Positive (Modal) Logic Beyond Distributivity

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Abstract

We present a duality for non-necessarily-distributive (modal) lattices and use this to study non-necessarily-distributive positive (modal) logic. Our duality is similar to Priestley duality and as such allows us to use similar tools and techniques to study corresponding logics. As a result, we prove Sahlqvist correspondence and canonicity for both the propositional positive logic as well as its modal extension.[1]

Keywords: duality, non-distributive positive logic, modal logic, Sahlqvist correspondence, Sahlqvist canonicity.

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Contents

[1] This paper is partially based on the Master's thesis [13].

1 Introduction

Dualities between modal algebras and modal spaces on the one hand and Heyting algebras and Esakia spaces on the other have been central to the study of modal and intermediate logics, [6, 8]. Many important results such as Sahlqvist canonicity and correspondence are based on duality techniques [36]. In [7], duality between modal algebras and modal spaces has been extended to a duality between modal distributive lattices (i.e. with distributive lattices taking the role of Boolean algebras) and modal Priestley spaces. Among other things, this led to a Sahlqvist theory for positive distributive modal logic.

When the algebraic side of a duality is based on Boolean algebras or distributive lattices, in the spatial side of the duality one works with the space of all prime filters of a given lattice. This is no longer the case when the base lattice is non-distributive. There have been many attempts to extend a duality for Boolean algebras and distributive lattices to the setting of all lattices, e.g. by Urquhart [37], Hartonas [23, 24], Gehrke and van Gool [16], and Goldblatt [21]. Each of these uses either a ternary relation, or two-sorted frames. While this has proven a fruitful and interesting approach towards duality, it is quite different from known dualities for propositional logics such as Stone and Priestley dualities. As a consequence, it can be difficult to modify existing tools and techniques from other propositional bases for these dualities.

An approach towards duality for (non-distributive) meet-semilattices was developed by Hofmann, Mislove and Stralka (HMS) [27], along the same lines of the proof of the Van Kampen-Pontryagin duality for locally compact abelian groups given in [35]. This was later generalised to a duality for lattices by Jipsen and Moshier [34]. In this approach the dual space is based not on prime filters, but all (proper) filters of a lattice. This perspective is closely related to the possibility semantics of modal logic [29] and to choice-free duality for Boolean algebras [4], where again one works with the space of all proper filters. Such an approach was also developed for ortholattices by Goldblatt [19] and extended later by Bimbo [5]. We also refer to the very recent work [28] for the possibility semantics for modal ortholattices.

Our aim in this paper is to investigate not necessarily distributive positive logics. We introduce these logics and call them simply positive (modal) logics. The duality we use is a restriction of HMS duality. Our choice of duality is based on the fact that, in our view, this duality is closest to dualities used in distributive cases. This is also demonstrated by the results in this paper which show that classic results such as Sahlqvist theory can be adapted from the distributive case to a non-distributive one.

Our approach is analogous to Esakia duality for Heyting algebras. We recall that a Priestley space is a partially ordered compact space satisfying the Priestley separation axiom

$$x \not\leq y \text{ implies that there is a clopen upset } U \text{ such that } x \in U \text{ and } y \notin U.$$

These spaces provide duality for distributive latices via the lattice of clopen upsets. In the HMS duality we work with similar structures, but instead of a partially ordered compact space we have a meet-semilattice with a compact topology and instead of clopen upsets we work with clopen filters. Then the HMS analogue of the Priestley separation axiom is

$$x \not\leq y \text{ implies that there is a clopen filter } U \text{ such that } x \in U \text{ and } y \notin U.$$

These spaces provide duality for meet-semilatices via clopen filters. Recall that an Esakia space is a Priestley space where for every clopen upsets U and V the Heyting implication $U \to V$ is also a clopen upset (a more standard condition equivalent to the former states that $\downarrow U$ is clopen for every clopen U). In analogy with this an HMS space is a *Lattice space* if for every clopen filters U and V their join in the lattice of filters (the least filter containing U and V), i.e. $\{x \mid x \geq a \wedge b \text{ for } a \in U \text{ and } b \in V\}$ is a clopen filter.

This also allows us to define a new Kripke like semantics for positive logics. The analogue of a Kripke frame is a meet-semilattice and formulas are interpreted as filters. This new semantics is a generalisation of the team semantics of [26] and of the modal information semantics of [3].

In the case of distributive lattices and Heyting algebras the lattice of all upsets is isomorphic to the canonical extension. The representation of lattices as clopen filters leads to two kinds of completions of a lattice: (1) by taking point generated upsets of the dual space of L we obtain the filter completion of L and (2) by taking all filters we obtain a new completion of L that we call the F^2-completion. The canonical extension of L is a completion which is situated between these two, although as we notice it is not a sublattice of F^2-completion. Our main results are preservation and correspondence results. Using a duality technique similar to that of Sambin and Vaccaro [36] we show that every sequent is preserved by filter completions and that every Sahlqvist formula is preserved by the double F^2-completion. The former provides a purely topological proof of the result by Baker and Hales [2] that every variety of lattices is closed under ideal completions. An alternative approach to Sahlqvist correspondence and canonicity for non-distributive logics has been undertaken in [10]. But this approach is purely algebra based and is not concerned with the relational semantics and duality developed in this paper.

We extend our results to a modal extension of (not necessarily distributive) positive logic. We expand the base logic with two unary modalities, $\Box$ and $\Diamond$, that are interpreted via a relation in the usual way. Interestingly, as a consequence of the non-standard interpretation of joins, while $\Box$ distributes over finite meets, $\Diamond$ does not distributive over finite joins but is merely monotone. A similar phenomenon in the context of modal intuitionistic logic has been exposed in [31]. The two modalities are related via one of Dunn's duality axioms for distributive positive modal logic [14]. The non-standard interpretation of joins makes Dunn's other duality axiom unsuitable in our context. We restrict our attention to the serial case, thus adding the seriality axiom, which allows us to obtain a duality between relational and algebraic semantics. This is similar to the distributive case and, in presence of seriality, uses only one of Dunn's axioms.

We extend the results for positive logic to the modal setting, and obtain a duality for its algebraic semantics, and Sahlqvist correspondence and canonicity results. Using this, we obtain a sound and complete semantics for the extension of the logic that legislates $\Diamond$ to distributive over finite joins.

With this paper we hope to be laying a groundwork for the theory of non-distributive modal logics. As discussed in the conclusion, there are many interesting directions for future research, ranging from intermediate positive logics (that lie between non-distributive and distributive positive logic) to deriving more results for the modal logic presented in this paper to extending non-distributive positive logic with different types of modalities, and beyond.

2 Duality for Lattices

In [27] Hofmann, Mislove and Stralka proved a Stone-type duality for meet-semilattices with a top element. The dual spaces are also given by meet-semilattices with top, but equipped with a Stone topology. Variations of this duality for the categories of bounded meet-semilattices (with both top and bottom) and unbounded meet-semilattices were given in [11, 9].

In this section, we recall the definition of a meet-semilattice and of several types of filters. We give the duality for them, reformulated in a way resembling Priestley duality. We then investigate the restriction to a duality for the category of lattices, similar to the restriction of Priestley duality to Esakia duality. Finally, we use this duality to describe three types of completions of a lattice.

2.1 Meet-Semilattices and Bounded Meet-Semilattices

We recall the definitions of meet-semilattices (which we shall generally refer to as simply *semilattices*) and bounded meet-semilattices, homomorphism, and the categories they form. We explore their connection by exposing a dual adjunction between these categories.

2.1 Definition. A *meet-semilattice* is a poset $(X, \leq)$ in which every pair of elements x and y have a greatest lower bound, denoted by $x \wedge y$. We sometimes write $(X, \wedge)$ for a meet-semilattice. The underlying poset order can then be recovered via $x \leq y$ if and only if $x \wedge y = x$. A *(meet-semilattice) homomorphism* from $(X, \wedge)$ to $(X', \wedge')$ is a function $f : X \to X'$ that satisfies $f(x \wedge y) = f(x) \wedge' f(y)$ for all $x, y \in X$. We write SL for the category of meet-semilattices and homomorphisms.

Since all semilattices we work with in this paper are meet-semilattices, we will omit the prefix "meet-" and simply refer to them as semilattices. Next we equip a semilattice with an upper and lower bound.

2.2 Definition. A semilattice $(X, \wedge)$ is called *bounded* if there exist elements $\top$ and $\bot$ in X, called *top* and *bottom*, such that $x \wedge \top = x$ and $x \wedge \bot = \bot$ for all $x \in X$. We denote a bounded semilattice by $(X, \top, \bot, \wedge)$. A *bounded semilattice homomorphism* is a top- and bottom-preserving semilattice homomorphism. We write BSL for the category of bounded semilattices and their homomorphisms.

We sometimes identify a (bounded) semilattice with its state-space, and refer to X instead of the tuple $(X, \top, \bot, \wedge)$. It will then be clear from context whether we are dealing with a semilattice or a bounded one.

Let $(X, \wedge)$ be a semilattice and $a \subseteq X$. Then we define the *upward closure* of a by $\uparrow a := \{y \in X \mid x \leq y \text{ for some } x \in a\}$ and we say that a is *upward closed* or an *upset* if $\uparrow a = a$. If $a = \{x\}$ is a singleton we write $\uparrow x$ instead of $\uparrow\{x\}$. We define similarly the downward closure and downsets.

2.3 Definition. A *filter* in a semilattice $(X, \wedge)$ is an upset $a \subseteq X$ that is closed under $\wedge$. It is called *proper* if $a \neq X$. A filter is called *principal* if it is empty, or of the form $\uparrow x$ for some $x \in X$.

2.4 Remark. Observe that we view the empty filter as a principal filter as well. This will streamline notation when discussing frame semantics in Section 3. $\lhd$

The collection of filters of $(X, \wedge)$ forms a bounded semilattice with intersection as meet and X and $\emptyset$ as top and bottom elements. We denote this bounded semilattices by $\mathcal{F}(X, \wedge)$. Furthermore, if $f : (X, \wedge) \to (X', \wedge')$ is a semilattice homomorphism, then $f^{-1} : \mathcal{F}(X', \wedge') \to \mathcal{F}(X, \wedge)$ is a bounded semilattice homomorphism, and setting $\mathcal{F} f = f^{-1}$ yields a contravariant functor $\mathcal{F} : \mathsf{SL} \to \mathsf{BSL}$.

Conversely, if $(A, \top, \bot, \wedge)$ is a bounded semilattice then the collection of proper non-empty filters with intersection forms a semilattice, denoted by $\mathcal{F}_b A$. Again, defining $\mathcal{F}_b f = f^{-1}$ for bounded homomorphisms yields a contravariant functor $\mathcal{F}_b : \mathsf{BSL} \to \mathsf{SL}$.

2.5 Proposition. *The functors $\mathcal{F}$ and $\mathcal{F}_b$ establish a dual adjunction between SL and BSL.*

Proof. Define the units $\eta : id_{\mathsf{SL}} \to \mathscr{F}_b\mathscr{F}$ and $\theta : id_{\mathsf{BSL}} \to \mathscr{F}\mathscr{F}_b$ on components by $\eta_X(x) = \{a \in \mathscr{F}X \mid x \in a\}$ and $\theta_X(x) = \{a \in \mathscr{F}_bX \mid x \in a\}$. It is easy to see that both $\eta_X(x)$ and $\theta_X(x)$ are filters. Moreover, $\theta_X(x)$ is proper and non-empty because $\emptyset \notin \theta_X(x)$ and $X \in \theta_X(x)$ for all x. A routine verification shows that both η and θ define natural transformations. In order to show that $\mathscr{F}$ and $\mathscr{F}_b$ form a dual adjunction it suffices to verify that they satisfy the following triangle equalities:

$$\begin{tikzcd} \mathscr{F} \arrow[r, "\theta_{\mathscr{F}}"] \arrow[rr, bend right, "id_{\mathscr{F}}"'] & \mathscr{F}\mathscr{F}_b\mathscr{F} \arrow[r, "\mathscr{F}\eta"] & \mathscr{F} \end{tikzcd} \qquad \begin{tikzcd} \mathscr{F}_b \arrow[r, "\eta_{\mathscr{F}_b}"] \arrow[rr, bend right, "id_{\mathscr{F}_b}"'] & \mathscr{F}_b\mathscr{F}\mathscr{F}_b \arrow[r, "\mathscr{F}_b\theta"] & \mathscr{F}_b \end{tikzcd}$$

We verify the left one, the right being similar. Let X be a semilattice and $a \in \mathscr{F}X$. Then we have:

$$x \in \mathscr{F}\eta_X(\theta_{\mathscr{F}X}(a)) \quad \text{iff} \quad \eta_X(x) \in \theta_{\mathscr{F}X}(a) \quad \text{iff} \quad a \in \eta_X(x) \quad \text{iff} \quad x \in a$$

so $\theta_{\mathscr{F}X}(\mathscr{F}\eta_X(a)) = a$. This completes the proof. □

2.6 Remark. Observe that filters of a semilattice correspond bijectively to homomorphisms into $2 = \{\top, \bot\}$, the two-element chain. If a is a filter in $(X, \wedge)$ then the characteristic map $\chi_a : X \to 2$ given by $\chi_a(x) = \top$ iff $x \in a$ is a homomorphism, and for every such homomorphism the preimage of $\top$ is a filter. Similarly, proper non-empty filters of a bounded semilattice correspond to bounded homomorphisms into 2. So the dual adjunction between SL and BSL is given by the dualising object 2. ◁

If $(X, \wedge)$ is a semilattice that happens to have all finite joins (including a bottom element), then we have a second way of obtaining a bounded semilattice. Namely, the principal filters of $(X, \wedge)$ form a sublattice of $\mathscr{F}(X, \wedge)$. The empty filter is principal by convention and X is principal because it is of the form $\uparrow\bot$, where $\bot$ is the bottom element of $(X, \wedge)$. Furthermore, if $p = \uparrow x$ and $q = \uparrow y$ are principal filters, then their intersection is principal as well, because $p \cap q = \uparrow(x \vee y)$.

This observation will help us elucidate the connection between the duality for lattices (whose dual spaces have finite joins) and the filter extension of a lattice (in Section 2.4). Moreover, we will show that using principal filters are interpretants for positive formulae yields stronger canonicity results (albeit with a more restrictive semantics) in Section 4.5.

2.2 Duality for Bounded Meet-Semilattices

We now work our way towards a duality for bounded semilattices. We do so by taking the dual adjunction between semilattices and bounded semilattices, and equipping semilattices with a Stone topology. This duality has appeared before in [11, Sections 2.4] in a more abstract desguise and is a variation of the duality given by Hofmann, Mislove and Stralka in Chapter I of [27].

2.7 Definition. An *M-space* is a tuple $\mathbb{X} = (X, \wedge, \tau)$ such that

(M_1) $(X, \wedge)$ is a semilattice;

(M_2) (X, τ) is a compact topological space;

(M_3) $(X, \wedge, \top)$ satisfies the *HMS separation axiom*:

for all $x, y \in X$, if $x \not\leq y$ then there exists a clopen filter a such that $x \in a$ and $y \notin a$.

An *M-morphism* is a continuous semilattice homomorphism. We write MSpace for the category of M-spaces and M-morphisms.

Condition (M_3) is a variation of the Priestley separation axiom. It immediately implies that any M-space is Hausdorff. Furthermore, it can be shown that every M-space is zero-dimensional in the same way as for Priestley spaces. For future reference, we derive some other useful properties of M-spaces.

2.8 Lemma. *A filter c in an M-space $\mathbb{X} = (X, \wedge, \tau)$ is closed if and only if it is principal.*

Proof. If c is the empty filter then it is principal by convention, and the empty principal filter is automatically closed because any empty set in a topological space is closed.

So let c be a non-empty closed filter and suppose towards a contradiction that it is not principal. Then for each $x \in c$ there exists some $y \in c$ such that $x \not\leq y$. Therefore, using (M_3), for each $x \in c$ we can find a clopen filter containing x such that $c \not\subseteq b_x$. Then $c \subseteq \bigcup_{x \in c} b_x$ is an open cover of c, and by compactness we can find a finite subcover, say $c \subseteq b_{x_1} \cup \cdots \cup b_{x_n}$. By construction, for each b_{x_i} we can find a $y_i \in a$ such that $y_i \notin b_{x_i}$. But this implies $y_1 \wedge \cdots \wedge y_n \in c$ because c is a filter, while $y_1 \wedge \cdots \wedge y_n \notin b_{x_i}$ for all $1 \leq i \leq n$, a contradiction.

Next, suppose c is non-empty and principal, i.e. $c = \uparrow x$ for some $x \in X$. If $y \notin c$ then $x \not\leq y$, so there exists a clopen filter a containing x and not containing y. But then $c = \uparrow x \subseteq a$, so $X \setminus a$ is an open neighbourhood of y disjoint from c. Therefore c is closed. □

2.9 Lemma. *Every closed filter of an M-space $\mathbb{X} = (X, \wedge, \tau)$ is the intersection of clopen filters.*

Proof. If c is the empty filter then the statement is trivial. If c is non-empty, then by Lemma 2.8 it is of the form $\uparrow x$ for some $x \in X$. Suppose $y \notin c$, then $x \not\leq y$ so we can find a clopen filter a containing x but not y. Since $x \in a$ we have $c \subseteq a$. Since y was chosen arbitrarily, the claim follows. □

2.10 Lemma. *Let c be a closed subset of an M-space $\mathbb{X} = (X, \wedge, \tau)$. Then $\uparrow c$ is closed as well.*

Proof. If $y \notin \uparrow c$ then for each $x \in c$ we have $x \not\leq y$, hence a clopen filter a_x containing x but not y. Then $c \subseteq \bigcup_{x \in c} a_x$, so by compactness we find a finite subcover, say, $c \subseteq a_1 \cup \cdots \cup a_n$. Since all the a_i are upward closed, we have $\uparrow c \subseteq a_1 \cup \cdots \cup a_n$. By construction, none of the a_i contain y, so $X \setminus (a_1 \cup \cdots \cup a_n)$ is an open neighbourhood of y disjoint from $\uparrow c$. □

We claim that every bounded semilattice gives rise to an M-space. We now denote a bounded semilattice by A, since we think of them as (a variety of) algebras.

2.11 Proposition. *Let A be a bounded semilattice and write $\mathscr{F}_b A$ for the set of non-empty proper filters of X. Generate a topology τ_A on $\mathscr{F}_b A$ by*

$$\{\theta_A(a) \mid a \in A\} \cup \{\theta_A(a)^c \mid a \in A\}, \tag{1}$$

where $\theta_A(a)^c = \mathscr{F}_b A \setminus \theta_A(a)$. Then $(\mathscr{F}_b A, \cap, \tau_A)$ is an M-space.

Proof. We already know that $(\mathscr{F}_b A, \cap)$ is a semilattice. Condition (M_3) follows easily: If $x, y \in \mathscr{F}_b A$ are such that $x \not\subseteq y$ then there is an element $a \in A$ such that $a \in x$ and $a \notin y$, and $\theta_A(a)$ is a clopen filter separating x and y.

To show that $(\mathscr{F}_b A, \tau_A)$ is compact, it suffices to show that every open cover of $\mathscr{F}_b A$ of subsets in the subbase (1) has a finite subcover. So suppose

$$\mathscr{F}_b A \subseteq \bigcup_{i \in I} \theta_A(a_i) \cup \bigcup_{j \in J} \theta_A(b_j)^c \tag{2}$$

where I and J are index sets and $a_i, b_j \in A$. Consider the filter $F = {\uparrow}\{b_{j_1} \wedge \cdots \wedge b_{j_n} \mid n \in \omega$ and $j_1, \dots, j_n \in J\}$. Note that F is nonempty because the case $n = 0$ entails that the empty meet (which is $\top$) is in F. If F is not proper then we must have $\bot \in F$, so there must exist $j_1, \dots, j_n \in J$ such that $b_{j_1} \wedge \cdots \wedge b_{j_n} = \bot$. Since filters are closed under meets, no proper filter can contain all of $b_{j_1}, \dots, b_{j_n}$, and hence $\theta_A(b_{j_1}) \cup \cdots \cup \theta_A(b_{j_n})$ is a finite subcover.

If F is proper, then $F \in \mathcal{F}_b A$. By construction $F \notin \theta_A(b_j)^c$ for all $j \in J$, so it must be contained in $\theta_A(a_{i'})$ for some $i' \in I$. This implies $a_{i'} \in F$, hence there are $j_1, \dots, j_n \in J$ such that $b_{j_1} \wedge \cdots \wedge b_{j_n} \leq a_{i'}$. But this implies that

$$\theta_A(a_{i'}) \cup \theta_A(b_{j_1})^c \cup \cdots \cup \theta_A(b_{j_n})^c$$

is a finite subcover of the one in (2). Indeed, any filter that is in none of $\theta_A(b_{j_1})^c, \dots, \theta_A(b_{j_n})^c$ must contain $b_{j_1}, \dots, b_{j_n}$, and hence also a. □

If A is a bounded semilattice, then we denote the M-space constructed in Proposition 2.11 by $\mathcal{F}_{top} A$. This assignment extends to a contravariant functor

$$\mathcal{F}_{top} : \mathsf{BSL} \to \mathsf{MSpace},$$

where we define $\mathcal{F}_{top} h = h^{-1}$ for bounded semilattice homomorphisms h.

Conversely, if $\mathbb{X} = (X, \wedge, \tau)$ is an M-space, then we denote by $\mathcal{F}_{clp}\mathbb{X}$ the collection of clopen filters. This forms a bounded semilattice with meet given by intersection, and X and $\emptyset$ as top and bottom. It gives rise to a contravariant functor

$$\mathcal{F}_{clp} : \mathsf{MSpace} \to \mathsf{BSL}$$

by setting $\mathcal{F}_{clp} f = f^{-1}$ for M-morphisms.

2.12 Theorem. *The functors $\mathcal{F}_{clp}$ and $\mathcal{F}_{top}$ establish a dual equivalence* $\mathsf{MSpace} \equiv^{\mathrm{op}} \mathsf{BSL}$.

Proof. We prove that $\eta : id_{\mathsf{MSpace}} \to \mathcal{F}_{top}\mathcal{F}_{clp}$ and $\theta : id_{\mathsf{BSL}} \to \mathcal{F}_{clp}\mathcal{F}_{top}$, defined on components by $\eta_{\mathbb{X}}(x) = \{a \in \mathcal{F}_{clp}\mathbb{X} \mid x \in a\}$ and $\theta_A(a) = \{x \in \mathcal{F}_{top}A \mid a \in x\}$, are natural isomorphisms.

Let us start with the former. Naturality is routine again, so we focus on showing that for each $\mathbb{X} \in \mathsf{MSpace}$, $\eta_{\mathbb{X}}$ is an isomorphism. So let $\mathbb{X}$ be an arbitrary M-space. If $x, y \in \mathbb{X}$ and $x \neq y$ then either $x \not\leq y$ or $y \not\leq x$, so by (M3) we find $\eta_{\mathbb{X}}(x) \neq \eta_{\mathbb{X}}(y)$, and hence $\eta_{\mathbb{X}}$ is injective. To see that $\eta_{\mathbb{X}}$ is surjective, let $F \in \mathcal{F}_{top}\mathcal{F}_{clp}\mathbb{X}$. Then $\bigcap F$ is a closed filter of $\mathbb{X}$, because it is the intersection of clopen filters, so by Lemma 2.8 it is of the form ${\uparrow}x$ for some $x \in \mathbb{X}$. We claim that $F = \eta_{\mathbb{X}}(x)$. By construction of x we have $x \in a$ for all $a \in F$, so $F \subseteq \eta_{\mathbb{X}}(x)$. Now if a is any clopen filter with $x \in a$, then we have $\bigcap F = {\uparrow}x \subseteq a$ and by a straightforward compactness argument we find $b_1, \dots, b_n \in F$ such that $b_1 \cap \cdots \cap b_n \subseteq a$. Since F is a filter we have $b_1 \cap \cdots \cap b_n \in F$, and hence $a \in F$. Finally, $\eta_{\mathbb{X}}$ preserves meets because

$$\begin{aligned}\eta_{\mathbb{X}}(x \wedge y) &= \{a \in \mathcal{F}_{clp}\mathbb{X} \mid x \wedge y \in a\} = \{a \in \mathcal{F}_{clp}\mathbb{X} \mid x \in a \text{ and } y \in a\} \\ &= \{a \in \mathcal{F}_{clp}\mathbb{X} \mid x \in a\} \cap \{a \in \mathcal{F}_{clp}\mathbb{X} \mid y \in a\} = \eta_{\mathbb{X}}(x) \cap \eta_{\mathbb{X}}(y)\end{aligned}$$

and it is continuous because for each $a \in \mathcal{F}_{clp}\mathbb{X}$ we have

$$\eta_{\mathbb{X}}^{-1}(\theta_{\mathcal{F}_{clp}\mathbb{X}}(a)) = \{x \in \mathbb{X} \mid \eta_{\mathbb{X}}(x) \in \theta_{\mathcal{F}_{clp}\mathbb{X}}(a)\} = \{x \in \mathbb{X} \mid a \in \eta_{\mathbb{X}}(x)\} = \{x \in \mathbb{X} \mid x \in a\} = a$$

and hence $\eta_{\mathbb{X}}^{-1}(\theta_{\mathcal{F}_{clp}\mathbb{X}}(a)^c) = \mathbb{X} \setminus a$. Since any bijective homomorphism preserves and reflects meets, and a bijective continuous function between Stone spaces is a homeomorphism, we find that $\eta_{\mathbb{X}}$ is an isomorphism in MSpace.

Next we establish that θ is an isomorphism on components, again leaving the verification of naturality to the reader. Let A be a bounded semilattice. It is easy to see that θ_A is well defined. If $a \neq b$, then without loss of generality we may assume $a \not\leq b$. We find that $\uparrow a \in \mathcal{F}_{top}A$ is a filter such that $\uparrow a \in \theta_A(a)$ while $\uparrow a \notin \theta_A(b)$. Therefore θ_A is injective. Preservation of meets is similar to preservation of meets for $\eta_{\mathbb{X}}$. Moreover, $\theta_A(\top) = \mathcal{F}_{top}A$ (the top element of $\mathcal{F}_{clp}\mathcal{F}_{top}A$) and $\theta_A(\bot) = \emptyset$ (the bottom element of $\mathcal{F}_{clp}\mathcal{F}_{top}A$) because the filters in $\mathcal{F}_{top}A$ are non-empty and proper, so they all contain $\top$ and none contains $\bot$. For surjectivity, let $F \in \mathcal{F}_{clp}\mathcal{F}_{top}A$ be any clopen filter. Then using an argument similar to the proof of Lemma 2.9, we have $F = \bigcap\{\theta_A(a) \mid a \in A, F \subseteq \theta_A(a)\}$. Compactness yields a finite number $a_1, \ldots, a_n \in A$ such that $F = \theta_A(a_1) \cap \cdots \cap \theta_A(a_n)$. But then $F = \theta_A(a_1 \wedge \cdots \wedge a_n)$, so θ_A is indeed surjective. Thus we have found that θ_A is a bijective BSL-morphism, and since BSL is a variety of algebras this implies that it is an isomorphism. □

2.13 Corollary. *Let $\mathbb{X}$ be an M-space. Then $\mathbb{X}$ has arbitrary non-empty meets. In particular, $\mathbb{X}$ has a bottom element.*

Proof. By Theorem 2.12 $\mathbb{X}$ is isomorphic to $\mathcal{F}_{top}A = (\mathcal{F}_b A, \cap, \tau_A)$ for some bounded semilattice A. The lemma follows from the facts that the intersection of an arbitrary nonempty set of non-empty proper filters is again a nonempty proper filter (as it must contain $\top$ and not $\bot$), and that meets in $\mathcal{F}_{top}A$ are given by intersections. □

2.3 Restriction to Lattices

Our next goal is to restrict the duality for bounded semilattices to a duality for lattices. By a *lattice* we will always mean a *bounded* lattice, i.e. a bounded semilattice with binary joins. The restriction of HMS duality to lattices was first studied by Jipsen and Moshier [34]. Here we follow a different version first explored in [13], since we find the interpretation of join that we discuss below simpler than in [34]. We also point out that [34] formulate their duality in terms of spectral spaces whereas we prefer to work with M-spaces.

A different restriction of HMS duality was used in [22], where it was restricted to a duality for implicative semilattices and used to study modal extensions of the meet-implication fragment of intuitionistic logic.

We start by making the observation that, if $(X, \wedge)$ is a semilattice, then $\mathcal{F}(X, \wedge)$ is not only a bounded semilattice, but it is also complete. Indeed, the arbitrary intersection of filters forms a filter again. Therefore, in particular, it is a complete lattice, with joins defined as the meet of all upper bounds. That is, if $F \subseteq \mathcal{F}(X, \wedge)$ is a collection of filters, then $\bigcurlyvee F = \bigcap\{a \in \mathcal{F}(X, \wedge) \mid b \subseteq a \text{ for all } b \in F\}$. We can characterise joins of non-empty filters as follows.

2.14 Lemma. *Let $(X, \wedge)$ be a semilattice. The join of a non-empty set $F \subseteq \mathcal{F}(X, \wedge)$ of non-empty filters of $(X, \wedge)$ can be defined as*

$$\bigcurlyvee F = \uparrow\{x_1 \wedge \cdots \wedge x_n \mid n \in \omega \text{ and } x_1, \ldots, x_n \in \bigcup F\}.$$

Proof. By definition the right hand side is the smallest filter containing all filters in F. □

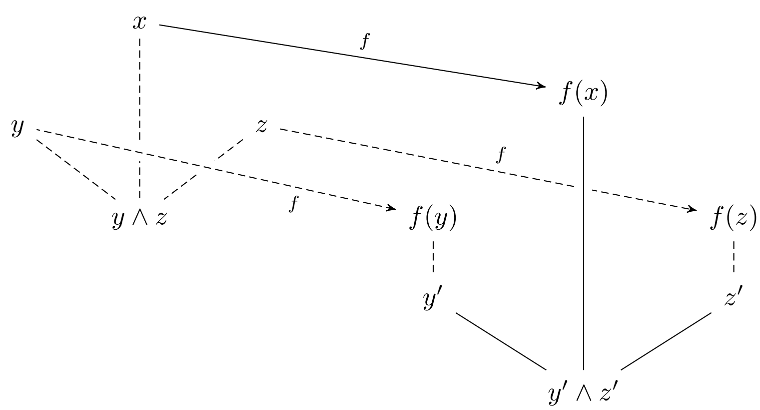
In particular, if F only contains the empty filter then we find that $\bigcurlyvee F = \uparrow\emptyset = \emptyset$. We abbreviate $\bigcurlyvee\{a, b\}$ by $a \curlyvee b$. Note that $a \curlyvee b = \uparrow\{x \wedge y \mid x \in a, y \in b\}$ (provided a and b are not the empty filter), and this is simply the smallest filter containing a and b. Moreover, for any set F of filters we have

$$\bigcurlyvee F = \bigcup\{a_1 \curlyvee \cdots \curlyvee a_n \mid a_1, \ldots, a_n \in F\}. \tag{3}$$

If the semilattice $\mathcal{F}(X, \wedge)$ happens to be distributive then we may omit the upward closure $\uparrow$, but in general we will not be in this situation.

While this yields a restriction of the functor $\mathcal{F} : \mathsf{SL} \to \mathsf{BSL}$ to $\mathsf{SL} \to \mathsf{Lat}$ on objects, it does not yet work for morphisms. Indeed, we need extra conditions on the morphisms between semilattices to ensure that their inverses preserve joins.

2.15 Definition. An *L-morphism* between semilattices $(X, \wedge)$ and $(X', \wedge')$ is a semilattice homomorphism $f : (X, \wedge) \to (X', \wedge')$ that satisfies for all $x \in X$ and $y', z' \in X'$: If $y' \wedge z' \leq f(x)$, then there exist $y, z \in X$ such that $y' \leq' f(y)$ and $z' \leq' f(z)$ and $y \wedge z \leq x$. In a diagram:



We write LFrm for the category of semilattices and L-morphisms.

The category LFrm will be used in Section 3 to interpret non-distributive positive logic. We verify that the inverse of an L-morphisms is indeed a lattice homomorphism.

2.16 Proposition. *If $f : (X, \wedge) \to (X', \wedge')$ is an L-morphism, then $f^{-1} : \mathcal{F}(X', \wedge') \to \mathcal{F}(X, \wedge)$ is a lattice homomorphism.*

Proof. We already know that f^{-1} is a bounded semilattice homomorphism, so we only have to show that it preserves binary joins. That is, we show that for arbitrary filters $a', b' \in \mathcal{F}(X', \wedge')$ we have

$$f^{-1}(a' \curlyvee b') = f^{-1}(a') \curlyvee f^{-1}(b'). \tag{4}$$

If a' or b' is the empty filter, then the equality is trivial, so suppose both are non-empty. The inclusion $\supseteq$ follows from the fact that $f^{-1}(a' \curlyvee b')$ is a filter that contains both $f^{-1}(a')$ and $f^{-1}(b')$. Conversely, suppose $x \in f^{-1}(a' \curlyvee b')$. Then $f(x) \in a' \curlyvee b'$ so there exist $y' \in a'$ and $z' \in b'$ such that $y' \wedge z' \leq f(x)$. By definition of L-frame morphism, we can find $y, z \in X$ such that $y' \leq f(y)$ and $z' \leq f(z)$ and $y \wedge z \leq x$. This means that $y \in f^{-1}(a')$ and $z \in f^{-1}(b')$, and hence $x \in f^{-1}(a') \curlyvee f^{-1}(b')$. $\square$

So $\mathcal{F}$ restricts to a contravariant functor $\mathcal{F} : \mathsf{LFrm} \to \mathsf{Lat}$. Interestingly, the converse holds as well; we can restrict $\mathcal{F}_b$ to a contravariant functor $\mathsf{Lat} \to \mathsf{LFrm}$.

2.17 Proposition. *Let $h : L \to L'$ be a lattice homomorphism. Then $h^{-1} : \mathcal{F}_b L' \to \mathcal{F}_b L$ is an L-morphism.*

Proof. We already know that h^{-1} is a semilattice homomorphism, so we only have to show that it satisfies the additional condition from Definition 2.15. Let $p' \in \mathcal{F}L'$ and $q, r \in \mathcal{F}L$ and suppose $q \cap r \subseteq h^{-1}(p')$. Let $q' := {\uparrow}h[q]$ and $r' := {\uparrow}h[r]$. Then it is easy to verify that q' and r' are filters (because q and r are), and by construction $q \subseteq h^{-1}(q')$ and $r \subseteq h^{-1}(r')$. It remains to show that $q' \cap r' \subseteq p'$. Let $a' \in L'$ be such that $a' \in q' \cap r'$. Since $a' \in q'$ there exists $a \in q$ such that $h(a) \leq a'$. Since $a' \in r'$ there exists $b \in r$ such that $h(b) \leq a'$. But then $a \vee b \in q \cap r$, so by assumption $h(a \vee b) \in p'$. This implies $a' \in p'$, because $h(a \vee b) = h(a) \vee h(b) \leq a'$ and p' is a filter (hence up-closed). □

The relation between the categories SL and LFrm is similar to that between the categories Pos, of posets and order-preserving functions, and $\mathsf{IntKrip}$, of posets and bounded morphisms. (The latter category is called Krip because the posets in it are often referred to as intuitionistic Kripke frames.) See also figure 1.

$$\begin{tikzcd}
\mathsf{SL} \arrow[r, bend left, "\mathcal{F}"] & \mathsf{BSL} \arrow[l, bend left, "\mathcal{F}_b"] \\
\mathsf{LFrm} \arrow[u] \arrow[r, bend left, "\mathcal{F}"] & \mathsf{Lat} \arrow[u] \arrow[l, bend left, "\mathcal{F}_b"]
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathsf{Pos} \arrow[r, bend left, "up"] & \mathsf{DL} \arrow[l, bend left, "pf"] \\
\mathsf{IntKrip} \arrow[u] \arrow[r, bend left, "up"] & \mathsf{HA} \arrow[u] \arrow[l, bend left, "pf"]
\end{tikzcd}$$

Figure 1: Dualities for various classes of (semi)lattices. The upper rows are dual adjunctions. The functor up takes a poset to its lattice of upsets, and pf takes a lattice to its ordered set of prime filters.

We now define the spaces and morphisms that will give a duality for lattices. They are a variation on the "PUP spaces" from [13].

2.18 Definition. A *lattice space*, or *L-space* for short, is an M-space $\mathbb{X} = (X, \wedge, \tau)$ such that $a \curlyvee b$ is a clopen filter whenever a and b are clopen filters. An L-space morphism is an M-space morphism that is simultaneously an L-morphism. We write LSpace for the category of L-spaces and L-space morphisms.

We prove the following lemma for future reference.

2.19 Lemma. *Let $\mathbb{X} = (X, \wedge, \tau)$ be an L-space. Then a filter b of $\mathbb{X}$ is open if and only if it is the join of clopen filters.*

Proof. Suppose b is an open filter. Using compactness and (M_3) it can be shown that for each $x \in b$ we can find a clopen filter a such that $x \in a \subseteq b$. As a consequence $b = \bigcup_{x \in b} a_x$, and since b is the smallest filter containing all of the a_x we have $b = \bigcurlyvee_{x \in b} a_x$. Conversely, suppose $b = \bigcurlyvee_{i \in I} a_i$, where each a_i is a clopen filter. Then b is a filter by definition. As a consequence of (3) we have

$$b = \bigcup \{a_{i_1} \curlyvee \cdots \curlyvee a_{i_n} \mid n \in \omega \text{ and } i_1, \ldots, i_n \in I\}.$$

Since $\mathbb{X}$ is an L-space this is the union of clopen sets, hence it is open. □

2.20 Theorem. *The duality for bounded semilattices from Theorem 2.12 restricts to a duality*

$$\mathsf{LSpace} \equiv^{\mathrm{op}} \mathsf{Lat}.$$

Proof. We only have to verify that the restriction of $\mathcal{F}_{clp}$ to LSpace lands in Lat, and the restriction of $\mathcal{F}_{top}$ to Lat lands in LSpace. The former follows from the fact that the clopen filters of an L-space are closed under $\curlyvee$, together with Proposition 2.16.

For the latter, suppose that L is a lattice and let $\theta_L(a)$ and $\theta_L(b)$ be two arbitrary clopen filters of $\mathcal{F}_{top}L$. Writing x, y, z for elements in $\mathcal{F}_{top}L$, we have

$$\theta_L(a) \curlyvee \theta_L(b) = {\uparrow}\{x \cap y \mid x \in \theta_L(a), y \in \theta_L(b)\} = \theta_L(a \vee b)$$

Let us elaborate on the last equality. If $a = \bot$ or $b = \bot$ then the proof is obvious, so suppose this is not the case. If $x \in \theta_L(a)$ and $y \in \theta_L(b)$ then $a \in x$ and $b \in y$, so $a \vee b \in x \cap y$. So $z \supseteq x \cap y$ implies $a \vee b \in z$, and therefore we have "$\subseteq$". Conversely, if $z \in \theta_L(a \vee b)$ then we need to find $x \in \theta_L(a)$ and $y \in \theta_L(b)$ such that $x \cap y \subseteq z$. Let $x = {\uparrow}a \in \theta_L(a)$ and $y = {\uparrow}b \in \theta_L(b)$. Then $d \in x \cap y$ implies $a \leq d$ and $b \leq d$, hence $a \vee b \leq d$. Since $z \in \theta_L(a \vee b)$ this implies $d \in z$, and therefore $x \cap y \subseteq z$. This proves "$\supseteq$". The restriction on morphisms follows from Proposition 2.17. $\square$

An overview of the dualities for semilattices and lattices and the analogy with intuitionistic logic is given in Figure 2, in Section 3.3.

2.4 Lattice Completions

In the final subsection of this section, we investigate how L-spaces give rise to several types of lattice completions. First we show how the well-known filter completion and canonical extension can be obtained as lattices of suitable filters of an L-space. Afterward, we discuss a third extension which arises from L-spaces in a natural way. To the best of our knowledge, this extension has not appeared in the literature before. The following diagram gives an overview of the situation.

$$\begin{tikzcd}
 & \text{Lattice} \arrow[d] & \\
 & \text{L-space} \arrow[dl, "\text{principal filters}"', "\text{= closed filters}"] \arrow[d, "\text{saturated filters}"] \arrow[dr, "\text{all filters}"] & \\
\text{Filter completion} & \text{Canonical extension} & F^2\text{-completion}
\end{tikzcd}$$

We begin by recalling the definition of a completion of a lattice.

2.21 Definition. A *completion* of a lattice L is a pair (e, C) where C is a complete lattice and $e : L \to C$ is a lattice embedding. An element in C is called *open* if it is the join of elements in the image of e, and *closed* if it is the meet of elements in the image of e.

2.22 Definition. For a lattice L, denote by $\mathcal{F}^\partial L$ the set of non-empty filters of L, ordered by reverse inclusion. Then $\mathcal{F}^\partial L$ forms a complete lattice with meet $\bigcurlyvee$ and join $\bigcap$. The sets $\{\top\}$ and L serve as top and bottom, respectively. If we define the embedding $i : L \to \mathcal{F}^\partial L$ by $a \mapsto {\uparrow}a$, we obtain the completion $(i, \mathcal{F}^\partial L)$ of L, called the *filter completion*.

Let $\mathbb{X} = (X, \wedge, \tau)$ be an M-space. Let $\mathcal{F}_k\mathbb{X}$ denote the collection of closed filter of $\mathbb{X}$. (Recall from Lemma 2.8 that these are precisely the principal filters.) Since the arbitrary intersection of a collection of closed filters is again a closed filter, $\mathcal{F}_k\mathbb{X}$ forms a complete semilattice with meet $\bigcap$. The top and bottom element are given by X and $\emptyset$, respectively. The join can then be defined as the smallest upper bound in the inclusion order. Specifically, the join of two closed filters c_1 and c_2 is given by $c_1 \curlyvee c_2$. Writing $c_i = {\uparrow}x_i$, this is equal to $c_1 \curlyvee c_2 = {\uparrow}(x_1 \wedge x_2)$. Interestingly, the join of an arbitrary collection $F \subseteq \mathcal{F}_k\mathbb{X}$ is not necessarily given by $\bigcurlyvee F$, because this need not be closed. Rather, the join in $\mathcal{F}_k\mathbb{X}$, denoted by $\overline{\bigcurlyvee}$, is given by

$$\overline{\bigcurlyvee}\{{\uparrow}x_i \mid i \in I\} = {\uparrow}(\bigwedge\{x_i \mid i \in I\}) \tag{5}$$

if the join does not include the empty set. The join of a set $\{\emptyset\} \cup \{{\uparrow}x_i \mid i \in I\}$ that does contain the empty set is given by

$$\overline{\bigcurlyvee}(\{\emptyset\} \cup \{{\uparrow}x_i \mid i \in I\}) = {\uparrow}(\bigwedge\{x_i \mid i \in I\}) \tag{6}$$

where we presume the meet on the right hand side to give the empty set if $I = \emptyset$. Note that we make use of the fact that the semilattice underlying an M-space has all non-empty meets.

2.23 Theorem. *The filter completion of a lattice is isomorphic to the complete lattice of closed filters of its dual L-space. That is, for any lattice L, we have*

$$\mathcal{F}^{\partial}L \cong \mathcal{F}_k\mathcal{F}_{top}L.$$

Proof. Define $\xi : \mathcal{F}^{\partial}L \to \mathcal{F}_k\mathcal{F}_{top}L : p \mapsto \{x \in \mathcal{F}_{top}L \mid p \subseteq x\}$. Clearly this is well defined, since $\xi(p)$ is a principal filter for each $p \in \mathcal{F}^{\partial}L$, hence a closed filter. Suppose $p \neq q$ are two filters in $\mathcal{F}^{\partial}L$. Without loss of generality assume $p \not\subseteq q$. Then $q \notin \xi(p)$ and $q \in \xi(q)$, so $\xi(p) \neq \xi(q)$ and therefore ξ is injective.

To see that ξ is surjective, suppose given a closed filter $c \in \mathcal{F}_k\mathcal{F}_{top}L$. If c is non-empty, then by Lemma 2.8 it is of the form ${\uparrow}x$ for some $x \in \mathcal{F}_{top}L$. But $\mathcal{F}_{top}L$ contains all non-empty proper filters, so x is also in $\mathcal{F}^{\partial}L$. It now follows from the definition that $c = \xi(x)$. If c is empty then $c = \emptyset = \xi(L)$. Indeed, $\xi(L)$ is empty because no filter in $\mathcal{F}_{top}L$ contains the bottom element of L, hence none of them are supersets of L. This proves surjectivity.

Next we claim that ξ is a complete lattice homomorphism. It preserves the top element because $\xi(\{\top_L\}) = \mathcal{F}_{top}L$, and the bottom element since $\xi(L) = \emptyset$. To see that ξ preserves all meets, recall that the meet of elements in $\mathcal{F}^{\partial}L$ is given by $\bigcurlyvee$. For all $F \subseteq \mathcal{F}^{\partial}L$ such that $L \notin F$ we have

$$\xi(\bigcurlyvee F) = \{x \in \mathcal{F}_{top}L \mid \bigcurlyvee F \subseteq x\} = \{x \in \mathcal{F}_{top}L \mid p \subseteq x \text{ for all } p \in F\} = \bigcap\{\xi(p) \mid p \in F\}.$$

If $L \in F$ then $\bigcurlyvee F = L$. Since $\xi(L) = \emptyset$ we then find $\xi(\bigcurlyvee F) = \xi(L) = \bigcap\{\xi(p) \mid p \in F\}$. Finally, note it follows immediately from (5) and (6) that

$$\xi(\bigcap F) = \overline{\bigcurlyvee}\{\xi(p) \mid p \in F\},$$

so ξ also preserves all joins. It follows that ξ is a complete lattice homomorphism. $\square$

The inclusion map $i : L \to \mathcal{F}^{\partial}L$ corresponds to the unit $\theta_L : L \to \mathcal{F}_k\mathcal{F}_{top}L : a \mapsto \{x \in \mathcal{F}_{top}L \mid a \in x\}$.

The next completion we investigate is the well-known *canonical extension*.

2.24 Definition. Let L be a lattice. A completion (e, C) is called *dense* if every element of C can be written as the join of meets of elements in L, and as the meet of joins of elements in L. It is called *compact* if for any set A of closed elements of C and B of open elements of C, $\bigwedge A \leq \bigvee B$ if and only if there are finite subsets $A' \subseteq A$ and $B' \subseteq B$ such that $\bigwedge A' \leq \bigvee B'$.

A *canonical extension* of L is a completion that is dense and compact.

It is known that every lattice has a canonical extension [15, Proposition 2.6], and that any two canonical extension of a lattice L are isomorphic by a unique isomorphism that commutes with the embeddings of L [15, Proposition 2.7]. So we can talk of *the* canonical extension of a lattice. We investigate how to obtain a description of the canonical extension of a lattice L using its dual L-space. This is similar to the topological description of canonical extensions found in [34].

2.25 Definition. Let $\mathbb{X} = (X, \wedge, \tau)$ be an L-space. By a *saturated filter* we mean a filter b on $\mathbb{X}$ that is the intersection of all open filters that contain it. We write $\mathcal{F}_{sat}\mathbb{X}$ for the collection of saturated filters. Clearly, the arbitrary intersection of saturated filters is again saturated, so $\mathcal{F}_{sat}\mathbb{X}$ forms a complete semilattice with top X and bottom $\emptyset$.

Note that every open filter is saturated. Every closed filters is the intersection of all clopen filters containing it, so closed filters are saturated as well. Since $\mathcal{F}_{sat}\mathbb{X}$ is a complete semilattice it is also a complete lattice, and the join of a set F of saturated filters is given by $\bigcurlyvee_{\mathrm{s}} F = \bigcap\{b \in \mathcal{F}_{sat}\mathbb{X} \mid \bigcup F \subseteq b\}$. Note that in general $\bigcurlyvee F \subseteq \bigcurlyvee_{\mathrm{s}} F$. If F consists entirely of open filters, then $\bigcurlyvee F$ is open again, hence saturated, so joins of open filters in $\mathcal{F}_{sat}\mathbb{X}$ are computed using $\bigcurlyvee$.

We now prove that the saturated filters can be used to describe the canonical extension of a lattice. This is the L-space counterpart of the characterisation of the canonical extension using spectral spaces, as given in [34, Theorem 4.1].

2.26 Theorem. *Let L be a lattice. The complete lattice $\mathcal{F}_{sat}\mathcal{F}_{top}L$ with embedding $\theta_L : L \to \mathcal{F}_{sat}\mathcal{F}_{top}L : a \mapsto \{x \in \mathcal{F}_{top}L \mid a \in x\}$ describes the canonical extension of L.*

Proof. It suffices to show that the completion is dense and compact. Since the clopen filters are the images of L in $\mathcal{F}_{sat}\mathcal{F}_{top}L$ under θ_L, and the open filters of $\mathcal{F}_{sat}\mathcal{F}_{top}L$ coincide precisely with joins (and unions) of clopen filters, the open elements of the embedding are simply the open filters (where the latter "open" means open as a subset of the space $\mathcal{F}_{top}L$). Similarly, as a consequence of Lemma 2.9 the closed element of the completion are the closed filters.

We first show that the completion is dense. By definition, every element in $\mathcal{F}_{sat}\mathcal{F}_{top}L$ is the intersection of open elements. Since $\uparrow x$ is closed for each $x \in \mathcal{F}_{top}L$, it follows that each filter d in $\mathcal{F}_{sat}\mathcal{F}_{top}L$ is the union of all closed elements it contains. This automatically implies that d is the smallest saturated filter containing the set of closed filters contained in d, hence d is the join of closed elements. Therefore $(\theta_L, \mathcal{F}_{sat}\mathcal{F}_{top}L)$ is dense.

Next we show that $\mathcal{F}_{sat}\mathcal{F}_{top}L$ is compact. By [15, Lemma 2.4] it suffices to show that for any $A, B \subseteq L$ we have $\bigcap\{\theta_L(a) \mid a \in A\} \subseteq \bigcurlyvee\{\theta_L(b) \mid b \in B\}$ iff there exist finite $A' \subseteq A$ and $B' \subseteq B$ such that $\bigcap\{\theta_L(a) \mid a \in A'\} \subseteq \bigcurlyvee\{\theta_L(b) \mid b \in B'\}$. (Note that we can use $\bigcurlyvee$ here because we are taking the join of opens.) The implication from right to left is obvious. So suppose $\bigcap\{\theta_L(a) \mid a \in A\} \subseteq \bigcurlyvee\{\theta_L(b) \mid b \in B\}$. Then $\bigcurlyvee\{\theta_L(b) \mid b \in B\}$ is open, so by compactness there exists a finite subset $A' \subseteq A$ such that $\bigcap\{\theta_L(a) \mid a \in A'\} \subseteq \bigcurlyvee\{\theta_L(b) \mid b \in B\}$. Since the join of opens is computed as usual, we can rewrite the right hand side to obtain an open cover

$$\bigcap\{\theta_L(a) \mid a \in A'\} \subseteq \bigcup\{\theta_L(b_1) \curlyvee \cdots \curlyvee \theta_L(b_n) \mid n \in \omega \text{ and } b_1, \ldots, b_n \in B\}.$$

Using compactness and the fact that the right-hand union is over a directed set, we find $b_1, \ldots, b_n \in B$ such that $\bigcap\{\theta_L(a) \mid a \in A') \subseteq \theta_L(b_1) \curlyvee \cdots \curlyvee \theta_L(b_n)$. Setting $B' = \{b_1, \ldots, b_n\}$ yields the result. $\square$

Our final completion is defined in terms of L-spaces, rather than lattices.

2.27 Definition. The *double filter completion* or F^2*-completion* of a lattice L is the lattice $\mathcal{F}\mathcal{F}_b L$ with embedding θ. In other words, it is the complete lattice of all filters of an L-space (not just the clopen or closed or saturated ones).

An immediate question is whether we can characterise the double filter completion algebraically. We leave this as an interesting direction for further research.

3 Non-Distributive Positive Logic

We use the duality and dual adjunction from Section 2 to give a Kripke-style semantics for non-distributive positive logic. Inspired by the fact that the filters of a semilattice form a lattice, we use semilattices as frames and (principal) filters as valuations. We use these to study (non necessarily distributive) positive logic.

We start this section by giving an axiomatisation of our logic. By design the algebraic semantics is simply given by lattices. Then, in Section 3.2 we formally define frames and models, and give several (classes of) examples. We prove that the frame semantics is sound. In Section 3.3 we use the duality from Section 2 to derive completeness of the basic logic with respect to several classes of frames. We give the standard translation into a suitable first-order logic and prove Sahlqvist correspondence in Section 3.4, where we also work out specific examples of correspondence results. Finally, we prove Sahlqvist canonicity in Section 3.5. This gives rise to completeness results and a new proof of Baker and Hales' theorem which states that every variety of lattices is closed under filter extensions. Moreover, it follows that every variety of lattices is also closed under F^2-extensions.

To distinguish the various notions of entailment each has their own notation, which are summarised in Table 1. We denote the interpretation of a formula φ in a lattice $\mathscr{A}$ and in a frame $\mathfrak{M}$ by $(\!|\varphi|\!)_{\mathscr{A}}$ and $[\![\varphi]\!]^{\mathfrak{M}}$, respectively.

Notation	Purpose	Location
$\varphi \vdash \psi$	Syntactic entailment	Def. 3.1
$\varphi \Vdash\!\!\!\!\!\!\!\!\; \psi$	Algebraic entailment	Def. 3.3
$\varphi \Vdash \psi$	Semantic entailment	Def. 3.6
$\varphi \Vdash_{\mathsf{LSpace}} \psi$	Topological semantic entailment	Def. 3.20
$\varphi \models \psi$	First-order entailment	Sec. 3.4 & 3.5

Table 1: Different notions of entailment.

3.1 Logic and Algebraic Semantics

Let $\mathbf{L}$(Prop) denote the language of positive propositional logic. That is, $\mathbf{L}$(Prop) is generated by the grammar

$$\varphi ::= p \mid \top \mid \bot \mid \varphi \wedge \varphi \mid \varphi \vee \varphi,$$

where p ranges over some arbitrary but fixed set Prop of proposition letters. If no confusion arises we will omit reference to Prop and simply write $\mathbf{L}$.

For lack of a strong enough implication, we define the minimal logic $\mathcal{L}$ based on $\mathbf{L}$ as a collection of *consequence pairs*. This is based on Dunn's axiomatisation of positive modal logic [14], leaving out the modal axioms and distributivity. Formally, a consequence pair is simply an expression of the form $\varphi \trianglelefteq \psi$, where φ and ψ are formulae in $\mathbf{L}$. The intuitive reading of $\varphi \trianglelefteq \psi$ is: "If φ holds, then so does ψ."

3.1 Definition. Let $\boldsymbol{\mathcal{L}}$ be the smallest set of consequence pairs closed under the following axioms and rules: *top* and *bottom*

$$\varphi \trianglelefteq \top, \qquad \bot \trianglelefteq \varphi,$$

reflexivity and *transitivity*

$$\varphi \trianglelefteq \varphi, \qquad \frac{\varphi \trianglelefteq \psi \quad \psi \trianglelefteq \chi}{\varphi \trianglelefteq \chi},$$

the *conjunction rules*

$$\varphi \wedge \psi \trianglelefteq \varphi, \qquad \varphi \wedge \psi \trianglelefteq \psi, \qquad \frac{\chi \trianglelefteq \varphi \quad \chi \trianglelefteq \psi}{\chi \trianglelefteq \varphi \wedge \psi},$$

and the *disjunction rules*

$$\varphi \trianglelefteq \varphi \vee \psi, \qquad \psi \trianglelefteq \varphi \vee \psi, \qquad \frac{\varphi \trianglelefteq \chi \quad \psi \trianglelefteq \chi}{\varphi \vee \psi \trianglelefteq \chi}.$$

If Γ is a set of consequence pairs then we let $\boldsymbol{\mathcal{L}}(\Gamma)$ denote the smallest set of consequence pairs closed under the axioms and rules mentioned above and those in Γ. We write $\varphi \vdash_\Gamma \psi$ if $\varphi \trianglelefteq \psi \in \boldsymbol{\mathcal{L}}(\Gamma)$ and $\varphi \dashv\vdash_\Gamma \psi$ if both $\varphi \vdash_\Gamma \psi$ and $\psi \vdash_\Gamma \varphi$. If Γ is the empty set then we simply write $\varphi \vdash \psi$ and $\varphi \dashv\vdash \psi$.

The algebraic semantics of the logic $\mathbf{L}$ are simply lattices. We establish this formally.

3.2 Definition. Let A be a lattice with operations $\top_A, \bot_A, \wedge_A, \vee_A$, and induced order $\leq_A$. A *lattice model* is a pair $\mathscr{A} = (A, \sigma)$ consisting of a lattice A and an assignment $\sigma : \mathrm{Prop} \to A$ of the proposition letters. We define the interpretation $(\!|\varphi|\!)_{\mathscr{A}}$ of an $\mathbf{L}$-formula φ in $\mathscr{A}$ recursively via

$$\begin{aligned}
&(\!|p|\!)_{\mathscr{A}} = \sigma(p) \qquad && (\!|\top|\!)_{\mathscr{A}} = \top_A \qquad && (\!|\varphi \wedge \psi|\!)_{\mathscr{A}} = (\!|\varphi|\!)_{\mathscr{A}} \wedge_A (\!|\psi|\!)_{\mathscr{A}} \\
& && (\!|\bot|\!)_{\mathscr{A}} = \bot_A && (\!|\varphi \vee \psi|\!)_{\mathscr{A}} = (\!|\varphi|\!)_{\mathscr{A}} \vee_A (\!|\psi|\!)_{\mathscr{A}}.
\end{aligned}$$

We say that a lattice A *validates* a consequence pair $\varphi \trianglelefteq \psi$ if $(\!|\varphi|\!)_{\mathscr{A}} \leq_A (\!|\psi|\!)_{\mathscr{A}}$ for all lattice models $\mathscr{A}$ based on A, notation: $A \Vdash \varphi \trianglelefteq \psi$. If Γ is a set of consequence pairs then we write $\mathsf{Lat}(\Gamma)$ for the full subcategory of Lat whose objects validate all consequence pairs in Γ.

3.3 Definition. Let $\Gamma \cup \{\varphi \trianglelefteq \psi\}$ be a set of consequence pairs. Write $\varphi \Vdash_\Gamma \psi$ if $(\!|\varphi|\!)_{\mathscr{A}} \leq_A (\!|\psi|\!)_{\mathscr{A}}$ for every lattice model $\mathscr{A} = (A, \sigma)$ such that $A \in \mathsf{Lat}(\Gamma)$. If Γ is empty we write $\varphi \Vdash \psi$ instead of $\varphi \Vdash_\emptyset \psi$.

Observe that $\dashv\vdash_\Gamma$ is an equivalence relation on $\mathbf{L}$. Write $L(\Gamma)$ for the set of $\dashv\vdash_\Gamma$-equivalence classes of $\mathbf{L}$, and denote by $[\varphi]$ the equivalence class of φ in $L(\Gamma)$. Then it follows from the rules in Definition 3.1 that $L(\Gamma)$ carries a lattice structure, where $\top_L = [\top]$, $\bot_L = [\bot]$, $[\varphi] \wedge_L [\psi] = [\varphi \wedge \psi]$ and $[\varphi] \wedge_L [\psi] = [\varphi \vee \psi]$. Moreover, $L(\Gamma)$ is in $\mathsf{Lat}(\Gamma)$, and setting $\sigma_L : \mathrm{Prop} \to L(\Gamma) : p \mapsto [p]$ yields lattice model $\mathscr{L}_\Gamma = (L(\Gamma), \sigma_L)$ which acts as the Lindenbaum-Tarski algebra. It follows from an induction on the structure of φ that $(\!|\varphi|\!)_{\mathscr{L}_\Gamma} = [\varphi]$ for all $\mathbf{P}$-formulae φ.

3.4 Lemma. *We have $\varphi \Vdash_\Gamma \psi$ if and only if $(\!|\varphi|\!)_{\mathscr{L}_\Gamma} \leq_L (\!|\psi|\!)_{\mathscr{L}_\Gamma}$.*

Proof. The direction from left to right holds by definition. Conversely, if $A \in \mathsf{Lat}(\Gamma)$ and $\mathscr{A} = (A, \sigma_A)$ is a lattice model, then the assignment $[p] \mapsto \sigma_A(p)$ extends to a lattice homomorphism $i : \mathscr{L}_\Gamma \to \mathscr{A}$ such that $[\varphi] = (\!|\varphi|\!)_{\mathscr{A}}$. (This is well defined because A validates all consequence pairs in Γ.) Then $(\!|\varphi|\!)_{\mathscr{L}_\Gamma} \leq_L (\!|\psi|\!)_{\mathscr{L}_\Gamma}$ implies $[\varphi] \leq_L [\psi]$ and monotonicity of i entails $(\!|\varphi|\!)_{\mathscr{A}} \leq_L (\!|\psi|\!)_{\mathscr{A}}$. Therefore $\varphi \Vdash_\Gamma \psi$. $\square$

3.5 Theorem. *We have $\varphi \vdash_\Gamma \psi$ if and only if $\varphi \Vdash_\Gamma \psi$.*

Proof. As a consequence of Lemma 3.4 it suffices to show that $\varphi \vdash_\Gamma \psi$ if and only if $(\!|\varphi|\!)_{\mathscr{L}_\Gamma} \leq_L (\!|\psi|\!)_{\mathscr{L}_\Gamma}$. It follows from the conjunction rules, reflexivity and transitivity that $\varphi \vdash_\Gamma \psi$ if and only if $\varphi \wedge \psi \dashv\vdash_\Gamma \varphi$. Therefore we have $\varphi \vdash_\Gamma \psi$ if and only if $[\varphi \wedge \psi] = [\varphi]$ in $\mathscr{L}_\Gamma$, and since $[\varphi \wedge \psi] = [\varphi] \wedge_L [\psi]$ this holds if and only if $[\varphi] \leq_L [\psi]$. Recalling that $[\varphi] = (\!|\varphi|\!)_{\mathscr{L}_\Gamma}$ completes the proof. $\square$

3.2 Frame Semantics

As we have seen, the collection of filters of a semilattice forms a lattice. Therefore we can use semilattices as a generalisation of (intuitionistic) Kripke frames to interpret non-distributive positive logic. As interpretants of the formulae we choose either all filters of the semilattice, or the principal filters.

3.6 Definition. A *lattice Kripke frame*, or *L-frame* for short, is a pair $(X, \leq)$ where X is a set and $\leq$ is a partial order on X such that each pair of elements $x, y \in X$ has a greatest lower bound, denoted by $x \wedge y$. A *valuation* for an L-frame $(X, \leq)$ is a function $V : \mathrm{Prop} \to \mathscr{F}(X, \leq)$ which assigns to each proposition letter a filter of $(X, \leq)$. An L-frame together with a valuation is called an *L-model*. The interpretation of an $\mathbf{L}$-formula φ at a state x in an L-model $\mathfrak{M} = (X, \leq, V)$ is defined recursively via

$$
\begin{array}{rcl}
\mathfrak{M}, x \Vdash \top & & \text{always} \\
\mathfrak{M}, x \Vdash \bot & & \text{never} \\
\mathfrak{M}, x \Vdash p & \text{iff} & x \in V(p) \\
\mathfrak{M}, x \Vdash \varphi \wedge \psi & \text{iff} & \mathfrak{M}, x \Vdash \varphi \text{ and } \mathfrak{M}, x \Vdash \psi \\
\mathfrak{M}, x \Vdash \varphi \vee \psi & \text{iff} & \mathfrak{M}, x \Vdash \varphi \text{ or } \mathfrak{M}, x \Vdash \psi \text{ or} \\
& & \exists y, z \in X \text{ s.t. } \mathfrak{M}, y \Vdash \varphi \text{ and } \mathfrak{M}, z \Vdash \psi \text{ and } y \wedge z \leq x
\end{array}
$$

We write $[\![\varphi]\!]^{\mathfrak{M}} := \{x \in X \mid \mathfrak{M}, x \Vdash \varphi\}$ for the *truth set* of φ in $\mathfrak{M}$. If the frame is fixed and we want to emphasise the role of the valuation in the interpretation, we will write $V(\varphi)$ instead of $[\![\varphi]\!]^{\mathfrak{M}}$. The *theory* of x is denoted by $\mathrm{th}_{\mathfrak{M}}(x) := \{\varphi \in \mathbf{L} \mid \mathfrak{M}, x \Vdash \varphi\}$.

We write $\mathfrak{M}, x \Vdash \varphi \trianglelefteq \psi$ if $x \in [\![\varphi]\!]^{\mathfrak{M}}$ implies $x \in [\![\psi]\!]^{\mathfrak{M}}$, and $\mathfrak{M} \Vdash \varphi \trianglelefteq \psi$ if $\mathfrak{M}, x \Vdash \varphi \trianglelefteq \psi$ for all states x in $\mathfrak{M}$. Note that $\mathfrak{M} \Vdash \varphi \trianglelefteq \psi$ iff $[\![\varphi]\!]^{\mathfrak{M}} \subseteq [\![\psi]\!]^{\mathfrak{M}}$. If $\mathfrak{X}$ is an L-frame, we let $\mathfrak{X}, x \Vdash \varphi \trianglelefteq \psi$ if $x \in V(\varphi)$ implies $x \in V(\psi)$ for all valuations V of $\mathfrak{X}$, and $\mathfrak{X} \Vdash \varphi \trianglelefteq \psi$ if $\mathfrak{X}, x \Vdash \varphi \trianglelefteq \psi$ for all states x of $\mathfrak{X}$. We say that $\mathfrak{M}$ or $\mathfrak{X}$ *validates* $\varphi \trianglelefteq \psi$ if $\mathfrak{M} \Vdash \varphi \trianglelefteq \psi$ or $\mathfrak{X} \Vdash \varphi \trianglelefteq \psi$, respectively.

If Γ is a set of consequence pairs, then we let $\mathsf{LFrm}(\Gamma)$ denote the full subcategory of LFrm whose objects validate all consequence pairs in Γ. We write $\varphi \Vdash_\Gamma \psi$ if $\mathfrak{X} \Vdash \varphi \trianglelefteq \psi$ for all $\mathfrak{X} \in \mathsf{LFrm}(\Gamma)$. If $\Gamma = \emptyset$ then we write $\varphi \Vdash \psi$ instead of $\varphi \Vdash_\emptyset \psi$.

Indeed, L-frames are simply semilattices. When used as frame semantics, we usually write them as a set with a partial order, rather than a set with a conjunction, to stress that they can be viewed as relational structures.

For any L-frame $\mathfrak{F} = (X, \leq)$, the collection $\mathfrak{F}^* := \mathscr{F}(X, \leq)$ forms a lattice, called the *complex algebra* of $\mathfrak{F}$. Since valuations of $\mathfrak{F}$ correspond bijectively to assignments of $\mathfrak{F}^*$, we can define the *complex algebra* of an L-model $\mathfrak{M} = (X, \leq, V)$ by $\mathfrak{M}^* = (\mathscr{F}(X, \leq), V)$. A routine induction on the structure of φ then proves the following lemma.

3.7 Lemma. *For every L-model $\mathfrak{M}$ and* **L***-formula φ we have*

$$[\![\varphi]\!]^{\mathfrak{M}} = (\!|\varphi|\!)_{\mathfrak{M}^*}.$$

As an immediate corollary we obtain the following persistence result. This is similar to persistence of intuitionistic formulae in intuitionistic Kripke frames, except for the fact that we require formulae to be interpreted as filters rather than upsets.

3.8 Proposition (Persistence)**.** *Let $\mathfrak{M} = (X, \leq, V)$ be an L-model. For each $\varphi \in \mathbf{L}$ the truth set $[\![\varphi]\!]^{\mathfrak{M}}$ of φ is a filter of $(X, \leq)$.*

Proof. Immediate consequence of Lemma 3.7. □

3.9 Theorem (Soundness)**.** *If $\varphi \vdash_\Gamma \psi$ then $\varphi \Vdash_\Gamma \psi$.*

Proof. If $\mathfrak{M}$ is a model that validates all consequence pairs in Γ, then $\mathfrak{M}^* \in \mathsf{Lat}(\Gamma)$. Since $\varphi \vdash_\Gamma \psi$, Theorem 3.5 yields $\varphi \Vdash_\Gamma \psi$, and hence $(\!|\varphi|\!)_{\mathfrak{M}^*} \leq (\!|\psi|\!)_{\mathfrak{M}^*}$. Lemma 3.7 now implies $[\![\varphi]\!]^{\mathfrak{M}} \leq [\![\psi]\!]^{\mathfrak{M}}$, so that $\mathfrak{M}$ validates $\varphi \trianglelefteq \psi$. Since $\mathfrak{M}$ was chosen arbitrarily, this proves $\varphi \Vdash_\Gamma \psi$. □

3.10 Example. We list some examples of (classes of) L-frames.

1. Any linearly ordered set is an L-frame. Filters in such frames are simply upsets. For example, $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{R}$ with the natural ordering are all L-frames.

2. Any Scott domain is a semilattice, hence an L-frame.

3. If X is a set, then the collections $\mathscr{P}^+X$, $\mathscr{P}_\omega X$ and $\mathscr{P}_\omega^+X$ of non-empty, finite, and finite non-empty subsets of X, respectively, form semilattices. In each case, the meet is given by set-theoretic union. The latter is the free semilattice over X, and filters of $\mathscr{P}_\omega^+X$ correspond bijectively with subsets of X. ◁

Before discussing more examples, we identify an important subclass of L-frames and L-models. In some cases, we can interpret every formula as a principal filter. (Recall that we legislated the empty filter to be principal as well.) These are significant because, as we have seen in Section 2.4, the principal filters of an L-space $\mathbb{X}$ give rise to the filter extension of the lattice dual to $\mathbb{X}$.

3.11 Definition. An L-frame $(X, \leq)$ is called *principal* if it has all finite joins, including a bottom element which is denoted by 0. A *principal valuation* V for an L-frame $(X, \leq)$ is a valuation such that $V(p)$ is a principal filter of $(X, \leq)$ for each proposition letter p. A *principal L-model* is a principal L-frame together with a principal valuation.

We have the following persistence results.

3.12 Proposition. *Let $\mathfrak{M} = (X, \leq, V)$ be a principal L-model. Then $[\![\varphi]\!]^{\mathfrak{M}}$ a principal filter for every $\varphi \in \mathbf{L}$.*

Proof. It follows from Lemma 3.7 that $[\![\varphi]\!]^{\mathfrak{M}}$ is a filter for each formula φ, so we only have to show that it is principal. For the base cases this is obvious: $V(p)$ is principal by definition and $[\![\bot[\![^{\mathfrak{M}} = \emptyset$ and $[\![\top]\!]^{\mathfrak{M}} = \uparrow 0$. For the step cases it suffices to observe that if $[\![\varphi_1]\!]^{\mathfrak{M}} = \uparrow x_1$ and $[\![\varphi_2]\!]^{\mathfrak{M}} = \uparrow x_2$, then $[\![\varphi_1 \wedge \varphi_2]\!]^{\mathfrak{M}} = \uparrow(x_1 \vee x_2)$ and $[\![\varphi_1 \wedge \varphi_2]\!]^{\mathfrak{M}} = \uparrow(x_1 \vee x_2)$. In other words, the collection of principal filters of a principal L-model is closed under $\cap$ and $\curlyvee$. $\square$

We give some examples of classes of principal L-models. Since every principal L-model is in particular an L-model, this simultaneously extends our collection of examples of L-models.

3.13 Example. The L-frame $(\mathbb{N}, \leq)$ from Example 3.10(1) is principal but $(\mathbb{Z}, \leq)$ and $(\mathbb{R}, \leq)$ are not, because they do not have minimal elements. Every filter of $(\mathbb{N}, \leq)$ is principal, so every L-model based on $(\mathbb{N}, \leq)$ is principal. An example of a non-principal filter of $(\mathbb{R}, \leq)$ is $\{x \in \mathbb{R} \mid 4 < x\}$. $\lhd$

3.14 Example. Another interesting class of examples of principal L-frames is given by rooted trees of finite depth. Filters in such frames are always principal, and a valuation indicates that a property p is true at a node x and henceforth. As usual, a node satisfies $\varphi \wedge \psi$ if it satisfies both φ and ψ. In such structures, a node x satisfies $\varphi \vee \psi$ if it has two cousins y and z whose "youngest" common ancestor is also an ancestor of x, such that y satisfies φ and z satisfies ψ. $\lhd$

3.15 Example. We briefly recall a simplified version of team semantics for propositional logics. This underlies many versions of modal dependence and independence logics, such as the ones studied in [25, 32, 38, 39]. Consider the language $\mathbf{T}(\mathrm{Prop})$ given by $\varphi ::= p \mid \neg p \mid \varphi \wedge \varphi \mid \varphi \vee \varphi$. These can be interpreted in models consisting of a set X and a valuation $\Pi : \mathrm{Prop} \to \mathcal{P}X$ of the proposition letters. However, rather than assigning truth of formulae to elements of X, truth is defined for *subsets* of X (the teams). Let $\mathfrak{M} = (X, \Pi)$ be such a model and $T \subseteq X$ a team, then we let

$$\begin{aligned} \mathfrak{M}, T \Vdash_t p \quad &\text{iff} \quad T \subseteq V(p) \\ \mathfrak{M}, T \Vdash_t \neg p \quad &\text{iff} \quad T \cap V(p) = \emptyset \\ \mathfrak{M}, T \Vdash_t \varphi \wedge \psi \quad &\text{iff} \quad \mathfrak{M}, T \Vdash_t \varphi \text{ and } \mathfrak{M}, T \Vdash_t \psi \\ \mathfrak{M}, T \Vdash_t \varphi \vee \psi \quad &\text{iff} \quad \exists T_1, T_2 \subseteq T \text{ s.t. } T_1 \cup T_2 = T \text{ and } \mathfrak{M}, T_1 \Vdash_t \varphi \text{ and } \mathfrak{M}, T_2 \Vdash_t \psi \end{aligned}$$

We can add $\top$ and $\bot$ by defining them to be always true and always false, respectively.

Interestingly, the interpretation looks a lot like that in Definition 3.6. Let us make this precise. For a set Prop of proposition letters, let $\neg\mathrm{Prop} = \{\neg p \mid p \in \mathrm{Prop}\}$. Then, given a team model $\mathfrak{M} = (X, \Pi)$, we can define a principal L-model $\mathfrak{M}' = (\mathcal{P}X, \supseteq, V)$, where $V : \mathrm{Prop} \cup \neg\mathrm{Prop} \to \mathcal{F}(X, \supseteq)$ is the valuation defined by $V(p) = \{a \in \mathcal{P}X \mid a \subseteq \Pi(p)\}$ and $V(\neg p) = \{a \in \mathcal{P}X \mid a \cap \Pi(p) = \emptyset\}$. Then the meet of $\mathfrak{M}'$ is given by set-theoretic union, and it is easy to see that V is a principal valuation. Moreover, $\mathbf{L}(\mathrm{Prop} \cup \neg\mathrm{Prop})$ is simply the extension of $\mathbf{T}(\mathrm{Prop})$ with a top and bottom element, and for each team model $\mathfrak{M}$, team T, and formula $\varphi \in \mathbf{L}(\mathrm{Prop} \cup \neg\mathrm{Prop})$ we have

$$\mathfrak{M}, T \Vdash_t \varphi \quad \text{iff} \quad \mathfrak{M}', T \Vdash \varphi. \qquad \lhd$$

3.16 Example. We investigate the relation between non-distributive positive logic interpreted in L-frames and *modal information logic* [3]. Modal information logic is the extension of propositional classical logic with two binary modal operators $\langle \mathsf{inf} \rangle$ and $\langle \mathsf{sup} \rangle$. These are interpreted in Kripke models $\mathfrak{M} = (X, R, V)$ where R is a pre-order on X as follows:

$$\mathfrak{M}, x \Vdash \langle \mathsf{inf} \rangle(\varphi, \psi) \quad \text{iff} \quad \exists y, z \in X \text{ s.t. } x = \inf(y, z) \text{ and } \mathfrak{M}, y \Vdash \varphi \text{ and } \mathfrak{M}, z \Vdash \psi$$
$$\mathfrak{M}, x \Vdash \langle \mathsf{sup} \rangle(\varphi, \psi) \quad \text{iff} \quad \exists y, z \in X \text{ s.t. } x = \sup(y, z) \text{ and } \mathfrak{M}, y \Vdash \varphi \text{ and } \mathfrak{M}, z \Vdash \psi$$

Note that we need not require that every pair of states has an infimum and a supremum, nor that it is unique. The definition simply uses the fact that they might exist. Observe that we can recover the usual modal diamond via $\Diamond\varphi = \langle \mathsf{inf} \rangle(\varphi, \top)$. Moreover, we can define a temporal diamond $\blacklozenge$ as $\blacklozenge\varphi = \langle \mathsf{sup} \rangle(\varphi, \top)$.

Clearly, every L-model is a model for modal information logic. Interestingly, the interpretation of $\langle \mathsf{inf} \rangle$ is closely aligned to our interpretation of joins; the only difference is that the infimum is allowed to be *below* the state under consideration. Taking this into account, our interpretation of joins in an L-model $\mathfrak{M} = (X, \leq, V)$ coincides with

$$\varphi \vee \psi = \blacklozenge(\langle \mathsf{inf} \rangle(\varphi, \psi)),$$

where $\vee$ is the non-classical join. $\lhd$

3.17 Remark. The partial translation of $\mathbf{L}$ into modal information logic may give rise to an analogue of the Gödel-McKinsey-Tarski translation [17, 33]. The role of $\Box$ can possibly be replaced by $\blacklozenge\langle \mathsf{inf} \rangle(-, -)$, as this ensures that the interpretation of a formula is a filter. We flag this as an interesting direction for future research. $\lhd$

3.18 Definition. An *L-model morphism* from $(X, \leq, V)$ to $(X', \leq', V')$ is an L-morphism (Definition 2.15) $f : (X, \leq) \to (X', \leq')$ that satisfies $V = f^{-1} \circ V'$. We write LMod for the category of L-models and L-model morphisms, and pLMod for its full subcategory of principal L-models.

As desired, L-model morphisms preserve and reflect truth of $\mathbf{L}$-formulae.

3.19 Proposition. *Let $f : \mathfrak{M} \to \mathfrak{M}'$ be an L-model morphism. Then for all states x of $\mathfrak{M}$ and all $\varphi \in \mathbf{L}$,*

$$\mathfrak{M}, x \Vdash \varphi \quad \textit{iff} \quad \mathfrak{M}', f(x) \Vdash \varphi.$$

Proof. Routine induction on the structure of φ. $\Box$

3.3 Descriptive Frames and Completeness

We have already seen a duality for lattices, by means of L-spaces. L-spaces can be viewed as topologised L-frames. In fact, since every L-space $\mathbb{X}$ is of the form $\mathscr{F}_{top}A$ for some lattice A, it follows that $\mathbb{X}$ has a bottom element (given by $\{\top_A\}$) and binary joins (given by $\curlyvee$), so that we may also view them as topologised *principal* L-frames. In this subsection we define how to interpret $\mathbf{L}$-formulae in L-spaces, and show how this gives rise to completeness for $\mathcal{L}$.

We denote L-spaces and L-spaces with a valuation by $\mathbb{X}$ and $\mathbb{M}$. Their non-topologised counterparts are L-frames and L-models, and are denoted by $\mathfrak{X}$ and $\mathfrak{M}$. If $\mathbb{X}$ is an L-space, then we write $\kappa\mathbb{X}$ for its underlying (principal) L-frame.

3.20 Definition. A *clopen valuation* for an L-space $\mathbb{X}$ is an assignment $V : \mathrm{Prop} \to \mathscr{F}_{clp}\mathbb{X}$, which assigns to each proposition letter a clopen filter of $\mathbb{X}$. We call a pair $\mathbb{M} = (\mathbb{X}, V)$ of an L-space and a clopen valuation an *L-space model*.

The interpretation $[\![\varphi]\!]^{\mathbb{M}}$ of an $\mathbf{L}$-formula φ in an L-space model $\mathbb{M} = (\mathbb{X}, V)$ is defined as in the underlying L-model $(\kappa\mathbb{X}, V)$. The L-space model $\mathbb{M}$ validates a consequence pair $\varphi \trianglelefteq \psi$ if

$[\![\varphi]\!]^{\mathbb{M}} \subseteq [\![\psi]\!]^{\mathbb{M}}$, notation: $\mathbb{M} \Vdash \varphi \trianglelefteq \psi$. We say that an L-space $\mathbb{X}$ validates $\varphi \trianglelefteq \psi$ if every L-space model based on it validates $\varphi \trianglelefteq \psi$. Finally, we write $\varphi \Vdash_{\mathsf{LSpace}} \psi$ if every L-space validates $\varphi \trianglelefteq \psi$.

3.21 Lemma. *Let $\mathbb{X}$ be an L-space, A its dual lattice, and $\varphi, \psi \in \mathbf{L}$. Then*

1. *$\mathbb{X} \Vdash \varphi$ iff $A \Vdash \varphi$;*
2. *$\mathbb{X} \Vdash \varphi \trianglelefteq \psi$ iff $A \Vdash \varphi \trianglelefteq \psi$.*

Proof. The first item follows from the fact that clopen valuations of $\mathbb{X}$ correspond bijectively to assignments of the proposition letters for A, together with a routine induction on the structure of φ. The second item follows immediately from the first. □

We can now prove completeness of $\mathbf{L}$ with respect to several classes of frames.

3.22 Theorem. *The logic $\mathcal{L}$ is (sound and) complete with respect to the classes of*

1. *L-spaces;*
2. *principal L-frames;*
3. *all L-frames.*

Proof. Soundness follows from Theorem 3.9. For completeness, we show that $\varphi \nvdash \psi$ implies $\varphi \nVdash \psi$. So suppose $\varphi \nvdash \psi$, then by Theorem 3.5 we can find a lattice A that does not validate $\varphi \trianglelefteq \psi$. As a consequence of Lemma 3.21 the L-space $\mathbb{X}$ dual to A does not validate $\varphi \trianglelefteq \psi$, so there must exist a clopen valuation V such that $(\mathbb{X}, V) \nVdash \varphi \trianglelefteq \psi$. But this implies that $(\kappa\mathbb{X}, V) \nVdash \varphi \trianglelefteq \psi$. Since $\kappa\mathbb{X}$ is a (principal) L-frame and every clopen filter of an L-space is principal (due to Lemma 2.8) we have found a (principal) L-model not validating $\varphi \trianglelefteq \psi$. □

It is now natural to wonder weather we can prove similar theorems for extensions of $\mathcal{L}$ with a set Γ of consequence pairs. We will answer this question positively in Section 3.5.

3.23 Remark. We can alternatively describe L-spaces as *descriptive L-frames*. That is, as L-frames with extra structure, similar to descriptive intuitionistic Kripke frames. Since we prefer to work with L-spaces, we only briefly sketch this perspective.

A *general L-frame* is a tuple $(X, \leq, A)$ such that $(X, \leq)$ is an L-frame and A is a collection of filters of $(X, \leq)$ containing X and $\emptyset$, and closed under $\cap$ and $\curlyvee$. The sets in A are called *admissible filters*. Let $-A = \{X \setminus a \mid a \in A\}$. A general L-frame $(X, \leq, A)$ is called

- *refined* if for all $x, y \in X$ such that $x \nleq y$ there exists an $a \in A$ such that $x \in a$ and $y \notin a$;
- *compact* if $\bigcap C \neq \emptyset$ for each $C \subseteq A \cup -A$ with the finite intersection property;
- *descriptive* if it is refined and compact.

A general L-morphism from $(X, \leq, A)$ to $(X', \leq', A')$ is an L-morphism $f : (X, \leq) \to (X', \leq')$ such that $f^{-1}(a') \in A$ for all $a' \in A'$. Write $\mathsf{D\text{-}LFrm}$ for the category of descriptive L-frames and general L-morphisms. Then we have

$$\mathsf{D\text{-}LFrm} \cong \mathsf{LSpace}.$$

Proof sketch. Suppose that $\mathbb{X} = (X, \wedge, \tau)$ is an L-space and define $\mathfrak{X} = (X, \leq, A_\tau)$, where $\leq$ is the order induced by $\wedge$ and A_τ is the collection of clopen filters of $\mathbb{X}$. Then $(X, \leq)$ is an L-frame, $\mathfrak{X}$ is compact because $\mathbb{X}$ is compact, $\mathfrak{X}$ is refined because $\mathbb{X}$ satisfies the HMS separation axiom, and A_τ is closed under $\curlyvee$ by assumption.

Next let $\mathfrak{X} = (X, \leq, A)$ be a descriptive L-frame, write $\wedge$ for the conjunction induced by $\leq$ and let τ_A be the topology on X generated by $A \cup -A$. Let $\mathbb{X} = (X, \wedge, \tau_A)$. Then $(X, \wedge)$ is a semilattice because $(X, \leq)$ is an L-frame, (X, τ) is compact because $\mathfrak{X}$ is compact, $\mathbb{X}$ satisfies the HMS separation axiom because $\mathfrak{X}$ is refined.

It is clear that the two constructions above define a bijection between the underlying semilattice/L-frame. We will show that moreover $\tau_{A_\tau} = \tau$ and $A = A_{\tau_A}$, thus proving the isomorphism on morphisms. This also shows that the second construction is well defined, as it follows that the clopen filters of τ_A are closed under $\curlyvee$.

It follows from the HMS separation axiom that the clopen filters and their complements form a subbase for τ. Therefore, if $b \in \tau$ then it is also in the topology generated by $A_\tau \cup -A_\tau$, so $b \in \tau_{A_\tau}$. Conversely, since $A \cup -A \subseteq \tau$ it follows that $b \in \tau_{A_\tau}$ implies $b \in \tau$. So $\tau = \tau_{A_\tau}$.

To see that $A = A_{\tau_A}$, we note that trivially $A \subseteq A_{\tau_A}$. So suppose $b \in A_{\tau_A}$, i.e. b is a clopen filter in τ_A. If $b = \emptyset$ then the statement is obvious, so suppose $b \neq \emptyset$. Then in particular it is a closed filter in an M-space, so by Lemma 2.8 we have $b = {\uparrow}x$ for some $x \in X$. For each $y \in X$ with $y \notin b$ we the have $x \not\leq y$, so by refinedness of $\mathfrak{X}$ there exists some $a \in A$ containing x but not y. This implies that $b = \bigcap\{a \in A \mid x \in a\}$, so that $-b \cap \bigcap A = \emptyset$. It follows from compactness of $\mathfrak{X}$ that there is a finite $A' \subseteq A$ such that $-b \cap \bigcap A' = \emptyset$, so that $b = \bigcup A'$. Since b is a filter we have $b = \bigcurlyvee\{a \mid a \in A'\}$ and since A is closed under $\curlyvee$ it follows that $b \in A$. We conclude that $A = A_{\tau_A}$, as desired.

The isomorphism on morphisms follows from unravelling the definitions. $\square$

Continuing the analogy with intuitionistic logic, descriptive L-frames relate to L-spaces in the same way intuitionistic Kripke frames relate to Esakia spaces. The various dualities are depicted in Figure 2, together with the analogous diagram from intuitionistic logic.

$$\begin{tikzcd}
 & \mathsf{MSpace} \arrow[r, dash, "\text{Thm. 2.12}", "\equiv^{\mathrm{op}}"'] & \mathsf{SL} \\
\mathsf{D\text{-}LFrm} \arrow[r, dash, "\text{Rem. 3.23}", "\cong"'] & \mathsf{LSpace} \arrow[u] \arrow[r, dash, "\text{Thm. 2.20}", "\equiv^{\mathrm{op}}"'] & \mathsf{Lat} \arrow[u]
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \mathsf{Pries} \arrow[r, dash, "\text{Priestley}", "\equiv^{\mathrm{op}}"'] & \mathsf{DL} \\
\mathsf{D\text{-}Krip} \arrow[r, dash, "\cong"'] & \mathsf{Esakia} \arrow[u] \arrow[r, dash, "\text{Esakia}", "\equiv^{\mathrm{op}}"'] & \mathsf{HA} \arrow[u]
\end{tikzcd}$$

Figure 2: Dualities for non-distributive positive logic and for intuitionistic logic.

3.4 The First-Order Translation and Sahlqvist Correspondence

In this section we define the standard translation of $\mathbf{L}$ into a suitable first-order logic. We use this to derive a Sahlqvist correspondence result. We prove that for every consequence pair $\psi \trianglelefteq \chi$, the collection of L-frames validate $\psi \trianglelefteq \chi$ are first-order definable. Our proof of the correspondence result follows a standard proof from normal modal logic, such as found in [6, Section 3.6]. Thus, it showcases how our duality for lattices allows us to transfer classical techniques to the positive non-distributive setting. However, it is complicated (or rather, made more interesting) by the non-standard interpretation of disjunctions.

3.24 Definition. Let **FOL** be the single-sorted first-order language which has a unary predicate P_p for every proposition letter p, and a binary relation symbol R.

Intuitively, the relation symbol of our first-order language accounts of the poset structure of L-frames. It is used in the translation of disjunctions.

If x, y and z are first-order variables, then we can express that x is above every lower bound of y and z in the ordering induced by the relation symbol R using a first-order sentence. In order to streamline notation we abbreviate this as follows:

$$\mathsf{abovemeet}(x; y, z) := \forall w((wRy \wedge wRz) \to wRx).$$

If $x, y_1, \ldots, y_n$ is a finite set of variables and $n \geq 1$ then we define $\mathsf{abovemeet}(x; y_1, \ldots, y_n)$ in the obvious way.

We are now ready to define the standard translation.

3.25 Definition. Let x be a first-order variable. Define the standard translation $\mathrm{st}_x : \mathbf{L} \to \mathbf{FOL}$ recursively via

$$\begin{aligned}
\mathrm{st}_x(p) &= P_p x \\
\mathrm{st}_x(\top) &= (x = x) \\
\mathrm{st}_x(\bot) &= (x \neq x) \\
\mathrm{st}_x(\varphi \wedge \psi) &= \mathrm{st}_x(\varphi) \wedge \mathrm{st}_x(\psi) \\
\mathrm{st}_x(\varphi \vee \psi) &= \mathrm{st}_x(\varphi) \vee \mathrm{st}_x(\psi) \vee \exists y \exists z(\mathsf{abovemeet}(x; y, z) \wedge \mathrm{st}_y(\varphi) \wedge \mathrm{st}_z(\psi))
\end{aligned}$$

Furthermore, we define the standard translation of a consequence pair $\varphi \trianglelefteq \psi$ as

$$\mathrm{st}_x(\varphi \trianglelefteq \psi) = \mathrm{st}_x(\varphi) \to \mathrm{st}_x(\psi).$$

Every L-model $\mathfrak{M} = (X, \leq, V)$ gives rise to a first-order structure for **FOL**: $\leq$ accounts for the interpretation of the binary relation symbol, and the interpretation of the unary predicates is given via the valuations of the proposition letters. We write $\mathfrak{M}^\circ$ for the L-model $\mathfrak{M}$ conceived of as a first-order structure for **FOL**.

3.26 Proposition. *We have*

$$\mathfrak{M}, w \Vdash \varphi \quad \textit{iff} \quad \mathfrak{M}^\circ \models \mathrm{st}_x(\varphi)[w].$$

Proof. This follows from a routine induction on the structure of φ. The propositional case holds by definition of the P_i. The cases $\varphi = \top$ and $\varphi = \bot$ hold by definition. The case $\varphi = \psi_1 \wedge \psi_2$ is trivial. For $\varphi = \psi_1 \vee \psi_2$, we have

$$\begin{aligned}
&\mathfrak{M}, w \Vdash \psi_1 \vee \psi_2 \\
&\qquad \text{iff} \quad \mathfrak{M}, w \Vdash \psi_1 \text{ or } \mathfrak{M}, w \Vdash \psi_2 \text{ or} \\
&\qquad\qquad \exists y, z \in X \text{ s.t. } y \wedge z \leq x \text{ and } \mathfrak{M}, y \Vdash \psi_1 \text{ and } \mathfrak{M}, z \Vdash \psi_2 && \text{(Def)} \\
&\qquad \text{iff} \quad \mathfrak{M}^\circ \models \mathrm{st}_x(\psi_1)[w] \text{ or } \mathfrak{M}^\circ \models \mathrm{st}_x(\psi_2)[w] \text{ or} \\
&\qquad\qquad \mathfrak{M}^\circ \models \exists y \exists z(\mathsf{abovemeet}(x; y, z) \wedge \mathrm{st}_y(\psi_1) \wedge \mathrm{st}_z(\psi_2))[w] && \text{(IH)} \\
&\qquad \text{iff} \quad \mathfrak{M}^\circ \models \mathrm{st}_x(\psi_1 \vee \psi_2)[w]
\end{aligned}$$

This completes the proof. $\square$

3.27 Corollary. *Let $\mathfrak{M}$ be an L-model. Then we have:*

1. $\mathfrak{M} \Vdash \varphi$ *iff* $\mathfrak{M}^\circ \models \forall x(\mathrm{st}_x(\varphi))$*;*
2. $\mathfrak{M}, w \Vdash \varphi \trianglelefteq \psi$ *iff* $\mathfrak{M}^\circ \models \mathrm{st}_x(\varphi \trianglelefteq \psi)[w]$*;*
3. $\mathfrak{M} \Vdash \varphi \trianglelefteq \psi$ *iff* $\mathfrak{M}^\circ \models \forall x(\mathrm{st}_x(\varphi \trianglelefteq \psi))$*.*

In order to obtain similar results as in Corollary 3.27 for frames, we need to quantify the unary predicates in **FOL** corresponding to the proposition letters. We can do so in a second-order language, say, **SOL**. However, getting a second-order correspondent for a consequence pair $\varphi \trianglelefteq \psi$ that is satisfied in a frame if and only if $\varphi \trianglelefteq \psi$ is, is not as easy as simply quantifying over all possible interpretations of the unary predicates. That is, we cannot simply add $\forall P_1 \cdots \forall P_n$ in front of $\mathrm{st}_x(\varphi \trianglelefteq \psi)$. Indeed, we wish to only take those interpretations into account that arise from a valuation of the proposition letters *as filters*.

Thus we wish to quantify over interpretations of the unary predicates corresponding to filters in the underlying frame. We can force this by adding conditions that ensure that the P's are interpreted as filters in the antecedent of the implication $\mathrm{st}_x(\varphi) \to \mathrm{st}_x(\psi)$. Then the implication is vacuously true for "illegal" interpretations of the unary predicates. This intuition motivates the following definition of the *second-order translation* of a consequence pair.

3.28 Definition. Let $p_1, \ldots, p_n$ be the proposition letters occurring in ψ and χ, and let $P_1, \ldots, P_n$ denote their corresponding unary predicates. For each P_i, abbreviate

$$\mathsf{isfil}(P_i) = \forall x \forall y \forall z((P_i y \wedge P_i z \wedge \mathsf{abovemeet}(x; y, z)) \to P_i x).$$

Using this abbreviation, we define the *second order translation* of a consequence pair $\psi \trianglelefteq \chi$ by

$$\mathrm{so}(\psi \trianglelefteq \chi) = \forall P_1 \cdots \forall P_n \forall x\big((\mathsf{isfil}(P_1) \wedge \cdots \wedge \mathsf{isfil}(P_n) \wedge \mathrm{st}_x(\psi)) \to \mathrm{st}_x(\chi)\big). \tag{7}$$

To disburden notation, we will often abbreviate $\mathsf{isfil}(P_1) \wedge \cdots \wedge \mathsf{isfil}(P_n)$ as ISFIL.

Since all unary predicates in $\mathrm{so}(\psi \trianglelefteq \chi)$ are bound (i.e. in the scope of a quantifier), the formula $\mathrm{so}(\psi \trianglelefteq \chi)$ can be interpreted in a first-order structure with a single relation. Therefore, every L-model $\mathfrak{X}$ gives rise to a structure $\mathfrak{X}^\circ$ for **SOL** in which we can interpret second order translations.

3.29 Lemma. *For all L-frames $\mathfrak{X} = (X, \leq)$ and all consequence pairs $\psi \trianglelefteq \chi$ we have*

$$\mathfrak{X} \Vdash \psi \trianglelefteq \chi \quad \textit{iff} \quad \mathfrak{X}^\circ \models \mathrm{so}(\psi \trianglelefteq \chi).$$

Proof. Suppose $\mathfrak{X} \Vdash \psi \trianglelefteq \chi$. Then for every valuation V of the proposition letters we have $V(\psi) \subseteq V(\chi)$. If any of the P_i is interpreted as a subset of X that is not a filter, then the implication inside the quantifiers in (15) is automatically true, because the antecedent is false. If all P_i are interpreted as filters, then the implication holds because of the assumption.

The converse is similar. □

Next, we show how one can use the second-order translation to obtain local correspondence results. We first define what we mean by local correspondence.

3.30 Definition. Let $\varphi \trianglelefteq \psi$ be a consequence pair and $\alpha(x)$ a first-order formula with free variable x. Then we say that $\varphi \trianglelefteq \psi$ and $\alpha(x)$ are *local frame correspondents* if for any L-frame $\mathfrak{X}$ and any state w we have

$$\mathfrak{X}, w \Vdash \varphi \trianglelefteq \psi \quad \text{iff} \quad \mathfrak{X} \models \alpha(x)[w].$$

Since our language is positive, every formula is upward monotone. That is, extending the valuation increases the truth set of formulae.

3.31 Lemma. *Let $\mathfrak{X}$ be an L-frame and let V and V' be valuations for $\mathfrak{X}$ such that $V(p) \subseteq V'(p)$ for all $p \in \mathrm{Prop}$. Then for all $\varphi \in \mathbf{L}$ we have $V(\varphi) \subseteq V'(\varphi)$.*

Proof. Straightforward induction on the structure of φ. □

As a consequence of this lemma, a frame validates $\top \trianglelefteq \chi$ if and only if it validates χ', where χ' is obtained from χ by replacing all proposition letters with $\bot$. This, in turn, implies that $\mathrm{so}(\top \trianglelefteq \chi')$ is a first-order correspondent of $\top \trianglelefteq \chi$, since the lack of proposition letters in χ' implies that there are no second-order quantifiers in $\mathrm{so}(\top \trianglelefteq \chi')$.

Furthermore, consequence pairs of the form $\bot \trianglelefteq \chi$ are vacuously valid on all frames. Both these cases (with either $\top$ or $\bot$ as the antecedent of the consequence pair) are not very interesting. In the next theorem we prove that all other consequence pairs have a local first-order correspondent as well. Since we have already discussed what happens if the antecedent is $\top$ or $\bot$, we may preclude these cases from the proof of our theorem.

3.32 Theorem. *Any consequence pair $\psi \trianglelefteq \chi$ of $\mathbf{L}$-formulae locally corresponds to a first-order formula with one free variable.*

Proof. We know that $\mathfrak{X} \Vdash \psi \trianglelefteq \chi$ if and only if $\mathfrak{X}^\circ \models \mathrm{so}(\psi \trianglelefteq \chi)$. Our strategy for obtaining a first-order correspondent is to remove all second-order quantifiers from the second-order translation. We assume that this expression has been processed such that no two quantifiers bind the same variable.

If the antecedent is equivalent to either $\top$ or $\bot$ then we already know that the statement is true, so we assume that it is not. Moreover, in this case we can replace the antecedent with an equivalent formula that does not contain $\top$ or $\bot$. So we may assume that the antecedent does not involve $\top$ or $\bot$.

Let $p_1, \ldots, p_n$ be the propositional variables occurring in ψ, and write $P_1, \ldots, P_n$ for their corresponding unary predicates. We assume that every proposition letter that occurs in χ also occurs is ψ, for otherwise we may replace it by $\bot$ to obtain a formula which is equivalent in terms of validity on frames.

Step 1. We start by pre-processing the formula $\mathrm{so}(\psi \trianglelefteq \chi)$ some more. We make use of the fact that, after applying the second-order translation, we have classical laws such as distributivity.

Step 1A. Use equivalences of the form

$$(\exists w(\alpha(w)) \wedge \beta) \leftrightarrow \exists w(\alpha(w) \wedge \beta), \qquad (\exists w(\alpha(w)) \vee \beta) \leftrightarrow \exists w(\alpha(w) \vee \beta),$$

and

$$(\exists w(\alpha(w)) \to \beta) \leftrightarrow \forall w(\alpha(w) \to \beta)$$

to pull out all quantifiers that arise in $\mathrm{st}_x(\psi)$. Let $Y := \{y_1, \ldots, y_m\}$ denote the set of (bound) variables that occur in the antecedent of the implication from the second-order translation. We end up with a formula of the form

$$\forall P_1 \cdots \forall P_n \forall x \forall y_1 \cdots \forall y_m \big((\underbrace{\mathsf{isfil}(P_1) \wedge \cdots \wedge \mathsf{isfil}(P_n)}_{\mathrm{ISFIL}} \wedge \bar{\psi}) \to \mathrm{st}_x(\chi) \big).$$

In this formula, $\bar{\psi}$ is made up of formulae of the form $P_i z$ and $\mathsf{abovemeet}(z; z', z'')$ by using $\wedge$ and $\vee$, where $z, z', z'' \in Y \cup \{x\}$.

Step 1B. Use distributivity (of first-order classical logic) to pull out the disjunctions from $\mathrm{ISFIL} \wedge \bar{\psi}$. That is, we rewrite $\mathrm{ISFIL} \wedge \bar{\psi}$ as a (finite) disjunction

$$\mathrm{ISFIL} \wedge \bar{\psi} = \bigvee \Big(\mathrm{ISFIL} \wedge \underbrace{P_{i_1} y_{i_1} \wedge \cdots \wedge P_{i_\ell} y_{i_\ell}}_{\mathrm{AT}} \wedge \underbrace{\mathsf{abovemeet}(y_{j_1}; y'_{j_1}, y''_{j_1}) \wedge \cdots \wedge \mathsf{abovemeet}(y_{j_k}; y'_{j_k}, y''_{j_k})}_{\mathrm{REL}} \Big)$$

All the y's with any sort of subscript come from the set $Y \cup \{x\}$.

Step 1C. Finally, use equivalences of the form

$$((\alpha \vee \beta) \to \gamma) \leftrightarrow ((\alpha \to \gamma) \wedge (\beta \to \gamma))$$

and

$$\forall \ldots (\alpha \wedge \beta) \leftrightarrow ((\forall \ldots \alpha) \wedge (\forall \ldots \beta))$$

to rewrite $\mathrm{so}(\varphi \trianglelefteq \psi)$ into a conjunction of formulae of the form

$$\forall P_1 \cdots \forall P_n \forall y_1 \cdots \forall y_m \big(\mathrm{ISFIL} \wedge \mathrm{AT} \wedge \mathrm{REL} \to \chi \big). \tag{8}$$

Step 2. Next we focus on each of the formulae of the form given in (8) individually. We read off minimal instances of the P_i making the antecedent true. Intuitively, these correspond to the smallest valuations for the p_i making the antecedent true.

For each proposition letter P_i, let $P_i y_{i_1}, \ldots, P_i y_{i_k}$ be the occurrences of P_i in AT in the antecedent of (8). Intuitively, we define the valuation of p_i to be the filter generated by the (interpretations of) $y_{i_1}, \ldots, y_{i_k}$. Formally,

$$\sigma(P_i) := \lambda u.\mathsf{abovemeet}(u; y_{i_1}, \ldots, y_{i_k}).$$

(If $k = 0$, i.e. there is no variable y with $P_i y$, then we let $\sigma(P_i) = \lambda u.(u \neq u)$.) Then for each L-model $\mathfrak{M}$ and states $x', y'_1, \ldots, y'_m$ in $\mathfrak{M}$ we have

$$\mathfrak{M}^\circ \models \mathrm{AT} \wedge \mathrm{REL}[x, y'_1, \ldots, y'_m] \quad \text{implies} \quad \mathfrak{M}^\circ \models \forall y(\sigma(P_i)y \to P_i y).$$

If we replace each unary predicate P in (16) with $\sigma(P)$, then all of the "isfil" formulas become true, as do all of the formulae in AT. Writing $[\sigma(P)/P]\,\mathrm{st}_x(\chi)$ for the formula obtained from $\mathrm{st}_x(\chi)$ by replacing each instance of a unary predicate P with $\sigma(P)$, we arrive at the first-order formula

$$\forall x \forall y_1 \cdots \forall y_m \big(\mathrm{REL} \to [\sigma(P_i)/P_i]\,\mathrm{st}_x(\chi) \big) \tag{9}$$

Step 3. Finally, we claim that for every L-frame $\mathfrak{X}$, $\mathfrak{X}^\circ$ validates (8) if and only if it validates (9). The implication from left to right is simply an instantiation of the quantifiers as filters. For the converse, assume that $\mathfrak{M}$ is some model based on $\mathfrak{X}$, so that $\mathfrak{M}^\circ$ is an extension of $\mathfrak{X}^\circ$ giving the interpretations of the unary predicates as filters. We may disregard the case where any of them is not a filter as that would make the antecedent in (16) false, hence the whole implication true. Let $x', y'_1, \ldots, y'_m$ be states in $\mathfrak{M}$ and assume that

$$\mathfrak{M}^\circ \models \mathrm{ISFIL} \wedge \mathrm{AT} \wedge \mathrm{REL}[x', y'_1, \ldots, y'_m]. \tag{10}$$

We need to show that $\mathfrak{M}^\circ \models \mathrm{st}_x(\chi)[x', y'_1, \cdots, y'_m]$. It follows from the assumption that (9) holds that $\mathfrak{M}^\circ \models [\sigma(P)/P]\,\mathrm{st}_x(\chi)[x', y'_1, \cdots, y'_m]$. Moreover, as a consequence of (10) we have $\mathfrak{M}^\circ \models \forall y(\sigma(P)(y) \to Py)$ for all $P \in \{P_1, \ldots, P_n\}$. Using Lemma 3.31 it follows that $\mathfrak{M}^\circ \models \mathrm{st}_x(\chi)[x', y'_1, \cdots, y'_m]$, as desired. $\square$

Let us work out some explicit examples so we can see the proof of the theorem in action.

3.33 Example. Consider the formula $p \wedge (q \vee q') \trianglelefteq (p \wedge q) \vee (p \wedge q')$. This corresponds to distributivity; the reverse consequence pair is always valid. We temporarily abbreviate $\chi := (p \wedge q) \vee (p \wedge q')$. The second-order translation of this formula is

$$\begin{aligned} \mathrm{so}(p \wedge (q \vee q') \trianglelefteq \chi) = \forall P \forall Q \forall Q' \forall x \big(\big[&\mathsf{isfil}(P) \wedge \mathsf{isfil}(Q) \wedge \mathsf{isfil}(Q') \\ &\wedge Px \wedge (Qx \vee Q'x \vee \exists y \exists y' (\mathsf{abovemeet}(x; y, y') \wedge Qy \wedge Q'y')) \big] \to \mathrm{st}_x(\chi) \big) \end{aligned}$$

As per instructions, we rewrite this to

$$\begin{aligned} &\mathrm{so}(p \wedge (q \vee q') \trianglelefteq \chi) \\ &\quad = \forall P \forall Q \forall Q' \forall x \forall y \forall y' \big((\mathrm{ISFIL} \wedge Px \wedge Qx) \to \mathrm{st}_x(\chi) \big) \\ &\qquad \wedge \forall P \forall Q \forall Q' \forall x \forall y \forall y' \big((\mathrm{ISFIL} \wedge Px \wedge Q'x) \to \mathrm{st}_x(\chi) \big) \\ &\qquad \wedge \forall P \forall Q \forall Q' \forall x \forall y \forall y' \big((\mathrm{ISFIL} \wedge Px \wedge \mathsf{abovemeet}(x; y, y') \wedge Qy \wedge Q'y') \to \mathrm{st}_x(\chi) \big). \end{aligned} \tag{11}$$

Tor each of the conjoints we can find a first-order correspondent. Let us focus first on the part, $\forall P \forall Q \forall Q' \forall x \forall y \forall y' \big((\mathrm{ISFIL} \wedge Px \wedge Qx) \to \mathrm{st}_x(\chi) \big)$. Then we have $\sigma(P) = \sigma(Q) = \lambda u.x \leq u$ and $\sigma(Q') = \lambda u.(u \neq u)$. The standard translation of χ is

$$(Px \wedge Qx) \vee (Px \wedge Q'x) \vee \exists z \exists z' (\mathsf{abovemeet}(x; z, z') \wedge Pz \wedge Qz \wedge Pz' \wedge Q'z').$$

This yields

$$[\sigma(P_i)/P_i]\chi = (x \leq x \wedge x \leq x) \vee \cdots$$

which is vacuously true because the first disjoint is true. As a consequence, the first conjoint of (11) is vacuously true. Hence it does not contribute non-trivially to the local correspondent, and we may ignore it. Similarly, the second conjoint in (11) is vacuously true.

For the third conjoint we obtain $\sigma(P) = \lambda u.x \leq u$, $\sigma(Q) = \lambda u.y \leq u$ and $\sigma(Q') = \lambda u.y' \leq u$. Plugging these into the antecedent yields

$$\begin{aligned} [\sigma(P_i)/P_i]\chi = &(y \leq x) \vee (y' \leq x) \\ &\vee \exists z \exists z' \big(\mathsf{abovemeet}(x; z, z') \wedge (x \leq z) \wedge (y \leq z) \wedge (x \leq z') \wedge (y' \leq z') \big). \end{aligned}$$

Thus we find the following first-order correspondent:

$$\begin{aligned} \forall x \forall y \forall y' \big(\mathsf{abovemeet}(x; y, y') \to \big[&(y \leq x) \vee (y' \leq x) \\ &\vee \exists z \exists z' (\mathsf{abovemeet}(x; z, z') \wedge (x \leq z) \wedge (y \leq z) \\ &\qquad \wedge (x \leq z') \wedge (y' \leq z')) \big] \big). \end{aligned}$$

This can be reformulated as follows:

$$\forall x \forall y \forall y' \big((y \wedge y' \leq x) \to \big[(y \leq x) \vee (y' \leq x) \vee \exists z \exists z' ((x = z \wedge z') \wedge (y \leq z) \wedge (y' \leq z')) \big] \big).$$

If the frame is principal this reduces further. If $y \leq x$ then taking $z = x$ and $z' = x \vee y'$ yields states above y and y' respectively whose meet is x. Similar if $y' \leq x$. We get: if $y \wedge y' \leq x$ then there are z, z' such that $y \leq z$ and $y' \leq z'$ and $z \wedge z' = x$. In other words, the lattice is distributive. $\triangleleft$

3.34 Example. Next consider the modularity axiom

$$((p_1 \wedge p_3) \vee p_2) \wedge p_3 \trianglelefteqslant (p_1 \wedge p_3) \vee (p_2 \wedge p_3).$$

Writing χ for the right hand side of the consequence pair, after applying step 1 of the proof of Theorem 3.32 we have

$$\begin{aligned} &\forall P_1 \forall P_2 \forall P_3 \forall x \forall y \forall z \big((\mathrm{ISFIL} \wedge P_1 x \wedge P_3 x) \to \mathrm{st}_x(\chi) \big) \\ \wedge\ &\forall P_1 \forall P_2 \forall P_3 \forall x \forall y \forall z \big((\mathrm{ISFIL} \wedge P_2 x \wedge P_3 x) \to \mathrm{st}_x(\chi) \big) \\ \wedge\ &\forall P_1 \forall P_2 \forall P_3 \forall x \forall y \forall z \big((\mathrm{ISFIL} \wedge P_1 y \wedge P_3 y \wedge P_2 z \wedge P_3 x \wedge \mathsf{abovemeet}(x; y, z)) \to \mathrm{st}_x(\chi) \big) \end{aligned} \tag{12}$$

We now compute the $\sigma(P_i)$ of the individual disjoints, and instantiate these in the standard translation of χ, which is

$$\mathrm{st}_x(\chi) = (P_1 x \wedge P_3 x) \vee (P_2 x \wedge P_3 x) \vee \exists s \exists t \big(\mathsf{abovemeet}(x; s, t) \wedge P_1 s \wedge P_3 s \wedge P_2 t \wedge P_3 t \big).$$

We begin with the first conjoint. Here $\sigma(P_1) = \sigma(P_3) = \lambda u.x \leq u$ and $\sigma(P_2) = \lambda u.u \neq u$. Substituting this in $\mathrm{st}_x(\chi)$ automatically makes it true, because the first conjoint of $[\sigma(P_i)/P_i]\, \mathrm{st}_x(\chi)$ then reads $(x \leq x) \wedge (x \leq x)$. Since this is automatically true we can ignore it in our first-order correspondent. Similar reasoning allows us to disregard the second conjoint of (12).

In the third conjoint we obtain $\sigma(P_1) = \lambda u.y \leq u$, $\sigma(P_2) = \lambda u.z \leq u$ and $\sigma(P_3) = \lambda u.\mathsf{abovemeet}(u; x, y)$. Substituting these in the third conjoint gives

$$\begin{aligned} &\forall x \forall y \forall z \big[\\ &\big(\mathrm{ISFIL} \wedge (y \leq y) \wedge \mathsf{abovemeet}(y; x, y) \wedge (z \leq z) \wedge \mathsf{abovemeet}(x; x, y) \wedge \mathsf{abovemeet}(x; y, z) \big) \\ &\ \to \big(((y \leq x) \wedge \mathsf{abovemeet}(x; x, y)) \vee ((z \leq x) \wedge \mathsf{abovemeet}(x; x, y)) \\ &\qquad \vee \exists s \exists t (\mathsf{abovemeet}(x; s, t) \wedge (y \leq s) \wedge \mathsf{abovemeet}(s; x, y) \wedge (z \leq t) \wedge \mathsf{abovemeet}(t; x, y)) \big) \big] \end{aligned}$$

Leaving out everything that is trivially true, this reduces to

$$\forall x \forall y \forall z \big(y \wedge z \leq x \to [(y \leq x) \vee (z \leq x) \vee \exists s \exists t ((s \wedge t \leq x) \wedge (y \leq s) \wedge (z \leq t) \wedge (x \wedge y \leq t))] \big)$$

In words, an L-frame $(X, \leq)$ validates modularity if for all $x, y, z \in X$ such that $y \wedge z \leq x$ either

- $y \leq x$; or
- $z \leq x$; or
- there exist $s, t \in X$ above y and z, respectively, such that $s \wedge t \leq x$ and $x \wedge y \leq t$. $\lhd$

3.5 Sahlqvist Canonicity

In this section we use the Sahlqvist correspondence result from the previous section to prove automated completeness results. Our approach resembles the one taken by Sambin and Vaccaro [36], and again demonstrates how classical techniques can be used in our non-distributive setting. We use the well-known notion of d-persistent to obtain completeness proofs of extensions of $\mathcal{L}$ in a similar way as in Theorem 3.22. Recall that we denote the L-frame underlying an L-space $\mathbb{X}$ by $\kappa\mathbb{X}$.

3.35 Definition. A consequence pair $\varphi \trianglelefteqslant \psi$ is called *d-persistent* if

$$\mathbb{X} \Vdash \varphi \trianglelefteqslant \psi \quad \text{implies} \quad \kappa\mathbb{X} \Vdash \varphi \trianglelefteqslant \psi$$

for all descriptive frames $\mathbb{X}$.

Clearly, if all consequence pairs in a set Γ are d-persistent, then the same proof as for Theorem 3.22 yields completeness of $\mathcal{L}(\Gamma)$ with respect to all (principal) L-frames validating all consequence pairs in Γ.

The main result of this section states that all consequence pairs of **L**-formulae are d-persistent. Before proving this, we collect some preliminary lemmas. These are essentially reformulations of Lemmas 4.2.7 and 4.2.8 from [13]. We fix an arbitrary L-space $\mathbb{X}$. If U is a valuation for $\mathbb{X}$ then we denote by $U(\varphi)$ the truth set of φ in $(\mathbb{X}, U)$. Furthermore, if U and V are valuations then $U \cap V$ is the valuation defined by $(U \cap V)(p) = U(p) \cap V(p)$. Finally, we say that V is a closed valuation for $\kappa\mathbb{X}$ if for each proposition letter p, the set $V(p)$ is a closed filter of $\mathbb{X}$.

3.36 Lemma. *Let φ be any formula, and U, V any valuations for an L-frame $\mathfrak{X} = (X, \leq)$. Then*

$$(U \cap V)(\varphi) \subseteq U(\varphi) \cap V(\varphi).$$

Proof. We use induction on the structure of φ. If $\varphi = \top$ or $\varphi = \bot$ then the statement is trivial. If $\varphi = p$ then it holds by definition. If $\varphi = \varphi_1 \wedge \varphi_2$ then we have

$$\begin{aligned}(U \cap V)(\varphi_1 \wedge \varphi_2) &= (U \cap V)(\varphi_1) \cap (U \cap V)(\varphi_2) \\ &\subseteq U(\varphi_1) \cap V(\varphi_1) \cap U(\varphi_2) \cap V(\varphi_2) \\ &\subseteq U(\varphi_1 \wedge \varphi_2) \cap V(\varphi_1 \wedge \varphi_2).\end{aligned}$$

Finally, if $\varphi = \varphi_1 \vee \varphi_2$ then

$$\begin{aligned}(U \cap V)(\varphi_1 \vee \varphi_2) &= (U \cap V)(\varphi_1) \curlyvee (U \cap V)(\varphi_2) \\ &\subseteq (U(\varphi_1) \cap V(\varphi_1)) \curlyvee (U(\varphi_2) \cap V(\varphi_2)) \\ &\subseteq (U(\varphi_1) \curlyvee U(\varphi_2)) \cap (V(\varphi_1) \curlyvee V(\varphi_2)) \\ &= U(\varphi_1 \vee \varphi_2) \cap V(\varphi_1 \vee \varphi_2).\end{aligned}$$

The first inclusion follows from the induction hypothesis. The second inclusion follows from the fact that the two filters on the third line both contain the sets generating the filter on the second line. $\square$

The following lemma is a version of the intersection lemma [36].

3.37 Lemma. *Let $\mathbb{X} = (X, \leq, \tau)$ be an L-space. Let V be any closed valuation for $\mathfrak{X}$ and write $V \lessdot U$ if U is a clopen valuation extending V. Then for all $\varphi \in \mathbf{L}$ we have*

$$V(\varphi) = \bigcap_{V \lessdot U} U(\varphi).$$

Proof. We prove this by induction on the structure of φ. If $\varphi = \top$ or $\varphi = \bot$ then the result is obvious. If $\varphi = p$ then the result follows from Lemma 2.9.

Suppose $\varphi = \varphi_1 \wedge \varphi_2$. Then we have

$$\begin{aligned}V(\varphi_1 \wedge \varphi_2) = V(\varphi_1) \cap V(\varphi_2) &= \bigcap_{V \lessdot U} U(\varphi_1) \cap \bigcap_{V \lessdot U} U(\varphi_2) \\ &= \bigcap_{V \lessdot U} (U(\varphi_1) \cap U(\varphi_2)) = \bigcap_{V \lessdot U} U(\varphi_1 \wedge \varphi_2)\end{aligned}$$

Lastly, suppose $\varphi = \varphi_1 \vee \varphi_2$. For this induction step we need to work a bit harder. View $\mathbb{X}$ as the dual of the lattice A. Then every clopen filter of $\mathbb{X}$ is of the form $\theta_A(a) = \{p \in \mathcal{F}_{top}A \mid a \in p\}$ for some $a \in A$, because the lattice of clopen filters of $\mathbb{X}$ is isomorphic to A via θ_A. To avoid unnecessary notational clutter, we suppress the subscript A from θ_A.

The inclusion $V(\varphi_1 \vee \varphi_2) \subseteq \bigcap_{V \lessdot U} U(\varphi_1 \vee \varphi_2)$ is obvious, so we focus on the other inclusion. For each clopen valuation U, $U(\varphi_1)$ and $U(\varphi_2)$ are clopen filters, and hence they are of the form $\theta(a_U)$ and $\theta(b_U)$ for some $a_U, b_U \in A$. Moreover, $U(\varphi_1 \vee \varphi_2) = \theta(a_U) \curlyvee \theta(b_U) = \theta(a_U \vee b_U)$. Suppose p is an element of $\mathbb{X}$, viewed as a filter of A, and $p \in U(\varphi_1 \vee \varphi_2) = \theta(a_U \vee b_U)$ for all $V \lessdot U$, so $a_U \vee b_U \in p$ for all clopen valuations U extending V. We wish to show that $p \in V(\varphi_1 \vee \varphi_2)$. So we need to find filters $q \in V(\varphi_1)$ and $r \in V(\varphi_2)$ such that $q \cap r \subseteq p$. Let q be the filter generated by the set $\{a_U \in A \mid V \lessdot U\}$ and r the filter generated by $\{b_U \in A \mid V \lessdot U\}$.

These are nonempty since there is a clopen valuation $U > V$ with $a_U = b_U = \top$. Suppose q is not a proper filter. Then there are $U_0, \dots, U_n$ such that $a_{U_0} \wedge \dots \wedge a_{U_n} = \bot$. Hence we get $V(\varphi_1) \subseteq \bigcap_{i=0}^{n} U_i(\varphi_1) = \theta(a_{U_0} \wedge \dots \wedge a_{U_n}) = \emptyset$. Then $V(\varphi_1 \vee \varphi_2) = V(\varphi_2)$ and we can just apply the induction hypothesis. Same for r.

Then $q \in \bigcap_{V \lessdot U} U(\varphi_1) = V(\varphi_1)$ and similarly $r \in V(\varphi_2)$. Suppose $c \in A$ is such that $c \in q \cap r$. Then there are $U_1, \dots, U_n$ such that $a_{U_1} \wedge \cdots \wedge a_{U_n} \leq c$ and $U_{n+1}, \dots, U_m$ such that $b_{U_{n+1}} \wedge \cdots \wedge b_{U_m} \leq c$. Then $(a_{U_1} \wedge \cdots \wedge a_{U_m}) \vee (b_{U_1} \wedge \cdots \wedge b_{U_m}) \leq c$. Clearly $U_1 \cap \cdots \cap U_n \cap U_{n+1} \cap \cdots \cap U_m$ is also a clopen filter extending V, and hence

$$
\begin{aligned}
p \in\ & (U_1 \cap \cdots \cap U_m)(\varphi_1 \vee \varphi_2) \\
&= (U_1 \cap \cdots \cap U_m)(\varphi_1) \curlyvee (U_1 \cap \cdots \cap U_m)(\varphi_2) \\
&\subseteq (U_1(\varphi_1) \cap \cdots \cap U_m(\varphi_1)) \curlyvee (U_1(\varphi_2) \cap \cdots \cap U_m(\varphi_2)) && \text{(Lemma 3.36)} \\
&= (\theta(a_{U_1}) \cap \cdots \cap \theta(a_{U_m})) \curlyvee (\theta(b_{U_1}) \cap \cdots \cap \theta(b_{U_m})) && (U_i(\varphi_1) = \theta(a_{U_i}),\ U_i(\varphi_2) = \theta(b_{U_i})) \\
&= \theta(a_{U_1} \wedge \cdots \wedge a_{U_m}) \curlyvee \theta(b_{U_1} \wedge \cdots \wedge b_{U_m}) \\
&= \theta((a_{U_1} \wedge \cdots \wedge a_{U_m}) \vee (b_{U_1} \wedge \cdots \wedge b_{U_m}))
\end{aligned}
$$

Therefore $(a_{U_1} \wedge \cdots \wedge a_{U_m}) \vee (b_{U_1} \wedge \cdots \wedge b_{U_m}) \in p$, and hence $c \in p$. This proves $q \cap r \subseteq p$, and therefore $p \in V(\varphi_1 \vee \varphi_2)$. □

We now have all the tools to prove a Sahlqvist canonicity theorem.

3.38 Theorem. *Any consequence pair $\psi \trianglelefteq \chi$ of* **L***-formulae is d-persistent.*

Proof. If the antecedent is equivalent to $\bot$ then the statement is vacuously true. If the antecedent is equivalent to $\top$ then (in terms of validity on frames) the consequence pair $\top \trianglelefteq \chi$ is equivalent to a consequence pair without proposition letters, so that the statement is true as well. In all other cases, the antecedent is equivalent to a formula that does not contain $\top$ or $\bot$, so we assume that ψ does not contain $\top$ or $\bot$.

Let $\mathbb{X}$ be an L-space that validates $\psi \trianglelefteq \chi$. We wish to show that $\kappa\mathbb{X} \Vdash \psi \trianglelefteq \chi$. As a consequence of Theorem 3.32, it suffices to show that $(\kappa\mathbb{X})^\circ \models \mathrm{so}(\varphi \trianglelefteq \psi)$. As in Step 1 in the proof of Theorem 3.32, we can rewrite $\mathrm{so}(\varphi \trianglelefteq \psi)$ to a conjunction of formulae of the form

$$\forall P_1 \cdots \forall P_n \forall x \forall y_1 \cdots \forall y_m (\mathrm{ISFIL} \wedge \mathrm{AT} \wedge \mathrm{REL} \to \mathrm{st}_x(\chi)),$$

where $P_1, \dots, P_n$ are the unary predicates corresponding to proposition letters in $\psi \trianglelefteq \chi$ and $y_1, \dots, y_m$ are the variables arising from the second-order translation. So it suffices to show that these are all validated in $(\kappa\mathbb{X})^\circ$. Observe that valuations of $\kappa\mathbb{X}$ correspond bijectively with interpretations of the unary predicates that make ISFIL true. So it suffices to show that for all (not necessarily clopen) valuations V for $\kappa\mathbb{X}$ and all $x', y'_1, \dots, y'_m$ we have

$$(\kappa\mathbb{X}, V)^\circ \models \mathrm{ISFIL} \wedge \mathrm{AT} \wedge \mathrm{REL}[x', y'_1, \dots, y'_m] \quad \text{implies} \quad (\kappa\mathbb{X}, V)^\circ \models \mathrm{st}_x(\chi)[x', y'_1, \dots, y'_n].$$

Here, by $(\kappa\mathbb{X}, V)^\circ$ we mean the first-order structure $(\kappa\mathbb{X})^\circ$ that interprets the unary predicates via the valuation V and variables via the assignment given in square brackets.

So suppose we have an instantiation $x', y'_1, \dots, y'_m$ of the variables. For each proposition letter p_i, let $V_m(p_i)$ be the filter generated by the set $\{y \mid P_i y \text{ occurs in AT }\}$. Then V_m defines a principal, hence closed valuation of the proposition letters in ψ and χ. We claim that

$$\begin{aligned} &(\kappa\mathbb{X}, V_m)^\circ \models \mathrm{ISFIL} \wedge \mathrm{AT} \wedge \mathrm{REL} \to \mathrm{st}_x(\chi)[x', y'_1, \dots, y'_m] \quad \text{iff} \\ &\quad (\kappa\mathbb{X}, V)^\circ \models \mathrm{ISFIL} \wedge \mathrm{AT} \wedge \mathrm{REL} \to \mathrm{st}_x(\chi)[x', y'_1, \dots, y'_m] \quad \text{for all valuations } V. \end{aligned} \tag{13}$$

The direction from right to left is trivial. For the converse, suppose the LHS holds and suppose $(\kappa\mathbb{X}, V)^\circ \models \mathrm{ISFIL} \wedge \mathrm{AT} \wedge \mathrm{REL}[x', y'_1, \dots, y'_m]$. Since $(\kappa\mathbb{X}, V) \models \mathrm{AT}[x', y'_1, \dots, y'_m]$ it follows that V is an extension of V_m. Since validity of REL is independent of the valuation, and the valuation V_m assigns a filter to each proposition letter, we have $(\kappa\mathbb{X}, V_m)^\circ \models \mathrm{ISFIL} \wedge \mathrm{AT} \wedge \mathrm{REL}[x', y'_1, \dots, y'_m]$. By assumption this implies $(\kappa\mathbb{X}, V_m)^\circ \models \mathrm{st}_x(\chi)[x', y'_1, \dots, y'_m]$, and since V is an extension of V_m we find $(\kappa\mathbb{X}, V)^\circ \models \mathrm{st}_x(\chi)[x', y'_1, \dots, y'_m]$. So the left to right condition of (13) also holds.

So it suffices to prove $(\kappa\mathbb{X}, V_m)^\circ \models \mathrm{ISFIL} \wedge \mathrm{AT} \wedge \mathrm{REL} \to \mathrm{st}_x(\chi)[x', y'_1, \dots, y'_m]$. By design of V_m we have $(\kappa\mathbb{X}, V_m)^\circ \models \mathrm{ISFIL} \wedge \mathrm{AT} \wedge \mathrm{REL}[x', y'_1, \dots, y'_m]$, so our goal further reduces to proving

$$(\kappa\mathbb{X}, V_m)^\circ \models \mathrm{st}_x(\chi)[x', y'_1, \dots, y'_m]. \tag{14}$$

Observe that the variables $y_1, \dots, y_m$ do not occur in $\mathrm{st}_x(\chi)$ (we have not substituted anything yet), so (14) holds if and only if $(\kappa\mathbb{X}, V_m), x' \Vdash \chi$. That is, if and only if $x' \in V_m(\chi)$. As a consequence of Lemma 3.37 it suffices to show that $x' \in U(\chi)$ for all clopen valuations of $\mathbb{X}$ extending V_m. But this follows from our assumptions: Since U is a clopen valuation extending V_m we have $(\kappa\mathbb{X}, U)^\circ \models \mathrm{ISFIL} \wedge \mathrm{AT} \wedge \mathrm{REL}[x', y'_1, \dots, y'_m]$. This implies $(\kappa\mathbb{X}, U)^\circ \models \mathrm{ISFIL} \wedge \bar{\psi}[x', y'_1, \dots, y'_m]$ (recall that $\bar{\psi}$ is the reformulation of $\mathrm{st}_x(\psi)$ with all quantifiers pulled out), and the instantiations $x', y'_1, \dots, y'_m$ now witness the fact that $(\kappa\mathbb{X}, U)^\circ \models \mathrm{st}_x(\psi)$. This implies $(\kappa\mathbb{X}, U) \Vdash \psi$. So by the assumption that $\mathbb{X}$ validates $\psi \trianglelefteq \chi$ we get $(\kappa\mathbb{X}, U) \Vdash \chi$, as desired. □

3.39 Theorem. *Let Γ be a set of consequence pairs. Then the logic $\mathcal{L}(\Gamma)$ is sound and complete with respect to the following classes of frames:*

- $\mathsf{D\text{-}LFrm}(\Gamma)$ *(descriptive frames validating Γ);*
- $\mathsf{PLFrm}(\Gamma)$ *(principal L-frames validating Γ);*
- $\mathsf{LFrm}(\Gamma)$ *(L-frames validating Γ).*

Proof. Straightforward adaptation of the proof of Theorem 3.22 using d-persistence. □

In particular, this shows that we get a sound and complete semantics for distributive and modular lattices. We also obtain the following generalisation of Baker and Hales' result [2], which states that every variety of lattices is closed under ideal completions. In order not to overload the paper we do not define ideal completions here and refer the reader to [13, Section 4.3] for all the details including on how the next theorem implies the Baker and Hales's result.

3.40 Theorem. *Every variety of lattices is closed under taking filter completions and F^2-completions.*

Proof. Suppose a lattice A satisfies an equation $\varphi = \psi$. Then it validates $\varphi \trianglelefteq \psi$ and $\psi \trianglelefteq \varphi$. Writing $\mathbb{X}$ for the L-space dual to A, Lemma 3.21 implies that $\mathbb{X} \Vdash \varphi \trianglelefteq \psi$ and $\psi \trianglelefteq \varphi$. Since every consequence pair is d-persistent the lattice of (principal) filters of $\mathbb{X}$ validate $\varphi \trianglelefteq \psi$ and $\psi \trianglelefteq \varphi$. But this precisely means that the filter extension of A and the double filter extension of A validate $\varphi = \psi$. □

3.41 Remark. The d-persistence that we discuss here allows us to move from valuations that interpret proposition letters as clopen filters to valuation that assign to each proposition letter an arbitrary filter. It is analogous to d-persistence in intuitionistic and modal logics [6, 8]. In the classical setting d-persistence allows one to move from clopen valuations to arbitrary valuations, and in the intuitionistic case from valuations into clopen upsets to valuations into all upsets. We point out that, while in the distributive setting this corresponds algebraically to canonical extensions, in our setting the corresponding algebraic structure is the F^2-completion. ◁

4 Normal Modal Extensions

We investigate the extension of non-distributive positive logic with two modal operators, $\Box$ and $\Diamond$, interpreted via a relation in the usual way (see e.g. [6, Definition 1.20]). As our point of departure we take L-frames with an additional relation. We stipulate sufficient conditions on the relation to ensure persistence, but we do not enforce any axioms on the modalities. It will turn out that $\Box$ preserves finite conjunctions as usual. However, as a consequence of the non-standard interpretation of disjunctions, $\Diamond$ is non-normal. This is reminiscent of the modal extension of intuitionistic logic investigated by Kojima [31], where also $\Box$ is normal and $\Diamond$ is not. The interaction axioms relating $\Box$ and $\Diamond$ are closely aligned to Dunn's axioms for (distributive) positive modal logic [14], see Remark 4.7.

After investigating the modal logic from a semantic point of view, we use our newly developed intuition to give syntactic definition of the logic in Section 4.2. This is sound by design, and the algebraic semantics is given by lattices with operators. In Section 4.3 we provide a duality for the algebraic semantics by means of modal L-spaces. We show that each modal L-space has an underlying modal L-frame, and as a consequence we obtain completeness for the basic logic.

Subsequently, in Section 4.4 we prove Sahlqvist correspondence. The methods used are analogous to those in Sections 3.4, but it is no longer the case that any consequence pair is Sahlqvist. We identify as Sahlqvist consequence pairs precisely the negation-free Sahlqvist formulae from normal modal logic [6, Definition 3.51], where the implication is replaced by $\trianglelefteq$. We then use this to prove Sahlqvist canonicity in Section 4.5, in the same way as in 3.5. As a consequence of the more restricted notion of a Sahlqvist consequence pair, we now distinguish two types of persistence: p-persistence and d-persistence. The former is persistence to the underlying principal frame, where proposition letters are interpreted as principal filters. For d-persistence we do not impose any extra conditions on the valuation. It will turn out that all consequence pairs are p-persistent, and all Sahlqvist consequence pairs are d-persistent.

One may wonder whether it would be more natural to insist that $\Diamond$ be normal as well. We do not because the additional conditions required to ensure that $\Diamond$ is normal complicate the presentation of the semantics and duality. Moreover, in order to make $\Diamond$ normal we only need to extend our basic system with the consequence pair

$$\Diamond(p \vee q) \trianglelefteq \Diamond p \vee \Diamond q,$$

which is Sahlqvist! So we obtain a local first-order correspondent and automated completeness via our Sahlqvist theorems from Sections 4.4 and 4.5. We explicitly compute the local correspondent in Example 4.37.

4.1 Relational Meet-Frames

Let $\mathbf{L}_{\Box\Diamond}$ be the language generated by the grammar

$$\varphi ::= p \mid \top \mid \bot \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \Box\varphi \mid \Diamond\varphi,$$

where p ranges over some set Prop of proposition letters. A *modal consequence pair* is an expression of the form $\varphi \trianglelefteq \psi$, where $\varphi, \psi \in \mathbf{L}_{\Box\Diamond}$. We derive an appropriate notion of modal L-frame, such that the truth set of each formula is guaranteed to be a filter.

4.1 Definition. A *modal L-frame* is a tuple $(X, \leq, R)$ where $(X, \leq)$ is an L-frame and R is a binary relation on X such that:

1. If $x \leq y$ and yRz then there exists a $w \in X$ such that xRw and $w \leq y$;
2. If $x \leq y$ and xRw then there exists a $z \in X$ such that yRz and $w \leq z$;
3. If $(x \wedge y)Rz$ then there exist $v, w \in X$ such that xRv and yRw and $v \wedge w \leq z$;
4. If xRv and yRw then $(x \wedge y)R(v \wedge w)$;
5. For all $x \in X$ there exists a $y \in X$ such that xRy.

A *modal L-model* is a a modal L-frame together with a valuation V that assigns to each proposition letter a filter of $(X, \leq)$.

The interpretation of $\mathbf{L}_{\Box\Diamond}$-formulae in a modal L-model $\mathfrak{M}$ is defined via the clauses from Definition 4.1 and

$$\begin{aligned} \mathfrak{M}, x \Vdash \Box\varphi \quad &\text{iff} \quad \forall y \in X, xRy \text{ implies } \mathfrak{M}, y \Vdash \varphi \\ \mathfrak{M}, x \Vdash \Diamond\varphi \quad &\text{iff} \quad \exists y \in X \text{ such that } xRy \text{ and } \mathfrak{M}, y \Vdash \varphi \end{aligned}$$

Satisfaction and validity of formulae and modal consequence pairs are defined as expected. In particular, if $\mathcal{K}$ is a class of modal L-frames and $\varphi \trianglelefteq \psi$ is a modal consequence pair, then we write $\varphi \Vdash_{\mathcal{K}} \psi$ if the consequence pair $\varphi \trianglelefteq \psi$ is valid on all frames in $\mathcal{K}$.

The four conditions of a modal L-frame are depicted in Figure 3. If $(X, \leq, R)$ is a modal L-frame, then we denote the set of R-successors of a state $x \in X$ by $R[x] := \{y \in X \mid xRy\}$. Also, for a subset $a \subseteq X$ we define

$$[R]a = \{x \in X \mid R[x] \subseteq a\}, \qquad \langle R\rangle a = \{x \in X \mid R[x] \cap a \neq \emptyset\}.$$

Then the first two conditions of Definition 4.1 say that $x \leq y$ implies that $R[x] \sqsubseteq R[y]$, where $\sqsubseteq$ denotes the Egli-Milner order on $\mathscr{P}X$ [1, Definition 6.2.2]. Furthermore, if $\mathfrak{M} = (X, \leq, R, V)$ is a modal L-model then we have

$$[\![\Box\varphi]\!]^{\mathfrak{M}} = [R][\![\varphi]\!]^{\mathfrak{M}}, \qquad [\![\Diamond\varphi]\!]^{\mathfrak{M}} = \langle R\rangle[\![\varphi]\!]^{\mathfrak{M}}.$$

Next we prove persistence.

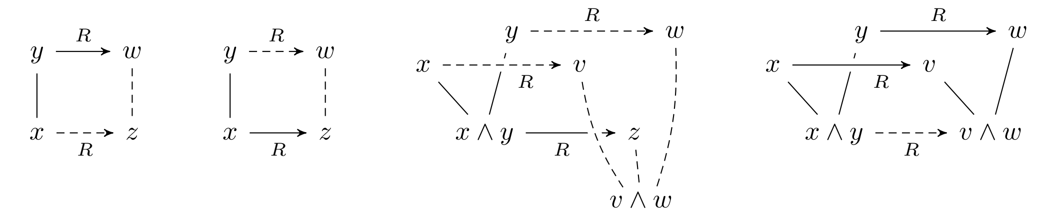


Figure 3: The four conditions of a modal L-frame. Lines denote the poset order, with high nodes being bigger. Arrows denote the relation R.

4.2 Proposition. *Let $\mathfrak{M} = (X, \leq, R, V)$ be a modal L-model. Then for each $\varphi \in \mathbf{L}_{\Box\Diamond}$ the set $[\![\varphi]\!]^{\mathfrak{M}}$ is a filter in $(X, \leq)$.*

Proof. The proof proceeds by induction on the structure of φ. The only non-trivial cases are the modal cases. We prove the statement for $\varphi = \Diamond\psi$, the case $\varphi = \Box\psi$ is similar.

So suppose $\varphi = \Diamond\psi$, $\mathfrak{M}, x \Vdash \Diamond\psi$ and $x \leq y$. Then there exists an R-successor z of x satisfying ψ, and by (2) we can find an R-successor w of y such that $z \leq y$. By the induction hypothesis we then find $\mathfrak{M}, w \Vdash \psi$ and therefore $\mathfrak{M}, y \Vdash \Diamond\psi$.

Next, suppose that both x and y satisfy $\Diamond\psi$. Then there exist $v \in R[x]$ and $w \in R[y]$ both satisfying ψ. By (4) we then have $(x \wedge y)R(v \wedge w)$ and as a consequence of the induction hypothesis $\mathfrak{M}, v \wedge w \Vdash \psi$. Therefore $\mathfrak{M}, x \wedge y \Vdash \Diamond\psi$. We conclude that $[\![\Diamond\psi]\!]^{\mathfrak{M}}$ is a filter in $(X, \leq)$. $\Box$

4.3 Remark. We could have slightly weakened condition 4 by requiring the existence of some $(x \wedge y)$-successor above $v \wedge w$. We use the current formulation because it aligns more closely to the notion of a modal L-space. $\lhd$

Morphisms between modal L-frames and -models are a combination of L-morphisms and bounded morphisms.

4.4 Definition. A *bounded L-morphism* from $(X, \leq, R)$ to $(X', \leq', R')$ is a function $f : X \to X'$ such that $f : (X, \leq) \to (X', \leq')$ is an L-morphism and for all $x, y \in X$ and $z' \in X'$:

1. If xRy then $f(x)R'f(y)$;
2. If $f(x)R'z'$ then there exists a $z \in X$ such that xRz and $f(z) \leq z'$;
3. If $f(x)R'z'$ then there exists a $w \in X$ such that xRw and $z' \leq' f(w)$.

A bounded L-morphism between models is bounded L-morphism between the underlying frames that preserves and reflects truth of proposition letters.

The bounded L-morphism conditions are depicted in Figure 4.

4.5 Proposition. *Let $\mathfrak{M} = (X, \leq, R, V)$ and $\mathfrak{M}' = (X', \leq', R', V')$ be two modal L-models. If $f : \mathfrak{M} \to \mathfrak{M}'$ is a bounded L-morphism, $x \in X$ and $\varphi \in \mathbf{L}_{\Box\Diamond}$, then*

$$\mathfrak{M}, x \Vdash \varphi \quad \textit{iff} \quad \mathfrak{M}', f(x) \Vdash \varphi.$$

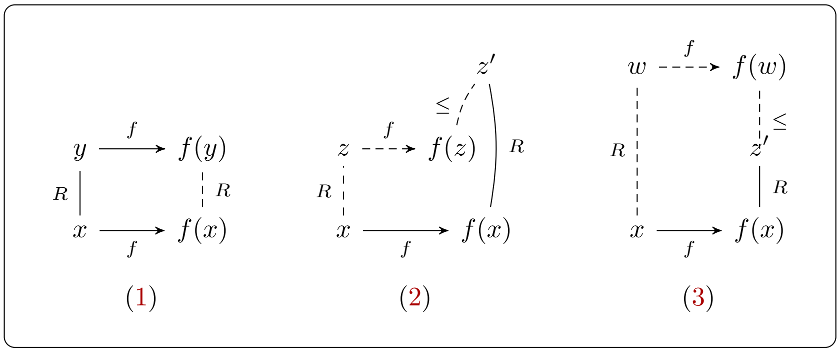


Figure 4: The conditions of a bounded L-morphism.

Proof. This follows from a routine induction on the structure of φ. We showcase the modal cases of the proof. Suppose $\varphi = \Box\psi$. It follows immediately from Definition 4.4(1) that $\mathfrak{M}', f(x) \Vdash \Box\psi$ implies $\mathfrak{M}, x \Vdash \Box\psi$. So suppose $\mathfrak{M}, x \Vdash \Box\psi$. If y' is an R'-successor of $f(x)$, then there exists some $z \in X$ such that xRz and $f(z) \leq y'$. This implies $\mathfrak{M}, z \Vdash \psi$ and by induction $\mathfrak{M}', f(z) \Vdash \psi$. Persistence then yields $\mathfrak{M}', y' \Vdash \psi$. Therefore $\mathfrak{M}', f(x) \Vdash \Box\psi$.

Next, let $\varphi = \Diamond\psi$. Then the preservation from left to right follows from Definition 4.4(1). Conversely, if $\mathfrak{M}', f(x) \Vdash \Diamond\psi$, then there exists a $y' \in X'$ such that $f(x)Ry'$ and $\mathfrak{M}', y' \Vdash \psi$. By (3) we can find some $w \in X$ such that xRw and $y' \leq f(w)$. Persistence implies $\mathfrak{M}', f(w) \Vdash \psi$ and induction yields $\mathfrak{M}, w \Vdash \psi$. Therefore $\mathfrak{M}, x \Vdash \Diamond\psi$. $\square$

We give a number of modal consequence pairs that are valid in every modal L-frame. These motivate the definition of a modal lattice in Section 4.2.

4.6 Lemma. *Let $(X, \leq, R)$ be a modal L-frame. Then the following consequence pairs are all valid:*

$\top \trianglelefteq \Box\top$	$\top \trianglelefteq \Diamond\top$	$\Diamond\bot \trianglelefteq \bot$	*(top and bottom)*
$\Box(\varphi \wedge \psi) \trianglelefteq \Box\varphi \wedge \Box\psi$	$\Diamond\varphi \trianglelefteq \Diamond(\varphi \vee \psi)$		*(monotonicity)*
$\Box\varphi \wedge \Box\psi \trianglelefteq \Box(\varphi \wedge \psi)$			*(normality)*
$\Diamond\varphi \wedge \Box\psi \trianglelefteq \Diamond(\varphi \wedge \psi)$			*(duality)*

Proof. All of these follow immediately from the definition of the interpretation of $\Box$ and $\Diamond$. In particular, they do not rely on any of the conditions from Definition 4.1. $\square$

Observe that the consequence pair $\top \trianglelefteq \Diamond\top$ corresponds to seriality, i.e. the frame condition that every state has an R-successor. In presence of $\top \trianglelefteq \Box\top$ and the duality axiom it is equivalent to $\Box\varphi \trianglelefteq \Diamond\varphi$.

4.7 Remark. The duality axiom in Lemma 4.6 corresponds to one of Dunn's duality axioms for positive modal logic [14]. It seems that the non-standard interpretation of joins makes Dunn's other duality axiom, $\Box(\varphi \vee \psi) \trianglelefteq \Box\varphi \vee \Diamond\psi$, unsuitable in our context. On the other hand, we had to stipulate seriality in order to prove Lemma 4.22, and this is not assumed by Dunn. We flag the investigation of the connection between the various axioms relating $\Box$ and $\Diamond$ as an interesting direction for further research. $\lhd$

Just like in the propositional case in Section 3.2, we can identify a class of frames where formulae can be interpreted exclusively as *principal* filters.

4.8 Definition. A *principal modal L-frame* is a tuple $(X, \leq, R)$ such that

0. $(X, \leq)$ has binary joins and all non-empty meets;
1. If $x \leq y$ and yRz then there exists a $w \in X$ such that xRw and $w \leq z$;
2. If $x \leq y$ and xRw then there exists a $z \in X$ such that yRz and $w \leq z$;
3. If $(\bigwedge x_i)Rz$, where the i range over some index set I, then there exist z_i such that x_iRz_i for all $i \in I$ and $\bigwedge z_i \leq z$;
4. If x_iRy_i, where i ranges over some non-empty index set I, then $(\bigwedge x_i)R(\bigwedge y_i)$;
5. For all $x \in X$ there exists a $y \in X$ such that xRy.

(Observe that clearly a principal modal L-frame is also a modal L-frame.) A *principal modal L-model* is a principal modal L-frame together with a principal valuation, and $\mathbf{L}_{\Box\Diamond}$-formulae are interpreted in the same way as in modal L-models. As for modal L-frames, if $\mathfrak{K}$ is a class of principal modal L-frames and $\varphi \trianglelefteq \psi$ is a modal consequence pair, then we write $\varphi \Vdash_{\mathfrak{K}} \psi$ if the consequence pair $\varphi \trianglelefteq \psi$ is valid on all frames in $\mathfrak{K}$.

Observe that these conditions subsume the ones from Definition 4.1, so principal modal L-frames form a subclass of the modal L-frames (and similar for the corresponding models). As desired, the truth set of a $\mathbf{L}_{\Box\Diamond}$-formula in a principal modal L-model is always given by a principal filter.

4.9 Proposition. *Let $\mathfrak{M} = (X, \leq, R, V)$ be a principal modal L-model. Then for all $\varphi \in \mathbf{L}_{\Box\Diamond}$ the truth set $[\![\varphi]\!]^{\mathfrak{M}}$ is a principal filter in $(X, \leq)$.*

Proof. The proof proceeds by induction on the structure of φ. The propositional cases are the same as in Proposition 3.12. We give the induction step for $\varphi = \Diamond\psi$ and leave the case $\varphi = \Box\psi$ to the reader.

So let $\varphi = \Diamond\psi$ and suppose that $[\![\psi]\!]^{\mathfrak{M}}$ is a principal filter (this is the induction hypothesis). If $[\![\Diamond\psi]\!]^{\mathfrak{M}}$ is empty then we are done. If not, then we can prove that it is upward closed in the same way as in Proposition 4.2. Moreover, it follows immeidately from (4) that $\bigwedge[\![\Diamond\psi]\!]^{\mathfrak{M}} \in [\![\Diamond\psi]\!]^{\mathfrak{M}}$, so $[\![\Diamond\psi]\!]^{\mathfrak{M}}$ is indeed a principal filter. □

4.2 Logic and Modal Lattices

Guided by the validities from Lemma 4.6, we define the logic $\mathcal{L}_{\Box\Diamond}$ as follows.

4.10 Definition. Let $\mathcal{L}_{\Box\Diamond}$ be the smallest set of modal consequence pairs closed under the axioms and rules from Definition 3.1, and under: *modal top* and *bottom*

$$\top \trianglelefteq \Box\top, \qquad \top \trianglelefteq \Diamond\top, \qquad \Diamond\bot \trianglelefteq \bot,$$

Becker's rules

$$\frac{\varphi \trianglelefteq \psi}{\Box\varphi \trianglelefteq \Box\psi}, \qquad \frac{\varphi \trianglelefteq \psi}{\Diamond\varphi \trianglelefteq \Diamond\psi},$$

linearity for $\Box$

$$\Box\varphi \wedge \Box\psi \trianglelefteq \Box(\varphi \wedge \psi),$$

and the *duality axiom*

$$\Diamond\varphi \wedge \Box\psi \trianglelefteq \Diamond(\varphi \wedge \psi).$$

If Γ is a set of modal consequence pairs then we let $\boldsymbol{\mathcal{L}}_{\Box\Diamond}(\Gamma)$ denote the smallest set of modal consequence pairs closed under the axioms and rules mentioned above and those in Γ. We write $\varphi \vdash_\Gamma \psi$ if $\varphi \trianglelefteq \psi \in \boldsymbol{\mathcal{L}}_{\Box\Diamond}(\Gamma)$ and $\varphi \dashv\vdash_\Gamma \psi$ if both $\varphi \vdash_\Gamma \psi$ and $\psi \vdash_\Gamma \varphi$. If Γ is the empty set then we simply write $\varphi \vdash \psi$ and $\varphi \dashv\vdash \psi$.

Observe that Becker's rule together with linearity for $\Box$ implies that $\Box$ is a normal modal operator. The logic gives rise to the following algebraic semantics, given by modal lattices.

4.11 Definition. A *modal lattice* is a tuple $(A, \Box, \Diamond)$ consisting of a lattice A and two maps $\Box, \Diamond : A \to A$ satisfying for all $a, b \in A$:

$$\begin{array}{lll} \top = \Box\top & \top = \Diamond\top & \Diamond\bot = \bot \\ \Box(a \wedge b) = \Box a \wedge \Box b & \Diamond a \leq \Diamond(a \vee b) & \\ \Diamond a \wedge \Box b \leq \Diamond(a \wedge b) & & \end{array}$$

A *modal lattice homomorphism* from $(A, \Box, \Diamond)$ to $(A', \Box', \Diamond')$ is a lattice homomorphism $h : A \to A'$ such that $h(\Box a) = \Box' h(a)$ and $h(\Diamond a) = \Diamond' h(a)$ for all $a \in A$. We write MLat for the category of modal lattice and modal lattice homomorphisms.

4.12 Example. Let $\mathfrak{X} = (X, \leq, R)$ be a modal L-frame. Then $\mathfrak{X}^\ddagger = (\mathcal{F}(X, \leq), [R], \langle R \rangle)$ is a modal lattice. If $\mathfrak{X} = (X, \leq, R)$ is a principal modal L-frame then $\mathfrak{X}^\dagger = (\mathcal{F}_p(X, \leq), [R], \langle R \rangle)$ is a modal lattice. $\triangleleft$

Formulae $\varphi \in \mathbf{L}_{\Box\Diamond}$ can be interpreted in a modal lattice $\mathcal{A} = (A, \Box, \Diamond)$ with an assignment $\sigma : \mathrm{Prop} \to A$. Analogous to Section 3.1, the interpretation of proposition letters is given by the assignment, and the connectives and modalities as interpreted via their counterparts in $\mathcal{A}$. This gives rise to validity of formulae and modal consequence pairs in a modal lattice $\mathcal{A}$.

If $\mathfrak{M} = (\mathfrak{X}, V)$ is a modal L-model then V is an assignment for $\mathfrak{X}^\ddagger$ and we write $\mathfrak{M}^\ddagger = (\mathfrak{X}^\ddagger, V)$. If $\mathfrak{M} = (\mathfrak{X}, V)$ is a principal modal L-model then V is an assignment for $\mathfrak{X}^\dagger$ and we let $\mathfrak{M}^\dagger = (\mathfrak{X}^\dagger, V)$. We obtain the following counterpart of Lemma 3.7.

4.13 Lemma. *Let $\mathfrak{M}$ be a modal L-model and $\mathfrak{N}$ a principal modal L-model. Then*

$$[\![\varphi]\!]^{\mathfrak{M}} = (\!|\varphi|\!)_{\mathfrak{M}^\ddagger} \quad \text{and} \quad [\![\varphi]\!]^{\mathfrak{N}} = (\!|\varphi|\!)_{\mathfrak{N}^\dagger}$$

for all $\varphi \in \mathbf{L}_{\Box\Diamond}$.

We write $\varphi \Vdash_\Gamma \psi$ if any modal lattice that validates all consequence pairs in Γ also validates $\varphi \trianglelefteq \psi$. Then we can prove the next theorem in the same way as in Section 3.1.

4.14 Theorem. *Let $\Gamma \cup \{\varphi \trianglelefteq \psi\}$ be a set of modal consequence pairs. Then we have $\varphi \vdash_\Gamma \psi$ if and only if $\varphi \Vdash_\Gamma \psi$.*

4.3 Modal L-spaces and Duality

We prove a duality for modal lattices by means of L-spaces with an additional relation. First we show that such structures have an underlying (principal) modal L-frame, and then we proceed to the duality.

4.15 Definition. A *modal L-space* is a tuple $(X, \leq, \tau, R)$ such that

1. $(X, \leq, \tau)$ is an L-space;

2. R is a binary relation on X such that for every $x \in X$ there exists a $y \in X$ with xRy;
3. If a is a clopen filter, then so are $[R]a$ and $\langle R \rangle a$;
4. For all $x, y \in X$ we have xRy if and only if for all $a \in \mathscr{F}_{clp}\mathbb{X}$:
   - If $x \in [R]a$ then $y \in a$;
   - If $y \in a$ then $x \in \langle R \rangle a$.

Truth and validity in modal L-spaces is defined as usual, using clopen valuations.

The third item is a condition often seen in the definition of general frames. Item (4) is our counterpart of the tightness condition, and has previously been used in [7, Section 2].

Next, we prove that each modal L-space has an underlying (principal) modal L-frame. We need the following lemma for this.

4.16 Lemma. *Let $\mathbb{X} = (X, \leq, \tau, R)$ be a modal L-space. Then $R[x]$ is closed for all $x \in X$.*

Proof. Suppose $y \notin R[x]$. Then there exists a clopen filter a such that either $x \in [R]a$ and $y \notin a$, or $y \in a$ and $x \notin \langle R \rangle a$. In the first case $X \setminus a$ is a clopen neighbourhood of y disjoint from $R[x]$. In the second case a is a clopen neighbourhood of y disjoint from $R[x]$. □

4.17 Proposition. *Let $\mathbb{X} = (X, \leq, \tau, R)$ be a modal L-space. Then $(X, \leq, R)$ is a principal modal L-frame.*

Proof. We know that L-spaces have all non-empty meets, and hence also binary joins, so (0) is satisfied. Furthermore, it is stipulated that every $x \in X$ has an R-successor so (5) is satisfied as well. We verify the other conditions from Definition 4.8. We start with (4), so that we can use it when proving the others.

Condition 4. Suppose $x_i R y_i$, where i ranges over some non-empty index set I. By the tightness condition of modal L-spaces, in order to prove $(\bigwedge x_i)R(\bigwedge y_i)$ it suffices to show that for all clopen filters a, $\bigwedge x_i \in [R]a$ implies $\bigwedge y_i \in a$ and $\bigwedge y_i \in a$ implies $\bigwedge x_i \in \langle R \rangle a$.

First assume $\bigwedge x_i \in [R]a$. Since $[R]a$ is a clopen filter and $\bigwedge x_i \leq x_j$ for each $j \in I$ we have $x_j \in [R]a$, so $R[x_j] \subseteq a$. By assumption $x_j R y_j$ so $y_j \in a$ for all $j \in I$. Since a is a clopen filter it is principal, and therefore $\bigwedge y_i \in a$.

Next, suppose $\bigwedge y_i \in a$. Since a is a filter and $\bigwedge y_i \leq y_j$ we have $y_j \in a$ for all $j \in I$. This implies $x_j \in \langle R \rangle a$ for all $j \in I$, and since $\langle R \rangle a$ is a clopen filter, hence principal, we find $\bigwedge x_i \in \langle R \rangle a$.

Condition 1. Let $x \leq y$ and yRz. Suppose towards a contradiction that there exists no $w \in X$ such that xRw and $w \leq z$. Now let $x' = \bigwedge R[x]$ be the minimal element in $R[x]$ (which is an R-successor of x by (4)). Then $x' \not\leq z$, so we can find a clopen filter a containing x' such that $z \notin a$. This implies $R[x] \subseteq a$, so that $x \in [R]a$, but $y \notin [R]a$ because yRz and $z \notin a$. As $x \leq y$ this violates the fact that $[R]a$ is a filter.

Condition 2. Let $x \leq y$ and xRw. Suppose towards a contradiction that there exists no $z \in X$ such that yRz and $w \leq z$. Then $R[y] \cap {\uparrow}w = \emptyset$. Both $R[y]$ and ${\uparrow}z$ are closed, as a consequence of Lemmas 4.16 and 2.10 and the fact that singletons in a Stone spaces are always closed. Therefore we can find a clopen filter a containing ${\uparrow}z$ which is disjoint from $R[y]$. This implies that $x \in \langle R \rangle a$ while $y \notin \langle R \rangle a$. Since $x \leq y$ this contradicts the fact that $\langle R \rangle a$ is a filter.

Condition 3. Finally, let $\{x_i \mid i \in I\}$ be some non-empty collection of elements of X and suppose $(\bigwedge x_i)Rz$. Since $\bigwedge x_i \leq x_j$ for all $j \in I$, condition (2) implies that $R[x_j] \neq \emptyset$ for all

$j \in I$. As a consequence of (4) there is a smallest element $z_j := \bigwedge R[x_j]$ in each $R[x_j]$. We claim that $\bigwedge z_j \leq z$. Suppose not, then there is a clopen filter a containing $\bigwedge z_j$ such that $z \notin a$. This implies $x_j \in [R]a$ for all $j \in I$, but $\bigwedge x_i \notin [R]a$ because $(\bigwedge x_i)Rz$ and $z \notin a$. But this contradicts the fact that $[R]a$ is principal filter. $\square$

Since we know that each modal L-space has an underlying modal L-frame, we can now conveniently define the morphisms between them as follows.

4.18 Definition. A *modal L-space morphism* from $(X, \leq, \tau, R)$ to $(X', \leq', \tau', R')$ is a function $f : X \to X'$ such that $f : (X, \leq, \tau) \to (X', \leq', \tau')$ is an L-space morphism and $f : (X, \leq, R) \to (X', \leq', R')$ is a modal L-morphism. We denote the resulting category by $\mathsf{MLSpace}$.

We work our way towards a duality between modal lattices and modal L-spaces.

4.19 Proposition. *If $\mathbb{X} = (X, \leq, \tau, R)$ is a modal L-space then $(\mathcal{F}_{clp}\mathbb{X}, [R], \langle R\rangle)$ is a modal lattice. Moreover, if $f : \mathbb{X} \to \mathbb{X}'$ is a modal L-space morphism, then*

$$\mathcal{F}_{clp}f = f^{-1} : (\mathcal{F}_{clp}\mathbb{X}', [R'], \langle R'\rangle) \to (\mathcal{F}_{clp}\mathbb{X}, [R], \langle R\rangle)$$

is a modal lattice homomorphism.

Proof. The maps $[R], \langle R\rangle$ are functions on $\mathcal{F}_{clp}\mathbb{X}$ by definition. It follows from Proposition 4.17 and Lemma 4.6 that they satisfy the conditions from Definition 4.11.

If f is a modal L-space morphism then in particular it is an L-space morphism, so $\mathcal{F}_{clp}f$ is a lattice homomorphism from $\mathcal{F}_{clp}\mathbb{X}'$ to $\mathcal{F}_{clp}\mathbb{X}$. So we only have to show that f^{-1} preserves the modalities. This can be proven in the same way as in Proposition 4.5. $\square$

4.20 Definition. Let $\mathscr{A} = (A, \Box, \Diamond)$ be a modal lattice. Then we define the binary relation R_A on $\mathcal{F}_bA$ by

$$pR_Aq \quad \text{iff} \quad \Box^{-1}(p) \subseteq q \subseteq \Diamond^{-1}(p).$$

4.21 Lemma. *Let $(A, \Box, \Diamond)$ be a modal lattice and $p \in \mathcal{F}_bA$. Then $\Box^{-1}(p)$ is a filter in $\mathcal{F}_bA$ and $pR_A\Box^{-1}(p)$.*

Proof. The set $\Box^{-1}(p)$ is non-empty because $\top = \Box\top \in p$ implies $\top \in \Box^{-1}(p)$. It is a filter because $\Box : A \to A$ preserves conjunctions. Moreover, we claim that $\bot \notin \Box^{-1}(p)$. If $\bot \in \Box^{-1}(p)$ then $\Box\bot \in p$, hence $\Diamond\top \wedge \Box\bot \in p$ (because $\Diamond\top = \top \in p$). But $\Diamond\top \wedge \Box\bot \leq \Diamond(\top \wedge \bot) \leq \bot$ so $\bot \in p$, a contradiction.

In order to show $pR_A\Box^{-1}(p)$ we need to prove $\Box^{-1}(p) \subseteq \Box^{-1}(p) \subseteq \Diamond^{-1}(p)$. The left inclusion is trivial, so we only have to prove the right one. Suppose $a \in \Box^{-1}(p)$. Then $\Box a \in p$. Since $\Diamond\top \in p$ for every nonempty filter and because $\Diamond\top \wedge \Box a \leq \Diamond(\top \wedge a) = \Diamond a$, this implies $\Diamond a \in p$. So $a \in \Diamond^{-1}(p)$. $\square$

4.22 Lemma. *Let $(A, \Box, \Diamond)$ be a modal lattice. Then for each $a \in A$ we have*

$$[R_A]\theta_A(a) = \theta_A(\Box a) \quad \textit{and} \quad \langle R_A\rangle\theta_A(a) = \theta_A(\Diamond a).$$

Proof. Suppose $p \in [R_A]\theta_A(a)$. By Lemma 4.21 we have $pR_A\Box^{-1}(p)$ and by assumption $a \in \Box^{-1}(p)$. This implies that $\Box a \in p$ and therefore $p \in \theta_A(\Box a)$. For the reverse inclusion, suppose $p \in \theta_A(\Box a)$. Then $a \in \Box^{-1}(p)$, so pR_Aq implies $a \in q$, and hence $p \in [R_A]\theta_A(a)$.

Next suppose $p \in \langle R_A\rangle\theta_A(a)$. Then there exists a filter q such that pR_Aq and $a \in q$. By definition of R_A this implies $a \in \Diamond^{-1}(p)$ and hence $\Diamond a \in p$, so $p \in \theta_A(\Diamond a)$. Conversely, suppose $p \in \theta_A(\Diamond a)$. Let q be the filter generated by $\Box^{-1}(p)$ and a. We claim that $c \in q$ implies $\Diamond c \in p$.

To see this, note that for each $c \in q$ there exists some $d \in \Box^{-1}(p)$ such that $d \wedge a \leq c$. We then have $\Box d \in p$, and $\Diamond a \in p$ by assumption, so $\Box d \wedge \Diamond a \in p$. Since $\Box d \wedge \Diamond a \leq \Diamond(d \wedge a) \leq \Diamond c$ we find $\Diamond c \in p$. The filter q is nonempty because it contains a, and proper because $\bot \in q$ would imply $\bot = \Diamond \bot \in p$, which contradicts the assumption that p is in $\mathscr{F}_b A$. Furthermore, we have $\Box^{-1}(p) \subseteq q$ by definition of q and we just derived that $c \in q$ implies $\Diamond c \in p$, so that $q \subseteq \Diamond^{-1}(p)$. This proves $pR_A q$. By design $a \in q$ so q witnesses the fact that $p \in \langle R_A \rangle \theta_A(a)$. $\square$

4.23 Lemma. *If $\mathcal{A} = (A, \Box, \Diamond)$ is a modal lattice, then $\mathcal{A}_* := (\mathscr{F}_b A, \subseteq, \tau_A, R_A)$ is a modal L-space.*

Proof. We verify the condition from Definition 4.15. Item (1) follows from Theorem 2.20. Item (2) follows from Lemma 4.21. Item (3) follows from Lemma 4.22. Item (4) follows from the definition of R_A and the fact that each clopen filter is of the form $\theta_A(a)$. $\square$

4.24 Lemma. *Let $h : \mathcal{A} \to \mathcal{A}'$ be a modal lattice homomorphism. Then $\mathscr{F}_b h = h^{-1} : \mathcal{A}'_* \to \mathcal{A}_*$ is a modal L-space morphism.*

Proof. It follows from the duality between lattices and L-spaces that h^{-1} is an L-space morphism, so we only have to verify the three conditions from Definition 4.4. We write R' and R for the relations from $\mathcal{A}'_*$ and $\mathcal{A}_*$.

1. Let p' and q' be filters of $\mathcal{A}'$ (elements of $\mathcal{A}'_*$) such that $p'R'q'$. In order to prove that $h^{-1}(p')Rh^{-1}(q')$ we have to show that $\Box^{-1}(h^{-1}(p')) \subseteq h^{-1}(q') \subseteq \Diamond^{-1}(h^{-1}(p'))$. Let $a \in \Box^{-1}(h^{-1}(p'))$. Then $\Box a \in h^{-1}(p')$ so $\Box'(h(a)) = h(\Box a) \in p'$. Therefore $h(a) \in (\Box')^{-1}(p')$, and since $p'R'q'$ this implies $h(a) \in q'$, so that $a \in h^{-1}(q')$.

   Next, if $a \in h^{-1}(q')$ then $h(a) \in q'$, so $h(\Diamond a) = \Diamond' h(a) \in p'$ Therefore $\Diamond a \in h^{-1}(p')$ so that $a \in \Diamond^{-1}(h^{-1}(p'))$.

2. Suppose $h^{-1}(p')Rq$. Since $h^{-1}(p')$ has an R-successor, we can show in the same way as in Lemma 4.22 that $\Diamond \top \in h^{-1}(p')$. This implies $\Diamond' \top' = \Diamond' h(\top) = h(\Diamond \top) \in p'$. Lemma 4.21 then implies that $p'R'(\Box')^{-1}(p')$. So it suffices to show that $h^{-1}((\Box')^{-1}(p')) \subseteq q$. To this end, suppose $a \in h^{-1}((\Box')^{-1}(p'))$. Then $h(\Box a) = \Box' h(a) \in p'$, so $\Box a \in h^{-1}(p')$. Since $h^{-1}(p')Rq$ this implies $a \in q$, as desired.

3. Suppose $h^{-1}(p')Rq$. Then $\uparrow h[q]$ is a filter (since h is a lattice homomorphism it is non-empty and closed under meets). Let q' be the filter generated by $\uparrow h[q]$ and $(\Box')^{-1}(p')$. Then $q \subseteq h^{-1}(q')$ by construction, so it suffices to show that $p'R'q'$. We have $(\Box')^{-1}(p') \subseteq q'$ by definition, so we only have to show that $q' \subseteq (\Diamond')^{-1}(p')$. Let $a' \in q'$. Then there are $b' \in (\Box')^{-1}(p')$ and $c \in q$ such that $b' \wedge' h(c) \leq a'$. We find

   $$\Box' b' \wedge h(\Diamond c) = \Box' b' \wedge \Diamond' h(c) \leq \Diamond a'.$$

   By construction we have $\Box' b' \in p'$. Furthermore, $c \in q$ implies $\Diamond c \in h^{-1}(p')$ and hence $h(\Diamond c) \in p'$. Therefore $\Diamond a' \in p'$, and consequently $p'R'q'$. $\square$

Using the above lemmas we now establish a duality for modal lattices.

4.25 Theorem. *The duality between* Lat *and* LSpace *lifts to a duality*

$$\mathsf{MLat} \equiv^{\mathrm{op}} \mathsf{MLSpace}.$$

Proof. Let $\mathbb{X} = (X, \leq, \tau, R)$ be a modal L-space. This gives rise to the modal lattice $\mathbb{X}^* = (\mathcal{F}_{clp}\mathbb{X}, [R], \langle R \rangle)$, which in turn yields a modal L-space $(\mathbb{X}^*)_* = (\mathcal{F}_b\mathcal{F}_{clp}\mathbb{X}, \subseteq, \tau_{\mathcal{F}_{clp}\mathbb{X}}, R_{\mathcal{F}_{clp}\mathbb{X}})$. As a consequence of the duality for lattices (Theorem 2.20) we know that $(X, \leq, \tau)$ is isomorphic to $(\mathcal{F}_b\mathcal{F}_{clp}\mathbb{X}, \subseteq, \tau_{\mathcal{F}_{clp}\mathbb{X}})$ via $x \mapsto \eta_{\mathbb{X}}(x) = \{a \in \mathcal{F}_{clp}\mathbb{X} \mid x \in a\}$. So we only have to show that R and $R_{\mathcal{F}_{clp}\mathbb{X}}$ coincide. This can be seen as follows:

$$\begin{aligned} xRy \quad &\text{iff} \quad \forall a \in \mathcal{F}_{clp}\mathbb{X}\big(x \in [R]a \text{ implies } y \in a \text{ and } y \in a \text{ implies } \langle R \rangle a\big) \\ &\text{iff} \quad \forall a \in \mathcal{F}_{clp}\mathbb{X}\big([R]a \in \eta_{\mathbb{X}}(x) \text{ implies } a \in \eta_{\mathbb{X}}(y) \text{ and } a \in \eta_{\mathbb{X}}(y) \text{ implies } \langle R \rangle a \in \eta_{\mathbb{X}}(x)\big) \\ &\text{iff} \quad \forall a \in \mathcal{F}_{clp}\mathbb{X}\big([R]^{-1}(\eta_{\mathbb{X}}(x)) \subseteq \eta_{\mathbb{X}}(y) \subseteq \langle R \rangle^{-1}(\eta_{\mathbb{X}}(x))\big) \\ &\text{iff} \quad \eta_{\mathbb{X}}(x) R_{\mathcal{F}_{clp}\mathbb{X}} \eta_{\mathbb{X}}(y) \end{aligned}$$

Next, let $\mathcal{A} = (A, \Box, \Diamond)$ be a modal lattice, $\mathcal{A}_* = (\mathcal{F}_b A, R_A)$ and $(\mathcal{A}_*)^* = (\mathcal{F}_{clp}\mathcal{F}_b A, [R_A], \langle R_A \rangle)$. Then the duality for lattices from Theorem 2.20 tells us that A and $\mathcal{F}_{clp}\mathcal{F}_b A$ are isomorphic via $a \mapsto \theta_A(a) = \{p \in \mathcal{F}_b A \mid a \in p\}$, so we just have to show that $\Box$ coincides with $[R_A]$ and $\Diamond$ coincides with $\langle R_A \rangle$. That is, we have to show that $\theta_A(\Box a) = [R_A]\theta_A(a)$ and $\theta_A(\Diamond a) = \langle R_A \rangle \theta_A(a)$. But we have already proven that in Lemma 4.22.

The two paragraphs above establish the duality on objects. The duality for morphisms follows immediately from Theorem 2.20 and the fact that all our categories are concrete. □

4.26 Definition. Let $\mathbb{X} = (X, \leq, \tau, R)$ be a modal L-space. Then we write $\pi\mathbb{X} = (X, \leq, R)$ for the underlying principal modal L-frame, and $\kappa\mathbb{X} = (X, \leq, R)$ for the underlying principal modal L-frame regarded as a modal L-frame.

While they may appear the same, the difference between $\pi\mathbb{X}$ and $\kappa\mathbb{X}$ lies in the valuations they allow for. While valuations of $\pi\mathbb{X}$ necessarily interpret proposition letters as principal filters, a valuation for $\kappa\mathbb{X}$ can assign any filter to a proposition letter. As a consequence, both frames differ in terms of *validity*. We will see in Section 4.5 that the move from $\mathbb{X}$ to $\pi\mathbb{X}$ preserves validity of all modal consequence pairs, and the move from $\mathbb{X}$ to $\kappa\mathbb{X}$ preserves validity of all Sahlqvist consequence pairs.

As we did for F^2-completions of lattices in Section 2.4, we define the F^2-completion of a modal lattice via the duality. We also define the filter completion in this way. Recall that if $\mathfrak{X}$ is a modal L-frame, then by $\mathfrak{X}^{\ddagger}$ we denote its modal lattice of filters, and if $\mathfrak{X}$ is a principal modal L-frame then $\mathfrak{X}^{\dagger}$ is the modal lattice of principal filters.

4.27 Definition. Let $\mathcal{A} = (A, \Box, \Diamond)$ be a modal lattice.

1. The *filter completion* of $\mathcal{A}$ is defined as $(\pi\mathcal{A}_*)^{\dagger}$.
2. We define the F^2*-completion* of $\mathcal{A}$ as $(\kappa\mathcal{A}_*)^{\ddagger}$.

In both cases the inclusion from $\mathcal{A}$ into its completion is given by θ_A, which sends $a \in A$ to $\{p \in \mathcal{F}_{top}(\mathcal{A}_*) \mid a \in p\}$. It follows from Lemma 4.22 that this map preserves $\Box$ and $\Diamond$.

4.4 Sahlqvist Correspondence

In this section we extend the results from Section 3.4 to obtain Sahlqvist correspondence for modal L-frames. Interestingly, our definition of a Sahlqvist consequence pair is closely aligned to Sahlqivst formulae from normal modal logic (see e.g. [6, Definition 3.51]).

To account for the additional relation in the definition of a modal L-model, we work with a first-order logic with an extra binary relation symbol (compared to Section 3.4). That is, we let

$\mathbf{FOL}_2$ be the first-order logic with a unary predicate for each proposition letter and two binary predicates S (corresponding to the partial order) and R (corresponding to the modal relation). Every modal L-model $\mathfrak{M}$ gives rise to a first-order structure $\mathfrak{M}^\circ$ for $\mathbf{FOL}_2$ in the obvious way. We extend the standard translation from Definition 3.25 to a translation $\mathrm{st}_x : \mathbf{L}_{\Box\Diamond} \to \mathbf{FOL}_2$ by adding the clauses

$$\mathrm{st}_x(\Box\varphi) = \forall y(xRy \to \mathrm{st}_y(\varphi))$$
$$\mathrm{st}_x(\Diamond\varphi) = \exists y(xRy \wedge \mathrm{st}_y(\varphi))$$

We then have the following counterparts of Proposition 4.28 and Corollary 4.29

4.28 Proposition. *Let $\mathfrak{M}$ be a modal L-model, w a state in $\mathfrak{M}$ and φ an $\mathbf{L}_{\Box\Diamond}$-formula. Then*

$$\mathfrak{M}, w \Vdash \varphi \quad \textit{iff} \quad \mathfrak{M}^\circ \models \mathrm{st}_x(\varphi)[w].$$

4.29 Corollary. *Let $\mathfrak{M}$ be a modal L-model and $\varphi, \psi \in \mathbf{L}_{\Box\Diamond}$. Then we have:*

1. *$\mathfrak{M} \Vdash \varphi$ iff $\mathfrak{M}^\circ \models \forall x(\mathrm{st}_x(\varphi))$;*
2. *$\mathfrak{M}, w \Vdash \varphi \trianglelefteq \psi$ iff $\mathfrak{M}^\circ \models \mathrm{st}_x(\varphi \trianglelefteq \psi)[w]$;*
3. *$\mathfrak{M} \Vdash \varphi \trianglelefteq \psi$ iff $\mathfrak{M}^\circ \models \forall x(\mathrm{st}_x(\varphi \trianglelefteq \psi))$.*

We obtain similar results for modal L-frames by extending the second-order translation. Write $\mathbf{SOL}_2$ for the second-order logic with the same predicates as $\mathbf{FOL}_2$ where we allow quantification over unary predicates.

4.30 Definition. Let $p_1, \dots, p_n$ be the proposition letters occurring in the $\mathbf{L}_{\Box\Diamond}$-formulae ψ and χ, and write $P_1, \dots, P_n$ denote their corresponding unary predicates. The *second order translation* of a consequence pair $\psi \trianglelefteq \chi$ by

$$\mathrm{so}(\psi \trianglelefteq \chi) = \forall P_1 \cdots \forall P_n \forall x\big((\mathsf{isfil}(P_1) \wedge \cdots \wedge \mathsf{isfil}(P_n) \wedge \mathrm{st}_x(\psi)) \to \mathrm{st}_x(\chi)\big). \tag{15}$$

Since all unary predicates in $\mathrm{so}(\varphi)$ are bounded, it can be interpreted in a first-order structure with two relations. So every modal L-model $\mathfrak{X}$ gives rise to a structure $\mathfrak{X}^\circ$ for $\mathbf{SOL}_2$-formulae with no free predicates.

4.31 Lemma. *For all modal L-frames $\mathfrak{X} = (X, \leq, R)$ and all modal consequence pairs $\psi \trianglelefteq \chi$ we have*

$$\mathfrak{X} \Vdash \psi \trianglelefteq \chi \quad \textit{iff} \quad \mathfrak{X}^\circ \models \mathrm{so}(\psi \trianglelefteq \chi).$$

Proof. Similar to the proof of Lemma 3.29. □

Finally, we still have monotonicity of all connectives of $\mathbf{L}_{\Box\Diamond}$, so the following analogue of Lemma 3.31 goes through without problems.

4.32 Lemma. *Let $\mathfrak{X}$ be a modal L-frame and let V and V' be valuations for $\mathfrak{X}$ such that $V(p) \subseteq V'(p)$ for all $p \in \mathrm{Prop}$. Then for all $\varphi \in \mathbf{L}$ we have $V(\varphi) \subseteq V'(\varphi)$.*

We are now ready to define Sahlqvist consequence pairs and prove a correspondence result. We make use of the following notion of a boxed atom.

4.33 Definition. A *boxed atom* is a formula of the form

$$\Box^n p := \underbrace{\Box \cdots \Box}_{n \text{ times}} p,$$

where p is a proposition letter.

If R is a relation, then we write R^n for the n-fold composition of R. That is, xR^ny if there exist $x_0, \ldots, x_{n+1}$ such that $x = x_0$, $y = x_{n+1}$ and x_iRx_{i+1} for all $i \in \{0, \ldots, n\}$. With this definition, truth of $\Box^n p$ in a modal L-model $\mathfrak{M} = (X, \leq, R, V)$ can be given as

$$\mathfrak{M}, x \Vdash \Box^n p \quad \text{iff} \quad \forall y \in X, xR^ny \text{ implies } \mathfrak{M}, y \Vdash p.$$

We legislate xR^0y iff $x = y$. Then the interpretation of $\Box^0 p$ simply coincides with p.

4.34 Definition. A *Sahlqvist antecedent* is a formula made from boxed atoms, $\top$ and $\bot$ by freely using $\wedge$, $\vee$ and $\Diamond$.

4.35 Theorem. *Let $\psi \in \mathbf{L}_{\Box\Diamond}$ be a Sahlqvist antecedent, and let $\chi \in \mathbf{L}$ be any formula. Then $\psi \trianglelefteq \chi$ locally corresponds to a first-order formula on frames that is effectively computable from the sequent.*

Proof. We employ the same strategy as in the proof of Theorem 3.32. The second half of the proof is identical to that of Theorem 3.32, so we focus on the first half.

We assume that this expression has been processed such that no two quantifiers bind the same variable. Let $p_1, \ldots, p_n$ be the propositional variables occurring in ψ, and write $P_1, \ldots, P_n$ for their corresponding unary predicates. We assume that every proposition letter that occurs in χ also occurs is ψ, for otherwise we may replace it by $\bot$ to obtain a formula which is equivalent in terms of validity on frames.

Step 1. We start by pre-processing the formula $\mathrm{so}(\psi \trianglelefteq \chi)$ some more. We make use of the fact that, after applying the second-order translation, we have classical laws such as distributivity.

Step 1A. Use equivalences of the form

$$(\exists w(\alpha(w)) \wedge \beta) \leftrightarrow \exists w(\alpha(w) \wedge \beta), \qquad (\exists w(\alpha(w)) \vee \beta) \leftrightarrow \exists w(\alpha(w) \vee \beta),$$

and

$$(\exists w(\alpha(w)) \to \beta) \leftrightarrow \forall w(\alpha(w) \to \beta)$$

to pull out all existential quantifiers that arise in $\mathrm{st}_x(\psi)$. Let $Y := \{y_1, \ldots, y_m\}$ denote the set of (bound) variables that arise in the antecedent of the implication from the second-order translation. We end up with a formula of the form

$$\forall P_1 \cdots \forall P_n \forall x \forall y_1 \cdots \forall y_m \big((\underbrace{\mathsf{isfil}(P_1) \wedge \cdots \wedge \mathsf{isfil}(P_n)}_{\text{ISFIL}} \wedge \bar{\psi}) \to \mathrm{st}_x(\chi) \big).$$

In this formula, $\bar{\psi}$ is made up of

- boxed atoms: formulae of the form $\forall z(z'R^nz \to P_iz)$ (where P_iz falls under this umbrella as $\forall z(z'R^0z \to P_iz)$);
- top and bottom: formulae of the form $(x = x)$ and $(x \neq x)$;

- relations of the form zRz'; and
- formulae of the form $\mathsf{abovemeet}(z; z', z'')$

by using $\wedge$ and $\vee$, where $z, z', z'' \in Y \cup \{x\}$.

Step 1B. Use distributivity (of first-order classical logic) to pull out the disjunctions from ISFIL $\wedge \bar{\psi}$. That is, we rewrite ISFIL $\wedge \bar{\psi}$ as a (finite) disjunction

$$\text{ISFIL} \wedge \bar{\psi} = \bigvee \Big(\text{ISFIL} \wedge \text{BOX-AT} \wedge \text{REL} \Big)$$

where BOX-AT contains atoms and boxed atoms and REL contains relations of the form zRz' and $\mathsf{abovemeet}(z; z', z'')$.

Step 1C. Finally, use equivalences of the form

$$((\alpha \vee \beta) \to \gamma) \leftrightarrow ((\alpha \to \gamma) \wedge (\beta \to \gamma))$$

and

$$\forall \dots (\alpha \wedge \beta) \leftrightarrow ((\forall \dots \alpha) \wedge (\forall \dots \beta))$$

to rewrite $\mathrm{so}(\varphi \trianglelefteq \psi)$ into a conjunction of formulae of the form

$$\forall P_1 \cdots \forall P_n \forall y_1 \cdots \forall y_m \Big(\text{ISFIL} \wedge \text{BOX-AT} \wedge \text{REL} \to \chi \Big). \tag{16}$$

Step 2. Next we focus on each of the formulae of the form given in (16) individually. We read off minimal instances of the P_i making the antecedent true. Intuitively, these correspond to the smallest valuations for the p_i making the antecedent true.

For each proposition letter P_i, let $\forall y_{i_1}(z_{i_1} R^{n_{i_1}} y_{i_1} \to P_i y_{i_1}), \dots, \forall y_{i_1}(z_{i_k} R^{n_k} y_{i_k} \to P_i y_{i_k})$ be the occurrences of P_i in BOX-AT in the antecedent of (16). Intuitively, we define the valuation of p_i to be the filter generated by the (interpretations of) $y_{i_1}, \dots, y_{i_k}$. Formally,

$$\sigma(P_i) := \bigvee \Big\{ \exists w_{j_1} \cdots \exists w_{j_\ell} \big(z_{j_1} R^{n_{j_1}} w_{j_1} \wedge \cdots \wedge z_{j_\ell} R^{n_{j_\ell}} w_{j_\ell} \wedge \mathsf{abovemeet}(u; w_{j_1}, \dots, w_{j_\ell}) \big) \mid \emptyset \neq \{j_1, \dots, j_\ell\} \subseteq \{i_1, \dots, i_k\} \Big\}.$$

(If $k = 0$, i.e. there are no boxed atoms involving P_i in the formula, then we let $\sigma(P_i) = \lambda u.(u \neq u)$.) If there are no $R^{n_{i_j}}$-successors of z_{i_j} then the corresponding boxed atom is vacuously true, so it does not affect the "interpretation" of P_i. In order to reflect this in the expression of $\sigma(P_i)$, we take the join over all subsets of $\{i_1, \dots, i_k\}$.

The remainder of the proof is analogous to the proof of Theorem 3.32. □

In the next example we apply the algorithm of the proof of Theorem 4.35 to two simple consequence pairs, $p \trianglelefteq \Diamond p$ and $\Box p \trianglelefteq p$. The shows the mechanism of the proof in action. Moreover, it demonstrates that the duality between $\Box$ and $\Diamond$ is weaker than in the classical case, because the formulae locally correspond to different frame conditions.

4.36 Example. The second-order translation of $p \trianglelefteq \Diamond p$ is

$$\forall P \forall x (\mathsf{isfil}(P) \wedge Px \to \exists y (xRy \wedge Py))$$

This is already of the desired shape, so we proceed to Step 2. We find $\sigma(P) = \lambda u.x \leq u$. Substituting this gives the first-order formula $\forall x(\mathsf{isfil}(P) \wedge (x \leq x) \to \exists y(xRy \wedge (x \leq y)))$. Note that the antecedent of the formula is always true, so the (simplified) local correspondent of $p \trianglelefteq \Diamond p$ is

$$\forall x \exists y(xRy \wedge x \leq y).$$

Next consider $\Box p \trianglelefteq p$. The second-order translation is

$$\forall P \forall x(\mathsf{isfil}(P) \wedge \forall y(xRy \to Py) \to Px).$$

Then $\sigma(P) = \lambda u.\exists y(xRy \wedge y \leq u)$. Instantiating this makes the antecedent of the outer implication vacuously true, so that we get the local correspondent

$$\forall x \exists y(xRy \wedge y \leq x). \qquad \lhd$$

Next, we use Theorem 4.35 to enforce normality for the diamond operator. It follows from Lemma 4.6 that the consequence pair $\Diamond p \vee \Diamond q \trianglelefteq \Diamond(p \vee q)$ is valid in every modal L-frame, so we focus on its converse. We arrive at a frame condition closely related to the one identified in [13, Definition 5.1.4].

4.37 Example. If we want $\Diamond$ to preserve joins we need to add the modal consequence pair

$$\Diamond(p \vee q) \trianglelefteq \Diamond p \vee \Diamond q \tag{17}$$

to our system. This is a Sahlqvist pair, so we can use the algorithm from Theorem 4.35 to the first-order frame condition ensuring its validity.

The standard translation of the antecedent is

$$\mathrm{st}_x(\Diamond(p \vee q)) = \exists y(xRy \wedge (Py \vee Qy \vee \exists z \exists z'(\mathsf{abovemeet}(y; z, z') \wedge Pz \wedge Qz')))$$

Processing the formula, we obtain the following second-order translation:

$$\begin{aligned} &\forall P \forall Q \forall x \forall y \forall z \forall z'(\mathrm{ISFIL} \wedge xRy \wedge Py \to \mathrm{st}_x(\chi)) \\ \wedge\, &\forall P \forall Q \forall x \forall y \forall z \forall z'(\mathrm{ISFIL} \wedge xRy \wedge Qy \to \mathrm{st}_x(\chi)) \\ \wedge\, &\forall P \forall Q \forall x \forall y \forall z \forall z'(\mathrm{ISFIL} \wedge xRy \wedge \mathsf{abovemeet}(y; , z, z') \wedge Pz \wedge Qz' \to \mathrm{st}_x(\chi)) \end{aligned} \tag{18}$$

As usual, we process the lines one by one, and instantiate the minimal instantiations of P and Q into $\mathrm{st}_x(\chi)$. First compute

$$\begin{aligned} \mathrm{st}_x(\chi) = \exists s(xRs \wedge Ps) \vee \exists t(xRt \wedge Qt) \\ \vee\, \exists v \exists v'(\mathsf{abovemeet}(x; v, v') \wedge \exists w(vRw \wedge Pw) \wedge \exists w'(v'Rw' \wedge Qw')) \end{aligned}$$

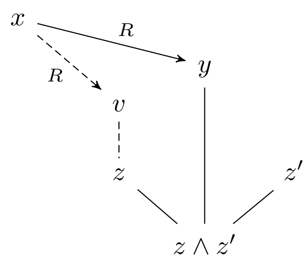
In the first line of (18) we get $\sigma(P) = \lambda u.y \leq u$ and $\sigma(Q) = \lambda u.u \neq u$. Then $[\sigma(P)/P, \sigma(Q)/Q]\,\mathrm{st}_x(\chi)$ holds, because its first disjoint is valid. The second line of (18) yields a similar situation. So both these give rise to first-order formulae that are always valid.

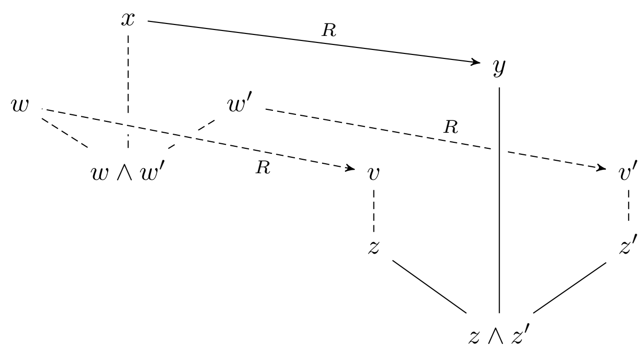
The third line of (18) gives $\sigma(P) = \lambda u.z \leq u$ and $\sigma(Q) = \lambda u.z' \leq u$. Instantiating this we get that a frame $(X, \leq)$ validates (17) if and only if for all $x, y, z, z' \in X$ such that xRy and $z \wedge z' \leq y$ we have either

- there exists a v such that xRv and $z \leq v$; or
- there exists a v' such that xRv' and $z' \leq v'$; or

- there exist v, v', w, w' such that $z \leq v$, $z' \leq v'$, $w \wedge w' \leq x$, wRv and $w'Rv'$.

The first and third item can be depicted as follows:





The second item can be depicted in a similar way as the first one. $\lhd$

4.5 Sahlqvist Canonicity

As we have seen, for each modal L-space $\mathbb{X}$ we have two flavours of "underlying frame," principal and non-principal modal L-frames. This gives rise to two different notions of persistence. We stress that our approach is along the same lines as classical Sahlqvist canonicity results [36, 6].

4.38 Definition. Let $\mathbb{X}$ be a modal L-space. A modal consequence pair $\psi \trianglelefteq \chi$ is called *p-persistent* if $\mathbb{X} \Vdash \psi \trianglelefteq \chi$ implies $\pi\mathbb{X} \Vdash \psi \trianglelefteq \chi$, and *d-persistent* if $\mathbb{X} \Vdash \psi \trianglelefteq \chi$ implies $\kappa\mathbb{X} \Vdash \psi \trianglelefteq \chi$.

In this section we show that every modal consequence pair is p-persistent, and that all Sahlqvist consequence pairs are d-persistent. We begin by working towards p-persistence. As we will see, this simply follows from an extension of Lemma 3.37.

4.39 Lemma. *Let $(X, \leq, R)$ be a modal L-frame and $\{a_i \mid i \in I\}$ an I-indexed collection of subsets of X. Then we have $[R]\bigcap_{i \in I} a_i = \bigcap_{i \in I}[R]a_i$.*

Proof. If $x \in [R]\bigcap_{i \in I} a_i$ then $R[x] \subseteq \bigcap_{i \in I} a_i \subseteq a_i$ for all $i \in I$, so $x \in [R]a_i$ for all $i \in I$ and hence $x \in \bigcap_{i \in I}[R]a_i$. Conversely, if $x \in \bigcap_{i \in I}[R]a_i$ then $x \in [R]a_i$ for all i, so $R[x] \subseteq a_i$ for all i. This implies $R[x] \subseteq \bigcap_{i \in I} a_i$ so that $x \in [R]\bigcap_{i \in I} a_i$. $\square$

4.40 Lemma. *Let $\mathbb{X} = (X, \leq, \tau, R)$ be a modal L-space and let V be a closed valuation. Then for all $\varphi \in \mathbf{L}_{\Box\Diamond}$ we have*

$$V(\varphi) = \bigcap_{V \lessdot U} U(\varphi).$$

Proof. The proof proceeds by induction on the structure of φ. The propositional cases are the same as in Lemma 3.37. If $\varphi = \Box\psi$ then we have

$$V(\Box\psi) = [R]V(\psi) = [R]\Big(\bigcap_{V \lessdot U} U(\psi)\Big) = \bigcap_{V \lessdot U}[R]U(\psi) = \bigcap_{V \lessdot U} U(\Box\psi).$$

Next suppose $\varphi = \Diamond\psi$. The inclusion $V(\Diamond\psi) \subseteq \bigcap_{V \lessdot U} U(\Diamond\psi)$ is clear because of monotonicity of the diamond. For the converse, assume $x \in \bigcap_{V \lessdot U} U(\Diamond\psi)$. We need to find some y such that xRy and $y \in V(\psi) = \bigcap_{V \lessdot U} U(\psi)$. So we want to find an element in

$$R[x] \cap \bigcap_{V \lessdot U} U(\psi).$$

Since all sets in this intersection are closed, it suffices to show that it has the finite intersection property. Given a finite subcollection $R[x], U_1, \ldots, U_n$. Then $V \lessdot (U_1 \cap \cdots \cap U_n)$, so $x \in (U_1 \cap \cdots \cap U_n)(\Diamond\psi)$ and hence there exists a state y' such that xRy' and $y' \in (U_1 \cap \cdots \cap U_n)(\psi)$. This witnesses the finite intersection property. $\square$

Since every principal valuation is closed, it follows immediately that every consequence pair which is valid on a modal L-space $\mathbb{X}$ is also valid in the underlying *principal* L-frame $\pi\mathbb{X}$.

4.41 Theorem. *Every modal consequence pair is p-persistent.*

Proof. The principal valuations for $\pi\mathbb{X}$ are precisely the closed valuations of $\mathbb{X}$. Let V be such a closed valuation. If $\mathbb{X} \Vdash \varphi \trianglelefteq \psi$ then $U(\varphi) \subseteq U(\psi)$ for every clopen valuation U for $\mathbb{X}$. This implies

$$V(\varphi) = \bigcap_{V \lessdot U} U(\varphi) \subseteq \bigcap_{V \lessdot U} U(\psi) = V(\psi)$$

and therefore $\pi\mathbb{X} \Vdash \varphi \trianglelefteq \psi$. $\square$

As an immediate consequence we obtain (soundness and) completeness of $\mathcal{L}_{\Box\Diamond}(\Gamma)$ with respect to the class of principal modal L-frames that validate all modal consequence pairs in Γ.

4.42 Theorem. *Let Γ be a set of modal consequence pairs and write* $\mathsf{PMLFrm}(\Gamma)$ *for the class of principal modal L-frames that validate all consequence pairs in Γ. Then for all $\varphi, \psi \in \mathbf{L}_{\Box\Diamond}$ we have*

$$\varphi \vdash_\Gamma \psi \quad \textit{iff} \quad \varphi \Vdash_{\mathsf{PMLFrm}(\Gamma)} \psi.$$

Moreover, we obtain a modal counterpart of Baker and Hales' theorem:

4.43 Theorem. *Every variety of modal lattices is closed under filter extensions.*

Proof. It suffices to show that filter extensions preserve inequalities. When writing inequalities, we may use the set of proposition letters as variables, so that an inequality is of the form $\varphi \leq \psi$, where φ and ψ are $\mathbf{L}_{\Box\Diamond}$-formulae. Moreover, a modal lattice $\mathcal{A} = (A, \Box, \Diamond)$ satisfies the inequality $\varphi \leq \psi$ if and only if the modal consequence pair $\varphi \trianglelefteq \psi$ is valid on $\mathcal{A}$. So it suffices to show that filter extensions preserve validity of modal consequence pairs.

So suppose $\mathcal{A} \Vdash \varphi \trianglelefteq \psi$. Then the dual modal L-space $\mathcal{A}_*$ validates $\varphi \trianglelefteq \psi$, and by Theorem 4.41 we have $\pi\mathcal{A}_* \Vdash \varphi \trianglelefteq \psi$. This implies $(\pi\mathcal{A}_*)^\dagger \Vdash \varphi \trianglelefteq \psi$, and since this is exactly the filter extension of $\mathcal{A}$ we have proven the theorem. $\square$

Next we focus on d-persistence. While we do not have an explicit example of a modal consequence pair that is not d-persistent, we have the following conjecture. We expect that a proof can be given inspired by [20].

4.44 Conjecture. *The McKinsey axiom $\Box\Diamond p \trianglelefteq \Diamond\Box p$ is not d-persistent.*

The following lemma plays a key role in the Sahlqvist canonicity theorem.

4.45 Lemma. *Let $\mathbb{X} = (X, \leq, \tau, R)$ be a modal L-space.*

1. *If $a \subseteq X$ is closed under binary meets, then so is $R[a]$.*
2. *For every closed subset c of $\mathbb{X}$, the set $R[c]$ is closed.*
3. *For every $x \in X$, the set ${\uparrow}R^n[x]$ is a closed filter.*

Proof. (1) Let $y, y' \in R[a]$. Then there exist $x, x' \in a$ such that xRy and $x'Ry'$. Then $(x \wedge x')R(y \wedge y')$. Since a is closed under binary meets we have $x \wedge x' \in a$, and therefore $y \wedge y' \in R[a]$.

(2) Suppose c is a closed subset of $\mathbb{X}$ and $y \notin R[c]$. We construct an open neighbourhood of y disjoint from $R[c]$. Since $y \notin R[c]$ we have $y \notin R[x]$ for all $x \in c$. So for each $x \in c$ we can find a set a_x that is either a clopen filter or a the complement of a clopen filter, such that $y \in a_x$ and $R[x] \cap a_x = \emptyset$. As a consequence $R[x] \subseteq (X \setminus a_x)$, and hence $x \in [R](X \setminus a_x)$ for each $x \in c$.

If a_x is the complement of a clopen filter then $(X \setminus a_x)$ is a clopen filter so $[R](X \setminus a_x)$ is clopen by definition. If a_x is a clopen filter then $[R](X \setminus a_x) = X \setminus \langle R \rangle a_x$, which is also clopen. So we have an open cover

$$c \subseteq \bigcup_{x \in c} [R](X \setminus a_x).$$

Since c is closed, hence compact, there exists a finite subcover, say,

$$c \subseteq [R](X \setminus a_1) \cup \cdots \cup [R](X \setminus a_n).$$

Since we have $z \in [R](X \setminus a_i)$ iff $R[z] \subseteq (X \setminus a_i)$ for all $z \in X$, this implies $R[c] \subseteq (X \setminus a_1) \cup \cdots \cup (X \setminus a_n)$. Then $a_1 \cap \cdots \cap a_n$ is a clopen neighbourhood of y disjoint from $R[c]$, as required.

(3) As a consequence of Definition 4.1(4) and item (1) $R^n[x]$ is closed under meets for all natural numbers n. This implies that $\uparrow R^n[x]$ is a filter. It follows from items (2), Lemma 2.10 and Lemma 4.16 that it is also closed. □

4.46 Theorem. *Every Sahlqvist consequence pair is d-persistent.*

Proof. The proof is essentially the same as that of Theorem 3.38. Using Lemma 4.45 one can show that minimal valuations are closed. □

As a corollary we obtain completeness with respect to classes of modal L-fames for extensions of $\mathcal{L}_{\Box\Diamond}$ with Sahlqvist consequence pairs.

4.47 Theorem. *Let Γ be a set of Sahlqvist consequence pairs and write $\mathsf{MLFrm}(\Gamma)$ for the class of modal L-frames that validate all consequence pairs in Γ. Then for all $\varphi, \psi \in \mathbf{L}_{\Box\Diamond}$ we have*

$$\varphi \vdash_\Gamma \psi \quad \textit{iff} \quad \varphi \Vdash_{\mathsf{MLFrm}(\Gamma)} \psi.$$

4.48 Remark. Similarly to Remark 3.41 we point out that our notion of d-persistence corresponds algebraically not to canonical extensions, as in the setting of distributive modal logics, but to F^2-completions. The reason for this is that our d-persistence allows us to move from valuations of clopen filters to valuations of all filters. Algebraically this corresponds exactly to F^2-completion, while canonical extensions correspond to taking saturated filters (Theorem 2.26). ◁

5 Conclusion

We have given a new duality for bounded (not necessarily distributive) lattices which more closely resembles Stone-type dualities than existing dualities. It arises as a restriction of a known duality for the category of bounded meet-semilattices given by Hofmann, Mislove and Stralka [27]. The relation between our duality and the duality by Hofmann, Mislove and Stralka is similar to the relation between Esakia duality and Priestley duality. It can also be seen as a Stone-type analogue of Jipsen and Moshier's spectral duality for lattices [34].

One of the advantages of the duality for bounded lattices presented in this paper is that it more closely resembles known dualities used in (modal) logic. Consequently, it allows us to use similar tools and techniques. To showcase this, we proved Sahlqvist correspondence and canonicity results along the lines of [6].

Furthermore, we extended the duality to a duality for a modal extension based on positive (non-distributive) logic with a $\Box$ and $\Diamond$. While these are interpreted using a relation in the way as in normal modal logic over a classical base, the changed interpretation of joins in our frame caused $\Diamond$ to no longer be join-preserving. This interesting phenomenon has also been observed in the context of modal intuitionistic logic [31].

There are many intriguing avenues for further research, some of which we list below.

Algebraic characterisation of the double filter completion. As noted at the end of Section 2.4, it is not yet know if and how we can characterise the double filter extension algebraically. While it follows from the definition that it is compact, it seems likely that the double filter completion is only "half dense."

Finite model property. While it is easy to derive the finite model property for positive (non-distributive) logic, the same result for the modal extension presented in this paper appears to be non-trivial.

Relation to ortho(modular) lattices. Ortholattices and orthomodular lattices provide other interesting classes of (not necessarily distributive) lattices with operators. However, in ortholattices the orthocomplement is turning joins into meets. Duality for these structures has been discussed by Goldblatt [18, 19] and Bimbo [5]. In [13, Chapter 6] a duality developed in this paper is extended to modal operators that turn joins into meets. These are called ∇-algebras there. Recently modal ortholattices have been studied in [28]. We leave it an an interesting open problem to see whether the Sahlqvist correspondence and canonicity results of this paper can be extended to ∇-algebras. It is also open whether these technique could be extended to orthomodular lattices [30]. This is especially interesting as orthomodalur lattices provide algebraic structures of quantum logic [12] and therefore these methods could be relevant in the study of quantum logic.

Different modalities. Yet another question is what other modal extensions of positive (non-distributive) logic we can define. In particular, it would be interesting to define a form of neighbourhood semantics based on the L-frames used in this paper and investigate the behaviour of the resulting modalities.

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