LECTURE NOTES IN ALGEBRAIC LOGIC

T. MORASCHINI

These are some very informal lecture notes I prepared for a short course at the University of Amsterdam. Please do not circulate them, as this version has not been sufficiently revised.

1. Beyond equational completeness theorems

Notational conventions. In order to simplify the notation, we shall work with a fixed denumerable set of variables $Var = \{x_n : n \in \omega\}$. This means that any syntactic entity mentioned in these notes (formulas, equations, logics etc.) is assumed to be expressed with variables among Var. Accordingly, when we write $x, y, z \dots$ for variables, it should be understood that these are variables in Var. Finally, given an algebraic language $\mathscr{L}$, we denote by $Fm_{\mathscr{L}}$ and $\boldsymbol{Fm}_{\mathscr{L}}$, respectively, the corresponding set of formulas and term algebra with variables in Var. When $\mathscr{L}$ is clear from the context we shall omit the subscript in $Fm_{\mathscr{L}}$ and $\boldsymbol{Fm}_{\mathscr{L}}$ and write simply Fm and $\boldsymbol{Fm}$. Lastly, the classes of algebras we consider are always assumed consist of algebras of the same similarity type. In the same spirit, many time we use the expression "for every algebra" in the sense of "for every algebra in the relevant similarity type".

Logic and algebra are often related by means of equational completeness theorems, which state that a certain logic $\vdash$ is complete with respect to a class of algebras K. Perhaps the most prototypical example of these is the equational completeness theorem of *classical propositional logic* **CPC** stating that for every set of formulas $\Gamma \cup \{\varphi\} \subseteq Fm$,

$$\begin{aligned} \Gamma \vdash_{\mathbf{CPC}} \varphi \Longleftrightarrow & \text{ for every Boolean algebra } \boldsymbol{A} \text{ and homomorphism } h\colon \boldsymbol{Fm} \to \boldsymbol{A}, \\ & \text{if } h(\gamma) = 1^{\boldsymbol{A}} \text{ for all } \gamma \in \Gamma, \text{ then } h(\varphi) = 1^{\boldsymbol{A}}. \end{aligned} \tag{1}$$

In order to simplify the notation, given a class of algebras K, let $\vDash_{\mathsf{K}}$ be the equational consequence defined for every set of equations $\Theta \cup \{\varepsilon \approx \delta\}$, as follows:

$$\begin{aligned} \Theta \vDash_{\mathsf{K}} \varepsilon \approx \delta \Longleftrightarrow & \text{ for every } \boldsymbol{A} \in \mathsf{K} \text{ and homomorphism } h\colon \boldsymbol{Fm} \to \boldsymbol{A}, \\ & \text{if } h(\varphi) = h(\psi) \text{ for all } \varphi \approx \psi \in \Theta, \text{ then } h(\varepsilon) = h(\delta). \end{aligned}$$

The relation $\vDash_{\mathsf{K}}$ is known as the *equational consequence relative to* K.

Accordingly, (1) can be written more succinctly as

$$\Gamma \vdash_{\mathbf{CPC}} \varphi \Longleftrightarrow \{\gamma \approx 1 : \gamma \in \Gamma\} \vDash_{\mathsf{BA}} \varphi \approx 1,$$

where BA is the class of Boolean algebras. Similarly, the equational completeness theorem of *intuitionistic propositional logic* **IPC** with respect to the class of Heyting algebras HA states that for all $\Gamma \cup \{\varphi\} \subseteq Fm$,

$$\Gamma \vdash_{\mathbf{IPC}} \varphi \Longleftrightarrow \{\gamma \approx 1 : \gamma \in \Gamma\} \vDash_{\mathsf{HA}} \varphi \approx 1. \tag{2}$$

Date: June 17, 2020.

Another familiar example of equational completeness theorem comes from the field of modal logic. Let $\mathfrak{F}$ be the class of all Kripke frames. Following [32], we associate two distinct logics with $\mathfrak{F}$. The *global modal logic* $\mathbf{K}_g$ as

$$\begin{aligned}\varGamma \vdash_{\mathbf{K}_g} \varphi \iff & \text{ for every } \langle W, R\rangle \in \mathfrak{F} \text{ and evaluation } v \text{ in } \langle W, R\rangle,\\ & \text{if } w, v \Vdash \varGamma \text{ for all } w \in W, \text{ then } w, v \Vdash \varphi \text{ for all } w \in W,\end{aligned}$$

for every set of modal formulas $\varGamma \cup \{\varphi\}$. On the other hand, the *local modal logic* $\mathbf{K}_l$ is defined setting

$$\begin{aligned}\varGamma \vdash_{\mathbf{K}_l} \varphi \iff & \text{ for every } \langle W, R\rangle \in \mathfrak{F}, w \in W, \text{ and evaluation } v \text{ in } \langle W, R\rangle,\\ & \text{if } w, v \Vdash \varGamma, \text{ then } w, v \Vdash \varphi.\end{aligned}$$

Observe that $\mathbf{K}_g$ and $\mathbf{K}_l$ are not the same logic, since

$$x \vdash_{\mathbf{K}_g} \Box x \text{ and } x \nvdash_{\mathbf{K}_l} \Box x. \tag{3}$$

Actually, it is not hard to show that $\mathbf{K}_g$ is strictly stronger than $\mathbf{K}_l$. For the moment, we shall focus on the global modal logic $\mathbf{K}_g$, whose completeness theorem with respect the the class MA of modal algebras states that

$$\varGamma \vdash_{\mathbf{K}_g} \varphi \iff \{\gamma \approx 1 \colon \gamma \in \varGamma\} \vDash_{\mathsf{MA}} \varphi \approx 1.$$

However, $\mathbf{K}_l$ will also have a cameo in this course.

While equational completeness theorems represent an important aspect of the relation between algebra and logic, they are not sufficient to explain the relation between **CPC** and BA (resp. between **IPC** and HA, or $\mathbf{K}_g$ and MA). To make this claim precise, we shall first identify a precise notion of logic and of equational completeness theorem.

To this end, given a set A, a relation $\vdash \subseteq \mathcal{P}(A) \times A$ is said to be a *consequence relation* on A when for every $X \cup Y \cup \{a\} \subseteq A$,

(i) if $x \in X$, then $X \vdash x$; and
(ii) if $X \vdash y$ for all $y \in Y$ and $Y \vdash a$, then $X \vdash a$.

The relation $X \vdash a$ should be read, intuitively, as "X proves a" or "a follows from X". In this reading, the demand expressed by condition (i) is rather natural, while (ii) should be viewed as a Cut principle.

Exercise 1.1. A *closure operator* on a set A is a map $C \colon \mathcal{P}(A) \to \mathcal{P}(A)$ such that for every $X, Y \subseteq A$,

$$X \subseteq C(C(X)) \subseteq C(X) \text{ and (if } X \subseteq Y, \text{ then } C(X) \subseteq C(Y)).$$

A *closure system* on A is a family $\mathcal{C} \subseteq \mathcal{P}(A)$ containing A and closed under arbitrary intersections. Convince yourself that consequence relations, closure operators, and closure systems are equivalent presentations of the same concept.[1] This observation will be used many times in what follows. ⊠

Definition 1.2. A *logic* is a consequence relation $\vdash$ on the set of formulas $Fm_{\mathscr{L}}$ (of some algebraic language $\mathscr{L}$) that, moreover, is *substitution invariant* in the sense that for every substitution (a.k.a. endomorphism of $\boldsymbol{Fm}$) and every set of formulas $\varGamma \cup \{\varphi\}$,

$$\text{if } \varGamma \vdash \varphi, \text{ then } \sigma[\varGamma] \vdash \sigma(\varphi).$$

[1]Notice that closure systems are complete lattices, whence every consequence relation is naturally associated to a complete lattice. Actually, it can be proved that every complete lattice is isomorphic to a closure system and, therefore, comes from a consequence relation.

Remark 1.3. As mentioned above, $\Gamma \vdash \varphi$ should be read as "Γ proves φ" or "φ follows from Γ". The requirement that $\vdash$ is substitution invariant, instead, is intended to capture the idea that logical inferences are true only in virtue of their form (as opposed to their content). ⊠

One can easily check that **CPC**, **IPC**, $\mathbf{K}_g$ and $\mathbf{K}_l$ are logics in our sense. Moreover, almost everything which has been called "deductive system" in the literature is associated with a consequence relation that is substitution invariant and, therefore, a logic.

Exercise 1.4. One may wonder why logics should in general be identified with consequence relations, and not just with their sets of tautologies (as it is sometimes done in the realm of modal logic). The reason is that different logics may have the same set of tautologies. A typical example is given by the modal logics $\mathbf{K}_g$ and $\mathbf{K}_l$. As shown in (3), these logics are different. Prove that, despite this fact, they have the same set of tautologies, i.e., formulas provable from an empty set of premises. ⊠

In order to clarify what is an equational completeness theorem, for every set of equations $\boldsymbol{\tau}(x)$ and set of formulas $\Gamma \cup \{\varphi\}$, we shall abbreviate

$$\{\varepsilon(\varphi) \approx \delta(\varphi) \colon \varepsilon \approx \delta \in \boldsymbol{\tau}\} \text{ as } \boldsymbol{\tau}(\varphi), \text{ and } \bigcup_{\gamma \in \Gamma} \boldsymbol{\tau}(\gamma) \text{ as } \boldsymbol{\tau}[\Gamma].$$

Definition 1.5. A logic $\vdash$ is said to admit an *equational completeness theorem* if there are a set of equations $\boldsymbol{\tau}(x)$ and a class K of algebras such that for all $\Gamma \cup \{\varphi\} \subseteq Fm$,

$$\Gamma \vdash \varphi \Longleftrightarrow \boldsymbol{\tau}[\Gamma] \vDash_{\mathsf{K}} \boldsymbol{\tau}(\varphi).$$

In this case K is said to be a *$\boldsymbol{\tau}$-algebraic semantics* (or simply an *algebraic semantics*) for $\vdash$.

This notion was introduced in [10] and studied for instance in [17, 35]. In this parlance, (1) states that the class of Boolean algebras BA is a $\boldsymbol{\tau}$-algebraic semantics for **CPC** where $\boldsymbol{\tau} = \{x \approx 1\}$. Similarly, the class of Heyting algebras (resp. modal algebras) is a $\boldsymbol{\tau}$-algebraic semantics for **IPC** (resp. $\mathbf{K}_g$). One may be tempted to conjecture that the relation between **CPC** and BA amounts to the fact that BA is an algebraic semantics for **CPC**. As we anticipated, however, this is *not* the case. To explain why the relation between **CPC** and BA cannot be explained in terms of completeness theorems only, it is convenient to recall classical result relating **CPC** and **IPC**:

Theorem 1.6 (Glivenko)**.** [26] *For every set of formulas $\Gamma \cup \{\varphi\}$,*

$$\Gamma \vdash_{\mathbf{CPC}} \varphi \Longleftrightarrow \{\neg\neg\gamma \colon \gamma \in \Gamma\} \vdash_{\mathbf{IPC}} \neg\neg\varphi.$$

Together with (2), Glivenko's Theorem implies that for every set of formulas $\Gamma \cup \{\varphi\}$,

$$\Gamma \vdash_{\mathbf{CPC}} \varphi \Longleftrightarrow \boldsymbol{\tau}[\Gamma] \vDash_{\mathsf{HA}} \boldsymbol{\tau}(\varphi),$$

where $\boldsymbol{\tau}(x) \coloneqq \{\neg\neg x \approx 1\}$. Consequently, the class of Heyting algebras is also an algebraic semantics for **CPC**. In other words, **CPC** admits an equational completeness theorem not only with respect to the class of Boolean algebras, but also with respect to that of Heyting algebras. This means that the univocal relation between **CPC** and the class of Boolean algebras cannot be explained in terms of the existence of completeness theorems only. On the contrary, as we will see, the origin of this relation should be recognized in a deeper phenomenon, known as *algebraizability* [10, 22].

Exercise 1.7. One may wonder whether the fact that **CPC** has many distinct algebraic semantics cannot be amended by restricting our attention to $\boldsymbol{\tau}$-algebraic semantics where $\boldsymbol{\tau} = \{x \approx 1\}$. This is not the case, as this exercise asks you to check. Let $\boldsymbol{A}$ be the three-element algebra $\langle \{0, 1, a\}; \wedge, \vee, \neg, 0, 1\rangle$ where $\langle A; \wedge, \vee\rangle$ is the lattice with order $0 < a < 1$ and $\neg \colon A \to A$ is the map described by the rule

$$\neg 0 = \neg a = 1 \text{ and } \neg 1 = 0.$$

Clearly, $\boldsymbol{A}$ is not a Boolean algebra (actually, there is no three-element Boolean algebra at all). Prove that $\{\boldsymbol{A}\}$ is $\boldsymbol{\tau}$-algebraic semantics for **CPC** where $\boldsymbol{\tau} = \{x \approx 1\}$. Hint: use the fact that the two-element Boolean algebra is a homomorphic image of $\boldsymbol{A}$. ⊠

2. Varieties and quasi-varieties

In order to introduce a robust theory of the algebraization of logic, which could account for the relation between **CPC** and BA, it is convenient to recall some basic facts from the theory of varieties [6, 18] and quasi-varieties [27, 33]. To this end, we denote by $\mathbb{I}, \mathbb{H}, \mathbb{S}, \mathbb{P}, \mathbb{P}_{\text{U}}$, and $\mathbb{P}_{\text{SD}}$ the class operators producing, respectively, isomorphic copies, homomorphic images, subalgebras, direct products, ultraproducts, and subdirect products. We assume that, when applied to empty families of algebeas, product-like constructions produce trivial algebras. Accordingly, for every class of algebras K, the classes $\mathbb{P}(\mathsf{K})$, $\mathbb{P}_{\text{U}}(\mathsf{K})$, and $\mathbb{P}_{\text{SD}}(\mathsf{K})$ contain all trivial algebras. Moreover, we assume some familiarity with basic universal algebra, for which we refer to [6, 18].

A class of algebras is said to be a *variety* when it can be axiomatized by means of (universally quantified) equations.

Theorem 2.1 (Birkhoff)**.** [6, Thm. 4.41] *A class of algebras is a variety if and only if it is closed under $\mathbb{H}, \mathbb{S}$, and $\mathbb{P}$.*

A *quasi-equation* is an expression Φ of the form

$$\bigwedge_{i<n} \varphi_i \approx \psi_i \to \varepsilon \approx \delta,$$

i.e., an implication between a finite set of equations $\{\varphi_i \approx \psi_i \colon i < n\}$ and an equation $\varepsilon \approx \delta$. Observe that if $n = 0$, then Φ is an equation. Consequently, equations can be identified with quasi-equations with empty antecedent. A quasi-equation Φ is *valid* in an algebra $\boldsymbol{A}$ when its universal closure is. In this case, we write $\boldsymbol{A} \vDash \Phi$. Furthermore, Φ is *valid* in a class of algebras K when it is valid in all its members, in which case we write $\mathsf{K} \vDash \Phi$. Notice that the validity of a quasi-equation in K can be expressed in terms of its relative equational consequence $\vDash_{\mathsf{K}}$, because

$$\mathsf{K} \vDash \bigwedge_{i<n} \varphi_i \approx \psi_i \to \varepsilon \approx \delta \Longleftrightarrow \{\varphi_i \approx \psi_i \colon i < n\} \vDash_{\mathsf{K}} \varepsilon \approx \delta.$$

A *quasi-variety* is a class of algebras axiomatizable by means of quasi-equations. Of course, all varieties are quasi-varieties, but the converse is not true in general, as we shall see in Example 2.7. Our first aim is to provide a characterization of quasi-varieties similar to that of Birkhoff's Theorem 2.1. To this end, it is convenient to recall some basic concepts. Let $\boldsymbol{A}$ be an algebra. A *local subgraph* $\boldsymbol{X}$ of $\boldsymbol{A}$ is a finite subset $X \subseteq A$ endowed with the restriction of finitely many basic operations of $\boldsymbol{A}$ to X. Observe that $\boldsymbol{X}$ is a finite *partial* algebra of finite type (even if $\boldsymbol{A}$ might be of infinite type).

Lemma 2.2. *Let $\mathsf{K} \cup \{\boldsymbol{A}\}$ be a class of algebras. If all local subgraphs of $\boldsymbol{A}$ can be embedded into some member of K, then $\boldsymbol{A} \in \mathbb{ISP}_{\mathrm{U}}(\mathsf{K})$.*

Exercise 2.3. Complete the following sketch of the proof of Lemma 2.2. Let I be the set of local subgraphs of $\boldsymbol{A}$. By assumption, for every $\boldsymbol{B} \in I$ there is an algebra $\boldsymbol{C_B} \in \mathsf{K}$ in which $\boldsymbol{B}$ embeds. Given $\boldsymbol{B} \in I$, define

$$J_{\boldsymbol{B}} \coloneqq \{\boldsymbol{D} \in I \colon \boldsymbol{D} \text{ embeds into } \boldsymbol{C_B}\}.$$

Observe that the family $\mathcal{F} \coloneqq \{J_{\boldsymbol{B}} \colon \boldsymbol{B} \in I\}$ has the finite intersection property. Consequently, $\mathcal{F}$ can be extended to an ultrafilter U on I. As $\boldsymbol{A}$ embeds naturally into $\prod_{\boldsymbol{B} \in I} \boldsymbol{C_B}/U$, we conclude that $\boldsymbol{A} \in \mathbb{ISP}_{\mathrm{U}}(\mathsf{K})$. ⊠

We are now ready to prove the quasi-variety counterpart of Theorem 2.1.

Theorem 2.4 (Maltsev)**.** *A class of algebras is a quasi-variety if and only if it is closed under $\mathbb{I}, \mathbb{S}, \mathbb{P}$, and $\mathbb{P}_{\mathrm{U}}$.*

Proof. The "only if" part follows from the fact that the validity of quasi-equations is preserved by the class operators $\mathbb{I}, \mathbb{S}, \mathbb{P}$, and $\mathbb{P}_{\mathrm{U}}$. To prove the converse, consider a class of algebras K closed under $\mathbb{I}, \mathbb{S}, \mathbb{P}$, and $\mathbb{P}_{\mathrm{U}}$. Moreover, let Σ be the set of quasi-equations valid in K. Our aim is to prove that any algebra satisfying Σ belongs to K.

To this end, consider an algebra $\boldsymbol{A}$ such that $\boldsymbol{A} \vDash \Sigma$. Let also $\boldsymbol{B}$ be a local subgraph of $\boldsymbol{A}$. By definition, $\boldsymbol{B}$ is a finite set $\{b_1, \dots, b_n\}$ endowed with the restriction of finitely many basic operations $f_1, \dots, f_m$ of $\boldsymbol{A}$ to B. Define

$$\begin{aligned}\mathcal{D}^+(\boldsymbol{B}) \coloneqq \{f_i(x_{k_1}, \dots, x_{k_t}) \approx x_j \colon i \leqslant m \text{ and } j, k_1, \dots, k_t \leqslant n \\ \text{and } f_i^{\boldsymbol{A}}(b_{k_1}, \dots, b_{k_t}) = b_j\}\end{aligned}$$

and

$$\mathcal{D}^-(\boldsymbol{B}) \coloneqq \{x_i \not\approx x_j \colon i, j \leqslant n \text{ and } b_i \neq b_j\}.$$

Observe that both $\mathcal{D}^+(\boldsymbol{B})$ and $\mathcal{D}^-(\boldsymbol{B})$ are finite sets. Then take an enumeration

$$\mathcal{D}^-(\boldsymbol{B}) = \{\varepsilon_1 \not\approx \delta_1, \dots, \varepsilon_k \not\approx \delta_k\}.$$

For each $i \leqslant k$, consider the quasi-equation

$$\Phi_i \coloneqq \bigwedge \mathcal{D}^+(\boldsymbol{B}) \to \varepsilon_i \approx \delta_i.$$

As witnessed by the local subgraph $\boldsymbol{B}$, the quasi-equation Φ_i fails in $\boldsymbol{A}$. Since $\boldsymbol{A} \vDash \Sigma$, this implies that Φ_i fails in some $\boldsymbol{C}_i \in \mathsf{K}$ under some assignment

$$x_1 \longmapsto c_1^i, \dots, x_n \longmapsto c_n^i.$$

Bearing this in mind, it is easy to see that the local subgraph $\boldsymbol{B}$ embeds in the direct product $\boldsymbol{C}_1 \times \dots \times \boldsymbol{C}_k$ under the assignment

$$b_1 \longmapsto \langle c_1^1, \dots, c_1^k \rangle, \dots, b_n \longmapsto \langle c_n^1, \dots, c_n^k \rangle.$$

As K is closed under $\mathbb{P}$, we conclude that $\boldsymbol{B}$ embeds into a member of K. Thus, every local subgraph of $\boldsymbol{A}$ can be embedded into a member of K. By Lemma 2.2, we get $\boldsymbol{A} \in \mathbb{ISP}_{\mathrm{U}}(\mathsf{K})$. Since K is closed under $\mathbb{I}, \mathbb{S}$, and $\mathbb{P}_{\mathrm{U}}$, we conclude that $\boldsymbol{A} \in \mathsf{K}$. ⊠

Exercise 2.5. Let K be a class of algebras closed under $\mathbb{I}, \mathbb{S}$, and $\mathbb{P}$ and take a proper class of variables

$$Var^{+} = \{x_{\alpha} : \alpha \text{ is an ordinal}\}.$$

Let also Eq^{+} be the class of equations with variables in Var^{+} and $\vDash^{+}_{\mathsf{K}}$ be the consequence relation defined for every *set* $\Theta \cup \{\varepsilon \approx \delta\} \subseteq Eq^{+}$ with variables in $\{\alpha_{\alpha} : \alpha < \lambda\}$ as

$$\begin{aligned}\Theta \vDash_{\mathsf{K}} \varepsilon \approx \delta \Longleftrightarrow & \text{ for every } \boldsymbol{A} \in \mathsf{K} \text{ and homomorphism } h \colon \boldsymbol{Fm}(\lambda) \to \boldsymbol{A},\\ & \text{if } h(\varphi) = h(\psi) \text{ for all } \varphi \approx \psi \in \Theta, \text{ then } h(\varepsilon) = h(\delta),\end{aligned}$$

where $\boldsymbol{Fm}(\lambda)$ is the algebra of formulas with variables in $\{\alpha_{\alpha} : \alpha < \lambda\}$. Formally, $\vDash^{+}_{\mathsf{K}}$ can be viewed as a sequence of sets (one for each infinite cardinal λ).

Show that K is a quasi-variety if and only if the equational consequence $\vDash^{+}_{\mathsf{K}}$ is *finitary* in the sense that for every set of equations $\Theta \cup \{\varepsilon \approx \delta\}$ with variables in Var^{+},

$$\text{if } \Theta \vDash^{+}_{\mathsf{K}} \varepsilon \approx \delta, \text{ then } \Delta \vDash_{\mathsf{K}} \varepsilon \approx \delta \text{ for some finite } \Delta \subseteq \Theta.$$

For the "only if" part: use Theorem 2.4 (the proof requires also an ultraproduct construction). For the "if" one, one can reason as follows. Suppose that $\vDash^{+}_{\mathsf{K}}$ is finitary. In view of Theorem 2.4, it suffices to show that K is closed under $\mathbb{P}_{\mathrm{U}}$. To this end, take a ultraproduct $\boldsymbol{A}$ of some family $\{\boldsymbol{A}_i : i \in I\}$ of algebras in K.

We claim that for every set equations $\Theta \cup \{\varepsilon \approx \delta\}$ in variables Var^{+} such that $\Theta \vDash^{+}_{\mathsf{K}} \varepsilon \approx \delta$, the algebra $\boldsymbol{A}$ satisfies the infinitary quasi-equation

$$\bigwedge \Theta \to \varepsilon \approx \delta.$$

To prove this, notice that, because of the finitarity of $\vDash^{+}_{\mathsf{K}}$, there is a finite $\Delta \subseteq \Theta$ such that $\Delta \vDash_{\mathsf{K}} \varepsilon = \delta$, i.e., K satisfies the quasi-equation $\bigwedge \Delta \to \varepsilon \approx \delta$. As quasi-equations are preserved under ultraproducts, $\boldsymbol{A}$ satisfies the quasi-equation $\bigwedge \Delta \to \varepsilon \approx \delta$. Consequently, it also satisfies the infinitary quasi-equation $\bigwedge \Theta \to \varepsilon \approx \delta$, thus establishing the lemma.

Now, let $\lambda = |A|$ and $\boldsymbol{Fm}(\lambda)$ be the formula algebra with variables in $\{x_{\alpha} : \alpha < \lambda\}$. Clearly, there is a surjective homomorphism $h : \boldsymbol{Fm}(\lambda) \to \boldsymbol{A}$. Its kernel $\mathrm{Ker}(h)$ is a set of pairs of formulas and, therefore, it can be viewed as a set of equations. Furthermore, for every pair of formulas $\langle \varepsilon, \delta \rangle \in (Fm(\lambda) \times Fm(\lambda)) \smallsetminus \mathrm{Ker}(h)$, we have that $\boldsymbol{A}$ rejects the infinitary quasi-equation $\bigwedge \mathrm{Ker}(h) \to \varepsilon \approx \delta$, whence, by the claim, $\mathrm{Ker}(h) \nvDash^{+}_{\mathsf{K}} \varepsilon \approx \delta$. Consequently, there is an algebra $\boldsymbol{B}_{\langle \varepsilon, \delta \rangle} \in \mathsf{K}$ and a homomorphism $g_{\langle \varepsilon, \delta \rangle} \colon \boldsymbol{Fm}(\lambda) \to \boldsymbol{B}_{\langle \varepsilon, \delta \rangle}$ such that $g_{\langle \varepsilon, \delta \rangle}(\varphi) = g_{\langle \varepsilon, \delta \rangle}(\psi)$ for all $\varphi \approx \psi \in \mathrm{Ker}(h)$ and $g_{\langle \varepsilon, \delta \rangle}(\varepsilon) \neq g_{\langle \varepsilon, \delta \rangle}(\delta)$. Recall that $h[Fm(\lambda)] = A$. Bearing this in mind, let

$$g \colon \boldsymbol{A} \to \prod_{\langle \varepsilon, \delta \rangle \in (Fm(\lambda) \times Fm(\lambda)) \smallsetminus \mathrm{Ker}(h)} \boldsymbol{B}_{\langle \varepsilon, \delta \rangle}$$

be the map that sends an element $h(\varphi)$ of $\boldsymbol{A}$ to the sequence

$$\langle g_{\langle \varepsilon, \delta \rangle}(\varphi) : \langle \varepsilon, \delta \rangle \in (Fm(\lambda) \times Fm(\lambda)) \smallsetminus \mathrm{Ker}(h) \rangle.$$

It is not hard to see that h is a well-defined embedding, whence $\boldsymbol{A} \in \mathbb{ISP}(\mathsf{K}) \subseteq \mathsf{K}$, as desired. ⊠

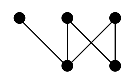
Given a class of algebras K, we denote by $\mathbb{V}(\mathsf{K})$ and $\mathbb{Q}(\mathsf{K})$, respectively, the smallest variety and quasi-variety containing K. These can be described as follows:

Corollary 2.6 (Tarski & Maltsev)**.** *For every class of algebras* K*,*

$$\mathbb{V}(\mathsf{K}) = \mathbb{HSP}(\mathsf{K}) \text{ and } \mathbb{Q}(\mathsf{K}) = \mathbb{ISPP}_{\text{U}}(\mathsf{K}).$$

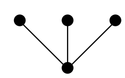
Proof sketch. In view of Theorems 2.1 and 2.4 it suffices to show that $\mathbb{HSP}(\mathsf{K})$ and $\mathbb{ISPP}_{\text{U}}(\mathsf{K})$ are closed, respectively, under $\mathbb{H}, \mathbb{S}, \mathbb{P}$ and under $\mathbb{I}, \mathbb{S}, \mathbb{P}, \mathbb{P}_{\text{U}}$. This is a consequence of standard properties of class operators, see for instance [18]. ⊠

Example 2.7. As promised, we are in the position to exhibit the example of a quasi-variety that is not a variety. To this end, we assume the reader is familiar with *Esakia duality* for Heyting algebras [21]. Let $\boldsymbol{A}$ be the finite Heyting algebra whose dual Esakia space is the following poset $\boldsymbol{A}_*$ endowed with the discrete topology:



Or aim is to show that $\mathbb{Q}(\boldsymbol{A})$ is a quasi-variety that is not a variety.

That $\mathbb{Q}(\boldsymbol{A})$ is a quasi-variety is obvious. To prove that it is not a variety, we reason as follows. First, consider the finite Heyting algebra $\boldsymbol{B}$ whose dual Esakia space is the the rooted poset $\boldsymbol{B}_*$ (endowed with the discrete topology) depicted below:

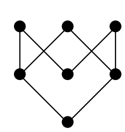


Notice that, as $\boldsymbol{B}_*$ is rooted, the algebra $\boldsymbol{B}$ is subdirectly irreducible. Moreover, observe that, as $\boldsymbol{B}_*$ is an upset of $\boldsymbol{A}_*$, by Esakia duality we obtain $\boldsymbol{B} \in \mathbb{H}(\boldsymbol{A}) \subseteq \mathbb{V}(\boldsymbol{A})$. Therefore, to prove that $\mathbb{Q}(\boldsymbol{A})$ is not a variety, it suffices to show that $\boldsymbol{B} \notin \mathbb{Q}(\boldsymbol{A})$. Suppose the contrary, with a view to contradiction. By Corollary 2.6, $\boldsymbol{B} \in \mathbb{Q}(\boldsymbol{A}) = \mathbb{ISPP}_{\text{U}}(\boldsymbol{A})$. As $\boldsymbol{A}$ is a finite algebra, $\mathbb{P}_{\text{U}}(\boldsymbol{A}) = \mathbb{I}(\boldsymbol{A})$, whence

$$\boldsymbol{B} \in \mathbb{ISP}(\boldsymbol{A}) \subseteq \mathbb{IP}_{\text{SD}}\mathbb{S}(\boldsymbol{A}).$$

As $\boldsymbol{B}$ is subdirectly irreducible, we conclude that $\boldsymbol{B} \in \mathbb{IS}(\boldsymbol{A})$. By Esakia duality, this means that $\boldsymbol{B}_*$ is a p-morphic image of $\boldsymbol{A}_*$. But a quick inspection of the posets $\boldsymbol{A}_*$ and $\boldsymbol{B}_*$ shows that this is impossible, a contradiction. Hence, we conclude that $\mathbb{Q}(\boldsymbol{A})$ is a quasi-variety that is not a variety, as desired. ⊠

Exercise 2.8. Show that the finite Heyting algebra, whose Esakia dual is the following poset (endowed with the discrete topology), generates a quasi-variety that is not a variety.



Of course, take inspiration from Example 2.7. ⊠

Quasi-varieties K need not be closed under homomorphic images. Consequently, if $\boldsymbol{A} \in \mathsf{K}$ and θ is a congruence of $\boldsymbol{A}$, one should not in principle expect that $\boldsymbol{A}/\theta \in \mathsf{K}$. Accordingly, a congruence θ of an algebra $\boldsymbol{B}$ is said to be a K-*congruence* when $\boldsymbol{B}/\theta \in \mathsf{K}$. The poset of K-congruences of $\boldsymbol{B}$ is denoted by $\mathrm{Con}_{\mathsf{K}}\boldsymbol{B}$ (notice that $\boldsymbol{B}$ is not assumed to be in K). Observe that if K is a variety and $\boldsymbol{B} \in \mathsf{K}$, then $\mathrm{Con}_{\mathsf{K}}\boldsymbol{B}$ coincides with the lattice of all congruences of $\boldsymbol{B}$, which we denote by $\mathrm{Con}\boldsymbol{B}$.

Given two algebras $\boldsymbol{A}$ and $\boldsymbol{B}$, we write $\boldsymbol{A} \leqslant \boldsymbol{B}$ to denote the fact that $\boldsymbol{A}$ embeds into $\boldsymbol{B}$.

Lemma 2.9. *Let* K *be a quasi-variety and* $\boldsymbol{A}$ *an algebra. Then* $\mathrm{Con}_{\mathsf{K}}\boldsymbol{A}$ *is a closure system on* $A \times A$*.*

Proof. Quasi-varieties contain trivial algebras, whence the total congruence of $\boldsymbol{A}$ belongs to $\mathrm{Con}_{\mathsf{K}}\boldsymbol{A}$. Thus, the poset $\mathrm{Con}_{\mathsf{K}}\boldsymbol{A}$ has a maximum. Consequently, to conclude the proof, it suffices to show that $\mathrm{Con}_{\mathsf{K}}\boldsymbol{A}$ is closed under intersections of arbitrary non-empty families. To this end, consider a non-empty family $\{\theta_i : i \in I\} \subseteq \mathrm{Con}_{\mathsf{K}}\boldsymbol{A}$. Observe that

$$\boldsymbol{A}/\bigcap_{i\in I}\theta_i \leqslant \prod_{i\in I}\boldsymbol{A}/\theta_i,$$

under the embedding defined by the rule

$$a/\bigcap_{i\in I}\theta_i \longmapsto \langle a/\theta_i : i \in I\rangle.$$

Since $\{\theta_i : i \in I\} \subseteq \mathrm{Con}_{\mathsf{K}}\boldsymbol{A}$, this implies $\boldsymbol{A}/\bigcap_{i\in I}\theta_i \in \mathbb{ISP}(\mathsf{K})$. Furthermore, as K is a quasi-variety, it is closed under $\mathbb{I}, \mathbb{S}$, and $\mathbb{P}$. We conclude that $\boldsymbol{A}/\bigcap_{i\in I}\theta_i \in \mathsf{K}$ and, therefore, $\bigcap_{i\in I}\theta_i \in \mathrm{Con}_{\mathsf{K}}\boldsymbol{A}$. ⊠

Corollary 2.10. *Quasi-varieties contain free algebras.*

Proof. Let K be a quasi-variety. Given a cardinal λ, let $\boldsymbol{Fm}(\lambda)$ be the algebra of formulas with variables $\{x_\alpha : \alpha < \lambda\}$. The free algebra over K generated by $\{x_\alpha : \alpha < \lambda\}$ is $\boldsymbol{Fm}(\lambda)/\theta$ where $\theta = \bigcap \mathrm{Con}_{\mathsf{K}}\boldsymbol{Fm}(\lambda)$. From Lemma 2.9 it follows $\theta \in \mathrm{Con}_{\mathsf{K}}\boldsymbol{Fm}(\lambda)$, whence $\boldsymbol{Fm}(\lambda)/\theta \in \mathsf{K}$, as desired. ⊠

Remark 2.11. The reader might have noticed that the proof of Lemma 2.9 depends only on the fact that K is closed under $\mathbb{I}, \mathbb{S}$, and $\mathbb{P}$. Classes of algebras closed under $\mathbb{I}, \mathbb{S}$, and $\mathbb{P}$ have been called *prevarieties*. These can be axiomatized by *proper classes* of infinitary quasi-equations. The demand that proper classes can be replaced by sets in the axiomatization of prevarieties is equivalent to *Vopěnka Principle* in set theory [27, Prop. 2.3.18] (see also [1]), but this is a different story. ⊠

In view of Lemma 2.9, if K is a quasi-variety and $\boldsymbol{A} \in \mathsf{K}$, then the operation

$$\mathrm{Cg}^{\boldsymbol{A}}_{\mathsf{K}} : \mathcal{P}(A \times A) \to \mathcal{P}(A \times A)$$

of K-congruence generation on $\boldsymbol{A}$ is a closure operator. If K is a variety and $\boldsymbol{A} \in \mathsf{K}$, then $\mathrm{Cg}^{\boldsymbol{A}}_{\mathsf{K}}$ coincides with the operation of congruence generation on $\boldsymbol{A}$, which we denote by $\mathrm{Cg}^{\boldsymbol{A}}$ (as it depends on $\boldsymbol{A}$ only). Given $a, c \in A$, we abbreviate $\mathrm{Cg}^{\boldsymbol{A}}_{\mathsf{K}}(\{\langle a, c\rangle\})$ as $\mathrm{Cg}^{\boldsymbol{A}}_{\mathsf{K}}(a, c)$. Bearing this in mind, a K-congruence θ of $\boldsymbol{A}$ is said to be *principal* if there exists a pair $\langle a, c\rangle \in A \times A$ such that

$$\theta = \mathrm{Cg}^{\boldsymbol{A}}_{\mathsf{K}}(a, c).$$

In order to shed light on the structure of lattices of K-congruences, recall that an element a of lattice $\boldsymbol{L}$ is said to be *compact* when for every set $X \subseteq L$ whose supremum exists in $\boldsymbol{L}$,

if $a \leqslant \bigvee X$, then there are finitely many $x_1, \dots, x_n \in X$ such that $a \leqslant x_1 \vee \dots \vee x_n$.

Then a lattice $\boldsymbol{L}$ is said to be *algebraic* if it is complete and all its elements are joins of compact ones.

Theorem 2.12. *Let* K *be a quasi-variety and* $\boldsymbol{A} \in \mathsf{K}$*. Then* $\mathrm{Con}_{\mathsf{K}}\boldsymbol{A}$ *is an algebraic lattice, whose compact elements are finitely generated* K*-congruences of* $\boldsymbol{A}$*, i.e., those of the form* $\mathrm{Cg}_{\mathsf{K}}^{\boldsymbol{A}}(X)$ *for a finite* $X \subseteq A \times A$*.*

Proof. That $\mathrm{Con}_{\mathsf{K}}\boldsymbol{A}$ is complete was proved in Lemma 2.9. Furthermore, it is clear that every element of $\mathrm{Con}_{\mathsf{K}}\boldsymbol{A}$ is a join of finitely generated K-congruences. Therefore, to conclude the proof, it suffices to show that the compact elements of $\mathrm{Con}_{\mathsf{K}}\boldsymbol{A}$ are precisely the finitely generated K-congruences. This easy (and tedious) exercise is left to the (patient) reader. ⊠

Given an algebra $\boldsymbol{A}$, we denote by Id_A the identity congruence of $\boldsymbol{A}$. Let K be a quasi-variety and $\boldsymbol{A} \in \mathsf{K}$. Then $\boldsymbol{A}$ is said to be *relatively subdirectly irreducible in* K if Id_A is completely meet-irreducible in $\mathrm{Con}_{\mathsf{K}}\boldsymbol{A}$., i.e., for every $\{\theta_i \colon i \in I\} \subseteq \mathrm{Con}_{\mathsf{K}}\boldsymbol{A}$,

$$\text{if } \bigcap_{i\in I}\theta_i = \mathrm{Id}_A\text{, then there is } i \in I \text{ such that } \theta_i = \mathrm{Id}_A.$$

Notice that this is *equivalent* to the assertion that there is a smallest congruence θ in $\mathrm{Con}_{\mathsf{K}}\boldsymbol{A} \smallsetminus \{\mathrm{Id}_A\}$. In this case, θ is generated by every pair $\langle a, c\rangle \in \theta$ such that $a \neq c$, in the sense that

$$\theta = \mathrm{Cg}_{\mathsf{K}}^{\boldsymbol{A}}(a, c).$$

This fact will be used repeatedly later on. An algebra $\boldsymbol{A}$ is said to be *subdirectly irreducible* (in the absolute sense) when Id_A it completely meet-irreducible in $\mathrm{Con}\boldsymbol{A}$. This is the notion you probably met in the realm of Heyting and modal algebras. The next result generalizes Birkhoff's subdirect representation theorem to the setting of quasi-varieties.

Theorem 2.13. *Every member of a quasi-variety* K *is a subdirect product of algebras that are relatively subdirectly irreducible in* K*.*

Proof. Similar to that of Birkhoff's subdirect representation theorem [6, Thm. 3.24]. ⊠

Let K be a quasi-variety. An algebra $\boldsymbol{A} \in \mathsf{K}$ is said to be *relatively simple in* K if $\mathrm{Con}_{\mathsf{K}}\boldsymbol{A}$ has exactly two elements. Notice that all relatively simple algebras are relatively subdirectly irreducible. An algebra $\boldsymbol{A}$ is said to be *simple* (in the absolute sense) when $\mathrm{Con}\boldsymbol{A}$ has exactly two elements. Given a quasi-variety K, we denote by $\mathsf{K}_{\mathrm{RSI}}$ (resp. K_{RS}) the class of its relatively subdirectly irreducible (resp. relatively simple) members. Furthermore, given an arbitrary class of algebras K (not necessarily a quasi-variety), we denote by K_{SI} (resp. K_{S}) the class of its subdirectly irreducible (resp. simple) members. Notice that if K is a variety, $\mathsf{K}_{\mathrm{RSI}} = \mathsf{K}_{\mathrm{SI}}$ and $\mathsf{K}_{\mathrm{RS}} = \mathsf{K}_{\mathrm{S}}$. The following observation helps locating $\mathsf{K}_{\mathrm{RSI}}$ inside a quasi-variety K.

Lemma 2.14. *If* K *is a class of algebras, then* $\mathbb{Q}(\mathsf{K})_{\mathrm{RSI}} \subseteq \mathbb{ISP}_{\mathrm{U}}(\mathsf{K})$*.*

Proof. Consider an algebra $\boldsymbol{A} \in \mathbb{Q}(\mathsf{K})_{\mathrm{RSI}}$. Recall from Corollary 2.6 that $\mathbb{Q}(\mathsf{K}) = \mathbb{ISPP}_{\mathrm{U}}(\mathsf{K})$. As it is easy to see that $\mathbb{ISPP}_{\mathrm{U}}(\mathsf{K}) = \mathbb{IP}_{\mathrm{SD}}\mathbb{SP}_{\mathrm{U}}(\mathsf{K})$, this implies $\boldsymbol{A} \in \mathbb{IP}_{\mathrm{SD}}\mathbb{SP}_{\mathrm{U}}(\mathsf{K})$. Accordingly, there is a subdirect embedding $f \colon \boldsymbol{A} \to \prod_{i\in I}\boldsymbol{B}_i$ for some $\{\boldsymbol{B}_i \colon i \in I\} \subseteq \mathbb{SP}_{\mathrm{U}}(\mathsf{K})$. For every $j \in I$, let θ_j be the kernel of the homomorphism $\pi_j \circ f \colon \boldsymbol{A} \to \boldsymbol{B}_i$, where $\pi_j \colon \prod_{i\in I}\boldsymbol{B}_i \to \boldsymbol{B}_j$ is the natural projection. As $\boldsymbol{A}/\theta_j \leqslant \boldsymbol{B}_j \in \mathbb{Q}(\mathsf{K})$ and $\mathbb{Q}(\mathsf{K})$ is closed under subalgebras, θ_j is a $\mathbb{Q}(\mathsf{K})$-congruence of $\boldsymbol{A}$. Moreover, since f is injective, $\bigcap_{i\in I}\theta_i = \mathrm{Id}_A$. Since $\boldsymbol{A}$ is relatively subdirectly irreducible in $\mathbb{Q}(\mathsf{K})$, there is $i \in I$ such that $\theta_i = \mathrm{Id}_A$. Consequently, $\boldsymbol{A}$ embeds into $\boldsymbol{B}_i$, whence $\boldsymbol{A} \in \mathbb{IS}(\boldsymbol{B}_i) \subseteq \mathbb{ISP}_{\mathrm{U}}(\mathsf{K})$. ⊠

Exercise 2.15. Let $\boldsymbol{A}$ be the Heyting algebra described in Example 2.7. Clearly, $\boldsymbol{A}$ is not subdirectly irreducible in the standard sense, as the dual of $\boldsymbol{A}$ is not rooted. Prove that, however, $\boldsymbol{A}$ is relatively subdirectly irreducible in $\mathbb{Q}(\boldsymbol{A})$. Accordingly, algebras that are not subdirectly irreducible in the absolute sense might become relatively subdirectly irreducible in some quasi-variety.

Hint: observe that, in view of Theorem 2.13, $\boldsymbol{A}$ is the subdirect product of some $\{\boldsymbol{B}_i : i \in I\}$ that are relatively subdirectly irreducible in $\mathbb{Q}(\boldsymbol{A})$. Furthermore, by Lemma 2.14, each $\boldsymbol{B}_i$ belongs to $\mathbb{IS}(\boldsymbol{A})$ (recall that, as $\boldsymbol{A}$ is finite, $\mathbb{IP}_{\text{U}}(\boldsymbol{A}) = \mathbb{I}(\boldsymbol{A})$). Use these facts to conclude that $\boldsymbol{A}$ is relatively subdirectly irreducible in $\mathbb{Q}(\boldsymbol{A})$. ⊠

We conclude this section with a technical lemma, which will be needed later on. For the moment you may safely skip to the next section, and come back to it when it is needed.

Lemma 2.16. *Let* K *be a quasi-variety,* $\boldsymbol{A}$ *an algebra, and* $a, c \in A$. *Then*

$$\mathrm{Cg}^{\boldsymbol{A}}_{\mathsf{K}}(a,c) = \bigcup\{\mathrm{Cg}^{\boldsymbol{B}}_{\mathsf{K}}(a,c) \colon \boldsymbol{B} \in \mathbb{S}(\boldsymbol{A}) \textit{ is finitely generated and } a,c \in B\}.$$

Proof. The inclusion from right to left is an easy exercise. In order to prove the converse, consider the following relation on A:

$$R := \bigcup\{\mathrm{Cg}^{\boldsymbol{B}}_{\mathsf{K}}(a,c) \colon \boldsymbol{B} \in \mathbb{S}(\boldsymbol{A}) \text{ is finitely generated and } a,c \in B\}.$$

It is easy to see that R is a congruence of $\boldsymbol{A}$ containing the pair $\langle a, c\rangle$ (why?). In order to check that R is also a K-congruence of $\boldsymbol{A}$, consider a quasi-equation

$$\bigwedge_{i<n} \varphi_i \approx \psi_i \to \varepsilon \approx \delta$$

valid in K. Observe that only finitely many variables, say $x_1, \dots, x_m$, occur in the above quasi-equation. Then consider $b_1, \dots, b_m \in A$ such that

$$\langle \varphi_i^{\boldsymbol{A}}(b_1, \dots, b_m), \psi_i^{\boldsymbol{A}}(b_1, \dots, b_m)\rangle \in R,$$

for all $i < n$. For all $i < n$, there is a finitely generated subalgebra $\boldsymbol{B}_i$ of $\boldsymbol{A}$ such that $a, c \in B$ and

$$\langle \varphi_i^{\boldsymbol{A}}(b_1, \dots, b_m), \psi_i^{\boldsymbol{A}}(b_1, \dots, b_m)\rangle \in \mathrm{Cg}^{\boldsymbol{B}_i}_{\mathsf{K}}(a,c). \tag{4}$$

Then there exists a finitely generated subalgebra $\boldsymbol{C}$ of $\boldsymbol{A}$ such that

$$\{b_1, \dots, b_m, a, c\} \cup \bigcup_{i<n} B_i \subseteq C.$$

Because of (4) and $\boldsymbol{B}_i \leqslant \boldsymbol{C}$, we get

$$\langle \varphi_i^{\boldsymbol{A}}(b_1, \dots, b_m), \psi_i^{\boldsymbol{A}}(b_1, \dots, b_m)\rangle \in \mathrm{Cg}^{\boldsymbol{C}}_{\mathsf{K}}(a,c).$$

Since $b_1, \dots, b_m$ and $\mathrm{Cg}^{\boldsymbol{C}}_{\mathsf{K}}(a,c)$ is a K-congruence, this implies

$$\langle \varepsilon^{\boldsymbol{A}}(b_1, \dots, b_m), \delta^{\boldsymbol{A}}(b_1, \dots, b_m)\rangle \in \mathrm{Cg}^{\boldsymbol{C}}_{\mathsf{K}}(a,c) \subseteq R.$$

Consequently, we conclude that R is a K-congruence of $\boldsymbol{A}$ containing $\langle a, c\rangle$. This implies that $\mathrm{Cg}^{\boldsymbol{A}}_{\mathsf{K}}(a,c) \subseteq R$, as desired. ⊠

3. The algebraization of logic

Convention. Recall that, formally speaking, a logic is a relation $\vdash \subseteq \mathcal{P}(Fm) \times Fm$. Nonetheless, sometimes we shall take the liberty of writing $\gamma_1, \dots, \gamma_n \vdash \varphi$ instead of $\{\gamma_1, \dots, \gamma_n\} \vdash \varphi$. Similarly, we sometimes write $\Gamma, \varphi \vdash \psi$ as a shorthand for $\Gamma \cup \{\varphi\} \vdash \psi$. Lastly, for every set of formulas $\Gamma \cup \Delta \cup \{\varphi, \psi\}$, we shall write:

(i) $\Gamma \vdash \Delta$, when $\Gamma \vdash \delta$ for every $\delta \in \Delta$;
(ii) $\varphi \dashv\vdash \psi$, when $\varphi \vdash \psi$ and $\psi \vdash \varphi$;
(iii) $\Gamma \dashv\vdash \Delta$, when $\Gamma \vdash \Delta$ and $\Delta \vdash \Gamma$.

The same conventions apply to relative equational consequences $\vDash_{\mathsf{K}}$.

In this section we shall introduce a robust theory of algebraization which will account for the relation between **CPC** and Boolean algebras and, more in general, between logic and algebra. To simplify the discussion, we shall restrict our attention to *finitary* logics only,[2] i.e., logics $\vdash$ such that for every $\Gamma \cup \{\varphi\} \subseteq Fm$,

$$\text{if } \Gamma \vdash \varphi \text{, then there is a finite } \Delta \subseteq \Gamma \text{ such that } \Delta \vdash \varphi.$$

The logics **CPC**, **IPC**, $\mathbf{K}_g$, and $\mathbf{K}_l$ are easily seen to be finitary, and the same applies to most logics in the literature.

Let us begin by a closer examination of the relation which connects **CPC** with the variety of Boolean algebras BA. As we mentioned, one half of this relation is to be found in the equational completeness theorem stating that for every set of formulas $\Gamma \cup \{\varphi\}$,

$$\Gamma \vdash_{\mathbf{CPC}} \varphi \iff \boldsymbol{\tau}[\Gamma] \vDash_{\mathsf{BA}} \boldsymbol{\tau}(\varphi), \tag{5}$$

where $\boldsymbol{\tau}(x) = \{x \approx 1\}$. In plain words, **CPC** can be interpreted into $\vDash_{\mathsf{BA}}$ by means of the set of equations $\boldsymbol{\tau}(x)$.

The other half of the relation between **CPC** and BA consists in the fact that this interpretation can be *reversed*, in the sense that $\vDash_{\mathsf{BA}}$ can be also interpreted into **CPC**. To clarify this point, for every set of formulas $\Delta(x, y)$ and set of equations $\Theta \cup \{\varepsilon \approx \delta\}$, we shall abbreviate

$$\{\varphi(\varepsilon, \delta) \colon \varphi(x, y) \in \Delta\} \text{ as } \Delta(\varepsilon, \delta), \text{ and } \bigcup_{\varphi \approx \psi \in \Theta} \Delta(\varphi, \psi) \text{ as } \Delta[\Theta].$$

Defining $\Delta(x, y) \coloneqq \{x \to y, y \to x\}$, we obtain that for all set of equations $\Theta \cup \{\varepsilon \approx \delta\}$,

$$\Delta[\Theta] \vdash_{\mathbf{CPC}} \Delta(\varepsilon, \delta) \iff \Theta \vDash_{\mathsf{BA}} \varepsilon \approx \delta. \tag{6}$$

In plain words, $\vDash_{\mathsf{BA}}$ can be interpreted into **CPC** by means of the set of formulas $\Delta(x, y)$.

Lastly, these two interpretations are provably *inverse one to the other*, in the sense that for every pair of formulas φ and ψ,

$$\varphi \dashv\vdash_{\mathbf{CPC}} \Delta[\boldsymbol{\tau}(\varphi)] \text{ and } \varphi \approx \psi \mathrel{=\!\mid\!\vDash_{\mathsf{BA}}} \boldsymbol{\tau}[\Delta(\varphi, \psi)]. \tag{7}$$

Conditions (5, 6, 7) globally express the fact that **CPC** and $\vDash_{\mathsf{BA}}$ are *equivalent* [7] in the sense that they can be interpreted one into the other by interpretations which are provably inverse one to the other. The notion of an algebraizable logic will make this demand precise for arbitrary logics and quasi-varieties.

[2] For a theory of algebraization encompassing infinitary logics, see for instance [22].

Exercise 3.1. Convince yourself that conditions (5, 6, 7) keep being true once **CPC** is replaced by **IPC** (resp. by $\mathbf{K}_g$) and BA by HA (resp. by MA). Thus, also **IPC** and HA (resp. $\mathbf{K}_g$ and MA) are "equivalent" in the above sense. ⊠

Definition 3.2. A finitary logic $\vdash$ is said to be *algebraizable* if there are a set of equations $\boldsymbol{\tau}(x)$, a set of formulas $\Delta(x, y)$, and a quasi-variety K such that

$$\Gamma \vdash \varphi \Longleftrightarrow \boldsymbol{\tau}[\Gamma] \vDash_{\mathsf{K}} \boldsymbol{\tau}(\varphi) \tag{Alg1}$$

$$\Delta[\Theta] \vdash \Delta(\varepsilon, \delta) \Longleftrightarrow \Theta \vDash_{\mathsf{K}} \varepsilon \approx \delta \tag{Alg2}$$

$$\varphi \dashv\vdash \Delta[\boldsymbol{\tau}(\varphi)] \tag{Alg3}$$

$$\varepsilon \approx \delta \mathrel{=\!\!|\!\vDash_{\mathsf{K}}} \boldsymbol{\tau}[\Delta(\varepsilon, \delta)] \tag{Alg4*}$$

for every set of formulas $\Gamma \cup \{\varphi\}$ and set of equations $\Theta \cup \{\varepsilon \approx \delta\}$. In this case, K is said to be an *equivalent algebraic semantics* for $\vdash$. In addition, $\boldsymbol{\tau}, \Delta$, and K are said to *witness* the algebraizability of $\vdash$.

The notion of an algebraizable logic originates in [10], but modern treatments can be found, for instance, in [20, 22, 24, 23]. In view of Exercise 3.1, the logics **CPC**, **IPC** and $\mathbf{K}_g$ are algebraizable with equivalent algebraic semantics, respectively, BA, HA, and MA.

In the next exercise, you are asked to check that the definition of an algebraizable logic can be formulated in a more concise way.

Exercise 3.3. Prove that conditions (Alg2) and (Alg3) in the definition of an algebraizable logic are redundant. Moreover, convince yourself that, as $\vDash_{\mathsf{K}}$ is also substitution invariant, (Alg4*) can be replaced by the demand that

$$x \approx y \mathrel{=\!\!|\!\vDash_{\mathsf{K}}} \boldsymbol{\tau}[\Delta(x, y)].$$

Accordingly, a finitary logic $\vdash$ is algebraizable precisely when there are a set of equations $\boldsymbol{\tau}(x)$, a set of formulas $\Delta(x, y)$, and a quasi-variety K such that

$$\Gamma \vdash \varphi \Longleftrightarrow \boldsymbol{\tau}[\Gamma] \vDash_{\mathsf{K}} \boldsymbol{\tau}(\varphi) \tag{Alg1}$$

$$x \approx y \mathrel{=\!\!|\!\vDash_{\mathsf{K}}} \boldsymbol{\tau}[\Delta(x, y)] \tag{Alg4}$$

for every set of formulas $\Gamma \cup \{\varphi\}$.

Convince yourself that the definition of algebraizability can be equivalently expressed in terms of (Alg2) and the requirement that $x \dashv\vdash \Delta[\boldsymbol{\tau}(x)]$. ⊠

Lemma 3.4. *The set of equations $\boldsymbol{\tau}$ and the set of formulas Δ witnessing the algebraizability of an algebraizable logic $\vdash$ can be always chosen finite.*

Proof. Suppose that the algebraizability of $\vdash$ is witnessed by some $\boldsymbol{\tau}$, Δ, and K. As K is a quasi-variety, the consequence relation $\vDash_{\mathsf{K}}$ is finitary (Exercise 2.5). Therefore, in view of (Alg4), there exists a finite subset $\Delta_0 \subseteq \Delta(x, y)$ such that

$$x \approx y \mathrel{=\!\!|\!\vDash_{\mathsf{K}}} \boldsymbol{\tau}[\Delta_0(x, y)].$$

Thus, (Alg1) and (Alg4) hold for $\boldsymbol{\tau}$, Δ_0, and K. By Exercise 3.3, we conclude that $\boldsymbol{\tau}$, Δ_0, and K witness the algebraizability of $\vdash$. A similar argument, this times using the finitarity of $\vdash$, shows that $\boldsymbol{\tau}$ can also be chosen finite. ⊠

At first sight, the definition of an algebraizable logic seems to depend on the choice of many entities, namely $\boldsymbol{\tau}$, Δ, and K. Our first aim is to show that, when existing, these are uniquely determined (in the cases of $\boldsymbol{\tau}$ and Δ up to provable equivalence).

Lemma 3.5. *Let $\vdash$ be an algebraizable logic whose algebraizability is witnessed by some set of equations $\boldsymbol{\tau}_1$, set of formulas Δ_1, and quasi-variety K_1. For every set of equations $\boldsymbol{\tau}_2$, set of formulas Δ_2, and quasi-variety K_2 witnessing the algebraizability of $\vdash$, the following conditions hold:*

(i) $\mathsf{K}_1 = \mathsf{K}_2$;
(ii) $\boldsymbol{\tau}_1$ *and* $\boldsymbol{\tau}_2$ *are provably equivalent modulo* K_i, *i.e.,* $\boldsymbol{\tau}_1(x) \mathrel{=\!\mid\!\models}_{\mathsf{K}_i} \boldsymbol{\tau}_2(x)$, *for* $i = 1, 2$;
(iii) Δ_1 *and* Δ_2 *are provably equivalent modulo* $\vdash$, *i.e.,* $\Delta_1(x, y) \dashv\vdash \Delta_2(x, y)$.

Proof. (iii): By symmetry, it suffices to prove $\Delta_1(x, y) \vdash \Delta_2(x, y)$. To this end, consider $\varphi \in \Delta_2$. Clearly, $x \approx y \vDash_{\mathsf{K}_1} \varphi(x, x) \approx \varphi(x, y)$. By (Alg2) this implies

$$\Delta_1(x, y) \vdash \Delta_1(\varphi(x, x), \varphi(x, y)). \tag{8}$$

On the other hand, observe that $\emptyset \vDash_{\mathsf{K}_2} x \approx x$. By applying (Alg2) this yields $\emptyset \vdash \Delta_2(x, x)$. Since $\varphi(x, y) \in \Delta_2(x, y)$, we obtain

$$\emptyset \vdash \varphi(x, x). \tag{9}$$

Lastly, observe that $\boldsymbol{\tau}_1(x), x \approx y \vDash_{\mathsf{K}_1} \boldsymbol{\tau}_1(y)$. By applying condition (Alg4), we get $\boldsymbol{\tau}_1(x), \boldsymbol{\tau}_1[\Delta_1(x, y)] \vDash_{\mathsf{K}_1} \boldsymbol{\tau}_1(y)$. Moreover, an application of (Alg1) yields $x, \Delta_1(x, y) \vdash y$. By substitution invariance, we conclude

$$\varphi(x, x), \Delta_1(\varphi(x, x), \varphi(x, y)) \vdash \varphi(x, y).$$

Together with (8, 9), this yields $\Delta_1(x, y) \vdash \varphi(x, y)$, whence $\Delta_1(x, y) \vdash \Delta_2(x, y)$, as desired.

(i): By symmetry, it suffices to show that $\mathsf{K}_1 \subseteq \mathsf{K}_2$. As K_2 is a quasi-variety, this amounts to proving that the quasi-equations valid in K_2 are also valid in K_1. To prove this, suppose that

$$\mathsf{K}_2 \vDash \bigwedge_{i<n} \varphi_i \approx \psi_i \to \varepsilon \approx \delta,$$

that is $\{\varphi_i \approx \psi_i : i < n\} \vDash_{\mathsf{K}_2} \varepsilon \approx \delta$. By (Alg2), this implies

$$\bigcup_{i<n} \Delta_2(\varphi_i, \psi_i) \vdash \Delta_2(\varepsilon, \delta).$$

Therefore, with an application of (iii), we obtain

$$\bigcup_{i<n} \Delta_1(\varphi_i, \psi_i) \vdash \Delta_1(\varepsilon, \delta).$$

By (Alg2) we conclude that $\{\varphi_i \approx \psi_i : i < n\} \vDash_{\mathsf{K}_1} \varepsilon \approx \delta$, i.e.,

$$\mathsf{K}_1 \vDash \bigwedge_{i<n} \varphi_i \approx \psi_i \to \varepsilon \approx \delta.$$

(ii): Since we proved that $\mathsf{K}_1 = \mathsf{K}_2$, it suffices to prove that statement for $i = 1$. From (Alg3) it follows $\Delta_1[\boldsymbol{\tau}_1(x)] \dashv\vdash \Delta_2[\boldsymbol{\tau}_2(x)]$. Since we proved that $\Delta_1(x, y) \dashv\vdash \Delta_2(x, y)$, this amounts to $\Delta_1[\boldsymbol{\tau}_1(x)] \dashv\vdash \Delta_1[\boldsymbol{\tau}_2(x)]$. By (Alg2) we conclude $\boldsymbol{\tau}_1(x) \mathrel{=\!\mid\!\models}_{\mathsf{K}_1} \boldsymbol{\tau}_2(x)$. ⊠

As we mentioned in Section 1, it is common that a logic has several algebraic semantics. On the other hand, we are now in the position of proving that algebraizability accounts for the univocal relation between **CPC** and the variety of Boolean algebras, in the sense that the latter is the unique equivalent algebraic semantics of **CPC**. More in general, Lemma 3.5(i) yields the following:

Corollary 3.6. *Every algebraizable logic has a unique equivalent algebraic semantics.*

One may wonder whether every quasi-variety is the equivalent algebraic semantics of an algebraizable logic. The following exercise provides a negative answer to this question.[3]

Exercise 3.7. A variety K that satisfies $f(x, \dots, x) \approx x$ for each of its basic operations f is said to be *idempotent*. For instance, all varieties of lattices are idempotent. Prove that a non-trivial idempotent variety cannot be the equivalent algebraic semantics of any algebraizable logic.

Hint: suppose, with a view to contradiction, that there exists a non-trivial idempotent variety K which, moreover, is the equivalent algebraic semantics of some algebraizable logic $\vdash$. First show that $\emptyset \vdash x$. By substitution invariance, derive $\emptyset \vdash \Delta(x, y)$. Conclude that K is trivial, a contradiction. ⊠

Let $\vdash$ be a finitary logic. A finitary logic $\vdash'$ is said to be an *extension* of $\vdash$ if it is formulated in the same language as $\vdash$ and for every set of formulas $\Gamma \cup \{\varphi\}$,

$$\text{if } \Gamma \vdash \varphi, \text{ then } \Gamma \vdash' \varphi.$$

A logic $\vdash'$ is said to be an *axiomatic extension* of $\vdash$ when it is formulated in the same language as $\vdash$ and there exists a set of formulas Σ, closed under substitutions, such that for every $\Gamma \cup \{\varphi\} \subseteq Fm$,

$$\Gamma \vdash' \varphi \iff \Gamma \cup \Sigma \vdash \varphi.$$

Clearly, all axiomatic extensions of $\vdash$ are also extensions of $\vdash$, while the converse is not true in general. Axiomatic extensions of **IPC** have been called *intermediate logics* (or *superintuitionistic logics*) in the literature.

Exercise 3.8. Prove that algebraizability persists in extensions. More precisely, show that if the algebraizability of a logic $\vdash$ is witnessed by some finite $\boldsymbol{\tau}$, Δ, and K, then the algebraizability of an extension $\vdash'$ of $\vdash$ is witnessed by $\boldsymbol{\tau}$, Δ, and some subquasi-variety of K. ⊠

The poset of extensions of $\vdash$ (ordered under inclusion) forms a lattice, which we denoted by $\mathcal{E}xt(\vdash)$. Similarly, given a quasi-variety K, let $\mathcal{Q}(\mathsf{K})$ be the poset of subquasi-varieties of K which, again, turns out to be a lattice. A subquasi-variety W of K is said to be a *relative subvariety of* K when if it is axiomatized relatively to K by a set of equations. Notice that if K is a variety, then relative subvarieties and subvarieties (in the absolute sense) coincide. In view of the next result, the extensions of an algebraizable logic can be studied through the lenses of the subquasi-varieities of its equivalent algebraic semantics, which, in turn, are amenable to the methods of model theory, universal algebra, and duality theory.

Theorem 3.9. *Let $\vdash$ be an algebraizable logic and K its equivalent algebraic semantics. The lattices $\mathcal{E}xt(\vdash)$ and $\mathcal{Q}(\mathsf{K})$ are dually isomorphic under the map that sends an extension of $\vdash$ to its equivalent algebraic semantics. This dual isomorphism restricts to one between the lattice of axiomatic extensions of $\vdash$ and that of relative subvarieties of K.*

[3]From the point of view of universal algebra, however, the demand that a variety is the equivalent algebraic semantics of some algebraizable logic is very weak. For instance, as shown in [36], varieties of this kind need not satisfy any non-trivial congruence identity in $\{\wedge, \vee, \circ\}$ such as congruence distributivity, modularity, permutability etc., see also [16, 15].

Proof sketch. In view of Exercise 3.8, the extensions of $\vdash$ are algebraizable and their equivalent algebraic semantics are subquasi-varieties of K. Dualizing the proof of this fact one obtains that every subquasi-variety of K is the equivalent algebraic semantics of some extension of $\vdash$. These observations induce two maps $f\colon \mathcal{E}xt(\vdash) \longleftrightarrow \mathcal{Q}(\mathsf{K})\colon g$ that are order-reversing and one inverse to the other. Thus, we conclude that $\mathcal{E}xt(\vdash)$ and $\mathcal{Q}(\mathsf{K})$ are dually isomorphic. ⊠

Remark 3.10. The above result generalizes the well-known fact that the lattice of intermediate logics (i.e., axiomatic extensions of **IPC**) is dually isomorphic to that of varieties of Heyting algebras. ⊠

4. The ubiquitous isomorphism between filters and congruences

It is very common the congruences of logically motivated algebras (such as Boolean, Heyting, and modal algebras) correspond to suitable lattice filters. Among the many instances of this phenomenon, let us recall that the lattice of congruences $\mathrm{Con}\boldsymbol{A}$ of a Heyting algebra $\boldsymbol{A}$ is isomorphic to the lattice $\mathrm{Fi}\boldsymbol{A}$ of filters of $\boldsymbol{A}$ under under the maps that send a congruence $\phi \in \mathrm{Con}\boldsymbol{A}$ to the filter

$$F_\phi := \{a \in A \colon \langle a, 1\rangle \in \phi\},$$

and a filter $G \in \mathrm{Fi}\boldsymbol{A}$ to a congruence

$$\theta_G = \{\langle a, c\rangle \in A \times A \colon a \to^{\boldsymbol{A}} c, c \to^{\boldsymbol{A}} a \in G\}.$$

Similarly, the lattice of congruences of a modal algebra $\boldsymbol{A}$ is isomorphic to the lattice $\mathrm{Op}\boldsymbol{A}$ of *open filters* of $\boldsymbol{A}$, i.e., filters of $\boldsymbol{A}$ closed under the modal operation $\Box^{\boldsymbol{A}}$, under an isomorphism similar to the one described above.

However, the correspondence between congruences and special subsets of the universe of algebras (such as filters or open filters) is not a peculiarity of the algebra of logic. On the contrary, similar correspondences are ubiquitous in classical algebras as well, where congruences of rings correspond to ideals, and congruences of groups to normal subgroups. As we shall see, all these correspondences can be explained as special instances of the algebraization phenomenon.

Evidence in favour of the non-mathematical claim that *every* correspondence between congruences and special subsets is a consequence of algebraizability can be found, for instance, in [2, 28] and in the series of papers [37, 3, 4, 5, 38]. In this section, we shall prove a modest version of this claim, namely that a logic $\vdash$ is algebraizable with equivalent algebraic semantics a quasi-variety K if and only if for all $\boldsymbol{A} \in \mathsf{K}$, the lattice $\mathrm{Con}_{\mathsf{K}}\boldsymbol{A}$ is isomorphic to the lattice of certain special subsets of $\boldsymbol{A}$ (Theorem 4.8). To this end, we need to introduce some new concept.

Definition 4.1. Given a logic $\vdash$ and an algebra $\boldsymbol{A}$, a set $F \subseteq A$ is said to be a *deductive filter* of $\vdash$ on $\boldsymbol{A}$ if it is closed under the interpretation of the rules valid in $\vdash$, that is

$$\begin{aligned}&\text{if } \Gamma \vdash \varphi, \text{ then for every homomorphism } h\colon \boldsymbol{Fm} \to \boldsymbol{A},\\ &\qquad\qquad \text{if } h[\Gamma] \subseteq F, \text{ then } h(\varphi) \in F,\end{aligned}$$

for every $\Gamma \cup \{\varphi\}$. When ordered under inclusion, the set of deductive filters of $\vdash$ on $\boldsymbol{A}$ becomes a closure system, which we denote by $\mathcal{F}i_{\vdash}\boldsymbol{A}$.

Exercise 4.2. Prove that the deductive filters of **IPC** on a Heyting algebra $\boldsymbol{A}$ are precisely the lattice filters of $\boldsymbol{A}$. To this end, you may wish to employ a Hilbert-style calculus for **IPC**. This is because the demand of "being a deductive filter of **IPC**" can be equivalently phrased as that of "being closed under the interpretation of the axioms and rules of a Hilbert-style calculus for **IPC**".

Similarly, prove that the deductive filters of $\mathbf{K}_g$ on a modal algebra $\boldsymbol{A}$ are exactly the open filters of $\boldsymbol{A}$. Again, you may wish to use a Hilbert-style calculus for $\mathbf{K}_g$. Such a calculus can be obtained by adding to any Hilbert-style presentation of **CPC** the rules

$$\begin{aligned} \emptyset &\rhd \Box(x \to y) \to (\Box x \to \Box y) \\ x, x \to y &\rhd y \\ x &\rhd \Box x. \end{aligned}$$

⊠

The set of *endomorphism* of an algebra $\boldsymbol{A}$ will be denoted by $\mathrm{End}(\boldsymbol{A})$.

Lemma 4.3. *Let $\vdash$ be a logic, K a quasi-variety, $\boldsymbol{A}$ an algebra, and $\sigma \in \mathrm{End}(\boldsymbol{A})$.*

(i) *If $F \in \mathcal{F}i_{\vdash}\boldsymbol{A}$, then $\sigma^{-1}[F] \in \mathcal{F}i_{\vdash}\boldsymbol{A}$.*
(ii) *If $\theta \in \mathrm{Con}_{\mathsf{K}}\boldsymbol{A}$, then $\sigma^{-1}[\theta] \in \mathrm{Con}_{\mathsf{K}}\boldsymbol{A}$, where $\sigma^{-1}[\theta]$ stands for*

$$\{\langle a, c\rangle \in A \times A \colon \langle \sigma(a), \sigma(c)\rangle \in \theta\}.$$

Proof. We detail only (i) and leave (ii) to the reader. Suppose that $F \in \mathcal{F}i_{\vdash}\boldsymbol{A}$. Then consider $\Gamma \cup \{\varphi\} \subseteq Fm$ and a homomorphism $h \colon \boldsymbol{Fm} \to \boldsymbol{A}$ such that $h[\Gamma] \subseteq \sigma^{1}[F]$. Clearly, $\sigma \circ h[\Gamma] \subseteq F$. Since $\sigma \circ h$ is a homomorphism from $\boldsymbol{Fm}$ to $\boldsymbol{A}$, and $F \in \mathcal{F}i_{\vdash}\boldsymbol{A}$, this implies $\sigma \circ h(\varphi) \subseteq F$. We conclude $h(\varphi) \subseteq \sigma^{1}[F]$. ⊠

In view of the above result, given a logic $\vdash$, a quasi-variety K, and an algebra $\boldsymbol{A}$, it makes sense to define the following lattice expansions

$$\begin{aligned} \mathcal{F}i_{\vdash}\boldsymbol{A}^{+} &:= \langle \mathcal{F}i_{\vdash}\boldsymbol{A}; \leqslant, \{\sigma^{-1} \colon \sigma \in \mathrm{End}(\boldsymbol{A})\}\rangle \\ \mathrm{Con}_{\mathsf{K}}\boldsymbol{A}^{+} &:= \langle \mathrm{Con}_{\mathsf{K}}\boldsymbol{A}; \leqslant, \{\sigma^{-1} \colon \sigma \in \mathrm{End}(\boldsymbol{A})\}\rangle. \end{aligned}$$

Accordingly, an isomorphism from $\mathcal{F}i_{\vdash}\boldsymbol{A}^{+}$ to $\mathrm{Con}_{\mathsf{K}}\boldsymbol{A}^{+}$ is a lattice isomorphism Φ from the lattice $\mathcal{F}i_{\vdash}\boldsymbol{A}$ to the lattice $\mathrm{Con}_{\mathsf{K}}\boldsymbol{A}$ that, moreover, commutes with inverse endomorphisms in the sense that

$$\Phi(\sigma^{-1}[F]) = \sigma^{-1}[\Phi(F)], \text{ for every } \sigma \in \mathrm{End}(\boldsymbol{A}).$$

The next result states that every algebraizable logic induces an isomorphism between congruences and special subsets of the universes of algebras, i.e., deductive filters.

Lemma 4.4. *Let $\vdash$ be an algebraizable logic with equivalent algebraic semantics K. Then $\mathcal{F}i_{\vdash}\boldsymbol{A}^{+} \cong \mathrm{Con}_{\mathsf{K}}\boldsymbol{A}^{+}$, for every algebra $\boldsymbol{A}$.*

Proof. Let $\boldsymbol{\tau}$ and Δ witness the algebraizability of $\vdash$ with respect to K. Then consider an algebra $\boldsymbol{A}$. We shall define two maps

$$\theta_{(\cdot)} \colon \mathcal{F}i_{\vdash}\boldsymbol{A} \longleftrightarrow \mathrm{Con}_{\mathsf{K}}\boldsymbol{A} \colon F_{(\cdot)}$$

as follows: for every $G \in \mathcal{F}i_{\vdash}\boldsymbol{A}$ and $\phi \in \mathrm{Con}_{\mathsf{K}}\boldsymbol{A}$, set

$$\begin{aligned} \theta_G &:= \{\langle a, c\rangle \in A \times A \colon \Delta^{\boldsymbol{A}}(a, c) \subseteq F\} \\ F_{\phi} &:= \{a \in A \colon \boldsymbol{\tau}^{\boldsymbol{A}}(a) \subseteq \phi\}. \end{aligned}$$

Claim 4.5. *The map $\theta_{(\cdot)}\mathcal{F}i_{\vdash}\boldsymbol{A} \to \mathrm{Con}_{\mathsf{K}}\boldsymbol{A}$ is well defined.*

Proof. We begin by proving that θ_G is a congruence of $\boldsymbol{A}$. To this end, observe that from (Alg1) and (Alg4) it follows that

$$\begin{aligned}\emptyset &\vdash \Delta(x,x)\\ \Delta(x,y) &\vdash \Delta(y,x)\\ \Delta(x,y),\Delta(y,z) &\vdash \Delta(x,z)\\ \Delta(x_1,y_1),\dots,\Delta(x_n,y_n) &\vdash \Delta(f(x_1,\dots,x_n),f(y_1,\dots,y_n))\end{aligned}$$

for every basic n-ary operation f of $\boldsymbol{A}$. As F is a deductive filter of $\vdash$ on $\boldsymbol{A}$, this implies that for all $a,b,c \in A$,

$$\begin{aligned}&\Delta^{\boldsymbol{A}}(a,a) \subseteq F\\ \text{if } &\Delta^{\boldsymbol{A}}(a,b) \subseteq F, \text{ then } \Delta^{\boldsymbol{A}}(b,a) \subseteq F\\ \text{if } &\Delta^{\boldsymbol{A}}(a,b) \cup \Delta^{\boldsymbol{A}}(b,c) \subseteq F, \text{ then } \Delta^{\boldsymbol{A}}(a,c) \subseteq F\end{aligned}$$

and that for all $a_1,\dots,a_n,c_1,\dots,c_n \in A$,

$$\text{if } \Delta^{\boldsymbol{A}}(a_1,c_1) \cup \dots \cup \Delta^{\boldsymbol{A}}(a_n,c_n) \subseteq F, \text{ then } \Delta^{\boldsymbol{A}}(f(\vec{a},f(\vec{c})) \subseteq F.$$

By the definition of θ_G, the above displays imply that θ_G is an equivalence relation on A compatible with the basic operations of $\boldsymbol{A}$. Thus, θ_G is a congruence of $\boldsymbol{A}$.

In order to prove that θ_G is also a K-congruence of $\boldsymbol{A}$, we need to show that $\boldsymbol{A}/\theta_G \in \mathsf{K}$. Since K is a quasi-variety, it suffices to check that the quasi-equations valid in K are also valid in $\boldsymbol{A}/\theta_G$. To this end, suppose that

$$\mathsf{K} \vDash \bigwedge_{i<n} \varphi_i \approx \psi_i \to \varepsilon \approx \delta \tag{10}$$

and consider a tuple $\vec{a} \in A$ such that $\varphi_i^{\boldsymbol{A}}(\vec{a})/\theta_G = \phi_i^{\boldsymbol{A}}(\vec{a})/\theta_G$ for all $i < n$. From the definition of θ_G it follows that

$$\Delta^{\boldsymbol{A}}(\varphi_1(\vec{a}),\psi_1(\vec{a})) \cup \dots \cup \Delta^{\boldsymbol{A}}(\varphi_n(\vec{a}),\psi_n(\vec{a})) \subseteq F. \tag{11}$$

Now, from (10) and (Alg2) it follows

$$\Delta(\varphi_1,\psi_1),\dots,\Delta(\varphi_n,\psi_n) \vdash \Delta(\varepsilon,\delta).$$

Together with (11) and the fact that F is a deductive filter of $\vdash$ on $\boldsymbol{A}$, this implies $\Delta^{\boldsymbol{A}}(\varepsilon(\vec{a}),\delta(\vec{a})) \subseteq F$. Therefore, $\langle \varepsilon^{\boldsymbol{A}}(\vec{a}),\delta^{\boldsymbol{A}}(\vec{a})\rangle \in \theta_G$. We conclude that $\boldsymbol{A}/\theta_G$ validates the quasi-equation in (10), as desired. Thus, θ_G is a K-congruence of $\boldsymbol{A}$. ⊠

Exercise 4.6. Prove that also the map $F_{(\cdot)}\colon \mathrm{Con}_{\mathsf{K}}\boldsymbol{A} \to \mathcal{F}i_{\vdash}\boldsymbol{A}$ is well defined.

To conclude the proof, it only remains to show that the maps $\theta_{(\cdot)}$ and $F_{(\cdot)}$ are isomorphism one inverse to the other between $\mathcal{F}i_{\vdash}\boldsymbol{A}^{+}$ and $\mathrm{Con}_{\mathsf{K}}\boldsymbol{A}^{+}$. Consider $G \in \mathcal{F}i_{\vdash}\boldsymbol{A}$ and $a \in A$. We have

$$a \in F_{\theta_G} \iff \boldsymbol{\tau}^{\boldsymbol{A}}(a) \subseteq \theta_G \iff \Delta^{\boldsymbol{A}}[\boldsymbol{\tau}^{\boldsymbol{A}}(a)] \subseteq G.$$

But, in view of condition (Alg3) and the fact that G is a deductive filter of $\vdash$ on $\boldsymbol{A}$,

$$\Delta^{\boldsymbol{A}}[\boldsymbol{\tau}^{\boldsymbol{A}}(a)] \subseteq G \iff a \in G.$$

Consequently, we obtain $G = F_{\theta_G}$. A similar argument shows that $\phi = \theta_{F_\phi}$, for every $\phi \in \mathrm{Con}_{\mathsf{K}}\boldsymbol{A}$. Consequently, $\theta_{(\cdot)}$ and $F_{(\cdot)}$ are bijections one inverse to the other.

Furthermore, observe that $\theta_{(\cdot)}$ and $F_{(\cdot)}$ are clearly order-preserving. Then, consider $G, H \in \mathcal{F}i_{\vdash}\boldsymbol{A}$ such that $\theta_G \subseteq \theta_H$. Since $F_{(\cdot)}$ is order-preserving and $\theta_{(\cdot)}$ and $F_{(\cdot)}$ are inverse one to the other, $G = F_{\theta_G} \subseteq F_{\theta_H} = H$. Consequently, $\theta_{(\cdot)}$ is order-preserving and reflecting and, therefore, a lattice isomorphism.

To prove that $\theta_{(\cdot)}$ preserves inverse endomorphism, consider $G \in \mathcal{F}i_{\vdash}\boldsymbol{A}$, $a, c \in A$, and $\sigma \in \mathrm{End}(\boldsymbol{A})$. We have

$$\begin{aligned}\langle a, c\rangle \in \sigma^{-1}\theta_G &\iff \langle \sigma(a), \sigma(c)\rangle \in \theta_G \iff \Delta^{\boldsymbol{A}}(\sigma(a), \sigma(c)) \subseteq G\\ &\iff \sigma[\Delta^{\boldsymbol{A}}(a, c)] \subseteq G \iff \Delta^{\boldsymbol{A}}(a, c) \subseteq \sigma^{-1}[G]\\ &\iff \langle a, c\rangle \in \theta_{\sigma^{-1}[G]}.\end{aligned}$$

Thus, $\theta_{(\cdot)} \colon \mathcal{F}i_{\vdash}\boldsymbol{A}^{+} \to \mathrm{Con}_{\mathsf{K}}\boldsymbol{A}^{+}$ is a well defined isomorphism, whose inverse is $F_{(\cdot)}$. ⊠

Notably, the isomorphism between deductive filters and congruences in Lemma 4.4 can be turn into a characterization of algebraizability, as we proceed to explain. First, recall that every logic $\vdash$, being a consequence relation, can be naturally associated with a closure system on Fm, namely the so-called lattice of *theories* of $\vdash$, i.e., sets of formulas Γ such that for every $\varphi \in Fm$, if $\Gamma \vdash \varphi$, then $\varphi \in \Gamma$. We denote the lattice of theories of $\vdash$ by $\mathcal{T}h(\vdash)$. Similarly, a *theory* of the relative equational consequence $\vDash_{\mathsf{K}}$ is a set of equations Θ such that for every equation $\varepsilon \approx \delta$, if $\Theta \vDash_{\mathsf{K}} \varepsilon \approx \delta$, then $\varepsilon \approx \delta \in \Theta$. The lattice of theories of $\vDash_{\mathsf{K}}$ is denoted by $\mathcal{T}h(\vDash_{\mathsf{K}})$. The following simple, but very important, exercise asks you to clarify the relations between theories, deductive filters, and K-congruences.

Exercise 4.7. Prove that $\mathcal{T}h(\vdash)$ and $\mathcal{T}h(\vDash_{\mathsf{K}})$ are, respectively, $\mathcal{F}i_{\vdash}\boldsymbol{Fm}$ and $\mathrm{Con}_{\mathsf{K}}\boldsymbol{Fm}$. Of course, to prove that $\mathcal{T}h(\vDash_{\mathsf{K}})$ and $\mathrm{Con}_{\mathsf{K}}\boldsymbol{Fm}$ are the same entity, one need to identify the equations $\varphi \approx \psi$ with pairs of formulas $\langle \varphi, \psi\rangle$. This identification will occur many times in these notes. ⊠

In view to the above exercise, it makes sense to write $\mathcal{T}h(\vdash)^{+}$ and $\mathcal{T}h(\vDash_{\mathsf{K}})^{+}$. Bearing this in mind, we are now ready to prove the *isomorphism theorem* of algebraizable logics:

Theorem 4.8 (Blok & Pigozzi). *Let $\vdash$ be a finitary logic and K a quasi-variety. The following conditions are equivalent:*

(i) *$\vdash$ is algebraizable with equivalent algebraic semantics K;*
(ii) *$\mathcal{F}i_{\vdash}\boldsymbol{A}^{+} \cong \mathrm{Con}_{\mathsf{K}}\boldsymbol{A}^{+}$, for every algebra $\boldsymbol{A}$;*
(iii) *$\mathcal{T}h(\vdash)^{+} \cong \mathcal{T}h(\vDash_{\mathsf{K}})^{+}$.*

Proof. The implication (i)⇒(ii) is precisely Lemma 4.4. Furthermore, in view of Exercise 4.7, (iii) is a special case of (ii). Therefore, in only remains to prove the implication (iii)⇒(i). To this end, let $\Phi \colon \mathcal{T}h(\vdash)^{+} \to \mathcal{T}h(\vDash_{\mathsf{K}})^{+}$ be the isomorphism given by (iii). Let also Eq be the set of equations and

$$\mathrm{Cn}_{\vdash} \colon \mathcal{P}(Fm) \to \mathcal{P}(Fm) \text{ and } \mathrm{Cn}_{\mathsf{K}} \colon \mathcal{P}(Eq) \to \mathcal{P}(Eq)$$

be the closure operators associated with $\vdash$ and $\vDash_{\mathsf{K}}$, respectively. The proof proceeds through a series of technical claims.

Claim 4.9. *For every set of formula Γ, set of equations Θ, and substitution σ,*

$$\begin{aligned}\Phi(\mathrm{Cn}_{\vdash}(\sigma[\Gamma])) &= \mathrm{Cn}_{\mathsf{K}}(\sigma[\Phi(\mathrm{Cn}_{\vdash}\Gamma)])\\ \Phi^{-1}[\mathrm{Cn}_{\mathsf{K}}(\sigma[\Theta])] &= \mathrm{Cn}_{\vdash}(\sigma[\Phi^{-1}(\mathrm{Cn}_{\mathsf{K}}(\Theta))]).\end{aligned}$$

Proof. We detail the proof of the first equality only, as that of the latter is analogous. First, when properly deciphered, the inclusion

$$\Phi(\mathrm{Cn}_{\vdash}(\Gamma)) \subseteq \sigma^{-1}[\mathrm{Cn}_{\mathsf{K}}(\sigma[\Phi(\mathrm{Cn}_{\vdash}(\Gamma))])]$$

is easy. As $\mathrm{Cn}_{\vdash}(\Gamma) \in \mathcal{T}h(\vdash)$ and Φ^{-1} is order-preserving and commutes with inverse substitutions, from the above display it follows

$$\begin{aligned}\mathrm{Cn}_{\vdash}(\Gamma) = \Phi^{-1}\Phi(\mathrm{Cn}_{\vdash}(\Gamma)) &\subseteq \Phi^{-1}(\sigma^{-1}[\mathrm{Cn}_{\mathsf{K}}(\sigma[\Phi(\mathrm{Cn}_{\vdash}(\Gamma))])]) \\ &\subseteq \sigma^{-1}[\Phi^{-1}(\mathrm{Cn}_{\mathsf{K}}(\sigma[\Phi(\mathrm{Cn}_{\vdash}(\Gamma))]))].\end{aligned}$$

Consequently,

$$\sigma[\mathrm{Cn}_{\vdash}(\Gamma)] \subseteq \Phi^{-1}(\mathrm{Cn}_{\mathsf{K}}(\sigma[\Phi(\mathrm{Cn}_{\vdash}(\Gamma))])).$$

As the right-hand side of the above display belongs to $\mathcal{T}h(\vdash)$, we obtain

$$\mathrm{Cn}_{\vdash}(\sigma[\mathrm{Cn}_{\vdash}(\Gamma)]) \subseteq \Phi^{-1}(\mathrm{Cn}_{\mathsf{K}}(\sigma[\Phi(\mathrm{Cn}_{\vdash}(\Gamma))]))$$

which, in turn, implies

$$\Phi(\mathrm{Cn}_{\vdash}(\sigma[\Gamma])) \subseteq \Phi(\mathrm{Cn}_{\vdash}(\sigma[\mathrm{Cn}_{\vdash}(\Gamma)])) \subseteq \mathrm{Cn}_{\mathsf{K}}(\sigma[\Phi(\mathrm{Cn}_{\vdash}(\Gamma))]).$$

This establishes the left to right inclusion of the first equality in the statement.

To prove the other inclusion, observe that $\Gamma \subseteq \sigma^{-1}[\mathrm{Cn}_{\vdash}(\sigma[\Gamma])]$. Notice that, by substitution invariance, $\sigma^{-1}[\mathrm{Cn}_{\vdash}(\sigma[\Gamma])]$ is a theory of $\vdash$. Consequently,

$$\mathrm{Cn}_{\vdash}(\Gamma) \subseteq \mathrm{Cn}_{\vdash}((\sigma^{-1}[\mathrm{Cn}_{\vdash}(\sigma[\Gamma])]) = \sigma^{-1}[\mathrm{Cn}_{\vdash}(\sigma[\Gamma])].$$

At the left and right-hand sides of the above displays are theories of $\vdash$,

$$\Phi(\mathrm{Cn}_{\vdash}(\Gamma)) \subseteq \Phi(\sigma^{-1}[\mathrm{Cn}_{\vdash}(\sigma[\Gamma])]) = \sigma^{-1}[\Phi(\mathrm{Cn}_{\vdash}(\sigma[\Gamma]))].$$

Consequently, $\sigma[\Phi(\mathrm{Cn}_{\vdash}(\Gamma))] \subseteq \Phi(\mathrm{Cn}_{\vdash}(\sigma[\Gamma]))$, as desired. ⊠

Now, let σ_x (resp. $\sigma_{x,y}$) be the substitution sending all variables to x (resp. sending all variables other than y to x, and leaving untouched y). We define

$$\boldsymbol{\tau}(x) := \sigma_x[\Phi(\mathrm{Cn}_{\vdash}(\{x\}))] \text{ and } \Delta(x,y) := \sigma_{x,y}[\Phi^{-1}(\mathrm{Cn}_{\mathsf{K}}(\{x \approx y\}))].$$

Observe that $\boldsymbol{\tau}$ and Δ are, respectively, a set of equations and a set of formulas. Our aim is to prove that $\boldsymbol{\tau}$, Δ, and K witness the algebraizability of $\vdash$.

Claim 4.10. *For every formula φ,*

$$\Phi(\mathrm{Cn}_{\vdash}(\{\varphi\})) = \mathrm{Cn}_{\mathsf{K}}(\boldsymbol{\tau}(\varphi)).$$

Proof. First, observe that by substitution invariance, for all $\Gamma \subseteq Fm$, $\Theta \subseteq Eq$, and substitution σ,

$$\mathrm{Cn}_{\vdash}(\sigma[\mathrm{Cn}_{\vdash}(\Gamma)]) = \mathrm{Cn}_{\vdash}(\sigma[\Gamma]) \text{ and } \mathrm{Cn}_{\mathsf{K}}(\sigma[\mathrm{Cn}_{\mathsf{K}}(\Theta)]) = \mathrm{Cn}_{\mathsf{K}}(\sigma[\Theta]). \tag{12}$$

Bearing this in mind, we obtain

$$\Phi(\mathrm{Cn}_{\vdash}(\{x\})) = \Phi(\mathrm{Cn}_{\vdash}(\{\sigma_x(x)\})) = \Phi(\mathrm{Cn}_{\vdash}(\sigma_x[\mathrm{Cn}_{\vdash}(\{x\})])).$$

Together with Claim 4.9, this implies

$$\Phi(\mathrm{Cn}_{\vdash}(\{x\})) = \mathrm{Cn}_{\mathsf{K}}(\sigma_x[\Phi(\mathrm{Cn}_{\vdash}(\{x\}))]) = \mathrm{Cn}_{\mathsf{K}}(\boldsymbol{\tau}(x)). \tag{13}$$

Now, consider a formula φ and let σ be any substitution such that $\sigma(x) = \varphi$. By (12) we obtain

$$\Phi(\mathrm{Cn}_{\vdash}(\{\varphi\})) = \Phi(\mathrm{Cn}_{\vdash}(\{\sigma(x)\})) = \Phi(\mathrm{Cn}_{\vdash}(\sigma[\mathrm{Cn}_{\vdash}(\{x\})])).$$

Furthermore, by Claim 4.9 and (13),

$$\Phi(\mathrm{Cn}_{\vdash}(\sigma[\mathrm{Cn}_{\vdash}(\{x\})])) = \mathrm{Cn}_{\mathsf{K}}(\sigma[\Phi(\mathrm{Cn}_{\vdash}(\{x\}))]) = \mathrm{Cn}_{\mathsf{K}}(\sigma[\mathrm{Cn}_{\mathsf{K}}(\boldsymbol{\tau}(x))]).$$

Finally, from (12) it follows

$$\mathrm{Cn}_{\mathsf{K}}(\sigma[\mathrm{Cn}_{\mathsf{K}}(\boldsymbol{\tau}(x))]) = \mathrm{Cn}_{\mathsf{K}}(\sigma[\boldsymbol{\tau}(x)]) = \mathrm{Cn}_{\mathsf{K}}(\boldsymbol{\tau}(\varphi)).$$

From the last three display we conclude $\Phi(\mathrm{Cn}_{\vdash}(\{\varphi\})) = \mathrm{Cn}_{\mathsf{K}}(\boldsymbol{\tau}(\varphi))$. ⊠

Claim 4.11. *For every set of formulas Γ,*

$$\Phi(\mathrm{Cn}_{\vdash}(\Gamma)) = \mathrm{Cn}_{\mathsf{K}}(\boldsymbol{\tau}[\Gamma]).$$

Proof. From Claim 4.10 and the fact that Φ preserves arbitrary joins it follows

$$\begin{aligned}\Phi(\mathrm{Cn}_{\vdash}(\Gamma)) &= \Phi(\bigvee_{\gamma\in\Gamma} \mathrm{Cn}_{\vdash}(\{\gamma\})) = \bigvee_{\gamma\in\Gamma} \Phi(\mathrm{Cn}_{\vdash}(\{\gamma\})) = \bigvee_{\gamma\in\Gamma} \mathrm{Cn}_{\mathsf{K}}(\boldsymbol{\tau}(\gamma))\\ &= \mathrm{Cn}_{\mathsf{K}}(\bigcup_{\gamma\in\Gamma} \mathrm{Cn}_{\mathsf{K}}(\boldsymbol{\tau}(\gamma))) = \mathrm{Cn}_{\mathsf{K}}(\bigcup_{\gamma\in\Gamma} \boldsymbol{\tau}(\gamma)) = \boldsymbol{\tau}[\Gamma].\end{aligned}$$

⊠

In order to prove that $\boldsymbol{\tau}$, Δ, and K witness the algebraizability of $\vdash$, it suffices to check conditions (Alg1) and (Alg4). To prove (Alg4), consider a set of formulas $\Gamma \cup \{\varphi\}$. Applying the fact that Φ is an order isomorphism and Claim 4.11, we obtain

$$\begin{aligned}\Gamma \vdash \varphi &\iff \mathrm{Cn}_{\vdash}(\{\varphi\}) \subseteq \mathrm{Cn}_{\vdash}(\Gamma) \iff \Phi(\mathrm{Cn}_{\vdash}(\{\varphi\})) \subseteq \Phi(\mathrm{Cn}_{\vdash}(\Gamma))\\ &\iff \mathrm{Cn}_{\mathsf{K}}(\boldsymbol{\tau}(\varphi)) \subseteq \mathrm{Cn}_{\mathsf{K}}(\boldsymbol{\tau}[\Gamma]) \iff \boldsymbol{\tau}[\Gamma] \vDash_{\mathsf{K}} \boldsymbol{\tau}(\varphi).\end{aligned}$$

In order to prove (Alg4), observe that

$$\begin{aligned}\mathrm{Cn}_{\mathsf{K}}(\{x \approx y\}) &= \mathrm{Cn}_{\mathsf{K}}(\sigma_{x,y}[\{x \approx y\}])\\ &= \mathrm{Cn}_{\mathsf{K}}(\sigma_{x,y}[\mathrm{Cn}_{\mathsf{K}}(\{x \approx y\})])\\ &= \Phi\Phi^{-1}(\mathrm{Cn}_{\mathsf{K}}(\sigma_{x,y}[\mathrm{Cn}_{\mathsf{K}}(\{x \approx y\})]))\\ &= \Phi(\mathrm{Cn}_{\vdash}(\sigma_{x,y}[\Phi^{-1}(\mathrm{Cn}_{\mathsf{K}}(\{x \approx y\}))]))\\ &= \Phi(\mathrm{Cn}_{\vdash}(\Delta(x,y))\\ &= \mathrm{Cn}_{\mathsf{K}}(\boldsymbol{\tau}[\Delta(x,y)]).\end{aligned}$$

The equalities above are justified as follows. The second follows from (12), the fourth from Claim 4.9, and the sixth from Claim 4.11. From the above display it immediately follows (Alg4), thus concluding the proof. ⊠

Exercise 4.12. Let G be the variety of groups, formulated in the language $\langle +, -, 0\rangle$. We define a logic $\vdash_{\mathsf{G}}$ as follows:

$$\Gamma \vdash_{\mathsf{G}} \varphi \iff \{\gamma \approx 0 \colon \gamma \in \Gamma\} \vDash_{\mathsf{G}} \varphi \approx 0.$$

Check that $\vdash_{\mathsf{G}}$ is algebraizable with equivalent algebraic semantics the variety of groups G. Furthermore, show that the deductive filters of $\vdash_{\mathsf{G}}$ on a group $\boldsymbol{A}$ are exactly the normal subgroups of $\boldsymbol{A}$. Consequently, derive from Theorem 4.8 the well-known correspondence between normal subgroups and congruences typical of group theory. Convince yourself that a similar explanation can be offered for the correspondence between ideals and congruences in ring theory. ⊠

While part of the interest of Theorem 4.8 lies in the fact that it provides an explanation of the ubiquitous correspondence between special subsets of the universes of algebras and congruences, the same theorem provides a useful criterion to *disprove* that a logic is algebraizable, as shown in the next exercise.

Exercise 4.13. Complete the following proof sketch of the fact that the local modal logic $\mathbf{K}_l$ is not algebraizable. While doing this, you are allowed to use the fact that $\mathbf{K}_l$ can be axiomatized by the Hilbert-style calculus obtained as follows. As set of axioms we take the least set Σ of modal formulas such that:

(i) Σ contains all classical tautologies;
(ii) Σ contains the formula $\Box(x \to y) \to (\Box x \to \Box y)$;
(iii) Σ is closed under substitutions;
(iv) Σ is closed under modus ponens and necessitation.

As sole rule of inference, we take modus ponens.

Use this Hilbert-style calculus for $\mathbf{K}_l$ to prove that the deductive filters of $\mathbf{K}_l$ on a modal algebra $\boldsymbol{A}$ are precisely the lattice filters of $\boldsymbol{A}$. Then take a sufficiently large simple modal algebra $\boldsymbol{B}$. Observe that $\boldsymbol{B}$ has "many" lattice filters, but only "few" congruences. Combine this observation with Theorem 4.8 to conclude that $\mathbf{K}_l$ cannot be algebraizable. ⊠

5. The deduction-detachment theorem

The well-known *deduction-detachment theorem* of **IPC** states that

$$\Gamma, \varphi \vdash_{\mathbf{IPC}} \psi \iff \Gamma \vdash_{\mathbf{IPC}} \varphi \to \psi,$$

for every set of formulas $\Gamma \cup \{\varphi, \psi\}$. While the same deduction-detachment theorem holds for $\mathbf{K}_l$, it fails for $\mathbf{K}_g$, as witnessed by the fact that

$$x \vdash_{\mathbf{K}_g} \Box x \text{ and } \emptyset \nvdash_{\mathbf{K}_g} x \to \Box x.$$

It is therefore sensible to wonder whether $\mathbf{K}_g$ has any sort of deduction-detachment theorem at all. This is indeed the case, provided that we weaken our expectations on what a deduction-detachment theorem should be. More precisely, $\mathbf{K}_g$ has a *local deduction-detachment theorem* stating that

$$\Gamma, \varphi \vdash_{\mathbf{K}_g} \psi \iff \text{ there is } n \in \omega \text{ such that } \Gamma \vdash_{\mathbf{K}_g} (\varphi \wedge \Box\varphi \wedge \cdots \wedge \Box^n \varphi) \to \psi.$$

The expression "local" attached to the above deduction-detachment theorem is intended to indicate that the formula witnessing the deduction-detachment theorem may depend on the deduction-detachment under consideration.

In the next sections we shall see that the theory of algebraizable logics allows to make sense of (local) deduction-detachment theorems in general, as well as to shed light on their algebraic interpretation. We begin by considering a general form of deduction-detachment theorem.

Definition 5.1. A logic $\vdash$ has a *deduction-detachment theorem* (DDT) if there exists a finite set of formulas $I(x, y)$ such that for every set of formulas $\Gamma \cup \{\varphi, \psi\}$,

$$\Gamma, \varphi \vdash \psi \iff \Gamma \vdash I(\varphi, \psi).$$

Clearly, **IPC** has a DDT in this sense, witnessed by the set $I(x,y) := \{x \to y\}$. In order to review also an example from the setting of modal logic, let $\mathbf{K4}_g$ be the axiomatic extension of $\mathbf{K}_g$ obtained by adding (substitution instances of) the so-called *K4 axiom* $\Box x \to \Box\Box x$. Notice that for every set of formulas $\Gamma \cup \{\varphi, \psi\}$,

$$\Gamma, \varphi \vdash_{\mathbf{K4}_g} \psi \iff \Gamma \vdash_{\mathbf{K4}_g} (\varphi \land \Box\varphi) \to \psi.$$

Consequently, $\mathbf{K4}_g$ has a DDT witnessed by the set $I(x,y) = \{(x \land \Box x) \to y\}$. In view of the next exercise, the DDT persists in all intermediate logics, as well as in all axiomatic extensions of $\mathbf{K4}_g$.

Exercise 5.2. Prove that if a logic $\vdash$ has a DDT witnessed by a set $I(x,y)$, then so do all the axiomatic extensions of $\vdash$. ⊠

The aim of this section is to individuate the algebraic counterpart of the deduction-detachment theorem. To this end, let us first extend the notion of a deduction-detachment theorems to relative equational consequences.

Definition 5.3. Let K be a quasi-variety. The relative equational consequence $\vDash_{\mathsf{K}}$ has a *deduction-detachment theorem* (DDT) if there exists a finite set of equations $\Psi(x,y,z,v)$ such that for every set of equations $\Theta \cup \{\varphi \approx \psi, \varepsilon \approx \delta\}$,

$$\Theta, \varphi \approx \psi \vDash_{\mathsf{K}} \varepsilon \approx \delta \iff \Theta \vDash_{\mathsf{K}} \Psi(\varphi, \psi, \varepsilon, \delta).$$

The next result states that crossing the mirror between logic and algebra preserves the validity of the deduction-detachment theorem.

Lemma 5.4. *An algebraizable logic $\vdash$ has a DDT if and only if the equational consequence $\vDash_{\mathsf{K}}$ relative to its equivalent algebraic semantics K has one.*

Proof. We shall detail the proof of the "only if" part, as the other one is analogous. To this end, assume that $\vdash$ has a DDT witnessed by a finite set of formulas $I(x,y)$. For every $n \in \omega$, we define a set $I_n(x_1, \dots, x_n, y)$ recursively by the following rule:

$$I_0(y) := \{y\}$$
$$I_{k+1}(x_1, \dots, x_{k+1}, y) := \bigcup \{I(x_1, \varphi) \colon \varphi \in I_k(x_2, \dots, x_k, y)\}.$$

A straightforward inductive argument shows that for every $n \in \omega$ and every set of formulas $\Gamma \cup \{\psi\} \cup \{\varphi_i \colon i < n\}$,

$$\Gamma \cup \{\varphi_i \colon i < n\} \vdash \psi \iff \Gamma \vdash I_n(\varphi_0, \dots, \varphi_{n-1}, \psi). \tag{14}$$

Now, in view of Lemma 3.4, the algebraizability of $\vdash$ is witnessed by some finite sets $\boldsymbol{\tau}$ and Δ. As Δ is finite, it has the form $\{\alpha_i \colon i < n\}$ for some $n \in \omega$. Then, consider the following set of equations

$$\Phi(x,y,z,v) := \bigcup_{i<n} \boldsymbol{\tau}[I_n(\alpha_0(x,y), \dots, \alpha_{n-1}(x,y), \alpha_i(z,v))].$$

Observe that Φ is finite, as Δ and $\boldsymbol{\tau}$ are. We shall see that Φ witnesses a DDT for $\vDash_{\mathsf{K}}$. To this end, consider a set of equations $\Theta \cup \{\varphi \approx \psi, \varepsilon \approx \delta\}$. We have

$$\begin{aligned}
\Theta, \varphi \approx \psi \vDash_{\mathsf{K}} \varepsilon \approx \delta &\iff \Delta[\Theta], \Delta(\varphi, \psi) \vdash \Delta(\varepsilon, \delta) \\
&\iff \Delta[\Theta] \cup \{\alpha_i(\varphi, \psi) \colon i < n\} \vdash \alpha_j(\varepsilon, \delta), \text{ for all } j < n \\
&\iff \Delta[\Theta] \vdash \bigcup_{i<n} I_n(\alpha_1(\varphi, \psi), \dots, \alpha_n(\varphi, \psi), \alpha_i(\varepsilon, \delta)) \\
&\iff \boldsymbol{\tau}[\Delta[\Theta]] \vDash_{\mathsf{K}} \bigcup_{i<n} \boldsymbol{\tau}[I_n(\alpha_1(\varphi, \psi), \dots, \alpha_n(\varphi, \psi), \alpha_i(\varepsilon, \delta))] \\
&\iff \Theta \vDash_{\mathsf{K}} \Phi(\varphi, \psi, \varepsilon, \delta).
\end{aligned}$$

The above equivalences are proved using (Alg1), ..., (Alg4) and (14). Hence, we conclude that Φ witnesses a DDT for $\vDash_{\mathsf{K}}$, as desired. ⊠

In order to individuate the algebraic counterpart of the deduction-detachment theorem, we need to introduce some more concept.

Definition 5.5. A quasi-variety K is said to have *equationally definable principal relative congruences* (EDPRC) when there exists a finite set of equations $\Phi(x, y, z, v)$ such that for every $\boldsymbol{A} \in \mathsf{K}$ and $a, b, c, d \in A$,

$$\langle a, b\rangle \in \mathrm{Cg}_{\mathsf{K}}^{\boldsymbol{A}}(c, d) \iff \boldsymbol{A} \vDash \Phi(c, d, a, b).$$

When K is a variety, $\mathrm{Cg}_{\mathsf{K}}^{\boldsymbol{A}}$ can be replaced by $\mathrm{Cg}^{\boldsymbol{A}}$ in the above display, K is said to have *equationally definable principal congruences* (EDPC).

Example 5.6. Observe that for every Heyting algebra $\boldsymbol{A}$ and $a, b, c, d \in A$,

$$\langle a, b\rangle \in \mathrm{Cg}^{\boldsymbol{A}}(c, d) \iff ((c \to d) \wedge (d \to c)) \leqslant ((a \to b) \wedge (b \to a))$$

where $x \leqslant y$ is a shorthand for $x \wedge y \approx x$. Consequently, the variety of Heyting algebras has EDPC witnessed by the set of equations

$$\Phi(x, y, z, v) := \{((x \to y) \wedge (y \to x)) \leqslant ((z \to v) \wedge (v \to z))\}.$$

Similarly, for every K4-algebra $\boldsymbol{A}$ and $a, b, c, d \in A$,

$$\langle a, b\rangle \in \mathrm{Cg}^{\boldsymbol{A}}(c, d) \iff ((c \leftrightarrow d) \wedge \Box(c \leftrightarrow d)) \leqslant ((a \leftrightarrow b) \wedge \Box(a \leftrightarrow b)).$$

Consequently, the variety of K4-algebras has EDPC witnessed by the set of equations

$$\Phi(x, y, z, v) := \{((x \leftrightarrow y) \wedge \Box(x \leftrightarrow y)) \leqslant ((z \leftrightarrow v) \wedge \Box(z \leftrightarrow v))\}.$$ ⊠

As shown in the next result, EDPRC is the algebraic counterpart of the DDT. The equivalence between EDPRC and the DDT originates in the unpublished, but widely circulated, memoir [14].

Theorem 5.7 (Blok & Pigozzi)**.** *An algebraizable logic has a DDT if and only if its equivalent algebraic semantics has EDPRC.*

In view of Lemma 5.4, to establish the above result, it suffices to prove the following:

Lemma 5.8. *A quasi-variety K has EDPRC if and only if its relative equational consequence $\vDash_{\mathsf{K}}$ has a DDT.*

Proof. In order to prove the "if" part, suppose that $\vDash_{\mathsf{K}}$ has a DDT witnessed by a finite set of equations $\Phi(x, y, z, v)$. We shall see that Φ witnesses also EDPRC for K. In view of Lemma 2.16, it suffices to show that for every *finitely generated* $\boldsymbol{A} \in \mathsf{K}$ and $a, b, c, d \in A$,

$$\langle a, b\rangle \in \mathrm{Cg}^{\boldsymbol{A}}_{\mathsf{K}}(c, d) \Longleftrightarrow \boldsymbol{A} \vDash \Phi(c, d, a, b). \tag{15}$$

To this end, first suppose that $\langle a, b\rangle \in \mathrm{Cg}^{\boldsymbol{A}}_{\mathsf{K}}(c, d)$. Since $\boldsymbol{A}$ is finitely generated, we can choose some generators $e_1, \dots, e_n$ for it. Clearly, there are $\varphi_a, \varphi_b, \varphi_c$, and φ_d in variables $x_1, \dots, x_n$ such that

$$\begin{aligned}
\varphi_a^{\boldsymbol{A}}(e_1, \dots, e_n) &= a\\
\varphi_b^{\boldsymbol{A}}(e_1, \dots, e_n) &= b\\
\varphi_c^{\boldsymbol{A}}(e_1, \dots, e_n) &= c\\
\varphi_d^{\boldsymbol{A}}(e_1, \dots, e_n) &= d.
\end{aligned}$$

Furthermore, consider the set of equations

$$\Theta := \{\varepsilon(x_1, \dots, x_n) \approx \delta(x_1, \dots, x_n) \colon \varepsilon^{\boldsymbol{A}}(e_1, \dots, e_n) = \delta^{\boldsymbol{A}}(e_1, \dots, e_n)\}.$$

From the fact that $\langle a, b\rangle \in \mathrm{Cg}^{\boldsymbol{A}}_{\mathsf{K}}(c, d)$ it easily follows

$$\Theta, \varphi_c \approx \varphi_d \vDash_{\mathsf{K}} \varphi_a \approx \varphi_b.$$

As Φ witnesses a DDT for $\vDash_{\mathsf{K}}$, this implies

$$\Theta \vDash_{\mathsf{K}} \Phi(\varphi_c, \varphi_d, \varphi_a, \varphi_b).$$

Since $\boldsymbol{A} \vDash \Theta[\![e_1, \dots, e_n]\!]$, this yields $\boldsymbol{A} \vDash \Phi(\varphi_c, \varphi_d, \varphi_a, \varphi_b)[\![e_1, \dots, e_n]\!]$, i.e., $\boldsymbol{A} \vDash \Phi(c, d, a, b)$.

To prove the converse, notice that from $\Phi(x, y, z, v) \vDash_{\mathsf{K}} \Phi(x, y, z, v)$ and the assumption that Φ witnesses a DDT for $\vDash_{\mathsf{K}}$, it follows

$$x \approx y, \Phi(x, y, z, v) \vDash_{\mathsf{K}} z \approx v. \tag{16}$$

Suppose that $\boldsymbol{A} \vDash \Phi(c, d, a, b)$. Then the set of premises of the above deduction-detachment is valid in $\boldsymbol{A}/\mathrm{Cg}^{\boldsymbol{A}}_{\mathsf{K}}(c, d)$ under the assignment

$$\begin{aligned}
x &\longmapsto c/\mathrm{Cg}^{\boldsymbol{A}}_{\mathsf{K}}(c, d) & y &\longmapsto d/\mathrm{Cg}^{\boldsymbol{A}}_{\mathsf{K}}(c, d)\\
z &\longmapsto a/\mathrm{Cg}^{\boldsymbol{A}}_{\mathsf{K}}(c, d) & v &\longmapsto b/\mathrm{Cg}^{\boldsymbol{A}}_{\mathsf{K}}(c, d).
\end{aligned}$$

By (16) we obtain $a/\mathrm{Cg}^{\boldsymbol{A}}_{\mathsf{K}}(c, d) = b/\mathrm{Cg}^{\boldsymbol{A}}_{\mathsf{K}}(c, d)$, i.e., $\langle a, b\rangle \in \mathrm{Cg}^{\boldsymbol{A}}_{\mathsf{K}}(c, d)$. As this establishes (15), we conclude that K has EDPRC.

To prove the "only if" part, suppose that K has EDPRC witnessed by a finite set of equations $\Phi(x, y, z, v)$. We shall see that Φ witnesses also a DDT for $\vDash_{\mathsf{K}}$. To this end, consider a set of equations $\Theta \cup \{\varphi \approx \psi, \varepsilon \approx \delta\}$. Notice that the set of equations Θ can be though as a set of pairs of formulas. Bearing this in mind, we define

$$\eta := \mathrm{Cg}^{\boldsymbol{Fm}}_{\mathsf{K}}(\Theta) \text{ and } \boldsymbol{A} := \boldsymbol{Fm}/\eta.$$

We have

$$\begin{aligned}\Theta, \varphi \approx \psi \vDash_{\mathsf{K}} \varepsilon \approx \delta &\Longleftrightarrow \langle \varepsilon, \delta \rangle \in \mathrm{Cg}^{\boldsymbol{Fm}}_{\mathsf{K}}(\Theta \cup \{\langle \varphi, \psi \rangle\})\\ &\Longleftrightarrow \langle \varepsilon/\eta, \delta/\eta \rangle \in \mathrm{Cg}^{\boldsymbol{A}}_{\mathsf{K}}(\varphi/\eta, \psi/\eta)\\ &\Longleftrightarrow \boldsymbol{A} \vDash \Phi(\varphi/\eta, \psi/\eta, \varepsilon/\eta, \delta/\eta)\\ &\Longleftrightarrow \Phi(\varphi, \psi, \varepsilon, \delta) \subseteq \eta\\ &\Longleftrightarrow \Theta \vDash_{\mathsf{K}} \Phi(\varphi, \psi, \varepsilon, \delta).\end{aligned}$$

The above equivalences are justified as follows. The first and the latter follows from the fact that $\mathcal{T}h(\vDash_{\mathsf{K}}) = \mathrm{Con}_{\mathsf{K}}\boldsymbol{Fm}$ (Exercise 4.7). The third equivalence holds because Φ witnesses EDPRC for K. The second and the fourth equality are easy observations. ⊠

As we mentioned, all axiomatic extensions of $\mathbf{K4}_g$ have a DDT. It is therefore sensible to wonder whether this is true for all axiomatic extensions of $\mathbf{K}_g$. This is not the case, as we shall see in the next section. Indeed, a full characterization of axiomatic extensions of $\mathbf{K}_g$ with a DDT was obtained in [8]. We shall state it without proof for the reader's curiosity.

Theorem 5.9 (Blok & Köhler & Pigozzi)**.** *An axiomatic extension $\vdash$ of $\mathbf{K}_g$ has a DDT if and only there is some $n \in \omega$ such that*

$$\emptyset \vdash (x \wedge \Box x \wedge \cdots \wedge \Box^n x) \to (x \wedge \Box x \wedge \cdots \wedge \Box^{n+1} x).$$

6. Quasi-varieties with EDPRC

In this section we shall investigate the basic structure theory of quasi-varieties with EDPRC. In view of Theorem 5.7, this amounts to studying consequences and characterizations the deduction-detachment theorem. Our discussion is based on [30], but further references (for the case of varieties only) include the series of papers [9, 8, 12, 13].

Our first aim is to show that EDPRC can be rephrased as a property of the lattices of K-congruences, where K is the quasi-variety under consideration (Theorem 6.3). To this end, recall that a *semilattice* is an algebra $\langle A; * \rangle$ such that $*$ is an idempotent, commutative, and associative binary operation. Every semilattice $\boldsymbol{A}$ can be associated with two partial orders on A, namely the *meet-order* $\leqslant_\wedge$ and the *join-order* $\leqslant_\vee$, defined respectively by the following rules

$$a \leqslant_\wedge c \Longleftrightarrow a *^{\boldsymbol{A}} c = a \text{ and } a \leqslant_\vee c \Longleftrightarrow a *^{\boldsymbol{A}} c = c.$$

Clearly, $\leqslant_\wedge$ and $\leqslant_\vee$ are dual one to the other.

Example 6.1. Let K be a quasi-variety and $\boldsymbol{A} \in \mathsf{K}$. Recall from Theorem 2.12 that $\mathrm{Con}_{\mathsf{K}}\boldsymbol{A}$ is an algebraic lattice, whose compact elements are the finitely generated K-congruences of $\boldsymbol{A}$. Notice that the compact elements of an algebraic lattice are always closed under binary joins. Consequently, the set $\mathrm{Comp}_{\mathsf{K}}\boldsymbol{A}$ of compact elements of $\mathrm{Con}_{\mathsf{K}}\boldsymbol{A}$, when endowed under the join operation of $\mathrm{Con}_{\mathsf{K}}\boldsymbol{A}$, becomes a semilattice. ⊠

A *Brouwerian semilattice* is an algebra $\boldsymbol{A} = \langle A; *, \to \rangle$ such that $\langle A; * \rangle$ is a semilattice and for every $a, b, c \in A$,

$$a *^{\boldsymbol{A}} b \leqslant_\wedge c \Longleftrightarrow a \leqslant_\wedge b \to^{\boldsymbol{A}} c.$$

Remark 6.2. Brouwerian semilattices are most famous for coinciding with subalgebras of the $\langle \wedge, \to \rangle$-reducts of Heyting algebras (these are sometimes called $\langle \wedge, \to \rangle$-*subreducts* of Heyting algebras). That subalgebras of the $\langle \wedge, \to \rangle$-reducts of Heyting algebras are Brouwerian semilattices is a straightforward comprobation. On the other hand, given a Brouwerian semilattice $\boldsymbol{A}$, let $\mathcal{D}(\boldsymbol{A})$ be the Heyting algebra of the downsets *with respect to the meet-order* of $\boldsymbol{A}$. It is not hard (yet neither straightforward) to show that the map $\gamma \colon \boldsymbol{A} \to \mathcal{D}(\boldsymbol{A})$ given by the rule

$$\gamma(a) := \{F \in \mathcal{D}(\boldsymbol{A}) \colon a \in F\}$$

is an embedding of $\boldsymbol{A}$ into the $\langle \wedge, \to \rangle$-reduct of $\mathcal{D}(\boldsymbol{A})$. ⊠

For the present purpose, the interest of Brouwerian semilattices lies in the fact that these can be used to characterize EDPRC.

Theorem 6.3 (Köhler & Pigozzi)**.** *A quasi-variety* K *has EDPRC if and only if for every algebra* $\boldsymbol{A} \in \mathsf{K}$ *the semilattice* $\mathrm{Comp}_{\mathsf{K}}\boldsymbol{A}$ *can be endowed with a binary operation* $\to$ *such that* $\langle \mathrm{Comp}_{\mathsf{K}}\boldsymbol{A}; \vee, \to \rangle$ *is a Brouwerian semilattice.*

In order to prove the above theorem, we rely on the following observation.

Lemma 6.4. *Let* $\langle A; * \rangle$ *be a semilattice generated by a set* $X \subseteq A$*. Suppose that for every* $x, y \in X$ *there exists an element* $x \multimap y \in A$ *such that for all* $z \in A$*,*

$$x * z \leqslant_{\wedge} y \Longleftrightarrow z \leqslant_{\wedge} x \multimap y.$$

Then there is a binary operation $\to$ *on* A *such that* $\langle A; *, \to \rangle$ *is a Brouwerian semilattice.*

Proof. First, consider $x \in X$ and $y \in Y$. As X generates $\langle A; * \rangle$, there are $y_1, \dots, y_n \in X$ such that $y = y_1 * \cdots * y_n$. We set

$$x \to y := (x \multimap y_1) * \cdots * (x \multimap y_n).$$

Observe that for all $z \in A$,

$$\begin{aligned} z * x \leqslant_{\wedge} y &\Longleftrightarrow z * x \leqslant_{\wedge} y_1 * \cdots * y_n \\ &\Longleftrightarrow z * x \leqslant_{\wedge} y_i \text{ for all } i = 1, \dots, n \\ &\Longleftrightarrow z \leqslant_{\wedge} x \to y_i \text{ for all } i = 1, \dots, n \\ &\Longleftrightarrow z \leqslant_{\wedge} (x \multimap y_1) * \cdots * (x \multimap y_n) \\ &\Longleftrightarrow z \leqslant_{\wedge} x \to y. \end{aligned}$$

As X generates $\langle A; * \rangle$, to conclude the proof, it suffices to show that $(x_1 * \cdots * x_n) \to y$ exists for all $n \geqslant 1$ and $x_1, \dots, x_n \in X$. We shall reason by induction on n. The case where $n = 1$ was established above. Then we consider the case where $n = k + 1$ for some $k \geqslant 1$. Consider $x_1, \dots, x_{k+1} \in X$ and $y \in X$. Using the inductive hypothesis and the case $n = 1$, we can define

$$(x_1 * \cdots * x_{k+1}) \to y := x_{k+1} \to ((x_1 * \cdots * x_k) \to y).$$

Applying the inductive hypothesis and the case $n = 1$, we obtain that for every $z \in A$,

$$\begin{aligned} z * (x_1 * \cdots * x_{k+1}) \leqslant_{\wedge} y &\Longleftrightarrow (z * x_{k+1}) * (x_1 * \cdots * x_k) \leqslant_{\wedge} y \\ &\Longleftrightarrow z * x_{k+1} \leqslant_{\wedge} (x_1 * \cdots * x_k) \to y \\ &\Longleftrightarrow z \leqslant_{\wedge} x_{k+1} \to ((x_1 * \cdots * x_k) \to y) \\ &\Longleftrightarrow z \leqslant_{\wedge} (x_1 * \cdots * x_{k+1}) \to y, \end{aligned}$$

concluding the proof. ⊠

We are now ready to come back to the proof of the main theorem.

Proof of Theorem 6.3. To prove the "only if" part, suppose that K has EDPRC and consider $\boldsymbol{A} \in \mathsf{K}$. Observe that the semilattice $\langle \mathrm{Comp}_{\mathsf{K}}\boldsymbol{A}; \vee\rangle$ is generated by the principal K-congruences of $\boldsymbol{A}$. Thus, in view of Lemma 6.4, it suffices to show that for every $a, b, c, d \in A$, there exists $\theta \in \mathrm{Comp}_{\mathsf{K}}\boldsymbol{A}$ such that for all $\phi \in \mathrm{Comp}_{\mathsf{K}}\boldsymbol{A}$,

$$\phi \vee \mathrm{Cg}^{\boldsymbol{A}}_{\mathsf{K}}(c,d) \leqslant_{\wedge} \mathrm{Cg}^{\boldsymbol{A}}_{\mathsf{K}}(b,d) \Longleftrightarrow \phi \leqslant_{\wedge} \theta.$$

Notice that, in this case, the above display can be rephrased as

$$\mathrm{Cg}^{\boldsymbol{A}}_{\mathsf{K}}(a,b) \subseteq \phi \vee \mathrm{Cg}^{\boldsymbol{A}}_{\mathsf{K}}(c,d) \Longleftrightarrow \theta \subseteq \phi. \tag{17}$$

Let $\Phi(x,y,z,v)$ be the finite set of equations witnessing EDPRC for K. Observe that $\Phi^{\boldsymbol{A}}(c,d,a,b)$ can be viewed as a finite set of pairs of elements in A. Bearing this in mind, we can define a compact K-congruence of $\boldsymbol{A}$ as follows:

$$\theta \coloneqq \mathrm{Cg}^{\boldsymbol{A}}_{\mathsf{K}}(\Phi^{\boldsymbol{A}}(c,d,a,b)).$$

Bearing in mind that Φ witnesses EDPRC for K, we obtain

$$\begin{aligned}
\mathrm{Cg}^{\boldsymbol{A}}_{\mathsf{K}}(a,b) \subseteq \phi \vee \mathrm{Cg}^{\boldsymbol{A}}_{\mathsf{K}}(c,d) &\Longleftrightarrow \langle a,b\rangle \in \phi \vee \mathrm{Cg}^{\boldsymbol{A}}_{\mathsf{K}}(c,d)\\
&\Longleftrightarrow \langle a/\phi, b/\phi\rangle \in \mathrm{Cg}^{\boldsymbol{A}/\phi}_{\mathsf{K}}(c/\phi, d/\phi)\\
&\Longleftrightarrow \Phi^{\boldsymbol{A}}(c,d,a,b) \subseteq \phi\\
&\Longleftrightarrow \mathrm{Cg}^{\boldsymbol{A}/\phi}_{\mathsf{K}}(\Phi^{\boldsymbol{A}}(c,d,a,b)) \subseteq \phi\\
&\Longleftrightarrow \theta \subseteq \phi.
\end{aligned}$$

This establishes (17), as desired.

Then we turn to prove the "if" part. Let $\boldsymbol{F}$ be the free algebra of K with a denumerable set of free generators $x, y, z, v, w_1, w_2, \dots$. By assumption $\mathrm{Comp}_{\mathsf{K}}\boldsymbol{F}$ can be endowed with a binary operation $\to$ such that $\langle \mathrm{Comp}_{\mathsf{K}}\boldsymbol{F}; \vee, \to\rangle$ is a Brouwerian semilattice. As the K congruences $\mathrm{Cg}^{\boldsymbol{F}}_{\mathsf{K}}(x,y)$ and $\mathrm{Cg}^{\boldsymbol{F}}_{\mathsf{K}}(z,v)$ are finitely generated, they belong to $\mathrm{Comp}_{\mathsf{K}}\boldsymbol{F}$. Thus,

$$\mathrm{Cg}^{\boldsymbol{F}}_{\mathsf{K}}(z,v) \to \mathrm{Cg}^{\boldsymbol{F}}_{\mathsf{K}}(x,y) \in \mathrm{Comp}_{\mathsf{K}}\boldsymbol{F}.$$

As the above K-congruence is finitely generated, there is a finite set of pairs of formulas

$$\Psi = \{\langle \varepsilon_i(x,y,z,v,w_1,\dots,w_n), \delta_i(x,y,z,v,w_1,\dots,w_n)\rangle \colon i < m\}$$

such that

$$\mathrm{Cg}^{\boldsymbol{F}}_{\mathsf{K}}(z,v) \to \mathrm{Cg}^{\boldsymbol{F}}_{\mathsf{K}}(x,y) = \mathrm{Cg}^{\boldsymbol{F}}_{\mathsf{K}}(\Psi).$$

Define

$$\Phi(x,y,z,v) \coloneqq \{\varepsilon_i(x,y,z,v,x,\dots,x) \approx \delta_i(x,y,z,v,x,\dots,x) \colon i < m\}.$$

We shall see that Φ witnesses EDPRC for K.

In view of Lemma 2.16, it suffices to prove that for every finitely generated $\boldsymbol{A} \in \mathsf{K}$ and $a, b, c, d \in A$,

$$\langle a,b\rangle \in \mathrm{Cg}^{\boldsymbol{A}}_{\mathsf{K}}(c,d) \Longleftrightarrow \boldsymbol{A} \vDash \Phi(c,d,a,b).$$

To this end, consider a finitely generated $\boldsymbol{A} \in \mathsf{K}$ and $a, b, c, d \in A$. Since $\boldsymbol{A}$ is finitely generated, there is a homomorphism $f \colon \boldsymbol{F} \to \boldsymbol{A}$ such that $f(w_i) = c$ for all $i = 1, \dots, n$

and $f(x) = c$, $f(y) = d$, $f(z) = a$, and $f(v) = b$. Let $\mathrm{Ker}(f)$ be the kernel of f. Notice that

$$\langle a, b\rangle \in \mathrm{Cg}^{\boldsymbol{A}}_{\mathsf{K}}(c, d) \Longleftrightarrow \langle z, v\rangle \in \mathrm{Ker}(f) \vee \mathrm{Cg}^{\boldsymbol{F}}_{\mathsf{K}}(x, y),$$

where $\vee$ is computed in $\mathrm{Con}_{\mathsf{K}}\boldsymbol{F}$. Furthermore, we have

$$\begin{aligned}
&\mathrm{Cg}^{\boldsymbol{F}}_{\mathsf{K}}(z, v) \to \mathrm{Cg}^{\boldsymbol{F}}_{\mathsf{K}}(x, y) \subseteq \mathrm{Ker}(f)\\
\Longleftrightarrow\ &\Psi \subseteq \mathrm{Ker}(f)\\
\Longleftrightarrow\ &f(\delta_i(x, y, z, v, w_1, \dots, w_n)) = f(\varepsilon_i(x, y, z, v, w_1, \dots, w_n)), \text{ for all } i = 1, \dots, m\\
\Longleftrightarrow\ &\delta^{\boldsymbol{A}}_i(c, d, a, b, c, \dots, c) = \varepsilon^{\boldsymbol{A}}_i(c, d, a, b, c, \dots, c), \text{ for all } i = 1, \dots, m\\
\Longleftrightarrow\ &\boldsymbol{A} \vDash \Phi(c, d, a, b).
\end{aligned}$$

In view of the two display above, to conclude the proof, it suffices to show that

$$\langle z, v\rangle \in \mathrm{Ker}(f) \vee \mathrm{Cg}^{\boldsymbol{F}}_{\mathsf{K}}(x, y) \Longleftrightarrow \mathrm{Cg}^{\boldsymbol{F}}_{\mathsf{K}}(z, v) \to \mathrm{Cg}^{\boldsymbol{F}}_{\mathsf{K}}(x, y) \subseteq \mathrm{Ker}(f).$$

This is follows from the following equivalences:

$$\begin{aligned}
\langle z, v\rangle \in \mathrm{Ker}(f) \vee \mathrm{Cg}^{\boldsymbol{F}}_{\mathsf{K}}(x, y) &\Longleftrightarrow \mathrm{Cg}^{\boldsymbol{F}}_{\mathsf{K}}(z, v) \subseteq \mathrm{Cg}^{\boldsymbol{F}}_{\mathsf{K}}(\Sigma) \vee \mathrm{Cg}^{\boldsymbol{F}}_{\mathsf{K}}(x, y) \text{ for a finite } \Sigma \subseteq \mathrm{Ker}(f)\\
&\Longleftrightarrow \mathrm{Cg}^{\boldsymbol{F}}_{\mathsf{K}}(\Sigma) \vee \mathrm{Cg}^{\boldsymbol{F}}_{\mathsf{K}}(x, y) \leqslant_{\wedge} \mathrm{Cg}^{\boldsymbol{F}}_{\mathsf{K}}(z, v) \text{ for a finite } \Sigma \subseteq \mathrm{Ker}(f))\\
&\Longleftrightarrow \mathrm{Cg}^{\boldsymbol{F}}_{\mathsf{K}}(\Sigma) \leqslant_{\wedge} \mathrm{Cg}^{\boldsymbol{F}}_{\mathsf{K}}(x, y) \to \mathrm{Cg}^{\boldsymbol{F}}_{\mathsf{K}}(z, v) \text{ for a finite } \Sigma \subseteq \mathrm{Ker}(f)\\
&\Longleftrightarrow \mathrm{Cg}^{\boldsymbol{F}}_{\mathsf{K}}(x, y) \to \mathrm{Cg}^{\boldsymbol{F}}_{\mathsf{K}}(z, v) \subseteq \mathrm{Cg}^{\boldsymbol{F}}_{\mathsf{K}}(\Sigma) \text{ for a finite } \Sigma \subseteq \mathrm{Ker}(f)\\
&\Longleftrightarrow \mathrm{Cg}^{\boldsymbol{F}}_{\mathsf{K}}(x, y) \to \mathrm{Cg}^{\boldsymbol{F}}_{\mathsf{K}}(z, v) \subseteq \mathrm{Ker}(f),
\end{aligned}$$

which are justified as follows. The first and the last follow, respectively, from the fact that $\mathrm{Cg}^{\boldsymbol{F}}_{\mathsf{K}}(z, v)$ and $\mathrm{Cg}^{\boldsymbol{F}}_{\mathsf{K}}(x, y) \to \mathrm{Cg}^{\boldsymbol{F}}_{\mathsf{K}}(z, v)$ are compact. The second and the fourth are obvious. The third follows from the fact that the congruences $\mathrm{Cg}^{\boldsymbol{F}}_{\mathsf{K}}(x, y), \mathrm{Cg}^{\boldsymbol{F}}_{\mathsf{K}}(z, v)$, and $\mathrm{Cg}^{\boldsymbol{F}}_{\mathsf{K}}(\Sigma)$ are compact and $\langle \mathrm{Comp}_{\mathsf{K}}\boldsymbol{F}; \vee, \to\rangle$ is a Brouwerian semilattice. ⊠

Given a logic $\vdash$ and an algebra $\boldsymbol{A}$, we denote by $\mathrm{Comp}_{\vdash}\boldsymbol{A}$ the set of compact elements of $\mathcal{F}i_{\vdash}\boldsymbol{A}$. Observe that they are closed under the join operation of $\mathcal{F}i_{\vdash}\boldsymbol{A}$.

Corollary 6.5. *Let $\vdash$ be an algebraizable logic with equivalent algebraic semantics K. Then $\vdash$ has a DDT if and only if for every $\boldsymbol{A} \in \mathsf{K}$, the semilattice $\mathrm{Comp}_{\vdash}\boldsymbol{A}$ can be endowed with a binary operation $\to$ such that $\langle \mathrm{Comp}_{\vdash}\boldsymbol{A}; \vee, \to\rangle$ is a Brouwerian semilattice.*

Proof. By Theorem 6.3, $\vdash$ has a DDT if and only if K has EDPRC. Thus the result follows from Theorem 6.3 and the fact that $\mathrm{Con}_{\mathsf{K}}\boldsymbol{A} \cong \mathcal{F}i_{\vdash}\boldsymbol{A}$ for every $\boldsymbol{A} \in \mathsf{K}$ (Theorem 4.8). ⊠

Notably, EDPRC has also interesting consequences on the behaviour of arbitrary (i.e., not necessarily compact) K-congruences.

Definition 6.6. A quasi-variety K is said to be *relatively congruence distributive* when $\mathrm{Con}_{\mathsf{K}}\boldsymbol{A}$ is as distributive lattice, for every $\boldsymbol{A} \in \mathsf{K}$.

Our next task is to prove the following:

Theorem 6.7 (Köhler & Pigozzi)**.** *Quasi-varieties with EDPRC are relatively congruence distributive.*

To this end, recall that *filter* of a semilattice $\langle A; *\rangle$ is a non-empty subset $I \subseteq A$ such that I is an upset of the the meet-order which, moreover, is closed under $*$. Furthermore, notice that the set $\mathrm{Comp}\boldsymbol{L}$ of compact elements of an algebraic lattice $\boldsymbol{L}$ is closed under binary joins of $\boldsymbol{L}$. We rely on the following well-known observation.

Lemma 6.8. *Every algebraic lattice $\boldsymbol{A}$ is isomorphic to the lattice of filters of the semilattice $\langle \mathrm{Comp}\boldsymbol{A}; \vee\rangle$.*

Proof sketch. Let $\mathrm{Fi}(\mathrm{Comp}\boldsymbol{A})$ be the lattice of filters of $\langle \mathrm{Comp}\boldsymbol{A}; \vee\rangle$. Then consider the map

$$\gamma\colon \boldsymbol{A} \to \mathrm{Fi}(\mathrm{Comp}\boldsymbol{A})$$

defined by the rule

$$\gamma(a) = \{c \in \mathrm{Comp}\boldsymbol{A} \colon c \leqslant a\}.$$

Using the definition of an algebraic lattice, It is easy to show that γ is an isomorphism. ⊠

We are now ready to come back to the proof of the main result.

Proof of Theorem 6.7. Let K be a quasi-variety with EDPRC and $\boldsymbol{A} \in \mathsf{K}$. First recall from Theorem 2.12 that $\mathrm{Con}_{\mathsf{K}}\boldsymbol{A}$ is an algebraic lattice. Thus, $\mathrm{Con}_{\mathsf{K}}\boldsymbol{A}$ is isomorphic to the lattice of ideals of $\langle \mathrm{Comp}_{\mathsf{K}}\boldsymbol{A}; \vee\rangle$ by Lemma 6.8. In view of Theorem 6.3, this means that $\mathrm{Con}_{\mathsf{K}}\boldsymbol{A}$ is isomorphic to the lattice of filters of the semilattice underlying some Brouwerian semilattice. Accordingly, to conclude the proof, it suffices to prove that the latter lattice is distributive. This is left to you as the next exercise. ⊠

Exercise 6.9. Let $\boldsymbol{A}$ be a Brouwerian semilattice. A *filter* of $\boldsymbol{A}$ is a filter of the semilattice underlying $\boldsymbol{A}$. Prove that, when ordered under inclusion, the collection of filters of $\boldsymbol{A}$ forms a distributive lattice. ⊠

Exercise 6.10. Only after solving the above exercise, experiment with the following alternative solution, which relies on algebraizability. Let BSL be the class of Brouwerian semilattices. Even if this is not obvious at first sight, BSL is a variety (you can, for instance, check that it is closed under $\mathbb{H}, \mathbb{S}$, and $\mathbb{P}$). Furthermore, every Brouwerian semilattice has a term-definable maximum $1 := x \to x$ in the meet-order. Bearing this in mind, consider the logic $\vdash$ defined by the following rule:

$$\Gamma \vdash \varphi \iff \{\gamma \approx 1 \colon \gamma \in \Gamma\} \vDash_{\mathsf{BSL}} \varphi \approx 1.$$

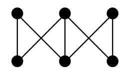
Use the fact that Brouwerian semilattices are the $\langle \wedge, \to\rangle$-subreducts of Heyting algebras, to conclude that $\vdash$ is the $\langle \wedge, \to\rangle$-fragment of **IPC**. As a consequence, infer that $\vdash$ has the same DDT as **IPC**.

Now, prove that $\vdash$ is algebraizable with equivalent algebraic semantics BSL. Use this fact and that $\vdash$ has a DDT to show that BSL is congruence distributive. Furthermore, show that the deductive filters of $\vdash$ on a Brouwerian semilattice $\boldsymbol{A}$ are precisely the filters of $\boldsymbol{A}$. Infer from Theorem 4.8 that the lattice of filters of every Brouwerian semilattice is distributive. ⊠

Corollary 6.11. *Let $\vdash$ be an algebraizable logic with equivalent algebraic semantics K. If $\vdash$ has a DDT, then every $\boldsymbol{A} \in \mathsf{K}$, the lattice $\mathcal{F}i_{\vdash}\boldsymbol{A}$ is distributive.*

Proof. Immediate from Theorems 5.7, 4.8, and 6.7. ⊠

Exercise 6.12. Let $\boldsymbol{A}$ be the finite Heyting algebra defined in Example 2.7. Prove that $\mathbb{Q}(\boldsymbol{A})$ is not relatively congruence distributive using the following hint. Let $\boldsymbol{B}$ be the Heyting algebra whose Esakia dual is the following poset (endowed with the discrete topology):



First, using Esakia duality, convince yourself that $\boldsymbol{B} \in \mathbb{Q}(\boldsymbol{A})$. Subsequently, compute the structure of the lattice $\mathrm{Con}_{\mathbb{Q}(\boldsymbol{A})}\boldsymbol{B}$. In view of Lemma 2.14, you might wish to reason as follows: a congruence θ of $\boldsymbol{B}$ belongs to $\mathrm{Con}_{\mathbb{Q}(\boldsymbol{A})}\boldsymbol{B}$ if and only if $\boldsymbol{B}/\theta$ is s subdirect product of algebras in $\mathbb{IS}(\boldsymbol{A})$. A further simplification is offered by Esakia duality, which tells you that the congruences of $\boldsymbol{B}$ corresponds to the upsets of the poset depicted above (but be careful: not all of them will be $\mathbb{Q}(\boldsymbol{A})$-congruences!). Conclude that $\mathrm{Con}_{\mathbb{Q}(\boldsymbol{A})}\boldsymbol{B}$ is not distributive, whence $\mathbb{Q}(\boldsymbol{A})$ is not relatively congruence distributive.

This shows that, while all varieties of Heyting algebras are congruence distributive (for instance, because they have EDPC), it is not true that all quasi-varieties of Heyting algebras are relatively congruence distributive.

Recall from Exercise 5.2 that all axiomatic extensions of **IPC** have a DDT. In view of the above exercise, this result cannot be extended to arbitrary extensions of **IPC**. To prove this, consider any quasi-variety K of Heyting algebras that is not relatively congruence distributive (for instance, the one in the above exercise). By Theorem 3.9, K is the equivalent algebraic semantics of an extension $\vdash$ of **IPC**. Furthermore, as K is not relatively congruence distributive, it lacks EDPRC by Theorem 6.7. Thus, by Theorem 5.7, we conclude that $\vdash$ has no DDT, as desired.

Before moving on, let us contemplate for a moment on the above observation, which exemplifies one of the strengths of Theorem 5.7. By finding an intelligible algebraic counterpart of the deduction-detachment theorem, we are now in the position of *disproving*–by means of transparent algebraic arguments–the existence of *any* deduction-detachment theorem. In the what follows we will see other applications of this kind, namely a proof of the fact that $\mathbf{K}_g$ lacks any DDT.

Recall that a class of algebras K is said to be *elementary* when it is axiomatizable by means of (first order) sentences. Furthermore, K is said to be a *universal*, when it is axiomatizable by means of universal sentences, i.e., sentences of the form $\forall \vec{x}\varphi$, where φ is quantifier-free. It is well known that a class of algebras is elementary precisely when it is closed under $\mathbb{I}, \mathbb{S}$, and $\mathbb{P}_{\mathrm{U}}$.

Theorem 6.13. *If K is a quasi-variety with EDPRC, then $\mathsf{K}_{\mathrm{RSI}}$ and K_{RS} are two elementary classes. Moreover, when K_{RS} is extended by adding all trivial algebras, it becomes a universal class.*

Proof. Recall that, given an algebra $\boldsymbol{A} \in \mathsf{K}$, we denote by Id_A the identity relation on A. Recall also that $\boldsymbol{A}$ belongs to $\mathsf{K}_{\mathrm{RSI}}$ precisely when Id_A is completely meet-irreducible in $\mathrm{Con}_{\mathsf{K}}\boldsymbol{A}$, i.e., when there is a congruence $\theta \in \mathrm{Con}_{\mathsf{K}}\boldsymbol{A} \smallsetminus \{\mathrm{Id}_A\}$ such that every $\phi \in \mathrm{Con}_{\mathsf{K}}\boldsymbol{A} \smallsetminus \{\mathrm{Id}_A\}$ is greater or equal then θ. In this case, θ must be generated by any pair $\langle a, b\rangle \in \theta$ such that $a \neq b$. Consequently, θ is a principal K-congruence of $\boldsymbol{A}$. This means that $\boldsymbol{A}$ belongs to RSI if and only if there are two distinct elements $a, b \in A$ such that for every pair of distinct $c, d \in A$,

$$\langle a, b\rangle \in \mathrm{Cg}^{\boldsymbol{A}}_{\mathsf{K}}(c, d).$$

Using the set Φ witnessing EDPRC, the above display can be rendered as $\boldsymbol{A} \vDash \Phi(c, d, a, b)$. Consequently, $\mathsf{K}_{\mathrm{RSI}}$ is axiomatized by the universal closure of the quasi-equations axiomatizing K plus the sentence

$$\exists x, y(x \not\approx y \,\&\, \forall z, v(z \not\approx v \to \Phi(z, v, x, y))).$$

We conclude that $\mathsf{K}_{\mathrm{RSI}}$ is an elementary class.

In order to prove that K_{RS} is an elementary class, observe that an algebra $\boldsymbol{A} \in \mathsf{K}$ belongs to K_{RS} precisely when $\mathrm{Con}_{\mathsf{K}}\boldsymbol{A}$ has two elements, namely Id_A and $A \times A$. This means that there are two distinct elements $a, c \in A$ (so that $\mathrm{Id}_A \neq A \times A$) and that every pair $b, c, \in A$ of distinct elements generates $A \times A$. The latter demand can be phrased as follows: $\boldsymbol{A} \vDash \Phi(b, c, d, e)$, for all distinct $b, c, d, e \in A$ such that $b \neq c$. Consequently, K_{SI} is axiomatizable by means of the universal closure of the quasi-equations axiomatizing K plus the sentence

$$\exists x, y(x \not\approx y) \,\&\, \forall z, v, w, u(z \not\approx v \to \Phi(z, v, w, u)).$$

Finally, the class obtained by adding all trivial algebras to K_{RS} is axiomatized by the universal closure of the quasi-equations axiomatizing K and the universal sentence

$$\forall z, v, w, u(z \not\approx v \to \Phi(z, v, w, u)). \qquad \boxtimes$$

The above result provides a useful criterion to disprove that a quasi-variety has EDPRC.

Example 6.14. As an exemplification, we shall see that the variety MA of modal algebras lacks EDPC. In view of Theorem 6.13, it suffices to show that the class of simple modal algebras is not elementary. To this end, recall that a modal algebra $\boldsymbol{A}$ is simple when it is nontrivial and for every element $a \in A \smallsetminus \{1\}$ there is $n \in \omega$ such that

$$a \wedge \Box a \wedge \cdots \wedge \Box^n a = 0. \tag{18}$$

This is proved for instance in [31], but can also be inferred easily from the correspondence between open filters and congruences in modal algebras.

Now, for every $n \in \omega$, consider the Kripke frame $\langle W_n, R_n \rangle$ with universe $W_n = \{x_1, \dots, x_n\}$ and accessibility relation R_n defined as follows:

$$\langle y, z \rangle \in R_n \iff \text{ either } y = z \text{ or } (y = x_i \text{ and } z = x_{i+1}) \text{ or } (y = x_n \text{ and } z = x_1).$$

Furthermore, let $\boldsymbol{A}_n$ be the complex algebra of $\langle W_n, R_n \rangle$. Observe that $\boldsymbol{A}_n$ is simple, since (18) holds for every $a \in A$. Notice that, for every $n \in \omega$ one can find an element $a_n \in A_{n+1}$ such that

$$a_n \wedge \Box^{\boldsymbol{A}_{n+1}} a_n \wedge \cdots \wedge \underbrace{\Box^{\boldsymbol{A}_{n+1}} \dots \Box^{\boldsymbol{A}_{n+1}}}_{n\text{-times}} a_n \neq 0^{\boldsymbol{A}_{n+1}}. \tag{19}$$

Suppose, with a view to contradiction, that the class of simple modal algebras is elementary. Then consider a non-principal ultrafilter U on ω and take the ultrapower

$$\boldsymbol{B} := \prod_{n \in \omega} \boldsymbol{A}_n / U.$$

As first order sentences persist in ultraproducts, $\boldsymbol{B}$ is also simple. Then consider the sequence

$$b := \langle 0 \rangle ^\frown \langle a_n \colon n \in \omega \rangle \in \prod_{n \in \omega} \boldsymbol{A}_n.$$

Since $\boldsymbol{B}$ is simple, there is $m \in \omega$ such that

$$b/U \wedge \Box^{\boldsymbol{B}} b/U \wedge \cdots \wedge \underbrace{\Box^{\boldsymbol{B}} \dots \Box^{\boldsymbol{B}}}_{m\text{-times}} b/U = 0^{\boldsymbol{B}}.$$

This means that

$$\{0\} \cup \{n+1 \in \omega \colon a_n \wedge \Box^{\boldsymbol{A}_{n+1}} a_n \wedge \cdots \wedge \underbrace{\Box^{\boldsymbol{A}_{n+1}} \dots \Box^{\boldsymbol{A}_{n+1}}}_{n\text{-times}} a_n = 0^{\boldsymbol{A}_{n+1}}\} \in U.$$

In view of (18), the left hand side of the above display is finite. Thus, U contains a finite subset of ω. As U is non-principal, it also contains the Fréchet filter, thus contradicting the fact that U has the finite intersection property. We conclude that the class of simple modal algebras is not elementary. By Theorem 6.13, this implies that the variety of modal algebras lacks EDPC. ⊠

Corollary 6.15. *The logic $\mathbf{K}_g$ does not have any DDT.*

Proof. In the above example we showed that the equivalent algebraic semantics of $\mathbf{K}_g$ (namely, the variety of modal algebras) lacks EDPC. By Theorem 5.7, we conclude that $\mathbf{K}_g$ lacks any DDT. ⊠

The above result can be viewed as a confirmation of the intuition that the local deduction-detachment theorem of $\mathbf{K}_g$ and the deduction-detachment theorem of **IPC** are indeed instances of different metalogical principles. It is therefore meaningful to investigate further the local deduction-detachment theorem of $\mathbf{K}_g$. But, before doing this, let us explain how, in the presence of EDPC, the theory of Jankov formulas can be reconstructed for arbitrary signatures.

7. Jankov formulas revisited

Definition 7.1. A *splitting pair* in a lattice $\boldsymbol{A}$ is a pair $\langle a, c\rangle \in A \times A$ such that $a \nleqslant c$ and $A = {\uparrow}a \cup {\downarrow}c$.

The notion of a splitting pair can be traced back at least to [39]. We shall investigate splitting pairs in lattices of subvarieties. To this end, given a variety K, let $\Lambda(\mathsf{K})$ be the lattice of subvarieties of K. The next result describes the structure of splitting pairs in $\Lambda(\mathsf{K})$.

Lemma 7.2. *Let K be a variety and $\mathsf{V}, \mathsf{W} \in \Lambda(\mathsf{K})$. Then $\langle \mathsf{V}, \mathsf{W}\rangle$ is a splitting pair in $\Lambda(\mathsf{K})$ if and only if there exists an algebra $\boldsymbol{A}$ of K such that $\mathsf{V} = \mathbb{V}(\boldsymbol{A})$ and W is the largest subvariety of K omitting $\boldsymbol{A}$. In this case, $\boldsymbol{A}$ can be chosen subdirectly irreducible, and W is axiomatized relatively to K by a single equation.*

Proof. The "if" part is straightforward. To prove the "only if" part (and the second part of the statement), suppose that $\langle \mathsf{V}, \mathsf{W}\rangle$ is a splitting pair in $\Lambda(\mathsf{K})$. As $\mathsf{V} \nsubseteq \mathsf{W}$ and varieties are determined by their subdirectly irreducible members, there is a subdirectly irreducible algebra $\boldsymbol{A} \in \mathsf{V} \smallsetminus \mathsf{W}$. As $\boldsymbol{A} \notin \mathsf{W}$, we obtain $\mathbb{V}(\boldsymbol{A}) \nsubseteq \mathsf{W}$. Together with the fact that $\langle \mathsf{V}, \mathsf{W}\rangle$ is a splitting pair, this yields $\mathsf{V} \subseteq \mathbb{V}(\boldsymbol{A})$. Since $\boldsymbol{A} \in \mathsf{V}$, we conclude that $\mathsf{V} = \mathbb{V}(\boldsymbol{A})$.

We shall see that W is the largest subvariety of K omitting $\boldsymbol{A}$. That W omits $\boldsymbol{A}$ should be clear. To prove that W is the largest such subvariety of K, consider $\mathsf{X} \in \Lambda(\mathsf{K})$ such that

$\boldsymbol{A} \notin \mathsf{X}$. As $\boldsymbol{A} \in \mathsf{V}$, this implies $\mathsf{V} \nsubseteq \mathsf{X}$. Since $\langle \mathsf{V}, \mathsf{W} \rangle$ is a splitting pair, we conclude that $\mathsf{X} \subseteq \mathsf{W}$. Thus, W is the largest subvariety of K omitting $\boldsymbol{A}$.

It only remains to prove that W is axiomatized by a single equation. To this end, observe that $\boldsymbol{A} \notin \mathsf{W}$. Therefore, there is an equation $\varphi \approx \psi$ which holds in W and fails in $\boldsymbol{A}$. Let X be the subvariety of K axiomatized by this equation. Clearly, $\mathsf{W} \subseteq \mathsf{X}$. On the other hand, as X omits $\boldsymbol{A}$ and W is the largest such subvariety of K, we also have $\mathsf{X} \subseteq \mathsf{W}$. Accordingly, we conclude that $\mathsf{X} = \mathsf{W}$ and, therefore, that $\varphi \approx \psi$ axiomatizes W. ⊠

Let K be a variety. In view of the above result, a subdirectly irreducible algebra $\boldsymbol{A} \in \mathsf{K}$ is said to be *splitting in* K if there exists the largest subvariety W of K omitting $\boldsymbol{A}$. In this case, $\langle \mathbb{V}(\boldsymbol{A}), \mathsf{W} \rangle$ is a splitting pair in $\Lambda(\mathsf{K})$ and, in view of Lemma 7.2 all splitting pairs in $\Lambda(\mathsf{K})$ are of this form. Furthermore, the single equation axiomatizing W relatively to K is said to be the *splitting equation* of $\boldsymbol{A}$.

The situation should be familiar from the theory of *Jankov formulas* in Heyting algebras [29], where every finite subdirectly irreducible algebra $\boldsymbol{A}$ is splitting and the splitting equation of $\boldsymbol{A}$ is obtained by equating to 1 the Jankov formula of $\boldsymbol{A}$. Let us stress that the notion of a splitting algebra depends on the variety under consideration, i.e., even if an algebra is splitting in some variety K, the same algebra can fail to be splitting in proper subvarieties of K.

In this section we shall generalize this observation to many varieties other than Heyting algebras. Our discussion in based on [9, 34]. We shall also rely on the following well-know result from universal algebra [6, Thm. 5.10].

Theorem 7.3 (Jónsson)**.** *Let* K *be a class of algebras. If* $\mathbb{V}(\mathsf{K})$ *is congruence distributive, then* $\mathbb{V}(\mathsf{K})_{\mathrm{SI}} \subseteq \mathbb{HSP}_{\mathrm{U}}(\mathsf{K})$.

Lemma 7.4 (McKenzie)**.** *If* K *is a congruence distributive variety generated by its finite members, the splitting algebras of* K *are finite.*

Proof. Suppose, with a view to contradiction, that K has an infinite splitting member $\boldsymbol{A}$. As K is congruence distributive, we can apply Theorem 7.3, obtaining that $\mathbb{V}(\boldsymbol{B})_{\mathrm{SI}} \subseteq \mathbb{HSP}_{\mathrm{U}}(\boldsymbol{B})$, for every algebra $\boldsymbol{B} \in \mathsf{K}$. Consequently, $\mathbb{V}(\boldsymbol{B})_{\mathrm{SI}}$ is a set of finite algebras, for every finite $\boldsymbol{B} \in \mathsf{K}$. As $\boldsymbol{A}$ is infinite and subdirectly irreducible, this implies that $\boldsymbol{A} \notin \mathbb{V}(\boldsymbol{B})$ for every finite member $\boldsymbol{B}$ of K. Furthermore, as $\boldsymbol{A}$ is a splitting algebra, there exists the largest subvariety W of K omitting $\boldsymbol{A}$. In particular, W contains all finite members of K. Since K is generated by its finite members, this implies $\mathsf{K} = \mathsf{W}$. But this means that $\boldsymbol{A} \in \mathsf{K} = \mathsf{W}$, a contradiction. ⊠

Lemma 7.5 (Blok & Pigozzi)**.** *If* K *is a variety of finite type with EDPC, then all its finite subdirectly irreducible members are splitting. Moreover, for every finite subdirectly irreducible algebra* $\boldsymbol{A} \in \mathsf{K}$ *there is a finite set of equations* $\mathcal{J}(\boldsymbol{A})$ *which axiomatizes the largest subvariety of* K *omitting* $\boldsymbol{A}$ *and such that for every* $\boldsymbol{B} \in \mathsf{K}$,

$$\boldsymbol{B} \vDash \mathcal{J}(\boldsymbol{A}) \iff \boldsymbol{A} \notin \mathbb{HS}(\boldsymbol{B}).$$

Proof. Let $\Phi(x, y, z, v)$ be the set of equations witnessing EDPC for K. We define recursively a finite set of equations $\Phi_k(x_1, y_1, \dots, x_k, \dots, y_k, z, v)$ for every positive $k \in \omega$ by the following rule:

$$\Phi_1 := \Phi(x_1, y_1, z, v)$$

$$\Phi_{n+1} := \{\varepsilon(x_{n+1}, y_{n+1}, \varphi, \psi) \approx \delta(x_{n+1}, y_{n+1}, \varphi, \psi) \colon \varepsilon \approx \delta \in \Phi \text{ and } \varphi \approx \psi \in \Phi_n\}.$$

A routinary induction shows that for every $k \in \omega$, $\boldsymbol{B} \in \mathsf{K}$, and $a, b, c_1, \ldots, c_k, d_1, \ldots, d_k \in B$,

$$\langle a, b\rangle \in \mathrm{Cg}^{\boldsymbol{B}}(\{\langle c_1, d_1\rangle, \ldots, \langle c_k, d_k\rangle\}) \iff \boldsymbol{B} \vDash \Phi_k(c_1, d_1, \ldots, c_k, d_k, a, b). \tag{20}$$

Now, let $\boldsymbol{A} \in \mathsf{K}$ be a finite subdirectly irreducible member of K. As K is of finite type, we can enumerate its language as $f_1, \ldots, f_n$. Moreover, we also take an enumeration $\{a_1, \ldots, a_n\}$ of A. Then, consider the following set of equations in variables $x_1, \ldots, x_x$:

$$\Sigma := \{f_i(x_{k_1}, \ldots, x_{k_t}) \approx x_j \colon i \leqslant n \text{ and } k_1, \ldots, k_t, j \leqslant m \text{ and } f_i^{\boldsymbol{A}}(a_{k_1}, \ldots, a_{k_t}) = a_j)\}.$$

Since Σ is finite, we can take an enumeration

$$\Sigma = \{\varphi_1 \approx \psi_1, \ldots, \varphi_k \approx \psi_k\}.$$

Moreover, since $\boldsymbol{A}$ is subdirectly irreducible, there are $0 < i < j \leqslant n$ such that every congruence of $\boldsymbol{A}$ other than the identity extends $\mathrm{Cg}^{\boldsymbol{A}}(a_i, a_j)$. We define

$$\mathcal{J}(\boldsymbol{A}) := \Phi_k(\varphi_1, \psi_1, \ldots, \varphi_k, \psi_k, x_i, x_j).$$

Consider an arbitrary algebra $\boldsymbol{B} \in \mathsf{K}$. Bearing in mind that $\mathrm{Cg}^{\boldsymbol{A}}(a_i, a_j)$ is the smallest non-identity congruence of $\boldsymbol{A}$, we obtain that $\boldsymbol{A} \in \mathbb{HS}(\boldsymbol{B})$ if and only if there are $b_1, \ldots, b_n \in B$ generating a subalgebra $\boldsymbol{C}$ of $\boldsymbol{B}$ such that

$$\langle b_i, b_j\rangle \notin \mathrm{Cg}^{\boldsymbol{C}}(\Sigma^{\boldsymbol{B}}(b_1, \ldots, b_n)).$$

In view of (20) this amounts to the demand that for all $b_1, \ldots, b_n \in B$,

$$\boldsymbol{B} \nvDash \mathcal{J}(\boldsymbol{A})[\![b_1, \ldots, b_n]\!].$$

We conclude that

$$\boldsymbol{B} \vDash \mathcal{J}(\boldsymbol{A}) \iff \boldsymbol{A} \notin \mathbb{HS}(\boldsymbol{B}).$$

It only remains to prove that $\mathcal{J}(\boldsymbol{A})$ axiomatizes the largest subvariety of K omitting $\boldsymbol{A}$. To this end, let W be the subvariety of K axiomatized by $\mathcal{J}(\boldsymbol{A})$. By the above display, W omits $\boldsymbol{A}$. Then consider an arbitrary subvariety V of K omitting $\boldsymbol{A}$. Clearly, $\boldsymbol{A} \notin \mathbb{HS}(\boldsymbol{B})$, for all $\boldsymbol{B} \in \mathsf{V}$. By the above display, we conclude $\mathsf{V} \vDash \mathcal{J}(\boldsymbol{A})$, whence $\mathsf{V} \subseteq \mathsf{W}$. As a consequence, W is the largest subvariety of K omitting $\boldsymbol{A}$. ⊠

We are now ready to derive the main result of the section:

Theorem 7.6. *Let K be a variety of finite type with EDPC generated by its finite members. The splitting algebras of K are exactly its finite subdirectly irreducible members. Moreover, for every finite subdirectly irreducible algebra $\boldsymbol{A} \in \mathsf{K}$ there is a finite set of equations $\mathcal{J}(\boldsymbol{A})$ which axiomatizes the largest subvariety of K omitting $\boldsymbol{A}$ and such that for every $\boldsymbol{B} \in \mathsf{K}$,*

$$\boldsymbol{B} \vDash \mathcal{J}(\boldsymbol{A}) \iff \boldsymbol{A} \notin \mathbb{HS}(\boldsymbol{B}).$$

Proof. First notice that K is congruence distributive by Theorem 6.7. Bearing this in mind, the result follows immediately from Lemmas 7.4 and 7.5. ⊠

Example 7.7. As we mentioned, the variety HA of Heyting algebras has EDPC. Moreover, it is well known that HA is generated by its finite members (this is sometimes referred to as the *finite model property*). Since the type of HA is finite, HA falls in the scope of Theorem 7.6, which, in this case, specializes to the well-known construction of Jankov formulas of finite subdirectly irreducible Heyting algebras. ⊠

Exercise 7.8. Prove that if K is a variety of finite type with EDPC, then the following holds: if there is an infinite antichain in the $\mathbb{SH}$-order on finite subdirectly irreducible members of K, then K has a continuum of subvarieties. You probably know an instance of this application in the realm of Heyting algebras, where it is used to show that HA has a continuum of subvarieties. ☒

8. The local deduction-detachment theorem

We conclude our journey by consider the local deduction-detachment theorem typical of $\mathbf{K}_g$. Our discussion in based on [11, 19]. The essence of the local deduction-detachment theorem is captured by the following definition.

Definition 8.1. A logic $\vdash$ has a *local deduction-detachment theorem* (LDDT) if there is a family $\{L_i(x,y)\colon i \in I\}$ of finite sets of formulas such that for every $\Gamma \cup \{\varphi, \psi\} \subseteq Fm$,

$$\Gamma, \varphi \vdash \psi \iff \text{there is } i \in I \text{ such that } \Gamma \vdash L_i(\varphi, \psi).$$

Clearly, logics with a DDT have also a LDDT. Furthermore, the LDDT persists in axiomatic extensions. As expected, $\mathbf{K}_g$ has a local deduction-detachment theorem, witnessed by the family of sets of formulas

$$\{\{(x \wedge \Box x \wedge \cdots \wedge \Box^n x) \to y\}\colon n \in \omega\}.$$

The algebraic counterpart of the LDDT is the following property of congruences:

Definition 8.2. A quasi-variety K is said to have the *relative congruence extension property* (RCEP) if for every $\boldsymbol{B} \leqslant \boldsymbol{A} \in \mathsf{K}$ and $\theta \in \mathrm{Con}_{\mathsf{K}}\boldsymbol{B}$ there exists $\phi \in \mathrm{Con}_{\mathsf{K}}\boldsymbol{A}$ such that $\theta = \phi \cap (B \times B)$. When K is a variety, $\mathrm{Con}_{\mathsf{K}}\boldsymbol{A}$ and $\mathrm{Con}_{\mathsf{K}}\boldsymbol{B}$ can be replaced, respectively, by $\mathrm{Con}\boldsymbol{A}$ and $\mathrm{Con}\boldsymbol{B}$, and K is said to have the *congruence extension property* (CEP).

Theorem 8.3 (Czelakowski)**.** *An algebraizanle logic $\vdash$ has a LDDT if and only if its equivalent algebraic semantics K has the RCEP.*

Proof. Let $\boldsymbol{\tau}$ and Δ be finite sets that, together with K, witness the algebraizability of $\vdash$. In order to prove the "only if" part, suppose that $\vdash$ has a LDDT witnessed by a family $\{L_i^1(x,y)\colon i \in I_1\}$ of finite sets of formulas. For every $k \geqslant 1$, we shall define a family

$$\mathcal{L}_k = \{L_i^k(x_1, \dots, x_k, y)\colon i \in I_k\}$$

of finite sets of formulas. Fist, let $\mathcal{L}_1$ be the family witnessing the LDDT for $\vdash$. Then set

$$\mathcal{L}_{k+1} \coloneqq \{L_{i_1}^1(x_{k+1}, \alpha_1) \cup \cdots \cup L_{i_m}^1(x_{k+1}, \alpha_m)\colon i_1, \dots, i_m \in I_1 \text{ and } \{\alpha_1, \dots, \alpha_m\} \in \mathcal{L}_k\}.$$

Clearly, we can write each $\mathcal{L}_k$ as $\{L_i^k(x_1, \dots, x_k, y)\colon i \in I_k\}$ for a suitable set of indexes I_k. A simple inductive argument shows that for all $\Gamma \cup \{\varphi_1, \dots, \varphi_k, \psi\} \subseteq \Gamma$,

$$\Gamma, \varphi_1, \dots, \varphi_k \vdash \psi \iff \text{there is } i \in I_k \text{ such that } \Gamma \vdash L_i^k(\varphi_1, \dots, \varphi_k, \varphi).$$

Take an enumeration $\Delta = \{\delta_1, \dots, \delta_m\}$. For all $\langle i_1, \dots, i_m\rangle \in (I_m)^m$, we define a set of equations

$$\Phi_{i_1,\dots,i_m}(x,y,z,v) \coloneqq \bigcup_{1 \leqslant j \leqslant m} \boldsymbol{\tau}[L_{i_j}^m(\delta_1(x,y), \dots, \delta_m(x,y), \delta_j(z,v))].$$

We shall prove that for every $\boldsymbol{A} \in \mathsf{K}$ and $a, b, c, d \in A$,

$$\langle a, b\rangle \in \mathrm{Cg}_{\mathsf{K}}^{\boldsymbol{A}}(c,d) \iff \text{there is } \langle i_1, \dots, i_m\rangle \in (I_m)^m \text{ s.t. } \boldsymbol{A} \vDash \Phi_{i_1,\dots,i_m}(c,d,a,b). \tag{21}$$

In view of Lemma 2.16, it suffices to prove the above display for the finitely generated members of K. To this end, let $\boldsymbol{A} \in \mathsf{K}$ be n-generated and $a, b, c, d \in A$. Then let $\boldsymbol{T}_n$ be the absolutely free term algebra in variables $x_1, \dots, x_n$. Since $\boldsymbol{A}$ is n-generated, there is a surjective homomorphism $f\colon \boldsymbol{T}_n \to \boldsymbol{A}$. Notice that the kernel Σ of f can be viewed as a set of equations. Moreover, let φ_a, φ_b, φ_c, and φ_d be formulas send, respectively, to a, b, c, and d by f. We have

$$
\begin{aligned}
\langle a, b\rangle \in \mathrm{Cg}^{\boldsymbol{A}}_{\mathsf{K}}(c, d) &\iff \Sigma, \varphi_c \approx \varphi_d \vDash_{\mathsf{K}} \varphi_a \approx \varphi_b\\
&\iff \text{for all } 1 \leqslant j \leqslant m,\ \Delta[\Sigma], \delta_1(\varphi_c, \varphi_d), \dots, \delta_m(\varphi_c, \varphi_d) \vdash \delta_j(\varphi_a, \varphi_b)\\
&\iff \text{for all } j = 1, \dots, m \text{ there is } i_j \in I_m \text{ such that}\\
&\qquad \Delta[\Sigma] \vdash L^m_{i_j}(\delta_1(\varphi_c, \varphi_d), \dots, \delta_m(\varphi_c, \varphi_d), \delta_j(\varphi_a, \varphi_b))\\
&\iff \text{there is } \langle i_1, \dots, i_m\rangle \in (I_m)^m,\\
&\qquad \Delta[\Sigma] \vdash \bigcup_{1 \leqslant j \leqslant m} L^m_{i_j}(\delta_1(\varphi_c, \varphi_d), \dots, \delta_m(\varphi_c, \varphi_d), \delta_j(\varphi_a, \varphi_b))\\
&\iff \text{there is } \langle i_1, \dots, i_m\rangle \in (I_m)^m,\\
&\qquad \boldsymbol{\tau}[\Delta[\Sigma]] \vDash_{\mathsf{K}} \boldsymbol{\tau}[\bigcup_{1 \leqslant j \leqslant m} L^m_{i_j}(\delta_1(\varphi_c, \varphi_d), \dots, \delta_m(\varphi_c, \varphi_d), \delta_j(\varphi_a, \varphi_b))]\\
&\iff \Sigma \vDash_{\mathsf{K}} \Phi_{i_1, \dots, i_m}(\varphi_c, \varphi_d, \varphi_a, \varphi_b), \text{ for some } \langle i_1, \dots, i_m\rangle \in (I_m)^m\\
&\iff \boldsymbol{A} \vDash \Phi_{i_1, \dots, i_m}(c, d, a, b), \text{ for some } \langle i_1, \dots, i_m\rangle \in (I_m)^m.
\end{aligned}
$$

This establishes (21).

Finally, consider $\boldsymbol{B} \leqslant \boldsymbol{A} \in \mathsf{K}$ and $\theta \in \mathrm{Con}_{\mathsf{K}}\boldsymbol{B}$. Define $\phi \coloneqq \mathrm{Cg}^{\boldsymbol{A}}_{\mathsf{K}}(\theta)$. Clearly, $\theta \subseteq \phi \cap (B \times B)$. In order to prove the converse, we first show, by induction on n, that for every $\langle c_1, d_1\rangle, \dots, \langle c_n, d_n\rangle \in \theta$,

$$(B \times B) \cap \mathrm{Cg}^{\boldsymbol{A}}_{\mathsf{K}}(\{\langle c_1, d_1\rangle, \dots, \langle c_n, d_n\rangle\}) \subseteq \theta.$$

The case where $n = 1$ follows from (21). For the inductive step, suppose that $\langle a, b\rangle \in (B \times B) \cap \mathrm{Cg}^{\boldsymbol{A}}_{\mathsf{K}}(\{\langle c_1, d_1\rangle, \dots, \langle c_{k+1}, d_{k+1}\rangle\})$. Clearly,

$$\langle a/\eta, b/\eta\rangle \in \mathrm{Cg}^{\boldsymbol{A}/\eta}_{\mathsf{K}}(c_{k+1}/\eta, d_{k+1}/\eta)$$

where $\eta \coloneqq \mathrm{Cg}^{\boldsymbol{A}}_{\mathsf{K}}(\{\langle c_1, d_1\rangle, \dots, \langle c_k, d_k\rangle\})$. Together with (21), the above display implies that there exists $\langle i_1, \dots, i_m\rangle \in (I_m)^m$ such that

$$\Phi^{\boldsymbol{A}}_{i_1, \dots, i_m}(c_{k+1}, d_{k+1}, a, b) \subseteq \eta.$$

By inductive hypothesis, $\Phi^{\boldsymbol{B}}_{i_1, \dots, i_m}(c_{k+1}, d_{k+1}, a, b) \subseteq \theta$, whence

$$\boldsymbol{B}/\theta \vDash \Phi_{i_1, \dots, i_m}(c_{k+1}/\theta, d_{k+1}/\theta, a/\theta, b/\theta).$$

Together with (21) and $c_{k+1}/\theta = d_{k+1}/\theta$, this implies $\langle a, b\rangle \in \theta$. This concludes the inductive argument. Now, consider any $\langle a, b\rangle \in \phi \cap (B \times B)$. By Theorem 2.12, there are $\langle c_1, d_1\rangle, \dots, \langle c_n, d_n\rangle \in \theta$ such that $\langle a, b\rangle \in (B \times B) \cap \mathrm{Cg}^{\boldsymbol{A}}_{\mathsf{K}}(\{\langle c_1, d_1\rangle, \dots, \langle c_{k+1}, d_{k+1}\rangle\})$. As we saw, this implies $\langle a, b\rangle \in \theta$. Thus, we conclude that $\theta = (B \times B) \cap \phi$ and, therefore, that K has the RCEP.

Then we turn to prove the "if" part". Suppose that K has the RCEP. Let $\{L_i \colon i \in I\}$ be the family of all finite sets of formulas $L_i(x, y)$ such that $x, L_i(x, y) \vdash y$. We shall see that this family witnesses the LDDT for $\vdash$. To this end, consider $\Gamma \cup \{\varphi, \psi\} \subseteq Fm$. Clearly, if there is $i \in I$ such that $\Gamma \vdash L_i(\varphi, \psi)$, then $\Gamma, \varphi \vdash \psi$. To prove the converse, suppose that $\Gamma, \varphi \vdash \psi$. Define

$$\theta := \mathrm{Cg}_{\mathsf{K}}^{\boldsymbol{Fm}}(\boldsymbol{\tau}[\Gamma]) \text{ and } \boldsymbol{A} := \boldsymbol{Fm}/\theta.$$

Furthermore, let $\boldsymbol{B}$ be the subalgebra of $\boldsymbol{A}$ generated by φ/θ and ψ/θ. Since K has the RCEP, the K-congruence $\mathrm{Cg}_{\mathsf{K}}^{\boldsymbol{B}}(\boldsymbol{\tau}(\varphi/\theta))$ of $\boldsymbol{B}$ is the restriction of some congruence $\phi \in \mathrm{Con}_{\mathsf{K}}\boldsymbol{A}$. As $\Gamma, \varphi \vdash \psi$, we have $\boldsymbol{\tau}[\Gamma], \boldsymbol{\tau}(\varphi) \vDash_{\mathsf{K}} \boldsymbol{\tau}(\psi)$. Together with the definition of $\boldsymbol{A}$ and that fact that $\mathrm{Cg}_{\mathsf{K}}^{\boldsymbol{B}}(\boldsymbol{\tau}(\varphi/\theta)) \subseteq \phi$, this implies $\boldsymbol{\tau}(\psi/\theta) \subseteq \phi$. Since $\mathrm{Cg}_{\mathsf{K}}^{\boldsymbol{B}}(\boldsymbol{\tau}(\varphi/\theta)) = (B \times B) \cap \phi$, we conclude that

$$\boldsymbol{\tau}(\psi/\theta) \subseteq \mathrm{Cg}_{\mathsf{K}}^{\boldsymbol{B}}(\boldsymbol{\tau}(\varphi/\theta)).$$

Let $\boldsymbol{Fm}(x, y)$ be the subalgebra of $\boldsymbol{Fm}$ generated by x and y. As $\boldsymbol{B}$ is generated by φ/θ and ψ/θ, there is a surjective homomorphism $f \colon \boldsymbol{Fm}(x, y) \to \boldsymbol{B}$ such that $f(x) = \varphi/\theta$ and $f(y) = \psi/\theta$. Let $\mathrm{Ker}(f)$ be the kernel of f. Since $\boldsymbol{\tau}(\psi/\theta) \subseteq \mathrm{Cg}_{\mathsf{K}}^{\boldsymbol{B}}(\boldsymbol{\tau}(\varphi/\theta))$, we have

$$\mathrm{Ker}(f), \boldsymbol{\tau}(x) \vDash_{\mathsf{K}} \boldsymbol{\tau}(y).$$

As the relative equational consequence $\vDash_{\mathsf{K}}$ is finitary (Exercise 2.5), there is a finite subset $\Sigma \subseteq \mathrm{Ker}(f)$ such that $\Sigma, \boldsymbol{\tau}(x) \vDash_{\mathsf{K}} \boldsymbol{\tau}(y)$. In particular, this yields $\Delta[\Sigma], \Delta[\boldsymbol{\tau}(x)] \vdash \Delta[\boldsymbol{\tau}(y)]$ and, therefore, $\Delta[\Sigma], x \vdash y$. As the set $\Delta[\Sigma]$ is finite and in variables x and y, it has the form $L_i(x, y)$ for some $i \in I$. Therefore, to conclude the proof, it suffices to show that $\Gamma \vdash \Delta[\Sigma(\varphi, \psi)]$, i.e., that $\boldsymbol{\tau}[\Gamma] \vDash_{\mathsf{K}} \Sigma(\varphi, \psi)$. Equivalently, we need to prove $\Sigma(\varphi, \psi) \subseteq \mathrm{Cg}_{\mathsf{K}}^{\boldsymbol{Fm}}(\boldsymbol{\tau}[\Gamma]) = \theta$. To this end, consider $\langle \alpha, \beta \rangle \in \Sigma$. Since $\Sigma \subseteq \mathrm{Ker}(f)$, we $f(\alpha) = f(\beta)$. Moreover,

$$\begin{aligned}
\alpha(\varphi, \psi)/\theta &= \alpha^{\boldsymbol{B}}(\varphi/\theta, \psi/\theta) = \alpha^{\boldsymbol{B}}(f(x), f(y)) = f(\alpha) \\
\beta(\varphi, \psi)/\theta &= \beta^{\boldsymbol{B}}(\varphi/\theta, \psi/\theta) = \beta^{\boldsymbol{B}}(f(x), f(y)) = f(\beta).
\end{aligned}$$

Thus, we obtain $\langle \alpha(\varphi, \psi), \beta(\varphi, \psi) \rangle \in \theta$. This concludes the proof that $\Sigma(\varphi, \psi) \subseteq \theta$. ⊠

Remark 8.4. A more verbose proof of Theorem 8.3 could have been split in two halves, the first showing that an algebraizable logic has a LDDT if and only is the equational consequence $\vDash_{\mathsf{K}}$ relative to its equivalent algebraic semantics K has one (cf. Lemma 5.4), and the second showing that $\vDash_{\mathsf{K}}$ has a LDDT precisely when K has the RCEP.

At this stage, some considerations are in order. First, EDPRC requires that principal relative congruences can be defined by means of a finite set of equations. It is therefore sensible to wonder how restrictive is the focus on equations (as opposed to arbitrary first order formulas). In order to answer this question, recall that a quasi-variety K is said to have *definable principal relative congruences* (DPRC) when there is a first order formula $\varphi(x, y, z, v)$ such that for every $\boldsymbol{A} \in \mathsf{K}$ and $a, b, c, d \in A$,

$$\langle a, b \rangle \in \mathrm{Cg}_{\mathsf{K}}^{\boldsymbol{A}}(c, d) \iff \boldsymbol{A} \vDash \varphi(c, d, a, b).$$

Theorem 8.5. *A quasi-variety has EDPRC if and only if it is relatively congruence distributive and it has DPRC and the RCEP.*

The "only if" part of the above result follows from Theorem 6.7 and the (obvious) observation that EDPRC implies the RCEP. The converse was supplied by Fried, Grätzer, and Quackenbush in [25]. As most logically motivated quasi-varieties have the RCEP and are relatively congruence distributive, in the algebra of logic it is very often that case that EDPRC is equivalent to DPRC. For instance, as all varieties of modal algebras have the CEP and are congruence distributive, EDPC and DPC coincide for them.

Having individuated the algebraic counterparts of the local and global deduction-detachment theorems, this is a good place where to finish end our discussion. Notice that the precise formulation of the equivalences between DDT and EDPRC (resp. LDDT and RCEP) was made possible by the notion algebraizability, which provided the right notion of equivalence between logic and algebra. Similarly, it is worth stressing that these results answer in a satisfactory way the question of what should a (local) deduction-detachment theorem be in general (and in a way that is independent on the language in which a logic is formulated). Of course, there would be much more to say (e.g., that relative congruence distributivity is the algebraic counterpart of the proof by cases property, and that EDPRC is also connected with generalizations of reductio ad absurdum), but, for good or bad, every course should finish somewhere.

References

[1] J. Adámek. How many variables does a quasivariety need? *Algebra Universalis*, (27):44–48, 1990.
[2] P. Aglianò and A. Ursini. Ideals and other generalizations of congruence classes. *J. Austral. Math. Soc. Ser. A*, 53(1):103–115, 1992.
[3] P. Agliano and A. Ursini. On subtractive varieties II: general properties. *Algebra Universalis*, 36:222–259, 1996.
[4] P. Agliano and A. Ursini. On subtractive varieties III: from ideals to congruences. *Algebra Universalis*, 37(3):296–333, 1997.
[5] P. Agliano and A. Ursini. On subtractive varieties IV: definability of principal ideals. *Algebra Universalis*, 38(3):355–389, 1997.
[6] C. Bergman. *Universal Algebra: Fundamentals and Selected Topics*. Chapman & Hall Pure and Applied Mathematics. Chapman and Hall/CRC, 2011.
[7] W. J. Blok and B. Jónsson. Equivalence of consequence operations. *Studia Logica*, 83(1–3):91–110, 2006.
[8] W. J. Blok, P. Köhler, and D. Pigozzi. On the structure of varieties with equationally definable principal congruences II. *Algebra Universalis*, 18:334–379, 1984.
[9] W. J. Blok and D. Pigozzi. On the structure of varieties with equationally definable principal congruences I. *Algebra Universalis*, 15:195–227, 1982.
[10] W. J. Blok and D. Pigozzi. *Algebraizable logics*, volume 396 of *Mem. Amer. Math. Soc.* A.M.S., Providence, January 1989.
[11] W. J. Blok and D. Pigozzi. Local deduction theorems in algebraic logic. In H. Andréka, J. D. Monk, and I. Németi, editors, *Algebraic Logic*, volume 54 of *Colloq. Math. Soc. János Bolyai*, pages 75–109. North-Holland, Amsterdam, 1991.
[12] W. J. Blok and D. Pigozzi. On the structure of varieties with equationally definable principal congruences III. *Algebra Universalis*, 32:545–608, 1994.
[13] W. J. Blok and D. Pigozzi. On the structure of varieties with equationally definable principal congruences IV. *Algebra Universalis*, 31:1–35, 1994.
[14] W. J. Blok and D. Pigozzi. Abstract algebraic logic and the deduction theorem. *Available in internet* *http://orion.math.iastate.edu/dpigozzi/*, 1997. Manuscript.
[15] W. J. Blok and J. G. Raftery. On congruence modularity in varieties of logic. *Algebra Universalis*, 45(1):15–21, 2001.
[16] W. J. Blok and J. G. Raftery. Assertionally equivalent quasivarieties. *International Journal of Algebra and Computation*, 18(4):589–681, 2008.
[17] W. J. Blok and J. Rebagliato. Algebraic semantics for deductive systems. *Studia Logica, Special Issue on Abstract Algebraic Logic, Part II*, 74(5):153–180, 2003.
[18] S. Burris and H. P. Sankappanavar. *A course in Universal Algebra*. Available in internet https://www.math.uwaterloo.ca/~snburris/htdocs/ualg.html, the millennium edition, 2012.
[19] J. Czelakowski. Local deductions theorems. *Studia Logica*, 45:377–391, 1986.

[20] J. Czelakowski. *Protoalgebraic logics*, volume 10 of *Trends in Logic—Studia Logica Library*. Kluwer Academic Publishers, Dordrecht, 2001.

[21] L. Esakia. *Heyting Algebras. Duality Theory*. Springer, English translation of the original 1985 book. 2019.

[22] J. M. Font. *Abstract Algebraic Logic - An Introductory Textbook*, volume 60 of *Studies in Logic - Mathematical Logic and Foundations*. College Publications, London, 2016.

[23] J. M. Font and R. Jansana. *A general algebraic semantics for sentential logics*, volume 7 of *Lecture Notes in Logic*. A.S.L., second edition 2017 edition, 2009. First edition 1996. Electronic version freely available through Project Euclid at projecteuclid.org/euclid.lnl/1235416965.

[24] J. M. Font, R. Jansana, and D. Pigozzi. A survey on abstract algebraic logic. *Studia Logica, Special Issue on Abstract Algebraic Logic, Part II*, 74(1–2):13–97, 2003. With an "Update" in 91 (2009), 125–130.

[25] E. Fried, G. Grätzer, and R. Quackenbush. Uniform congruence schemes. *Algebra Universalis*, 10:176–189, 1980.

[26] V. I. Glivenko. Sur quelques points de la logique de M. Brouwer. *Academie Royal de Belgique Bulletin*, 15:183–188, 1929.

[27] V. A. Gorbunov. *Algebraic theory of quasivarieties*. Siberian School of Algebra and Logic. Consultants Bureau, New York, 1998. Translated from the Russian.

[28] H. P. Gumm and A. Ursini. Ideals in universal algebras. *Algebra Universalis*, 19(1):45–54, 1984.

[29] V. A. Jankov. The construction of a sequence of strongly independent superintuitionistic propositional calculi. *Soviet Mathematics Doklady*, 9:806–807, 1968.

[30] P. Köhler and D. Pigozzi. Varieties with equationally definable principal congruences. *Algebra Universalis*, 11:213–219, 1980.

[31] M. Kracht. *Tools and techniques in modal logic*, volume 142 of *Studies in Logic and the Foundations of Mathematics*. North-Holland Publishing Co., Amsterdam, 1999.

[32] M. Kracht. *Modal consequence relations*, volume 3, chapter 8 of the Handbook of Modal Logic. Elsevier Science Inc., New York, NY, USA, 2006.

[33] A. I. Mal'cev. *The metamathematics of algebraic systems, collected papers: 1936-1967*. Amsterdam, North-Holland Pub. Co., 1971.

[34] R. McKenzie. Equational bases and nonmodular lattice varieties. *Transactions of the Americal Mathematical Society*, 174:1–43, 1972.

[35] T. Moraschini. On equational completeness theorems. Manuscript, available online, 2020.

[36] T. Moraschini and J. G. Raftery. On prevarieties of logic. *Algebra Universalis*, (80), 2019.

[37] A. Ursini. On subtractive varieties I. *Algebra Universalis*, 31:204–222, 1994.

[38] A. Ursini. On subtractive varieties. V. Congruence modularity and the commutators. *Algebra Universalis*, 43(1):51–78, 2000.

[39] P. Whitman. Splittings of a lattice. *American Journal of Mathematics*, 65:179–196, 1943.

Department of Philosophy, Faculty of Philosophy, University of Barcelona, Carrer de Montalegre 6, 08001, Barcelona, Spain

Email address: tommaso.moraschini@ub.edu