LECTURE NOTES ON THE ALGEBRA OF LOGIC

T. MORASCHINI

1. Algebras and equations

We begin by reviewing some fundamentals of general algebraic systems. For more information, the reader may consult [2, 6].

Definition 1.1.

(i) A *type* is a map $\rho \colon \mathcal{F} \to \mathbb{N}$, where $\mathcal{F}$ is a set of function symbols. In this case, $\rho(f)$ is said to be the *arity* of the function symbol f, for every $f \in \mathcal{F}$. Function symbols of arity zero are called *constants*.

(ii) An *algebra* of type ρ is a pair $\boldsymbol{A} = \langle A; F \rangle$ where A is a nonempty set and $F = \{f^{\boldsymbol{A}} : f \in \mathcal{F}\}$ is a set of operations on A whose arity is determined by ρ, in the sense that each $f^{\boldsymbol{A}}$ has arity $\rho(f)$. The set A is called the *universe* of $\boldsymbol{A}$.

When $\mathcal{F} = \{f_1, \dots, f_n\}$, we shall write $\langle A; f_1^{\boldsymbol{A}}, \dots, f_n^{\boldsymbol{A}} \rangle$ instead of $\langle A; F \rangle$. In this case, we often drop the superscripts, and write simply $\langle A; f_1, \dots, f_n \rangle$.

Classical examples of algebras are groups and rings. For instance, the type of groups ρ_G consists of a binary symbol $+$, a unary symbol $-$ and a constant symbol 0. Then a group is an algebra $\langle G; +, -, 0 \rangle$ of type ρ_G in which $+$ is associative, 0 is a neutral element for $+$ and $-$ produces inverses.

Lattices, Heyting algebras and modal algebras are also algebras in the above sense. For instance, the type of lattices ρ_L consists of two binary symbols $\wedge$ and $\vee$ and a lattice is an algebra $\langle A; \wedge, \vee \rangle$ of type ρ_L that satisfies the idempotent, commutative, associative and absorption laws. Similarly, the type of Heyting algebras ρ_H consists of three binary operations symbols $\wedge, \vee$ and $\to$ and of two constant symbols 0 and 1. Then a Heyting algebra is an algebra $\langle A; \wedge, \vee, \to, 0, 1 \rangle$ such that $\langle A; \wedge, \vee, 0, 1 \rangle$ is a bounded lattice and, for every $a, b, c \in A$,

$$a \wedge b \leqslant c \iff a \leqslant b \to c. \qquad (\textit{residuation law})$$

Boolean algebras can be viewed as the Heyting algebras that satisfy the following equational version of the *excluded middle law*:

$$x \vee (x \to 0) \approx 1.$$

In this case, the complement operation $\neg x$ can be defined as $x \to 0$.

Perhaps less obviously, even algebraic structures whose operations are apparently *external* can be viewed as algebras in the sense of the above definition. For instance, modules over a ring $\boldsymbol{R}$ can be viewed as algebras whose type ρ_R extends that of groups with the unary symbols $\{\lambda_r : r \in R\}$. From this point of view, a module over $\boldsymbol{R}$ is an

Date: June 3, 2021.

algebra $\langle G; +, -, 0, \{\lambda_r : r \in R\}\rangle$ of type ρ_R such that $\langle G; +, -, 0\rangle$ is an abelian group and, for every $r, s \in R$ and $a, c \in G$,

$$\begin{aligned}
\lambda_r(a+c) &= \lambda_r(a) + \lambda_r(c)\\
\lambda_{r+s}(a) &= \lambda_r(a) + \lambda_s(a)\\
\lambda_r(\lambda_s(a)) &= \lambda_{r\cdot s}(a)\\
\lambda_1(a) &= a.
\end{aligned}$$

Given a type $\rho\colon \mathcal{F} \to \mathbb{N}$ and a set of variables X disjoint from $\mathcal{F}$, the set of *terms of type ρ over X* is the least set $T_\rho(X)$ such that

(i) $X \subseteq T_\rho(X)$;
(ii) if $c \in \mathcal{F}$ is a constant, then $c \in T_\rho(X)$; and
(iii) if $\varphi_1, \dots, \varphi_{\rho(f)} \in T_\rho(X)$ and $f \in \mathcal{F}$, then $f\varphi_1 \dots \varphi_{\rho(f)} \in T_\rho(X)$.

For the sake of readability, we shall often write $f(\varphi_1, \dots, \varphi_{\rho(f)})$ instead of $f\varphi_1 \dots \varphi_{\rho(f)}$. Similarly, if f is a binary operation $+$, we often write $\varphi_1 + \varphi_2$ instead of $f(\varphi_1, \varphi_2)$.

Definition 1.2. Let $\rho\colon \mathcal{F} \to \mathbb{N}$ be a type and X a set of variables disjoint from $\mathcal{F}$. The *term algebra $\boldsymbol{T}_\rho(X)$* of type ρ over X is the unique algebra of type ρ whose universe is $T_\rho(X)$ and with basic n-ary operations f defined, for every $\varphi_1, \dots, \varphi_n \in T_\rho(X)$, as

$$f^{\boldsymbol{T}_\rho(X)}(\varphi_1, \dots, \varphi_n) \coloneqq f(\varphi_1, \dots, \varphi_n).$$

When no confusion might arise, we drop the subscript and write $T(X)$ instead of $T_\rho(X)$. Term algebras have the following fundamental property.

Proposition 1.3. *Let $\boldsymbol{A}$ be an algebra of type ρ and X a set of variables. Every function $f\colon X \to A$ extends uniquely to a homomorphism $f^*\colon \boldsymbol{T}_\rho(X) \to \boldsymbol{A}$.*

Proof. The unique extension f^* is defined, for every $\varphi(x_{\alpha_1}, \dots x_{\alpha_n}) \in T_\rho(X)$, as

$$f^*(\varphi) = \varphi^{\boldsymbol{A}}(f(x_{\alpha_1}), \dots, f(x_{\alpha_n})). \qquad \boxtimes$$

Exercise 1.4. Prove the above proposition. $\boxtimes$

Given a term $\varphi \in T_\rho(X)$, we write $\varphi(x_1, \dots, x_n)$ to indicate that the variables occurring in φ are among $x_1, \dots, x_n$. Furthermore, given an algebra $\boldsymbol{A}$ of type ρ and elements $a_1, \dots, a_n \in A$, we define an element

$$\varphi^{\boldsymbol{A}}(a_1, \dots, a_n)$$

of A, by recursion on the construction of φ, as follows:

(i) if φ is a variable x_i, then $\varphi^{\boldsymbol{A}}(a_1, \dots, a_n) \coloneqq a_i$;
(ii) if φ is a constant c, then $c^{\boldsymbol{A}}$ is the interpretation of c in $\boldsymbol{A}$;
(iii) if $\varphi = f(\psi_1, \dots, \psi_m)$, then

$$\varphi^{\boldsymbol{A}}(a_1, \dots, a_n) \coloneqq f^{\boldsymbol{A}}(\psi_1^{\boldsymbol{A}}(a_1, \dots, a_n), \dots, \psi_m^{\boldsymbol{A}}(a_1, \dots, a_n)).$$

An *equation of type ρ over X* is an expression of the form $\varphi \approx \psi$, where $\varphi, \psi \in T_\rho(X)$. We denote by $E_\rho(X)$ the set of equations of type ρ over X. Such an equation $\varphi \approx \psi$ is *valid* in an algebra $\boldsymbol{A}$ of type ρ, if

$$\varphi^{\boldsymbol{A}}(a_1, \dots, a_n) = \psi^{\boldsymbol{A}}(a_1, \dots, a_n), \text{ for every } a_1, \dots, a_n \in A,$$

in which case we say that $\boldsymbol{A}$ *satisfies* $\varphi \approx \psi$.

For instance, groups are precisely the algebras of type ρ_G that satisfy the equations

$$x + (y + z) \approx (x + y) + z \quad x + 0 \approx x \quad 0 + x \approx x \quad x + -x \approx 0 \quad -x + x \approx 0.$$

Similarly, lattices are the algebras of type ρ_L that satisfy the equations

$$\begin{array}{llr} x \wedge x \approx x & x \vee x \approx x & (\textit{idempotent laws}) \\ x \wedge y \approx y \wedge x & x \vee y \approx y \vee x & (\textit{commutative laws}) \\ x \wedge (y \wedge z) \approx (x \wedge y) \wedge z & x \vee (y \vee z) \approx (x \vee y) \vee z & (\textit{associative laws}) \\ x \wedge (y \vee x) \approx x & x \vee (y \wedge x) \approx x. & (\textit{absorption laws}) \end{array}$$

From now on, we will work with a fixed denumerable set of variables

$$Var = \{x_n : n \in \mathbb{N}\}.$$

Accordingly, when we write $x, y, z \dots$ for variables, it should be understood that these are variables in Var.

2. Basic constructions

Algebras of the same type are called *similar* and can be compared by means of maps that preserve their structure.

Definition 2.1. Given two similar algebras $\boldsymbol{A}$ and $\boldsymbol{B}$, a *homomorphism* from $\boldsymbol{A}$ to $\boldsymbol{B}$ is a map $f \colon A \to B$ such that, for every n-ary operation g of the common type and $a_1, \dots, a_n \in A$,

$$f(g^{\boldsymbol{A}}(a_1, \dots, a_n)) = g^{\boldsymbol{B}}(f(a_1), \dots, f(a_n)).$$

An injective homomorphism is called an *embedding* and, if there exists an embedding from $\boldsymbol{A}$ to $\boldsymbol{B}$, we say that $\boldsymbol{A}$ *embeds* into $\boldsymbol{B}$. Lastly, a surjective embedding is called an *isomorphism*. Accordingly, $\boldsymbol{A}$ and $\boldsymbol{B}$ are said to be *isomorphic* if there exists an isomorphism between them, in which case we write $\boldsymbol{A} \cong \boldsymbol{B}$.

A simple induction on the construction of terms shows that, for every pair of algebras $\boldsymbol{A}$ and $\boldsymbol{B}$ of type ρ and every term $\varphi(x_1, \dots, x_n)$ of ρ, if f is a homomorphism from $\boldsymbol{A}$ to $\boldsymbol{B}$, then

$$f(\varphi^{\boldsymbol{A}}(a_1, \dots, a_n)) = \varphi^{\boldsymbol{B}}(f(a_1), \dots, f(a_n)),$$

for every $a_1, \dots, a_n \in A$. Therefore homomorphisms preserve not only basic operations, but also arbitrary terms.

In the particular case where $\boldsymbol{A}$ and $\boldsymbol{B}$ are lattices, a homomorphism from $\boldsymbol{A}$ to $\boldsymbol{B}$ is a map $f \colon A \to B$ such that, for every $a, c \in A$,

$$f(a \wedge^{\boldsymbol{A}} c) = f(a) \wedge^{\boldsymbol{B}} f(c) \text{ and } f(a \vee^{\boldsymbol{A}} c) = f(a) \vee^{\boldsymbol{B}} f(c).$$

For instance, the inclusion map from the lattice $\langle \mathbb{N}; \leqslant \rangle$ into the lattice $\langle \mathbb{Z}; \leqslant \rangle$ is an injective homomorphism, that is, an embedding. Similarly, given two sets $Y \subseteq X$, the inclusion map from the powerset lattice $\langle \mathcal{P}(Y); \subseteq \rangle$ to the powerset lattice $\langle \mathcal{P}(X); \subseteq \rangle$ is also an embedding. On the other hand, if $Y \subsetneq X$, the map

$$(-) \cap Y \colon \mathcal{P}(X) \to \mathcal{P}(Y)$$

that sends every $Z \subseteq X$ to $Z \cap Y$ is a noninjective homomorphism from $\langle \mathcal{P}(X); \subseteq \rangle$ to $\langle \mathcal{P}(Y); \subseteq \rangle$.

Definition 2.2. Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be algebras of the same type $\rho\colon \mathcal{F} \to \mathbb{N}$. Then $\boldsymbol{A}$ is said to be a *subalgebra* of $\boldsymbol{B}$ if $A \subseteq B$ and $f^{\boldsymbol{A}}$ is the restriction of $f^{\boldsymbol{B}}$ to A, for every $f \in \mathcal{F}$. In this case, we write $\boldsymbol{A} \leqslant \boldsymbol{B}$.

Given a class of algebras K, let

$$\mathbb{I}(\mathsf{K}) := \{\boldsymbol{A} : \boldsymbol{A} \cong \boldsymbol{B} \text{ for some } \boldsymbol{B} \in \mathsf{K}\}$$
$$\mathbb{S}(\mathsf{K}) := \{\boldsymbol{A} : \boldsymbol{A} \leqslant \boldsymbol{B} \text{ for some } \boldsymbol{B} \in \mathsf{K}\}.$$

When $\mathsf{K} = \{\boldsymbol{A}\}$, we write $\mathbb{I}(\boldsymbol{A})$ and $\mathbb{S}(\boldsymbol{A})$ as a shorthand for $\mathbb{I}(\{\boldsymbol{A}\})$ and $\mathbb{S}(\{\boldsymbol{A}\})$, respectively. The following observation is an immediate consequence of the definitions.

Proposition 2.3. *Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be algebras of the same type. Then $\boldsymbol{A} \in \mathbb{IS}(\boldsymbol{B})$ if and only if there exists an embedding $f\colon \boldsymbol{A} \to \boldsymbol{B}$. In this case, $\boldsymbol{A}$ is isomorphic to the unique subalgebra of $\boldsymbol{B}$ with universe $f[A]$.*

As we mentioned, homomorphisms can be used to compare similar algebras.

Definition 2.4. Given two similar algebras $\boldsymbol{A}$ and $\boldsymbol{B}$, we say that $\boldsymbol{A}$ is a *homomorphic image* of $\boldsymbol{B}$ if there exists a surjective homomorphism $f\colon \boldsymbol{B} \to \boldsymbol{A}$.

Accordingly, given a class of algebras K, we set

$$\mathbb{H}(\mathsf{K}) := \{\boldsymbol{A} : \boldsymbol{A} \text{ is a homomorphic image of some } \boldsymbol{B} \in \mathsf{K}\}.$$

As usual, when $\mathsf{K} = \{\boldsymbol{A}\}$, we write $\mathbb{H}(\boldsymbol{A})$ as a shorthand for $\mathbb{H}(\{\boldsymbol{A}\})$.

Observe that every (not necessarily surjective) homomorphism $f\colon \boldsymbol{A} \to \boldsymbol{B}$ induces a homomorphic image of $\boldsymbol{A}$.

Proposition 2.5. *If $f\colon \boldsymbol{A} \to \boldsymbol{B}$ is a homomorphism, then $f[A]$ is the universe of a subalgebra of $\boldsymbol{B}$ that, moreover, is a homomorphic image of $\boldsymbol{A}$.*

Proof. Observe that $f[A]$ is nonempty, because A is. Then consider an n-ary function symbol g of the common type of $\boldsymbol{A}$ and $\boldsymbol{B}$ and $b_1, \dots, b_n \in f[A]$. Clearly, there are $a_1, \dots, a_n \in A$ such that $f(a_i) = b_i$, for every $i \leqslant n$. Since f is a homomorphism from $\boldsymbol{A}$ to $\boldsymbol{B}$, we obtain

$$g^{\boldsymbol{B}}(b_1, \dots, b_n) = g^{\boldsymbol{B}}(f(a_1), \dots, g(a_n)) = f(g^{\boldsymbol{A}}(a_1, \dots, a_n)) \in f[A].$$

Hence, we conclude that $f[A]$ is the universe of a subalgebra $f[\boldsymbol{A}]$ of $\boldsymbol{B}$.

Furthermore, $f\colon \boldsymbol{A} \to f[\boldsymbol{A}]$ is a homomorphism, because for every basic n-ary function symbol g of the common type and $a_1, \dots, a_n \in A$,

$$f(g^{\boldsymbol{A}}(a_1, \dots, a_n)) = g^{\boldsymbol{B}}(f(a_1), \dots, f(a_n)) = g^{f[\boldsymbol{A}]}(f(a_1), \dots, f(a_n)),$$

where the first equality follows from the assumption that $f\colon \boldsymbol{A} \to \boldsymbol{B}$ is a homomorphism. Since the map $f\colon A \to f[A]$ is surjective, we conclude that $f[\boldsymbol{A}] \in \mathbb{H}(\boldsymbol{A})$. ⊠

In view of the above result, when $f\colon \boldsymbol{A} \to \boldsymbol{B}$ is a homomorphism, we denote by $f[\boldsymbol{A}]$ the unique subalgebra of $\boldsymbol{B}$ with universe $f[A]$.

For instance, let $f\colon \mathbb{Z} \to \mathbb{R}$ be the absolute value map, that is, the function defined by the rule

$$f(n) := \text{the absolute value of } n.$$

Observe that f is a nonsurjective homomorphism from the lattice of integers to that of reals. Furthermore, the homomorphic image $f[\langle \mathbb{Z}; \leqslant \rangle]$ of $\langle \mathbb{Z}; \leqslant \rangle$ is the lattice of natural numbers $\langle \mathbb{N}; \leqslant \rangle$, which, in turn, is a subalgebra of lattice of reals.

Notably, the homomorphic images of an algebra $\boldsymbol{A}$ can be "internalized" as special equivalence relations on $\boldsymbol{A}$ as follows.

Definition 2.6. A *congruence* of an algebra $\boldsymbol{A}$ is an equivalence relation θ on A such that, for every basic n-ary opeartion f of $\boldsymbol{A}$ and $a_1, \dots, a_n, c_1, \dots, c_n \in A$,

$$\text{if } \langle a_1, c_1\rangle, \dots, \langle a_n, c_n\rangle \in \theta, \text{ then } \langle f^{\boldsymbol{A}}(a_1, \dots, a_n), f^{\boldsymbol{A}}(c_1, \dots, c_n)\rangle \in \theta. \tag{1}$$

In this case, we often write $a \equiv_\theta c$ as a shorthand for $\langle a, c\rangle \in \theta$. The poset of congruences of $\boldsymbol{A}$ ordered under the inclusion relation will be denoted by $\mathsf{Con}(\boldsymbol{A})$.

A simple induction on the construction of terms shows that, for every congruence θ of $\boldsymbol{A}$ and every term $\varphi(x_1, \dots, x_n)$,

$$\text{if } \langle a_1, c_1\rangle, \dots, \langle a_n, c_n\rangle \in \theta, \text{ then } \langle \varphi^{\boldsymbol{A}}(a_1, \dots, a_n), \varphi^{\boldsymbol{A}}(c_1, \dots, c_n)\rangle \in \theta,$$

for every $a_1, \dots, a_n \in A$. Therefore, congruences preserve not only basic operations, but also arbitrary terms. Furthermore, a simple argument shows that $\mathsf{Con}(\boldsymbol{A})$ is a complete (indeed algebraic) lattice whose maximum is the total relation $A \times A$ and whose minimum is the identity relation $\mathrm{id}_A \coloneqq \{\langle a, a\rangle : a \in A\}$.

Example 2.7 (Heyting algebras)**.** Recall that a *filter* of a Heyting algebra $\boldsymbol{A}$ is a nonempty upset $F \subseteq A$ closed under binary meets. We denote by $\mathsf{Fi}(\boldsymbol{A})$ the poset of filters of $\boldsymbol{A}$ ordered under the inclusion relation. It is easy to see $\mathsf{Fi}(\boldsymbol{A})$ is a complete lattice. Furthermore, the lattices $\mathsf{Fi}(\boldsymbol{A})$ and $\mathsf{Con}(\boldsymbol{A})$ are isomorphic via the inverse isomorphisms

$$\boldsymbol{\Omega}^{\boldsymbol{A}}(-) \colon \mathsf{Fi}(\boldsymbol{A}) \to \mathsf{Con}(\boldsymbol{A}) \;\text{ and }\; \boldsymbol{\tau}^{\boldsymbol{A}}(-) \colon \mathsf{Con}(\boldsymbol{A}) \to \mathsf{Fi}(\boldsymbol{A})$$

defined by the rules

$$\begin{aligned}\boldsymbol{\Omega}^{\boldsymbol{A}}(F) &\coloneqq \{\langle a, c\rangle \in A \times A : a \to c, c \to a \in F\}\\ \boldsymbol{\tau}^{\boldsymbol{A}}(\theta) &\coloneqq \{a \in A : \langle a, 1\rangle \in \theta\}.\end{aligned}$$

Because of this, every congruence θ of a Heyting algebra $\boldsymbol{A}$ is induced by some filter F, in the sense that $\theta = \boldsymbol{\Omega}^{\boldsymbol{A}} F$. ⊠

Example 2.8 (Modal algebras)**.** A *modal algebra* is an algebra $\boldsymbol{A} = \langle A; \wedge, \vee, \neg, \Box, 0, 1\rangle$ such that $\langle A; \wedge, \vee, \neg, 0, 1\rangle$ is a Boolean algebra and $\Box$ is a unary operation such that

$$\Box(a \wedge c) = \Box a \wedge \Box c \;\text{ and }\; \Box 1 = 1,$$

for every $a, c \in A$. An *open filter* of a modal algebra $\boldsymbol{A}$ is a filter of the Boolean reduct of $\boldsymbol{A}$ that, moreover, is closed under the operation $\Box$. The poset of open filters of $\boldsymbol{A}$ ordered under the inclusion relation will be denoted by $\mathsf{Op}(\boldsymbol{A})$. It forms a complete lattice. Furthermore, the lattices $\mathsf{Op}(\boldsymbol{A})$ and $\mathsf{Con}(\boldsymbol{A})$ are isomorphic via the inverse isomorphisms described in Example 2.7. Because of this, every congruence of a modal algebra $\boldsymbol{A}$ has the form $\theta = \boldsymbol{\Omega}^{\boldsymbol{A}} F$, for some open filter F. ⊠

Example 2.9 (Groups)**.** Similarly, it is well known that the lattice of congruences of a group is isomorphic to that of its normal subgroups. Because of this, every congruence of a group is induced by some normal subgroup. ⊠

As we mentioned, there is a tight correspondence between the homomorphic images and the congruences of an algebra $\boldsymbol{A}$. On the one hand, every congruence θ of $\boldsymbol{A}$ gives rise to a homomorphic image $\boldsymbol{A}/\theta$ of $\boldsymbol{A}$. Let $\mathcal{F}$ be the set of function symbols of $\boldsymbol{A}$. Given

$\theta \in \mathsf{Con}(\boldsymbol{A})$ and a basic n-ary function symbol $f \in \mathcal{F}$, let $f^{\boldsymbol{A}/\theta}$ be the n-ary operation on A/θ defined by the rule

$$f^{\boldsymbol{A}/\theta}(a_1/\theta, \dots, a_n/\theta) \coloneqq f^{\boldsymbol{A}}(a_1, \dots, a_n)/\theta.$$

Notice that $f^{\boldsymbol{A}/\theta}$ is well-defined, by condition (1). As a consequence, the structure

$$\boldsymbol{A}/\theta \coloneqq \langle A/\theta; \{f^{\boldsymbol{A}/\theta} : f \in \mathcal{F}\}\rangle$$

is a well-defined algebra of the type as $\boldsymbol{A}$. Furthermore, $\boldsymbol{A}/\theta \in \mathbb{H}(\boldsymbol{A})$, because the map $\pi_\theta \colon A \to A/\theta$, defined, for every $a \in A$, as $\pi_\theta(a) \coloneqq a/\theta$, is a surjective homomorphism from $\boldsymbol{A}$ to $\boldsymbol{A}/\theta$. To prove this, consider $a_1, \dots, a_n \in A$. We have

$$\begin{aligned}
\pi_\theta(f^{\boldsymbol{A}}(a_1, \dots, a_n)) &= f^{\boldsymbol{A}}(a_1, \dots, a_n)/\theta \\
&= f^{\boldsymbol{A}/\theta}(a_1/\theta, \dots, a_n/\theta) \\
&= f^{\boldsymbol{A}/\theta}(\pi_\theta(a_1), \dots, \pi_\theta(a_n)),
\end{aligned}$$

where the second equality follows from the definition of the operation $f^{\boldsymbol{A}/\theta}$.

Corollary 2.10. *If θ is a congruence of an algebra $\boldsymbol{A}$, then $\boldsymbol{A}/\theta$ is a well-defined homomorphic image of $\boldsymbol{A}$.*

In view of the above result, every congruence θ of an algebra $\boldsymbol{A}$ induces a homomorphic image of $\boldsymbol{A}$, namely $\boldsymbol{A}/\theta$. The converse is also true, as we proceed to explain.

Definition 2.11. The *kernel* of a homomorphism $f \colon \boldsymbol{A} \to \boldsymbol{B}$ is the binary relation

$$\mathsf{Ker}(f) \coloneqq \{\langle a, c\rangle \in A \times A : f(a) = f(c)\}.$$

Proposition 2.12. *The kernel of a homomorphism $f \colon \boldsymbol{A} \to \boldsymbol{B}$ is a congruence of $\boldsymbol{A}$.*

Proof. It is obvious that $\mathsf{Ker}(f)$ is an equivalence relation on A. Therefore, to prove that $\mathsf{Ker}(f)$ is a congruence of $\boldsymbol{A}$, it suffices to show that it preserves the basic operations of $\boldsymbol{A}$. Consider a basic n-ary operation g of $\boldsymbol{A}$ and $a_1, \dots, a_n, c_1, \dots, c_n \in A$ such that $\langle a_1, c_1\rangle, \dots, \langle a_n, c_n\rangle \in \mathsf{Ker}(f)$. By the definition of $\mathsf{Ker}(f)$,

$$f(a_i) = f(c_i), \text{ for every } i \leqslant n.$$

It follows that $g^{\boldsymbol{B}}(f(a_1), \dots, f(a_n)) = g^{\boldsymbol{B}}(f(c_1), \dots, f(c_n))$. Since $f \colon \boldsymbol{A} \to \boldsymbol{B}$ is a homomorphism, this yields

$$f(g^{\boldsymbol{A}}(a_1, \dots, a_n)) = g^{\boldsymbol{B}}(f(a_1), \dots, f(a_n)) = g^{\boldsymbol{B}}(f(c_1), \dots, f(c_n)) = f(g^{\boldsymbol{A}}(c_1, \dots, c_n)).$$

Hence, we conclude that $\langle g^{\boldsymbol{A}}(a_1, \dots, a_n), g^{\boldsymbol{A}}(c_1, \dots, c_n)\rangle \in \mathsf{Ker}(f)$, as desired. ⊠

The behaviour of kernels is governed by the next principle.

Fundamental Homomorphism Theorem 2.13. *If $f \colon \boldsymbol{A} \to \boldsymbol{B}$ is a homomorphism with kernel θ, then there exists a unique embedding $g \colon \boldsymbol{A}/\theta \to \boldsymbol{B}$ such that $f = g \circ \pi_\theta$.*

Proof. We begin by proving the existence of g. Let $g \colon A/\theta \to B$ be the map defined as $g(a/\theta) \coloneqq f(a)$, for every $a \in A$. To show that g is well-defined, consider $a, c \in A$ such that $a/\theta = c/\theta$. Since $\theta = \mathsf{Ker}(f)$, this means that $f(a) = f(c)$, as desired. Furthermore, the definition of g guarantees that $f = g \circ \pi_\theta$.

Now, observe g is injective, because, for every $a, c \in A$ such that $g(a/\theta) = g(c/\theta)$, we have $f(a) = f(c)$, that is, $\langle a, c\rangle \in \mathsf{Ker}(f) = \theta$ and, therefore, $a/\theta = c/\theta$. Moreover, for every basic n-ary operation p of $\boldsymbol{A}$ and $a_1, \dots, a_n \in A$, we have

$$\begin{aligned} g(p^{\boldsymbol{A}/\theta}(a_1/\theta, \dots, a_n/\theta)) &= g(p^{\boldsymbol{A}}(a_1, \dots, a_n)/\theta) \\ &= f(p^{\boldsymbol{A}}(a_1, \dots, a_n)) \\ &= p^{\boldsymbol{B}}(f(a_1), \dots, f(a_n)) \\ &= p^{\boldsymbol{B}}(g(a_1/\theta), \dots, g(a_n/\theta)). \end{aligned}$$

The first equality above follows from the definition of $\boldsymbol{A}/\theta$, the second and the last from the definition of g and the third from the assumption that $f\colon \boldsymbol{A} \to \boldsymbol{B}$ is a homomorphism. Hence, we conclude that $g\colon \boldsymbol{A}/\theta \to \boldsymbol{B}$ is a homomorphism and, therefore, an embedding, as desired.

The uniqueness of g follows from the fact that, if a map g^* satisfies the condition in the statement of the theorem, then, for every $a \in A$,

$$f(a) = g^* \circ \pi_\theta(a) = g^*(a/\theta),$$

that is, g^* coincides with g. ⊠

Corollary 2.14. *If $f\colon \boldsymbol{A} \to \boldsymbol{B}$ is a homomorphism, then $f[\boldsymbol{A}] \cong \boldsymbol{A}/\mathsf{Ker}(f)$. In particular, if f is surjective, $\boldsymbol{B} \cong \boldsymbol{A}/\mathsf{Ker}(f)$.*

Proof. In the proof of the Fundamental Homomorphism Theorem we showed that the map $g\colon A/\mathsf{Ker}(f) \to B$, defined by the rule $g(a/\mathsf{Ker}(f)) := f(a)$, is an embedding of $\boldsymbol{A}/\mathsf{Ker}(f)$ into $\boldsymbol{B}$. As g can be viewed as a surjective embedding of $\boldsymbol{A}/\mathsf{Ker}(f)$ into $f[\boldsymbol{A}]$, we conclude that $f[\boldsymbol{A}] \cong \boldsymbol{A}/\mathsf{Ker}(f)$. ⊠

At this stage, it should be clear that if θ is a congruence on an algebra $\boldsymbol{A}$, then $\pi_\theta\colon \boldsymbol{A} \to \boldsymbol{A}/\theta$ is a surjective homomorphism whose kernel is θ. Similarly, if $f\colon \boldsymbol{A} \to \boldsymbol{B}$ is a surjective homomorphism, then $\boldsymbol{A}/\mathsf{Ker}(f) \cong \boldsymbol{B}$, by Corollary 2.14. As a consequence, for every class of algebras K,

$$\mathbb{H}(\mathsf{K}) = \mathbb{I}\{\boldsymbol{A}/\theta : \boldsymbol{A} \in \mathsf{K} \text{ and } \theta \in \mathsf{Con}(\boldsymbol{A})\}. \tag{2}$$

Now, recall that the Cartesian product of a family of sets $\{A_i : i \in I\}$ is the set

$$\prod_{i\in I} A_i := \{f\colon I \to \bigcup_{i \in I} A_i \ :\ f(i) \in A_i, \text{ for all } i \in I\}.$$

In particular, if I is empty, then $\prod_{i\in I} A_i$ is the singleton containing only the empty map.

Definition 2.15. The *direct product* of a family of similar algebras $\{\boldsymbol{A}_i : i \in I\}$ is the unique algebra of the common type whose universe is the Cartesian product $\prod_{i\in I} A_i$ and such that, for every basic n-ary operation symbol f and every $\vec{a}_1, \dots, \vec{a}_n \in \prod_{i\in I} A_i$,

$$f^{\prod_{i\in I}\boldsymbol{A}_i}(\vec{a}_1, \dots, \vec{a}_n)(i) = f^{\boldsymbol{A}_i}(\vec{a}_1(i), \dots, \vec{a}_n(i)), \text{ for every } i \in I.$$

We denote this algebra by $\prod_{i\in I} \boldsymbol{A}_i$.

In this case, for every $j \in I$, the projection map on the j-th component $p_j\colon \prod_{i\in I} A_i \to A_j$, defined by the rule $p_j(\vec{a}) := \vec{a}(j)$, is a surjective homomorphism from $\prod_{i\in I} \boldsymbol{A}_i$ to $\boldsymbol{A}_j$.

Given a class of similar algebras K, we set

$$\mathbb{P}(\mathsf{K}) := \{\boldsymbol{A} : \boldsymbol{A} \text{ is a direct product of a family } \{\boldsymbol{B}_i : i \in I\} \subseteq \mathsf{K}\}.$$

As usual, when $\mathsf{K} = \{\boldsymbol{A}\}$, we write $\mathbb{P}(\boldsymbol{A})$ as a shorthand for $\mathbb{P}(\{\boldsymbol{A}\})$.

Notice that up to isomorphism, there exists a unique one-element algebra of a given type. Because of this, one-element algebras are called *trivial*. Accordingly, when the set of indexes I is empty, the direct product $\prod_{i \in I} \boldsymbol{A}_i$ is the trivial algebra of the given type. It follows that $\mathbb{P}(\mathsf{K})$ contains always a trivial algebra, for every class of similar algebras K.

Example 2.16 (Powerset algebras)**.** Boolean algebras of the form $\langle \mathcal{P}(X); \cap, \cup, -, \emptyset, X \rangle$ are called *powerset Boolean algebras*. Let $\boldsymbol{B}$ be the two-element Boolean algebra and observe that $\mathbb{IP}(\boldsymbol{B})$ is the class of algebras isomorphic to some powerset Boolean algebra. To prove this, observe that every powerset Boolean algebra $\mathcal{P}(X)$ is isomorphic to a direct product of $\boldsymbol{B}$ via the *characteristic function* $f_X \colon \mathcal{P}(X) \to \prod_{x \in X} \boldsymbol{B}_x$, defined by the rule

$$f(Y)(x) := \begin{cases} 1 & \text{if } x \in Y \\ 0 & \text{if } x \notin Y, \end{cases}$$

where $Y \in \mathcal{P}(X)$ and $x \in X$. By the same token, every direct product $\prod_{i \in I} \boldsymbol{B}_i$ of $\boldsymbol{B}$ is isomorphic to the powerset Boolean algebra $\mathcal{P}(I)$ via the isomorphism f_I. ⊠

We close this section by reviewing the subdirect product construction.

Definition 2.17. A subalgebra $\boldsymbol{B}$ of a direct product $\prod_{i \in I} \boldsymbol{A}_i$ is said to be a *subdirect product* of $\{\boldsymbol{A}_i : i \in I\}$ if the projection map p_i is surjective, for every $i \in I$. Similarly, an embedding $f \colon \boldsymbol{B} \to \prod_{i \in I} \boldsymbol{A}_i$ is said to be *subdirect* when $f[\boldsymbol{B}]$ is a subdirect product of the family $\{\boldsymbol{A}_i : i \in I\}$.

Given a class of similar algebras K, we set

$$\mathbb{P}_{\mathrm{SD}}(\mathsf{K}) := \{\boldsymbol{A} : \boldsymbol{A} \text{ is a subdirect direct product of a family } \{\boldsymbol{B}_i : i \in I\} \subseteq \mathsf{K}\}.$$

As usual, when $\mathsf{K} = \{\boldsymbol{A}\}$, we write $\mathbb{P}_{\mathrm{SD}}(\boldsymbol{A})$ as a shorthand for $\mathbb{P}_{\mathrm{SD}}(\{\boldsymbol{A}\})$. Clearly, $\mathbb{P}_{\mathrm{SD}}(\mathsf{K}) \subseteq \mathbb{SP}(\mathsf{K})$. Furthermore, $\mathbb{P}_{\mathrm{SD}}(\mathsf{K})$ contains always a trivial algebra.

Example 2.18 (Distributive lattices)**.** Let DL be the class of distributive lattices and $\boldsymbol{B}$ be the two-element distributive lattice. Birkhoff's Representation Theorem states that $\mathsf{DL} = \mathbb{IP}_{\mathrm{SD}}(\boldsymbol{B})$. The inclusion $\mathbb{IP}_{\mathrm{SD}}(\boldsymbol{B}) \subseteq \mathsf{DL}$ follows from the fact that DL is closed under $\mathbb{I}, \mathbb{S}$ and $\mathbb{P}$. For the other inclusion, consider a distributive lattice $\boldsymbol{A}$ and let I be the set of its prime filters. By Birkhoff's Representation Theorem, the map

$$\gamma \colon \boldsymbol{A} \to \prod_{F \in I} \boldsymbol{B}_F,$$

defined, for every $a \in A$ and $F \in I$, by the rule

$$\gamma(a)(F) := \begin{cases} 1 & \text{if } a \in F \\ 0 & \text{if } a \notin F, \end{cases}$$

is a well-defined subdirect embedding. ⊠

Example 2.19 (Boolean algebras)**.** Similarly, Stone's Representation Theorem states that the class of Boolean algebras coincides with $\mathbb{IP}_{\mathrm{SD}}(\boldsymbol{B})$, where $\boldsymbol{B}$ the two-element Boolean algebra. ⊠

The next result provides a general recipe to construct subdirect products.

Proposition 2.20. *Let $\boldsymbol{A}$ be an algebra and $\{\theta_i : i \in I\} \subseteq \mathsf{Con}(\boldsymbol{A})$. Then the map*
$$f\colon \boldsymbol{A}/\bigcap_{i\in I}\theta_i \to \prod_{i\in I}\boldsymbol{A}/\theta_i,$$
defined, for every $a \in A$ and $j \in I$, as
$$f(a/\bigcap_{i\in I}\theta_i)(j) := a/\theta_j,$$
is a subdirect embedding.

Proof. For the sake of readability, set $\boldsymbol{B} := \boldsymbol{A}/\bigcap_{i\in I}\theta_i$. To prove that f is injective, consider $a, c \in A$ such that $\langle a, c\rangle \notin \bigcap_{i\in I}\theta_i$. Then there exists $j \in I$ such that $\langle a, c\rangle \notin \theta_j$ and, therefore,
$$f(a/\bigcap_{i\in I}\theta_i)(j) := a/\theta_j \neq c/\theta_j = f(c/\bigcap_{i\in I}\theta_i)(j).$$
It follows that $f(a/\bigcap_{i\in I}\theta_i) \neq f(c/\bigcap_{i\in I}\theta_i)$. Thus, f is injective. Moreover, by the definition of f, the composition $p_i \circ f\colon \boldsymbol{B} \to \boldsymbol{A}/\theta_i$ is surjective, for every $i \in I$.

It only remains to prove that f is a homomorphism. Consider an n-ary basic operation g and $a_1, \dots, a_n \in A$. For every $j \in I$, we have
$$\begin{aligned}
f(g^{\boldsymbol{B}}(a_1/\bigcap_{i\in I}\theta_i, \dots, a_n/\bigcap_{i\in I}\theta_i))(j) &= f(g^{\boldsymbol{A}}(a_1, \dots, a_n)/\bigcap_{i\in I}\theta_i)(j)\\
&= g^{\boldsymbol{A}}(a_1, \dots, a_n)/\theta_j\\
&= g^{\boldsymbol{A}/\theta_j}(a_1/\theta_j, \dots, a_n/\theta_j)\\
&= g^{\boldsymbol{A}/\theta_j}(f(a_1/\bigcap_{i\in I}\theta_i)(j), \dots, f(a_n/\bigcap_{i\in I}\theta_i)(j))\\
&= g^{\prod_{i\in I}\boldsymbol{A}/\theta_i}(f(a_1/\bigcap_{i\in I}\theta_i), \dots, f(a_n/\bigcap_{i\in I}\theta_i))(j).
\end{aligned}$$
It follows that
$$f(g^{\boldsymbol{B}}(a_1/\bigcap_{i\in I}\theta_i, \dots, a_n/\bigcap_{i\in I}\theta_i)) = g^{\prod_{i\in I}\boldsymbol{A}/\theta_i}(f(a_1/\bigcap_{i\in I}\theta_i), \dots, f(a_n/\bigcap_{i\in I}\theta_i)). \qquad \boxtimes$$

3. Propositional logics and equational completeness theorems

For general information on propositional logics we refer the reader to [8, 9, 10, 11]. Recall that a *closure operator* on a set A is a map $C\colon \mathcal{P}(A) \to \mathcal{P}(A)$ such that, for every $X \subseteq Y \subseteq A$,
$$X \subseteq C(X) = C(C(X)) \text{ and } C(X) \subseteq C(Y).$$
Given a closure operator C on A, a subset $X \subseteq A$ is said to be *closed* if $X = C(X)$. A *closure system* on A is a family $\mathcal{C} \subseteq \mathcal{P}(A)$ that contains A and such that $\bigcap \mathcal{F}$, for every nonempty $\mathcal{F} \subseteq \mathcal{C}$. Closure operators and systems on A are two faces of the same coin. More precisely, if the family of closed sets of a closure operator on A is a closure system on A. On the other hand, if $\mathcal{C}$ is a closure system on A, then the map $C\colon \mathcal{P}(A) \to \mathcal{P}(A)$, defined by the rule
$$C(X) := \bigcap\{Y \in \mathcal{C} : X \subseteq Y\},$$
is a closure operator on A. These transformations between closure operators and systems on A are one inverse to the other.

Exercise 3.1. Prove that these transformations are well-defined and one inverse to the other. ⊠

Another way of presenting closure operators or systems is by means of the following concept.

Definition 3.2. A *consequence relation* on a set A is a binary relation $\vdash\, \subseteq \mathcal{P}(A) \times A$ such that, for every $X \cup Y \cup \{a\} \subseteq A$,

(i) if $a \in X$, then $X \vdash a$; and
(ii) if $X \vdash y$ for all $y \in Y$ and $Y \vdash a$, then $X \vdash a$.

Furthermore, $\vdash$ is said to be *finitary* when, for every $X \cup \{a\} \subseteq A$,

$$\text{if } X \vdash a, \text{ there exists a finite } Y \subseteq X \text{ such that } Y \vdash a.$$

Remark 3.3. The relation $X \vdash a$ should be read, intuitively, as "X proves a" or "a follows from X". In this reading, the demand expressed by condition (i) is rather natural, while (ii) is an abstract of the Cut rule. ⊠

Formally speaking, a consequence relation on a set A is a binary relation $\vdash\, \subseteq \mathcal{P}(A) \times A$. However, to simplify the notation, we will often write $a_1, \dots, a_n \vdash c$ as a shorthand for $\{a_1, \dots, a_n\} \vdash c$. Similarly, we will use $X, a \vdash c$ as a shorthand for $X \cup \{a\} \vdash c$. Lastly, for every set of formulas $X \cup Y \cup \{a, c\}$, we write

(i) $X \vdash Y$, when $X \vdash y$ for every $y \in Y$;
(ii) $a \dashv\vdash c$, when $a \vdash c$ and $c \vdash a$; and
(iii) $X \dashv\vdash Y$, when $X \vdash Y$ and $Y \vdash X$.

Definition 3.4. Let $\vdash$ be a consequence relation on a set A. A *theory* of $\vdash$ is a subset $X \subseteq A$ such that, for every $a \in A$, if $X \vdash a$, then $a \in X$. The set of theories of A will be denoted by $\mathcal{T}h(\vdash)$.

It is easy to see that $\mathcal{T}h(\vdash)$ is a closure system on A. Moreover, given a closure operator C on A, the following is a consequence relation on A:

$$\{\langle X, a\rangle \in \mathcal{P}(A) \times A : a \in C(X)\}.$$

Together with the correspondence between closure systems and operators, these transformations induce a one-to-one correspondence between consequence relations, closure operators and closure systems on A.

Exercise 3.5. Prove these facts. ⊠

In the context of logic, the term algebra $\boldsymbol{T}_\rho(Var)$ is often called the *algebra of formulas* (of type ρ) and its elements are referred to as *formulas*. An *endomorphism* of an algebra $\boldsymbol{A}$ is a homomorphism whose domain and codomain is $\boldsymbol{A}$. Endomorphisms of the algebra of formulas play a fundamental role in logic.

Definition 3.6. A *substitution* of type ρ is an endomorphism σ of $\boldsymbol{T}_\rho(Var)$.

When the type ρ is clear from the context, we will simply say that σ is a substitution.

In view of Proposition 1.3 and of the fact that Var is a set of generators for $\boldsymbol{T}_\rho(Var)$, every function $\sigma \colon Var \to T_\rho(Var)$ can be uniquely extended to a substitution σ^+ of type ρ, namely the function defined by the rule

$$\varphi(x_1, \dots, x_n) \longmapsto \varphi(\sigma(x_1), \dots, \sigma(x_n)).$$

Because of this, substitutions of type ρ can be presented by exhibiting functions $\sigma \colon Var \to T_\rho(Var)$.

Definition 3.7. A *logic* of type ρ is a consequence relation $\vdash$ on the set of formulas $T_\rho(Var)$ that, moreover, is *substitution invariant* in the sense that for every substitution σ of type ρ and every set of formulas $\Gamma \cup \{\varphi\} \subseteq T_\rho(Var)$,

$$\text{if } \Gamma \vdash \varphi, \text{ then } \sigma[\Gamma] \vdash \sigma(\varphi).$$

Remark 3.8. As mentioned above, $\Gamma \vdash \varphi$ should be read as "Γ proves φ" or "φ follows from Γ". The requirement that $\vdash$ is substitution invariant, instead, is intended to capture the idea that logical inferences are true only in virtue of their form (as opposed to their content). ⊠

Example 3.9 (Hilbert calculi)**.** We work within a fixed, but arbitrary, type ρ. A *rule* is an expression of the form $\Gamma \rhd \varphi$, where $\Gamma \cup \{\varphi\} \subseteq T_\rho(Var)$. In this case, Γ is said to be the set of *premises* of the rule and φ the *conclusion*. When $\Gamma = \emptyset$, the rule $\Gamma \rhd \varphi$ is sometimes called an *axiom*. A *Hilbert calculus* is a set of rules.

Every Hilbert calculus H induces a logic, as we proceed to explain. Consider a set of formulas $\Gamma \cup \{\varphi\} \subseteq T_\rho(Var)$. A *proof of φ from Γ in* H is a well-ordered sequence $\langle \psi_\alpha : \alpha \leqslant \gamma \rangle$ of formulas $\psi_\alpha \in T_\rho(Var)$ whose last element ψ_γ is φ and such that, for every $\alpha \leqslant \gamma$, either $\psi_\alpha \in \Gamma$ or there exist a substitution σ and a rule $\Delta \rhd \delta$ in H such that the formulas in $\sigma[\Delta]$ occur in the initial segment $\langle \psi_\beta : \beta < \alpha \rangle$ and $\psi_\alpha = \sigma(\delta)$.

The logic $\vdash_{\mathsf{H}}$ induced by H is defined, for every $\Gamma \cup \{\varphi\} \subseteq T_\rho(Var)$, as

$$\Gamma \vdash_{\mathsf{H}} \varphi \iff \text{there exists a proof of } \varphi \text{ from } \Gamma \text{ in } \mathsf{H}.$$

As expected, $\vdash_{\mathsf{H}}$ is a logic in the sense of Definition 3.7. Furthermore, it is the least logic $\vdash$ such that $\Gamma \vdash \varphi$, for every rule $\Gamma \rhd \varphi$ in H.

A logic $\vdash$ is said to be *axiomatized* by a Hilbert calculus H when it coincides with $\vdash_{\mathsf{H}}$. Notice that every logic $\vdash$ is vacuously axiomatized by the Hilbert calculus

$$\{\Gamma \rhd \varphi : \Gamma \vdash \varphi\}.$$

Because of this, axiomatizations in terms of Hilbert calculi H acquire special interest when H is finite or, at least, recursive. ⊠

When no confusion shall arise, given a sequence $\vec{a}$ and a set A, we write $\vec{a} \in A$ to indicate that the elements of the sequence $\vec{a}$ belong to A. The following concept is instrumental to exhibit further examples of logics.

Definition 3.10. Let K be a class of similar algebras We define a binary relation $\vDash_{\mathsf{K}} \subseteq \mathcal{P}(E_\rho(Var)) \times E_\rho(Var)$ as follows:

$$\begin{aligned} \Theta \vDash_{\mathsf{K}} \varepsilon \approx \delta \iff & \text{for every } \boldsymbol{A} \in \mathsf{K} \text{ and every } \vec{a} \in A, \\ & \quad \text{if } \varphi^{\boldsymbol{A}}(\vec{a}) = \psi^{\boldsymbol{A}}(\vec{a}) \text{ for all } \varphi \approx \psi \in \Theta, \text{ then } \varepsilon^{\boldsymbol{A}}(\vec{a}) = \delta^{\boldsymbol{A}}(\vec{a}). \end{aligned}$$

The relation $\vDash_{\mathsf{K}}$ is known as the *equational consequence relative to* K.

Example 3.11 (Equationally defined logics)**.** We work within a fixed, but arbitrary, type ρ. Given a set of equations $\boldsymbol{\tau}(x)$ in a single variable x and a set of formulas $\Gamma \cup \{\varphi\} \subseteq T(Var)$, we abbreviate

$$\{\varepsilon(\varphi) \approx \delta(\varphi) \colon \varepsilon \approx \delta \in \boldsymbol{\tau}\} \text{ as } \boldsymbol{\tau}(\varphi), \text{ and } \bigcup_{\gamma \in \Gamma} \boldsymbol{\tau}(\gamma) \text{ as } \boldsymbol{\tau}[\Gamma].$$

Given a class of algebras K and a set of equations $\boldsymbol{\tau}(x)$, we define a logic $\vdash_{\mathsf{K},\tau}$ as follows: for every $\Gamma \cup \{\varphi\} \subseteq T(Var)$,

$$\Gamma \vdash_{\mathsf{K},\boldsymbol{\tau}} \varphi \iff \boldsymbol{\tau}[\Gamma] \vDash_{\mathsf{K}} \boldsymbol{\tau}(\varphi).$$

It is easy to prove that $\vdash_{\mathsf{K},\tau}$ is indeed a logic in the sense of Definition 3.7. Notice that, in this case, $\vdash$ is related to K by a *completeness theorem* witnessed by the set of equations $\boldsymbol{\tau}(x)$ that allows to translate formulas into equations and, therefore, to interpret $\vdash_{\mathsf{K},\tau}$ into $\vDash_{\mathsf{K}}$.

For instance, the completeness theorem of classical propositional logic **CPC** with respect to the class of Boolean algebras BA states precisely that **CPC** coincides with $\vdash_{\mathsf{BA},\boldsymbol{\tau}}$ where $\boldsymbol{\tau} = \{x \approx 1\}$. Similarly, the completeness theorem of intuitionistic propositional logic **IPC** with respect to the class of Heyting algebras HA states precisely that **IPC** coincides with $\vdash_{\mathsf{HA},\boldsymbol{\tau}}$ where $\boldsymbol{\tau} = \{x \approx 1\}$. Because of this, **CPC** and **IPC** can be defined as follows: for every set of formulas $\Gamma \cup \{\varphi\}$ of the appropriate type,

$$\Gamma \vdash_{\mathbf{CPC}} \varphi \iff \boldsymbol{\tau}[\Gamma] \vDash_{\mathsf{BA}} \boldsymbol{\tau}(\varphi)$$
$$\Gamma \vdash_{\mathbf{IPC}} \varphi \iff \boldsymbol{\tau}[\Gamma] \vDash_{\mathsf{HA}} \boldsymbol{\tau}(\varphi),$$

where $\boldsymbol{\tau} = \{x \approx 1\}$. ⊠

The relation between logic and algebra is often explained in terms of the existence of equational completeness theorems. The following definition makes this concept precise. As we will see, however, equational completeness theorems alone are not sufficient to account for the relation between logic and algebra.

Definition 3.12. A logic $\vdash$ is said to admit an *equational completeness theorem* if there are a set of equations $\boldsymbol{\tau}(x)$ and a class K of algebras such that for all $\Gamma \cup \{\varphi\} \subseteq T(Var)$,

$$\Gamma \vdash \varphi \iff \boldsymbol{\tau}[\Gamma] \vDash_{\mathsf{K}} \boldsymbol{\tau}(\varphi).$$

In this case, $\vdash$ coincides with $\vdash_{\mathsf{K},\boldsymbol{\tau}}$ and K is said to be a $\boldsymbol{\tau}$*-algebraic semantics* (or simply an *algebraic semantics*) for $\vdash$.

This notion was introduced in [4] and studied in depth in [5, 15, 16]. For instance, the classes of Boolean and Heyting algebras are, respectively, $\boldsymbol{\tau}$-algebraic semantics for **CPC** and **IPC** where $\boldsymbol{\tau} = \{x \approx 1\}$.

Another familiar example of equational completeness theorem arises from the field of modal logic. Let Fr be the class of all Kripke frames. We can associate two distinct logics with Fr, see for instance [13, 14]. The *global consequence* $\mathbf{K}_g$ of the modal system $\mathbf{K}$ is the logic defined, for every set of modal formulas $\Gamma \cup \{\varphi\}$, as follows:

$$\begin{aligned}\Gamma \vdash_{\mathbf{K}_g} \varphi \iff & \text{ for every } \langle W, R\rangle \in \mathsf{Fr} \text{ and evaluation } v \text{ in } \langle W, R\rangle,\\ & \text{ if } w, v \Vdash \Gamma \text{ for all } w \in W, \text{ then } w, v \Vdash \varphi \text{ for all } w \in W.\end{aligned}$$

On the other hand, the *local consequence* $\mathbf{K}_\ell$ of the modal system $\mathbf{K}$ is defined, for every set of modal formulas $\Gamma \cup \{\varphi\}$, as follows:

$$\begin{aligned}\Gamma \vdash_{\mathbf{K}_\ell} \varphi \iff & \text{ for every } \langle W, R\rangle \in \mathsf{Fr}, w \in W, \text{ and evaluation } v \text{ in } \langle W, R\rangle,\\ & \text{ if } w, v \Vdash \Gamma, \text{ then } w, v \Vdash \varphi.\end{aligned}$$

It is easy to see that $\mathbf{K}_g$ and $\mathbf{K}_\ell$ are logics. Moreover, they are distinct, because

$$x \vdash_{\mathbf{K}_g} \Box x \text{ and } x \nvdash_{\mathbf{K}_\ell} \Box x. \tag{3}$$

Exercise 3.13. Prove that $\mathbf{K}_g$ and $\mathbf{K}_\ell$ are logics. Notice also that the modal system $\mathbf{K}$ is not a logic itself, because it is not a consequence relation. Indeed, there are two ways to turn $\mathbf{K}$ into a logic, namely, $\mathbf{K}_g$ and $\mathbf{K}_\ell$. ⊠

Exercise 3.14. Prove that $\mathbf{K}_g$ and $\mathbf{K}_\ell$ have the same *theorems*, i.e., formulas provable from the empty set. Prove also that the set of their theorems is the modal system $\mathbf{K}$. This indicates that, even in the modal setting, logics should not be identified with their sets of theorems. ⊠

The global consequence $\mathbf{K}_g$ is related to the class MA of modal algebras by the following equational completeness theorem.

Theorem 3.15. *For every set $\Gamma \cup \{\varphi\}$ of modal formulas,*

$$\Gamma \vdash_{\mathbf{K}_g} \varphi \iff \boldsymbol{\tau}[\Gamma] \vDash_{\mathsf{MA}} \boldsymbol{\tau}(\varphi),$$

where $\boldsymbol{\tau} = \{x \approx 1\}$. Consequently, the class of modal algebras is a $\boldsymbol{\tau}$-algebraic semantics for $\mathbf{K}_g$.

In order to prove it, recall that a filter on a Boolean algebra $\boldsymbol{A}$ is said to be *proper* when it differs from A. Moreover, a proper filter U of $\boldsymbol{A}$ is said to be a *ultrafilter* of $\boldsymbol{A}$ if it is maximal among the proper filters of $\boldsymbol{A}$ or, equivalently, if

$$a \in U \text{ or } \neg a \in U, \text{ for every } a \in A.$$

While the following result holds in ZFC, it cannot be proved in ZF (although it is strictly weaker then the axiom of choice).

Ultrafilter Lemma 3.16. *Every proper filter on a Boolean algebra can be extended to a ultrafilter.*

We are now ready to prove Theorem 3.15.

Proof sketch. It suffices to prove that

$$\Gamma \nvdash_{\mathbf{K}_g} \varphi \iff \boldsymbol{\tau}[\Gamma] \nvDash_{\mathsf{MA}} \boldsymbol{\tau}(\varphi).$$

Suppose first that $\Gamma \nvdash_{\mathbf{K}_g} \varphi$. Then there are a Kripke frame $\langle W, R\rangle$, an evaluation v in it and a world u such that

$$w, v \Vdash \Gamma \text{ for all } w \in W \quad\text{and}\quad u, v \nVdash \varphi. \tag{4}$$

Then consider the complex algebra of $\langle W, R\rangle$, that is, the structure

$$\boldsymbol{A} := \langle \mathcal{P}(W); \cap, \cup, -, \Box, \emptyset, W\rangle,$$

where $-$ is set theoretic complement and, for every $V \subseteq W$,

$$\Box V := \{w \in W : \text{if } \langle w, t\rangle \in R, \text{ then } t \in V\}.$$

It is easy to prove that $\boldsymbol{A}$ is a modal algebra. Then consider the unique homomorphism $f\colon \boldsymbol{T}(Var) \to \boldsymbol{A}$ such that

$$f(x) = \{w \in W : w, v \Vdash x\},$$

for every $x \in Var$. A simple induction of the construction of terms shows that, for every formula ψ,

$$f(\psi) = \{w \in W : w, v \Vdash \psi\}.$$

Together with (4), this yields

$$f[\Gamma] \subseteq \{W\} \quad\text{and}\quad f(\varphi) \neq W.$$

Hence, we conclude that $\boldsymbol{\tau}[\Gamma] \nvDash_{\mathsf{MA}} \boldsymbol{\tau}(\varphi)$.

To prove the converse, suppose that $\boldsymbol{\tau}[\Gamma] \nvDash_{\mathsf{MA}} \boldsymbol{\tau}(\varphi)$. Then there are a modal algebra $\boldsymbol{A}$ and a homomorphism $f\colon \boldsymbol{T}(Var) \to \boldsymbol{A}$ such that

$$f[\Gamma] \subseteq \{1\} \text{ and } f(\varphi) \neq 1.$$

Then consider the Kripke frame dual to $\boldsymbol{A}$, that is, the structure $\langle W, R\rangle$, where W is the set of ultrafilters of $\boldsymbol{A}$ and R the binary relation on W defined as follows:

$$R := \{\langle U, V\rangle \in W \times W : \{a \in A : \Box a \in U\} \subseteq V\}.$$

Let then $v\colon Var \to \mathcal{P}(W)$ be the evaluation in $\langle W, R\rangle$ defined by the rule

$$v(x) := \{U \in W : f(x) \in U\}.$$

An easy induction on the construction of terms shows that, for every formula ψ,

$$f(\psi) = \{U \in W : U, v \Vdash \psi\}. \tag{5}$$

Now, since $f(\varphi) \neq 1$, the Ultrafilter Lemma guarantees the existence of an ultrafilter F such that $f(\varphi) \notin F$. Furthermore, as every ultrafilter contain 1, from $f[\Gamma] \subseteq \{1\}$ it follows that $f[\Gamma] \subseteq U$, for all $U \in W$. In short,

$$f[\Gamma] \subseteq U \text{ for all } U \in W \text{ and } f(\varphi) \notin F.$$

Together with (5), this yields

$$U, v \Vdash \Gamma \text{ for all } U \in W \text{ and } F, v \nVdash \varphi.$$

Hence, we conclude that $\Gamma \nvdash_{\mathbf{K}_g} \varphi$. ⊠

At this stage, it is tempting to conjecture that the relation between logic and algebra can be explained in terms of equational completeness theorems only. As we anticipated, however, this is not the case. For instance, the relation between **CPC** and BA cannot be explained in terms of completeness theorems only, because the class of Heyting algebras HA is also an algebraic semantics for **CPC**. To explain why, it is convenient to recall the following classical result relating **CPC** and **IPC** [12].

Givenko's Theorem 3.17. *For every set of formulas $\Gamma \cup \{\varphi\}$,*

$$\Gamma \vdash_{\mathbf{CPC}} \varphi \iff \{\neg\neg\gamma\colon \gamma \in \Gamma\} \vdash_{\mathbf{IPC}} \neg\neg\varphi.$$

As a consequence, we obtain the desired result.

Corollary 3.18. *The class of Heyting algebras is an algebraic semantics for* **CPC**.

Proof. For every set of formulas $\Gamma \cup \{\varphi\}$, we have

$$\begin{aligned}\Gamma \vdash_{\mathbf{CPC}} \varphi &\iff \{\neg\neg\gamma\colon \gamma \in \Gamma\} \vdash_{\mathbf{IPC}} \neg\neg\varphi \\ &\iff \{\neg\neg\gamma \approx 1\colon \gamma \in \Gamma\} \vDash_{\mathsf{HA}} \neg\neg\varphi \approx 1.\end{aligned}$$

The first equivalent above is Glivenko's Theorem, while the second is a consequence of the completeness theorem of **IPC** with respect to HA. As a consequence, the class of Heyting algebras is a $\boldsymbol{\tau}$-algebraic semantics for **IPC**, where $\boldsymbol{\tau} = \{\neg\neg x \approx 1\}$. ⊠

This means that the univocal relation between **CPC** and the class of Boolean algebras cannot be explained in terms of the existence of completeness theorems only. As we shall see, this relation arises from a deeper phenomenon, known as *algebraizability* [4, 9].

Exercise 3.19. One may wonder whether the fact that **CPC** has many distinct algebraic semantics cannot be amended by restricting our attention to $\boldsymbol{\tau}$-algebraic semantics where $\boldsymbol{\tau} = \{x \approx 1\}$. This is not the case, as this exercise asks you to check. Let $\boldsymbol{A}$ be the three-element algebra $\langle\{0, 1, a\}; \wedge, \vee, \neg, 0, 1\rangle$ where $\langle A; \wedge, \vee\rangle$ is the lattice with order $0 < a < 1$ and $\neg\colon A \to A$ is the map described by the rule

$$\neg 0 = \neg a = 1 \text{ and } \neg 1 = 0.$$

Clearly, $\boldsymbol{A}$ is not a Boolean algebra (as there is no three-element Boolean algebra). Prove that $\{\boldsymbol{A}\}$ is $\boldsymbol{\tau}$-algebraic semantics for **CPC** where $\boldsymbol{\tau} = \{x \approx 1\}$. Hint: use the fact that the two-element Boolean algebra is a homomorphic image of $\boldsymbol{A}$. ⊠

Indeed the existence of equational completeness theorems between a logic and a class of algebras turns out to be a very weak relation, as shown in [5, 15]. For instance, while many interesting logics lack a natural equational completeness theorem, they still admit a nonstandard one. This is the case of $\mathbf{K}_\ell$, as we proceed to explain.

A logic $\vdash$ is said to be *protoalgebraic* if there exists a set $\Delta(x, y)$ of formulas such that $\emptyset \vdash \Delta(x, x)$ and $x, \Delta(x, y) \vdash y$. Notice that all logics $\vdash$ with a binary connective $\to$ such that $\emptyset \vdash x \to x$ and $x, x \to y \vdash y$ are protoalgebraic, as witnessed by the set $\Delta := \{x \to y\}$. Furthermore, a logic $\vdash$ is said to be *nontrivial* if $x \nvdash y$.

Theorem 3.20 (M. [15, Thm. 9.3])**.** *A nontrivial protoalgebraic logic. $\vdash$ has an algebraic semantics if and only it there are two distinct formulas φ and ψ that are logically equivalent in the sense that*

$$\delta(\varphi, \vec{z}) \dashv\vdash \delta(\psi, \vec{z}), \textit{ for all } \delta(x, \vec{z}) \in T(\mathit{Var}).$$

As a consequence, we obtain the following.

Corollary 3.21. *The logic $\mathbf{K}_\ell$ has an algebraic semantics.*

Proof. Clearly, $\mathbf{K}_\ell$ is nontrivial and protoalgebraic. Furthermore, the formulas x and $x \wedge x$ are distinct, but logical equivalent in $\mathbf{K}_\ell$. Therefore, $\mathbf{K}_\ell$ has an algebraic semantics in view of Theorem 3.20. ⊠

On the other hand, $\mathbf{K}_\ell$ lacks any natural equational completeness theorem.

Theorem 3.22 (M. [15, Cor. 9.7])**.** *No class of modal algebras is an algebraic semantics for $\mathbf{K}_\ell$.*

Proof. We begin by proving that, for all $\varphi, \psi \in T(\mathit{Var})$,

$$\mathsf{MA} \vDash \varphi \approx \psi \iff \varphi \dashv\vdash_{\mathbf{K}_\ell} \psi. \tag{6}$$

To this end, observe that

$$\begin{aligned} \mathsf{MA} \vDash \varphi \approx \psi &\iff \mathsf{MA} \vDash \varphi \leftrightarrow \psi \approx 1 \\ &\iff \emptyset \vdash_{\mathbf{K}_g} \varphi \leftrightarrow \psi \\ &\iff \emptyset \vdash_{\mathbf{K}_\ell} \varphi \leftrightarrow \psi \\ &\iff \varphi \dashv\vdash_{\mathbf{K}_\ell} \psi. \end{aligned}$$

The above equivalence are justified as follows. The first is an easy property of Boolean algebras, the second is a consequence of Theorem 3.15, the third holds because $\mathbf{K}_g$ and $\mathbf{K}_\ell$ have the same theorems (see Exercise 3.14) and the last one because $\mathbf{K}_\ell$ has a standard deduction theorem.

Now, suppose, with a view to contradiction, that $\mathbf{K}_\ell$ has a $\boldsymbol{\tau}$-algebraic semantics $\mathsf{K} \subseteq \mathsf{MA}$. This implies that there exists an equation $\varepsilon \approx \delta \in \boldsymbol{\tau}$ such that $\mathsf{MA} \nvDash \varepsilon \approx \delta$. Thus, in view of the above display, we can assume, by symmetry, that $\varepsilon \nvdash_{\mathbf{K}_\ell} \delta$. This means that there are a Kripke frame $\mathbb{X} = \langle X, R \rangle$, an element $w \in X$ and a valuation v in $\mathbb{X}$ such that $w, v \Vdash \varepsilon$ and $w, v \nVdash \delta$.

Let $\mathbb{X}^+ = \langle X^+; R^+ \rangle$ be the Kripke frame obtained by adding a new point w^+ to $\mathbb{X}$ and defining the relation R^+ as follows:

$$\langle p, q \rangle \in R^+ \iff p = w^+ \text{ or } \langle p, q \rangle \in R.$$

Let also v^+ be the unique evaluation in $\mathbb{X}^+$ such that for every $y \in \mathit{Var}$ and $q \in \mathbb{X}^+$:

$$q, v^+ \Vdash y \iff \text{ either } (q \in X \text{ and } q, v \Vdash y) \text{ or } q = w^+.$$

From the definition of $\mathbb{X}^+$ and v^+ it follows that

$$q, v^+ \Vdash \varphi \iff q, v \Vdash \varphi$$

for all $\varphi \in T(\mathit{Var})$ and $q \in \mathbb{X}$. Consequently, as $w, v \Vdash \varepsilon$ and $w, v \nVdash \delta$,

$$w^+, v^+ \Vdash x \text{ and } w^+, v^+ \nVdash \Box(\varepsilon \to \delta).$$

This implies

$$x \nvdash_{\mathbf{K}_\ell} \Box(\varepsilon \to \delta).$$

On the other hand, clearly $\emptyset \vdash_{\mathbf{K}_\ell} \Box(\delta \to \delta)$. Consequently,

$$x, \Box(\delta \to \delta) \nvdash_{\mathbf{K}_\ell} \Box(\varepsilon \to \delta). \tag{7}$$

Now, observe that, for every $\varphi, \psi \in T(\mathit{Var})$,

$$\varepsilon(x) \approx \delta(x), \varphi(\Box(\delta \to \delta)) \approx \psi(\Box(\delta \to \delta)) \vDash_{\mathsf{K}} \varphi(\Box(\varepsilon \to \delta)) \approx \psi(\Box(\varepsilon \to \delta)).$$

Since $\varepsilon \approx \delta \in \boldsymbol{\tau}(x)$, this implies

$$\boldsymbol{\tau}(x), \boldsymbol{\tau}(\Box(\delta \to \delta)) \vDash_{\mathsf{K}} \boldsymbol{\tau}(\Box(\varepsilon \to \delta)).$$

Since K is a $\boldsymbol{\tau}$-algebraic semantics for $\mathbf{K}_\ell$, this yields $x, \Box(\delta \to \delta) \vdash_{\mathbf{K}_\ell} \Box(\varepsilon \to \delta)$, a contradiction with (7). ⊠

While, in view of Theorem 3.22, most logics have an algebraic semantics, examples of logics lacking any algebraic semantics are known since [3].

Exercise 3.23. Prove that $\mathbf{K}_\ell$ has a standard deduction theorem, i.e., that for every set of formulas $\Gamma \cup \{\psi, \varphi\}$,

$$\Gamma, \psi \vdash_{\mathbf{K}_\ell} \varphi \iff \Gamma \vdash_{\mathbf{K}_\ell} \psi \to \varphi.$$

Prove that this is not the case for $\mathbf{K}_g$. ⊠

Exercise 3.24. Prove that no class of distributive lattices is an algebraic semantics for the $\langle \wedge, \vee \rangle$-fragment $\mathbf{CPC}_{\wedge\vee}$ of $\mathbf{CPC}$. Hint: use the fact that every equation in a single variable holds in the class of distributive lattices.

Furthermore, prove that $\mathbf{CPC}_{\wedge\vee}$ has a nonstandard algebraic semantics. To this end, consider the three-element algebra $\boldsymbol{A} = \langle \{0^+, 0^-, 1\}; \wedge, \vee \rangle$ whose binary commutative operations are defined by the following tables

$\wedge$	0^-	0^+	1
0^-	0^+	0^+	0^+
0^+		0^-	0^+
1			1

$\vee$	0^-	0^+	1
0^-	0^+	0^+	1
0^+		0^-	1
1			1

and prove that $\{\boldsymbol{A}\}$ is a $\boldsymbol{\tau}$-algebraic semantics for $\boldsymbol{\tau} = \{x \approx x \wedge x\}$. Conclude that $\mathbf{CPC}_{\wedge\vee}$ is another example of logic that admits a nonstandard algebraic semantics, but lacks a standard one. ⊠

4. Ultraproducts and the finite embeddability property

In order to understand the relation between logic and algebra, we need to take a short detour in universal algebra and the theory of quasi-varieties. We begin by reviewing a product-like construction known as *ultraproduct* [1, 7]. First, recall that ultrafilters on powerset Boolean algebras $\mathcal{P}(X)$ are also called *ultrafilters on* X. Then let $\{\boldsymbol{A}_i : i \in I\}$ be a family of similar algebras. The *equalizer* $[\![\vec{a} = \vec{c}]\!]$ of a pair of elements $\vec{a}, \vec{c} \in \prod_{i \in I} A_i$ is the set of indexes on which the sequences $\vec{a}$ and $\vec{c}$ agree, that is,

$$[\![\vec{a} = \vec{c}]\!] := \{i \in I : \vec{a}(i) = \vec{c}(i)\}.$$

Moreover, given an ultrafilter U on the index set I, let θ_U be the binary relation on the Cartesian product $\prod_{i \in I} A_i$ defined as

$$\theta_U := \{\langle \vec{a}, \vec{c} \rangle : [\![\vec{a} = \vec{c}]\!] \in U\}.$$

Proposition 4.1. *If $\{\boldsymbol{A}_i : i \in I\}$ is a family of similar algebras and U an ultrafilter on I, then θ_U is a congruence of $\prod_{i \in I} \boldsymbol{A}_i$.*

Proof. We begin by proving that θ_U is an equivalence relation on $\prod_{i \in I} A_i$. To this end, consider $\vec{a}, \vec{b}, \vec{c} \in \prod_{i \in I} A_i$. We have

$$[\![\vec{a} = \vec{a}]\!] = \{i \in I : \vec{a}(i) = \vec{a}(i)\} = I.$$

Observe that $I \in U$, since U is a nonempty upset of $\mathcal{P}(I)$. Together with the above display, this yields $[\![\vec{a} = \vec{a}]\!] \in U$ and, therefore, $\langle \vec{a}, \vec{a} \rangle \in \theta_U$. It follows that θ_U is reflexive. To prove that it is symmetric, suppose that $\langle \vec{a}, \vec{c} \rangle \in \theta_U$. Then $[\![\vec{a} = \vec{c}]\!] \in U$. Since $[\![\vec{a} = \vec{c}]\!] = [\![\vec{c} = \vec{a}]\!]$, this implies $[\![\vec{c} = \vec{a}]\!] \in U$ and, therefore, $\langle \vec{c}, \vec{a} \rangle \in \theta_U$. Lastly, to prove that θ_U is transitive, suppose that $\langle \vec{a}, \vec{b} \rangle, \langle \vec{b}, \vec{c} \rangle \in \theta_U$, that is, $[\![\vec{a} = \vec{b}]\!], [\![\vec{b} = \vec{c}]\!] \in U$. Since U is closed under binary meets,

$$[\![\vec{a} = \vec{b}]\!] \cap [\![\vec{b} = \vec{c}]\!] \in U$$

Clearly, $[\![\vec{a} = \vec{b}]\!] \cap [\![\vec{b} = \vec{c}]\!] \subseteq [\![\vec{a} = \vec{c}]\!]$. Since U is an upset of $\mathcal{P}(I)$, we obtain that $[\![\vec{a} = \vec{c}]\!] \in U$, whence $\langle \vec{a}, \vec{c} \rangle \in \theta_U$. We conclude that θ_U is an equivalence relation.

To prove that θ_U is a congruence, it only remains to show that it preserves the basic operations. Accordingly, let f be a basic n-ary operation and $\vec{a}_1, \dots, \vec{a}_n, \vec{c}_1, \dots, \vec{c}_n \in \prod_{i \in I} A_i$ such that

$$\langle \vec{a}_1, \vec{c}_1 \rangle, \dots, \langle \vec{a}_n, \vec{c}_n \rangle \in \theta_U.$$

By definition of θ_U, this amounts to $[\![\vec{a}_1 = \vec{c}_1]\!], \dots, [\![\vec{a}_n = \vec{c}_n]\!] \in U$. Since U is a filter, it is closed under finite meets, whence

$$[\![\vec{a}_1 = \vec{c}_1]\!] \cap \cdots \cap [\![\vec{a}_n = \vec{c}_n]\!] \in U. \tag{8}$$

We will show that

$$[\![\vec{a}_1 = \vec{c}_1]\!] \cap \cdots \cap [\![\vec{a}_n = \vec{c}_n]\!] \subseteq [\![f^{\prod_{i \in I} \boldsymbol{A}_i}(\vec{a}_1, \dots, \vec{a}_n) = f^{\prod_{i \in I} \boldsymbol{A}_i}(\vec{c}_1, \dots, \vec{c}_n)]\!]. \tag{9}$$

To this end, consider $j \in [\![\vec{a}_1 = \vec{c}_1]\!] \cap \cdots \cap [\![\vec{a}_n = \vec{c}_n]\!]$. We have

$$\vec{a}_1(j) = \vec{c}_1(j), \dots, \vec{a}_n(j) = \vec{c}_n(j).$$

Consequently,

$$\begin{aligned} f^{\prod_{i\in I} \boldsymbol{A}_i}(\vec{a}_1, \dots, \vec{a})(j) &= f^{\boldsymbol{A}_j}(\vec{a}_1(j), \dots, \vec{a}_n(j)) \\ &= f^{\boldsymbol{A}_j}(\vec{c}_1(j), \dots, \vec{c}_n(j)) \\ &= f^{\prod_{i\in I} \boldsymbol{A}_i}(\vec{c}_1, \dots, \vec{c})(j), \end{aligned}$$

that is, $j \in [\![f^{\prod_{i\in I} \boldsymbol{A}_i}(\vec{a}_1, \dots, \vec{a}_n) = f^{\prod_{i\in I} \boldsymbol{A}_i}(\vec{c}_1, \dots, \vec{c}_n)]\!]$. This establishes (9). Since U is an upset of $\mathcal{P}(I)$, from (8) and (9) it follows

$$[\![f^{\prod_{i\in I} \boldsymbol{A}_i}(\vec{a}_1, \dots, \vec{a}_n) = f^{\prod_{i\in I} \boldsymbol{A}_i}(\vec{c}_1, \dots, \vec{c}_n)]\!] \in U.$$

Hence, we conclude that $\langle f^{\prod_{i\in I} \boldsymbol{A}_i}(\vec{a}_1, \dots, \vec{a}_n), f^{\prod_{i\in I} \boldsymbol{A}_i}(\vec{c}_1, \dots, \vec{c}_n)\rangle \in \theta_U$, as desired. ⊠

In view of the above result, we can make the following definition.

Definition 4.2. An *ultraproduct* of a family of similar algebras $\{\boldsymbol{A}_i : i \in I\}$ is an algebra of the form $\prod_{i\in I} \boldsymbol{A}_i/\theta_U$, for some ultrafilter U on I.

Given a class of similar algebras K, we set

$$\mathbb{P}_{\mathrm{U}}(\mathsf{K}) := \{\boldsymbol{A} : \boldsymbol{A} \text{ is an ultraproduct of a family } \{\boldsymbol{B}_i : i \in I\} \subseteq \mathsf{K}\}.$$

Notice that $\mathbb{P}_{\mathrm{U}}(\mathsf{K}) \subseteq \mathbb{HP}(\mathsf{K})$. Furthermore, as usual, when $\mathsf{K} = \{\boldsymbol{A}\}$, we write $\mathbb{P}_{\mathrm{U}}(\boldsymbol{A})$ as a shorthand for $\mathbb{P}_{\mathrm{U}}(\{\boldsymbol{A}\})$.

Exercise 4.3. Prove that if U is not free (that is, it is principal), then $\prod_{i\in I} \boldsymbol{A}_i/\theta_U$ is isomorphic to some $\boldsymbol{A}_i$. Conclude that if I is finite, then $\prod_{i\in I} \boldsymbol{A}_i/\theta_U$ belongs to $\mathbb{I}\{\boldsymbol{A}_i : i \in I\}$. Because of this, interesting ultraproducts arise from free ultrafilters only. ⊠

Exercise 4.4. Prove that K is a finite set of finite algebras, $\mathbb{P}_{\mathrm{U}}(\mathsf{K}) \subseteq \mathbb{I}(\mathsf{K})$. ⊠

The importance of ultraproducts is largely due to the following result [6, Thm. V.2.9].

Łoś' Theorem 4.5. *Let $\{\boldsymbol{A}_i : i \in I\}$ be a family of similar algebras, U an ultrafilter on I and $\phi(x_1, \dots, x_n)$ a first order formula. For every $\vec{a}_1, \dots, \vec{a}_n \in \prod_{i\in I} A_i$,*

$$\prod_{i\in I} \boldsymbol{A}_i/\theta_U \vDash \phi(\vec{a}_1/\theta_U, \dots, \vec{a}_n/\theta_U) \iff \{i \in I : \boldsymbol{A}_i \vDash \phi(\vec{a}_1(i), \dots, \vec{a}_n(i))\} \in U.$$

Corollary 4.6. *Let $\{\boldsymbol{A}_i : i \in I\}$ be a family of similar algebras, U an ultrafilter on I and ϕ a sentence. If ϕ is valid in all the $\boldsymbol{A}_i$, then it is valid in $\prod_{i\in I} \boldsymbol{A}_i/\theta_U$.*

In view Łos' Theorem, ultraproducts are instrumental to construct nonstandard models of first order theories. For instance, let $\mathbb{N} = \langle \mathbb{N}; s, +, \cdot, 0\rangle$ be the standard model of Peano Arithmetic. If U is an ultrafilter on $\mathbb{N}$, the ultraproduct $\prod_{n\in\mathbb{N}} \mathbb{N}_n/U$ is *elementarily equivalent* to $\mathbb{N}$, that is, it satisfies the same sentences as $\mathbb{N}$. On the other hand, it is not hard to see that if U is free, $\prod_{n\in\mathbb{N}} \mathbb{N}_n/U$ is uncountable and, therefore, contains many "infinite" (or nonstandard) natural numbers.

For the present purpose, however, we will not need the full strength of Łoś' Theorem and, therefore, we shall omit its proof. Instead, we shall focus on a particular embedding theorem for ultraproducts that depends on the following notion.

Definition 4.7. A *local subgraph* $\mathbb{X}$ of an algebra $\boldsymbol{A}$ is a finite subset $X \subseteq A$ endowed with the restriction of finitely many basic operations of $\boldsymbol{A}$ to X.

In this case, $\mathbb{X}$ is a finite *partial* algebra of finite type (even when the type of $\boldsymbol{A}$ is infinite).

Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be similar algebras and $\mathbb{X}$ a local subgraph of $\boldsymbol{A}$. A map $f\colon X \to B$ is said to be an *embedding* of $\mathbb{X}$ into $\boldsymbol{B}$ if it is injective and, for every basic n-ary operation g of the type of $\mathbb{X}$ and $a_1, \dots, a_n \in X$ such that $g^{\boldsymbol{A}}(a_1, \dots, a_n) \in X$,

$$f(g^{\boldsymbol{A}}(a_1, \dots, a_n)) = g^{\boldsymbol{B}}(f(a_1), \dots, f(a_n)).$$

Theorem 4.8. *Let $\mathsf{K} \cup \{\boldsymbol{A}\}$ be a class of similar algebras. If every local subgraph of $\boldsymbol{A}$ can be embedded into some member of K, then $\boldsymbol{A} \in \mathbb{ISP}_{\mathrm{U}}(\mathsf{K})$.*

Proof. Let I be the set of local subgraphs of $\boldsymbol{A}$. By assumption, for every $\mathbb{X} \in I$ there are an algebra $\boldsymbol{B}_{\mathbb{X}} \in \mathsf{K}$ and an embedding $h_{\mathbb{X}}\colon \mathbb{X} \to \boldsymbol{B}_{\mathbb{X}}$. We define a partial order $\sqsubseteq$ on I as follows:

$$\mathbb{X} \sqsubseteq \mathbb{Y} \iff X \subseteq Y \text{ and the type of } \mathbb{Y} \text{ extends that of } \mathbb{X}.$$

Then, for every $\mathbb{X} \in I$, define

$$J_{\mathbb{X}} := \{\mathbb{Y} \in I \colon \mathbb{X} \sqsubseteq \mathbb{Y}\}.$$

Moreover, let $\mathcal{F}$ be the filter of $\mathcal{P}(I)$ generated by $\{J_{\mathbb{X}} \colon \mathbb{X} \in I\}$. Recall that

$$\mathcal{F} = \{Y \subseteq I : J_{\mathbb{X}_1} \cap \dots \cap J_{\mathbb{X}_n} \subseteq Y, \text{ for some } \mathbb{X}_1, \dots, \mathbb{X}_n \in I\}.$$

We will prove that $\mathcal{F}$ is proper. To this end, consider $\mathbb{X}_1, \dots, \mathbb{X}_n \in I$. Then let $\mathbb{Y}$ be the local subgraph of $\boldsymbol{A}$ with universe $Y := X_1 \cup \dots \cup X_n$ and whose type in the union of the types of the various $\mathbb{X}_i$. Then

$$\mathbb{X}_i \sqsubseteq \mathbb{Y}, \text{ for every } i \leqslant n,$$

that is, $\mathbb{Y} \in J_{\mathbb{X}_1} \cap \dots \cap J_{\mathbb{X}_n}$. It follows that $\emptyset \notin \mathcal{F}$ and, therefore, that $\mathcal{F}$ is proper. As $\mathcal{F}$ is a proper filter, by the Ultrafilter Lemma, it can be extended to an ultrafilter U on I.

Now, consider a map

$$f\colon A \to \prod_{\mathbb{X} \in I} B_{\mathbb{X}}$$

such that $f(a)(\mathbb{X}) = h_{\mathbb{X}}(a)$, for every $a \in A$ and $\mathbb{X} \in I$ such that $a \in \mathbb{X}$. Moreover, let

$$f^*\colon A \to \prod_{\mathbb{X} \in I} B_{\mathbb{X}} / \theta_U$$

be the map defined by the rule

$$f^*(a) := f(a)/\theta_U.$$

We will show f^* is an embedding of $\boldsymbol{A}$ into $\prod_{\mathbb{X} \in I} \boldsymbol{B}_{\mathbb{X}} / \theta_U$.

In order to prove that f^* is injective, consider a pair of distinct elements $a, c \in A$. Consider a local subgraph $\mathbb{Y}$ of $\boldsymbol{A}$ containing a and c. We will show that

$$J_{\mathbb{Y}} \subseteq \{\mathbb{X} \in I : f(a)(\mathbb{X}) \neq f(c)(\mathbb{X})\} \tag{10}$$

Consider $\mathbb{X} \in J_{\mathbb{Y}}$. Then $\mathbb{Y} \sqsubseteq \mathbb{X}$ and, therefore, $a, c \in Y \subseteq X$. Since $a, c \in X$, we have

$$f(a)(\mathbb{X}) = h_{\mathbb{X}}(a) \text{ and } f(c)(\mathbb{X}) = h_{\mathbb{X}}(c).$$

Furthermore, $h_{\mathbb{X}}(a) \neq h_{\mathbb{X}}(c)$, because $h_{\mathbb{X}}$ is injective and $a \neq c$. This yields $f(a)(\mathbb{X}) \neq f(c)(\mathbb{X})$, establishing (10).

Recall that the definition of U guarantees that $J_{\mathbb{Y}} \in \mathcal{F} \subseteq U$. Therefore, since U is an upset of $\mathcal{P}(I)$, we can apply (10) obtaining

$$I \smallsetminus [\![f(a) = f(c)]\!] = \{\mathbb{X} \in I : f(a)(\mathbb{X}) \neq f(c)(\mathbb{X})\} \in U.$$

Since U is a proper filter, this implies

$$[\![f(a) = f(c)]\!] \notin U$$

and, therefore,

$$f^*(a) = f(a)/\theta_U \neq f(c)/\theta_U = f^*(c).$$

Hence, we conclude that f^* is injective.

To prove that it is a homomorphism, consider a basic n-ary operation g and $a_1, \dots, a_n \in A$. Then consider a local subgraph $\mathbb{Y}$ of $\boldsymbol{A}$ whose universe contains $a_1, \dots, a_n, g^{\boldsymbol{A}}(a_1, \dots, a_n)$ and whose type contains g. We will prove that

$$J_{\mathbb{Y}} \subseteq [\![f(g^{\boldsymbol{A}}(a_1, \dots, a_n)) = g^{\prod_{\mathbb{X} \in I} \boldsymbol{B}_{\mathbb{X}}}(f(a_1), \dots, f(a_n))]\!]. \tag{11}$$

Consider $\mathbb{V} \in J_{\mathbb{Y}}$. Since $\mathbb{Y} \sqsubseteq \mathbb{V}$, the type of $\mathbb{V}$ contains g and $a_1, \dots, a_n, g^{\boldsymbol{A}}(a_1, \dots, a_n) \in V$. Since $a_1, \dots, a_n, g^{\boldsymbol{A}}(a_1, \dots, a_n) \in V$, we have

$$\begin{aligned} f(a_1)(\mathbb{V}) &= h_{\mathbb{V}}(a_1) \\ &\vdots \\ f(a_n)(\mathbb{V}) &= h_{\mathbb{V}}(a_n) \\ f(g^{\boldsymbol{A}}(a_1, \dots, a_n))(\mathbb{V}) &= h_{\mathbb{V}}(g^{\boldsymbol{A}}(a_1, \dots, a_n)). \end{aligned}$$

Furthermore, as the type of $\mathbb{V}$ contains g,

$$h_{\mathbb{V}}(g^{\boldsymbol{A}}(a_1, \dots, a_n)) = g^{\boldsymbol{B}_{\mathbb{V}}}(h_{\mathbb{V}}(a_1), \dots, h_{\mathbb{V}}(a_n)).$$

From the above displays it follows

$$f(g^{\boldsymbol{A}}(a_1, \dots, a_n))(\mathbb{V}) = g^{\boldsymbol{B}_{\mathbb{V}}}(f(a_1)(\mathbb{V}), \dots, f(a_n)(\mathbb{V})) = g^{\prod_{\mathbb{X} \in I} \boldsymbol{B}_{\mathbb{X}}}(f(a_1), \dots, f(a_n))(\mathbb{V}),$$

that is, $\mathbb{V} \in [\![f(g^{\boldsymbol{A}}(a_1, \dots, a_n)) = g^{\prod_{\mathbb{X} \in I} \boldsymbol{B}_{\mathbb{X}}}(f(a_1), \dots, f(a_n))]\!]$. This establishes (11). Lastly, as $J_{\mathbb{Y}} \in U$ and U is an upset of $\mathcal{P}(I)$, condition (11) implies

$$[\![f(g^{\boldsymbol{A}}(a_1, \dots, a_n)) = g^{\prod_{\mathbb{X} \in I} \boldsymbol{B}_{\mathbb{X}}}(f(a_1), \dots, f(a_n))]\!] \in U,$$

and, therefore,

$$\begin{aligned} f^*(g^{\boldsymbol{A}}(a_1, \dots, a_n)) &= f(g^{\boldsymbol{A}}(a_1, \dots, a_n))/\theta_U \\ &= g^{\prod_{\mathbb{X} \in I} \boldsymbol{B}_{\mathbb{X}}}(f(a_1), \dots, f(a_n))/\theta_U \\ &= g^{\prod_{\mathbb{X} \in I} \boldsymbol{B}_{\mathbb{X}}/\theta_U}(f(a_1)/\theta_U, \dots, f(a_n)/\theta_U) \\ &= g^{\prod_{\mathbb{X} \in I} \boldsymbol{B}_{\mathbb{X}}/\theta_U}(f^*(a_1), \dots, f^*(a_n)). \end{aligned}$$

Hence, we conclude that f^* is a homomorphism and, therefore, an embedding of $\boldsymbol{A}$ into $\prod_{\mathbb{Y} \in I} \boldsymbol{B}_{\mathbb{Y}}/\theta_U$. As a consequence,

$$\boldsymbol{A} \in \mathbb{ISP}_{\mathbb{U}}(\{\boldsymbol{B}_{\mathbb{X}} : \mathbb{X} \in I\}) \subseteq \mathbb{ISP}_{\mathbb{U}}(\mathsf{K}). \qquad \boxtimes$$

Corollary 4.9. *Every algebra embeds into an ultraproduct of its finitely generated subalgebras.*

REFERENCES

[1] J. L. Bell and A. B. Slomson. *Models and ultraproducts: An introduction*. North-Holland, Amsterdam, 1971. Second revised printing.

[2] C. Bergman. *Universal Algebra: Fundamentals and Selected Topics*. Chapman & Hall Pure and Applied Mathematics. Chapman and Hall/CRC, 2011.

[3] W. J. Blok and P. Köhler. Algebraic semantics for quasi-classical modal logics. *The Journal of Symbolic Logic*, 48:941–964, 1983.

[4] W. J. Blok and D. Pigozzi. *Algebraizable logics*, volume 396 of *Mem. Amer. Math. Soc.* A.M.S., Providence, January 1989.

[5] W. J. Blok and J. Rebagliato. Algebraic semantics for deductive systems. *Studia Logica, Special Issue on Abstract Algebraic Logic, Part II*, 74(5):153–180, 2003.

[6] S. Burris and H. P. Sankappanavar. *A course in Universal Algebra*. Available online https://www.math.uwaterloo.ca/~snburris/htdocs/ualg.html, the millennium edition, 2012.

[7] C. C. Chang and H. J. Keisler. *Model Theory*, volume 73 of *Studies in Logic*. North-Holland, Amsterdam, third edition, 1990.

[8] J. Czelakowski. *Protoalgebraic logics*, volume 10 of *Trends in Logic—Studia Logica Library*. Kluwer Academic Publishers, Dordrecht, 2001.

[9] J. M. Font. *Abstract Algebraic Logic - An Introductory Textbook*, volume 60 of *Studies in Logic - Mathematical Logic and Foundations*. College Publications, London, 2016.

[10] J. M. Font and R. Jansana. *A general algebraic semantics for sentential logics*, volume 7 of *Lecture Notes in Logic*. A.S.L., second edition 2017 edition, 2009. First edition 1996. Electronic version freely available through Project Euclid at projecteuclid.org/euclid.lnl/1235416965.

[11] J. M. Font, R. Jansana, and D. Pigozzi. A survey on abstract algebraic logic. *Studia Logica, Special Issue on Abstract Algebraic Logic, Part II*, 74(1–2):13–97, 2003. With an "Update" in 91 (2009), 125–130.

[12] V. I. Glivenko. Sur quelques points de la logique de M. Brouwer. *Academie Royal de Belgique Bulletin*, 15:183–188, 1929.

[13] M. Kracht. *Tools and techniques in modal logic*, volume 142 of *Studies in Logic and the Foundations of Mathematics*. North-Holland Publishing Co., Amsterdam, 1999.

[14] M. Kracht. *Modal consequence relations*, volume 3, chapter 8 of the Handbook of Modal Logic. Elsevier Science Inc., New York, NY, USA, 2006.

[15] T. Moraschini. On equational completeness theorems. Submitted manuscript, available online, 2020.

[16] J. G. Raftery. The equational definability of truth predicates. *Reports on Mathematical Logic*, (41):95–149, 2006.

DEPARTMENT OF PHILOSOPHY, UNIVERSITY OF BARCELONA
Email address: tommaso.moraschini@ub.edu