Algebraic Logic

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Contents

CHAPTER 1

Universal algebra

1.1 Algebras and equations

We begin by reviewing some fundamentals of general algebraic systems.

Definition 1.1.

(i) A *type* is a map $\rho\colon \mathcal{F} \to \mathbb{N}$, where $\mathcal{F}$ is a set of function symbols. In this case, $\rho(f)$ is said to be the *arity* of the function symbol f, for every $f \in \mathcal{F}$. Function symbols of arity zero are called *constants*.

(ii) An *algebra* of type ρ is a pair $\boldsymbol{A} = \langle A; F \rangle$ where A is a nonempty set and $F = \{f^{\boldsymbol{A}} : f \in \mathcal{F}\}$ is a set of operations on A whose arity is determined by ρ, in the sense that each $f^{\boldsymbol{A}}$ has arity $\rho(f)$. The set A is called the *universe* of $\boldsymbol{A}$.

When $\mathcal{F} = \{f_1, \dots, f_n\}$, we shall write $\langle A; f_1^{\boldsymbol{A}}, \dots, f_n^{\boldsymbol{A}} \rangle$ instead of $\langle A; F \rangle$. In this case, we often drop the superscripts, and write simply $\langle A; f_1, \dots, f_n \rangle$.

Classical examples of algebras are groups and rings. For instance, the type of groups ρ_G consists of a binary symbol $+$, a unary symbol $-$, and a constant symbol 0. Then a group is an algebra $\langle G; +, -, 0 \rangle$ of type ρ_G in which $+$ is associative, 0 is a neutral element for $+$, and $-$ produces inverses.

Lattices, Heyting algebras, and modal algebras are also algebras in the above sense. For instance, the type of lattices ρ_L consists of two binary symbols $\wedge$ and $\vee$ and a lattice is an algebra $\langle A; \wedge, \vee \rangle$ of type ρ_L that satisfies the idempotent, commutative, associative, and absorption laws. Similarly, the type of Heyting algebras ρ_H consists of three binary operations symbols $\wedge, \vee$, and $\to$ and of two constant symbols 0 and 1. Then a Heyting algebra is an algebra $\langle A; \wedge, \vee, \to, 0, 1 \rangle$ such that $\langle A; \wedge, \vee, 0, 1 \rangle$ is a bounded lattice and, for every $a, b, c \in A$,

$$a \wedge b \leqslant c \iff a \leqslant b \to c. \qquad \textit{(residuation law)}$$

Boolean algebras can be viewed as the Heyting algebras that satisfy the following equational version of the *excluded middle law*:

$$x \vee (x \to 0) \approx 1.$$

In this case, the complement operation $\neg x$ can be defined as $x \to 0$.

Perhaps less obviously, even algebraic structures whose operations are apparently *external* can be viewed as algebras in the sense of the above definition. For instance, modules over a ring $\boldsymbol{R}$ can be viewed as algebras whose type ρ_R extends that of groups with the unary symbols $\{\lambda_r : r \in R\}$. From this point of view, a module over $\boldsymbol{R}$ is an algebra $\langle G; +, -, 0, \{\lambda_r : r \in R\}\rangle$ of type ρ_R such that $\langle G; +, -, 0\rangle$ is an abelian group and, for every $r, s \in R$ and $a, c \in G$,

$$\begin{aligned} \lambda_r(a+c) &= \lambda_r(a) + \lambda_r(c) \\ \lambda_{r+s}(a) &= \lambda_r(a) + \lambda_s(a) \\ \lambda_r(\lambda_s(a)) &= \lambda_{r \cdot s}(a) \\ \lambda_1(a) &= a. \end{aligned}$$

Given a type $\rho \colon \mathcal{F} \to \mathbb{N}$ and a set of variables X disjoint from $\mathcal{F}$, the set of *terms of type ρ over X* is the least set $T_\rho(X)$ such that

(i) $X \subseteq T_\rho(X)$;

(ii) if $c \in \mathcal{F}$ is a constant, then $c \in T_\rho(X)$; and

(iii) if $\varphi_1, \dots, \varphi_{\rho(f)} \in T_\rho(X)$ and $f \in \mathcal{F}$, then $f\varphi_1 \dots \varphi_{\rho(f)} \in T_\rho(X)$.

For the sake of readability, we shall often write $f(\varphi_1, \dots, \varphi_{\rho(f)})$ instead of $f\varphi_1 \dots \varphi_{\rho(f)}$. Similarly, if f is a binary operation $+$, we often write $\varphi_1 + \varphi_2$ instead of $f(\varphi_1, \varphi_2)$.

Given a term $\varphi \in T_\rho(X)$, we write $\varphi(x_1, \dots, x_n)$ to indicate that the variables occurring in φ are among $x_1, \dots, x_n$. Furthermore, given an algebra $\boldsymbol{A}$ of type ρ and elements $a_1, \dots, a_n \in A$, we define an element

$$\varphi^{\boldsymbol{A}}(a_1, \dots, a_n)$$

of A, by recursion on the construction of φ, as follows:

(i) if φ is a variable x_i, then $\varphi^{\boldsymbol{A}}(a_1, \dots, a_n) \coloneqq a_i$;

(ii) if φ is a constant c, then $c^{\boldsymbol{A}}$ is the interpretation of c in $\boldsymbol{A}$;

(iii) if $\varphi = f(\psi_1, \dots, \psi_m)$, then

$$\varphi^{\boldsymbol{A}}(a_1, \dots, a_n) \coloneqq f^{\boldsymbol{A}}(\psi_1^{\boldsymbol{A}}(a_1, \dots, a_n), \dots, \psi_m^{\boldsymbol{A}}(a_1, \dots, a_n)).$$

An *equation of type ρ over X* is an expression of the form $\varphi \approx \psi$, where $\varphi, \psi \in T_\rho(X)$. Such an equation $\varphi \approx \psi$ is *valid* in an algebra $\boldsymbol{A}$ of type ρ, if

$$\varphi^{\boldsymbol{A}}(a_1, \dots, a_n) = \psi^{\boldsymbol{A}}(a_1, \dots, a_n), \text{ for every } a_1, \dots, a_n \in A,$$

in which case we say that $\boldsymbol{A}$ *satisfies* $\varphi \approx \psi$.

For instance, groups are precisely the algebras of type ρ_G that satisfy the equations

$$x + (y+z) \approx (x+y) + z \quad x + 0 \approx x \quad 0 + x \approx x \quad x + -x \approx 0 \quad -x + x \approx 0.$$

Similarly, lattices are the algebras of type ρ_L that satisfy the equations

$$\begin{array}{llr} x \wedge x \approx x & x \vee x \approx x & (\textit{idempotent laws}) \\ x \wedge y \approx y \wedge x & x \vee y \approx y \vee x & (\textit{commutative laws}) \\ x \wedge (y \wedge z) \approx (x \wedge y) \wedge z & x \vee (y \vee z) \approx (x \vee y) \vee z & (\textit{associative laws}) \\ x \wedge (y \vee x) \approx x & x \vee (y \wedge x) \approx x. & (\textit{absorption laws}) \end{array}$$

1.2 Basic constructions

Algebras of the same type are called *similar* and can be compared by means of maps that preserve their structure.

Definition 1.2. Given two similar algebras $\boldsymbol{A}$ and $\boldsymbol{B}$, a *homomorphism* from $\boldsymbol{A}$ to $\boldsymbol{B}$ is a map $f\colon A \to B$ such that, for every n-ary operation g of the common type and $a_1, \dots, a_n \in A$,

$$f(g^{\boldsymbol{A}}(a_1, \dots, a_n)) = g^{\boldsymbol{B}}(f(a_1), \dots, f(a_n)).$$

An injective homomorphism is called an *embedding* and, if there exists an embedding from $\boldsymbol{A}$ to $\boldsymbol{B}$, we say that $\boldsymbol{A}$ *embeds* into $\boldsymbol{B}$. Lastly, a surjective embedding is called an *isomorphism*. Accordingly, $\boldsymbol{A}$ and $\boldsymbol{B}$ are said to be *isomorphic* if there exists an isomorphism between them, in which case we write $\boldsymbol{A} \cong \boldsymbol{B}$.

A simple induction on the construction of terms shows that, for every pair of algebras $\boldsymbol{A}$ and $\boldsymbol{B}$ of type ρ and every term $\varphi(x_1, \dots, x_n)$ of ρ, if f is a homomorphism from $\boldsymbol{A}$ to $\boldsymbol{B}$, then

$$f(\varphi^{\boldsymbol{A}}(a_1, \dots, a_n)) = \varphi^{\boldsymbol{B}}(f(a_1), \dots, f(a_n)),$$

for every $a_1, \dots, a_n \in A$. Therefore homomorphisms preserve not only basic operations, but also arbitrary terms.

In the particular case where $\boldsymbol{A}$ and $\boldsymbol{B}$ are lattices, a homomorphism from $\boldsymbol{A}$ to $\boldsymbol{B}$ is a map $f\colon A \to B$ such that, for every $a, c \in A$,

$$f(a \wedge^{\boldsymbol{A}} c) = f(a) \wedge^{\boldsymbol{B}} f(c) \text{ and } f(a \vee^{\boldsymbol{A}} c) = f(a) \vee^{\boldsymbol{B}} f(c).$$

For instance, the inclusion map from the lattice $\langle \mathbb{N}; \leqslant \rangle$ into the lattice $\langle \mathbb{Z}; \leqslant \rangle$ is an injective homomorphism, that is, an embedding. Similarly, given two sets $Y \subseteq X$, the inclusion map from the powerset lattice $\langle \mathcal{P}(Y); \subseteq \rangle$ to the powerset lattice $\langle \mathcal{P}(X); \subseteq \rangle$ is also an embedding. On the other hand, if $Y \subsetneq X$, the map

$$(-) \cap Y\colon \mathcal{P}(X) \to \mathcal{P}(Y)$$

that sends every $Z \subseteq X$ to $Z \cap Y$ is a noninjective homomorphism from $\langle \mathcal{P}(X); \subseteq \rangle$ to $\langle \mathcal{P}(Y); \subseteq \rangle$.

Definition 1.3. Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be algebras of the same type $\rho\colon \mathcal{F} \to \mathbb{N}$. Then $\boldsymbol{A}$ is said to be a *subalgebra* of $\boldsymbol{B}$ if $A \subseteq B$ and $f^{\boldsymbol{A}}$ is the restriction of $f^{\boldsymbol{B}}$ to A, for every $f \in \mathcal{F}$. In this case, we write $\boldsymbol{A} \leqslant \boldsymbol{B}$.

Given a class of algebras K, let

$$\mathbb{I}(\mathsf{K}) := \{\boldsymbol{A} : \boldsymbol{A} \cong \boldsymbol{B} \text{ for some } \boldsymbol{B} \in \mathsf{K}\}$$
$$\mathbb{S}(\mathsf{K}) := \{\boldsymbol{A} : \boldsymbol{A} \leqslant \boldsymbol{B} \text{ for some } \boldsymbol{B} \in \mathsf{K}\}.$$

When $\mathsf{K} = \{\boldsymbol{A}\}$, we write $\mathbb{I}(\boldsymbol{A})$ and $\mathbb{S}(\boldsymbol{A})$ as a shorthand for $\mathbb{I}(\{\boldsymbol{A}\})$ and $\mathbb{S}(\{\boldsymbol{A}\})$, respectively. The following observation is an immediate consequence of the definitions.

Proposition 1.4. *Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be algebras of the same type. Then $\boldsymbol{A} \in \mathbb{IS}(\boldsymbol{B})$ if and only if there exists an embedding $f\colon \boldsymbol{A} \to \boldsymbol{B}$. In this case, $\boldsymbol{A}$ is isomorphic to the unique subalgebra of $\boldsymbol{B}$ with universe $f[A]$.*

As we mentioned, homomorphisms can be used to compare similar algebras.

Definition 1.5. Given two similar algebras $\boldsymbol{A}$ and $\boldsymbol{B}$, we say that $\boldsymbol{A}$ is a *homomorphic image* of $\boldsymbol{B}$ if there exists a surjective homomorphism $f \colon \boldsymbol{B} \to \boldsymbol{A}$.

Accordingly, given a class of algebras K, we set

$$\mathbb{H}(\mathsf{K}) := \{\boldsymbol{A} : \boldsymbol{A} \text{ is a homomorphic image of some } \boldsymbol{B} \in \mathsf{K}\}.$$

As usual, when $\mathsf{K} = \{\boldsymbol{A}\}$, we write $\mathbb{H}(\boldsymbol{A})$ as a shorthand for $\mathbb{H}(\{\boldsymbol{A}\})$.

Observe that every (not necessarily surjective) homomorphism $f \colon \boldsymbol{A} \to \boldsymbol{B}$ induces a homomorphic image of $\boldsymbol{A}$.

Proposition 1.6. *If $f \colon \boldsymbol{A} \to \boldsymbol{B}$ is a homomorphism, then $f[A]$ is the universe of a subalgebra of $\boldsymbol{B}$ that, moreover, is a homomorphic image of $\boldsymbol{A}$.*

Proof. Observe that $f[A]$ is nonempty, because A is. Then consider an n-ary function symbol g of the common type of $\boldsymbol{A}$ and $\boldsymbol{B}$ and $b_1, \dots, b_n \in f[A]$. Clearly, there are $a_1, \dots, a_n \in A$ such that $f(a_i) = b_i$, for every $i \leqslant n$. Since f is a homomorphism from $\boldsymbol{A}$ to $\boldsymbol{B}$, we obtain

$$g^{\boldsymbol{B}}(b_1, \dots, b_n) = g^{\boldsymbol{B}}(f(a_1), \dots, g(a_n)) = f(g^{\boldsymbol{A}}(a_1, \dots, a_n)) \in f[A].$$

Hence, we conclude that $f[A]$ is the universe of a subalgebra $f[\boldsymbol{A}]$ of $\boldsymbol{B}$.

Furthermore, $f \colon \boldsymbol{A} \to f[\boldsymbol{A}]$ is a homomorphism, because for every basic n-ary function symbol g of the common type and $a_1, \dots, a_n \in A$,

$$f(g^{\boldsymbol{A}}(a_1, \dots, a_n)) = g^{\boldsymbol{B}}(f(a_1), \dots, f(a_n)) = g^{f[\boldsymbol{A}]}(f(a_1), \dots, f(a_n)),$$

where the first equality follows from the assumption that $f \colon \boldsymbol{A} \to \boldsymbol{B}$ is a homomorphism. Since the map $f \colon A \to f[A]$ is surjective, we conclude that $f[\boldsymbol{A}] \in \mathbb{H}(\boldsymbol{A})$. ⊠

In view of the above result, when $f \colon \boldsymbol{A} \to \boldsymbol{B}$ is a homomorphism, we denote by $f[\boldsymbol{A}]$ the unique subalgebra of $\boldsymbol{B}$ with universe $f[A]$.

For instance, let $f \colon \mathbb{Z} \to \mathbb{R}$ be the absolute value map, that is, the function defined by the rule

$$f(n) := \text{the absolute value of } n.$$

Observe that f is a nonsurjective homomorphism from the lattice of integers to that of reals. Furthermore, the homomorphic image $f[\langle \mathbb{Z}; \leqslant \rangle]$ of $\langle \mathbb{Z}; \leqslant \rangle$ is the lattice of natural numbers $\langle \mathbb{N}; \leqslant \rangle$, which, in turn, is a subalgebra of lattice of reals.

Notably, the homomorphic images of an algebra $\boldsymbol{A}$ can be "internalized" as special equivalence relations on $\boldsymbol{A}$ as follows.

Definition 1.7. A *congruence* of an algebra $\boldsymbol{A}$ is an equivalence relation θ on A such that, for every basic n-ary opearation f of $\boldsymbol{A}$ and $a_1, \dots, a_n, c_1, \dots, c_n \in A$,

$$\text{if } \langle a_1, c_1 \rangle, \dots, \langle a_n, c_n \rangle \in \theta, \text{ then } \langle f^{\boldsymbol{A}}(a_1, \dots, a_n), f^{\boldsymbol{A}}(c_1, \dots, c_n) \rangle \in \theta. \tag{1.1}$$

In this case, we often write $a \equiv_\theta c$ as a shorthand for $\langle a, c \rangle \in \theta$. The poset of congruences of $\boldsymbol{A}$ ordered under the inclusion relation will be denoted by $\mathsf{Con}(\boldsymbol{A})$.

A simple induction on the construction of terms shows that, for every congruence θ of $\boldsymbol{A}$ and every term $\varphi(x_1, \dots, x_n)$,

$$\text{if } \langle a_1, c_1\rangle, \dots, \langle a_n, c_n\rangle \in \theta, \text{ then } \langle \varphi^{\boldsymbol{A}}(a_1, \dots, a_n), \varphi^{\boldsymbol{A}}(c_1, \dots, c_n)\rangle \in \theta,$$

for every $a_1, \dots, a_n \in A$. Therefore, congruences preserve not only basic operations, but also arbitrary terms. Furthermore, a simple argument shows that $\mathsf{Con}(\boldsymbol{A})$ is an inductive closure system and, therefore, an algebraic lattice whose maximum is the total relation $A \times A$ and whose minimum is the identity relation $\mathrm{id}_A := \{\langle a, a\rangle : a \in A\}$.

Example 1.8 (Boolean algebras)**.** Recall that a *filter* of a Boolean algebra $\boldsymbol{A}$ is a nonempty upset $F \subseteq A$ closed under binary meets. We denote by $\mathsf{Fi}(\boldsymbol{A})$ the poset of filters of $\boldsymbol{A}$ ordered under the inclusion relation. It is easy to see $\mathsf{Fi}(\boldsymbol{A})$ is an inductive closure system and, therefore, an algebraic lattice. Furthermore, the lattices $\mathsf{Fi}(\boldsymbol{A})$ and $\mathsf{Con}(\boldsymbol{A})$ are isomorphic via the inverse isomorphisms

$$\boldsymbol{\Omega}^{\boldsymbol{A}}(-) \colon \mathsf{Fi}(\boldsymbol{A}) \to \mathsf{Con}(\boldsymbol{A}) \;\text{ and }\; \boldsymbol{\tau}^{\boldsymbol{A}}(-) \colon \mathsf{Con}(\boldsymbol{A}) \to \mathsf{Fi}(\boldsymbol{A})$$

defined by the rules

$$\begin{aligned}\boldsymbol{\Omega}^{\boldsymbol{A}}(F) &:= \{\langle a, c\rangle \in A \times A : a \to c, c \to a \in F\}\\ \boldsymbol{\tau}^{\boldsymbol{A}}(\theta) &:= \{a \in A : \langle a, 1\rangle \in \theta\}.\end{aligned}$$

Because of this, every congruence θ of a Boolean algebra $\boldsymbol{A}$ is induced by some filter F, in the sense that $\theta = \boldsymbol{\Omega}^{\boldsymbol{A}}F$. This correspondence between filters and congruences generalizes straightforwardly to all Heyting algebras. ⊠

Example 1.9 (Modal algebras)**.** A *modal algebra* is an algebra $\boldsymbol{A} = \langle A; \wedge, \vee, \neg, \Box, 0, 1\rangle$ such that $\langle A; \wedge, \vee, \neg, 0, 1\rangle$ is a Boolean algebra and $\Box$ is a unary operation such that

$$\Box(a \wedge c) = \Box a \wedge \Box c \;\text{ and }\; \Box 1 = 1,$$

for every $a, c \in A$. An *open filter* of a modal algebra $\boldsymbol{A}$ is a filter of the Boolean reduct of $\boldsymbol{A}$ that, moreover, is closed under the operation $\Box$. The poset of open filters of $\boldsymbol{A}$ ordered under the inclusion relation will be denoted by $\mathsf{Op}(\boldsymbol{A})$. It forms an inductive closure system and, therefore, an algebraic lattice. Furthermore, the lattices $\mathsf{Op}(\boldsymbol{A})$ and $\mathsf{Con}(\boldsymbol{A})$ are isomorphic via the inverse isomorphisms described in Example 1.8. Because of this, every congruence of a modal algebra $\boldsymbol{A}$ has the form $\theta = \boldsymbol{\Omega}^{\boldsymbol{A}}F$, for some open filter F. ⊠

Example 1.10 (Groups)**.** Similarly, it is well known that the lattice of congruences of a group is isomorphic to that of its normal subgroups. Because of this, every congruence of a group is induced by some normal subgroup. ⊠

As we mentioned, there is a tight correspondence between the homomorphic images and the congruences of an algebra $\boldsymbol{A}$. On the one hand, every congruence θ of $\boldsymbol{A}$ gives rise to a homomorphic image $\boldsymbol{A}/\theta$ of $\boldsymbol{A}$. Let $\mathcal{F}$ be the set of function symbols of $\boldsymbol{A}$. Given $\theta \in \mathsf{Con}(\boldsymbol{A})$ and a basic n-ary function symbol $f \in \mathcal{F}$, let $f^{\boldsymbol{A}/\theta}$ be the n-ary operation on A/θ defined by the rule

$$f^{\boldsymbol{A}/\theta}(a_1/\theta, \dots, a_n/\theta) := f^{\boldsymbol{A}}(a_1, \dots, a_n)/\theta.$$

Notice that $f^{\boldsymbol{A}/\theta}$ is well-defined, by condition (1.1). As a consequence, the structure

$$\boldsymbol{A}/\theta \coloneqq \langle A/\theta; \{f^{\boldsymbol{A}/\theta} : f \in \mathcal{F}\}\rangle$$

is a well-defined algebra of the type as $\boldsymbol{A}$. Furthermore, $\boldsymbol{A}/\theta \in \mathbb{H}(\boldsymbol{A})$, because the map $\pi_\theta \colon A \to A/\theta$, defined, for every $a \in A$, as $\pi_\theta(a) \coloneqq a/\theta$, is a surjective homomorphism from $\boldsymbol{A}$ to $\boldsymbol{A}/\theta$. To prove this, consider $a_1, \dots, a_n \in A$. We have

$$\begin{aligned}\pi_\theta(f^{\boldsymbol{A}}(a_1, \dots, a_n)) &= f^{\boldsymbol{A}}(a_1, \dots, a_n)/\theta \\ &= f^{\boldsymbol{A}/\theta}(a_1/\theta, \dots, a_n/\theta) \\ &= f^{\boldsymbol{A}/\theta}(\pi_\theta(a_1), \dots, \pi_\theta(a_n)),\end{aligned}$$

where the second equality follows from the definition of the operation $f^{\boldsymbol{A}/\theta}$.

Corollary 1.11. *If θ is a congruence of an algebra $\boldsymbol{A}$, then $\boldsymbol{A}/\theta$ is a well-defined homomorphic image of $\boldsymbol{A}$.*

In view of the above result, every congruence θ of an algebra $\boldsymbol{A}$ induces a homomorphic image of $\boldsymbol{A}$, namely $\boldsymbol{A}/\theta$. The converse is also true, as we proceed to explain.

Definition 1.12. The *kernel* of a homomorphism $f \colon \boldsymbol{A} \to \boldsymbol{B}$ is the binary relation

$$\mathsf{Ker}(f) \coloneqq \{\langle a, c\rangle \in A \times A : f(a) = f(c)\}.$$

Proposition 1.13. *The kernel of a homomorphism $f \colon \boldsymbol{A} \to \boldsymbol{B}$ is a congruence of $\boldsymbol{A}$.*

Proof. It is obvious that $\mathsf{Ker}(f)$ is an equivalence relation on A. Therefore, to prove that $\mathsf{Ker}(f)$ is a congruence of $\boldsymbol{A}$, it suffices to show that it preserves the basic operations of $\boldsymbol{A}$. Consider a basic n-ary operation g of $\boldsymbol{A}$ and $a_1, \dots, a_n, c_1, \dots, c_n \in A$ such that $\langle a_1, c_1\rangle, \dots, \langle a_n, c_n\rangle \in \mathsf{Ker}(f)$. By the definition of $\mathsf{Ker}(f)$,

$$f(a_i) = f(c_i), \text{ for every } i \leqslant n.$$

It follows that $g^{\boldsymbol{B}}(f(a_1), \dots, f(a_n)) = g^{\boldsymbol{B}}(f(c_1), \dots, f(c_n))$. Since $f \colon \boldsymbol{A} \to \boldsymbol{B}$ is a homomorphism, this yields

$$f(g^{\boldsymbol{A}}(a_1, \dots, a_n)) = g^{\boldsymbol{B}}(f(a_1), \dots, f(a_n)) = g^{\boldsymbol{B}}(f(c_1), \dots, f(c_n)) = f(g^{\boldsymbol{A}}(c_1, \dots, c_n)).$$

Hence, we conclude that $\langle g^{\boldsymbol{A}}(a_1, \dots, a_n), g^{\boldsymbol{A}}(c_1, \dots, c_n)\rangle \in \mathsf{Ker}(f)$, as desired. ⊠

The behaviour of kernels is governed by the next principle.

Fundamental Homomorphism Theorem 1.14. *If $f \colon \boldsymbol{A} \to \boldsymbol{B}$ is a homomorphism with kernel θ, then there exists a unique embedding $g \colon \boldsymbol{A}/\theta \to \boldsymbol{B}$ such that $f = g \circ \pi_\theta$.*

Proof. We begin by proving the existence of g. Let $g \colon A/\theta \to B$ be the map defined as $g(a/\theta) \coloneqq f(a)$, for every $a \in A$. To show that g is well-defined, consider $a, c \in A$ such that $a/\theta = c/\theta$. Since $\theta = \mathsf{Ker}(f)$, this means that $f(a) = f(c)$, as desired. Furthermore, the definition of g guarantees that $f = g \circ \pi_\theta$.

Now, observe g is injective, because, for every $a, c \in A$ such that $g(a/\theta) = g(c/\theta)$, we have $f(a) = f(c)$, that is, $\langle a, c \rangle \in \mathsf{Ker}(f) = \theta$ and, therefore, $a/\theta = c/\theta$. Moreover, for every basic n-ary operation p of $\boldsymbol{A}$ and $a_1, \dots, a_n \in A$, we have

$$\begin{aligned} g(p^{\boldsymbol{A}/\theta}(a_1/\theta, \dots, a_n/\theta)) &= g(p^{\boldsymbol{A}}(a_1, \dots, a_n)/\theta) \\ &= f(p^{\boldsymbol{A}}(a_1, \dots, a_n)) \\ &= p^{\boldsymbol{B}}(f(a_1), \dots, f(a_n)) \\ &= p^{\boldsymbol{B}}(g(a_1/\theta), \dots, g(a_n/\theta)). \end{aligned}$$

The first equality above follows from the definition of $\boldsymbol{A}/\theta$, the second and the last from the definition of g, and the third from the assumption that $f\colon \boldsymbol{A} \to \boldsymbol{B}$ is a homomorphism. Hence, we conclude that $g\colon \boldsymbol{A}/\theta \to \boldsymbol{B}$ is a homomorphism and, therefore, an embedding, as desired.

The uniqueness of g follows from the fact that, if a map g^* satisfies the condition in the statement of the theorem, then, for every $a \in A$,

$$f(a) = g^* \circ \pi_\theta(a) = g^*(a/\theta),$$

that is, g^* coincides with g. ⊠

Corollary 1.15. *If $f\colon \boldsymbol{A} \to \boldsymbol{B}$ is a homomorphism, then $f[\boldsymbol{A}] \cong \boldsymbol{A}/\mathsf{Ker}(f)$. In particular, if f is surjective, $\boldsymbol{B} \cong \boldsymbol{A}/\mathsf{Ker}(f)$.*

Proof. In the proof of the Fundamental Homomorphism Theorem we showed that the map $g\colon A/\mathsf{Ker}(f) \to B$, defined by the rule $g(a/\mathsf{Ker}(f)) := f(a)$, is an embedding of $\boldsymbol{A}/\mathsf{Ker}(f)$ into $\boldsymbol{B}$. As g can be viewed as a surjective embedding of $\boldsymbol{A}/\mathsf{Ker}(f)$ into $f[\boldsymbol{A}]$, we conclude that $f[\boldsymbol{A}] \cong \boldsymbol{A}/\mathsf{Ker}(f)$. ⊠

At this stage, it should be clear that if θ is a congruence on an algebra $\boldsymbol{A}$, then $\pi_\theta\colon \boldsymbol{A} \to \boldsymbol{A}/\theta$ is a surjective homomorphism whose kernel is θ. Similarly, if $f\colon \boldsymbol{A} \to \boldsymbol{B}$ is a surjective homomorphism, then $\boldsymbol{A}/\mathsf{Ker}(f) \cong \boldsymbol{B}$, by Corollary 1.15. As a consequence, for every class of algebras K,

$$\mathbb{H}(\mathsf{K}) = \mathbb{I}\{\boldsymbol{A}/\theta : \boldsymbol{A} \in \mathsf{K} \text{ and } \theta \in \mathsf{Con}(\boldsymbol{A})\}. \tag{1.2}$$

Now, recall that the Cartesian product of a family of sets $\{A_i : i \in I\}$ is the set

$$\prod_{i \in I} A_i := \{f\colon I \to \bigcup_{i \in I} A_i \ :\ f(i) \in A_i, \text{ for all } i \in I\}.$$

In particular, if I is empty, then $\prod_{i \in I} A_i$ is the singleton containing only the empty map.

Definition 1.16. The *direct product* of a family of similar algebras $\{\boldsymbol{A}_i : i \in I\}$ is the unique algebra of the common type whose universe is the Cartesian product $\prod_{i \in I} A_i$ and such that, for every basic n-ary operation symbol f and every $\vec{a}_1, \dots, \vec{a}_n \in \prod_{i \in I} A_i$,

$$f^{\prod_{i \in I} \boldsymbol{A}_i}(\vec{a}_1, \dots, \vec{a}_n)(i) = f^{\boldsymbol{A}_i}(\vec{a}_1(i), \dots, \vec{a}_n(i)), \text{ for every } i \in I.$$

We denote this algebra by $\prod_{i \in I} \boldsymbol{A}_i$.

In this case, for every $j \in I$, the projection map on the j-th component $p_j \colon \prod_{i \in I} A_i \to A_j$, defined by the rule $p_j(\vec{a}) := \vec{a}(j)$, is a surjective homomorphism from $\prod_{i \in I} \boldsymbol{A}_i$ to $\boldsymbol{A}_j$.

Given a class of similar algebras K, we set

$$\mathbb{P}(\mathsf{K}) := \{\boldsymbol{A} : \boldsymbol{A} \text{ is a direct product of a family } \{\boldsymbol{B}_i : i \in I\} \subseteq \mathsf{K}\}.$$

As usual, when $\mathsf{K} = \{\boldsymbol{A}\}$, we write $\mathbb{P}(\boldsymbol{A})$ as a shorthand for $\mathbb{P}(\{\boldsymbol{A}\})$.

Notice that up to isomorphism, there exists a unique one-element algebra of a given type. Because of this, one-element algebras are called *trivial*. Accordingly, when the set of indexes I is empty, the direct product $\prod_{i \in I} \boldsymbol{A}_i$ is the trivial algebra of the given type. It follows that $\mathbb{P}(\mathsf{K})$ contains always a trivial algebra, for every class of similar algebras K.

Example 1.17 (Powerset algebras)**.** Boolean algebras of the form $\langle \mathcal{P}(X); \cap, \cup, -, \emptyset, X\rangle$ are called *powerset Boolean algebras*. Let $\boldsymbol{B}$ be the two-element Boolean algebra and observe that $\mathbb{IP}(\boldsymbol{B})$ is the class of algebras isomorphic to some powerset Boolean algebra. To prove this, observe that every powerset Boolean algebra $\mathcal{P}(X)$ is isomorphic to a direct product of $\boldsymbol{B}$ via the *characteristic function* $f_X \colon \mathcal{P}(X) \to \prod_{x \in X} \boldsymbol{B}_x$, defined by the rule

$$f(Y)(x) := \begin{cases} 1 & \text{if } x \in Y \\ 0 & \text{if } x \notin Y, \end{cases}$$

where $Y \in \mathcal{P}(X)$ and $x \in X$. By the same token, every direct product $\prod_{i \in I} \boldsymbol{B}_i$ of $\boldsymbol{B}$ is isomorphic to the powerset Boolean algebra $\mathcal{P}(I)$ via the isomorphism f_I. ⊠

We close this section by reviewing the subdirect product construction.

Definition 1.18. A subalgebra $\boldsymbol{B}$ of a direct product $\prod_{i \in I} \boldsymbol{A}_i$ is said to be a *subdirect product* of $\{\boldsymbol{A}_i : i \in I\}$ if the projection map p_i is surjective, for every $i \in I$. Similarly, an embedding $f \colon \boldsymbol{B} \to \prod_{i \in I} \boldsymbol{A}_i$ is said to be *subdirect* when $f[\boldsymbol{B}]$ is a subdirect product of the family $\{\boldsymbol{A}_i : i \in I\}$.

Given a class of similar algebras K, we set

$$\mathbb{P}_{\text{SD}}(\mathsf{K}) := \{\boldsymbol{A} : \boldsymbol{A} \text{ is a subdirect direct product of a family } \{\boldsymbol{B}_i : i \in I\} \subseteq \mathsf{K}\}.$$

As usual, when $\mathsf{K} = \{\boldsymbol{A}\}$, we write $\mathbb{P}_{\text{SD}}(\boldsymbol{A})$ as a shorthand for $\mathbb{P}_{\text{SD}}(\{\boldsymbol{A}\})$. Clearly, $\mathbb{P}_{\text{SD}}(\mathsf{K}) \subseteq \mathbb{SP}(\mathsf{K})$. Furthermore, $\mathbb{P}_{\text{SD}}(\mathsf{K})$ contains always a trivial algebra.

Example 1.19 (Distributive lattices)**.** Let DL be the class of distributive lattices and $\boldsymbol{B}$ be the two-element distributive lattice. Birkhoff's Representation Theorem states that $\mathsf{DL} = \mathbb{IP}_{\text{SD}}(\boldsymbol{B})$. The inclusion $\mathbb{IP}_{\text{SD}}(\boldsymbol{B}) \subseteq \mathsf{DL}$ follows from the fact that DL is closed under $\mathbb{I}, \mathbb{S}$, and $\mathbb{P}$. For the other inclusion, consider a distributive lattice $\boldsymbol{A}$ and let I be the set of its prime filters. By Birkhoff's Representation Theorem, the map

$$\gamma \colon \boldsymbol{A} \to \prod_{F \in I} \boldsymbol{B}_F,$$

defined, for every $a \in A$ and $F \in I$, by the rule

$$\gamma(a)(F) := \begin{cases} 1 & \text{if } a \in F \\ 0 & \text{if } a \notin F, \end{cases}$$

is a well-defined subdirect embedding. ⊠

Example 1.20 (Boolean algebras)**.** Similarly, Stone's Representation Theorem states that the class of Boolean algebras coincides with $\mathbb{IP}_{\text{SD}}(\boldsymbol{B})$, where $\boldsymbol{B}$ the two-element Boolean algebra. $\boxtimes$

The next result provides a general recipe to construct subdirect products.

Proposition 1.21. *Let $\boldsymbol{A}$ be an algebra and $\{\theta_i : i \in I\} \subseteq \mathsf{Con}(\boldsymbol{A})$. Then the map*

$$f \colon \boldsymbol{A}/\bigcap_{i\in I}\theta_i \to \prod_{i\in I}\boldsymbol{A}/\theta_i,$$

defined, for every $a \in A$ and $j \in I$, as

$$f(a/\bigcap_{i\in I}\theta_i)(j) := a/\theta_j,$$

is a subdirect embedding.

Proof. For the sake of readability, set $\boldsymbol{B} := \boldsymbol{A}/\bigcap_{i\in I}\theta_i$. To prove that f is injective, consider $a, c \in A$ such that $\langle a, c\rangle \notin \bigcap_{i\in I}\theta_i$. Then there exists $j \in I$ such that $\langle a, c\rangle \notin \theta_j$ and, therefore,

$$f(a/\bigcap_{i\in I}\theta_i)(j) := a/\theta_j \neq c/\theta_j = f(c/\bigcap_{i\in I}\theta_i)(j).$$

It follows that $f(a/\bigcap_{i\in I}\theta_i) \neq f(c/\bigcap_{i\in I}\theta_i)$. Thus, f is injective. Moreover, by the definition of f, the composition $p_i \circ f \colon \boldsymbol{B} \to \boldsymbol{A}/\theta_i$ is surjective, for every $i \in I$.

It only remains to prove that f is a homomorphism. Consider an n-ary basic operation g and $a_1, \dots, a_n \in A$. For every $j \in I$, we have

$$\begin{aligned}
f(g^{\boldsymbol{B}}(a_1/\bigcap_{i\in I}\theta_i, \dots, a_n/\bigcap_{i\in I}\theta_i))(j) &= f(g^{\boldsymbol{A}}(a_1, \dots, a_n)/\bigcap_{i\in I}\theta_i)(j) \\
&= g^{\boldsymbol{A}}(a_1, \dots, a_n)/\theta_j \\
&= g^{\boldsymbol{A}/\theta_j}(a_1/\theta_j, \dots, a_n/\theta_j) \\
&= g^{\boldsymbol{A}/\theta_j}(f(a_1/\bigcap_{i\in I}\theta_i)(j), \dots, f(a_n/\bigcap_{i\in I}\theta_i)(j)) \\
&= g^{\prod_{i\in I}\boldsymbol{A}/\theta_i}(f(a_1/\bigcap_{i\in I}\theta_i), \dots, f(a_n/\bigcap_{i\in I}\theta_i))(j).
\end{aligned}$$

It follows that

$$f(g^{\boldsymbol{B}}(a_1/\bigcap_{i\in I}\theta_i, \dots, a_n/\bigcap_{i\in I}\theta_i)) = g^{\prod_{i\in I}\boldsymbol{A}/\theta_i}(f(a_1/\bigcap_{i\in I}\theta_i), \dots, f(a_n/\bigcap_{i\in I}\theta_i)). \qquad \boxtimes$$

1.3 Prevarieties

Definition 1.22. A class of similar algebras closed under $\mathbb{I}, \mathbb{S}$, and $\mathbb{P}$ is a said to be a *prevariety*.

Given a class of similar algebras K, the least prevariety extending K is $\mathbb{ISP}(\mathsf{K})$ and is called the prevariety *generated* by K. For instance, in view of Examples 1.19, the class of distributive lattices is the prevariety generated by the two-element distributive lattice.

Similarly, the class of Boolean algebras is the prevariety generated by the two-element Boolean algebra (see Example 1.20, if necessary).

Our aim will be to prove that prevarieties are precisely the classes of algebras axiomatized by a certain kind of infinitary formulas. To this end, we rely on the following notational convention. When no confusion shall arise, given a sequence $\vec{a}$ and a set A, we write $\vec{a} \in A$ to indicate that the elements of the sequence $\vec{a}$ belong to A.

Definition 1.23. A *generalized quasi-equation* of type ρ is an expression Φ of the form

$$\Big(\mathop{\&}_{i \in I} \varphi_i(\vec{x}) \approx \psi_i(\vec{x}) \Big) \Longrightarrow \varepsilon(\vec{x}) \approx \delta(\vec{x}),$$

where $\{\varphi_i \approx \psi_i : i \in I\} \cup \{\varepsilon \approx \delta\}$ is a set of equations of type ρ. Then Φ is *valid* in an algebra $\boldsymbol{A}$ of type ρ when so is its universal closure, that is, for every $\vec{a} \in A$,

$$\text{if } (\varphi_i^{\boldsymbol{A}}(\vec{a}) = \psi_i^{\boldsymbol{A}}(\vec{a}), \text{ for all } i \in I), \text{ then } \varepsilon^{\boldsymbol{A}}(\vec{a}) = \delta^{\boldsymbol{A}}(\vec{a}).$$

In this case, we often say that $\boldsymbol{A}$ *satisfies* Φ and write $\boldsymbol{A} \vDash \Phi$.

Notice that, in the above definition, the set of indexes I can be arbitrarily large and that the same applies to the sequence of variables $\vec{x}$ that appear in the equations of Φ. This motivates the following.

Definition 1.24. A generalized quasi-equation is said to be

(i) a *quasi-equation* when the index set I is finite; and

(ii) an *equation* when the index set I is empty.

Remark 1.25. It might seem that we are using the term *equations* to refer to two distinct kinds of expressions, namely those of the form $\varepsilon \approx \delta$ and $\emptyset \Longrightarrow \varepsilon \approx \delta$. This is not a problem, however, because these expressions are synonyms, in the sense that an algebra satisfies $\varepsilon \approx \delta$ if and only if it satisfies $\emptyset \Longrightarrow \varepsilon \approx \delta$. Because of this, we will continue to denote equations by $\varepsilon \approx \delta$, while keeping in mind that they are special instances of generalized quasi-equations. ⊠

Definition 1.26. Let $\rho \colon \mathcal{F} \to \mathbb{N}$ be a type and X a set of variables disjoint from $\mathcal{F}$. The *term algebra* $\boldsymbol{T}_\rho(X)$ of type ρ over X is the unique algebra of type ρ whose universe is $T_\rho(X)$ and with basic n-ary operations f defined, for every $\varphi_1, \dots, \varphi_n \in T_\rho(X)$, as

$$f^{\boldsymbol{T}_\rho(X)}(\varphi_1, \dots, \varphi_n) \coloneqq f(\varphi_1, \dots, \varphi_n).$$

When no confusion might arise, we drop the subscript and write $T(X)$ instead of $T_\rho(X)$. Term algebras have the following fundamental property.

Proposition 1.27. *Let $\boldsymbol{A}$ be an algebra of type ρ and X a set of variables. Every function $f \colon X \to A$ extends uniquely to a homomorphism $f^* \colon \boldsymbol{T}_\rho(X) \to \boldsymbol{A}$.*

Proof. The unique extension f^* is defined, for every $\varphi(x_{\alpha_1}, \dots x_{\alpha_n}) \in T_\rho(X)$, as

$$f^*(\varphi) = \varphi^{\boldsymbol{A}}(f(x_{\alpha_1}), \dots, f(x_{\alpha_n})).$$ ⊠

Exercise 1.28. Prove the above proposition. ⊠

Theorem 1.29. *A class of similar algebras is a prevariety if and only if it can be axiomatized by a class of generalized quasi-equations.*

Proof. The "if" part follows from the fact that the validity of generalized quasi-equations persists under the formation of isomorphic copies, subalgebras, and direct product. To prove the converse, consider a prevariety K and let Σ be the class of generalized quasi-equations valid in it. Let K^+ be the class of algebras in which the generalized quasi-equations in Σ are valid. Clearly, $\mathsf{K} \subseteq \mathsf{K}^+$. To prove the other inclusion, consider an algebra $\boldsymbol{A} \in \mathsf{K}^+$. Let also X be a set of variables for which there exists a surjective map $f \colon X \to A$. By Proposition 1.27, f extends to a surjective homomorphism $f^* \colon \boldsymbol{T}(X) \to \boldsymbol{A}$. Together with Corollary 1.15, this yields

$$\boldsymbol{A} \cong \boldsymbol{T}(X)/\mathsf{Ker}(f^*). \tag{1.3}$$

Now, consider an arbitrary pair $\langle \varphi, \psi \rangle \in (T(X) \times T(X)) \smallsetminus \mathsf{Ker}(f^*)$. Notice that the elements of $T(X) \times T(X)$ are ordered pairs of terms and, therefore, can be viewed as equations under the identification of $\langle \varepsilon, \delta \rangle$ with $\varepsilon \approx \delta$. In this way, $\mathsf{Ker}(f^*)$ becomes a set of equations in variables X. Bearing this in mind, consider the generalized quasi-equation

$$\Phi := \Big(\& \mathsf{Ker}(f^*) \Big) \Longrightarrow \varphi \approx \psi.$$

We will prove that Φ fails in $\boldsymbol{A}$. For the sake of readability we will denote by $\vec{x}$ the sequence of all variables in X. Observe that every element $\varepsilon \in T(X)$ is of the form $\varepsilon(\vec{x})$. Then consider the assignment $f \colon X \to A$. We will denote by $f(\vec{x})$ the sequence obtained by applying f component-wise to $\vec{x}$. For every pair $\langle \varepsilon, \delta \rangle \in \mathsf{Ker}(f^*)$, we have

$$\varepsilon^{\boldsymbol{A}}(f(\vec{x})) = \varepsilon^{\boldsymbol{A}}(f^*(\vec{x})) = f^*(\varepsilon(\vec{x})) = f^*(\delta(\vec{x})) = \delta^{\boldsymbol{A}}(f^*(\vec{x})) = \delta^{\boldsymbol{A}}(f(\vec{x})).$$

The equalities above can be justified as follows. The first and the last holds because f^* extends f, the second and the fourth because $f^* \colon \boldsymbol{T}(X) \to \boldsymbol{A}$ is a homomorphism, and the third because $\langle \varepsilon, \delta \rangle \in \mathsf{Ker}(f^*)$. On the other hand, since $\langle \varphi, \psi \rangle \notin \mathsf{Ker}(f^*)$, a similar argument shows

$$\varphi^{\boldsymbol{A}}(f(\vec{x})) \neq \psi^{\boldsymbol{A}}(f(\vec{x})).$$

Thus, $\boldsymbol{A}$ refutes Φ, as desired.

Since $\boldsymbol{A} \in \mathsf{K}^+$, this implies that there exists some algebra $\boldsymbol{C}_{\varphi,\psi} \in \mathsf{K}$ and an assignment $g_{\varphi,\psi} \colon X \to C_{\varphi,\psi}$ such that

$$\varepsilon^{\boldsymbol{C}_{\varphi,\psi}}(g_{\varphi,\psi}(\vec{x})) = \delta^{\boldsymbol{C}_{\varphi,\psi}}(g_{\varphi,\psi}(\vec{x})), \text{ for all } \langle \varepsilon, \delta \rangle \in \mathsf{Ker}(f^*), \text{ and } \varphi^{\boldsymbol{C}_{\varphi,\psi}}(g_{\varphi,\psi}(\vec{x})) \neq \psi^{\boldsymbol{C}_{\varphi,\psi}}(g_{\varphi,\psi}(\vec{x})).$$

Recall that $g_{\varphi,\psi}$ extends uniquely to a homomorphism $g^*_{\varphi,\psi} \colon \boldsymbol{T}(X) \to \boldsymbol{C}_{\varphi,\psi}$. Moreover, from the above display it follows

$$g^*_{\varphi,\psi}(\varepsilon) = g^*_{\varphi,\psi}(\delta), \text{ for all } \langle \varepsilon, \delta \rangle \in \mathsf{Ker}(f^*), \text{ and } g^*_{\varphi,\psi}(\varphi) \neq g^*_{\varphi,\psi}(\psi).$$

Consequently,

$$\mathsf{Ker}(f^*) \subseteq \mathsf{Ker}(g^*_{\varphi,\psi}) \text{ and } \langle \varphi, \psi \rangle \notin \mathsf{Ker}(g^*_{\varphi,\psi}).$$

It follows that

$$\mathsf{Ker}(f^*) = \bigcap \{ \mathsf{Ker}(g^*_{\varphi,\psi}) : \langle \varphi, \psi \rangle \in (T(X) \times T(X)) \smallsetminus \mathsf{Ker}(f^*) \}.$$

By Proposition 1.21, this yields

$$\boldsymbol{T}(X)/\mathsf{Ker}(f^*) \in \mathbb{IP}_{\mathrm{SD}}(\{\boldsymbol{T}(X)/\mathsf{Ker}(g^*_{\varphi,\psi}) : \langle \varphi, \psi \rangle \in (T(X) \times T(X)) \smallsetminus \mathsf{Ker}(f^*)\}). \quad (1.4)$$

Moreover, from Corollary 1.15 and the fact that K is closed under $\mathbb{I}$ and $\mathbb{S}$ it follows that

$$\boldsymbol{T}(X)/\mathsf{Ker}(g^*_{\varphi,\psi}) \in \mathbb{IS}(\boldsymbol{C}_{\varphi,\psi}) \subseteq \mathsf{K},$$

for every $\langle \varphi, \psi \rangle \in (T(X) \times T(X)) \smallsetminus \mathsf{Ker}(f^*)$. Consequently, (1.4) simplifies to

$$\boldsymbol{T}(X)/\mathsf{Ker}(f^*) \in \mathbb{IP}_{\mathrm{SD}}(\mathsf{K}) \subseteq \mathsf{K},$$

where the last inclusion follows from the fact that K is a prevariety. Together with (1.3), this yields $\boldsymbol{A} \in \mathbb{I}(\mathsf{K}) \subseteq \mathsf{K}$. ⊠

Remark 1.30. In view of Theorem 1.29, prevarieties are classes of algebras axiomatized by classes of generalized quasi-equations. It is therefore natural to wonder whether there exists a prevariety that cannot be axiomatized by a set (as opposed to proper class) of generalized quasi-equations. It turns out that the answer to this question depends on the set theory we live in, as the nonexistence of such a prevariety is equivalent to *Vopěnka's Principle*. ⊠

Nonetheless, prevarieties axiomatizable by a set of generalized quasi-equations admit a relatively transparent description, as we proceed to explain. Given an infinite cardinal κ and a class of algebras K, let

$$\mathbb{U}_\kappa(\mathsf{K}) := \{\boldsymbol{A} : \boldsymbol{B} \in \mathsf{K}, \text{ for all } \kappa\text{-generated } \boldsymbol{B} \leqslant \boldsymbol{A}\}.$$

Definition 1.31. Let κ be an infinite cardinal. A *κ-generalized quasi-variety* is a prevariety closed under $\mathbb{U}_\kappa$.

When $\kappa = \aleph_0$, we often say that K is simply a *generalized quasi-variety*. Given a class of similar algebras K, the least κ-generalized quasi-variety extending K is $\mathbb{U}_\kappa\mathbb{ISP}(\mathsf{K})$ and is called the κ-generalized quasi-variety *generated* by K.

Theorem 1.32. *Let κ be an infinite cardinal. A class of similar algebras is a κ-generalized quasi-variety if and only if it can be axiomatized by a set of generalized quasi-equations in which at most κ variables occur.*

Proof. The "if" part follows from the fact that the validity of generalized quasi-equations in $\leqslant \kappa$ variables persist under the $\mathbb{I}, \mathbb{S}, \mathbb{P}, \mathbb{U}_\kappa$. To prove the converse, consider a κ-generalized quasi-variety K. Then let X be a set of variables of cardinality κ and Σ the class of generalized quasi-equations written with variables in X. Since X is a set, so is Σ. It only remains to prove that K coincides with the class K^+ of algebras satisfying the generalized quasi-equations in Σ. Clearly, $\mathsf{K} \subseteq \mathsf{K}^+$. To prove the other inclusion, consider an algebra $\boldsymbol{A} \in \mathsf{K}^+$. We need to prove that $\boldsymbol{A} \in \mathsf{K}$. Since K is closed under $\mathbb{U}_\kappa$, it suffices to show that all the κ-generated subalgebras of $\boldsymbol{A}$ belong to K.

Accordingly, let $\boldsymbol{B}$ be a κ-generated subalgebra of $\boldsymbol{A}$ and $Y \subseteq B$ a set of generators for $\boldsymbol{B}$ of size $\leqslant \kappa$. There exists a surjective map $f\colon X \to Y$. By Proposition 1.27, f extends to a surjective homomorphism $f^*\colon \boldsymbol{T}(X) \to \boldsymbol{B}$. Now, we repeat the argument in the proof of Theorem 1.29, obtaining $\boldsymbol{B} \in \mathsf{K}$, as desired. ⊠

Corollary 1.33. *A prevariety can be axiomatized by a set of generalized quasi-equations if and only if it is a κ-generalized quasi-variety, for some infinite cardinal κ.*

*Exercise** 1.34. Let K be a class of similar algebras and κ an infinite cardinal. Prove that the prevariety and the κ-generalized quasi-variety generated by K are, respectively, $\mathbb{ISP}(\mathsf{K})$ and $\mathbb{U}_\kappa\mathbb{ISP}(\mathsf{K})$. ⊠

1.4 Ultraproducts

Let $\boldsymbol{A}$ be a Boolean algebra. A nonempty subset $F \subseteq A$ is said to be a *filter* of $\boldsymbol{A}$ if it is an upset closed under binary meets. A filter is said to be *proper* when it differs from A. Lastly, a proper filter U of $\boldsymbol{A}$ is said to be a *ultrafilter* of $\boldsymbol{A}$ if it is maximal among the proper filters of $\boldsymbol{A}$ or, equivalently, if

$$a \in U \text{ or } \neg a \in U, \text{ for every } a \in A.$$

While the following result holds in ZFC, it cannot be proved in ZF (although it is strictly weaker then the axiom of choice).

Ultrafilter Lemma 1.35. *Every proper filter on a Boolean algebra can be extended to a ultrafilter.*

Ultrafilters on powerset Boolean algebras $\mathcal{P}(X)$ are also called *ultrafilters on* X. In this section we will use them to define a product-like construction known as *ultraproduct*. To this end, let $\{\boldsymbol{A}_i : i \in I\}$ be a family of similar algebras. The *equalizer* of a pair of elements $\vec{a}, \vec{c} \in \prod_{i\in I} A_i$ is the set of indexes on which the sequences $\vec{a}$ and $\vec{c}$ agree, that is,

$$[\![\vec{a} = \vec{c}]\!] := \{i \in I : \vec{a}(i) = \vec{c}(i)\}.$$

Moreover, given an ultrafilter U on the index set I, let θ_U be the binary relation on the Cartesian product $\prod_{i\in I} A_i$ defined as

$$\theta_U := \{\langle \vec{a}, \vec{c}\rangle : [\![\vec{a} = \vec{c}]\!] \in U\}.$$

Proposition 1.36. *If $\{\boldsymbol{A}_i : i \in I\}$ is a family of similar algebras and U an ultrafilter on I, then θ_U is a congruence of $\prod_{i\in I} \boldsymbol{A}_i$.*

Proof. We begin by proving that θ_U is an equivalence relation on $\prod_{i\in I} A_i$. To this end, consider $\vec{a}, \vec{b}, \vec{c} \in \prod_{i\in I} A_i$. We have

$$[\![\vec{a} = \vec{a}]\!] = \{i \in I : \vec{a}(i) = \vec{a}(i)\} = I.$$

Observe that $I \in U$, since U is a nonempty upset of $\mathcal{P}(I)$. Together with the above display, this yields $[\![\vec{a} = \vec{a}]\!] \in U$ and, therefore, $\langle \vec{a}, \vec{a}\rangle \in \theta_U$. It follows that θ_U is reflexive. To prove that it is symmetric, suppose that $\langle \vec{a}, \vec{c}\rangle \in \theta_U$. Then $[\![\vec{a} = \vec{c}]\!] \in U$. Since $[\![\vec{a} = \vec{c}]\!] = [\![\vec{c} = \vec{a}]\!]$, this implies $[\![\vec{c} = \vec{a}]\!] \in U$ and, therefore, $\langle \vec{c}, \vec{a}\rangle \in \theta_U$. Lastly, to prove that θ_U is transitive, suppose that $\langle \vec{a}, \vec{b}\rangle, \langle \vec{b}, \vec{c}\rangle \in \theta_U$, that is, $[\![\vec{a} = \vec{b}]\!], [\![\vec{b} = \vec{c}]\!] \in U$. Since U is closed under binary meets,

$$[\![\vec{a} = \vec{b}]\!] \cap [\![\vec{b} = \vec{c}]\!] \in U$$

Clearly, $[\![\vec{a} = \vec{b}]\!] \cap [\![\vec{b} = \vec{c}]\!] \subseteq [\![\vec{a} = \vec{c}]\!]$. Since U is an upset of $\mathcal{P}(I)$, we obtain that $[\![\vec{a} = \vec{c}]\!] \in U$, whence $\langle \vec{a}, \vec{c} \rangle \in \theta_U$. We conclude that θ_U is an equivalence relation.

To prove that θ_U is a congruence, it only remains to show that it preserves the basic operations. Accordingly, let f be a basic n-ary operation and $\vec{a}_1, \dots, \vec{a}_n, \vec{c}_1, \dots, \vec{c}_n \in \prod_{i \in I} A_i$ such that

$$\langle \vec{a}_1, \vec{c}_1 \rangle, \dots, \langle \vec{a}_n, \vec{c}_n \rangle \in \theta_U.$$

By definition of θ_U, this amounts to $[\![\vec{a}_1 = \vec{c}_1]\!], \dots, [\![\vec{a}_n = \vec{c}_n]\!] \in U$. Since U is a filter, it is closed under finite meets, whence

$$[\![\vec{a}_1 = \vec{c}_1]\!] \cap \dots \cap [\![\vec{a}_n = \vec{c}_n]\!] \in U. \tag{1.5}$$

We will show that

$$[\![\vec{a}_1 = \vec{c}_1]\!] \cap \dots \cap [\![\vec{a}_n = \vec{c}_n]\!] \subseteq [\![f^{\prod_{i \in I} \boldsymbol{A}_i}(\vec{a}_1, \dots, \vec{a}_n) = f^{\prod_{i \in I} \boldsymbol{A}_i}(\vec{c}_1, \dots, \vec{c}_n)]\!]. \tag{1.6}$$

To this end, consider $j \in [\![\vec{a}_1 = \vec{c}_1]\!] \cap \dots \cap [\![\vec{a}_n = \vec{c}_n]\!]$. We have

$$\vec{a}_1(j) = \vec{c}_1(j), \dots, \vec{a}_n(j) = \vec{c}_n(j).$$

Consequently,

$$\begin{aligned} f^{\prod_{i \in I} \boldsymbol{A}_i}(\vec{a}_1, \dots, \vec{a})(j) &= f^{\boldsymbol{A}_j}(\vec{a}_1(j), \dots, \vec{a}_n(j)) \\ &= f^{\boldsymbol{A}_j}(\vec{c}_1(j), \dots, \vec{c}_n(j)) \\ &= f^{\prod_{i \in I} \boldsymbol{A}_i}(\vec{c}_1, \dots, \vec{c})(j), \end{aligned}$$

that is, $j \in [\![f^{\prod_{i \in I} \boldsymbol{A}_i}(\vec{a}_1, \dots, \vec{a}_n) = f^{\prod_{i \in I} \boldsymbol{A}_i}(\vec{c}_1, \dots, \vec{c}_n)]\!]$. This establishes (1.6). Since U is an upset of $\mathcal{P}(I)$, from (1.5) and (1.6) it follows

$$[\![f^{\prod_{i \in I} \boldsymbol{A}_i}(\vec{a}_1, \dots, \vec{a}_n) = f^{\prod_{i \in I} \boldsymbol{A}_i}(\vec{c}_1, \dots, \vec{c}_n)]\!] \in U.$$

Hence, we conclude that $\langle f^{\prod_{i \in I} \boldsymbol{A}_i}(\vec{a}_1, \dots, \vec{a}_n), f^{\prod_{i \in I} \boldsymbol{A}_i}(\vec{c}_1, \dots, \vec{c}_n) \rangle \in \theta_U$, as desired. ⊠

In view of the above result, we can make the following definition.

Definition 1.37. An *ultraproduct* of a family of similar algebras $\{\boldsymbol{A}_i : i \in I\}$ is an algebra of the form $\prod_{i \in I} \boldsymbol{A}_i / \theta_U$, for some ultrafilter U on I.

Given a class of similar algebras K, we set

$$\mathbb{P}_{\mathrm{U}}(\mathsf{K}) := \{\boldsymbol{A} : \boldsymbol{A} \text{ is an ultraproduct of a family } \{\boldsymbol{B}_i : i \in I\} \subseteq \mathsf{K}\}.$$

Notice that $\mathbb{P}_{\mathrm{U}}(\mathsf{K}) \subseteq \mathbb{HP}(\mathsf{K})$. Furthermore, as usual, when $\mathsf{K} = \{\boldsymbol{A}\}$, we write $\mathbb{P}_{\mathrm{U}}(\boldsymbol{A})$ as a shorthand for $\mathbb{P}_{\mathrm{U}}(\{\boldsymbol{A}\})$.

Exercise 1.38. Prove that if U is not free (that is, it is principal), then $\prod_{i \in I} \boldsymbol{A}_i / \theta_U$ is isomorphic to some $\boldsymbol{A}_i$. Conclude that if I is finite, then $\prod_{i \in I} \boldsymbol{A}_i / \theta_U$ belongs to $\mathbb{I}\{\boldsymbol{A}_i : i \in I\}$. Because of this, interesting ultraproducts arise from free ultrafilters only. ⊠

The importance of ultraproducts is tightly related to the following fundamental result.

Theorem 1.39 (Łos). *Let $\{\boldsymbol{A}_i : i \in I\}$ be a family of similar algebras, U an ultrafilter on I, and $\phi(x_1, \dots, x_n)$ a first-order formula. For every $\vec{a}_1, \dots, \vec{a}_n \in \prod_{i \in I} A_i$,*

$$\prod_{i \in I} \boldsymbol{A}_i / \theta_U \vDash \phi(\vec{a}_1/\theta_U, \dots, \vec{a}_n/\theta_U) \Longleftrightarrow \{i \in I : \boldsymbol{A}_i \vDash \phi(\vec{a}_1(i), \dots, \vec{a}_n(i))\} \in U.$$

Corollary 1.40. *Let $\{\boldsymbol{A}_i : i \in I\}$ be a family of similar algebras, U an ultrafilter on I, and ϕ a first-order sentence. If ϕ is valid in all the $\boldsymbol{A}_i$, then it is valid in $\prod_{i \in I} \boldsymbol{A}_i / \theta_U$.*

In view Łos' Theorem, ultraproducts are instrumental to construct nonstandard models of first-order theories. For instance, let $\mathbb{N} = \langle \mathbb{N}; s, +, \cdot, 0 \rangle$ be the standard model of Peano Arithmetic. If U is a free ultrafilter on $\mathbb{N}$, the ultraproduct $\prod_{n \in \mathbb{N}} \mathbb{N}_n / U$ is *elementarily equivalent* to $\mathbb{N}$, that is, it satisfies the same first-order sentences as $\mathbb{N}$. On the other hand, it is not hard to see that $\prod_{n \in \mathbb{N}} \mathbb{N}_n / U$ is uncountable and, therefore, contains many "infinite" (or nonstandard) natural numbers.

For the present purpose, however, we will not need the full strength of Łos Theorem and, therefore, we shall omit its proof. Instead, we shall focus on a particular embedding theorem for ultraproducts that depends on the following notion.

Definition 1.41. A *local subgraph* $\mathbb{X}$ of an algebra $\boldsymbol{A}$ is a finite subset $X \subseteq A$ endowed with the restriction of finitely many basic operations of $\boldsymbol{A}$ to X.

In this case, $\mathbb{X}$ is a finite *partial* algebra of finite type (even when the type of $\boldsymbol{A}$ is infinite).

Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be similar algebras and $\mathbb{X}$ a local subgraph of $\boldsymbol{A}$. A map $f \colon X \to B$ is said to be an *embedding* of $\mathbb{X}$ into $\boldsymbol{B}$ if it is injective and, for every basic n-ary operation g of the type of $\mathbb{X}$ and $a_1, \dots, a_n \in X$ such that $g^{\boldsymbol{A}}(a_1, \dots, a_n) \in X$,

$$f(g^{\boldsymbol{A}}(a_1, \dots, a_n)) = g^{\boldsymbol{B}}(f(a_1), \dots, f(a_n)).$$

Theorem 1.42. *Let $\mathsf{K} \cup \{\boldsymbol{A}\}$ be a class of similar algebras. If every local subgraph of $\boldsymbol{A}$ can be embedded into some member of K, then $\boldsymbol{A} \in \mathbb{ISP}_{\mathrm{U}}(\mathsf{K})$.*

Proof. Let I be the set of local subgraphs of $\boldsymbol{A}$. By assumption, for every $\mathbb{X} \in I$ there are an algebra $\boldsymbol{B}_{\mathbb{X}} \in \mathsf{K}$ and an embedding $h_{\mathbb{X}} \colon \mathbb{X} \to \boldsymbol{B}_{\mathbb{X}}$. We define a partial order $\sqsubseteq$ on I as follows:

$$\mathbb{X} \sqsubseteq \mathbb{Y} \Longleftrightarrow X \subseteq Y \text{ and the type of } \mathbb{Y} \text{ extends that of } \mathbb{X}.$$

Then, for every $\mathbb{X} \in I$, define

$$J_{\mathbb{X}} \coloneqq \{\mathbb{Y} \in I \colon \mathbb{X} \sqsubseteq \mathbb{Y}\}.$$

Moreover, let $\mathcal{F}$ be the filter of $\mathcal{P}(I)$ generated by $\{J_{\mathbb{X}} \colon \mathbb{X} \in I\}$. Recall that

$$\mathcal{F} = \{Y \subseteq I : J_{\mathbb{X}_1} \cap \dots \cap J_{\mathbb{X}_n} \subseteq Y, \text{ for some } \mathbb{X}_1, \dots, \mathbb{X}_n \in I\}.$$

We will prove that $\mathcal{F}$ is proper. To this end, consider $\mathbb{X}_1, \dots, \mathbb{X}_n \in I$. Then let $\mathbb{Y}$ be the local subgraph of $\boldsymbol{A}$ with universe $Y \coloneqq X_1 \cup \dots \cup X_n$ and whose type in the union of the types of the various $\mathbb{X}_i$. Then

$$\mathbb{X}_i \sqsubseteq \mathbb{Y}, \text{ for every } i \leqslant n,$$

that is, $\mathbb{Y} \in J_{\mathbb{X}_1} \cap \cdots \cap J_{\mathbb{X}_n}$. It follows that $\emptyset \notin \mathcal{F}$ and, therefore, that $\mathcal{F}$ is proper. As $\mathcal{F}$ is a proper filter, by the Ultrafilter Lemma, it can be extended to an ultrafilter U on I.

Now, consider a map

$$f \colon A \to \prod_{\mathbb{X} \in I} B_{\mathbb{X}}$$

such that $f(a)(\mathbb{X}) = h_{\mathbb{X}}(a)$, for every $a \in A$ and $\mathbb{X} \in I$ such that $a \in \mathbb{X}$. Moreover, let

$$f^* \colon A \to \prod_{\mathbb{X} \in I} B_{\mathbb{X}}/\theta_U$$

be the map defined by the rule

$$f^*(a) \coloneqq f(a)/\theta_U.$$

We will show f^* is an embedding of $\boldsymbol{A}$ into $\prod_{\mathbb{X} \in I} \boldsymbol{B}_{\mathbb{X}}/\theta_U$.

In order to prove that f^* is injective, consider a pair of distinct elements $a, c \in A$. Consider a local subgraph $\mathbb{Y}$ of $\boldsymbol{A}$ containing a and c. We will show that

$$J_{\mathbb{Y}} \subseteq \{\mathbb{X} \in I : f(a)(\mathbb{X}) \neq f(c)(\mathbb{X})\} \tag{1.7}$$

Consider $\mathbb{X} \in J_{\mathbb{Y}}$. Then $\mathbb{Y} \sqsubseteq \mathbb{X}$ and, therefore, $a, c \in Y \subseteq X$. Since $a, c \in X$, we have

$$f(a)(\mathbb{X}) = h_{\mathbb{X}}(a) \quad \text{and} \quad f(c)(\mathbb{X}) = h_{\mathbb{X}}(c).$$

Furthermore, $h_{\mathbb{X}}(a) \neq h_{\mathbb{X}}(c)$, because $h_{\mathbb{X}}$ is injective and $a \neq c$. This yields $f(a)(\mathbb{X}) \neq f(c)(\mathbb{X})$, establishing (1.7).

Recall that the definition of U guarantees that $J_{\mathbb{Y}} \in \mathcal{F} \subseteq U$. Therefore, since U is an upset of $\mathcal{P}(I)$, we can apply (1.7) obtaining

$$I \smallsetminus [\![f(a) = f(c)]\!] = \{\mathbb{X} \in I : f(a)(\mathbb{X}) \neq f(c)(\mathbb{X})\} \in U.$$

Since U is a proper filter, this implies

$$[\![f(a) = f(c)]\!] \notin U$$

and, therefore,

$$f^*(a) = f(a)/\theta_U \neq f(c)/\theta_U = f^*(c).$$

Hence, we conclude that f^* is injective.

To prove that it is a homomorphism, consider a basic n-ary operation g and $a_1, \ldots, a_n \in A$. Then consider a local subgraph $\mathbb{Y}$ of $\boldsymbol{A}$ whose universe contains $a_1, \ldots, a_n, g^{\boldsymbol{A}}(a_1, \ldots, a_n)$ and whose type contains g. We will prove that

$$J_{\mathbb{Y}} \subseteq [\![f(g^{\boldsymbol{A}}(a_1, \ldots, a_n)) = g^{\prod_{\mathbb{X} \in I} \boldsymbol{B}_{\mathbb{X}}}(f(a_1), \ldots, f(a_n))]\!]. \tag{1.8}$$

Consider $\mathbb{V} \in J_{\mathbb{Y}}$. Since $\mathbb{Y} \sqsubseteq \mathbb{V}$, the type of $\mathbb{V}$ contains g and $a_1, \ldots, a_n, g^{\boldsymbol{A}}(a_1, \ldots, a_n) \in V$. Since $a_1, \ldots, a_n, g^{\boldsymbol{A}}(a_1, \ldots, a_n) \in V$, we have

$$\begin{gathered} f(a_1)(\mathbb{V}) = h_{\mathbb{V}}(a_1) \\ \vdots \\ f(a_n)(\mathbb{V}) = h_{\mathbb{V}}(a_n) \\ f(g^{\boldsymbol{A}}(a_1, \ldots, a_n))(\mathbb{V}) = h_{\mathbb{V}}(g^{\boldsymbol{A}}(a_1, \ldots, a_n)). \end{gathered}$$

Furthermore, as the type of $\mathbb{V}$ contains g,

$$h_{\mathbb{V}}(g^{\boldsymbol{A}}(a_1, \dots, a_n)) = g^{\boldsymbol{B}_{\mathbb{V}}}(h_{\mathbb{V}}(a_1), \dots, h_{\mathbb{V}}(a_n)).$$

From the above displays it follows

$$f(g^{\boldsymbol{A}}(a_1, \dots, a_n))(\mathbb{V}) = g^{\boldsymbol{B}_{\mathbb{V}}}(f(a_1)(\mathbb{V}), \dots, f(a_n)(\mathbb{V})) = g^{\prod_{\mathbb{X}\in I}\boldsymbol{B}_{\mathbb{X}}}(f(a_1), \dots, f(a_n))(\mathbb{V}),$$

that is, $\mathbb{V} \in [\![f(g^{\boldsymbol{A}}(a_1, \dots, a_n)) = g^{\prod_{\mathbb{X}\in I}\boldsymbol{B}_{\mathbb{X}}}(f(a_1), \dots, f(a_n))]\!]$. This establishes (1.8). Lastly, as $J_{\mathbb{Y}} \in U$ and U is an upset of $\mathcal{P}(I)$, condition (1.8) implies

$$[\![f(g^{\boldsymbol{A}}(a_1, \dots, a_n)) = g^{\prod_{\mathbb{X}\in I}\boldsymbol{B}_{\mathbb{X}}}(f(a_1), \dots, f(a_n))]\!] \in U,$$

and, therefore,

$$\begin{aligned} f^*(g^{\boldsymbol{A}}(a_1, \dots, a_n)) &= f(g^{\boldsymbol{A}}(a_1, \dots, a_n))/\theta_U \\ &= g^{\prod_{\mathbb{X}\in I}\boldsymbol{B}_{\mathbb{X}}}(f(a_1), \dots, f(a_n))/\theta_U \\ &= g^{\prod_{\mathbb{X}\in I}\boldsymbol{B}_{\mathbb{X}}/\theta_U}(f(a_1)/\theta_U, \dots, f(a_n)/\theta_U) \\ &= g^{\prod_{\mathbb{X}\in I}\boldsymbol{B}_{\mathbb{X}}/\theta_U}(f^*(a_1), \dots, f^*(a_n)). \end{aligned}$$

Hence, we conclude that f^* is a homomorphism and, therefore, an embedding of $\boldsymbol{A}$ into $\prod_{\mathbb{Y}\in I}\boldsymbol{B}_{\mathbb{Y}}/\theta_U$. As a consequence,

$$\boldsymbol{A} \in \mathbb{ISP}_{\mathrm{U}}(\{\boldsymbol{B}_{\mathbb{X}} : \mathbb{X} \in I\}) \subseteq \mathbb{ISP}_{\mathrm{U}}(\mathsf{K}). \qquad \boxtimes$$

Corollary 1.43. *Every algebra embeds into an ultraproduct of its finitely generated subalgebras.*

Example 1.44 (Lattices)**.** Let Latt be the class of all lattices and $\mathsf{Latt}^{<\omega}$ that of finite lattices. We will show that $\mathsf{Latt} = \mathbb{ISP}_{\mathrm{U}}(\mathsf{Latt}^{<\omega})$. The inclusion from right to left follows from the fact that Latt is closed under $\mathbb{I}, \mathbb{S}$, and $\mathbb{P}_{\mathrm{U}}$. For the other inclusion, consider a lattice$\boldsymbol{A}$. We know that every local subgraph $\mathbb{X}$ of $\boldsymbol{A}$ can be embedded into the Dedekind-MacNeille completion of the subposet of $\boldsymbol{A}$ with universe X. As the Dedekind-MacNeille completion of a finite poset is finite, it follows that $\mathbb{X}$ can be emdebbed into a finite lattice. As a consequence, every local subgraph of $\boldsymbol{A}$ can be embedded into some finite lattice. By Theorem 1.42, this implies that $\boldsymbol{A} \in \mathbb{ISP}_{\mathrm{U}}(\mathsf{Latt}^{<\omega})$, as desired. $\boxtimes$

1.5 Quasi-varieties

At this stage it is natural to wonder whether it is possible to characterize classes of algebras axiomatizable by quasi-equations (Definition 1.24) in terms of closure under certain class operators. The answer is affirmative, as we proceed to explain.

Definition 1.45. A prevariety closed under $\mathbb{P}_{\mathrm{U}}$ is a said to be a *quasi-variety*.

The aim of this section is to prove the following classical result.

Maltsev's Theorem 1.46. *A class of similar algebras is a quasi-variety if and only if it can be axiomatized by a set of quasi-equations.*

Proof. The "only if" part follows from the fact that the validity of quasi-equations is preserved by the class operators $\mathbb{I}, \mathbb{S}, \mathbb{P}$, and $\mathbb{P}_{\mathrm{U}}$. To prove the converse, consider a prevariety K closed under $\mathbb{P}_{\mathrm{U}}$. Moreover, let *Var* be a denumerable set of variables and Σ the set of quasi-equations, with variables in *Var*, valid in K. Let also K^+ be the class of algebras axiomatized by Σ. Our aim is to prove that $\mathsf{K} = \mathsf{K}^+$.

The inclusion $\mathsf{K} \subseteq \mathsf{K}^+$ is straightforward. To prove the other one, consider an algebra $\boldsymbol{A} \in \mathsf{K}^+$. In order to prove that $\boldsymbol{A} \in \mathsf{K}^+$, it suffices to show that every local subgraph of $\boldsymbol{A}$ embeds in some members of K. This is because, in this case, $\boldsymbol{A} \in \mathbb{ISP}_{\mathrm{U}}(\mathsf{K})$, by Theorem 1.42. Since K is closed under $\mathbb{I}, \mathbb{S}$, and $\mathbb{P}_{\mathrm{U}}$, this implies $\boldsymbol{A} \in \mathsf{K}$, as desired.

Then consider a local subgraph $\mathbb{X}$ of $\boldsymbol{A}$. By definition, $\mathbb{X}$ consists of a finite set $\{a_1, \dots, a_n\}$ endowed with the restriction of finitely many basic operations $f_1, \dots, f_m$ of $\boldsymbol{A}$ to X. Fix n distinct variables $x_1, \dots, x_n \in Var$, corresponding to the elements $a_1, \dots, a_n$ of X. The *positive* and *negative atomic diagrams* of $\mathbb{X}$ are, respectively,

$$\mathcal{D}^+(\mathbb{X}) := \{f_i(x_{k_1}, \dots, x_{k_s}) \approx x_j \colon i \leqslant m \text{ and } k_1, \dots, k_s, j \leqslant n \text{ and } f_i^{\boldsymbol{A}}(a_{k_1}, \dots, a_{k_s}) = a_j\}$$
$$\mathcal{D}^-(\mathbb{X}) := \{x_i \not\approx x_j \colon i, j \leqslant n \text{ and } a_i \neq a_j\}.$$

Observe that both $\mathcal{D}^+(\mathbb{X})$ and $\mathcal{D}^-(\mathbb{X})$ are finite sets. Then take an enumeration

$$\mathcal{D}^-(\mathbb{X}) = \{\varepsilon_1 \not\approx \delta_1, \dots, \varepsilon_t \not\approx \delta_t\}.$$

Moreover, for each $i \leqslant t$, consider the quasi-equation

$$\Phi_i := \left(\mathop{\&} \mathcal{D}^+(\mathbb{X}) \right) \Longrightarrow \varepsilon_i \approx \delta_i.$$

As witnessed by the natural assignment

$$x_1 \longmapsto a_1, \dots, x_n \longmapsto a_n,$$

the quasi-equations $\Phi_1, \dots, \Phi_t$ fail in $\boldsymbol{A}$. Since they are written with variables in *Var* and $\boldsymbol{A}$ satisfies all the quasi-equations with variables in *Var* valid in K, this implies that each Φ_i fails in some $\boldsymbol{B}_i \in \mathsf{K}$ under an assignment

$$x_1 \longmapsto b^i_1, \dots, x_n \longmapsto b^i_n. \tag{1.9}$$

Now, consider the map $h \colon X \to (B_1 \times \cdots \times B_t)$, defined by the rule

$$a_1 \longmapsto \langle b^1_1, \dots, b^t_1 \rangle, \dots, a_n \longmapsto \langle b^1_n, \dots, b^t_n \rangle.$$

We will prove that h is an embedding of $\mathbb{X}$ into $\boldsymbol{B}_1 \times \cdots \times \boldsymbol{B}_t$. To prove that h is injective, consider two distinct elements $a_p, a_q \in X$. Then the formula $x_p \not\approx x_q$ belongs to the negative atomic diagram of $\mathbb{X}$. Then there exists $i \leqslant t$ such that

$$\Phi_i = \left(\mathop{\&} \mathcal{D}^+(\mathbb{X}) \right) \Longrightarrow x_p \approx x_q.$$

Since Φ_i fails in $\boldsymbol{B}_i$ under the assignment in (1.9), we obtain $b^i_p \neq b^i_q$. As a consequence,

$$h(a_p)(i) = b^i_p \neq b^i_q = h(a_q)(i)$$

and, therefore, $h(a_p) \neq h(a_q)$. Hence, h is injective. To prove that it preserves the partial operations, consider a basic s-ary operation f_j in the type of $\mathbb{X}$ and $a_{k_1}, \dots, a_{k_s} \in X$ such that $f_j^{\boldsymbol{A}}(a_{k_1}, \dots, a_{k_s}) \in X$. Then there exists some $p \leqslant n$ such that $a_p = f_j^{\boldsymbol{A}}(a_{k_1}, \dots, a_{k_s})$. Moreover, the equation

$$f_j(x_{k_1}, \dots, x_{k_s}) \approx x_p$$

belongs to the positive atomic diagram $\mathcal{D}^+(\mathbb{X})$ of $\mathbb{X}$. As each quasi-equation Φ_i fails under the assignment in (1.9), the same assignment satisfies the antecedent of Φ_i, namely $\mathcal{D}^+(\mathbb{X})$. It follows that

$$f_j^{\boldsymbol{B}_i}(b^i_{k_1}, \dots, b^i_{k_s}) = b^i_p, \text{ for each } i \leqslant t.$$

As a consequence, for every $i \leqslant t$,

$$\begin{aligned} h(f_j^{\boldsymbol{A}}(a_{k_1}, \dots, a_{k_s}))(i) &= h(a_p)(i) \\ &= b^i_p \\ &= f_j^{\boldsymbol{B}_i}(b^i_{k_1}, \dots, b^i_{k_s}) \\ &= f_j^{\boldsymbol{B}_i}(h(a_{k_1})(i), \dots, h(a_{k_s})(i)) \\ &= f_j^{\boldsymbol{B}_1 \times \cdots \times \boldsymbol{B}_t}(h(a_{k_1}), \dots, h(a_{k_s}))(i). \end{aligned}$$

Thus, $h(f_j^{\boldsymbol{A}}(a_{k_1}, \dots, a_{k_s})) = f_j^{\boldsymbol{B}_1 \times \cdots \times \boldsymbol{B}_t}(h(a_{k_1}), \dots, h(a_{k_s}))$. We conclude that $h \colon \mathbb{X} \to (\boldsymbol{B}_1 \times \cdots \times \boldsymbol{B}_t)$ is an embedding. Since $\boldsymbol{B}_1, \dots, \boldsymbol{B}_t \in \mathsf{K}$ and K is closed under $\mathbb{P}$, the direct product $\boldsymbol{B}_1 \times \cdots \times \boldsymbol{B}_t$ belongs to K. Hence, $\mathbb{X}$ embeds into some member of K, as desired. ⊠

Exercise 1.47. Prove that if a quasi-equation Φ is valid in a class of similar algebras K, then it is also valid in $\mathbb{P}_{\mathrm{U}}(\mathsf{K})$. ⊠

Given a class of similar algebras K, the least quasi-variety extending K exists and will be denoted by $\mathbb{Q}(\mathsf{K})$ and called the quasi-variety *generated* by K.

Proposition 1.48 (Maltsev)**.** *For every class of algebras* K*,*

$$\mathbb{Q}(\mathsf{K}) = \mathbb{ISPP}_{\mathrm{U}}(\mathsf{K}).$$

Proof. The inclusion $\mathbb{ISPP}_{\mathrm{U}}(\mathsf{K}) \subseteq \mathbb{Q}(\mathsf{K})$ is obvious. To prove the other, consider $\boldsymbol{A} \in \mathbb{Q}(\mathsf{K})$. By Maltsev's Theorem, $\mathbb{Q}(\mathsf{K})$ is the class of all algebras satisfying the quasi-equations valid in K. The proof of the hard part of Maltsev's Theorem show that $\boldsymbol{A} \in \mathbb{ISP}_{\mathrm{U}}\mathbb{P}(\mathsf{K})$. Therefore, it only remains to show that $\mathbb{P}_{\mathrm{U}}\mathbb{P}(\mathsf{K}) \subseteq \mathbb{ISPP}_{\mathrm{U}}(\mathsf{K})$. But this is an easy exercise on class operators (the details are sketched below).

Consider an algebra $\boldsymbol{B} \in \mathbb{P}_{\mathrm{U}}\mathbb{P}(\mathsf{K})$. There exists an index set I, an ultrafilter U on I, and a family of algebras $\{\boldsymbol{B}_j : j \in J_i\}$ for each $i \in I$ such that

$$\boldsymbol{B} = \Big(\prod_{i \in I}\Big(\prod_{j \in J_i} \boldsymbol{B}_j\Big)\Big)/\theta_U.$$

Let J be the set of all maps $f \colon I \to \bigcup_{i \in I} J_i$ such that $f(i) \in J_i$. Moreover, let

$$g \colon B \to \prod_{f \in J}\Big(\prod_{i \in I} B_{f(i)}\Big)$$

be the map defined by the rule $g(b)(f)(i) := b(i)(f(i))$. It is not hard to check that the map

$$g^* : B \to \Big(\prod_{f \in J} \Big(\prod_{i \in I} B_{f}(i)\Big)\Big)/\theta_U$$

that sends an element $b \in B$ to $f(b)/\theta_U$ is an embedding, whence $\boldsymbol{B} \in \mathbb{ISPP}_{\mathrm{U}}(\mathsf{K})$. $\boxtimes$

Corollary 1.49. *If* K *be a finite set of finite similar algebras, then* $\mathbb{Q}(\mathsf{K}) = \mathbb{ISP}_{\mathrm{U}}(\mathsf{K})$, *that is, the quasi-variety and the prevariety generated by* K *coincide.*

Proof. Since K is a finite set of finite algebras, $\mathbb{P}_{\mathrm{U}}(\mathsf{K}) \subseteq \mathbb{I}(\mathsf{K})$. As a consequence, we obtain $\mathbb{ISP}(\mathsf{K}) = \mathbb{ISPP}_{\mathrm{U}}(\mathsf{K})$. Together with Proposition 1.48, this yields $\mathbb{Q}(\mathsf{K}) = \mathbb{ISP}(\mathsf{K})$. $\boxtimes$

Exercise 1.50. Prove that if K is a class of similar algebras, then $\mathbb{P}_{\mathrm{U}}\mathbb{P}(\mathsf{K}) \subseteq \mathbb{ISPP}_{\mathrm{U}}(\mathsf{K})$. Hint: use the sketch in the last part of Proposition 1.48. $\boxtimes$

Fix a denumerable set of variables X. The *quasi-equational theory* of a class of similar algebras K is the set of quasi-equations with variables in X valid in K.

Example 1.51 (Lattices)**.** Recall from Example 1.44 that $\mathsf{Latt} = \mathbb{ISP}_{\mathrm{U}}(\mathsf{Latt}^{<\omega})$. As a consequence, $\mathsf{Latt} = \mathbb{Q}(\mathsf{Latt}^{<\omega})$. Thus, a quasi-equation is valid in Latt if and only if it is valid in $\mathsf{Latt}^{<\omega}$. Since the class of lattices is finitely axiomatizable, this implies that the quasi-equational theory of Latt is decidable.

At this stage, it is natural to wonder whether Latt is also the prevariety generated by $\mathsf{Latt}^{<\omega}$. This is not the case, as we proceed to explain. First, consider a lattice $\boldsymbol{A}$ with precisely two congruences, namely id_A and $A \times A$. For instance, we can take $\boldsymbol{A}$ to the the poset of equivalence relations on an infinite set. Then suppose, with a view to contradiction, that $\mathsf{Latt} = \mathbb{ISP}(\mathsf{Latt}^{<\omega})$. Since $\mathbb{ISP}(\mathsf{Latt}^{<\omega}) = \mathbb{IP}_{\mathrm{SD}}\mathbb{S}(\mathsf{Latt}^{<\omega}) = \mathbb{IP}_{\mathrm{SD}}(\mathsf{Latt}^{<\omega})$, there exists a subdirect embedding $f\colon \boldsymbol{A} \to \prod_{i \in I} \boldsymbol{B}_i$ where $\{\boldsymbol{B}_i : i \in I\}$ is a family of finite lattices. For the sake of simplicity, we can assume that f is the identity map and that $\boldsymbol{A}$ is a subdirect product of $\{\boldsymbol{B}_i : i \in I\}$. Then consider consider $j \in I$. Since $\boldsymbol{A}$ is a subdirect product of $\{\boldsymbol{B}_i : i \in I\}$, the projection map $p_j\colon \boldsymbol{A} \to \boldsymbol{B}_j$ is surjective. As $\boldsymbol{A}$ is simple, $\mathsf{Ker}(p_j)$ is either id_A and $A \times A$. If it is id_A, then p_j is injective and, therefore, $\boldsymbol{B}_j$ is infinite, which is false. Then $\mathsf{Ker}(p_j) = A \times A$. But this implies that $\boldsymbol{B}_j$ is the trivial lattice. It follows that each $\boldsymbol{B}_i$ is trivial, whence so is $\prod_{i \in I} \boldsymbol{B}_i$. But this contradicts the assumption that $\boldsymbol{A}$ is an infinite subalgebra of $\prod_{i \in I} \boldsymbol{B}_i$. We conclude that $\mathbb{ISP}(\mathsf{Latt}^{<\omega}) \subsetneq \mathsf{Latt}$ and, therefore, that there are generalized quasi-equations that are valid in $\mathsf{Latt}^{<\omega}$, but not in Latt.

The contrasts with the case of distributive lattices. Indeed the class DL of distributive lattices is the prevariety generated by its finite members. Even more is true: DL is the prevariety generated by the two-element distributive lattice. Similarly, the class of Boolean algebras is the prevariety generated the two-element Boolean algebra (and, therefore, by finite Boolean algebras). $\boxtimes$

Given a set of variables X and a type ρ, we denote by $E_\rho(X)$ the set of equations of type ρ with variables in X.

Definition 1.52. Let K be a class of similar algebras and X a set of variables. We define a binary relation $\vDash_{\mathsf{K}}^{X} \subseteq \mathcal{P}(E_\rho(X)) \times E_\rho(X)$ as follows:

$$\begin{aligned} \Theta \vDash_{\mathsf{K}}^{X} \varepsilon \approx \delta \Longleftrightarrow & \text{ for every } \boldsymbol{A} \in \mathsf{K} \text{ and every } \vec{a} \in A, \\ & \text{if } \varphi^{A}(\vec{a}) = \psi^{A}(\vec{a}) \text{ for all } \varphi \approx \psi \in \Theta, \text{ then } \varepsilon^{A}(\vec{a}) = \delta^{A}(\vec{a}). \end{aligned}$$

The relation $\vDash_{\mathsf{K}}^{X}$ is known as the *equational consequence relative to* K (with variables in X).

Notice that the equational consequence relative to K describes the validity of generalized quasi-equations in K, in the sense that

$$\Theta \vDash_{\mathsf{K}}^{X} \varepsilon \approx \delta \Longleftrightarrow \mathsf{K} \vDash \bigwith \Theta \Longrightarrow \varepsilon \approx \delta.$$

When the set X is understood, we drop the superscript and write $\vDash_{\mathsf{K}}$ instead of $\vDash_{\mathsf{K}}^{X}$. Moreover, when $\mathsf{K} = \{\boldsymbol{A}\}$ we write $\vDash_{\boldsymbol{A}}^{X}$ as a shorthand for $\vDash_{\{\boldsymbol{A}\}}^{X}$.

Example 1.53 (Lattices)**.** Since the class of distributive lattices DL is the prevariety generated by the two-element distributive lattice $\boldsymbol{B}$, the equational consequences relative to DL and to $\boldsymbol{B}$ coincide, for every set of variables X. On the other hand, as the class Latt of all lattices is not the prevariety generated by the class $\mathsf{Latt}^{<\omega}$ of finite lattices, the equational consequence relative to Latt and $\mathsf{Latt}^{<\omega}$ do not coincide in general. ⊠

Example 1.54 (Boolean algebras)**.** Recall that the class BA of Boolean algebras is the prevariety generated by the two-element Boolean algebras $\boldsymbol{B}$. Consequently, the equational consequences relative to BA and $\boldsymbol{B}$ coincide, for every set of variables X. Notice that these relative equational consequences are related to Classical Propositional Logic **CPC** by the following completeness theorem: for every set of variables X and $\Gamma \cup \{\varphi\} \subseteq T(X)$,

$$\begin{aligned} \Gamma \vdash_{\mathbf{CPC}} \varphi & \Longleftrightarrow \{\gamma \approx 1 : \gamma \in \Gamma\} \vDash_{\mathsf{BA}} \varphi \approx 1 \\ & \Longleftrightarrow \{\gamma \approx 1 : \gamma \in \Gamma\} \vDash_{\boldsymbol{B}} \varphi \approx 1. \end{aligned}$$

The first of the above equivalences can be proved using the Lindenbaum-Tarski method, while the second follows from the Ultrafilter Lemma. ⊠

Notably, quasi-varieties coincide with the prevarieties whose relative equational consequence is finitary.

Theorem 1.55. *A* K *prevariety is a quasi-variety if and only if the equational consequence relative to* K *is finitary on every set of variables* X*, in the sense that*

$$\textit{if } \Theta \vDash_{\mathsf{K}}^{X} \varepsilon \approx \delta, \textit{ there exists a finite } \Sigma \subseteq \Theta \textit{ such that } \Sigma \vDash_{\mathsf{K}}^{X} \varepsilon \approx \delta.$$

The proof of the above result is based on an ultraproduct construction, whose standard application is a proof of the Compactness Theorem of first-order logic (semantically defined). Here we will use it to prove the compactness (a.k.a. finitarity) of relative equational consequences.

*Exercise** 1.56. Prove Theorem 1.55. To do this, use the following proof strategy:

(i) First, suppose that the equational consequence relative to K is finitary (on every set of variables). Use this fact to show that every generalized quasi-equation valid in K can be finitized, i.e., transformed into a quasi-equation valid in K.

(ii) Then prove that ultraproducts preserve the validity of quasi-equations (Exercise 1.47). Use this fact to conclude that all the generalized quasi-equations valid in K are also valid in $\mathbb{P}_{\mathsf{U}}(\mathsf{K})$.

(iii) Lastly, use Theorem 1.29 to conclude that $\mathbb{P}_{\mathsf{U}}(\mathsf{K}) \subseteq \mathsf{K}$, as desired.

(iv) The converse implication is the standard ultraproduct argument. Suppose that K is closed under $\mathbb{P}_{\mathsf{U}}$. Then consider $\Theta \cup \{\varepsilon \approx \delta\} \subseteq E(X)$ such that $\Sigma \nvDash_{\mathsf{K}} \varepsilon \approx \delta$, for every finite $\Sigma \subseteq \Theta$. Then, for every finite $\Sigma \subseteq \Theta$ there are $\boldsymbol{A}_\Sigma \in \mathsf{K}$ and an assignment $h_\Sigma \colon X \to \boldsymbol{A}_\Sigma$ that falsify the quasi-equation $\mathop{\&} \Sigma \Longrightarrow \varepsilon \approx \delta$.

(v) Let I be the set of finite subsets of Θ. For each $\Sigma \in I$, define

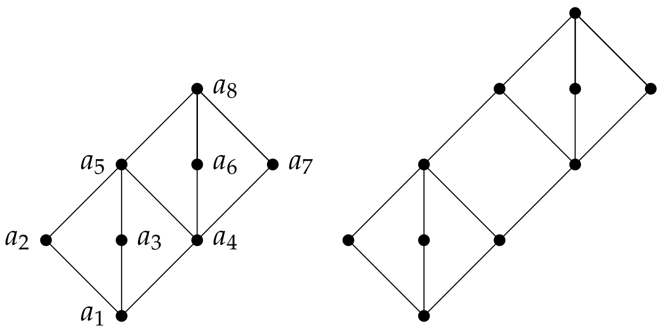
$$J_\Sigma \coloneqq \{\Delta \in I : \Sigma \subseteq \Delta\}.$$

Prove that there exists a ultrafilter U on I extending the set $\{J_\Sigma : \Sigma \in I\}$.

(vi) Then consider the ultraproduct $\prod_{\Sigma \in I} \boldsymbol{A}_\Sigma / \theta_U$ and observe that it belongs to K by assumption. Let $h \colon X \to \prod_{\Sigma \in I} \boldsymbol{A}_\Sigma / \theta_U$ be the natural assignment induced by the various h_Σ. Prove that it rejects the generalized quasi-equation $\mathop{\&} \Theta \Longrightarrow \varepsilon \approx \delta$. Then conclude that $\Theta \nvDash_{\mathsf{K}} \varepsilon \approx \delta$ and, therefore, that $\vDash_{\mathsf{K}}^{X}$ is finitary. ⊠

1.6 Relative congruences

Quasi-varieties need not be closed under homomorphic images. For instance, consider the lattices $\boldsymbol{A}$ and $\boldsymbol{B}$ depicted below. We will show that the quasi-variety $\mathbb{Q}(\boldsymbol{B})$ is not closed under $\mathbb{H}$.



$\boldsymbol{A}$ $\boldsymbol{B}$

To this end, let $\mathcal{D}^+(\boldsymbol{A})$ be the positive atomic diagram of $\boldsymbol{A}$ written with the variables $x_1, \dots, x_8$ corresponding to the elements $a_1, \dots, a_8$ and consider the quasi-equation

$$\Phi = \mathop{\&} \mathcal{D}^+(\boldsymbol{A}) \Longrightarrow x_1 \approx x_8.$$

Notice that $\boldsymbol{B}$ validates Φ. To prove this, consider an assignment $f \colon \{x_1, \dots, x_8\} \to B$ that validates $\mathcal{D}^+(\boldsymbol{A})$ in $\boldsymbol{B}$. Using the definition of $\mathcal{D}^+(\boldsymbol{A})$, it is easy to see that the map $h\colon A \to B$ that sends a_i to $f(a_i)$ is a homomorphism from $\boldsymbol{A}$ to $\boldsymbol{B}$. Since $\boldsymbol{A}$ is simple, $\mathsf{Ker}(h)$ is either id_A or $A \times A$. Notice that there is no embedding of $\boldsymbol{A}$ into $\boldsymbol{B}$. Therefore, $\mathsf{Ker}(h)$ cannot be the identity relation. It follows that $\mathsf{Ker}(h) = A \times A$. In particular, $\langle a_1, a_8 \rangle \in \mathsf{Ker}(h)$ and, therefore, $f(x_1) = h(a_1) = f(a_8) = f(x_8)$. Hence, we conclude that $\boldsymbol{B} \vDash \Phi$, as desired. Moreover, Φ fails in $\boldsymbol{A}$, as witnessed by the assignment $x_i \longmapsto a_i$.

In brief, Φ holds in $\boldsymbol{B}$ but fails in $\boldsymbol{A}$. By Maltsev's Theorem, we conclude that $\boldsymbol{A}$ does not belong to the quasi-variety $\mathbb{Q}(\boldsymbol{B})$ generated by $\boldsymbol{B}$. On the other hand, $\boldsymbol{A}$ is a homomorphic image of $\boldsymbol{B}$ (obtained by glueing two pairs of elements of $\boldsymbol{B}$). Thus, the quasi-variety $\mathbb{Q}(\boldsymbol{B})$ is not closed under $\mathbb{H}$.

More in general, if K is a quasi-variety and θ a congruence of some $\boldsymbol{A} \in \mathsf{K}$, the algebra $\boldsymbol{A}/\theta$ need not belong to K. This makes the following concept attractive.

Definition 1.57. Let $\mathsf{K} \cup \{\boldsymbol{A}\}$ be a class of similar algebras. A congruence $\theta \in \mathsf{Con}(\boldsymbol{A})$ is said to be a K-*congruence* of $\boldsymbol{A}$ if $\boldsymbol{A}/\theta \in \mathsf{K}$. We denote the poset of K-congruences of $\boldsymbol{A}$, ordered under the inclusion relation, by $\mathsf{Con}_{\mathsf{K}}(\boldsymbol{A})$.

Notice that if K is closed under $\mathbb{H}$, then $\mathsf{Con}_{\mathsf{K}}(\boldsymbol{A}) = \mathsf{Con}(\boldsymbol{A})$, for all $\boldsymbol{A} \in \mathsf{K}$ (see (1.2), if necessary). Under some minimal assumptions, the converse is also true.

Proposition 1.58. *Let K be a class of similar algebras closed under $\mathbb{I}$. Then K is closed under $\mathbb{H}$ if and only if $\mathsf{Con}(\boldsymbol{A}) = \mathsf{Con}_{\mathsf{K}}(\boldsymbol{A})$, for all $\boldsymbol{A} \in \mathsf{K}$.*

Proof. First, suppose that $\mathsf{Con}(\boldsymbol{A}) = \mathsf{Con}_{\mathsf{K}}(\boldsymbol{A})$, for all $\boldsymbol{A} \in \mathsf{K}$. Then consider an algebra $\boldsymbol{B} \in \mathbb{H}(\mathsf{K})$. In view of (1.2), there exists $\boldsymbol{A} \in \mathsf{K}$ and a congruence θ of $\boldsymbol{A}$ such that $\boldsymbol{B} \cong \boldsymbol{A}/\theta$. Since $\theta \in \mathsf{Con}\boldsymbol{A} = \mathsf{Con}_{\mathsf{K}}\boldsymbol{A}$, then $\boldsymbol{A}/\theta \in \mathsf{K}$. As K is closed under the formation of isomorphic copies, we conclude that $\boldsymbol{B} \in \mathsf{K}$, as desired.

Conversely, suppose that K is closed under $\mathbb{H}$. Consider $\boldsymbol{A} \in \mathsf{K}$. Since $\boldsymbol{A}/\theta$ is a homomorphic image of $\boldsymbol{A}$, we obtain $\boldsymbol{A}/\theta \in \mathbb{H}(\boldsymbol{A}) \subseteq \mathbb{H}(\mathsf{K}) \subseteq \mathsf{K}$. Thus, $\theta \in \mathsf{Con}_{\mathsf{K}}\boldsymbol{A}$. ⊠

Relative congruences and subdirect products are related as follows.

Proposition 1.59. *Let K be a class of algebras of type ρ closed under $\mathbb{I}$. Then K is closed under $\mathbb{P}_{\mathrm{SD}}$ if and only if $\mathsf{Con}_{\mathsf{K}}(\boldsymbol{A})$ is a closure system on $A \times A$, for all algebras $\boldsymbol{A}$ of type ρ.*

Proof. Suppose first that K is closed under $\mathbb{P}_{\mathrm{SD}}$. Then let $\boldsymbol{A}$ be an algebra of type ρ. Since K is closed under $\mathbb{P}_{\mathrm{SD}}$ it contains a trivial algebra (the subdirect product of the empty family). Therefore, as K is closed under $\mathbb{I}$ by assumption, it contains all trivial algebras and, in particular, $\boldsymbol{A}/(A \times A)$. Thus, $A \times A \in \mathsf{Con}_{\mathsf{K}}\boldsymbol{A}$. Then consider a nonempty family $\{\theta_i : i \in I\} \subseteq \mathsf{Con}_{\mathsf{K}}(\boldsymbol{A})$. By Proposition 1.21,

$$\boldsymbol{A}/\bigcap_{i \in I} \theta_i \in \mathbb{I}\mathbb{P}_{\mathrm{SD}}(\{\boldsymbol{A}/\theta_i : i \in I\}).$$

Observe that $\{\boldsymbol{A}/\theta_i : i \in I\} \subseteq \mathsf{K}$, since the various θ_i are K-congruences of $\boldsymbol{A}$. Together with the above display and the assumption that K is closed under $\mathbb{I}$ and $\mathbb{P}_{\mathrm{SD}}$, this yields $\boldsymbol{A}/\bigcap_{i \in I} \theta_i \in \mathsf{K}$, whence $\bigcap_{i \in I} \theta_i \in \mathsf{Con}_{\mathsf{K}}(\boldsymbol{A})$. It follows that $\mathsf{Con}_{\mathsf{K}}(\boldsymbol{A})$ is a closure system.

To prove the converse, suppose that $\mathsf{Con}_{\mathsf{K}}(\boldsymbol{A})$ is a closure system on $A \times A$, for all algebras $\boldsymbol{A}$ of type ρ. Then consider a subdirect product $\boldsymbol{A}$ of a family $\{\boldsymbol{B}_i : i \in I\} \subseteq \mathsf{K}$. Then the canonical projection $p_i \colon \boldsymbol{A} \to \boldsymbol{B}_i$ is a surjective homomorphism, for every

$i \in I$. We will show that $\{\mathsf{Ker}(p_i) : i \in I\} \subseteq \mathsf{Con}_{\mathsf{K}}\boldsymbol{A}$. To this end, consider $i \in I$. Since $p_i \colon \boldsymbol{A} \to \boldsymbol{B}_i$ is surjective, we have $\boldsymbol{A}/\mathsf{Ker}(p_i) \cong \boldsymbol{B}_i$, by Corollary 1.15. Since $\boldsymbol{B}_i \in \mathsf{K}$ and K is closed under $\mathbb{I}$, this yields $\boldsymbol{A}/\mathsf{Ker}(p_i) \in \mathsf{K}$ and, therefore, $\mathsf{Ker}(p_i) \in \mathsf{Con}_{\mathsf{K}}\boldsymbol{A}$, as desired.

Now, recall that $\mathsf{Con}_{\mathsf{K}}\boldsymbol{A}$ is a closure system, by assumption. Therefore, from $\{\mathsf{Ker}(p_i) : i \in I\} \subseteq \mathsf{Con}_{\mathsf{K}}\boldsymbol{A}$ it follows $\bigcap_{i\in I} \mathsf{Ker}(p_i) \in \mathsf{Con}_{\mathsf{K}}\boldsymbol{A}$. We will prove that $\bigcap_{i\in I} \mathsf{Ker}(p_i)$ is the identity relation on A. Consider two distinct $a, c \in A$. Since $\boldsymbol{A}$ is a subdirect product of $\{\boldsymbol{B}_i : i \in I\}$, we have $\boldsymbol{A} \leqslant \prod_{i\in I} \boldsymbol{B}_i$. Then there must be some $j \in I$ such that $a(j) \neq c(j)$. As a consequence, $\langle a, c\rangle \notin \mathsf{Ker}(p_j)$, whence $\langle a, c\rangle \notin \bigcap_{i\in I} \mathsf{Ker}(p_i)$. We conclude that

$$\bigcap_{i\in I} \mathsf{Ker}(p_i) = \mathrm{id}_A.$$

Therefore, $\mathrm{id}_A \in \mathsf{Con}_{\mathsf{K}}\boldsymbol{A}$, whence $\boldsymbol{A}/\mathrm{id}_A \in \mathsf{K}$. Since K is closed under $\mathbb{I}$ and $\boldsymbol{A} \cong \boldsymbol{A}/\mathrm{id}_A$, we obtain $\boldsymbol{A} \in \mathsf{K}$, as desired. ⊠

Corollary 1.60. *If* K *is a prevariety, then* $\mathsf{Con}_{\mathsf{K}}\boldsymbol{A}$ *is a closure system, for every algebra* $\boldsymbol{A}$ *of the same type as* K.

In the case of quasi-varieties something more is true.

Proposition 1.61. *Let* K *be a quasi-variety. If* $\boldsymbol{A}$ *is an algebra of the same type as* K*, then* $\mathsf{Con}_{\mathsf{K}}\boldsymbol{A}$ *is an inductive closure system and, therefore, an algebraic lattice.*

Proof. In view of Corollary 1.60, it suffices to show that the union ϕ of a nonempty upward directed family $\{\theta_i : i \in I\} \subseteq \mathsf{Con}_{\mathsf{K}}(\boldsymbol{A})$ is still a K-congruence of $\boldsymbol{A}$. It is clear that ϕ is a congruence of $\boldsymbol{A}$. Therefore, we only detail a proof of the fact that $\boldsymbol{A}/\phi \in \mathsf{K}$.

In view of Maltsev's Theorem, it suffices to show that all quasi-equations valid in K are also valid in $\boldsymbol{A}/\phi$. Accordingly, consider a quasi-equation

$$\Phi = (\varphi_1 \approx \psi_1 \,\&\, \dots \,\&\, \varphi_n \approx \psi_n) \Longrightarrow \varepsilon \approx \delta$$

valid in K. Moreover, let $\vec{a} \in A$ be such that $\varphi_j^{\boldsymbol{A}/\phi}(\vec{a}/\phi) = \psi_j^{\boldsymbol{A}/\phi}(\vec{a}/\phi)$, for every $j \leqslant n$. Then consider $j \leqslant n$. Since

$$\varphi_j^{\boldsymbol{A}}(\vec{a})/\phi = \varphi_j^{\boldsymbol{A}/\phi}(\vec{a}/\phi) = \psi_j^{\boldsymbol{A}/\phi}(\vec{a}/\phi) = \psi_j^{\boldsymbol{A}}(\vec{a})/\phi$$

and $\phi = \bigcup_{i\in I} \theta_i$, there exists $i_j \in I$ such that $\langle \varphi_j^{\boldsymbol{A}}(\vec{a}), \psi_j^{\boldsymbol{A}}(\vec{a})\rangle \in \theta_{i_j}$. Thus,

$$\langle \varphi_1^{\boldsymbol{A}}(\vec{a}), \psi_1^{\boldsymbol{A}}(\vec{a})\rangle \in \theta_{i_1}, \dots, \langle \varphi_n^{\boldsymbol{A}}(\vec{a}), \psi_n^{\boldsymbol{A}}(\vec{a})\rangle \in \theta_{i_n}.$$

Since the family $\{\theta_i : i \in I\}$ is upward directed, there exists $k \in I$ such that $\theta_{i_1}, \dots, \theta_{i_n} \subseteq \theta_k$. Therefore,

$$\langle \varphi_1^{\boldsymbol{A}}(\vec{a}), \psi_1^{\boldsymbol{A}}(\vec{a})\rangle, \dots, \langle \varphi_n^{\boldsymbol{A}}(\vec{a}), \psi_n^{\boldsymbol{A}}(\vec{a})\rangle \in \theta_k.$$

This implies

$$\varphi_j^{\boldsymbol{A}/\theta_k}(\vec{a}/\theta_k) = \varphi_j^{\boldsymbol{A}}(\vec{a})/\theta_k = \psi_j^{\boldsymbol{A}}(\vec{a})/\theta_k = \psi_j^{\boldsymbol{A}/\theta_k}(\vec{a}/\theta_k), \text{ for every } j \leqslant n.$$

Since $\boldsymbol{A}/\theta_k \in \mathsf{K}$, we know that this algebra validates the quasi-equation Φ. Together with the above display, this yields

$$\langle \varepsilon^{\boldsymbol{A}}(\vec{a}), \delta^{\boldsymbol{A}}(\vec{a})\rangle \in \theta_k \subseteq \phi.$$

It follows that $\varepsilon^{\boldsymbol{A}/\phi}(\vec{a}/\phi) = \delta^{\boldsymbol{A}/\phi}(\vec{a}/\phi)$. Therefore, $\boldsymbol{A}/\phi \vDash \Phi$, as desired. ⊠

Corollary 1.62. *Let K be a quasi-variety and $\boldsymbol{A} \in \mathsf{K}$. Every K-congruence of $\boldsymbol{A}$ is the intersection of a family of K-congruences of $\boldsymbol{A}$ that are completely meet irreducible in $\mathsf{Con}_{\mathsf{K}}\boldsymbol{A}$.*

Proof. Every element of an algebraic lattice is a meet of a family of completely meet irreducible elements. Therefore, the result follows immediately from Proposition 1.61. ⊠

1.7 Subdirect decomposition

In this section, we shall present a general decomposition of for algebraic structures in terms of subdirect products. Because of this, it makes sense to isolate the building blocks of subdirect products, that is, the algebras that cannot be obtained as subdirect products of algebras other than themselves.

Definition 1.63. Let K be a quasi-variety. A member $\boldsymbol{A}$ of K is said to be *subdirectly irreducible relative to* K when for every subdirect emebedding $f\colon \boldsymbol{A} \to \prod_{i\in I} \boldsymbol{B}_i$ with $\{\boldsymbol{B}_i : i \in I\} \subseteq \mathsf{K}$, there exists some $i \in I$ such that the composition $p_i \circ f\colon \boldsymbol{A} \to \boldsymbol{B}_i$ is an isomorphism. The class of all subdirectly irreducible algebras relative to K will be denoted by $\mathsf{K}_{\mathrm{RSI}}$.

An algebra $\boldsymbol{A}$ is said to be *subdirectly irreducible* (in the absolute sense) when it is subdirectly irreducible relative to the quasi-variety of all algebras of its type.

The next result connects subdirect irreduciblity with congruence lattices.

Proposition 1.64. *Let K be a quasi-variety. An algebra $\boldsymbol{A} \in \mathsf{K}$ is subdirectly irreducible relative to K if and only if id_A is completely meet irreducible in $\mathsf{Con}_{\mathsf{K}}\boldsymbol{A}$.*

Proof. Suppose first that id_A is not completely meet irreducible. Then there exists a family $\{\theta_i : i \in I\} \subseteq \mathsf{Con}_{\mathsf{K}}\boldsymbol{A} \smallsetminus \{\mathrm{id}_A\}$. By Proposition 1.21, there exists a subdirect embedding $f\colon \boldsymbol{A} \to \prod_{i\in I} \boldsymbol{A}/\theta_i$. We will show that $p_i \circ f$ is not injective, for every $i \in I$. To this end, consider $i \in I$. We have $\mathsf{Ker}(p_i \circ f) = \theta_i$. Since θ_i is not the identity, we obtain $\mathsf{Ker}(p_i \circ f) \neq \mathrm{id}_A$, whence $p_i \circ f$ is not injective, as desired. It follows that $\boldsymbol{A}$ is not subdirectly irreducible relative to K.

To prove the other implication, suppose that id_A is completely meet irreducible in $\mathsf{Con}_{\mathsf{K}}\boldsymbol{A}$. Then consider a subdirect embedding $f\colon \boldsymbol{A} \to \prod_{i\in I} \boldsymbol{B}_i$ with $\{\boldsymbol{B}_i : i \in I\} \subseteq \mathsf{K}$. For each $i \in I$, we consider the congruence $\mathsf{Ker}(p_i \circ f)$ of $\boldsymbol{A}$. Since $p_i \circ f$ is sujective (because f is subdirect), we can apply Corollary 1.15, obtaining that $\boldsymbol{A}/\mathsf{Ker}(p_i \circ f) \cong \boldsymbol{B}_i$ and, therefore, $\boldsymbol{A}/\mathsf{Ker}(p_i \circ f) \in \mathsf{K}$. It follows that $\mathsf{Ker}(p_i \circ f) \in \mathsf{Con}_{\mathsf{K}}\boldsymbol{A}$. We will show that

$$\mathrm{id}_A = \bigcap_{i\in I} \mathsf{Ker}(p_i \circ f). \tag{1.10}$$

To this end, consider $a, c \in A$. We have

$$\begin{aligned}
\langle a, c\rangle \in \mathrm{id}_A &\iff a = c \\
&\iff f(a) = f(c) \\
&\iff f(a)(i) = f(c)(i), \text{ for every } i \in I \\
&\iff p_i \circ f(a) = p_i \circ f(c), \text{ for every } i \in I \\
&\iff \langle a, c\rangle \in \mathsf{Ker}(p_i \circ f), \text{ for every } i \in I \\
&\iff \langle a, c\rangle \in \bigcap_{i\in I} \mathsf{Ker}(p_i \circ f),
\end{aligned}$$

where the second equivalence follows from the injectivity of f. From (1.10) and the assumption that id_A is completely meet irreducible in $\mathsf{Con}_{\mathsf{K}}\boldsymbol{A}$ it follows that there exists $i \in I$ such that $\mathrm{id}_A = \mathsf{Ker}(p_i \circ f)$. It follows that the homomorphism $p_i \circ f \colon \boldsymbol{A} \to \boldsymbol{B}_i$ is injective. As it is also surjective, because f is subdirect, we conclude that it is an isomorphism. Hence, $\boldsymbol{A}$ is subdirectly irreducible relative to K. ⊠

Corollary 1.65. *An algebra $\boldsymbol{A}$ is subdirectly irreducible if and only if id_A is completely meet irreducible in $\mathsf{Con}\boldsymbol{A}$.*

Remark 1.66. Let K be a quasi-variety and $\boldsymbol{A} \in \mathsf{K}$. In view of Proposition 1.64, $\boldsymbol{A}$ is subdirectly irreducible relative to K precisely when there exists $\phi \in \mathsf{Con}_{\mathsf{K}}(\boldsymbol{A}) \smallsetminus \{\mathrm{id}_A\}$ such that every element of $\mathsf{Con}_{\mathsf{K}}(\boldsymbol{A}) \smallsetminus \{\mathrm{id}_A\}$ extends ϕ. In this case, ϕ is sometimes called the *relative monolith* of $\boldsymbol{A}$.

To prove the above equivalence, observe that if such a ϕ exists, then id_A is clearly completely meet irreducible in $\mathsf{Con}_{\mathsf{K}}(\boldsymbol{A})$ and, therefore, $\boldsymbol{A}$ is subdirectly irreducible relative to K, by Proposition 1.64. Conversely, suppose that $\boldsymbol{A}$ is subdirectly irreducible and, therefore, that id_A is completely meet irreducible in $\mathsf{Con}_{\mathsf{K}}(\boldsymbol{A})$. Then the K-congruence

$$\phi := \bigcap\{\theta \in \mathsf{Con}_{\mathsf{K}}(\boldsymbol{A}) : \theta \neq \mathrm{id}_A\}$$

is different from identity relation id_A. Furthermore, every element of $\mathsf{Con}_{\mathsf{K}}(\boldsymbol{A}) \smallsetminus \{\mathrm{id}_A\}$ extends ϕ, as desired. ⊠

A special kind of relative subdirectly irreducible algebras is the following.

Definition 1.67. Let K be a quasi-variety. An algebra $\boldsymbol{A} \in \mathsf{K}$ is *simple relative to* K if it has exactly two K-congruences.

In this case, $\mathsf{Con}_{\mathsf{K}}(\boldsymbol{A}) = \{\mathrm{id}_A, A \times A\}$ and $\mathrm{id}_A \neq A \times A$, whence $\boldsymbol{A}$ is nontrivial. Moreover, $\mathsf{Con}_{\mathsf{K}}(\boldsymbol{A})$ is the two-element chain with minimum id_A and, therefore, id_A is completely meet irreducible in $\mathsf{Con}_{\mathsf{K}}(\boldsymbol{A})$. Therefore, by Proposition 1.64, we obtain the following.

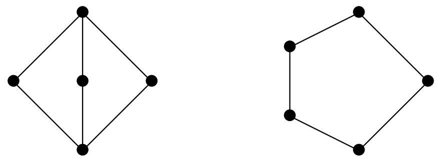
Corollary 1.68. *If an algebra $\boldsymbol{A}$ is simple relative to a quasi-variety K, it is also subdirectly irreducible relative to K.*

As for the case of subdirect irreducibility, the notion of a simple algebra admits an absolute variant. More precisely, an algebra $\boldsymbol{A}$ is *simple* (in the absolute sense) if $\mathsf{Con}(\boldsymbol{A})$ has precisely two elements.

Example 1.69 (Distributive lattices)**.** Recall that the class of distributive lattices DL coincides with $\mathbb{IP}_{\mathsf{SD}}(\boldsymbol{B})$, where $\boldsymbol{B}$ is the two-element distributive lattice. It follows that the class of (relative) subdirectly irreducible members of DL is included into $\mathbb{I}(\boldsymbol{B})$. Furthermore, notice that $\mathsf{Con}\boldsymbol{B}$ is a two-element chain with maximum $B \times B$ and minimum id_B. Consequently, $\boldsymbol{B}$ is simple and, therefore, subdirectly irreducible. It follows that $\mathbb{I}(\boldsymbol{B})$ is the class of all (relative) subdirectly irreducible members of DL.

A similar argument shows that the (relative) subdirectly irreducible members of the class of Boolean algebras are the two-element chains. ⊠

Exercise 1.70. The following lattices are called, respectively, $\boldsymbol{M}_3$ and $\boldsymbol{N}_5$.



Their importance is related to two classical result. The first, due to Dedekind, states that a lattice is nonmodular if and only if $\boldsymbol{N}_5$ embeds into it. The second, due to Birkhoff, states that a lattice fails to be distributive precisely when $\boldsymbol{M}_3$ or $\boldsymbol{N}_5$ can be embedded into it.

Find the congruence lattices of $\boldsymbol{M}_3$ and $\boldsymbol{N}_5$ and use them to convince yourself that both $\boldsymbol{M}_3$ and $\boldsymbol{N}_5$ are subdirectly irreducible. Prove also that $\boldsymbol{M}_3$ is simple, but $\boldsymbol{N}_5$ is not. ⊠

Example 1.71 (Heyting algebras)**.** A filter on a Heyting algebra $\boldsymbol{A}$ is a nonempty upset closed under binary meets. The poset of all filters of $\boldsymbol{A}$ ordered under the inclusion relation will be denoted by $\mathsf{Fi}(\boldsymbol{A})$. Recall that $\mathsf{Fi}(\boldsymbol{A})$ is isomorphic to $\mathsf{Con}(\boldsymbol{A})$ via the map $\boldsymbol{\Omega}^{\boldsymbol{A}} \colon \mathsf{Fi}(\boldsymbol{A}) \to \mathsf{Con}(\boldsymbol{A})$, defined in Example 1.8. We shall prove that a Heyting algebra $\boldsymbol{A}$ is subdirectly irreducible if and only if there exists an element $a \in A \smallsetminus \{1\}$ such that $A = \{1\} \cup {\downarrow}a$.

Suppose first that $\boldsymbol{A}$ is subdirectly irreducible. In view of Remark 1.66, there exists the least congruence θ of $\boldsymbol{A}$ different from the identity. Since $\mathsf{Fi}(\boldsymbol{A}) \cong \mathsf{Con}(\boldsymbol{A})$, there exists also the least filter F of $\boldsymbol{A}$ different from the minimum $\{1\}$ of $\mathsf{Fi}(\boldsymbol{A})$. Since $F \neq \{1\}$, there exists some $a \in F \smallsetminus \{1\}$. As ${\uparrow}a$ is a filter different from $\{1\}$ and F is the least such, we obtain $F \subseteq {\uparrow}a$. On the other hand, as $a \in F$, the other inclusion holds, whence $F = {\uparrow}a$. Then consider an element $c \in A \smallsetminus \{1\}$. Since ${\uparrow}c$ is a filter different from $\{1\}$ and F is the least such, we obtain ${\uparrow}a = F \subseteq {\uparrow}c$, that is, $c \leqslant a$. Hence, $A = \{1\} \cup {\downarrow}a$, as desired.

To prove the converse, suppose that there exists an element $a \in A \smallsetminus \{1\}$ such that $A = \{1\} \cup {\downarrow}a$. In this case, ${\uparrow}a$ is the least filter of $\boldsymbol{A}$ different from the minimum filter $\{1\}$. Since $\mathsf{Fi}(\boldsymbol{A}) \cong \mathsf{Con}(\boldsymbol{A})$, there exists also the least congruence of $\boldsymbol{A}$ different from the identity relation. But, by Remark 1.66, this implies that $\boldsymbol{A}$ is subdirectly irreducible. ⊠

The last ingredient of the general representation theorem is the following.

Correspondence Theorem 1.72. *Let* K *be a quasi-variety and* $\boldsymbol{A} \in \mathsf{K}$*. Given a* K*-congruence* θ *of* $\boldsymbol{A}$*, the subposet* ${\uparrow}\theta$ *of* $\mathsf{Con}_{\mathsf{K}}(\boldsymbol{A})$ *is isomorphic to* $\mathsf{Con}_{\mathsf{K}}(\boldsymbol{A}/\theta)$ *under the map* $f \colon {\uparrow}\theta \to \mathsf{Con}_{\mathsf{K}}(\boldsymbol{A}/\theta)$*, defined by the rule*

$$f(\phi) := \{\langle a/\theta, c/\theta\rangle \in A/\theta \times A/\theta : \langle a, c\rangle \in \phi\}.$$

Proof. We claim that, for every $\phi \in \mathsf{Con}_{\mathsf{K}}(\boldsymbol{A})$ such that $\theta \subseteq \phi$ and $a, c \in A$,

$$\langle a, c\rangle \in \phi \Longleftrightarrow \langle a/\theta, c/\theta\rangle \in f(\phi).$$

The implication from left to right is an immediate consequence of the definition of f. To prove the other implication, suppose that $\langle a/\theta, c/\theta\rangle \in f(\phi)$. By definition of $f(\phi)$, there is a pair $\langle b, d\rangle \in \phi$ such that $a/\theta = b/\theta$ and $c/\theta = d/\theta$. Since ϕ is an equivalence relation extending θ, this implies $\langle a, c\rangle \in \phi$, as desired.

Then we turn to prove that f is well-defined. Consider $\phi \in \mathsf{Con}_{\mathsf{K}}(\boldsymbol{A})$ such that $\theta \subseteq \phi$. Since ϕ is an equivalence relation on A, the definition of f guarantees that $f(\phi)$ is an equivalence relation on A/θ. Then consider a basic n-ary operation g and $a_1, \dots, a_n, c_1, \dots, c_n \in A$ such that

$$\langle a_1/\theta, c_1/\theta \rangle, \dots, \langle a_n/\theta, c_n/\theta \rangle \in f(\phi).$$

By the claim, $\langle a_1, c_1 \rangle, \dots, \langle a_n, c_n \rangle \in \phi$. Since ϕ is a congruence of $\boldsymbol{A}$, this yields

$$\langle g^{\boldsymbol{A}}(a_1, \dots, a_n), g^{\boldsymbol{A}}(c_1, \dots, c_n) \rangle \in \phi.$$

Hence,

$$\begin{aligned} g^{\boldsymbol{A}/\theta}(a_1/\theta, \dots, a_n/\theta) &= g^{\boldsymbol{A}}(a_1, \dots, a_n)/\theta \\ &\equiv_{f(\phi)} g^{\boldsymbol{A}}(c_1, \dots, c_n)/\theta \\ &= g^{\boldsymbol{A}/\theta}(c_1/\theta, \dots, c_n/\theta) \end{aligned}$$

and, therefore, $f(\phi)$ is a congruence of $\boldsymbol{A}/\theta$. The proof that $f(\phi)$ is also a K-congruence of $\boldsymbol{A}/\theta$ is left an an exercise. We conclude that f is well-defined.

Then consider $\phi, \eta \in \mathsf{Con}_{\mathsf{K}}(\boldsymbol{A})$ such that $\theta \subseteq \phi, \eta$. We have

$$\phi \subseteq \eta \iff f(\phi) \subseteq f(\eta).$$

The implication from left to right is an immediate consequence of the definition of f. To prove the other implication, suppose that $f(\phi) \subseteq f(\eta)$ and consider a pair $\langle a, c \rangle \in \phi$. We have $\langle a/\theta, c/\theta \rangle \in f(\phi) \subseteq f(\eta)$. With an application of the claim, we obtain $\langle a, c \rangle \in \eta$, as desired. Hence, f is an order embedding.

To prove that it is surjective, consider a K-congruence ϕ of $\boldsymbol{A}/\theta$ and let

$$\eta := \{\langle a, c \rangle \in A \times A : \langle a/\theta, c/\theta \rangle \in \phi\}.$$

Since ϕ is an equivalence relation on A/θ, the definition of η guarantees that η is an equivalence relation on A. Then consider an n-ary operation g and $a_1, \dots, a_n, c_1, \dots, c_n \in A$ such that $\langle a_1, c_1 \rangle, \dots, \langle a_n, c_n \rangle \in \eta$. We have $\langle a_1/\theta, c_1/\theta \rangle, \dots, \langle a_n/\theta, c_n/\theta \rangle \in \phi$. Since ϕ is a congruence on $\boldsymbol{A}/\theta$, this yields

$$g^{\boldsymbol{A}}(a_1, \dots, a_n)/\theta = g^{\boldsymbol{A}/\theta}(a_1/\theta, \dots, a_n/\theta) \equiv_{\phi} g^{\boldsymbol{A}/\theta}(c_1/\theta, \dots, c_n/\theta) = g^{\boldsymbol{A}}(c_1, \dots, c_n)/\theta$$

and, therefore, $\langle g^{\boldsymbol{A}}(a_1, \dots, a_n), g^{\boldsymbol{A}}(c_1, \dots, c_n) \rangle \in \eta$. We conclude that η is a congruence of $\boldsymbol{A}$. The proof that η is also a K-congruence of $\boldsymbol{A}$ is left as an exercise.

To prove that it extends θ, consider a pair $\langle a, c \rangle \in \theta$. Then $a/\theta = c/\theta$ and, since ϕ is reflexive, $\langle a/\theta, c/\theta \rangle \in \phi$. It follows that $\langle a, c \rangle \in \phi$, as desired. Thus, $\phi \in {\uparrow}\theta$. Furthermore, the definition of f implies that $f(\eta) = \phi$. Hence, f is surjective and, therefore, an isomorphism. ⊠

Exercise 1.73. Two parts of the above proof (both related to K-congruences) were left as an exercise. Complete the missing details. ⊠

The following representation theorem is as an application of lattice theory to general algebra, its main ingredient being the observation that every element of an algebraic lattice can be obtained as a meet of completely meet irreducible ones.

Subdirect Decomposition Theorem 1.74. *If* K *is a quasi-variety, then* $\mathsf{K} = \mathbb{IP}_{\text{SD}}(\mathsf{K}_{\text{RSI}})$.

Proof. Consider an algebra $\boldsymbol{A} \in \mathsf{K}$. By Corollary 1.62, there exists a family $\{\theta_i : i \in I\} \subseteq \mathsf{Con}_{\mathsf{K}}(\boldsymbol{A})$ such that each θ_i is completely meet irreducible in $\mathsf{Con}_{\mathsf{K}}(\boldsymbol{A})$ and, moreover,

$$\text{id}_A = \bigcap_{i \in I} \theta_i.$$

By Proposition 1.21, $\boldsymbol{A} \in \mathbb{IP}_{\text{SD}}(\{\boldsymbol{A}/\theta_i : i \in I\})$.

Now, we know that each $\boldsymbol{A}/\theta_i$ belongs to K, because θ_i is a K-congruence of $\boldsymbol{A}$. Therefore, to conclude the proof, it only remains to show that $\boldsymbol{A}/\theta_i$ is subdirectly irreducible relative to K. By the Correspondence Theorem, $\mathsf{Con}_{\mathsf{K}}(\boldsymbol{A}/\theta_i)$ is isomorphic to the upset generated by θ_i in $\mathsf{Con}_{\mathsf{K}}\boldsymbol{A}$. Moreover, this isomorphism sends θ_i to id_{A/θ_i}. Therefore, from the assumption that θ_i is completely meet irreducible in $\mathsf{Con}_{\mathsf{K}}(\boldsymbol{A})$, it follows that id_{A/θ_i} is completely meet irreducible in $\mathsf{Con}_{\mathsf{K}}(\boldsymbol{A}/\theta_i)$. By Proposition 1.64, we conclude that $\boldsymbol{A}/\theta_i$ is subdirectly irreducible relative to K. ⊠

Corollary 1.75. *Every algebra is isomorphic to a subdirect product of subdirectly irreducible algebras.*

The Subdirect Decomposition Theorem was discovered by Birkhoff in the form of the above corollary. Its posterior formulation for quasi-varieties is due to Maltsev.

Example 1.76 (Subdirect decomposition)**.** Birkhoff's representation of distributive lattices as subdirect products of the two-element chain and Stone's representation of Boolean algebras as subdirect products of the two-element Boolean algebra are special instances of the Subdirect Decomposition Theorem. Another such application is the observation that every Heyting algebra is isomorphic to a subdirect products of subdirectly irreducible Heyting algebras, where the latter were described in Example 1.71. Lastly, an example from classical algebra is the following: every finite Abelian group is a direct product of cyclic groups of prime power order. The latter happen to be precisely the finite subdirectly irreducible Abelian groups. ⊠

Since relative subdirectly irreducible algebras form the building blocks of quasi-varieties, it is natural to wonder how do they arise. The next results provides an answer in terms of the generators of a quasi-variety.

Proposition 1.77. *If* K *is a class of similar algebras, then* $\mathbb{Q}(\mathsf{K})_{\text{RSI}} \subseteq \mathbb{ISP}_{\text{U}}(\mathsf{K})$.

Proof. Consider an algebra $\boldsymbol{A} \in \mathbb{Q}(\mathsf{K})_{\text{RSI}}$. Recall from Corollary 1.49 that $\mathbb{Q}(\mathsf{K}) = \mathbb{ISPP}_{\text{U}}(\mathsf{K})$. Moreover, it is easy to see that $\mathbb{ISPP}_{\text{U}}(\mathsf{K}) = \mathbb{IP}_{\text{SD}}\mathbb{SP}_{\text{U}}(\mathsf{K})$, whence $\mathbb{Q}(\mathsf{K}) = \mathbb{IP}_{\text{SD}}\mathbb{SP}_{\text{U}}(\mathsf{K})$. In particular, this implies $\boldsymbol{A} \in \mathbb{IP}_{\text{SD}}\mathbb{SP}_{\text{U}}(\mathsf{K})$. Accordingly, there exists a subdirect embedding $f \colon \boldsymbol{A} \to \prod_{i \in I} \boldsymbol{B}_i$ with $\{\boldsymbol{B}_i \colon i \in I\} \subseteq \mathbb{SP}_{\text{U}}(\mathsf{K})$. As $\boldsymbol{A}$ is subdirectly irreducible relative to $\mathbb{Q}(\mathsf{K})$ and $\mathbb{SP}_{\text{U}}(\mathsf{K}) \subseteq \mathbb{Q}(\mathsf{K})$, there exists $i \in I$ such that $p_i \circ f \colon \boldsymbol{A} \to \boldsymbol{B}_i$ is an isomorphism. Thus, $\boldsymbol{A} \in \mathbb{I}(\boldsymbol{B}) \subseteq \mathbb{ISP}_{\text{U}}(\mathsf{K})$. ⊠

Corollary 1.78. *If* K *is a finite set of finite similar algebras, then* $\mathbb{Q}(\mathsf{K})_{\text{RSI}} \subseteq \mathbb{IS}(\mathsf{K})$.

Example 1.79 (Closure under $\mathbb{H}$)**.** Consider the lattices $\boldsymbol{A}$ and $\boldsymbol{B}$ defined at the beginning of Section 1.6. There, we prove that the quasi-variety $\mathbb{Q}(\boldsymbol{B})$ is not closed under $\mathbb{H}$. We are now in the position of offering a simpler proof that, moreover, can be easily adapted to other cases.

First, observe that $\boldsymbol{A} \in \mathbb{H}(\boldsymbol{B})$ and, therefore, it suffices to show that $\boldsymbol{A} \notin \mathbb{Q}(\boldsymbol{B})$. Suppose the contrary. Since $\boldsymbol{A}$ is simple and, therefore, subdirectly irreducible, it must be also subdirectly irreducible relative to $\mathbb{Q}(\boldsymbol{B})$. Therefore, $\boldsymbol{A} \in \mathbb{IS}(\boldsymbol{B})$, by Corollary 1.78. This contradicts the fact that there cannot be any embedding of $\boldsymbol{A}$ into $\boldsymbol{B}$ (look at their Hasse diagrams to convince you of this). ⊠

*Exercise** 1.80 (Modal algebras). Modal algebras and open filters were defined in Example 1.9, where it is also explained that the lattice $\mathsf{Op}(\boldsymbol{A})$ of open filters of a modal algebra $\boldsymbol{A}$ is isomorphic to $\mathsf{Con}(\boldsymbol{A})$ under the maps $\boldsymbol{\Omega}^{\boldsymbol{A}}$ and $\boldsymbol{\tau}^{\boldsymbol{A}}$. Given a modal algebra $\boldsymbol{A}$, an element $a \in A$, and a natural number $n \in \mathbb{N}$, we define recursively an element $\Box^n a$ of $\boldsymbol{A}$ as follows:

$$\Box^0 a := a \text{ and } \Box^{k+1} a := \Box^{\boldsymbol{A}} \Box^k a.$$

Similarly, we define an element $\boxplus^n a$ as follows:

$$\boxplus^0 a := a \text{ and } \boxplus^{k+1} a := \Box^{k+1} a \wedge \boxplus^k a.$$

Thus, for instance, $\boxplus^2 a = \Box\Box a \wedge \Box a \wedge a$, where $\Box$ binds stronger than $\wedge$.

Prove that a modal algebra $\boldsymbol{A}$ is subdirectly irreducible if and only if there exists an element $a \in A \smallsetminus \{1\}$ such that for every $c \in A \smallsetminus \{1\}$ there exists some $n \in \mathbb{N}$ such that $\boxplus^n c \leqslant a$. To do this, take inspiration from the proof of the characterization of subdirectly irreducible Heyting algebras given in Example 1.71. While doing so, be careful to the fact that arbitrary filters of $\boldsymbol{A}$ need not be open! Indeed, if F is an open filter of $\boldsymbol{A}$ and $a \in F$, then $\boxplus^n a \in F$, for every $n \in \mathbb{N}$. More precisely, you will need to use (and prove) the fact that the open filter of $\boldsymbol{A}$ generated by a subset $X \subseteq A$ is

$$\{1\} \cup \{a \in A : \boxplus^m (c_1 \wedge \cdots \wedge c_n) \leqslant a : m, n \in \mathbb{N}, 1 \leqslant n, \text{ and } c_1, \ldots, c_n \in X\}. \quad ⊠$$

1.8 Categories, functors, and natural transformations

Intuitively, a *category* is a class of mathematical structures (called *objects*) with structure preserving maps (called *arrows*) between them. The arrows of a category are endowed with a composition operation that abstracts the behaviour of the usual composition of functions and with identity arrows that abstract the behaviour of the usual identity functions.

While reading the following definition, you might wish to keep in mind the example of the category of topological spaces, whose objects are all the topological spaces and whose arrows are the continuous functions. In this category composition is the usual composition of functions and identity arrows are identity functions.

Definition 1.81. A category C consists of a class of objects Obj and a class Arr of arrows with two class functions

$$\mathsf{dom}\colon \mathsf{Arr} \to \mathsf{Obj} \text{ and } \mathsf{cdom}\colon \mathsf{Arr} \to \mathsf{Obj}$$

assigning a *domain* $A = \mathsf{dom}(f)$ and a *codomain* $B = \mathsf{cdom}(f)$ with each arrow f (in which case we write $f\colon A \to B$) in such a way that

(i) for each pair of arrows $f\colon A \to B$ and $g\colon B \to C$, there exists a special arrow $g \circ f\colon A \to C$, called the *composition* of f and g; and

(ii) for each object A there exists a special arrow $1_A\colon A \to A$, called the *identity* arrow on A, such that

- *composition is associative*: for every triple of arrows $f\colon A \to B, g\colon B \to C$, and $h\colon C \to D$,
$$h \circ (g \circ f) = (h \circ g) \circ f; \text{ and}$$
- *identity arrows can be cancelled*: for every pair of arrows $f\colon A \to B$ and $g\colon C \to A$,
$$f \circ 1_A = f \text{ and } 1_A \circ g = g.$$

The notion of a category is general enough to encompass most prominent collections of mathematical objects. We already pointed at the example of topological spaces with continuous maps. To mention a few more, every prevariety K can be viewed as a category whose objects are the members of K and whose arrows are the homomorphisms between them. In this category, which we also denote by K, composition and identity arrows are defined in the standard way. For the present purpose, the following example will play a fundamental role.

Example 1.82 (Bounded distributive lattices)**.** A *bounded distributive lattice* is an algebra $\boldsymbol{A} = \langle A; \wedge, \vee, 0, 1\rangle$ such that $\langle A; \wedge, \vee\rangle$ is a distributive lattice with maximum 1 and minimum 0. Bounded distributive lattices form a quasi-variety that we denote by BDL. Given two bounded distributive lattices $\boldsymbol{A}$ and $\boldsymbol{B}$, a homomorphism $f\colon \boldsymbol{A} \to \boldsymbol{B}$ is a map $f\colon A \to B$ such that for every $a, c \in A$,
$$f(a \wedge^{\boldsymbol{A}} c) = f(a) \wedge^{\boldsymbol{B}} f(c) \quad f(a \vee^{\boldsymbol{A}} c) = f(a) \vee^{\boldsymbol{B}} f(c) \quad f(0^{\boldsymbol{A}}) = 0^{\boldsymbol{B}} \quad f(1^{\boldsymbol{A}}) = 1^{\boldsymbol{B}}.$$
As we mentioned, BDL can be viewed as a category whose objects are bounded distributive lattices and whose arrows are homomorphisms between them. ⊠

Another useful example is the following.

Example 1.83 (Posets)**.** The class Pos of all (possibly empty) posets can be viewed as a category whose objects are posets and whose arrows are order preserving maps between them. In this category, composition and identity arrows are defined in the usual way. ⊠

At this stage, it is important to stress that the arrows of a category need not be functions (they just need to behave as such). For instance, given a category C, the *opposite category* C^{op} is the category obtained from C by inverting the direction of arrows. Formally speaking, the class of objects of C^{op} is that of C, the domain function of C^{op} is the codomain function of C, the codomain function of C^{op} is the domain function of C, the identity arrows of C^{op} are the same as those of C, and the composition $f \circ g$ in C^{op} of two arrows f and g is the arrow $g \circ f$ of C. Notice that $\mathsf{C}^{opop} = \mathsf{C}$.

Given a category C, we denote by $\mathsf{Obj}(\mathsf{C})$ and $\mathsf{Arr}(\mathsf{C})$, respectively, the classes of objects and arrows of C.

Definition 1.84. Let C and D be categories. A *covariant functor* $\mathcal{F}$ from C to D consists of two class functions
$$\mathcal{F}_{\mathsf{Obj}}\colon \mathsf{Obj}(\mathsf{C}) \to \mathsf{Obj}(\mathsf{D}) \text{ and } \mathcal{F}_{\mathsf{Arr}}\colon \mathsf{Arr}(\mathsf{C}) \to \mathsf{Arr}(\mathsf{D})$$
that satisfy the following requirements:

- *preservation of domains and codomains*: if $f: A \to B$ is an arrow of C, then the domain and the codomain of $\mathcal{F}_{\mathsf{Arr}}(f)$ are, respectively, $\mathcal{F}_{\mathsf{Obj}}(A)$ and $\mathcal{F}_{\mathsf{Obj}}(B)$;
- *preservation of identity arrows*: if A is an object of C, then $\mathcal{F}_{\mathsf{Arr}}(1_A) = 1_{\mathcal{F}_{\mathsf{Obj}}(A)}$;
- *preservation of composition*: for every pair of arrows $f: A \to B$ and $g: B \to C$ in C,

$$\mathcal{F}_{\mathsf{Arr}}(g \circ f) = \mathcal{F}_{\mathsf{Arr}}(g) \circ \mathcal{F}_{\mathsf{Arr}}(f).$$

In this case, we often drop the subscripts from $\mathcal{F}_{\mathsf{Obj}}$ and $\mathcal{F}_{\mathsf{Arr}}$ and write simply $\mathcal{F}$.

A *contravariant* functor from C to D is a covariant functor from C to D^{op}. Notice that, in this case, if $f: A \to B$ is an arrow in C, then $\mathcal{F}(f): \mathcal{F}(B) \to \mathcal{F}(A)$ is an arrow in D. We write $\mathcal{F}: \mathsf{C} \to \mathsf{D}$ to indicate that $\mathcal{F}$ is a functor (covariant or contravariant) from C to D.

Example 1.85 (Prime spectra)**.** We shall define a contravariant functor $(-)_*: \mathsf{BDL} \to \mathsf{Pos}$. First, given a bounded distributive lattice $\boldsymbol{A}$, let $\boldsymbol{A}_*$ be the poset of its prime filters ordered under the inclusion relation (sometimes called the *prime spectrum* of $\boldsymbol{A}$). Clearly, $\boldsymbol{A}_*$ is an object of Pos. Then, given two bounded distributive lattices $\boldsymbol{A}$ and $\boldsymbol{B}$ and a homomorphism $f: \boldsymbol{A} \to \boldsymbol{B}$, let $f_*: \boldsymbol{B}_* \to \boldsymbol{A}_*$ be the map defined, for every $F \in \boldsymbol{B}_*$, as $f_*(F) := f^{-1}[F]$. Notice that, if $f_*: \boldsymbol{B}_* \to \boldsymbol{A}_*$ is indeed an arrow of Pos, then the application $(-)_*: \mathsf{BDL} \to \mathsf{Pos}$ is a contravariant functor, because it reverses domains and codomains and preserves identity arrows and composition.

Therefore, it only remains to prove that if $f: \boldsymbol{A} \to \boldsymbol{B}$ is a homomorphism between bounded distributive lattices, then $f_*: \boldsymbol{B}_* \to \boldsymbol{A}_*$ is an arrow of Pos, that is, a well-defined order preserving map from $\boldsymbol{B}_*$ to $\boldsymbol{A}_*$. The fact that it is order preserving is obvious. To prove that it is well-defined, consider a prime filter F of $\boldsymbol{B}$. We need to show that $f_*(F)$ is a prime filter of $\boldsymbol{A}$.

We begin by showing that $f_*(F)$ is a filter of $\boldsymbol{A}$. As F is nonempty, so is $f_*(F) = f^{-1}[F]$. To prove that $f_*(F)$ is an upset of $\boldsymbol{A}$, consider $a, c \in A$ such that $a \in f_*(F)$ and $a \leqslant^{\boldsymbol{A}} c$. Since f is order preserving (being a lattice homomorphism), $f(a) \leqslant^{\boldsymbol{B}} f(c)$. Moreover, by assumption, $a \in f_*(F) = f^{-1}[F]$, whence $f(a) \in F$. Since F is an upset of $\boldsymbol{B}$, we obtain $f(c) \in F$ and, therefore, $c \in f^{-1}[F] = f_*(F)$. Thus, $f_*(F)$ is an upset of $\boldsymbol{A}$. Lastly, to prove that $f_*(F)$ is closed under binary meets, consider $a, c \in f_*(F)$. We have $f(a), f(c) \in F$ and, since F is a filter of $\boldsymbol{B}$, $f(a) \wedge^{\boldsymbol{B}} f(c) \in F$. Since f is a homomorphism, $f(a \wedge^{\boldsymbol{A}} c) = f(a) \wedge^{\boldsymbol{B}} f(c) \in F$, whence $a \wedge^{\boldsymbol{A}} c \in f^{-1}[F] = f_*(F)$. Hence, $f_*(F)$ is a filter of $\boldsymbol{A}$.

To prove that it is prime, it only remains to show that $f_*(F)$ is proper and that, for every $a, c \in A$, if $a \vee^{\boldsymbol{A}} c \in f_*(F)$, then $a \in f_*(F)$ or $c \in f_*(F)$. First, observe that $0^{\boldsymbol{B}} \notin F$ (because F proper) and $f(0^{\boldsymbol{A}}) = 0^{\boldsymbol{B}}$ (because f is a homomorphism), then $f(0^{\boldsymbol{A}}) = 0^{\boldsymbol{B}} \notin F$, whence $0^{\boldsymbol{A}} \notin f^{-1}[F] = f_*(F)$. We conclude that $f_*(F)$ is proper. Then consider $a, c \in A$ such that $a \vee^{\boldsymbol{A}} c \in f_*(F)$. Since f is a homomorphism,

$$f(a) \vee^{\boldsymbol{B}} f(c) = f(a \vee^{\boldsymbol{A}} c) \in F.$$

Since F is prime, $f(a) \in F$ or $f(c)$, that is, $a \in f_*(F)$ or $c \in f_*(F)$. ⊠

Example 1.86 (Upsets)**.** Recall that, given a poset $\mathbb{X}$, the collection $\mathsf{Up}(\mathbb{X})$ of its upsets can be viewed as a bounded distributive lattice

$$\mathbb{X}^* := \langle \mathsf{Up}(\mathbb{X}); \cap, \cup, \emptyset, X \rangle.$$

We will define a contravariant functor $(-)^*\colon \mathsf{Pos} \to \mathsf{BDL}$ exploiting this observation.

Clearly, if $\mathbb{X}$ is a poset, then $\mathbb{X}^*$ is an object of BDL. Moreover, given two posets $\mathbb{X}$ and $\mathbb{Y}$ and an order preserving map $f\colon \mathbb{X} \to \mathbb{Y}$, let $f^*\colon \mathbb{Y}^* \to \mathbb{X}^*$ be the map defined, for every $U \in \mathsf{Up}(\mathbb{Y})$, as $f^*(U) := f^{-1}[U]$. Clearly, f_* is well-defined, because inverse images of upsets under order preserving maps are still upsets. Moreover, as the inverse image map $f^{-1}\colon \mathcal{P}(Y) \to \mathcal{P}(X)$ preserves intersections and unions, f^* is a lattice homomorphism. As $f^{-1}(\emptyset) = \emptyset$ and $f^{-1}[Y] = X$, we conclude that $f^*\colon \mathbb{Y}^* \to \mathbb{X}^*$ is a well-defined homomorphism of bounded lattices and, therefore, an arrow of BDL. It follows that $(-)^*\colon \mathsf{Pos} \to \mathsf{BDL}$ is indeed a contravariant functor. ⊠

The following notion is useful to compare pair of functors with the same domain and codomain.

Definition 1.87. Let $\mathcal{F}\colon \mathsf{C} \to \mathsf{D}$ and $\mathcal{G}\colon \mathsf{C} \to \mathsf{D}$ be covariant functors. A *natural transformation* η from $\mathcal{F}$ to $\mathcal{G}$ is a is a collection $\{\eta_A : A \in \mathsf{Obj}(\mathsf{C})\}$ such that each $\eta_A\colon \mathcal{F}(A) \to \mathcal{G}(A)$ is an arrow in D and, for every arrow $f\colon B \to C$ in C,

$$\eta_C \circ \mathcal{F}(f) = \mathcal{G}(f) \circ \eta_B.$$

We write $\eta\colon \mathcal{F} \to \mathcal{G}$ to indicate that ε is a natural transformation from $\mathcal{F}$ to $\mathcal{G}$.

Notice that, for every category C, there exists a (covariant) *identity functor* $id_{\mathsf{C}}\colon \mathsf{C} \to \mathsf{C}$ that behaves as the identity on the objects and arrows of C. Furthermore, if $\mathcal{F}\colon \mathsf{C} \to \mathcal{D}$ and $\mathcal{G}\colon \mathsf{D} \to \mathsf{E}$ are functors, then the natural composition $\mathcal{G} \circ \mathcal{F}$ is a functor from C to E.

Example 1.88 (Canonical extensions)**.** Since the functors $(-)_*\colon \mathsf{BDL} \to \mathsf{Pos}$ and $(-)^*\colon \mathsf{Pos} \to \mathsf{BDL}$ are contravariant, their composition

$$(-)^* \circ (-)_*\colon \mathsf{BDL} \to \mathsf{BDL}$$

is covariant. We shall define a natural transformation $\varepsilon\colon id_{\mathsf{BDL}} \to (-)^* \circ (-)_*$ as follows. Observe that, for a bounded distributive lattice $\boldsymbol{A}$, the structure $(\boldsymbol{A}_*)^*$ is the bounded distributive lattice of upsets of the poset of prime filters of $\boldsymbol{A}$. By Birkhoff's representation theorem for distributive lattices, we know that the map $\varepsilon_{\boldsymbol{A}}\colon \boldsymbol{A} \to (\boldsymbol{A}_*)^*$, defined by the rule

$$\varepsilon_{\boldsymbol{A}}(a) := \{F \in \boldsymbol{A}_* : a \in F\},$$

is a well-defined embedding of bounded distributive lattices and, therefore, an arrow in BDL. Notice that the pair $\langle \boldsymbol{A}, \varepsilon_{\boldsymbol{A}} \rangle$ is the *canonical extension* of $\boldsymbol{A}$. It is easy to prove that the collection $\varepsilon := \{\varepsilon_{\boldsymbol{A}} : \boldsymbol{A} \in \mathsf{DBL}\}$ is indeed a natural transformation from id_{BDL} to $(-)^* \circ (-)_*$. ⊠

Another useful example of a natural transformation is the following.

Example 1.89. Again, since the functors $(-)_*\colon \mathsf{BDL} \to \mathsf{Pos}$ and $(-)^*\colon \mathsf{Pos} \to \mathsf{BDL}$ are contravariant, their composition

$$(-)_* \circ (-)^*\colon \mathsf{Pos} \to \mathsf{Pos}$$

is covariant. We shall define a natural transformation $\eta\colon id_{\mathsf{Pos}} \to (-)_* \circ (-)^*$ as follows. Recall that, for a poset $\mathbb{X}$, the structure $(\mathbb{X}^*)_*$ is the poset of prime filters of the distributive lattice of upsets of $\mathbb{X}$. Bearing this in mind, consider the map $\eta_{\mathbb{X}} \colon \mathbb{X} \to (\mathbb{X}^*)_*$, defined by the rule

$$\eta_{\mathbb{X}}(x) := \{U \in \mathsf{Up}(\mathbb{X}) : x \in U\}.$$

To prove that $\eta_{\mathbb{X}}$ is well-defined, consider an element $x \in X$. We need to prove that $\eta_{\mathbb{X}}(x)$ is a prime filter of $\mathbb{X}^*$. As $\eta_{\mathbb{X}}(x)$ is the principal upset of $\mathbb{X}^*$ generated by $\uparrow x$, it a filter of $\mathbb{X}^*$. Furthermore, $\eta_{\mathbb{X}}(x)$ is proper, since $\emptyset \notin \eta_{\mathbb{X}}(x)$. To prove that $\eta_{\mathbb{X}}(x)$ is prime, consider $U, V \in \mathbb{X}^*$ such that $U \cup V \in \eta_{\mathbb{X}}(x)$. Then $x \in U \cup V$ and, therefore, $x \in U$ or $x \in V$. As a consequence, $U \in \eta_{\mathbb{X}}(x)$ or $V \in \eta_{\mathbb{X}}(x)$, as desired. Thus, $\eta_{\mathbb{X}}(x)$ is a prime filter of $\mathbb{X}^*$ and, therefore, $\eta_{\mathbb{X}}$ is well-defined. The fact that $\eta_{\mathbb{X}} \colon \mathbb{X} \to (\mathbb{X}^*)_*$ is order preserving is an immediate consequence of its definition. Therefore, we conclude that $\eta_{\mathbb{X}}$ is an arrow of Pos, as desired.

It is easy to prove that the collection $\eta := \{\eta_{\mathbb{X}} : \mathbb{X} \in \mathsf{Pos}\}$ is indeed a natural transformation from id_{Pos} to $(-)_* \circ (-)^*$. ⊠

Remark 1.90. The categorically minded reader might wish to observe that $(-)_* \colon \mathsf{BDL} \to \mathsf{Pos}$ and $(-)^* \colon \mathsf{Pos} \to \mathsf{BDL}$ is a dual adjunction with unit η and counit ε. ⊠

Exercise 1.91. Check that the examples are indeed natural transformations (you only need to check that ε and η satisfy the commutativity condition in the definition of a natural transformation). ⊠

1.9 Priestley duality: the finite case

An arrow $f \colon A \to B$ in a category C is said to be a *categorical isomorphism* in C when it is invertible, that is, there exists an arrow $g \colon B \to A$ such that $1_A = g \circ f$ and $1_B = f \circ g$. Notice that if $f \colon \boldsymbol{A} \to \boldsymbol{B}$ is an arrow in BDL, then

$$f \text{ is a categorical isomorphism in } \mathsf{BDL} \iff f \text{ is an algebraic isomorphism.}$$

The implication from left to right follows from the fact that if f is a categorical isomorphism, then there exists a homomorphism $g \colon \boldsymbol{B} \to \boldsymbol{A}$ such that $g \circ f$ and $f \circ g$ are the identity maps on A and B, respectively. It follows that f has an inverse, namely g, and, therefore, f is bijective. Thus, f is a bijective homomorphism and, therefore, an algebraic isomorphism. Conversely, if f is an algebraic isomorphism, then $f^{-1} \colon \boldsymbol{B} \to \boldsymbol{A}$ is a well-defined homomorphism such that $1_A = f^{-1} \circ f$ and $1_B = f \circ f^{-1}$. As a consequence, f is a categorical isomorphism. Similarly, in Pos categorical isomorphisms coincide with order isomorphisms.

The categorical notion of an isomorphism can be used to introduce a notion of equivalence between categories. Intuitively, two categories C and D are equivalent when there translations between them, i.e., functors $\mathcal{F} \colon \mathsf{C} \to \mathsf{D}$ and $\mathcal{G} \colon \mathsf{D} \to \mathsf{C}$, that, when composed, are essentially the identity functors on C and D.

Definition 1.92. An *equivalence* between two categories C and D consists of two covariant functors $\mathcal{F} \colon \mathsf{C} \to \mathsf{D}$ and $\mathcal{G} \colon \mathsf{D} \to \mathsf{C}$ and natural transformations $\varepsilon \colon (\mathcal{F} \circ \mathcal{G}) \to id_{\mathsf{D}}$ and $\eta \colon id_{\mathsf{C}} \to (\mathcal{G} \circ \mathcal{F})$ such that, for every object A of D and B of C, the arrows

$$\varepsilon_A \colon \mathcal{F} \circ \mathcal{G}(A) \to A \text{ and } \eta_B \colon B \to \mathcal{G} \circ \mathcal{F}(B)$$

are isomorphisms in D and C, respectively. In this case, C and D are said to be *equivalent*.

A *dual equivalence* between two categories C and D is of an equivalence between C and D^{op}. In full, it consists of two contravariant functors $\mathcal{F}\colon \mathsf{C} \to \mathsf{D}$ and $\mathcal{G}\colon \mathsf{D} \to \mathsf{C}$ and natural transformations $\varepsilon\colon (\mathcal{F} \circ \mathcal{G}) \to id_{\mathsf{D}}$ and $\eta\colon id_{\mathsf{C}} \to (\mathcal{G} \circ \mathcal{F})$ such that, for every object A of D and B of C, the arrows

$$\varepsilon_A\colon A \to \mathcal{F} \circ \mathcal{G}(A) \text{ and } \eta_B\colon B \to \mathcal{G} \circ \mathcal{F}(B)$$

are isomorphisms in D and C. In this case, we say that C and D are *dually equivalent*.

Our aim is to prove that the category $\mathsf{BDL}^{<\omega}$ of finite bounded distributive lattices with homomorphisms between them is dually equivalent to the category $\mathsf{Pos}^{<\omega}$ of finite posets with order preserving maps between them.

Priestley's Finite Duality 1.93. *The categories* $\mathsf{BDL}^{<\omega}$ *and* $\mathsf{Pos}^{<\omega}$ *are dually equivalent.*

Proof. Consider the contravariant functors $(-)_*\colon \mathsf{BDL} \to \mathsf{Pos}$ and $(-)^*\colon \mathsf{Pos} \to \mathsf{BDL}$, defined in Examples 1.85 and 1.86. Notice that if $\boldsymbol{A}$ is a finite bounded distributive lattice, then $\boldsymbol{A}_*$ is finite, being a set of subsets of A. Similarly, if $\mathbb{X}$ is a finite poset, then $\mathbb{X}^*$ is a finite bounded lattice, being a set of subsets of X. Therefore, natural restrictions

$$(-)_*\colon \mathsf{BDL}^{<\omega} \to \mathsf{Pos}^{<\omega} \text{ and } (-)^*\colon \mathsf{Pos}^{<\omega} \to \mathsf{BDL}^{<\omega}$$

are well-defined contravariant functors. Then consider the natural transformations

$$\varepsilon\colon id_{\mathsf{BDL}^{<\omega}} \to (-)^* \circ (-)_* \text{ and } \eta\colon id_{\mathsf{Pos}^{<\omega}} \to (-)_* \circ (-)^*,$$

defined in Examples 1.88 and 1.89. To conclude the proof, it only remains to prove that, for every finite distributive lattice $\boldsymbol{A}$ and every finite poset $\mathbb{X}$, the following maps are isomorphisms

$$\varepsilon_{\boldsymbol{A}}\colon \boldsymbol{A} \to (\boldsymbol{A}_*)^* \text{ and } \eta_{\mathbb{X}}\colon \mathbb{X} \to (\mathbb{X}^*)_*.$$

We start from the case of $\varepsilon_{\boldsymbol{A}}\colon \boldsymbol{A} \to (\boldsymbol{A}_*)^*$. Birkhoff's representation theorem for distributive lattices guarantees that $\varepsilon_{\boldsymbol{A}}$ is an embedding. We need to use the fact that $\boldsymbol{A}$ is finite, to prove that $\varepsilon_{\boldsymbol{A}}$ is also surjective and, therefore, an isomorphism. To this end, consider an arbitrary element $U \in (\boldsymbol{A}_*)^*$, that is, is an upset of the poset of prime filters of $\boldsymbol{A}$. If U is the empty set, then

$$\varepsilon_{\boldsymbol{A}}(0^{\boldsymbol{A}}) = \{F \in \boldsymbol{A}_* : a \in F\} = \emptyset = U,$$

where the last equality follows from the fact that prime filters are proper. Then we consider the case where U is nonempty. Being an upset of a finite poset, U is itself finite. Then consider an enumeration $U = \{F_1, \dots, F_n\}$. Here, $F_1, \dots, F_n$ are prime filters of $\boldsymbol{A}$. Since $\boldsymbol{A}$ is finite, each F_i is principal, that is, there exists $a_i \in A$ such that $F_i = {\uparrow}a_i$. Then consider the element

$$c := a_1 \vee \dots \vee a_n$$

of $\boldsymbol{A}$. Clearly, c belongs to each F_i, whence $U \subseteq \varepsilon_{\boldsymbol{A}}(c)$. Then consider a prime filter $F \in \varepsilon_{\boldsymbol{A}}(c)$. As $\boldsymbol{A}$ is finite, there F is principal. Then there exists $a \in A$ such that $F = {\uparrow}a$. Thus, from $c \in F$ it follows

$$a \leqslant c = a_1 \vee \dots \vee a_n.$$

Therefore, $a_1 \vee \dots \vee a_n \in F$. Since F is prime, there is $i \leqslant n$ such that $a_i \in F$. Since $F_i = {\uparrow}a_i$, this implies $F_i \subseteq F$. As $F_i \in U$ and U is an upset in the poset of prime filters of $\boldsymbol{A}$, this implies $F \in U$. Thus, $\varepsilon_{\boldsymbol{A}}(c) \subseteq U$, whence $\varepsilon_{\boldsymbol{A}}(c) = U$. Thus, we conclude that $\varepsilon_{\boldsymbol{A}}$ is surjective and, therefore, an isomorphism.

To prove that $\eta_{\mathbb{X}} \colon \mathbb{X} \to (\mathbb{X}^*)_*$ is an isomorphism, we begin by showing that it is an order embedding. We already know that $\eta_{\mathbb{X}}$ is order preserving (because it is an arrow in Pos). Then consider $x, y \in X$ such that $\eta_{\mathbb{X}}(x) \subseteq \eta_{\mathbb{X}}(y)$. Then observe that

$$\uparrow x \in \{U \in \mathsf{Up}(\mathbb{X}) : x \in U\} = \eta_{\mathbb{X}}(x).$$

Thus, $\uparrow x \in \eta_{\mathbb{X}}(x) \subseteq \eta_{\mathbb{X}}(y) = \{U \in \mathsf{Up}(\mathbb{X}) : y \in U\}$. Therefore, $y \in \uparrow x$, that is, $x \leqslant y$. Hence, we conclude that $\eta_{\mathbb{X}}$ is an order embedding. Therefore, to prove that it is an isomorphism, it suffices to show that it is surjective. Consider a prime filter F of $\mathsf{Up}(\mathbb{X})$. Since $\mathsf{Up}(\mathbb{X})$ is a finite lattice, F is principal, that is, there is $U \in \mathsf{Up}(\mathbb{X})$ such that

$$F = \{V \in \mathsf{Up}(\mathbb{X}) : U \subseteq V\}.$$

Since F is prime, it is proper, whence $U \neq \emptyset$. Then take an enumeration $U = \{x_1, \dots, x_n\}$. Notice that $\uparrow x_1, \dots, \uparrow x_n$ are elements of the lattice $\mathsf{Up}(\mathbb{X})$ and

$$\uparrow x_1 \vee^{\mathbb{X}^*} \cdots \vee^{\mathbb{X}^*} \uparrow x_n = \uparrow x_1 \cup \cdots \cup \uparrow x_n = U \in F.$$

Since F is a prime filter of $\mathbb{X}^*$, there exists $i \leqslant n$ such that $\uparrow x_i \in F$. Then $U \subseteq \uparrow x_i$. Together with $x_i \in U$, this yields $U = \uparrow x_i$. Then

$$\begin{aligned}
\eta_{\mathbb{X}}(x_i) &= \{V \in \mathsf{Up}(\mathbb{X}) : x_i \in V\} \\
&= \{V \in \mathsf{Up}(\mathbb{X}) : \uparrow x_i \in V\} \\
&= \{V \in \mathsf{Up}(\mathbb{X}) : U \in V\} \\
&= F.
\end{aligned}$$

Hence, we conclude that $\eta_{\mathbb{X}}$ is surjective and, therefore, an isomorphism. ⊠

Corollary 1.94. *Up to isomorphism, finite distributive lattices are precisely the algebras of the form* $\langle \mathsf{Up}(\mathbb{X}); \cap, \cup, \emptyset, X\rangle$ *for some finite poset* $\mathbb{X}$.

Priestley's Finite Duality can be adapted to finite Boolean algebras as follows. Let $\mathsf{BA}^{<\omega}$ be the category of finite Boolean algebras with homomorphisms between them and $\mathsf{Set}^{<\omega}$ the category of finite sets with functions between them. Notice that Boolean algebras can be viewed as bounded distributive lattices and that sets can be viewed as posets endowed with the identity relation. In what follows we will use these identifications repeatedly.

Let $\boldsymbol{A}$ be a finite Boolean algebra. As $\boldsymbol{A}$ is a bounded distributive lattice, we can consider the poset $\boldsymbol{A}_*$ of its ultrafilters (that is, of its prime filters). As the ultrafilters of $\boldsymbol{A}$ are precisely its maximal proper filters, the order relation of the poset $\boldsymbol{A}_*$ is the identity. Therefore, $\boldsymbol{A}_*$ can be identified with its universe, which is a finite set and, therefore, an object of Set. Similarly, if $f \colon \boldsymbol{A} \to \boldsymbol{B}$ is a homomorphism between Boolean algebras, then $f^* \colon \boldsymbol{B}_* \to \boldsymbol{A}_*$ is clearly a function between finite sets. Therefore, the functor $(-)_* \colon \mathsf{BDL} \to \mathsf{Pos}$ can be restricted to a functor $(-)_* \colon \mathsf{BA}^{<\omega} \to \mathsf{Set}^{<\omega}$.

On the other hand, consider a finite set X. We can identify X with the poset $\mathbb{X}$ obtained by endowing X with the identity relation. Under this identification, $\mathsf{Up}(\mathbb{X}) = \mathcal{P}(X)$. Therefore, $\mathbb{X}^*$ is the powerset lattice

$$\langle \mathcal{P}(X); \cap, \cup, \emptyset, X\rangle,$$

which, in turn, can be viewed as a Boolean algebra in which complementation is defined as $\neg V := X \smallsetminus V$. Moreover, if $f\colon X \to Y$ is a function between finite sets, then f can be viewed as an order preserving map between the posets $\mathbb{X}$ to $\mathbb{Y}$, obtained endowing X and Y with the identity relations. Then $f^*\colon \mathbb{Y}^* \to \mathbb{X}^*$ is a homomorphism of bounded lattices from the Boolean algebra $\mathbb{Y}^*$ to $\mathbb{X}^*$. To prove that f^* is a homomorphism of Boolean algebras, it only remains to prove that it preserves complements, but this follows from the uniqueness of complements in bounded distributive lattices. Thus, the functor $(-)^*\colon \mathsf{Pos} \to \mathsf{BDL}$ can be restricted to a functor $(-)^*\colon \mathsf{Set}^{<\omega} \to \mathsf{BA}^{<\omega}$.

The functors

$$(-)_*\colon \mathsf{BA}^{<\omega} \to \mathsf{Set}^{<\omega} \text{ and } (-)^*\colon \mathsf{Set}^{<\omega} \to \mathsf{BA}^{<\omega}$$

form a dual equivalence.

Stone's Finite Duality 1.95. *The categories* $\mathsf{BA}^{<\omega}$ *and* $\mathsf{Set}^{<\omega}$ *are dually equivalent.*

Proof. The restrictions of the natural transformations in the proof of Priestley's Finite Duality can be viewed as natural transformations

$$\varepsilon\colon id_{\mathsf{BA}^{<\omega}} \to (-)^* \circ (-)_* \text{ and } \eta\colon id_{\mathsf{Set}^{<\omega}} \to (-)_* \circ (-)^*,$$

Therefore it only remains to prove that $\varepsilon_{\boldsymbol{A}}$ and η_A are isomorphisms for every finite Boolean algebra $\boldsymbol{A}$ and every finite set X. By Priestley's Finite Duality, we know that $\varepsilon_{\boldsymbol{A}}$ is bijective. Since it is a homomorphism, we conclude that it is an isomorphism. The same argument shows that η_X is also an isomorphism. ⊠

Corollary 1.96. *Up to isomorphism, finite Boolean algebras are precisely the algebras of the form* $\langle \mathcal{P}(X); \cap, \cup, -, \emptyset, X\rangle$ *for some finite set* X.

Remark 1.97. Priestley and Stone dualities can be extended to the infinite case, provided that posets and sets are equipped with suitable topologies. More precisely, Priestley's duality establishes a dual equivalence between the category of bounded distributive lattices and the category certain ordered topological spaces, known as Priestley spaces. Similarly, Stone duality establishes a dual equivalence between the category of Boolean algebra and the category certain topological spaces, known as Stone spaces. ⊠

1.10 Free algebras

Definition 1.98. Let $\mathsf{K} \cup \{\boldsymbol{A}\}$ be a class of similar algebras and $X \subseteq A$. Then $\boldsymbol{A}$ is said to be *free for* K *over* X if X is a set of generators for $\boldsymbol{A}$ and, for every $\boldsymbol{B} \in \mathsf{K}$ and every function $f\colon X \to B$, there exists a homomorphism $f^+\colon \boldsymbol{A} \to \boldsymbol{B}$ extending f.

Notice that in this case f^+ is unique, because X is a set of generators of $\boldsymbol{A}$ and if two homomorphisms $g, h\colon \boldsymbol{A} \to \boldsymbol{B}$ agree on X, then $g = h$.

Example 1.99 (Term algebras)**.** Consider a type ρ and a set of variables X. Recall from Proposition 1.27 that the term algebra $\boldsymbol{T}_\rho(X)$ has the following property: for every algebra $\boldsymbol{A}$ of type ρ and every function $f\colon X \to A$, there exists a homomorphism $f^+\colon \boldsymbol{T}_\rho(X) \to \boldsymbol{A}$ extending f. Since X is a set of generators of $\boldsymbol{T}_\rho(X)$, we conclude that the term algebra $\boldsymbol{T}_\rho(X)$ is free for the class of all algebras of type ρ over X. ⊠

When working with a given class of algebras K, it is convenient to strengthen the definition of a free algebra for K over X by requiring that it belongs to K.

Definition 1.100. Let $\mathsf{K} \cup \{\boldsymbol{A}\}$ be a class of similar algebras and $X \subseteq A$. Then $\boldsymbol{A}$ is a said to be *free in* K *over* X when it is free for K over X and, moreover, $\boldsymbol{A} \in \mathsf{K}$. In this case, we say that X is a set of free generators for $\boldsymbol{A}$.

Under this assumption, free algebras are uniquely determined by the cardinality of their sets of free generators.

Proposition 1.101. *Let* K *be a class of similar algebras. If* $\boldsymbol{A}$ *and* $\boldsymbol{B}$ *are free in* K*, respectively, over* X_A *and* X_B *and* $|X_A| = |X_B|$*, then* $\boldsymbol{A} \cong \boldsymbol{B}$*.*

Proof. Consider a bijection $f \colon X_A \to X_B$. Since $\boldsymbol{B} \in \mathsf{K}$ and $\boldsymbol{A}$ is free for K over X_A, there exists a homomorphism $f^+ \colon \boldsymbol{A} \to \boldsymbol{B}$ extending f. Then consider the inverse function $f^{-1} \colon X_B \to X_A$. Similarly, this map extends to a homomorphism $f^{-1+} \colon \boldsymbol{B} \to \boldsymbol{A}$.

Now, consider the homomorphisms

$$f^{-1+} \circ f^+ \colon \boldsymbol{A} \to \boldsymbol{A} \text{ and } f^+ \circ f^{-1+} \colon \boldsymbol{B} \to \boldsymbol{B}.$$

Observe that the restriction of $f^{-1+} \circ f^+$ to the set of generators X_A of $\boldsymbol{A}$ is $f^{-1} \circ f$, that is, the identity map. We will use this observation to prove that $f^{-1+} \circ f^+$ is the identity map on $\boldsymbol{A}$. To this end, consider an arbitrary element $c \in A$. Since X_A is a set of generators of $\boldsymbol{A}$, there exists a term $\varphi(x_1, \dots, x_n)$ and $a_1, \dots, a_n \in X_A$ such that

$$c = \varphi^{\boldsymbol{A}}(a_1, \dots, a_n).$$

We have

$$\begin{aligned} f^{-1+} \circ f^+(c) &= f^{-1+} \circ f^+(\varphi^{\boldsymbol{A}}(a_1, \dots, a_n)) \\ &= \varphi^{\boldsymbol{A}}(f^{-1+} \circ f^+(a_1), \dots, f^{-1+} \circ f^+(a_n)) \\ &= \varphi^{\boldsymbol{A}}(a_1, \dots, a_n) \\ &= c. \end{aligned}$$

The second equality above follows from the fact that homomorphisms preserve complex terms and the third from the observation that $f^{-1+} \circ f^+$ is the identity map on the set of generators X_A. Hence, we conclude that $f^{-1+} \circ f^+ \colon \boldsymbol{A} \to \boldsymbol{A}$ is the identity map on $\boldsymbol{A}$. Similarly, $f^+ \circ f^{-1+}$ is the identity map on $\boldsymbol{B}$. But this implies that f^+ is bijective and, therefore, an isomorphism. ⊠

In view of the above result, given a class of similar algebras K and a cardinal κ, there exists (up to isomorphism) at most one algebra that is free in K over a set of size κ. Because of this, when it exists, we will call it the *free* κ*-generated algebra* of K and denote it by $\boldsymbol{F}_{\mathsf{K}}(\kappa)$.

The aim of this section is to provide a concrete description of free n-generated bounded distributive lattices and Boolean algebras. To this end, it is convenient to introduce the following categorical concept.

Definition 1.102. Let C be a category and $A_1, \dots, A_n, A$ objects of C.

(i) A is said to be a *coproduct* of $A_1, \ldots, A_n$ in C if there are arrows

$$f_1 \colon A_1 \to A, \ldots, f_n \colon A_n \to A$$

such that, for every element B of C and arrows $g_1 \colon A_1 \to B, \ldots, g_n \colon A_n \to B$, there exists a unique arrow $h \colon A \to B$ such that

$$h \circ f_1 = g_1, \ldots, h \circ f_n = g_n.$$

(ii) A is said to be a *product* of $A_1, \ldots, A_n$ in C if it is a coproduct of $A_1, \ldots, A_n$ in C^{op}, that is, if there are arrows

$$f_1 \colon A \to A_1, \ldots, f_n \colon A \to A_n$$

such that, for every element B of C and arrows $g_1 \colon B \to A_1, \ldots, g_n \colon B \to A_n$, there exists a unique arrow $h \colon B \to A$ such that

$$f_1 \circ h = g_1, \ldots, f_n \circ h = g_n.$$

When they exist, the product and coproduct of $A_1, \ldots, A_n$ in C are unique up to categorical isomorphism. Because of this, we will speak of *the* product (resp. coproduct) of $A_1, \ldots, A_n$ in C.

We say that a category C has products (resp. coproducts) when the product (resp. cooproduct) of $A_1, \ldots, A_n$ in C exists, for all objects $A_1, \ldots, A_n$ of C.

Exercise 1.103. Prove the above assertion on the uniqueness of products and coproducts. ⊠

Example 1.104 (Finite distributive lattices)**.** Products in $\mathsf{Pos}^{<\omega}$ are the usual Cartesian products of posets. To prove this, consider posets $\mathbb{X}_1, \ldots, \mathbb{X}_n$. For each $i \leqslant n$, let

$$p_i \colon (\mathbb{X}_1 \times \cdots \times \mathbb{X}_n) \to \mathbb{X}_i$$

be the the projection map on the j-th component. Notice that p_i is order preserving and, therefore, an arrow of Pos. Moreover, for every order preserving maps $g_1 \colon \mathbb{Y} \to \mathbb{X}_1, \ldots, g_n \colon \mathbb{Y} \to \mathbb{X}_n$, there exists a unique order preserving map $h \colon \mathbb{Y} \to (\mathbb{X}_1 \times \cdots \times \mathbb{X}_n)$ such that $p_1 \circ h = g_1, \ldots, p_n \circ h = g_n$, namely the map defined by the rule

$$h(y)(i) \coloneqq g_i(y), \text{ for all } i \leqslant n.$$

Therefore, we conclude that $\prod_{i \in I} \mathbb{X}_i$ is the product of the various $\mathbb{X}_i$ in Pos.

We can use this observation to describe coproducts in DBL. For consider finite $\boldsymbol{A}_1, \ldots, \boldsymbol{A}_n$ bounded distributive lattices. By Priestley's Finite Duality, the categories $\mathsf{BDL}^{<\omega}$ and $\mathsf{Pos}^{<\omega}$ are dually equivalent, that is, are essentially the same except from the fact that the order of their arrows is reversed. Therefore, in order to compute the coproduct of $\boldsymbol{A}_1, \ldots, \boldsymbol{A}_n$ in BDL, it is enough to transform $\boldsymbol{A}_1, \ldots, \boldsymbol{A}_n$ into posets using the functor $(-)_*$, compute their product in Pos, and transform it into a bounded distributive lattice using the functor $(-)^*$. In brief, the coproduct of $\boldsymbol{A}_1, \ldots, \boldsymbol{A}_n$ in BDL is $(\boldsymbol{A}_{1*} \times \cdots \times \boldsymbol{A}_{n*})^*$. This observation will be instrumental to describe free n-generated bounded distributive lattices. ⊠

Example 1.105 (Finite Boolean algebras)**.** Similarly, products in $\mathsf{Set}^{<\omega}$ are just Cartesian products. By Stone's Finite Duality, this implies that the coproduct of the Boolean algebras $\boldsymbol{A}_1, \dots, \boldsymbol{A}_n$ is the powerset Boolean algebra $(\boldsymbol{A}_{1*} \times \dots \times \boldsymbol{A}_{n*})^*$. ⊠

Let K be a prevariety. We say that K is *locally finite* when its finitely generated members are finite. Moreover, we denote by $\mathsf{K}^{<\omega}$ the category whose objects are the finite members of K and whose arrows are the homomorphisms between them.

Proposition 1.106. *Let K be locally finite prevariety containing a nontrivial algebra. If $\boldsymbol{F}_{\mathsf{K}}(1)$ exists and $\mathsf{K}^{<\omega}$ has coproducts, then for every positive integer n, the algebra $\boldsymbol{F}_{\mathsf{K}}(n)$ exists and is (up to isomorphism) the coproduct in $\mathsf{K}^{<\omega}$ of n copies of $\boldsymbol{F}_{\mathsf{K}}(1)$.**

Proof. Let $\boldsymbol{A}$ be the coproduct in $\mathsf{K}^{<\omega}$of n copies of $\boldsymbol{F}_{\mathsf{K}}(1)$. Then there are homomorphisms

$$f_1 \colon \boldsymbol{F}_{\mathsf{K}}(1) \to \boldsymbol{A}, \dots, f_n \colon \boldsymbol{F}_{\mathsf{K}}(1) \to \boldsymbol{A}.$$

Let x be the free generator of $\boldsymbol{F}_{\mathsf{K}}(1)$ and define

$$x_1 := f_1(x), \dots, x_n := f_n(x)$$

and $X := \{x_1, \dots, x_n\}$. In view of Proposition 1.101, if we can prove that

(i) $\boldsymbol{A}$ is free in K over X; and

(ii) $|X| = n$,

then $\boldsymbol{A}$ is (up to isomorphism) $\boldsymbol{F}_{\mathsf{K}}(n)$ and, therefore, we are done. We shall prove conditions (i) and (ii) separately.

(i): First we show that X is a set of generators for $\boldsymbol{A}$. Suppose, on the contrary, that the subalgebra $\boldsymbol{B}$ of $\boldsymbol{A}$ generated by X differs from $\boldsymbol{A}$. Since x is a generator of $\boldsymbol{F}_{\mathsf{K}}(1)$ and $f_i(x) \in X$, then $f_i[A] \leqslant B$. Therefore, each f_i can be viewed as a homomorphism $f_i \colon \boldsymbol{F}_{\mathsf{K}}(1) \to \boldsymbol{B}$. As $\boldsymbol{B}$ is finite and belongs to K (the latter, because $\boldsymbol{B} \in \mathbb{S}(\boldsymbol{A}) \subseteq \mathbb{S}(\mathsf{K}) \subseteq \mathsf{K}$), we can use the property of the coproduct to obtain a homomorphism $h \colon \boldsymbol{A} \to \boldsymbol{B}$ that is the identity on X. Notice that, since $\boldsymbol{B} \leqslant \boldsymbol{A}$, we can view h as a homomorphism $h \colon \boldsymbol{A} \to \boldsymbol{A}$. Moreover, h is not the identity, since $h[A] \subseteq B \subsetneq A$.

Now, consider again the homomorphisms $f_i \colon \boldsymbol{F}_{\mathsf{K}}(1) \to \boldsymbol{A}$. By the definition of a coproduct, there exist a unique homomorphism $p \colon \boldsymbol{A} \to \boldsymbol{A}$ such that $p \circ f_i = f_i$, for all $i \leqslant n$. Clearly, the identity map 1_A on $\boldsymbol{A}$ can serve as p. However, notice that $h \circ f_i(x) = h(x_i) = x_i = f_i(x)$, where the first equality follows from the fact that h is the identity on X. Since $\boldsymbol{F}_{\mathsf{K}}(1)$ is generated by x, we conclude that $h \circ f_i = f_i$, for every $i \leqslant n$. Therefore, by the uniqueness of p, we obtain $1_A = h$. But this contradicts the fact that h is not the identity map. Hence, we conclude that X is a set of generators for $\boldsymbol{A}$.

Then consider an algebra $\boldsymbol{B} \in \mathsf{K}$ and a function $h \colon X \to B$. Consider the subalgebra $\boldsymbol{C}$ of $\boldsymbol{B}$ generated by $h[X]$. Since K is locally finite, $\boldsymbol{C}$ is finite. Since x is a free generator for $\boldsymbol{F}_{\mathsf{K}}(1)$, for each $i \leqslant n$ there exists a homomorphism $g_i \colon \boldsymbol{F}_{\mathsf{K}}(1) \to \boldsymbol{C}$ such that

*Actually, something more general is true. If K is a nontrivial prevariety and $\kappa > 0$ a cardinal, then $\boldsymbol{F}_{\mathsf{K}}(\kappa)$ exists and is the coproduct in K of κ copies of $\boldsymbol{F}_{\mathsf{K}}(1)$. When, moreover, K is locally finite, $\boldsymbol{F}_{\mathsf{K}}(n)$ is the coproduct in $\mathsf{K}^{<\omega}$ of n copies of $\boldsymbol{F}_{\mathsf{K}}(1)$. The proof of these results, however, would take us on a detour.

$g_i(x) = h(x_i)$. Since $\boldsymbol{A}$ is a coproducts of n copies of $\boldsymbol{F}_{\mathsf{K}}(1)$ in $\mathsf{K}^{<\omega}$ there exists a homomorphism $h^+ \colon \boldsymbol{A} \to \boldsymbol{C}$ such that

$$h^+(x_i) = h^+ \circ f_i(x) = g_i(x) = h(x_i), \text{ for every } i \leqslant n.$$

It follows that h^+ extends h. Moreover, as $\boldsymbol{C} \leqslant \boldsymbol{B}$, the extension h^+ can be viewed as a homomorphism $h^+ \colon \boldsymbol{A} \to \boldsymbol{B}$. We conclude that $\boldsymbol{A}$ is free for K over X. Since $\boldsymbol{A} \in \mathsf{K}$, by assumption, it is also free in K for X, as desired.

(ii): To prove that $|X| = n$, consider a nontrivial member $\boldsymbol{B}$ of K. We can assume that $\boldsymbol{B}$ has at least n elements, otherwise we replace it by the product $\boldsymbol{B} \times \cdots \times \boldsymbol{B}$ of n-copies of $\boldsymbol{B}$. We can also assume that $\boldsymbol{B}$ is finite, otherwise we consider distinct $b_1, \ldots, b_n \in B$ and replace $\boldsymbol{B}$ by the subalgebra generated by $\{b_1, \ldots, b_n\}$ (which is finite, because K is locally finite). Since x is a free generator for $\boldsymbol{F}_{\mathsf{K}}(1)$, for each $i \leqslant n$ there exists a homomorphism $g_i \colon \boldsymbol{F}_{\mathsf{K}}(1) \to \boldsymbol{B}$ such that $g_i(x) = b_i$. As $\boldsymbol{A}$ is the coproduct of n copies of $\boldsymbol{F}_{\mathsf{K}}(1)$ there exists a homomorphism $h \colon \boldsymbol{A} \to \boldsymbol{B}$ such that

$$h(x_i) = h \circ f_i(x) = g_i(x) = b_i, \text{ for all } i \leqslant n.$$

Since the various b_i are different, so must be the various x_i. Hence, we conclude that $|X| = n$. ⊠

Example 1.107 (Free bounded distributive lattices)**.** Consider the thee-element chain $\boldsymbol{C}_3$, viewed as a bounded distributive lattice. It is easy to see that it is the free one-generated bounded distributive lattice with free generator its middle element. In short, $\boldsymbol{F}_{\mathsf{BDL}}(1) = \boldsymbol{C}_3$. Now, recall that BDL is a locally finite prevariety (this can be proved, for instance, using conjunctive normal forms). Moreover, we already showed that coproducts in $\mathsf{BDL}^{<\omega}$ exist. Therefore, we can apply Proposition 1.106, obtaining that, for every positive integer n, the n-generated free bounded distributive lattice $\boldsymbol{F}_{\mathsf{BDL}}(n)$ is the coproduct of n copies of $\boldsymbol{C}_3$ in $\mathsf{BDL}^{<\omega}$. In view of Example 1.104,

$$\boldsymbol{F}_{\mathsf{BDL}}(n) \cong (\boldsymbol{C}_{3*} \times \cdots \times \boldsymbol{C}_{3*})^*.$$

Let us describe the above structure in more detail. First, notice that the poset of prime filters of $\boldsymbol{C}_3$ is the two element chain. Therefore, the Cartesian product of n copies of $\boldsymbol{C}_{3*}$ is (up to isomorphism) the powerset lattice of an n element set. Thus,

$$\boldsymbol{F}_{\mathsf{BDL}}(n) \cong (\mathcal{P}(\{1, \ldots, n\}))^*. \qquad ⊠$$

Example 1.108 (Free Boolean algebras)**.** Consider the four-element Boolean algebra $\boldsymbol{A}$. It is easy to see that it is the free one-generated Boolean algebra with free generator any of its middle elements. In short, $\boldsymbol{F}_{\mathsf{BA}}(1) = \boldsymbol{A}$. An argument similar to the one described above shows that $\boldsymbol{F}_{\mathsf{BA}}(n)$ is the coproduct of n copies of $\boldsymbol{A}$ in $\mathsf{BA}^{<\omega}$, that is,

$$\boldsymbol{F}_{\mathsf{BA}}(n) \cong (\boldsymbol{A}_* \times \cdots \times \boldsymbol{A}_*)^*.$$

As the set of ultrafilters of $\boldsymbol{A}$ has just two elements, the Cartesian product of n copies of $\boldsymbol{A}_*$ is a set of 2^n elements. Thus, $\boldsymbol{F}_{\mathsf{BA}}(n) \cong \mathcal{P}(\{1, \ldots, 2^n\})$. ⊠

We shall see prevarieties contain free algebras. More precisely, we have the following.

Theorem 1.109. *Let K be a prevariety and $\boldsymbol{A} \in \mathsf{K}$. If $0 < \kappa \leqslant |A|$, then $\boldsymbol{F}_{\mathsf{K}}(\kappa)$ exists.*

Proof. Suppose that $0 < \kappa \leqslant |A|$. We shall identify κ with a set of variables

$$X = \{x_\alpha : \alpha < \kappa\}.$$

Bearing this in mind, consider the term algebra $\boldsymbol{T}(X)$. Moreover, let

$$\theta := \bigcap\{\mathsf{Ker}(f) : f\colon \boldsymbol{T}(X) \to \boldsymbol{B} \text{ is a homomorphism and } \boldsymbol{B} \in \mathsf{K}\}.$$

Observe that θ is a congruence of $\boldsymbol{T}(X)$, because it is an intersection of congruences. By Proposition 1.21,

$$\boldsymbol{T}(X)/\theta \in \mathbb{IP}_{\text{SD}}(\{\boldsymbol{T}(X)/\mathsf{Ker}(f) : f\colon \boldsymbol{T}(X) \to \boldsymbol{B} \text{ is a homomorphism and } \boldsymbol{B} \in \mathsf{K}\}).$$

Since $\boldsymbol{T}(X)/\mathsf{Ker}(f) \in \mathbb{IS}(\boldsymbol{B})$, for every homomorphism $f\colon \boldsymbol{T}(X) \to \boldsymbol{B}$, this yields

$$\boldsymbol{T}(X)/\theta \in \mathbb{IP}_{\text{SD}}\mathbb{S}(\mathsf{K}) \subseteq \mathsf{K},$$

where the last inclusion follows from the assumption that K is a prevariety. Therefore, to conclude the proof, it suffices to show that $\boldsymbol{T}(X)/\theta$ is free for K over X/θ and that $|X/\theta| = \kappa$.

To prove that $|X/\theta| = \kappa$, consider an injection $f\colon X \to A$ (which exists because, by assumption, $|X| = \kappa \leqslant |A|$). From Proposition 1.27, f can be extended to a homomorphism $f^+\colon \boldsymbol{T}(X) \to \boldsymbol{A}$. Since f is an injection on X, we obtain that $|X/\mathsf{Ker}(f^+)| = |X| = \kappa$. Since $\theta \subseteq \mathsf{Ker}(f^+)$, this yields that also $|X/\mathsf{Ker}(f^+)| = \kappa$.

Then we turn to prove that $\boldsymbol{T}(X)/\theta$ has the universal property typical of free algebras. To this end, consider a function $f\colon X/\theta \to \boldsymbol{B}$ with $\boldsymbol{B} \in \mathsf{K}$. Let $\pi_\theta\colon X \to X/\theta$ be the natural projection and consider the composition $f \circ \pi_\theta : X \to B$. By Proposition 1.27, it can be extended to a homomorphism $(f \circ \pi_\theta)^+\colon \boldsymbol{T}(X) \to \boldsymbol{B}$. Now, we define a map $f^+\colon \boldsymbol{T}(X)/\theta \to \boldsymbol{B}$ by the rule

$$f^+(\varphi/\theta) := (f \circ \pi_\theta)^+(\varphi).$$

This map is well-defined because the definition of θ guarantees that $\theta \subseteq \mathsf{Ker}((f \circ \pi_\theta)^+)$. Furthermore, it is easily seen to be a homomorphism. Therefore, it only remains to prove that it extends f. Consider $x_\alpha \in X$. We have

$$f^+(x_\alpha/\theta) = (f \circ \pi_\theta)^+(x_\alpha) = f \circ \pi_\theta(x_\alpha) = f(x_\alpha),$$

where the second equality holds because $(f \circ \pi_\theta)^+$ extends $f \circ \pi_\theta$.

Lastly, as $\boldsymbol{T}(X)$ is generated by X, the quotient $\boldsymbol{T}(X)/\theta$ is generated by X/θ. Hence, we conclude that $\boldsymbol{T}(X)/\theta$ is free in K for X/θ and that X/θ has size κ, that is, $\boldsymbol{T}(X)/\theta \cong \boldsymbol{F}_{\mathsf{K}}(\kappa)$. ☒

Remark 1.110. Observe that the congruence θ defined in the above proof is the relation

$$\{\langle \varphi, \psi\rangle \in T(X) \times T(X) : \mathsf{K} \vDash \varphi \approx \psi\}.$$

We will use this fact in the next section. ☒

1.11 Varieties

We conclude our journey in universal algebra by obtaining a description of classes of algebras axiomatizable by equations in terms of closure under certain class operators.

Definition 1.111. A class of similar algebras is said to be a *variety* when it is closed under $\mathbb{H}, \mathbb{S}$, and $\mathbb{P}$.

Notice that every variety is a quasi-variety. To prove this, observe that, for every class of similar algebras K,

$$\mathbb{I}(\mathsf{K}) \subseteq \mathbb{H}(\mathsf{K}) \text{ and } \mathbb{P}_{\mathrm{U}}(\mathsf{K}) \subseteq \mathbb{HP}(\mathsf{K}.$$

Consequently, if K is a variety, then it is also closed under $\mathbb{I}$ and $\mathbb{P}_{\mathrm{U}}$ and, therefore, is a quasi-variety. Similarly, we know that all quasi-varieties are prevarieties. The reverse inclusions do not hold in general. For recall from Example 1.51 that the prevariety generated by the class of all finite lattices is not a quasi-variety and from Example 1.79 that there are quasi-varieties that are not closed under $\mathbb{H}$ and, therefore, are not varieties.

The aim this section is to prove the following classical description of varieties.

Birkhoff's Theorem 1.112. *A class of similar algebras is a variety if and only if it can be axiomatized by a set of equations.*

Proof. As usual, one implication is an easy exercise: since the validity of equations persists under the formation of homomorphic images, subalgebras, and direct products, every class of algebras axiomatized by a set of equations is a variety.

To prove the converse, consider a variety K. Then let Y be a denumerable set of variables and let K^+ the class of algebras axiomatized by the set of equations, with variables in Y, valid in K. Clearly, $\mathsf{K} \subseteq \mathsf{K}^+$. To prove the other inclusion, consider an algebra $\boldsymbol{A} \in \mathsf{K}^+$. Then let X be a set of variables with a bijection $f\colon X \to A$. By Proposition 1.27, f can be extended to a surjective homomorphism $f^+\colon \boldsymbol{T}(X) \to \boldsymbol{A}$. Let θ be the congruence of $\boldsymbol{T}(X)$ defined in the proof of Theorem 1.109. Since K is a prevariety, the proof of Theorem 1.109 shows that $\boldsymbol{T}(X)/\theta \in \mathsf{K}$. Since K is a variety, this implies that $\mathbb{H}(\boldsymbol{T}(X)/\theta) \subseteq \mathsf{K}$. Therefore, to conclude the proof, it will be enough to show that $\boldsymbol{A} \in \mathbb{H}(\boldsymbol{T}(X)/\theta)$.

To this end, consider the map $g\colon \boldsymbol{T}(X)/\theta \to \boldsymbol{A}$ defined, for every $\varphi \in T(X)$, as

$$g(\varphi/\theta) := f^+(\varphi).$$

We will prove that g is a well-defined surjective homomorphism. To prove that it is well-defined, consider $\varphi(x_1, \dots, x_n), \psi(x_1, \dots, x_n) \in T(X)$ such that $\langle \varphi, \psi \rangle \in \theta$. By Remark 1.110,

$$\mathsf{K} \vDash \varphi(x_1, \dots, x_n) \approx \psi(x_1, \dots, x_n).$$

Let $y_1, \dots, y_n$ be distinct variables in Y. Clearly, $\mathsf{K} \vDash \varphi(y_1, \dots, y_n) \approx \psi(y_1, \dots, y_n)$. As $\boldsymbol{A}$ satisfies all the equations, with variables in Y, valid in K, we conclude that $\boldsymbol{A}$ satisfies $\varphi(y_1, \dots, y_n) \approx \psi(y_1, \dots, y_n)$ and, therefore,

$$\boldsymbol{A} \vDash \varphi(x_1, \dots, x_n) \approx \psi(x_1, \dots, x_n).$$

Consequently,

$$f^+(\varphi(x_1,\dots,x_n)) = \varphi^{\boldsymbol{A}}(f^+(x_1),\dots,f^+(x_n)) = \psi^{\boldsymbol{A}}(f^+(x_1),\dots,f^+(x_n)) = f^+(\psi(x_1,\dots,x_n)).$$

Hence, we conclude that g is well-defined. Furthermore, the surjectivity of f^+ implies that g is also surjective. Therefore, it only remains to show that g is a homomorphism. Consider a basic n-ary operation p and $\varphi_1,\dots,\varphi_n \in T(X)$. We have

$$\begin{aligned} g(p^{\boldsymbol{T}(X)/\theta}(\varphi_1/\theta,\dots,\varphi_n/\theta)) &= g(p^{\boldsymbol{T}(X)}(\varphi_1),\dots,\varphi_n)/\theta) \\ &= f^+(p^{\boldsymbol{T}(X)}(\varphi_1),\dots,\varphi_n)) \\ &= p^{\boldsymbol{A}}(f^+(\varphi_1),\dots,f^+(\varphi_n)) \\ &= p^{\boldsymbol{A}}(g(\varphi_1/\theta),\dots,g(\varphi_n/\theta)). \end{aligned}$$

Hence, we conclude that $g\colon \boldsymbol{T}(X)/\theta \to \boldsymbol{A}$ is a homomorphism, as desired. ⊠

Given a class of similar algebras K, the least variety extending K exists and will be denoted by $\mathbb{V}(\mathsf{K})$ and called the variety *generated* by K.

Corollary 1.113 (Tarski)**.** *If K is a class of similar algebras, then $\mathbb{V}(\mathsf{K}) = \mathbb{HSP}(\mathsf{K})$.*

Proof. In view of Birkhoff's Theorem, it suffices to show that $\mathbb{HSP}(\mathsf{K})$ is the class of algebras K^+ axiomatized by the set of equations (in a given denumerable set of variables) valid in K. Clearly, $\mathbb{HSP}(\mathsf{K}) \subseteq \mathsf{K}^+$, since the validity of equations persists under $\mathbb{H}, \mathbb{S}$, and $\mathbb{P}$. To prove the other inclusion, consider $\boldsymbol{A} \in \mathsf{K}$. The proof of Birkhoff's Theorem shows that $\boldsymbol{A} \in \mathbb{H}(\boldsymbol{T}(X)/\theta)$, for a sufficiently large set of variables X and the congruence θ defined in the proof of Theorem 1.109. The latter theorem shows that $\boldsymbol{T}(X)/\theta$ belongs to the prevariety generated by K, that is, $\mathbb{ISP}(\mathsf{K})$. Hence, $\boldsymbol{A} \in \mathbb{HISP}(\mathsf{K}) \subseteq \mathbb{HSP}(\mathsf{K})$, as desired. ⊠

In view of Birkhoff's Theorem examples of varieties include the classes of lattices, distributive of lattices, modular lattices, Boolean algebras, Heyting algebras, and modal algebras, because all these classes can be axiomatized by equations. By the same token, varieties are also ubiquitous in classical algebra: for instance, the classes of (commutative) rings, (abelian) groups, and monoids can be axiomatized by equations and, therefore, are varieties.

Remark 1.114*.* Since varieties are closed under $\mathbb{H}$, the relative and absolute notions of a congruence coincide for varieties. More precisely, if K is a variety and $\boldsymbol{A} \in \mathsf{K}$, then $\mathsf{Con}(\boldsymbol{A}) = \mathsf{Con}_{\mathsf{K}}(\boldsymbol{A})$. ⊠

CHAPTER 2

Intuitionistic and modal logics

2.1 Propositional logics

Recall that a *closure operator* on a set A is a map $C\colon \mathcal{P}(A) \to \mathcal{P}(A)$ such that, for every $X \subseteq Y \subseteq A$,

$$X \subseteq C(X) = C(C(X)) \text{ and } C(X) \subseteq C(Y).$$

Given a closure operator C on A, a subset $X \subseteq A$ is said to be *closed* if $X = C(X)$. A *closure system* on A is a family $\mathcal{C} \subseteq \mathcal{P}(A)$ that contains A and such that $\bigcap \mathcal{F}$, for every nonempty $\mathcal{F} \subseteq \mathcal{C}$. Closure operators and systems on A are two faces of the same coin. More precisely, if the family of closed sets of a closure operator on A is a closure system on A. On the other hand, if $\mathcal{C}$ is a closure system on A, then the map $C\mathcal{P}(A) \to \mathcal{P}(A)$, defined by the rule

$$C(X) := \bigcap \{Y \in \mathcal{C} : X \subseteq Y\},$$

is a closure operator on A. These transformations between closure operators and systems on A are one inverse to the other. Another way of presenting closure operators or systems is by means of the following concept.

Definition 2.1. A *consequence relation* on a set A is a binary relation $\vdash \subseteq \mathcal{P}(A) \times A$ such that, for every $X \cup Y \cup \{a\} \subseteq A$,

(i) if $a \in X$, then $X \vdash a$; and

(ii) if $X \vdash y$ for all $y \in Y$ and $Y \vdash a$, then $X \vdash a$.

Remark 2.2. The relation $X \vdash a$ should be read, intuitively, as "X proves a" or "a follows from X". In this reading, the demand expressed by condition (i) is rather natural, while (ii) is an abstract of the Cut rule. ☒

Formally speaking, a consequence relation on a set A is a binary relation $\vdash \subseteq \mathcal{P}(A) \times A$. However, to simplify the notation, we will often write $a_1, \ldots, a_n \vdash c$ as a shorthand for $\{a_1, \ldots, a_n\} \vdash c$. Similarly, we will use $X, a \vdash c$ as a shorthand for $X \cup \{A\} \vdash c$. Lastly, for every set of formulas $X \cup Y \cup \{a, c\}$, we write

(i) $X \vdash Y$, when $X \vdash y$ for every $y \in Y$;

(ii) $a \dashv\vdash c$, when $a \vdash c$ and $c \vdash a$; and

(iii) $X \dashv\vdash Y$, when $X \vdash Y$ and $Y \vdash X$.

Definition 2.3. Let $\vdash$ be a consequence relation on a set A. A *theory* of $\vdash$ is a subset $X \subseteq A$ such that, for every $a \in A$, if $X \vdash a$, then $a \in X$. The set of theories of A will be denoted by $\mathcal{T}h(\vdash)$.

It is easy to see that $\mathcal{T}h(\vdash)$ is a closure system on A. Moreover, given a closure operator C on A, the following is a consequence relation on A:

$$\{\langle X, a\rangle \in \mathcal{P}(A) \times A : X \vdash a\}.$$

Together with the correspondence between closure systems and operators, these transformations induce a one-to-one correspondence between consequence relations, closure operators, and closure systems on A.

From now on we work with a fixed denumerable set of variables

$$Var = \{x_0, x_1, x_2, x_3, \dots\}.$$

For the sake of simplicity, we will sometimes write $x, y, z \dots$ for the elements of Var.

In the context of logic, the term algebra $\boldsymbol{T}_\rho(Var)$ is often called the *algebra of formulas* (of type ρ) and its elements are referred to as *formulas*. An *endomorphism* of an algebra $\boldsymbol{A}$ is a homomorphism whose domain and codomain is $\boldsymbol{A}$. Endomorphism the algebra of formulas play a fundamental role in logic.

Definition 2.4. A *substitution* of type ρ is an endomorphism σ of $\boldsymbol{T}_\rho(Var)$.

When the type ρ is clear from the context, we will simply say that σ is a substitution.

In view of Proposition 1.27 and of the fact that Var is a set of generators for $\boldsymbol{T}_\rho(Var)$, every function $\sigma\colon Var \to T_\rho(Var)$ can be uniquely extended to a substitution σ^+ of type ρ, namely the function defined by the rule

$$\varphi(x_1, \dots, x_n) \longmapsto \varphi(\sigma(x_1), \dots, \sigma(x_n)).$$

Because of this, substitutions of type ρ can be presented by exhibiting functions $\sigma\colon Var \to T_\rho(Var)$.

Definition 2.5. A *logic* of type ρ is a consequence relation $\vdash$ on the set of formulas $T_\rho(Var)$ that, moreover, is *substitution invariant* in the sense that for every substitution σ of type ρ and every set of formulas $\Gamma \cup \{\varphi\} \subseteq T_\rho(Var)$,

$$\text{if } \Gamma \vdash \varphi, \text{ then } \sigma[\Gamma] \vdash \sigma(\varphi).$$

Remark 2.6. As mentioned above, $\Gamma \vdash \varphi$ should be read as "Γ proves φ" or "φ follows from Γ". The requirement that $\vdash$ is substitution invariant, instead, is intended to capture the idea that logical inferences are true only in virtue of their form (as opposed to their content). ⊠

Exercise 2.7. Let K be a class of similar algebras. Prove that the equational consequence $\vDash_{\mathsf{K}}^{Var}$ relative to K is a substitution invariant consequence relation on the set of equations $E(Var)$. To do so, you will need to guess which is the correct meaning of "substitution invariant" here. ⊠

In the rest of this section we will provide a series of examples of logics.

Example 2.8 (Equationally defined logics)**.** We work within a fixed, but arbitrary, type ρ. Given a set of equations $\boldsymbol{\tau}(x)$ in a single variable x and a set of formulas $\varGamma \cup \{\varphi\} \subseteq T(Var)$, we abbreviate

$$\{\varepsilon(\varphi) \approx \delta(\varphi) \colon \varepsilon \approx \delta \in \boldsymbol{\tau}\} \text{ as } \boldsymbol{\tau}(\varphi), \text{ and } \bigcup_{\gamma \in \varGamma} \boldsymbol{\tau}(\gamma) \text{ as } \boldsymbol{\tau}[\varGamma].$$

Given a class of algebras K and a set of equations $\boldsymbol{\tau}(x)$, we define a logic $\vdash_{\mathsf{K},\boldsymbol{\tau}}$ as follows: for every $\varGamma \cup \{\varphi\} \subseteq T(Var)$,

$$\varGamma \vdash_{\mathsf{K},\boldsymbol{\tau}} \varphi \iff \boldsymbol{\tau}[\varGamma] \vDash_{\mathsf{K}} \boldsymbol{\tau}(\varphi).$$

It is easy to prove that $\vdash_{\mathsf{K},\boldsymbol{\tau}}$ is indeed a logic in the sense of Definition 2.5. Notice that, in this case, $\vdash$ is related to K by a *completeness theorem* witnessed by the set of equations $\boldsymbol{\tau}(x)$ that allows to translate formulas into equations and, therefore, to interpret $\vdash_{\mathsf{K},\boldsymbol{\tau}}$ into $\vDash_{\mathsf{K}}$. For instance, the completeness theorem of classical propositional logic **CPC** with respect to the variety of Boolean algebras BA states precisely that **CPC** coincides with $\vdash_{\mathsf{BA},\boldsymbol{\tau}}$ where $\boldsymbol{\tau} = \{x \approx 1\}$. ⊠

Example 2.9 (Logics preserving degrees of truth)**.** An *ordered algebra* is a pair $\langle \boldsymbol{A}; \leqslant \rangle$ where $\boldsymbol{A}$ is an algebra and $\leqslant$ a partial order on its universe. Algebras with a lattice reduct (such as Boolean, Heyting, and modal algebras) can be viewed as ordered algebras, when endowed with their natural lattice ordering.

The *logic preserving degrees of truth* of a class K of similar ordered algebras is defined as follows: for every $\varGamma \cup \{\varphi\} \subseteq T(Var)$,

$$\begin{aligned} \varGamma \vdash_{\mathsf{K}}^{\leqslant} \varphi \iff & \text{for every } \langle \boldsymbol{A}; \leqslant \rangle \in \mathsf{K}, \text{homomorphism } f \colon \boldsymbol{T}(Var) \to \boldsymbol{A}, \text{ and } a \in A, \\ & \text{if } a \leqslant f(\gamma) \text{ for every } \gamma \in \varGamma, \text{ then } a \leqslant f(\varphi). \end{aligned}$$

Again, it is easy to prove that $\vdash_{\mathsf{K}}^{\leqslant}$ is indeed a logic in the sense of Definition 2.5.

To grasp the intuitive meaning of the above definition, you should think that the members $\langle \boldsymbol{A}; \leqslant \rangle$ of K are algebras whose elements are *truth values*, ordered from the less true to the truer. In this reading, homomorphisms $f \colon \boldsymbol{T}(Var) \to \boldsymbol{A}$ can be viewed as maps that assign a truth value, namely $f(\varphi)$, with every formula φ. Accordingly, the relation $\varGamma \vdash_{\mathsf{K}}^{\leqslant} \varphi$ indicates that under any assignment of truth values to the formulas in $\varGamma \cup \{\varphi\}$, the truth value of the conclusion φ is at least as true as the overall truth value of the premises $\varGamma$. In short, the overall degree of truth of $\varGamma$ is preserved (that is, is smaller than or equal to) the degree of truth of φ. ⊠

Example 2.10 (Hilbert calculi)**.** We work within a fixed, but arbitrary, type ρ. A *rule* is an expression of the form $\varGamma \rhd \varphi$, where $\varGamma \cup \{\varphi\} \subseteq T_\rho(Var)$. In this case, $\varGamma$ is said to be the set of *premises* of the rule and φ the *conclusion*. When $\varGamma = \emptyset$, the rule $\varGamma \rhd \varphi$ is sometimes called an *axiom*. A *Hilbert calculus* is a set of rules.

Every Hilbert calculus H induces a logic, as we proceed to explain. Consider a set of formulas $\varGamma \cup \{\varphi\} \subseteq T_\rho(Var)$. A *proof of φ from $\varGamma$ in* H is a well-ordered sequence $\langle \psi_\alpha : \alpha \leqslant \gamma \rangle$ of formulas $\psi_\alpha \in T_\rho(Var)$ whose last element ψ_γ is φ and such that, for every $\alpha < \gamma$, either $\psi_\alpha \in \varGamma$ or there exists a substitution σ and a rule $\varDelta \rhd \delta$ in H such that the formulas in $\sigma[\varDelta]$ occur in the initial segment $\langle \psi_\beta : \beta < \alpha \rangle$ and $\psi_\alpha = \sigma(\delta)$.

The logic $\vdash_{\mathsf{H}}$ induced by H is defined, for every $\Gamma \cup \{\varphi\} \subseteq T_\rho(Var)$, as

$$\Gamma \vdash_{\mathsf{H}} \varphi \iff \text{there exists a proof of } \varphi \text{ from } \Gamma \text{ in } \mathsf{H}.$$

As expected, $\vdash_{\mathsf{H}}$ is a logic in the sense of Definition 2.5. Furthermore, it is the least logic $\vdash$ such that $\Gamma \vdash \varphi$, for every rule $\Gamma \rhd \varphi$ in H.

A logic $\vdash$ is said to be *axiomatized* by a Hilbert calculus H when it coincides with $\vdash_{\mathsf{H}}$. Notice that every logic $\vdash$ is vacuously axiomatized by the Hilbert calculus

$$\{\Gamma \rhd \varphi : \Gamma \vdash \varphi\}.$$

Because of this, axiomatizations in terms of Hilbert calculi H acquire special interest when H is finite or, at least, recursive. ⊠

2.2 Intuitionistic logic

Motivated by the philosophy of constructivism in mathematics, intuitionistic logic identifies the principles of constructive reasoning. In this section we review its algebra-based and Kripke semantics. To this end, let ρ_I be the type comprising three binary symbols $\wedge, \vee, \rightarrow$ and two constants 0 and 1.

Definition 2.11. Intuitionistic propositional logic **IPC** is the logic of type ρ_I axiomatized by the following Hilbert calculus, denoted by IPC:

$$\begin{aligned}
&\emptyset \rhd 1\\
&\emptyset \rhd x \rightarrow (y \rightarrow x)\\
&\emptyset \rhd x \rightarrow (y \rightarrow (x \wedge y))\\
&\emptyset \rhd (x \wedge y) \rightarrow x\\
&\emptyset \rhd (x \wedge y) \rightarrow y\\
&\emptyset \rhd x \rightarrow (x \vee y)\\
&\emptyset \rhd y \rightarrow (x \vee y)\\
&\emptyset \rhd (x \vee y) \rightarrow ((x \rightarrow z) \rightarrow ((y \rightarrow z) \rightarrow z))\\
&\emptyset \rhd (x \rightarrow y) \rightarrow ((x \rightarrow (y \rightarrow z)) \rightarrow (x \rightarrow z))\\
&\emptyset \rhd 0 \rightarrow x\\
x, x \rightarrow y &\rhd y.
\end{aligned}$$

Remark 2.12. Since the set of premises of every rule in IPC is finite, for every $\Gamma \cup \{\varphi\} \subseteq T(Var)$, we have

$$\Gamma \vdash_{\mathbf{IPC}} \varphi \iff \text{there exists a } \textit{finite} \text{ proof of } \varphi \text{ from } \Gamma \text{ in } \mathsf{IPC}. \quad ⊠$$

Definition 2.13. We denote by HA the variety of Heyting algebras.

Our first goal is to prove the completeness of **IPC** with respect to HA.

Theorem 2.14. *Let $\boldsymbol{\tau} = \{x \approx 1\}$. For every $\Gamma \cup \{\varphi\} \subseteq T(Var)$, we have*

$$\Gamma \vdash_{\mathbf{IPC}} \varphi \iff \boldsymbol{\tau}[\Gamma] \vDash_{\mathsf{HA}} \boldsymbol{\tau}(\varphi).$$

Proof. We begin by claiming that $\boldsymbol{\tau}[\sigma[\Gamma]] \vDash_{\mathsf{HA}} \boldsymbol{\tau}(\sigma(\varphi))$, for every substitution σ and rule $\Gamma \rhd \varphi$ in IPC. It is an easy exercise to prove that $\boldsymbol{\tau}[\Gamma] \vDash_{\mathsf{HA}} \boldsymbol{\tau}(\varphi)$, for every rule $\Gamma \rhd \varphi$ in IPC. Then consider a rule $\Gamma \rhd \varphi$ in IPC and a substitution σ. Moreover, let $\boldsymbol{A} \in \mathsf{HA}$ and $f \colon \boldsymbol{T}(\mathit{Var}) \to \boldsymbol{A}$ a homomorphism such that $f(\sigma(\gamma)) = 1$, for every $\gamma \in \Gamma$. Recall that σ is an endomorphism of $\boldsymbol{T}(\mathit{Var})$ and, therefore, $f \circ \sigma \colon \boldsymbol{T}(\mathit{Var}) \to \boldsymbol{A}$ is a homomorphism. Since $\boldsymbol{\tau}[\Gamma] \vDash_{\mathsf{HA}} \boldsymbol{\tau}(\varphi)$ and $f(\sigma(\gamma)) = 1$, for every $\gamma \in \Gamma$, this yields $f(\sigma(\varphi)) = 1$. Hence, we conclude that $\boldsymbol{\tau}[\sigma[\Gamma]] \vDash_{\mathsf{HA}} \boldsymbol{\tau}(\sigma(\varphi))$, as desired. This establishes the claim.

Now, suppose that $\Gamma \vdash_{\mathbf{IPC}} \varphi$. Then there exists a finite proof $\langle \psi_1, \dots, \psi_n \rangle$ of φ from Γ in IPC. We will prove that $\boldsymbol{\tau}[\Gamma] \vDash_{\mathsf{HA}} \boldsymbol{\tau}(\psi_m)$ by complete induction on m. Suppose that $\boldsymbol{\tau}[\Gamma] \vDash_{\mathsf{HA}} \boldsymbol{\tau}(\psi_k)$, for every $k < m$. By definition of a proof, one of the following conditions holds:

(i) $\psi_m \in \Gamma$; or

(ii) there exists a substitution σ and a rule $\Delta \rhd \delta$ in IPC such that the formulas in $\sigma[\Delta]$ occur in the initial segment $\langle \psi_k : k < m \rangle$ and $\psi_m = \sigma(\delta)$.

If $\psi_m \in \Gamma$, then obviously $\boldsymbol{\tau}[\Gamma] \vDash_{\mathsf{HA}} \boldsymbol{\tau}(\psi_m)$. The suppose that condition (ii) holds. By the claim, $\boldsymbol{\tau}[\sigma[\Delta]] \vDash_{\mathsf{HA}} \boldsymbol{\tau}(\psi_m)$. Furthermore, by the inductive hypothesis, $\boldsymbol{\tau}[\Gamma] \vDash_{\mathsf{HA}} \boldsymbol{\tau}[\sigma[\Delta]]$. Since the relation $\vDash_{\mathsf{HA}}$ satisfies the Cut principle, we conclude that $\boldsymbol{\tau}[\Gamma] \vDash_{\mathsf{HA}} \boldsymbol{\tau}(\psi_m)$. This concludes the inductive argument. Since $\varphi = \psi_n$, we conclude that $\boldsymbol{\tau}[\Gamma] \vDash_{\mathsf{HA}} \boldsymbol{\tau}(\varphi)$.

To prove the converse, we reason by contraposition. Accordingly, suppose that $\Gamma \nvdash_{\mathbf{IPC}} \varphi$. Using the Hilbert calculus IPC it is easy (but tedious) to prove that the relation

$$\theta := \{\langle \psi, \gamma \rangle \in T(\mathit{Var}) \times T(\mathit{Var}) : \Gamma \vdash_{\mathbf{IPC}} (\psi \to \gamma) \wedge (\gamma \to \psi)\}$$

is a congruence on the algebra of formulas $\boldsymbol{T}(\mathit{Var})$.

The quotient $\boldsymbol{T}(\mathit{Var})/\theta$ is sometimes called the *Lindenbaum-Tarski algebra* of Γ. Again, using the Hilbert calculus IPC it is easy to prove that $\boldsymbol{T}(\mathit{Var})/\theta$ is a Heyting algebra. For instance, to prove that the operation $\wedge$ is idempotent in $\boldsymbol{T}(\mathit{Var})/\theta$ one needs to show that, for every $\gamma \in T(\mathit{Var})$,

$$\Gamma \vdash_{\mathbf{IPC}} (\gamma \to (\gamma \wedge \gamma)) \wedge ((\gamma \wedge \gamma) \to \gamma).$$

Lastly, consider the canonical homomorphism $\pi \colon \boldsymbol{T}(\mathit{Var}) \to \boldsymbol{T}(\mathit{Var})/\theta$, defined by the rule $\pi(\psi) := \psi/\theta$. Then consider a formula $\gamma \in \Gamma$. Using the axiom $\emptyset \rhd x \to (y \to x)$, we obtain

$$\Gamma \vdash_{\mathbf{IPC}} \gamma \to (1 \to \gamma).$$

By modus ponens and the fact that $\gamma \in \Gamma$, we conclude that

$$\Gamma \vdash_{\mathbf{IPC}} 1 \to \gamma.$$

A similar argument, which relies on the axiom $\emptyset \rhd 1$, yields $\Gamma \vdash_{\mathbf{IPC}} \gamma \to 1$. Lastly, from the axiom $\emptyset \rhd x \to (y \to (x \wedge y))$, we obtain

$$\Gamma \vdash_{\mathbf{IPC}} (1 \to \gamma) \to ((\gamma \to 1) \to ((1 \to \gamma) \wedge (\gamma \to 1))).$$

Since $\Gamma \vdash_{\mathbf{IPC}} 1 \to \gamma$ and $\Gamma \vdash_{\mathbf{IPC}} \gamma \to 1$, we can apply modus ponens to the above display, obtaining

$$\Gamma \vdash_{\mathbf{IPC}} (1 \to \gamma) \wedge (\gamma \to 1).$$

This means that $\langle \gamma, 1 \rangle \in \theta$, that is, $\pi(\gamma) = \gamma/\theta = 1/\theta$. Hence, $\pi[\Gamma] \subseteq \{1/\theta\}$. On the other hand, $\pi(\varphi) \neq 1/\theta$. For suppose the contrary, with a view to contradiction. Then $\Gamma \vdash_{\mathbf{IPC}} (\varphi \to 1) \wedge (1 \to \varphi)$. Using the axiom $\emptyset \rhd (x \wedge y) \to y$ and modus ponens, we obtain $\Gamma \vdash_{\mathbf{IPC}} 1 \to \varphi$. Therefore, using the axiom $\emptyset \rhd 1$ and modus ponens, we obtain that $\Gamma \vdash_{\mathbf{IPC}} \varphi$, a contradiction. Hence, we conclude that $\pi(\varphi) \neq 1/\theta$. Putting all the pieces together, we have a Heyting algebra $\boldsymbol{A} := \boldsymbol{T}(Var)/\theta$ and a homomorphism $\pi\colon \boldsymbol{T}(Var) \to \boldsymbol{A}$ such that $\pi(\gamma) = 1$ for every $\gamma \in \Gamma$ and $\pi(\varphi) \neq 1$. Hence, we conclude that $\boldsymbol{\tau}[\Gamma] \nvDash_{\mathsf{HA}} \boldsymbol{\tau}(\varphi)$, as desired. ⊠

Corollary 2.15. *The logics* **IPC** *and* $\vdash_{\mathsf{HA},\boldsymbol{\tau}}$*, where* $\boldsymbol{\tau} = \{x \approx 1\}$*, coincide.*

*Exercise** 2.16. This exercise describes an instructive strategy for proving the decidability of **IPC**. First, show that every local subgraph $\mathbb{X}$ of a Heyting algebra $\boldsymbol{A}$ can be embedded into a finite Heyting algebra. To this end, follow the next steps:

(i) Let $\boldsymbol{B}$ be the bounded sublattice of $\boldsymbol{A}$ generated by $\mathbb{X}$. Notice that $\boldsymbol{B}$ is finite, because it is a finitely generated bounded distributive lattice. However, $\boldsymbol{B}$ need not be a subalgebra of $\boldsymbol{A}$.

(ii) Show that the bounded lattice $\boldsymbol{B}$ can be endowed with a binary operation $\to$ that turns it into a finite Heyting algebra $\boldsymbol{B}^+$. Hint: try to define

$$a \to^{\boldsymbol{B}^+} c := \bigvee^{\boldsymbol{B}} \{b \in B : b \wedge^{\boldsymbol{B}} a \leqslant c\}.$$

(iii) Show that the inclusion map is an embedding of $\mathbb{X}$ into $\boldsymbol{B}^+$.

Now, use this observation to conclude that every Heyting algebra embeds into a ultraproduct of finite Heyting algebras. Then proceed as follows:

(iv) Infer that the class of Heyting algebras is the quasi-variety generated by the class of finite Heyting algebras.

(v) Use this fact and the completeness theorem for **IPC** to show that **IPC** is decidable, in the sense that the following set is recursive:

$$\{\langle \Gamma, \varphi \rangle : \Gamma \cup \{\varphi\} \subseteq T(Var) \text{ is finite and } \Gamma \vdash_{\mathbf{IPC}} \varphi\}.$$

The last step requires an argument similar to the one that we used to conclude that the quasi-equational theory of lattices is decidable. ⊠

As Heyting algebras have a lattice reduct, we can view them as ordered algebras endowed with their lattice ordering. Notably, we obtain the following.

Theorem 2.17. **IPC** *is the logic preserving degrees of truth of* HA.

Proof. Consider $\Gamma \cup \{\varphi\} \subseteq T(Var)$. We need to prove that

$$\Gamma \vdash_{\mathbf{IPC}} \varphi \iff \text{for every } \langle \boldsymbol{A}; \leqslant \rangle \in \mathsf{HA}, \text{ homomorphism } f \colon \boldsymbol{T}(Var) \to \boldsymbol{A}, \text{ and } a \in A, \\ \text{if } a \leqslant f(\gamma) \text{ for every } \gamma \in \Gamma, \text{ then } a \leqslant f(\varphi).$$

To prove the implication from right to left, we reason by contraposition. Accordingly, suppose that $\Gamma \nvdash_{\mathbf{IPC}} \varphi$. By Theorem 2.14, there exists some $\boldsymbol{A} \in \mathsf{HA}$ and a homomorphism $f \colon \boldsymbol{T}(Var) \to \boldsymbol{A}$ such that $f(\gamma) = 1$, for every $\gamma \in \Gamma$, and $f(\varphi) \neq 1$. Since 1 is the maximum of $\boldsymbol{A}$, this amounts to $1 \leqslant f(\gamma)$, for every $\gamma \in \Gamma$, and $1 \nleqslant f(\varphi)$.

To prove the implication from left to right, suppose that $\Gamma \vdash_{\mathbf{IPC}} \varphi$ and consider a Heyting algebra $\boldsymbol{A}$, a homomorphism $f \colon \boldsymbol{T}(Var) \to \boldsymbol{A}$, and an element $a \in A$ such that $a \leqslant f(\gamma)$, for every $\gamma \in \Gamma$. Since $\uparrow a$ is a filter of $\boldsymbol{A}$, we can use the isomorphism $\boldsymbol{\Omega}^{\boldsymbol{A}} \colon \mathsf{Fi}(\boldsymbol{A}) \to \mathsf{Con}(\boldsymbol{A})$, described in Example 1.8, to produce a congruence

$$\boldsymbol{\Omega}^{\boldsymbol{A}}{\uparrow} a = \{\langle b, c\rangle \in A \times A : a \leqslant b \to c, c \to b\}$$

of $\boldsymbol{A}$. Then let $\pi \colon \boldsymbol{A} \to \boldsymbol{\Omega}^{\boldsymbol{A}}{\uparrow} a$ be the canonical homomorphism defined by the rule $\pi(c) := c/\boldsymbol{\Omega}^{\boldsymbol{A}}{\uparrow} a$. Notice that the composition $\pi \circ f \colon \boldsymbol{T}(Var) \to \boldsymbol{A}$ is also a homomorphism.

We will prove that $\pi \circ f(\gamma) = 1$, for every $\gamma \in \Gamma$. To this end, consider $\gamma \in \Gamma$ and recall that $a \leqslant f(\gamma)$. Using the residuation law, we obtain

$$a \leqslant 1 \to f(\gamma) \iff a \wedge 1 \leqslant f(\gamma) \iff a \leqslant f(\gamma).$$

Since, by assumption $a \leqslant f(\gamma)$, we conclude that $a \leqslant 1 \to f(\gamma)$. Moreover, with another application of the residuation law,

$$a \leqslant f(\gamma) \to 1 \iff a \wedge f(\gamma) \leqslant 1.$$

Since 1 is the maximum of $\boldsymbol{A}$, this yields $a \leqslant f(\gamma) \to 1$. Thus, $a \leqslant 1 \to f(\gamma), f(\gamma) \to 1$ and, therefore,

$$\langle 1, f(\gamma)\rangle \in \boldsymbol{\Omega}^{\boldsymbol{A}}{\uparrow} a.$$

But this means precisely that $\pi \circ f(\gamma) = 1$. Thus, $\pi \circ f(\gamma) = 1$, for every $\gamma \in \Gamma$.

Now, from $\Gamma \vdash_{\mathbf{IPC}} \varphi$ and Theorem 2.14 it follows $\boldsymbol{\tau}[\Gamma] \vDash_{\mathsf{HA}} \boldsymbol{\tau}(\varphi)$. Since $\pi \circ f(\gamma) = 1$, for every $\gamma \in \Gamma$, this implies $\pi \circ f(\varphi) = 1$, that is,

$$a \leqslant f(\varphi) \to 1, 1 \to f(\varphi).$$

By applying the residuation law to $a \leqslant 1 \to f(\varphi)$, we obtain $a \leqslant f(\varphi)$, as desired. ⊠

Remark 2.18. Natural adaptations of the proofs of Theorems 2.14 and 2.17 establish the completeness theorem of classical propositional logic **CPC** with respect to the variety BA of Boolean algebras and that **CPC** is the logic preserving degrees of truth of BA. ⊠

2.3 Kripke semantics and Esakia's duality

Intuitionistic propositional logic admits also a Kripke semantics. As it will become clear, it is tightly related to the following adaptation of Birkhoff's representation of bounded distributive lattices. Given a poset $\mathbb{X}$, consider the algebra

$$\mathbb{X}^{+} := \langle \mathsf{Up}(\mathbb{X}); \cap, \cup, \to, \emptyset, X\rangle,$$

where $\to$ is the binary operation defined, for every $U, V \in \mathsf{Up}(\mathbb{X})$, as follows:

$$U \to V := X \smallsetminus {\downarrow} (U \smallsetminus V).$$

Notice that $\mathbb{X}^+$ is a Heyting algebra. This is because $\langle \mathsf{Up}(\mathbb{X}); \cap, \cup, \emptyset, X\rangle$ is obviously a bounded lattice and, for every $U, V, W \in \mathsf{Up}(\mathbb{X})$, the residuation law holds:

$$\begin{aligned} U \cap V \subseteq W &\iff \text{for every } x \in U, {\uparrow}x \cap V \subseteq W \\ &\iff U \subseteq X \smallsetminus {\downarrow} (V \smallsetminus W) \\ &\iff U \subseteq V \to W. \end{aligned}$$

Furthermore, every Heyting algebra embeds into one of this form.

Theorem 2.19 (Kripke)**.** *If $\boldsymbol{A}$ is a Heyting algebra, then the map $\varepsilon_{\boldsymbol{A}} \colon \boldsymbol{A} \to (\boldsymbol{A}_*)^+$, defined by the rule*

$$\varepsilon_{\boldsymbol{A}}(a) := \{F \in \boldsymbol{A}_* : a \in F\},$$

is an embedding.

Proof. Birkhoff's representation theorem for bounded distributive lattices guarantees that $\varepsilon_{\boldsymbol{A}}$ is a well-defined bounded lattice embedding. Therefore, it only remains to prove that it preserves $\to$. Consider $a, c \in A$ and $F \in \boldsymbol{A}_*$. We need to prove that

$$\varepsilon_{\boldsymbol{A}}(a \to^{\boldsymbol{A}} c) = \varepsilon_{\boldsymbol{A}}(a) \to^{(\boldsymbol{A}_*)^+} \varepsilon_{\boldsymbol{A}}(c). \tag{2.1}$$

Accordingly, consider $F \in \boldsymbol{A}_*$. Suppose, with a view to contradiction, that there exists some $F \in \varepsilon_{\boldsymbol{A}}(a \to^{\boldsymbol{A}} c) \smallsetminus \varepsilon_{\boldsymbol{A}}(a) \to^{(\boldsymbol{A}_*)^+} \varepsilon_{\boldsymbol{A}}(c)$. From $F \in \varepsilon_{\boldsymbol{A}}(a \to^{\boldsymbol{A}} c)$ it follows $a \to^{\boldsymbol{A}} c \in F$ and from $F \notin \varepsilon_{\boldsymbol{A}}(a) \to^{(\boldsymbol{A}_*)^+} \varepsilon_{\boldsymbol{A}}(c)$ that there exists $G \in \boldsymbol{A}_*$ such that $F \subseteq G$, $a \in G$, and $c \notin G$. Since $a \to^{\boldsymbol{A}} c \in F \subseteq G$ and $a \in G$, we obtain $a \wedge^{\boldsymbol{A}} (a \to^{\boldsymbol{A}} c) \in G$. Now, using the residuation law in $\boldsymbol{A}$, we obtain $a \wedge^{\boldsymbol{A}} (a \to^{\boldsymbol{A}} c) \leqslant c$. Together with the fact that G is an upset, this implies $c \in G$, a contradiction. This establishes the inclusion from left to right in (2.6).

To prove the implication from left to right, we reason by contraposition. Consider $F \in \boldsymbol{A}_*$ such that $F \notin \varepsilon_{\boldsymbol{A}}(a \to^{\boldsymbol{A}} c)$, that is, $a \to^{\boldsymbol{A}} c \notin F$. Then consider the filter G of $\boldsymbol{A}$ generated by $F \cup \{a\}$. We have that

$$\begin{aligned} c \in G &\iff \text{there are } b_1, \dots, b_n \in F \text{ such that } a \wedge^{\boldsymbol{A}} b_1 \wedge^{\boldsymbol{A}} \cdots \wedge^{\boldsymbol{A}} b_n \leqslant c \\ &\iff \text{there are } b_1, \dots, b_n \in F \text{ such that } b_1 \wedge^{\boldsymbol{A}} \cdots \wedge^{\boldsymbol{A}} b_n \leqslant a \to^{\boldsymbol{A}} c \\ &\iff a \to^{\boldsymbol{A}} c \in F. \end{aligned}$$

Since $a \to^{\boldsymbol{A}} c \notin F$, we conclude that $c \notin G$. As $\boldsymbol{A}$ is a distributive lattice, every proper filter of $\boldsymbol{A}$ is an intersection of prime filters. Together with the fact that G is a proper filter such that $c \notin G$, this implies the existence of a prime filter $H \supseteq G$ such that $c \notin H$. Therefore, H is a prime filter of $\boldsymbol{A}$ containing a but not c, that is, $G \in \varepsilon_{\boldsymbol{A}}(a) \smallsetminus \varepsilon_{\boldsymbol{A}}(c)$. Since $F \subseteq G$, we conclude that $F \in {\downarrow}(\varepsilon_{\boldsymbol{A}}(a) \smallsetminus \varepsilon_{\boldsymbol{A}}(c))$ and, therefore, that

$$F \notin \boldsymbol{A}_* \smallsetminus {\downarrow}(\varepsilon_{\boldsymbol{A}}(a) \smallsetminus \varepsilon_{\boldsymbol{A}}(c)) = \varepsilon_{\boldsymbol{A}}(a) \to^{(\boldsymbol{A}_*)^+} \varepsilon_{\boldsymbol{A}}(c).$$ ⊠

Corollary 2.20. *Finite Heyting algebras are (up to isomorphism) algebras of the form $\mathbb{X}^+$ for some finite poset $\mathbb{X}$.*

Proof. If $\mathbb{X}$ is a finite poset, then $\mathbb{X}^+$ is clearly a finite Heyting algebras. Conversely, suppose that $\boldsymbol{A}$ is a finite Heyting algebra. By Theorem 2.19, the map $\varepsilon_{\boldsymbol{A}} \colon \boldsymbol{A} \to (\boldsymbol{A}_*)^+$ is an embedding. Furthermore, $\varepsilon_{\boldsymbol{A}}$ is surjective, by Finite Priestley's Duality. Thus, $\boldsymbol{A} \cong (\boldsymbol{A}_*)^+$, Since $\boldsymbol{A}$ is finite, $\boldsymbol{A}_*$ is a finite poset and we are done. ⊠

Exercise 2.21. Prove that a finite Heyting algebra is subdirectly irreducible if and only if it isomorphic to $\mathbb{X}^+$ for some finite poset $\mathbb{X}$ with a minimum. ⊠

As we mentioned, this representation theorem is connected with the Kripke semantics of **IPC**, as we proceed to explain. A *valuation* in a poset $\mathbb{X}$ is a function $v \colon Var \to \mathsf{Up}(\mathbb{X})$. Given a valuation v on a poset $\mathbb{X}$, we define a notion of *validity* of a formula $\varphi \in T(Var)$ at a point $w \in X$ under v, in symbols $w, v \Vdash \varphi$, by recursion on the construction of φ as follows. For variables $x \in Var$ we set

$$w, v \Vdash x \iff w \in v(x), \text{ for } x \in Var$$

and for constant symbols

$$w, v \Vdash 1 \text{ and } w, v \nVdash 0.$$

For complex formulas we set

$$\begin{aligned} w, v \Vdash \alpha \wedge \beta &\iff w, v \Vdash \alpha \text{ and } w, v \Vdash \beta \\ w, v \Vdash \alpha \vee \beta &\iff w, v \Vdash \alpha \text{ or } w, v \Vdash \beta \\ w, v \Vdash \alpha \to \beta &\iff \text{ for every } u \in X \text{ such that } w \leqslant u, \text{ if } u, v \Vdash \alpha, \text{ then } u, v \Vdash \beta. \end{aligned}$$

Given a set of formulas Γ, we write $w, v \Vdash \Gamma$ to indicate that $w, v \Vdash \gamma$, for every $\gamma \in \Gamma$.

We define two logics $\vdash^{\ell}_{\mathsf{Pos}}$ and $\vdash^{g}_{\mathsf{Pos}}$ associated with the class of all posets as follows: for every $\Gamma \cup \{\varphi\} \subseteq T(Var)$,

$$\begin{aligned} \Gamma \vdash^{\ell}_{\mathsf{Pos}} \varphi \iff &\text{for every poset } \mathbb{X}, \text{ every valuation } v \text{ in } \mathbb{X}, \text{ and every } w \in X, \\ &\text{if } w, v \Vdash \Gamma, \text{ then } w, v \Vdash \varphi; \text{ and} \\ \Gamma \vdash^{g}_{\mathsf{Pos}} \varphi \iff &\text{for every poset } \mathbb{X} \text{ and every valuation } v \text{ in } \mathbb{X}, \\ &\text{if } w, v \Vdash \Gamma \text{ for every } w \in X, \text{ then } w, v \Vdash \varphi \text{ for every } w \in X. \end{aligned}$$

The logics $\vdash^{\ell}_{\mathsf{Pos}}$ and $\vdash^{g}_{\mathsf{Pos}}$ are known, respectively, as the *local* and the *global consequence relations* associated with the class of posets.

Theorem 2.22. *For every set of formulas* $\Gamma \cup \{\varphi\} \subseteq T(Var)$,

$$\Gamma \vdash_{\mathbf{IPC}} \varphi \iff \Gamma \vdash^{\ell}_{\mathsf{Pos}} \varphi \iff \Gamma \vdash^{g}_{\mathsf{Pos}} \varphi.$$

Proof. As usual, the nontrivial part of the proof consists in showing that if $\Gamma \nvdash_{\mathbf{IPC}} \varphi$, then $\Gamma \nvdash^{\ell}_{\mathsf{Pos}} \varphi$ and $\Gamma \nvdash^{\ell}_{\mathsf{Pos}} \varphi$. From the definition of $\vdash^{\ell}_{\mathsf{Pos}}$ and $\vdash^{g}_{\mathsf{Pos}}$ it follows immediately that $\Gamma \nvdash^{g}_{\mathsf{Pos}} \varphi$ implies $\Gamma \nvdash^{\ell}_{\mathsf{Pos}} \varphi$. Therefore, it suffices to show that $\Gamma \nvdash_{\mathbf{IPC}} \varphi$ implies $\Gamma \vdash^{g}_{\mathsf{Pos}} \varphi$.

Accordingly, suppose that $\Gamma \nvdash_{\mathbf{IPC}} \varphi$. By Theorem 2.14 there exist a Heyting algebra $\boldsymbol{A}$ and a homomorphism $f \colon \boldsymbol{T}(Var) \to \boldsymbol{A}$ such that $f(\gamma) = 1$, for all $\gamma \in \Gamma$, and $f(\varphi) \neq 1$. By Theorem 2.19, the map $\varepsilon_{\boldsymbol{A}} \colon \boldsymbol{A} \to (\boldsymbol{A}_*)^+$ is also a homomorphism. Therefore, so is the composition $\varepsilon_{\boldsymbol{A}} \circ f \colon \boldsymbol{T}(Var) \to (\boldsymbol{A}_*)^+$.

Since the maximum 1 of $\boldsymbol{A}$ belongs to every prime filter of $\boldsymbol{A}$, we obtain

$$\varepsilon_{\boldsymbol{A}}(f(\gamma)) = \boldsymbol{A}_*, \text{ for every } \gamma \in \Gamma. \tag{2.2}$$

On the other hand, since $f(\varphi) \neq 1$, by the Prime Filter Theorem, there exists a prime filter F of $\boldsymbol{A}$ such that $f(\varphi) \notin F$. Consequently, $F \notin \varepsilon_{\boldsymbol{A}}(f(\varphi))$ and, therefore,

$$\varepsilon_{\boldsymbol{A}}(f(\varphi)) \neq \boldsymbol{A}_*. \tag{2.3}$$

Now, consider the poset $\boldsymbol{A}_*$ and the valuation $v \colon Var \to \mathsf{Up}(\boldsymbol{A}_*)$ defined by the rule

$$v(x) := \varepsilon_{\boldsymbol{A}}(f(x)).$$

We claim that, for every formula $\psi \in T(Var)$,

$$\varepsilon_{\boldsymbol{A}}(f(\psi)) = \{F \in \boldsymbol{A}_* : F, v \Vdash \psi\}.$$

The proof goes by induction on the construction of ψ. For variables $x \in Var$ and the constant symbols 0 and 1 we have

$$\begin{aligned}
\varepsilon_{\boldsymbol{A}}(f(x)) &= v(x) = \{F \in \boldsymbol{A}_* : F, v \Vdash x\} \\
\varepsilon_{\boldsymbol{A}}(f(0)) &= \varepsilon_{\boldsymbol{A}}(0^{\boldsymbol{A}}) = \emptyset = \{F \in \boldsymbol{A}_* : F, v \Vdash 0\} \\
\varepsilon_{\boldsymbol{A}}(f(1)) &= \varepsilon_{\boldsymbol{A}}(1^{\boldsymbol{A}}) = \boldsymbol{A}_* = \{F \in \boldsymbol{A}_* : F, v \Vdash 1\}.
\end{aligned}$$

Then we turn to the induction step. We have three cases:

(i) $\psi = \psi_1 \wedge \psi_2$;

(ii) $\psi = \psi_1 \vee \psi_2$;

(iii) $\psi = \psi_1 \to \psi_2$.

(i): In this case, we have

$$\begin{aligned}
\varepsilon_{\boldsymbol{A}}(f(\psi_1 \wedge \psi_2)) &= \varepsilon_{\boldsymbol{A}}(f(\psi_1)) \cap \varepsilon_{\boldsymbol{A}}(f(\psi_2)) \\
&= \{F \in \boldsymbol{A}_* : F, v \Vdash \psi_1\} \cap \{F \in \boldsymbol{A}_* : F, v \Vdash \psi_2\} \\
&= \{F \in \boldsymbol{A}_* : F, v \Vdash \psi_1 \text{ and } F, v \Vdash \psi_2\} \\
&= \{F \in \boldsymbol{A}_* : F, v \Vdash \psi_1 \wedge \psi_2\}.
\end{aligned}$$

The first equality above follows from the fact that $\varepsilon_{\boldsymbol{A}} \circ f \colon \boldsymbol{T}(Var) \to (\boldsymbol{A}_*)^+$ is a homomorphism, the second from the inductive hypothesis, the third is straightforward, while the last one is a consequence of the definition of $\Vdash$.

(ii): Analogous to case (i).

(iii): In this case, we have

$$\begin{aligned}
\varepsilon_{\boldsymbol{A}}(f(\psi_1 \to \psi_2)) &= \boldsymbol{A}_* \smallsetminus {\downarrow}(\varepsilon_{\boldsymbol{A}}(f(\psi_1)) \smallsetminus \varepsilon_{\boldsymbol{A}}(f(\psi_2))) \\
&= \{F \in \boldsymbol{A}_* : \text{for all } G \in \boldsymbol{A}_* \text{ such that } F \subseteq G, \\
&\qquad \text{if } G \in \varepsilon_{\boldsymbol{A}}(f(\psi_1)), \text{ then } G \in \varepsilon_{\boldsymbol{A}}(f(\psi_2))\} \\
&= \{F \in \boldsymbol{A}_* : \text{for all } G \in \boldsymbol{A}_* \text{ such that } F \subseteq G, \\
&\qquad \text{if } G, v \Vdash \psi_1, \text{ then } G, v \Vdash \psi_2\} \\
&= \{F \in \boldsymbol{A}_* : F, v \Vdash \psi_1 \to \psi_2\}.
\end{aligned}$$

Again, the first equality above follows from the fact that $\varepsilon_{\boldsymbol{A}} \circ f \colon \boldsymbol{T}(Var) \to (\boldsymbol{A}_*)^+$ is a homomorphism, the second is straightforward, the third from the inductive hypothesis, and the last one is a consequence of the definition of $\Vdash$. This concludes the proof of the claim.

By applying the claim to conditions (2.2) and (2.3), we obtain that

$$F, v \Vdash \Gamma, \text{ for all } F \in \boldsymbol{A}_*, \text{ but there exists } F \in \boldsymbol{A}_* \text{ such that } F, v \nVdash \varphi.$$

Hence, we conclude that $\Gamma \nvdash^{g}_{\mathsf{Pos}} \varphi$, as desired. ⊠

The correspondence between Heyting algebras and posets described above can enhanced to a dual equivalence of categories. As usual, for the sake of simplicity, we shall restrict our attention to the finite case.

Definition 2.23. Given two posets $\mathbb{X}$ and $\mathbb{Y}$, an order preserving map $f \colon \mathbb{X} \to \mathbb{Y}$ is said to be a *p-morphism* if, for every $x \in X$ and $y \in Y$,

$$\text{if } f(x) \leqslant^{\mathbb{Y}} y, \text{ there exists } z \in X \text{ such that } x \leqslant^{\mathbb{X}} z \text{ and } f(z) = y.$$

We denote by Esa the category of finite posets with p-morphisms as arrows and by $\mathsf{HA}^{<\omega}$ that of finite Heyting algebras with homomorphisms as arrows.

Esakia's Finite Duality 2.24. *The categories* $\mathsf{HA}^{<\omega}$ *and* Esa *are dually equivalent.*

Proof. We begin by proving that the functor $(-)^* \colon \mathsf{DBL}^{<\omega} \to \mathsf{Pos}^{<\omega}$ can be restricted to a functor $(-)^* \colon \mathsf{HA}^{<\omega} \to \mathsf{Esa}$. To this end, it suffices to prove that $f_* \colon \boldsymbol{B}_* \to \boldsymbol{A}_*$ is a p-morphism, for every homomorphism $f \colon \boldsymbol{A} \to \boldsymbol{B}$ between Heyting algebras.

Accordingly, consider a homomorphism $f \colon \boldsymbol{A} \to \boldsymbol{B}$ between Heyting algebras. Clearly, f_* is order preserving. Then consider $F \in \boldsymbol{B}_*$ and $G \in \boldsymbol{A}_*$ such that $f_*(F) \subseteq G$, that is, $f^{-1}(F) \subseteq G$. Let H be the filter of $\boldsymbol{B}$ generated by $F \cup f[G]$ and I the ideal of $\boldsymbol{B}$ generated by $f[A \smallsetminus G]$.

We claim that $H \cap I = \emptyset$. For suppose the contrary, with a view to contradiction. Then there exists an element $a \in H \cap I$. Since $a \in H$, there are $b_1, \dots, b_n \in F$ and $c_1, \dots, c_n \in G$ such that

$$b_1 \wedge \cdots \wedge b_n \wedge f(c_1) \wedge \cdots \wedge f(c_n) \leqslant a.$$

Since F and G are closed under binary meets, $b := b_1 \wedge \cdots \wedge b_n \in F$ and $c := c_1 \wedge \cdots \wedge c_n \in G$. Moreover, as f is a homomorphism, $f(c) = f(c_1 \wedge \cdots \wedge c_n) = f(c_1) \wedge \cdots \wedge f(c_n)$. Therefore, the above display amounts to $b \wedge f(c) \leqslant a$, where $d \in F$ and $c \in G$.

Similarly, since $a \in I$, there are $d_1, \dots, d_n \in A \smallsetminus G$ such that $a \leqslant f(d_1) \vee \cdots \vee f(d_n)$. Since G is a prime filter, $A \smallsetminus G$ is an ideal, whence $d := d_1 \vee \cdots \vee d_n \in A \smallsetminus G$. Furthermore, since f is a homomorphism, we have $f(d_1) \vee \cdots \vee f(d_n) = f(d)$. Therefore, we conclude that $a \leqslant f(d)$, where $d \in A \smallsetminus G$. As a consequence, we get

$$b \wedge f(c) \leqslant a \leqslant f(d).$$

Using the residuation law and the fact that f is a homomorphism, we obtain $b \leqslant f(c) \to f(d) = f(c \to d)$. Since $b \in F$, this implies $f(c \to d) \in F$. Now, recall from the assumptions that $f^{-1}(F) \subseteq G$. Therefore, $c \to d \in G$. Since $c \in G$ and $c \wedge (c \to d) \leqslant d$, this implies $d \in G$, a contradiction. This concludes the proof of the claim.

By the claim, we can apply the Prime Filter Theorem, obtaining a prime filter H^+ of $\boldsymbol{A}$ that extends H but is disjoint from I. Clearly, $F \subseteq H \subseteq H^+$. Therefore, to conclude the proof, it only remains to show that $f^{-1}(H^+) = G$. By definition of H, we have $f[G] \subseteq H \subseteq H^+$, whence $G \subseteq f^{-1}(H^+)$. To prove the other inclusion, consider an element $a \in A \smallsetminus G$. By definition of I, we have $f(a) \in I$. Since H^+ and I are disjoint, $f(a) \notin H^+$ and, therefore, $a \notin f^{-1}(H)$. Hence, we conclude that $f^{-1}(H^+) = G$, as desired. It follows that $f_* \colon \boldsymbol{B}_* \to \boldsymbol{A}_*$ is a well-defined p-morphism and, therefore, that the natural restriction $(-)_* \colon \mathsf{HA}^{<\omega} \to \mathsf{Esa}$ is a contravariant functor.

Then we turn to define a contravariant functor $(-)^+ \colon \mathsf{Esa} \to \mathsf{HA}^{<\omega}$. Given a poset $\mathbb{X}$, we already defined the Heyting algebra $\mathbb{X}^+$. Moreover, given a p-morphism $f \colon \mathbb{X} \to \mathbb{Y}$ between finite posets, let $f^+ \colon \mathbb{Y}^+ \to \mathbb{X}^+$ be the map defined as $f^+ := f^*$. Since $(-)^* \colon \mathsf{Pos} \to \mathsf{BDL}^{<\omega}$ is a well-defined functor, $f^* \colon \mathbb{Y}^+ \to \mathbb{X}^+$ is a homomorphism of finite bounded lattices.

To prove that it is also a homomorphism of Heyting algebras, consider $U, V \in \mathsf{Up}(\mathbb{Y})$. We will show that

$$f^{-1}(U \to^{\mathbb{Y}^+} V) = f^{-1}(U) \to^{\mathbb{X}^+} f^{-1}(V).$$

We begin by proving the inclusion from right to left. To this end, we reason by contraposition. Consider $x \in X$. If $x \notin f^{-1}(U \to^{\mathbb{Y}^+} V)$, then $f(x) \notin U \to^{\mathbb{Y}^+} V = Y \smallsetminus {\downarrow}(U \smallsetminus V)$. Then there exists $y \in Y$ such that $f(x) \leqslant^{\mathbb{Y}} y$ and $y \in U \smallsetminus V$. Since $f \colon \mathbb{X} \to \mathbb{Y}$ is a p-morphism, there exists $z \in X$ such that $x \leqslant^{\mathbb{X}} z$ such that $f(z) = y \in U \smallsetminus V$. Then $z \in f^{-1}(U) \smallsetminus f^{-1}(V)$. Hence, we conclude that $x \in {\downarrow}(f^{-1}(U) \smallsetminus f^{-1}(V))$ and, therefore, $x \notin X \smallsetminus {\downarrow}(f^{-1}(U) \smallsetminus f^{-1}(V)) = f^{-1}(U) \to^{\mathbb{X}^+} f^{-1}(V)$.

To prove the inclusion from left to right, we also reason by contraposition. Consider $x \in X$ such that $x \notin f^{-1}(U) \to^{\mathbb{X}^+} f^{-1}(V) = X \smallsetminus {\downarrow}(f^{-1}(U) \smallsetminus f^{-1}(V))$. Then there exists $y \in X$ such that $x \leqslant^{\mathbb{X}} y$ and $y \in f^{-1}(U) \smallsetminus f^{-1}(V)$. As a consequence, $f(y) \in U \smallsetminus V$. Furthermore, as f is an order preserving map and $x \leqslant^{\mathbb{X}} y$, we have $f(x) \leqslant^{\mathbb{Y}} f(y)$. Thus, $f(x) \in {\downarrow}(U \smallsetminus V)$. It follows that $x \notin f^{-1}(Y \smallsetminus {\downarrow}(U \smallsetminus V)) = f^{-1}({}^{-1}(U \to^{\mathbb{Y}^+} V))$. Hence, we conclude that $f^* \colon \mathbb{Y}^+ \to \mathbb{X}^+$ is a homomorphism of Heyting algebras and, therefore, that $(-)^+ \colon \mathsf{Esa} \to \mathsf{HA}^{<\omega}$ is a well-defined contravariant functor.

Lastly, recall from Theorem 2.19 that $\varepsilon_{\boldsymbol{A}} \colon \boldsymbol{A} \to (\boldsymbol{A}_*)^+$ is a well-defined isomorphism, for every finite Heyting algebra $\boldsymbol{A}$. Furthermore, the family $\varepsilon := \{\varepsilon_{\boldsymbol{A}} : \boldsymbol{A} \in \mathsf{HA}^{<\omega}\}$ satisfies commutativity condition typical of natural transformations, because the analogous family did so in Finite Priestley's Duality. Hence, we conclude that ε is a natural isomorphism from the identity functor $id_{\mathsf{HA}^{<\omega}}$ to the composition $(-)^+ \circ (-)_* \colon \mathsf{HA} \to \mathsf{HA}$. Similarly, consider the family of maps $\eta := \{\eta_{\mathbb{X}} : \mathbb{X} \text{ is a finite poset}\}$, where the various $\eta_{\mathbb{X}} \colon \mathbb{X} \to (\mathbb{X}^+)_*$ are the order isomorphisms defined in Priestley's Finite Duality. From that duality we know that they satisfy the commutativity condition typical of natural transformations and, therefore, that η is a natural isomorphism from the identity functor id_{Esa} to the composition $(-)_* \circ (-)^+ \colon \mathsf{Esa} \to \mathsf{Esa}$. Hence, we conclude that the categories $\mathsf{HA}^{<\omega}$ and Esa are dually equivalent. ⊠

2.4 Modal logic

To this end, let ρ_M be the type comprising two unary symbols $\neg, \Box$, two binary symbols $\wedge, \vee$ and two constants 0 and 1. We will alsowrite $x \to y$ as an abbreviation for $\neg x \vee y$.

Definition 2.25. Let $\mathbf{K}$ be the least subset of $T_{\rho_M}(Var)$ such that:

(i) *Σ contains the tautologies of* **CPC**;

(ii) Σ *contains* $\Box(x \to y) \to (\Box x \to \Box y)$;

(iii) Σ is *closed under substitutions*: $\sigma(\varphi) \in \Sigma$, for every substitution σ and $\varphi \in \Sigma$;

(iv) Σ is *closed under modus ponens*: $\psi \in \Sigma$, for every $\varphi, \varphi \to \psi \in \Sigma$;

(v) Σ is *closed under necessitation*: $\Box\varphi \in \Sigma$, for every $\varphi \in \Sigma$.

K is sometimes called the *least normal modal logic*

Notice that, strictly speaking, **K** is not a logic, because it is a set of formulas, as opposed to a consequence relation. However, it is possible to associate two logics with **K**, as we proceed to explain.

Definition 2.26. Let $\mathbf{K}_\ell$ and $\mathbf{K}_g$ be the logics of type ρ_M defined as follows:

(i) The *local consequence* $\mathbf{K}_\ell$ of **K** is the logic axiomatized by the Hilbert calculus

$$\emptyset \rhd \varphi, \text{ for all } \varphi \in \mathbf{K} \qquad x, x \to y \rhd y.$$

(ii) The *global consequence* $\mathbf{K}_g$ of **K** is the logic axiomatized by the Hilbert calculus

$$\emptyset \rhd \varphi, \text{ for all } \varphi \in \mathbf{K} \qquad x, x \to y \rhd y \qquad x \rhd \Box x.$$

The rule $x \rhd \Box x$ is sometimes called the *necessitation rule*.

Notice that the natural adaptation of Remark 2.12 to $\mathbf{K}_\ell$ and $\mathbf{K}_g$ is also true.

Definition 2.27. We denote by MA the variety of modal algebras.

Our first goal is to prove the completeness of $\mathbf{K}_g$ and $\mathbf{K}_\ell$ with respect to MA.

Theorem 2.28. *Let* $\boldsymbol{\tau} = \{x \approx 1\}$. *For every* $\Gamma \cup \{\varphi\} \subseteq T(Var)$, *we have*

$$\Gamma \vdash_{\mathbf{K}_g} \varphi \iff \boldsymbol{\tau}[\Gamma] \vDash_{\mathsf{MA}} \boldsymbol{\tau}(\varphi).$$

Proof. Soundness is proved essentially in the same way as in the proof of Theorem 2.14. To prove the converse, we reason by contraposition. Accordingly, suppose that $\Gamma \nvdash_{\mathbf{K}_g} \varphi$. Using the Hilbert calculus K_g of $\mathbf{K}_g$ it is easy (but tedious) to prove that the relation

$$\theta := \{\langle\psi, \gamma\rangle \in T(Var) \times T(Var) : \Gamma \vdash_{\mathbf{K}_g} (\psi \to \gamma) \wedge (\gamma \to \psi)\}$$

is a congruence on the algebra of formulas $\boldsymbol{T}(Var)$. Let us detail, as an exemplification, why θ preserves $\Box$, as this will clarify the role of the necessitation rule. Consider a pair $\langle\psi, \gamma\rangle \in \theta$. Then $\Gamma \vdash_{\mathbf{K}_g} (\psi \to \gamma) \wedge (\gamma \to \psi)$. Using K_g we obtain

$$\Gamma \vdash_{\mathbf{K}_g} \psi \to \gamma \text{ and } \Gamma \vdash_{\mathbf{K}_g} \gamma \to \psi.$$

Using the necessitation rule, we obtain

$$\Gamma \vdash_{\mathbf{K}_g} \Box(\psi \to \gamma) \text{ and } \Gamma \vdash_{\mathbf{K}_g} \Box(\gamma \to \psi). \tag{2.4}$$

Moreover, since **K** contains $\Box(x \to y) \to (\Box x \to \Box y)$ and is closed under substitutions,

$$\varGamma \vdash_{\mathbf{K}_g} \Box(\psi \to \gamma) \to (\Box\psi \to \Box\gamma) \text{ and } \varGamma \vdash_{\mathbf{K}_g} \Box(\gamma \to \psi) \to (\Box\gamma \to \Box\psi).$$

Together with (2.4) and modus ponens, we obtain

$$\varGamma \vdash_{\mathbf{K}_g} \Box\psi \to \Box\gamma \quad \text{and} \quad \varGamma \vdash_{\mathbf{K}_g} \Box\gamma \to \Box\psi.$$

Hence, we conclude that $\langle \Box\psi, \Box\gamma \rangle \in \theta$.

The quotient $\boldsymbol{T}(Var)/\theta$ is sometimes called the *Lindenbaum-Tarski algebra* of $\varGamma$. Again, using the Hilbert calculus K_g it is easy to prove that $\boldsymbol{T}(Var)/\theta$ is a modal algebra.

Lastly, consider the canonical homomorphism $\pi \colon \boldsymbol{T}(Var) \to \boldsymbol{T}(Var)/\theta$, defined by the rule $\pi(\psi) := \psi/\theta$. As in the proof of Theorem 2.14, we obtain that $\pi(\gamma) = 1$, for every $\gamma \in \varGamma$, but $\pi(\varphi) \neq 1$. Hence, we conclude that $\boldsymbol{\tau}[\varGamma] \nvDash_{\mathsf{MA}} \boldsymbol{\tau}(\varphi)$. ⊠

Corollary 2.29. *The logics $\mathbf{K}_g$ and $\vdash_{\mathsf{MA},\boldsymbol{\tau}}$, where $\boldsymbol{\tau} = \{x \approx 1\}$, coincide.*

In order to establish the algebraic completeness theorem for $\mathbf{K}_\ell$, we rely on the following observations.

Proposition 2.30. *For every $\varGamma \cup \{\varphi\} \subseteq T(Var)$,*

$$\textit{if } \varGamma \vdash^{\leqslant}_{\mathsf{MA}} \varphi, \textit{ there exists some finite } \varDelta \subseteq \varGamma \textit{ such that } \varDelta \vdash^{\leqslant}_{\mathsf{MA}} \varphi.$$

Proof. Suppose that $\varGamma \vdash^{\leqslant}_{\mathsf{MA}} \varphi$. Then consider the substitution σ defined on variables as

$$\sigma(x_n) := x_{n+1}, \text{ for every } n \in \mathbb{N}.$$

As $\vdash^{\leqslant}_{\mathsf{MA}}$ is substitution invariant, $\sigma[\varGamma] \vdash^{\leqslant}_{\mathsf{MA}} \sigma(\varphi)$. This means that

$$\{x_0 \leqslant \sigma(\gamma) : \gamma \in \varGamma\} \vDash_{\mathsf{MA}} x_0 \leqslant \sigma(\varphi),$$

where $x \leqslant y$ is a shorthand for the equation $x \wedge y \approx x$. Since the class of modal algebra is a variety, it is closed under ultraproducts. By Theorem 1.55, this guarantees the existence of some finite $\varDelta \subseteq \varGamma$ such that

$$\{x_0 \leqslant \sigma(\gamma) : \gamma \in \varDelta\} \vDash_{\mathsf{MA}} x_0 \leqslant \sigma(\varphi).$$

In turn, this implies $\sigma[\varDelta] \vdash^{\leqslant}_{\mathsf{MA}} \sigma(\varphi)$, by the definition of $\vdash^{\leqslant}_{\mathsf{MA}}$.

Lastly, consider the substitution π, defined on variables as

$$\sigma(x_0) = x_0 \text{ and } \sigma(x_n) := x_{n-1}, \text{ for every positive integer } n.$$

Since $\vdash^{\leqslant}_{\mathsf{MA}}$ is substitution invariant, we have $\pi\sigma[\varDelta] \vdash^{\leqslant}_{\mathsf{MA}} \pi\sigma(\varphi)$. As $\varDelta = \pi\sigma[\varDelta]$ and $\varphi = \pi\sigma(\varphi)$, we conclude that $\varDelta \vdash^{\leqslant}_{\mathsf{MA}} \varphi$. ⊠

Proposition 2.31. *For every $n \in \mathbb{N}$,*

$$\varnothing \vdash_{\mathbf{K}_\ell} \Box^{n+1}(x \to y) \to \Box^n(\Box x \to \Box y).$$

Proof. It suffices to show that, for every $n \in \mathbb{N}$,

$$\Box^{n+1}(x \to y) \to \Box^n(\Box x \to \Box y) \in \mathbf{K}.$$

We reason by induction on n. The base case, where $n = 0$, follows immediately from the definition of $\mathbf{K}$. Then we consider the case where $n = m + 1$. By the inductive hypothesis,

$$\Box^{m+1}(x \to y) \to \Box^m(\Box x \to \Box y) \in \mathbf{K}.$$

Consequently, as $\mathbf{K}$ is closed under necessitation,

$$\Box(\Box^{m+1}(x \to y) \to \Box^m(\Box x \to \Box y)) \in \mathbf{K}.$$

Now, observe that, since $\Box(x \to y) \to (\Box x \to \Box y) \in \mathbf{K}$ and $\mathbf{K}$ is closed under substitutions,

$$\Box(\Box^{m+1}(x \to y) \to \Box^m(\Box x \to \Box y)) \to (\Box^{m+2}(x \to y) \to \Box^{m+1}(\Box x \to \Box x)) \in \mathbf{K}.$$

Since $\mathbf{K}$ is closed under modus ponens, from the last two displays it follows

$$\Box^{m+2}(x \to y) \to \Box^{m+1}(\Box x \to \Box x) \in \mathbf{K}.$$

This concludes the inductive proof. ⊠

The algebraic completeness theorem for $\mathbf{K}_\ell$ takes the following form.

Theorem 2.32. *$\mathbf{K}_\ell$ is the logic preserving degrees of truth of* MA.

Proof. Consider $\Gamma \cup \{\varphi\} \subseteq T(Var)$. We have to prove that

$$\Gamma \vdash_{\mathbf{K}_\ell} \varepsilon \iff \Gamma \vdash^{\leqslant}_{\mathsf{MA}} \varepsilon. \tag{2.5}$$

The implication from left to right follows from a standard soundness induction on the length of the proofs of the Hilbert calculus axiomatizing $\mathbf{K}_\ell$. To prove the other implication, suppose that $\Gamma \nvdash_{\mathbf{K}_\ell} \varepsilon$. Then consider the binary relation

$$\theta := \{\langle \psi, \gamma \rangle \in T(Var)^2 : \Gamma \vdash_{\mathbf{K}_\ell} \Box^n(\psi \to \gamma) \text{ and } \Gamma \vdash_{\mathbf{K}_\ell} \Box^n(\gamma \to \psi), \text{ for all } n \in \mathbb{N}\}.$$

Using the Hilbert calculus defining $\mathbf{K}_\ell$, it is possible to prove that θ is a congruence of $\boldsymbol{T}(Var)$. As an exemplification, we detail a proof that θ preserves $\Box$. To this end, consider $\langle \psi, \gamma \rangle \in \theta$ and consider $n \in \mathbb{N}$. Clearly, we have

$$\Gamma \vdash_{\mathbf{K}_\ell} \Box^{n+1}(\psi \to \gamma) \text{ and } \Gamma \vdash_{\mathbf{K}_\ell} \Box^{n+1}(\gamma \to \psi), \text{ for all } n \in \mathbb{N}.$$

Together with Proposition 2.31, this implies

$$\Gamma \vdash_{\mathbf{K}_\ell} \Box^n(\Box\psi \to \Box\gamma) \text{ and } \Gamma \vdash_{\mathbf{K}_\ell} \Box^n(\Box\gamma \to \Box\psi), \text{ for all } n \in \mathbb{N}.$$

Hence, we conclude that $\langle \Box\psi, \Box\gamma \rangle \in \theta$, as desired. Similarly, using the Hilbert calculus defining $\mathbf{K}_\ell$, it is possible to prove that $\boldsymbol{T}(Var)/\theta$ is a modal algebra.

Then consider the natural homomorphism $\pi \colon \boldsymbol{T}(Var) \to \boldsymbol{T}(Var)/\theta$. We will use to prove it that $\Delta \nvdash^{\leqslant}_{\mathsf{MA}} \varphi$, for every finite $\Delta \subseteq \Gamma$. Accordingly, consider a finite $\Delta \subseteq \Gamma$. We can assume that Δ is nonempty, otherwise we extend it with the formula 1 which is provable from the emptyset in $\mathbf{K}_\ell$. Then consider an enumeration $\Delta = \{\gamma_1, \ldots, \gamma_n\}$. In order to prove that $\Delta \nvdash^{\leqslant}_{\mathsf{MA}} \varphi$, it suffices to show that

$$\pi(\gamma_1) \wedge \cdots \wedge \pi(\gamma_n) \nleqslant \pi(\varphi).$$

Suppose, with a view to contradiction, that $\pi(\gamma_1) \wedge \cdots \wedge \pi(\gamma_n) \leqslant \pi(\varphi)$. Since π is a homomorphism, this means that $\pi((\gamma_1 \wedge \cdots \wedge \gamma_n) \vee \varphi) = \pi(\varphi)$. As a consequence,

$$\Gamma \vdash_{\mathbf{K}_\ell} ((\gamma_1 \wedge \cdots \wedge \gamma_n) \vee \varphi) \to \varphi.$$

Recall that $\Gamma \vdash_{\mathbf{K}_\ell} \gamma_1 \wedge \cdots \wedge \gamma_n$, because $\Delta \subseteq \Gamma \cup \{1\}$ and $\Gamma \vdash_{\mathbf{K}_\ell} 1$. Therefore, we obtain $\Gamma \vdash_{\mathbf{K}_\ell} (\gamma_1 \wedge \cdots \wedge \gamma_n) \vee \varphi$. Together with modus ponens and the above display, this implies $\Gamma \vdash_{\mathbf{K}_\ell} \varphi$, a contradiction. Hence, we conclude that $\Delta \nvdash^{\leqslant}_{\mathsf{MA}} \varphi$, for every finite $\Delta \subseteq \Gamma$. By Proposition , this implies that $\Gamma \nvdash^{\leqslant}_{\mathsf{MA}} \varphi$, as desired. ⊠

Corollary 2.33. *The logics $\mathbf{K}_\ell$ and $\mathbf{K}_g$ are different.*

Proof. Consider the four-element Boolean algebra $\boldsymbol{A}$ with universe $\{a, c, 0, 1\}$. We endow it with a unary operation $\Box$ defined as follows:

$$\Box(b) := \begin{cases} 1 & \text{if } b = 1 \\ 0 & \text{otherwise.} \end{cases}$$

The resulting structure $\boldsymbol{B}$ is a modal algebra. Consider any homomorphism $f \colon \boldsymbol{T}(Var) \to \boldsymbol{B}$ such that $f(x) := a$. We have

$$f(\Box x) = \Box^{\boldsymbol{B}} f(x) = \Box^{\boldsymbol{B}} a = 0 < a = f(x).$$

Since, by Theorem 2.32, $\mathbf{K}_\ell$ is the logic preserving degrees of truth of HA, this implies $x \nvdash_{\mathbf{K}_\ell} \Box x$. Furthermore, since the Hilbert calculus that axiomatizes $\mathbf{K}_g$ contains the necessitation rule, we obtain $x \vdash_{\mathbf{K}_g} \Box x$. Hence, we conclude that the logics $\mathbf{K}_\ell$ and $\mathbf{K}_g$ are indeed different. ⊠

2.5 Kripke semantics and duality

Both the local and global consequences of $\mathbf{K}$ admit a Kripke semantics, which is tightly related to the following adaptation of Stone's representation of Boolean algebras. A *Kripke frame* is a graph, that is, a pair $\mathbb{X} = \langle X; R \rangle$, where X is a set and R a binary relation on it. Given a Kripke frame $\mathbb{X}$, consider the algebra

$$\mathbb{X}^+ := \langle \mathcal{P}(X); \cap, \cup, -, \Box, \emptyset, X \rangle,$$

where $\Box$ is the binary operation defined, for every $U \in \mathcal{P}(X)$, as follows:

$$\Box U := \{x \in X : \text{for every } y \in X, \text{ if } \langle x, y \rangle \in R, \text{ then } y \in U\}.$$

Notice that $\mathbb{X}^+$ is a modal algebra. This is because $\langle \mathcal{P}(X); \cap, \cup, -, \emptyset, X \rangle$ is a powerset Boolean algebra and, for every $U, V \in \mathcal{P}(X)$,

$$\Box 1 = \Box X = \{x \in X : \text{for every } y \in X, \text{ if } \langle x, y \rangle \in R, \text{ then } y \in X\} = X = 1.$$

and

$$\Box U \cap \Box V = \{x \in X : \text{for every } y \in X, \text{ if } \langle x, y\rangle \in R, \text{ then } y \in U \cap V\} = \Box(U \cap V).$$

Furthermore, every modal algebra embeds into one of this form. To prove it, given a modal algebra $\boldsymbol{A}$, we set

$$\boldsymbol{A}_+ \coloneqq \langle \boldsymbol{A}_*; R_{\boldsymbol{A}}\rangle$$

where $\boldsymbol{A}_*$ is the set of ultrafilters of $\boldsymbol{A}$ and $R_{\boldsymbol{A}}$ is the binary relation on $\boldsymbol{A}_*$ defined by the rule

$$\langle F, G\rangle \in R_{\boldsymbol{A}} \Longleftrightarrow \Box^{-1}(F) \subseteq G.$$

Theorem 2.34 (Jónsson & Tarski)**.** *If $\boldsymbol{A}$ is a modal algebra, then the map $\varepsilon_{\boldsymbol{A}} \colon \boldsymbol{A} \to (\boldsymbol{A}_*)^+$, defined by the rule*

$$\varepsilon_{\boldsymbol{A}}(a) \coloneqq \{F \in \boldsymbol{A}_* : a \in F\},$$

is an embedding.

Proof. Stone's representation theorem for Boolean algebras guarantees that $\varepsilon_{\boldsymbol{A}}$ is a well-defined Boolean algebra embedding. Therefore, it only remains to prove that it preserves $\Box$. Consider $a \in A$ and $F \in \boldsymbol{A}_*$. We need to prove that

$$\varepsilon_{\boldsymbol{A}}(\Box^{\boldsymbol{A}} a) = \Box^{(\boldsymbol{A}_+)^+} \varepsilon_{\boldsymbol{A}}(a). \tag{2.6}$$

Accordingly, consider $F \in \boldsymbol{A}_*$. Suppose first that $F \in \varepsilon_{\boldsymbol{A}}(\Box^{\boldsymbol{A}} a)$. Then consider $G \in \boldsymbol{A}_*$ such that $\langle F, G\rangle \in R_{\boldsymbol{A}}$, that is, $\Box^{-1}(F) \subseteq G$. Since $F \in \varepsilon_{\boldsymbol{A}}(\Box^{\boldsymbol{A}} a)$, we have $\Box^{\boldsymbol{A}} a \in F$ and, therefore, $a \in \Box^{-1}(F) \subseteq G$. As a consequence, $G \in \varepsilon_{\boldsymbol{A}}(a)$. Hence, we conclude that $F \in \varepsilon_{\boldsymbol{A}}(a)$.

To prove the other inclusion, we reason by contraposition. Consider $F \in \boldsymbol{A}_* \smallsetminus \varepsilon_{\boldsymbol{A}}(\Box^{\boldsymbol{A}} a)$. Then $\Box^{\boldsymbol{A}} a \notin F$. Then consider the set $G \coloneqq \Box^{-1}(F)$. We will prove that G is a filter of $\boldsymbol{A}$. First, observe that G is nonempty, because $\Box 1 = 1 \in F$ and, therefore, $1 \in G$. Then consider $b, c \in A$. Suppose that $b \in G$ and $b \leqslant c$. Since $b \leqslant c$ and $\Box$ commutes with binary meets, we have

$$\Box b \wedge \Box c = \Box(b \wedge c) = \Box b,$$

that is, $\Box b \leqslant \Box c$. From $b \in G$ it follows $\Box b \in F$. Since F is an upset and $\Box b \leqslant \Box c$, we obtain $\Box c \in F$ and, therefore, $c \in G$. Hence, G is an upset. To prove that G is closed under binary meets, suppose that $b, c \in G$. Then $\Box b, \Box c \in F$. Since F is a filter $\Box(b \wedge c) = \Box b \wedge \Box c \in F$, whence $b \wedge c \in G$. Hence, we conclude that G is a filter of $\boldsymbol{A}$.

Now, consider the ideal ${\downarrow} a$. Observe that $G \cap {\downarrow} a = \emptyset$, otherwise $a \in G \subseteq \Box^{-1}(F)$, against the assumption that $\Box a \in F$. By the Ultrafilter Lemma, there exists a ultrafilter U of $\boldsymbol{A}$ that extends G and does not contain a. In particular, $\Box^{-1}(F) = G \subseteq U$ and $U \notin \varepsilon_{\boldsymbol{A}}(a)$. Notice that $\Box^{-1}(F) \subseteq U$ means that $\langle F, U\rangle \in R_{\boldsymbol{A}}$. Hence, we conclude that $F \notin \Box^{(\boldsymbol{A}_+)^+} \varepsilon_{\boldsymbol{A}}(a)$. ⊠

Corollary 2.35. *Finite modal algebras are (up to isomorphism) algebras of the form $\mathbb{X}^+$ for some finite Kripke frame $\mathbb{X}$.*

Proof. If $\mathbb{X}$ is a finite Kripke, then $\mathbb{X}^+$ is clearly a finite modal algebras. Conversely, suppose that $\boldsymbol{A}$ is a finite modal algebra. By Theorem 2.34, the map $\varepsilon_{\boldsymbol{A}} \colon \boldsymbol{A} \to (\boldsymbol{A}_+)^+$ is an embedding. Furthermore, $\varepsilon_{\boldsymbol{A}}$ is surjective, by Finite Stone's Duality. Thus, $\boldsymbol{A} \cong (\boldsymbol{A}_+)^+$, Since $\boldsymbol{A}$ is finite, $\boldsymbol{A}_+$ is a finite Kripke frame and we are done. ⊠

A *valuation* in a Kripke frame $\mathbb{X}$ is a function $v\colon Var \to \mathcal{P}(X)$. Given a valuation v in a Kripke frame $\mathbb{X}$, we define a notion of *validity* of a formula $\varphi \in T(Var)$ at a point $w \in X$ under v, in symbols $w, v \Vdash \varphi$, by recursion on the construction of φ as follows. For variables $x \in Var$ we set

$$w, v \Vdash x \iff w \in v(x), \text{ for } x \in Var$$

and for constant symbols

$$w, v \Vdash 1 \text{ and } w, v \Vdash 0.$$

For complex formulas we set

$$\begin{aligned}
w, v \Vdash \alpha \wedge \beta &\iff w, v \Vdash \alpha \text{ and } w, v \Vdash \beta \\
w, v \Vdash \alpha \vee \beta &\iff w, v \Vdash \alpha \text{ or } w, v \Vdash \beta \\
w, v \Vdash \neg\alpha &\iff w, v \nVdash \alpha \\
w, v \Vdash \Box\alpha &\iff u, v \Vdash \alpha, \text{ for every } u \in X \text{ such that } \langle w, u\rangle \in R.
\end{aligned}$$

Given a set of formulas Γ, we write $w, v \Vdash \Gamma$ to indicate that $w, v \Vdash \gamma$, for every $\gamma \in \Gamma$.

We define two logics $\vdash^{\ell}_{\mathsf{Frm}}$ and $\vdash^{g}_{\mathsf{Frm}}$ associated with the class of all Kripke frames as follows: for every $\Gamma \cup \{\varphi\} \subseteq T(Var)$,

$$\begin{aligned}
\Gamma \vdash^{\ell}_{\mathsf{Frm}} \varphi &\iff \text{for every Kripke frame } \mathbb{X}, \text{ every valuation } v \text{ in } \mathbb{X}, \text{ and every } w \in X, \\
&\qquad \text{if } w, v \Vdash \Gamma, \text{ then } w, v \Vdash \varphi; \text{ and} \\
\Gamma \vdash^{g}_{\mathsf{Frm}} \varphi &\iff \text{for every Kripke frame } \mathbb{X} \text{ and every valuation } v \text{ in } \mathbb{X}, \\
&\qquad \text{if } w, v \Vdash \Gamma \text{ for every } w \in X, \text{ then } w, v \Vdash \varphi \text{ for every } w \in X.
\end{aligned}$$

The logics $\vdash^{\ell}_{\mathsf{Frm}}$ and $\vdash^{g}_{\mathsf{Frm}}$ are known, respectively, as the *local* and the *global consequence relations* associated with the class of Kripke frames.

Theorem 2.36. *For every set of formulas $\Gamma \cup \{\varphi\} \subseteq T(Var)$,*

$$\Gamma \vdash_{\mathbf{K}_\ell} \varphi \iff \Gamma \vdash^{\ell}_{\mathsf{Frm}} \varphi \quad \text{and} \quad \Gamma \vdash_{\mathbf{K}_g} \varphi \iff \Gamma \vdash^{g}_{\mathsf{Frm}} \varphi.$$

Proof. Consider $\Gamma \cup \{\varphi\} \subseteq T(Var)$. We begin by proving that

$$\Gamma \vdash_{\mathbf{K}_\ell} \varphi \iff \Gamma \vdash^{\ell}_{\mathsf{Frm}} \varphi. \tag{2.7}$$

The implication from left to right follows from a standard soundness argument. To prove the other implication, we reason by contraposition. Suppose that $\Gamma \nvdash_{\mathbf{K}_\ell} \varphi$. In view of Theorem 2.32, there exist a modal algebra $\boldsymbol{A}$, a homomorphism $f\colon \boldsymbol{T}(Var) \to \boldsymbol{A}$, and an element $a \in A$ such that

$$a \leqslant f(\gamma), \text{ for every } \gamma \in \Gamma, \text{ and } a \nleqslant f(\varphi).$$

Then the filter $\uparrow a$ and $\downarrow f(\varphi)$ are, respectively, a filter and an ideal of $\boldsymbol{A}$ that, moreover, are disjoint. By the Ultrafilter Lemma, there exists a ultrafilter F such that $a \in F$ and $f(\varphi) \notin F$. Consider the homomorphism $\varepsilon_{\boldsymbol{A}}\colon \boldsymbol{A} \to (\boldsymbol{A}_+)^+$. We have

$$F \in \varepsilon_{\boldsymbol{A}}(f(\gamma)), \text{ for all } \gamma \in \Gamma, \text{ and } F \notin \varepsilon_{\boldsymbol{A}}(f(\varphi)). \tag{2.8}$$

Now, consider the valuation $v\colon Var \to \mathcal{P}(\boldsymbol{A}_*)$ is the Kripke frame $\boldsymbol{A}_+$. We claim that, for every formula $\psi \in T(Var)$,

$$\varepsilon_{\boldsymbol{A}}(f(\psi)) = \{G \in \boldsymbol{A}_* : G, v \Vdash \psi\}.$$

We reason by induction on the construction of ψ. The base case is handled as in the case of **IPC** and the same applies to the inductive cases for the connectives $\wedge$ and $\vee$. Therefore, it only remains to consider the following cases:

(i) $\psi = \neg\alpha$;

(ii) $\psi = \Box\alpha$.

(i): We have

$$\begin{aligned}\varepsilon_{\boldsymbol{A}}(f(\neg\alpha)) &= \neg^{(\boldsymbol{A}_+)^+}\varepsilon_{\boldsymbol{A}}(f(\alpha))\\ &= \boldsymbol{A}_* \smallsetminus \varepsilon_{\boldsymbol{A}}(f(\alpha))\\ &= \{G \in \boldsymbol{A}_* : G, v \nVdash \alpha\}\\ &= \{G \in \boldsymbol{A}_* : G, v \Vdash \neg\alpha\}.\end{aligned}$$

The above equalities can be justified as follows. The first follows holds because $\varepsilon_{\boldsymbol{A}} \circ f\colon \boldsymbol{T}(Var) \to (\boldsymbol{A}_+)^+$ is a homomorphism, the second follows from the definition of $\neg$ in $(\boldsymbol{A}_+)^+$, the third from the inductive hypothesis, and the last one from the definition of $\Vdash$.

(ii): In this case, we have

$$\begin{aligned}\varepsilon_{\boldsymbol{A}}(f(\Box\alpha)) &= \Box^{(\boldsymbol{A}_+)^+}\varepsilon_{\boldsymbol{A}}(f(\alpha))\\ &= \{G \in \boldsymbol{A}_* : \text{for all } H \in \boldsymbol{A}_*, \text{ if } \langle G, H\rangle \in R_{\boldsymbol{A}}, \text{ then } H \in \varepsilon_{\boldsymbol{A}}(f(\alpha))\}\\ &= \{G \in \boldsymbol{A}_* : \text{for all } H \in \boldsymbol{A}_*, \text{ if } \langle G, H\rangle \in R_{\boldsymbol{A}}, \text{ then } H, v \Vdash \alpha\}\\ &= \{G \in \boldsymbol{A}_* : G, v \Vdash \Box\alpha\}.\end{aligned}$$

The above equalities can be justified as follows. The first follows holds because $\varepsilon_{\boldsymbol{A}} \circ f\colon \boldsymbol{T}(Var) \to (\boldsymbol{A}_+)^+$ is a homomorphism, the second follows from the definition of $\neg$ in $(\boldsymbol{A}_+)^+$, the third from the inductive hypothesis, and the last one from the definition of $\Vdash$. This concludes the proof of the claim.

From the claim and (2.8) it follows that

$$F, v \Vdash \Gamma \text{ and } F, v \nVdash \varphi.$$

Hence, we conclude that $\Gamma \nvdash^{\ell}_{\mathsf{Frm}} \varphi$, as desired. This establishes (2.7).

The proof of the equivalence

$$\Gamma \vdash_{\mathbf{K}_g} \varphi \Longleftrightarrow \Gamma \vdash^{g}_{\mathsf{Frm}} \varphi \tag{2.9}$$

is analogous. Accordingly, we shall sketch only the proof of the implication from right to left. As usual, we reason by contraposition. Suppose that $\Gamma \nvdash_{\mathbf{K}_g} \varphi$. By Theorem 2.28 there exist a modal algebra $\boldsymbol{A}$ and a homomorphism $f\colon \boldsymbol{T}(Var) \to \boldsymbol{A}$ such that

$$f(\gamma) = 1, \text{ for every } \gamma \in \Gamma, \text{ and } f(\varphi) \neq 1.$$

By the Ultrafilter Lemma, this implies that

$$\varepsilon_{\boldsymbol{A}}(f(\gamma)) = \boldsymbol{A}_*, \text{ for every } \gamma \in \Gamma, \text{ and } \varepsilon_{\boldsymbol{A}}(f(\varphi)) \neq \boldsymbol{A}_*.$$

Then we consider the valuation v in $\boldsymbol{A}_+$ defined in the previous case. From the claim and the above display it follows that

$$F, v \Vdash \Gamma, \text{ for every } \gamma \in \Gamma, \text{ and there exists } F \in \boldsymbol{A}_* \text{ such that } F, v \nVdash \varphi.$$

Hence, we conclude that $\Gamma \nvdash^{g}_{\mathsf{Frm}} \varphi$. ⊠

The relation between finite Kripke frames and finite modal algebras hinted in Corollary 2.35 can be enhanced to a dual equivalence.

*Exercise** 2.37. Let $\mathbb{X} = \langle X; R_X \rangle$ and $\mathbb{Y} = \langle Y; R_Y \rangle$ be Kripke frames. A *bounded morphism* from $\mathbb{X}$ to $\mathbb{Y}$ is a function $f\colon X \to Y$ the satisfies the following conditions:

(i) for every $x, y \in X$, if $\langle x, y \rangle \in R_X$, then $\langle f(x), f(y) \rangle \in R_Y$;

(ii) for every $x \in X$ and $y \in Y$, if $\langle f(x), y \rangle \in R_Y$, there exists $z \in X$ such that $\langle x, z \rangle \in R_X$ and $f(z) = y$.

Prove that if $f\colon \mathbb{X} \to \mathbb{Y}$ is a bounded morphism between Kripke frames, the inverse image map $f^{-1}\colon \mathbb{Y}^+ \to \mathbb{X}^+$ is a homomorphism of modal algebras. Similarly, show that if $f\colon \boldsymbol{A} \to \boldsymbol{B}$ is a homomorphism of modal algebras, then the inverse image map $f^{-1}\colon \boldsymbol{B}_+ \to \boldsymbol{A}_+$ is a bounded morphism between Kripke frames.

Lastly, discuss (very briefly) why this observation can be used to prove that the category of finite modal algebras with homomorphisms as arrows is dually equivalent to that of finite Kripke frames with bounded morphisms as arrows. ⊠

CHAPTER 3

The algebra of logic

3.1 Algebraic semantics

Definition 3.1. A class of algebras K is said to be an *algebraic semantics* for a logic $\vdash$ if there exists a set of equations $\boldsymbol{\tau}(x)$ such that $\vdash$ coincides with $\vdash_{\mathsf{K},\boldsymbol{\tau}}$. In this case, we say that K is a $\boldsymbol{\tau}$*-algebraic semantics* for $\vdash$.

In other words, K has is an algebraic semantics for $\vdash$ when it is related to $\vdash$ by an equational completeness theorem. For instance, the classes of Boolean, Heyting, and modal algebras are algebraic semantics for **CPC**, **IPC**, and $\mathbf{K}_g$.

While equational completeness theorems are ubiquitous in logic, as we shall see, they are not sufficient to explain what connects a given logic with a *unique* distinguished class of algebras. For instance, in view of Glivenko's Theorem, the class of Heyting algebras is also an algebraic semantics for **CPC**, as we proceed to explain. In the next proofs we will use repeatedly the fact that every Heyting algebra $\boldsymbol{A}$ satisfies the equations

$$\neg x \approx \neg\neg\neg x \;\text{ and }\; x \wedge \neg x \approx 0.$$

Lemma 3.2. *The following equations hold in every Heyting algebra:*

$$\neg\neg(x \wedge y) \approx \neg\neg x \wedge \neg\neg y \quad \neg\neg(x \vee y) \approx \neg(\neg x \wedge \neg y) \quad \neg\neg(x \to y) \approx \neg\neg x \to \neg\neg y.$$

Proof. Consider a Heyting algebra $\boldsymbol{A}$ and $a, c \in A$. We begin by proving the first equality in the statement. We have $a \wedge c \leqslant a$. Since $\neg$ reverses the order,

$$\neg a \leqslant \neg(a \wedge c) = \neg\neg\neg(a \wedge c).$$

By the residuation law, we obtain $\neg a \wedge \neg\neg(a \wedge c) \leqslant 0$. With another application of the residuation law, $\neg\neg(a \wedge c) \leqslant \neg\neg a$. Similarly, $\neg\neg(a \wedge c) \leqslant \neg\neg c$, whence $\neg\neg(a \wedge c) \leqslant \neg\neg a \wedge \neg\neg c$. On the other hand, using the residuation law, we obtain

$$\begin{aligned}
\neg\neg a \wedge \neg\neg c \leqslant \neg\neg(a \wedge c) &\iff \neg\neg a \wedge \neg\neg c \wedge \neg(a \wedge c) \leqslant 0\\
&\iff \neg\neg a \wedge \neg(a \wedge c) \leqslant \neg\neg\neg c = \neg c\\
&\iff \neg\neg a \wedge \neg(a \wedge c) \wedge c \leqslant 0\\
&\iff \neg(a \wedge c) \wedge c \leqslant \neg\neg\neg a = \neg a\\
&\iff \neg(a \wedge c) \wedge (a \wedge c) \leqslant 0.
\end{aligned}$$

Since the last inequality above holds, we conclude that $\neg\neg a \wedge \neg\neg c \leqslant \neg\neg(a \wedge c)$, as desired.

To prove the second equality in the statement, observe that

$$\begin{aligned}\neg\neg(a \vee c) \leqslant \neg(\neg a \wedge \neg c) &\iff \neg\neg(a \vee c) \wedge \neg a \wedge \neg c \leqslant 0\\ &\iff \neg a \wedge \neg c \leqslant \neg\neg\neg(a \vee c) = \neg(a \vee c)\\ &\iff (a \vee c) \wedge \neg a \wedge \neg c \leqslant 0\\ &\iff (a \wedge \neg a \wedge \neg c) \vee (c \wedge \neg c \wedge \neg a) \leqslant 0,\end{aligned}$$

where the last equivalence follows from distributivity. Since the last inequality above holds, we conclude that $\neg\neg(a \vee c) \leqslant \neg(\neg a \wedge \neg c)$. To prove the other inequality, observe that $a \leqslant a \vee c$. Since $\neg$ reverses the order, $\neg(a \vee c) \leqslant \neg a$. Similarly, $\neg(a \vee c) \leqslant \neg c$ and, therefore, $\neg(a \vee c) \leqslant \neg a \wedge \neg c$. As $\neg$ reverses the order, we conclude that $\neg(\neg a \wedge \neg c) \leqslant \neg\neg(a \vee c)$.

To prove the third equality in the statement, we rely on the following equality:

$$\neg(a \to c) = \neg\neg a \wedge \neg c. \tag{3.1}$$

First, as$a \wedge \neg a = 0 \leqslant c$, with an application of the residuation law we obtain $\neg a \leqslant a \to c$. Moreover, since $\neg$ is order reversing, $\neg(a \to c) \leqslant \neg\neg a$. Similarly, from $a \wedge c \leqslant c$ and the residuation law, it follows $c \leqslant a \to c$. Since $\neg$ is order reversing, this yields $\neg(a \to c) \leqslant \neg c$. Hence, we conclude that $\neg(a \to c) \leqslant \neg\neg a \wedge \neg c$. To prove the other inequality, observe that, by residuation,

$$\begin{aligned}\neg\neg a \wedge \neg c \leqslant \neg(a \to c) &\iff \neg\neg a \wedge \neg c \wedge (a \to c) \leqslant 0\\ &\iff \neg c \wedge (a \to c) \leqslant \neg\neg\neg a = \neg a\\ &\iff a \wedge (a \to c) \wedge \neg c \leqslant 0.\end{aligned}$$

Notice that the latter equality above holds, because $a \wedge (a \to c) \leqslant c$ and $c \wedge \neg c = 0$. This establishes (3.1).

Lastly, using residuation and (3.1), we obtain

$$\begin{aligned}\neg\neg(a \to c) \leqslant \neg\neg a \to \neg\neg c &\iff \neg(\neg\neg a \wedge \neg c) \leqslant \neg\neg a \to \neg\neg c\\ &\iff \neg(\neg\neg a \wedge \neg c) \wedge (\neg\neg a \wedge \neg c) \leqslant 0.\end{aligned}$$

As the latter equality above holds, we conclude that $\neg\neg(a \to c) \leqslant \neg\neg a \to \neg\neg c$. Similarly, using residuation and (3.1), we obtain

$$\begin{aligned}\neg\neg a \to \neg\neg c \leqslant \neg\neg(a \to c) &\iff (\neg\neg a \to \neg\neg c) \wedge \neg(a \to c) \leqslant 0\\ &\iff (\neg\neg a \to \neg\neg c) \wedge \neg\neg a \wedge \neg c \leqslant 0\\ &\iff (\neg\neg a \to \neg\neg c) \wedge \neg\neg a \leqslant \neg\neg c\\ &\iff \neg\neg a \to \neg\neg c \leqslant \neg\neg a \to \neg\neg c.\end{aligned}$$

As the latter equality above holds, we conclude that $\neg\neg a \to \neg\neg c \leqslant \neg\neg(a \to c)$ and, therefore, that $\neg\neg a \to \neg\neg c = \neg\neg(a \to c)$. ☒

Definition 3.3. An element a of a Heyting algebra $\boldsymbol{A}$ is said to be *regular* if $a = \neg\neg a$. We denote by $\mathsf{R}(\boldsymbol{A})$ the set of regular elements of $\boldsymbol{A}$.

Observe that the above definition is redundant in the sense that an element $a \in A$ is regular when $\neg\neg a \leqslant a$. This is because, by the residuation law,

$$a \leqslant \neg\neg a \iff a \leqslant (a \to 0) \to 0 \iff a \wedge (a \to 0) \leqslant 0 \iff a \to 0 \leqslant a \to 0.$$

Since the latter inequality holds, we conclude that $a \leqslant \neg\neg a$ always holds.

Proposition 3.4. *If $\boldsymbol{A}$ is a Heyting algebra, then the structure*

$$\langle \mathsf{R}(\boldsymbol{A}); \wedge, +, \to, 0, 1\rangle,$$

where $x + y := \neg(\neg x \wedge \neg y)$, is a Boolean algebra. Furthermore, the map

$$\neg\neg \colon \boldsymbol{A} \to \mathsf{R}(\boldsymbol{A})$$

is a well-defined surjective homomorphism.

Proof. Throughout the proof, we denote by $\leqslant$ the order relation of $\boldsymbol{A}$, as opposed to that of $\mathsf{R}(\boldsymbol{A})$. We begin by proving that $\mathsf{R}(\boldsymbol{A})$ is closed under the operations $\wedge, +, \to, 0$, and 1. The fact that it contains 0 and 1 is clear. Then consider $a, c \in A$ regular. From Lemma 3.2 it follows

$$a \wedge c = \neg\neg a \wedge \neg\neg c = \neg\neg(a \wedge c)$$
$$a \vee c = \neg\neg a \vee \neg\neg c = \neg(\neg a \wedge \neg c) = \neg\neg\neg(\neg a \wedge \neg c) = \neg\neg(a \vee c)$$
$$a \to c = \neg\neg a \to \neg\neg c = \neg\neg(a \to c).$$

Hence, $\langle \mathsf{R}(\boldsymbol{A}); \wedge, +, \to, 0, 1\rangle$ is a well-defined algebra.

We turn to prove that it is a Boolean algebra. We begin by proving that $\langle \mathsf{R}(\boldsymbol{A}); \wedge, +, 0, 1\rangle$ is a bounded lattice. Clearly, $\wedge$ is idempotent, commutative, and associative. We shall see that the same holds for $+$. To this end, consider $a, b, c \in \mathsf{R}(\boldsymbol{A})$. The operation $+$ is idempotent, because

$$a + a = \neg(\neg a \wedge \neg a) = \neg\neg a = a.$$

As $+$ is clearly commutative, we only detail the proof that $+$ is associative. We have

$$\begin{aligned}(a + b) + c &= \neg(\neg\neg(\neg a \wedge \neg b) \wedge \neg c)\\ &= \neg(\neg\neg(\neg a \wedge \neg b) \wedge \neg\neg\neg c)\\ &= \neg\neg\neg(\neg a \wedge \neg b \wedge \neg c)\\ &= \neg(\neg a \wedge \neg b \wedge \neg c).\end{aligned}$$

The equalities above are justified as follows: the first follows from the definition of $+$, the second and the fourth from the validity of the equation $\neg x \approx \neg\neg\neg x$ in every Heyting algebra, and the third from Lemma 3.2. A similar proof shows that $a + (b + c) = \neg(\neg a \wedge \neg b \wedge \neg c)$. Hence, we conclude that $(a + b) + c = a + (b + c)$.

In order to prove that $\langle \mathsf{R}(\boldsymbol{A}); \wedge, +\rangle$ is a lattice, it only remains to show that it satisfies the absorption laws. Using Lemma 3.2 and the fact that a and c are regular, we obtain

$$a + (a \wedge c) = \neg(\neg a \wedge \neg(a \wedge c)) = \neg\neg a \vee \neg\neg(a \wedge c) = a \vee (\neg\neg a \wedge \neg\neg c) = a \vee (a \wedge c) = a.$$

To prove the other absorption law, observe that, since $a \wedge \neg a = 0$, we have $a \wedge \neg a \wedge \neg c \leqslant 0$. By the residuation law, $a \leqslant \neg(\neg a \wedge \neg c) = a + c$. Thus, $a \wedge (a + c) = a$. Hence,

$\langle \mathsf{R}(\boldsymbol{A}); \wedge, + \rangle$ is a lattice. Clearly, 0 and 1 are, respectively, its minimum and maximum elements.

Lastly, as $\boldsymbol{A}$ satisfies the residuation law, so does $\langle \mathsf{R}(\boldsymbol{A}); \wedge, +, \to, 0, 1 \rangle$. Hence, $\langle \mathsf{R}(\boldsymbol{A}); \wedge, +, \to, 0, 1 \rangle$ is a Heyting algebra. To prove that it is a Boolean algebra, consider $a \in \mathsf{R}(\boldsymbol{A})$. Using the fact that a is regular, we obtain

$$a + \neg a = \neg(\neg a \wedge \neg\neg a) = \neg(\neg a \wedge a) = \neg 0 = 1.$$

Hence, $\langle \mathsf{R}(\boldsymbol{A}); \wedge, +, \to, 0, 1 \rangle$ is a Boolean algebra.

It only remains to show that $\neg\neg \colon \boldsymbol{A} \to \mathsf{R}(\boldsymbol{A})$ is a well-defined surjective homomorphism. First, this map is well-defined because the equation $\neg x \approx \neg\neg\neg x$ holds in every Heyting algebra. Furthermore, it is surjective, because $\neg\neg a = a$, for every regular element $a \in A$. To prove that it is a homomorphism, consider $a, c \in A$. From Lemma 3.2 it follows that

$$\neg\neg(a \wedge c) = \neg\neg a \wedge \neg\neg c$$

$$\neg\neg(a \vee c) = \neg(\neg a \wedge \neg c) = \neg(\neg\neg\neg a \wedge \neg\neg\neg c) = \neg\neg a + \neg\neg c$$

$$\neg\neg(a \to c) = \neg\neg a \to \neg\neg c.$$

Since $\neg\neg 0 = 0$ and $\neg\neg 1 = 1$, the map $\neg\neg \colon \boldsymbol{A} \to \mathsf{R}(\boldsymbol{A})$ is a homomorphism. ⊠

We are now ready to prove the main theorem of this section, which provides a double negation translation between classical and constructive logic.

Glivenkos' Theorem 3.5. *For every set of formulas $\Gamma \cup \{\varphi\}$,*

$$\Gamma \vdash_{\mathbf{CPC}} \varphi \iff \{\neg\neg\gamma : \gamma \in \Gamma\} \vdash_{\mathbf{IPC}} \neg\neg\varphi.$$

Proof. First, suppose that $\{\neg\neg\gamma : \gamma \in \Gamma\} \vdash_{\mathbf{IPC}} \neg\neg\varphi$. Since **CPC** extends **IPC**, this implies $\{\neg\neg\gamma : \gamma \in \Gamma\} \vdash_{\mathbf{CPC}} \neg\neg\varphi$. Furthermore, as $x \dashv\vdash_{\mathbf{CPC}} \neg\neg x$, this yields $\Gamma \vdash_{\mathbf{CPC}} \varphi$.

To prove the converse, suppose that $\Gamma \vdash_{\mathbf{CPC}} \varphi$. Let $\tau := \{x \approx 1\}$. As **CPC** coincides with $\vdash_{\mathsf{BA},\boldsymbol{\tau}}$, we obtain

$$\boldsymbol{\tau}[\Gamma] \vDash_{\mathsf{HA}} \boldsymbol{\tau}(\varphi). \tag{3.2}$$

Now, consider a Heyting algebra $\boldsymbol{A}$ and a homomorphism $f \colon \boldsymbol{T}(Var) \to \boldsymbol{A}$ such that $f(\neg\neg\gamma) = 1$, for all $\gamma \in \Gamma$. Then consider the homomorphism $\neg\neg \colon \boldsymbol{A} \to \mathsf{R}(\boldsymbol{A})$, given by Proposition 3.4. For every $\gamma \in \Gamma$, we have

$$\neg\neg f(\gamma) = f(\neg\neg\gamma) = 1.$$

Since $\mathsf{R}(\boldsymbol{A})$ is a Boolean algebra and $\neg\neg \circ f \colon \boldsymbol{T}(Var) \to \mathsf{R}(\boldsymbol{A})$ a homomorphism, from (3.2) it follows that $\neg\neg f(\varphi) = 1$. But this means that $f(\neg\neg\varphi) = 1$, because f is a homomorphism. Hence, we conclude that

$$\{\boldsymbol{\tau}(\neg\neg\gamma) : \gamma \in \Gamma\} \vDash_{\mathsf{HA}} \boldsymbol{\tau}(\neg\neg\varphi).$$

Together with Theorem 2.14, this yields $\{\neg\neg\gamma : \gamma \in \Gamma\} \vdash_{\mathbf{IPC}} \neg\neg\varphi$, as desired. ⊠

Corollary 3.6. *The variety of Heyting algebras is an algebraic semantics for* **CPC**.

Proof. Consider the set of equations $\boldsymbol{\tau}(x) := \{\neg\neg x \approx 1\}$. From Glivenko's Theorem and Theorem 2.14 it follows that, for every $\Gamma \cup \{\varphi\} \subseteq T(Var)$,

$$\begin{aligned}\Gamma \vdash_{\mathbf{CPC}} \varphi &\iff \{\neg\neg\gamma : \gamma \in \Gamma\} \vdash_{\mathbf{IPC}} \neg\neg\varphi \\ &\iff \{\neg\neg\gamma \approx 1 : \gamma \in \Gamma\} \vDash_{\mathsf{HA}} \neg\neg\varphi \approx 1 \\ &\iff \boldsymbol{\tau}[\Gamma] \vDash_{\mathsf{HA}} \boldsymbol{\tau}(\varphi).\end{aligned}$$

Hence, we conclude that HA is an algebraic semantics for **CPC**. ⊠

This result shows that **CPC** has various algebraic semantics, e.g., the varieties of Boolean and Heyting algebras. This confirms that equational completeness theorems alone cannot explain the relation between a logic and a *unique* distinguished class of algebras. In fact, equational completeness theorems may exhibit a very counter-intuitive behaviour. For instance, while it is possible to prove that $\mathbf{K}_\ell$ has an artificial algebraic semantics, we have the following.

Theorem 3.7. *No class of modal algebras is an algebraic semantics for* $\mathbf{K}_\ell$.

Proof. From Theorem 2.32 it follows that, for all $\varphi, \psi \in T(Var)$,

$$\mathsf{MA} \vDash \varphi \approx \psi \iff \varphi \dashv\vdash_{\mathbf{K}_\ell} \psi.$$

Suppose, with a view to contradiction, that $\mathbf{K}_\ell$ has a $\boldsymbol{\tau}$-algebraic semantics $\mathsf{K} \subseteq \mathsf{MA}$. This implies that there exists an equation $\varepsilon \approx \delta \in \boldsymbol{\tau}$ such that $\mathsf{MA} \nvDash \varepsilon \approx \delta$. Thus, in view of the above display, we can assume, by symmetry, that $\varepsilon \nvdash_{\mathbf{K}_\ell} \delta$. This means that there are a Kripke frame $\mathbb{X} = \langle X, R\rangle$, an element $w \in X$, and a valuation v in $\mathbb{X}$ such that $w, v \Vdash \varepsilon$ and $w, v \nVdash \delta$.

Let $\mathbb{X}^+ = \langle X^+; R^+\rangle$ be the Kripke frame obtained by adding a new point w^+ to $\mathbb{X}$ and defining the relation R^+ as follows:

$$\langle p, q\rangle \in R^+ \iff p = w^+ \text{ or } \langle p, q\rangle \in R.$$

Let also v^+ be the unique evaluation in $\mathbb{X}^+$ such that for every $y \in Var$ and $q \in \mathbb{X}^+$:

$$q, v^+ \Vdash y \iff \text{ either } (q \in X \text{ and } q, v \Vdash y) \text{ or } q = w^+.$$

From the definition of $\mathbb{X}^+$ and v^+ it follows that

$$q, v^+ \Vdash \varphi \iff q, v \Vdash \varphi$$

for all $\varphi \in T(Var)$ and $q \in \mathbb{X}$. Consequently, as $w, v \Vdash \varepsilon$ and $w, v \nVdash \delta$,

$$w^+, v^+ \Vdash x \text{ and } w^+, v^+ \nVdash \Box(\varepsilon \to \delta).$$

This implies

$$x \nvdash_{\mathbf{K}_\ell} \Box(\varepsilon \to \delta).$$

On the other hand, clearly $\emptyset \vdash_{\mathbf{K}_\ell} \Box(\delta \to \delta)$. Consequently,

$$x, \Box(\delta \to \delta) \nvdash_{\mathbf{K}_\ell} \Box(\varepsilon \to \delta). \tag{3.3}$$

Now, observe that, for every $\varphi, \psi \in T(Var)$,

$$\varepsilon(x) \approx \delta(x), \varphi(\Box(\delta \to \delta)) \approx \psi(\Box(\delta \to \delta)) \vDash_{\mathsf{K}} \varphi(\Box(\varepsilon \to \delta)) \approx \psi(\Box(\varepsilon \to \delta)).$$

Since $\varepsilon \approx \delta \in \boldsymbol{\tau}(x)$, this implies

$$\boldsymbol{\tau}(x), \boldsymbol{\tau}(\Box(\delta \to \delta)) \vDash_{\mathsf{K}} \boldsymbol{\tau}(\Box(\varepsilon \to \delta)).$$

Since K is a $\boldsymbol{\tau}$-algebraic semantics for $\mathbf{K}_\ell$, this yields $x, \Box(\delta \to \delta) \vdash_{\mathbf{K}_\ell} \Box(\varepsilon \to \delta)$, a contradiction with (3.3). ⊠

3.2 Algebraizable logics