vdash$ on $\boldsymbol{T}(Var)$ guarantees that $\varphi = id(\varphi) \in \Gamma$. Hence, we conclude that Γ is a theory of $\vdash$. ⊠

The next observation is instrumental to describe deductive filters in concrete cases.

Proposition 7.4. *Let $\vdash$ be the logic axiomatized by an Hilbert calculus H. A subset F of the universe of an algebra $\boldsymbol{A}$ is a deductive filter of $\vdash$ on $\boldsymbol{A}$ if and only if F is closed under the interpretation of the rules in H, that is, for every rule $\Gamma \rhd \varphi$ in H and every homomorphism $f\colon \boldsymbol{T}(Var) \to \boldsymbol{A}$,*

$$\textit{if } f[\Gamma] \subseteq F\textit{, then } f(\varphi) \in F.$$

Proof. The "only if" part is straightforward. To prove the "if" one, suppose that F is closed under the interpretation of the rules in H. Then consider a set of formulas $\Gamma \cup \{\varphi\}$ such that $\Gamma \vdash \varphi$ and a homomorphism $f\colon \boldsymbol{T}(Var) \to \boldsymbol{A}$ such that $f[\Gamma] \subseteq F$. Since H axiomatizes $\vdash$, there exists a formal proof $\langle \psi_\alpha : \alpha \leqslant \gamma \rangle$ of φ from Γ in H. We will prove, by complete induction on α, that $f(\psi_\alpha) \in F$. Accordingly, assume that $\{f(\psi_\beta) : \beta < \alpha\} \subseteq F$. Then we have two cases: either $\psi_\alpha \in \Gamma$ or there exists a rule $\Sigma \rhd \delta$ in H and a substitution σ such that $\sigma[\Delta] \subseteq \{\psi_\beta : \beta < \alpha\}$ and $\sigma(\delta) = \psi_\alpha$. If $\psi_\alpha \in \Gamma$, then it is clear that $f(\psi_\alpha) \in f[\Gamma] \subseteq F$. Then we consider the second case. By the inductive hypothesis, we have

$$f(\sigma[\Delta]) \subseteq \{f(\psi_\beta) : \beta < \alpha\} \subseteq F.$$

Furthermore, consider the homomorphism $f \circ \sigma\colon \boldsymbol{T}(Var) \to \boldsymbol{A}$. From the above display it follows $f \circ \sigma[\Delta] \subseteq F$. Since F is closed under the interpretation of the rules in H (and, in particular, under $\Delta \rhd \delta$), this yields

$$f(\psi_\alpha) = f(\sigma(\delta)) = f \circ \sigma(\delta) \in F.$$

This concludes the inductive proof. Since $\varphi = \psi_\gamma$, we obtain that $f(\varphi) \in F$ and, therefore, that F is a deductive filter of $\vdash$ on $\boldsymbol{A}$. $\boxtimes$

Example 7.5 (Deductive filters)**.** We will prove that

(i) The deductive filters of **IPC** on a Heyting algebra $\boldsymbol{A}$ are the lattice filters of $\boldsymbol{A}$;
(ii) The deductive filters of **CPC** on a Boolean algebra $\boldsymbol{A}$ are the lattice filters of $\boldsymbol{A}$;
(iii) The deductive filters of $\mathbf{K}_g$ on a modal algebra $\boldsymbol{A}$ are the open filters of $\boldsymbol{A}$.

(i): Consider a subset F of A. Suppose first that F is a deductive filter of **IPC**. As $\emptyset \vdash_{\mathbf{IPC}} 1$ and F is a deductive filter of **IPC** on $\boldsymbol{A}$, we obtain $1 \in F$. Then consider $a, c \in A$. To prove that F is an upset, suppose that $a \in F$ and $a \leqslant c$. From the residuation law it follows

$$a \leqslant c \Longrightarrow 1 \wedge a \leqslant c \Longrightarrow 1 \leqslant a \to c \Longrightarrow a \to c = 1.$$

Since $1 \in F$, this yields $a, a \to c \in F$. Together with $x, y \vdash_{\mathbf{IPC}} x \wedge y$ and the assumption that F is a deductive filter of **IPC** on $\boldsymbol{A}$, this yields $c \in F$, as desired. Lastly, to prove that F is closed under binary meets, suppose that $a, c \in F$. Since F is a deductive filter of **IPC** on $\boldsymbol{A}$, the fact that $x, y \vdash_{\mathbf{IPC}} x \wedge y$ and $a, c \in F$ implies $a \wedge c \in F$. Hence, we conclude that F is a lattice filter of $\boldsymbol{A}$.

Conversely, suppose that F is a lattice filter of $\boldsymbol{A}$. In view of Proposition 7.4, it suffices to show that F is closed under the interpretation of the rules of an Hilbert calculus axiomatizing **IPC**. Accordingly, let H be the Hilbert calculus whose set of axioms is

$$\{\emptyset \rhd \varphi : \varphi \in T(Var) \text{ and } \emptyset \vdash_{\mathbf{IPC}} \varphi\}$$

and whose sole rule is modus ponens $x, x \to y \rhd y$. As H axiomatizes **IPC**, it only remains to prove that F is closed under the interpretation of its rules. Let then $\emptyset \rhd \varphi(x_1, \dots, x_n)$ be an axiom of H. Clearly, $\emptyset \vdash_{\mathbf{IPC}} \varphi(x_1, \dots, x_n)$. Since the class of Heyting algebras is an $\{x \approx 1\}$-algebraic semantics for **IPC**, it follows that, for all $a_1, \dots, a_n \in A$,

$$\varphi^{\boldsymbol{A}}(a_1, \dots, a_n) = 1 \in F.$$

The only rule in H is modus ponens $x, x \to y \rhd y$. To prove that F is closed under its interpretation, consider $a, c \in A$ such that $a, a \to c \in F$. Since F is closed under binary meets, $a \wedge (a \to c) \in F$. Moreover, by the residuation law,

$$a \to c \leqslant a \to c \Longleftrightarrow a \wedge (a \to c) \leqslant c.$$

Since $a \to c \leqslant a \to c$ always holds, we get $a \wedge (a \to c) \leqslant c$. As F is an upset that contains $a \wedge (a \to c)$, we obtain that $c \in F$. This shows that F is a deductive filter of **IPC** on $\boldsymbol{A}$.

(ii): Analogous to the proof of (i).

(iii): Consider a subset F of A. Suppose first that F is a deductive filter of $\mathbf{K}_g$ on $\boldsymbol{A}$. The proof that F is a lattice filter of $\boldsymbol{A}$ is analogous to the one detailed in the case of (i). In order to prove that F is also closed under $\Box$, consider $a \in F$. Since $x \vdash_{\mathbf{K}_g} \Box x$, the fact that F is a deductive filter of $\mathbf{K}_g$ on $\boldsymbol{A}$ and $a \in F$ implies that $\Box a \in F$. Hence, we conclude that F is an open filter of $\boldsymbol{A}$, as desired.

To prove the converse, suppose that F is an open filter of $\boldsymbol{A}$. In view of Proposition 7.4, it suffices to show that F is closed under the interpretation of the rules in the Hilbert calculus axiomatizing $\mathbf{K}_g$. Accordingly, let H be the Hilbert calculus whose set of axioms is

$$\{\emptyset \rhd \varphi : \varphi \in T(Var) \text{ and } \emptyset \vdash_{\mathbf{K}_g} \varphi\}$$

and whose rules are modus ponens $x, x \to y \rhd y$ and necessitation $x \rhd \Box x$. As H axiomatizes $\mathbf{K}_g$, it only remains to show that F is closed under the interpretation of its rules. The proof that F is closed under the interpretation of the axioms in H and of modus ponens is analogous to the one detailed for the case of (i). Therefore, it only remains to prove that F is closed under the interpretation of the necessitation rule $x \rhd \Box x$. But this is an immediate consequence of the fact that the filter F is open. ⊠

Exercise 7.6. Prove that the deductive filters of $\mathbf{K}_\ell$ on a modal algebra $\boldsymbol{A}$ are precisely the lattice filters of $\boldsymbol{A}$. Use the fact that $\mathbf{K}_\ell$ can be axiomatized by the Hilbert calculus whose set of axioms is

$$\{\emptyset \rhd \varphi : \varphi \in T(Var) \text{ and } \emptyset \vdash_{\mathbf{K}_\ell} \varphi\}$$

and whose sole rule is modus ponens $x, x \to y \rhd y$. Hint: you may use the fact that $\mathbf{K}_\ell$ and $\mathbf{K}_g$ have the same theorems. ⊠

The notion of a deductive filter can be extended to relative equational consequences as follows.

Definition 7.7. Let $\mathsf{K} \cup \{\boldsymbol{A}\}$ be a class of similar algebras. A set $\theta \subseteq A \times A$ is said to be a *deductive filter* of $\vDash_{\mathsf{K}}$ on $\boldsymbol{A}$ when, for every $\Theta \cup \{\varepsilon \approx \delta\} \subseteq E(Var)$,

if $\Theta \vDash_{\mathsf{K}} \varphi \approx \psi$, then for every homomorphism $f \colon \boldsymbol{T}(Var) \to \boldsymbol{A}$,

if $\langle f(\varphi), f(\psi)\rangle \in \theta$ for all $\varphi \approx \psi \in \Theta$, then $\langle f(\varepsilon), f(\delta)\rangle \in \theta$.

Proposition 7.8. *Let K be a quasi-variety and $\boldsymbol{A}$ an algebra of the same type. The deductive filters of $\vDash_{\mathsf{K}}$ on $\boldsymbol{A}$ are precisely the K-congruences of $\boldsymbol{A}$.*

Proof. Consider a subset θ of $A \times A$. First suppose that θ is a deductive filters of $\vDash_{\mathsf{K}}$ on $\boldsymbol{A}$. Notice that

$$\emptyset \vDash_{\mathsf{K}} x \approx x \qquad x \approx y \vDash_{\mathsf{K}} y \approx x \qquad x \approx y, y \approx z \vDash_{\mathsf{K}} x \approx z.$$

Furthermore, for every basic n-ary operation f, we have

$$x_1 \approx y_1, \dots, x_n \approx y_n \vDash_{\mathsf{K}} f(x_1, \dots, x_n) \approx f(y_1, \dots, y_n).$$

Since θ is a deductive filter of $\vDash_{\mathsf{K}}$ on $\boldsymbol{A}$, the above displays guarantee that θ is a congruence of $\boldsymbol{A}$. To prove that it is also a K-congruence, it remains to show that $\boldsymbol{A}/\theta \in \mathsf{K}$. Since K is a quasi-variety, it suffices to prove that $\boldsymbol{A}$ satisfies all the quasi-equations valid in K. Accordingly, consider a quasi-equation $\mathop{\&} \Theta \Longrightarrow \varepsilon \approx \delta$ valid in K and let $f \colon \boldsymbol{T}(Var) \to \boldsymbol{A}/\theta$ be a homomorphism such that $f(\varphi) = f(\psi)$, for all $\varphi \approx \psi \in \Theta$. Since the canonical homomorphism $\pi \colon \boldsymbol{A} \to \boldsymbol{A}/\theta$ is surjective, we can apply Corollary 1.5, obtaining a homomorphism $g \colon \boldsymbol{T}(Var) \to \boldsymbol{A}$ such that $f = \pi \circ g$. For every $\varphi \approx \psi \in \Theta$, we have

$$g(\varphi)/\theta = \pi \circ g(\varphi) = f(\varphi) = f(\psi) = \pi \circ g(\psi) = g(\psi)/\theta$$

and, therefore, $\langle g(\varphi), g(\psi)\rangle \in \theta$. Since θ is a deductive filter of $\vDash_{\mathsf{K}}$ and $\Theta \vDash_{\mathsf{K}} \varepsilon \approx \delta$, this yields $\langle g(\varepsilon), g(\delta)\rangle \in \theta$. In turn, this implies

$$f(\varepsilon) = \pi \circ g(\varepsilon) = \pi \circ g(\delta) = f(\delta).$$

Hence, we conclude that $\boldsymbol{A}/\theta \in \mathsf{K}$, as desired.

To prove the converse, suppose that θ is a K-congruence of $\boldsymbol{A}$. Consider a set of equations $\Theta \cup \{\varepsilon \approx \delta\} \subseteq E(Var)$ such that $\Theta \vDash_{\mathsf{K}} \varepsilon \approx \delta$ and a homomorphism $f \colon \boldsymbol{T}(Var) \to \boldsymbol{A}$ such that $\langle f(\varphi), f(\psi)\rangle \in \theta$, for all $\varphi \approx \psi \in \Theta$. Now, let $\pi \colon \boldsymbol{A} \to \boldsymbol{A}/\theta$ be the canonical projection. We have $\pi \circ f(\varphi) = \pi \circ f(\psi)$, for all $\varphi \approx \psi \in \Theta$. Since $\boldsymbol{A}/\theta \in \mathsf{K}$, this yields $\pi \circ f(\varepsilon) = \pi \circ f(\delta)$, which is $\langle f(\varepsilon), f(\delta)\rangle \in \theta$. ⊠

Definition 7.9. Let K be a quasi-variety. A set of equations $\Theta \subseteq E(Var)$ is said to be a *theory* of $\vDash_{\mathsf{K}}$ when, for every $\varepsilon \approx \delta \in E(Var)$,

$$\text{if } \Theta \vDash_{\mathsf{K}} \varepsilon \approx \delta, \text{ then } \varepsilon \approx \delta \in \Theta.$$

When ordered under the inclusion relation, the theories of $\vDash_{\mathsf{K}}$ form a lattice that we denote by $\mathcal{T}h(\vDash_{\mathsf{K}})$.

Recall that, formally speaking, equations are ordered pairs of formulas, e.g., the expression $\varepsilon \approx \delta$ is a suggestive notation for the ordered pair $\langle \varepsilon, \delta \rangle$. The following result builds on this observation.

Proposition 7.10. *If K is a quasi-variety, the lattice $\mathcal{T}h(\vDash_{\mathsf{K}})$ of theories of $\vDash_{\mathsf{K}}$ coincides with the lattice $\mathsf{Con}_{\mathsf{K}}(\boldsymbol{T}(Var))$ of K-congruences of the formula algebra $\boldsymbol{T}(Var)$.*

Proof. An argument analogous to the one detailed in the proof of Proposition 7.3 shows that $\mathcal{T}h(\vDash_{\mathsf{K}})$ coincides with the lattice of deductive filters of $\vDash_{\mathsf{K}}$ on $\boldsymbol{T}(Var)$. But, in view of Proposition 7.8, the latter coincides with $\mathsf{Con}_{\mathsf{K}}(\boldsymbol{T}(Var))$. ⊠

Deductive filters are closed under inverse endomorphisms, as we proceed to explain. First, the set of *endomorphism* of an algebra $\boldsymbol{A}$ will be denoted by $\mathsf{End}(\boldsymbol{A})$. Then, given an endomorphism σ and a congruence θ of $\boldsymbol{A}$, we set

$$\sigma^{-1}[\theta] := \{\langle a, c \rangle \in A \times A \colon \langle \sigma(a), \sigma(c) \rangle \in \theta\}.$$

Lemma 7.11. *Let $\vdash$ be a logic, K a quasi-variety, $\boldsymbol{A}$ an algebra and $\sigma \in \mathsf{End}(\boldsymbol{A})$.*

(i) *If $F \in \mathsf{Fi}_{\vdash}(\boldsymbol{A})$, then $\sigma^{-1}[F] \in \mathsf{Fi}_{\vdash}(\boldsymbol{A})$.*
(ii) *If $\theta \in \mathsf{Con}_{\mathsf{K}}(\boldsymbol{A})$, then $\sigma^{-1}[\theta] \in \mathsf{Con}_{\mathsf{K}}(\boldsymbol{A})$.*

Proof. (i): Suppose that $F \in \mathsf{Fi}_{\vdash}(\boldsymbol{A})$. Then consider $\Gamma \cup \{\varphi\} \subseteq T(Var)$ such that $\Gamma \vdash \varphi$ and a homomorphism $f \colon \boldsymbol{T}(Var) \to \boldsymbol{A}$ such that $f[\Gamma] \subseteq \sigma^{-1}[F]$. Clearly, $\sigma \circ f[\Gamma] \subseteq F$. Since $\sigma \circ f \colon \boldsymbol{T}(Var) \to \boldsymbol{A}$ is a homomorphism and $F \in \mathsf{Fi}_{\vdash}(\boldsymbol{A})$, this implies $\sigma \circ f(\varphi) \subseteq F$. Hence, we conclude $f(\varphi) \subseteq \sigma^{-1}[F]$, as desired.

(ii): Recall from Proposition 7.8 that $\mathsf{Con}_{\mathsf{K}}(\boldsymbol{A})$ is the set of deductive filters of $\vDash_{\mathsf{K}}$ on $\boldsymbol{A}$. Because of this, we can mimic the proof detailed for condition (i) and obtain that $\sigma^{-1}[\theta] \in \mathsf{Con}_{\mathsf{K}}(\boldsymbol{A})$, for every $\theta \in \mathsf{Con}_{\mathsf{K}}(\boldsymbol{A})$. ⊠

In view of Lemma 7.11, given a logic $\vdash$ and an algebra $\boldsymbol{A}$, the lattice $\mathsf{Fi}_{\vdash}(\boldsymbol{A})$ of deductive filters of $\vdash$ on $\boldsymbol{A}$ can be expanded with the unary operations $\{\sigma^{-1} : \sigma \in \mathsf{End}(\boldsymbol{A})\}$. Similarly, given a quasi-variety K, the lattice $\mathsf{Con}_{\mathsf{K}}(\boldsymbol{A})$ of K-congruences of $\boldsymbol{A}$ can also be expanded with the unary operations $\{\sigma^{-1} : \sigma \in \mathsf{End}(\boldsymbol{A})\}$. Accordingly, we set

$$\mathsf{Fi}_{\vdash}(\boldsymbol{A})^{+} := \langle \mathsf{Fi}_{\vdash}(\boldsymbol{A}); \wedge, \vee, \{\sigma^{-1} : \sigma \in \mathsf{End}(\boldsymbol{A})\}\rangle$$
$$\mathsf{Con}_{\mathsf{K}}(\boldsymbol{A})^{+} := \langle \mathsf{Con}_{\mathsf{K}}(\boldsymbol{A}); \wedge, \vee, \{\sigma^{-1} : \sigma \in \mathsf{End}(\boldsymbol{A})\}\rangle.$$

The above structures can be viewed as algebras whose type comprises two binary operations $\wedge$ and $\vee$ an a family of unary operations $\{\sigma^{-1} : \sigma \in \mathsf{End}(\boldsymbol{A})\}$. From this perspective, an isomorphism from $\mathsf{Fi}_{\vdash}(\boldsymbol{A})^{+}$ to $\mathsf{Con}_{\mathsf{K}}(\boldsymbol{A})^{+}$ is a lattice isomorphism $\Phi \colon \mathsf{Fi}_{\vdash}(\boldsymbol{A}) \to \mathsf{Con}_{\mathsf{K}}(\boldsymbol{A})$ that commutes with inverse endomorphisms, in the sense that

$$\Phi(\sigma^{-1}[F]) = \sigma^{-1}[\Phi(F)], \text{ for every } \sigma \in \mathsf{End}(\boldsymbol{A}).$$

Recall from Propositions 7.3 and 7.10 that

$$\mathcal{T}h(\vdash) = \mathsf{Fi}_{\vdash}(\boldsymbol{T}(Var)) \text{ and } \mathcal{T}h(\vDash_{\mathsf{K}}) = \mathsf{Con}_{\mathsf{K}}(\boldsymbol{T}(Var)).$$

Because of this, when $\boldsymbol{A} = \boldsymbol{T}(Var)$, we will denote $\mathsf{Fi}_{\vdash}(\boldsymbol{A})^{+}$ and $\mathsf{Con}_{\mathsf{K}}(\boldsymbol{A})^{+}$ by

$$\mathcal{T}h(\vdash)^{+} \text{ and } \mathcal{T}h(\vDash_{\mathsf{K}})^{+}.$$

The importance of these structures will become apparent in the next section.

References

[1] J. Adámek. How many variables does a quasivariety need? *Algebra Universalis*, (27):44–48, 1990.

[2] J. L. Bell and A. B. Slomson. *Models and ultraproducts: An introduction*. North-Holland, Amsterdam, 1971. Second revised printing.

[3] C. Bergman. Structural completeness in algebra and logic. In *Algebraic logic (Budapest, 1988)*, volume 54 of *Colloq. Math. Soc. János Bolyai*, pages 59–73. North-Holland, Amsterdam, 1991.

[4] C. Bergman. *Universal Algebra: Fundamentals and Selected Topics*. Chapman & Hall Pure and Applied Mathematics. Chapman and Hall/CRC, 2011.

[5] N. Bezhanishvili and T. Moraschini. Citkin's description of hereditarily structurally complete intermediate logics via Esakia duality. *Submitted manuscript, available online*, 2020.

[6] W. J. Blok and P. Köhler. Algebraic semantics for quasi-classical modal logics. *The Journal of Symbolic Logic*, 48:941–964, 1983.

[7] W. J. Blok and D. Pigozzi. *Algebraizable logics*, volume 396 of *Mem. Amer. Math. Soc.* A.M.S., Providence, January 1989.

[8] W. J. Blok and D. Pigozzi. Algebraic semantics for universal Horn logic without equality. In A. Romanowska and J. D. H. Smith, editors, *Universal Algebra and Quasigroup Theory*, pages 1–56. Heldermann, Berlin, 1992.

[9] W. J. Blok and J. G. Raftery. Assertionally equivalent quasivarieties. *International Journal of Algebra and Computation*, 18(4):589–681, 2008.

[10] W. J. Blok and J. Rebagliato. Algebraic semantics for deductive systems. *Studia Logica, Special Issue on Abstract Algebraic Logic, Part II*, 74(5):153–180, 2003.

[11] S. Burris. Boolean constructions. In *Universal algebra and lattice theory (Puebla, 1982)*, volume 1004 of *Lecture Notes in Mathematics*, pages 67–90. Springer, Berlin, 1983.

[12] S. Burris and H. P. Sankappanavar. *A course in Universal Algebra*. Available online `https://www.math.uwaterloo.ca/~snburris/htdocs/ualg.html`, the millennium edition, 2012.

[13] A. Chagrov and M. Zakharyaschev. *Modal Logic*, volume 35 of *Oxford Logic Guides*. Oxford University Press, 1997.

[14] C. C. Chang and H. J. Keisler. *Model Theory*, volume 73 of *Studies in Logic*. North-Holland, Amsterdam, third edition, 1990.

[15] A. Citkin. On structurally complete superintuitionistic logics. *Soviet Mathematics Doklady*, 19:816–819, 1978.

[16] A. Citkin. Structurally complete superintuitionistic logics and primitive varieties of pseudoBoolean algebras. *Mat. Issled. Neklass. Logiki*, 98:134–151, 1987.

[17] J. Czelakowski. *Protoalgebraic logics*, volume 10 of *Trends in Logic—Studia Logica Library*. Kluwer Academic Publishers, Dordrecht, 2001.

[18] J. J. Dukarm. Morita equivalence of algebraic theories. *Colloquium Mathematicum*, 55:11–17, 1988.

[19] W. Dziobiak, A. V. Kravchenko, and P. J. Wojciechowski. Equivalents for a quasivariety to be generated by a single structure. *Studia Logica*, 91(1):113–123, 2009.

[20] L. Esakia. Topological Kripke models. *Soviet Math. Dokl.*, 15:147–151, 1974.

[21] L. Esakia. *Heyting Algebras. Duality Theory*. Springer, English translation of the original 1985 book. 2019.

[22] J. M. Font. *Abstract Algebraic Logic - An Introductory Textbook*, volume 60 of *Studies in Logic - Mathematical Logic and Foundations*. College Publications, London, 2016.

[23] J. M. Font and R. Jansana. *A general algebraic semantics for sentential logics*, volume 7 of *Lecture Notes in Logic*. A.S.L., second edition 2017 edition, 2009. First edition 1996. Electronic version freely available through Project Euclid at `projecteuclid.org/euclid.lnl/1235416965`.

[24] J. M. Font, R. Jansana, and D. Pigozzi. A survey on abstract algebraic logic. *Studia Logica, Special Issue on Abstract Algebraic Logic, Part II*, 74(1–2):13–97, 2003. With an "Update" in 91 (2009), 125–130.

[25] J. M. Font and V. Verdú. Algebraic logic for classical conjunction and disjunction. *Studia Logica, Special Issue on Algebraic Logic*, 50:391–419, 1991.

[26] P. Freyd. Algebra valued functors in general and tensor products in particular. *Colloq. Math.*, 14:89–106, 1966.

[27] V. I. Glivenko. Sur quelques points de la logique de M. Brouwer. *Academie Royal de Belgique Bulletin*, 15:183–188, 1929.

[28] V. A. Gorbunov. *Algebraic theory of quasivarieties*. Siberian School of Algebra and Logic. Consultants Bureau, New York, 1998. Translated from the Russian.

[29] A. Grzegorczyk. Undecidability of some topological theories. *Fundamenta Mathematicae*, 38:137–152, 1951.

[30] S. Jaśkowski. Recherches sur le systeme de la logique intuitioniste. In *Actes du Congrès International des Philosophie Scientifique, Paris*, volume VI, pages 58–61, 1936.

[31] M. Kracht. *Tools and techniques in modal logic*, volume 142 of *Studies in Logic and the Foundations of Mathematics*. North-Holland Publishing Co., Amsterdam, 1999.

[32] M. Kracht. *Modal consequence relations*, volume 3, chapter 8 of the Handbook of Modal Logic. Elsevier Science Inc., New York, NY, USA, 2006.

[33] A. I. Mal'cev. *The metamathematics of algebraic systems, collected papers: 1936-1967*. Amsterdam, North-Holland Pub. Co., 1971.

[34] R. McKenzie. An algebraic version of categorical equivalence for varieties and more general algebraic categories. In P. Aglianò and R. Magari, editors, *Logic and algebra*, volume 180 of *Lecture Notes in Pure and Appl. Math.*, pages 211–243. Dekker, New York, 1996.

[35] J. C. C. McKinsey. A solution of the decision problem for the Lewis systems S2 and S4, with an application to topology. *The Journal of Symbolic Logic*, 6:117–134, 1941.

[36] J. C. C. McKinsey and A. Tarski. The algebra of topology. *Annals of Mathematics*, 45:141–191, 1944.

[37] T. Moraschini. A computational glimpse at the Leibniz and Frege hierarchies. *Annals of Pure and Applied Logic*, 169(1):1–20, January 2018.

[38] T. Moraschini. A logical and algebraic characterization of adjunctions between generalized quasi-varieties. *Journal of Symbolic Logic*, 83(3):899–919, 2018.

[39] T. Moraschini. On the complexity of the Leibniz hierarchy. *Annals of Pure and Applied Logic*, 170(7):805–824, July 2019.

[40] T. Moraschini. On equational completeness theorems. *Journal of Symbolic Logic*, 2021. To appear.

[41] T. Moraschini and J. G. Raftery. On prevarieties of logic. *Algebra Universalis*, (80), 2019.

[42] T. Moraschini, J. G. Raftery, and J. J. Wannenburg. Singly generated quasivarieties and residuated stuctures. *Mathematical Logic Quarterly*, 66(2):150–172, 2020.

[43] J. G. Raftery. The equational definability of truth predicates. *Reports on Mathematical Logic*, (41):95–149, 2006.

[44] J. G. Raftery. Admissible Rules and the Leibniz Hierarchy. *Notre Dame Journal of Formal Logic*, 57(4):569–606, 2016.

[45] H. Rasiowa. *An algebraic approach to non-classical logics*, volume 78 of *Studies in Logic and the Foundations of Mathematics*. North-Holland, Amsterdam, 1974.

[46] V. V. Rybakov. Hereditarily structurally complete modals logics. *The Journal of Symbolic Logic*, 60(1):266–288, March 1995.

[47] V. V. Rybakov. *Admissibility of logical inference rules*, volume 136 of *Studies in Logic*. Elsevier, Amsterdam, etc., 1997.

[48] T. Skolem. Logisch-kombinatorische untersuchungen über die erfüllbarkeit und beweisbarkeit mathematischen sätze nebst einem theoreme über dichte mengen. *Videnskabsakademiet i Kristiania, Skrifter I*, (4):1–36, 1920. Also in "Selected Works in Logic by Th. Skolem", ed. by Jens Erik Fenstak, Scand. Univ. Books, Universitetsforlaget, Oslo, 1970, pp. 103-136. [The first section is translated in: From Frege to Gödel, van Heijenoort, Harvard Univ. Press, 1971, 252-263.].

[49] R. Wójcicki. *Theory of logical calculi. Basic theory of consequence operations*, volume 199 of *Synthese Library*. Reidel, Dordrecht, 1988.

[50] A. Wroński. On cardinalities of matrices strongly adequate for the intuitionistic propositional logic. *Reports on Mathematical Logic*, 2:55–62, 1974.

DEPARTMENT OF PHILOSOPHY, UNIVERSITY OF BARCELONA
Email address: tommaso.moraschini@ub.edu