THE LOGIC OF THE FINITE COMBS

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1. Introduction

2. Preliminaries

In this section, we review the notation and basic concepts and results that we will use throughout this paper. For an in-depth study of bi-IPC and bi-Heyting algebras, see, e.g., [12, 13, 14, 15, 16]. As a main source for universal algebra we use [1, 5].

2.1. **Sets and posets.** Given a set A, $|A|$ denotes its cardinality. ω is both the set of nonnegative integers and the first infinite ordinal, while $\mathbb{Z}^+$ denotes the set of positive integers. Given $n \in \omega$, the notation $i \leqslant n$ will always mean that $i \in \{0, \dots, n\}$. If A and B are disjoint sets, we write $A \uplus B$ for their union.

A *poset* $\mathfrak{F} = (X, \leqslant)$ is a nonempty set X equipped with a binary relation $\leqslant$ which is reflexive, transitive, and anti-symmetric. Two points $x, y \in \mathfrak{F}$ are $\leqslant$-*comparable* if $x \leqslant y$ or $y \leqslant x$. Otherwise, they are said to be $\leqslant$-*incomparable*, which we denote by $x \perp y$. When the relation $\leqslant$ is clear form the context, we shall omit the prefix "$\leqslant$", and simply say that x and y are comparable (or incomparable). If $W, U \subseteq X$ are such that every element of W is incomparable to every element of U, we write $W \perp U$. If $W = \{z\}$ is a singleton, we simply write $z \perp U$.

We write $max(U)$ for the set the *maximal* elements of U (viewed as a subposet of $\mathfrak{F}$), and if U has a *supremum*, we denote it by $sup(U)$. If, moreover, $sup(U)$ is contained in U, we call it the *maximum* (or *greatest element*) of U, and write $Max(U)$ instead. Similarly, we define $min(U)$, $inf(U)$, and $Min(U)$.

If, when restricted to U, $\leqslant$ is a *total order* (i.e., any two points in U are comparable), then U is called a $\leqslant$-*chain* (or simply a *chain*). Moreover, if both $Max(U)$ and $Min(U)$ exist, then U is a *bounded chain*. If, for some $n \in \mathbb{Z}^+$, a chain U contains exactly n many elements, we say that U is an *n-chain*.

x is said to be an *immediate predecessor* of y, denoted by $x \prec y$, if $x < y$ and no point of $\mathfrak{F}$ lies between them (i.e., if $z \in \mathfrak{F}$ is such that $x \leqslant z \leqslant y$, then either $x = z$ or $y = z$). If this is the case, we call y an *immediate successor* of x. If x has a unique immediate successor in $\mathfrak{F}$, we denote it by x^+. Given a subset $U \subseteq X$, we denote the set of points which have immediate successors in U by

$$\prec U := \{x \in \mathfrak{F} \colon \exists u \in U \ (x \prec u)\}.$$

If $U = \{u\}$, we simply write $\prec u$. When we write an expression of the form

$$(x_1 R_1 x_2 \dots R_{n-1} x_n) \in U,$$

where $n \in \omega$ and $R_1, \dots R_{n-1} \in \{\leqslant, <, \prec\}$, we mean that both $x_1 R_1 x_2 \dots R_{n-1} x_n$ and $x_1, \dots, x_n \in U$ hold. Hence, when we write $x_1 R_1 x_2 \dots R_{n-1} x_n \in U$, we only require that $x_n \in U$.

We denote the *upset generated* by U by

$$\uparrow U := \{x \in \mathfrak{F} : \exists u \in U \ (u \leqslant x)\},$$

and if $U = \uparrow U$, then U is called an *upset*. If $U = \{u\}$, we simply write $\uparrow u$ and call it a *principal upset*. We shall often make use of the following notation

$$\mathring{\uparrow} u := (\uparrow u) \setminus \{u\} = \{x \in \mathfrak{F} \colon u < x\}.$$

The notion of a *downset* and the arrow operators $\downarrow$ and $\mathring{\downarrow}$ are defined analogously. A set that is both an upset and a downset is an *updownset*. We denote the set of upsets of $\mathfrak{F}$ by $Up(\mathfrak{F})$, of downsets by $Do(\mathfrak{F})$, and of updownsets by $UpDo(\mathfrak{F})$. We will always use the convention that the "arrow operators" defined above bind stronger than other set theoretic operations. For example, the expressions $\uparrow U \smallsetminus V$ and $\mathring{\downarrow} x \cap V$ are to be read as $(\uparrow U) \smallsetminus V$ and $(\mathring{\downarrow} x) \cap V$, respectively.

When $x, y \in \mathfrak{F}$, we denote:

- $[x, y] := \uparrow x \cap \downarrow y = \{u \in \mathfrak{F} \colon x \leqslant u \leqslant y\}$;
- $[x, y[:= \uparrow x \cap \mathring{\downarrow} y = \{u \in \mathfrak{F} \colon x \leqslant u < y\}$;
- $]x, y] := \mathring{\uparrow} x \cap \downarrow y = \{u \in \mathfrak{F} \colon x < u \leqslant y\}$;
- $]x, y[:= \mathring{\uparrow} x \cap \mathring{\uparrow} y = \{u \in \mathfrak{F} \colon x < u < y\}$.

Definition 2.1. Let $[y, x]$ be a bounded chain in a poset $\mathfrak{F}$. We say that this is an *isolated chain* (in $\mathfrak{F}$) if

$$\uparrow y \setminus [y, x] = \mathring{\uparrow} x \text{ and } \downarrow x \setminus [y, x] = \mathring{\downarrow} y.$$

2.2. **Bi-intuitionistic propositional logic.** Given a formula φ, we write $\neg\varphi$ and $\sim\varphi$ as a shorthand for $\varphi \to \bot$ and $\top \leftarrow \varphi$. The *Bi-intuitionistic propositional calculus* $\mathsf{bi\text{-}IPC}$ is the least set of formulas in the language $\wedge, \vee, \to, \leftarrow, \top, \bot$ (built up from a denumerable set $Prop$ of variables) that contains IPC and the eight axioms below, and is closed under modus ponens, uniform substitutions, and the *double negation rule* (DN for short): "from φ infer $\neg \sim\varphi$".

1. $p \to \big(q \vee (p \leftarrow q)\big)$,
2. $(p \leftarrow q) \to \sim(p \to q)$,
3. $\big((p \leftarrow q) \leftarrow r\big) \to (p \leftarrow q \vee r)$,
4. $\neg(p \leftarrow q) \to (p \to q)$,
5. $\big(p \to (q \leftarrow q)\big) \to \neg p$,
6. $\neg p \to \big(p \to (q \leftarrow q)\big)$,
7. $\big((p \to p) \leftarrow q\big) \to \sim q$,
8. $\sim q \to \big((p \to p) \leftarrow q\big)$,

It turns out that $\mathsf{bi\text{-}IPC}$ is a conservative extension of IPC. Furthermore, we may identify the *classical propositional calculus* CPC with the proper extension of $\mathsf{bi\text{-}IPC}$ obtained by adding the *law of excluded middle* $p \vee \neg p$. Notably, in CPC the co-implication $\leftarrow$ is term-definable by the other connectives, since $(p \leftarrow q) \leftrightarrow (p \wedge \neg q) \in \mathsf{CPC}$. Consequently, the DN rule becomes superfluous, as it translates to "from ϕ infer ϕ".

A set of formulas L closed under the three inference rules (modus ponens, uniform substitution, and DN) is called a *super-bi-intuitionistic logic* if it contains $\mathsf{bi\text{-}IPC}$. Given a formula ϕ and a super-bi-intuitionistic logic L, we say that ϕ is a *theorem* of L, denoted by $L \vdash \phi$, if $\phi \in L$. Otherwise, write $L \nvdash \phi$. We call L *consistent* if $L \nvdash \bot$, and *inconsistent* otherwise. Given another super-bi-intuitionistic logic L', we say that L' is an *extension* of L if $L \subseteq L'$. Consistent extensions of $\mathsf{bi\text{-}IPC}$ are called *bi-intermediate logics*, and it can be shown that L is a bi-intermediate logic iff $\mathsf{bi\text{-}IPC} \subseteq L \subseteq \mathsf{CPC}$. Finally, given a set of formulas Σ, we denote by $L + \Sigma$ the least (with respect to inclusion) bi-intuitionistic logic containing $L \cup \Sigma$. If Σ is a singleton $\{\phi\}$, we simply write $L + \phi$. Given another formula ψ, we say that ϕ and ψ are *L-equivalent* if $L \vdash \phi \leftrightarrow \psi$.

2.3. **Varieties of algebras.** We denote by $\mathbb{H}, \mathbb{S}, \mathbb{I}, \mathbb{P}$, and $\mathbb{P}_{\mathrm{U}}$ the class operators of closure under homomorphic images, subalgebras, isomorphic copies, direct products, and ultraproducts, respectively. A variety V is a class of (similar) algebras closed under homomorphic images, subalgebras, and (direct) products. By Birkhoff's Theorem, varieties coincide with classes of algebras that can be axiomatized by sets of equations (see, e.g., [5, Thm. II.11.9]). The smallest variety $\mathbb{V}(\mathsf{K})$ containing a class K of algebras is called the *variety generated by* K and coincides with $\mathbb{HSP}(\mathsf{K})$. If $\mathsf{K} = \{\mathfrak{A}\}$, we simply write $\mathbb{V}(\mathfrak{A})$.

Given an algebra $\mathfrak{A}$, we denote by $Con(\mathfrak{A})$ its congruence lattice. An algebra $\mathfrak{A}$ is said to be *subdirectly irreducible*, or SI for short, (resp. *simple*) if $Con(\mathfrak{A})$ has a second least element (resp. has exactly two elements: the identity relation Id_A and the total relation A^2). Consequently, every simple algebra is subdirectly irreducible.

Given a class K of algebras, we denote by $\mathsf{K}^{<\omega}$, K_{SI}, and $\mathsf{K}_{SI}^{<\omega}$ the classes of finite members of K, SI members of K, and SI members of K which are finite, respectively. In view of the Subdirect Decomposition Theorem, if K is a variety, then $\mathsf{K} = \mathbb{V}(\mathsf{K}_{SI})$ (see, e.g., [5, Thm. II.8.6]).

Definition 2.2. A variety V is said to:

(i) be *semi-simple* if its SI members are simple;
(ii) be *locally finite* if its finitely generated members are finite;
(iii) have the *finite model property* (FMP for short) if it is generated by its finite members;
(iv) be *congruence distributive* if every member of V has a distributive lattice of congruences;
(v) have *equationally definable principal congruences* (EDPC for short) if there exists a conjunction $\Phi(x, y, z, v)$ of finitely many equations such that for every $\mathfrak{A} \in \mathsf{V}$ and all $a, b, c, d \in A$,

$$(c, d) \in \Theta^{\mathfrak{A}}(a, b) \iff \mathfrak{A} \models \Phi(a, b, c, d),$$

where $\Theta^{\mathfrak{A}}(a, b)$ is the least congruence of $\mathfrak{A}$ that identifies a and b;
(vi) be a *discriminator variety* if there exists a *discriminator term* $t(x, y, z)$ for V, i.e., a ternary term such that for every $\mathfrak{A} \in \mathsf{V}_{SI}$ and all $a, b, c \in A$, we have

$$t^{\mathfrak{A}}(a, b, c) = \begin{cases} c & \text{if } a = b, \\ a & \text{if } a \neq b. \end{cases}$$

The next result collects some of the relations between these properties.

Proposition 2.3. *If V is a variety and K a class of similar algebras, then the following conditions hold:*
(i) *if V is locally finite, then its subvarieties have the FMP;*
(ii) *V has the FMP iff $\mathsf{V} = \mathbb{V}(\mathsf{K}_{SI}^{<\omega})$;*
(iii) *if V has EDPC, then V is congruence distributive and $\mathbb{HS}(\mathsf{K}) = \mathbb{SH}(\mathsf{K})$ for all $\mathsf{K} \subseteq \mathsf{V}$;*
(iv) Jónsson's Lemma: *if $\mathbb{V}(\mathsf{K})$ is congruence distributive, then $\mathbb{V}(\mathsf{K})_{SI} \subseteq \mathbb{HSP}_{\mathrm{U}}(\mathsf{K})$;*
(v) *if V is discriminator, then it is semi-simple and it has EDPC.*

Proof. Condition (i) holds because every variety is generated by its finitely generated members (see, e.g., [1, Thm. 4.4]), while condition (ii) is an immediate consequence of the definition of the FMP together with the Subdirect Decomposition Theorem. The first part of condition (iii) was established in [11] and the second in [7]. For condition (iv), see, e.g., [5, Thm. VI.6.8]. Lastly, for the first part of condition (v) see, e.g., [5, Lem. IV.9.2(b)] and for the second [4, Exa. 6 p. 200]. $\square$

The following result provides a useful description of locally finite varieties of finite type (for a proof, see [2]).

Theorem 2.4. *A variety V of a finite type is locally finite iff*

$$\forall m \in \omega, \exists k(m) \in \omega, \forall \mathfrak{A} \in \mathsf{V}_{SI} \; \big(\mathfrak{A} \textit{ is } m\textit{-generated} \implies |A| \leqslant k(m)\big).$$

2.4. **Bi-Heyting algebras.**

Definition 2.5. A *bi-Heyting algebra* is a Heyting algebra $\mathfrak{A}$ whose order-dual is also a Heyting algebra. Equivalently, $\mathfrak{A}$ is both a Heyting and a co-Heyting algebra, i.e., $\mathfrak{A}$ is a bounded distributive lattice such that for every $a, b \in A$, there are elements $a \to b, a \leftarrow b \in A$ satisfying

$$\big(c \leqslant a \to b \iff a \wedge c \leqslant b\big) \text{ and } \big(a \leftarrow b \leqslant c \iff a \leqslant b \vee c\big),$$

for all $c \in A$. In this case, we use the abbreviations $\neg a := a \to 0$ and $\sim a := 1 \leftarrow a$.

It is well known that the class $\mathsf{bi\text{-}HA}$ of bi-Heyting algebras is a variety. The following properties of bi-Heyting algebras will be useful throughout.

Proposition 2.6. *If $\mathfrak{A} \in \mathsf{bi\text{-}HA}$ and $a, b, c \in \mathfrak{A}$, then:*

1. $a \to b = \bigvee\{d \in A : a \wedge d \leqslant b\}$,
2. $a \to b = 1 \iff a \leqslant b$,
3. $\neg a = 1 \iff a = 0$,
4. $a \wedge \neg a = 0$,
5. $a \leftarrow b = \bigwedge\{d \in A : a \leqslant d \vee b\}$,
6. $a \leftarrow b = 0 \iff a \leqslant b$,
7. $\sim a = 0 \iff a = 1$,
8. $a \vee \sim a = 1$.

Example 2.7. Here we present some standard examples of bi-Heyting algebras.

(i) Every finite Heyting algebra $\mathfrak{A}$ can be viewed as a bi-Heyting algebra, by simply defining $a \leftarrow b := \bigwedge\{d \in A \colon a \leqslant d \vee b\}$. Since this is a meet of finitely many elements, the operation $\leftarrow$ is well defined on $\mathfrak{A}$, and it can be easily shown that it satisfies the dual residuation law stated in Definition 2.5.
(ii) Every Boolean algebra $\mathfrak{A}$ can be viewed as a bi-Heyting algebra, by defining the co-implication as $a \leftarrow b := a \wedge \neg b$.
(iii) Given a poset $\mathfrak{F} = (W, \leqslant)$, then $(Up(\mathfrak{F}), \cup, \cap, \rightarrow, \leftarrow, \emptyset, W)$ is a bi-Heyting algebra, where the implications are defined by

$$U \rightarrow V := W \smallsetminus {\downarrow}(U \smallsetminus V) \text{ and } U \leftarrow V := {\uparrow}(U \smallsetminus V).$$

A *valuation* on a bi-Heyting algebra $\mathfrak{A}$ is a map $v \colon Prop \rightarrow A$, where $Prop$ is the denumerable set of propositional variables of our language. Any such valuation can be uniquely extended to a bi-Heyting algebra morphism from the term algebra to $\mathfrak{A}$. We say that a formula ϕ is *valid* on $\mathfrak{A}$, denoted by $\mathfrak{A} \models \phi$, if $v(\phi) = 1$ for all valuations v on $\mathfrak{A}$. On the other hand, if $v(\phi) \neq 1$ for some valuation v on $\mathfrak{A}$, we say that $\mathfrak{A}$ *refutes* ϕ (via v), and write $\mathfrak{A} \not\models \phi$. If K is a class of bi-Heyting algebras such that $\mathfrak{A} \models \phi$ for all $\mathfrak{A} \in \mathsf{K}$, we write $\mathsf{K} \models \phi$. Otherwise, write $\mathsf{K} \not\models \phi$.

Using the well-known Lindenbaum-Tarski construction (see, e.g., [6, 10]) we obtain the following equivalence: $\mathsf{bi\text{-}IPC} \vdash \phi$ iff $\mathsf{bi\text{-}HA} \models \phi$. This phenomenon, known as the algebraic completeness of $\mathsf{bi\text{-}IPC}$, can be extended to all other super-bi-intuitionistic logics. Let L be such a logic, and denote the *variety of* L by $\mathsf{V}_L := \{\mathfrak{A} \in \mathsf{bi\text{-}HA} \colon \mathfrak{A} \models L\}$. On the other hand, given a subvariety $\mathsf{V} \subseteq \mathsf{bi\text{-}HA}$, we denote its *logic* by $L_{\mathsf{V}} := Log(\mathsf{V}) = \{\phi \colon \mathsf{V} \models \phi\}$. Again using the standard Lindenbaum-Tarski construction, it can be shown that L is *sound* and *complete* with respect to V_L, i.e., for all formulas ϕ, we have $L \vdash \phi$ iff $\mathsf{V}_L \models \phi$. It follows that this correspondence between extensions of $\mathsf{bi\text{-}IPC}$ and subvarieties of bi-Heyting algebras is one-to-one, and therefore the following theorem can now be easily proved.

Theorem 2.8. *If L is a super-bi-intuitionistic logic, then the lattice of extensions of L is dually isomorphic to the lattice of subvarieties of* V_L. *Equivalently, if* V *is a variety of bi-Heyting algebras, then the lattice of subvarieties of* V *is dually isomorphic to the lattice of extensions of* L_{V}.

2.5. **Bi-Esakia spaces.** Given an ordered topological space $\mathcal{X}$, we denote its set of open sets by $Op(\mathcal{X})$, closed sets by $Cl(\mathcal{X})$, clopen sets by $Cp(\mathcal{X})$, clopen upsets by $CpUp(\mathcal{X})$, and closed updownsets by $ClUpDo(\mathcal{X})$.

Definition 2.9. Let $\mathfrak{F} = (W, \leqslant)$ and $\mathfrak{G} = (G, \leqslant)$ be posets. A map $f \colon W \rightarrow G$ is called a *bi-p-morphism*, denoted by $f \colon \mathfrak{F} \rightarrow \mathfrak{G}$, if it satisfies the following conditions:

- *Order preserving:* $\forall x, y \in W \ \big(x \leqslant y \implies f(x) \leqslant f(y)\big)$;
- *Up:* $\forall x \in W, \forall u \in G \ \big(f(x) \leqslant u \implies \exists y \in {\uparrow}x \ (f(y) = u)\big)$;
- *Down:* $\forall x \in W, \forall v \in G \ \big(v \leqslant f(x) \implies \exists z \in {\downarrow}x \ (f(z) = v)\big)$.

If f is moreover surjective, then we say that $\mathfrak{G}$ is a *bi-p-morphic image* of $\mathfrak{F}$ (via f), denoted by $f \colon \mathfrak{F} \twoheadrightarrow \mathfrak{G}$.

Proposition 2.10. *If* $f \colon \mathfrak{F} \rightarrow \mathfrak{G}$ *is a bi-p-morphism, then the following conditions hold:*

(i) $f[{\uparrow}x] = {\uparrow}f(x)$ *and* $f[{\downarrow}x] = {\downarrow}f(x)$, *for all* $x \in \mathfrak{F}$;
(ii) $f[max(\mathfrak{F})] \subseteq max(\mathfrak{G})$ *and* $f[min(\mathfrak{F})] \subseteq min(\mathfrak{G})$;
(iii) *if both* $Max(\mathfrak{F})$ *and* $Max(\mathfrak{G})$ *exist, then* $f\big(Max(\mathfrak{F})\big) = Max(\mathfrak{G})$ *and* f *is necessarily surjective.*

Proof. Condition (i) follows immediately from the definition of f, while the other two are direct consequences of (i). □

Definition 2.11. A triple $\mathcal{X} = (X, \tau, \leqslant)$ is a *bi-Esakia space* if it is both an Esakia and a co-Esakia space, i.e., (X, τ) is a topological space equipped with a partial order $\leqslant$, and satisfies the following conditions:

- (X,τ) is compact;
- $\forall U \in Cp(\mathcal{X}) \ \big({\uparrow}U, {\downarrow}U \in Cp(\mathcal{X})\big)$;

- *Priestley separation axiom* (PSA for short):

$$\forall x, y \in X \ \big(x \nleqslant y \implies \exists U \in CpUp(\mathcal{X}) \ (x \in U \text{ and } y \notin U)\big).$$

Given U as in the above display, we say that U *separates* x from y.

A map $f\colon \mathcal{X} \to \mathcal{Y}$ is a *bi-Esakia morphism* if it is a continuous bi-p-morphism between bi-Esakia spaces. If f is moreover bijective, then $\mathcal{X}$ and $\mathcal{Y}$ are said to be *isomorphic*, denoted by $\mathcal{X} \cong \mathcal{Y}$. Finally, we define the following operations on $CpUp(\mathcal{X})$:

$$\begin{aligned}
U \to V &:= X \smallsetminus {\downarrow}(U \smallsetminus V) = \{x \in X \colon {\uparrow}x \cap U \subseteq V\},\\
U \leftarrow V &:= {\uparrow}(U \smallsetminus V) = \{x \in X \colon {\downarrow}x \cap U \nsubseteq V\},\\
\neg U &:= U \to \varnothing = X \smallsetminus {\downarrow}U,\\
\sim U &:= X \leftarrow U = {\uparrow}(X \smallsetminus U).
\end{aligned}$$

Example 2.12. Every finite poset can be viewed as a bi-Esakia space, when equipped with the discrete topology. In fact, since (bi-)Esakia spaces are Hausdorff, this is the only way to view a finite poset as a bi-Esakia space. Furthermore, since maps between spaces equipped with the discrete topology are always continuous, it follows that every bi-p-morphism between finite posets is a bi-Esakia morphism.

The celebrated Esakia duality restricts to a duality between the category of bi-Heyting algebras and bi-Heyting morphisms, and that of bi-Esakia spaces and bi-Esakia morphisms [8] (for a proof, see [12]). Here we just recall the main constructions establishing this duality. Given a bi-Heyting algebra $\mathfrak{A}$, we denote its *bi-Esakia dual* by $\mathfrak{A}_* := (A_*, \tau, \subseteq)$, where A_* is the set of prime filters of $\mathfrak{A}$ and τ is the topology generated by the subbasis

$$\{\varphi(a) \colon a \in A\} \cup \{A_* \smallsetminus \varphi(a) \colon a \in A\},$$

where $\varphi(a) := \{F \in A_* \colon a \in F\}$. Moreover, it can be shown that $CpUp(\mathfrak{A}_*) = \{\varphi(a) \colon a \in A\}$. Furthermore, if $f\colon \mathfrak{A} \to \mathfrak{B}$ is a bi-Heyting morphism, then its dual is the restricted inverse image map $f_* := f^{-1}[-]\colon \mathfrak{B}_* \to \mathfrak{A}_*$.

Conversely, given a bi-Esakia space $\mathcal{X}$ we denote its *bi-Heyting* (or *algebraic*) *dual* by $\mathcal{X}^* := (CpUp(\mathcal{X}), \cup, \cap, \to, \leftarrow, \varnothing, X)$, and if $f\colon \mathcal{X} \to \mathcal{Y}$ is a bi-Esakia morphism, then its dual is the restricted inverse image map $f^* := f^{-1}[-]\colon \mathcal{Y}^* \to \mathcal{X}^*$. We note that $\mathfrak{A}$ and $(\mathfrak{A}_*)^*$ are isomorphic as bi-Heyting algebras, while $\mathcal{X}$ and $(\mathcal{X}^*)_*$ are isomorphic as bi-Esakia spaces.

The following result collects some useful properties of bi-Esakia spaces.

Proposition 2.13. *The following conditions hold for a bi-Esakia space $\mathcal{X}$:*

(i) *$\mathcal{X}$ is Hausdorff;*
(ii) *$\mathcal{X}$ is 0-dimensional, that is, it has a clopen basis;*
(iii) *if $Z \in Cl(\mathcal{X})$, then ${\downarrow}Z, {\uparrow}Z \in Cl(\mathcal{X})$. Consequently, ${\downarrow}x, {\uparrow}x \in Cl(\mathcal{X})$ for all $x \in X$;*
(iv) *$min(\mathcal{X})$ is a closed set;*
(v) *if $x \in X$, then there are $y \in min(\mathcal{X})$ and $z \in max(\mathcal{X})$ satisfying $y \leqslant x \leqslant z$;*
(vi) *if $x, y \in X$ and $x < y$, then there exists a gap between x and y, that is, there are $a, b \in X$ such that $x \leqslant a \prec b \leqslant y$;*
(vii) *if $C \subseteq X$ is a chain, then it has both an infimum and a supremum in $\mathcal{X}$, and the closure of C is a bounded chain;*
(viii) *$\neg \sim U = \{x \in X \colon {\downarrow}{\uparrow}x \subseteq U\}$, for all $U \in CpUp(\mathcal{X})$.*

Proof. The results stated in the first four conditions are either well-known results for Esakia spaces, or their order-dual versions (for co-Esakia spaces). For a proof of the former, see [9]. We now prove (v). By spelling out the definition of $\neg \sim U$, we have

$$\neg \sim U = X \smallsetminus {\downarrow}{\uparrow}(X \smallsetminus U) = \big\{x \in X \colon \forall y \in X \ \big(\exists z \in X \smallsetminus U \ (z \leqslant y) \implies x \nleqslant y\big)\big\}.$$

Suppose that $x \in \neg \sim U$ and let $u \in {\downarrow}{\uparrow}x$, so there exists $v \in {\uparrow}x$ such that $u \leqslant v$. By the equality above, $x \leqslant v$ entails that for all $z \in X \smallsetminus U$, we have $z \nleqslant v$. Hence u must be in U and we conclude ${\downarrow}{\uparrow}x \subseteq U$. Thus, $\neg \sim U \subseteq \{x \in X \colon {\downarrow}{\uparrow}x \subseteq U\}$.

To prove the right to left inclusion, suppose $\downarrow\uparrow x \subseteq U$, for some $x \in X$. Let $y \in X$ be such that there exists a $z \in (X \smallsetminus U) \cap \downarrow y$. If $x \leqslant y$, then we would have $z \in \downarrow\uparrow x \subseteq U$, a contradiction. Thus $x \nleqslant y$, and we conclude $x \in \neg\sim U$, as desired. □

Throughout, when we refer to condition (iv) above, we will simply say "a bi-Esakia space has enough gaps".

Next we define the three standard methods of generating new bi-Esakia spaces from old ones. Let $\mathcal{X} = (X, \tau, R)$, $\mathcal{Y} = (Y, \pi, S)$, $\mathcal{X}_1 = (X_1, \tau_1, R_1), \ldots, \mathcal{X}_n = (X_n, \tau_n, R_n)$ be bi-Esakia spaces. We say that:

(i) $\mathcal{Y}$ is a *bi-generated subframe* of $\mathcal{X}$ if $Y \in ClUpDo(\mathcal{X})$, π is the subspace topology, and $S = Y^2 \cap R$;
(ii) $\mathcal{Y}$ is a *bi-Esakia (morphic) image* of $\mathcal{X}$, denoted by $\mathcal{X} \twoheadrightarrow \mathcal{Y}$, if there exists a surjective bi-Esakia morphism from $\mathcal{X}$ onto $\mathcal{Y}$;
(iii) $\mathcal{X} = \biguplus_{i=1}^{n} \mathcal{X}_i$ is the *disjoint union* of the collection $\{\mathcal{X}_1, \ldots, \mathcal{X}_n\}$ if (X, R) is the disjoint union $\biguplus_{i=1}^{n}(X_i, R_i)$ of the various posets and (X, τ) is the topological sum of the (X_i, τ_i).

As is the case with the analogous notions for Esakia spaces, the above definitions can be translated (using the bi-Esakia duality) into the terminology of bi-Heyting algebras (for a proof, see [12]).

Proposition 2.14. *Let $\{\mathfrak{A}, \mathfrak{B}\} \cup \{\mathfrak{A}_1, \ldots, \mathfrak{A}_n\}$ and $\{\mathcal{X}_1, \ldots, \mathcal{X}_n\}$ be finite sets of bi-Heyting algebras and bi-Esakia spaces, respectively. The following conditions hold:*

(i) *$\mathfrak{B}$ is a homomorphic image of $\mathfrak{A}$ iff $\mathfrak{B}_*$ is (isomorphic to) a bi-generated subframe of $\mathfrak{A}_*$;*
(ii) *$\mathfrak{B}$ is (isomorphic to) a subalgebra of $\mathfrak{A}$ iff $\mathfrak{B}_*$ is a bi-Esakia morphic image of $\mathfrak{A}_*$;*
(iii) *$(\prod_{i=1}^{n} \mathfrak{A}_i)_* \cong \biguplus_{i=1}^{n} \mathfrak{A}_{i*}$ and $(\biguplus_{i=1}^{n} \mathcal{X}_i)^* \cong \prod_{i=1}^{n} \mathcal{X}_i^*$.*

Let $\mathcal{X}$ be a bi-Esakia space. A map $V \colon Prop \to CpUp(\mathcal{X})$ is called a *valuation* on $\mathcal{X}$, and the pair $\mathfrak{M} := (\mathcal{X}, V)$ a *bi-Esakia model* (on $\mathcal{X}$). Moreover, we define the *validity* of a formula ϕ in $\mathcal{X}$ by $\mathcal{X} \models \phi$ iff $\mathcal{X}^* \models \phi$. In other words, $V(\phi) = X$, for all valuations V on $\mathcal{X}$. Otherwise, write $\mathcal{X} \not\models \phi$. Since the validity of a formula is preserved under taking homomorphic images, subalgebras, and direct products of bi-Heyting algebras, it follows from the previous proposition that the validity of a formula is preserved under taking bi-generated subframes, bi-Esakia morphic images, and finite disjoint unions of bi-Esakia spaces.

Finally, we review the Coloring Theorem, a result that provides a characterization of the finitely generated bi-Heyting algebras using properties of their bi-Esakia duals. To this end, we need to define the notions of bi-bisimulation equivalences and colorings on bi-Esakia spaces.

Definition 2.15. Let $\mathcal{X} = (X, \tau, \leqslant)$ be a bi-Esakia space and E an equivalence relation on X. We say that E is a *bi-bisimulation equivalence* on $\mathcal{X}$ if it satisfies the following conditions:

- *Up:* $\forall w, w', v' \in X \left(wEw' \text{ and } w' \leqslant v' \implies \exists v \in [\![v']\!]_E \ (w \leqslant v)\right)$;
- *Down:* $\forall w, w', u' \in X \left(wEw' \text{ and } u' \leqslant w' \implies \exists u \in [\![u']\!]_E \ (u \leqslant w)\right)$;
- *Refined:* Any two non-E-equivalent elements of X are *separated* by an *E-saturated* clopen upset, that is, for every $w, v \in X$, if $(w, v) \notin E$ then there exists $U \in CpUp(\mathcal{X})$ such that $E[U] = \{x \in X \colon \exists u \in U \ (uEx)\} = U$, and exactly one of w and v is contained in U.

We call a bi-bisimulation equivalence E on $\mathcal{X}$ *trivial* if $E = X^2$, and *proper* otherwise.

Let $\mathfrak{M} = (\mathcal{X}, V)$ be a bi-Esakia model and $p_1, \ldots, p_n \in Prop$ a finite number of fixed distinct propositional variables. To every point $w \in \mathfrak{M}$, we associate the sequence $col(w) := i_1 \ldots i_n$ defined by

$$i_k := \begin{cases} 1 & \text{if } w \models p_k, \\ 0 & \text{if } w \not\models p_k, \end{cases}$$

for $k \in \{1, \ldots, n\}$. We call $col(w)$ the *color* of w (relative to $p_1, \ldots, p_n$), and call a valuation $V \colon \{p_1, \ldots, p_n\} \to CpUp(\mathcal{X})$ a *coloring* of $\mathfrak{M}$.

Now, note that thinking of a bi-Heyting algebra $\mathfrak{A}$ endowed with some fixed elements $a_1, \ldots, a_n$ is the same as equipping $\mathfrak{A}$ with a valuation $v \colon \{p_1, \ldots, p_n\} \to A$ defined by $v(p_i) :=$

a_i, for $i \leqslant n$. The dual of the tuple $(\mathfrak{A}, v)$ is the bi-Esakia model $\mathfrak{M} = (\mathfrak{A}_*, V)$, where V satisfies $V(p_i) = \varphi(a_i) = \{x \in A_* : g_i \in x\}$, and this valuation clearly defines a coloring of $\mathfrak{M}$.

The following theorem can be proven using the same argument as the analogous result for Heyting algebras (see, e.g., [3, Thm. 3.1.5]). Notice the use of the notation $\mathfrak{A} = \langle a_1, \ldots, a_n \rangle$ for "$\mathfrak{A}$ is generated (as a bi-Heyting algebra) by $\{a_1, \ldots, a_n\}$".

Theorem 2.16 (Coloring Theorem). *Let $\mathfrak{A}$ be a bi-Heyting algebra, $a_1, \ldots, a_n \in A$ a finite number of fixed elements, and $(\mathcal{X}, V)$ the corresponding colored bi-Esakia model. Then $\mathfrak{A} = \langle a_1, \ldots, a_n \rangle$ iff every proper bi-bisimulation equivalence E on $\mathcal{X}$ identifies points of different colors.*

Lemma 2.17. *If $\mathfrak{A}$ is an n-generated bi-Heyting algebra, for some $n \in \mathbb{Z}^+$, and $H \subseteq \mathfrak{A}_*$ is an isolated chain, then $|H| \leqslant 2^n$.*

2.6. **Bi-Gödel Algebras and their Duals.** The *bi-intuitionistic linear calculus* (or *bi-Gödel-Dummett's logic*) is the bi-intermediate logic

$$\mathsf{bi\text{-}LC} := \mathsf{bi\text{-}IPC} + (p \to q) \vee (q \to p).$$

This terminology hints that $\mathsf{bi\text{-}LC}$ is a bi-intuionistic analogue of the linear calculus LC, the intermediate logic axiomatized by the same axiom over IPC. In view of Theorem 2.8, $\mathsf{bi\text{-}LC}$ is algebraized by the variety

$$\mathsf{bi\text{-}GA} := \mathsf{V}_{\mathsf{bi\text{-}LC}} = \{\mathfrak{A} \in \mathsf{bi\text{-}HA} : \mathfrak{A} \models (p \to q) \vee (q \to p)\},$$

whose elements are be called *bi-Gödel algebras*. Furthermore, there exists a dual isomorphism between the lattice $\Lambda(\mathsf{bi\text{-}LC})$ of consistent extensions of $\mathsf{bi\text{-}LC}$ and that of nontrivial subvarieties of $\mathsf{bi\text{-}GA}$. Recall that a *co-tree* is a poset with a greatest element (called the *co-root*) and whose principal upsets are chains, and that a disjoint union of co-trees is called a *co-forest*. Moreover, a *bi-Esakia co-forest* (respectively, *bi-Esakia co-tree*) is a bi-Esakia space whose underlying poset is a co-forest (respectively, co-tree).

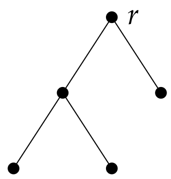

FIGURE 1. A co-tree.

Theorem 2.18. *Let $\mathfrak{A} \in \mathsf{bi\text{-}HA}$. Then $\mathfrak{A}$ is a bi-Gödel algebra iff $\mathfrak{A}_*$ is a bi-Esakia co-forest. Moreover, $\mathfrak{A}$ is an SI bi-Gödel algebra iff $\mathfrak{A}_*$ is a bi-Esakia co-tree.*

Let $\mathfrak{A}$ be a finite and SI bi-Gödel algebra. Then the *Jankov formula of* $\mathfrak{A}$ is given by

$$\mathcal{J}(\mathfrak{A}) := \neg{\sim}\Gamma \to \neg \bigwedge \{p_a \leftarrow p_b : (a,b) \in A^2 \text{ and } a \nleq b\},$$

where

$$\begin{aligned}\Gamma := &\bigwedge\{p_{a\vee b} \leftrightarrow (p_a \vee p_b) : (a,b) \in A^2\} \wedge \bigwedge\{p_{a\wedge b} \leftrightarrow (p_a \wedge p_b) : (a,b) \in A^2\} \wedge \\ &\bigwedge\{p_{a\to b} \leftrightarrow (p_a \to p_b) : (a,b) \in A^2\} \wedge \bigwedge\{p_{a\leftarrow b} \leftrightarrow (p_a \leftarrow p_b) : (a,b) \in A^2\} \wedge \\ &\wedge \{p_0 \leftrightarrow \bot\} \wedge \{p_1 \leftrightarrow \top\}.\end{aligned}$$

Lemma 2.19. *(Jankov lemma) If $\mathfrak{B} \in \mathsf{bi\text{-}GA}$ and $\mathfrak{A} \in \mathsf{bi\text{-}GA}^{<\omega}_{SI}$, then $\mathfrak{B} \not\models \mathcal{J}(\mathfrak{A})$ iff $\mathfrak{A} \in \mathbb{SH}(\mathfrak{B})$. If $\mathfrak{B}$ is moreover SI, then $\mathfrak{B} \not\models \mathcal{J}(\mathfrak{A})$ iff $\mathfrak{A} \in \mathbb{S}(\mathfrak{B})$.*

Throughout this paper, the most common usage of the Jankov Lemma takes the form of the following corollary. Notice the use of the notation $\mathcal{J}(\mathcal{Y})$ as a shorthand for $\mathcal{J}(\mathcal{Y}^*)$.

Corollary 2.20. *Let $\mathcal{X}$ and $\mathcal{Y}$ be bi-Esakia co-trees. If $\mathcal{Y}$ is finite, then $\mathcal{X} \not\models \mathcal{J}(\mathcal{Y})$ iff $\mathcal{Y}$ is a bi-Esakia image of $\mathcal{X}$.*

Given a finite and SI bi-Gödel algebra $\mathfrak{A}$, we denote the *subframe formula of* $\mathfrak{A}$ by

$$\beta(\mathfrak{A}) := \neg \sim\Gamma \to \neg \bigwedge \{p_a \leftarrow p_b \colon (a,b) \in A^2 \text{ and } a \nleqslant b\},$$

where

$$\Gamma := \bigwedge\{p_{a\vee b} \leftrightarrow (p_a \vee p_b) \colon (a,b) \in A^2\} \wedge \bigwedge\{p_{a\leftarrow b} \leftrightarrow (p_a \leftarrow p_b) \colon (a,b) \in A^2\}.$$

Before we present a version of the Jankov Lemma for subframe formulas, recall that an *order-embedding* is an order-invariant map between posets.

Lemma 2.21. *(Subframe Jankov Lemma) If* $\mathfrak{B} \in \mathsf{bi\text{-}GA}$ *and* $\mathfrak{A} \in \mathsf{bi\text{-}GA}_{SI}^{<\omega}$*, then* $\mathfrak{B} \not\models \beta(\mathfrak{A})$ *iff* $\mathfrak{A}_*$ *order-embeds into* $\mathfrak{B}_*$*.*

3. The Logic of the Finite Combs

3.1. **The Combs and the Axioms.** For each positive integer n, we define the *n-comb* $\mathfrak{C}_n := (C_n, \leqslant)$ as the finite bi-Esakia co-tree depicted in Figure 2.

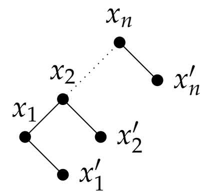

FIGURE 2. The n-comb $\mathfrak{C}_n$.

We denote the class of finite combs by $FC := \{\mathfrak{C}_n \colon n \in \mathbb{Z}^+\}$, and its logic by $Log(FC) := \{\varphi \colon \forall n \in \mathbb{Z}^+ \ (\mathfrak{C}_n \models \varphi)\}$. The next results make clear why the finite combs are of interest.

Theorem 3.1. *An extension L of* $\mathsf{bi\text{-}LC}$ *is locally tabular iff* $\mathcal{J}(\mathfrak{C}_n^*) \in L$*, for some positive integer n.*

Corollary 3.2. *An extension L of* $\mathsf{bi\text{-}LC}$ *is locally tabular iff* $L \nsubseteq Log(FC)$*. Consequently,* $Log(FC)$ *is the only pre-locally tabular extension of* $\mathsf{bi\text{-}LC}$*.*

The main goal of this paper is to prove that the logic of the finite combs is not only finitely axiomatizable, but it also enjoys the FMP. As a consequence of this, the problem of determining if a given variety of bi-Gödel algebras is locally finite is decidable.

To achieve this goal, we will prove that $Log(FC)$ coincides with the logic

$$L := \mathsf{bi\text{-}LC} + \beta_1 + \neg\beta_2^\partial + \mathcal{J}_1 + \mathcal{J}_2 + \mathcal{J}_3,$$

where and $\mathcal{J}_1 := \mathcal{J}(\mathfrak{F}_1)$, $\mathcal{J}_2 := \mathcal{J}(\mathfrak{F}_2)$, $\mathcal{J}_3 := \mathcal{J}(\mathfrak{F}_3)$, $\beta_1 := \beta(\mathfrak{F}_4)$, and

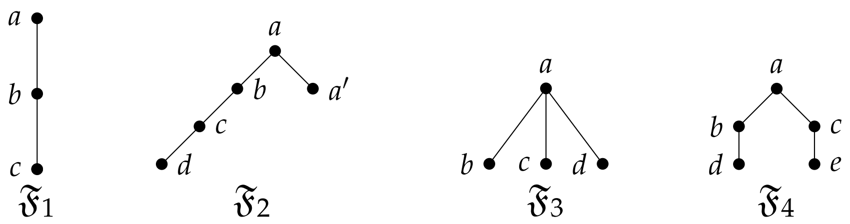

FIGURE 3. The co-trees $\mathfrak{F}_1, \mathfrak{F}_2, \mathfrak{F}_3$, and $\mathfrak{F}_4$.

$$\beta_2^\partial := \bigvee_{i=0}^{2} \left[(q_i \leftarrow \bigwedge_{j\neq i} q_j) \leftarrow \bigwedge_{j\neq i} q_j\right] \leftarrow \bigwedge_{i=0}^{2} q_i.$$

The formula $\neg\beta_2^\partial$, henceforth called the *axiom of 2-bounded branching*, enjoys the following property:

Proposition 3.3. *If $\mathfrak{F}$ is a finite co-tree, then $\mathfrak{F} \models \neg\beta_2^\partial$ iff $|{\prec}x| \leqslant 2$, for all $x \in \mathfrak{F}$.*

Proof. Just dualize the proof of the analogous result for trees (see [Gabbay]). □

Proposition 3.4. *If $\mathcal{X}$ is a bi-Esakia co-tree, then $\mathcal{X} \models \neg\beta_2^\partial$ implies $|{\prec}x| \leqslant 2$, for all $x \in \mathcal{X}$.*

Proof. Let $\mathcal{X}$ be a bi-Esakia co-tree such that $\mathcal{X} \models \neg\beta_2^\partial$, and take some $x \in \mathcal{X}$. Suppose, with a view towards contradiction, that ${\prec}x$ contains three distinct elements, y_0, y_1, y_2. By the PSA, for each $i \leqslant 2$ there exists a clopen upset U_i satisfying $U_i \cap \{y_0, y_1, y_2\} = \{y_i\}$. Let V be a valuation on $\mathcal{X}$ such that $V(q_i) = U_i$, for each $i \leqslant 2$.

As $\mathcal{X} \models \neg\beta_2^\partial$ by hypothesis, we must have $(\mathcal{X}, V), x \not\models \beta_2^\partial$, so $y_0 < x$ and the fact that $y_0 \not\models \bigwedge_{i=0}^2 q_i$ entail

$$y_0 \not\models \bigvee_{i=0}^{2} [(q_i \leftarrow \bigwedge_{j\neq i} q_j) \leftarrow \bigwedge_{j\neq i} q_j].$$

In particular, $y_0 \not\models (q_0 \leftarrow q_1 \wedge q_2) \leftarrow q_1 \wedge q_2$. But now, since $y_0 \not\models q_1 \wedge q_2$, this implies $y_0 \not\models q_0$, contradicting our assumption on the valuation V. □

Our proof that $Log(FC) = L$ will consist of the following: first we show that not only does $Log(FC)$ extend L, but also that the finite SI bi-Gödel algebras which validate L coincide with those which validate $Log(FC)$; after unravelling the intricate structure of the (dual spaces of) finitely generated SI algebras in V_L, we use their properties to prove that L enjoys the FMP; finally, by combining these facts, we conclude that indeed $Log(FC) = L$.

We need a technical lemma.

Lemma 3.5. *Let $f\colon \mathfrak{F} \to \mathfrak{G}$ be a bi-p-morphism between co-trees. Then the image of the co-root of $\mathfrak{F}$ is the co-root of $\mathfrak{G}$, and if $x \prec y \in \mathfrak{F}$, then either $f(x) = f(y)$ or $f(x) \prec f(y)$.*

Proof. The first part of the statement is an immediate consequence of Proposition 2.10.(iii). Assume $x \prec y \in \mathfrak{F}$. As bi-p-morphisms are order preserving, $x \prec y$ entails $f(x) \leqslant f(y)$. If $f(x) = f(y)$ we are done, so let us suppose that $f(x) < f(y)$. Suppose as well that $f(x) \leqslant u \leqslant f(y)$, for some $u \in \mathfrak{G}$. By the up condition (see the Definition 2.9), there exists $z \in {\uparrow}x$ satisfying $f(z) = u$. Notice that ${\uparrow}x = \{x\} \uplus {\uparrow}y$, since $x \prec y$ and the principal upsets of $\mathfrak{F}$ are chains. If $z = x$, then $f(x) = f(z) = u$. If $z \in {\uparrow}y$, then $f(y) \leqslant f(z) = u \leqslant f(y)$, and thus $f(y) = u$. We conclude $f(x) \prec f(y)$. □

Proposition 3.6. *$Log(FC)$ extends the logic L.*

Proof. By the definitions of these logics, it suffices to show that every finite comb validates the axioms of L. Let n be a positive integer, and consider the n-comb $\mathfrak{C}_n$. By the aforementioned defining property of the axiom of 2-bounded branching, it is clear that $\mathfrak{C}_n \models \neg\beta_2^\partial$. Moreover, by the Subframe Jankov Lemma 2.21, it is easy to see that $\mathfrak{C}_n \models \beta_1$.

We now prove that $\mathfrak{C}_n \models \mathcal{J}_1$. Let us suppose otherwise, so by the Jankov Lemma 2.20, there exists a surjective bi-Esakia morphism $f\colon \mathfrak{C}_n \twoheadrightarrow \mathfrak{F}_1$. By Lemma 3.5, we must have $f(x_n') \in \{a, b\}$. But this is already a contradiction, since x_n' is a minimal point of $\mathfrak{C}_n$, so it must be mapped to a minimal point of $\mathfrak{F}_1$ (see Proposition 2.10.(ii)), and neither a nor b are minimal in $\mathfrak{F}_1$. Thus, $\mathfrak{C}_n \models \mathcal{J}_1$.

To see that $\mathfrak{C}_n \models \mathcal{J}_2$, we again assume otherwise. By the Jankov Lemma 2.20, there exists a surjective bi-Esakia morphism $f\colon \mathfrak{C}_n \twoheadrightarrow \mathfrak{F}_2$. Notice that this assumption forces $n > 2$. As f is surjective and b is not a minimal point, the fact that minimal points must be mapped to minimal points, together with Lemma 3.5, implies that the set $\{x_i \in \mathfrak{C}_n : i < n \text{ and } f(x_i) = b\}$ is nonempty. Now, we simply take x_i belonging to this set, and note that aforementioned lemma forces $f(x_i') \in \{b, c\}$, a contradiction, since x_i' is minimal but neither b nor c are so. It follows that $\mathfrak{C}_n \models \mathcal{J}_2$.

Finally, let us suppose that $\mathfrak{C}_n \not\models \mathcal{J}_3$ and arrive at a contradiction. By the Jankov Lemma 2.20, our assumption yields a surjective bi-Esakia morphism $f\colon \mathfrak{C}_n \twoheadrightarrow \mathfrak{F}_3$. Since the image of x_n is necessarily a, the set $\{x_i \in \mathfrak{C}_n : f(x_i) = a\}$ is nonempty, and therefore has a least element, by the structure of $\mathfrak{C}_n$. Let x_m be this least element, and note that its definition forces

$|f[\downarrow x_m] \cap \{b, c, d\}| \leqslant 2$. But the down condition of bi-p-morphisms (see Definition 2.9) applied to $f(x_m) = a$ implies $|f[\downarrow x_m] \cap \{b, c, d\}| = 3$, a contradiction. We conclude $\mathfrak{C}_n \models \mathcal{J}_3$, thus finishing the proof that $L \subseteq Log(FC)$. $\square$

Next we show that the finite SI algebras which validate L coincide with those which validate $Log(FC)$. As $L \subseteq Log(FC)$ was established above, it is clear that $\mathsf{V}_{FC} \subseteq \mathsf{V}_L$ (where $V_{FC} := \mathsf{V}_{Log(FC)}$), hence $(\mathsf{V}_{FC})^{<\omega}_{SI} \subseteq (\mathsf{V}_L)^{<\omega}_{SI}$ follows.

To prove the reverse inclusion, we need to define yet another notable class of finite bi-Esakia co-trees, the class $F\c{C}$ of *finite combs with handle* (*finite hcombs*, for short). For each integer n, we define the *n-comb with handle* (*n-hcomb*, for short) $\mathfrak{C}'_n := (C'_n, \leqslant)$ as the finite bi-Esakia co-tree depicted in Figure 4. Notice that the 0-hcomb is just a singleton.

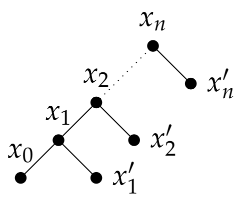

FIGURE 4. The n-hcomb $\mathfrak{C}'_n$.

Notice that for each integer n, there is a clear surjective bi-p-morphism from the $(n+1)$-comb $\mathfrak{C}_{n+1}$ to the n-hcomb $\mathfrak{C}'_n$. Equivalently, by the bi-Esakia duality, the algebraic dual of $\mathfrak{C}'_n$ is a subalgebra of the dual of $\mathfrak{C}_n$. It follows that the algebraic duals of the finite hcombs (which are all SI bi-Gödel algebras, by Theorem 2.18) are all contained in $(\mathsf{V}_{FC})^{<\omega}_{SI}$.

Lemma 3.7. *If $\mathfrak{A} \in (\mathsf{V}_L)^{<\omega}_{SI}$, then $\mathfrak{A}_*$ is either a finite comb, or a finite hcomb.*

Proof. Let $\mathfrak{A} \in (\mathsf{V}_L)^{<\omega}_{SI}$ and suppose that $\mathcal{X} := \mathfrak{A}_*$ is not a finite comb. As $\mathcal{X}$ is a finite co-tree, it has finite depth $d \in \mathbb{Z}^+$. If $d = 1$, then $\mathcal{X}$ is a singleton, and therefore can be identified with $\mathfrak{C}'_0$, a finite hcomb.

Suppose now that $d > 1$, and denote the co-root of $\mathcal{X}$ by y_1. Our assumption $d > 1$ clearly entails that y_1 is not minimal, so the fact that $\mathcal{X}$ is finite now yields that $\prec x_1$ is nonempty. Note that if y_1 has exactly one immediate predecessor, y_2, then this point must be minimal, otherwise we could define a clear surjective bi-p-morphism from $\mathcal{X}$ onto $\mathfrak{F}_1$, contradicting our assumption that $\mathcal{X} \models \mathcal{J}_1$. But then $X = \{y_1, y_2\}$ can be identified with the 1-comb, contradicting our assumption. Therefore, y_1 must have more than one immediate predecessor, and since we assumed $\mathcal{X} \models \neg\beta^{\partial}_2$, we conclude that it has exactly two. Denote $\prec y_1 = \{y'_1, y_2\}$. If $X = \{y_1, y'_1, y_2\}$ (equivalently, if $d = 2$), we are done, since $\mathcal{X}$ can be identified with the 1-hcomb $\mathfrak{C}'_1$.

Let us now suppose that $d > 2$.

Claim 3.7.1. *For every $y \in X$, if $dp(y) < d - 1$, then either $y \in \min(\mathcal{X})$ or y has exactly two immediate predecessors, only one of which is minimal.*

Proof of Claim. We prove this by strong induction on $i < d - 1$. Notice that the base case was established above, and that if $d = 3$, we only needed to prove the base case. Accordingly, we assume that $d > 3$ and take some $y \in \mathcal{X}$ satisfying $1 < dp(y) = i < d - 1$. Moreover, by our induction hypothesis (i.e., the assumption that every point in $\mathcal{X}$ whose depth is less than i must be either minimal, or have exactly two immediate predecessors, only one of which is minimal), we know that $\mathcal{X}$ looks like the poset represented in Figure 5, where y_i is not minimal.

As there are exactly two points in $\mathcal{X}$ which have depth i, namely, y_i and y'_{i-1}, our assumption that $dp(y) = i$ forces y to be one of the aforementioned points. If y is minimal, i.e., if $y = y'_{i-1}$, we are done, so let us suppose that y is not minimal, i.e., that $y = y_i$. We will now prove that y_i has exactly two immediate predecessors, only one of which is minimal. Since y_i is not minimal and $\mathcal{X}$ is finite, y_i must have an immediate predecessor, z. To see that this point cannot be the

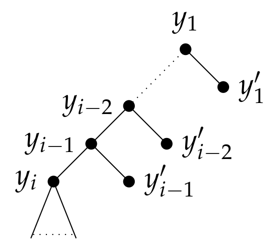

FIGURE 5. The poset $\mathcal{X}$.

sole element of $\prec y_i$, let us assume otherwise. As $dp(z) = dp(y_i) + 1 = i + 1 < d = dp(\mathcal{X})$, we can infer from the known structure of $\mathcal{X}$ (see Figure 5) that z cannot be minimal, i.e., that $\downarrow z \setminus \{z\} \neq \emptyset$. But now there is a clear surjective bi-p-morphism $f\colon \mathcal{X} \twoheadrightarrow \mathfrak{F}_2$, defined by $f[\uparrow y_{i-1}] := \{a\}$, $f[min(\mathcal{X}) \cap (\downarrow y_1 \setminus \downarrow y_i)] := \{a'\}$, $f(y_i) := b$, $f(z) := c$, and $f[\downarrow z \setminus \{z\}] := \{d\}$, contradicting $\mathcal{X} \models \mathcal{J}_2$. Therefore, z cannot be the only immediate predecessor of y_i, and since we assumed $\mathcal{X} \models \neg\beta_2^\partial$, we must have $\prec y_i = \{z, u\}$, for some $u \in \mathcal{X}$.

Finally, let us note that $dp(z) = dp(u) = dp(y_i) + 1 = i + 1 < d$, so by the known structure of $\mathcal{X}$ (see Figure 5), at least one point in $\prec y_i$ must be non-minimal. Denote this point by y_{i+1}, and notice that if the other immediate predecessor of y_i distinct from y_{i+1}, which we will denote by y'_i, was also non-minimal, we could clearly define an order-embedding from $\mathfrak{F}_4$ into $\mathcal{X}$, contradicting $\mathcal{X} \models \beta_1$. We conclude that indeed y_i has exactly two immediate predecessors, only one of which is minimal, as desired □

By the Claim, we now know that $\mathcal{X}$ looks like the poset represented in Figure 6, where y_{d-1} is not minimal and has depth $d-1$.

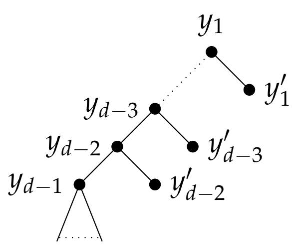

FIGURE 6. The poset $\mathcal{X}$.

As y_{d-1} is not minimal and $\mathcal{X}$ is finite, it must have immediate predecessors, which will have depth $d = dp(\mathcal{X})$, and are therefore necessarily minimal. Since we assumed that $\mathcal{X}$ is not a finite comb, y_{d-1} must have more than one immediate predecessor (otherwise we could identify $\mathcal{X}$ with $\mathfrak{C}_{d-1}$), so $\mathcal{X} \models \neg\beta_2^\partial$ now implies that y_{d-1} has exactly two such predecessors. It is now clear that we can identify $\mathcal{X}$ with $\mathfrak{C}'_{d-1}$, a finite hcomb, as desired. □

Proposition 3.8. $(\mathsf{V}_{FC})_{SI}^{<\omega} = (\mathsf{V}_L)_{SI}^{<\omega}$*, and the dual space of any algebra in this class is either a finite comb, or a finite hcomb.*

Proof. As previously discussed, the left to right inclusion is immediate from the fact $L \subseteq Log(FC)$ proved in Proposition 3.6. The reverse inclusion and the second part of the statement follow from the previous lemma, since both the finite combs and the finite combs with handle are contained in $(\mathsf{V}_{FC})_{SI}^{<\omega}$. □

With our first goal fulfilled, we note that proving that L enjoys the FMP now suffices to establish the equality $L = Log(FC)$, because in this case, the following holds true:

$$\mathsf{V}_{FC} \subseteq V_L = \mathbb{V}((\mathsf{V}_L)_{SI}^{<\omega}) = \mathbb{V}((\mathsf{V}_{FC})_{SI}^{<\omega}) \subseteq \mathsf{V}_{FC}.$$

3.2. **The Finitely Generated SI Algebras of V_{FC}.** Our strategy to prove that L has the FMP consists in characterizing the bi-Esakia duals of the finitely generated SI algebras of V_L, and subsequently use this robust characterization to develop a method which from an arbitrary formula φ and an arbitrary infinite finitely generated $\mathfrak{A} \in (\mathsf{V}_L)_{SI}$ refuting φ, it constructs a finite comb or hcomb that also refutes φ.

Throughout this subsection, we work with a fixed but arbitrary infinite bi-Esakia co-tree $\mathcal{X}$ which validates L, whose algebraic dual is n-generated as a bi-Heyting algebra, for some $n \in \mathbb{Z}^+$. We denote the co-root of $\mathcal{X}$ by r. By our previous discussion on colored bi-Esakia spaces, we have some fixed propositional variables $p_1, \dots, p_n$ that define the coloring of $\mathcal{X}$.

Our first step is to show that $\mathcal{X}$ has a "comb-like" structure, in the sense that there exists a distinguished minimal point m such that every point $x \in \mathcal{X}$ either lies in $\uparrow m$, or is a minimal point with an immediate successor x^+ in $\uparrow m$. We need a couple of lemmas.

Lemma 3.9. *In $\mathcal{X}$, non-minimal points must be comparable.*

Proof. Let us suppose, with a view to contradiction, that x and y are non-minimal but incomparable. Take some $x' \in \mathring{\downarrow} x$ and $y' \in \mathring{\downarrow} y$. We use the defining properties of co-trees to show $\{x', x\} \perp \{y', y\}$, noting that by symmetry, it suffices prove $x' \perp \{y', y\}$.

Firstly, notice that if $x' \leqslant y'$, then $x' < y$, so $x, y \in \uparrow x'$. As principal upsets in $\mathcal{X}$ are chains, this contradicts our assumption that x and y are incomparable. On the other hand, if $y' \leqslant x'$, then $x', y \in \uparrow y'$, so x' and y must be comparable. As previously stated, we cannot have $x' \leqslant y$, and if $y \leqslant x'$, then $y \leqslant x$, another contradiction. Thus, we have $x' \perp \{y', y\}$.

Now, since $x \perp y$, it follows that both points are distinct from the co-root of $\mathcal{X}$, r. Hence, we have $x' < x < r > y > y'$. This, together with $\{x', x\} \perp \{y', y\}$, makes clear the existence of an order-embedding from $\mathfrak{F}_4$ (see Figure 3) into $\mathcal{X}$. Equivalently, $\mathcal{X} \not\models \beta(\mathfrak{F}_4)$, by the Subframe Jankov Lemma 2.21. But this contradicts $\mathcal{X} \models L$, since $\beta(\mathfrak{F}_4) = \beta_1$ is an axiom of L. $\square$

Lemma 3.10. *If $\mathcal{X}$ is an infinite bi-Esakia co-tree validating L, then there exists $m \in min(\mathcal{X})$ such that*

$$\forall y \in X \setminus \uparrow m,\ \exists x \in \underset{\circ}{\uparrow} m\ (\uparrow y = \{y\} \cup \uparrow x).$$

Proof. Let us consider the set $H := \{\uparrow x \colon x \in \mathcal{X} \text{ and } \downarrow x \text{ contains a 3-chain}\}$, and note that it must contain $\uparrow r$. For suppose that $\downarrow r$ does not contain a 3-chain. As $\mathcal{X}$ is infinite and r its greatest element, this implies that r has infinitely many immediate predecessors, all of which are minimal. But this contradicts Proposition 3.4, since $\mathcal{X} \models L$ and $\neg\beta_2^{\partial}$ is an axiom of L. Thus, $\uparrow r \in H$, and H is nonempty. Furthermore, by the definition of H, if $\uparrow x, \uparrow y \in H$ then both x and y are non-minimal points. By the previous lemma, they must be $\leqslant$-comparable, hence $\uparrow x$ and $\uparrow y$ are $\subseteq$-comparable. We conclude that H is a $\subseteq$-chain.

It is clear that H being a $\subseteq$-chain implies that $\cup H$ is a $\leqslant$-chain in the bi-Esakia space $\mathcal{X}$, hence $u := inf(\cup H)$ exists, by Proposition 2.13. There are three possibilities for this point.

- <u>**Case A:**</u> $u \notin \cup H$ and $u \in min(\mathcal{X})$.
  In this case, we take $m := u$.

- <u>**Case B:**</u> $u \notin \cup H$ and $u \notin min(\mathcal{X})$.
  In this case, as u is not minimal and $\downarrow u$ does not contain a 3-chain, we have $\varnothing \neq \mathring{\downarrow} u \subseteq min(\mathcal{X})$. We take some $m \in \mathring{\downarrow} u$.

- <u>**Case C:**</u> $u \in \cup H$.
  In this case, $\downarrow u$ contains a 3-chain, but cannot contain a 4-chain, as u is the infimum of $\cup H$. Moreover, by Lemma 3.9, and since points with a common immediate successor are necessarily incomparable, $\prec u$ contains a single non-minimal point, which we will denote by m^+. This point has at most two immediate predecessors by Proposition 3.4, all of which must be minimal, since $\downarrow u$ does not contain a 4-chain. We take m to be one of these points.

Notice that we always have $u \in \uparrow m$. We now prove that in all three possible cases, m satisfies the desired condition. For suppose otherwise, i.e., that there exists $y \in X \setminus \uparrow m$ such that $\uparrow y \neq \{y\} \cup \uparrow x$, for every $x \in \underset{\circ}{\uparrow} m$. Clearly, $m \neq y$, and since $\uparrow m$ and $\uparrow y$ are non-disjoint closed chains (they both contain r), $\uparrow m \cap \uparrow y$ is a closed chain in a bi-Esakia space, and thus has a least element $x := Min(\uparrow m \cap \uparrow y)$, by Proposition 2.13. Since m is minimal and we assumed $y \notin \uparrow m$, x must be strictly above both m and y. From our assumption on y, in particular, that $\uparrow y \neq \{y\} \cup \uparrow x$, we can see that $y < x$ forces the existence of a point z satisfying $y < z < x$.

We claim that z must be incomparable to the infimum of $\cup H$, u. By the definition of x, $y < z < x$ already implies $z \notin \uparrow m$, so $u \in \uparrow m$ yields $u \nleq z$. In particular, we have $z \neq u$. To see that $z \nleq u$ also holds, note that if A or B holds, then $\downarrow u$ does not contain a 3-chain, so $y < z \neq u$ entails $z \nleq u$. If C holds, then as m^+ is the unique point in $\overset{\circ}{\downarrow} u$ which is not minimal, $z < u$ would force $z = m^+$, since $y < z$, which would contradict the fact $z \notin \uparrow m$ established above. But $m^+ \in \uparrow m$, so $z = m^+$ and $y < z < x = Min(\uparrow m \cap \uparrow y)$ would yield a contradiction.

But now, the fact that z is a non-minimal point incomparable to u, together with Lemma 3.9, entails that u must be necessarily minimal. Therefore, the only possibility for m is the one detailed in Case A, that is, $m = u = inf(\cup H) \notin \cup H$. Observe that in this case, $\cup H$ is an infinite descending chain, hence $m < x$ (see the definition of x) implies that there exists a point v satisfying $m < v < x$. Since $z \notin \uparrow m$, $v \nleq z$ follows, and as $v < x = Min(\uparrow m \cap \uparrow y)$ and $y < z < x$, we have $z \nleq v$. Thus, v and z are non-minimal incomparable points, contradicting Lemma 3.9. □

Proposition 3.11. *$\mathcal{X}$ has a distinguished minimal element m which satisfies $X = \underset{\circ}{\uparrow} m \uplus min(\mathcal{X})$. Moreover, for any other minimal element y distinct from m, y^+ exists and is contained in $\underset{\circ}{\uparrow} m$.*

Proof. That $\underset{\circ}{\uparrow} m$ and $min(\mathcal{X})$ are disjoint sets contained in X is trivial. If $y \notin \underset{\circ}{\uparrow} m$, then by the previous lemma, there must exist $x \in \underset{\circ}{\uparrow} m$ such that $\uparrow y = \{y\} \cup \uparrow x$. Clearly, we have $x = y^+$, and it is not hard to see that by Lemma 3.9 forces y to be a minimal point. □

Remark 3.12. We note that the distinguished minimal point m mentioned above is not necessarily unique. If the conditions of Case B or C hold (see the proof of Lemma 3.7), then, by Proposition 3.4, we have at most two choices for m. In what follows, we shall work with a fixed m, and the chain $\uparrow m$ will sometimes be referred to as "the diagonal of $\mathcal{X}$".

Corollary 3.13. *The following conditions hold true:*

(i) *$\underset{\circ}{\uparrow} m$ is an open set;*
(ii) *if $x \in \underset{\circ}{\uparrow} m$, then there exists a clopen upset U such that $x \in U \subseteq \underset{\circ}{\uparrow} m$;*
(iii) *if $x \in \underset{\circ}{\uparrow} m$ and $y \in min(\mathcal{X})$, then there exists a clopen upset contained in $\underset{\circ}{\uparrow} m$ which separates x from y.*

Proof. (i) As $\mathcal{X}$ is a bi-Esakia space, $min(\mathcal{X})$ is closed (see Proposition 2.13), so $X = \underset{\circ}{\uparrow} m \uplus min(\mathcal{X})$ entails that $\underset{\circ}{\uparrow} m$ is open.

(ii) As $\underset{\circ}{\uparrow} m$ is open by above, and $\mathcal{X}$ is 0-dimensional, it follows that for any $x \in \underset{\circ}{\uparrow} m$, there must exist a clopen U' such that $x \in U' \subseteq \underset{\circ}{\uparrow} m$. Clearly, $\uparrow U$ is a clopen upset satisfying the desired conditions.

(iii) This is an immediate consequence of (ii). □

Equipped with the previous proposition, we can already see that the structure of $\mathcal{X}$ is somewhat "comb-like".

We will now characterize the points in the diagonal of $\mathcal{X}$, $\uparrow m$. If $x \in \underset{\circ}{\uparrow} m$ has no immediate predecessor in $\uparrow m$, then we call x an *up limit* (of $\mathcal{X}$). If $x \in \uparrow m \setminus \{r\}$ has no immediate successor, then x is a *down limit* (of $\mathcal{X}$). Notice that these points are also limit points, in the topological sense, of some appropriate sets. For example, if x is an up limit, then it must be a limit point of the chain $[y, x]$, for every $y \in [m, x[$. To see this, observe that $[y, x] = {\uparrow}y \cap {\downarrow}x$ is a closed set containing x, and x cannot be an isolated point of $[y, x]$, as this would entail that $[y, x[$ is a closed chain in $\mathcal{X}$. But in a bi-Esakia space, closed chains must be bounded (see Proposition 2.13), and the definition of up limits ensures that the chain $[y, x[$ cannot have a greatest element. In a similar way, we can prove that a down limit x must be a limit point of the chain $[x, y]$, for every $y \in]x, r]$.

The next result lists some properties of points in the diagonal which do not fall under at least one of the previous definitions.

Lemma 3.14. *For $y, x, z \in \mathcal{X}$, the following conditions hold true:*
- (i) *if $(y \prec x) \in {\uparrow}m$, then ${\uparrow}x$ is a clopen;*
- (ii) *if $(x \prec z) \in \mathring{\uparrow}m$, then ${\downarrow}x$ is a clopen;*
- (iii) *if $(y \prec x \prec z) \in {\uparrow}m$, then $\{x\}$ is a clopen.*

Proof. (i) Suppose $(y \prec x) \in {\uparrow}m$. As $\mathring{\uparrow}m$ is open by Corollary 3.13.(i) and ${\downarrow}y$ is closed, it follows that $\mathring{\uparrow}m \setminus {\downarrow}y = {\uparrow}x$ is open. So the fact that ${\uparrow}x$ is also closed yields the desired result.

(ii) Suppose $(x \prec z) \in \mathring{\uparrow}m$. As $\mathring{\uparrow}m$ is open and ${\uparrow}z$ is closed, we have that $\mathring{\uparrow}m \setminus {\uparrow}z =]m, x]$ is a nonempty open set. It follows that ${\downarrow}]m, x] = {\downarrow}x$ is open, and since we already know that it is closed, we are done.

(iii) By the two previous conditions, if $(y \prec x \prec z) \in {\uparrow}m$, then both ${\uparrow}x$ and ${\downarrow}x$ are clopen. Therefore, ${\uparrow}x \cap {\downarrow}x = \{x\}$ is clopen. $\square$

Our next goal is to show that every up limit has exactly two immediate predecessors, which, by the definition of up limits, lie necessarily outside of the diagonal ${\uparrow}m$. To this end, we need a couple of technical lemmas, that will also prove useful in what follows.

Recall that every bi-Esakia space has enough gaps (see Proposition 2.13). This fact will be used repeatedly without further reference in what follows.

Lemma 3.15. *If $[y, x] \subseteq {\uparrow}m$ has more that 2^n elements, then $min(\mathcal{X}) \cap ({\downarrow}x \setminus {\downarrow}y)$ is nonempty.*

Proof. Suppose that $[y, x]$ has more than 2^n elements and is contained in ${\uparrow}m$. As $[y, x]$ is a chain in a co-tree, we always have both

$$ {\uparrow}y \setminus [y, x] = \mathring{\uparrow}x \text{ and } \mathring{\downarrow}y \subseteq {\downarrow}x \setminus [y, x]. $$

Now, by Lemma 2.17, $[y, x]$ cannot be an isolated chain (see Definition 2.1). By our previous comment, this means that ${\downarrow}x \setminus [y, x] \nsubseteq \mathring{\downarrow}y$, so there must exist a point $z \in]y, x]$ such that ${\downarrow}z \nsubseteq [y, x] \cup \mathring{\downarrow}y$. Let $u \in {\downarrow}z \setminus ([y, x] \cup \mathring{\downarrow}y)$, and note that $u \notin {\uparrow}m$, since otherwise $u, y \in {\uparrow}m$ would entail $u \leqslant y$ or $y \leqslant u$, and both conditions clearly contradict our assumption on u. It now follows from $\mathcal{X} = \mathring{\uparrow}m \uplus min(\mathcal{X})$ (see Proposition 3.11) that u is necessarily minimal and that $z = u^+$. Clearly, $u \in min(\mathcal{X}) \cap ({\downarrow}x \setminus {\downarrow}y)$. $\square$

Corollary 3.16. *If $[y, x] \subseteq {\uparrow}m$ has more that 2^n elements, then there exists $y' \in]y, x]$ such that ${\prec}y' \setminus \mathring{\uparrow}m$ is nonempty.*

Proof. This is clear from the proof of the previous result. $\square$

Lemma 3.17. *If $x \in \mathcal{X}$ is an up limit, then it must have immediate predecessors.*

Proof. Let us suppose that ${\prec}x = \emptyset$, for some up limit x, and arrive at a contradiction. As ${\downarrow}x$ is closed, it follows from Proposition 2.13 that $min({\downarrow}x)$ is closed as well. We list $\{x_i\}_{i \in I} := min({\downarrow}x)$, noting that clearly $\{x_i\}_{i \in I} \subseteq min(\mathcal{X})$. Now, since we assumed that x is an up limit, it follows that for every $y \in]m, x[$, $[y, x]$ is an infinite chain. Using the previous lemma, we conclude that $min({\downarrow}x)$ contains points distinct from m, since $\emptyset \neq min(\mathcal{X}) \cap ({\downarrow}x \setminus {\downarrow}y) \subseteq min({\downarrow}x)$ and clearly $m \in {\downarrow}y$.

For each $i \in I' := \{j \in I : x_j \neq m\}$, $x_i \in min(\mathcal{X}) \setminus \{m\}$ implies, by Proposition 3.11, that x_i^+ exists and is contained in $\mathring{\uparrow}m$. Moreover, since we assumed ${\prec}x = \emptyset$, it is clear that $x_i^+ < x$, and so there exists a gap $x_i^+ \leqslant a_i \prec b_i < x$. By applying Lemma 3.14.(ii) to $(a_i \prec b_i) \in \underset{\circ}{\uparrow} m$, we see that $\downarrow a_i$ is a clopen which contains both x_i and m. It follows that the family $\{\downarrow a_i\}_{i \in I'}$ is an open cover of the closed set $\{x_i\}_{i \in I} = min(\downarrow x)$. By compactness, there exists a finite subcover $\{\downarrow a_{i_1}, \ldots, \downarrow a_{i_t}\}$, but since the $a_{i_1}, \ldots, a_{i_t}$ are all contained in the chain $\uparrow m$, it is clear that there exists some $j \in \{i_1, \ldots, i_t\} \subseteq I'$ satisfying

$$min(\downarrow x) = \{x_i\}_{i \in I} \subseteq \bigcup_{l=1}^{t} \downarrow a_{i_l} = \downarrow a_j.$$

By our definition, $a_j < x$, hence the assumption that x is an up limit yields that $[a_j, x]$ is an infinite chain, which is moreover contained in $\uparrow m$. Again by the previous lemma, $min(\mathcal{X}) \cap (\downarrow x \setminus \downarrow a_j)$ must be nonempty. But this set is clearly contained in $min(\downarrow x)$ and disjoint from $\downarrow a_j$, contradicting $min(\downarrow x) \subseteq \downarrow a_j$. $\square$

We now know that up limits must have immediate predecessors. By Proposition 3.4, our assumption $\mathcal{X} \models \neg \beta_2^\partial$ already guarantees that points in $\mathcal{X}$ have at most two such predecessors. So, to achieve our current goal, it suffices to prove that no up limit can have a sole immediate predecessor. Our proof relies on the fact that the bi-Esakia co-tree $\mathcal{X}$ is equipped with a coloring, given by the propositional variables $p_1, \ldots, p_n$. Notice that each element $x \in \mathcal{X}$ is contained in the clopen corresponding to $col(x)$, that is, the clopen

$$C_x := \Big(\bigcap_{i=1}^{n} \{V(p_i) \colon x \in V(p_i)\}\Big) \setminus \Big(\bigcup_{i=1}^{n} \{V(p_i) \colon x \notin V(p_i)\}\Big).$$

It is easy to see that not only every point in C_x has the same color, but also that the intersection of this clopen with the diagonal $\uparrow m$ is an interval, as C_x can be written as a finite intersection of upsets and downsets. It follows that if $y, z \in \uparrow m \cap C_x$ are such that $y \leqslant z$, then $[y, z] \subseteq C_x$.

Lemma 3.18. *If $(y \leqslant x) \in \uparrow m$, then at least one of the following conditions must fail:*

(i) *$col(y) = col(x)$;*
(ii) *$\prec y \setminus \underset{\circ}{\uparrow} m \neq \varnothing$;*
(iii) *the set $M := \big[(\downarrow x \setminus \downarrow y) \cup (\prec y \setminus \underset{\circ}{\uparrow} m)\big] \cap min(\mathcal{X})$ only contains points of the same color, and has at least two elements.*

Proof. Suppose that $(y \leqslant x) \in \uparrow m$ are such that all of the above conditions hold true. We show that this forces

$$E := [y, x]^2 \cup M^2 \cup Id_X$$

to be a proper bi-bisimulation equivalence (see Definition 2.15) that only identifies points of the same color, thus contradicting the Coloring Theorem 2.16.

Firstly, note that every point in M has an immediate successor contained in $[y, x]$. For if $u \in M$ is distinct from m, then it follows from Proposition 3.11 that u^+ exists and is contained in $\underset{\circ}{\uparrow} m$. Clearly, if $u \in \prec y \setminus \underset{\circ}{\uparrow} m$, then $u^+ = y$ and we are done. On the other hand, if $u \in \downarrow x \setminus \downarrow y$, then $u \in \downarrow x$ implies $u^+ \leqslant x$ since $\uparrow u$ is a chain and by the definition of u^+, while $u \notin \downarrow y$ entails $y < u^+$, since y and u^+ must be comparable, as they both lie in the chain $\uparrow m$, but $u^+ \leqslant y$ cannot happen. Thus, $u^+ \in]y, x]$, as desired. As for the possibility $m \in M$, notice that this case can only occur when y is the immediate successor of m, so our claim is trivially fulfilled.

Let us now show that E contradicts the Coloring Theorem. That this is a proper equivalence relation which only identifies points of the same color is clear from the definition of E and conditions (i) (recall that this condition implies that $[y, x] \subseteq C_x$) and (iii). To see that E satisfies the up condition, i.e., that

$$\text{if } uEu' \text{ and } v \in \underset{\circ}{\uparrow} u, \text{ then } \exists v' \in \uparrow u' \text{ st. } vEv',$$

let us take u, u', v in the above conditions. For the sake of non-triviality, we also assume that $(v, u) \notin E$ and $u \neq u'$, hence either $u, u' \in [y, x]$ or $u, u' \in M$. If $u, u' \in [y, x]$, then $u \leqslant x, v$ implies that x and v are comparable, so the assumption $(v, u) \notin E$, i.e., that $v \notin [y, x]$, forces $x < v$. Thus, $u' \leqslant x < v$, and we can take $v' := v$.

If $u, u' \in M$, then by our first comment in this proof, both u^+ and u'^+ exist and are contained in $[y, x]$, hence $u^+ E u'^+$. Now, by the definition of u^+ and that of a co-tree, from $u < v$ we get that $u^+ \leqslant v$, hence $u^+ \in [y, x]$ implies that x and v are comparable. If $v \leqslant x$, then $u^+ \leqslant v$ yields $v \in [y, x]$, so vEu'^+ and $v' := u'^+$ satisfies the desired conditions. If $x \leqslant v$, then we have $u'^+ \leqslant x \leqslant v$, and we take $v' := v$.

Next we show that E satisfies the down condition, that is,

$$\text{if } uEu' \text{ and } v \in \overset{\circ}{\downarrow} u, \text{ then } \exists v' \in {\downarrow} u' \text{ st. } vEv'.$$

Again, without loss of generality, we take u and u' such that $u \neq u'$, uEu', and $v \in \overset{\circ}{\downarrow} u$ such that $(v, u) \notin E$. Moreover, since points in M are necessarily minimal, it suffices to consider the case when $u, u' \in [y, x]$. As this is a chain, we have $u \leqslant u'$ or $u' \leqslant u$. If $u \leqslant u'$, or, more in general, if $v < u'$, we can simply take $v' := v$ and we are done. Suppose now that $u' \leqslant u$ and $v \nleqslant u'$. Notice that these assumptions, together with $(v, u) \notin E$ (which in this case, translates to $v \notin [y, x]$), imply that $v \in ({\downarrow} u \setminus {\downarrow} u') \cap min(\mathcal{X})$, since we have a clear description of ${\downarrow} u$ as

$${\downarrow} u = [u', u] \uplus \overset{\circ}{\downarrow} u' \uplus \left[({\downarrow} u \setminus {\downarrow} u') \cap min(\mathcal{X}) \right].$$

As $({\downarrow} u \setminus {\downarrow} u') \cap min(\mathcal{X}) \subseteq M$, it follows $v \in M$. By our assumption that condition (ii) holds, i.e., that ${\prec} y \setminus \underset{\circ}{\uparrow} m \neq \emptyset$, y has an immediate predecessor, v', which by the definition of M is contained in this set. Hence, vEv', by our previous comment. But now, since $u' \in [y, x]$, we have $y \leqslant u'$, hence $v' < u'$ and we are done.

It remains to show that E is refined, that is,

$$\text{if } (u, v) \notin E, \text{ then there exists an } E\text{-saturated clopen upset } U \text{ st. } |U \cap \{u, v\}| = 1.$$

We proceed by cases.

- **Case:** $u \in M$ and $v = m \notin M$

  If $m \notin M$, then $m < y$ and y is not the immediate successor of m, a point that might not exist. If it does, then we must have $m^+ < y$, so there exists a gap $m^+ \leqslant a \prec b \leqslant y$. By Lemma 3.14.(ii), $(a \prec b) \in \underset{\circ}{\uparrow} m$ entails that ${\downarrow} a$ is a clopen, which clearly contains m and is disjoint from M. It is also clear that ${\uparrow}{\downarrow} a$ satisfies the desired conditions, as it is a clopen upset containing $\{m\} \cup [y, x]$ (since $m \in {\downarrow} a$ and $a \leqslant y$) and disjoint from M, a set that contains u.

  On the other hand, if m^+ does not exist, then since $m < y$, there exists a gap $m < a \prec b \leqslant y$ such that $(a \prec b) \in \underset{\circ}{\uparrow} m$, and we can repeat our previous argument.

- **Case:** $u \in min(\mathcal{X})$ and $v \in min(\mathcal{X}) \setminus (M \cup \{m\})$

  By Proposition 3.11, v^+ exists and is contained in $\underset{\circ}{\uparrow} m$. Moreover, by the definition of M, $v \notin M$ clearly implies that $v^+ \notin [y, x]$, so either $v^+ < y$ or $x < v^+$. If $v^+ < y$, we take a gap $v^+ \leqslant a \prec b \leqslant y$, noting that $(a \prec b) \in \underset{\circ}{\uparrow} m$. By Lemma 3.14.(ii), ${\downarrow} a$ must be a clopen set, which not only contains v, but is also clearly disjoint from M. It follows that ${\uparrow}{\downarrow} a$ is a clopen upset containing $\{v\} \cup [y, x]$ and disjoint from M. Now, as u and v are distinct minimal points, we have $v \nleqslant u$, so by the PSA, there exists a clopen upset V' separating v from u. Furthermore, as V' is an upset containing v, the assumption $v \prec v^+ < y$ entails $[y, x] \subseteq V'$. We take $V := {\uparrow}{\downarrow} a \cap V'$, noting that it clearly satisfies the desired conditions.

  On the other hand, if $x < v^+$, then clearly $v \nleqslant x$ by the structure of $\mathcal{X}$, so by two applications of the PSA and taking the intersection of the resulting clopen upsets, there exists a clopen upset V containing v but omitting both x and u. It is clear that $V \cap ([y, x] \cup M \cup \{u\}) = \emptyset$, hence we are done.

- **Case:** $u \in \underset{\circ}{\uparrow} m$ and $v \in min(\mathcal{X})$

  Simply note that, by Corollary 3.13.(iii), we can always find a clopen upset $U \subseteq \underset{\circ}{\uparrow} m$ which contains $\{u\} \cup [y, x]$ and is clearly disjoint from $min(\mathcal{X})$, a set that contains $\{v\} \cup M$.

- **Case:** $u, v \in \underset{\circ}{\uparrow} m$ and $|[y, x] \cap \{u, v\}| \leqslant 1$

Without loss of generality, we assume $u < v$. If $v < y$, then we take a gap $u \leqslant a \prec b \leqslant v$, so by Lemma 3.14.(i), $\uparrow b$ is a clopen upset, which moreover contains $\{v\} \cup [y, x]$ and is disjoint from $\{u\} \cup M$.

If $u < y \leqslant v$, then we take a gap $u \leqslant a \prec b \leqslant y$, so by Lemma 3.14.(i), $\uparrow b$ is a clopen upset, which moreover contains $\{v\} \cup [y, x]$ and is disjoint from $\{u\} \cup M$.

If $y \leqslant u \leqslant x < v$, then we take a gap $x \leqslant a \prec b \leqslant v$, so by Lemma 3.14.(i), $\uparrow b$ is a clopen upset, which moreover contains v and is disjoint from $\{u\} \cup [y, x] \cup M$.

If $x < u < v$, then we take a gap $u \leqslant a \prec b \leqslant v$, so by Lemma 3.14.(i), $\uparrow b$ is a clopen upset, which moreover contains v and is disjoint from $\{u\} \cup [y, x] \cup M$. □

Proposition 3.19. *If $x \in \mathcal{X}$ is an up limit, then it has exactly two immediate predecessors, which must lie outside of $\uparrow m$.*

Proof. We start by noting that, by Proposition 3.4 and Lemma 3.17, we already know that $0 < |{\prec}x| \leqslant 2$. We suppose that ${\prec}x = \{x_1\}$, for some x_1, and arrive at a contradiction, thus proving $|{\prec}x| = 2$.

Recall that x_1 is contained in the clopen corresponding to $col(x_1)$, i.e., C_{x_1}, and consider the set $min({\downarrow}x) \setminus C_{x_1}$. As ${\downarrow}x$ is closed, so is $min({\downarrow}x)$ (see Proposition 2.13), hence $min({\downarrow}x) \setminus C_{x_1}$ is also closed. We claim that $min({\downarrow}x) \setminus C_{x_1}$ is contained in ${\downarrow}a$, for some $a \in [m, x[$. For suppose that m is the only point in $min({\downarrow}x) \setminus C_{x_1}$. Then clearly $a := m$ would satisfy our claim. On the other hand, if $min({\downarrow}x) \setminus C_{x_1}$ contains points distinct from m, then we can repeat the same argument used in the proof of Lemma 3.17, using gaps and compactness, to find such a point $a \in [m, x[$ whose downset contains $min({\downarrow}x) \setminus C_{x_1}$.

Now, as x is an up limit, we know that it must be a limit point of the closed chain $[a, x]$. Because of this, the intersection of the clopen C_x with the set $[a, x[$ must be nonempty, since C_x contains x. Let z be a point in this intersection, and notice that, by the definition of up limit, $[z, x]$ is an infinite chain, that moreover is contained in the clopen C_x. It now follows from Corollary 3.16 that we can find a point $y \in]z, x[$ such that ${\prec}y \setminus \underset{\circ}{\uparrow}m \neq \emptyset$.

But now, y and x satisfy the following conditions: (i) $col(y) = col(x)$, since $y \in [z, x] \subseteq C_x$; (ii) ${\prec}y \setminus \underset{\circ}{\uparrow}m \neq \emptyset$, by above; and (iii) the set $M := \big[({\downarrow}x \setminus {\downarrow}y) \cup ({\prec}y \setminus \underset{\circ}{\uparrow}m)\big] \cap min(\mathcal{X})$ only contains points of the same color and has at least two elements, since we established above that $min({\downarrow}x) \setminus C_{x_1} \subseteq {\downarrow}a$, so $a < y$ clearly implies $M \subseteq C_{x_1}$, and M contains $({\prec}y \setminus \underset{\circ}{\uparrow}m) \cup \{x_1\}$ (that $x_1 \notin ({\prec}y \setminus \underset{\circ}{\uparrow}m)$ is clear from $y < x$). This contradicts Lemma 3.18, hence we conclude that ${\prec}x$ cannot be a singleton, as desired. □

Our next goal is to show that down limits do not exist.

For the next lemma, we shall use the following definition: if x is a down limit, then

$$M_x := \begin{cases} {\prec}x \setminus \underset{\circ}{\uparrow}m & \text{if } x \neq m, \\ \{x\} & \text{if } x = m. \end{cases}$$

Lemma 3.20. *If x is a down limit, $z \in \underset{\circ}{\uparrow}x$, and C is a clopen containing M_x, then there exists a point $z' \in]x, z[$ such that*

$$min({\downarrow}z' \setminus {\downarrow}x) \cup M_x \subseteq C.$$

Proof. Let us consider the set $W := min({\downarrow}z \setminus {\downarrow}x) \setminus C$. Notice that both $m \in {\downarrow}x$ and $min({\downarrow}z \setminus {\downarrow}x) \subseteq min(\mathcal{X})$ follow from the assumption that x is a down limit, together with the fact $X = \underset{\circ}{\uparrow}m \uplus min(\mathcal{X})$ (see Proposition 3.11). We claim that W is closed. Observe that, by the known structure of $\mathcal{X}$, we have a clear description

$$X \setminus W = {\downarrow}x \cup C \cup]x, z] \cup ({\downarrow}r \setminus {\downarrow}z) = \underset{\circ}{\uparrow}m \cup min({\downarrow}x) \cup C \cup ({\downarrow}r \setminus {\downarrow}z).$$

To establish our claim, we show that every point in $X \setminus W$ has an open neighbourhood disjoint from W. For suppose $u \in X \setminus W$. We will use the right most union of the above display. The cases $u \in \underset{\circ}{\uparrow}m$ and $u \in C$ are clear, since the former set is open by Corollary 3.13, while the latter is clopen by assumption, and both sets are clearly disjoint from

$$min({\downarrow}z \setminus {\downarrow}x) \setminus C \subseteq min(\mathcal{X}) = X \setminus \underset{\circ}{\uparrow}m.$$

The case $u \in {\downarrow} r \setminus {\downarrow} z$ is also easy to see, since this implies $u \notin {\downarrow} z$, i.e., $u \nleqslant z$, and the PSA yields a clopen upset neighbourhood of u omitting z. This clopen upset must be clearly disjoint from ${\downarrow} z$, and in particular, from $W \subseteq {\downarrow} z$.

Now, note that if $x = m$, then we are done, since in this case, we have $min({\downarrow} x) = \{m\} = M_x \subseteq C$. Accordingly, we assume, without loss of generality, that $x \neq m$, hence $m < x$ holds, and that $u \in min({\downarrow} x) \setminus C$. Furthermore, by our assumption $M_x = {\prec} x \setminus \underset{\circ}{\uparrow} m \subseteq C$ and since $min({\downarrow} x) \subseteq min(\mathcal{X}) = X \setminus \underset{\circ}{\uparrow} m$, we have $\neg(u \prec x)$. If $u = m$, then $m < x$ yields a gap $m \leqslant a \prec b \leqslant x$. But since we just established that $\neg(m \prec x)$, one of the previous inequalities must be strict. If $m < a$, then $(a \prec b) \in \underset{\circ}{\uparrow} m$ and it follows from Lemma 3.14.(ii) that ${\downarrow} a$ is a clopen neighbourhood of m clearly disjoint from W. On the other hand, if $b < x$, then there exists a gap $m < b \leqslant c \prec d \leqslant x$, hence $(c \prec d) \in \underset{\circ}{\uparrow} m$ and ${\downarrow} c$ is a clopen neighbourhood of m clearly disjoint from W. Finally, we consider the case $u \in min({\downarrow} x) \setminus (C \cup \{m\})$. Since u is then a minimal point distinct from m, it follows from Proposition 3.11 that u^+ exists and is contained in $\underset{\circ}{\uparrow} m$, and moreover satisfies $u^+ < x$, as we established $\neg(u \prec x)$ above. This strict inequality yields a gap $u^+ \leqslant a \prec b \leqslant x$, and it is clear that $(a \prec b) \in \underset{\circ}{\uparrow} m$, since $m < u^+$. By Lemma 3.14.(ii), ${\downarrow} a$ is a clopen neighbourhood of u clearly disjoint from W. We conclude that W is indeed a closed set, as desired.

List $\{y_i\}_{i \in I} := min({\downarrow} z \setminus {\downarrow} x) \setminus C = W$, which is now established as a closed set of minimal points all distinct from m. Because of this, Proposition 3.11 implies that for each $i \in I$, y_i^+ exists and is contained in $\underset{\circ}{\uparrow} m$. Clearly $y_i^+ \leqslant z$ holds, since $y_i \in {\downarrow} z$ and by the definition of y_i^+. Furthermore, as $y_i \notin {\downarrow} x$ and both x and y_i^+ lie in the chain ${\uparrow} m$, we must have $x < y_i^+$, hence there is a gap $x < a_i \prec b_i \leqslant y_i^+$ (where the fact that $x < a_i$ is strict comes from the assumption that m is a down limit). By the structure of $\mathcal{X}$, it is clear that $y_i \nleqslant a_i$, so the PSA yields a clopen upset V_i separating y_i from a_i. Thus, $\{V_i\}_{i \in I}$ is an open cover of the closed set $\{y_i\}_{i \in I}$. By compactness, there exists a finite subcover $\{V_1, \ldots, V_t\}$, i.e., $\{y_i\}_{i \in I} \subseteq \bigcup_{j=1}^{t} V_j$. As this is a finite union of clopens, it is clopen as well, hence ${\uparrow} m \cap (\bigcup_{j=1}^{t} V_j)$ is a closed chain, which, by Proposition 2.13, must have a minimum z_0. Notice that, for each $j \leqslant t$, V_j contains y_j^+, since it is an upset which contains y_j by definition. As $(y_j^+ \leqslant z) \in {\uparrow} m$ always holds, it is clear that $z_0 \leqslant z$. Moreover, there is an $l \leqslant t$ for which $z_0 \in V_l$, so by the definition of this clopen, we have $x < a_l < z_0$. But now, from $W = min({\downarrow} z \setminus {\downarrow} x) \setminus C \subseteq \bigcup_{j=1}^{t} V_j$ and

$$x < a_l < z_0 = Min\big({\uparrow} m \cap (\bigcup_{j=1}^{t} V_j)\big) \leqslant z,$$

it follows that ${\downarrow} a_l \cap W = \emptyset$ and $min({\downarrow} a_l \setminus {\downarrow} x) \subseteq min({\downarrow} z \setminus {\downarrow} x)$. Therefore, the inclusion $min({\downarrow} a_l \setminus {\downarrow} x) \subseteq C$ holds, and the point $z' := a_l \in]x, z[$ satisfies the desired conditions. □

Lemma 3.21. *The point m cannot be a down limit.*

Proof. We suppose, with a view to contradiction, that m is a down limit. This implies, in particular, that m is a limit point of the closed chain ${\uparrow} m$. Since C_m, the clopen corresponding to $col(m)$, is a clopen neighbourhood of m, the previous comment yields a point $z \in \underset{\circ}{\uparrow} m \cap C_m$, and it now follows from the definitions of down limits and of C_m that $[m, z]$ is an infinite chain contained in C_m. By Lemma 3.20, there exists a point $z' \in]m, z[$ such that

$$min({\downarrow} z' \setminus {\downarrow} m) \cup \{m\} = min({\downarrow} z') \subseteq C_m.$$

But we also know that $[m, z'] \subseteq [m, z] \subseteq C_m$, hence ${\downarrow} z' \subseteq C_m$ follows.

As $[m, z']$ is an infinite chain contained in C_m and we assumed that m is a down limit, it is not hard to see that we can apply Corollary 3.16 twice, yielding points $(y < x) \in]m, z']$ satisfying the following conditions: (i) $col(y) = col(x) = col(m)$; (ii) ${\prec} y \setminus \underset{\circ}{\uparrow} m \neq \emptyset$; (iii) the set $\big[({\downarrow} x \setminus {\downarrow} y) \cup ({\prec} y \setminus \underset{\circ}{\uparrow} m)\big] \cap min(\mathcal{X})$ has at least two elements and is contained in ${\downarrow} z' \subseteq C_m$. By Lemma 3.18, we found the desired contradiction. □

Lemma 3.22. *If $x \in \mathcal{X}$ is a down limit, then it has exactly two immediate predecessors, which must lie outside of $\underset{\circ}{\uparrow} m$.*

Proof. Let us first consider the case when $x = m^+$ is a down limit. We suppose, with a view to contradiction, that m is the unique immediate predecessor of m^+. This entails $\prec m^+ \setminus \underset{\circ}{\uparrow} m = \{m\} \subseteq C_m$, where C_m is the clopen corresponding to $col(m)$. From the assumption that m^+ is a down limit, it follows that m^+ is also a limit point of the closed chain $\uparrow m^+$, and therefore, that there exists a point $z \in C_{m^+} \cap \underset{\circ}{\uparrow} m^+$. By Lemma 3.20, we have a point $z' \in]m^+, z]$ which satisfies

$$min(\downarrow z' \setminus \downarrow m^+) \cup (\prec m^+ \setminus \underset{\circ}{\uparrow} m) \subseteq C_m.$$

Notice that $[m^+, z']$ is an infinite chain contained in C_{m^+}, and that, by Corollary 3.16, there must exist $y \in]m^+, z']$ such that $\prec y \setminus \underset{\circ}{\uparrow} m \neq \varnothing$. But now, $(m^+ < y) \in \uparrow m$ are points which satisfy the following conditions: (i) $col(m^+) = col(y)$, since $[m^+, y] \subseteq [m^+, z'] \subseteq C_{m^+}$; (ii) $\prec m^+ \setminus \underset{\circ}{\uparrow} m \neq \varnothing$, since $m \in \prec m^+ \setminus \underset{\circ}{\uparrow} m$; (iii) the set

$$M := [(\downarrow y \setminus \downarrow m^+) \cup (\prec m^+ \setminus \underset{\circ}{\uparrow} m)] \cap min(\mathcal{X}) = min(\downarrow y \setminus \downarrow m^+) \cup \{m\}$$

has at least two elements, since clearly $\varnothing \neq \prec y \setminus \underset{\circ}{\uparrow} m \subseteq M \setminus \{m\}$, and is moreover contained in C_m, as $y \leqslant z'$ and the definition of z' entail $min(\downarrow y \setminus \downarrow m^+) \subseteq min(\downarrow z' \setminus \downarrow m^+) \subseteq C_m$. As the above conditions contradict Lemma 3.18, we conclude that m cannot be the sole immediate predecessor of m^+ if the latter point is a down limit, as desired.

Next we consider the case $x \neq m^+$. Note that if $\prec x \cap \uparrow m \neq \varnothing$, i.e., if x is an up limit, then the desired result already follows from Proposition 3.19. Accordingly, we assume without loss of generality, that there is $x^- \in \uparrow m$ such that $x^- \prec x$. We claim that the set $\prec x \setminus \underset{\circ}{\uparrow} m$ must be nonempty. For suppose otherwise. Then for some $z \in \underset{\circ}{\uparrow} x$, since $\varnothing$ is a clopen containing $\prec x \setminus \underset{\circ}{\uparrow} m$, it follows from Lemma 3.20 that there is a point $z' \in]x, z[$ satisfying $min(\downarrow z' \setminus \downarrow x) \subseteq \varnothing$. But x is assumed to be a down limit, hence the chain $[x, z']$ must be infinite, and the previous inclusion forces $[x, z']$ to be an infinite isolated chain, contradicting Lemma 2.17.

We now know that $\prec x \setminus \underset{\circ}{\uparrow} m \neq \varnothing$, and since we are considering the case when x has an immediate predecessor x^- in the diagonal, it follows from Proposition 3.4 that $\prec x \setminus \underset{\circ}{\uparrow} m$ contains a single element, which we will denote by x_1. We now can use a very similar argument as for the case $x = m^+$ detailed above to arrive at a contradiction. We sketch this argument below. As x is a down limit, there is a point $z \in \underset{\circ}{\uparrow} x$ of the same color. By Lemma 3.20, we can find $z' \in]x, z]$ such that

$$min(\downarrow z' \setminus \downarrow x) \cup \{x_1\} \subseteq C_{x_1}.$$

But now, Corollary 3.16 yields a point $y \in]x, z']$ that, together with x, contradicts Lemma 3.18. □

We are finally ready to prove that down limits do not exist.

Proposition 3.23. *There are no down limits in $\mathcal{X}$.*

Proof. Suppose that $x \in \mathcal{X}$ is a down limit. By the previous result, $\prec x \setminus \underset{\circ}{\uparrow} m = \prec x = \{x_1, x_2\}$, for some distinct $x_1, x_2 \in min(\mathcal{X})$. By Lemma 3.18 (applied to $(x \leqslant x) \in \uparrow m$), it is clear that $col(x_1) \neq col(x_2)$. Furthermore, since x is assumed to be a down limit and $\prec x \setminus \underset{\circ}{\uparrow} m$ is contained in the clopen $C_{x_1} \cup C_{x_2}$, it follows from Lemma 3.20 that there is a point $z \in \underset{\circ}{\uparrow} x$, which we can assume has the same color as x, such that

$$min(\downarrow z \setminus \downarrow x) \cup \{x_1, x_2\} \subseteq C_{x_1} \cup C_{x_2}$$

For $i \leqslant 2$, we define $M_i := \big(min(\downarrow z \setminus \downarrow x) \cap C_{x_i}\big) \cup \{x_i\}$, and set

$$E := [x, z]^2 \cup M_1^2 \cup M_2^2 \cup Id_X.$$

We prove that E is a proper bi-bisimulation equivalence (see Definition 2.15) that only identifies points of the same color, thus contradicting the Coloring Theorem 2.16. That E is a proper equivalence relation is clear. Moreover, since $[x, z] \subseteq C_x$ and $M_i \subseteq C_{x_i}$, for $i \leqslant 2$, it is clear that

E only identifies points of the same color. The proof that E satisfies the up and down conditions is easy and follows closely the argument used to establish these conditions that we detailed in the proof of Lemma 3.18, hence we skip it.

It remains to show that E is refined, that is, for $u, v \in \mathcal{X}$,

$$\text{if } (u,v) \notin E, \text{ then there exists an } E\text{-saturated clopen upset } U \text{ st. } |U \cap \{u,v\}| = 1.$$

We proceed by cases.

- **Case:** $u \in M_1$ and $v \in M_2$
  We take $U := \uparrow C_{x_1}$, noting that this clopen upset clearly contains $M_1 \cup [x,z]$ (since $x_1 \in M_1 \subseteq C_{x_1}$ and $x_1 < x$) and is disjoint from M_2 (since $M_2 \subseteq C_{x_2}$ and by the definition of coloring, $C_{x_1} \neq C_{x_2}$ forces $C_{x_1} \cap C_{x_2} = \emptyset$).

- **Case:** $u \in min(\mathcal{X})$ and $v \in min(\mathcal{X}) \setminus (M_1 \cup M_2)$
  Recall that by their definitions, we have $M_1 \cup M_2 = min({\downarrow} z \setminus {\downarrow} x) \cup {\prec} x$. So, our assumption on v implies either $v \in min({\downarrow} x) \setminus {\prec} x$ or $v \in min(\mathcal{X}) \cap ({\downarrow} r \setminus {\downarrow} z)$.

  If $v \in min({\downarrow} x) \setminus {\prec} x$, then we must have $x \neq m^+$, since otherwise we would have $min({\downarrow} x) = {\prec} x$, by the structure of $\mathcal{X}$. It now follows from the previous lemma that x must be an up limit, and as such, from $v < x$ we can certainly find a gap $v < a \prec b < x$ satisfying $(a \prec b) \in \underset{\circ}{\uparrow} m$. By Lemma 3.14.(ii), ${\downarrow} a$ is a clopen neighbourhood of v, clearly disjoint from $M_1 \cup M_2$. It is easy to see that ${\uparrow}{\downarrow} a$ contains $\{v\} \cup [x,z]$ and is disjoint from $M_1 \cup M_2$. Now, since u and v are non-E-equivalent minimal points, they must satisfy $v \nleqslant u$. By the PSA, there exists a clopen upset V separating v from u, and since $v < x$ by hypothesis, the fact that V is an upset yields $[x,z] \subseteq V$. We take $V' := {\uparrow}{\downarrow} a \cap V$ as our E-saturated clopen upset, as it clearly contains $\{v\} \cup [x,z]$ and is disjoint from $M_1 \cup M_2 \cup \{u\}$.

  If, on the other hand, we have $v \in min(\mathcal{X}) \cap ({\downarrow} r \setminus {\downarrow} z)$, then by applying the PSA to both $v \nleqslant z$ and $v \nleqslant u$ and taking an appropriate intersection, we can find a clopen upset V which contains v but omits both z and u. This upset is clearly E-saturated, since it does not contain z.

- **Case:** $u \in min(\mathcal{X})$ and $v \in \underset{\circ}{\uparrow} m$
  Let $v' := Min\{v, x\}$, which exists since $v, x \in {\uparrow} m$, a chain. Notice that $v' \in \underset{\circ}{\uparrow} m$, so our assumption $u \in min(\mathcal{X})$ together with Corollary 3.13.(ii) yields a clopen upset $V \subseteq \underset{\circ}{\uparrow} m$ which contains v'. By the definition of v', we have $[x,z] \subseteq V$, while $V \subseteq \underset{\circ}{\uparrow} m$ forces

  $$V \cap (M_1 \cup M_2 \cup \{u\}) = \emptyset.$$

  Thus, V satisfies the desired conditions.

- **Case:** $u, v \in \underset{\circ}{\uparrow} m$ and $|[x,z] \cap \{u,v\}| \leqslant 1$
  Without loss of generality, we assume $u < v$. If $v < x$, then we take a gap $u \leqslant a \prec b \leqslant v$, so by Lemma 3.14.(i), ${\uparrow} b$ is a clopen upset, which moreover contains $\{v\} \cup [x,z]$ and is disjoint from $\{u\} \cup M_1 \cup M_2$.

  If $u < x \leqslant v$, then we take a gap $u \leqslant a \prec b \leqslant x$, so by Lemma 3.14.(i), ${\uparrow} b$ is a clopen upset, which moreover contains $\{v\} \cup [x,z]$ and is disjoint from $\{u\} \cup M_1 \cup M_2$.

  If $x \leqslant u \leqslant z < v$, then we take a gap $z \leqslant a \prec b \leqslant v$, so by Lemma 3.14.(i), ${\uparrow} b$ is a clopen upset, which moreover contains v and is disjoint from $\{u\} \cup [x,z] \cup M_1 \cup M_2$.

  If $z < u < v$, then we take a gap $u \leqslant a \prec b \leqslant v$, so by Lemma 3.14.(i), ${\uparrow} b$ is a clopen upset, which moreover contains v and is disjoint from $\{u\} \cup [x,z] \cup M_1 \cup M_2$.

  In all the possibilities detailed above, $V := {\uparrow} b$ is an E-saturated clopen upset separating v from u, as desired.

As E contradicts the Coloring Theorem 2.16, we conclude that $\mathcal{X}$ has no down limits. □

Lemma 3.24. *If $(y \prec x \prec z) \in \underset{\circ}{\uparrow} m$, then x must have a unique immediate predecessor lying outside of ${\uparrow} m$.*

Proof. By Proposition 3.4, $y \prec x$ already entails that $|{\prec}x \setminus {\uparrow}m| \leqslant 1$, so it suffices to show ${\prec}x \setminus {\uparrow}m \neq \emptyset$. Accordingly, we suppose that this set is empty, and show that this forces $\mathcal{X} \not\models \mathcal{J}_2$, thus contradicting $\mathcal{X} \models L$.

By Lemma 3.14, $(y \prec x \prec z) \in \underset{\circ}{\uparrow}m$ implies that all of $\{x\}$, $\underset{\circ}{\uparrow}x = {\uparrow}z$, and $\overset{\circ}{\downarrow}x = {\downarrow}y$ are clopen. Set $V_a := {\uparrow}z$, $V_{a'} := X \setminus ({\downarrow}x \cup {\uparrow}z)$, and $V_b := \{x\}$, noting that they are pairwise disjoint clopens.

Claim 3.24.1. *The set $V_{a'}$ is nonempty.*

Proof of Claim: If ${\prec}r = \emptyset$, then $x \prec z$ entails that $[z, r]$ is an infinite chain. As infinite chains cannot be isolated chains by Lemma 2.17, $V_{a'} \neq \emptyset$ follows. On the other hand, if ${\prec}r \neq \emptyset$, then it is necessary that the set ${\prec}r \setminus {\uparrow}m$, which clearly is contained in $V_{a'}$, be nonempty. To see this, suppose otherwise, i.e., that $\emptyset \neq {\prec}r \subseteq {\uparrow}m$. It follows that there is a point $r^- \in {\uparrow}m$ such that $r^- \prec r$. Notice that the assumption that $\mathcal{X}$ is infinite, together with Proposition 3.4, implies that $r^- \in \underset{\circ}{\uparrow}m$. Therefore, by Corollary 3.13.(ii), there exists a clopen upset U satisfying $r^- \in U \subseteq \underset{\circ}{\uparrow}m$. Moreover, from Lemma 3.14.(i), we see that $(r^- \prec r) \in {\uparrow}m$ entails that ${\uparrow}r = \{r\}$ is a clopen. Set $U_a := \{r\}$, $U_b := U \setminus \{r\}$, and $U_c := {\downarrow}U_b \setminus U_b$, and note that they are pairwise disjoint nonempty clopens (U_c must contain m, since $m \notin U_b \subseteq \underset{\circ}{\uparrow}m$), that cover $\mathcal{X}$. It is not hard to see that there is a clear surjective bi-Esakia morphism $f\colon \mathcal{X} \twoheadrightarrow \mathfrak{F}_1$ (see Figure 3) defined by $f[U_a] := \{a\}$, $f[U_b] := \{b\}$, and $f[U_c] := \{c\}$. Equivalently, $\mathcal{X} \not\models \mathcal{J}(\mathfrak{F}_1)$, by the Jankov Lemma 2.20. But this contradicts $\mathcal{X} \models L$, as $\mathcal{J}(\mathfrak{F}_1) = \mathcal{J}_1$ is an axiom of L. □

Now, as we assumed $y \in \underset{\circ}{\uparrow}m$, by Lemma 3.13.(ii) there is a clopen upset $U \in \underset{\circ}{\uparrow}m$ such that $y \in U \subseteq \underset{\circ}{\uparrow}m$. We set $V_c := U \setminus {\uparrow}x$ and $V_d := {\downarrow}V_c \setminus V_c$, noting that $m \in V_d$ holds. It is not hard to see that $\{V_a, V_{a'}, V_b, V_c, V_d\}$ is a family of nonempty pairwise disjoint clopens that covers $\mathcal{X}$, and that there is a clear surjective bi-Esakia morphism $f\colon \mathcal{X} \twoheadrightarrow \mathfrak{F}_2$ (see Figure 3) defined by $f[V_a] := \{a\}$ $f[V_{a'}] := \{a'\}$, $f[V_b] := \{b\}$, $f[V_c] := \{c\}$, and $f[V_d] := \{d\}$. But now, the Jankov Lemma 2.20 contradicts $\mathcal{X} \models L$. □

Corollary 3.25. *If ${\prec}r \neq \emptyset$, then ${\prec}r \setminus {\uparrow}m \neq \emptyset$.*

Proof. This statement was established during the proof of the previous Claim. □

We now have a fairly good understanding of the structure of our co-tree $\mathcal{X}$. It has a diagonal ${\uparrow}m$ which has no infinite descending chain (since down limits do not exist), and is therefore well-ordered, and every point not contained in the diagonal must be minimal, with a unique immediate successor contained in ${\uparrow}m$. Furthermore, every point in $\underset{\circ}{\uparrow}m^+$ which is not an up limit has exactly one immediate predecessor lying outside of the diagonal, while up limits have exactly two such predecessors.

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