TREES AND SPECTRA OF HEYTING ALGEBRAS

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Abstract. A poset is *Esakia representable* when it is isomorphic to the prime spectrum of a Heyting algebra. Notably, every Esakia representable poset is also the spectrum of a commutative ring with unit. The problem of describing the Esakia representable posets was raised in 1985 and remains open to this day. We recall that a *forest* is a disjoint union of trees and that a *root system* is the order dual of a forest. It is shown that a root system is Esakia representable if and only if it satisfies a simple order theoretic condition, known as "having enough gaps", and each of its nonempty chains has an infimum. This strengthens Lewis's characterization of the root systems which are spectra of commutative rings with unit. While a similar characterization of arbitrary Esakia representable forests seems currently out of reach, we show that a *well-ordered* forest is Esakia representable if and only if it has enough gaps and each of its nonempty chains has a supremum.

1. Introduction

The *prime spectrum* of a commutative ring with unit is the poset of its prime ideals. The *representation problem*, raised by Kaplansky in [15, pp. 5–7], asks for a characterization of the posets isomorphic to the prime spectra of commutative rings with unit. A similar problem was raised by Grätzer in [10, Problem 34, p. 156] in the context of order theory. We recall that the *prime spectrum* of a bounded distributive lattice is the poset of its prime filters. Grätzer's problem asks for a characterization of the posets isomorphic to the prime spectra of bounded distributive lattices.

Notably, the two problems coincide because commutative rings with unit and bounded distributive lattices have the same prime spectra (see, e.g., [20, Thm. 1.1]). More precisely, Hochster showed that the prime spectra of commutative rings with unit endowed with the Zariski topology are precisely the spectral spaces [12] (see also [6]) and Stone did the same for the prime spectra of bounded distributive lattices [24]. Because of this, we say that a poset is *representable* when it is isomorphic to the prime spectrum of a commutative ring with unit (equiv. of a bounded distributive lattice). In this parlance, the representation problem asks for a characterization of the representable posets.

Some conditions equivalent to the representability of a poset are known. For instance, Joyal [14] and Speed [22, Thm. p. 85] showed that a poset is representable if and only if it is profinite. Moreover, in view of Priestley duality [18, 19], a poset is representable precisely when it can be endowed with a topology that turns it into a Priestley space. However, these characterizations provide little information on the inner structure of representable posets, which is why the representation problem remains elusive to this day.

One of the main positive results on the inner structure of representable posets is due to Lewis [16, Thm. 3.1]. We recall that a poset is a *tree* when it is rooted and its principal downsets are chains and that it is a *root system* when it is a disjoint union of order duals of trees. Lewis showed that a root system X is representable if and only if each of its nonempty chains has an infimum

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and X has *enough gaps*, where the latter means that if $x < y$, there exist $z, v \in [x, y]$ such that z is an immediate predecessor of v. Since the class of representable posets is closed under the formation of order duals, we obtain that a *forest* (i.e., a disjoint union of trees) is representable if and only if it has enough gaps and each of its nonempty chains has a supremum.

In this paper, we focus on the representation problem for a prominent class of bounded distributive lattices, namely, *Heyting algebras*. These are the bounded distributive lattices in which the meet operation has an adjoint, sometimes called implication. Heyting algebras arise in different areas of mathematics, including:

(i) topology: the lattice of open sets of any topological space is a Heyting algebra;
(ii) domain theory: each continuous distributive lattice is a Heyting algebra;
(iii) topos theory: the subobject classifier of any topos is a Heyting algebra;
(iv) algebra: any distributive algebraic lattice is a Heyting algebra;
(v) constructivism and logic: the algebraic models of intuitionistic logic are Heyting algebras;
(vi) order theory: Heyting algebras are the most common generalization of Boolean algebras.

In 1985, Esakia raised the problem of describing the posets isomorphic to the prime spectra of Heyting algebras [9, Appendix A.5]. Accordingly, we say that a poset is *Esakia representable* when it is isomorphic to the prime spectrum of a Heyting algebra. While every Esakia representable poset is representable in the traditional sense (because every Heyting algebra is a bounded distributive lattice), the converse does not hold in general: for instance, the poset depicted in Figure 1 is representable, but not Esakia representable (see [3, Example 5.6]).

After four decades, the problem of describing the Esakia representable posets remains open. In addition, this problem cannot be reduced to the one of describing the representable posets because no concrete way of isolating the Esakia representable posets from the class of all the representable posets is known. Nonetheless, some progress has been made and, recently, a characterization of the Esakia representable root systems whose maximal chains are either finite or of order type dual to $\omega + 1$ was obtained (see [2, Cor. 6.20] and its proof).

In this paper, we extend this result by providing a description of all the Esakia representable root systems. More precisely, we show that a root system is Esakia representable if and only if has enough gaps and each of its nonempty chains has an infimum (Theorem 3.5). As a corollary, we obtain Lewis' classical description of the representable root systems. We recall that the Heyting algebras whose prime spectrum is a root system have been called *Gödel algebras* [11] (see also [13, Thm. 2.4]). Therefore, our result takes the form of a characterization of the prime spectra of Gödel algebras.

Contrarily to the case of arbitrary representable posets, the class of Esakia representable posets is not closed under order duals. In particular, the tree depicted in Figure 1 is not Esakia representable, although its order dual is Esakia representable because it has enough gaps and its nonempty chains have infima. Notice that the tree in Figure 1 contains an infinite descending chain. We will show that Lewis' description of the representable forests can be extended to Esakia representable forests by prohibiting the presence of such chains. More precisely, a forest is said to be *well-ordered* when it has no infinite descending chain. We show that a well-ordered forest is Esakia representable if and only if each of its nonempty chains has a supremum (Theorem 4.3).

At the heart of our proof stands a novel compactness argument which combines intuitions from combinatorics, algebra, and topology and highlights the higher complexity of Esakia representable forests, as opposed to arbitrary representable forests. It remains an open problem to give a full characterization of arbitrary (i.e., not necessarily well-ordered) Esakia representable forests.

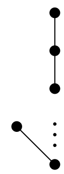

FIGURE 1. A representable tree that is not Esakia representable.

2. Representable posets

Among bounded distributive lattices, a special role is played by Heyting algebras [1, 9, 21]. We recall that a bounded distributive lattice A is said to be a *Heyting algebra* when it can be expanded with a binary operation $\to$ such that for every $a, b, c \in A$,

$$a \wedge b \leqslant c \iff a \leqslant b \to c.$$

In this case, this expansion is unique and $b \to c = \max\{a \in A : a \wedge b \leqslant c\}$.

In view of *Priestley* and *Esakia dualities* [8, 9, 18, 19], the problem of describing the spectra of bounded distributive lattices and Heyting algebras can be phrased in purely topological terms, as we proceed to illustrate. Given a poset $\langle X; \leqslant \rangle$ and $Y \subseteq X$, let

$${\uparrow}Y \coloneqq \{x \in X : \exists y \in Y \text{ s.t. } y \leqslant x\} \text{ and } {\downarrow}Y \coloneqq \{x \in X : \exists y \in Y \text{ s.t. } x \leqslant y\}.$$

The set Y is said to be an *upset* (resp. *downset*) if $Y = {\uparrow}Y$ (resp. $Y = {\downarrow}Y$). When $Y = \{x\}$, we will write ${\uparrow}x$ and ${\downarrow}x$ instead of ${\uparrow}\{x\}$ and ${\downarrow}\{x\}$. We denote the set of clopen upsets of an ordered topological space X by $\mathsf{ClUp}(X)$.

Definition 2.1. An ordered topological space $X = \langle X; \leqslant, \tau \rangle$ is a *Priestley space* when it is compact and satisfies the *Priestley separation axiom*: for every $x, y \in X$,

$$\text{if } x \nleqslant y, \text{ there exists } U \in \mathsf{ClUp}(X) \text{ such that } x \in U \text{ and } y \notin U.$$

If, in addition, ${\downarrow}U$ is open for every open set $U \subseteq X$, then X is said to be an *Esakia space*.

Let A be a bounded distributive lattice. A set $F \subseteq A$ is a *prime filter* of A when it is a nonempty proper upset such that for every $a, b \in A$,

$$(a, b \in F \Longrightarrow a \wedge b \in F) \text{ and } (a \vee b \in F \Longrightarrow a \in F \text{ or } b \in F).$$

The *prime spectrum* of A is the poset $\langle \mathsf{Pr}(A); \subseteq \rangle$, where $\mathsf{Pr}(A)$ is the set of prime filters of A.

Now, for each $a \in A$ let

$$\gamma_A(a) \coloneqq \{F \in \mathsf{Pr}(A) : a \in F\}.$$

Then the triple $A_+ \coloneqq \langle \mathsf{Pr}(A); \subseteq, \tau \rangle$, where τ is the topology on $\mathsf{Pr}(A)$ generated by the subbase

$$\{\gamma_A(a) : a \in A\} \cup \{\gamma_A(a)^c : a \in A\},$$

is a Priestley space. If, moreover, A is a Heyting algebra, then A_+ is an Esakia space. On the other hand, given a Priestley space X, the structure $X^+ \coloneqq \langle \mathsf{ClUp}(X); \cap, \cup, \emptyset, X \rangle$ is a bounded distributive lattice. If, in addition, X is an Esakia space, then X^+ is a Heyting algebra in which the operation $\to$ is defined as

$$U \to V \coloneqq \{x \in X : U \cap {\uparrow}x \subseteq V\}.$$

It is a consequence of Priestley and Esakia dualities that these transformations are one inverse to the other, in the sense that

$$A \cong (A_+)^+ \text{ and } X \cong \left(X^+\right)_+. \tag{1}$$

Recall that a poset is *representable* (resp. *Esakia representable*) when it is isomorphic to the prime spectrum of a bounded distributive lattice (resp. Heyting algebra). The next observation is a consequence of the isomorphisms in condition (1).

Theorem 2.2. *The following conditions hold:*

(i) *A poset is representable iff it can be endowed with a topology that turns it into a Priestley space;*
(ii) *A poset is Esakia representable iff it can be endowed with a topology that turns it into an Esakia space.*

While the structure of (Esakia) representable posets remains largely unknown, they need to satisfy a number of nontrivial properties. Given a poset $\langle X; \leqslant \rangle$ and $x, y \in X$, we say that x is an *immediate predecessor* of y when $x < y$ and there exists no $z \in X$ such that $x < z < y$. We write $x \prec y$ to indicate that this is the case.

Definition 2.3. A poset $X = \langle X; \leqslant \rangle$ is said to

(i) have *enough gaps* when for every $x, y \in X$ such that $x < y$, there exist $x' \geqslant x$ and $y' \leqslant y$ such that $x' \prec y'$;
(ii) be *Dedekind complete* when every nonempty chain in X has a supremum and an infimum.

A subset U of a poset X is *order open* when it belongs to the least family $\mathcal{O}$ of subsets of X such that:

(i) $\{x\}^c \in \mathcal{O}$ for every $x \in X$;
(ii) if $U \in \mathcal{O}$, then $(\uparrow(U^c))^c, (\downarrow(U^c))^c \in \mathcal{O}$;
(iii) $\mathcal{O}$ is closed under finite intersections and arbitrary unions.

Definition 2.4. A poset X is said to be *order compact* when for every family $\{U_i : i \in I\}$ of order open sets,

$$\text{if } \bigcup_{i \in I} U_i = X, \text{ there exists a finite } J \subseteq I \text{ such that } \bigcup_{j \in J} U_j = X.$$

Proposition 2.5. *Representable posets have enough gaps and are both Dedekind complete and order compact.*

Proof. For the fact that representable posets have enough gaps and are Dedekind complete, see [15, pp. 5–7]. On the other hand, every representable poset X is order compact because the order open sets are open in any topology that turns X into a Priestley space and Priestley spaces are compact (a slightly weaker statement can be found in [17, p. 822, condition (H)]. ⊠

The converse of Proposition 2.5 does not hold, however, as shown in [17, Example 2.1]. We will also rely on the following observation.

Proposition 2.6. *The following conditions hold:*

(i) *The class of representable posets is closed under disjoint unions and order duals;*
(ii) *The class of Esakia representable posets is closed under disjoint unions.*

Proof. Condition (ii) is Proposition 5.1(6) in the Appendix of [9]. Therefore, we turn to prove Condition (i). The fact that the class of representable posets is closed under order duals follows immediately from Theorem 2.2(i) and the fact that if $\langle X; \leqslant, \tau \rangle$ is a Priestley space, so is $\langle X; \geqslant, \tau \rangle$. On the other hand, closure under disjoint unions holds by [17, Thm. 4.1]. ⊠

Notice that the class of Esakia representable posets is not closed under order duals because the poset in Figure 1 is not Esakia representable [3, Example 5.6], although its order dual is [2, Cor. 6.20].

Given a pair of sets X and Y, we will write $X \subseteq_\omega Y$ to indicate that X is a finite subset of Y. We will rely on the following easy observation.

Lemma 2.7. *Let X be a poset and $Y, Z \subseteq_\omega X$. Then $(\uparrow Y \cap \downarrow Z)^c$ is an order open set of X.*

Proof. We will show that Y^c and Z^c are order open. By symmetry it suffices to prove that Y^c is order open. If $Y = \emptyset$, then $Y^c = X$ is the intersection of the empty family. As the family of order open sets is closed under finite (possibly empty) intersections, we are done. Then we consider the case where $Y \neq \emptyset$. Consider an enumeration $Y = \{y_1, \dots, y_n\}$. Since the sets $\{y_1\}^c, \dots, \{y_n\}^c$ are order open, so is their intersection $Y^c = \{y_1\}^c \cap \dots \cap \{y_n\}^c$. Hence, Y^c and Z^c are order open sets as desired. As a consequence, $(\uparrow Y)^c$ and $(\downarrow Z)^c$ are order open too and so is their union $(\uparrow Y)^c \cup (\downarrow Z)^c$. Since $(\uparrow Y)^c \cup (\downarrow Z)^c = (\uparrow Y \cap \downarrow Z)^c$, we are done. ⊠

Throughout the paper, we denote the set of all ordinals by Ord. Furthermore, given a poset $\langle X; \leqslant \rangle$ and a set $Y \subseteq X$, we denote the sets of maximal and minimal elements of the subposet $\langle Y; \leqslant \rangle$ by $\max Y$ and $\min Y$, respectively. Furthermore, when they exist, we let $\sup Y$ and $\inf Y$ be the supremum and infimum of Y, respectively.

3. Esakia representable root systems

Definition 3.1. A poset is said to be

(i) a *tree* when it is rooted and its principal downsets are chains;
(ii) a *forest* when it is isomorphic to the disjoint union of a family of trees;
(iii) a *root system* when it is the order dual of a forest.

One of the main positive results on the representation problem is the next theorem of Lewis.

Theorem 3.2 ([16, Thm. 3.1])**.** *A root system is representable iff it has enough gaps and each of its nonempty chains has an infimum.*

In this section, we strengthen this result by showing that it still holds in the context of Esakia representable posets. To this end, we recall that a Heyting algebra is a *Gödel algebra* [11] when it validates the equation

$$(x \to y) \vee (y \to x) \approx 1$$

or, equivalently, it is isomorphic to a subdirect product of chains [13, Thm. 1.2]. From a logical standpoint, the importance of Gödel algebras comes from the fact that they algebraize the *Gödel-Dummett logic* [7] in the sense of [4] (see, e.g., [5]). Notably, Gödel algebras can be characterized in term of the shape of their spectra.

Theorem 3.3 ([13, Thm. 2.4])**.** *A Heyting algebra is a Gödel algebra iff its prime spectrum is a root system.*

From Theorems 2.2(ii) and 3.3 we deduce the following.

Corollary 3.4. *A poset is isomorphic to the prime spectrum of a Gödel algebra iff it is an Esakia representable root system.*

The aim of this section is to establish the following description of the Esakia representable root systems (equiv. of the prime spectra of Gödel algebras).

Theorem 3.5. *A root system is Esakia representable iff it has enough gaps and each of its nonempty chains has an infimum.*

We remark that Theorem 3.2 is an immediate consequence of Theorem 3.5. More precisely, the implication from left to right in Theorem 3.2 holds by Proposition 2.5, while the other implication holds by Theorem 3.5 and the fact that every Esakia representable poset is representable. Furthermore, a weaker version of Theorem 3.5, stating that the result holds for the root systems whose maximal chains are either finite or of order type dual to $\omega + 1$, can be deduced from [2, Cor. 6.20].

Proof of Theorem 3.5. In view of Proposition 2.5, it suffices to prove the implication from right to left. To this end, it will be enough to show that the following condition holds for every poset X whose order dual is a tree:

$$\begin{gathered}\text{if } X \text{ has enough gaps and each of its nonempty chains has an infimum,}\\ \text{then } X \text{ is Esakia representable.}\end{gathered} \tag{2}$$

For suppose that condition (2) holds for the order duals of trees and consider a root system X with enough gaps and in which each nonempty chain has an infimum. Since X is a root system, it is the disjoint union of a family of posets $\{X_i : i \in I\}$ whose order duals are trees. Furthermore, each X_i has enough gaps as well as infima of nonempty chains. Therefore, each X_i is Esakia representable by condition (2). Hence, the disjoint union X is also Esakia representable by Proposition 2.6(ii) as desired.

Therefore, we turn to prove condition (2). Consider a poset $X = \langle X; \leqslant \rangle$ with enough gaps, in which every nonempty chain has an infimum, and whose order dual is a tree. Let then τ be the topology on X generated by the subbase

$$\mathcal{S} \coloneqq \{{\downarrow}x : \exists y \in X \text{ s.t. } x \prec y\} \cup \{({\downarrow}x)^c : \exists y \in X \text{ s.t. } x \prec y\}.$$

We will show that $X = \langle X; \leqslant, \tau \rangle$ is an Esakia space. The proof proceeds through a series of claims.

Claim 3.6. *The topological space X is compact.*

Proof of the Claim. Suppose the contrary, with a view to contradiction. By Alexander's subbase theorem there exists an open cover $\mathcal{C} \subseteq \mathcal{S}$ of X without any finite subcover. To this end, we will define recursively a sequence $\{x_\alpha : \alpha \in \mathsf{Ord}\}$ of elements of X such that for every ordinal α,

(i) $({\downarrow}x_\alpha)^c \in \mathcal{C}$;
(ii) $x_\beta < x_\gamma$ for every $\gamma < \beta \leqslant \alpha$.

Clearly, the validity of condition (ii) for every ordinal α implies that X is a proper class, which is the desired contradiction.

Consider an ordinal α and suppose that we already defined a sequence $\{x_\beta : \beta < \alpha\}$ of elements of X such that

(L1) $({\downarrow}x_\beta)^c \in \mathcal{C}$ for each $\beta < \alpha$;
(L2) $x_\beta < x_\gamma$ for every $\gamma < \beta < \alpha$.

We will prove that the set $Y \coloneqq \{x_\beta : \beta < \alpha\}$ has an infimum in X. If $Y = \emptyset$, then $\inf Y$ is the maximum of X, which exists because X is the order dual of a tree. The we consider the case where $Y \neq \emptyset$. In view of condition (L2), the set Y is a chain. As nonempty chains have infima by assumption, we conclude that $\inf Y$ exists.

Since $\mathcal{C}$ covers X, there exists $U \in \mathcal{C}$ such that $\inf Y \in U$. Furthermore, as $\mathcal{C} \subseteq \mathcal{S}$, there also exists $z \in X$ such that

(C1) z has an immediate successor;
(C2) either $U = {\downarrow}z$ or $U = ({\downarrow}z)^c$.

We will show that the case where $U = {\downarrow}z$ never happens. Suppose the contrary, with a view to contradiction. We have two cases: either $\inf Y \in Y$ or $\inf Y \notin Y$. First, suppose that $\inf Y \in Y$. Since $\inf Y \in U = {\downarrow}z$, we have $X = U \cup ({\downarrow}\inf Y)^c$. As $U \in \mathcal{C}$ and $\mathcal{C}$ lacks a finite subcover by assumption, this yields $({\downarrow}\inf Y)^c \notin \mathcal{C}$. On the other hand, from $\inf Y \in Y$ and condition (L1) it follows that $({\downarrow}\inf Y)^c \in \mathcal{C}$, a contradiction. Then we consider the case where $\inf Y \notin Y$. Together with the fact that ${\uparrow}\inf Y$ is a chain (because the order dual of X is a tree), this implies that $\inf Y$ does not have immediate successors. By condition (C1) we obtain $\inf Y \neq z$. Therefore, from $\inf Y \in U = {\downarrow}z$ it follows that $\inf Y < z$. As ${\uparrow}\inf Y$ is a chain and $Y = \{x_\beta : \beta < \alpha\}$, there exists

$\beta < \alpha$ such that $x_\beta < z$. By condition (L1) we have $(\downarrow x_\beta)^c \in \mathcal{C}$ which, together with $x_\beta < z$ and $\downarrow z = U \in \mathcal{C}$, implies that $\{(\downarrow x_\beta)^c, \downarrow z\}$ is a finite subcover of $\mathcal{C}$, a contradiction. Therefore, we conclude that $U \neq \downarrow z$ as desired. By condition (C2) this means that $U = (\downarrow z)^c$.

We will prove that $z < x_\beta$ for every $\beta < \alpha$. Suppose, on the contrary, that there exists $\beta < \alpha$ such that $z = x_\beta$ or $z \nleqslant x_\beta$. From $\inf Y \in U = (\downarrow z)^c$ it follows that $\inf Y \nleqslant z$. Since $x_\beta \in Y$, this yields $x_\beta \nleqslant z$. Together with the assumption that either $z = x_\beta$ or $z \nleqslant x_\beta$, this implies $z \nleqslant x_\beta$. Consequently, x_β and z are incomparable. As the order dual of X is a tree, this guarantees that $\downarrow x_\beta \cap \downarrow z = \emptyset$. Hence,

$$(\downarrow x_\beta)^c \cup (\downarrow z)^c = (\downarrow x_\beta \cap \downarrow z)^c = \emptyset^c = X.$$

Since $(\downarrow z)^c = U \in \mathcal{C}$ and $(\downarrow x_\beta)^c \in \mathcal{C}$ (the latter by condition (L1)), we obtain that $\{(\downarrow x_\beta)^c, (\downarrow z)^c\}$ is a finite subcover of $\mathcal{C}$, a contradiction. Hence, we conclude that $z < x_\beta$ for every $\beta < \alpha$. Thus, letting $x_\alpha \coloneqq z$, we obtain $x_\alpha < x_\beta$ for every $\beta < \alpha$. Since $(\downarrow x_\alpha)^c = (\downarrow z)^c = U \in \mathcal{C}$, the elements in the sequence $\{x_\beta : \beta \leqslant \alpha\}$ satisfy conditions (i) and (ii) as desired.

This completes the recursive definition of the sequence $\{x_\alpha : \alpha \in \mathsf{Ord}\}$ and produces the desired contradiction. ⊠

Claim 3.7. *The ordered topological space X satisfies Priestley separation axiom.*

Proof of the Claim. Consider $x, y \in X$ such that $x \nleqslant y$. If y has an immediate successor, we have $\downarrow y, (\downarrow y)^c \in \mathcal{S}$ by the definition of $\mathcal{S}$. In this case, $(\downarrow y)^c$ is a clopen upset containing x and missing y as desired. Then we consider the case where y does not have immediate successors. Notice that y is not the maximum of X, otherwise we would have $x \leqslant y$, which is false. Therefore, $\uparrow y \smallsetminus \{y\} \neq \emptyset$. Furthermore, $\uparrow y \smallsetminus \{y\}$ is a chain because the order dual of X is a tree. Now, since $\uparrow y \smallsetminus \{y\}$ is a nonempty chain, it has an infimum by assumption. As y lacks immediate successors, this infimum must be y itself. As a consequence, from $x \nleqslant y$ it follows that there exists $z > y$ such that $x \nleqslant z$. As X has enough gaps, there exists also an element $y^+ \in X$ with an immediate successor and such that $y \leqslant y^+ < z$. Consequently, $\downarrow y^+, (\downarrow y^+)^c \in \mathcal{S}$ by the definition of $\mathcal{S}$. Furthermore, $x \nleqslant y^+$ because $x \nleqslant z$ and $y^+ \leqslant z$. Thus, $(\downarrow y^+)^c$ is a clopen upset containing x and missing y. ⊠

From Claims 3.6 and 3.7 it follows that X is a Priestley space. In order to prove that it is also an Esakia space, we need to show that the downset of every open set is also open. To this end, let $\mathcal{B}$ be the base for the topology of X consisting of all the finite intersections of the elements of the subbase $\mathcal{S}$. As every open set U is the union of a family $\{U_i : i \in I\} \subseteq \mathcal{B}$ and

$$\downarrow U = \bigcup_{i \in I} \downarrow U_i,$$

it will be enough to prove that the downset of every element of $\mathcal{B}$ is open.

Consider $U_1, \dots, U_n \in \mathcal{S}$. We need to show that $\downarrow(U_1 \cap \dots \cap U_n)$ is open. We may assume that $U_1 \cap \dots \cap U_n \neq \emptyset$, otherwise $\downarrow(U_1 \cap \dots \cap U_n) = \emptyset$ and we are done. By the definition of $\mathcal{S}$ for every $m \leqslant n$ there exists $x_m \in X$ such that either $U_m = \downarrow x_m$ or $U_m = (\downarrow x_m)^c$. Let $Y \coloneqq \{x_m : U_m = \downarrow x_m\}$ and let Y^c be the complement of Y relative to $\{x_m : m \leqslant n\}$. Observe that

$$U_1 \cap \dots \cap U_n = \bigcap_{x_m \in Y} \downarrow x_m \cap \bigcap_{y_m \in Y^c} (\downarrow x_m)^c = \bigcap_{x_m \in Y} \downarrow x_m \cap (\downarrow(Y^c))^c. \tag{3}$$

We have two cases: either $Y = \emptyset$ or $Y \neq \emptyset$. First, suppose that $Y = \emptyset$. In view of the above equalities, we have $U_1 \cap \dots \cap U_n = (\downarrow(Y^c))^c$. As $U_1 \cap \dots \cap U_n \neq \emptyset$ by assumption, the upset $(\downarrow(Y^c))^c$ is nonempty and, therefore, contains the maximum $\top$ of X. Consequently, $\top \in U_1 \cap \dots \cap U_n$ and, therefore, $\downarrow(U_1 \cap \dots \cap U_n) = X$ is an open set.

Then we consider the case where $Y \neq \emptyset$. We will prove that Y is a chain. For if Y contained two incomparable elements x_k and x_m, we would have

$$U_1 \cap \cdots \cap U_n \subseteq U_k \cap U_m = {\downarrow} x_k \cap {\downarrow} x_m = \emptyset,$$

where the last equality follows from the assumption that x_k and x_m are incomparable and the order dual of X is a tree. But this contradicts the assumption that $U_1 \cap \cdots \cap U_n \neq \emptyset$.

Now, since Y is a finite nonempty chain, it has a minimum y. Consequently, condition (3) can be simplified as follows:

$$U_1 \cap \cdots \cap U_n = {\downarrow} y \cap ({\downarrow} Y^c)^c. \tag{4}$$

We will prove that $y \in U_1 \cap \cdots \cap U_n$. In view of Condition (4), it suffices to show that $y \in ({\downarrow} Y^c)^c$. Suppose the contrary, with a view to contradiction. Then there exists $x_m \in Y^c$ such that $y \leqslant x_m$. Consequently, ${\downarrow} y \cap ({\downarrow} x_m)^c = \emptyset$. Together with condition (4) and $x_m \in Y^c$, this implies $U_1 \cap \cdots \cap U_n = \emptyset$, a contradiction. Hence, we conclude that $y \in U_1 \cap \cdots \cap U_n$.

As a consequence, we obtain that ${\downarrow} y \subseteq {\downarrow}(U_1 \cap \cdots \cap U_n)$. Since the reverse inclusion holds by condition (4), we conclude that ${\downarrow}(U_1 \cap \cdots \cap U_n) = {\downarrow} y$. From $y \in Y$ and the definition of Y it follows that ${\downarrow} y = U_m$ for some $m \leqslant n$. Therefore, ${\downarrow}(U_1 \cap \cdots \cap U_n) = {\downarrow} y = U_m$. As $U_m \in \mathcal{S}$, we conclude that ${\downarrow}(U_1 \cap \cdots \cap U_n)$ is an open set. ⊠

4. Esakia representable well-ordered forests

Recall from Proposition 2.6(i) that the class of representable posets is closed under order duals. Therefore, Theorem 3.2 can also be viewed as a characterization of the representable forests. More precisely, we have following.

Theorem 4.1. *A forest is representable iff it has enough gaps and each of its nonempty chains has a supremum.*

It is therefore natural to wonder whether the above result holds for Esakia representable forests too. However, this is not the case because the tree depicted in Figure 1 is not Esakia representable (see [3, Example 5.6]), although it has enough gaps and each of its nonempty chains has an supremum. Notice that the tree in Figure 1 contains an *infinite descending chain*

$$\cdots < x_n < \cdots < x_2 < x_1 < x_0.$$

Our main results states that the above description of the representable forests can be extended to Esakia representable forests by prohibiting the presence of such chains.

Definition 4.2. A forest is *well-ordered* when it lacks infinite descending chains, that is, it does not contain any subposet isomorphic to the order dual of $\langle \mathbb{N}; \leqslant \rangle$.

Notice that every well-ordered forest has enough gaps. Therefore, our main result takes the following form.

Theorem 4.3. *A well-ordered forest is Esakia representable iff each of its nonempty chains has a supremum.*

Let $X = \langle X; \leqslant \rangle$ be a well-ordered forest. We recall for each $x \in X$ there exists a unique ordinal α such that $\langle {\downarrow} x \smallsetminus \{x\}; < \rangle$ is isomorphic to $\langle \alpha; \in \rangle$. The ordinal α is called the *order type* of x and will be denoted by $\mathsf{h}(x)$. Given $Y \subseteq X$ and an ordinal α, we let

$$\begin{aligned} \mathsf{h}(X) &\coloneqq \text{ the least ordinal } \alpha \text{ s.t. } \mathsf{h}(x) \leqslant \alpha \text{ for every } x \in X; \\ X_\alpha &\coloneqq \{x \in X : \mathsf{h}(x) = \alpha\}; \\ X_{*\alpha} &\coloneqq \{x \in X : \mathsf{h}(x) * \alpha\} \text{ for } * \in \{\leqslant, <, \geqslant, >\}; \\ {\uparrow}_\alpha Y &\coloneqq X_{\leqslant \alpha} \cap {\uparrow} Y. \end{aligned}$$

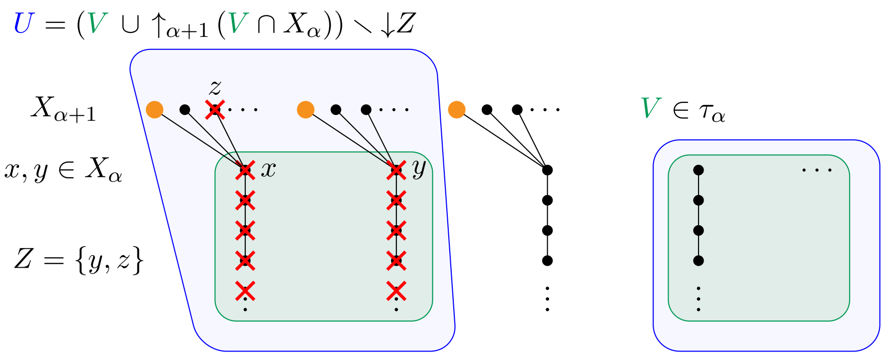

FIGURE 2. A member of $\mathcal{S}_{\alpha+1}$ of the form described in condition (iii). For each $v \in X_\alpha$ we coloured in orange the corresponding element v^+ of $X_{\alpha+1}$. Furthermore, we coloured in green the set $V \in \tau_\alpha$. Lastly, $Z = \{y, z\}$ is a finite subset of P_α. Then the set $U = (V \cup {\uparrow_{\alpha+1}} (V \cap X_\alpha)) \smallsetminus {\downarrow} Z$ is obtained by considering the blue shape and removing the elements crossed in red from it.

The implication from left to right in Theorem 4.3 holds by Proposition 2.5. The rest of the paper is devoted to proving the implication from right to left. As in the case of Theorem 3.5, it suffices to prove this implication for well-ordered trees (as opposed to arbitrary well-ordered forests). Therefore, from now on we fix an arbitrary well-ordered tree $X = \langle X; \leqslant \rangle$ in which every nonempty chain has a supremum. Our aim is to prove that X is Esakia representable. To this end, we will define a topology τ_α on $X_{\leqslant \alpha}$ for each ordinal α and show that $\langle X; \leqslant, \tau_{\mathsf{h}(X)} \rangle$ is indeed an Esakia space (observe that $X = X_{\leqslant \mathsf{h}(X)}$).

First, let τ_0 be the unique topology on the singleton $X_{\leqslant 0}$. For the successor case, suppose that we already defined a topology τ_α on $X_{\leqslant \alpha}$ for some ordinal α. Then let

$$P_\alpha \coloneqq \{x \in X_\alpha : \exists y \in X_{\alpha+1} \text{ s.t. } x < y\}$$

and for each $x \in P_\alpha$ choose an element $x^+ \in X_{\alpha+1}$ such that $x < x^+$. Moreover, let

$$S_{\alpha+1} \coloneqq X_{\alpha+1} \smallsetminus \{x^+ : x \in P_\alpha\}.$$

Lastly, let $\tau_{\alpha+1}$ be the topology on $X_{\leqslant \alpha+1}$ generated by the subbase $\mathcal{S}_{\alpha+1}$ comprising the sets

(i) $\{x\}$ for every $x \in S_{\alpha+1}$;
(ii) ${\downarrow} x$ for every $x \in P_\alpha$;
(iii) $(V \cup {\uparrow_{\alpha+1}} (V \cap X_\alpha)) \smallsetminus {\downarrow} Z$ for every $V \in \tau_\alpha$ and every $Z \subseteq_\omega P_\alpha \cup S_{\alpha+1}$ (see Figure 2).

For the limit case, let α be a limit ordinal and suppose that we already defined a topology τ_β on $X_{\leqslant \beta}$ for each $\beta < \alpha$. Then let τ_α be the topology on $X_{\leqslant \alpha}$ generated by the subbase

$$\mathcal{S}_\alpha \coloneqq \{V \cup {\uparrow_\alpha} (V \cap X_\beta) : \beta < \alpha \text{ and } V \in \tau_\beta\}.$$

The next observation will be used later on.

Lemma 4.4. *For each pair of ordinals $\beta < \alpha$ and $U \in \tau_\beta$ we have $U \cup {\uparrow_\alpha} (U \cap X_\beta) \in \mathcal{S}_\alpha$.*

Proof. Let $U_\alpha \coloneqq U \cup {\uparrow_\alpha} (U \cap X_\beta)$. The proof proceeds by induction on α. The case where $\alpha = 0$ holds vacuously because there exists no $\beta < 0$. For the successor case, we suppose that the statement holds for α and we will prove that it also holds for $\alpha + 1$. Consider $\beta < \alpha + 1$. We have two cases: either $\beta = \alpha$ or $\beta < \alpha$.

First, suppose that $\beta = \alpha$. Then $U \in \tau_\beta = \tau_\alpha$ by assumption. Therefore, condition (iii) in the definition of $\mathcal{S}_{\alpha+1}$ and the assumption that $U \in \tau_\alpha$ guarantee that

$$U_{\alpha+1} = U \cup {\uparrow_{\alpha+1}} (U \cap X_\alpha) \in \mathcal{S}_{\alpha+1}.$$

Then we consider the case where $\beta < \alpha$. By the inductive hypothesis we have $U_\alpha \in \tau_\alpha$. Thus, condition (iii) in the definition of $\mathcal{S}_{\alpha+1}$ and guarantees that

$$U_\alpha \cup {\uparrow_{\alpha+1}} (U_\alpha \cap X_\alpha) \in \mathcal{S}_{\alpha+1}. \tag{5}$$

We claim that

$$U_{\alpha+1} = U_\alpha \cup {\uparrow_{\alpha+1}} (U_\alpha \cap X_\alpha). \tag{6}$$

Together with condition (5), this would imply $U_{\alpha+1} \in \mathcal{S}_{\alpha+1}$ as desired.

To prove condition (6), consider $x \in U_{\alpha+1} = U \cup {\uparrow_{\alpha+1}} (U \cap X_\beta)$. If $x \in U$, then $x \in U_\alpha$ too by the definition of U_α and we are done. Then we consider the case where $x \in {\uparrow_{\alpha+1}} (U \cap X_\beta)$. We have two cases: either $x \in {\uparrow_\alpha} (U \cap X_\beta)$ or $x \in X_{\alpha+1}$. If $x \in {\uparrow_\alpha} (U \cap X_\beta)$, then $x \in U_\alpha$ by the definition of U_α and we are done. Then we consider the case where $x \in X_{\alpha+1}$. Let y be the unique member of $X_\alpha \cap {\downarrow} x$. Since $x \in {\uparrow_{\alpha+1}} (U \cap X_\beta)$ and $\beta \leqslant \alpha$, we have $y \in X_\alpha \cap {\uparrow_\alpha} (U \cap X_\beta)$. By the definition of U_α this yields $y \in U_\alpha \cap X_\alpha$ and, therefore, $x \in {\uparrow_{\alpha+1}} (U_\alpha \cap X_\alpha)$ as desired.

It only remains to prove the inclusion from right to left in condition (6). Consider $x \in U_\alpha \cup {\uparrow_{\alpha+1}} (U_\alpha \cap X_\alpha)$. We have two cases: either $x \in U_\alpha$ or $x \in {\uparrow_{\alpha+1}} (U_\alpha \cap X_\alpha)$. First, suppose that $x \in U_\alpha = U \cup {\uparrow_\alpha} (U \cap X_\beta)$. If $x \in U$, then $x \in U_{\alpha+1}$ by the definition of $U_{\alpha+1}$ and we are done. While if $x \in {\uparrow_\alpha} (U \cap X_\beta)$, then $x \in {\uparrow_{\alpha+1}} (U \cap X_\beta) \subseteq U_{\alpha+1}$ as desired. Then we consider the case where $x \in {\uparrow_{\alpha+1}} (U_\alpha \cap X_\alpha)$. There exists $y \in U_\alpha \cap X_\alpha$ such that $y \leqslant x$. Since $U \in \tau_\beta$ by assumption, we have $U \subseteq X_{\leqslant \beta}$. Together with $\beta < \alpha$ and $y \in X_\alpha$, this yields $y \notin U$. Therefore, from $y \in U_\alpha = U \cup {\uparrow_\alpha} (U \cap X_\beta)$ it follows that there exists $z \in U \cap X_\beta$ such that $z \leqslant y$. As $y \leqslant x$ and $x \in X_{\leqslant \alpha+1}$, we conclude that $x \in {\uparrow_{\alpha+1}} (U \cap X_\beta)$. Hence, $x \in U_{\alpha+1}$ as desired. This establishes condition (6) and concludes the analysis of the successor case.

Lastly, consider the case where α is a limit ordinal. Since $\beta < \alpha$ and $U \in \tau_\beta$, the definition of $\mathcal{S}_\alpha$ ensures that $U_\alpha \in \mathcal{S}_\alpha$. ⊠

We shall now define a function that will play an important role in the compactness proof. For every $x \in X$ and ordinal $\alpha \geqslant \mathsf{h}(x)$ we define and element $f_x(\alpha) \in X$ by recursion as

$$f_x(\mathsf{h}(x)) \coloneqq x;$$

$$f_x(\alpha+1) \coloneqq \begin{cases} (f_x(\alpha))^+ & \text{if } f_x(\alpha) \in P_\alpha; \\ f_x(\alpha) & \text{otherwise;} \end{cases}$$

$$f_x(\alpha) \coloneqq \bigvee \{f_x(\beta) : \mathsf{h}(x) \leqslant \beta < \alpha\} \text{ when } \alpha \text{ is a limit ordinal.}$$

Informally, we will regard f_x as a function from $\{\alpha \in \mathsf{Ord} : \mathsf{h}(x) \leqslant \alpha\}$ to X (although its domain is not a set). Furthermore, given a pair of ordinals α and β, we write

$$[\alpha, \beta] \coloneqq \{\gamma \in \mathsf{Ord} : \alpha \leqslant \gamma \leqslant \beta\} \text{ and } [\alpha, \beta) \coloneqq \{\gamma \in \mathsf{Ord} : \alpha \leqslant \gamma < \beta\}.$$

Lemma 4.5. *For every $x \in X$ the function f_x is well defined and order preserving.*

Proof. It suffices to prove that for every ordinal $\alpha \geqslant \mathsf{h}(x)$ the restriction $f_x \colon [\mathsf{h}(x), \alpha] \to X$ is well defined and order preserving. The proof works by induction starting at $\mathsf{h}(x)$. The base case and the successor case are straightforward. Then we consider the case where α is a limit ordinal such that $\mathsf{h}(x) < \alpha$. By the inductive hypothesis $f_x \colon [\mathsf{h}(x), \alpha) \to X$ is well defined and order preserving. Consequently, $\{f_x(\beta) : \mathsf{h}(x) \leqslant \beta < \alpha\}$ is a chain which, moreover, is nonempty because $\mathsf{h}(x) < \alpha$. Therefore, this chain has a supremum $f_x(\alpha)$ in X by assumption. Hence, $f_x \colon [\mathsf{h}(x), \alpha] \to X$ is also well defined and order preserving. ⊠

We will make use of the following properties of the function f_x.

Lemma 4.6. *The following conditions hold for every $x, y \in X$ and ordinal $\alpha \geqslant \mathsf{h}(x)$:*

(i) $f_x(\alpha) \in \max X_{\leqslant \alpha}$;
(ii) $f_x(\alpha + 1) \notin S_{\alpha+1}$;
(iii) *for every $y \leqslant f_x(\alpha)$ such that $\mathsf{h}(x) \leqslant \mathsf{h}(y)$ we have $y = f_x(\mathsf{h}(y))$;*
(iv) $\mathsf{h}(x) \leqslant \mathsf{h}(f_x(\alpha))$ *and* $f_x(\alpha) = f_x(\mathsf{h}(f_x(\alpha)))$;
(v) *for every $\beta \in [\mathsf{h}(f_x(\alpha)), \alpha]$ we have $f_x(\alpha) = f_x(\beta)$.*

Proof. In this proof will make extensive use of the fact that f_x is order preserving (see Lemma 4.5).

A straightforward induction on α establishes condition (i). Condition (ii) follows from (i) and the definition of f_x. To prove condition (iii), assume that $y \leqslant f_x(\alpha)$ and $\mathsf{h}(x) \leqslant \mathsf{h}(y)$. We will prove that $\mathsf{h}(y) \leqslant \alpha$. Suppose, on the contrary, that $\alpha < \mathsf{h}(y)$. By condition (i) we have $f_x(\alpha) \in \max X_{\leqslant \alpha}$. Together with $\alpha < \mathsf{h}(y)$, this yields $y \nleqslant f_x(\alpha)$, a contradiction. Since $\mathsf{h}(y) \leqslant \alpha$ and $\mathsf{h}(x) \leqslant \mathsf{h}(y)$, we obtain $f_x(\mathsf{h}(y)) \leqslant f_x(\alpha)$. On the other hand, $y \leqslant f_x(\alpha)$ by assumption. Therefore, the elements y and $f_x(\mathsf{h}(y))$ are comparable because X is a tree. By condition (i) we have $f_x(\mathsf{h}(y)) \in \max X_{\leqslant \mathsf{h}(y)}$. This yields $y \nless f_x(\mathsf{h}(y))$ and $f_x(\mathsf{h}(y)) \nless y$. As y and $f_x(\mathsf{h}(y))$ are comparable, we conclude that $y = f_x(\mathsf{h}(y))$ as desired. Then we turn to prove condition (iv). As $\mathsf{h}(x) \leqslant \alpha$, we also have $x = f_x(\mathsf{h}(x)) \leqslant f_x(\alpha)$, whence $\mathsf{h}(x) \leqslant \mathsf{h}(f_x(\alpha))$. By applying condition (iii) to $y \coloneqq f_x(\alpha)$ we obtain $f_x(\alpha) = f_x(\mathsf{h}(f_x(\alpha)))$. Lastly, condition (v) is an immediate consequence of condition (iv) and the fact that f_x is order preserving. ⊠

Corollary 4.7. *Let $x \in X$ and α an ordinal such that $\mathsf{h}(x) \leqslant \alpha + 1$. If $f_x(\alpha + 1) \in U$ for some $U \in \mathcal{S}_{\alpha+1}$, there exist $V \in \tau_\alpha$ and $Z \subseteq_\omega P_\alpha \cup S_{\alpha+1}$ such that*

$$U = (V \cup {\uparrow_{\alpha+1}} (V \cap X_\alpha)) \smallsetminus {\downarrow} Z.$$

Proof. As U is a member of $\mathcal{S}_{\alpha+1}$, it satisfies one of the conditions (i)–(iii) in the definition of $\mathcal{S}_{\alpha+1}$. If U satisfies condition (iii), we are done. Then suppose that U does not satisfy condition (iii), with a view to contradiction. In this case, U satisfies either condition (i) or condition (ii). If U satisfies condition (i), there exists $y \in S_{\alpha+1}$ such that $U = \{y\}$. Hence, $f_x(\alpha + 1) \in U = \{y\}$ and, therefore, $f_x(\alpha + 1) = y \in S_{\alpha+1}$, a contradiction with Lemma 4.6(ii). On the other hand, if U satisfies condition (ii), there exists $y \in P_\alpha$ such that $U = {\downarrow} y$. Therefore, $f_x(\alpha + 1) \in U = {\downarrow} y$. Since $y \in P_\alpha$, we have $y \notin \max X_{\leqslant \alpha+1}$, whence $f_x(\alpha + 1) \notin X_{\leqslant \alpha+1}$, a contradiction with Lemma 4.6(i). ⊠

5. The main lemma

The next result plays a central role in the proof that the topological space $\langle X; \tau_{\mathsf{h}(X)} \rangle$ is compact.

Main Lemma 5.1. *Let $x \in X$ and α be an ordinal such that $\mathsf{h}(x) \leqslant \alpha$. If $f_x(\alpha) \in U$ for some $U \in \mathcal{S}_\alpha$, there exist*

$$v \leqslant x, \quad Y \subseteq_\omega X_{>\mathsf{h}(x)} \cap {\uparrow_\alpha} y, \quad \text{and} \quad Z \subseteq_\omega X_{<\alpha} \cap {\uparrow} y$$

such that ${\uparrow_\alpha} v \smallsetminus ({\uparrow_\alpha} Y \cup {\downarrow} Z) \subseteq U$ and $\mathsf{h}(y)$ is either zero or a successor ordinal.

Proof. It holds that $\mathsf{h}(x) \leqslant \alpha$ by assumption. We proceed by induction on the left subtraction $\alpha - \mathsf{h}(x)$, i.e., the only ordinal β such that $\mathsf{h}(x) + \beta = \alpha$.

Base case. In the base case, $\alpha - \mathsf{h}(x) = 0$ and, therefore, $\mathsf{h}(x) = \alpha$. Together with the definition of f_x, this yields $x = f_x(\mathsf{h}(x)) = f_x(\alpha)$. Consequently, Lemma 4.6(i) implies $x \in \max X_{\leqslant\alpha}$, whence $\uparrow_\alpha x = \{x\}$. Suppose first that either $\mathsf{h}(x) = 0$ or $\mathsf{h}(x)$ is a successor ordinal. Letting $v \coloneqq x$, $Y \coloneqq \emptyset$, and $Z \coloneqq \emptyset$ and using the assumption that $x = f_x(\alpha) \in U$, we obtain

$$\uparrow_\alpha v \smallsetminus (\uparrow_\alpha Y \cup \downarrow Z) = \uparrow_\alpha x \smallsetminus \emptyset = \{x\} \smallsetminus \emptyset = \{x\} \subseteq U$$

and we are done. Then we consider the case where $\alpha = \mathsf{h}(x)$ is a limit ordinal. As $U \in \mathcal{S}_\alpha$ by assumption, the definition of $\mathcal{S}_\alpha$ implies that there exist $\beta < \alpha$ and $V \in \tau_\beta$ such that $U = V \cup \uparrow_\alpha (V \cap X_\beta)$. From $\beta < \alpha$ and $V \in \tau_\beta$ it follows that $V \cap X_\alpha = \emptyset$ (because $V \subseteq X_{\leqslant\beta}$). As $\mathsf{h}(x) = \alpha$, this yields $x \notin V$. Together with the assumptions that $x = f_x(\alpha) \in U$ and $U = V \cup \uparrow_\alpha (V \cap X_\beta)$, this implies $x \in \uparrow_\alpha (V \cap X_\beta)$. Consequently, there exists $v^* \leqslant x$ such that $v^* \in V \cap X_\beta$. Therefore,

$$\uparrow_\alpha v^* \subseteq \uparrow_\alpha (V \cap X_\beta) \subseteq U.$$

Now, recall that $\beta < \alpha$ and that α is a limit ordinal. Therefore, there exists a successor ordinal γ such that $\beta \leqslant \gamma < \alpha$. Furthermore, as $\mathsf{h}(x) = \alpha$, $\mathsf{h}(v^*) = \beta$, and $v^* \leqslant x$, there exists $v \in X$ such that $v^* \leqslant v \leqslant x$ and $\mathsf{h}(v) = \gamma$. In view of the above display and $v^* \leqslant v$, by letting $Y \coloneqq \emptyset$ and $Z \coloneqq \emptyset$, we conclude that

$$\uparrow_\alpha v \smallsetminus (\uparrow_\alpha Y \cup \downarrow Z) = \uparrow_\alpha v \smallsetminus \emptyset = \uparrow_\alpha v \subseteq \uparrow_\alpha v^* \subseteq U.$$

As $\mathsf{h}(v) = \gamma$ is a successor ordinal, we are done.

Successor case. In the successor case of the induction, $\alpha - \mathsf{h}(x)$ is a successor ordinal $\beta + 1$ and $\alpha = \mathsf{h}(x) + \beta + 1$. By assumption we have

$$f_x(\mathsf{h}(x) + \beta + 1) = f_x(\alpha) \in U \quad \text{and} \quad U \in \mathcal{S}_\alpha = \mathcal{S}_{\mathsf{h}(x)+\beta+1}.$$

Therefore, we can apply Corollary 4.7 obtaining

$$U = \left(V \cup \uparrow_\alpha \left(V \cap X_{\mathsf{h}(x)+\beta}\right)\right) \smallsetminus \downarrow \bar{Z} \tag{7}$$

for some $V \in \tau_{\mathsf{h}(x)+\beta}$ and $\bar{Z} \subseteq_\omega P_{\mathsf{h}(x)+\beta} \cup S_\alpha$.

Claim 5.2. $f_x(\mathsf{h}(x) + \beta) \in V$.

Proof of the Claim. Recall that $f_x(\alpha) \in U \subseteq V \cup \uparrow_\alpha \left(V \cap X_{\mathsf{h}(x)+\beta}\right)$. Therefore, we have two cases: either $f_x(\alpha) \in V$ or $f_x(\alpha) \in \uparrow_\alpha \left(V \cap X_{\mathsf{h}(x)+\beta}\right)$. First, suppose that $f_x(\alpha) \in V$. Then

$$f_x(\alpha) \in V \subseteq X_{\leqslant \mathsf{h}(x)+\beta},$$

where the last inclusion holds because $V \in \tau_{\mathsf{h}(x)+\beta}$. From from the above display and $\mathsf{h}(x)+\beta < \alpha$ it follows that $\mathsf{h}(f_x(\alpha)) \leqslant \mathsf{h}(x) + \beta < \alpha$. By Lemma 4.6(v) we conclude that $f_x(\mathsf{h}(x) + \beta) = f_x(\alpha) \in V$ as desired. Then we consider the case where $f_x(\alpha) \in \uparrow_\alpha \left(V \cap X_{\mathsf{h}(x)+\beta}\right)$. There exists $y \in V \cap X_{\mathsf{h}(x)+\beta}$ such that $y \leqslant f_x(\alpha)$. Since $y \leqslant f_x(\alpha)$ and $\mathsf{h}(x) \leqslant \mathsf{h}(y)$, we can apply Lemma 4.6(iii), obtaining $y = f_x(\mathsf{h}(y))$. As $y \in X_{\mathsf{h}(x)+\beta}$ and, therefore, $\mathsf{h}(y) = \mathsf{h}(x) + \beta$, we conclude that $f_x(\mathsf{h}(x) + \beta) = f_x(\mathsf{h}(y)) = y \in V$. ⊠

Now, recall that $\mathcal{S}_{\mathsf{h}(x)+\beta}$ is a subbase for $\tau_{\mathsf{h}(x)+\beta}$ and that $V \in \tau_{\mathsf{h}(x)+\beta}$. As $f_x(\mathsf{h}(x) + \beta) \in V$ by Claim 5.2, there exist $W_1, \dots, W_n \in \mathcal{S}_{\mathsf{h}(x)+\beta}$ such that

$$f_x(\mathsf{h}(x) + \beta) \in W_1 \cap \cdots \cap W_n \subseteq V. \tag{8}$$

Claim 5.3. *There exist $v \leqslant x$, $Y^* \subseteq_\omega X_{>\mathsf{h}(x)} \cap \uparrow_{\mathsf{h}(x)+\beta} v$, and $Z^* \subseteq_\omega X_{<\mathsf{h}(x)+\beta} \cap \uparrow v$ such that*

$$\uparrow_{\mathsf{h}(x)+\beta} v \smallsetminus \left(\uparrow_{\mathsf{h}(x)+\beta} Y^* \cup \downarrow Z^*\right) \subseteq W_1 \cap \cdots \cap W_n \subseteq V$$

and $\mathsf{h}(v)$ is either zero or a successor ordinal.

Proof of the Claim. By applying the inductive hypothesis to $W_1, \dots, W_n \in \mathcal{S}_{\mathsf{h}(x)+\beta}$ and condition (8), we obtain that for every $m \leqslant n$ there exist

$$v_m \leqslant x, \quad Y_m \subseteq_\omega X_{>\mathsf{h}(x)} \cap {\uparrow}_{\mathsf{h}(x)+\beta} v_m, \text{ and } Z_m \subseteq_\omega X_{<\mathsf{h}(x)+\beta} \cap {\uparrow} v_m$$

such that

$${\uparrow}_{\mathsf{h}(x)+\beta} v_m \smallsetminus \left({\uparrow}_{\mathsf{h}(x)+\beta} Y_m \cup {\downarrow} Z_m\right) \subseteq W_m \tag{9}$$

and $\mathsf{h}(y_m)$ is either zero or a successor ordinal. As X is a tree and $v_1, \dots, v_n \leqslant x$, the set $\{v_m : m \leqslant n\}$ is a nonempty chain and, therefore, has a maximum v. Then, letting

$$Y^* \coloneqq (Y_1 \cup \dots \cup Y_n) \cap {\uparrow} v \text{ and } Z^* \coloneqq (Z_1 \cup \dots \cup Z_m) \cap {\uparrow} v,$$

we obtain

$$Y^* \subseteq_\omega X_{>\mathsf{h}(x)} \cap {\uparrow}_{\mathsf{h}(x)+\beta} v \text{ and } Z^* \subseteq_\omega X_{<\mathsf{h}(x)+\beta} \cap {\uparrow} v.$$

Furthermore, $v \leqslant x$ and $\mathsf{h}(v)$ is either zero or a successor ordinal. Therefore, it only remains to prove that

$${\uparrow}_{\mathsf{h}(x)+\beta} v \smallsetminus \left({\uparrow}_{\mathsf{h}(x)+\beta} Y^* \cup {\downarrow} Z^*\right) \subseteq W_1 \cap \dots \cap W_n \subseteq V.$$

Since $W_1 \cap \dots \cap W_n \subseteq V$ by condition (8), it suffices to show that ${\uparrow}_{\mathsf{h}(x)+\beta} v \smallsetminus \left({\uparrow}_{\mathsf{h}(x)+\beta} Y^* \cup {\downarrow} Z^*\right) \subseteq W_1 \cap \dots \cap W_n$. To this end, consider $z \in {\uparrow}_{\mathsf{h}(x)+\beta} v \smallsetminus \left({\uparrow}_{\mathsf{h}(x)+\beta} Y^* \cup {\downarrow} Z^*\right)$ and $m \leqslant n$. We need to show that $z \in W_m$. In view of condition (9), the definition of Y^* and Z^*, and $v_m \leqslant v$, it will be enough to show that $z \notin {\uparrow}_{\mathsf{h}(x)+\beta} Y_m \cup {\downarrow} Z_m$. We begin by proving that $z \notin {\uparrow}_{\mathsf{h}(x)+\beta} Y_m$. Suppose the contrary, with a view to contradiction. Then $\mathsf{h}(z) \leqslant \mathsf{h}(x) + \beta$ and there exists $y \in Y_m$ such that $y \leqslant z$. Since $v \leqslant z$ and X is a tree, the elements v and y must be comparable. We have two cases: either $v \leqslant y$ or $y < v$. If $v \leqslant y$, then $y \in Y^*$ because $y \in Y_m$. Therefore, $z \in {\uparrow}_{\mathsf{h}(x)+\beta} Y^*$, a contradiction with $z \in {\uparrow}_{\mathsf{h}(x)+\beta} v \smallsetminus \left({\uparrow}_{\mathsf{h}(x)+\beta} Y^* \cup {\downarrow} Z^*\right)$. Then we consider the case where $y < v$. As $v \leqslant x$, this implies $\mathsf{h}(y) < \mathsf{h}(x)$, a contradiction with $y \in Y_m \subseteq X_{>\mathsf{h}(x)}$. Hence, we conclude that $z \notin {\uparrow}_{\mathsf{h}(x)+\beta} Y_m$. Then we turn to prove that $z \notin {\downarrow} Z_m$. Suppose the contrary, with a view to contradiction. Then there exists $y \in Z_m$ such that $z \leqslant y$. Since $v \leqslant z$, we obtain $v \leqslant y \in Z_m$. By the definition of Z^* we obtain $y \in Z^*$ and, therefore, $z \in {\downarrow} Z^*$, a contradiction with $z \in {\uparrow}_{\mathsf{h}(x)+\beta} v \smallsetminus \left({\uparrow}_{\mathsf{h}(x)+\beta} Y^* \cup {\downarrow} Z^*\right)$. This establishes the above display. ⊠

Now, consider the sets

$$Y \coloneqq Y^* \cup \left(X_\alpha \cap \bar{Z} \cap {\uparrow} v\right) \text{ and } Z \coloneqq Z^* \cup \left(X_{\mathsf{h}(x)+\beta} \cap {\downarrow} \bar{Z} \cap {\uparrow} v\right).$$

From Claim 5.3 it follows that $v \leqslant x$ and that $\mathsf{h}(v)$ is either zero or a successor ordinal. Furthermore, as $\bar{Z}$ and Y^* are finite (the latter by Claim 5.3), the set Y is also finite. Lastly, as X is a tree and $\bar{Z}$ is finite, ${\downarrow}\bar{Z}$ is a union of finitely many chains. Therefore, $X_{\mathsf{h}(x)+\beta} \cap {\downarrow}\bar{Z}$ is a finite set. As Z^* is finite by Claim 5.3, we conclude that Z is also finite. Therefore, it only remains to show that

$$Y \subseteq X_{>\mathsf{h}(x)} \cap {\uparrow}_\alpha v, \quad Z \subseteq X_{<\alpha} \cap {\uparrow} v, \text{ and } {\uparrow}_\alpha v \smallsetminus ({\uparrow}_\alpha Y \cup {\downarrow} Z) \subseteq U.$$

By Claim 5.3 we have $Y^* \subseteq X_{>\mathsf{h}(x)} \cap {\uparrow}_{\mathsf{h}(x)+\beta} v \subseteq X_{>\mathsf{h}(x)} \cap {\uparrow}_\alpha v$ and $Z^* \subseteq X_{<\mathsf{h}(x)+\beta} \cap {\uparrow} v \subseteq X_{<\alpha} \cap {\uparrow} v$. Together with $\alpha = \mathsf{h}(x) + \beta + 1$ and the definition of Y and Z, this guarantees the validity of the first two conditions in the above display. Therefore, it only remains to prove that ${\uparrow}_\alpha v \smallsetminus ({\uparrow}_\alpha Y \cup {\downarrow} Z) \subseteq U$. By condition (7) this amounts to

$${\uparrow}_\alpha v \smallsetminus ({\uparrow}_\alpha Y \cup {\downarrow} Z) \subseteq \left(V \cup {\uparrow}_\alpha \left(V \cap X_{\mathsf{h}(x)+\beta}\right)\right) \smallsetminus {\downarrow} \bar{Z}. \tag{10}$$

Consider $z \in {\uparrow}_\alpha v \smallsetminus ({\uparrow}_\alpha Y \cup {\downarrow} Z)$. Then $v \leqslant z \in X_{\leqslant \alpha}$ and $z \notin {\uparrow}_\alpha Y \cup {\downarrow} Z$. Since $\alpha = \mathsf{h}(x) + \beta + 1$ and $z \in X_{\leqslant \alpha}$, we have two cases: either $z \in X_{\leqslant \mathsf{h}(x)+\beta}$ or $z \in X_\alpha$. First, suppose that $z \in X_{\leqslant \mathsf{h}(x)+\beta}$. Then

$$z \in \left(X_{\leqslant \mathsf{h}(x)+\beta} \cap {\uparrow}_\alpha v\right) \smallsetminus ({\uparrow}_\alpha Y \cup {\downarrow} Z) \subseteq {\uparrow}_{\mathsf{h}(x)+\beta} v \smallsetminus \left({\uparrow}_{\mathsf{h}(x)+\beta} Y^* \cup {\downarrow} Z^*\right) \subseteq V,$$

where the first inclusion holds because $Y^* \subseteq Y$ and $Z^* \subseteq Z$, and the last by Claim 5.3. Therefore, in order to conclude that z belongs to the right hand side of condition (10), it suffices to show that $z \notin {\downarrow}\bar{Z}$. Suppose the contrary, with a view to contradiction. Then there exists $y \in \bar{Z}$ such that $z \leqslant y$. Since $v \leqslant z \in X_{\leqslant \mathsf{h}(x)+\beta}$ and $\bar{Z} \subseteq P_{\mathsf{h}(x)+\beta} \cup S_\alpha$, there exists $y^* \in X_{\mathsf{h}(x)+\beta}$ such that $v \leqslant z \leqslant y^* \leqslant y$. Hence, $y^* \in X_{\mathsf{h}(x)+\beta} \cap {\downarrow}\bar{Z} \cap {\uparrow}v \subseteq Z$, where the last inclusion holds by the definition of Z. Together with $z \leqslant y^*$, this implies $z \in {\downarrow}Z$, which is false.

Then we consider the case where $z \in X_\alpha$. Let y be the unique element of $X_{\mathsf{h}(x)+\beta} \cap {\downarrow}z$. We will prove that $y \in V$. By Claim 5.3 it suffices to show that

$$y \in {\uparrow}_{\mathsf{h}(x)+\beta} v \smallsetminus \left({\uparrow}_{\mathsf{h}(x)+\beta} Y^* \cup {\downarrow}Z^*\right).$$

From $z \in X_\alpha$, $v \leqslant x$, and $\mathsf{h}(x) < \alpha$ it follows that $\mathsf{h}(v) < \mathsf{h}(z)$. Since $v \leqslant z$, this implies $v < z$. Moreover, as X is a tree, from $v, y < z$ it follows that v and y are comparable. Since y is the unique immediate predecessor of z by definition and $v < z$, we conclude that $v \leqslant y$. Hence, $y \in {\uparrow}_{\mathsf{h}(x)+\beta} v$. Now, observe that $y \notin {\uparrow}Y^*$, otherwise we would have $z \in {\uparrow}_\alpha Y$, a contradiction. Moreover, observe that $y \notin {\downarrow}Z^*$ because $y \in X_{\mathsf{h}(x)+\beta}$ and $Z^* \subseteq X_{<\mathsf{h}(x)+\beta}$ by Claim 5.3. This establishes the above display and, therefore, that $y \in V$. Together with $y \leqslant z$, $y \in X_{\mathsf{h}(x)+\beta}$, and $z \in X_\alpha$, this yields $z \in {\uparrow}_\alpha \left(V \cap X_{\mathsf{h}(x)+\beta}\right)$. Therefore, in order to prove that z belongs to the right hand side of condition (10), it only remains to show that $z \notin {\downarrow}\bar{Z}$. Suppose the contrary, with a view of contradiction. Then there exists $u \in \bar{Z}$ such that $z \leqslant u$. Together with $z \in X_\alpha$ and $u \in \bar{Z} \subseteq P_{\mathsf{h}(x)+\beta} \cup S_\alpha$, this implies $z = u \in \bar{Z}$. Hence, $z \in X_\alpha \cap \bar{Z} \cap {\uparrow}v$. By the definition of Y this yields $z \in {\uparrow}_\alpha Y$, a contradiction.

Limit case. Finally, we consider the case where $\alpha - \mathsf{h}(x)$ is a limit ordinal. In this case, α is also a limit ordinal. Consequently, from $U \in \mathcal{S}_\alpha$ it follows that $U = V \cup {\uparrow}_\alpha (V \cap X_\beta)$ for some $\beta < \alpha$ and $V \in \tau_\beta$. We have two cases: either $\beta < \mathsf{h}(x)$ or $\mathsf{h}(x) \leqslant \beta$.

First, suppose that $\beta < \mathsf{h}(x)$. Then $\beta < \mathsf{h}(x) \leqslant \mathsf{h}(f_x(\alpha))$. Together with $V \subseteq X_{\leqslant\beta}$ (because $V \in \tau_\beta$), this implies $f_x(\alpha) \notin V$. On the other hand, $f_x(\alpha) \in U = V \cup {\uparrow}_\alpha (V \cap X_\beta)$ by assumption. Therefore, $f_x(\alpha) \in {\uparrow}_\alpha (V \cap X_\beta)$. Then there exists $z \in V \cap X_\beta$ such that $z \leqslant f_x(\alpha)$. As X is a tree, from $x, z \leqslant f_x(\alpha)$ it follows that x and z are comparable. Since $z \in X_\beta$ and $\beta < \mathsf{h}(x)$, we deduce that $z < x$. Thus,

$${\uparrow}_\alpha x \subseteq {\uparrow}_\alpha z \subseteq {\uparrow}_\alpha (V \cap X_\beta) \subseteq U.$$

Now, let $Y \coloneqq \emptyset$ and $Z \coloneqq \emptyset$. Furthermore, if $\mathsf{h}(x)$ is zero or a successor ordinal, let $v \coloneqq x$. While if $\mathsf{h}(x)$ is a limit ordinal, recall that $z < x$ and let v be any element strictly between z and x whose height is a successor ordinal. In both cases, we are done.

Then we consider the case where $\mathsf{h}(x) \leqslant \beta$. We will prove that $f_x(\beta) \in V$. Recall that $f_x(\alpha) \in U = V \cup {\uparrow}_\alpha (V \cap X_\beta)$. Then we have two cases: either $f_x(\alpha) \in V$ or $f_x(\alpha) \in {\uparrow}_\alpha (V \cap X_\beta)$. If $f_x(\alpha) \in V$, from $V \subseteq X_{\leqslant\beta}$ it follows that $\mathsf{h}(f_x(\alpha)) \leqslant \beta$. Together with $\beta \leqslant \alpha$ and Lemma 4.6(v), this yields $f_x(\beta) = f_x(\alpha) \in V$ as desired. Then we consider the case where $f_x(\alpha) \in {\uparrow}_\alpha (V \cap X_\beta)$. There exists $z \in V$ such that $\mathsf{h}(z) = \beta$ and $z \leqslant f_x(\alpha)$. By Lemma 4.5(iii) we have $f_x(\beta) = z \in V$. This establishes that $f_x(\beta) \in V$ as desired.

As $\mathcal{S}_\beta$ is a subbase for the topology τ_β, from $f_x(\beta) \in V \in \tau_\beta$ it follows that there exist $W_1, \dots, W_n \in \mathcal{S}_\beta$ such that $f_x(\beta) \in W_1 \cap \dots \cap W_n$. Since $\mathsf{h}(x) \leqslant \beta < \alpha$, we have $\beta - \mathsf{h}(x) < \alpha - \mathsf{h}(x)$. Therefore, we can apply the inductive hypothesis obtaining that for each $m \leqslant n$ there exist

$$v_m \leqslant x, \quad Y_m \subseteq_\omega X_{>\mathsf{h}(x)} \cap {\uparrow}_\beta v_m, \quad Z_m \subseteq_\omega X_{<\beta} \cap {\uparrow}v_m$$

such that

$${\uparrow}_\beta v_m \smallsetminus ({\uparrow}_\beta Y_m \cup {\downarrow}Z_m) \subseteq W_m$$

and $\mathsf{h}(v_m)$ is either zero or a successor ordinal. As X is a tree and $v_1, \dots, v_n \leqslant x$, the set $\{v_m : m \leqslant n\}$ is a nonempty chain and, therefore, has a maximum v. Then, letting

$$Y \coloneqq (Y_1 \cup \dots \cup Y_n) \cap {\uparrow} v \text{ and } Z \coloneqq (Z_1 \cup \dots \cup Z_m) \cap {\uparrow} v,$$

we obtain

$$Y \subseteq_\omega X_{>\mathsf{h}(x)} \cap {\uparrow}_\beta v \text{ and } Z \subseteq_\omega X_{<\beta} \cap {\uparrow} v. \tag{11}$$

Furthermore,

$${\uparrow}_\beta v \smallsetminus ({\uparrow}_\beta Y \cup {\downarrow} Z) \subseteq W_1 \cap \dots \cap W_n \subseteq V \tag{12}$$

and $\mathsf{h}(v)$ is either zero or a successor ordinal.

From condition (11) and $\beta \leqslant \alpha$ it follows that

$$Y \subseteq_\omega X_{>\mathsf{h}(x)} \cap {\uparrow}_\alpha v \text{ and } Z \subseteq_\omega X_{<\alpha} \cap {\uparrow} v.$$

Since $\mathsf{h}(v)$ is either zero or a successor ordinal and $U = V \cup {\uparrow}_\alpha (V \cap X_\beta)$, it only remains to show that

$${\uparrow}_\alpha v \smallsetminus ({\uparrow}_\alpha Y \cup {\downarrow} Z) \subseteq V \cup {\uparrow}_\alpha (V \cap X_\beta).$$

To this end, let $z \in X_{\leqslant\alpha}$ be such that $v \leqslant z$ and $z \notin {\uparrow}_\alpha Y \cup {\downarrow} Z$. We have two cases: either $z \in X_{\leqslant\beta}$ or $z \notin X_{\leqslant\beta}$. In the former case, we have $z \in {\uparrow}_\beta v \smallsetminus ({\uparrow}_\beta Y \cup {\downarrow} Z)$. By condition (12) we conclude that $z \in V$ as desired. Then we consider the case where $z \notin X_{\leqslant\beta}$, i.e., $\mathsf{h}(z) > \beta$. Let y be the unique element of ${\downarrow} z \cap X_\beta$. As X is a tree and $v, y \leqslant z$, we deduce that either $y < v$ or $v \leqslant y$. However, the former case cannot happen because $y \in X_\beta$, $v \leqslant x$, and $\mathsf{h}(x) \leqslant \beta$. Hence, $v \leqslant y$ and, therefore, $y \in {\uparrow}_\beta v$. Moreover, $y \notin {\uparrow}_\beta Y$ because $z \notin {\uparrow}_\alpha Y$ and $y \leqslant z \in X_{\leqslant\alpha}$. Lastly, $y \notin {\downarrow} Z$ because $y \in X_\beta$ and $Z \subseteq X_{<\beta}$. Therefore, $y \in {\uparrow}_\beta v \smallsetminus ({\uparrow}_\beta Y \cup {\downarrow} Z)$. From condition (12) it follows that $y \in V$. Thus, $y \in V \cap X_\beta$. Together with $y \leqslant z \in X_{\leqslant\alpha}$, this implies $z \in {\uparrow}_\alpha (V \cap X_\beta)$. ⊠

6. Compactness

The aim of this section is to prove the following.

Theorem 6.1. *The topological space $\langle X; \tau_{\mathsf{h}(X)} \rangle$ is compact.*

To this end, let $\mathcal{C}$ be an open covering of X. We need to show that $\mathcal{C}$ has a finite subcover. By Alexander's subbase theorem we may assume $\mathcal{C} \subseteq \mathcal{S}_{\mathsf{h}(X)}$. The construction of a finite subcover proceeds through a series of technical observations.

Proposition 6.2. *For every $x \in X$ there exists $\mathcal{V}_x \subseteq_\omega \mathcal{C}$ such that ${\downarrow} x \subseteq \bigcup \mathcal{V}_x$.*

Proof. We proceed by induction on $\mathsf{h}(x)$. If $\mathsf{h}(x) = 0$, then x is the root of X and the claim follows from ${\downarrow} x = \{x\}$. If $\mathsf{h}(x) = \alpha + 1$, there exists $y \in X_\alpha$ such that ${\downarrow} x = \{x\} \cup {\downarrow} y$. As $\mathsf{h}(y) = \alpha < \alpha + 1$, by the inductive hypothesis there exists $\mathcal{V}_y \subseteq_\omega \mathcal{C}$ such that ${\downarrow} y \subseteq \bigcup \mathcal{V}_y$. Let $U \in \mathcal{C}$ be such that $x \in U$. Letting $\mathcal{V}_x \coloneqq \mathcal{V}_y \cup \{U\}$, we conclude that $\mathcal{V}_x \subseteq_\omega \mathcal{C}$ and ${\downarrow} x \subseteq \bigcup \mathcal{V}_x$.

Finally, suppose that $\mathsf{h}(x)$ is a limit ordinal and consider $U \in \mathcal{C}$ such that $x \in U$. We begin with the following observation.

Claim 6.3. *There exists $y < x$ such that $[y, x] \subseteq U$.*

Proof of the Claim. We will prove that for every $\alpha \leqslant \mathsf{h}(X)$,

$$\text{if } x \in W \text{ for some } W \in \mathcal{S}_\alpha, \text{ there exists } y < x \text{ such that } [y, x] \subseteq W.$$

Since $x \in U \in \mathcal{C} \subseteq \mathcal{S}_{\mathsf{h}(X)}$ by assumption, Claim 6.3 follows immediately from the above display in the case where $\alpha = \mathsf{h}(X)$ and $W = U$.

We proceed by induction on α. The case where $\alpha = 0$ is straightforward because the assumption that $\mathsf{h}(x)$ is a limit ordinal guarantees that $x \notin X_{\leqslant 0}$ and, therefore, $x \notin \bigcup \mathcal{S}_0$. Then we consider the case where α is a successor ordinal $\beta + 1$. Suppose that $x \in W \in \mathcal{S}_{\beta+1}$. Since $\mathsf{h}(x)$ is a limit ordinal, we have $x \notin X_{\beta+1}$. Therefore, the definition of $\mathcal{S}_{\beta+1}$ and $x \in W \in \mathcal{S}_{\beta+1}$ ensures that either $W = {\downarrow} z$ for some $z \in P_\beta$ or $W = V \cup {\uparrow}_{\leqslant \beta+1}(V \cap X_\beta)$ for some $V \in \tau_\beta$. First, suppose that $W = {\downarrow} z$. As $\mathsf{h}(x)$ is a limit ordinal, there exists $y < x$. Since $x \in W = {\downarrow} z$, we obtain $[y, x] \subseteq W$ as desired. Then we consider the case where $W = V \cup {\uparrow}_{\leqslant \beta+1}(V \cap X_\beta)$ for some $V \in \tau_\beta$. Together with $x \in W \smallsetminus X_{\beta+1}$, this yields $x \in V$. As $V \in \tau_\beta$ and $\mathcal{S}_\beta$ is a subbase for τ_β, there exist $V_1, \dots, V_n \in \mathcal{S}_\beta$ such that $x \in V_1 \cap \dots \cap V_n = V \subseteq W$. Then the inductive hypothesis ensures that there exist $y_1, \dots, y_n < x$ such that $[y_1, x] \subseteq V_1, \dots, [y_n, x] \subseteq V_n$. As $y_1, \dots, y_n < x$ and X is a tree, the set $\{y_1, \dots, y_n\}$ is a nonempty chain and, therefore, has a maximum y. We have $y < x$ and $[y, x] \subseteq V_1 \cap \dots \cap V_n \subseteq W$ as desired.

Lastly, we consider the case where α is a limit ordinal. Suppose that $x \in W \in \mathcal{S}_\alpha$. Then $W = V \cup {\uparrow}(V \cap X_\beta)$ for some $\beta < \alpha$ and $V \in \tau_\beta$ by the definition of $\mathcal{S}_\alpha$. Together with $x \in W$, this yields $x \in V \cup {\uparrow}(V \cap X_\beta)$. If $x \in {\uparrow}(V \cap X_\beta)$, there exists $y \in V \cap X_\beta$ such that $y < x$. Therefore, $[y, x] \subseteq {\uparrow}(V \cap X_\beta) \subseteq W$ and we are done. Then we consider the case where $x \in V$. Together with $V \subseteq W$, the assumption that $V \in \tau_\beta$ and that $\mathcal{S}_\beta$ is a subbase for τ_β implies the existence of $V_1, \dots, V_n \in \mathcal{S}_\beta$ such that $x \in V_1 \cap \dots \cap V_n = V \subseteq W$. Since $\beta < \alpha$, we can apply the inductive hypothesis obtaining $y_1, \dots, y_n < x$ such that $[y_1, x] \subseteq V_1, \dots, [y_n, x] \subseteq V_n$. As before, letting y be the maximum of $\{y_1, \dots, y_n\}$, we obtain $y < x$ and $[y, x] \subseteq V_1 \cap \dots \cap V_n \subseteq W$. ⊠

By the Claim there exists $y < x$ such that $[y, x] \subseteq U$. As $\mathsf{h}(y) < \mathsf{h}(x)$, we can apply the inductive hypothesis, obtaining that there exists $\mathcal{V}_y \subseteq_\omega \mathcal{C}$ such that ${\downarrow} y \subseteq \bigcup \mathcal{V}_y$. Therefore, letting $\mathcal{V}_x := \mathcal{V}_y \cup \{U\}$, we obtain that $\mathcal{V}_x \subseteq_\omega \mathcal{C}$ and ${\downarrow} x = [y, x] \cup {\downarrow} y \subseteq \bigcup \mathcal{V}_x$. ⊠

The heart of the compactness proof is the following observation.

Proposition 6.4. *For each ordinal α there exist $\mathcal{U}^\alpha \subseteq_\omega \mathcal{C}$ and an antichain $F^\alpha \subseteq_\omega X$ such that*

$$X \smallsetminus {\uparrow} F^\alpha \subseteq \bigcup \mathcal{U}^\alpha \text{ and there are no } x \in F^\alpha, \beta < \alpha, \text{ and } y \in F^\beta \text{ such that } x < y. \tag{13}$$

Furthermore, if $\alpha = \beta + 1$, then $F^\alpha \subseteq {\uparrow} F^\beta \smallsetminus F^\beta$.

For the sake of readability, we will postpone the proof of the above proposition to the end of this section. Instead, we shall now explain how the above proposition can be used to prove that $\mathcal{C}$ has a finite subcover.

Corollary 6.5. *There exists an ordinal α such that for every ordinal $\gamma \geqslant \alpha$ it holds that $F^\gamma \subseteq \bigcup_{\beta < \gamma} F^\beta$.*

Proof. Suppose the contrary, i.e., that for every ordinal α there exists an ordinal $\alpha' \geqslant \alpha$ such that $F^{\alpha'} \nsubseteq \bigcup_{\beta < \alpha'} F^\beta$. Then for each ordinal α we define an ordinal α^* as follows. First, we let $0^* := 0'$. Then consider an ordinal $\alpha > 0$ and assume that γ^* has been defined for each $\gamma < \alpha$. We let

$$\alpha^* := (\sup(\{\alpha\} \cup \{\gamma^* : \gamma < \alpha\}) + 1)'.$$

It is easy to see that for every pair of ordinals $\alpha < \beta$ we have $\alpha^* < \beta^*$ and that for each ordinal α,

$$\alpha \leqslant \alpha^* \quad \text{and} \quad F^{\alpha^*} \nsubseteq \bigcup_{\beta < \alpha^*} F^\beta. \tag{14}$$

In view of the right hand side of the above display, for every ordinal α there exists

$$x_\alpha \in F^{\alpha^*} \smallsetminus \bigcup_{\beta < \alpha^*} F^\beta.$$

We will prove that $x_\alpha \neq x_\beta$ for each pair of distinct ordinals α and β. Suppose that $\alpha \neq \beta$. By symmetry we may assume $\alpha < \beta$. As we mentioned, this implies $\alpha^* < \beta^*$. As $x_\alpha \in F^{\alpha^*}$, we obtain $x_\alpha \notin F^{\beta^*} \smallsetminus \bigcup_{\gamma < \beta^*} F^\beta$. Since $x_\beta \in F^{\beta^*} \smallsetminus \bigcup_{\gamma < \beta^*} F^\beta$, we conclude that $x_\alpha \neq x_\beta$ as desired.

Hence, $\{x_\alpha : \alpha \text{ is an ordinal}\}$ is a proper class. But this contradicts the assumption that X is a set containing each x_α. ⊠

We are now ready to show that $\mathcal{C}$ has a finite subcover. Suppose the contrary, with a view of contradiction, i.e., that

$$\text{there is no } \mathcal{U} \subseteq_\omega \mathcal{C} \text{ such that } X \subseteq \bigcup \mathcal{U}. \tag{15}$$

Recall from Corollary 6.5 that there exists an ordinal α such that

$$F^\gamma \subseteq \bigcup_{\beta<\gamma} F^\beta \text{ for each ordinal } \gamma \geqslant \alpha.$$

We will show that the set $F^{\alpha+1}$ is nonempty. Suppose the contrary, with a view to contradiction. Then the left hand side of condition (13) yields $X \subseteq \bigcup \mathcal{U}^{\alpha+1}$. Therefore, the finite family $\mathcal{U} := \mathcal{U}^{\alpha+1}$ contradicts condition (15). Hence, we conclude that $F^{\alpha+1} \neq \emptyset$.

Then there exists $y \in F^{\alpha+1}$. In view of the above display, there also exists $\beta \leqslant \alpha$ such that $y \in F^\beta$. As $\alpha+1$ is a successor ordinal, the last part of Proposition 6.4 implies $y \in F^{\alpha+1} = {\uparrow}F^\alpha \smallsetminus F^\alpha$. Therefore, there exists $x \in F^\alpha$ such that $x < y$. Since F^β is an antichain by Proposition 6.4 and $y \in F^\beta$, we obtain $x \notin F^\beta$. Together with $x \in F^\alpha$, this yields $\alpha \neq \beta$. Thus, from $\beta \leqslant \alpha$ it follows that $\beta < \alpha$. As $x \in F^\alpha$, $y \in F^\beta$, and $x < y$, this contradicts the right hand side of condition (13). Hence, we conclude that $\mathcal{C}$ has a finite subcover as desired. Therefore, in order to establish Theorem 6.1, it only remains to prove Proposition 6.4.

Proof of Proposition 6.4. As $\mathcal{C}$ covers X, for each $x \in X$ there exists $U_x \in \mathcal{C}$ such that $f_x(\mathsf{h}(X)) \in U_x$. Since $\mathcal{C} \subseteq \mathcal{S}_{\mathsf{h}(X)}$, we can apply the Main Lemma 5.1, obtaining $v_x \in X$ such that $v_x \leqslant x$ and $Y_x \subseteq_\omega {\uparrow}v_x \cap X_{>\mathsf{h}(x)}$ and $Z_x \subseteq_\omega X_{<\mathsf{h}(X)} \cap {\uparrow}v_x$ such that

$$\uparrow v_x \smallsetminus ({\uparrow}Y_x \cup {\downarrow}Z_x) \subseteq U_x. \tag{16}$$

Furthermore, $\mathsf{h}(v_x)$ is either zero or a successor ordinal. In addition, for each $x \in X$ there exists $\mathcal{V}_x \subseteq_\omega \mathcal{C}$ such that

$$\downarrow x \subseteq \bigcup \mathcal{V}_x \tag{17}$$

by Proposition 6.2. The objects v_x, U_x, Y_x, Z_x, and $\mathcal{V}_x$ will be used repeatedly in the proof, which proceeds by induction on α.

Base case. If $\alpha = 0$, we let $\mathcal{U}^0 := \emptyset$ and define F^0 as the singleton containing the root of X. Then $X \smallsetminus {\uparrow}F^0 = X \smallsetminus X = \emptyset \subseteq \bigcup \mathcal{U}^0$ and the other conditions in the statement of Proposition 6.4 are clearly satisfied.

Successor case. Consider a successor ordinal $\alpha+1$. By the inductive hypothesis there exist $\mathcal{U}^\alpha \subseteq_\omega \mathcal{C}$ and an antichain $F^\alpha \subseteq_\omega X$ satisfying condition (13). We let

$$\begin{aligned}
A^{\alpha+1} &:= \big\{x : x \in Y_y \cap {\uparrow}y \text{ for some } y \in F^\alpha \text{ and there are no } \beta \leqslant \alpha \text{ and } z \in F^\beta \text{ s.t. } x < z\big\};\\
F^{\alpha+1} &:= \min A^{\alpha+1};\\
\mathcal{U}^{\alpha+1} &:= \mathcal{U}^\alpha \cup \{U_x \colon x \in F^\alpha\} \cup \{U : \text{there exist } y \in F^\alpha \text{ and } x \in Z_y \text{ s.t. } U \in \mathcal{V}_x\}.
\end{aligned}$$

As F^α is finite and so is Y_y for each $y \in F^\alpha$, the set $A^{\alpha+1}$ is also finite. Consequently, $F^{\alpha+1}$ is a finite antichain. On the other hand, as $\mathcal{U}^\alpha$ and F^α are finite and so is Z_y for each $y \in F^\alpha$ as well as $\mathcal{V}_x$ for each $x \in Z_y$, the set $\mathcal{U}^{\alpha+1}$ is also finite. Furthermore, $\mathcal{U}^{\alpha+1} \subseteq \mathcal{C}$ because $\mathcal{U}^\alpha \subseteq \mathcal{C}$ by the inductive hypothesis and $\{U_x\} \cup \mathcal{V}_x \subseteq \mathcal{C}$ for each $x \in X$ by assumption. Hence, $\mathcal{U}^{\alpha+1} \subseteq_\omega \mathcal{C}$ and $F^{\alpha+1} \subseteq_\omega X$, where $F^{\alpha+1}$ is also an antichain. Therefore, it only remains to prove that $\mathcal{U}^{\alpha+1}$ and $F^{\alpha+1}$ satisfy condition (13) and the last part of Proposition 6.4.

Claim 6.6. *We have* $X \smallsetminus {\uparrow}F^{\alpha+1} \subseteq \bigcup \mathcal{U}^{\alpha+1}$.

Proof of the Claim. Let $x \in X \smallsetminus {\uparrow}F^{\alpha+1}$. By the inductive hypothesis we have $X \smallsetminus {\uparrow}F^{\alpha} \subseteq \bigcup \mathcal{U}^{\alpha}$. Therefore, if $x \notin {\uparrow}F^{\alpha}$, then $x \in X \smallsetminus {\uparrow}F^{\alpha} \subseteq \bigcup \mathcal{U}^{\alpha} \subseteq \bigcup \mathcal{U}^{\alpha+1}$, where the last inclusion follows from the assumption that $\mathcal{U}^{\alpha} \subseteq \mathcal{U}^{\alpha+1}$. Then we consider the case where $x \in {\uparrow}F^{\alpha}$. There exists $y \in F^{\alpha}$ such that $y \leqslant x$. We have two cases: either $x \in {\downarrow}Z_y$ or $x \notin {\downarrow}Z_y$. First, suppose that $x \in {\downarrow}Z_y$. Then there exists $z \in Z_y$ such that $x \leqslant z$. By condition (17) we have ${\downarrow}z \subseteq \bigcup \mathcal{V}_z$. Since $x \leqslant z$, this yields $x \in \bigcup \mathcal{V}_z$. On the other hand, from $z \in Z_y$ and $y \in F^{\alpha}$ it follows that $\mathcal{V}_z \subseteq \mathcal{U}^{\alpha+1}$. Hence, $x \in \bigcup \mathcal{U}^{\alpha+1}$ as desired. Then we consider the case where $x \notin {\downarrow}Z_y$. Again, we have two cases: either $x \notin {\uparrow}Y_y$ or $x \in {\uparrow}Y_y$. First, suppose that $x \notin {\uparrow}Y_y$. Together with $v_y \leqslant y \leqslant x$ and $x \notin {\downarrow}Z_y$, this yields $x \in {\uparrow}v_y \smallsetminus ({\uparrow}Y_y \cup {\downarrow}Z_y)$. By condition (16) this implies $x \in U_y$. Since $y \in F^{\alpha}$, we have $U_y \in \mathcal{U}^{\alpha+1}$ and, therefore, $x \in \bigcup \mathcal{U}^{\alpha+1}$ as desired. It only remains to consider the case where $x \in {\uparrow}Y_y$. We will show that this cases never happens, in the sense that it leads to a contradiction. First, as $x \in {\uparrow}Y_y$, there exists $z \in Y_y$ such that $z \leqslant x$. We will prove that $z \in A^{\alpha+1}$. Since X is a tree and $y, z \leqslant x$, the elements y and z must be comparable. As $Y_y \subseteq X_{>\mathsf{h}(y)}$ and $z \in Y_y$, we deduce $y < z$. Therefore, $z \in Y_y \cap {\uparrow}y$ and $y \in F^{\alpha}$. Consequently, to prove that $z \in A^{\alpha+1}$, it only remains to show that there are no $\beta \leqslant \alpha$ and $w \in F^{\beta}$ such that $z < w$. Suppose, on the contrary, that there exist such β and w. From $y < z < w$ it follows that $y < w$. Recall that $\beta \leqslant \alpha$. Then either $\beta < \alpha$ or $\beta = \alpha$. The case where $\beta < \alpha$ cannot happen because F^{α} satisfies the right hand side of condition (13) and $y \in F^{\alpha}, w \in F^{\beta}$, and $y < w$. Therefore, we obtain $\alpha = \beta$. As a consequence, $y, w \in F^{\alpha}$ because $y \in F^{\alpha}$ and $w \in F^{\beta}$. Together with $y < w$, this contradicts the assumption that F^{α} is an antichain. Hence, we conclude that $z \in A^{\alpha+1}$. Since the set $A^{\alpha+1}$ is finite and $F^{\alpha+1} = \min A^{\alpha+1}$, this yields $z \in {\uparrow}F^{\alpha+1}$. As $z \leqslant x$, we obtain $x \in {\uparrow}F^{\alpha+1}$, a contradiction with the assumption that $x \in X \smallsetminus {\uparrow}F^{\alpha+1}$. ⊠

By the Claim 6.6 the set $F^{\alpha+1}$ satisfies the left hand side of condition (13). The right hand side of the same conditions holds by the definition of $F^{\alpha+1}$. Therefore, it only remains to prove the last part of Proposition 6.4, namely, that $F^{\alpha+1} \subseteq {\uparrow}F^{\alpha} \smallsetminus F^{\alpha}$. To this end, consider $x \in F^{\alpha+1}$. By the definition of $F^{\alpha+1}$ we have $x \in Y_y \cap {\uparrow}y$ for some $y \in F^{\alpha}$. Therefore, $x \in {\uparrow}F^{\alpha}$. It only remains to prove that $x \notin F^{\alpha}$. From $x \in Y_y \subseteq X_{>\mathsf{h}(y)}$ and $x \geqslant y$ it follows that $x < y$. As F^{α} is an antichain containing y, this implies $x \notin F^{\alpha}$.

Limit case. For each nonempty $Y \subseteq X$ let

$$\sup{}^{*} Y \coloneqq \{x \in X : x \text{ is the supremum of a maximal chain } Z \subseteq Y\}.$$

Suppose that α is a limit ordinal. By the inductive hypothesis for each $\beta < \alpha$ there exist $\mathcal{U}^{\beta} \subseteq_{\omega} \mathcal{C}$ and an antichain $F^{\beta} \subseteq_{\omega} X$ satisfying condition (13). We let

$$F \coloneqq \bigcup_{\beta<\alpha} F^{\beta} \cup \bigcup_{\beta<\alpha} \left(X \smallsetminus {\uparrow}F^{\beta}\right) \quad \text{and} \quad F^{*} \coloneqq {\downarrow}(\sup{}^{*} F)\,.$$

Notice that F is nonempty because it contains the root of X (the latter belongs to F^0 by construction and $F^0 \subseteq F$). Therefore, every maximal chain in F is nonempty and, therefore, has a supremum in X by assumption.

The proof relies on a series of technical observation.

Claim 6.7. *The sets F and F^{*} are nonempty downsets of X.*

Proof of the Claim. We begin by proving that F is a nonempty downset of X. First, F is nonempty because $0 < \alpha$ and $F^0 \subseteq F$ is the singleton containing the root of X. To prove that F is a downset, for every ordinal $\beta < \alpha$ let

$$G^{\beta} \coloneqq F^{\beta} \cup \left(X \smallsetminus {\uparrow}F^{\beta}\right).$$

We show that each G^β is a downset. Consider $x \in G^\beta$ and $y < x$. We need to prove that $y \in G^\beta$. There are two cases: either $x \in F^\beta$ or $x \in X \smallsetminus {\uparrow}F^\beta$. Suppose that $x \in F^\beta$. Then y cannot belong to ${\uparrow}F^\beta$, otherwise there exists $z \in F^\beta$ such that $z \leqslant y < x$. Since $x, z \in F^\beta$, this contradicts the assumption that F^β is an antichain. Hence, $y \in X \smallsetminus {\uparrow}F^\beta \subseteq G^\beta$ as desired. Then we consider the case where $x \in X \smallsetminus {\uparrow}F^\beta$. Since $X \smallsetminus {\uparrow}F^\beta$ is a downset and $y \leqslant x$, we obtain $y \in X \smallsetminus {\uparrow}F^\beta \subseteq G^\beta$ too. Hence, each G^β is a downset. As $F = \bigcup_{\beta<\alpha} G^\beta$, we conclude that F is a downset. Lastly, F^* is a nonempty downset by definition. ⊠

Claim 6.8. *The poset $\langle F^*; \leqslant\rangle$ is order compact and each of its nonempty chains has a supremum in F^*.*

Proof of the Claim. Recall that F^* is a nonempty downset of X by Claim 6.7. Together with the assumption that X is a tree with enough gaps, this yields that F^* is also a tree with enough gaps. We will show that each of its nonempty chains has a supremum in F^*. Together with Theorem 4.1, this implies that X is representable and, therefore, order compact by Proposition 2.5. As such, in order to conclude the proof, it suffices to show that in F^* each nonempty chain has a supremum.

All suprema in the rest of the proof will be computed in X, unless said otherwise. Consider a nonempty chain $C = \{x_i : i \in I\}$ in F^*. We will prove that C has a supremum in the poset $\langle F^*; \leqslant\rangle$. We begin by showing that

$$x_i = \sup\left(F \cap {\downarrow}x_i\right) \quad \text{for each } i \in I. \tag{18}$$

Consider $i \in I$. If $x_i \in F$, clearly $x_i = \sup(F \cap {\downarrow}x_i)$ and we are done. Then consider the case where $x_i \notin F$. Since $x_i \in F^* = {\downarrow}(\sup^* F)$, there exists $y \in \sup^* F$ such that $x_i \leqslant y$. Furthermore, $y = \sup(F \cap {\downarrow}y)$ because $y \in \sup^* F$. We have two cases: either $x_i = y$ or $x_i \neq y$. If $x_i = y$, we have $x_i = y = \sup(F \cap {\downarrow}y) = \sup(F \cap {\downarrow}x_i)$ and we are done. Then we consider the case where $x_i \neq y$. As $x_i \leqslant y$, we have $x_i < y$. Since $y = \sup(F \cap {\downarrow}y)$, this guarantees the existence of $z \in F$ such that $z \leqslant y$ and $z \nleqslant x_i$. As X is a tree and $x_i, z \leqslant y$, the elements x_i and z must be comparable. Together with $z \nleqslant x_i$, this yields $x_i \leqslant z$. Since $z \in F$ and F is a downset by Claim 6.7, we obtain $x_i \in F$, a contradiction. This establishes condition (18).

Then consider the set

$$D := \bigcup_{i \in I} \left(F \cap {\downarrow}x_i\right).$$

We will prove that D is a nonempty chain. First, recall that the root of X belongs to F^0 and, therefore, to F by construction. Thus, D is nonempty. Then consider $y, z \in D$. By the definition of D there exist $i, j \in I$ such that $y \leqslant x_i$ and $z \leqslant x_j$. Since C is a chain, by symmetry we may assume that $x_i \leqslant x_j$. Therefore, $y, z \leqslant x_j$. As X is a tree, we conclude that y and z are comparable. Hence, D is a nonempty chain as desired. Consequently, $\sup D$ exists by the assumptions on X. Together with the definitions of C and D and with condition (18), this yields that also $\sup C$ exists and coincides with $\sup D$. Furthermore, the definition of D ensures that $D \subseteq F$. Since D can be extended to a maximal chain in F by Zorn's Lemma, we obtain $\sup C = \sup D \in {\downarrow}(\sup^* F) = F^*$. Thus, the supremum of C computed in X exists and belongs to F^*. Clearly, this coincides with the supremum of C computed in F^*. Therefore, we conclude that C has a supremum also in the poset $\langle F^*; \leqslant\rangle$ as desired. ⊠

Recall that for each $x \in X$ we have $v_x \leqslant x$ and that $\mathsf{h}(x)$ is either zero or a successor ordinal.

Claim 6.9. *For every $x \in \sup^* F$ there exist an ordinal $\gamma_x < \alpha$, an element $y_x \in X$, an order open subset V_x of $\langle F^*; \leqslant\rangle$, and $\mathcal{W}_x \subseteq_\omega \mathcal{C}$ satisfying the following conditions:*

(i) *$v_x \leqslant y_x \leqslant x$;*
(ii) *V_x is disjoint both from ${\uparrow}(F^{\gamma_x} \smallsetminus {\uparrow}y_x)$ and ${\uparrow}Y_x$;*
(iii) *$x \in V_x \subseteq U_x \cup \bigcup \mathcal{W}_x \cup \bigcup \mathcal{U}^{\gamma_x}$.*

Proof of the Claim. Consider $x \in \sup^* F$. We will prove that there exist

$$\gamma_x < \alpha \text{ and } y_x \in F^{\gamma_x} \cup (X \smallsetminus {\uparrow} F^{\gamma_x}) \text{ such that } v_x \leqslant y_x \leqslant x. \tag{19}$$

This will establish condition (i). Recall that x is the supremum of a maximal chain of $\langle F; \leqslant \rangle$ because $x \in \sup^* F$. We have two cases: either $\mathsf{h}(x)$ is a limit ordinal or not. First, suppose that $\mathsf{h}(x)$ is not a limit ordinal. Since x is the supremum of a nonempty chain of $\langle F; \leqslant \rangle$, this yields $x \in F$. Consequently, there exists $\gamma_x < \alpha$ such that $x \in F^{\gamma_x} \cup (X \smallsetminus {\uparrow} F^{\gamma_x})$. Therefore, letting $y_x \coloneqq x$, we are done. Then we consider the case where $\mathsf{h}(x)$ is a limit ordinal. As $\mathsf{h}(v_x)$ is either zero or a successor ordinal and $v_x \leqslant x$, this yields $v_x < x$. Since x is the supremum of chain of $\langle F; \leqslant \rangle$, there exist $\gamma_x < \alpha$ and $y_x \in F^{\gamma_x} \cup (X \smallsetminus {\uparrow} F^{\gamma_x})$ such that $y_x \leqslant x$ and $y_x \nleqslant v_x$. As X is a tree and $y_x, v_x \leqslant x$, we deduce that either $y_x \leqslant v_x$ or $v_x \leqslant y_x$. Together with $y_x \nleqslant v_x$, this yields $v_x \leqslant y_x$ and establishes the above display.

Let

$$V'_x \coloneqq {\uparrow}(F^{\gamma_x} \smallsetminus {\uparrow} y_x) \cup {\uparrow} Y_x \cup {\downarrow}(Z_x \smallsetminus {\uparrow} x) \text{ and } V_x \coloneqq F^* \smallsetminus V'_x.$$

Notice that V_x satisfies condition (ii) by definition. We will prove that V_x is an order open set of the poset $\langle F^*; \leqslant \rangle$. First, observe that $V_x \subseteq F^*$ by the definition of V_x. Then consider the sets

$$A \coloneqq F^* \cap ((F^{\gamma_x} \smallsetminus {\uparrow} y_x) \cup Y_x) \text{ and } B \coloneqq \max(F^* \cap {\downarrow}(Z_x \smallsetminus {\uparrow} z)).$$

We shall see that $A, B \subseteq_\omega F^*$. Since F^{γ_x} and Y_x are fine, we obtain $A \subseteq_\omega F^*$. On the other hand, as X is a tree and Z_x finite, the set ${\downarrow}(Z_x \smallsetminus {\uparrow} z)$ is the union of n chains for some nonnegative integer n. Consequently, $|B| \leqslant n$ and, therefore, $B \subseteq_\omega F^*$ as desired. From $A, B \subseteq_\omega F^*$ and Lemma 2.7 it follows that $F^* \smallsetminus ({\uparrow} A \cup {\downarrow} B)$ is an order open set of $\langle F^*; \leqslant \rangle$.

To prove that V_x is also an order open set of $\langle F^*; \leqslant \rangle$, we rely on the equalities

$$F^* \cap ({\uparrow}(F^{\gamma_x} \smallsetminus {\uparrow} y_x) \cup {\uparrow} Y_x) = F^* \cap {\uparrow} A \text{ and } F^* \cap {\downarrow}(Z_x \smallsetminus {\uparrow} x) = F^* \cap {\downarrow} B. \tag{20}$$

First, observe that

$$\begin{aligned} F^* \cap ({\uparrow}(F^{\gamma_x} \smallsetminus {\uparrow} y_x) \cup {\uparrow} Y_x) &= F^* \cap {\uparrow}((F^{\gamma_x} \smallsetminus {\uparrow} y_x) \cup Y_x) \\ &= F^* \cap {\uparrow}(F^* \cap ((F^{\gamma_x} \smallsetminus {\uparrow} y_x) \cup Y_x)) \\ &= F^* \cap {\uparrow} A, \end{aligned}$$

where the first equality is straightforward, the second holds because F^* is a downset of X, and the third holds by the definition of A. This establishes the left hand side of condition (20). Then we turn to prove the right hand side of the same condition. The inclusion from right to left is an immediate consequence of the definition of B. To prove the other inclusion, consider $z \in F^* \cap {\downarrow}(Z_x \smallsetminus {\uparrow} x)$. By Zorn's lemma there exists a maximal chain $C \subseteq F^* \cap {\downarrow}(Z_x \smallsetminus {\uparrow} x)$ such that $z \in C$. Since C is a nonempty chain of F^*, it has a supremum $\sup C$ in $\langle F^*; \leqslant \rangle$ by Claim 6.8. We will prove that $\sup C \in F^* \cap {\downarrow}(Z_x \smallsetminus {\uparrow} x)$. Since $\sup C \in F^*$, it suffices to show that $\sup C \in F^* \in {\downarrow}(Z_x \smallsetminus {\uparrow} x)$. Recall that Z_x is finite. Therefore, so is $Z_x \smallsetminus {\uparrow} x$. Furthermore, $Z_x \smallsetminus {\uparrow} x$ is nonempty because $z \in {\downarrow}(Z_x \smallsetminus {\uparrow} x)$. Then consider an enumeration $Z_x \smallsetminus {\uparrow} x = \{z_1, \dots, z_n\}$. We will show that $C \subseteq {\downarrow} z_i$ for some $i \leqslant n$. Suppose the contrary, with a view to contradiction. Then for each $i \leqslant n$ there exists $c_i \in C$ such that $c_i \nleqslant z_i$. As C is a chain, the set $\{c_1, \dots, c_n\}$ has a maximum c. Clearly, we have $c \nleqslant z_1, \dots, z_n$, a contradiction with the assumption that $C \subseteq {\downarrow}(Z_x \smallsetminus {\uparrow} x)$. Hence, there exists $i \leqslant n$ such that $C \subseteq {\downarrow} z_i$. Consequently, $\sup C \leqslant z_i$. Since $z_i \in Z_x \smallsetminus {\uparrow} x$, we obtain $\sup C \in {\downarrow}(Z_x \smallsetminus {\uparrow} x)$ as desired. From $\sup C \in (F^* \cap {\downarrow}(Z_x \smallsetminus {\uparrow} x))$ and the maximality of the chain C it follows that $\sup C \in \max(F^* \cap {\downarrow}(Z_x \smallsetminus {\uparrow} x)) = B$. Together with $z \in C$, this yields $z \in {\downarrow} B$. As $z \in F^*$, we conclude that $z \in F^* \cap {\downarrow} B$, establishing condition (20).

Lastly, observe that

$$
\begin{aligned}
V_x &= F^* \smallsetminus (\uparrow (F^{\gamma_x} \smallsetminus \uparrow y_x) \cup \uparrow Y_x \cup \downarrow (Z_x \smallsetminus \uparrow x)) \\
&= F^* \smallsetminus ((F^* \cap \uparrow ((F^{\gamma_x} \smallsetminus \uparrow y_x) \cup Y_x)) \cup (F^* \cap \downarrow (Z_x \smallsetminus \uparrow x))) \\
&= F^* \smallsetminus ((F^* \cap \uparrow A) \cup (F^* \cap \downarrow B)) \\
&= F^* \smallsetminus (\uparrow A \cup \downarrow B),
\end{aligned}
$$

where the first equality holds by the definition of V_x, the second and the last are straightforward, and the third holds by condition (20). Therefore, since $F^* \smallsetminus (\uparrow A \cup \downarrow B)$ is an order open set of $\langle F^*; \leqslant \rangle$, we conclude that so is V_x.

Therefore, it only remains to construct $\mathcal{W}_x \subseteq_\omega \mathcal{C}$ so that condition (iii) holds. Let

$$\mathcal{W}_x \coloneqq \{U : U \in \mathcal{V}_z \text{ for some } z \in Z_x\}.$$

Since Z_x is finite and $\mathcal{V}_z \subseteq_\omega \mathcal{C}$ for each $z \in Z_x$, we obtain $\mathcal{W}_x \subseteq_\omega \mathcal{C}$. Then we turn to prove condition (iii).

We begin by showing that $x \in V_x$. Suppose the contrary, with a view to contradiction. Since $x \in \sup^* F \subseteq F^*$ by assumption, we obtain $x \in F^* \smallsetminus V_x \subseteq V'_x$. From the definition of V'_x it follows that

$$\text{either } \; x \in \uparrow (F^{\gamma_x} \smallsetminus \uparrow y_x) \; \text{ or } \; x \in \uparrow Y_x \; \text{ or } \; x \in \downarrow (Z_x \smallsetminus \uparrow x).$$

First, suppose $x \in \uparrow (F^{\gamma_x} \smallsetminus \uparrow y_x)$. Then there exists $z \in F^{\gamma_x} \smallsetminus \uparrow y_x$ such that $z \leqslant x$. Since X is a tree and $y_x, z \leqslant x$ (for $y_x \leqslant x$, see condition (19)), we deduce that either $z \leqslant y_x$ or $y_x \leqslant z$. As $z \in F^{\gamma_x} \smallsetminus \uparrow y_x$, this amounts to $z < y_x$. In view of condition (19), either $y_x \in F^{\gamma_x}$ or $y_x \in X \smallsetminus \uparrow F^{\gamma_x}$. We will show that both cases lead to a contradiction. If $y_x \in F^{\gamma_x}$, we have $y_x, z \in F^{\gamma_x}$. Together with $z < y_x$, this contradicts the assumption that F^{γ_x} is an antichain. On the other hand, if $y_x \in X \smallsetminus \uparrow F^{\gamma_x}$, we obtain a contradiction with $z < y_x$ and $z \in F^{\gamma_x}$. Lastly, the case where $x \in \uparrow Y_x$ leads to a contradiction because $Y_x \subseteq X_{>\mathsf{h}(x)}$, and the case $x \in \downarrow (Z_x \smallsetminus \uparrow x)$ is obviously impossible. Hence, we conclude that $x \in V_x$.

Therefore, to conclude the proof, it only remains to show that

$$V_x \subseteq U_x \cup \bigcup \mathcal{W}_x \cup \bigcup \mathcal{U}^{\gamma_x}.$$

Consider $y \in V_x$. There are two cases: either $y \in \downarrow Z_x$ or $y \notin \downarrow Z_x$. First, suppose that $y \in \downarrow Z_x$. Then there exists $z \in Z_x$ such that $y \leqslant z$. Therefore, $\mathcal{V}_z \subseteq \mathcal{W}_x$ by the definition of $\mathcal{W}_x$. From condition (17) and $y \leqslant z$ it follows that $y \in \downarrow z \subseteq \bigcup \mathcal{V}_z \subseteq \bigcup \mathcal{W}_x$ as desired. Then we consider the case where $y \notin \downarrow Z_x$. Again, we have two cases: either $y \notin \uparrow F^{\gamma_x}$ or $y \in \uparrow F^{\gamma_x}$. If $y \notin \uparrow F^{\gamma_x}$, we have $y \in X \smallsetminus \uparrow F^{\gamma_x}$. Therefore, the fact that $\mathcal{U}^{\gamma_x}$ and F^{γ_x} satisfy condition (13) ensures that $y \in \bigcup \mathcal{U}^{\gamma_x}$ and we are done. Lastly, we consider the case where $y \in \uparrow F^{\gamma_x}$. Since $y \in V_x$ by assumption and $V_x \subseteq (\uparrow Y_x)^c \cap (\uparrow (F^{\gamma_x} \smallsetminus \uparrow y_x))^c$ by the definition of V_x, we have $y \notin \uparrow Y_x$ and $y \notin \uparrow (F^{\gamma_x} \smallsetminus \uparrow y_x)$. Together with $y \in \uparrow F^{\gamma_x}$, the latter yields $y \in \uparrow y_x$. Therefore, $y \in \uparrow y_x$, $y \notin \uparrow Y_x$, and $y \notin \downarrow Z_x$. Since $v_x \leqslant y_x$ by condition (19), this yields $y \in \uparrow v_x \smallsetminus (\uparrow Y_x \cup \downarrow Z_x)$. By condition (16) we conclude that $y \in U_x$ as desired. ⊠

Recall that Claim 6.9 associates a set V_x with every $x \in \sup^* F$. Using these sets, we obtain the following:

Claim 6.10. *There exist $G \subseteq_\omega \sup^* F$ and $\Gamma \subseteq_\omega \alpha$ such that*

$$F^* \subseteq \bigcup_{x \in G} V_x \cup \bigcup_{\beta \in \Gamma} \left(X \smallsetminus \uparrow F^\beta \right).$$

Proof of the Claim. First, we show that

$$F^* \subseteq \bigcup_{x \in \sup^* F} V_x \cup \bigcup_{\beta < \alpha} \left(X \smallsetminus \uparrow F^\beta \right). \tag{21}$$

To prove this, consider $y \in F^* = \downarrow(\sup^* F)$. Then there exists $x \in \sup^* F$ such that $y \leqslant x$. If $y = x$, Claim 6.9(iii) ensures $y \in V_y$ and we are done. Then we consider the case where $y < x$. Since x is the supremum of a maximal chain of $\langle F; \leqslant \rangle$, there exist $\beta < \alpha$ and $z \in F^\beta \cup \left(X \smallsetminus \uparrow F^\beta \right)$ such that $z \leqslant x$ and $z \nleqslant y$. As X is a tree and $y, z \leqslant x$, the elements y and z must be comparable. Together with $z \nleqslant y$, this yields $y < z$. We will prove that $y \notin \uparrow F^\beta$. Recall that $z \in F^\beta \cup \left(X \smallsetminus \uparrow F^\beta \right)$. We will consider the cases where $z \in F^\beta$ and $z \in X \smallsetminus \uparrow F^\beta$ separately. First, suppose that $z \in F^\beta$. Since F^β is an antichain containing z and $y < z$, we obtain $y \notin \uparrow F^\beta$ as desired. On the other hand, if $z \in X \smallsetminus \uparrow F^\beta$, then $y \notin \uparrow F^\beta$ because $y < z$. This concludes the proof that $y \notin \uparrow F^\beta$. Therefore, $y \in X \smallsetminus \uparrow F^\beta$ with $\beta < \alpha$, establishing the above display.

Now, observe that the following are order open sets of $\langle F^*; \leqslant \rangle$:

(i) V_x for each $x \in \sup^* F$;
(ii) $F^* \cap \left(X \smallsetminus \uparrow F^\beta \right)$ for each $\beta < \alpha$.

The sets in condition (i) are order open by Claim 6.9. To prove that the sets in condition (ii) are also order open, consider $\beta < \alpha$. Since F^* is a downset of X, we have $F^* \cap \left(X \smallsetminus \uparrow F^\beta \right) = \left(\uparrow \left(F^\beta \cap F^* \right) \right)^c$, where upsets and complements are computed in F^*. Therefore, it suffices to show that $\left(\uparrow \left(F^\beta \cap F^* \right) \right)^c$ is an order open set of $\langle F^*; \leqslant \rangle$. The latter follows from Lemma 2.7 and the fact that $F^\beta \cap F^*$ is finite (because so is F^β).

Since the sets in conditions (i) and (ii) are order open sets of $\langle F^*; \leqslant \rangle$ and this poset is order compact by Claim 6.8, from condition (21) it follows that there exist $G \subseteq_\omega \sup^* F$ and $\Gamma \subseteq_\omega \alpha$ satisfying the statement of the claim. ⊠

Using the sets G and Γ given by Claim 6.10 and the sets $\mathcal{W}_x$ and the ordinals γ_x given by Claim 6.9, we let

$$\begin{aligned}
A^\alpha &\coloneqq \{x : x \in Y_y \text{ for some } y \in G \text{ and there are no } \beta < \alpha \text{ and } z \in F^\beta \text{ s.t. } x < z\};\\
F^\alpha &\coloneqq \min A^\alpha;\\
\mathcal{U}^\alpha &\coloneqq \{U : \text{there are } y \in G \text{ and } x \in Z_y \text{ s.t. } U \in \mathcal{V}_x\} \cup \{U : U \in \mathcal{W}_x \text{ for some } x \in G\} \cup \\
&\qquad \{U : U \in \mathcal{U}^{\gamma_x} \text{ for some } x \in G\} \cup \left\{U : U \in \mathcal{U}^\beta \text{ for some } \beta \in \Gamma\right\} \cup \{U_x \colon x \in G\}.
\end{aligned}$$

Since G is finite by Claim 6.10 and so is Y_y for each $y \in G$, the set A^α is also finite. Consequently, F^α is a finite antichain. Moreover, $\mathcal{U}^\alpha$ is finite because so are the sets of the form Z_x, $\mathcal{V}_x$, $\mathcal{W}_x$, and $\mathcal{U}^\beta$ for each $x \in X$ and $\beta < \alpha$ (for the case of $\mathcal{W}_x$, see Claim 6.9) as well as the sets G and Γ by Claim 6.10. Furthermore, $\mathcal{U}^\alpha \subseteq \mathcal{C}$ because $\mathcal{V}_x, \mathcal{W}_x, \mathcal{U}^\beta, \{U_x\} \subseteq \mathcal{C}$ for each $x \in X$ and $\beta < \alpha$ (for the case of $\mathcal{W}_x$, see Claim 6.9). Hence, $\mathcal{U}^\alpha \subseteq_\omega \mathcal{C}$ and $F^\alpha \subseteq_\omega X$, where F^α is also an antichain.

Observe that the last part of Proposition 6.4 holds vacuously because α is a limit ordinal. Therefore, it only remains to prove condition (13). The right hand side of this condition holds by the definition of A^α and the fact that $F^\alpha \subseteq A^\alpha$. Therefore, we turn to prove the left hand side of condition (13), that is,

$$X \smallsetminus \uparrow F^\alpha \subseteq \bigcup \mathcal{U}^\alpha. \tag{22}$$

Consider $x \in X \smallsetminus \uparrow F^\alpha$. We have two cases: either $x \in F^*$ or $x \notin F^*$. First, suppose that $x \in F^*$. By Claim 6.10

$$\text{either } x \in V_y \text{ for some } y \in G \;\text{ or }\; x \in X \smallsetminus \uparrow F^\beta \text{ for some } \beta \in \Gamma.$$

We begin with the case where $x \in V_y$ for some $y \in G$. Since $G \subseteq \sup^* F$ by Claim 6.10, we obtain $y \in \sup^* F$. Hence, we can apply Claim 6.9(iii) and the assumption that $x \in V_y$, obtaining

$$x \in V_y \subseteq U_y \cup \bigcup \mathcal{W}_y \cup \bigcup \mathcal{U}^{\gamma_y}.$$

As $y \in G$, the definition of $\mathcal{U}^\alpha$ guarantees that the right hand side of the above display is included in $\bigcup \mathcal{U}^\alpha$. Hence, $x \in \bigcup \mathcal{U}^\alpha$ as desired. Then we consider the case where $x \in X \smallsetminus {\uparrow} F^\beta$ for some $\beta \in \Gamma$. Since $\beta \in \Gamma \subseteq \alpha$, we have $\beta < \alpha$. Therefore, β satisfies condition (13). Consequently, from $x \in X \smallsetminus {\uparrow} F^\beta$ it follows that $x \in \bigcup \mathcal{U}^\beta$. As $\beta \in \Gamma$, the definition of $\mathcal{U}^\alpha$ guarantees that $\mathcal{U}^\beta \subseteq \mathcal{U}^\alpha$. Consequently, $\bigcup \mathcal{U}^\beta \subseteq \bigcup \mathcal{U}^\alpha$. Since $x \in \bigcup \mathcal{U}^\beta$, we obtain $x \in \bigcup \mathcal{U}^\alpha$ as desired. This concludes the analysis of the case where $x \in F^*$.

Therefore, we may assume that $x \in X \smallsetminus F^*$. For future reference, it is useful to state the following consequences of this assumption:

$$x \in {\uparrow} F^\beta \text{ for every } \beta < \alpha \ \text{ and } \ x \notin \sup^* F. \tag{23}$$

Claim 6.11. *There exist $y^* \in \sup^* F$ and $z^* \in G$ such that $y^* \leqslant x$ and $y^* \in V_{z^*}$.*

Proof of the Claim. The left hand side of condition (23) guarantees that for each $\beta < \alpha$ there exists $y_\beta \in F^\beta$ such that $y_\beta \leqslant x$. Since X is a tree and α a limit ordinal, the set $C \coloneqq \{y_\beta : \beta < \alpha\}$ is a nonempty chain in F. Since C is a chain and X a tree, the set

$$C^* \coloneqq F \cap {\downarrow} C$$

is also a chain in F. Furthermore, from the definition of y^* and C^* it follows that $y^* = \sup C^*$. Therefore, in order to prove that $y^* \in \sup^* F$, it suffices to show that the chain C^* is maximal in F. Suppose the contrary, with a view to contradiction. Then there exists $w \in F \smallsetminus C^*$ such that $C^* \cup \{w\}$ is still a chain. By the definition of F there exists $\beta < \alpha$ such that either $w \in F^\beta$ or $w \in X \smallsetminus {\uparrow} \left(F^\beta\right)$. First, suppose that $w \in F^\beta$. Since w and y_β are distinct elements of $C^* \cup \{w\}$, we obtain that either $w < y_\beta$ or $y_\beta < w$. Together with $w, y_\beta \in F^\beta$, this contradicts the assumption that F^β is an antichain. Then we consider the case where $w \in X \smallsetminus {\uparrow} F^\beta$. Since $y_\beta \in F^\beta$ and w and y_β are two elements of the chain $C^* \cup \{w\}$, this implies $w < y_\beta$. As $y_\beta \in C$ and $w \in F$, we conclude that $w \in F \cap {\downarrow} C = C^*$, a contradiction. This establishes that the chain C^* is maximal in F and, therefore, $y^* \in \sup^* F$.

It only remains to prove that $y^* \in V_{z^*}$ for some $z^* \in G$. To this end, observe that from $y^* \in \sup^* F$ and Claim 6.10 it follows that

$$y^* \in F^* \subseteq \bigcup_{z \in G} V_z \cup \bigcup_{\beta \in \Gamma} \left(X \smallsetminus {\uparrow} F^\beta\right).$$

Since $y_\beta \in F^\beta$ and $y_\beta \leqslant y^*$ for every $\beta < \alpha$ by construction and $\Gamma \subseteq \alpha$, this yields $y^* \in \bigcup_{z \in G} V_z$. Therefore, there exists $z^* \in G$ such that $y^* \in V_{z^*}$. ⊠

Now, let $y^* \in \sup^* F$ and $z^* \in G \subseteq \sup^* F$ be the elements given by Claim 6.11. Furthermore, let $y_{z^*} \in X$ be the element given by Claim 6.9. Lastly, recall that v_{z^*} is the element associated with z^* at the beginning of the proof of Proposition 6.4.

Claim 6.12. *We have that $v_{z^*} \leqslant x$.*

Proof of the Claim. By Claim 6.11 we have $y^* \in V_{z^*}$. Together with Claim 6.9(ii), this yields $y^* \notin {\uparrow}(F^{\gamma_{z^*}} \smallsetminus {\uparrow} y_{z^*})$. On the other hand, $x \in {\uparrow} F^{\gamma_{z^*}}$ by the left hand side of condition (23). Therefore, there exists $w \in F^{\gamma_{z^*}}$ such that $w \leqslant x$. Moreover, $y^* \leqslant x$ by Claim 6.11. Since X is a tree, from $y^*, w \leqslant x$ it follows that y^* and w are comparable. Since $y^* \in \sup^* F$, the element y^* is the supremum of a maximal chain C in F. From the maximality of C and the assumption that $w \in F^{\gamma_{z^*}} \subseteq F$, it follows that $y^* < w$ is impossible (otherwise $C \cup \{w\}$ would be a chain in F larger than C). Therefore, we conclude that $w \leqslant y^*$. Together with $w \in F^{\gamma_{z^*}}$ and $y^* \notin {\uparrow}(F^{\gamma_{z^*}} \smallsetminus {\uparrow}y_{z^*})$, this yields $y_{z^*} \leqslant w$. As $w \leqslant x$, we obtain $y_{z^*} \leqslant x$. Lastly, by Claim 6.9(i) we have $v_{z^*} \leqslant y_{z^*}$ and, therefore, $v_{z^*} \leqslant x$ as desired. ⊠

We are now ready to conclude the proof, i.e., we establish the left hand side of condition (22) for $x \in X \smallsetminus {\uparrow}F^\alpha$, $x \in X \smallsetminus F^*$. We have two cases: either $x \in {\downarrow}Z_{z^*}$ or $x \notin {\downarrow}Z_{z^*}$. First, suppose that $x \in {\downarrow}Z_{z^*}$. Then there exists $w \in Z_{z^*}$ such that $x \leqslant w$. By condition (17) we have $x \in {\downarrow}w \subseteq \bigcup \mathcal{V}_w$. Since $w \in Z_{z^*}$ and $z^* \in G$ by Claim 6.11, we obtain $\mathcal{V}_w \subseteq \mathcal{U}^\alpha$ and, therefore, $x \in \bigcup \mathcal{U}^\alpha$ as desired. Then we consider the case where $x \notin {\downarrow}Z_{z^*}$. Again, we have two cases: either $x \notin {\uparrow}Y_{z^*}$ or $x \in {\uparrow}Y_{z^*}$. First, suppose that $x \notin {\uparrow}Y_{z^*}$. Together with Claim 6.12, this yields $x \in {\uparrow}v_{z^*} \smallsetminus ({\uparrow}Y_{z^*} \cup {\downarrow}Z_{z^*})$. By condition (16) this implies $x \in U_{z^*}$. As $z^* \in G$ by Claim 6.11, the definition of $\mathcal{U}^\alpha$ guarantees that $U_{z^*} \in \mathcal{U}^\alpha$. Consequently, $x \in U_{z^*} \subseteq \bigcup \mathcal{U}^\alpha$ as desired.

Lastly, we consider the case where $x \in {\uparrow}Y_{z^*}$. We will show that this case never happens, i.e., that it leads to a contradiction. First, there exists $w \in Y_{z^*}$ such that $w \leqslant x$. We will prove that $w \in A^\alpha$. Observe that $w \in Y_{z^*}$ and $z^* \in G$ by Claim 6.11. Consequently, to prove that $w \in A^\alpha$, it only remains to show that there are no $\beta < \alpha$ and $t \in F^\beta$ such that $w < t$. Suppose, on the contrary, that there exist such β and t. Recall that $y^* \leqslant x$ by Claim 6.11 and that $w \leqslant x$. Since X is a tree, this yields that y^* and w must be comparable. We have two cases: either $y^* < w$ or $w \leqslant y^*$. First, suppose that $y^* < w$. Together with $w < t$, this yields $y^* < t$. Since $y^* \in \sup^* F$ by Claim 6.11, we know that y^* is the supremum of a maximal chain C in F. As $y^* < t$ and $t \in F^\beta \subseteq F$, we obtain a contradiction with the maximality of C. Then we consider the case where $w \leqslant y^*$. As $w \in Y_{z^*}$, we obtain $y^* \in {\uparrow}Y_{z^*}$. Recall that from Claim 6.11 that $y^* \in V_{z^*}$. Together with $y^* \in {\uparrow}Y_{z^*}$, this contradicts Claim 6.9(ii). Hence, we conclude that $w \in A^\alpha$. As the set A^α is finite and $F^\alpha = \min A^\alpha$, from $w \in A^\alpha$ and $w \leqslant x$ it follows that $x \in {\uparrow}F^\alpha$, contradicting the assumption that $x \in X \smallsetminus {\uparrow}F^\alpha$. This establishes the left hand side of condition (13), thus concluding the argument. ⊠

7. Priestley separation axiom

The aim of this section is to prove the following.

Theorem 7.1. *The ordered topological space $\langle X; \leqslant, \tau_{\mathsf{h}(X)} \rangle$ is a Priestley space.*

In view of Theorem 6.1, the space $\langle X; \tau_{\mathsf{h}(X)} \rangle$ is compact. Therefore, to establish the above theorem, it suffices to show that $\langle X; \leqslant, \tau_{\mathsf{h}(X)} \rangle$ satisfies Priestley separation axiom. The rest of this section is devoted to this task.

Proposition 7.2. *The ordered topological space $\langle X; \leqslant, \tau_{\mathsf{h}(X)} \rangle$ satisfies Priestley separation axiom.*

Proof. We will prove that for every ordinal α and $x, y \in X_{\leqslant \alpha}$ such that $x \nleqslant y$ there exists a clopen upset of the ordered topological space $\langle X_{\leqslant \alpha}; \leqslant, \tau_\alpha \rangle$ such that $x \in U$ and $y \notin U$. The statement will then follow immediately from the case where $\alpha = \mathsf{h}(X)$. During the proof, we will often use $X_{\leqslant \alpha}$ as a shorthand for $\langle X_{\leqslant \alpha}; \leqslant, \tau_\alpha \rangle$. The proof proceeds by induction on α.

Base case. The case where $\alpha = 0$ is straightforward because $X_{\leqslant \alpha}$ is the singleton containing the root of X.

Successor case. Suppose that the statement holds for an ordinal α and consider $x, y \in X_{\leqslant \alpha+1}$ such that $x \nleqslant y$. Then for each $z \in \{x, y\} \subseteq X_{\leqslant \alpha+1}$ let

$$\bar{z} := \begin{cases} z & \text{if } z \in X_{\leqslant \alpha}; \\ \text{the immediate predecessor of } z & \text{if } z \in X_{\alpha+1}. \end{cases}$$

Clearly, $\bar{z} \leqslant z$ and $z \in X_{\leqslant \alpha}$. We have two cases: either $\bar{x} \nleqslant \bar{y}$ or $\bar{x} \leqslant \bar{y}$.

First, suppose that $\bar{x} \nleqslant \bar{y}$. Since $\bar{x}, \bar{y} \in X_{\leqslant\alpha}$ and $\bar{x} \nleqslant \bar{y}$, we can apply the inductive hypothesis obtaining a clopen upset V of $X_{\leqslant\alpha}$ such that $\bar{x} \in V$ and $\bar{y} \notin V$. Then let

$$U \coloneqq V \cup {\uparrow_{\alpha+1}} (V \cap X_\alpha).$$

We will prove that U is a clopen upset of $X_{\leqslant\alpha+1}$. From the assumption that V is an upset of $X_{\leqslant\alpha}$ it follows that U is an upset of $X_{\leqslant\alpha+1}$. Furthermore, the fact that V is an open set of $X_{\leqslant\alpha}$ and the definition of $\mathcal{S}_{\alpha+1}$ guarantee that U is an open set of $X_{\leqslant\alpha+1}$. Therefore, it only remains to show that U is a closed set of $X_{\leqslant\alpha+1}$. Since V is an upset of $X_{\leqslant\alpha}$ and X a tree, we have

$$X \smallsetminus (V \cup {\uparrow}(V \cap X_\alpha)) = (X_{\leqslant\alpha} \smallsetminus V) \cup {\uparrow}(X_\alpha \smallsetminus V).$$

Using the definition of U and restricting to $X_{\leqslant\alpha+1}$ both sides of the above equality, we obtain

$$X_{\leqslant\alpha+1} \smallsetminus U = X_{\leqslant\alpha+1} \smallsetminus \big(V \cup {\uparrow_{\alpha+1}} (V \cap X_\alpha)\big) = (X_{\leqslant\alpha} \smallsetminus V) \cup {\uparrow_{\alpha+1}} ((X_{\leqslant\alpha} \smallsetminus V) \cap X_\alpha).$$

As $X_{\leqslant\alpha} \smallsetminus V \in \tau_\alpha$ by assumption, the definition of $\mathcal{S}_{\alpha+1}$ guarantees that the right hand side of the above display is an open set of $X_{\leqslant\alpha+1}$. Hence, U is a closed set of $X_{\leqslant\alpha+1}$. This establishes that U is a clopen upset of $X_{\leqslant\alpha+1}$.

Therefore, it only remains to prove that $x \in U$ and $y \notin U$. Recall that $\bar{x} \in V$ and $\bar{x} \leqslant x \in X_{\leqslant\alpha+1}$. As U is an upset of $X_{\leqslant\alpha+1}$ containing V, we obtain $x \in U$. To prove that $y \notin U$, we consider separately two cases: either $y \in X_{\leqslant\alpha}$ or $y \in X_{\alpha+1}$. First suppose that $y \in X_{\leqslant\alpha}$. Then $y = \bar{y}$. As $\bar{y} \notin V$ by assumption, we also have $y \notin V$. Together with $y \in X_{\leqslant\alpha}$, this yields $y \notin V \cup {\uparrow_{\alpha+1}} (V \cap X_\alpha) = U$ as desired. Then we consider the case where $y \in X_{\alpha+1}$. We have $y \notin V$ because $V \subseteq X_{\leqslant\alpha}$. Moreover, $y \notin {\uparrow_{\alpha+1}} (V \cap X_\alpha)$ because $\bar{y}$, which is the only predecessor of y of height α, does not belong to V by assumption. Hence, we conclude that $y \notin U$.

It only remains to consider the case where $\bar{x} \leqslant \bar{y}$. As $\bar{y} \leqslant y$ and $x \nleqslant y$, we have $\bar{x} \neq x$. By the definition of $\bar{x}$ this implies $x \in X_{\alpha+1}$ and $\bar{x} \in X_\alpha$. Therefore, from $\bar{x} \leqslant \bar{y} \in X_{\leqslant\alpha}$ it follows that $\bar{x} = \bar{y}$. Hence, $\bar{y} = \bar{x} \in X_\alpha$. We have three subcases:

$$\text{either } \; y \in P_\alpha \; \text{ or } \; y \in S_{\alpha+1} \; \text{ or } \; y \notin P_\alpha \cup S_{\alpha+1}.$$

Suppose first that $y \in P_\alpha$. We will prove that ${\downarrow}y$ is a clopen set of $X_{\leqslant\alpha+1}$. Since $y \in P_\alpha$, the definition of $\mathcal{S}_{\alpha+1}$ guarantees that ${\downarrow}y$ is an open set of $X_{\leqslant\alpha+1}$. By the same token the set

$$(X_{\leqslant\alpha} \cup {\uparrow_{\alpha+1}} (X_{\leqslant\alpha} \cap X_\alpha)) \smallsetminus {\downarrow}y$$

is also an open of $X_{\leqslant\alpha+1}$, which is easily seen to coincide with $X_{\leqslant\alpha+1} \smallsetminus {\downarrow}y$. Therefore, ${\downarrow}y$ is a clopen set of $X_{\leqslant\alpha+1}$. Together with $x \nleqslant y$, this implies that $X_{\leqslant\alpha+1} \smallsetminus {\downarrow}y$ is a clopen upset of $X_{\leqslant\alpha+1}$ containing x but not y and we are done.

Then we consider the case where $y \in S_{\alpha+1}$. As before, it suffices to show that ${\downarrow}y$ is a clopen set of $X_{\leqslant\alpha+1}$. The fact that ${\downarrow}y$ is closed is proved as in the previous case. To prove that it is open, observe that $\bar{y} \in P_\alpha$ because $y \in S_{\alpha+1}$. From the definition of $\mathcal{S}_{\alpha+1}$ and the assumption that $y \in S_{\alpha+1}$ and $\bar{y} \in P_\alpha$ it follows that both $\{y\}$ and ${\downarrow}\bar{y}$ are open sets of $X_{\leqslant\alpha+1}$. Therefore, ${\downarrow}y = \{y\} \cup {\downarrow}\bar{y}$ is an open set of $X_{\leqslant\alpha+1}$ as desired.

Lastly, we consider the case where $y \notin P_\alpha \cup S_{\alpha+1}$. We will prove that $x \in S_{\alpha+1}$. Suppose the contrary, with a view to contradiction. As $x \in X_{\alpha+1}$ and $\bar{x} \in X_\alpha$, from $x \notin S_{\alpha+1}$ it follows that $x = \bar{x}^+$. Moreover, from $\bar{y} = \bar{x} \in X_\alpha$ and $\bar{y} \leqslant y \in X_{\leqslant\alpha+1}$ it follows that either $y \in \{\bar{y}, \bar{y}^+\} \cup S_{\alpha+1}$. As $y \notin S_{\alpha+1}$ by assumption, we get $y \in \{\bar{y}, \bar{y}^+\}$. Moreover, from $\bar{y} = \bar{x} < x$ and $\bar{y} \in X_\alpha$ it follows that $\bar{y} \in P_\alpha$. Together with $y \in \{\bar{y}, \bar{y}^+\}$ and the assumption that $y \notin P_\alpha$, this yields $y = \bar{y}^+$. Since $\bar{x} = \bar{y}$ and $\bar{x}^+ = x$, we obtain $y = x$, a contradiction with $x \nleqslant y$. Hence, we conclude that $x \in S_{\alpha+1}$.

We will use this fact to prove that $\{x\}$ is a clopen upset of $X_{\leqslant\alpha+1}$ containing x but not y. As $x \nleqslant y$ and x is a maximal element of $X_{\leqslant\alpha+1}$ (the latter because $x \in X_{\alpha+1}$), it suffices to show that $\{x\}$ is a clopen set of $X_{\leqslant \alpha+1}$. Since $x \in S_{\alpha+1}$, the definition of $\mathcal{S}_{\alpha+1}$ guarantees that $\{x\}$ is an open set of $X_{\leqslant \alpha+1}$. To prove that it is also closed, observe that

$$X_{\leqslant \alpha+1} \smallsetminus \{x\} = ((X_{\leqslant \alpha} \cup {\uparrow_{\alpha+1}} (X_{\leqslant \alpha} \cap X_\alpha)) \smallsetminus {\downarrow} x) \cup {\downarrow} \bar{x}$$

because $x \in X_{\alpha+1}$ and $\bar{x}$ is the unique immediate predecessor of x. Furthermore, as $\bar{x} \in P_\alpha$ and $x \in S_{\alpha+1}$, the right hand side of the above display is the union of two members of $\mathcal{S}_{\alpha+1}$. Hence, $\{x\}$ is a closed set of $X_{\leqslant \alpha+1}$ as desired.

Limit case. Suppose that α is a limit ordinal and consider $x, y \in X_{\leqslant \alpha}$ such that $x \nleqslant y$. We will prove that there exist $\beta < \alpha$ and $x^* \in X_{\leqslant \beta}$ such that $x^* \leqslant x$ and $x^* \nleqslant y$. If $x \in X_{<\alpha}$, we are done letting $x^* \coloneqq x$ and $\beta \coloneqq \mathsf{h}(x)$. Then we consider the case where $x \in X_\alpha$. Since α is a limit ordinal and every nonempty chain in X has a supremum, from $x \in X_\alpha$ it follows that x is the supremum of the nonempty chain ${\downarrow} x \smallsetminus \{x\}$. As $x \nleqslant y$, this implies that there exists $x^* \in {\downarrow} x \smallsetminus \{x\}$ such that $x^* \nleqslant y$. Letting $\beta \coloneqq \mathsf{h}(x^*)$ and observing that $\beta < \alpha$, we are done.

Now, consider the nonempty chain $C \coloneqq X_{\leqslant \beta} \cap {\downarrow} y$. By assumption the supremum y^* of C exists and, moreover, belongs to $X_{\leqslant \beta}$ because $C \subseteq X_{\leqslant \beta}$. Since $x^* \nleqslant y$ and $y^* \leqslant y$, we have $x^* \nleqslant y^*$. Recall that $\beta < \alpha$. As $x^*, y^* \in X_{\leqslant \beta}$ and $x^* \nleqslant y^*$, the inductive hypothesis guarantees the existence of a clopen upset U of $X_{\leqslant \beta}$ such that $x^* \in U$ and $y^* \notin U$. Since α is a limit ordinal, the definition of $\mathcal{S}_\alpha$ ensures that both

$$U \cup {\uparrow_\alpha} (U \cap X_\beta) \quad \text{and} \quad (X_{\leqslant \beta} \smallsetminus U) \cup {\uparrow_\alpha} (X_\beta \smallsetminus U)$$

are open sets of $X_{\leqslant \alpha}$. As U is an upset of $X_{\leqslant \beta}$, the set on the left hand side of the above display coincides with ${\uparrow_\alpha} U$. Similarly, the set of the right hand side of the display is $X_{\leqslant \alpha} \smallsetminus {\uparrow_\alpha} U$ because X is a tree and U an upset of $X_{\leqslant \beta}$. Therefore, ${\uparrow_\alpha} U$ is a clopen upset of $X_{\leqslant \alpha}$.

Lastly, from $x^* \in U$ and $x^* \leqslant x \in X_{\leqslant \alpha}$ it follows that $x \in {\uparrow_\alpha} U$. Therefore, it only remains to prove that $y \notin {\uparrow_\alpha} U$. Since ${\uparrow_\alpha} U = U \cup {\uparrow_\alpha} (U \cap X_\beta)$, it suffices to show that $y \notin U$ and $y \notin {\uparrow_\alpha} (U \cap X_\beta)$. Suppose the contrary, with a view to contradiction. We have two cases: either $y \in U$ or $y \in {\uparrow_\alpha} (U \cap X_\beta)$. First, suppose that $y \in U$. Then $y = y^*$ because $y \in U \subseteq X_{\leqslant \beta}$ and y^* is the supremum of ${\downarrow} y \cap X_{\leqslant \beta}$. But this implies $y^* = y \in U$, which is false. Then we consider the case where $y \in {\uparrow_\alpha} (U \cap X_\beta)$. The definition of y^* guarantees that $y^* \in U \cap X_\beta$, a contradiction with $y^* \notin U$. Hence, we conclude that $y \notin {\uparrow_\alpha} U$. ⊠

8. The end

In order to conclude the proof of Theorem 4.3, we need to show that $\langle X; \leqslant, \tau_{\mathsf{h}(X)} \rangle$ is an Esakia space. As $\langle X; \leqslant, \tau_{\mathsf{h}(X)} \rangle$ is a Priestley space by Theorem 7.1, it only remains to prove that the downset of every open set is still open. Therefore, the following observation concludes the proof of Theorem 4.3.

Proposition 8.1. *For every $U \in \tau_{\mathsf{h}(X)}$ we have ${\downarrow} U \in \tau_{\mathsf{h}(X)}$.*

Proof. The proof hinges on the following claim:

Claim 8.2. *Let α be an ordinal and $x \in X_{\leqslant \alpha} \smallsetminus \max X_{\leqslant \alpha}$. Then ${\downarrow} x \in \mathcal{S}_\alpha$.*

Proof of the Claim. The proof of the claim proceeds by induction on α.

Base case. The case where $\alpha = 0$ holds vacuously because $X_{\leqslant 0} \smallsetminus \max X_{\leqslant 0} = \emptyset$ and, therefore, $x \in X_{\leqslant 0} \smallsetminus \max X_{\leqslant 0}$ is impossible.

Successor case. Suppose that $x \in X_{\leqslant \alpha+1} \smallsetminus \max X_{\leqslant \alpha+1}$. We have two cases: either $x \in \max X_{\leqslant \alpha}$ or $x \notin \max X_{\leqslant \alpha}$. First, suppose that $x \in \max X_{\leqslant \alpha}$. Since $x \notin \max X_{\leqslant \alpha+1}$, this implies $x \in P_\alpha$. Consequently, ${\downarrow} x \in \mathcal{S}_{\alpha+1}$ by the definition of $\mathcal{S}_{\alpha+1}$. Then we consider the case where $x \notin \max X_{\leqslant \alpha}$. Together with the assumption that $x \in X_{\leqslant \alpha+1} \smallsetminus \max X_{\leqslant \alpha+1}$, this yields $x \in X_{<\alpha}$. As $x \notin \max X_{\leqslant \alpha}$, we can infer $x \in X_{\leqslant \alpha} \smallsetminus \max X_{\leqslant \alpha}$. Consequently, we can apply the inductive hypothesis, obtaining ${\downarrow} x \in \mathcal{S}_\alpha$. By the definition of $\mathcal{S}_{\alpha+1}$ we have

$$
{\downarrow} x \cup {\uparrow}_{\alpha+1}(X_\alpha \cap {\downarrow} x) \in \mathcal{S}_{\alpha+1}.
$$

Furthermore, from $x \in X_{<\alpha}$ it follows that $X_\alpha \cap {\downarrow} x = \emptyset$. Therefore, the above display simplifies to ${\downarrow} x \in \mathcal{S}_{\alpha+1}$ and we are done.

Limit case. Let $x \in X_{\leqslant \alpha} \smallsetminus \max X_{\leqslant \alpha}$ and assume that α is a limit ordinal. As $x \notin \max X_{\leqslant \alpha}$, we have $\mathsf{h}(x) < \alpha$. We will prove that

$$
x \in X_{\leqslant \mathsf{h}(x)+1} \smallsetminus \max X_{\leqslant \mathsf{h}(x)+1}.
$$

It is clear that $x \in X_{\leqslant \mathsf{h}(x)+1}$. Therefore, it suffices to prove that $x \notin \max X_{\leqslant \mathsf{h}(x)+1}$. Suppose the contrary, with a view to contradiction. From $x \in \max X_{\leqslant \mathsf{h}(x)+1}$ and the fact that x has order type $\mathsf{h}(x)$ it follows that x is a maximal element of X. Together with $\mathsf{h}(x) \leqslant \alpha$, this yields $x \in \max X_{\leqslant \alpha}$, a contradiction. This establishes the above display.

Recall that $\mathsf{h}(x) < \alpha$. Since α is a limit ordinal, this yields $\mathsf{h}(x) + 1 < \alpha$. Therefore, we can apply the inductive hypothesis to the above display, obtaining ${\downarrow} x \in \mathcal{S}_{\mathsf{h}(x)+1}$. Since α is a limit ordinal, the definition of $\mathcal{S}_\alpha$ guarantees that

$$
{\downarrow} x \cup {\uparrow}_\alpha(X_{\mathsf{h}(x)+1} \cap {\downarrow} x) \in \mathcal{S}_\alpha.
$$

As ${\downarrow} x \subseteq X_{\leqslant \mathsf{h}(x)}$, we have $X_{\mathsf{h}(x)+1} \cap {\downarrow} x = \emptyset$. Therefore, the above display simplifies to ${\downarrow} x \in \mathcal{S}_\alpha$. ⊠

Now, we turn to prove the main statement. Let $U \in \tau_{\mathsf{h}(X)}$. Clearly, we have

$$
{\downarrow} U = U \cup \bigcup \{ {\downarrow} w : w \in {\downarrow} U \colon w \notin \max X \}.
$$

As $U \in \tau_{\mathsf{h}(X)}$ by assumption and ${\downarrow} w \in \tau_{\mathsf{h}(X)}$ for each $w \notin \max X$ by Claim 8.2, the right hand side of the above display belongs to the topology $\tau_{\mathsf{h}(X)}$. Hence, we conclude that ${\downarrow} U \in \tau_{\mathsf{h}(X)}$. ⊠

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