

# BI-INTERMEDIATE LOGICS OF TREES AND CO-TREES

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**ABSTRACT.** A bi-Heyting algebra validates the Gödel-Dummett axiom  $(p \rightarrow q) \vee (q \rightarrow p)$  iff the poset of its prime filters is a disjoint union of co-trees (i.e., order duals of trees). Bi-Heyting algebras of this kind are called *bi-Gödel algebras* and form a variety that algebraizes the extension bi-LC of bi-intuitionistic logic axiomatized by the Gödel-Dummett axiom. In this paper we initiate the study of the lattice  $\Lambda(\text{bi-LC})$  of extensions of bi-LC.

We develop the methods of Jankov-style formulas for bi-Gödel algebras and use them to prove that there are exactly continuum many extensions of bi-LC. We also show that all these extensions can be uniformly axiomatized by canonical formulas. Our main result is a characterization of the locally tabular extensions of bi-LC. We introduce a sequence of co-trees, called the *finite combs*, and show that a logic in  $\Lambda(\text{bi-LC})$  is locally tabular iff it contains at least one of the Jankov formulas associated with the finite combs. It follows that there exists the greatest non-locally tabular extension of bi-LC and consequently, a unique pre-locally tabular extension of bi-LC. These results contrast with the case of the intermediate logic axiomatized by the Gödel-Dummett axiom, which is known to have only countably many extensions, all of which are locally tabular.

## 1. INTRODUCTION

*Bi-intuitionistic logic* bi-IPC is the conservative extension of intuitionistic logic IPC obtained by adding a new binary connective  $\leftarrow$  to the language, called the *co-implication* (or exclusion, or subtraction), which is dual to  $\rightarrow$ . Thus, bi-IPC enjoys a symmetry that IPC lacks (each connective  $\wedge, \top, \rightarrow$  has its dual  $\vee, \perp, \leftarrow$ , respectively).

The Kripke semantics of bi-IPC [52] provides a transparent interpretation of co-implication: given a Kripke model  $\mathfrak{M}$ , a point  $x$  in  $\mathfrak{M}$ , and formulas  $\phi, \psi$ , we define

$$\mathfrak{M}, x \models \phi \leftarrow \psi \iff \exists y \leq x (\mathfrak{M}, y \models \phi \text{ and } \mathfrak{M}, y \not\models \psi).$$

This new connective gives bi-IPC significantly greater expressivity than IPC. For instance, if the points of a Kripke frame are interpreted as states in time, the language of bi-IPC is expressive enough to reason about the past, something that is not possible in IPC. With this example in mind, Wolter [58] extended Gödel's embedding of IPC into S4 to an embedding of bi-IPC into tense-S4. In particular, he proved a version of the Blok-Esakia Theorem [11, 26] stating that the lattice  $\Lambda(\text{bi-IPC})$  of *bi-intermediate logics* (i.e., consistent axiomatic extensions\* of bi-IPC) is isomorphic to that of consistent normal tense logics containing Grz.t (see also [16, 56]).

The greater symmetry of bi-IPC compared to IPC is reflected in the fact that bi-IPC is algebraized in the sense of [13] by the variety bi-HA of *bi-Heyting algebras* [51], i.e., Heyting algebras whose order duals are also Heyting algebras. As a consequence, the lattice  $\Lambda(\text{bi-IPC})$  is dually isomorphic to that of nontrivial varieties of bi-Heyting algebras. The latter, in turn, is amenable to the methods of universal algebra and duality theory since the category of bi-Heyting algebras is dually isomorphic to that of *bi-Esakia spaces* [25], see also [7].

The theory of bi-Heyting algebras was developed in a series of papers by Rauszer and others motivated by the connection with bi-intuitionistic logic (see, e.g., [2, 40, 50, 51, 52, 54]). However, bi-Heyting algebras also arise naturally in other fields of research such as topos theory [45, 46, 53]. Furthermore, the lattice of open sets of an Alexandrov space is always a bi-Heyting algebra, and so is the lattice of subgraphs of an arbitrary graph (see, e.g., [57]). Similarly, every quantum system can be associated with a complete bi-Heyting algebra [20].

The lattice  $\Lambda(\text{IPC})$  of intermediate logics (i.e., consistent extensions of IPC) has been thoroughly investigated (see, e.g., [17]). On the other hand, the lattice  $\Lambda(\text{bi-IPC})$  of bi-intermediate

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\*From now on we will use *extension* as a synonym of *axiomatic extension*.

logics lacks such an in-depth analysis, but for some recent developments see, e.g., [1, 10, 30, 31, 55]. In this paper we contribute to filling this gap by studying a simpler, yet nontrivial, sublattice of  $\Lambda(\text{bi-IPC})$ : the lattice of consistent extensions of the *bi-intuitionistic linear calculus* (or the bi-Gödel-Dummett's logic),

$$\text{bi-LC} := \text{bi-IPC} + (p \rightarrow q) \vee (q \rightarrow p).$$

Notably, the properties of  $\Lambda(\text{bi-IPC})$  and its extensions diverge significantly from those of its intuitionistic counterpart, i.e., the *intuitionistic linear calculus* (or the Gödel-Dummett's logic)  $\text{LC} := \text{IPC} + (p \rightarrow q) \vee (q \rightarrow p)$  [21, 29].

The choice of bi-LC as a case study was motivated by some of its properties that make it an interesting logic on its own. In particular, bi-LC is complete in the sense of Kripke semantics with respect to the class of *co-trees* (i.e., order duals of trees). Moreover, we prove that the bi-intuitionistic logic of linearly ordered Kripke frames is a proper extension of bi-LC (Theorem 4.26). This contrasts with the case of intermediate logics, where LC is both the logic of the class of linearly ordered Kripke frames and of co-trees. Because of this, the language of bi-IPC seems more appropriate to study tree-like structures than that of IPC. Furthermore, because of the symmetric nature of bi-intuitionistic logic, our results on extensions of bi-LC can be extended in a straightforward manner to the extensions of the bi-intermediate logic of trees by replacing in what follows every formula  $\varphi$  by its dual  $\neg\varphi^\partial$ , where  $\varphi^\partial$  is the formula obtained from  $\varphi$  by replacing each occurrence of  $\wedge, \top, \rightarrow$  by  $\vee, \perp, \leftarrow$  respectively, and every algebra of Kripke frame by its order dual.

Also, extensions of bi-LC admit a form of a classical *reductio ad absurdum* (Theorem 4.1). Recall that a deductive system  $\vdash$  is said to have a *classical inconsistency lemma* if, for every nonnegative integer  $n$ , there exists a finite set of formulas  $\Psi_n(p_1, \dots, p_n)$ , which satisfies the equivalence

$$\Gamma \cup \Psi_n(\varphi_1, \dots, \varphi_n) \text{ is inconsistent in } \vdash \iff \Gamma \vdash \{\varphi_1, \dots, \varphi_n\}, \quad (1)$$

for all sets of formulas  $\Gamma \cup \{\varphi_1, \dots, \varphi_n\}$  [49] (see also [15, 44, 43]). As expected, the only intermediate logic having a classical inconsistency lemma is CPC (with  $\Psi_n(p_1, \dots, p_n) := \{\neg(p_1 \wedge \dots \wedge p_n)\}$ ). This contrasts with the case of bi-intermediate logics, where every member of  $\Lambda(\text{bi-LC})$  has a classical inconsistency lemma witnessed by

$$\Phi_n := \{\sim \neg \sim (p_1 \wedge \dots \wedge p_n)\}$$

(here,  $\neg\varphi$  and  $\sim\varphi$  are shorthands for  $\varphi \rightarrow \perp$  and  $\top \leftarrow \varphi$ , respectively). Accordingly, logics in  $\Lambda(\text{bi-LC})$  exhibit a certain balance between the classical and intuitionistic behavior of negation connectives.

The main contributions of the paper can be summarized as follows. In order to classify extensions of bi-LC, we develop theories of Jankov, subframe and canonical formulas for them. We then employ Jankov formulas to obtain a characterization of splittings in  $\Lambda(\text{bi-LC})$  and to show that this lattice has the cardinality of the continuum (Theorems 4.11 and 4.17), cf. [8]. This contrasts with the case of  $\Lambda(\text{LC})$  which is well known to be a chain of order type  $(\omega + 1)^\partial$  [17]. Moreover, we show that canonical formulas provide a uniform axiomatization for all the extensions of bi-LC (Theorem 4.7). Lastly, subframe formulas can be used to describe the fine structure of co-trees, by governing the embeddability of finite co-trees into arbitrary co-forests (Lemma 4.24).

By combining the defining properties of subframe and Jankov formulas, we establish the main result of this paper: a characterization of locally tabular extensions of bi-LC (Theorem 5.1). More precisely, we show that an extension  $L$  of bi-LC is locally tabular iff  $L$  contains at least one of the Jankov formulas associated with *finite combs* (a particular class of co-trees defined in Figure 4). It follows that the logic of finite combs is the greatest non-locally tabular extension of bi-LC. Recall that a logic is called *pre-locally tabular* if it is not locally tabular, but all of its proper extensions are. It is a consequence of our main result that the logic of finite combs is the only pre-locally tabular extension of bi-LC (Corollary 5.31). As a consequence, we obtain that bi-LC is not locally tabular, which contrasts with the well-known fact that LC is locally tabular [33].

## 2. PRELIMINARIES

In this section, we review the basic concepts and results that we will need throughout this paper. For an in-depth study of bi-IPC and bi-Heyting algebras, see, e.g., [42, 50, 51, 52, 57]. As a main source for universal algebra we use [3, 14]. Henceforth,  $|X|$  denotes the cardinality of a set  $X$ ,  $\omega$  denotes the set of nonnegative integers,  $\mathbb{Z}^+$  the set of positive integers, and given  $n \in \omega$ , the notation  $i \leq n$  will always mean that  $i \in \{0, \dots, n\}$ .

**2.1. Bi-intuitionistic propositional logic.** Given a formula  $\varphi$ , we write  $\neg\varphi$  and  $\sim\varphi$  as a shorthand for  $\varphi \rightarrow \perp$  and  $\top \leftarrow \varphi$ . The *Bi-intuitionistic propositional calculus* bi-IPC is the least set of formulas in the language  $\wedge, \vee, \rightarrow, \leftarrow, \top, \perp$  (built up from a denumerable set *Prop* of variables) that contains IPC and the eight axioms below, and is closed under modus ponens, uniform substitutions, and the *double negation rule* (DN for short): “from  $\varphi$  infer  $\neg\sim\varphi$ ”.

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| 1. $p \rightarrow (q \vee (p \leftarrow q))$ ,                             | 5. $(p \rightarrow (q \leftarrow q)) \rightarrow \neg p$ , |
| 2. $(p \leftarrow q) \rightarrow \sim(p \rightarrow q)$ ,                  | 6. $\neg p \rightarrow (p \rightarrow (q \leftarrow q))$ , |
| 3. $((p \leftarrow q) \leftarrow r) \rightarrow (p \leftarrow q \vee r)$ , | 7. $((p \rightarrow p) \leftarrow q) \rightarrow \sim q$ , |
| 4. $\neg(p \leftarrow q) \rightarrow (p \rightarrow q)$ ,                  | 8. $\sim q \rightarrow ((p \rightarrow p) \leftarrow q)$ , |

It turns out that bi-IPC is a conservative extension of IPC. Furthermore, we may identify the *classical propositional calculus* CPC with the proper extension of bi-IPC obtained by adding the *law of excluded middle*  $p \vee \neg p$ . Notably, in CPC the co-implication  $\leftarrow$  is term-definable by the other connectives, since  $(p \leftarrow q) \leftrightarrow (p \wedge \neg q) \in \text{CPC}$ . Consequently, the DN rule becomes superfluous, as it translates to “from  $\phi$  infer  $\phi$ ”.

A set of formulas  $L$  closed under the three inference rules (modus ponens, uniform substitution, and DN) is called a *super-bi-intuitionistic logic* if it contains bi-IPC. Given a formula  $\phi$  and a super-bi-intuitionistic logic  $L$ , we say that  $\phi$  is a *theorem* of  $L$ , denoted by  $L \vdash \phi$ , if  $\phi \in L$ . Otherwise, write  $L \not\vdash \phi$ . We call  $L$  *consistent* if  $L \not\vdash \perp$ , and *inconsistent* otherwise. Given another super-bi-intuitionistic logic  $L'$ , we say that  $L'$  is an *extension* of  $L$  if  $L \subseteq L'$ . Consistent extensions of bi-IPC are called *bi-intermediate logics*, and it can be shown that  $L$  is a bi-intermediate logic iff  $\text{bi-IPC} \subseteq L \subseteq \text{CPC}$ . Finally, given a set of formulas  $\Sigma$ , we denote by  $L + \Sigma$  the least (with respect to inclusion) bi-intuitionistic logic containing  $L \cup \Sigma$ . If  $\Sigma$  is a singleton  $\{\phi\}$ , we simply write  $L + \phi$ . Given another formula  $\psi$ , we say that  $\phi$  and  $\psi$  are *L-equivalent* if  $L \vdash \phi \leftrightarrow \psi$ .

**2.2. Varieties of algebras.** We denote by  $\mathbb{H}, \mathbb{S}, \mathbb{I}, \mathbb{P}$ , and  $\mathbb{P}_U$  the class operators of closure under homomorphic images, subalgebras, isomorphic copies, direct products, and ultraproducts, respectively. A variety  $\mathbb{V}$  is a class of (similar) algebras closed under homomorphic images, subalgebras, and (direct) products. By Birkhoff’s Theorem, varieties coincide with classes of algebras that can be axiomatized by sets of equations (see, e.g., [14, Thm. II.11.9]). The smallest variety  $\mathbb{V}(\mathbb{K})$  containing a class  $\mathbb{K}$  of algebras is called the *variety generated by*  $\mathbb{K}$  and coincides with  $\mathbb{HSP}(\mathbb{K})$ . If  $\mathbb{K} = \{\mathfrak{A}\}$ , we simply write  $\mathbb{V}(\mathfrak{A})$ .

Given an algebra  $\mathfrak{A}$ , we denote by  $\text{Con}(\mathfrak{A})$  its congruence lattice. An algebra  $\mathfrak{A}$  is said to be *subdirectly irreducible*, or SI for short, (resp. *simple*) if  $\text{Con}(\mathfrak{A})$  has a second least element (resp. has exactly two elements: the identity relation  $\text{Id}_{\mathfrak{A}}$  and the total relation  $A^2$ ). Consequently, every simple algebra is subdirectly irreducible.

Given a class  $\mathbb{K}$  of algebras, we denote by  $\mathbb{K}^{<\omega}$ ,  $\mathbb{K}_{SI}$ , and  $\mathbb{K}_{SI}^{<\omega}$  the classes of finite members of  $\mathbb{K}$ , SI members of  $\mathbb{K}$ , and SI members of  $\mathbb{K}$  which are finite, respectively. In view of the Subdirect Decomposition Theorem, if  $\mathbb{K}$  is a variety, then  $\mathbb{K} = \mathbb{V}(\mathbb{K}_{SI})$  (see, e.g., [14, Thm. II.8.6]).

**Definition 2.1.** A variety  $\mathbb{V}$  is said to:

- (i) be *semi-simple* if its SI members are simple;
- (ii) be *locally finite* if its finitely generated members are finite;
- (iii) have the *finite model property* (FMP for short) if it is generated by its finite members;
- (iv) be *congruence distributive* if every member of  $\mathbb{V}$  has a distributive lattice of congruences;

- (v) have *equationally definable principal congruences* (EDPC for short) if there exists a conjunction  $\Phi(x, y, z, v)$  of finitely many equations such that for every  $\mathfrak{A} \in \mathbf{V}$  and all  $a, b, c, d \in A$ ,

$$(c, d) \in \Theta^{\mathfrak{A}}(a, b) \iff \mathfrak{A} \models \Phi(a, b, c, d),$$

where  $\Theta^{\mathfrak{A}}(a, b)$  is the least congruence of  $\mathfrak{A}$  that identifies  $a$  and  $b$ ;

- (vi) be a *discriminator variety* if there exists a *discriminator term*  $t(x, y, z)$  for  $\mathbf{V}$ , i.e., a ternary term such that for every  $\mathfrak{A} \in \mathbf{V}_{SI}$  and all  $a, b, c \in A$ , we have

$$t^{\mathfrak{A}}(a, b, c) = \begin{cases} c & \text{if } a = b, \\ a & \text{if } a \neq b. \end{cases}$$

The next result collects some of the relations between these properties.

**Proposition 2.2.** *If  $\mathbf{V}$  is a variety and  $\mathbf{K}$  a class of similar algebras, then the following conditions hold:*

- (i) *if  $\mathbf{V}$  is locally finite, then its subvarieties have the FMP;*
- (ii)  *$\mathbf{V}$  has the FMP iff  $\mathbf{V} = \mathbf{V}(\mathbf{K}_{SI}^{\leq \omega})$ ;*
- (iii) *if  $\mathbf{V}$  has EDPC, then  $\mathbf{V}$  is congruence distributive and  $\mathbf{HS}(\mathbf{K}) = \mathbf{SH}(\mathbf{K})$  for all  $\mathbf{K} \subseteq \mathbf{V}$ ;*
- (iv) *Jónsson's Lemma: if  $\mathbf{V}(\mathbf{K})$  is congruence distributive, then  $\mathbf{V}(\mathbf{K})_{SI} \subseteq \mathbf{HSP}_U(\mathbf{K})$ ;*
- (v) *if  $\mathbf{V}$  is discriminator, then it is semi-simple and it has EDPC.*

*Proof.* Condition (i) holds because every variety is generated by its finitely generated members (see, e.g., [3, Thm. 4.4]), while condition (ii) is an immediate consequence of the definition of the FMP together with the Subdirect Decomposition Theorem. The first part of condition (iii) was established in [40] and the second in [19]. For condition (iv), see, e.g., [14, Thm. VI.6.8]. Lastly, for the first part of condition (v) see, e.g., [14, Lem. IV.9.2(b)] and for the second [12, Exa. 6 p. 200].  $\square$

The following result provides a useful description of locally finite varieties of finite type (for a proof, see [4]).

**Theorem 2.3.** *A variety  $\mathbf{V}$  of a finite type is locally finite iff*

$$\forall m \in \omega, \exists k(m) \in \omega, \forall \mathfrak{A} \in \mathbf{V}_{SI} \ (\mathfrak{A} \text{ is } m\text{-generated} \implies |A| \leq k(m)).$$

**2.3. Bi-Heyting algebras.** Given a subset  $U$  of a poset  $\mathfrak{F}$ , let  $\max(U)$  be the set the *maximal* elements of  $U$  viewed as a subposet of  $\mathfrak{F}$ , and if  $U$  has a *maximum* (i.e., a greatest element), we denote it by  $\text{Max}(U)$ . Similarly, we define  $\min(U)$  and  $\text{Min}(U)$ . We denote the *upset generated* by  $U$  by

$$\uparrow U := \{x \in \mathfrak{F} : \exists u \in U (u \leq x)\},$$

and if  $U = \uparrow U$ , then  $U$  is called an *upset*. If  $U = \{u\}$ , we simply write  $\uparrow u$  and call it a *principal upset*. We define the *downsets* of  $\mathfrak{F}$  in a similar way. A set that is both an upset and a downset is an *updownset*. We denote the set of upsets of  $\mathfrak{F}$  by  $\text{Up}(\mathfrak{F})$ , of downsets by  $\text{Do}(\mathfrak{F})$ , and of updownsets by  $\text{UpDo}(\mathfrak{F})$ . We will always use the convention that the “arrow operators” defined above bind stronger than other set theoretic operations. For example, the expressions  $\uparrow U \setminus V$  and  $\downarrow U \cap V$  are to be read as  $(\uparrow U) \setminus V$  and  $(\downarrow U) \cap V$ , respectively. Given two distinct points  $x, y \in \mathfrak{F}$ ,  $x$  is an *immediate predecessor* of  $y$ , denoted by  $x \prec y$ , if  $x \leq y$  and no point of  $\mathfrak{F}$  lies between them (i.e., if  $z \in \mathfrak{F}$  is such that  $x \leq z \leq y$ , then either  $x = z$  or  $y = z$ ). If this is the case, we call  $y$  an *immediate successor* of  $x$ .

**Definition 2.4.** A *bi-Heyting algebra* is a Heyting algebra  $\mathfrak{A}$  whose order-dual is also a Heyting algebra. Equivalently,  $\mathfrak{A}$  is both a Heyting and a co-Heyting algebra, i.e.,  $\mathfrak{A}$  is a bounded distributive lattice such that for every  $a, b \in A$ , there are elements  $a \rightarrow b, a \leftarrow b \in A$  satisfying

$$(c \leq a \rightarrow b \iff a \wedge c \leq b) \text{ and } (a \leftarrow b \leq c \iff a \leq b \vee c),$$

for all  $c \in A$ . In this case, we use the abbreviations  $\neg a := a \rightarrow 0$  and  $\sim a := 1 \leftarrow a$ .

It is well known that the class bi-HA of bi-Heyting algebras is a variety. The following properties of bi-Heyting algebras will be useful throughout.

**Proposition 2.5.** *If  $\mathfrak{A} \in \text{bi-HA}$  and  $a, b, c \in \mathfrak{A}$ , then:*

1.  $a \rightarrow b = \bigvee \{d \in A : a \wedge d \leq b\},$
2.  $a \rightarrow b = 1 \iff a \leq b,$
3.  $\neg a = 1 \iff a = 0,$
4.  $a \wedge \neg a = 0,$
5.  $a \leftarrow b = \bigwedge \{d \in A : a \leq d \vee b\},$
6.  $a \leftarrow b = 0 \iff a \leq b,$
7.  $\sim a = 0 \iff a = 1,$
8.  $a \vee \sim a = 1.$

**Example 2.6.** Here we present some standard examples of bi-Heyting algebras.

- (i) Every finite Heyting algebra  $\mathfrak{A}$  can be viewed as a bi-Heyting algebra, by simply defining  $a \leftarrow b := \bigwedge \{d \in A : a \leq d \vee b\}$ . Since this is a meet of finitely many elements, the operation  $\leftarrow$  is well defined on  $\mathfrak{A}$ , and it can be easily shown that it satisfies the dual residuation law stated in Definition 2.4.
- (ii) Every Boolean algebra  $\mathfrak{A}$  can be viewed as a bi-Heyting algebra, by defining the co-implication as  $a \leftarrow b := a \wedge \neg b$ .
- (iii) Given a poset  $\mathfrak{F} = (W, \leq)$ , then  $(Up(\mathfrak{F}), \cup, \cap, \rightarrow, \leftarrow, \emptyset, W)$  is a bi-Heyting algebra, where the implications are defined by

$$U \rightarrow V := W \setminus \downarrow(U \setminus V) \text{ and } U \leftarrow V := \uparrow(U \setminus V).$$

A *valuation* on a bi-Heyting algebra  $\mathfrak{A}$  is a map  $v: Prop \rightarrow A$ , where  $Prop$  is the denumerable set of propositional variables of our language. Any such valuation can be uniquely extended to a bi-Heyting algebra morphism from the term algebra to  $\mathfrak{A}$ . We say that a formula  $\phi$  is *valid* on  $\mathfrak{A}$ , denoted by  $\mathfrak{A} \models \phi$ , if  $v(\phi) = 1$  for all valuations  $v$  on  $\mathfrak{A}$ . On the other hand, if  $v(\phi) \neq 1$  for some valuation  $v$  on  $\mathfrak{A}$ , we say that  $\mathfrak{A}$  *refutes*  $\phi$  (via  $v$ ), and write  $\mathfrak{A} \not\models \phi$ . If  $K$  is a class of bi-Heyting algebras such that  $\mathfrak{A} \models \phi$  for all  $\mathfrak{A} \in K$ , we write  $K \models \phi$ . Otherwise, write  $K \not\models \phi$ .

Using the well-known Lindenbaum-Tarski construction (see, e.g., [17, 28]) we obtain the following equivalence:  $\text{bi-IPC} \vdash \phi$  iff  $\text{bi-HA} \models \phi$ . This phenomenon, known as the algebraic completeness of bi-IPC, can be extended to all other super-bi-intuitionistic logics. Let  $L$  be such a logic, and denote the *variety* of  $L$  by  $V_L := \{\mathfrak{A} \in \text{bi-HA} : \mathfrak{A} \models L\}$ . On the other hand, given a subvariety  $V \subseteq \text{bi-HA}$ , we denote its *logic* by  $L_V := \text{Log}(V) = \{\phi : V \models \phi\}$ . Again using the standard Lindenbaum-Tarski construction, it can be shown that  $L$  is *sound* and *complete* with respect to  $V_L$ , i.e., for all formulas  $\phi$ , we have  $L \vdash \phi$  iff  $V_L \models \phi$ . It follows that this correspondence between extensions of bi-IPC and subvarieties of bi-Heyting algebras is one-to-one, and therefore the following theorem can now be easily proved.

**Theorem 2.7.** *If  $L$  is a super-bi-intuitionistic logic, then the lattice of extensions of  $L$  is dually isomorphic to the lattice of subvarieties of  $V_L$ . Equivalently, if  $V$  is a variety of bi-Heyting algebras, then the lattice of subvarieties of  $V$  is dually isomorphic to the lattice of extensions of  $L_V$ .*

**2.4. Bi-Esakia spaces.** Given an ordered topological space  $\mathcal{X}$ , we denote its set of open sets by  $Op(\mathcal{X})$ , closed sets by  $Cl(\mathcal{X})$ , clopen sets by  $Cp(\mathcal{X})$ , clopen upsets by  $CpUp(\mathcal{X})$ , and closed updownsets by  $ClUpDo(\mathcal{X})$ .

**Definition 2.8.** Let  $\mathfrak{F} = (W, \leq)$  and  $\mathfrak{G} = (G, \leq)$  be posets. A map  $f: W \rightarrow G$  is called a *bi-p-morphism*, denoted by  $f: \mathfrak{F} \rightarrow \mathfrak{G}$ , if it satisfies the following conditions:

- *Order preserving:*  $\forall x, y \in W (x \leq y \implies f(x) \leq f(y));$
- *Up:*  $\forall x \in W, \forall u \in G (f(x) \leq u \implies \exists y \in \uparrow x (f(y) = u));$
- *Down:*  $\forall x \in W, \forall v \in G (v \leq f(x) \implies \exists z \in \downarrow x (f(z) = v)).$

If  $f$  is moreover surjective, then we say that  $\mathfrak{G}$  is a *bi-p-morphic image* of  $\mathfrak{F}$  (via  $f$ ), denoted by  $f: \mathfrak{F} \twoheadrightarrow \mathfrak{G}$ .

**Proposition 2.9.** *If  $f: \mathfrak{F} \rightarrow \mathfrak{G}$  is a bi-p-morphism, then the following conditions hold:*

- (i)  $f[\uparrow x] = \uparrow f(x)$  and  $f[\downarrow x] = \downarrow f(x)$ , for all  $x \in \mathfrak{F}$ ;
- (ii)  $f[\max(\mathfrak{F})] \subseteq \max(\mathfrak{G})$  and  $f[\min(\mathfrak{F})] \subseteq \min(\mathfrak{G})$ ;
- (iii) if both  $\max(\mathfrak{F})$  and  $\max(\mathfrak{G})$  exist, then  $f(\max(\mathfrak{F})) = \max(\mathfrak{G})$  and  $f$  is necessarily surjective.

*Proof.* Condition (i) follows immediately from the definition of  $f$ , while the other two are direct consequences of (i).  $\square$



**Definition 2.10.** A triple  $\mathcal{X} = (X, \tau, \leq)$  is a *bi-Esakia space* if it is both an Esakia and a co-Esakia space, i.e.,  $(X, \tau)$  is a topological space equipped with a partial order  $\leq$ , and satisfies the following conditions:

- $(X, \tau)$  is compact;
- $\forall U \in Cp(\mathcal{X}) \ (\uparrow U, \downarrow U \in Cp(\mathcal{X}))$ ;
- *Priestley separation axiom* (PSA for short):

$$\forall x, y \in X \ (x \not\leq y \implies \exists U \in CpUp(\mathcal{X}) \ (x \in U \text{ and } y \notin U)).$$

Given  $U$  as in the above display, we say that  $U$  *separates*  $x$  from  $y$ .

A map  $f: \mathcal{X} \rightarrow \mathcal{Y}$  is a *bi-Esakia morphism* if it is a continuous bi-p-morphism between bi-Esakia spaces. If  $f$  is moreover bijective, then  $\mathcal{X}$  and  $\mathcal{Y}$  are said to be *isomorphic*, denoted by  $\mathcal{X} \cong \mathcal{Y}$ . Finally, we define the following operations on  $CpUp(\mathcal{X})$ :

$$\begin{aligned} U \rightarrow V &:= X \setminus \downarrow(U \setminus V) = \{x \in X: \uparrow x \cap U \subseteq V\}, \\ U \leftarrow V &:= \uparrow(U \setminus V) = \{x \in X: \downarrow x \cap U \not\subseteq V\}, \\ \neg U &:= U \rightarrow \emptyset = X \setminus \downarrow U, \\ \sim U &:= X \leftarrow U = \uparrow(X \setminus U). \end{aligned}$$

**Example 2.11.** Every finite poset can be viewed as a bi-Esakia space, when equipped with the discrete topology. In fact, since (bi-)Esakia spaces are Hausdorff, this is the only way to view a finite poset as a bi-Esakia space. Furthermore, since maps between spaces equipped with the discrete topology are always continuous, it follows that every bi-p-morphism between finite posets is a bi-Esakia morphism.

The celebrated Esakia duality restricts to a duality between the category of bi-Heyting algebras and bi-Heyting morphisms, and that of bi-Esakia spaces and bi-Esakia morphisms [25] (for a proof, see [42]). Here we just recall the main constructions establishing this duality. Given a bi-Heyting algebra  $\mathfrak{A}$ , we denote its *bi-Esakia dual* by  $\mathfrak{A}_* := (A_*, \tau, \subseteq)$ , where  $A_*$  is the set of prime filters of  $\mathfrak{A}$  and  $\tau$  is the topology generated by the subbasis

$$\{\varphi(a): a \in A\} \cup \{A_* \setminus \varphi(a): a \in A\},$$

where  $\varphi(a) := \{F \in A_*: a \in F\}$ . Moreover, it can be shown that  $CpUp(\mathfrak{A}_*) = \{\varphi(a): a \in A\}$ . Furthermore, if  $f: \mathfrak{A} \rightarrow \mathfrak{B}$  is a bi-Heyting morphism, then its dual is the restricted inverse image map  $f_* := f^{-1}[-]: \mathfrak{B}_* \rightarrow \mathfrak{A}_*$ .

Conversely, given a bi-Esakia space  $\mathcal{X}$  we denote its *bi-Heyting* (or *algebraic*) *dual* by  $\mathcal{X}^* := (CpUp(\mathcal{X}), \cup, \cap, \rightarrow, \leftarrow, \emptyset, X)$ , and if  $f: \mathcal{X} \rightarrow \mathcal{Y}$  is a bi-Esakia morphism, then its dual is the restricted inverse image map  $f^* := f^{-1}[-]: \mathcal{Y}^* \rightarrow \mathcal{X}^*$ . We note that  $\mathfrak{A}$  and  $(\mathfrak{A}_*)^*$  are isomorphic as bi-Heyting algebras, while  $\mathcal{X}$  and  $(\mathcal{X}^*)^*$  are isomorphic as bi-Esakia spaces.

The following result collects some useful properties of bi-Esakia spaces.

**Proposition 2.12.** *The following conditions hold for a bi-Esakia space  $\mathcal{X}$ :*

- $\mathcal{X}$  is Hausdorff;
- if  $Z \in Cl(\mathcal{X})$ , then  $\downarrow Z, \uparrow Z \in Cl(\mathcal{X})$ . Consequently,  $\downarrow x, \uparrow x \in Cl(\mathcal{X})$  for all  $x \in X$ ;
- if  $x \in X$ , then there are  $y \in \min(\mathcal{X})$  and  $z \in \max(\mathcal{X})$  satisfying  $y \leq x \leq z$ ;
- $\neg \sim U = \{x \in X: \downarrow \uparrow x \subseteq U\}$ , for all  $U \in CpUp(\mathcal{X})$ .

*Proof.* The results stated in the first three conditions are either well-known results for Esakia spaces, or their order-dual versions (for co-Esakia spaces). For a proof of the former, see [27]. We now prove (iv). By spelling out the definition of  $\neg \sim U$ , we have

$$\neg \sim U = X \setminus \downarrow \uparrow (X \setminus U) = \{x \in X: \forall y \in X \ (\exists z \in X \setminus U \ (z \leq y) \implies x \not\leq y)\}.$$

Suppose that  $x \in \neg \sim U$  and let  $u \in \downarrow \uparrow x$ , so there exists  $v \in \uparrow x$  such that  $u \leq v$ . By the equality above,  $x \leq v$  entails that for all  $z \in X \setminus U$ , we have  $z \not\leq v$ . Hence  $u$  must be in  $U$  and we conclude  $\downarrow \uparrow x \subseteq U$ . Thus,  $\neg \sim U \subseteq \{x \in X: \downarrow \uparrow x \subseteq U\}$ .

To prove the right to left inclusion, suppose  $\downarrow\uparrow x \subseteq U$ , for some  $x \in X$ . Let  $y \in X$  be such that there exists a  $z \in (X \setminus U) \cap \downarrow y$ . If  $x \leq y$ , then we would have  $z \in \downarrow\uparrow x \subseteq U$ , a contradiction. Thus  $x \not\leq y$ , and we conclude  $x \in \neg \sim U$ , as desired.  $\square$

Next we define the three standard methods of generating new bi-Esakia spaces from old ones. Let  $\mathcal{X} = (X, \tau, R)$ ,  $\mathcal{Y} = (Y, \pi, S)$ ,  $\mathcal{X}_1 = (X_1, \tau_1, R_1), \dots, \mathcal{X}_n = (X_n, \tau_n, R_n)$  be bi-Esakia spaces. We say that:

- (i)  $\mathcal{Y}$  is a *bi-generated subframe* of  $\mathcal{X}$  if  $Y \in \text{CUpDo}(\mathcal{X})$ ,  $\pi$  is the subspace topology, and  $S = Y^2 \cap R$ ;
- (ii)  $\mathcal{Y}$  is a *bi-Esakia (morphic) image* of  $\mathcal{X}$ , denoted by  $\mathcal{X} \twoheadrightarrow \mathcal{Y}$ , if there exists a surjective bi-Esakia morphism from  $\mathcal{X}$  onto  $\mathcal{Y}$ ;
- (iii)  $\mathcal{X} = \uplus_{i=1}^n \mathcal{X}_i$  is the *disjoint union* of the collection  $\{\mathcal{X}_1, \dots, \mathcal{X}_n\}$  if  $(X, R)$  is the disjoint union  $\uplus_{i=1}^n (X_i, R_i)$  of the various posets and  $(X, \tau)$  is the topological sum of the  $(X_i, \tau_i)$ .

As is the case with the analogous notions for Esakia spaces, the above definitions can be translated (using the bi-Esakia duality) into the terminology of bi-Heyting algebras (for a proof, see [42]).

**Proposition 2.13.** *Let  $\{\mathfrak{A}, \mathfrak{B}\} \cup \{\mathfrak{A}_1, \dots, \mathfrak{A}_n\}$  and  $\{\mathcal{X}_1, \dots, \mathcal{X}_n\}$  be finite sets of bi-Heyting algebras and bi-Esakia spaces, respectively. The following conditions hold:*

- (i)  $\mathfrak{B}$  is a homomorphic image of  $\mathfrak{A}$  iff  $\mathfrak{B}_*$  is (isomorphic to) a bi-generated subframe of  $\mathfrak{A}_*$ ;
- (ii)  $\mathfrak{B}$  is (isomorphic to) a subalgebra of  $\mathfrak{A}$  iff  $\mathfrak{B}_*$  is a bi-Esakia morphic image of  $\mathfrak{A}_*$ ;
- (iii)  $(\prod_{i=1}^n \mathfrak{A}_i)_* \cong \uplus_{i=1}^n \mathfrak{A}_{i*}$  and  $(\uplus_{i=1}^n \mathcal{X}_i)^* \cong \prod_{i=1}^n \mathcal{X}_i^*$ .

Let  $\mathcal{X}$  be a bi-Esakia space. A map  $V: \text{Prop} \rightarrow \text{CpUp}(\mathcal{X})$  is called a *valuation* on  $\mathcal{X}$ , and the pair  $\mathfrak{M} := (\mathcal{X}, V)$  a *bi-Esakia model* (on  $\mathcal{X}$ ). Moreover, we define the *validity* of a formula  $\phi$  in  $\mathcal{X}$  by  $\mathcal{X} \models \phi$  iff  $\mathcal{X}^* \models \phi$ . In other words,  $V(\phi) = X$ , for all valuations  $V$  on  $\mathcal{X}$ . Otherwise, write  $\mathcal{X} \not\models \phi$ . Since the validity of a formula is preserved under taking homomorphic images, subalgebras, and direct products of bi-Heyting algebras, it follows from the previous proposition that the validity of a formula is preserved under taking bi-generated subframes, bi-Esakia morphic images, and finite disjoint unions of bi-Esakia spaces.

Finally, we review the Coloring Theorem, a result that provides a characterization of the finitely generated bi-Heyting algebras using properties of their bi-Esakia duals. To this end, we need to define the notions of bi-bisimulation equivalences and colorings on bi-Esakia spaces.

**Definition 2.14.** Let  $\mathcal{X} = (X, \tau, \leq)$  be a bi-Esakia space and  $E$  an equivalence relation on  $X$ . We say that  $E$  is a *bi-bisimulation equivalence* on  $\mathcal{X}$  if it satisfies the following conditions:

- *Up*:  $\forall w, w', v' \in X (wEw' \text{ and } w' \leq v' \implies \exists v \in \llbracket v' \rrbracket_E (w \leq v))$ ;
- *Down*:  $\forall w, w', u' \in X (wEw' \text{ and } u' \leq w' \implies \exists u \in \llbracket u' \rrbracket_E (u \leq w))$ ;
- *Refined*: Any two non- $E$ -equivalent elements of  $X$  are *separated* by an  $E$ -saturated clopen upset, that is, for every  $w, v \in X$ , if  $(w, v) \notin E$  then there exists  $U \in \text{CpUp}(\mathcal{X})$  such that  $E[U] = \{x \in X: \exists u \in U (uEx)\} = U$ , and exactly one of  $w$  and  $v$  is contained in  $U$ .

We call a bi-bisimulation equivalence  $E$  on  $\mathcal{X}$  *trivial* if  $E = X^2$ , and *proper* otherwise.

Let  $\mathfrak{M} = (\mathcal{X}, V)$  be a bi-Esakia model and  $p_1, \dots, p_n \in \text{Prop}$  a finite number of fixed distinct propositional variables. To every point  $w \in \mathfrak{M}$ , we associate the sequence  $\text{col}(w) := i_1 \dots i_n$  defined by

$$i_k := \begin{cases} 1 & \text{if } w \models p_k, \\ 0 & \text{if } w \not\models p_k, \end{cases}$$

for  $k \in \{1, \dots, n\}$ . We call  $\text{col}(w)$  the *color* of  $w$  (relative to  $p_1, \dots, p_n$ ), and call a valuation  $V: \{p_1, \dots, p_n\} \rightarrow \text{CpUp}(\mathcal{X})$  a *coloring* of  $\mathfrak{M}$ .

Now, note that thinking of a bi-Heyting algebra  $\mathfrak{A}$  endowed with some fixed elements  $a_1, \dots, a_n$  is the same as equipping  $\mathfrak{A}$  with a valuation  $v: \{p_1, \dots, p_n\} \rightarrow A$  defined by  $v(p_i) := a_i$ , for  $i \leq n$ . The dual of the tuple  $(\mathfrak{A}, v)$  is the bi-Esakia model  $\mathfrak{M} = (\mathfrak{A}_*, V)$ , where  $V$  satisfies  $V(p_i) = \varphi(a_i) = \{x \in A_*: g_i \in x\}$ , and this valuation clearly defines a coloring of  $\mathfrak{M}$ .

The following theorem can be proven using the same argument as the analogous result for Heyting algebras (see, e.g., [9, Thm. 3.1.5]). Notice the use of the notation  $\mathfrak{A} = \langle a_1, \dots, a_n \rangle$  for “ $\mathfrak{A}$  is generated (as a bi-Heyting algebra) by  $\{a_1, \dots, a_n\}$ ”.

**Theorem 2.15** (Coloring Theorem). *Let  $\mathfrak{A}$  be a bi-Heyting algebra,  $a_1, \dots, a_n \in A$  a finite number of fixed elements, and  $(\mathcal{X}, V)$  the corresponding colored bi-Esakia model. Then  $\mathfrak{A} = \langle a_1, \dots, a_n \rangle$  iff every proper bi-bisimulation equivalence  $E$  on  $\mathcal{X}$  identifies points of different colors.*

### 3. THE BI-INTUITIONISTIC LINEAR CALCULUS

The *bi-intuitionistic linear calculus* (or *bi-Gödel-Dummett’s logic*) is the bi-intermediate logic

$$\text{bi-LC} := \text{bi-IPC} + (p \rightarrow q) \vee (q \rightarrow p).$$

This terminology hints that bi-LC is a bi-intuitionistic analogue of the linear calculus LC, the intermediate logic axiomatized by the same axiom over IPC. In view of Theorem 2.7, bi-LC is algebraized by the variety

$$\text{bi-GA} := \text{V}_{\text{bi-LC}} = \{\mathfrak{A} \in \text{bi-HA} : \mathfrak{A} \models (p \rightarrow q) \vee (q \rightarrow p)\},$$

whose elements will be called *bi-Gödel algebras*. Furthermore, there exists a dual isomorphism between the lattice  $\Lambda(\text{bi-LC})$  of consistent extensions of bi-LC and that of nontrivial subvarieties of bi-GA. In this section, we will restrict the bi-Esakia duality to a duality between the bi-Gödel algebras and bi-Esakia co-forests. This allows us to obtain a transparent description of the SI members of bi-GA and, therefore, conclude that bi-GA is a discriminator variety.

To this end, recall that a *co-tree* is a poset with a greatest element (called the *co-root*) and whose principal upsets are chains, and that a disjoint union of co-trees is called a *co-forest*. Moreover, a *bi-Esakia co-forest* (respectively, *bi-Esakia co-tree*) is a bi-Esakia space whose underlying poset is a co-forest (respectively, co-tree).

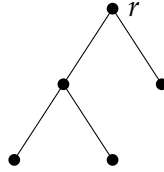


FIGURE 1. A co-tree.

**Theorem 3.1.** *Let  $\mathfrak{A} \in \text{bi-HA}$ . Then  $\mathfrak{A}$  is a bi-Gödel algebra iff  $\mathfrak{A}_*$  is a bi-Esakia co-forest.*

*Proof.* Observe that a bi-Heyting algebra  $\mathfrak{A}$  validates the axiom  $(p \rightarrow q) \vee (q \rightarrow p)$  iff so does its Heyting algebra reduct  $\mathfrak{A}^-$ . Since the latter condition is equivalent to  $\mathfrak{A}_*$  being a co-forest [32, Thm. 2.4], and as  $\mathfrak{A}_* = \mathfrak{A}_*^-$ , the result follows.  $\square$

Before we characterize the SI bi-Gödel algebras, let us recall the standard characterization of simple and SI bi-Heyting algebras by means of their bi-Esakia duals, as well as prove that the existence of a greatest prime filter of  $\mathfrak{A} \in \text{bi-HA}$  is a sufficient condition for  $\mathfrak{A}$  to be simple.

**Theorem 3.2.** *Let  $\mathfrak{A} \in \text{bi-HA}$ . Then the following conditions are equivalent:*

- (i)  $\mathfrak{A}$  is SI;
- (ii)  $(\text{Con}(\mathfrak{A}) \setminus \{Id_A\}, \subseteq)$  has a least element;
- (iii)  $(\text{ClUpDo}(\mathfrak{A}_*) \setminus \{A_*\}, \subseteq)$  has a greatest element.

*Proof.* Using Proposition 2.13, it can be shown that the lattice of congruences on  $\mathfrak{A}$  is dually isomorphic to that of closed updownsets of  $\mathfrak{A}_*$ , yielding the result.  $\square$

**Corollary 3.3.** *Let  $\mathfrak{A} \in \text{bi-HA}$ . Then the following conditions are equivalent:*

- (i)  $\mathfrak{A}$  is simple;
- (ii)  $\text{Con}(\mathfrak{A}) = \{Id_A, A^2\}$  and  $Id_A \neq A^2$ ;
- (iii)  $\text{ClUpDo}(\mathfrak{A}_*) = \{\emptyset, A_*\}$  and  $\emptyset \neq A_*$ .



*Proof.* This result follows immediately from the definition of a simple algebra and the aforementioned dual isomorphism between the lattice of congruences on  $\mathfrak{A}$  and that of closed updownsets of  $\mathfrak{A}_*$ .  $\square$

**Proposition 3.4.** *Let  $\mathfrak{A} \in \text{bi-HA}$ . If  $\mathfrak{A}_*$  has a greatest element, then  $\mathfrak{A}$  is simple.*

*Proof.* First, let us note that if  $\mathfrak{A}_*$  has a greatest element  $x$ , then  $A_* \neq \emptyset$  and every nonempty upset of  $\mathfrak{A}_*$  contains  $x$ . Since we also have  $\downarrow x = A_*$ , it now follows that  $\text{UpDo}(\mathfrak{A}_*) = \{\emptyset, A_*\}$ , hence also  $\text{ClUpDo}(\mathfrak{A}_*) = \{\emptyset, A_*\}$ . Therefore, by Corollary 3.3, we have that  $\mathfrak{A}$  is simple.  $\square$

*Remark 3.5.* The converse of Proposition 3.4 fails in general because  $\text{Up}(\mathfrak{F})$  is a simple bi-Heyting algebra for every nonempty finite connected poset  $\mathfrak{F}$ .

The next theorem lists equivalent conditions for a bi-Gödel algebra to be SI. In particular, it provides a transparent characterization of these algebras: they are exactly the bi-Heyting algebras whose duals are bi-Esakia co-trees. Recall that in a bounded distributive lattice  $\mathfrak{L}$ , the element 0 is said to be  $\wedge$ -irreducible if for all  $a, b \in L$ ,  $a \wedge b = 0$  implies  $a = 0$  or  $b = 0$ .

**Theorem 3.6.** *Let  $\mathfrak{A} \in \text{bi-GA}$ . Then the following conditions are equivalent:*

- (i)  $\mathfrak{A}$  is SI;
- (ii)  $\mathfrak{A}_*$  is a bi-Esakia co-tree;
- (iii)  $\mathfrak{A}_*$  has a greatest element;
- (iv)  $\mathfrak{A}$  is simple;
- (v)  $\mathfrak{A}$  is nontrivial and  $\neg a = 0$ , for all  $a \in A \setminus \{0\}$ ;
- (vi)  $\mathfrak{A}$  is nontrivial and 0 is  $\wedge$ -irreducible.

*Proof.*  $\boxed{\text{(i)} \implies \text{(ii)}}$  Let  $\mathfrak{A}$  be an SI bi-Gödel algebra. Since  $\mathfrak{A}_*$  is a bi-Esakia co-forest by Theorem 3.1, we can write  $A_* = \bigsqcup \{\downarrow x : x \in \max(\mathfrak{A}_*)\}$  as a disjoint union of co-trees. As  $\mathfrak{A}$  is SI, Theorem 3.2 entails that  $\text{ClUpDo}(\mathfrak{A}_*) \setminus \{A_*\}$  has a greatest element  $U$ . Since  $U$  is a proper downset, there must be a maximal point  $w$  of  $\mathfrak{A}_*$  not in  $U$ . Note that, since  $w$  is maximal and principal upsets are chains, it follows that  $\downarrow w$  is an upset of  $\mathfrak{A}_*$ . By definition, it is also a downset. Moreover, we know by Proposition 2.12 that  $\downarrow w$  is closed, hence it is a closed updownset not contained in  $U$ . By the definition of  $U$ , it follows that  $\downarrow w = A_*$ , so  $\mathfrak{A}_*$  is indeed a bi-Esakia co-tree.

$\boxed{\text{(ii)} \implies \text{(iii)} \implies \text{(iv)} \implies \text{(i)}}$  The first implication follows directly from the definition of a co-tree, the second is an immediate consequence of Proposition 3.4, while the third follows from the fact that simple algebras are SI.

$\boxed{\text{(iii)} \implies \text{(v)}}$  Suppose that  $\mathfrak{A}_*$  has a greatest element, i.e., that  $\mathfrak{A}$  has a greatest prime filter  $r$ . It is an immediate consequence of the Prime Filter Theorem and the definition of  $r$  that  $a \in A \setminus \{0\}$  iff  $a$  is contained in some prime filter iff  $a \in r$ . If  $a \in A \setminus \{0\}$ , then  $a \wedge \neg a = 0 \notin r$  entails  $\neg a \notin r$ , i.e.,  $\neg a = 0$  by the previous remark.

$\boxed{\text{(v)} \implies \text{(vi)}}$  Suppose that  $\mathfrak{A}$  satisfies condition (v), and  $a \wedge c = 0$ , for some  $a, c \in A$ . If  $a \neq 0$ , then  $\neg a = 0$ , so  $c \in \{b \in A : b \wedge a \leq 0\}$  now entails  $c = 0 = \neg a = \bigvee \{b \in A : b \wedge a \leq 0\}$ . Therefore, 0 is  $\wedge$ -irreducible.

$\boxed{\text{(vi)} \implies \text{(iii)}}$  If  $\mathfrak{A}$  is a nontrivial bi-Gödel algebra whose 0 is  $\wedge$ -irreducible, then it is easy to see that  $A \setminus \{0\}$  is not only a prime filter, but is in fact the greatest such filter.  $\square$

**Corollary 3.7.** *bi-GA is a discriminator variety. Consequently, it is semi-simple and has EDPC.*

*Proof.* We will prove that

$$t(x, y, z) := ((x + y) \wedge z) \vee (\neg(x + y) \wedge x)$$

is a discriminator term for bi-GA, where  $x + y := \neg((x \leftarrow y) \vee (y \leftarrow x))$ . Let  $\mathfrak{A} \in \text{bi-GA}_{SI}$  and  $a, b \in A$ . If  $a = b$ , then  $a \leftarrow b = 0 = b \leftarrow a$ , hence  $a + b = \neg 0 = 1$ . On the other hand, if  $a \neq b$ , we can assume without loss of generality that  $a \not\leq b$ . By Proposition 2.5, this is equivalent to

$a \leftarrow b \neq 0$ , and therefore  $0 < ((a \leftarrow b) \vee (b \leftarrow a))$ . As  $\mathfrak{A}$  is SI, it follows from Theorem 3.6 that  $a + b = \neg((a \leftarrow b) \vee (b \leftarrow a)) = 0$ . This discussion yields

$$a + b = \begin{cases} 1 & \text{if } a = b, \\ 0 & \text{if } a \neq b, \end{cases}$$

and it is now an easy check to see that the term  $t$  is a discriminator term for  $\mathfrak{A}$ . Hence, bi-GA is a discriminator variety. The second part of the statement follows from Proposition 2.2.(v).  $\square$

*Remark 3.8.* In [57] it is shown that a variety  $\mathbf{V}$  of bi-Heyting algebras is discriminator iff there exists some  $n \in \omega$  such that

$$\mathbf{V} \models (\neg \sim)^{n+1}x \approx (\neg \sim)^n x,$$

where  $(\neg \sim)^{n+1}x$  is recursively defined as  $(\neg \sim)^{n+1}x := \neg \sim((\neg \sim)^n x)$ . It is easy to see that in the case of bi-GA, the above display holds for  $n = 1$ .

**Corollary 3.9.** *If  $\mathfrak{A}$  is a subalgebra of an SI bi-Gödel algebra, or is an ultraproduct of a family of SI bi-Gödel algebras, then  $\mathfrak{A}$  is also SI.*

*Proof.* It is well known that in every discriminator variety  $\mathbf{V}$ ,  $\mathbf{V}_{SI}$  forms a universal class (see, e.g., [14, Thm. IX.9.4.(c)]). Since universal sentences are preserved under taking subalgebras and ultraproducts, the desired result now follows from the previous corollary.

To be more explicit, notice that Theorem 3.6 ensures that a bi-Gödel algebra  $\mathfrak{A}$  is SI iff  $\mathfrak{A}$  is nontrivial and 0 is  $\wedge$ -irreducible. It follows that  $\text{bi-GA}_{SI}$  can be axiomatized by a single universal sentence:

$$0 \neq 1 \text{ and } \forall x, y (x \wedge y = 0 \implies (x = 0 \text{ or } y = 0)). \quad \square$$

#### 4. STABLE, JANKOV AND SUBFRAME FORMULAS FOR BI-GÖDEL ALGEBRAS

In this section, we develop the theories of stable canonical and subframe formulas for bi-Gödel algebras. For an overview of these formulas and their use in superintuitionistic and modal logics we refer to [8] and [17], respectively. We use the class of stable canonical formulas to provide a uniform axiomatization of all extensions of bi-LC. By focusing on a particular subclass, that of the Jankov formulas, we fully characterize the splitting logics of the lattice  $\Lambda(\text{bi-LC})$ , and prove that  $|\Lambda(\text{bi-LC})| = 2^{\aleph_0}$ . As for the subframe formulas, we utilize them to establish a straightforward axiomatization of some notable extensions of bi-LC, such as the logic of co-trees of depth (respectively, width) less than an arbitrary  $n \in \mathbb{Z}^+$ . But our main use for these formulas will come in the following section.

**4.1. Stable canonical formulas and Jankov formulas.** Let  $\phi$  be a formula and  $\mathfrak{M} = (W, \leq, V)$  a Kripke model with a greatest element  $r$ . Note the following equivalences, for an arbitrary  $w \in \mathfrak{M}$ :

$$\begin{aligned} w \models \sim \neg \phi &\iff w \models \top \leftarrow \neg \phi \iff \exists v \in \downarrow w (v \models \top \text{ and } v \not\models \neg \phi) \\ &\iff \exists v \in \downarrow w (v \not\models \neg \phi) \iff \exists u \in \uparrow \downarrow w (u \models \phi). \end{aligned}$$

It is clear from the above display that  $w \models \sim \neg \phi$  implies that  $\phi$  holds in some point of  $\mathfrak{M}$ . Suppose now that  $v \models \phi$ , for some  $v \in \mathfrak{M}$ . By persistence, it follows  $r \models \phi$  since  $r$  is the greatest element of  $\mathfrak{M}$ , so the fact that  $r \in \uparrow \downarrow w$  now implies  $w \models \sim \neg \phi$ , again by the above display. Hence, we conclude that  $w \models \sim \neg \phi$  iff  $\phi$  holds in some point of  $\mathfrak{M}$ . Thus, in the setting of Kripke models with a greatest element (in particular, of models on co-trees),  $\sim \neg$  can be viewed as an analogue to the notion of the universal diamond from modal logic.

Similarly, let us note that

$$\begin{aligned} w \models \neg \sim \phi &\iff \forall v \in \uparrow w (v \not\models \sim \phi) \iff \forall v \in \uparrow w (v \not\models \top \leftarrow \phi) \\ &\iff \forall u \in \downarrow \uparrow w (u \not\models \top \text{ or } u \models \phi) \iff \forall u \in \downarrow \uparrow w (u \models \phi). \end{aligned}$$

As  $\mathfrak{M}$  has a greatest element  $r$ , we have  $W = \downarrow r = \downarrow \uparrow w$ . Thus, the equivalences above imply that  $w \models \neg \sim \phi$  iff  $\phi$  holds everywhere in  $\mathfrak{M}$ . Therefore, in the setting of Kripke models with a

greatest element (in particular, of models on co-trees),  $\neg \sim$  can be viewed as an analogue of the notion of the universal box from modal logic.

This discussion not only provides some intuition for what follows, by highlighting a similarity with the construction of the Jankov-Fine formulas for modal frames, but it helps us show that extensions of bi-LC admit a metalogical classical inconsistency lemma as in condition (1). These types of lemmas were formally introduced and studied in [49], see also [15, 44, 43]. Given an extension  $L$  of bi-LC, let us define a consequence relation  $\vdash_L$  in the following manner: given a set of formulas  $\Sigma \cup \{\varphi\}$ , we write  $\Sigma \vdash_L \varphi$  iff for every model  $\mathfrak{M} = \langle \mathcal{X}, V \rangle$  on a co-tree  $\mathcal{X}$  satisfying  $\mathcal{X} \models L$ , if  $\mathfrak{M} \models \Sigma$  then  $\mathfrak{M} \models \varphi$ .

**Theorem 4.1.** *Let  $L$  be an extension of bi-LC. If  $\Sigma \cup \{\varphi\}$  is a set of formulas, then*

$$\Sigma \cup \{\sim \neg \sim \varphi\} \vdash_L \perp \iff \Sigma \vdash_L \varphi.$$

*Consequently,  $L$  has a classical inconsistency lemma.*

*Proof.* Suppose first that  $\Sigma \cup \{\sim \neg \sim \varphi\} \vdash_L \perp$  and consider a model  $\mathfrak{M} = \langle \mathcal{X}, V \rangle$  such that  $\mathcal{X}$  is a co-tree validating  $L$  and  $\mathfrak{M} \models \Sigma$ . Since  $\mathfrak{M}$  is nonempty,  $\mathfrak{M} \not\models \perp$ . As  $\Sigma \cup \{\sim \neg \sim \varphi\} \vdash_L \perp$  and  $\mathfrak{M} \models \Sigma$ , this implies  $\mathfrak{M} \not\models \sim \neg \sim \varphi$ , i.e., that there exists  $x \in X$  such that  $x \not\models \sim \neg \sim \varphi$ . It now follows that  $x \models \neg \sim \varphi$ , and by our previous discussion on the connective  $\neg \sim$ , we conclude  $\mathfrak{M} \models \varphi$ .

For the converse, let us suppose  $\Sigma \vdash_L \varphi$ . We can show that  $\Sigma \cup \{\sim \neg \sim \varphi\} \vdash_L \perp$  holds by proving that  $\mathfrak{M} \not\models \Sigma \cup \{\sim \neg \sim \varphi\}$ , for every model  $\mathfrak{M} = \langle \mathcal{X}, V \rangle$  such that  $\mathcal{X}$  is a co-tree and  $\mathcal{X} \models L$ . Accordingly, let  $\mathfrak{M}$  be such a model and suppose that  $\mathfrak{M} \models \Sigma \cup \{\sim \neg \sim \varphi\}$ . By our assumption,  $\mathfrak{M} \models \Sigma$  implies  $\mathfrak{M} \models \varphi$ , and by our discussion on the connective  $\sim \neg$ ,  $\mathfrak{M} \models \sim \neg \sim \varphi$  entails the existence of a point  $x \in X$  such that  $x \models \sim \varphi$ . But now we have  $y \not\models \varphi$ , for some  $y \in \downarrow x$ , contradicting  $\mathfrak{M} \models \varphi$ . Therefore,  $\mathfrak{M} \not\models \Sigma \cup \{\sim \neg \sim \varphi\}$ , as desired.  $\square$

In view of Theorem 4.1, extensions of bi-LC exhibit an appealing balance between the classical and intuitionistic behavior of negation connectives, adding more motivation for studying this system.

We will now extend the method of stable canonical formulas [8] to the setting of bi-LC. Let  $\mathfrak{A} \in \text{bi-GA}_{S_I}^{\leq \omega}$  and  $D \subseteq A^2$ . For each  $a \in A$ , we introduce a fresh propositional variable  $p_a \in \text{Prop}$ . Let  $\Gamma$  be the formula describing the Heyting algebra structure of  $\mathfrak{A}$  fully, while the behavior of the operator  $\leftarrow$  is only given for elements of  $D$ , i.e.,

$$\begin{aligned} \Gamma := & \bigwedge \{p_{a \vee b} \leftrightarrow (p_a \vee p_b) : (a, b) \in A^2\} \wedge \bigwedge \{p_{a \wedge b} \leftrightarrow (p_a \wedge p_b) : (a, b) \in A^2\} \wedge \\ & \bigwedge \{p_{a \rightarrow b} \leftrightarrow (p_a \rightarrow p_b) : (a, b) \in A^2\} \wedge \bigwedge \{p_{a \leftarrow b} \leftrightarrow (p_a \leftarrow p_b) : (a, b) \in D\} \wedge \\ & \wedge \{p_0 \leftrightarrow \perp\} \wedge \{p_1 \leftrightarrow \top\}. \end{aligned}$$

We define the *stable canonical formula* associated with  $\mathfrak{A}$  and  $D$  by

$$\gamma(\mathfrak{A}, D) := \neg \sim \Gamma \rightarrow \neg \bigwedge \{p_a \leftarrow p_b : (a, b) \in A^2 \text{ and } a \not\leq b\}.$$

Moreover, given  $\mathfrak{B} \in \text{bi-GA}$  and a Heyting homomorphism  $h: \mathfrak{A} \rightarrow \mathfrak{B}$ , we call  $D$  a  $\leftarrow$ -stable domain of  $\mathfrak{A}$  if for all  $(a, b) \in D$ , we have  $h(a \leftarrow b) = h(a) \leftarrow h(b)$ . In this case, we say that  $h$  satisfies the  $\leftarrow$ -stable domain condition ( $\text{SDC}_{\leftarrow}$  for short) for  $D$ .

Before we discuss the fundamental properties of the stable canonical formulas, we recall four elementary facts about bi-Heyting algebras, that will be used without further reference in what follows.

**Lemma 4.2.** *Let  $\mathfrak{A} \in \text{bi-HA}$  and  $a, b \in A$ . Then:*

- (i)  $\neg a = 1 \iff a = 0$ ,
- (ii)  $\neg \sim a = 1 \iff a = 1$ ,
- (iii)  $a \rightarrow b = 1 \iff a \leq b$ ,
- (iv)  $a \leftarrow b \neq 0 \iff a \not\leq b$ .

**Lemma 4.3.** (*Stable Jankov Lemma*) Let  $\mathfrak{B} \in \text{bi-GA}$ . If  $\mathfrak{A} \in \text{bi-GA}_{SI}^{\leq \omega}$  and  $D \subseteq A^2$ , then  $\mathfrak{B} \not\models \gamma(\mathfrak{A}, D)$  iff there exists a Heyting algebra embedding  $h: \mathfrak{A} \hookrightarrow \mathfrak{C}$  satisfying the  $\text{SDC}_{\leftarrow}$  for  $D$ , for some  $\mathfrak{C} \in \mathbb{H}(\mathfrak{B})_{SI}$ .

*Proof.* We start by proving the left to right implication. Suppose that a bi-Gödel algebra  $\mathfrak{B}$  refutes  $\gamma(\mathfrak{A}, D)$ . By the Subdirect Decomposition Theorem, there exists  $\mathfrak{C} \in \mathbb{H}(\mathfrak{B})_{SI}$  such that  $\mathfrak{C} \not\models \gamma(\mathfrak{A}, D)$ . Let  $v$  be a valuation on  $\mathfrak{C}$  refuting  $\gamma(\mathfrak{A}, D)$ , i.e., such that

$$v(\gamma(\mathfrak{A}, D)) = v(\neg \sim \Gamma) \rightarrow v(\neg \bigwedge \{p_a \leftarrow p_b : (a, b) \in A^2 \text{ and } a \not\leq b\}) < 1.$$

Notice that this implies  $0 < v(\neg \sim \Gamma)$ , and that if  $(\mathfrak{C}_*, V)$  is the bi-Esakia model corresponding to  $(\mathfrak{C}, v)$  (via the bi-Esakia duality), then  $0 < v(\neg \sim \Gamma)$  iff  $V(\neg \sim \Gamma) \neq \emptyset$ , i.e.,  $\neg \sim \Gamma$  holds true in some point of  $(\mathfrak{C}_*, V)$ . By our previous discussion on the behaviour of  $\neg \sim$  on models with a greatest element, the fact that  $\mathfrak{C}$  is SI now entails that  $V(\Gamma) = C_*$ , i.e., that  $v(\Gamma) = 1$ .

Let us now define a map  $h: A \rightarrow C$  by setting  $h(a) := v(p_a)$ , for every  $a \in A$ . By the definitions of  $h$  and the formula  $\Gamma$ , it is easy to see that  $v(\Gamma) = 1$  iff  $h$  is a Heyting homomorphism which satisfies the  $\text{SDC}_{\leftarrow}$  for  $D$ , hence the only condition that remains to be shown is that  $h$  is injective. To this end, let  $a \neq b \in A$  and suppose, without loss of generality, that  $a \not\leq b$ . Notice that, since  $v(\gamma(\mathfrak{A}, D)) \neq 1$ , we must have

$$v(\neg \bigwedge \{p_x \leftarrow p_y : x, y \in A \text{ and } x \not\leq y\}) \neq 1,$$

which is equivalent to

$$\bigwedge \{v(p_x) \leftarrow v(p_y) : x, y \in A \text{ and } x \not\leq y\} \neq 0.$$

By the definition of  $h$ , it now follows that

$$\bigwedge \{h(x) \leftarrow h(y) : x, y \in A \text{ and } x \not\leq y\} \neq 0,$$

thus  $h(a) \leftarrow h(b) \neq 0$ , i.e.,  $h(a) \not\leq h(b)$ . We conclude that  $h$  is indeed a Heyting algebra embedding satisfying the  $\text{SDC}_{\leftarrow}$  for  $D$ , as desired.

For the converse, let  $\mathfrak{C} \in \mathbb{H}(\mathfrak{B})_{SI}$ ,  $h: \mathfrak{A} \hookrightarrow \mathfrak{C}$  be a Heyting algebra embedding that satisfies the  $\text{SDC}_{\leftarrow}$  for  $D$ , and  $v: Prop \rightarrow C$  a valuation satisfying  $v(p_a) = h(a)$ , for each  $a \in A$ . We shall prove that  $\mathfrak{C}$  refutes  $\gamma(\mathfrak{A}, D)$  via  $v$ , hence showing that  $\mathfrak{B} \not\models \gamma(\mathfrak{A}, D)$  holds (recall that the validity of a formula is preserved under taking homomorphic images).

By the definitions of  $h$  and  $\Gamma$ , it is clear that  $v(\Gamma) = 1$ , which is equivalent to  $\neg \sim v(\Gamma) = v(\neg \sim \Gamma) = 1$ . Now, let  $a, b \in A$  be such that  $a \not\leq b$ , i.e.,  $a \rightarrow b \neq 1$ . Since  $h$  is a Heyting algebra embedding, it follows that  $h(a \rightarrow b) = h(a) \rightarrow h(b) \neq 1$ , i.e.,  $h(a) \not\leq h(b)$ , which in turn is equivalent to  $h(a) \leftarrow h(b) = v(p_a) \leftarrow v(p_b) \neq 0$ . Thus, we have

$$0 \notin \{v(p_a) \leftarrow v(p_b) : a, b \in A \text{ and } a \not\leq b\}.$$

As  $\mathfrak{C}$  is SI by assumption,  $0_{\mathfrak{C}}$  is  $\wedge$ -irreducible (see Theorem 3.6), hence

$$v(\bigwedge \{p_a \leftarrow p_b : a, b \in A \text{ and } a \not\leq b\}) \neq 0.$$

Consequently,  $\neg v(\bigwedge \{p_a \leftarrow p_b : a, b \in A \text{ and } a \not\leq b\}) \neq 1$ , and we conclude that  $v(\gamma(\mathfrak{A}, D)) \neq 1$ . Therefore,  $\mathfrak{C}$  refutes  $\gamma(\mathfrak{A}, D)$  via  $v$ .  $\square$

The following lemma is essential for what follows. Notice that it makes crucial use of the fact that the HA-reduct of bi-GA is locally finite [33], much like the analogous version of this result for Heyting algebras relies on the local finiteness of the lattice reduct of HA. Henceforth, we will make extensive use of the fact that every finite Gödel algebra (i.e., a Heyting algebra that validates the Gödel-Dummett axiom) can be regarded as a finite bi-Gödel algebra (see Example 2.6).

**Lemma 4.4.** (*Stable Filtration Lemma*) Let  $\phi$  be a formula and  $\mathfrak{B} \in \text{bi-GA}$ . If  $\mathfrak{B} \not\models \phi$ , then there exists a finite Heyting subalgebra  $\mathfrak{A}$  of  $\mathfrak{B}$  such that  $\mathfrak{A} \in \text{bi-GA}$  and  $\mathfrak{A} \not\models \phi$ . If  $\mathfrak{B}$  is moreover SI, then so is  $\mathfrak{A}$ .

*Proof.* Suppose that  $\mathfrak{B} \not\models \phi$ . Then  $\phi^{\mathfrak{B}}(\bar{a}) \neq 1$  for some tuple  $\bar{a} \in B$ . Let

$$\Sigma := \{\psi(\bar{a}) : \psi \text{ is a subformula of } \phi\}$$

and  $\mathfrak{A}$  be the Heyting subalgebra of  $\mathfrak{B}$  generated by  $\Sigma$ . Since  $\Sigma$  is finite and bi-GA has a locally finite HA-reduct, it follows that  $\mathfrak{A}$  is a finite Heyting algebra, hence also a finite bi-Heyting algebra (see Example 2.6), although not necessarily a bi-Heyting subalgebra of  $\mathfrak{B}$ . Moreover, since bi-GA is axiomatized (relative to bi-IPC) by the Gödel-Dummett axiom, a  $\leftarrow$ -free formula, and  $\mathfrak{A}$  is a Heyting subalgebra of  $\mathfrak{B}$ , then clearly  $\mathfrak{B} \in \text{bi-GA}$  implies  $\mathfrak{A} \in \text{bi-GA}$ .

As  $\mathfrak{A}$  is a bi-Heyting algebra, it has a well-defined  $\leftarrow^{\mathfrak{A}}$  operation. And although this operation need not coincide with  $\leftarrow^{\mathfrak{B}}$ , it crucially does so when  $a \leftarrow^{\mathfrak{B}} b \in \Sigma$ . To see this, just note that  $A \subseteq B$  implies

$$a \leftarrow^{\mathfrak{B}} b = \bigwedge \{c \in B : a \leq c \vee b\} \leq \bigwedge \{c \in A : a \leq c \vee b\} = a \leftarrow^{\mathfrak{A}} b,$$

for all  $a, b \in A$ . Moreover, if  $a \leftarrow^{\mathfrak{B}} b \in A$ , then clearly  $a \leftarrow^{\mathfrak{B}} b \in \{c \in A : a \leq c \vee b\}$ , hence  $a \leftarrow^{\mathfrak{A}} b \leq a \leftarrow^{\mathfrak{B}} b$ . It now follows that if  $a \leftarrow^{\mathfrak{B}} b \in A$  (in particular, if  $a \leftarrow^{\mathfrak{B}} b \in \Sigma$ ), then  $a \leftarrow^{\mathfrak{B}} b = a \leftarrow^{\mathfrak{A}} b$ . Therefore, using a simple argument by induction on the complexity of  $\phi$ , we can conclude that  $\phi^{\mathfrak{B}}(\bar{a}) = \phi^{\mathfrak{A}}(\bar{a})$ , hence  $\phi^{\mathfrak{A}}(\bar{a}) \neq 1$ . Thus,  $\mathfrak{A} \not\models \phi$ , as desired.

Suppose now that  $\mathfrak{B}$  is SI, hence  $0_{\mathfrak{B}}$  is  $\wedge$ -irreducible by Theorem 3.6. Since  $\mathfrak{A}$  is a Heyting subalgebra of  $\mathfrak{B}$ , it is nontrivial and  $0_{\mathfrak{A}}$  must also be  $\wedge$ -irreducible. Again using Theorem 3.6, we conclude that  $\mathfrak{A}$  is SI.  $\square$

**Corollary 4.5.** *The variety bi-GA has the FMP.*

Equipped with the two previous lemmas, we can start our proof of the first main result of this subsection, which establishes a uniform axiomatization of all extensions of bi-LC by means of stable canonical formulas. This is in analogy with the intuitionistic case, see e.g., [7, 17]. However, a similar axiomatization technique for arbitrary bi-intermediate logics cannot be obtained, as we discuss below.

Fix a formula  $\phi \notin \text{bi-LC}$  and set  $n := |\text{Sub}(\phi)|$ . Since the HA-reduct of bi-GA is locally finite, there exists a bound  $c(\phi) \in \omega$  on the size of  $n$ -generated Heyting algebras belonging to this reduct. Accordingly, let  $\mathfrak{A}_1, \dots, \mathfrak{A}_{m(n)}$  be the list of (up to isomorphism) all  $n$ -generated SI bi-Gödel algebras such that  $|A_i| \leq c(\phi)$  and  $\mathfrak{A}_i \not\models \phi$ . Now, for each of these bi-Gödel algebras  $\mathfrak{A}_i$  refuting  $\phi$  via a valuation  $v$ , we set

$$D^{\leftarrow} := \{(a, b) \in \Theta^2 : a \leftarrow b \in \Theta\},$$

where  $\Theta := v[\text{Sub}(\phi)]$ . Consider a new list  $(\mathfrak{A}_1, D_1^{\leftarrow}), \dots, (\mathfrak{A}_{k(n)}, D_{k(n)}^{\leftarrow})$  (notice that  $k(n)$  need not be smaller than  $m(n)$ , since each  $\mathfrak{A}_i$  may refute  $\phi$  through distinct valuations), whose elements we call the *refutation patterns* for  $\phi$ . Keeping this discussion in mind, we have the following theorem:

**Theorem 4.6.** *If  $\mathfrak{B}$  is an SI bi-Gödel algebra, then:*

- (i)  $\mathfrak{B} \not\models \phi$  iff there exists  $i \leq k(n)$  and a Heyting algebra embedding  $h: \mathfrak{A}_i \hookrightarrow \mathfrak{B}$  satisfying the  $\text{SDC}_{\leftarrow}$  for  $D_i^{\leftarrow}$ ;
- (ii)  $\mathfrak{B} \models \phi \iff \mathfrak{B} \models \bigwedge_{i=1}^{k(n)} \gamma(\mathfrak{A}_i, D_i^{\leftarrow})$ .

*Proof.* (i) Firstly, note that right to left implication follows immediately from  $(\mathfrak{A}_i, D_i^{\leftarrow}) \not\models \phi$  and the definition of  $D_i^{\leftarrow}$ , since if a Heyting algebra embedding  $h: \mathfrak{A}_i \hookrightarrow \mathfrak{B}$  satisfies the  $\text{SDC}_{\leftarrow}$  for  $D_i^{\leftarrow}$ , then we clearly have  $\mathfrak{B} \not\models \phi$ . To prove the converse, suppose that  $\mathfrak{B} \not\models \phi$ . As  $\mathfrak{B} \in \text{bi-GA}_{SI}$ , it follows from Lemma 4.4 that there is a finite Heyting subalgebra  $\mathfrak{A}$  of  $\mathfrak{B}$  such that  $\mathfrak{A} \in \text{bi-GA}_{SI}$  and  $\mathfrak{A}$  refutes  $\phi$  via some valuation  $v$ . Thus, there exists a Heyting embedding  $h: \mathfrak{A} \hookrightarrow \mathfrak{B}$ , and by looking at the proof of Lemma 4.4, we not only see that  $\mathfrak{A}$  is  $n$ -generated for  $n = |\text{Sub}(\phi)|$  (as a Heyting algebra), but also that  $a \leftarrow b \in v[\text{Sub}(\phi)]$  implies  $h(a \leftarrow b) = h(a) \leftarrow h(b)$ , for all  $a, b \in A$ . It is now easy to see that  $h$  satisfies the  $\text{SDC}_{\leftarrow}$  for

$$D^{\leftarrow} := \{(a, b) \in v[\text{Sub}(\phi)]^2 : a \leftarrow b \in v[\text{Sub}(\phi)]\}.$$



Therefore, the pair  $(\mathfrak{A}, D^{\leftarrow})$  must be one of the  $(\mathfrak{A}_i, D_i^{\leftarrow})$  listed above, hence we showed that the right side of the desired equivalence holds, as desired.

(ii) This follows immediately from (i) and the Stable Jankov Lemma 4.3.  $\square$

As a consequence, stable canonical formulas can be used to axiomatize extensions of bi-LC.

**Theorem 4.7.** *Every extension of bi-LC is axiomatizable by stable canonical formulas. Moreover, if  $L$  is finitely axiomatized, then  $L$  is axiomatizable by finitely many stable canonical formulas.*

*Proof.* Suppose that  $L = \text{bi-LC} + \{\phi_i : i \in I\}$ , so we can assume without loss of generality that  $\text{bi-LC} \not\models \phi_i$ , for all  $i \in I$ . By the previous theorem, we know that for each  $i \in I$  there is a list of refutation patterns  $(\mathfrak{A}_{i,1}, D_{i,1}^{\leftarrow}), \dots, (\mathfrak{A}_{i,k(i)}, D_{i,k(i)}^{\leftarrow})$  such that

$$\text{bi-LC} + \phi_i = \text{bi-LC} + \bigwedge_{j=1}^{k(i)} \gamma(\mathfrak{A}_{i,j}, D_{i,j}^{\leftarrow}).$$

Thus, we have

$$L = \text{bi-LC} + \{\phi_i : i \in I\} = \text{bi-LC} + \left\{ \bigwedge_{j=1}^{k(i)} \gamma(\mathfrak{A}_{i,j}, D_{i,j}^{\leftarrow}) : i \in I \right\}.$$

The last part of the statement clearly follows from the previous equality.  $\square$

**Corollary 4.8.** *Let  $L' \subseteq L$  be extensions of bi-LC. Then  $L$  is axiomatizable over  $L'$  by stable canonical formulas. Moreover, if  $L$  is finitely axiomatized over  $L'$ , then  $L$  is axiomatizable over  $L'$  by finitely many stable canonical formulas.*

*Proof.* This is an immediate consequence of the proof of the previous theorem.  $\square$

We will now focus on a particular class of stable canonical formulas: the Jankov formulas [34, 35, 36]. For each  $\mathfrak{A} \in \text{bi-GA}_{SI}^{<\omega}$ , we call  $\mathcal{J}(\mathfrak{A}) := \gamma(\mathfrak{A}, A^2)$  the *Jankov formula* of  $\mathfrak{A}$ . We compile the defining properties of these formulas in the following lemma, and subsequently use them to characterize the splitting logics of the lattice  $\Lambda(\text{bi-LC})$ , as well as finding its cardinality.

**Lemma 4.9.** (*Jankov Lemma*) *If  $\mathfrak{B} \in \text{bi-GA}$  and  $\mathfrak{A} \in \text{bi-GA}_{SI}^{<\omega}$ , then the following conditions are equivalent:*

- (i)  $\mathfrak{B} \not\models \mathcal{J}(\mathfrak{A})$ ,
- (ii) *there exists  $\mathfrak{C} \in \mathbb{H}(\mathfrak{B})_{SI}$  and a bi-Heyting algebra embedding  $h: \mathfrak{A} \hookrightarrow \mathfrak{C}$ ,*
- (iii)  $\mathfrak{A} \in \text{SIH}(\mathfrak{B})$ ,
- (iv)  $\mathfrak{A} \in \text{IHS}(\mathfrak{B})$ .

*Proof.* Firstly, let us note that the equivalence (i)  $\iff$  (ii) is just a particular instance of Lemma 4.3, and that (ii) clearly implies (iii). The equivalence (iii)  $\iff$  (iv) follows from Proposition 2.2.(iii) and the fact that bi-GA has EDPC (see Corollary 3.7). Finally, (iv)  $\implies$  (i) follows from the easily checked fact that  $\mathfrak{A} \not\models \mathcal{J}(\mathfrak{A})$ , and by noting that the operators  $\mathbb{H}$  and  $\mathbb{S}$  preserve the validity of formulas.  $\square$

**Corollary 4.10.** *If  $\mathfrak{B} \in \text{bi-GA}_{SI}$ , then  $\mathbb{V}(\mathfrak{B})_{SI}^{<\omega} = \mathbb{IS}(\mathfrak{B})^{<\omega}$ .*

*Proof.* We start by noting that  $\mathbb{IS}(\mathfrak{B})^{<\omega} \subseteq \mathbb{V}(\mathfrak{B})_{SI}^{<\omega}$  follows directly from Corollary 3.9. To prove the other inclusion, we use the Jankov Lemma and the fact that the product of algebras preserves the validity of formulas to deduce that if  $\mathfrak{A} \in \mathbb{V}(\mathfrak{B})_{SI}^{<\omega}$ , then  $\mathfrak{B} \not\models \mathcal{J}(\mathfrak{A})$ , i.e.,  $\mathfrak{A} \in \text{SIH}(\mathfrak{B})$ . As bi-GA is a semi-simple variety (see Corollary 3.7) and simple algebras have no nontrivial homomorphic images,  $\mathfrak{B} \in \text{bi-GA}_{SI}$  now implies that  $\mathfrak{A} \in \mathbb{IS}(\mathfrak{B})^{<\omega}$ , as desired.  $\square$

Given a lattice  $\mathfrak{L}$  and elements  $a, b \in \mathfrak{L}$ , we call  $(a, b)$  a *splitting pair* for  $\mathfrak{L}$  if  $\mathfrak{L} = \uparrow a \uplus \downarrow b$  (we use the symbol  $\uplus$  to denote the union of sets which are pairwise disjoint). In particular, if  $\mathfrak{L} = \Lambda(\text{bi-LC})$  then  $a$  is said to be a *splitting logic*.

**Theorem 4.11.** (*Splitting Theorem*) *If  $L \in \Lambda(\text{bi-LC})$ , then:*

- (i)  $L$  is a splitting logic iff  $L$  is axiomatized by a single Jankov formula,
- (ii)  $L$  is a join of splitting logics iff  $L$  is axiomatized by Jankov formulas.

*Proof.* We start by noting that condition (i) clearly implies (ii), hence we only prove the former equivalence. Suppose that  $(L, L')$  is a splitting pair for  $\Lambda(\text{bi-LC})$ , for some  $L' \in \Lambda(\text{bi-LC})$ . As  $\text{bi-GA}$  is a congruence distributive variety (by Proposition 2.2 together with Corollary 3.7) with the FMP (see Corollary 4.5), it follows from a result by McKenzie [48] that  $L' = \text{Log}(\mathfrak{A})$ , for some  $\mathfrak{A} \in \text{bi-GA}_{SI}^{<\omega}$ . Using the definition of a splitting pair together with the fact  $\mathfrak{A} \not\models \mathcal{J}(\mathfrak{A})$ , it is easy to see that the equivalence  $\mathfrak{B} \models \mathcal{J}(\mathfrak{A})$  iff  $\mathfrak{B} \models L$  holds for all  $\mathfrak{B} \in \text{bi-GA}$ . Thus,  $L = \text{bi-LC} + \mathcal{J}(\mathfrak{A})$ .

Conversely, assume  $L = \text{bi-LC} + \mathcal{J}(\mathfrak{A})$  for some  $\mathfrak{A} \in \text{bi-GA}_{SI}^{<\omega}$ . Set  $L' := \text{Log}(\mathfrak{A})$  and notice that  $\mathfrak{A} \not\models \mathcal{J}(\mathfrak{A})$  implies  $L \not\subseteq L'$ . Now, take  $E \in \Lambda(\text{bi-LC})$  and suppose  $L \not\subseteq E$ , i.e.,  $\mathcal{J}(\mathfrak{A}) \not\subseteq E$ . By a simple application of the Jankov Lemma 4.9, this implies  $\mathfrak{A} \in V_E = \{\mathfrak{B} \in \text{bi-HA} : \mathfrak{B} \models E\}$ . Equivalently,  $E \subseteq \text{Log}(\mathfrak{A}) = L'$ . We just proved that for  $E \in \Lambda(\text{bi-LC})$ ,  $E \not\subseteq \uparrow L$  entails  $E \subseteq \downarrow L'$ , i.e., that  $\Lambda(\text{bi-LC}) = \uparrow L \uplus \downarrow L'$ . Therefore,  $(L, L')$  is a splitting pair for  $\Lambda(\text{bi-LC})$ .  $\square$

Translating the Splitting Theorem into algebraic terms yields a characterization of the *splitting algebras* of the variety  $\text{bi-GA}$ , that is, elements  $\mathfrak{A} \in \text{bi-GA}_{SI}$  for which there exists a largest subvariety  $V \subseteq \text{bi-GA}$  omitting  $\mathfrak{A}$ . In other words,  $(V(\mathfrak{A}), V)$  is a splitting pair for the lattice of nontrivial subvarieties of  $\text{bi-Gödel algebras } \Lambda(\text{bi-GA})$ .

**Theorem 4.12.** *The splitting algebras of  $\text{bi-GA}$  are exactly the finite SI bi-Gödel algebras.*

*Remark 4.13.* The equality between splitting algebras and finite SI algebras holds more in general for every variety of finite type with EDPC and the FMP, as shown in [12, Cor. 3.2] and [48].

It is well known that the analogue of the previous theorem holds for the variety of Heyting algebras (see, e.g., [17]): the splitting algebras of  $\text{HA}$  are exactly its finite SI elements. However, this is far from the case for  $\text{bi-Heyting algebras}$ ; a result by Wolter [58] shows that the only splitting algebras in  $\text{bi-HA}$  are the two-element and three-element chains. This is the main reason why the theories of stable canonical formulas cannot be developed for  $\text{bi-IPC}$ .

**4.2. The cardinality of  $\Lambda(\text{bi-LC})$ .** The goal of this section is to prove that the cardinality of the lattice  $\Lambda(\text{bi-LC})$  is  $2^{\aleph_0}$ . Accordingly, let us define a partial order  $\leq$  on the class  $\text{bi-GA}_{SI}^{<\omega}$  by  $\mathfrak{A} \leq \mathfrak{B}$  iff  $\mathfrak{A} \in \text{IHS}(\mathfrak{B})$ . Note that, by Corollary 4.10, we have

$$\mathfrak{A} \leq \mathfrak{B} \iff \mathfrak{A} \in \text{IS}(\mathfrak{B}).$$

We will show  $|\Lambda(\text{bi-LC})| = 2^{\aleph_0}$  by proving that there exists a countably infinite  $\leq$ -antichain  $\Omega \subseteq \text{bi-GA}_{SI}^{<\omega}$  (that is, the elements of  $\Omega$  are pairwise  $\leq$ -incomparable). That the existence of  $\Omega$  suffices to establish the desired equality follows easily from the next proposition, as we shall see in a moment.

**Proposition 4.14.** *Let  $\Omega \subseteq \text{bi-GA}_{SI}^{<\omega}$  be a countably infinite  $\leq$ -antichain. If  $\Omega_1, \Omega_2 \in \mathcal{P}(\Omega)$  are distinct, then*

$$\text{bi-LC} + \mathcal{J}(\Omega_1) \neq \text{bi-LC} + \mathcal{J}(\Omega_2),$$

where  $\mathcal{J}(\Omega_i) := \{\mathcal{J}(\mathfrak{A}) : \mathfrak{A} \in \Omega_i\}$ .

*Proof.* Without loss of generality, suppose that there exists  $\mathfrak{B} \in \Omega_1 \setminus \Omega_2$ . Since  $\mathfrak{B} \not\models \mathcal{J}(\mathfrak{B})$ , it is clear that  $\mathfrak{B} \not\models \text{bi-LC} + \mathcal{J}(\Omega_1)$ . On the other hand, if  $\mathfrak{B} \not\models \text{bi-LC} + \mathcal{J}(\Omega_2)$  then there is  $\mathfrak{A} \in \Omega_2$  such that  $\mathfrak{B} \not\models \mathcal{J}(\mathfrak{A})$ . By the Jankov Lemma, it follows  $\mathfrak{A} \leq \mathfrak{B}$ . But this is a contradiction, since  $\mathfrak{A}$  and  $\mathfrak{B}$  are in  $\Omega$ , an  $\leq$ -antichain. Therefore,  $\mathfrak{B} \models \text{bi-LC} + \mathcal{J}(\Omega_2)$ , and we conclude

$$\text{bi-LC} + \mathcal{J}(\Omega_1) \neq \text{bi-LC} + \mathcal{J}(\Omega_2). \quad \square$$

Suppose that we have  $\Omega$  satisfying the conditions of the previous proposition. As our language is countable, we know that there are at most continuum many extensions of  $\text{bi-LC}$ , that is,  $|\Lambda(\text{bi-LC})| \leq 2^{\aleph_0}$ . But we just proved that distinct subsets of the countably infinite  $\leq$ -antichain  $\Omega$  give rise to distinct extensions of  $\text{bi-LC}$ , hence it follows  $|\mathcal{P}(\Omega)| = 2^{\aleph_0} \leq |\Lambda(\text{bi-LC})|$ . Therefore, we get the desired equality.

We end this discussion by noting that we can use the bi-Esakia duality and Example 2.11 to translate the partial order  $\leq$  defined above into one on the class of finite co-trees:  $\mathfrak{F} \leq \mathfrak{G}$  iff  $\mathfrak{F}$  is a bi-p-morphic image of  $\mathfrak{G}$ . It is now clear that to find our desired  $\leq$ -antichain of finite SI bi-Gödel algebras, it suffices to find a countably infinite  $\leq$ -antichain of finite co-trees. In order to do this, we rely on the following observation. Recall that  $x \prec y$  denotes that  $y$  is an immediate successor of  $x$ .

**Lemma 4.15.** *Let  $f: \mathfrak{F} \rightarrow \mathfrak{G}$  be a bi-p-morphism between co-trees. If  $x \prec y \in \mathfrak{F}$ , then either  $f(x) = f(y)$  or  $f(x) \prec f(y)$ .*

*Proof.* Assume  $x \prec y$ . As bi-p-morphisms are order preserving,  $x \prec y$  entails  $f(x) \leq f(y)$ . If  $f(x) = f(y)$  we are done, so let us suppose that  $f(x) < f(y)$ . Suppose as well that  $f(x) \leq u \leq f(y)$ , for some  $u \in \mathfrak{G}$ . By the up condition (see the Definition 2.8), there exists  $z \in \uparrow x$  satisfying  $f(z) = u$ . Notice that  $\uparrow x = \{x\} \uplus \uparrow y$ , since  $x \prec y$  and the principal upsets of  $\mathfrak{F}$  are chains. If  $z = x$ , then  $f(x) = f(z) = u$ . If  $z \in \uparrow y$ , then  $f(y) \leq f(z) = u \leq f(y)$ , and thus  $f(y) = u$ . We conclude  $f(x) \prec f(y)$ .  $\square$

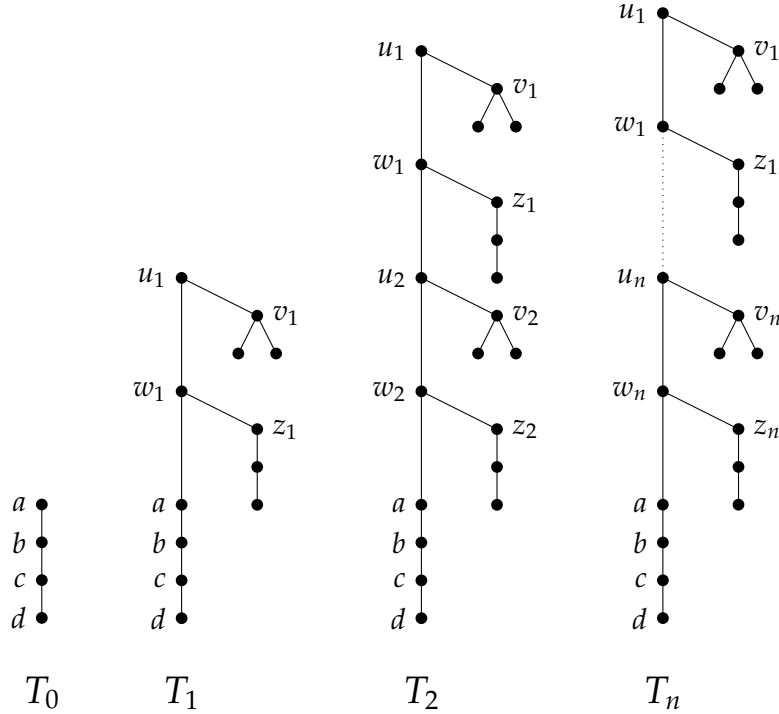


FIGURE 2. The co-trees  $T_0, T_1, T_2$ , and  $T_n$ .

Let  $\mathcal{T} := \{T_n : n \in \omega\}$  be the family of finite co-trees depicted in Figure 2. The next result proves that this is an  $\leq$ -antichain of finite co-trees.<sup>†</sup> Its proof makes extensive use, often without reference, of Proposition 2.9 (in particular, of the fact that if  $f: \mathfrak{F} \rightarrow \mathfrak{G}$  is a bi-p-morphism between co-trees, then minimal points must be mapped to minimal points, and the co-root of  $\mathfrak{F}$  must be mapped to the co-root of  $\mathfrak{G}$ ) and of Lemma 4.15.

**Proposition 4.16** (Hodkinson). *The set  $\mathcal{T} := \{T_n : n \in \omega\}$  is a countably infinite  $\leq$ -antichain of finite co-trees.*

*Proof.* Firstly, let us note that for  $n \in \mathbb{Z}^+$ , if  $m < n$  then  $|T_m| < |T_n|$ , hence clearly  $T_n \not\leq T_m$ . Moreover, if  $f: T_n \rightarrow T_0$  is a bi-p-morphism, then the co-root  $u_1$  of  $T_n$  must be mapped to the co-root of  $T_0$ . By Lemma 4.15, this entails that both nontrivial predecessors of  $v_1$ , which are

<sup>†</sup>This  $\leq$ -antichain was constructed by Ian Hodkinson (personal communication). We use a different method to show that this is in fact an  $\leq$ -antichain

minimal, must be mapped to nonminimal points, contradicting Proposition 2.9. So  $f$  cannot exist, and thus  $T_0 \not\leq T_n$ .

It remains to show  $T_m \not\leq T_n$ , for all  $m \in \mathbb{Z}^+$  such that  $m < n$ . We denote the elements of  $T_m$  by  $u'_1, v'_1, a', b'$ , etc., and those of  $T_n$  by  $u_1, v_1, a, b$ , etc. Set  $U'_m := \{u'_j \in T_m : j \leq m\}$  and  $U_n := \{u_i \in T_n : i \leq n\}$ . In a similar way, define the sets  $V'_m$  and  $V_n$ ,  $W'_m$  and  $W_n$ ,  $Z'_m$  and  $Z_n$ . Suppose, with a view to contradiction, that there exists a surjective bi-p-morphism  $f : T_n \twoheadrightarrow T_m$ , and take  $i \leq n$  and  $j \leq m$ .

Before we begin a proof by induction, let us establish a series of useful equalities involving these newly defined sets. First we show that

$$f[V_n] \cap (U'_m \cup W'_m \cup V'_m \cup \{a'\}) = \emptyset. \quad (2)$$

Since we have  $|\downarrow v_i| = 3 < \text{Min}\{|\downarrow u'_j|, |\downarrow w'_j|, |\downarrow a'|\}$ , the down condition of bi-p-morphisms (see Definition 2.8) already implies  $f(v_i) \notin U'_m \cup W'_m \cup \{a'\}$ . To see that  $f(v_i) \notin V'_m$ , suppose otherwise. By Lemma 4.15, this implies that both nontrivial predecessors of  $v_i$ , which are minimal, must be mapped to nonminimal points, a contradiction. This establishes the desired equality. We now prove

$$f[Z_n] \cap (U'_m \cup W'_m \cup V'_m \cup \{a'\}) = \emptyset. \quad (3)$$

Notice that  $|\downarrow z_i| = 3 < \text{Min}\{|\downarrow u'_j|, |\downarrow w'_j|, |\downarrow a'|\}$  holds, so the down condition forces  $f(z_i) \notin U'_m \cup W'_m \cup \{a'\}$ . Moreover,  $f(z_i) = v'_j$  would imply that  $f[\downarrow z_i] = \downarrow v'_j$ , again by the down condition, which clearly contradicts the fact that  $f$  is order preserving. Finally, to see that

$$f[U_n] \cap W'_m = \emptyset = f[W_n] \cap U'_m, \quad (4)$$

suppose  $f(u_i) = w'_j$ . It follows from Lemma 4.15 that  $f(v_i) \in \{w'_j, z'_j, \alpha(j)\}$ , where  $\alpha(j) := u'_{j+1}$  if  $j < m$ , and  $\alpha(j) := a'$  otherwise. Since this contradicts (2), we conclude  $f[U_n] \cap W'_m = \emptyset$ . Suppose now that  $f(w_i) = u'_j$ . By Lemma 4.15,  $f(z_i) \in \{u'_j, v'_j, w'_j\}$ , so (3) yields a contradiction. Thus, we have  $f[W_n] \cap U'_m = \emptyset$ .

We now prove by strong induction that for all  $0 < i \leq m$ ,

$$f(u_i) = u'_i \quad f(v_i) = v'_i \quad f(w_i) = w'_i \quad f(z_i) = z'_i.$$

Let  $i = 1$ . By Proposition 2.9, the co-root  $u_1$  of  $T_n$  must be mapped to the co-root  $u'_1$  of  $T_m$ . In other words,  $f(u_1) = u'_1$ . By Lemma 4.15,  $f(v_1) \in \{u'_1, v'_1, w'_1\}$  follows. So (2) now forces  $f(v_1) = v'_1$ . To see that  $f(w_1) = w'_1$ , notice that Lemma 4.15 and (4) already imply  $f(w_1) \in \{w'_1, v'_1\}$ , so it suffices to show that  $f(w_1) \neq v'_1$ . Suppose otherwise, i.e., that  $f(w_1) = v'_1$ . By the definition of a bi-p-morphism, we must have  $f[\downarrow w_1] = \downarrow v'_1$ . Since the aforementioned equalities  $f(u_1) = u_1$  and  $f(v_1) = v'_1$  entail  $f[\{u_1\} \cup \downarrow v_1] = \{u'_1\} \cup \downarrow v'_1$ , it now follows from the structures of both  $T_n$  and  $T_m$  that

$$f[T_n] = f[\{u_1\} \cup \downarrow v_1 \cup \downarrow w_1] = f[\{u_1\} \cup \downarrow v_1] \cup f[\downarrow w_1] = \{u'_1\} \cup \downarrow v'_1 \subsetneq T_m.$$

But this contradicts our assumption that  $f$  is surjective, hence we conclude  $f(w_1) = w'_1$ . Finally, Lemma 4.15 now implies  $f(z_1) \in \{w'_1, z'_1, \alpha(1)\}$  (where  $\alpha(1) = u'_2$  if  $1 < m$ , and  $\alpha(1) = a'$  otherwise), so (3) yields  $f(z_1) = z'_1$ , as desired.

If  $m = 1$ , we are done with our proof by induction, so let us assume otherwise. Then consider  $1 < i \leq m$  and suppose that for all  $j \in \mathbb{Z}^+$  such that  $j < i \leq m$ , the induction hypothesis holds true.

Since this entails  $f(w_{i-1}) = w'_{i-1}$ , Lemma 4.15, together with  $f[U_n] \cap W'_m = \emptyset$ , forces  $f(u_i) \in \{u'_i, z'_{i-1}\}$ . Let us show that  $u_i$  must be mapped to  $u'_i$ , by proving that  $f(u_i) = z'_{i-1}$  cannot happen. For suppose otherwise. Then  $f[\downarrow u_i] = \downarrow z'_{i-1}$ . By simply looking at the poset structure of  $T_n$  and  $T_m$ , we see that this equality implies  $a' \notin f[\downarrow u_i]$ . But our induction hypothesis and the definition of a bi-p-morphism ensure that no element in  $T_n \setminus \downarrow u_i$  is mapped to  $a'$ . It follows  $a' \notin f[\downarrow u_i] \cup f[T_n \setminus \downarrow u_i] = f[T_n]$ , contradicting our assumption that  $f$  is surjective. Thus, we must have  $f(u_i) = u'_i$ .

That  $f(w_i) = w'_i$  is proved in a very similar way:  $f(u_i) = u'_i$ , Lemma 4.15, and (4) imply  $f(w_i) \in \{v'_i, w'_i\}$ , and  $f(w_i) = v'_i$  cannot happen, since this would force  $a' \notin f[\downarrow w_i] = \downarrow v'_i$ , contradicting the surjectivity of  $f$ .

With the equalities  $f(u_i) = u'_i$  and  $f(w_i) = w'_i$  now established, we can use Lemma 4.15 together with condition (2) to show that  $f(v_i) = v'_i$ , and with condition (3) to show that  $f(z_i) = z'_i$ . This finishes our proof by induction.

As a consequence, we now know  $f(w_m) = w'_m$ . Since  $f[U_n] \cap W'_m = \emptyset$  by (4), Lemma 4.15 entails  $f(u_{m+1}) = \{a', z'_m\}$ . The same argument used above (to show that  $f(u_i) \neq z'_{i-1}$ , for all  $i \leq m$ ) ensures  $f(u_{m+1}) \neq z'_m$ . Thus, we must have  $f(u_{m+1}) = a'$ . But, by Lemma 4.15, this implies that both nontrivial predecessors of  $v_{m+1}$ , which are minimal, must be mapped to nonminimal points, a contradiction. Therefore,  $f$  cannot exist, and we showed  $T_m \not\leq T_n$ , as desired.  $\square$

By our previous discussion, the following theorem follows immediately.

**Theorem 4.17.** *The cardinality of the lattice  $\Lambda(\text{bi-LC})$  is  $2^{\aleph_0}$ .*

**4.3. Subframe formulas.** Let  $\mathfrak{A} \in \text{bi-GA}_{SI}^{<\omega}$  and introduce, for each  $a \in A$ , a fresh propositional variable  $p_a \in \text{Prop}$ . Let  $\Gamma$  be the formula describing the algebraic structure of the  $(\vee, \leftarrow)$ -reduct of  $\mathfrak{A}$ , that is,

$$\Gamma := \bigwedge \{p_{a \vee b} \leftrightarrow (p_a \vee p_b) : (a, b) \in A^2\} \wedge \bigwedge \{p_{a \leftarrow b} \leftrightarrow (p_a \leftarrow p_b) : (a, b) \in A^2\}.$$

We define the *subframe formula* of  $\mathfrak{A}$  by

$$\beta(\mathfrak{A}) := \neg \sim \Gamma \rightarrow \neg \bigwedge \{p_a \leftarrow p_b : (a, b) \in A^2 \text{ and } a \not\leq b\}.$$

In order to state the analogue of the Stable Jankov Lemma for subframe formulas, we need to introduce the notion of a  $(\vee, \leftarrow)$ -homomorphism between co-Heyting algebras, i.e., a map  $f: \mathfrak{A} \rightarrow \mathfrak{B}$  that preserves both  $\vee$  and  $\leftarrow$ . Notice that any such map must always preserve 0, since the equation  $x \leftarrow x \approx 0$  is valid on all co-Heyting algebras, and therefore

$$f(0) = f(a \leftarrow a) = f(a) \leftarrow f(a) = 0.$$

If  $f$  is moreover injective, then it is called a  $(\vee, \leftarrow)$ -embedding, denoted by  $f: \mathfrak{A} \hookrightarrow \mathfrak{B}$ .

As before, we use the facts stated in Lemma 4.2 without further reference in the following proof.

**Lemma 4.18.** (Subframe Jankov Lemma) *Let  $\mathfrak{B} \in \text{bi-GA}$ . If  $\mathfrak{A} \in \text{bi-GA}_{SI}^{<\omega}$ , then  $\mathfrak{B} \not\models \beta(\mathfrak{A})$  iff there exists a  $(\vee, \leftarrow)$ -embedding  $h: \mathfrak{A} \hookrightarrow \mathfrak{C}$ , for some  $\mathfrak{C} \in \mathbb{H}(\mathfrak{B})_{SI}$ .*

*Proof.* To show that the left to right implication holds, we can apply the argument used to prove the same direction of the Stable Jankov Lemma 4.3, but since now the formula  $\Gamma$  only describes the algebraic structure of the  $(\vee, \leftarrow)$ -reduct of  $\mathfrak{A}$ , the injective map  $h: A \rightarrow C$  is simply a  $(\vee, \leftarrow)$ -embedding, as desired.

Conversely, suppose there exists a  $(\vee, \leftarrow)$ -embedding  $h: \mathfrak{A} \hookrightarrow \mathfrak{C}$ , for some  $\mathfrak{C} \in \mathbb{H}(\mathfrak{B})_{SI}$ . Let  $v: \text{Prop} \rightarrow C$  be a valuation on  $\mathfrak{C}$  satisfying  $v(p_a) = h(a)$ , for all  $a \in A$ . We show that  $\mathfrak{C}$  refutes  $\beta(\mathfrak{A})$  via  $v$ , which is a sufficient condition for  $\mathfrak{B} \not\models \beta(\mathfrak{A})$ . Firstly, note that since  $h$  is a  $(\vee, \leftarrow)$ -homomorphism, we have for that all  $a, b \in A$ ,

$$v(p_{a \vee b}) = h(a \vee b) = h(a) \vee h(b) = v(p_a) \vee v(p_b),$$

hence  $v(p_{a \vee b} \leftrightarrow p_a \vee p_b) = 1$ . Similarly, we have

$$v(p_{a \leftarrow b}) = h(a \leftarrow b) = h(a) \leftarrow h(b) = v(p_a) \leftarrow v(p_b),$$

and thus  $v(p_{a \leftarrow b} \leftrightarrow p_a \leftarrow p_b) = 1$ . By the equalities above, we see that  $v(\Gamma) = 1$ , and therefore  $v(\neg \sim \Gamma) = \neg \sim v(\Gamma) = \neg \sim 1 = 1$ . Now, if  $a, b \in A$  are such that  $a \not\leq b$ , i.e.,  $a \leftarrow b \neq 0$ , then it follows that  $0 \neq h(a \leftarrow b) = v(p_{a \leftarrow b})$ , since  $h$  is an injective map that preserves 0. This proves that  $0 \notin \{v(p_a \leftarrow p_b) : a, b \in A \text{ and } a \not\leq b\}$ . As  $\mathfrak{C}$  is SI by assumption,  $0_{\mathfrak{C}}$  is  $\wedge$ -irreducible (see Theorem 3.6), and thus we obtain  $\bigwedge \{v(p_a \leftarrow p_b) : a, b \in A \text{ and } a \not\leq b\} \neq 0$ . Equivalently,

$$\neg \bigwedge \{v(p_a \leftarrow p_b) : a, b \in A \text{ and } a \not\leq b\} \neq 1,$$



and we conclude

$$v(\beta(\mathfrak{A})) = 1 \rightarrow \neg \bigwedge \{v(p_a \leftarrow p_b) : a, b \in A \text{ and } a \not\leq b\} \neq 1. \quad \square$$

Next we introduce partial co-Esakia morphisms, which enable us to translate the Subframe Jankov Lemma into terms of bi-Esakia spaces.

**Definition 4.19.** Let  $\mathcal{X}$  and  $\mathcal{Y}$  be co-Esakia spaces. A partial map  $f: \mathcal{X} \rightarrow \mathcal{Y}$  is a *partial co-Esakia morphism* if it satisfies the following conditions:

- (i)  $\forall x, z \in \text{dom}(f) (x \leq z \implies f(x) \leq f(z))$ ;
- (ii)  $\forall x \in \text{dom}(f), \forall y \in Y (y \leq f(x) \implies \exists z \in \downarrow x (f(z) = y))$ ;
- (iii)  $\forall x \in X (x \in \text{dom}(f) \iff \exists y \in Y (f[\downarrow x] = \downarrow y))$ ;
- (iv)  $\forall x \in X (f[\downarrow x] \in Cl(\mathcal{Y}))$ ;
- (v)  $\forall U \in CpUp(\mathcal{Y}) (\uparrow f^{-1}U \in CpUp(\mathcal{X}))$ .

**Proposition 4.20.** Let  $\mathfrak{A}$  and  $\mathfrak{B}$  be co-Heyting algebras, while  $\mathcal{X}$  and  $\mathcal{Y}$  are co-Esakia spaces.

- (i) If  $f: \mathcal{X} \rightarrow \mathcal{Y}$  is a partial co-Esakia morphism, then setting

$$f^*(U) := \uparrow f^{-1}U, \text{ for } U \in CpUp(\mathcal{Y}),$$

yields a  $(\vee, \leftarrow)$ -homomorphism  $f^*: \mathcal{Y}^* \rightarrow \mathcal{X}^*$  between co-Heyting algebras. If  $f$  is moreover surjective, then  $f^*$  is a  $(\vee, \leftarrow)$ -embedding;

- (ii) If  $h: \mathfrak{A} \rightarrow \mathfrak{B}$  is a  $(\vee, \leftarrow)$ -homomorphism between co-Heyting algebras, then setting

$$\text{dom}(h_*) := \{x \in B_* : h^{-1}x \in A_*\} \text{ and } h_*(x) := h^{-1}x, \text{ for } x \in \text{dom}(h_*),$$

yields a partial co-Esakia morphism  $h_*: \mathfrak{B}_* \rightarrow \mathfrak{A}_*$  between co-Esakia spaces. If  $h$  is moreover a  $(\vee, \leftarrow)$ -embedding, then  $h_*$  is surjective.

*Proof.* Both (i) and (ii) can be proven simply by order-dualizing the proofs of the analogous results for partial Esakia morphisms and  $(\wedge, \rightarrow)$ -homomorphisms between Heyting algebras (see, e.g., [6]).  $\square$

Before we present the Dual Subframe Lemma and some of its equivalent conditions, we need one more definition and a subsequent lemma.

**Definition 4.21.** A map  $f: \mathfrak{F} \rightarrow \mathfrak{G}$  between posets is called an *order-embedding* if it is *order-invariant*, that is, if

$$w \leq v \iff f(w) \leq f(v),$$

for all  $w, v \in \mathfrak{F}$ . In this case, we say that  $\mathfrak{F}$  *order-embeds into*  $\mathfrak{G}$  (via  $f$ ), and denote this by  $f: \mathfrak{F} \hookrightarrow \mathfrak{G}$ .

*Remark 4.22.* It is clear by definition that  $\mathfrak{F}$  order-embeds into  $\mathfrak{G}$  iff  $\mathfrak{F}$  can be viewed as a subposet of  $\mathfrak{G}$ .

**Lemma 4.23.** If  $\mathfrak{F}$  is a finite co-tree and  $\mathcal{X}$  a co-Esakia space, then  $\mathfrak{F}$  order-embeds into  $\mathcal{X}$  iff there exists a surjective partial co-Esakia morphism  $f: \mathcal{X} \twoheadrightarrow \mathfrak{F}$ .

*Proof.* This is exactly the order-dual version of [5, Thm. 3.6] (and thus we omit the proof).  $\square$

We are finally ready to translate the Subframe Jankov Lemma into the language of bi-Esakia co-forests.

**Lemma 4.24.** (Dual Subframe Jankov Lemma) If  $\mathfrak{B} \in \text{bi-GA}$  and  $\mathfrak{A} \in \text{bi-GA}_{SI}^{\leq \omega}$ , then the following conditions are equivalent:

1.  $\mathfrak{B} \not\models \beta(\mathfrak{A})$ ;
2. there exists a  $(\vee, \leftarrow)$ -embedding  $h: \mathfrak{A} \hookrightarrow \mathfrak{C}$ , for some  $\mathfrak{C} \in \text{H}(\mathfrak{B})_{SI}$ ;
3. there exists a surjective partial co-Esakia morphism  $f: \mathfrak{C}_* \twoheadrightarrow \mathfrak{A}_*$ , for some  $\mathfrak{C} \in \text{H}(\mathfrak{B})_{SI}$ ;
4.  $\mathfrak{A}_*$  order-embeds into  $\mathfrak{C}_*$ , for some  $\mathfrak{C} \in \text{H}(\mathfrak{B})_{SI}$ ;
5.  $\mathfrak{A}_*$  order-embeds into  $\mathfrak{B}_*$ ;
6. there exists a surjective partial co-Esakia morphism  $f: \mathfrak{B}_* \twoheadrightarrow \mathfrak{A}_*$ ;

7. there exists a  $(\vee, \leftarrow)$ -embedding  $h: \mathfrak{A} \rightarrow \mathfrak{B}$ .

*Proof.* The equivalence (1)  $\iff$  (2) is just the Subframe Jankov Lemma 4.18, while (2)  $\iff$  (3) follows immediately from the duality between  $(\vee, \leftarrow)$ -homomorphisms of co-Heyting algebras and partial co-Esakia morphisms stated in Proposition 4.20. Notice that this result also yields (6)  $\iff$  (7). As an immediate consequence of Lemma 4.23, we have that both (3)  $\iff$  (4) and (5)  $\iff$  (6) hold true.

Finally, to see that (4)  $\implies$  (5), let  $\mathfrak{C} \in \mathcal{H}(\mathfrak{B})_{SI}$  and note that if  $\mathfrak{A}_*$  order-embeds into a bi-generated subframe of  $\mathfrak{B}_*$ , such as  $\mathfrak{C}_*$ , then clearly  $\mathfrak{A}_*$  order-embeds into  $\mathfrak{B}_*$ .

Conversely, suppose that  $\mathfrak{A}_*$  order-embeds into  $\mathfrak{B}_*$ . Since  $\mathfrak{A}$  is nontrivial,  $\mathfrak{A}_*$  is nonempty, hence so is  $\mathfrak{B}_*$ . Therefore, we can write  $B_* = \bigsqcup_{i \in I} T_i$  as a nonempty disjoint union of maximal (with respect to inclusion) co-trees. By the definition of an order-embedding, it is clear that the co-tree  $\mathfrak{A}_*$  is mapped to a single co-tree  $T_i$ . Since we can view  $T_i$  as a bi-generated subframe of  $\mathfrak{B}_*$ , by equipping  $T_i$  with the subspace topology, we conclude by duality that (4) holds. Thus, we proved (5)  $\implies$  (4).  $\square$

Before we present some applications of subframe formulas, we need a few definitions. Let  $\mathfrak{F}$  be a co-tree and  $n \in \mathbb{Z}^+$ . If  $\mathfrak{F}$  has a chain with  $n$  elements, and all chains of  $\mathfrak{F}$  have at most  $n$  elements, we say that  $\mathfrak{F}$  has *depth*  $n$ , and write  $dp(\mathfrak{F}) = n$ . Otherwise, we say that  $\mathfrak{F}$  has *infinite depth*.

Furthermore, if  $\mathfrak{F}$  has an antichain (i.e., a subposet whose elements are pairwise incomparable) with  $n$  elements, and all antichains in  $\mathfrak{F}$  have at most  $n$  elements, we say that  $\mathfrak{F}$  has *width*  $n$ , and write  $wd(\mathfrak{F}) = n$ . Otherwise, we say that  $\mathfrak{F}$  has *infinite width*.

We prove that if  $n \in \mathbb{Z}^+$ , then the (bi-intuitionistic) logic of co-trees of depth (respectively, width) less than  $n$  can be axiomatized by a single subframe formula.

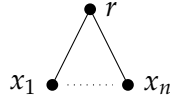


FIGURE 3. The  $n$ -co-fork  $\mathfrak{F}_n$ .

**Proposition 4.25.** Let  $n \in \mathbb{Z}^+$ ,  $\mathfrak{L}_n$  be the  $n$ -chain, and  $\mathfrak{F}_n$  be the  $n$ -co-fork (see Figure 3). If  $\mathcal{X}$  is bi-Esakia co-tree, then:

- (i)  $\mathcal{X} \models \beta(\mathfrak{L}_n^*) \iff dp(\mathcal{X}) < n$ . Equivalently,  $\text{bi-LC} + \beta(\mathfrak{L}_n^*)$  is the logic of co-trees of depth less than  $n$ ;
- (ii)  $\mathcal{X} \models \beta(\mathfrak{F}_n^*) \iff wd(\mathcal{X}) < n$ . Equivalently,  $\text{bi-LC} + \beta(\mathfrak{F}_n^*)$  is the logic of co-trees of width less than  $n$ .

*Proof.* The desired equivalences follow immediately from Lemma 4.24, noting that by the definition of an order-embedding, we clearly have that  $\mathfrak{L}_n$  does not order-embed into  $\mathcal{X}$  iff  $dp(\mathcal{X}) < n$ , while  $\mathfrak{F}_n$  does not order-embed into  $\mathcal{X}$  iff  $wd(\mathcal{X}) < n$ .

The last part of both statements are now an immediate consequence of the algebraic completeness of bi-LC and the bi-Esakia duality.  $\square$

The above result can also be used to characterize another notable class of algebras. A bounded distributive lattice is said to be *linear* if its partial order  $\leq$  is linear. Note that any such lattice can be viewed as a bi-Heyting algebra, as it has well-defined implications:

$$a \rightarrow b = \begin{cases} 1 & \text{if } a \leq b, \\ b & \text{if } b < a, \end{cases} \quad \text{and} \quad a \leftarrow b = \begin{cases} 0 & \text{if } a \leq b, \\ a & \text{if } b < a. \end{cases}$$

Thus, the terms linear distributive lattice, *linear Heyting algebra*, and *linear bi-Heyting algebra* are all equivalent. Using the bi-Esakia duality, it is easy to see that  $\mathfrak{A}$  is a nontrivial linear Heyting algebra iff (the underlying poset of its bi-Esakia dual)  $\mathfrak{A}_*$  is a nonempty chain iff  $\mathfrak{A}_*$  is a co-tree of width 1. In other words,  $\mathfrak{A}$  is an SI bi-Gödel algebra satisfying  $\beta(\mathfrak{F}_2^*)$ .

Let us denote by bi-LA the variety generated by linear bi-Heyting algebras. From the above discussion, we obtain the following.

**Theorem 4.26.** *The bi-intermediate logic of totally ordered Kripke frames is algebraized by bi-LA and coincides with bi-LC +  $\beta(\mathfrak{F}_2^*)$ .*

## 5. LOCALLY TABULAR EXTENSIONS OF bi-LC

A bi-intermediate logic  $L$  is said to be *locally tabular* if for every positive integer  $n$ , there are only finitely many non- $L$ -equivalent formulas built from the propositional variables  $p_1, \dots, p_n$ . Equivalently, if  $V_L$  is locally finite (i.e., if every finitely generated algebra in  $V_L$  is finite).

In this section we present a characterization of locally tabular extensions of bi-LC: they are exactly those which contain at least one of the Jankov formulas associated with the *finite combs* (a particular class of bi-Esakia co-trees defined below). It is an immediate consequence of this criterion that the logic of the finite combs is the only pre-locally tabular extension of bi-LC (recall that a logic is said to be *pre-locally tabular* if it is not locally tabular, but all of its proper extensions are so).

For each positive integer  $n$ , we define the  $n$ -comb  $\mathfrak{C}_n := (C_n, \leq)$  as the finite bi-Esakia co-tree depicted in Figure 4. Our aim in this section is to prove the following criterion:

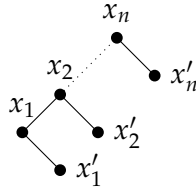


FIGURE 4. The  $n$ -comb  $\mathfrak{C}_n$ .

**Theorem 5.1.**  *$L \in \Lambda(\text{bi-LC})$  is locally tabular iff  $\mathcal{J}(\mathfrak{C}_n^*) \in L$ , for some positive integer  $n$ .*

The above theorem is proved in three steps:

- Step 1: we prove that for all  $n \in \mathbb{Z}^+$  and all  $\mathfrak{A} \in \text{bi-GA}$ ,  $\mathfrak{A} \not\models \mathcal{J}(\mathfrak{C}_n^*)$  iff  $\mathfrak{A} \not\models \beta(\mathfrak{C}_n^*)$ ;
- Step 2: we show that a variety  $V \subseteq \text{bi-GA}$  containing all the algebraic duals of the finite combs cannot be locally finite;
- Step 3: we establish the existence of a natural bound for the size of  $m$ -generated SI bi-Gödel whose bi-Esakia duals do not admit the  $n$ -comb  $\mathfrak{C}_n$  as a subposet.

The left to right implication of Step 1 is a straightforward consequence of the defining properties of the Jankov and subframe formulas established in the previous section. As for the reverse implication, we prove a sufficient condition for it to hold: “if  $\mathfrak{C}_n$  order-embeds into a bi-Esakia co-tree  $\mathcal{X}$ , then there exists a surjective bi-Esakia morphism  $f: \mathcal{X} \twoheadrightarrow \mathfrak{C}_n$ ”. The construction of the map  $f$  is very lengthy and technical, requiring a heavily combinatorial and careful manipulation of the clopens of  $\mathcal{X}$  that will be the  $f$ -preimages of the points in  $\mathfrak{C}_n$ .

Step 2 is arguably the easiest step. It can be shown by proving that  $\mathfrak{C}_n^*$  is 1-generated as a bi-Heyting algebra, for every  $n \in \mathbb{Z}^+$ . Since the finite combs are arbitrarily large bi-Esakia co-trees, the assumption that their algebraic duals are all contained in  $V$  now yields that there are arbitrarily large 1-generated algebras in  $V$ . Therefore, the 1-generated free  $V$ -algebra must be infinite, and thus  $V$  cannot be locally finite.

Step 3 makes extensive use of the Coloring Theorem 2.15. We first show that in a bi-Esakia co-forest, two particular equivalence relations must always be bi-bisimulation equivalences. By assuming that an SI bi-Gödel algebra  $\mathfrak{A}$  is  $m$ -generated and  $\mathfrak{A}_*$  does not admit  $\mathfrak{C}_n$  as a subposet, we can use the definitions of the aforementioned bi-bisimulation equivalences, together with the Coloring Theorem, to not only derive a bound for the depth of  $\mathfrak{A}_*$ , but also to establish the existence of a bound for the cardinality of  $\mathfrak{A}$ .

**5.1. Step 1.** Recall our convention that the “arrow operators” of generating upsets and downsets bind stronger than other set theoretic operations (e.g., expressions of the form  $\uparrow U \setminus V$  and  $\downarrow U \cap V$  are to be read as  $(\uparrow U) \setminus V$  and  $(\downarrow U) \cap V$ , respectively). We will start by proving two elementary lemmas about posets. A subset  $W$  of a poset is said to be *convex* if  $W = \downarrow W \cap \uparrow W$ . Since we always have  $W \subseteq \downarrow W \cap \uparrow W$ , this is equivalent to demanding that  $\downarrow W \cap \uparrow W \subseteq W$ .

**Lemma 5.2.** *Let  $\mathcal{X} = (X, \leq)$  be a poset and  $W, V \subseteq X$ . If  $W$  is convex, then:*

- (i)  $V \subseteq \downarrow W \setminus W$  implies  $V \cap \uparrow W = \emptyset$ ,
- (ii)  $V \subseteq \uparrow W \setminus W$  implies  $V \cap \downarrow W = \emptyset$ .

*Proof.* (i) Suppose, with a view towards contradiction, that  $x \in V \cap \uparrow W$ . As  $V \subseteq \downarrow W$ , it follows that  $x \in \downarrow W \cap \uparrow W$ , i.e., that  $x \in W$ , since  $W$  is convex. But then  $x \in V \cap W$ , contradicting the definition of  $V$ . Condition (ii) is proved analogously.  $\square$

**Lemma 5.3.** *Let  $\mathcal{X} = (X, \leq)$  be a poset and  $W, V \subseteq X$ . The following equivalences hold true:*

$$\downarrow W \cap \uparrow \downarrow V = \emptyset \iff \downarrow W \cap \downarrow V = \emptyset \iff \uparrow \downarrow W \cap \downarrow V = \emptyset.$$

*Proof.* By symmetry, it suffices to show the first equivalence, whose right to left implication follows immediately from the inclusion  $\downarrow V \subseteq \uparrow \downarrow V$ . We prove the reverse implication by contraposition. Suppose  $x \in \downarrow W \cap \uparrow \downarrow V$ , i.e., that  $x \in \downarrow W$  and  $y \leq x$ , for some  $y \in \downarrow V$ . Then we clearly have  $y \in \downarrow W \cap \downarrow V$ , as desired.  $\square$

For the remainder of this subsection, we fix an arbitrary positive integer  $n$  and a bi-Esakia co-tree  $\mathcal{X}$  that admits  $\mathfrak{C}_n$  as a subposet. Without loss of generality, we suppose that the co-root of  $\mathfrak{C}_n$ ,  $x_n$ , is also the co-root of  $\mathcal{X}$ . The bulk of this subsection will be dedicated to proving the next result. Notice the use of the symbol  $\uplus$  to denote the union of sets which are pairwise disjoint.

**Proposition 5.4.** *There are clopens  $X_1, \dots, X_n, X'_1, \dots, X'_n \subseteq \mathcal{X}$  satisfying:*

- (IH.i) *For all  $i \leq n$ ,  $X_i$  is a convex set containing  $x_i$ ,  $X'_i$  is a downset containing  $x'_i$ , and  $X_i \cap \uparrow \downarrow x'_j = \emptyset$  for all  $j$  such that  $i < j \leq n$ ;*
- (IH.ii)  $\downarrow X_1 = X_1 \uplus X'_1$  and  $\downarrow X_i = X_i \uplus \downarrow X_{i-1} \uplus X'_i$ , for all  $1 < i \leq n$ ;
- (IH.iii)  $\downarrow X_{i-1} \cap \uparrow X'_i = \emptyset$ , for all  $1 < i \leq n$ ;
- (IH.iv)  $\uparrow X'_1 = \uparrow X_1 \uplus X'_1$  and  $\uparrow X_i = \uparrow X_{i-1} \cap \uparrow X'_i$ , for all  $1 < i \leq n$ .

We will prove this by induction on  $m \leq n$ . Let us assume  $m = 1$ . By the properties of bi-Esakia spaces,

$$Y := \uparrow \downarrow \{x'_2, \dots, x'_n\} \cup \downarrow x'_1$$

is a closed set. Moreover, the structure of the  $n$ -comb and that of co-trees forces  $x_1 \notin Y$ . To see this, notice that if  $n = 1$  then clearly  $x_1 \notin Y = \downarrow x'_1$ , and if  $1 < i \leq n$ , then since  $\mathcal{X}$  is a co-tree,  $x_1 \in \uparrow \downarrow x'_i$  would imply either  $x_1 \leq x'_i$  or  $x'_i \leq x_1$ , contradicting the order of  $\mathfrak{C}_n$ . It follows that  $x_1$  is contained in the open set  $Y^c$ , and as bi-Esakia spaces are 0-dimensional, there exists a clopen  $V_1$  satisfying  $x_1 \in V_1 \subseteq Y^c$ . Again by the properties of bi-Esakia spaces, the following sets are all clopens:

$$U_1 := \downarrow V_1 \cap \uparrow V_1 \quad U'_1 := \downarrow U_1 \setminus U_1 \quad W_1 := \uparrow U'_1 \cap U_1 \quad W'_1 := \downarrow W_1 \setminus W_1.$$

By the above definitions, it is clear that  $\downarrow W_1 = W_1 \uplus \downarrow W'_1$ ,  $x_1 \in U_1$  and that  $x'_1 \notin U_1$ . As  $x'_1 \leq x_1$ , it follows  $x'_1 \in U'_1 = \downarrow U_1 \setminus U_1$ , hence  $x_1 \in \uparrow U'_1 \cap U_1 = W_1$ . Consequently, we have  $x'_1 \in W'_1$ . Now, to see that  $\downarrow W_1 \cap \uparrow \downarrow x'_j = \emptyset$ , for all  $j$  such that  $1 < j \leq n$ , just note that if this intersection was nonempty, then by our construction of  $W_1$  it would follow that  $V_1 \cap \uparrow \downarrow x'_j \neq \emptyset$ , contradicting the definition of  $V_1$ .

We now prove that  $\downarrow W_1 \cap \uparrow W_1 \subseteq W_1$  i.e., that  $W_1$  is convex. Suppose  $x \in \downarrow W_1 \cap \uparrow W_1$ , so there are  $y, z \in W_1 = \uparrow U'_1 \cap U_1$  satisfying  $y \leq x \leq z$ . As  $y \in \uparrow U'_1$ ,  $y \leq x$  entails  $x \in \uparrow U'_1$ , so to establish  $x \in W_1$  we only need to prove  $x \in U_1$ . By definition,  $U_1 = \downarrow V_1 \cap \uparrow V_1$ , so  $y, z \in U_1$  and  $y \leq x \leq z$  yield  $x \in \downarrow V_1 \cap \uparrow V_1 = U_1$ , as desired.

Using  $W_1 = \downarrow W_1 \cap \uparrow W_1$  and the definition of  $W'_1$ , it is easy to see that  $W'_1$  is a downset, i.e., that  $W'_1 = \downarrow W'_1$ . Consequently, we can write

$$\downarrow W_1 = W_1 \uplus \downarrow W'_1 = W_1 \uplus W'_1,$$

by above.

Finally, we prove that  $\uparrow W'_1 = \uparrow W_1 \uplus W'_1$ . To see that the sets on the right side of the equality are disjoint, just notice that  $W_1 \cap W'_1 = \emptyset$  by definition of these clopens, hence the fact that  $W_1$  is convex entails  $\uparrow W_1 \cap W'_1 = \emptyset$ , by Lemma 5.2. To establish the inclusion  $\uparrow W'_1 \subseteq \uparrow W_1 \uplus W'_1$ , suppose  $x \in \uparrow W'_1 \setminus W'_1$ , i.e., that there is  $y \in W'_1$  such that  $y \leq x$  and  $x \notin W'_1$ . Since  $W'_1 \subseteq \downarrow W_1$  by definition of  $W'_1$ , there exists  $z \in W_1$  satisfying  $y \leq z$ . As  $\mathcal{X}$  is a co-tree, we have  $y \leq x \leq z$  or  $y \leq z \leq x$ . If  $y \leq x \leq z \in W_1$ , then  $x \in \downarrow W_1 = W_1 \uplus W'_1$  by above, so  $x \notin W'_1$  yields  $x \in W_1 \subseteq \uparrow W_1$ . If  $y \leq z \leq x$ , then  $x \in \uparrow W_1$  since  $z \in W_1$ . We conclude  $\uparrow W'_1 \subseteq \uparrow W_1 \uplus W'_1$ .

To prove that the reverse inclusion holds, it suffices to show that  $\uparrow W_1 \subseteq \uparrow W'_1$ , since  $W'_1 \subseteq \uparrow W'_1$ . To this end, suppose  $x \in \uparrow W_1$ , i.e., that  $x$  lies above some  $y \in W_1 = \uparrow U'_1 \cap U_1$ . It follows that there is  $z \in U'_1$  such that  $z \leq y \leq x$ . Notice that  $U'_1 = \downarrow U_1 \setminus U_1$  entails  $z \notin U_1$ , hence  $z \notin W_1 \subseteq U_1$ . From  $z \leq y \in W_1$ , we can now infer  $z \in \downarrow W_1 \setminus W_1$ , i.e., that  $z \in W'_1$ . As  $z \leq x$ , it now follows that  $x \in \uparrow W'_1$ , as desired.

If  $n = 1$ , just set  $X_1 := W_1$  and  $X'_1 := W'_1$ , and we are done with the proof of Proposition 5.4. So, let us now assume that  $n > 1$ . Before continuing, we derive some consequences of the conditions in our induction hypothesis. They will not only help us in our proof by induction, but they are essential for proving the main result of this subsection.

**Lemma 5.5.** *Clopens  $X_1, \dots, X_m, X'_1, \dots, X'_m \subseteq \mathcal{X}$  that satisfy conditions (IH.i-iv) also satisfy the following conditions:*

1.  $\downarrow X_1 \subseteq \downarrow X_2 \subseteq \dots \subseteq \downarrow X_m$  and  $X'_i \subseteq \downarrow X_i$ , for all  $i \leq m$ ;
2.  $\uparrow X_m \subseteq \dots \subseteq \uparrow X_2 \subseteq \uparrow X_1$  and  $\uparrow X_i \subseteq \uparrow X'_i$ , for all  $i \leq m$ ;
3.  $\downarrow X_i \cap \uparrow X'_j = \uparrow \downarrow X_i \cap X'_j = \downarrow X_i \cap \uparrow X_j = \emptyset$ , for all  $i < j \leq m$ ;
4.  $\uparrow X_{i-1} \setminus X_{i-1} = \uparrow X_i = \uparrow X'_i \setminus X'_i$ , for all  $i \leq m$ ;
5. The sets  $X_1, \dots, X_m, X'_1, \dots, X'_m$  are pairwise disjoint.

*Proof.* Condition (1) is immediate from (IH.ii), while (2) clearly follows from (IH.iv).

(3) Suppose  $i < j \leq m$ . Since  $\downarrow X_i \subseteq \downarrow X_{j-1}$  by condition (1), then (IH.iii) implies

$$\emptyset = \downarrow X_i \cap \uparrow X'_j = \downarrow X_i \cap \uparrow \downarrow X'_j,$$

where the last equality follows from the fact that  $X'_j$  is a downset, by (IH.i). That the above intersections are empty already entails  $\emptyset = \uparrow \downarrow X_i \cap \downarrow X'_j = \uparrow \downarrow X_i \cap X'_j$  by Lemma 5.3, and that  $\downarrow X_i \cap \uparrow X_j = \emptyset$ , since  $\uparrow X_j \subseteq \uparrow X'_j$  by (IH.iv).

(4) Let us start by proving the inclusion  $\uparrow X_{i-1} \setminus X_{i-1} \subseteq \uparrow X_i$ . Suppose that  $x \in \uparrow X_{i-1} \setminus X_{i-1}$ , i.e.,  $x \notin X_{i-1}$  but there is  $y \in X_{i-1}$  such that  $y \leq x$ . As  $X_{i-1} \subseteq \downarrow X_i$  by (IH.ii), there must be a  $z \in X_i$  above  $y$ . As  $\mathcal{X}$  is a co-tree, we have  $y \leq z \leq x$  or  $y \leq x \leq z$ , with both cases yielding  $x \in \uparrow X_i$ . The former is clear since  $z \in X_i$ , and if  $y \leq x \leq z$ , then  $z \in X_i$  and (IH.ii) imply

$$x \in \downarrow X_i = X_i \uplus \downarrow X_{i-1} \uplus X'_i.$$

By assumption,  $x \notin X_{i-1}$  and  $x \in \uparrow X_{i-1}$ , so (IH.i) (in particular, that  $X_{i-1}$  is convex) entails  $x \notin \downarrow X_{i-1}$ . It follows that either  $x \in X_i$  or  $x \in X'_i$ . But  $x \in X'_i$  cannot happen, since  $\uparrow \downarrow X_{i-1} \cap X'_i = \emptyset$  by condition (3) proved above. Thus, we must have  $x \in X_i \subseteq \uparrow X_i$ , as desired.

To prove  $\uparrow X_i \subseteq \uparrow X_{i-1} \setminus X_{i-1}$ , notice that  $\uparrow X_i \subseteq \uparrow X_{i-1}$  by (2), so it suffices to show  $\uparrow X_i \cap X_{i-1} = \emptyset$ , which follows immediately from condition (3) proved above. The equality  $\uparrow X_i = \uparrow X'_i \setminus X'_i$  is proved analogously, hence we omit it.

Finally, by combining (IH.ii-iii) with conditions (3) and (4), it is clear that (5) is verified.  $\square$

We now resume our proof by induction. Suppose that we have clopens  $X_1, \dots, X_{m-1}, X'_1, \dots, X'_{m-1} \subseteq \mathcal{X}$  satisfying our induction hypothesis, that is, conditions (IH.i-iv). Using



a similar argument as for the base case, we can easily prove  $x'_m, x_m \notin \uparrow\downarrow\{x'_{m+1}, \dots, x'_n\}$ . Furthermore, by (IH.i), we know that  $X_{m-1} \cap \uparrow\downarrow x'_m = \emptyset$ . Notice that this equality clearly implies  $\downarrow X_{m-1} \cap \uparrow\downarrow x'_m = \emptyset$ , hence it follows from Lemma 5.3 that  $\uparrow\downarrow X_{m-1} \cap \downarrow x'_m = \emptyset$ . As  $x'_m \in \downarrow x'_m$ , we established  $x'_m \notin \uparrow\downarrow X_{m-1}$ . Since  $\mathcal{X}$  is a bi-Esakia space, there must be a clopen  $U'_m$  satisfying

$$x'_m \in U'_m \subseteq (\uparrow\downarrow\{x'_{m+1}, \dots, x'_n\} \cup \uparrow\downarrow X_{m-1})^c.$$

We know that  $x_m \in \uparrow\downarrow x'_m$  by the order of  $\mathfrak{C}_n$ , so the fact  $\downarrow X_{m-1} \cap \uparrow\downarrow x'_m = \emptyset$  proved above yields  $x_m \notin \downarrow X_{m-1}$ . But, since  $x_{m-1} \leq x_m$  and  $x_{m-1} \in X_{m-1}$ , by the structure of  $\mathfrak{C}_n$  and (IH.i), respectively, it follows  $x_m \in \uparrow X_{m-1}$ . As  $U'_m \cap \uparrow\downarrow X_{m-1} = \emptyset$  by definition of  $U'_m$ , we must have  $x_m \notin \downarrow U'_m$ , since otherwise there would exist  $y \in U'_m$  satisfying  $x_m \leq y$ , and therefore  $y \in U'_m \cap \uparrow X_{m-1} \subseteq U'_m \cap \uparrow\downarrow X_{m-1}$ , a contradiction. We conclude that there exists a clopen  $U_m$  such that

$$x_m \in U_m \subseteq (\uparrow\downarrow\{x'_{m+1}, \dots, x'_n\} \cup \downarrow X_{m-1} \cup \downarrow U'_m)^c.$$

As  $\mathcal{X}$  is a bi-Esakia space, the following sets are all clopens:

- $W_m := \downarrow U_m \cap \uparrow X_{m-1} \cap \uparrow\downarrow U'_m$ ;
- $W'_m := \downarrow U'_m \cap \downarrow W_m$ ;
- $W_i := X_i \cap \downarrow W_m$ , for  $i \leq m-1$ ;
- $W'_i := X'_i \cap \downarrow W_m$ , for  $i \leq m-1$ .

The following lemma establishes some crucial properties of our newly defined clopens.

**Lemma 5.6.** *The following conditions hold:*

1.  $W_m = \downarrow U_m \cap \uparrow W_{m-1} \cap \uparrow\downarrow W'_m$ ;
2.  $W_i$  is convex, for all  $i \leq m$ ;
3.  $\downarrow W_{m-1} \cap \uparrow W'_m = \downarrow W_{m-1} \cap W'_m = \uparrow\downarrow W_{m-1} \cap W'_m = \emptyset$ ;
4. For all  $i < m$ ,

$$\downarrow W_i = \downarrow X_i \cap \downarrow W_m \quad \text{and} \quad \downarrow W'_i = \downarrow X'_i \cap \downarrow W_m;$$

5. For all  $i < m$ ,

$$\uparrow W_i = (\uparrow X_i \cap \downarrow W_m) \cup \uparrow W_m \quad \text{and} \quad \uparrow W'_i = (\uparrow X'_i \cap \downarrow W_m) \cup \uparrow W_m.$$

*Proof.* (1) The right to left inclusion is clear, since by the definitions of  $W_{m-1}$  and  $W'_m$ , we have  $\uparrow W_{m-1} \subseteq \uparrow X_{m-1}$  and  $\uparrow W'_m \subseteq \uparrow\downarrow U'_m$ , hence

$$\downarrow U_m \cap \uparrow W_{m-1} \cap \uparrow\downarrow W'_m \subseteq \downarrow U_m \cap \uparrow X_{m-1} \cap \uparrow\downarrow U'_m = W_m.$$

To prove the reverse inclusion, let  $x \in W_m = \downarrow U_m \cap \uparrow X_{m-1} \cap \uparrow\downarrow U'_m$ , so there are  $y \in X_{m-1}$  and  $z \in \downarrow U'_m$  such that  $y, z \leq x$ . As  $x \in W_m$ , we have both

$$y \in X_{m-1} \cap \downarrow W_m = W_{m-1} \quad \text{and} \quad z \in \downarrow U'_m \cap \downarrow W_m = W'_m.$$

Since  $x \in \downarrow U_m$  by definition of  $W_m$ , it is now clear that  $x \in \downarrow U_m \cap \uparrow W_{m-1} \cap \uparrow\downarrow W'_m$ , as desired.

(2) That  $W_m$  is convex follows immediately from the easily checked fact that the intersection of a downset and an upset is always convex.

Now, for  $i < m$ , notice that if  $x \in \downarrow W_i \cap \uparrow W_i$ , there are  $y, z \in W_i = X_i \cap \downarrow W_m$  satisfying  $y \leq x \leq z$ . As  $X_i = \downarrow X_i \cap \uparrow X_i$  by (IH.i), it follows that  $x \in X_i$ , and since  $z \in \downarrow W_m$ , we conclude  $x \in X_i \cap \downarrow W_m = W_i$ . This shows  $\downarrow W_i \cap \uparrow W_i \subseteq W_i$ , i.e., that  $W_i$  is convex, as desired.

(3) We prove that  $\downarrow W_{m-1} \cap \uparrow\downarrow W'_m = \emptyset$ , and therefore that

$$\downarrow W_{m-1} \cap \downarrow W'_m = \emptyset = \uparrow\downarrow W_{m-1} \cap \downarrow W'_m,$$

by Lemma 5.3. As  $W'_m$  is the intersection of two downsets, and thus itself a downset, the above three intersections are exactly those we want to prove to be empty. We suppose, with a view towards contradiction, that  $x \in \downarrow W_{m-1} \cap \uparrow\downarrow W'_m = \downarrow W_{m-1} \cap \uparrow W'_m$ , i.e., that there are  $y \in W'_m$  and  $z \in W_{m-1}$  satisfying  $y \leq x \leq z$ . As  $W'_m \subseteq \downarrow U'_m$  and  $W_{m-1} \subseteq X_{m-1}$ ,  $y \leq z$  entails  $y \in \downarrow U'_m \cap \downarrow X_{m-1}$ . It follows that  $U'_m \cap \uparrow\downarrow X_{m-1} \neq \emptyset$ , contradicting the definition of  $U'_m \subseteq (\uparrow\downarrow X_{m-1})^c$ , as desired.

(4) Let us first prove  $\downarrow W_i = \downarrow X_i \cap \downarrow W_m$ , for  $i < m$ . The left to right inclusion is clear, since  $W_i = X_i \cap W_m$ . For the reverse inclusion, suppose that  $x \in \downarrow X_i \cap \downarrow W_m$ , i.e., that there are  $z \in X_i$

and  $y \in W_m$  such that  $x \leq y, z$ . As  $\mathcal{X}$  is a co-tree, we have  $x \leq z \leq y$  or  $x \leq y \leq z$ . The former clearly yields  $x \in \downarrow W_i$ , since in this case we have  $z \in X_i \cap \downarrow W_m = W_i$ . We now prove that  $x \leq y \leq z$  cannot happen. For if this was the case, we would have  $z \in X_i \cap \uparrow W_m \subseteq \downarrow X_i \cap \uparrow W_m$ , hence Lemma 5.5.(1) (in particular, that  $\downarrow X_i \subseteq \downarrow X_{m-1}$ ) yields

$$\downarrow X_{m-1} \cap \uparrow W_m \neq \emptyset.$$

Since  $W_m \subseteq \uparrow \downarrow U'_m$  by definition of  $W_m$ , it now follows that  $\downarrow X_{m-1} \cap \uparrow \downarrow U'_m \neq \emptyset$ . By Lemma 5.3, this is equivalent to  $\uparrow \downarrow X_{m-1} \cap \downarrow U'_m \neq \emptyset$ , which clearly implies  $\uparrow \downarrow X_{m-1} \cap U'_m \neq \emptyset$ , contradicting the definition of  $U'_m \subseteq (\uparrow \downarrow X_{m-1})^c$ .

To prove the nontrivial inclusion of  $\downarrow W'_i = \downarrow X'_i \cap \downarrow W_m$ , i.e., that  $\downarrow X'_i \cap \downarrow W_m \subseteq \downarrow W'_i$ , suppose that  $x \leq z, y$ , for some  $z \in X'_i$  and  $y \in W_m$ . As  $\mathcal{X}$  is a co-tree, we have either  $x \leq z \leq y$  or  $x \leq y \leq z$ . The former case immediately yields  $x \in \downarrow W'_i = \downarrow (X'_i \cap \downarrow W_m)$ , and the latter again yields a contradiction. To see this, notice that we would then have  $z \in \downarrow X_i \cap \uparrow W_m$ , since  $X'_i \subseteq \downarrow X_i$  by (IH.ii), but we just proved above that  $\downarrow X_i \cap \uparrow W_m \neq \emptyset$  cannot happen.

(5) We start by proving

$$\uparrow W_i \subseteq (\uparrow X_i \cap \downarrow W_m) \cup \uparrow W_m.$$

Suppose that  $x \in \uparrow W_i = \uparrow (X_i \cap \downarrow W_m)$ , so there is  $y \in X_i \cap \downarrow W_m$  such that  $y \leq x$ . As  $y \in \downarrow W_m$ , we have  $y \leq z$  for some  $z \in W_m$ . Since  $\mathcal{X}$  is a co-tree, it follows that  $y \leq x \leq z$  or  $y \leq z \leq x$ . If  $y \leq x \leq z$ , then  $x \in \uparrow X_i \cap \downarrow W_m$ , and if  $y \leq z \leq x$ , then  $x \in \uparrow W_m$ . Since both cases yield  $x \in (\uparrow X_i \cap \downarrow W_m) \cup \uparrow W_m$ , we are done.

Now, for the other inclusion, suppose that  $x \in \uparrow W_m$ , i.e., that  $z \leq x$  for some  $z \in W_m$ . By definition of this set, we know  $W_m \subseteq \uparrow X_{m-1}$ . Furthermore, by applying Lemma 5.5.(2) we get the inclusion  $W_m \subseteq \uparrow X_{m-1} \subseteq \uparrow X_i$ . Hence, there is  $y \in X_i \cap \downarrow W_m = W_i$  satisfying  $y \leq z \leq x$ , and therefore  $x \in \uparrow W_i$ . It is an easy to see that  $\uparrow X_i \cap \downarrow W_m \subseteq \uparrow W_i$ , thus establishing the desired inclusion.

That  $\uparrow W'_i = (\uparrow X'_i \cap \downarrow W_m) \cup \uparrow W_m$  is proved analogously.  $\square$

In order to satisfy (IH.ii), we would like to have the following equality:

$$\downarrow W_m = W_m \uplus \downarrow W_{m-1} \uplus W'_m.$$

Notice that  $W'_m$  is a downset by definition, and that  $W_m \subseteq \uparrow W_{m-1} \cap \uparrow W'_m$  by Lemma 5.6.(1). Consequently, from condition (3) of the same lemma we can infer that the 3 clopens on the right side of the above equality are in fact pairwise disjoint. To see this, notice that  $\downarrow W_{m-1} \cap \uparrow W'_m = \emptyset$  yields that  $\downarrow W_{m-1}$  is disjoint from  $W'_m$  and from  $W_m \subseteq \uparrow W'_m$ , and that  $\uparrow \downarrow W_{m-1} \cap W'_m = \emptyset$  yields that  $W'_m$  is disjoint from  $W_m \subseteq \uparrow W_{m-1}$ .

Although  $W_m \uplus \downarrow W_{m-1} \uplus W'_m \subseteq \downarrow W_m$  follows immediately from the definitions of  $W_{m-1}$  and  $W'_m$ , nothing in our construction ensures that the reverse inclusion holds true. In other words, there can be points lying strictly below  $W_m$  which are not contained in  $\downarrow W_{m-1} \cup W'_m$ , that is, the set  $\downarrow W_m \setminus (W_m \cup \downarrow W_{m-1} \cup W'_m)$  can be nonempty. We will now characterize these *inconvenient* points.

**Definition 5.7.** A point  $x \in \downarrow W_m \setminus (W_m \cup \downarrow W_{m-1} \cup W'_m)$  is a *2-point* if

$$\downarrow x \cap \downarrow W_{m-1} \neq \emptyset \neq \downarrow x \cap W'_m.$$

We will prove that 2-points (they are called as such because their downsets intersect with both  $\downarrow W_{m-1}$  and  $W'_m$ ) do not exist. But first, let us show a helpful equivalence.

**Lemma 5.8.** If  $x \in \downarrow W_m \setminus (W_m \cup \downarrow W_{m-1} \cup W'_m)$ , then

$$\downarrow x \cap \downarrow W_{m-1} \neq \emptyset \iff \downarrow x \cap W_{m-1} \neq \emptyset$$

*Proof.* The right to left implication is clear since  $W_{m-1} \subseteq \downarrow W_{m-1}$ . For the other direction, suppose that  $y \in \downarrow x \cap \downarrow W_{m-1}$ . As  $y \in \downarrow W_{m-1}$ , there exists  $z \in W_{m-1}$  such that  $y \leq z$ , and since  $\mathcal{X}$  is a co-tree, we have  $y \leq x \leq z$  or  $y \leq z \leq x$ . But  $y \leq x \leq z$  cannot happen, since  $z \in W_{m-1}$  and  $x \notin \downarrow W_{m-1}$  by assumption. Thus, we must have  $y \leq z \leq x$ , and it now follows that  $z \in \downarrow x \cap W_{m-1}$ , as desired.  $\square$

Now, let us suppose that  $x$  is a 2-point, so, in particular, we have  $x \in \downarrow W_m \setminus W_m$ . Using the previous lemma, our assumption on  $x$  yields

$$\downarrow x \cap W_{m-1} \neq \emptyset \neq \downarrow x \cap W'_m,$$

i.e., that  $x \in \uparrow W_{m-1} \cap \uparrow W'_m$ . As  $W_m \subseteq \downarrow U_m$  by definition of  $W_m$ ,  $x \in \downarrow W_m$  now entails

$$x \in \downarrow U_m \cap \uparrow W_{m-1} \cap \uparrow W'_m = W_m,$$

where the equality above follows from Lemma 5.6.(1). But this contradicts our assumption  $x \notin W_m$ . We conclude that 2-points do not exist, as desired.

**Definition 5.9.** A point  $x \in \downarrow W_m \setminus (W_m \cup \downarrow W_{m-1} \cup W'_m)$  is a *1-point* if

$$\downarrow x \cap \downarrow W_{m-1} \neq \emptyset = \downarrow x \cap W'_m.$$

Equivalently, if

$$\downarrow x \cap W_{m-1} \neq \emptyset = \downarrow x \cap W'_m.$$

The next lemma provides an equivalent characterization of 1-points, which are called as such because their downsets intersect with only one of  $\downarrow W_{m-1}$  and  $W'_m$  (namely, with  $\downarrow W_{m-1}$ ).

**Lemma 5.10.** (*1-point Lemma*) If  $x \in \mathcal{X}$ , then  $x$  is a 1-point iff  $x \in \uparrow W_{m-1} \setminus (W_{m-1} \cup \uparrow W_m)$ .

*Proof.* Let us first prove the left to right implication. Suppose that  $x$  is a 1-point, so by definition we have  $x \in \downarrow W_m \setminus (W_m \cup \downarrow W_{m-1})$  and  $\downarrow x \cap W_{m-1} \neq \emptyset$ , i.e., that  $x \in \uparrow W_{m-1}$ . This already shows that  $x \in \uparrow W_{m-1} \setminus W_{m-1}$ , so it remains to prove that  $x \notin \uparrow W_m$ . But we know  $x \in \downarrow W_m \setminus W_m$  by above, so the fact that  $W_m$  is convex proved in Lemma 5.6.(2) forces  $x \notin \uparrow W_m$ , by Lemma 5.2.

To prove the reverse implication, let us assume  $x \in \uparrow W_{m-1} \setminus (W_{m-1} \cup \uparrow W_m)$ . Since this already entails  $x \notin W_{m-1} \cup W_m$  and  $\downarrow x \cap W_{m-1} \neq \emptyset$ , to establish  $x$  as a 1-point it remains to show  $x \in \downarrow W_m \setminus W'_m$  and  $\downarrow x \cap W'_m = \emptyset$ . Equivalently, that  $x \in \downarrow W_m \setminus \uparrow W'_m$ . Since  $x \in \uparrow W_{m-1}$ , there exists  $y \in W_{m-1}$  such that  $y \leq x$ . By the definition of  $W_{m-1} = X_{m-1} \cap \downarrow W_m$ , we have  $y \leq z$  for some  $z \in W_m$ . As  $\mathcal{X}$  is a co-tree, it follows that  $y \leq z \leq x$  or  $y \leq x \leq z$ . The former cannot happen, since  $z \in W_m$  and by hypothesis  $x \notin \uparrow W_m$ . Hence, we have  $y \leq x \leq z$ , and we proved  $x \in \downarrow W_m$ . To see that  $x \notin \uparrow W'_m$ , just notice that  $W_m \subseteq \downarrow U_m$  by definition, so  $x \leq z \in W_m$  entails  $x \in \downarrow U_m$ . As  $x \in \uparrow W_{m-1}$  and  $x \notin W_m$  by assumption, the fact  $W_m = \downarrow U_m \cap \uparrow W_{m-1} \cap \uparrow W'_m$  proved in Lemma 5.6.(1) clearly implies  $x \notin \uparrow W'_m$ .  $\square$

In analogy with the previous definition, we define the 1'-points and provide an equivalent characterization for them, whose proof we skip since it uses a very similar argument to the one detailed above. Notice that 1'-points are called as such because they intersect with only one of  $\downarrow W_{m-1}$  and  $W'_m$  (namely, with  $W'_m$ ).

**Definition 5.11.** A point  $x' \in \downarrow W_m \setminus (W_m \cup \downarrow W_{m-1} \cup W'_m)$  is a *1'-point* if

$$\downarrow x' \cap \downarrow W_{m-1} = \emptyset \neq \downarrow x' \cap W'_m.$$

**Lemma 5.12.** (*1'-point Lemma*) If  $x' \in \mathcal{X}$ , then  $x'$  is a 1'-point iff  $x' \in \uparrow W'_m \setminus (W'_m \cup \uparrow W_m)$ .

The following auxiliary lemma provides some sufficient conditions for a point in  $\mathcal{X}$  to be a 1-point or a 1'-point. They will not only help us in the characterization of the inconvenient points, that is, the points in  $\downarrow W_m \setminus (W_m \cup \downarrow W_{m-1} \cup W'_m)$ , but we can also use these conditions to deduce easily that 1-points and 1'-points are incomparable.

**Lemma 5.13.** *The following conditions hold, for any  $y \in \mathcal{X}$ :*

1. If  $y \in (\downarrow W_m \setminus W_m) \cap \uparrow x$  or  $y \in (\uparrow W_{m-1} \setminus W_{m-1}) \cap \downarrow x$  for some 1-point  $x$ , then  $y$  is also a 1-point;
2. If  $y \in (\downarrow W_m \setminus W_m) \cap \uparrow x'$  or  $y \in (\uparrow W'_m \setminus W'_m) \cap \downarrow x'$  for some 1'-point  $x'$ , then  $y$  is also a 1'-point;
3. 1-points and 1'-points are incomparable.

*Proof.* (1) Suppose that  $y \in (\downarrow W_m \setminus W_m) \cap \uparrow x$ , for some 1-point  $x$ . We prove that  $y$  is contained in  $\uparrow W_{m-1} \setminus (W_{m-1} \cup \uparrow W_m)$ , i.e., that  $y$  is a 1-point, by Lemma 5.10. As  $W_m$  is convex by Lemma 5.6.(2), our assumption  $y \in \downarrow W_m \setminus W_m$  already entails  $y \notin \uparrow W_m$ , by Lemma 5.2. Furthermore, as 1-points are contained in  $\uparrow W_{m-1}$  by definition,  $x \leq y$  yields  $y \in \uparrow W_{m-1}$ . Finally, to see that

$y \notin W_{m-1}$ , notice that otherwise we would have  $x \in \downarrow W_{m-1}$ , contradicting the assumption that  $x$  is a 1-point. We conclude that  $y$  is indeed a 1-point, as desired.

Suppose now that  $y \in (\uparrow W_{m-1} \setminus W_{m-1}) \cap \downarrow x$  for some 1-point  $x$ . By the 1-point Lemma 5.10, we know that  $x \notin \uparrow W_m$ . Consequently, we must have  $y \notin \uparrow W_m$ , since we assumed  $y \leq x$ . It follows that  $y \in \uparrow W_{m-1} \setminus (W_{m-1} \cup \uparrow W_m)$ , i.e., that  $y$  is a 1-point, by the aforementioned lemma.

(2) This is proved analogously to condition (1) above, but in place of the fact that  $W_{m-1}$  is convex, we use that  $W'_m$  is a downset.

(3) Let  $x$  be a 1-point and  $x'$  a 1'-point. Recall that, by definition, both points lie in  $\downarrow W_m \setminus W_m$ . If  $x \leq x'$ , then  $x' \in (\downarrow W_m \setminus W_m) \cap \uparrow x$ , so  $x'$  is a 1-point by condition (1) proved above. If  $x' \leq x$ , then  $x \in (\downarrow W_m \setminus W_m) \cap \uparrow x'$ , hence  $x$  is a 1'-point by condition (2) proved above. Both cases yield a contradiction, since the definitions of 1-points and 1'-points are clearly mutually exclusive.  $\square$

We will now define the last class of points that we will need to fully characterize the set of inconvenient points  $\downarrow W_m \setminus (W_m \cup \downarrow W_{m-1} \cup W'_m)$ . They are called 0-points because, unsurprisingly, their downsets do not intersect with  $\downarrow W_{m-1}$  nor with  $W'_m$ .

**Definition 5.14.** A point  $x \in \downarrow W_m \setminus (W_m \cup \downarrow W_{m-1} \cup W'_m)$  is a 0-point if

$$\downarrow x \cap \downarrow W_{m-1} = \emptyset = \downarrow x \cap W'_m,$$

and  $x$  does not lie below a 1-point, nor below a 1'-point.

*Remark 5.15.* It is clear by the definition above that points lying below 0-points must also be 0-points.

We can now use the mutually exclusive nature of the definitions above (just notice which intersections are empty and which ones are not in their respective definitions) to fully characterize the points in  $\downarrow W_m \setminus (W_m \cup \downarrow W_{m-1} \cup W'_m)$ . They are, and this will be proved shortly, of five distinct forms:

- 1-points;
- points which are not 1-points but lie below a 1-point;
- 1'-points;
- points which are not 1'-points but lie below a 1'-point;
- 0-points.

Our solution to “deal” with these inconvenient points is the following: 1-points will be added to  $W_{m-1}$ ; points which are not 1-points but lie below one will be added to  $W'_{m-1}$ ; 1'-points, their downsets, and 0-points will all be added to  $W'_m$ .

Note that by the 1-point Lemma 5.10, the set of 1-points can be written as

$$Z := \uparrow W_{m-1} \setminus (W_{m-1} \cup \uparrow W_m),$$

and is therefore a clopen. Note as well that the set of inconvenient points which are not 1-points but lie below a 1-point takes the form of

$$\downarrow Z \setminus (Z \cup \downarrow W_{m-1}),$$

which is clearly a clopen as well. Finally, let us denote the set of 1'-points by  $Z'$  and the set of 0-points by  $Z_0$ . As  $\mathcal{X}$  is a bi-Esakia space, the following sets are all clopens:

- $V_{m-1} := W_{m-1} \cup Z$ ;
- $V'_{m-1} := W'_{m-1} \cup [\downarrow Z \setminus (Z \cup \downarrow W_{m-1})]$ ;
- $V'_m := \downarrow W_m \setminus (W_m \cup \downarrow V_{m-1})$ .

We will now prove that the definition of the set  $V'_m$  complies with our “solution” stated above, that is,  $V'_m$  is simply  $W'_m$  together with all the 0-points and all the points in the downsets of 1'-points.

**Lemma 5.16.** *The following equality holds:*

$$V'_m = W'_m \cup \downarrow Z' \cup Z_0.$$

*Proof.* We start by proving the left to right inclusion. Suppose that  $x \in V'_m$  i.e., that  $x \in \downarrow W_m \setminus (W_m \cup \downarrow V_{m-1})$ . If  $x \in W'_m$  we are done, so let us assume otherwise. It now follows from our assumption on  $x$  and the definition of  $V_{m-1} = W_{m-1} \cup Z$  that

$$x \in \downarrow W_m \setminus (W_m \cup \downarrow W_{m-1} \cup W'_m).$$

Since we proved that 2-points do not exist,  $\downarrow x \cap \downarrow W_{m-1} \neq \emptyset \neq \downarrow x \cap W'_m$  cannot happen. If

$$\downarrow x \cap \downarrow W_{m-1} \neq \emptyset = \downarrow x \cap W'_m,$$

then  $x$  would satisfy the definition of a 1-point, i.e.,  $x \in Z$ . But this contradicts our assumption  $x \notin \downarrow V_{m-1}$ , since  $Z \subseteq V_{m-1}$ . If

$$\downarrow x \cap \downarrow W_{m-1} = \emptyset \neq \downarrow x \cap W'_m,$$

then  $x$  satisfies the definition of a 1'-point, i.e.,  $x \in Z'$ , and we are done. Finally, suppose

$$\downarrow x \cap \downarrow W_{m-1} = \emptyset = \downarrow x \cap W'_m.$$

Since the case  $x \in \downarrow Z'$  is immediate, let us assume otherwise, i.e., that  $x$  does not lie below a 1'-point. Since we also assumed  $x \notin \downarrow V_{m-1} = \downarrow (W_{m-1} \cup Z)$ , in particular, that  $x$  cannot lie below a 1-point, it now follows that  $x$  satisfies the definition of a 0-point, i.e.,  $x \in Z_0$ , and we are done. We conclude  $V'_m \subseteq W'_m \cup \downarrow Z' \cup Z_0$ , as desired.

We now prove that  $W'_m \cup \downarrow Z' \cup Z_0 \subseteq V'_m = \downarrow W_m \setminus (W_m \cup \downarrow V_{m-1})$ . Let us start by showing

$$W'_m \cup \downarrow Z' \cup Z_0 \subseteq \downarrow W_m \setminus W_m.$$

That  $W'_m \subseteq \downarrow W_m \setminus W_m$  was already proved above (see the comment before the definition of 2-points), and that  $\downarrow Z' \cup Z_0 \subseteq \downarrow W_m \setminus W_m$  follows immediately from the definitions of 1'-points and 0-points.

To establish the desired inclusion, it remains to show  $W'_m \cup \downarrow Z' \cup Z_0 \subseteq \downarrow W_m \setminus \downarrow V_{m-1}$ . As we already know  $W'_m \cup \downarrow Z' \cup Z_0 \subseteq \downarrow W_m$ , it suffices to show

$$(W'_m \cup \downarrow Z' \cup Z_0) \cap \downarrow V_{m-1} = \emptyset$$

We will prove this by noting that

$$(W'_m \cup \downarrow Z' \cup Z_0) \cap \downarrow V_{m-1} = (W'_m \cap \downarrow V_{m-1}) \cup (\downarrow Z' \cap \downarrow V_{m-1}) \cup (Z_0 \cap \downarrow V_{m-1}),$$

and showing that the three intersections on the right side of the above equality are all empty. Recall that  $\downarrow V_{m-1} = \downarrow W_{m-1} \cup \downarrow Z$ , by definition. That  $W'_m \cap (\downarrow W_{m-1} \cup \downarrow Z) = \emptyset$  follows from Lemma 5.6.(3) and the definition of 1'-points. To see that  $\downarrow Z' \cap (\downarrow W_{m-1} \cup \downarrow Z) = \emptyset$ , recall that  $\downarrow Z \cap \downarrow W_{m-1}$  follows immediately from the definition of 1'-points, and note that as 1-points and 1'-points are incomparable (see Lemma 5.13.(3)), the fact that  $\mathcal{X}$  is a co-tree forces  $\downarrow Z \cap \downarrow Z' = \emptyset$ . Finally, that  $Z_0 \cap (\downarrow W_{m-1} \cup \downarrow Z) = \emptyset$  is immediate from the definition of 0-points.  $\square$

We are finally ready to finish our proof by induction. To improve the readability of what follows and for ease of reference, below we re-label some of our clopens, restating their definitions as well as some useful equalities:

- $V_i := W_i = X_i \cap \downarrow W_m$ , for all  $i < m - 1$ ;
- $V'_i := W'_i = X'_i \cap \downarrow W_m$ , for all  $i < m - 1$ ;
- $W_{m-1} = X_{m-1} \cap \downarrow W_m$ ;
- $W'_{m-1} = X'_{m-1} \cap \downarrow W_m$ ;
- $W'_m = \downarrow U'_m \cap \downarrow W_m$ ;
- $V_{m-1} = W_{m-1} \cup Z$ ;
- $V'_{m-1} = W'_{m-1} \cup [\downarrow Z \setminus (Z \cup \downarrow W_{m-1})]$ ;
- $V'_m = \downarrow W_m \setminus (W_m \cup \downarrow V_{m-1}) = W'_m \cup \downarrow Z' \cup Z_0$ ;
- $V_m = W_m = \downarrow U_m \cap \uparrow X_{m-1} \cap \uparrow \downarrow U'_m = \downarrow U_m \cap \uparrow W_{m-1} \cap \uparrow W'_m$ .

Recall that  $x_i \in U_i$ ,  $x'_i \in U'_i$  and that  $U'_i \subseteq (\uparrow \downarrow x'_j)^c$ , for all  $i < j \leq n$ . These facts will be used repeatedly in the next proof.



**Proposition 5.17.** *The clopens  $V_1, \dots, V_m, V'_1, \dots, V'_m \subseteq \mathcal{X}$  satisfy condition (IH.i), that is, for all  $i \leq m$ ,  $V_i$  is a convex set containing  $x_i$ ,  $V'_i$  is a downset containing  $x'_i$ , and  $V_i \cap \uparrow \downarrow x'_j = \emptyset$  for all  $j$  such that  $i < j \leq n$ .*

*Proof.* Firstly, we show that the statement holds when  $i = m$ . As  $x_{m-1} \in X_{m-1}$  by (IH.i) and  $x'_m \in U'_m$  by definition of  $U'_m$ , it follows  $x_m \in \uparrow X_{m-1} \cap \uparrow \downarrow U'_m$ , since  $x'_m \leq x_m$  by the order of  $\mathfrak{C}_n$ . As we also know  $x_m \in U_m$  by the definition of this set, we indeed have

$$x_m \in \downarrow U_m \cap \uparrow X_{m-1} \cap \uparrow \downarrow U'_m = W_m = V_m.$$

Furthermore, that this set is convex was already established in Lemma 5.6.(2).

We stated above that  $x'_m \in U'_m$ , so  $x'_m \leq x_m \in W_m$  now yields  $x'_m \in \downarrow U'_m \cap \downarrow W_m = W'_m$ . As  $W'_m \subseteq V'_m$  by Lemma 5.16, it follows  $x'_m \in V'_m$ . Note as well that this lemma also implies that  $V'_m$  is a downset, since it is characterized as the union of three downsets (recall that  $W'_m = \downarrow U'_m \cap \downarrow W_m$  and  $\downarrow Z'$  are downsets by definition, and that  $Z_0$  being a downset is an immediate consequence of the definitions of 0-points, since any point lying below a 0-point must also be a 0-point).

Finally, that  $\downarrow V_m \cap \uparrow \downarrow x'_j = \emptyset$  for all  $j$  such that  $m < j \leq n$ , follows easily from the definitions of  $V_m$  and  $U_m$ , since  $V_m = W_m \subseteq \downarrow U_m$  and  $U_m \subseteq (\uparrow \downarrow x'_j)^c$ .

Suppose now that  $i < m - 1$ . By (IH.i), we know that  $x_i \in X_i$  and  $x'_i \in X'_i$ . As  $x_m \in W_m$  by above, the order of  $\mathfrak{C}_n$  yields both  $x_i \in X_i \cap \downarrow W_m = V_i$  and  $x'_i \in X'_i \cap \downarrow W_m = V_i$ . Moreover, that  $V_i = W_i$  is convex was already established in Lemma 5.6.(2), and since  $X'_i$  is a downset by (IH.i), it is clear that  $V'_i = X'_i \cap \downarrow W_m$  is also a downset. It remains to show  $V_i \cap \uparrow \downarrow x'_j = \emptyset$  for all  $j$  such that  $i < j \leq n$ , which follows easily from the definition  $V_i = X_i \cap \downarrow W_m \subseteq X_i$  and from  $X_i \cap \uparrow \downarrow x'_j = \emptyset$  by (IH.i).

It remains to consider the case  $i = m - 1$ . By an argument similar to the one detailed for the previous case, we can easily show that  $x_{m-1} \in W_{m-1}$  and  $x'_{m-1} \in W'_{m-1}$ . As  $W_{m-1} \subseteq V_{m-1}$  and  $W'_{m-1} \subseteq V'_{m-1}$  by the definitions of  $V_{m-1}$  and  $V'_{m-1}$ , we have  $x_{m-1} \in V_{m-1}$  and  $x'_{m-1} \in V'_{m-1}$ . To see that  $V_{m-1} = W_{m-1} \cup Z$  is convex, i.e., that  $\downarrow V_{m-1} \cap \uparrow V_{m-1} \subseteq V_{m-1}$ , recall that  $W_{m-1}$  is convex by Lemma 5.6.(2) and that  $Z \subseteq \uparrow W_{m-1}$  by the definition of 1-points. It follows that

$$\begin{aligned} \downarrow V_{m-1} \cap \uparrow V_{m-1} &= (\downarrow W_{m-1} \cup \downarrow Z) \cap (\uparrow W_{m-1} \cup \uparrow Z) = (\downarrow W_{m-1} \cup \downarrow Z) \cap \uparrow W_{m-1} \\ &= (\downarrow W_{m-1} \cap \uparrow W_{m-1}) \cup (\downarrow Z \cap \uparrow W_{m-1}) = W_{m-1} \cup (\downarrow Z \cap \uparrow W_{m-1}). \end{aligned}$$

Thus, to show that  $V_{m-1}$  is convex, it suffices to prove the inclusion  $\downarrow Z \cap \uparrow W_{m-1} \subseteq W_{m-1} \cup Z = V_{m-1}$ . As  $Z$  is the set of 1-points, this inclusion is an immediate consequence of Lemma 5.13.(1), and we are done.

We now prove that  $V'_{m-1} = W'_{m-1} \cup [\downarrow Z \setminus (Z \cup \downarrow W_{m-1})]$  is a downset. Recall that  $X'_{m-1}$  is a downset by (IH.i), hence  $W'_{m-1} = X'_{m-1} \cap \downarrow W_m$  is also a downset. To see that  $\downarrow Z \setminus (Z \cup \downarrow W_{m-1})$  is a downset as well, we suppose otherwise and arrive at a contradiction. Assume that there are  $x$  and  $y$  satisfying  $x \in \downarrow Z \setminus (Z \cup \downarrow W_{m-1})$ ,  $y \notin \downarrow Z \setminus (Z \cup \downarrow W_{m-1})$ , and  $y \leq x$ . Notice that there must exist  $z \in Z$  such that  $x \leq z$ , hence  $y \leq x \leq z$  forces  $y \in \downarrow Z$ . This, together with our assumption on  $y$ , yields  $y \in Z \cup \downarrow W_{m-1}$ . If  $y \in Z$ , then we have  $x \in \downarrow Z \cap \uparrow Z$ . But we know that  $Z$  is convex (this is an immediate consequence of Lemma 5.13.(1)), so it follows  $x \in Z$ , a contradiction. If  $y \in \downarrow W_{m-1}$ , then there is  $w \in W_{m-1}$  such that  $y \leq w$ . Since  $\mathcal{X}$  is a co-tree,  $y \leq w, x$  entails  $y \leq x \leq w$  or  $y \leq x \leq w$ . The former cannot happen, since  $x \notin \downarrow W_{m-1}$  by assumption, so we must have  $y \leq w \leq x$ . It follows  $x \in (\uparrow W_{m-1} \setminus W_{m-1}) \cap \downarrow z$ , and since  $z$  is a 1-point, Lemma 5.13.(1) now yields that  $x$  is a 1-point, i.e., that  $x \in Z$ , another contradiction. We conclude that no such  $y$  can exist, that is, that  $\downarrow Z \setminus (Z \cup \downarrow W_{m-1})$  is a downset. Thus,  $V'_{m-1}$  is the union of two downsets, and therefore a downset, as desired.

Finally, we prove that  $\downarrow V_{m-1} \cap \uparrow \downarrow x'_j = \emptyset$ , for all  $j$  such that  $m - 1 < j \leq n$ . By the definition of  $V_{m-1}$ , we have

$$\downarrow V_{m-1} \cap \uparrow \downarrow x'_j = \downarrow (W_{m-1} \cup Z) \cap \uparrow \downarrow x'_j = (\downarrow W_{m-1} \cup \downarrow Z) \cap \uparrow \downarrow x'_j.$$

Since  $W_{m-1} = X_{m-1} \cap \downarrow W_m \subseteq X_{m-1}$  and we know  $X_{m-1} \cap \uparrow \downarrow x'_j = \emptyset$  by (IH.i), it suffices to show

$$\downarrow Z \cap \uparrow \downarrow x'_j = \emptyset.$$

If  $j > m$ , then this already follows from the equality  $V_m \cap \uparrow \downarrow x'_j = \emptyset$  proved above, since  $\downarrow Z \subseteq \downarrow W_m = \downarrow V_m$  by the definition of 1-points. If  $j = m$ , then as  $x'_m$  is contained in the downset  $W'_m$  by above, we can use the definition of 1-points, in particular, that  $\downarrow Z \cap W'_m = \emptyset$ , to conclude  $\downarrow Z \cap \uparrow \downarrow x'_m = \emptyset$ .  $\square$

In the next result, we will use the conventions  $W_0, X_0, W'_0, X'_0 \in \{\emptyset\}$ .

**Proposition 5.18.** *The clopens  $V_1, \dots, V_m, V'_1, \dots, V'_m \subseteq \mathcal{X}$  satisfy condition (IH.ii), that is, for all  $i \leq m$ , we have  $\downarrow V_i = V_i \uplus \downarrow V_{i-1} \uplus V'_i$ .*

*Proof.* We first show that  $\downarrow W_i = W_i \uplus \downarrow W_{i-1} \uplus W'_i$ , for all  $i \leq m-1$ . Recall that in this case, we have  $W_i = X_i \cap \downarrow W_m$  and  $W'_i = X'_i \cap \downarrow W_m$ , by the definitions of  $W_i$  and  $W'_i$ , respectively. Recall as well that  $\downarrow W_i = \downarrow X_i \cap \downarrow W_m$  and  $\downarrow W_{i-1} = \downarrow X_{i-1} \cap \downarrow W_m$  follow from Lemma 5.6.(4), and that we assumed  $\downarrow X_i = X_i \uplus \downarrow X_{i-1} \uplus X'_i$  in (IH.ii). Compiling all of these equalities yields

$$\begin{aligned} W_i \cup \downarrow W_{i-1} \cup W'_i &= (X_i \cap \downarrow W_m) \cup (\downarrow X_{i-1} \cap \downarrow W_m) \cup (X'_i \cap \downarrow W_m) \\ &= (X_i \cup \downarrow X_{i-1} \cup X'_i) \cap \downarrow W_m = \downarrow X_i \cap \downarrow W_m \\ &= \downarrow W_i. \end{aligned}$$

As the clopens  $X_i, \downarrow X_{i-1}, X'_i$  are pairwise disjoint by (IH.ii), and since they contain  $W_i, \downarrow W_{i-1}, W'_i$ , respectively, we conclude

$$\downarrow W_i = W_i \uplus \downarrow W_{i-1} \uplus W'_i \tag{5}$$

for all  $i \leq m-1$ , as desired. Since, by definition, we have  $V_j = W_j$  and  $V'_j = W'_j$  for all  $j < m-1$ , we just proved that the statement holds when  $i < m-1$ .

Next we prove the case where  $i = m-1$ , that is, we establish the equality

$$\downarrow V_{m-1} = V_{m-1} \uplus \downarrow V_{m-2} \uplus V'_{m-1}.$$

Equivalently (using our previous notation), we prove

$$\downarrow (W_{m-1} \cup Z) = (W_{m-1} \cup Z) \uplus \downarrow W_{m-2} \uplus (W'_{m-1} \cup [\downarrow Z \setminus (Z \cup \downarrow W_{m-1})]).$$

Using the equality  $\downarrow W_{m-1} = W_{m-1} \uplus \downarrow W_{m-2} \uplus W'_{m-1}$ , a particular instance of (5) proved above, we can re-write

$$\begin{aligned} (W_{m-1} \cup Z) \cup \downarrow W_{m-2} \cup (W'_{m-1} \cup [\downarrow Z \setminus (Z \cup \downarrow W_{m-1})]) &= \\ (W_{m-1} \cup \downarrow W_{m-2} \cup W'_{m-1}) \cup (Z \cup [\downarrow Z \setminus (Z \cup \downarrow W_{m-1})]) &= \\ \downarrow W_{m-1} \cup (Z \cup [\downarrow Z \setminus (Z \cup \downarrow W_{m-1})]). \end{aligned}$$

Since we clearly have  $\downarrow (W_{m-1} \cup Z) = \downarrow W_{m-1} \cup \downarrow Z = \downarrow W_{m-1} \cup Z \cup [\downarrow Z \setminus (Z \cup \downarrow W_{m-1})]$ , the above display now entails

$$\downarrow (W_{m-1} \cup Z) = (W_{m-1} \cup Z) \cup \downarrow W_{m-2} \cup (W'_{m-1} \cup [\downarrow Z \setminus (Z \cup \downarrow W_{m-1})]).$$

To finish the case  $i = m-1$ , it remains to show that the three sets on the right side of the above equality are pairwise disjoint. We again rely on

$$\downarrow W_{m-1} = W_{m-1} \uplus \downarrow W_{m-2} \uplus W'_{m-1}.$$

Not only this equality yields that  $W_{m-1}, \downarrow W_{m-2}$ , and  $W'_{m-1}$  are pairwise disjoint, but also that  $\downarrow W_{m-2} \subseteq \downarrow W_{m-1}$ . Since 1-points (i.e., the elements of  $Z$ ) are by definition inconvenient points (i.e., the elements of  $\downarrow W_m \setminus (W_m \cup \downarrow W_{m-1} \cup \downarrow W'_m)$ ), the previous inclusion ensures  $Z \cap \downarrow W_{m-2} = \emptyset$ . This proves  $(W_{m-1} \cup Z) \cap \downarrow W_{m-2} = \emptyset$ . Again using the inclusion  $\downarrow W_{m-2} \subseteq \downarrow W_{m-1}$ , we infer that

$$\downarrow W_{m-2} \cap [\downarrow Z \setminus (Z \cup \downarrow W_{m-1})] = \emptyset,$$

and since  $\downarrow W_{m-2} \cap W'_{m-1} = \emptyset$  was already established above, we conclude

$$\downarrow W_{m-2} \cap (W'_{m-1} \cup [\downarrow Z \setminus (Z \cup \downarrow W_{m-1})]) = \emptyset.$$

To see that  $(W_{m-1} \cup Z) \cap (W'_{m-1} \cup [\downarrow Z \setminus (Z \cup \downarrow W_{m-1})]) = \emptyset$ , notice that:  $W_{m-1}$  and  $W'_{m-1}$  are disjoint by above;  $W_{m-1}$  and  $Z$  are clearly disjoint from  $\downarrow Z \setminus (Z \cup \downarrow W_{m-1})$ ; and that  $Z \cap \downarrow W_{m-1} = \emptyset$  by the definition of 1-points, so the fact  $W'_{m-1} \subseteq \downarrow W_{m-1}$  (an immediate consequence of (5)) now yields  $Z \cap W'_{m-1} = \emptyset$ . Concluding, and returning to our current notation, we proved

$$\downarrow V_{m-1} = V_{m-1} \uplus \downarrow V_{m-2} \uplus V'_{m-1}.$$

Finally, the case where  $i = m$  follows immediately from the definitions. Recall that  $V_m = W_m$ ,  $V_{m-1} = W_{m-1} \cup Z$ ,  $W_{m-1} = X_{m-1} \cap \downarrow W_m$ ,  $V'_m = \downarrow W_m \setminus (W_m \cup \downarrow V_{m-1})$ , and that  $Z \subseteq \downarrow W_m$  by the definition of 1-points. This ensures

$$\downarrow W_m = W_m \cup \downarrow (W_{m-1} \cup Z) \cup (\downarrow W_m \setminus [W_m \cup \downarrow (W_{m-1} \cup Z)]),$$

that is,

$$\downarrow V_m = V_m \cup \downarrow V_{m-1} \cup V'_m.$$

That  $V'_m = \downarrow W_m \setminus (W_m \cup \downarrow V_{m-1})$  is disjoint from both  $W_m$  (i.e., from  $V_m$ ) and  $V_{m-1}$  is clear. To see that  $\downarrow V_{m-1} \cap V_m = \emptyset$ , i.e., that  $\downarrow (W_{m-1} \cup Z) \cap W_m = \emptyset$ , recall that  $\downarrow W_{m-1} \cap W_m = \emptyset$  is an immediate consequence of Lemma 5.6.(3) (since  $W_m \subseteq \uparrow W'_m$ ), and that  $\downarrow Z \cap W_m = \emptyset$  by the definition of 1-points. Thus, we have

$$\downarrow V_m = V_m \uplus \downarrow V_{m-1} \uplus V'_m. \quad \square$$

**Proposition 5.19.** *The clopens  $V_1, \dots, V_m, V'_1, \dots, V'_m \subseteq \mathcal{X}$  satisfy condition (IH.iii), that is, for all  $1 < i \leq m$ , we have  $\downarrow V_{i-1} \cap \uparrow V'_i = \emptyset$ .*

*Proof.* The case  $1 < i < m - 1$  follows immediately from the definitions and (IH.iii), since  $V_{i-1} = W_{i-1} = X_{i-1} \cap \downarrow W_m$ ,  $V'_i = W'_i = X'_i \cap \downarrow W_m$ , and  $\downarrow X_{i-1} \cap \uparrow X'_i = \emptyset$ .

Suppose that  $i = m - 1$ . As  $V_{m-2} = W_{m-2}$  and  $V'_{m-1} = W'_{m-1} \cup [\downarrow Z \setminus (Z \cup \downarrow W_{m-1})]$  by their definitions, what we want to prove is

$$\downarrow W_{m-2} \cap \uparrow (W'_{m-1} \cup [\downarrow Z \setminus (Z \cup \downarrow W_{m-1})]) = \emptyset.$$

Using the same argument as for the previous case, we can easily show that  $\downarrow W_{m-2}$  is disjoint from  $\uparrow W'_{m-1}$ . Therefore, it only remains to prove

$$\downarrow W_{m-2} \cap \uparrow [\downarrow Z \setminus (Z \cup \downarrow W_{m-1})] = \emptyset.$$

Let us assume otherwise, so there are  $w \in W_{m-2}$  and  $z \in \downarrow Z \setminus (Z \cup \downarrow W_{m-1})$  satisfying  $z \leq x \leq w$ , for some  $x$ . In particular, we have  $z \leq w$ . Using the inclusion  $\downarrow W_{m-2} \subseteq \downarrow W_{m-1}$  established in the proof of the previous result,  $z \leq w$  and  $w \in W_{m-2}$  yield  $z \in \downarrow W_{m-1}$ , contradicting our assumption  $z \notin \downarrow W_{m-1}$ . We conclude that there is no such  $x$ , i.e., that

$$\downarrow V_{m-2} \cap \uparrow V'_{m-1} = \downarrow W_{m-2} \cap \uparrow (W'_{m-1} \cup [\downarrow Z \setminus (Z \cup \downarrow W_{m-1})]) = \emptyset,$$

as desired.

Finally, we show that  $\downarrow V_{m-1} \cap \uparrow V'_m = \emptyset$ . Using the definition of  $V_{m-1}$  and the equality proved in Lemma 5.16, that the previous intersection is empty can be written as

$$\downarrow (W_{m-1} \cup Z) \cap \uparrow (W'_m \cup \downarrow Z' \cup Z_0) = \emptyset.$$

Equivalently,

$$(\downarrow W_{m-1} \cup \downarrow Z) \cap (\uparrow W'_m \cup \uparrow \downarrow Z' \cup \uparrow Z_0) = \emptyset.$$

That  $\downarrow W_{m-1}$  is disjoint from  $\uparrow W'_m$  was already established in Lemma 5.6.(3). To see that  $\downarrow W_{m-1} \cap \uparrow Z_0 = \emptyset$ , suppose otherwise, so there are  $w \in W_{m-1}$  and  $z \in Z_0$  satisfying  $z \leq x \leq w$ , for some  $x$ . This implies  $z \in \downarrow W_{m-1}$ , a contradiction, since 0-points (i.e., the elements of  $Z_0$ ) are defined to be in  $\downarrow W_m \setminus (W_m \cup \downarrow W_{m-1} \cup W'_m)$ . By Lemma 5.3,  $\downarrow W_{m-1} \cap \uparrow \downarrow Z' = \emptyset$  iff  $\downarrow W_{m-1} \cap \downarrow Z' = \emptyset$ , and the latter equality is immediate from the definition of 1'-points, since the downsets generated by the points in  $Z'$  are disjoint from  $\downarrow W_{m-1}$ . All of this establishes

$$\downarrow W_{m-1} \cap (\uparrow W'_m \cup \uparrow \downarrow Z' \cup \uparrow Z_0) = \emptyset,$$

so it remains to show

$$\downarrow Z \cap (\uparrow W'_m \cup \uparrow \downarrow Z' \cup \uparrow Z_0) = \emptyset.$$

Again using Lemma 5.3, and the fact that  $W'_m$  is a downset, we see that  $\downarrow Z \cap \uparrow W'_m = \emptyset$  iff  $\downarrow Z \cap W'_m = \emptyset$ . Just note that the latter equality is clear, since the downsets generated by 1-points are defined to be disjoint from  $W'_m$ . The aforementioned lemma also yields the equivalence  $\downarrow Z \cap \uparrow \downarrow Z' = \emptyset$  iff  $\downarrow Z \cap \downarrow Z' = \emptyset$ , whose right side condition can be easily deduced using the fact that 1-points and 1'-points are incomparable (see Lemma 5.13) together with the co-tree structure of  $\mathcal{X}$ . Finally, to establish that  $\downarrow Z$  and  $\uparrow Z_0$  are disjoint, simply recall that by definition, 0-points do not lie below 1-points. Therefore, we have proved

$$\downarrow Z \cap (\uparrow W'_m \cup \uparrow \downarrow Z' \cup \uparrow Z_0) = \emptyset,$$

and we are done.  $\square$

**Proposition 5.20.** *The clopens  $V_1, \dots, V_m, V'_1, \dots, V'_m \subseteq \mathcal{X}$  satisfy condition (IH.iv), that is, we have*

$$\uparrow V'_1 = \uparrow V_1 \uplus V'_1 \text{ and } \uparrow V_i = \uparrow V_{i-1} \cap \uparrow V'_i,$$

for all  $1 < i \leq m$ .

*Proof.* Let us start by noting that for all  $1 < i \leq m-1$ , we have

$$\begin{aligned} \uparrow W_{i-1} \cap \uparrow W'_i &= ((\uparrow X_{i-1} \cap \downarrow W_m) \cup \uparrow W_m) \cap ((\uparrow X'_i \cap \downarrow W_m) \cup \uparrow W_m) \\ &= ((\uparrow X_{i-1} \cap \downarrow W_m) \cap (\uparrow X'_i \cap \downarrow W_m)) \cup \uparrow W_m \\ &= ((\uparrow X_{i-1} \cap \uparrow X'_i) \cap \downarrow W_m) \cup \uparrow W_m \\ &= (\uparrow X_i \cap \downarrow W_m) \cup \uparrow W_m \\ &= \uparrow W_i, \end{aligned} \tag{6}$$

where the first and last equalities follow from Lemma 5.6.(5), the second and third follow from the distributivity of the set theoretic operations, while the fourth uses our assumption (IH.iv), namely, that  $\uparrow X_{i-1} \cap \uparrow X'_i = \uparrow X_i$ . As  $V_j = W_j$  and  $V'_j = V'_j$  for all  $j < m-1$ , we just proved  $\uparrow V_i = \uparrow V_{i-1} \cap \uparrow V'_i$  for all  $1 < i < m-1$ .

We now prove that  $\uparrow V_{m-1} = \uparrow V_{m-2} \cap \uparrow V'_{m-1}$ . The left to right inclusion is straightforward. Just note that  $Z \subseteq \uparrow W_{m-1}$  by the 1-point Lemma 5.10, so using the definition of  $V_{m-1} = W_{m-1} \cup Z$  yields

$$\uparrow V_{m-1} = \uparrow (W_{m-1} \cup Z) = \uparrow W_{m-1}.$$

Consequently, to show that the desired inclusion holds it suffices to prove  $\uparrow W_{m-1} \subseteq \uparrow V_{m-2} \cap \uparrow V'_{m-1}$ . Just recall that, by their respective definitions, we have  $V_{m-2} = W_{m-2}$  and  $W'_{m-1} \subseteq V'_{m-1}$ , hence the previous inclusion is now immediate from the equality

$$\uparrow W_{m-1} = \uparrow W_{m-2} \cap \uparrow W'_{m-1},$$

a particular instance of (6) proved above.

To see that  $\uparrow V_{m-2} \cap \uparrow V'_{m-1} \subseteq \uparrow V_{m-1}$ , let us suppose that  $x \in \uparrow V_{m-2} \cap \uparrow V'_{m-1}$ . Note that by Proposition 5.19, we know that  $\downarrow V_{m-2} \cap \uparrow V'_{m-1} = \emptyset$ . As  $V'_{m-1}$  is a downset by Proposition 5.17, the previous equality is equivalent to  $\downarrow V_{m-2} \cap \uparrow \downarrow V'_{m-1} = \emptyset$ , which in turn is equivalent to  $\uparrow \downarrow V_{m-2} \cap \downarrow V'_{m-1} = \emptyset$  by Lemma 5.3. In particular, this implies that if  $x \in \uparrow V_{m-2} \cap \uparrow V'_{m-1}$ , then  $x \notin V'_{m-1}$ , so we must have  $x \in \uparrow V'_{m-1} \setminus V'_{m-1}$ .

We now prove that  $\uparrow V'_{m-1} \setminus V'_{m-1} \subseteq \uparrow V_{m-1}$ , thus establishing our desired inclusion  $\uparrow V_{m-2} \cap \uparrow V'_{m-1} \subseteq \uparrow V_{m-1}$ . If  $x \in \uparrow V'_{m-1} \setminus V'_{m-1}$ , then there exists  $y \in V'_{m-1}$  such that  $y \leq x$ . As  $V'_{m-1} \subseteq \downarrow V_{m-1}$  by Proposition 5.18, there must be a  $z \in V_{m-1}$  such that  $y \leq z$ . Since  $\mathcal{X}$  is a co-tree, we have  $y \leq z \leq x$  or  $y \leq x \leq z$ , and both possibilities yield  $x \in \uparrow V_{m-1}$ . The former is clear, since  $z \in V_{m-1}$ . Assuming  $y \leq x \leq z$  entails  $x \in \downarrow V_{m-1}$ . Equivalently,

$$x \in V_{m-1} \uplus \downarrow V_{m-2} \uplus V'_{m-1}$$

by Proposition 5.18. By hypothesis,  $x \in \uparrow V'_{m-1} \setminus V'_{m-1}$ , so we not only know  $x \notin V'_{m-1}$ , but we can also infer  $x \notin \downarrow V_{m-2}$ , since  $\downarrow V_{m-2} \cap \uparrow V'_{m-1} = \emptyset$  by Proposition 5.19. Thus, we must have  $x \in V_{m-1} \subseteq \uparrow V_{m-1}$ , as desired. We conclude

$$\uparrow V_{m-1} = \uparrow V_{m-2} \cap \uparrow V'_{m-1}.$$

To finish the proof of the second part of the statement, it remains to show that

$$\uparrow V_m = \uparrow V_{m-1} \cap \uparrow V'_m.$$

Let us recall the characterization of  $V_m = W_m = \downarrow U_m \cap \uparrow W_{m-1} \cap \uparrow W'_m$  given in Lemma 5.6.(1), the definition of  $V_{m-1} = W_{m-1} \cup Z$  and the equality  $V'_m = W'_m \cup \downarrow Z' \cup Z_0$  established in Lemma 5.16. Notice that from the three previous equalities it is clear that the desired left to right inclusion holds true, so let us prove the reverse inclusion.

By the 1-point Lemma 5.10, we know that  $Z \subseteq \uparrow W_{m-1}$ . Similarly, the 1'-point Lemma 5.12 yields  $Z' \subseteq \uparrow W'_m$ . Consequently, we have both  $\uparrow Z \subseteq \uparrow W_{m-1}$  and  $\uparrow Z' \subseteq \uparrow W'_m$ , hence what we need to prove can be written as

$$\begin{aligned} \uparrow V_{m-1} \cap \uparrow V'_m &= \uparrow (W_{m-1} \cup Z) \cap \uparrow (W'_m \cup \downarrow Z' \cup Z_0) \\ &= \uparrow W_{m-1} \cap (\uparrow W'_m \cup \uparrow \downarrow Z' \cup \uparrow Z_0) \\ &\subseteq \uparrow W_m = \uparrow V_m. \end{aligned}$$

To see that  $\uparrow W_{m-1} \cap \uparrow W'_m \subseteq \uparrow W_m$ , suppose that  $x \in \uparrow W_{m-1} \cap \uparrow W'_m$ , so there exists  $w \in W_{m-1}$  such that  $w \leq x$ . As  $W_{m-1} = X_{m-1} \cap \downarrow W_m$  by the definition of  $W_{m-1}$ , there must be a  $y \in W_m$  satisfying  $w \leq y$ . So  $w \leq x, y$  yields  $x \leq y$  or  $y \leq x$ , since  $\mathcal{X}$  is a co-tree. If  $y \leq x$  we are done, as this implies  $x \in \uparrow W_m$ . If  $x \leq y$ , then notice that  $y \in W_m = \downarrow U_m \cap \uparrow W_{m-1} \cap \uparrow W'_m$  entails  $x \in \downarrow U_m$ . This, together with our assumption  $x \in \uparrow W_{m-1} \cap \uparrow W'_m$ , yields  $x \in \downarrow U_m \cap \uparrow W_{m-1} \cap \uparrow W'_m = W_m \subseteq \uparrow W_m$ , as desired.

Let us now prove that  $\uparrow W_{m-1} \cap \uparrow \downarrow Z' \subseteq \uparrow W_m$ . Suppose  $x \in \uparrow W_{m-1} \cap \uparrow \downarrow Z'$ , so, in particular, there are  $z' \in Z'$  and  $y \in \downarrow z'$  satisfying  $y \leq z', x$ . Since  $\mathcal{X}$  is a co-tree, this entails  $x \leq z'$  or  $z' \leq x$ . But  $x \leq z'$  cannot happen, as our assumption  $x \in \uparrow W_{m-1}$  would then force  $z' \in \uparrow W_{m-1}$ , contradicting the definition of 1'-points (more specifically, that their downsets are disjoint from  $W_{m-1}$ ). Thus, we must have  $z' \leq x$ . As mentioned above, the 1'-point Lemma 5.12 ensures  $z' \in \uparrow W'_m$ , so  $z' \leq x$  now implies  $x \in \uparrow W'_m$ . This, together with our assumption  $x \in \uparrow W_{m-1}$ , yields  $x \in \uparrow W_{m-1} \cap \uparrow W'_m$ . Since we proved in the previous paragraph that  $\uparrow W_{m-1} \cap \uparrow W'_m \subseteq \uparrow W_m$ , we are done.

It remains to show  $\uparrow W_{m-1} \cap \uparrow Z_0 \subseteq \uparrow W_m$ . Take  $x \in \uparrow W_{m-1} \cap \uparrow Z_0$  and suppose, with a view towards contradiction, that  $x \notin \uparrow W_m$ . Recall that  $Z_0 \subseteq \downarrow W_m$  by the definition of 0-points, so  $x \in \uparrow Z_0$  implies that there are  $z \in Z_0$  and  $w \in W_m$  such that  $z \leq x, w$ . As  $\mathcal{X}$  is a co-tree, we have  $x \leq w$  or  $w \leq x$ . But  $w \in W_m$  and we assumed  $x \notin \uparrow W_m$ , so we must have  $x \leq w$ . This shows  $x \in \downarrow W_m \setminus W_m$ . Furthermore, that  $z \leq x$  and  $z \in Z_0$  forces  $x \notin \downarrow W_{m-1} \cup W'_m$ , since 0-points are defined to lie outside of this union (recall that  $W'_m = \downarrow U'_m \cup \downarrow W_m$  is a downset by definition).

We have proved

$$x \in \downarrow W_m \setminus (W_m \cup \downarrow W_{m-1} \cup W'_m),$$

i.e., that  $x$  is an inconvenient point. This, together with our assumption  $x \in \uparrow W_{m-1}$ , implies  $\downarrow x \cap \downarrow W_{m-1} \neq \emptyset$ , by Lemma 5.8. Recall that 2-points, that is, inconvenient points satisfying

$$\downarrow u \cap \downarrow W_{m-1} \neq \emptyset \neq \downarrow u \cap W'_m,$$

do not exist. Therefore, it follows that  $x$  is an inconvenient point satisfying

$$\downarrow x \cap \downarrow W_{m-1} \neq \emptyset = \downarrow x \cap W'_m,$$

i.e.,  $x$  satisfies the definition of a 1-point. But now our assumption  $x \in \uparrow Z_0$  yields a contradiction, since by definition, no 0-point lies below a 1-point. We conclude that  $x \in \uparrow W_m = \uparrow V_m$ , as desired. Therefore, we have established

$$\uparrow V_m = \uparrow V_{m-1} \cap \uparrow V'_m,$$

and finished the proof that the second part of the statement holds.



Finally, let us show that  $\uparrow V'_1 = \uparrow V_1 \uplus V'_1$ . Notice that the definitions of  $V_1$  and  $V'_1$  depend on whether  $m > 2$  or  $m = 2$ . If  $m > 2$ , we need to prove that  $\uparrow V'_1 = \uparrow W'_1 = \uparrow W_1 \uplus W'_1$ . Using the definitions of these clopens, together with Lemma 5.6.(5) and (IH.iv), we have

$$\begin{aligned}\uparrow W'_1 &= (\uparrow X'_1 \cap \downarrow W_m) \cup \uparrow W_m = ((\uparrow X_1 \cup X'_1) \cap \downarrow W_m) \cup \uparrow W_m \\ &= ((\uparrow X_1 \cap \downarrow W_m) \cup (X'_1 \cap \downarrow W_m)) \cup \uparrow W_m \\ &= ((\uparrow X_1 \cap \downarrow W_m) \cup \uparrow W_m) \cup (X'_1 \cap \downarrow W_m) = \uparrow W_1 \cup W'_1.\end{aligned}$$

To see that  $\uparrow W_1$  and  $W'_1$  are in fact disjoint, notice that  $W'_1 \subseteq \downarrow W_1 \setminus W_1$  by Proposition 5.18 and that  $W_1$  is convex by Proposition 5.17, so  $\uparrow W_1 \cap W'_1 = \emptyset$  follows from Lemma 5.2. We have proved  $\uparrow V_1 = \uparrow V_1 \uplus V'_1$  if  $m > 2$ , as desired.

Suppose now that  $m = 2$ . Recall that we proved above  $\uparrow V'_{m-1} \setminus V'_{m-1} \subseteq \uparrow V_{m-1}$ . Since we are assuming that  $m = 2$ , this means  $\uparrow V'_1 \setminus V'_1 \subseteq \uparrow V_1$ , and it is now clear that  $\uparrow V'_1 \subseteq \uparrow V_1 \cup V'_1$ . To prove the reverse inclusion, noting that clearly  $V'_1 \subseteq \uparrow V'_1$ , it suffices to show  $\uparrow V_1 \subseteq \uparrow V'_1$ , i.e., that  $\uparrow(W_1 \cup Z) \subseteq \uparrow V'_1$ . Equivalently, that  $\uparrow W_1 \subseteq \uparrow V'_1$ , since  $Z \subseteq \uparrow W_1$  by the 1-point Lemma 5.10. By using the same argument as above, we can show that  $\uparrow W'_1 = \uparrow W_1 \cup W'_1$ , and thus infer  $\uparrow W_1 \subseteq \uparrow W'_1 \subseteq \uparrow V'_1$ , as desired. We have proved that the equality  $\uparrow V'_1 = \uparrow V_1 \cup V'_1$  holds, so it remains to show that the sets on the right side are disjoint. Just notice that as  $V'_1 \subseteq \downarrow V_1 \setminus V_1$  by Proposition 5.18 and  $V_1$  is convex by Proposition 5.17,  $\uparrow V_1 \cap V'_1 = \emptyset$  follows from Lemma 5.2.  $\square$

With the four previous results now proven, we have finished the induction step of our proof by induction, thus showing that Proposition 5.4 holds true. We are finally ready to prove the main result of this subsection:

**Theorem 5.21.** *Let  $n \in \mathbb{Z}^+$ . If  $\mathcal{X}$  is a bi-Esakia co-tree, then  $\mathcal{X}$  admits the  $n$ -comb  $\mathfrak{C}_n$  as a subposet iff  $\mathfrak{C}_n$  is a bi-Esakia morphic image of  $\mathcal{X}$ .*

*Proof.* We first prove the left to right implication. Suppose that  $\mathcal{X}$  admits  $\mathfrak{C}_n$  as a subposet. Without loss of generality, we can assume that the co-root of  $\mathfrak{C}_n$ ,  $x_n$ , is identified with the co-root of  $\mathcal{X}$ . By Proposition 5.4, there are clopens  $X_1, \dots, X_n, X'_1, \dots, X'_n \subseteq \mathcal{X}$  satisfying:

- (i) For all  $i \leq n$ ,  $X_i$  is a convex set containing  $x_i$ ,  $X'_i$  is a downset containing  $x'_i$ , and  $X_i \cap \uparrow \downarrow x'_j = \emptyset$  for all  $j$  such that  $i < j \leq n$ ;
- (ii)  $\downarrow X_1 = X_1 \uplus X'_1$  and  $\downarrow X_i = X_i \uplus \downarrow X_{i-1} \uplus X'_i$ , for all  $1 < i \leq n$ ;
- (iii)  $\downarrow X_{i-1} \cap \uparrow X'_i = \emptyset$ , for all  $1 < i \leq n$ ;
- (iv)  $\uparrow X'_1 = \uparrow X_1 \uplus X'_1$  and  $\uparrow X_i = \uparrow X_{i-1} \cap \uparrow X'_i$ , for all  $1 < i \leq n$ .

Notice that as  $x_n$  is the co-root of  $\mathcal{X}$ ,  $x_n \in X_n$  entails  $\downarrow X_n = \mathcal{X}$ . By successive applications of condition (ii) to this equality, we have

$$\begin{aligned}\mathcal{X} &= \downarrow X_n = X_n \uplus \downarrow X_{n-1} \uplus X'_n = (X_n \uplus X'_n) \uplus \downarrow X_{n-1} \\ &= (X_n \uplus X'_n) \uplus (X_{n-1} \uplus \downarrow X_{n-2} \uplus X'_{n-1}) \\ &= (X_n \uplus X'_n) \uplus (X_{n-1} \uplus X'_{n-1}) \uplus \downarrow X_{n-2} \\ &= (X_n \uplus X_{n-1} \uplus X'_n \uplus X'_{n-1}) \uplus \downarrow X_{n-2} \\ &= \dots = \biguplus_{i=1}^n X_i \uplus \biguplus_{i=1}^n X'_i.\end{aligned}$$

We define the map  $f: \mathcal{X} \rightarrow \mathfrak{C}_n$  by

$$f(z) := \begin{cases} x_i & \text{if } z \in X_i, \\ x'_i & \text{if } z \in X'_i, \end{cases}$$

and prove that it is a surjective bi-Esakia morphism. That  $f$  is well-defined follows immediately from  $\mathcal{X} = \biguplus_{i=1}^n X_i \uplus \biguplus_{i=1}^n X'_i$ . Moreover, since for each  $i \leq n$ ,  $X_i$  and  $X'_i$  are clopens containing  $x_i$  and  $x'_i$ , respectively, it is clear by definition that  $f$  is both continuous and surjective.

We now prove that  $f$  is order preserving. As  $X_n$  contains the co-root of  $\mathcal{X}$ ,  $x_n$ , then for any  $w \in \uparrow X_n$ , since we always have  $w \leq x_n$ , it follows  $w \in \uparrow X_n \cap \downarrow X_n$ . But  $X_n$  is convex by condition (i), hence we proved  $\uparrow X_n \subseteq X_n$ , and it is now clear that  $X_n = \uparrow X_n$ . This fact will be used to establish the last equality of the display below, whose other (non trivial) equalities follow from successive applications of Lemma 5.5.(4), in particular, of the fact that  $\uparrow X_j \setminus X_j = \uparrow X_{j+1}$ , for all  $j < n$ . For each  $i \leq n$ , we have

$$\begin{aligned} \uparrow X_i &= X_i \uplus (\uparrow X_i \setminus X_i) = X_i \uplus \uparrow X_{i+1} \\ &= X_i \uplus (X_{i+1} \uplus (\uparrow X_{i+1} \setminus X_{i+1})) \\ &= X_i \uplus X_{i+1} \uplus \uparrow X_{i+2} \\ &= X_i \uplus X_{i+1} \uplus (X_{i+2} \uplus (\uparrow X_{i+2} \setminus X_{i+2})) \\ &= X_i \uplus X_{i+1} \uplus X_{i+2} \uplus \uparrow X_{i+3} \\ &= \dots \\ &= X_i \uplus X_{i+1} \uplus \dots \uplus \uparrow X_n \\ &= X_i \uplus X_{i+1} \uplus \dots \uplus X_n. \end{aligned}$$

Recall that the aforementioned lemma also ensures  $\uparrow X'_i \setminus X'_i = \uparrow X_i$ , so the above display immediately yields

$$\uparrow X'_i = X'_i \uplus (\uparrow X'_i \setminus X'_i) = X'_i \uplus \uparrow X_i = X'_i \uplus X_i \uplus X_{i+1} \uplus \dots \uplus X_n.$$

Using the descriptions of  $\mathcal{X}$ ,  $\uparrow X_i$ , and  $\uparrow X'_i$  proved above, it is now easy to see that  $f$  is indeed order preserving. For suppose  $z \leq y \in \mathcal{X}$ . As  $\mathcal{X} = \uplus_{i=1}^n X_i \uplus \uplus_{i=1}^n X'_i$ , either  $z \in X_i$  or  $z \in X'_i$ , for some  $i \leq n$ . If  $z \in X_i$ , then

$$y \in \uparrow X_i = X_i \uplus X_{i+1} \uplus \dots \uplus X_n$$

implies  $y \in X_j$ , for some  $j \geq i$ , and therefore that  $f(z) = x_i \leq x_j = f(y)$ . If  $z \in X'_i$ , then

$$y \in \uparrow X'_i = X'_i \uplus X_i \uplus X_{i+1} \uplus \dots \uplus X_n$$

entails either  $y \in X'_i$  or  $y \in X_j$ , for some  $j \geq i$ . Thus, either  $f(z) = x'_i = f(y)$ , or  $f(z) = x'_i \leq x_j = f(y)$ . We conclude that  $f$  is order preserving.

Next we show that  $f$  satisfies the up condition (see the definition of a bi-p-morphism 2.8). To this end, suppose that  $f(z) \leq x$ , for some  $z \in \mathcal{X}$  and  $x \in \mathfrak{C}_n$ . If  $x$  is of the form  $x'_i$ , for some  $i \leq n$ , then  $x$  is a minimal point of  $\mathfrak{C}_n$  (recall our definition of the  $n$ -comb in Figure 4). So  $f(z) \leq x$  forces  $f(z) = x'_i$  and we are done.

If instead we have  $x = x_i$ , for some  $i \leq n$ , then  $f(z) \leq x_i$  and the order of  $\mathfrak{C}_n$  entail  $f(z) \in \{x_j, x'_j\}$ , for some  $j \leq i$ . By Lemma 5.5.(1), we know

$$X'_j \subseteq \downarrow X_j \subseteq \downarrow X_{j+1} \subseteq \dots \subseteq \downarrow X_i.$$

It is now clear that both possibilities for  $f(z) \in \{x_j, x'_j\}$  (equivalently, by the definition of  $f$ ,  $z \in X_j \cup X'_j$ ), yield  $z \in \downarrow X_i$ , i.e., that there exists a  $y \in X_i$  satisfying  $z \leq y$  and  $f(y) = x_i$ , as desired.

It remains to prove that  $f$  satisfies the down condition (see Definition 2.8). Let  $z \in \mathcal{X}$  and  $x \in \mathfrak{C}_n$ , and suppose  $x \leq f(z)$ . As the case  $x = f(z)$  is trivial, we can assume without loss of generality that  $x < f(z)$ . In particular, we are assuming that  $f(z)$  is not a minimal point of  $\mathfrak{C}_n$ , which forces  $f(z) = x_i$ , for some  $i \leq n$ . This has two immediate consequences:  $z \in X_i$ , by our definition of the map  $f$ ; and  $x = x'_i$  or  $x \in \{x_j, x'_j\}$  for some  $j < i$ , by the structure of  $\mathfrak{C}_n$ .

If  $x = x'_i$ , then the inclusion  $X_i \subseteq \uparrow X'_i$  (recall that  $\uparrow X'_i \setminus X'_i = \uparrow X_i$ , by Lemma 5.5.(4)) ensures the existence of a point  $y \in X'_i$  satisfying  $y \leq z \in X_i$  and  $f(y) = x'_i$ , as required.

On the other hand, if  $x \in \{x_j, x'_j\}$  for some  $j < i$ , then as

$$X_i \subseteq \uparrow X_i \subseteq \uparrow X_{i-1} \subseteq \dots \subseteq \uparrow X_j \subseteq \uparrow X'_j$$

follows from Lemma 5.5.(2), and thus  $z \in X_i \subseteq \uparrow X_j \subseteq \uparrow X'_j$ , we can easily find, for both possibilities on  $x \in \{x_j, x'_j\}$ , a point  $y \in \uparrow X'_j$  such that  $y \leq z$  and  $f(y) = x$ , as desired. This

finishes the proof that  $f$  satisfies the down condition, and we conclude that  $f$  is a surjective bi-Esakia morphism.

Therefore,  $\mathfrak{C}_n$  is a bi-Esakia morphic image of  $\mathcal{X}$ , and we proved the left to right implication of the desired equivalence.

The reverse implication is straightforward. Suppose that  $\mathfrak{C}_n$  is a bi-Esakia morphic image of  $\mathcal{X}$ . By the Dual Subframe Jankov Lemma 4.24, it is clear that  $\mathfrak{C}_n \not\models \beta(\mathfrak{C}_n^*)$ , since  $\mathfrak{C}_n$  trivially order-embeds into itself. As the validity of formulas is preserved under taking bi-Esakia morphic images, it follows  $\mathcal{X} \not\models \beta(\mathfrak{C}_n^*)$ , i.e., that  $\mathfrak{C}_n$  order-embeds into  $\mathcal{X}$ , again by the aforementioned lemma. Therefore,  $\mathfrak{C}_n$  can be regarded as a subposet of  $\mathcal{X}$ , as desired.  $\square$

We can now derive the following corollary, thus finishing the first step in the proof of our criterion.

**Corollary 5.22.** *Let  $n \in \mathbb{Z}^+$ . If  $\mathcal{X}$  is a bi-Esakia co-forest, then  $\mathcal{X} \not\models \beta(\mathfrak{C}_n^*)$  iff  $\mathcal{X} \not\models \mathcal{J}(\mathfrak{C}_n^*)$ . Equivalently, if  $\mathfrak{B} \in \text{bi-GA}$ , then  $\mathfrak{B} \not\models \beta(\mathfrak{C}_n^*)$  iff  $\mathfrak{B} \not\models \mathcal{J}(\mathfrak{C}_n^*)$ .*

*Proof.* We prove the second part of the statement, which is equivalent to the first part by duality. Let  $n$  be a positive integer and  $\mathfrak{B} \in \text{bi-GA}$ . Note the following equivalences:

$$\begin{aligned} \mathfrak{B} \not\models \beta(\mathfrak{C}_n^*) &\iff \mathfrak{C}_n \text{ order-embeds into } \mathfrak{D}_*, \text{ for some } \mathfrak{D} \in \mathbf{H}(\mathfrak{B})_{SI}, \\ &\iff \mathfrak{D} \not\models \mathcal{J}(\mathfrak{C}_n^*), \text{ for some } \mathfrak{D} \in \mathbf{H}(\mathfrak{B})_{SI}, \\ &\iff \mathfrak{B} \not\models \mathcal{J}(\mathfrak{C}_n^*) \end{aligned}$$

The first equivalence follows from (1)  $\iff$  (4) of the Dual Subframe Jankov Lemma 4.24. To see that the second equivalence holds, notice that as  $\mathfrak{D}$  is SI, and thus simple,  $\mathfrak{D}$  has no nontrivial homomorphic images and  $\mathfrak{D}_*$  is a co-tree. By the previous theorem,  $\mathfrak{C}_n$  order-embeds into  $\mathfrak{D}_*$  iff  $\mathfrak{C}_n$  is a bi-Esakia morphic image of  $\mathfrak{D}_*$ , which in turn, by duality and by our previous comment, is equivalent to  $\mathfrak{C}_n^* \in \mathbf{S}(\mathfrak{D}) = \mathbf{SH}(\mathfrak{D})$ . Equivalently,  $\mathfrak{D} \not\models \mathcal{J}(\mathfrak{C}_n^*)$ , by the Jankov Lemma 4.9, and we established the second equivalence.

Finally, the last equivalence is an immediate consequence of the aforementioned Jankov Lemma.  $\square$

**5.2. Step 2.** The nontrivial part of this step consists in showing that the algebraic duals of the finite combs are all 1-generated as bi-Heyting algebras. Before we prove this, we need a short lemma about bi-bisimulation equivalences on bi-Esakia spaces (see Definition 2.14).

**Lemma 5.23.** *Let  $\mathcal{X}$  be a bi-Esakia space and  $E$  a bi-bisimulation equivalence on  $\mathcal{X}$ . If  $x \leq y \leq z \in \mathcal{X}$ , then  $(x, z) \in E$  implies  $(x, y) \in E$ .*

*In particular, if  $\mathcal{X}$  is the  $n$ -comb  $\mathfrak{C}_n$  and  $(x_i, x_j) \in E$  for some  $i < j \leq n$ , then  $(x_i, x_{i+1}) \in E$ .*

*Proof.* Suppose  $x \leq y \leq z$  and  $(x, z) \in E$ . With a view towards contradiction, we assume  $(x, y) \notin E$ . By the refined condition of  $E$ , there exists an  $E$ -saturated clopen upset  $V$  that separates  $x$  and  $y$ . As  $x \leq y$ ,  $y$  must be the point contained in  $V$ , since  $V$  is an upset. But then  $y \leq z$  entails  $z \in V$ . As  $(x, z) \in E$  and  $V$  is  $E$ -saturated, we now have that  $x \in V$ , contradicting the definition of  $V$ .

The second part of the statement is clearly a particular instance of what we just proved, since if  $i < j \leq n$ , then the order of  $\mathfrak{C}_n$  entails  $x_i < x_{i+1} \leq x_j$ .  $\square$

**Proposition 5.24.**  $\mathfrak{C}_n^*$  is a 1-generated bi-Heyting algebra, for every  $n \in \mathbb{Z}^+$ .

*Proof.* Firstly, the algebraic dual of the 1-comb is the three element chain, which is generated as a bi-Heyting algebra by its only element distinct from 0 and 1. Let then  $n \geq 2$  and recall that by the Coloring Theorem 2.15, to show that  $\mathfrak{C}_n^*$  is 1-generated, i.e., that there exists  $U \in \text{Up}(\mathfrak{C}_n)$  such that  $\mathfrak{C}_n^* = \langle U \rangle$ , it suffices to show that every proper bi-bisimulation equivalence on  $\mathfrak{C}_n$  identifies points of different colors, where the coloring of  $\mathfrak{C}_n = (C_n, \leq)$  is defined by

$$\text{col}(w) = \begin{cases} 1 & \text{if } w \in U, \\ 0 & \text{if } w \notin U, \end{cases}$$

for all  $w \in C_n$ . To this end, let  $U := \{x_1\} \cup \uparrow\{x'_i \in C_n : i \text{ is even}\}$  and  $E$  be a proper bi-bisimulation equivalence on  $\mathfrak{C}_n$ .

We now prove by complete induction on  $i < n$  that if  $(x_i, x_{i+1}) \in E$ , then  $E$  must identify points of different colors. Suppose  $(x_i, x_{i+1}) \in E$  and that our induction hypothesis holds true for all  $j < i$ . By the definition of  $E$ , in particular, since it satisfies the down condition, the fact  $x'_{i+1} \leq x_{i+1}$  together with our assumption  $(x_i, x_{i+1}) \in E$  entails  $(w, x'_{i+1}) \in E$ , for some  $w \leq x_i$ . By the structure of  $\mathfrak{C}_n$ , the condition  $w \leq x_i$  can be split into three cases:  $w = x'_i$ , or  $w = x'_j$  for some  $j < i$ , or  $w = x_j$  for some  $j \leq i$ .

If  $w = x'_i$ , then  $(x'_i, x'_{i+1}) \in E$  and we are done, since  $\text{col}(x'_i) \neq \text{col}(x'_{i+1})$ , by our definition of the coloring of  $\mathfrak{C}_n$ .

If  $w = x'_j$  for some  $j < i$ , then applying the up condition of  $E$  to  $x'_j \leq x_j$  and  $(x'_j, x'_{i+1}) \in E$  yields  $(z, x_j) \in E$ , for some  $z \in \uparrow x'_{i+1}$ . If  $z = x'_{i+1}$ , then by the down condition, and noting that  $x'_{i+1}$  is minimal in  $\mathfrak{C}_n$ , it follows that  $\downarrow x_j \subseteq \llbracket x'_{i+1} \rrbracket_E$ , where  $\llbracket x'_{i+1} \rrbracket_E$  is the  $E$ -equivalence class of  $x'_{i+1}$ . Since  $x_1, x'_1 \in \downarrow x_j$ , the inclusion  $\downarrow x_j \subseteq \llbracket x'_{i+1} \rrbracket_E$  forces  $(x_1, x'_1) \in E$  and we are done, since  $\text{col}(x'_1) = 0 \neq 1 = \text{col}(x_1)$ . On the other hand, if  $z \neq x'_{i+1}$ , then we must have  $z = x_t$ , for some  $t \geq i+1$ . As  $(x_j, z) \in E$ , i.e.,  $(x_j, x_t) \in E$ , and since  $j < i < t$ , Lemma 5.23 entails  $(x_j, x_{j+1}) \in E$ , which falls under our induction hypothesis. Thus,  $E$  identifies points of different colors, as desired.

It remains to consider the case  $w = x_j$ , for  $j \leq i$ . Just note that in this case,  $(x_j, x'_{i+1}) \in E$  and the fact that  $x'_{i+1}$  is a minimal point entail  $\downarrow x_j \subseteq \llbracket x'_{i+1} \rrbracket_E$ , by the down condition of  $E$ . By the same reasoning as above, this implies that  $x_1$  and  $x'_1$ , two points of different colors, are  $E$ -equivalent, as desired.

We are finally ready to prove that if  $E$  is a proper bi-bisimulation equivalence on  $\mathfrak{C}_n$ , then  $E$  identifies points of different colors, and therefore the Coloring Theorem ensures that  $\mathfrak{C}_n^* = \langle U \rangle$  is a 1-generated bi-Heyting algebra. Let  $i < j \leq n$ . If  $(x_i, x_j) \in E$ , then Lemma 5.23 entails  $(x_i, x_{i+1}) \in E$ , and the result now follows by our previous discussion. If  $(x'_i, x'_j) \in E$  or  $(x_i, x'_j) \in E$ , using the same argument as the one detailed above (namely, when dealing with the case " $w = x'_j$ " in the proof by induction) yields  $E$ -equivalent points of different colors, as desired. The only cases that remain are either  $(x'_i, x_i) \in E$  or  $(x'_i, x_j) \in E$ , which are clear, since the minimality of  $x'_i$  implies either  $\downarrow x_i \subseteq \llbracket x'_i \rrbracket_E$  or  $\downarrow x_j \subseteq \llbracket x'_i \rrbracket_E$ , respectively, hence  $x'_1 E x_i E x_1$ . Since we have  $\text{col}(x'_1) \neq \text{col}(x_1)$ , the result follows.  $\square$

**Corollary 5.25.** *Let  $V$  be a variety of bi-Gödel algebras. If  $V$  contains all the algebraic duals of the finite combs, then  $V$  is not locally finite.*

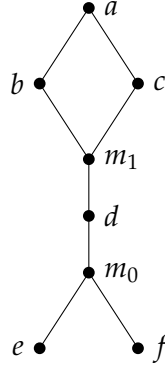
*Proof.* Let us note that, by definition, the finite combs are arbitrarily large finite co-trees. Hence, their bi-Heyting duals are arbitrarily large finite bi-Gödel algebras. If we assume that these duals are all contained in  $V$ , then it follows from the previous proposition that there are arbitrarily large finite 1-generated algebras in  $V$ . Therefore, the 1-generated free  $V$ -algebra must be infinite, and thus  $V$  cannot be locally finite.  $\square$

**5.3. Step 3.** The third and last result we will need to prove Theorem 5.1 consists in establishing, for arbitrary  $n, m \in \mathbb{Z}^+$ , the existence of a natural bound  $k(n, m)$  for the size of  $m$ -generated SI bi-Gödel algebras whose bi-Esakia duals do not admit the  $n$ -comb as a subposet. We need a few auxiliary lemmas.

**Definition 5.26.** Given a poset  $\mathfrak{F}$  and a chain  $H \subseteq \mathfrak{F}$  with a least element  $m_0$  and a greatest element  $m_1$ , we say that  $H$  is an *isolated chain* (in  $\mathfrak{F}$ ) if

$$\downarrow m_1 \setminus H = \downarrow m_0 \setminus \{m_0\} \text{ and } \uparrow m_0 \setminus H = \uparrow m_1 \setminus \{m_1\}.$$

**Example 5.27.** Consider the poset  $\mathfrak{F}$  depicted in Figure 5. The set  $H := \{m_0, d, m_1\}$  forms an isolated chain in  $\mathfrak{F}$ , since  $\downarrow m_1 \setminus H = \{e, f\} = \downarrow m_0 \setminus \{m_0\}$  and  $\uparrow m_0 \setminus H = \{b, c, a\} = \uparrow m_1 \setminus \{m_1\}$ . On the other hand, the chain  $G := \{m_1, b, a\}$  is not isolated in  $\mathfrak{F}$ , since, for example,  $c \in \downarrow a \setminus G$  but  $c \notin \downarrow m_1 \setminus \{m_1\}$ .

FIGURE 5. The poset  $\mathfrak{F}$ 

**Lemma 5.28.** *If  $\mathcal{X}$  is a bi-Esakia space and  $H \subseteq \mathcal{X}$  is an isolated chain, then the least equivalence relation identifying the points in  $H$ ,  $E := H^2 \cup \text{Id}_X$ , is a bi-bisimulation equivalence on  $\mathcal{X}$ .*

*Proof.* That  $E$  is an equivalence relation that satisfies the up and down conditions follows immediately from the definition of  $E$  and that of an isolated chain. It remains to show that  $E$  is refined, i.e., that every two non- $E$ -equivalent points are separated by an  $E$ -saturated clopen upset of  $\mathcal{X}$  (notice that, by the definition of  $E$ , a clopen upset  $U$  is  $E$ -saturated iff  $H \subseteq U$  or  $H \cap U = \emptyset$ ). To this end, let  $w, v \in X$  and suppose  $(w, v) \notin E$ . There are only two possible cases: either  $w, v \notin H$ , or, without loss of generality,  $w \in H$  and  $v \notin H$ .

We first suppose that  $w, v \notin H$ . Since  $(w, v) \notin E$ , we have  $w \neq v$ , and we can suppose without loss of generality that  $w \not\leq v$ . By the PSA, there exists  $U \in \text{CpUp}(\mathcal{X})$  satisfying  $w \in U$  and  $v \notin U$ . If  $w < m_1 := \text{Max}(H)$ , then  $H$  being an isolated chain in  $\mathcal{X}$  and  $w \notin H$  imply  $w < m_0 := \text{Min}(H)$ , hence we have  $H \subseteq U$ , since  $U$  is an upset containing  $w$ . Thus,  $U$  is an  $E$ -saturated clopen upset that separates  $w$  from  $v$ . On the other hand, if  $w \not\leq m_1$ , then by the PSA there exists  $V \in \text{CpUp}(\mathcal{X})$  such that  $w \in V$  and  $m_1 \notin V$ . Since  $V$  is an upset not containing  $m_1$ , it follows  $H \cap V = \emptyset$ , and it is easy to see that  $U \cap V$  is an  $E$ -saturated clopen upset that separates  $w$  from  $v$ , as desired.

Suppose now that  $w \in H$  and  $v \notin H$ . If  $m_1 \not\leq v$ , then we also have  $m_0 \not\leq v$ , by the definition of an isolated chain. The PSA now yields  $U \in \text{CpUp}(\mathcal{X})$  satisfying  $m_0 \in U$  and  $v \notin U$ . Since  $U$  is an upset containing  $m_0$ , we have  $H \subseteq U$  and clearly  $U$  satisfies our desired conditions. On the other hand, if  $m_1 \leq v$ , we must have  $m_1 < v$  because  $v \notin H$ . Consequently, in this case, we have  $v \not\leq m_1$ . Therefore, we can apply the PSA obtaining some  $V \in \text{CpUp}(\mathcal{X})$  such that  $v \in V$  and  $m_1 \notin V$ . Since  $V$  is an upset, it follows  $H \cap V = \emptyset$ , and we conclude that  $V$  is an  $E$ -saturated clopen upset that separates  $v$  from  $w$ , as desired.  $\square$

Recall that an *order-isomorphism* is an order-invariant bijection between posets (in other words, a surjective order-embedding), and that given two points  $w$  and  $v$  in a poset  $\mathfrak{F} = (W, \leq)$ , we denote  $[w, v] := \{x \in W : w \leq x \leq v\}$ . Notice that if  $\mathfrak{F}$  is a co-forest, then  $[w, v]$  is a chain.

**Lemma 5.29.** *Let  $\mathcal{X}$  be a bi-Esakia co-forest and  $w, v \in X$  two distinct points with a common immediate successor. If both  $\downarrow w$  and  $\downarrow v$  are finite, and there exists an order-isomorphism  $f : \downarrow w \rightarrow \downarrow v$ , then*

$$E := \{(x, y) \in X^2 : (x \in \downarrow w \text{ and } f(x) = y) \text{ or } (x \in \downarrow v \text{ and } f(y) = x)\} \cup \text{Id}_X$$

*is a bi-bisimulation equivalence on  $\mathcal{X}$ .*

*Proof.* We start by noting that, by its definition,  $E$  is clearly an equivalence relation. Furthermore, that  $E$  satisfies the down condition is immediate from the definition of  $E$  and that of an order-isomorphism. Since we assumed that  $w$  and  $v$  share an immediate successor, and since in a co-forest points have at most one immediate successor, it follows that  $E$  satisfies the up condition. To see this, let us denote the unique immediate successor of both  $w$  and  $v$  by  $u$ , and note that, for  $x \in \downarrow w$  (or  $x \in \downarrow v$ ), we have a description  $\uparrow x = [x, w] \uplus \uparrow u$  (respectively,  $\uparrow x = [x, v] \uplus \uparrow u$ ),



since the principal upsets of  $\mathcal{X}$  are chains. Using this description of  $\uparrow x$ , the definition of  $E$ , and that of an order-isomorphism, it is now clear that  $E$  satisfies the up condition.

We now show that  $E$  is refined, thus ensuring that  $E$  is a bi-bisimulation equivalence on  $\mathcal{X}$ . Let  $x, y \in X$  and suppose that  $(x, y) \notin E$ . So  $x \neq y$ , and we can suppose without loss of generality that  $x \not\leq y$ . We proceed by cases:

- **Case 1:**  $\{x, y\} \cap (\downarrow w \cup \downarrow v) = \emptyset$ ;

In this case, we have  $x \not\leq w$  and  $x \not\leq v$ . Since we also assumed  $x \not\leq y$ , by the PSA there are  $U_y, U_w, U_v \in \text{CpUp}(\mathcal{X})$  all containing  $x$ , and such that  $y \notin U_y$ ,  $w \notin U_w$ , and  $v \notin U_v$ . As  $U_w$  is an upset not containing  $w$ , we have  $\downarrow w \cap U_w = \emptyset$ . Similarly, it follows  $\downarrow v \cap U_v = \emptyset$ . Thus,  $U := U_y \cap U_w \cap U_v$  is an  $E$ -saturated (since  $U \cap (\downarrow w \cup \downarrow v) = \emptyset$ ) clopen upset separating  $x$  from  $y$ , as desired.

- **Case 2:**  $x \notin \downarrow w \cup \downarrow v$  and  $y \in \downarrow w \cup \downarrow v$ ;

By assumption, we have  $x \not\leq w$  and  $x \not\leq v$ , so by the PSA there are  $U_w, U_v \in \text{CpUp}(\mathcal{X})$ , both containing  $x$ , satisfying  $w \notin U_w$  and  $v \notin U_v$ . As  $U_w$  is an upset not containing  $w$ , we have  $\downarrow w \cap U_w = \emptyset$ . Similarly, it follows  $\downarrow v \cap U_v = \emptyset$ . Thus,  $U := U_w \cap U_v$  is an  $E$ -saturated (since  $U \cap (\downarrow w \cup \downarrow v) = \emptyset$ ) clopen upset separating  $x$  from  $y$ , since we assumed  $y \in \downarrow w \cup \downarrow v$ . We note that the previous argument can also be used when  $y \notin \downarrow w \cup \downarrow v$  and  $x \in \downarrow w \cup \downarrow v$ , by replacing  $x$  with  $y$ , and vice-versa.

- **Case 3:**  $x, y \in \downarrow w \cup \downarrow v$ .

Without loss of generality, we suppose that  $x \in \downarrow w$  and  $y \in \downarrow w$  (if  $y \in \downarrow v$  or  $x \in \downarrow v$ , we can replace  $y$  or  $x$  in the following argument by  $f^{-1}(y)$  or  $f^{-1}(x)$ , respectively, where  $f^{-1}$  is the inverse of the order-isomorphism  $f$ ). As  $\downarrow w$  is finite by hypothesis, we can enumerate  $\downarrow w \setminus \downarrow y := \{x_1, \dots, x_n\}$ . Notice that for all  $i \leq n$ , we have  $x_i \not\leq y$  by the definition of  $x_i$ , and  $x_i \not\leq f(y)$ , since  $f(y) \in \downarrow v$  and  $x_i \notin \downarrow v$  (recall that  $w$  and  $v$  are distinct points in a co-forest with a common immediate successor, hence we have  $\downarrow w \cap \downarrow v = \emptyset$ ). Using the same argument as in the previous cases,  $x_i \not\leq y$  and  $x_i \not\leq f(y)$  imply, by the PSA, that there exists  $U_i \in \text{CpUp}(\mathcal{X})$  satisfying  $x_i \in U_i$  and  $y, f(y) \notin U_i$ . As  $U_i$  is an upset, it follows that  $U_i \cap (\downarrow y \cup \downarrow f(y)) = \emptyset$ . Furthermore, by the definition of an order-isomorphism,  $x_i \not\leq y$  entails  $f(x_i) \not\leq f(y)$ , and since we have  $f(x_i) \in \downarrow v$  and  $y \in \downarrow w$ , it follows  $f(x_i) \not\leq y$ . Again, the PSA yields some  $V_i \in \text{CpUp}(\mathcal{X})$  satisfying  $f(x_i) \in V_i$  and  $V_i \cap (\downarrow y \cup \downarrow f(y)) = \emptyset$ . Let  $U := \bigcup_{i=1}^n U_i \cup \bigcup_{i=1}^n V_i$ , and note that this is a clopen upset satisfying

$$\{x_1, \dots, x_n, f(x_1), \dots, f(x_n)\} \subseteq U \text{ and } U \cap (\downarrow y \cup \downarrow f(y)) = \emptyset.$$

As we assumed  $x \in \downarrow w$  and  $x \not\leq y$ , it now follows  $x \in \{x_1, \dots, x_n\} = \downarrow w \setminus \downarrow y$ , and thus  $x \in U$ . By the way we defined  $U$  and  $E$ , we conclude that  $U$  is an  $E$ -saturated clopen upset separating  $x$  from  $y$ .

Therefore,  $E$  is indeed a bi-bisimulation equivalence on  $\mathcal{X}$ .  $\square$

We now have all the necessary tools to obtain the desired bound.

**Proposition 5.30.** *If  $n$  and  $m$  are positive integers, then there is a natural bound  $k(n, m)$  (only dependent on  $n$  and  $m$ ) on the size of  $m$ -generated SI bi-Gödel algebras whose bi-Esakia duals do not admit the  $n$ -comb  $\mathfrak{C}_n$  as a subposet.*

*Proof.* Let  $n$  and  $m$  be positive integers. Take a bi-Esakia co-tree  $\mathcal{X}$  which does not admit the  $n$ -comb as a subposet, and suppose that  $\mathcal{X}^*$  is  $m$ -generated as a bi-Heyting algebra, so there are  $U_1, \dots, U_m \in \text{CpUp}(\mathcal{X})$  such that  $\mathcal{X}^* = \langle U_1, \dots, U_m \rangle$ . By the Coloring Theorem 2.15, every proper bi-bisimulation equivalence on  $\mathcal{X}$  must identify points of different colors, where the coloring of  $\mathcal{X}$  is defined by  $V(p_i) = U_i$ , for  $i \leq m$ .

First we prove that if  $w \in \min(\mathcal{X})$  then  $|\uparrow w| \leq (m+1) \cdot n$ . Take  $w \in \min(\mathcal{X})$  and notice that, since the  $U_i$  are upsets, we can re-enumerate them in such a way as to satisfy

$$\uparrow w \cap U_1 \subseteq \dots \subseteq \uparrow w \cap U_m.$$

Set  $H_1 := \uparrow w \cap U_1$ ,  $H_i := \uparrow w \cap (U_i \setminus U_{i-1})$  for every  $i \in \{2, \dots, m\}$ , and  $H_{m+1} := \uparrow w \setminus U_m$ . It is clear that  $\uparrow w = \bigsqcup_{i=1}^{m+1} H_i$ . We now show that  $|H_i| \leq n$ , for all  $i \leq m+1$ . For suppose this

is not the case, i.e., that  $|H_i| > n$  for some  $i \leq m+1$ . As  $H_i$  is, by definition, contained in the chain  $\uparrow w$ ,  $H_i$  must also be a chain. Hence,  $|H_i| > n$  implies that there exists a strictly ascending sequence  $a_1 < \dots < a_n < a_{n+1}$  contained in  $H_i$ .

Let  $j \leq n$  and suppose that  $[a_j, a_{j+1}]$  is an isolated chain in  $\mathcal{X}$ . By the definitions of  $H_i$  and of our coloring of  $\mathcal{X}$ , the inclusion  $[a_j, a_{j+1}] \subseteq H_i$  forces all the points in this isolated chain to have the same color. But now Lemma 5.28 yields a proper (since  $a_j < a_{j+1}$ ) bi-bisimulation equivalence on  $\mathcal{X}$  which does not identify points of different colors, contradicting the Coloring Theorem 2.15. Thus, the chain  $[a_j, a_{j+1}]$  cannot be isolated.

Since  $\mathcal{X}$  is a co-tree, it is clear that both

$$\uparrow a_j \setminus [a_j, a_{j+1}] = \uparrow a_{j+1} \setminus \{a_{j+1}\} \text{ and } \downarrow a_j \setminus \{a_j\} \subseteq \downarrow a_{j+1} \setminus [a_j, a_{j+1}],$$

hold true. Therefore, the fact that the chain  $[a_j, a_{j+1}]$  is not isolated in  $\mathcal{X}$  (see Definition 5.26) implies

$$\downarrow a_{j+1} \setminus [a_j, a_{j+1}] \not\subseteq \downarrow a_j \setminus \{a_j\}.$$

Equivalently, there must exist  $x_j \in [a_j, a_{j+1}] \setminus \{a_j\}$  such that  $\downarrow x_j \setminus ([a_j, a_{j+1}] \cup \downarrow a_j) \neq \emptyset$ .

As  $j$  was arbitrary in the above discussion, we can now fix, for each  $j \leq n$ , an element  $x'_j \in \downarrow x_j \setminus ([a_j, a_{j+1}] \cup \downarrow a_j)$ . Thus, we have found a subposet of  $\mathcal{X}$ ,

$$(\{x_j : j \leq n\} \cup \{x'_j : j \leq n\}, \leq),$$

which is clearly a copy of the  $n$ -comb  $\mathfrak{C}_n$ , contradicting our hypothesis. Therefore, there can be no chain  $a_1 < \dots < a_n < a_{n+1}$  contained in  $H_i$ , and it follows  $|H_i| \leq n$  for all  $i \leq m+1$ .

Consequently, we conclude that  $\uparrow w = \bigsqcup_{i=1}^{m+1} H_i$  consists of at most  $m+1$  pieces, each of size at most  $n$ , that is,  $|\uparrow w| \leq (m+1) \cdot n$  as desired.

Since every point in a bi-Esakia space lies above a minimal one (see Proposition 2.12), it now follows from the definition of the depth of a co-tree that  $dp(\mathcal{X}) \leq (m+1) \cdot n$ . Notice that  $\mathcal{X}$  being a co-tree of finite depth entails that every point distinct from its co-root  $r$  has a unique immediate successor. Let  $\{w_i\}_{i \in I} \subseteq \min(\mathcal{X})$ , and suppose they all share their unique immediate successor,  $v$ . Note that there are only  $2^m$  distinct colors, and that  $i \neq j \in I$  implies  $col(w_i) \neq col(w_j)$ , otherwise Lemma 5.29 would contradict the Coloring Theorem. Thus, we have  $|I| \leq 2^m$  and  $|\downarrow v| \leq 2^m + 1$ .

Now, let  $u \in X$  be such that all of its strict predecessors are either minimal, or are immediate successors of minimal points. Set  $\{v_i\}_{i \in I} := \{y \in X : y \prec u\}$ , and notice that for all  $i \in I$ , we have  $|\downarrow v_i| \leq 2^m + 1$  by above. Moreover, since there are only  $2^m$  distinct colors, there exists a natural bound  $b(m)$  for the number of possible distinct colored configurations (by which we mean poset structure together with a coloring) of the posets  $\downarrow v_i$ . As the  $v_i$  all share their unique immediate successor, we cannot have that for  $i \neq j \in I$ ,  $\downarrow v_i$  and  $\downarrow v_j$  have both the same poset structure (i.e., there exists an order-isomorphism from  $\downarrow v_i$  to  $\downarrow v_j$ ) and coloring, otherwise Lemma 5.29 would contradict the Coloring Theorem. Hence  $|I| \leq b(m)$ , and we now have  $|\downarrow u| \leq (2^m + 1) \cdot b(m) + 1$ .

Since we have a natural bound for the depth of  $\mathcal{X}$ , we can now iterate the above argument a finite number of times (namely, at most  $(m+1) \cdot n$  times) to find a bound  $k_0(n, m) \in \omega$  for the size of  $X$ , i.e.,  $|X| = |\downarrow r| \leq k_0(n, m)$ . By the nature of the argument that led to this bound,  $k_0(n, m)$  depends only on  $n$  and  $m$ , and not on  $\mathcal{X}$ .

As there are only finitely many co-trees of size less than or equal to  $k_0(n, m)$ , it follows that there are only finitely many bi-Esakia co-trees which do not admit  $\mathfrak{C}_n$  as a subposet and whose algebraic dual is  $m$ -generated. Therefore, we can now find a natural bound  $k(n, m)$  (only dependent on  $k_0(n, m)$ ) for the size of the bi-Heyting duals of these bi-Esakia co-trees.  $\square$

**5.4. The proof of Theorem 5.1 and a criterion for local tabularity.** We are finally ready to prove Theorem 5.1.

*Proof.* Let  $L$  be an extension of bi-LC. We start by proving the contrapositive of the left to right implication. Accordingly, let us suppose that for all  $n \in \mathbb{Z}^+$ ,  $\mathcal{J}(\mathfrak{C}_n^*) \notin L$ , i.e., that  $V_L \not\models \mathcal{J}(\mathfrak{C}_n^*)$  by duality. It is now an immediate consequence of the Jankov Lemma 4.9 that  $V_L$  contains all

the algebraic duals of the finite combs. By Corollary 5.25,  $V_L$  is not locally finite, and thus  $L$  is not locally tabular, as desired.

To prove the reverse implication, suppose that  $\mathcal{J}(\mathfrak{C}_n^*) \in L$ , for some  $n \in \mathbb{Z}^+$ . By duality, this is equivalent to  $V_L \models \mathcal{J}(\mathfrak{C}_n^*)$ , which in turn is equivalent to  $V_L \models \beta(\mathfrak{C}_n^*)$  by Corollary 5.22. In particular, it now follows that for each positive  $m \in \omega$ , if  $\mathfrak{A}$  is an SI  $m$ -generated algebra contained in  $V_L$ , then  $\mathfrak{A} \models \beta(\mathfrak{C}_n^*)$ . By the Dual Subframe Jankov Lemma 4.24, this is equivalent to  $\mathfrak{A}_*$  not admitting  $\mathfrak{C}_n$  as a subposet. Since  $\mathfrak{A}$  satisfies all the conditions in the statement of Proposition 5.30, we now have that  $|A| \leq k(n, m)$ , and we can use Theorem 2.3 to conclude that  $V_L$  is locally finite, i.e., that  $L$  is locally tabular.  $\square$

We can now derive the following criterion for local tabularity. Let us denote the logic of the finite combs by  $\text{Log}(FC) := \{\varphi : \forall n \in \mathbb{Z}^+ (\mathfrak{C}_n \models \varphi)\}$ .

**Corollary 5.31.** *If  $L \in \Lambda(\text{bi-LC})$ , then the following conditions are equivalent:*

- (i)  $L$  is locally tabular;
- (ii)  $\mathcal{J}(\mathfrak{C}_n^*) \in L$ , for some  $n \in \mathbb{Z}^+$ ;
- (iii)  $\beta(\mathfrak{C}_n^*) \in L$ , for some  $n \in \mathbb{Z}^+$ ;
- (iv)  $L \not\subseteq \text{Log}(FC)$ .

Consequently,  $\text{Log}(FC)$  is the only pre-locally tabular extension of bi-LC.

*Proof.* The equivalence (i)  $\iff$  (ii) is just Theorem 5.1, while (ii)  $\iff$  (iii) follows immediately by duality and Corollary 5.22.

We now prove (ii)  $\iff$  (iv). Suppose  $\mathcal{J}(\mathfrak{C}_n^*) \in L$ , for some  $n \in \mathbb{Z}^+$ . Since  $\mathfrak{C}_n^* \not\models \mathcal{J}(\mathfrak{C}_n^*)$  by the Jankov Lemma 4.9, it is clear that  $L \not\subseteq \text{Log}(FC) \subseteq \text{Log}(\mathfrak{C}_n^*)$ . Conversely, if  $L \not\subseteq \text{Log}(FC)$ , i.e., if there exists a finite comb satisfying  $\mathfrak{C}_n \not\models L$ , then the Jankov Lemma yields  $\mathcal{J}(\mathfrak{C}_n^*) \in L$ , as desired.

The last part of the statement is an immediate consequence of the equivalence (i)  $\iff$  (iv).  $\square$

We close the paper by comparing some properties of the logic bi-LC (algebraized by bi-GA) with those of the thoroughly investigated linear calculus LC which is algebraized by the variety GA of Gödel algebras, i.e., the class of Heyting algebras satisfying the Gödel-Dummett axiom. In the table below, SRC is a short hand for *strongly rooted chain*, i.e., a chain with an isolated minimum. The fact that  $\Lambda(\text{bi-LC})$  is not a chain is an immediate consequence of the proof of Theorem 4.17, while the previous result clearly ensures that bi-LC is not locally tabular.

$\text{LC} = \text{IPC} + (p \rightarrow q) \vee (q \rightarrow p)$	$\text{bi-LC} = \text{bi-IPC} + (p \rightarrow q) \vee (q \rightarrow p)$
$\mathfrak{A} \in \text{GA}_{SI} \iff \mathfrak{A}_*$ is a SRC	$\mathfrak{A} \in \text{bi-GA}_{SI} \iff \mathfrak{A}_*$ is a co-tree
LC has the FMP	bi-LC has the FMP
LC is locally tabular	bi-LC is not locally tabular
$\Lambda(\text{LC})$ is a chain of order-type $(\omega + 1)^\partial$	$\Lambda(\text{bi-LC})$ is of size $2^{\aleph_0}$ and is not a chain

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