LECTURE NOTES ON THE ALGEBRA OF LOGIC

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1. Algebras and equations

We begin by reviewing some fundamentals of general algebraic systems. For more information, the reader may consult [3, 7].

Definition 1.1.

(i) A *type* is a map $\rho\colon \mathcal{F} \to \mathbb{N}$, where $\mathcal{F}$ is a set of function symbols. In this case, $\rho(f)$ is said to be the *arity* of the function symbol f, for every $f \in \mathcal{F}$. Function symbols of arity zero are called *constants*.

(ii) An *algebra* of type ρ is a pair $\boldsymbol{A} = \langle A; F\rangle$ where A is a nonempty set and $F = \{f^{\boldsymbol{A}} : f \in \mathcal{F}\}$ is a set of operations on A whose arity is determined by ρ, in the sense that each $f^{\boldsymbol{A}}$ has arity $\rho(f)$. The set A is called the *universe* of $\boldsymbol{A}$.

When $\mathcal{F} = \{f_1, \dots, f_n\}$, we shall write $\langle A; f_1^{\boldsymbol{A}}, \dots, f_n^{\boldsymbol{A}}\rangle$ instead of $\langle A; F\rangle$. In this case, we often drop the superscripts, and write simply $\langle A; f_1, \dots, f_n\rangle$.

Classical examples of algebras are groups and rings. For instance, the type of groups ρ_G consists of a binary symbol $+$, a unary symbol $-$, and a constant symbol 0. Then a group is an algebra $\langle G; +, -, 0\rangle$ of type ρ_G in which $+$ is associative, 0 is a neutral element for $+$, and $-$ produces inverses.

Lattices, Heyting algebras, and modal algebras are also algebras in the above sense. For instance, the type of lattices ρ_L consists of two binary symbols $\wedge$ and $\vee$ and a lattice is an algebra $\langle A; \wedge, \vee\rangle$ of type ρ_L that satisfies the idempotent, commutative, associative, and absorption laws. Similarly, the type of Heyting algebras ρ_H consists of three binary operations symbols $\wedge, \vee$, and $\to$ and of two constant symbols 0 and 1. Then a Heyting algebra is an algebra $\langle A; \wedge, \vee, \to, 0, 1\rangle$ such that $\langle A; \wedge, \vee, 0, 1\rangle$ is a bounded lattice and, for every $a, b, c \in A$,

$$a \wedge b \leqslant c \iff a \leqslant b \to c. \qquad \text{(}\textit{residuation law}\text{)}$$

Boolean algebras can be viewed as the Heyting algebras that satisfy the following equational version of the *excluded middle law*:

$$x \vee (x \to 0) \approx 1.$$

In this case, the complement operation $\neg x$ can be defined as $x \to 0$.

Perhaps less obviously, even algebraic structures whose operations are apparently *external* can be viewed as algebras in the sense of the above definition. For instance, modules over a ring $\boldsymbol{R}$ can be viewed as algebras whose type ρ_R extends that of groups with the unary symbols $\{\lambda_r : r \in R\}$. From this point of view, a module over $\boldsymbol{R}$ is an

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algebra $\langle G; +, -, 0, \{\lambda_r : r \in R\}\rangle$ of type ρ_R such that $\langle G; +, -, 0\rangle$ is an abelian group and, for every $r, s \in R$ and $a, c \in G$,

$$\lambda_r(a + c) = \lambda_r(a) + \lambda_r(c)$$
$$\lambda_{r+s}(a) = \lambda_r(a) + \lambda_s(a)$$
$$\lambda_r(\lambda_s(a)) = \lambda_{r\cdot s}(a)$$
$$\lambda_1(a) = a.$$

Given a type $\rho\colon \mathcal{F} \to \mathbb{N}$ and a set of variables X disjoint from $\mathcal{F}$, the set of *terms of type ρ over X* is the least set $T_\rho(X)$ such that

(i) $X \subseteq T_\rho(X)$;
(ii) if $c \in \mathcal{F}$ is a constant, then $c \in T_\rho(X)$; and
(iii) if $\varphi_1, \dots, \varphi_{\rho(f)} \in T_\rho(X)$ and $f \in \mathcal{F}$, then $f\varphi_1 \dots \varphi_{\rho(f)} \in T_\rho(X)$.

For the sake of readability, we shall often write $f(\varphi_1, \dots, \varphi_{\rho(f)})$ instead of $f\varphi_1 \dots \varphi_{\rho(f)}$. Similarly, if f is a binary operation $+$, we often write $\varphi_1 + \varphi_2$ instead of $f(\varphi_1, \varphi_2)$.

Definition 1.2. Let $\rho\colon \mathcal{F} \to \mathbb{N}$ be a type and X a set of variables disjoint from $\mathcal{F}$. The *term algebra* $\boldsymbol{T}_\rho(X)$ of type ρ over X is the unique algebra of type ρ whose universe is $T_\rho(X)$ and with basic n-ary operations f defined, for every $\varphi_1, \dots, \varphi_n \in T_\rho(X)$, as

$$f^{\boldsymbol{T}_\rho(X)}(\varphi_1, \dots, \varphi_n) := f(\varphi_1, \dots, \varphi_n).$$

When no confusion might arise, we drop the subscript and write $T(X)$ instead of $T_\rho(X)$. Term algebras have the following fundamental property.

Proposition 1.3. *Let $\boldsymbol{A}$ be an algebra of type ρ and X a set of variables. Every function $f\colon X \to A$ extends uniquely to a homomorphism $f^*\colon \boldsymbol{T}_\rho(X) \to \boldsymbol{A}$.*

Proof. The unique extension f^* is defined, for every $\varphi(x_{\alpha_1}, \dots x_{\alpha_n}) \in T_\rho(X)$, as

$$f^*(\varphi) = \varphi^{\boldsymbol{A}}(f(x_{\alpha_1}), \dots, f(x_{\alpha_n})). \qquad \boxtimes$$

Exercise 1.4. Prove the above proposition. $\boxtimes$

Given a term $\varphi \in T_\rho(X)$, we write $\varphi(x_1, \dots, x_n)$ to indicate that the variables occurring in φ are among $x_1, \dots, x_n$. Furthermore, given an algebra $\boldsymbol{A}$ of type ρ and elements $a_1, \dots, a_n \in A$, we define an element

$$\varphi^{\boldsymbol{A}}(a_1, \dots, a_n)$$

of A, by recursion on the construction of φ, as follows:

(i) if φ is a variable x_i, then $\varphi^{\boldsymbol{A}}(a_1, \dots, a_n) := a_i$;
(ii) if φ is a constant c, then $c^{\boldsymbol{A}}$ is the interpretation of c in $\boldsymbol{A}$;
(iii) if $\varphi = f(\psi_1, \dots, \psi_m)$, then

$$\varphi^{\boldsymbol{A}}(a_1, \dots, a_n) := f^{\boldsymbol{A}}(\psi_1^{\boldsymbol{A}}(a_1, \dots, a_n), \dots, \psi_m^{\boldsymbol{A}}(a_1, \dots, a_n)).$$

An *equation of type ρ over* X is an expression of the form $\varphi \approx \psi$, where $\varphi, \psi \in T_\rho(X)$. We denote by $E_\rho(X)$ the set of equations of type ρ over X. Such an equation $\varphi \approx \psi$ is *valid* in an algebra $\boldsymbol{A}$ of type ρ, if

$$\varphi^{\boldsymbol{A}}(a_1, \dots, a_n) = \psi^{\boldsymbol{A}}(a_1, \dots, a_n), \text{ for every } a_1, \dots, a_n \in A,$$

in which case we say that $\boldsymbol{A}$ *satisfies* $\varphi \approx \psi$.

For instance, groups are precisely the algebras of type ρ_G that satisfy the equations

$$x + (y + z) \approx (x + y) + z \quad x + 0 \approx x \quad 0 + x \approx x \quad x + -x \approx 0 \quad -x + x \approx 0.$$

Similarly, lattices are the algebras of type ρ_L that satisfy the equations

$$\begin{array}{llr} x \wedge x \approx x & x \vee x \approx x & (\textit{idempotent laws}) \\ x \wedge y \approx y \wedge x & x \vee y \approx y \vee x & (\textit{commutative laws}) \\ x \wedge (y \wedge z) \approx (x \wedge y) \wedge z & x \vee (y \vee z) \approx (x \vee y) \vee z & (\textit{associative laws}) \\ x \wedge (y \vee x) \approx x & x \vee (y \wedge x) \approx x. & (\textit{absorption laws}) \end{array}$$

From now on, we will work with a fixed denumerable set of variables

$$Var = \{x_n : n \in \mathbb{N}\}.$$

Accordingly, when we write $x, y, z \dots$ for variables, it should be understood that these are variables in Var.

2. Basic constructions

Algebras of the same type are called *similar* and can be compared by means of maps that preserve their structure.

Definition 2.1. Given two similar algebras $\boldsymbol{A}$ and $\boldsymbol{B}$, a *homomorphism* from $\boldsymbol{A}$ to $\boldsymbol{B}$ is a map $f \colon A \to B$ such that, for every n-ary operation g of the common type and $a_1, \dots, a_n \in A$,

$$f(g^{\boldsymbol{A}}(a_1, \dots, a_n)) = g^{\boldsymbol{B}}(f(a_1), \dots, f(a_n)).$$

An injective homomorphism is called an *embedding* and, if there exists an embedding from $\boldsymbol{A}$ to $\boldsymbol{B}$, we say that $\boldsymbol{A}$ *embeds* into $\boldsymbol{B}$. Lastly, a surjective embedding is called an *isomorphism*. Accordingly, $\boldsymbol{A}$ and $\boldsymbol{B}$ are said to be *isomorphic* if there exists an isomorphism between them, in which case we write $\boldsymbol{A} \cong \boldsymbol{B}$.

A simple induction on the construction of terms shows that, for every pair of algebras $\boldsymbol{A}$ and $\boldsymbol{B}$ of type ρ and every term $\varphi(x_1, \dots, x_n)$ of ρ, if f is a homomorphism from $\boldsymbol{A}$ to $\boldsymbol{B}$, then

$$f(\varphi^{\boldsymbol{A}}(a_1, \dots, a_n)) = \varphi^{\boldsymbol{B}}(f(a_1), \dots, f(a_n)),$$

for every $a_1, \dots, a_n \in A$. Therefore homomorphisms preserve not only basic operations, but also arbitrary terms.

In the particular case where $\boldsymbol{A}$ and $\boldsymbol{B}$ are lattices, a homomorphism from $\boldsymbol{A}$ to $\boldsymbol{B}$ is a map $f \colon A \to B$ such that, for every $a, c \in A$,

$$f(a \wedge^{\boldsymbol{A}} c) = f(a) \wedge^{\boldsymbol{B}} f(c) \text{ and } f(a \vee^{\boldsymbol{A}} c) = f(a) \vee^{\boldsymbol{B}} f(c).$$

For instance, the inclusion map from the lattice $\langle \mathbb{N}; \leqslant \rangle$ into the lattice $\langle \mathbb{Z}; \leqslant \rangle$ is an injective homomorphism, that is, an embedding. Similarly, given two sets $Y \subseteq X$, the inclusion map from the powerset lattice $\langle \mathcal{P}(Y); \subseteq \rangle$ to the powerset lattice $\langle \mathcal{P}(X); \subseteq \rangle$ is also an embedding. On the other hand, if $Y \subsetneq X$, the map

$$(-) \cap Y \colon \mathcal{P}(X) \to \mathcal{P}(Y)$$

that sends every $Z \subseteq X$ to $Z \cap Y$ is a noninjective homomorphism from $\langle \mathcal{P}(X); \subseteq \rangle$ to $\langle \mathcal{P}(Y); \subseteq \rangle$.

Definition 2.2. Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be algebras of the same type $\rho\colon \mathcal{F} \to \mathbb{N}$. Then $\boldsymbol{A}$ is said to be a *subalgebra* of $\boldsymbol{B}$ if $A \subseteq B$ and $f^{\boldsymbol{A}}$ is the restriction of $f^{\boldsymbol{B}}$ to A, for every $f \in \mathcal{F}$. In this case, we write $\boldsymbol{A} \leqslant \boldsymbol{B}$.

Given a class of algebras K, let

$$\mathbb{I}(\mathsf{K}) := \{\boldsymbol{A} : \boldsymbol{A} \cong \boldsymbol{B} \text{ for some } \boldsymbol{B} \in \mathsf{K}\}$$
$$\mathbb{S}(\mathsf{K}) := \{\boldsymbol{A} : \boldsymbol{A} \leqslant \boldsymbol{B} \text{ for some } \boldsymbol{B} \in \mathsf{K}\}.$$

When $\mathsf{K} = \{\boldsymbol{A}\}$, we write $\mathbb{I}(\boldsymbol{A})$ and $\mathbb{S}(\boldsymbol{A})$ as a shorthand for $\mathbb{I}(\{\boldsymbol{A}\})$ and $\mathbb{S}(\{\boldsymbol{A}\})$, respectively. The following observation is an immediate consequence of the definitions.

Proposition 2.3. *Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be algebras of the same type. Then $\boldsymbol{A} \in \mathbb{IS}(\boldsymbol{B})$ if and only if there exists an embedding $f\colon \boldsymbol{A} \to \boldsymbol{B}$. In this case, $\boldsymbol{A}$ is isomorphic to the unique subalgebra of $\boldsymbol{B}$ with universe $f[A]$.*

As we mentioned, homomorphisms can be used to compare similar algebras.

Definition 2.4. Given two similar algebras $\boldsymbol{A}$ and $\boldsymbol{B}$, we say that $\boldsymbol{A}$ is a *homomorphic image* of $\boldsymbol{B}$ if there exists a surjective homomorphism $f\colon \boldsymbol{B} \to \boldsymbol{A}$.

Accordingly, given a class of algebras K, we set

$$\mathbb{H}(\mathsf{K}) := \{\boldsymbol{A} : \boldsymbol{A} \text{ is a homomorphic image of some } \boldsymbol{B} \in \mathsf{K}\}.$$

As usual, when $\mathsf{K} = \{\boldsymbol{A}\}$, we write $\mathbb{H}(\boldsymbol{A})$ as a shorthand for $\mathbb{H}(\{\boldsymbol{A}\})$.

Observe that every (not necessarily surjective) homomorphism $f\colon \boldsymbol{A} \to \boldsymbol{B}$ induces a homomorphic image of $\boldsymbol{A}$.

Proposition 2.5. *If $f\colon \boldsymbol{A} \to \boldsymbol{B}$ is a homomorphism, then $f[A]$ is the universe of a subalgebra of $\boldsymbol{B}$ that, moreover, is a homomorphic image of $\boldsymbol{A}$.*

Proof. Observe that $f[A]$ is nonempty, because A is. Then consider an n-ary function symbol g of the common type of $\boldsymbol{A}$ and $\boldsymbol{B}$ and $b_1, \dots, b_n \in f[A]$. Clearly, there are $a_1, \dots, a_n \in A$ such that $f(a_i) = b_i$, for every $i \leqslant n$. Since f is a homomorphism from $\boldsymbol{A}$ to $\boldsymbol{B}$, we obtain

$$g^{\boldsymbol{B}}(b_1, \dots, b_n) = g^{\boldsymbol{B}}(f(a_1), \dots, g(a_n)) = f(g^{\boldsymbol{A}}(a_1, \dots, a_n)) \in f[A].$$

Hence, we conclude that $f[A]$ is the universe of a subalgebra $f[\boldsymbol{A}]$ of $\boldsymbol{B}$.

Furthermore, $f\colon \boldsymbol{A} \to f[\boldsymbol{A}]$ is a homomorphism, because for every basic n-ary function symbol g of the common type and $a_1, \dots, a_n \in A$,

$$f(g^{\boldsymbol{A}}(a_1, \dots, a_n)) = g^{\boldsymbol{B}}(f(a_1), \dots, f(a_n)) = g^{f[\boldsymbol{A}]}(f(a_1), \dots, f(a_n)),$$

where the first equality follows from the assumption that $f\colon \boldsymbol{A} \to \boldsymbol{B}$ is a homomorphism. Since the map $f\colon A \to f[A]$ is surjective, we conclude that $f[\boldsymbol{A}] \in \mathbb{H}(\boldsymbol{A})$. ⊠

In view of the above result, when $f\colon \boldsymbol{A} \to \boldsymbol{B}$ is a homomorphism, we denote by $f[\boldsymbol{A}]$ the unique subalgebra of $\boldsymbol{B}$ with universe $f[A]$.

For instance, let $f\colon \mathbb{Z} \to \mathbb{R}$ be the absolute value map, that is, the function defined by the rule

$$f(n) := \text{the absolute value of } n.$$

Observe that f is a nonsurjective homomorphism from the lattice of integers to that of reals. Furthermore, the homomorphic image $f[\langle \mathbb{Z}; \leqslant \rangle]$ of $\langle \mathbb{Z}; \leqslant \rangle$ is the lattice of natural numbers $\langle \mathbb{N}; \leqslant \rangle$, which, in turn, is a subalgebra of lattice of reals.

Notably, the homomorphic images of an algebra $\boldsymbol{A}$ can be "internalized" as special equivalence relations on $\boldsymbol{A}$ as follows.

Definition 2.6. A *congruence* of an algebra $\boldsymbol{A}$ is an equivalence relation θ on A such that, for every basic n-ary opearation f of $\boldsymbol{A}$ and $a_1, \dots, a_n, c_1, \dots, c_n \in A$,

$$\text{if } \langle a_1, c_1 \rangle, \dots, \langle a_n, c_n \rangle \in \theta, \text{ then } \langle f^{\boldsymbol{A}}(a_1, \dots, a_n), f^{\boldsymbol{A}}(c_1, \dots, c_n) \rangle \in \theta. \tag{1}$$

In this case, we often write $a \equiv_\theta c$ as a shorthand for $\langle a, c \rangle \in \theta$. The poset of congruences of $\boldsymbol{A}$ ordered under the inclusion relation will be denoted by $\mathsf{Con}(\boldsymbol{A})$.

A simple induction on the construction of terms shows that, for every congruence θ of $\boldsymbol{A}$ and every term $\varphi(x_1, \dots, x_n)$,

$$\text{if } \langle a_1, c_1 \rangle, \dots, \langle a_n, c_n \rangle \in \theta, \text{ then } \langle \varphi^{\boldsymbol{A}}(a_1, \dots, a_n), \varphi^{\boldsymbol{A}}(c_1, \dots, c_n) \rangle \in \theta,$$

for every $a_1, \dots, a_n \in A$. Therefore, congruences preserve not only basic operations, but also arbitrary terms. Furthermore, a simple argument shows that $\mathsf{Con}(\boldsymbol{A})$ is a complete (indeed algebraic) lattice whose maximum is the total relation $A \times A$ and whose minimum is the identity relation $\mathrm{id}_A := \{\langle a, a \rangle : a \in A\}$.

Example 2.7 (Heyting algebras)**.** Recall that a *filter* of a Heyting algebra $\boldsymbol{A}$ is a nonempty upset $F \subseteq A$ closed under binary meets. We denote by $\mathsf{Fi}(\boldsymbol{A})$ the poset of filters of $\boldsymbol{A}$ ordered under the inclusion relation. It is easy to see $\mathsf{Fi}(\boldsymbol{A})$ is a complete lattice. Furthermore, the lattices $\mathsf{Fi}(\boldsymbol{A})$ and $\mathsf{Con}(\boldsymbol{A})$ are isomorphic via the inverse isomorphisms

$$\boldsymbol{\Omega}^{\boldsymbol{A}}(-) \colon \mathsf{Fi}(\boldsymbol{A}) \to \mathsf{Con}(\boldsymbol{A}) \ \text{ and } \ \boldsymbol{\tau}^{\boldsymbol{A}}(-) \colon \mathsf{Con}(\boldsymbol{A}) \to \mathsf{Fi}(\boldsymbol{A})$$

defined by the rules

$$\begin{aligned} \boldsymbol{\Omega}^{\boldsymbol{A}}(F) &:= \{\langle a, c \rangle \in A \times A : a \to c, c \to a \in F\} \\ \boldsymbol{\tau}^{\boldsymbol{A}}(\theta) &:= \{a \in A : \langle a, 1 \rangle \in \theta\}. \end{aligned}$$

Because of this, every congruence θ of a Heyting algebra $\boldsymbol{A}$ is induced by some filter F, in the sense that $\theta = \boldsymbol{\Omega}^{\boldsymbol{A}} F$. ⊠

Example 2.8 (Modal algebras)**.** A *modal algebra* is an algebra $\boldsymbol{A} = \langle A; \wedge, \vee, \neg, \Box, 0, 1 \rangle$ such that $\langle A; \wedge, \vee, \neg, 0, 1 \rangle$ is a Boolean algebra and $\Box$ is a unary operation such that

$$\Box(a \wedge c) = \Box a \wedge \Box c \ \text{ and } \ \Box 1 = 1,$$

for every $a, c \in A$. An *open filter* of a modal algebra $\boldsymbol{A}$ is a filter of the Boolean reduct of $\boldsymbol{A}$ that, moreover, is closed under the operation $\Box$. The poset of open filters of $\boldsymbol{A}$ ordered under the inclusion relation will be denoted by $\mathsf{Op}(\boldsymbol{A})$. It forms a complete lattice. Furthermore, the lattices $\mathsf{Op}(\boldsymbol{A})$ and $\mathsf{Con}(\boldsymbol{A})$ are isomorphic via the inverse isomorphisms described in Example 2.7. Because of this, every congruence of a modal algebra $\boldsymbol{A}$ has the form $\theta = \boldsymbol{\Omega}^{\boldsymbol{A}} F$, for some open filter F. ⊠

Example 2.9 (Groups)**.** Similarly, it is well known that the lattice of congruences of a group is isomorphic to that of its normal subgroups. Because of this, every congruence of a group is induced by some normal subgroup. ⊠

As we mentioned, there is a tight correspondence between the homomorphic images and the congruences of an algebra $\boldsymbol{A}$. On the one hand, every congruence θ of $\boldsymbol{A}$ gives rise to a homomorphic image $\boldsymbol{A}/\theta$ of $\boldsymbol{A}$. Let $\mathcal{F}$ be the set of function symbols of $\boldsymbol{A}$. Given

$\theta \in \mathsf{Con}(\boldsymbol{A})$ and a basic n-ary function symbol $f \in \mathcal{F}$, let $f^{\boldsymbol{A}/\theta}$ be the n-ary operation on A/θ defined by the rule

$$f^{\boldsymbol{A}/\theta}(a_1/\theta, \dots, a_n/\theta) \coloneqq f^{\boldsymbol{A}}(a_1, \dots, a_n)/\theta.$$

Notice that $f^{\boldsymbol{A}/\theta}$ is well-defined, by condition (1). As a consequence, the structure

$$\boldsymbol{A}/\theta \coloneqq \langle A/\theta; \{f^{\boldsymbol{A}/\theta} : f \in \mathcal{F}\}\rangle$$

is a well-defined algebra of the type as $\boldsymbol{A}$. Furthermore, $\boldsymbol{A}/\theta \in \mathbb{H}(\boldsymbol{A})$, because the map $\pi_\theta \colon A \to A/\theta$, defined, for every $a \in A$, as $\pi_\theta(a) \coloneqq a/\theta$, is a surjective homomorphism from $\boldsymbol{A}$ to $\boldsymbol{A}/\theta$. To prove this, consider $a_1, \dots, a_n \in A$. We have

$$\begin{aligned}
\pi_\theta(f^{\boldsymbol{A}}(a_1, \dots, a_n)) &= f^{\boldsymbol{A}}(a_1, \dots, a_n)/\theta \\
&= f^{\boldsymbol{A}/\theta}(a_1/\theta, \dots, a_n/\theta) \\
&= f^{\boldsymbol{A}/\theta}(\pi_\theta(a_1), \dots, \pi_\theta(a_n)),
\end{aligned}$$

where the second equality follows from the definition of the operation $f^{\boldsymbol{A}/\theta}$.

Corollary 2.10. *If θ is a congruence of an algebra $\boldsymbol{A}$, then $\boldsymbol{A}/\theta$ is a well-defined homomorphic image of $\boldsymbol{A}$.*

In view of the above result, every congruence θ of an algebra $\boldsymbol{A}$ induces a homomorphic image of $\boldsymbol{A}$, namely $\boldsymbol{A}/\theta$. The converse is also true, as we proceed to explain.

Definition 2.11. The *kernel* of a homomorphism $f \colon \boldsymbol{A} \to \boldsymbol{B}$ is the binary relation

$$\mathsf{Ker}(f) \coloneqq \{\langle a, c\rangle \in A \times A : f(a) = f(c)\}.$$

Proposition 2.12. *The kernel of a homomorphism $f \colon \boldsymbol{A} \to \boldsymbol{B}$ is a congruence of $\boldsymbol{A}$.*

Proof. It is obvious that $\mathsf{Ker}(f)$ is an equivalence relation on A. Therefore, to prove that $\mathsf{Ker}(f)$ is a congruence of $\boldsymbol{A}$, it suffices to show that it preserves the basic operations of $\boldsymbol{A}$. Consider a basic n-ary operation g of $\boldsymbol{A}$ and $a_1, \dots, a_n, c_1, \dots, c_n \in A$ such that $\langle a_1, c_1\rangle, \dots, \langle a_n, c_n\rangle \in \mathsf{Ker}(f)$. By the definition of $\mathsf{Ker}(f)$,

$$f(a_i) = f(c_i), \text{ for every } i \leqslant n.$$

It follows that $g^{\boldsymbol{B}}(f(a_1), \dots, f(a_n)) = g^{\boldsymbol{B}}(f(c_1), \dots, f(c_n))$. Since $f \colon \boldsymbol{A} \to \boldsymbol{B}$ is a homomorphism, this yields

$$f(g^{\boldsymbol{A}}(a_1, \dots, a_n)) = g^{\boldsymbol{B}}(f(a_1), \dots, f(a_n)) = g^{\boldsymbol{B}}(f(c_1), \dots, f(c_n)) = f(g^{\boldsymbol{A}}(c_1, \dots, c_n)).$$

Hence, we conclude that $\langle g^{\boldsymbol{A}}(a_1, \dots, a_n), g^{\boldsymbol{A}}(c_1, \dots, c_n)\rangle \in \mathsf{Ker}(f)$, as desired. ⊠

The behaviour of kernels is governed by the next principle.

Fundamental Homomorphism Theorem 2.13. *If $f \colon \boldsymbol{A} \to \boldsymbol{B}$ is a homomorphism with kernel θ, then there exists a unique embedding $g \colon \boldsymbol{A}/\theta \to \boldsymbol{B}$ such that $f = g \circ \pi_\theta$.*

Proof. We begin by proving the existence of g. Let $g \colon A/\theta \to B$ be the map defined as $g(a/\theta) \coloneqq f(a)$, for every $a \in A$. To show that g is well-defined, consider $a, c \in A$ such that $a/\theta = c/\theta$. Since $\theta = \mathsf{Ker}(f)$, this means that $f(a) = f(c)$, as desired. Furthermore, the definition of g guarantees that $f = g \circ \pi_\theta$.

Now, observe g is injective, because, for every $a, c \in A$ such that $g(a/\theta) = g(c/\theta)$, we have $f(a) = f(c)$, that is, $\langle a, c\rangle \in \mathsf{Ker}(f) = \theta$ and, therefore, $a/\theta = c/\theta$. Moreover, for every basic n-ary operation p of $\boldsymbol{A}$ and $a_1, \dots, a_n \in A$, we have

$$\begin{aligned} g(p^{\boldsymbol{A}/\theta}(a_1/\theta, \dots, a_n/\theta)) &= g(p^{\boldsymbol{A}}(a_1, \dots, a_n)/\theta) \\ &= f(p^{\boldsymbol{A}}(a_1, \dots, a_n)) \\ &= p^{\boldsymbol{B}}(f(a_1), \dots, f(a_n)) \\ &= p^{\boldsymbol{B}}(g(a_1/\theta), \dots, g(a_n/\theta)). \end{aligned}$$

The first equality above follows from the definition of $\boldsymbol{A}/\theta$, the second and the last from the definition of g, and the third from the assumption that $f\colon \boldsymbol{A} \to \boldsymbol{B}$ is a homomorphism. Hence, we conclude that $g\colon \boldsymbol{A}/\theta \to \boldsymbol{B}$ is a homomorphism and, therefore, an embedding, as desired.

The uniqueness of g follows from the fact that, if a map g^* satisfies the condition in the statement of the theorem, then, for every $a \in A$,

$$f(a) = g^* \circ \pi_\theta(a) = g^*(a/\theta),$$

that is, g^* coincides with g. ⊠

Corollary 2.14. *If $f\colon \boldsymbol{A} \to \boldsymbol{B}$ is a homomorphism, then $f[\boldsymbol{A}] \cong \boldsymbol{A}/\mathsf{Ker}(f)$. In particular, if f is surjective, $\boldsymbol{B} \cong \boldsymbol{A}/\mathsf{Ker}(f)$.*

Proof. In the proof of the Fundamental Homomorphism Theorem we showed that the map $g\colon A/\mathsf{Ker}(f) \to B$, defined by the rule $g(a/\mathsf{Ker}(f)) := f(a)$, is an embedding of $\boldsymbol{A}/\mathsf{Ker}(f)$ into $\boldsymbol{B}$. As g can be viewed as a surjective embedding of $\boldsymbol{A}/\mathsf{Ker}(f)$ into $f[\boldsymbol{A}]$, we conclude that $f[\boldsymbol{A}] \cong \boldsymbol{A}/\mathsf{Ker}(f)$. ⊠

At this stage, it should be clear that if θ is a congruence on an algebra $\boldsymbol{A}$, then $\pi_\theta\colon \boldsymbol{A} \to \boldsymbol{A}/\theta$ is a surjective homomorphism whose kernel is θ. Similarly, if $f\colon \boldsymbol{A} \to \boldsymbol{B}$ is a surjective homomorphism, then $\boldsymbol{A}/\mathsf{Ker}(f) \cong \boldsymbol{B}$, by Corollary 2.14. As a consequence, for every class of algebras K,

$$\mathbb{H}(\mathsf{K}) = \mathbb{I}\{\boldsymbol{A}/\theta : \boldsymbol{A} \in \mathsf{K} \text{ and } \theta \in \mathsf{Con}(\boldsymbol{A})\}. \tag{2}$$

Now, recall that the Cartesian product of a family of sets $\{A_i : i \in I\}$ is the set

$$\prod_{i\in I} A_i := \{f\colon I \to \bigcup_{i \in I} A_i \ :\ f(i) \in A_i, \text{for all } i \in I\}.$$

In particular, if I is empty, then $\prod_{i\in I} A_i$ is the singleton containing only the empty map.

Definition 2.15. The *direct product* of a family of similar algebras $\{\boldsymbol{A}_i : i \in I\}$ is the unique algebra of the common type whose universe is the Cartesian product $\prod_{i\in I} A_i$ and such that, for every basic n-ary operation symbol f and every $\vec{a}_1, \dots, \vec{a}_n \in \prod_{i\in I} A_i$,

$$f^{\prod_{i\in I}\boldsymbol{A}_i}(\vec{a}_1, \dots, \vec{a}_n)(i) = f^{\boldsymbol{A}_i}(\vec{a}_1(i), \dots, \vec{a}_n(i)), \text{ for every } i \in I.$$

We denote this algebra by $\prod_{i\in I} \boldsymbol{A}_i$.

In this case, for every $j \in I$, the projection map on the j-th component $p_j\colon \prod_{i\in I} A_i \to A_j$, defined by the rule $p_j(\vec{a}) := \vec{a}(j)$, is a surjective homomorphism from $\prod_{i\in I} \boldsymbol{A}_i$ to $\boldsymbol{A}_j$.

Given a class of similar algebras K, we set

$$\mathbb{P}(\mathsf{K}) := \{\boldsymbol{A} : \boldsymbol{A} \text{ is a direct product of a family } \{\boldsymbol{B}_i : i \in I\} \subseteq \mathsf{K}\}.$$

As usual, when $\mathsf{K} = \{\boldsymbol{A}\}$, we write $\mathbb{P}(\boldsymbol{A})$ as a shorthand for $\mathbb{P}(\{\boldsymbol{A}\})$.

Notice that up to isomorphism, there exists a unique one-element algebra of a given type. Because of this, one-element algebras are called *trivial*. Accordingly, when the set of indexes I is empty, the direct product $\prod_{i \in I} \boldsymbol{A}_i$ is the trivial algebra of the given type. It follows that $\mathbb{P}(\mathsf{K})$ contains always a trivial algebra, for every class of similar algebras K.

Example 2.16 (Powerset algebras)**.** Boolean algebras of the form $\langle \mathcal{P}(X); \cap, \cup, -, \emptyset, X \rangle$ are called *powerset Boolean algebras*. Let $\boldsymbol{B}$ be the two-element Boolean algebra and observe that $\mathbb{IP}(\boldsymbol{B})$ is the class of algebras isomorphic to some powerset Boolean algebra. To prove this, observe that every powerset Boolean algebra $\mathcal{P}(X)$ is isomorphic to a direct product of $\boldsymbol{B}$ via the *characteristic function* $f_X \colon \mathcal{P}(X) \to \prod_{x \in X} \boldsymbol{B}_x$, defined by the rule

$$f(Y)(x) := \begin{cases} 1 & \text{if } x \in Y \\ 0 & \text{if } x \notin Y, \end{cases}$$

where $Y \in \mathcal{P}(X)$ and $x \in X$. By the same token, every direct product $\prod_{i \in I} \boldsymbol{B}_i$ of $\boldsymbol{B}$ is isomorphic to the powerset Boolean algebra $\mathcal{P}(I)$ via the isomorphism f_I. ⊠

We close this section by reviewing the subdirect product construction.

Definition 2.17. A subalgebra $\boldsymbol{B}$ of a direct product $\prod_{i \in I} \boldsymbol{A}_i$ is said to be a *subdirect product* of $\{\boldsymbol{A}_i : i \in I\}$ if the projection map p_i is surjective, for every $i \in I$. Similarly, an embedding $f \colon \boldsymbol{B} \to \prod_{i \in I} \boldsymbol{A}_i$ is said to be *subdirect* when $f[\boldsymbol{B}]$ is a subdirect product of the family $\{\boldsymbol{A}_i : i \in I\}$.

Given a class of similar algebras K, we set

$$\mathbb{P}_{\text{SD}}(\mathsf{K}) := \{\boldsymbol{A} : \boldsymbol{A} \text{ is a subdirect direct product of a family } \{\boldsymbol{B}_i : i \in I\} \subseteq \mathsf{K}\}.$$

As usual, when $\mathsf{K} = \{\boldsymbol{A}\}$, we write $\mathbb{P}_{\text{SD}}(\boldsymbol{A})$ as a shorthand for $\mathbb{P}_{\text{SD}}(\{\boldsymbol{A}\})$. Clearly, $\mathbb{P}_{\text{SD}}(\mathsf{K}) \subseteq \mathbb{SP}(\mathsf{K})$. Furthermore, $\mathbb{P}_{\text{SD}}(\mathsf{K})$ contains always a trivial algebra.

Example 2.18 (Distributive lattices)**.** Let DL be the class of distributive lattices and $\boldsymbol{B}$ be the two-element distributive lattice. Birkhoff's Representation Theorem states that $\mathsf{DL} = \mathbb{IP}_{\text{SD}}(\boldsymbol{B})$. The inclusion $\mathbb{IP}_{\text{SD}}(\boldsymbol{B}) \subseteq \mathsf{DL}$ follows from the fact that DL is closed under $\mathbb{I}, \mathbb{S}$, and $\mathbb{P}$. For the other inclusion, consider a distributive lattice $\boldsymbol{A}$ and let I be the set of its prime filters. By Birkhoff's Representation Theorem, the map

$$\gamma \colon \boldsymbol{A} \to \prod_{F \in I} \boldsymbol{B}_F,$$

defined, for every $a \in A$ and $F \in I$, by the rule

$$\gamma(a)(F) := \begin{cases} 1 & \text{if } a \in F \\ 0 & \text{if } a \notin F, \end{cases}$$

is a well-defined subdirect embedding. ⊠

Example 2.19 (Boolean algebras)**.** Similarly, Stone's Representation Theorem states that the class of Boolean algebras coincides with $\mathbb{IP}_{\text{SD}}(\boldsymbol{B})$, where $\boldsymbol{B}$ the two-element Boolean algebra. ⊠

The next result provides a general recipe to construct subdirect products.

Proposition 2.20. *Let $\boldsymbol{A}$ be an algebra and $\{\theta_i : i \in I\} \subseteq \mathsf{Con}(\boldsymbol{A})$. Then the map*
$$f \colon \boldsymbol{A}/\bigcap_{i\in I}\theta_i \to \prod_{i\in I}\boldsymbol{A}/\theta_i,$$
defined, for every $a \in A$ and $j \in I$, as
$$f(a/\bigcap_{i\in I}\theta_i)(j) := a/\theta_j,$$
is a subdirect embedding.

Proof. For the sake of readability, set $\boldsymbol{B} := \boldsymbol{A}/\bigcap_{i\in I}\theta_i$. To prove that f is injective, consider $a, c \in A$ such that $\langle a, c\rangle \notin \bigcap_{i\in I}\theta_i$. Then there exists $j \in I$ such that $\langle a, c\rangle \notin \theta_j$ and, therefore,
$$f(a/\bigcap_{i\in I}\theta_i)(j) := a/\theta_j \neq c/\theta_j = f(c/\bigcap_{i\in I}\theta_i)(j).$$
It follows that $f(a/\bigcap_{i\in I}\theta_i) \neq f(c/\bigcap_{i\in I}\theta_i)$. Thus, f is injective. Moreover, by the definition of f, the composition $p_i \circ f \colon \boldsymbol{B} \to \boldsymbol{A}/\theta_i$ is surjective, for every $i \in I$.

It only remains to prove that f is a homomorphism. Consider an n-ary basic operation g and $a_1, \dots, a_n \in A$. For every $j \in I$, we have
$$\begin{aligned}
f(g^{\boldsymbol{B}}(a_1/\bigcap_{i\in I}\theta_i, \dots, a_n/\bigcap_{i\in I}\theta_i))(j) &= f(g^{\boldsymbol{A}}(a_1, \dots, a_n)/\bigcap_{i\in I}\theta_i)(j)\\
&= g^{\boldsymbol{A}}(a_1, \dots, a_n)/\theta_j\\
&= g^{\boldsymbol{A}/\theta_j}(a_1/\theta_j, \dots, a_n/\theta_j)\\
&= g^{\boldsymbol{A}/\theta_j}(f(a_1/\bigcap_{i\in I}\theta_i)(j), \dots, f(a_n/\bigcap_{i\in I}\theta_i)(j))\\
&= g^{\prod_{i\in I}\boldsymbol{A}/\theta_i}(f(a_1/\bigcap_{i\in I}\theta_i), \dots, f(a_n/\bigcap_{i\in I}\theta_i))(j).
\end{aligned}$$
It follows that
$$f(g^{\boldsymbol{B}}(a_1/\bigcap_{i\in I}\theta_i, \dots, a_n/\bigcap_{i\in I}\theta_i)) = g^{\prod_{i\in I}\boldsymbol{A}/\theta_i}(f(a_1/\bigcap_{i\in I}\theta_i), \dots, f(a_n/\bigcap_{i\in I}\theta_i)). \qquad \boxtimes$$

3. Propositional logics and equational completeness theorems

For general information on propositional logics we refer the reader to [9, 11, 12]. Recall that a *closure operator* on a set A is a map $C \colon \mathcal{P}(A) \to \mathcal{P}(A)$ such that, for every $X \subseteq Y \subseteq A$,
$$X \subseteq C(X) = C(C(X)) \text{ and } C(X) \subseteq C(Y).$$
Given a closure operator C on A, a subset $X \subseteq A$ is said to be *closed* if $X = C(X)$. A *closure system* on A is a family $\mathcal{C} \subseteq \mathcal{P}(A)$ that contains A and such that $\bigcap \mathcal{F}$, for every nonempty $\mathcal{F} \subseteq \mathcal{C}$. Closure operators and systems on A are two faces of the same coin. More precisely, if the family of closed sets of a closure operator on A is a closure system on A. On the other hand, if $\mathcal{C}$ is a closure system on A, then the map $C \colon \mathcal{P}(A) \to \mathcal{P}(A)$, defined by the rule
$$C(X) := \bigcap\{Y \in \mathcal{C} : X \subseteq Y\},$$
is a closure operator on A. These transformations between closure operators and systems on A are one inverse to the other.

Exercise 3.1. Prove that these transformations are well-defined and one inverse to the other. ⊠

Another way of presenting closure operators or systems is by means of the following concept.

Definition 3.2. A *consequence relation* on a set A is a binary relation $\vdash \subseteq \mathcal{P}(A) \times A$ such that, for every $X \cup Y \cup \{a\} \subseteq A$,

(i) if $a \in X$, then $X \vdash a$; and
(ii) if $X \vdash y$ for all $y \in Y$ and $Y \vdash a$, then $X \vdash a$.

Remark 3.3. The relation $X \vdash a$ should be read, intuitively, as "X proves a" or "a follows from X". In this reading, the demand expressed by condition (i) is rather natural, while (ii) is an abstract of the Cut rule. ⊠

Formally speaking, a consequence relation on a set A is a binary relation $\vdash \subseteq \mathcal{P}(A) \times A$. However, to simplify the notation, we will often write $a_1, \dots, a_n \vdash c$ as a shorthand for $\{a_1, \dots, a_n\} \vdash c$. Similarly, we will use $X, a \vdash c$ as a shorthand for $X \cup \{a\} \vdash c$. Lastly, for every set of formulas $X \cup Y \cup \{a, c\}$, we write

(i) $X \vdash Y$, when $X \vdash y$ for every $y \in Y$;
(ii) $a \dashv\vdash c$, when $a \vdash c$ and $c \vdash a$; and
(iii) $X \dashv\vdash Y$, when $X \vdash Y$ and $Y \vdash X$.

Definition 3.4. Let $\vdash$ be a consequence relation on a set A. A *theory* of $\vdash$ is a subset $X \subseteq A$ such that, for every $a \in A$, if $X \vdash a$, then $a \in X$. The set of theories of A will be denoted by $\mathcal{T}h(\vdash)$.

It is easy to see that $\mathcal{T}h(\vdash)$ is a closure system on A. Moreover, given a closure operator C on A, the following is a consequence relation on A:

$$\{\langle X, a\rangle \in \mathcal{P}(A) \times A : X \vdash a\}.$$

Together with the correspondence between closure systems and operators, these transformations induce a one-to-one correspondence between consequence relations, closure operators, and closure systems on A.

Exercise 3.5. Prove these facts. ⊠

In the context of logic, the term algebra $\boldsymbol{T}_\rho(Var)$ is often called the *algebra of formulas* (of type ρ) and its elements are referred to as *formulas*. An *endomorphism* of an algebra $\boldsymbol{A}$ is a homomorphism whose domain and codomain is $\boldsymbol{A}$. Endomorphisms of the algebra of formulas play a fundamental role in logic.

Definition 3.6. A *substitution* of type ρ is an endomorphism σ of $\boldsymbol{T}_\rho(Var)$.

When the type ρ is clear from the context, we will simply say that σ is a substitution.

In view of Proposition 1.3 and of the fact that Var is a set of generators for $\boldsymbol{T}_\rho(Var)$, every function $\sigma\colon Var \to T_\rho(Var)$ can be uniquely extended to a substitution σ^+ of type ρ, namely the function defined by the rule

$$\varphi(x_1, \dots, x_n) \longmapsto \varphi(\sigma(x_1), \dots, \sigma(x_n)).$$

Because of this, substitutions of type ρ can be presented by exhibiting functions $\sigma\colon Var \to T_\rho(Var)$.

Definition 3.7. A *logic* of type ρ is a consequence relation $\vdash$ on the set of formulas $T_\rho(Var)$ that, moreover, is *substitution invariant* in the sense that for every substitution σ of type ρ and every set of formulas $\Gamma \cup \{\varphi\} \subseteq T_\rho(Var)$,

$$\text{if } \Gamma \vdash \varphi, \text{ then } \sigma[\Gamma] \vdash \sigma(\varphi).$$

Remark 3.8. As mentioned above, $\Gamma \vdash \varphi$ should be read as "Γ proves φ" or "φ follows from Γ". The requirement that $\vdash$ is substitution invariant, instead, is intended to capture the idea that logical inferences are true only in virtue of their form (as opposed to their content). ⊠

Example 3.9 (Hilbert calculi)**.** We work within a fixed, but arbitrary, type ρ. A *rule* is an expression of the form $\Gamma \rhd \varphi$, where $\Gamma \cup \{\varphi\} \subseteq T_\rho(Var)$. In this case, Γ is said to be the set of *premises* of the rule and φ the *conclusion*. When $\Gamma = \emptyset$, the rule $\Gamma \rhd \varphi$ is sometimes called an *axiom*. A *Hilbert calculus* is a set of rules.

Every Hilbert calculus H induces a logic, as we proceed to explain. Consider a set of formulas $\Gamma \cup \{\varphi\} \subseteq T_\rho(Var)$. A *proof of* φ *from* Γ *in* H is a well-ordered sequence $\langle \psi_\alpha : \alpha \leqslant \gamma \rangle$ of formulas $\psi_\alpha \in T_\rho(Var)$ whose last element ψ_γ is φ and such that, for every $\alpha \leqslant \gamma$, either $\psi_\alpha \in \Gamma$ or there exist a substitution σ and a rule $\Delta \rhd \delta$ in H such that the formulas in $\sigma[\Delta]$ occur in the initial segment $\langle \psi_\beta : \beta < \alpha \rangle$ and $\psi_\alpha = \sigma(\delta)$.

The logic $\vdash_{\mathsf{H}}$ induced by H is defined, for every $\Gamma \cup \{\varphi\} \subseteq T_\rho(Var)$, as

$$\Gamma \vdash_{\mathsf{H}} \varphi \iff \text{there exists a proof of } \varphi \text{ from } \Gamma \text{ in } \mathsf{H}.$$

As expected, $\vdash_{\mathsf{H}}$ is a logic in the sense of Definition 3.7. Furthermore, it is the least logic $\vdash$ such that $\Gamma \vdash \varphi$, for every rule $\Gamma \rhd \varphi$ in H.

A logic $\vdash$ is said to be *axiomatized* by a Hilbert calculus H when it coincides with $\vdash_{\mathsf{H}}$. Notice that every logic $\vdash$ is vacuously axiomatized by the Hilbert calculus

$$\{\Gamma \rhd \varphi : \Gamma \vdash \varphi\}.$$

Because of this, axiomatizations in terms of Hilbert calculi H acquire special interest when H is finite or, at least, recursive. ⊠

When no confusion shall arise, given a sequence $\vec{a}$ and a set A, we write $\vec{a} \in A$ to indicate that the elements of the sequence $\vec{a}$ belong to A. The following concept is instrumental to exhibit further examples of logics.

Definition 3.10. Let K be a class of similar algebras We define a binary relation $\vDash_{\mathsf{K}} \subseteq \mathcal{P}(E_\rho(Var)) \times E_\rho(Var)$ as follows:

$$\begin{aligned}\Theta \vDash_{\mathsf{K}} \varepsilon \approx \delta \iff & \text{ for every } \boldsymbol{A} \in \mathsf{K} \text{ and every } \vec{a} \in A,\\ & \text{ if } \varphi^{\boldsymbol{A}}(\vec{a}) = \psi^{\boldsymbol{A}}(\vec{a}) \text{ for all } \varphi \approx \psi \in \Theta, \text{ then } \varepsilon^{\boldsymbol{A}}(\vec{a}) = \delta^{\boldsymbol{A}}(\vec{a}).\end{aligned}$$

The relation $\vDash_{\mathsf{K}}$ is known as the *equational consequence relative to* K.

Example 3.11 (Equationally defined logics)**.** We work within a fixed, but arbitrary, type ρ. Given a set of equations $\boldsymbol{\tau}(x)$ in a single variable x and a set of formulas $\Gamma \cup \{\varphi\} \subseteq T(Var)$, we abbreviate

$$\{\varepsilon(\varphi) \approx \delta(\varphi) \colon \varepsilon \approx \delta \in \boldsymbol{\tau}\} \text{ as } \boldsymbol{\tau}(\varphi), \text{ and } \bigcup_{\gamma \in \Gamma} \boldsymbol{\tau}(\gamma) \text{ as } \boldsymbol{\tau}[\Gamma].$$

Given a class of algebras K and a set of equations $\boldsymbol{\tau}(x)$, we define a logic $\vdash_{\mathsf{K},\boldsymbol{\tau}}$ as follows: for every $\Gamma \cup \{\varphi\} \subseteq T(Var)$,

$$\Gamma \vdash_{\mathsf{K},\boldsymbol{\tau}} \varphi \Longleftrightarrow \boldsymbol{\tau}[\Gamma] \vDash_{\mathsf{K}} \boldsymbol{\tau}(\varphi).$$

It is easy to prove that $\vdash_{\mathsf{K},\boldsymbol{\tau}}$ is indeed a logic in the sense of Definition 3.7. Notice that, in this case, $\vdash$ is related to K by a *completeness theorem* witnessed by the set of equations $\boldsymbol{\tau}(x)$ that allows to translate formulas into equations and, therefore, to interpret $\vdash_{\mathsf{K},\boldsymbol{\tau}}$ into $\vDash_{\mathsf{K}}$.

For instance, the completeness theorem of classical propositional logic **CPC** with respect to the class of Boolean algebras BA states precisely that **CPC** coincides with $\vdash_{\mathsf{BA},\boldsymbol{\tau}}$ where $\boldsymbol{\tau} = \{x \approx 1\}$. Similarly, the completeness theorem of intuitionistic propositional logic **IPC** with respect to the class of Heyting algebras HA states precisely that **IPC** coincides with $\vdash_{\mathsf{HA},\boldsymbol{\tau}}$ where $\boldsymbol{\tau} = \{x \approx 1\}$. Because of this, **CPC** and **IPC** can be defined as follows: for every set of formulas $\Gamma \cup \{\varphi\}$ of the appropriate type,

$$\Gamma \vdash_{\mathbf{CPC}} \varphi \Longleftrightarrow \boldsymbol{\tau}[\Gamma] \vDash_{\mathsf{BA}} \boldsymbol{\tau}(\varphi)$$
$$\Gamma \vdash_{\mathbf{IPC}} \varphi \Longleftrightarrow \boldsymbol{\tau}[\Gamma] \vDash_{\mathsf{HA}} \boldsymbol{\tau}(\varphi),$$

where $\boldsymbol{\tau} = \{x \approx 1\}$. ⊠

The relation between logic and algebra is often explained in terms of the existence of equational completeness theorems. The following definition makes this concept precise. As we will see, however, equational completeness theorems alone are not sufficient to account for the relation between logic and algebra.

Definition 3.12. A logic $\vdash$ is said to admit an *equational completeness theorem* if there are a set of equations $\boldsymbol{\tau}(x)$ and a class K of algebras such that for all $\Gamma \cup \{\varphi\} \subseteq T(Var)$,

$$\Gamma \vdash \varphi \Longleftrightarrow \boldsymbol{\tau}[\Gamma] \vDash_{\mathsf{K}} \boldsymbol{\tau}(\varphi).$$

In this case K is said to be a $\boldsymbol{\tau}$-*algebraic semantics* (or simply an *algebraic semantics*) for $\vdash$.

This notion was introduced in [4] and studied in depth in [5, 19]. For instance, the classes of Boolean and Heyting algebras are, respectively, $\boldsymbol{\tau}$-algebraic semantics for **CPC** and **IPC** where $\boldsymbol{\tau} = \{x \approx 1\}$.

Another familiar example of equational completeness theorem arises from the field of modal logic. Let Fr be the class of all Kripke frames. We can associate two distinct logics with Fr, see for instance [16, 17]. The *global consequence* $\mathbf{K}_g$ of the modal system $\mathbf{K}$ is the logic defined, for every set of modal formulas $\Gamma \cup \{\varphi\}$, as follows:

$$\begin{aligned}\Gamma \vdash_{\mathbf{K}_g} \varphi \Longleftrightarrow\ & \text{for every } \langle W, R\rangle \in \mathsf{Fr} \text{ and evaluation } v \text{ in } \langle W, R\rangle,\\ & \text{if } w, v \Vdash \Gamma \text{ for all } w \in W, \text{ then } w, v \Vdash \varphi \text{ for all } w \in W.\end{aligned}$$

On the other hand, the *local consequence* $\mathbf{K}_\ell$ of the modal system $\mathbf{K}$ is defined, for every set of modal formulas $\Gamma \cup \{\varphi\}$, as follows:

$$\begin{aligned}\Gamma \vdash_{\mathbf{K}_\ell} \varphi \Longleftrightarrow\ & \text{for every } \langle W, R\rangle \in \mathsf{Fr}, w \in W, \text{ and evaluation } v \text{ in } \langle W, R\rangle,\\ & \text{if } w, v \Vdash \Gamma, \text{ then } w, v \Vdash \varphi.\end{aligned}$$

It is easy to see that $\mathbf{K}_g$ and $\mathbf{K}_\ell$ are logics. Moreover, they are distinct, because

$$x \vdash_{\mathbf{K}_g} \Box x \text{ and } x \nvdash_{\mathbf{K}_\ell} \Box x. \tag{3}$$

Exercise 3.13. Prove that $\mathbf{K}_g$ and $\mathbf{K}_\ell$ are logics. Notice also that the modal system $\mathbf{K}$ is not a logic itself, because it is not a consequence relation. Indeed, there are two ways to turn $\mathbf{K}$ into a logic, namely, $\mathbf{K}_g$ and $\mathbf{K}_\ell$. ⊠

Exercise 3.14. Prove that $\mathbf{K}_g$ and $\mathbf{K}_\ell$ have the same *theorems*, i.e., formulas provable from the empty set. Prove also that the set of their theorems is the modal system $\mathbf{K}$. This indicates that, even in the modal setting, logics should not be identified with their sets of theorems. ⊠

The global consequence $\mathbf{K}_g$ is related to the class MA of modal algebras by the following equational completeness theorem.

Theorem 3.15. *For every set $\Gamma \cup \{\varphi\}$ of modal formulas,*

$$\Gamma \vdash_{\mathbf{K}_g} \varphi \iff \boldsymbol{\tau}[\Gamma] \vDash_{\mathsf{MA}} \boldsymbol{\tau}(\varphi),$$

where $\boldsymbol{\tau} = \{x \approx 1\}$. Consequently, the class of modal algebras is a $\boldsymbol{\tau}$-algebraic semantics for $\mathbf{K}_g$.

In order to prove it, recall that a filter on a Boolean algebra $\boldsymbol{A}$ is said to be *proper* when it differs from A. Moreover, a proper filter U of $\boldsymbol{A}$ is said to be a *ultrafilter* of $\boldsymbol{A}$ if it is maximal among the proper filters of $\boldsymbol{A}$ or, equivalently, if

$$a \in U \text{ or } \neg a \in U, \text{ for every } a \in A.$$

While the following result holds in ZFC, it cannot be proved in ZF (although it is strictly weaker then the axiom of choice).

Ultrafilter Lemma 3.16. *Every proper filter on a Boolean algebra can be extended to a ultrafilter.*

We are now ready to prove Theorem 3.15.

Proof sketch. It suffices to prove that

$$\Gamma \nvdash_{\mathbf{K}_g} \varphi \iff \boldsymbol{\tau}[\Gamma] \nvDash_{\mathsf{MA}} \boldsymbol{\tau}(\varphi).$$

Suppose first that $\Gamma \nvdash_{\mathbf{K}_g} \varphi$. Then there are a Kripke frame $\langle W, R\rangle$, an evaluation v in it and a world u such that

$$w, v \Vdash \Gamma \text{ for all } w \in W \ \text{ and } \ u, v \nVdash \varphi. \tag{4}$$

Then consider the complex algebra of $\langle W, R\rangle$, that is, the structure

$$\boldsymbol{A} := \langle \mathcal{P}(W); \cap, \cup, -, \Box, \emptyset, W\rangle,$$

where $-$ is set theoretic complement and, for every $V \subseteq W$,

$$\Box V := \{w \in W : \text{if } \langle w, t\rangle \in R, \text{ then } t \in V\}.$$

It is easy to prove that $\boldsymbol{A}$ is a modal algebra. Then consider the unique homomorphism $f\colon \boldsymbol{T}(Var) \to \boldsymbol{A}$ such that

$$f(x) = \{w \in W : w, v \Vdash x\},$$

for every $x \in Var$. A simple induction of the construction of terms shows that, for every formula ψ,

$$f(\psi) = \{w \in W : w, v \Vdash \psi\}.$$

Together with (4), this yields

$$f[\Gamma] \subseteq \{W\} \ \text{ and } \ f(\varphi) \neq W.$$

Hence, we conclude that $\boldsymbol{\tau}[\Gamma] \nvDash_{\mathsf{MA}} \boldsymbol{\tau}(\varphi)$.

To prove the converse, suppose that $\boldsymbol{\tau}[\Gamma] \nvDash_{\mathsf{MA}} \boldsymbol{\tau}(\varphi)$. Then there are a modal algebra $\boldsymbol{A}$ and a homomorphism $f\colon \boldsymbol{T}(Var) \to \boldsymbol{A}$ such that

$$f[\Gamma] \subseteq \{1\} \text{ and } f(\varphi) \neq 1.$$

Then consider the Kripke frame dual to $\boldsymbol{A}$, that is, the structure $\langle W, R\rangle$, where W is the set of ultrafilters of $\boldsymbol{A}$ and R the binary relation on W defined as follows:

$$R := \{\langle U, V\rangle \in W \times W : \{a \in A : \Box a \in U\} \subseteq V\}.$$

Let then $v\colon Var \to \mathcal{P}(W)$ be the evaluation in $\langle W, R\rangle$ defined by the rule

$$v(x) := \{U \in W : f(x) \in U\}.$$

An easy induction on the construction of terms shows that, for every formula ψ,

$$f(\psi) = \{U \in W : U, v \Vdash \psi\}. \tag{5}$$

Now, since $f(\varphi) \neq 1$, the Ultrafilter Lemma guarantees the existence of an ultrafilter F such that $f(\varphi) \notin F$. Furthermore, as every ultrafilter contain 1, from $f[\Gamma] \subseteq \{1\}$ it follows that $f[\Gamma] \subseteq U$, for all $U \in W$. In short,

$$f[\Gamma] \subseteq U \text{ for all } U \in W \text{ and } f(\varphi) \notin F.$$

Together with (5), this yields

$$U, v \Vdash \Gamma \text{ for all } U \in W \text{ and } F, v \nVdash \varphi.$$

Hence, we conclude that $\Gamma \nvdash_{\mathbf{K}_g} \varphi$. ⊠

At this stage, it is tempting to conjecture that the relation between logic and algebra can be explained in terms of equational completeness theorems only. As we anticipated, however, this is not the case. For instance, the relation between **CPC** and BA cannot be explained in terms of completeness theorems only, because the class of Heyting algebras HA is also an algebraic semantics for **CPC**. To explain why, it is convenient to recall the following classical result relating **CPC** and **IPC**.

Theorem 3.17 (Glivenko [13])**.** *For every set of formulas $\Gamma \cup \{\varphi\}$,*

$$\Gamma \vdash_{\mathbf{CPC}} \varphi \iff \{\neg\neg\gamma \colon \gamma \in \Gamma\} \vdash_{\mathbf{IPC}} \neg\neg\varphi.$$

As a consequence, we obtain the desired result.

Corollary 3.18. *The class of Heyting algebras is an algebraic semantics for* **CPC**.

Proof. For every set of formulas $\Gamma \cup \{\varphi\}$, we have

$$\begin{aligned}\Gamma \vdash_{\mathbf{CPC}} \varphi &\iff \{\neg\neg\gamma \colon \gamma \in \Gamma\} \vdash_{\mathbf{IPC}} \neg\neg\varphi\\ &\iff \{\neg\neg\gamma \approx 1 \colon \gamma \in \Gamma\} \vDash_{\mathsf{HA}} \neg\neg\varphi \approx 1.\end{aligned}$$

The first equivalent above is Glivenko's Theorem, while the second is a consequence of the completeness theorem of **IPC** with respect to HA. As a consequence, the class of Heyting algebras is a $\boldsymbol{\tau}$-algebraic semantics for **IPC**, where $\boldsymbol{\tau} = \{\neg\neg x \approx 1\}$. ⊠

This means that the univocal relation between **CPC** and the class of Boolean algebras cannot be explained in terms of the existence of completeness theorems only. As we shall see, this relation arises from a deeper phenomenon, known as *algebraizability* [4, 11].

Exercise 3.19. One may wonder whether the fact that **CPC** has many distinct algebraic semantics cannot be amended by restricting our attention to $\boldsymbol{\tau}$-algebraic semantics where $\boldsymbol{\tau} = \{x \approx 1\}$. This is not the case, as this exercise asks you to check. Let $\boldsymbol{A}$ be the three-element algebra $\langle \{0, 1, a\}; \wedge, \vee, \neg, 0, 1\rangle$ where $\langle A; \wedge, \vee\rangle$ is the lattice with order $0 < a < 1$ and $\neg \colon A \to A$ is the map described by the rule

$$\neg 0 = \neg a = 1 \text{ and } \neg 1 = 0.$$

Clearly, $\boldsymbol{A}$ is not a Boolean algebra (as there is no three-element Boolean algebra). Prove that $\{\boldsymbol{A}\}$ is $\boldsymbol{\tau}$-algebraic semantics for **CPC** where $\boldsymbol{\tau} = \{x \approx 1\}$. Hint: use the fact that the two-element Boolean algebra is a homomorphic image of $\boldsymbol{A}$. ⊠

Indeed the existence of equational completeness theorems between a logic and a class of algebras turns out to be a very weak relation, as shown in [5, 19]. For instance, while many interesting logics lack a natural equational completeness theorem, they still admit a nonstandard one. This is the case of $\mathbf{K}_\ell$, as we proceed to explain.

A logic $\vdash$ is said to be *protoalgebraic* if there exists a set $\Delta(x, y)$ of formulas such that $\emptyset \vdash \Delta(x, x)$ and $x, \Delta(x, y) \vdash y$. Notice that all logics $\vdash$ with a binary connective $\to$ such that $\emptyset \vdash x \to x$ and $x, x \to y \vdash y$ are protoalgebraic, as witnessed by the set $\Delta := \{x \to y\}$. Furthermore, a logic $\vdash$ is said to be *nontrivial* if $x \nvdash y$.

Theorem 3.20 (M. [19, Thm. 9.3])**.** *A nontrivial protoalgebraic logic. $\vdash$ has an algebraic semantics if and only it there are two distinct formulas φ and ψ that are logically equivalent in the sense that*

$$\delta(\varphi, \vec{z}) \dashv\vdash \delta(\psi, \vec{z}), \textit{ for all } \delta(x, \vec{z}) \in T(Var).$$

As a consequence, we obtain the following.

Corollary 3.21. *The logic $\mathbf{K}_\ell$ has an algebraic semantics.*

Proof. Clearly, $\mathbf{K}_\ell$ is nontrivial and protoalgebraic. Furthermore, the formulas x and $x \wedge x$ are distinct, but logical equivalent in $\mathbf{K}_\ell$. Therefore, $\mathbf{K}_\ell$ has an algebraic semantics in view of Theorem 3.20. ⊠

On the other hand, $\mathbf{K}_\ell$ lacks any natural equational completeness theorem.

Theorem 3.22 (M. [19, Cor. 9.7])**.** *No class of modal algebras is an algebraic semantics for $\mathbf{K}_\ell$.*

Proof. We begin by proving that, for all $\varphi, \psi \in T(Var)$,

$$\mathsf{MA} \vDash \varphi \approx \psi \iff \varphi \dashv\vdash_{\mathbf{K}_\ell} \psi. \tag{6}$$

To this end, observe that

$$\begin{aligned} \mathsf{MA} \vDash \varphi \approx \psi &\iff \mathsf{MA} \vDash \varphi \leftrightarrow \psi \approx 1 \\ &\iff \emptyset \vdash_{\mathbf{K}_g} \varphi \leftrightarrow \psi \\ &\iff \emptyset \vdash_{\mathbf{K}_\ell} \varphi \leftrightarrow \psi \\ &\iff \varphi \dashv\vdash_{\mathbf{K}_\ell} \psi. \end{aligned}$$

The above equivalence are justified as follows. The first is an easy property of Boolean algebras, the second is a consequence of Theorem 3.15, the third holds because $\mathbf{K}_g$ and $\mathbf{K}_\ell$ have the same theorems (see Exercise 3.14) and the last one because $\mathbf{K}_\ell$ has a standard deduction theorem.

Now, suppose, with a view to contradiction, that $\mathbf{K}_\ell$ has a $\boldsymbol{\tau}$-algebraic semantics $\mathsf{K} \subseteq \mathsf{MA}$. This implies that there exists an equation $\varepsilon \approx \delta \in \boldsymbol{\tau}$ such that $\mathsf{MA} \nvDash \varepsilon \approx \delta$. Thus, in view of the above display, we can assume, by symmetry, that $\varepsilon \nvdash_{\mathbf{K}_\ell} \delta$. This means that there are a Kripke frame $\mathbb{X} = \langle X, R \rangle$, an element $w \in X$, and a valuation v in $\mathbb{X}$ such that $w, v \Vdash \varepsilon$ and $w, v \nVdash \delta$.

Let $\mathbb{X}^+ = \langle X^+; R^+ \rangle$ be the Kripke frame obtained by adding a new point w^+ to $\mathbb{X}$ and defining the relation R^+ as follows:

$$\langle p, q \rangle \in R^+ \iff p = w^+ \text{ or } \langle p, q \rangle \in R.$$

Let also v^+ be the unique evaluation in $\mathbb{X}^+$ such that for every $y \in Var$ and $q \in \mathbb{X}^+$:

$$q, v^+ \Vdash y \iff \text{ either } (q \in X \text{ and } q, v \Vdash y) \text{ or } q = w^+.$$

From the definition of $\mathbb{X}^+$ and v^+ it follows that

$$q, v^+ \Vdash \varphi \iff q, v \Vdash \varphi$$

for all $\varphi \in T(Var)$ and $q \in \mathbb{X}$. Consequently, as $w, v \Vdash \varepsilon$ and $w, v \nVdash \delta$,

$$w^+, v^+ \Vdash x \text{ and } w^+, v^+ \nVdash \Box(\varepsilon \to \delta).$$

This implies

$$x \nvdash_{\mathbf{K}_\ell} \Box(\varepsilon \to \delta).$$

On the other hand, clearly $\emptyset \vdash_{\mathbf{K}_\ell} \Box(\delta \to \delta)$. Consequently,

$$x, \Box(\delta \to \delta) \nvdash_{\mathbf{K}_\ell} \Box(\varepsilon \to \delta). \tag{7}$$

Now, observe that, for every $\varphi, \psi \in T(Var)$,

$$\varepsilon(x) \approx \delta(x), \varphi(\Box(\delta \to \delta)) \approx \psi(\Box(\delta \to \delta)) \vDash_{\mathsf{K}} \varphi(\Box(\varepsilon \to \delta)) \approx \psi(\Box(\varepsilon \to \delta)).$$

Since $\varepsilon \approx \delta \in \boldsymbol{\tau}(x)$, this implies

$$\boldsymbol{\tau}(x), \boldsymbol{\tau}(\Box(\delta \to \delta)) \vDash_{\mathsf{K}} \boldsymbol{\tau}(\Box(\varepsilon \to \delta)).$$

Since K is a $\boldsymbol{\tau}$-algebraic semantics for $\mathbf{K}_\ell$, this yields $x, \Box(\delta \to \delta) \vdash_{\mathbf{K}_\ell} \Box(\varepsilon \to \delta)$, a contradiction with (7). ⊠

Exercise 3.23. Prove that $\mathbf{K}_\ell$ has a standard deduction theorem, i.e., that for every set of formulas $\Gamma \cup \{\psi, \varphi\}$,

$$\Gamma, \psi \vdash_{\mathbf{K}_\ell} \varphi \iff \Gamma \vdash_{\mathbf{K}_\ell} \psi \to \varphi.$$

Prove that this is not the case for $\mathbf{K}_g$. ⊠

Exercise 3.24. Prove that no class of distributive lattices is an algebraic semantics for the $\langle \wedge, \vee \rangle$-fragment $\mathbf{CPC}_{\wedge\vee}$ of $\mathbf{CPC}$. Hint: use the fact that every equation in a single variable holds in the class of distributive lattices.

Furthermore, prove that $\mathbf{CPC}_{\wedge\vee}$ has a nonstandard algebraic semantics. To this end, consider the three-element algebra $\boldsymbol{A} = \langle \{0^+, 0^-, 1\}; \wedge, \vee \rangle$ whose binary commutative operations are defined by the following tables

$\wedge$	0^-	0^+	1
0^-	0^+	0^+	0^+
0^+		0^-	0^+
1			1

$\vee$	0^-	0^+	1
0^-	0^+	0^+	1
0^+		0^-	1
1			1

and prove that $\{\boldsymbol{A}\}$ is a $\boldsymbol{\tau}$-algebraic semantics for $\boldsymbol{\tau} = \{x \approx x \wedge x\}$. Conclude that $\mathbf{CPC}_{\wedge\vee}$ is another example of logic that admits a nonstandard algebraic semantics, but lacks a standard one. ⊠

4. Ultraproducts and the finite embeddability property

In order to explain the relation between logic and algebra, we need to take a short detour in universal algebra and the theory of quasi-varieties. We begin by reviewing a product-like construction known as *ultraproduct* [2, 8]. First, recall that ultrafilters on powerset Boolean algebras $\mathcal{P}(X)$ are also called *ultrafilters on* X. Then let $\{\boldsymbol{A}_i : i \in I\}$ be a family of similar algebras. The *equalizer* of a pair of elements $\vec{a}, \vec{c} \in \prod_{i \in I} A_i$ is the set of indexes on which the sequences $\vec{a}$ and $\vec{c}$ agree, that is,

$$\llbracket \vec{a} = \vec{c} \rrbracket := \{i \in I : \vec{a}(i) = \vec{c}(i)\}.$$

Moreover, given an ultrafilter U on the index set I, let θ_U be the binary relation on the Cartesian product $\prod_{i \in I} A_i$ defined as

$$\theta_U := \{\langle \vec{a}, \vec{c} \rangle : \llbracket \vec{a} = \vec{c} \rrbracket \in U\}.$$

Proposition 4.1. *If $\{\boldsymbol{A}_i : i \in I\}$ is a family of similar algebras and U an ultrafilter on I, then θ_U is a congruence of $\prod_{i \in I} \boldsymbol{A}_i$.*

Proof. We begin by proving that θ_U is an equivalence relation on $\prod_{i \in I} A_i$. To this end, consider $\vec{a}, \vec{b}, \vec{c} \in \prod_{i \in I} A_i$. We have

$$\llbracket \vec{a} = \vec{a} \rrbracket = \{i \in I : \vec{a}(i) = \vec{a}(i)\} = I.$$

Observe that $I \in U$, since U is a nonempty upset of $\mathcal{P}(I)$. Together with the above display, this yields $\llbracket \vec{a} = \vec{a} \rrbracket \in U$ and, therefore, $\langle \vec{a}, \vec{a} \rangle \in \theta_U$. It follows that θ_U is reflexive. To prove that it is symmetric, suppose that $\langle \vec{a}, \vec{c} \rangle \in \theta_U$. Then $\llbracket \vec{a} = \vec{c} \rrbracket \in U$. Since $\llbracket \vec{a} = \vec{c} \rrbracket = \llbracket \vec{c} = \vec{a} \rrbracket$, this implies $\llbracket \vec{c} = \vec{a} \rrbracket \in U$ and, therefore, $\langle \vec{c}, \vec{a} \rangle \in \theta_U$. Lastly, to prove that θ_U is transitive, suppose that $\langle \vec{a}, \vec{b} \rangle, \langle \vec{b}, \vec{c} \rangle \in \theta_U$, that is, $\llbracket \vec{a} = \vec{b} \rrbracket, \llbracket \vec{b} = \vec{c} \rrbracket \in U$. Since U is closed under binary meets,

$$\llbracket \vec{a} = \vec{b} \rrbracket \cap \llbracket \vec{b} = \vec{c} \rrbracket \in U$$

Clearly, $\llbracket \vec{a} = \vec{b} \rrbracket \cap \llbracket \vec{b} = \vec{c} \rrbracket \subseteq \llbracket \vec{a} = \vec{c} \rrbracket$. Since U is an upset of $\mathcal{P}(I)$, we obtain that $\llbracket \vec{a} = \vec{c} \rrbracket \in U$, whence $\langle \vec{a}, \vec{c} \rangle \in \theta_U$. We conclude that θ_U is an equivalence relation.

To prove that θ_U is a congruence, it only remains to show that it preserves the basic operations. Accordingly, let f be a basic n-ary operation and $\vec{a}_1, \dots, \vec{a}_n, \vec{c}_1, \dots, \vec{c}_n \in \prod_{i \in I} A_i$ such that

$$\langle \vec{a}_1, \vec{c}_1 \rangle, \dots, \langle \vec{a}_n, \vec{c}_n \rangle \in \theta_U.$$

By definition of θ_U, this amounts to $\llbracket \vec{a}_1 = \vec{c}_1 \rrbracket, \dots, \llbracket \vec{a}_n = \vec{c}_n \rrbracket \in U$. Since U is a filter, it is closed under finite meets, whence

$$\llbracket \vec{a}_1 = \vec{c}_1 \rrbracket \cap \dots \cap \llbracket \vec{a}_n = \vec{c}_n \rrbracket \in U. \tag{8}$$

We will show that

$$\llbracket \vec{a}_1 = \vec{c}_1 \rrbracket \cap \dots \cap \llbracket \vec{a}_n = \vec{c}_n \rrbracket \subseteq \llbracket f^{\prod_{i \in I} \boldsymbol{A}_i}(\vec{a}_1, \dots, \vec{a}_n) = f^{\prod_{i \in I} \boldsymbol{A}_i}(\vec{c}_1, \dots, \vec{c}_n) \rrbracket. \tag{9}$$

To this end, consider $j \in \llbracket \vec{a}_1 = \vec{c}_1 \rrbracket \cap \dots \cap \llbracket \vec{a}_n = \vec{c}_n \rrbracket$. We have

$$\vec{a}_1(j) = \vec{c}_1(j), \dots, \vec{a}_n(j) = \vec{c}_n(j).$$

Consequently,

$$\begin{aligned} f^{\prod_{i\in I} \boldsymbol{A}_i}(\vec{a}_1,\dots,\vec{a})(j) &= f^{\boldsymbol{A}_j}(\vec{a}_1(j),\dots,\vec{a}_n(j)) \\ &= f^{\boldsymbol{A}_j}(\vec{c}_1(j),\dots,\vec{c}_n(j)) \\ &= f^{\prod_{i\in I} \boldsymbol{A}_i}(\vec{c}_1,\dots,\vec{c})(j), \end{aligned}$$

that is, $j \in [\![f^{\prod_{i\in I} \boldsymbol{A}_i}(\vec{a}_1,\dots,\vec{a}_n) = f^{\prod_{i\in I} \boldsymbol{A}_i}(\vec{c}_1,\dots,\vec{c}_n)]\!]$. This establishes (9). Since U is an upset of $\mathcal{P}(I)$, from (8) and (9) it follows

$$[\![f^{\prod_{i\in I} \boldsymbol{A}_i}(\vec{a}_1,\dots,\vec{a}_n) = f^{\prod_{i\in I} \boldsymbol{A}_i}(\vec{c}_1,\dots,\vec{c}_n)]\!] \in U.$$

Hence, we conclude that $\langle f^{\prod_{i\in I} \boldsymbol{A}_i}(\vec{a}_1,\dots,\vec{a}_n), f^{\prod_{i\in I} \boldsymbol{A}_i}(\vec{c}_1,\dots,\vec{c}_n)\rangle \in \theta_U$, as desired. ⊠

In view of the above result, we can make the following definition.

Definition 4.2. An *ultraproduct* of a family of similar algebras $\{\boldsymbol{A}_i : i \in I\}$ is an algebra of the form $\prod_{i\in I} \boldsymbol{A}_i/\theta_U$, for some ultrafilter U on I.

Given a class of similar algebras K, we set

$$\mathbb{P}_{\mathrm{U}}(\mathsf{K}) := \{\boldsymbol{A} : \boldsymbol{A} \text{ is an ultraproduct of a family } \{\boldsymbol{B}_i : i \in I\} \subseteq \mathsf{K}\}.$$

Notice that $\mathbb{P}_{\mathrm{U}}(\mathsf{K}) \subseteq \mathbb{HP}(\mathsf{K})$. Furthermore, as usual, when $\mathsf{K} = \{\boldsymbol{A}\}$, we write $\mathbb{P}_{\mathrm{U}}(\boldsymbol{A})$ as a shorthand for $\mathbb{P}_{\mathrm{U}}(\{\boldsymbol{A}\})$.

Exercise 4.3. Prove that if U is not free (that is, it is principal), then $\prod_{i\in I} \boldsymbol{A}_i/\theta_U$ is isomorphic to some $\boldsymbol{A}_i$. Conclude that if I is finite, then $\prod_{i\in I} \boldsymbol{A}_i/\theta_U$ belongs to $\mathbb{I}\{\boldsymbol{A}_i : i \in I\}$. Because of this, interesting ultraproducts arise from free ultrafilters only. ⊠

Exercise 4.4. Prove that K is a finite set of finite algebras, $\mathbb{P}(\mathsf{K}) \subseteq \mathbb{I}(\mathsf{K})$. ⊠

The importance of ultraproducts is tightly related to the following fundamental result.

Łoś' Theorem 4.5. *Let $\{\boldsymbol{A}_i : i \in I\}$ be a family of similar algebras, U an ultrafilter on I, and $\phi(x_1,\dots,x_n)$ a first-order formula. For every $\vec{a}_1,\dots,\vec{a}_n \in \prod_{i\in I} A_i$,*

$$\prod_{i\in I} \boldsymbol{A}_i/\theta_U \vDash \phi(\vec{a}_1/\theta_U,\dots,\vec{a}_n/\theta_U) \Longleftrightarrow \{i \in I : \boldsymbol{A}_i \vDash \phi(\vec{a}_1(i),\dots,\vec{a}_n(i))\} \in U.$$

Corollary 4.6. *Let $\{\boldsymbol{A}_i : i \in I\}$ be a family of similar algebras, U an ultrafilter on I, and ϕ a first-order sentence. If ϕ is valid in all the $\boldsymbol{A}_i$, then it is valid in $\prod_{i\in I} \boldsymbol{A}_i/\theta_U$.*

In view Łos' Theorem, ultraproducts are instrumental to construct nonstandard models of first-order theories. For instance, let $\mathbb{N} = \langle \mathbb{N}; s, +, \cdot, 0\rangle$ be the standard model of Peano Arithmetic. If U is an ultrafilter on $\mathbb{N}$, the ultraproduct $\prod_{n\in\mathbb{N}} \mathbb{N}_n/U$ is *elementarily equivalent* to $\mathbb{N}$, that is, it satisfies the same first-order sentences as $\mathbb{N}$. On the other hand, it is not hard to see that if U is free, $\prod_{n\in\mathbb{N}} \mathbb{N}_n/U$ is uncountable and, therefore, contains many "infinite" (or nonstandard) natural numbers.

For the present purpose, however, we will not need the full strength of Łoś' Theorem and, therefore, we shall omit its proof. Instead, we shall focus on a particular embedding theorem for ultraproducts that depends on the following notion.

Definition 4.7. A *local subgraph* $\mathbb{X}$ of an algebra $\boldsymbol{A}$ is a finite subset $X \subseteq A$ endowed with the restriction of finitely many basic operations of $\boldsymbol{A}$ to X.

In this case, $\mathbb{X}$ is a finite *partial* algebra of finite type (even when the type of $\boldsymbol{A}$ is infinite).

Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be similar algebras and $\mathbb{X}$ a local subgraph of $\boldsymbol{A}$. A map $f \colon X \to B$ is said to be an *embedding* of $\mathbb{X}$ into $\boldsymbol{B}$ if it is injective and, for every basic n-ary operation g of the type of $\mathbb{X}$ and $a_1, \dots, a_n \in X$ such that $g^{\boldsymbol{A}}(a_1, \dots, a_n) \in X$,

$$f(g^{\boldsymbol{A}}(a_1, \dots, a_n)) = g^{\boldsymbol{B}}(f(a_1), \dots, f(a_n)).$$

Theorem 4.8. *Let $\mathsf{K} \cup \{\boldsymbol{A}\}$ be a class of similar algebras. If every local subgraph of $\boldsymbol{A}$ can be embedded into some member of K, then $\boldsymbol{A} \in \mathbb{ISP}_{\mathrm{U}}(\mathsf{K})$.*

Proof. Let I be the set of local subgraphs of $\boldsymbol{A}$. By assumption, for every $\mathbb{X} \in I$ there are an algebra $\boldsymbol{B}_{\mathbb{X}} \in \mathsf{K}$ and an embedding $h_{\mathbb{X}} \colon \mathbb{X} \to \boldsymbol{B}_{\mathbb{X}}$. We define a partial order $\sqsubseteq$ on I as follows:

$$\mathbb{X} \sqsubseteq \mathbb{Y} \iff X \subseteq Y \text{ and the type of } \mathbb{Y} \text{ extends that of } \mathbb{X}.$$

Then, for every $\mathbb{X} \in I$, define

$$J_{\mathbb{X}} := \{\mathbb{Y} \in I \colon \mathbb{X} \sqsubseteq \mathbb{Y}\}.$$

Moreover, let $\mathcal{F}$ be the filter of $\mathcal{P}(I)$ generated by $\{J_{\mathbb{X}} \colon \mathbb{X} \in I\}$. Recall that

$$\mathcal{F} = \{Y \subseteq I : J_{\mathbb{X}_1} \cap \dots \cap J_{\mathbb{X}_n} \subseteq Y, \text{ for some } \mathbb{X}_1, \dots, \mathbb{X}_n \in I\}.$$

We will prove that $\mathcal{F}$ is proper. To this end, consider $\mathbb{X}_1, \dots, \mathbb{X}_n \in I$. Then let $\mathbb{Y}$ be the local subgraph of $\boldsymbol{A}$ with universe $Y := X_1 \cup \dots \cup X_n$ and whose type in the union of the types of the various $\mathbb{X}_i$. Then

$$\mathbb{X}_i \sqsubseteq \mathbb{Y}, \text{ for every } i \leqslant n,$$

that is, $\mathbb{Y} \in J_{\mathbb{X}_1} \cap \dots \cap J_{\mathbb{X}_n}$. It follows that $\emptyset \notin \mathcal{F}$ and, therefore, that $\mathcal{F}$ is proper. As $\mathcal{F}$ is a proper filter, by the Ultrafilter Lemma, it can be extended to an ultrafilter U on I.

Now, consider a map

$$f \colon A \to \prod_{\mathbb{X} \in I} B_{\mathbb{X}}$$

such that $f(a)(\mathbb{X}) = h_{\mathbb{X}}(a)$, for every $a \in A$ and $\mathbb{X} \in I$ such that $a \in \mathbb{X}$. Moreover, let

$$f^* \colon A \to \prod_{\mathbb{X} \in I} B_{\mathbb{X}} / \theta_U$$

be the map defined by the rule

$$f^*(a) := f(a)/\theta_U.$$

We will show f^* is an embedding of $\boldsymbol{A}$ into $\prod_{\mathbb{X} \in I} \boldsymbol{B}_{\mathbb{X}} / \theta_U$.

In order to prove that f^* is injective, consider a pair of distinct elements $a, c \in A$. Consider a local subgraph $\mathbb{Y}$ of $\boldsymbol{A}$ containing a and c. We will show that

$$J_{\mathbb{Y}} \subseteq \{\mathbb{X} \in I : f(a)(\mathbb{X}) \neq f(c)(\mathbb{X})\} \tag{10}$$

Consider $\mathbb{X} \in J_{\mathbb{Y}}$. Then $\mathbb{Y} \sqsubseteq \mathbb{X}$ and, therefore, $a, c \in Y \subseteq X$. Since $a, c \in X$, we have

$$f(a)(\mathbb{X}) = h_{\mathbb{X}}(a) \quad \text{and} \quad f(c)(\mathbb{X}) = h_{\mathbb{X}}(c).$$

Furthermore, $h_{\mathbb{X}}(a) \neq h_{\mathbb{X}}(c)$, because $h_{\mathbb{X}}$ is injective and $a \neq c$. This yields $f(a)(\mathbb{X}) \neq f(c)(\mathbb{X})$, establishing (10).

Recall that the definition of U guarantees that $J_{\mathbb{Y}} \in \mathcal{F} \subseteq U$. Therefore, since U is an upset of $\mathcal{P}(I)$, we can apply (10) obtaining

$$I \smallsetminus [\![f(a) = f(c)]\!] = \{\mathbb{X} \in I : f(a)(\mathbb{X}) \neq f(c)(\mathbb{X})\} \in U.$$

Since U is a proper filter, this implies

$$[\![f(a) = f(c)]\!] \notin U$$

and, therefore,

$$f^*(a) = f(a)/\theta_U \neq f(c)/\theta_U = f^*(c).$$

Hence, we conclude that f^* is injective.

To prove that it is a homomorphism, consider a basic n-ary operation g and $a_1, \dots, a_n \in A$. Then consider a local subgraph $\mathbb{Y}$ of $\boldsymbol{A}$ whose universe contains $a_1, \dots, a_n, g^{\boldsymbol{A}}(a_1, \dots, a_n)$ and whose type contains g. We will prove that

$$J_{\mathbb{Y}} \subseteq [\![f(g^{\boldsymbol{A}}(a_1, \dots, a_n)) = g^{\prod_{\mathbb{X} \in I} \boldsymbol{B}_{\mathbb{X}}}(f(a_1), \dots, f(a_n))]\!]. \tag{11}$$

Consider $\mathbb{V} \in J_{\mathbb{Y}}$. Since $\mathbb{Y} \sqsubseteq \mathbb{V}$, the type of $\mathbb{V}$ contains g and $a_1, \dots, a_n, g^{\boldsymbol{A}}(a_1, \dots, a_n) \in V$. Since $a_1, \dots, a_n, g^{\boldsymbol{A}}(a_1, \dots, a_n) \in V$, we have

$$\begin{aligned} f(a_1)(\mathbb{V}) &= h_{\mathbb{V}}(a_1) \\ &\vdots \\ f(a_n)(\mathbb{V}) &= h_{\mathbb{V}}(a_n) \\ f(g^{\boldsymbol{A}}(a_1, \dots, a_n))(\mathbb{V}) &= h_{\mathbb{V}}(g^{\boldsymbol{A}}(a_1, \dots, a_n)). \end{aligned}$$

Furthermore, as the type of $\mathbb{V}$ contains g,

$$h_{\mathbb{V}}(g^{\boldsymbol{A}}(a_1, \dots, a_n)) = g^{\boldsymbol{B}_{\mathbb{V}}}(h_{\mathbb{V}}(a_1), \dots, h_{\mathbb{V}}(a_n)).$$

From the above displays it follows

$$f(g^{\boldsymbol{A}}(a_1, \dots, a_n))(\mathbb{V}) = g^{\boldsymbol{B}_{\mathbb{V}}}(f(a_1)(\mathbb{V}), \dots, f(a_n)(\mathbb{V})) = g^{\prod_{\mathbb{X} \in I} \boldsymbol{B}_{\mathbb{X}}}(f(a_1), \dots, f(a_n))(\mathbb{V}),$$

that is, $\mathbb{V} \in [\![f(g^{\boldsymbol{A}}(a_1, \dots, a_n)) = g^{\prod_{\mathbb{X} \in I} \boldsymbol{B}_{\mathbb{X}}}(f(a_1), \dots, f(a_n))]\!]$. This establishes (11). Lastly, as $J_{\mathbb{Y}} \in U$ and U is an upset of $\mathcal{P}(I)$, condition (11) implies

$$[\![f(g^{\boldsymbol{A}}(a_1, \dots, a_n)) = g^{\prod_{\mathbb{X} \in I} \boldsymbol{B}_{\mathbb{X}}}(f(a_1), \dots, f(a_n))]\!] \in U,$$

and, therefore,

$$\begin{aligned} f^*(g^{\boldsymbol{A}}(a_1, \dots, a_n)) &= f(g^{\boldsymbol{A}}(a_1, \dots, a_n))/\theta_U \\ &= g^{\prod_{\mathbb{X} \in I} \boldsymbol{B}_{\mathbb{X}}}(f(a_1), \dots, f(a_n))/\theta_U \\ &= g^{\prod_{\mathbb{X} \in I} \boldsymbol{B}_{\mathbb{X}}/\theta_U}(f(a_1)/\theta_U, \dots, f(a_n)/\theta_U) \\ &= g^{\prod_{\mathbb{X} \in I} \boldsymbol{B}_{\mathbb{X}}/\theta_U}(f^*(a_1), \dots, f^*(a_n)). \end{aligned}$$

Hence, we conclude that f^* is a homomorphism and, therefore, an embedding of $\boldsymbol{A}$ into $\prod_{\mathbb{Y} \in I} \boldsymbol{B}_{\mathbb{Y}}/\theta_U$. As a consequence,

$$\boldsymbol{A} \in \mathbb{ISP}_{\mathrm{U}}(\{\boldsymbol{B}_{\mathbb{X}} : \mathbb{X} \in I\}) \subseteq \mathbb{ISP}_{\mathrm{U}}(\mathsf{K}). \qquad \boxtimes$$

Corollary 4.9. *Every algebra embeds into an ultraproduct of its finitely generated subalgebras.*

Theorem 4.8 is related to the following property that provides an algebraic path to the strong finite model property typical of logic.

Definition 4.10. A class of similar algebras K is said to have the *finite embeddability property* (FEP, for short) if every local subgraph of a member of K can be embedded into a finite member of K.

A first order sentence is said to be *universal* if it has the form $\forall x_1, \ldots, x_n \varphi$ for some quantifier free formula φ. The *universal theory* of a class of algebras K is the set of universal sentences valid in K. A class of algebras is said to be *finitely axiomatizable* if it can be axiomatized by finitely many first order sentences.

Proposition 4.11. *If a finitely axiomatizable class of algebras has the FEP, its universal theory is decidable.*

Proof. Let K be a finitely axiomatizable class of algebras with the FEP. Recall that a set is recursive if and only if both it and its complement are recursively enumerable. Therefore, it suffices to prove that both the universal theory and the set of universal sentences that fail in K are recursively enumerable. The fact that the universal theory of K is recursively enumerable is an immediate consequence of the fact that K is finitely axiomatizable.

Therefore, it only remains to prove that the set of universal sentences that fail in K is also recursively enumerable. To this end, we define an algorithm as follows. Let Σ be a finite set of axioms for K and $\{f_1, \ldots, f_n\}$ the function symbols that appear in Σ. Given a universal sentence Φ, we enumerate the finite models $\boldsymbol{A}_1, \boldsymbol{A}_2 \ldots$ of Σ in the type $\{f_1, \ldots, f_n, g_1, \ldots, g_m\}$, where $g_1, \ldots, g_m$ are the function symbols that occur in Φ. This can be done mechanically, because Σ is finite. Our algorithm tests if Φ fails in some $\boldsymbol{A}_n$. If this is the case, it stops and answers that Φ does not belong to the universal theory of K, otherwise it runs forever.

It is clear that if the algorithm stops, then Φ does not belong to the universal theory of K. To prove the converse, consider a universal sentence $\forall \vec{x} \varphi$ that does not belong to the universal theory of K. Then there exist $\boldsymbol{A} \in \mathsf{K}$ and $a_1, \ldots, a_n \in A$ such that $\boldsymbol{A} \nvDash \varphi(a_1, \ldots, a_n)$. Let $\mathbb{X}$ be the local subgraph of $\boldsymbol{A}$ whose universe is $\{a_1, \ldots, a_n\}$ and whose basic partial operations are the restrictions to X of the function symbols occurring in φ. Since K has the FEP, there exist $n \in \mathbb{N}$ and an embedding $f \colon \mathbb{X} \to \boldsymbol{A}_n$. It follows that $\boldsymbol{A} \nvDash \varphi(f(a_1), \ldots, f(a_n))$, whence $\forall \vec{x} \varphi$ fails in $\boldsymbol{A}_n$, as desired. ⊠

Given a class of algebras K, we denote by $\mathsf{K}^{<\omega}$ the class of its finite members.

Proposition 4.12. *A class K of similar algebras closed under $\mathbb{I}, \mathbb{S}$ and $\mathbb{P}_{\mathrm{U}}$ has the FEP if and only if $\mathsf{K} = \mathbb{ISP}_{\mathrm{U}}(\mathsf{K}^{<\omega})$.*

Proof sketch. Suppose that K has the FEP. By Theorem 4.8, we obtain $\mathsf{K} \subseteq \mathbb{ISP}_{\mathrm{U}}(\mathsf{K}^{<\omega})$. The reverse inclusion follows from the assumption that K is closed under $\mathbb{I}, \mathbb{S}$ and $\mathbb{P}_{\mathrm{U}}$. To prove the converse, suppose that $\mathsf{K} \subseteq \mathbb{ISP}_{\mathrm{U}}(\mathsf{K}^{<\omega})$ and let $\mathbb{X}$ be a local subgraph of an algebra $\boldsymbol{A} \in \mathsf{K}$. Since $\mathsf{K} = \mathbb{ISP}_{\mathrm{U}}(\mathsf{K}^{<\omega})$, we may assume that $\boldsymbol{A}$ is a subalgebra of an ultraproduct $\prod_{i \in I} \boldsymbol{B}_i / U$ of finite members of K. The structure of the local subgraph of $\mathbb{X}$ can be described by an existential sentence Φ. As existential sentences persist in extensions, Φ holds in $\prod_{i \in I} \boldsymbol{B}_i / U$ as well. By Łoś' Theorem, there exists $i \in I$ such that $\boldsymbol{B}_i \vDash \Phi$, whence $\mathbb{X}$ embeds into $\boldsymbol{B}_i$. We conclude that K has the FEP, as desired. ⊠

Example 4.13 (Lattices)**.** We will prove that the class Latt of all lattices has the FEP. For consider a lattice $\boldsymbol{A}$ and let $\mathbb{X}$ be one of its local subgraphs. Let also

$$B := \{a_1 \wedge^{\boldsymbol{A}} \cdots \wedge^{\boldsymbol{A}} a_n : a_1, \ldots, a_n \in X \text{ and } n \in \mathbb{N}\}.$$

Since the operation $\wedge^{\boldsymbol{A}}$ is idempotent, commutative and associative, the set B is finite. furthermore, B can be viewed as a subposet of $\boldsymbol{A}$. Let $\boldsymbol{B}^+$ be the poset obtained extending $\langle B; \leqslant \rangle$ with a new top element. Since $\boldsymbol{B}^+$ is a finite meet-semilattice with maximum, it is also a lattice. Furthermore, $\mathbb{X}$ embeds into $\boldsymbol{B}^+$. Hence, Latt has the FEP. By Propositions 4.11 and 4.12, this implies that the universal theory of lattices is decidable and that $\mathsf{Latt} = \mathbb{ISP}_{\mathrm{U}}(\mathsf{Latt}^{<\omega})$. On the other hand, the first order theory of distributive lattices (and, therefore, of any nontrivial equational class of lattices) is undecidable [15]. ⊠

Example 4.14 (Heyting algebras)**.** We will prove that the class HA of Heyting algebras has the FEP. For consider a Heyting algebra $\boldsymbol{A}$ and let $\mathbb{X}$ be one of its local subgraphs. Then let $\boldsymbol{B}$ be the bounded sublattice of $\boldsymbol{A}$ generated by $\mathbb{X}$. Notice that $\boldsymbol{B}$ is finite, because it is a finitely generated bounded distributive lattice. Since $\boldsymbol{B}$ is a finite distributive lattice, it can be viewed as a finite Heyting algebra $\boldsymbol{B}^+$. Furthermore, it is easy to see that the identity map is an embedding of $\mathbb{X}$ into $\boldsymbol{B}^+$. Hence, we conclude that HA has the FEP. By Propositions 4.11 and 4.12, this implies that the universal theory of Heyting algebras is decidable and $\mathsf{HA} = \mathbb{ISP}_{\mathrm{U}}(\mathsf{HA}^{<\omega})$. On the other hand, the first order theory of nontrivial equational class of Heyting algebras other than that of Boolean algebras is known to be undecidable [6].

Notably, the fact that the universal theory of HA is decidable implies that **IPC** is also decidable, in the sense that, given a finite set of formulas $\{\gamma_1, \dots, \gamma_n, \varphi\}$, we can decide whether $\gamma_1, \dots, \gamma_n \vdash_{\mathbf{IPC}} \varphi$. This is because, in view of the fact that HA is a $\{x \approx 1\}$-algebraic semantics for **IPC**, we have

$$\begin{aligned} \gamma_1, \dots, \gamma_n \vdash_{\mathbf{IPC}} \varphi &\iff \gamma_1 \approx 1, \dots, \gamma_n \approx 1 \vDash_{\mathsf{HA}} \varphi \approx 1 \\ &\iff \mathsf{HA} \vDash \forall \vec{x}\, ((\gamma_1 \approx 1 \,\&\, \dots \,\&\, \gamma_n \approx 1) \implies \varphi \approx 1). \end{aligned}$$

As $\forall \vec{x}\, ((\gamma_1 \approx 1 \,\&\, \dots \,\&\, \gamma_n \approx 1) \implies \varphi \approx 1)$ is a universal sentence, we can decide whether it holds in HA or not, because the universal theory of HA is decidable. ⊠

Exercise 4.15. Prove that the class MA of modal algebras has the FEP. Conclude that the universal theory of MA is decidable and $\mathsf{MA} = \mathbb{ISP}_{\mathrm{U}}(\mathsf{MA}^{<\omega})$. ⊠

Exercise 4.16. Consider the consequence relation $\vdash$ on the set of sentences of a given algebraic language defined as follows:

$$\begin{aligned} \Gamma \vdash \varphi \iff & \text{ for every algebra } \boldsymbol{A} \text{ of the appropriate type,} \\ & \text{ if } \boldsymbol{A} \vDash \Gamma, \text{ then } \boldsymbol{A} \vDash \varphi. \end{aligned}$$

This exercise asks you to prove the Compactness Theorem for $\vdash$ using the ultraproduct construction. To this end, consider a set of sentences $\Gamma \cup \{\varphi\}$ such that $\Delta \nvdash \varphi$, for every finite $\Delta \subseteq \Gamma$. Then we can associate with every finite $\Delta \subseteq \Gamma$ an algebra $\boldsymbol{A}_\Delta$ such that

$$\boldsymbol{A}_\Delta \vDash \Delta \;\text{ and }\; \boldsymbol{A} \nvDash \varphi.$$

Prove that there exists an ultraproduct $\boldsymbol{B}$ of the family $\{\boldsymbol{A}_\Delta : \Delta \text{ is a finite subset of } \Gamma\}$ such that $\boldsymbol{B} \vDash \Gamma$ and $\boldsymbol{B} \nvDash \varphi$. Then conclude that $\Gamma \nvdash \varphi$, as desired. Hint: use Łoś' Theorem. ⊠

5. Quasi-varieties

For the theory of quasi-varieties, we refer the reader to [14, 18].

Definition 5.1. A class of similar algebras closed under $\mathbb{I}, \mathbb{S}, \mathbb{P}$ and $\mathbb{P}_{\mathrm{U}}$ is said to be a *quasi-variety*.

Examples of quasi-varieties include the classes of Boolean, Heyting and modal algebras, as well as the class of (bounded) distributive lattices and groups. Our aim will be to prove that quasi-varieties are precisely the classes of algebras axiomatized by the following kind of first order formulas.

Definition 5.2. A *quasi-equation* of type ρ is an expression Φ of the form

$$(\varphi_1 \approx \psi_1 \,\&\, \dots \,\&\, \varphi_n \approx \psi_n) \Longrightarrow \varepsilon \approx \delta,$$

where $\{\varphi_1 \approx \psi_1, \dots, \varphi_n \approx \psi_n, \varepsilon \approx \delta\}$ is a set of equations of type ρ. Then Φ is *valid* in an algebra $\boldsymbol{A}$ of type ρ when so is its universal closure, that is, for every $\vec{a} \in A$,

$$\text{if } \varphi_1^{\boldsymbol{A}}(\vec{a}) = \psi_1^{\boldsymbol{A}}(\vec{a}), \dots, \varphi_n^{\boldsymbol{A}}(\vec{a}) = \psi_n^{\boldsymbol{A}}(\vec{a}), \text{ then } \varepsilon^{\boldsymbol{A}}(\vec{a}) = \delta^{\boldsymbol{A}}(\vec{a}).$$

In this case, we often say that $\boldsymbol{A}$ *satisfies* Φ and write $\boldsymbol{A} \vDash \Phi$. Lastly, a quasi-equation is said to be an *equation* when its set of premises is empty.

Remark 5.3. It might seem that we are using the term *equations* to refer to two distinct kinds of expressions, namely those of the form $\varepsilon \approx \delta$ and $\emptyset \Longrightarrow \varepsilon \approx \delta$. This is not a problem, however, because these expressions are synonyms, in the sense that an algebra satisfies $\varepsilon \approx \delta$ if and only if it satisfies $\emptyset \Longrightarrow \varepsilon \approx \delta$. Because of this, we will continue to denote equations by $\varepsilon \approx \delta$, while keeping in mind that they are special instances of quasi-equations. ⊠

The aim of this section is to prove the following classical result.

Maltsev's Theorem 5.4. *A class of similar algebras is a quasi-variety if and only if it can be axiomatized by a set of quasi-equations.*

Proof. The "only if" part follows from the fact that the validity of quasi-equations is preserved by the class operators $\mathbb{I}, \mathbb{S}, \mathbb{P}$ and $\mathbb{P}_{\mathrm{U}}$. To prove the converse, consider a prevariety K closed under $\mathbb{P}_{\mathrm{U}}$. Moreover, let Σ the set of quasi-equations, with variables in *Var*, valid in K. Let also K^+ be the class of algebras axiomatized by Σ. Our aim is to prove that $\mathsf{K} = \mathsf{K}^+$.

The inclusion $\mathsf{K} \subseteq \mathsf{K}^+$ is straightforward. To prove the other one, consider an algebra $\boldsymbol{A} \in \mathsf{K}^+$. In order to prove that $\boldsymbol{A} \in \mathsf{K}^+$, it suffices to show that every local subgraph of $\boldsymbol{A}$ embeds in some members of K. This is because, in this case, $\boldsymbol{A} \in \mathbb{ISP}_{\mathrm{U}}(\mathsf{K})$, by Theorem 4.8. Since K is closed under $\mathbb{I}, \mathbb{S}$ and $\mathbb{P}_{\mathrm{U}}$, this implies $\boldsymbol{A} \in \mathsf{K}$, as desired.

Then consider a local subgraph $\mathbb{X}$ of $\boldsymbol{A}$. By definition, $\mathbb{X}$ consists of a finite set $\{a_1, \dots, a_n\}$ endowed with the restriction of finitely many basic operations $f_1, \dots, f_m$ of $\boldsymbol{A}$ to X. Fix n distinct variables $x_1, \dots, x_n \in Var$, corresponding to the elements $a_1, \dots, a_n$ of X. The *positive* and *negative atomic diagrams* of $\mathbb{X}$ are, respectively,

$$\mathcal{D}^+(\mathbb{X}) := \{f_i(x_{k_1}, \dots, x_{k_s}) \approx x_j \colon i \leqslant m \text{ and } k_1, \dots, k_s, j \leqslant n \text{ and } f_i^{\boldsymbol{A}}(a_{k_1}, \dots, a_{k_s}) = a_j\}$$
$$\mathcal{D}^-(\mathbb{X}) := \{x_i \not\approx x_j \colon i, j \leqslant n \text{ and } a_i \neq a_j\}.$$

Observe that both $\mathcal{D}^+(\mathbb{X})$ and $\mathcal{D}^-(\mathbb{X})$ are finite sets. Then take an enumeration

$$\mathcal{D}^-(\mathbb{X}) = \{\varepsilon_1 \not\approx \delta_1, \dots, \varepsilon_t \not\approx \delta_t\}.$$

Moreover, for each $i \leqslant t$, consider the quasi-equation

$$\Phi_i := \left(\bigwith \mathcal{D}^+(\mathbb{X}) \right) \Longrightarrow \varepsilon_i \approx \delta_i.$$

As witnessed by the natural assignment

$$x_1 \longmapsto a_1, \dots, x_n \longmapsto a_n,$$

the quasi-equations $\Phi_1, \dots, \Phi_t$ fail in $\boldsymbol{A}$. Since they are written with variables in *Var* and $\boldsymbol{A}$ satisfies all the quasi-equations with variables in *Var* valid in K, this implies that each Φ_i fails in some $\boldsymbol{B}_i \in \mathsf{K}$ under an assignment

$$x_1 \longmapsto b^i_1, \dots, x_n \longmapsto b^i_n. \tag{12}$$

Now, consider the map $h \colon X \to (B_1 \times \cdots \times B_t)$, defined by the rule

$$a_1 \longmapsto \langle b^1_1, \dots, b^t_1 \rangle, \dots, a_n \longmapsto \langle b^1_n, \dots, b^t_n \rangle.$$

We will prove that h is an embedding of $\mathbb{X}$ into $\boldsymbol{B}_1 \times \cdots \times \boldsymbol{B}_t$. To prove that h is injective, consider two distinct elements $a_p, a_q \in X$. Then the formula $x_p \not\approx x_q$ belongs to the negative atomic diagram of $\mathbb{X}$. Then there exists $i \leqslant t$ such that

$$\Phi_i = \Big(\mathop{\&} \mathcal{D}^+(\mathbb{X}) \Big) \Longrightarrow x_p \approx x_q.$$

Since Φ_i fails in $\boldsymbol{B}_i$ under the assignment in (12), we obtain $b^i_p \neq b^i_q$. As a consequence,

$$h(a_p)(i) = b^i_p \neq b^i_q = h(a_q)(i)$$

and, therefore, $h(a_p) \neq h(a_q)$. Hence, h is injective. To prove that it preserves the partial operations, consider a basic s-ary operation f_j in the type of $\mathbb{X}$ and $a_{k_1}, \dots, a_{k_s} \in X$ such that $f_j^{\boldsymbol{A}}(a_{k_1}, \dots, a_{k_s}) \in X$. Then there exists some $p \leqslant n$ such that $a_p = f_j^{\boldsymbol{A}}(a_{k_1}, \dots, a_{k_s})$. Moreover, the equation

$$f_j(x_{k_1}, \dots, x_{k_s}) \approx x_p$$

belongs to the positive atomic diagram $\mathcal{D}^+(\mathbb{X})$ of $\mathbb{X}$. As each quasi-equation Φ_i fails under the assignment in (12), the same assignment satisfies the antecedent of Φ_i, namely $\mathcal{D}^+(\mathbb{X})$. It follows that

$$f_j^{\boldsymbol{B}_i}(b^i_{k_1}, \dots, b^i_{k_s}) = b^i_p, \text{ for each } i \leqslant t.$$

As a consequence, for every $i \leqslant t$,

$$\begin{aligned} h(f_j^{\boldsymbol{A}}(a_{k_1}, \dots, a_{k_s}))(i) &= h(a_p)(i) \\ &= b^i_p \\ &= f_j^{\boldsymbol{B}_i}(b^i_{k_1}, \dots, b^i_{k_s}) \\ &= f_j^{\boldsymbol{B}_i}(h(a_{k_1})(i), \dots, h(a_{k_s})(i)) \\ &= f_j^{\boldsymbol{B}_1 \times \cdots \times \boldsymbol{B}_t}(h(a_{k_1}), \dots, h(a_{k_s}))(i). \end{aligned}$$

Thus, $h(f_j^{\boldsymbol{A}}(a_{k_1}, \dots, a_{k_s})) = f_j^{\boldsymbol{B}_1 \times \cdots \times \boldsymbol{B}_t}(h(a_{k_1}), \dots, h(a_{k_s}))$. We conclude that $h \colon \mathbb{X} \to (\boldsymbol{B}_1 \times \cdots \times \boldsymbol{B}_t)$ is an embedding. Since $\boldsymbol{B}_1, \dots, \boldsymbol{B}_t \in \mathsf{K}$ and K is closed under $\mathbb{P}$, the direct product $\boldsymbol{B}_1 \times \cdots \times \boldsymbol{B}_t$ belongs to K. Hence, $\mathbb{X}$ embeds into some member of K, as desired. ⊠

Exercise 5.5. In view of Łoś' Theorem quasi-equations persist in ultraproducts. If you are not familiar with the proof of Łoś' Theorem, offer a direct proof of this fact. ⊠

Given a class of similar algebras K, the least quasi-variety extending K exists and will be denoted by $\mathbb{Q}(\mathsf{K})$ and called the quasi-variety *generated* by K.

Proposition 5.6 (Maltsev)**.** *For every class of algebras* K,

$$\mathbb{Q}(\mathsf{K}) = \mathbb{ISPP}_{\mathrm{U}}(\mathsf{K}).$$

Proof. The inclusion $\mathbb{ISPP}_{\mathrm{U}}(\mathsf{K}) \subseteq \mathbb{Q}(\mathsf{K})$ is obvious. To prove the other, consider $\boldsymbol{A} \in \mathbb{Q}(\mathsf{K})$. By Maltsev's Theorem, $\mathbb{Q}(\mathsf{K})$ is the class of all algebras satisfying the quasi-equations valid in K. The proof of the hard part of Maltsev's Theorem show that $\boldsymbol{A} \in \mathbb{ISP}_{\mathrm{U}}\mathbb{P}(\mathsf{K})$. Therefore, it only remains to show that $\mathbb{P}_{\mathrm{U}}\mathbb{P}(\mathsf{K}) \subseteq \mathbb{ISPP}_{\mathrm{U}}(\mathsf{K})$. But this is an easy exercise on class operators (the details are sketched below).

Consider an algebra $\boldsymbol{B} \in \mathbb{P}_{\mathrm{U}}\mathbb{P}(\mathsf{K})$. There exists an index set I, an ultrafilter U on I, and a family of algebras $\{\boldsymbol{B}_j : j \in J_i\}$ for each $i \in I$ such that

$$\boldsymbol{B} = \Big(\prod_{i \in I}\Big(\prod_{j \in J_i}\boldsymbol{B}_j\Big)\Big)/\theta_U.$$

Let J be the set of all maps $f\colon I \to \bigcup_{i \in I} J_i$ such that $f(i) \in J_i$. Moreover, let

$$g\colon B \to \prod_{f \in J}\Big(\prod_{i \in I} B_{f(i)}\Big)$$

be the map defined by the rule $g(b)(f)(i) := b(i)(f(i))$. It is not hard to check that the map

$$g^{*}\colon B \to \Big(\prod_{f \in J}\Big(\prod_{i \in I} B_{f(i)}\Big)\Big)/\theta_U$$

that sends an element $b \in B$ to $f(b)/\theta_U$ is an embedding, whence $\boldsymbol{B} \in \mathbb{ISPP}_{\mathrm{U}}(\mathsf{K})$. ⊠

Corollary 5.7. *If* K *be a finite set of finite similar algebras, then* $\mathbb{Q}(\mathsf{K}) = \mathbb{ISP}(\mathsf{K})$.

Proof. Since K is a finite set of finite algebras, $\mathbb{P}_{\mathrm{U}}(\mathsf{K}) \subseteq \mathbb{I}(\mathsf{K})$ (see Exercise 4.4). As a consequence, we obtain $\mathbb{ISP}(\mathsf{K}) = \mathbb{ISPP}_{\mathrm{U}}(\mathsf{K})$. Together with Proposition 5.6, this yields $\mathbb{Q}(\mathsf{K}) = \mathbb{ISP}(\mathsf{K})$. ⊠

Exercise 5.8. Prove that if K is a class of similar algebras, then $\mathbb{P}_{\mathrm{U}}\mathbb{P}(\mathsf{K}) \subseteq \mathbb{ISPP}_{\mathrm{U}}(\mathsf{K})$. Hint: use the sketch in the last part of Proposition 5.6. ⊠

Example 5.9 (Quasi-varieties)**.** In view of Examples 4.13 and 4.14, we know that

$$\mathsf{Latt} = \mathbb{ISP}_{\mathrm{U}}(\mathsf{Latt}^{<\omega}) \text{ and } \mathsf{HA} = \mathbb{ISP}_{\mathrm{U}}(\mathsf{HA}^{<\omega}).$$

This implies that $\mathsf{Latt} \subseteq \mathbb{Q}(\mathsf{Latt}^{<\omega})$ and $\mathsf{HA} \subseteq \mathbb{Q}(\mathsf{HA}^{<\omega})$. Since both Latt and HA are closed under $\mathbb{I}, \mathbb{S}, \mathbb{P}$ and $\mathbb{P}_{\mathrm{U}}$, this yields

$$\mathsf{Latt} = \mathbb{Q}(\mathsf{Latt}^{<\omega}) \text{ and } \mathsf{HA} = \mathbb{Q}(\mathsf{HA}^{<\omega}).$$

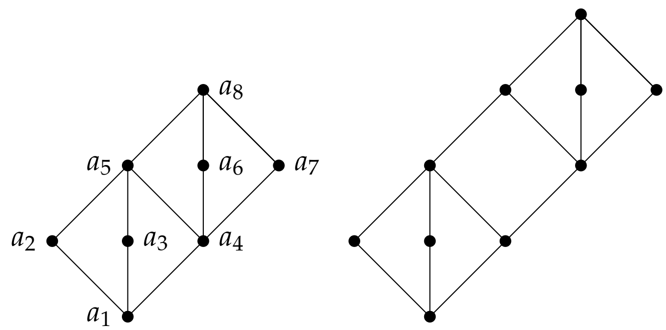
Let DL be the class of distributive lattices and $\boldsymbol{B}$ the two-element distributive lattice. In view of Example 2.18, we have $\mathsf{DL} = \mathbb{IP}_{\mathrm{SD}}(\boldsymbol{B})$. Clearly, $\mathbb{IP}_{\mathrm{SD}}(\boldsymbol{B}) \subseteq \mathbb{ISP}(\boldsymbol{B}) \subseteq \mathbb{Q}(\boldsymbol{B})$. On the other hand, since DL is closed under $\mathbb{I}, \mathbb{S}, \mathbb{P}$ and $\mathbb{P}_{\mathrm{U}}$, we obtain $\mathbb{Q}(\boldsymbol{B}) \subseteq \mathsf{DL}$. Hence, $\mathsf{DL} = \mathbb{Q}(\boldsymbol{B})$. Similarly, the class of Boolean algebras is the quasi-variety generated by the two-element Boolean algebra (see 2.19, if necessary). ⊠

Exercise 5.10. Prove that there is no finite Heyting algebra $\boldsymbol{A}$ such that $\mathsf{HA} = \mathbb{Q}(\boldsymbol{A})$, cf. with $\mathsf{HA} = \mathbb{Q}(\mathsf{HA}^{<\omega})$. Prove that, however, there exists an infinite Heyting algebra $\boldsymbol{A}$ such that $\mathsf{HA} = \mathbb{Q}(\boldsymbol{A})$, see for instance [20]. ⊠

6. Relative congruences

Quasi-varieties need not be closed under homomorphic images, as shown in the next examples.

Example 6.1 (Lattices)**.** Consider the lattices $\boldsymbol{A}$ and $\boldsymbol{B}$ depicted below. We will show that the quasi-variety $\mathbb{Q}(\boldsymbol{B})$ is not closed under $\mathbb{H}$.



$\boldsymbol{A}$ $\boldsymbol{B}$

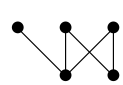
To this end, let $\mathcal{D}^+(\boldsymbol{A})$ be the positive atomic diagram of $\boldsymbol{A}$ written with the variables $x_1, \dots, x_8$ corresponding to the elements $a_1, \dots, a_8$ and consider the quasi-equation

$$\Phi = \mathop{\&} \mathcal{D}^+(\boldsymbol{A}) \Longrightarrow x_1 \approx x_8.$$

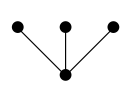
Notice that $\boldsymbol{B}$ validates Φ. To prove this, consider an assignment $f \colon \{x_1, \dots, x_8\} \to B$ that validates $\mathcal{D}^+(\boldsymbol{A})$ in $\boldsymbol{B}$. Using the definition of $\mathcal{D}^+(\boldsymbol{A})$, it is easy to see that the map $h \colon A \to B$ that sends a_i to $f(a_i)$ is a homomorphism from $\boldsymbol{A}$ to $\boldsymbol{B}$. Since $\boldsymbol{A}$ is simple, $\mathsf{Ker}(h)$ is either id_A or $A \times A$. Notice that there is no embedding of $\boldsymbol{A}$ into $\boldsymbol{B}$. Therefore, $\mathsf{Ker}(h)$ cannot be the identity relation. It follows that $\mathsf{Ker}(h) = A \times A$. In particular, $\langle a_1, a_8 \rangle \in \mathsf{Ker}(h)$ and, therefore, $f(x_1) = h(a_1) = f(a_8) = f(x_8)$. Hence, we conclude that $\boldsymbol{B} \vDash \Phi$, as desired. Moreover, Φ fails in $\boldsymbol{A}$, as witnessed by the assignment $x_i \longmapsto a_i$.

In brief, Φ holds in $\boldsymbol{B}$ but fails in $\boldsymbol{A}$. By Maltsev's Theorem, we conclude that $\boldsymbol{A}$ does not belong to the quasi-variety $\mathbb{Q}(\boldsymbol{B})$ generated by $\boldsymbol{B}$. On the other hand, $\boldsymbol{A}$ is a homomorphic image of $\boldsymbol{B}$ (obtained by glueing two pairs of elements of $\boldsymbol{B}$). Thus, the quasi-variety $\mathbb{Q}(\boldsymbol{B})$ is not closed under $\mathbb{H}$. ⊠

Example 6.2 (Heyting algebras)**.** We assume the reader is familiar with *Esakia duality* for Heyting algebras [10]. Let $\boldsymbol{A}$ be the finite Heyting algebra whose dual Esakia space is the following poset $\boldsymbol{A}_*$ endowed with the discrete topology.



Or aim is to show that $\mathbb{Q}(\boldsymbol{A})$ is not closed under $\mathbb{H}$. To this end, consider the finite Heyting algebra $\boldsymbol{B}$ whose dual Esakia space is the the rooted poset $\boldsymbol{B}_*$ (endowed with the discrete topology) depicted below.

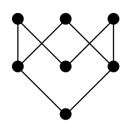


Notice that, as $\boldsymbol{B}_*$ is rooted, the algebra $\boldsymbol{B}$ is subdirectly irreducible. Moreover, observe that, as $\boldsymbol{B}_*$ is an upset of $\boldsymbol{A}_*$, by Esakia duality we obtain $\boldsymbol{B} \in \mathbb{H}(\boldsymbol{A})$. Therefore, it suffices to show that $\boldsymbol{B} \notin \mathbb{Q}(\boldsymbol{A})$. Suppose the contrary, with a view to contradiction. By Corollary 5.7,

$$\boldsymbol{B} \in \mathbb{ISP}(\boldsymbol{A}) \subseteq \mathbb{IP}_{\text{SD}}\mathbb{S}(\boldsymbol{A}).$$

As $\boldsymbol{B}$ is subdirectly irreducible, we conclude that $\boldsymbol{B} \in \mathbb{IS}(\boldsymbol{A})$. By Esakia duality, this means that $\boldsymbol{B}_*$ is a p-morphic image of $\boldsymbol{A}_*$. But a quick inspection of the posets $\boldsymbol{A}_*$ and $\boldsymbol{B}_*$ shows that this is impossible, a contradiction. Hence, we conclude that $\mathbb{Q}(\boldsymbol{A})$ does not contain $\boldsymbol{B}$ and, therefore, is not closed under $\mathbb{H}$. ⊠

Exercise 6.3. Let $\boldsymbol{A}$ be the Heyting algebra whose Esakia dual is the poset depicted below.



Prove that $\mathbb{Q}(\boldsymbol{A})$ is not closed under $\mathbb{H}$. ⊠

More in general, if K is a quasi-variety and θ a congruence of some $\boldsymbol{A} \in \mathsf{K}$, the algebra $\boldsymbol{A}/\theta$ need not belong to K. This makes the following concept attractive.

Definition 6.4. Let $\mathsf{K} \cup \{\boldsymbol{A}\}$ be a class of similar algebras. A congruence $\theta \in \mathsf{Con}(\boldsymbol{A})$ is said to be a K-*congruence* of $\boldsymbol{A}$ if $\boldsymbol{A}/\theta \in \mathsf{K}$. We denote the poset of K-congruences of $\boldsymbol{A}$, ordered under the inclusion relation, by $\mathsf{Con}_{\mathsf{K}}(\boldsymbol{A})$.

Proposition 6.5. *If* K *is a quasi-variety, then* $\mathsf{Con}_{\mathsf{K}}(\boldsymbol{A})$ *is a complete lattice in which meets are intersections, for every algebra* $\boldsymbol{A}$ *of the type of* K.

Proof. Then let $\boldsymbol{A}$ be an algebra of the type of K. Since K is a quasi-variety, it is closed under $\mathbb{P}_{\text{SD}}$ and it contains a trivial algebra (the subdirect product of the empty family). Therefore, as K is closed under $\mathbb{I}$ by assumption, it contains all trivial algebras and, in particular, $\boldsymbol{A}/(A \times A)$. Thus, $A \times A \in \mathsf{Con}_{\mathsf{K}}(\boldsymbol{A})$. Then consider a nonempty family $\{\theta_i : i \in I\} \subseteq \mathsf{Con}_{\mathsf{K}}(\boldsymbol{A})$. By Proposition 2.20,

$$\boldsymbol{A}/\bigcap_{i \in I}\theta_i \in \mathbb{IP}_{\text{SD}}(\{\boldsymbol{A}/\theta_i : i \in I\}).$$

Observe that $\{\boldsymbol{A}/\theta_i : i \in I\} \subseteq \mathsf{K}$, since the various θ_i are K-congruences of $\boldsymbol{A}$. Together with the above display and the assumption that K is closed under $\mathbb{I}$ and $\mathbb{P}_{\text{SD}}$, this yields $\boldsymbol{A}/\bigcap_{i \in I}\theta_i \in \mathsf{K}$, whence $\bigcap_{i \in I}\theta_i \in \mathsf{Con}_{\mathsf{K}}(\boldsymbol{A})$. It follows that $\mathsf{Con}_{\mathsf{K}}(\boldsymbol{A})$ has a minimum, namely $A \times A$, and infima of nonempty families. Therefore, arbitrary infima exist in $\mathsf{Con}_{\mathsf{K}}(\boldsymbol{A})$ and, therefore, $\mathsf{Con}_{\mathsf{K}}(\boldsymbol{A})$ is a complete lattice. ⊠

Corollary 6.6. *If* K *is a quasi-variety and* $\boldsymbol{A}$ *an algebra of the same type, the map*

$$\mathrm{Cg}^{\boldsymbol{A}}_{\mathsf{K}} \colon \mathcal{P}(A \times A) \to \mathcal{P}(A \times A)$$

that associates the least K*-congruence of* $\boldsymbol{A}$ *containing a subset* X *with a subset* $X \subseteq A \times A$ *is a closure operator on* $A \times A$.

Remark 6.7. The reader might have noticed that the proof of Proposition 6.5 depends only on the fact that K is closed under $\mathbb{I}, \mathbb{S}$, and $\mathbb{P}$. Classes of algebras closed under $\mathbb{I}, \mathbb{S}$, and $\mathbb{P}$

have been called *prevarieties*. These can be axiomatized by *proper classes* of infinitary quasi-equations. The demand that proper classes can be replaced by sets in the axiomatization of prevarieties is equivalent to an independent set theoretical principle known as *Vopěnka Principle* [14, Prop. 2.3.18] (see also [1]), but this is a different story. ⊠

An element a of lattice $\boldsymbol{A}$ is said to be *compact* when for every subset X of A whose supremum exists in $\boldsymbol{A}$, the following condition holds:

$$\text{if } a \leqslant \bigvee X \text{, there is a finite } Y \subseteq X \text{ such that } a \leqslant \bigvee Y.$$

A complete lattice is said to be *algebraic* if all its elements are joins of compact ones.

Theorem 6.8. *Let* K *be a quasi-variety and* $\boldsymbol{A}$ *and algebra of the same type. Then* $\mathsf{Con}_{\mathsf{K}}(\boldsymbol{A})$ *is an algebraic lattice, whose compact elements are the finitely generated* K*-congruences of* $\boldsymbol{A}$*, i.e., the* K*-congruences of* $\boldsymbol{A}$ *of the form* $\mathrm{Cg}_{\mathsf{K}}^{\boldsymbol{A}}(X)$ *for a finite* $X \subseteq A \times A$*.*

Proof. Recall $\mathsf{Con}_{\mathsf{K}}(\boldsymbol{A})$ is a complete lattice, by Proposition 7.15. Furthermore, it is obvious that every element of $\mathsf{Con}_{\mathsf{K}}(\boldsymbol{A})$ is a join of finitely generated K-congruences. of $\boldsymbol{A}$ Therefore, it only remains to show that the compact elements of $\mathsf{Con}_{\mathsf{K}}(\boldsymbol{A})$ are precisely the finitely generated K-congruences of $\boldsymbol{A}$. This easy exercise is left to the reader. ⊠

Exercise 6.9. Let K be a quasi-variety and $\boldsymbol{A}$ and algebra of the same type. Prove that the compact elements of $\mathsf{Con}_{\mathsf{K}}(\boldsymbol{A})$ are precisely the finitely generated K-congruences of $\boldsymbol{A}$. ⊠

Given an algebra $\boldsymbol{A}$, we denote by Id_A the identity congruence of $\boldsymbol{A}$. Let K be a quasi-variety and $\boldsymbol{A} \in \mathsf{K}$. Then $\boldsymbol{A}$ is said to be *relatively subdirectly irreducible in* K if Id_A is completely meet-irreducible in $\mathrm{Con}_{\mathsf{K}}\boldsymbol{A}$., i.e., for every $\{\theta_i : i \in I\} \subseteq \mathrm{Con}_{\mathsf{K}}\boldsymbol{A}$,

$$\text{if } \bigcap_{i \in I} \theta_i = \mathrm{Id}_A \text{, then there is } i \in I \text{ such that } \theta_i = \mathrm{Id}_A.$$

Notice that this is *equivalent* to the assertion that there is a smallest congruence θ in $\mathrm{Con}_{\mathsf{K}}\boldsymbol{A} \smallsetminus \{\mathrm{Id}_A\}$. In this case, θ is generated by every pair $\langle a, c\rangle \in \theta$ such that $a \neq c$, in the sense that

$$\theta = \mathrm{Cg}_{\mathsf{K}}^{\boldsymbol{A}}(a, c).$$

This fact will be used repeatedly later on. An algebra $\boldsymbol{A}$ is said to be *subdirectly irreducible* (in the absolute sense) when Id_A it completely meet-irreducible in $\mathrm{Con}\boldsymbol{A}$. This is the notion you probably met in the realm of Heyting and modal algebras. The next result generalizes Birkhoff's subdirect representation theorem to the setting of quasi-varieties.

Theorem 6.10. *Every member of a quasi-variety* K *is a subdirect product of algebras that are relatively subdirectly irreducible in* K*.*

Proof. Similar to that of Birkhoff's subdirect representation theorem [3, Thm. 3.24]. ⊠

Let K be a quasi-variety. An algebra $\boldsymbol{A} \in \mathsf{K}$ is said to be *relatively simple in* K if $\mathrm{Con}_{\mathsf{K}}\boldsymbol{A}$ has exactly two elements. Notice that all relatively simple algebras are relatively subdirectly irreducible. An algebra $\boldsymbol{A}$ is said to be *simple* (in the absolute sense) when $\mathrm{Con}\boldsymbol{A}$ has exactly two elements. Given a quasi-variety K, we denote by $\mathsf{K}_{\mathrm{RSI}}$ (resp. K_{RS}) the class of its relatively subdirectly irreducible (resp. relatively simple) members. Furthermore, given an arbitrary class of algebras K (not necessarily a quasi-variety), we denote by K_{SI} (resp. K_{S}) the class of its subdirectly irreducible (resp. simple) members. Notice that if K is a variety, $\mathsf{K}_{\mathrm{RSI}} = \mathsf{K}_{\mathrm{SI}}$ and $\mathsf{K}_{\mathrm{RS}} = \mathsf{K}_{\mathrm{S}}$. The following observation helps locating $\mathsf{K}_{\mathrm{RSI}}$ inside a quasi-variety K.

Lemma 6.11. *If* K *is a class of algebras, then* $\mathbb{Q}(\mathsf{K})_{\text{RSI}} \subseteq \mathbb{ISP}_{\text{U}}(\mathsf{K})$.

Proof. Consider an algebra $\boldsymbol{A} \in \mathbb{Q}(\mathsf{K})_{\text{RSI}}$. Recall from Corollary **??** that $\mathbb{Q}(\mathsf{K}) = \mathbb{ISPP}_{\text{U}}(\mathsf{K})$. As it is easy to see that $\mathbb{ISPP}_{\text{U}}(\mathsf{K}) = \mathbb{IP}_{\text{SD}}\mathbb{SP}_{\text{U}}(\mathsf{K})$, this implies $\boldsymbol{A} \in \mathbb{IP}_{\text{SD}}\mathbb{SP}_{\text{U}}(\mathsf{K})$. Accordingly, there is a subdirect embedding $f \colon \boldsymbol{A} \to \prod_{i \in I} \boldsymbol{B}_i$ for some $\{\boldsymbol{B}_i \colon i \in I\} \subseteq \mathbb{SP}_{\text{U}}(\mathsf{K})$. For every $j \in I$, let θ_j be the kernel of the homomorphism $\pi_j \circ f \colon \boldsymbol{A} \to \boldsymbol{B}_i$, where $\pi_j \colon \prod_{i \in I} \boldsymbol{B}_i \to \boldsymbol{B}_j$ is the natural projection. As $\boldsymbol{A}/\theta_j \leqslant \boldsymbol{B}_j \in \mathbb{Q}(\mathsf{K})$ and $\mathbb{Q}(\mathsf{K})$ is closed under subalgebras, θ_j is a $\mathbb{Q}(\mathsf{K})$-congruence of $\boldsymbol{A}$. Moreover, since f is injective, $\bigcap_{i \in I} \theta_i = \text{Id}_A$. Since $\boldsymbol{A}$ is relatively subdirectly irreducible in $\mathbb{Q}(\mathsf{K})$, there is $i \in I$ such that $\theta_i = \text{Id}_A$. Consequently, $\boldsymbol{A}$ embeds into $\boldsymbol{B}_i$, whence $\boldsymbol{A} \in \mathbb{IS}(\boldsymbol{B}_i) \subseteq \mathbb{ISP}_{\text{U}}(\mathsf{K})$. ⊠

Exercise 6.12. Let $\boldsymbol{A}$ be the Heyting algebra described in Example **??**. Clearly, $\boldsymbol{A}$ is not subdirectly irreducible in the standard sense, as the dual of $\boldsymbol{A}$ is not rooted. Prove that, however, $\boldsymbol{A}$ is relatively subdirectly irreducible in $\mathbb{Q}(\boldsymbol{A})$. Accordingly, algebras that are not subdirectly irreducible in the absolute sense might become relatively subdirectly irreducible in some quasi-variety.

Hint: observe that, in view of Theorem 6.10, $\boldsymbol{A}$ is the subdirect product of some $\{\boldsymbol{B}_i \colon i \in I\}$ that are relatively subdirectly irreducible in $\mathbb{Q}(\boldsymbol{A})$. Furthermore, by Lemma 6.11, each $\boldsymbol{B}_i$ belongs to $\mathbb{IS}(\boldsymbol{A})$ (recall that, as $\boldsymbol{A}$ is finite, $\mathbb{IP}_{\text{U}}(\boldsymbol{A}) = \mathbb{I}(\boldsymbol{A})$). Use these facts to conclude that $\boldsymbol{A}$ is relatively subdirectly irreducible in $\mathbb{Q}(\boldsymbol{A})$. ⊠

In the case of quasi-varieties something more is true.

Proposition 6.13. *Let* K *be a quasi-variety. If* $\boldsymbol{A}$ *is an algebra of the same type as* K*, then* $\mathsf{Con}_{\mathsf{K}}\boldsymbol{A}$ *is an inductive closure system and, therefore, an algebraic lattice.*

Proof. In view of Corollary **??**, it suffices to show that the union ϕ of a nonempty upward directed family $\{\theta_i : i \in I\} \subseteq \mathsf{Con}_{\mathsf{K}}(\boldsymbol{A})$ is still a K-congruence of $\boldsymbol{A}$. It is clear that ϕ is a congruence of $\boldsymbol{A}$. Therefore, we only detail a proof of the fact that $\boldsymbol{A}/\phi \in \mathsf{K}$.

In view of Maltsev's Theorem, it suffices to show that all quasi-equations valid in K are also valid in $\boldsymbol{A}/\phi$. Accordingly, consider a quasi-equation

$$\Phi = (\varphi_1 \approx \psi_1 \,\&\, \dots \,\&\, \varphi_n \approx \psi_n) \Longrightarrow \varepsilon \approx \delta$$

valid in K. Moreover, let $\vec{a} \in A$ be such that $\varphi_j^{\boldsymbol{A}/\phi}(\vec{a}/\phi) = \psi_j^{\boldsymbol{A}/\phi}(\vec{a}/\phi)$, for every $j \leqslant n$. Then consider $j \leqslant n$. Since

$$\varphi_j^{\boldsymbol{A}}(\vec{a})/\phi = \varphi_j^{\boldsymbol{A}/\phi}(\vec{a}/\phi) = \psi_j^{\boldsymbol{A}/\phi}(\vec{a}/\phi) = \psi_j^{\boldsymbol{A}}(\vec{a})/\phi$$

and $\phi = \bigcup_{i \in I} \theta_i$, there exists $i_j \in I$ such that $\langle \varphi_j^{\boldsymbol{A}}(\vec{a}), \psi_j^{\boldsymbol{A}}(\vec{a}) \rangle \in \theta_{i_j}$. Thus,

$$\langle \varphi_1^{\boldsymbol{A}}(\vec{a}), \psi_1^{\boldsymbol{A}}(\vec{a}) \rangle \in \theta_{i_1}, \dots, \langle \varphi_n^{\boldsymbol{A}}(\vec{a}), \psi_n^{\boldsymbol{A}}(\vec{a}) \rangle \in \theta_{i_n}.$$

Since the family $\{\theta_i : i \in I\}$ is upward directed, there exists $k \in I$ such that $\theta_{i_1}, \dots, \theta_{i_n} \subseteq \theta_k$. Therefore,

$$\langle \varphi_1^{\boldsymbol{A}}(\vec{a}), \psi_1^{\boldsymbol{A}}(\vec{a}) \rangle, \dots, \langle \varphi_n^{\boldsymbol{A}}(\vec{a}), \psi_n^{\boldsymbol{A}}(\vec{a}) \rangle \in \theta_k.$$

This implies

$$\varphi_j^{\boldsymbol{A}/\theta_k}(\vec{a}/\theta_k) = \varphi_j^{\boldsymbol{A}}(\vec{a})/\theta_k = \psi_j^{\boldsymbol{A}}(\vec{a})/\theta_k = \psi_j^{\boldsymbol{A}/\theta_k}(\vec{a}/\theta_k), \text{ for every } j \leqslant n.$$

Since $\boldsymbol{A}/\theta_k \in \mathsf{K}$, we know that this algebra validates the quasi-equation Φ. Together with the above display, this yields

$$\langle \varepsilon^{\boldsymbol{A}}(\vec{a}), \delta^{\boldsymbol{A}}(\vec{a}) \rangle \in \theta_k \subseteq \phi.$$

It follows that $\varepsilon^{\boldsymbol{A}/\phi}(\vec{a}/\phi) = \delta^{\boldsymbol{A}/\phi}(\vec{a}/\phi)$. Therefore, $\boldsymbol{A}/\phi \vDash \Phi$, as desired. ⊠

Corollary 6.14. *Let K be a quasi-variety and $\boldsymbol{A} \in \mathsf{K}$. Every K-congruence of $\boldsymbol{A}$ is the intersection of a family of K-congruences of $\boldsymbol{A}$ that are completely meet irreducible in $\mathsf{Con}_{\mathsf{K}}\boldsymbol{A}$.*

Proof. Every element of an algebraic lattice is a meet of a family of completely meet irreducible elements. Therefore, the result follows immediately from Proposition 6.13. ⊠

7. Subdirect decomposition

In this section, we shall present a general decomposition of for algebraic structures in terms of subdirect products. Because of this, it makes sense to isolate the building blocks of subdirect products, that is, the algebras that cannot be obtained as subdirect products of algebras other than themselves.

Definition 7.1. Let K be a quasi-variety. A member $\boldsymbol{A}$ of K is said to be *subdirectly irreducible relative to* K when for every subdirect emebedding $f\colon \boldsymbol{A} \to \prod_{i \in I} \boldsymbol{B}_i$ with $\{\boldsymbol{B}_i : i \in I\} \subseteq \mathsf{K}$, there exists some $i \in I$ such that the composition $p_i \circ f\colon \boldsymbol{A} \to \boldsymbol{B}_i$ is an isomorphism. The class of all subdirectly irreducible algebras relative to K will be denoted by $\mathsf{K}_{\mathsf{RSI}}$.

An algebra $\boldsymbol{A}$ is said to be *subdirectly irreducible* (in the absolute sense) when it is subdirectly irreducible relative to the quasi-variety of all algebras of its type.

The next result connects subdirect irreduciblity with congruence lattices.

Proposition 7.2. *Let K be a quasi-variety. An algebra $\boldsymbol{A} \in \mathsf{K}$ is subdirectly irreducible relative to K if and only if id_A is completely meet irreducible in $\mathsf{Con}_{\mathsf{K}}\boldsymbol{A}$.*

Proof. Suppose first that id_A is not completely meet irreducible. Then there exists a family $\{\theta_i : i \in I\} \subseteq \mathsf{Con}_{\mathsf{K}}\boldsymbol{A} \smallsetminus \{\mathrm{id}_A\}$ whose intersection is id_A. By Proposition 2.20, there exists a subdirect embedding $f\colon \boldsymbol{A} \to \prod_{i \in I} \boldsymbol{A}/\theta_i$. We will show that $p_i \circ f$ is not injective, for every $i \in I$. To this end, consider $i \in I$. We have $\mathsf{Ker}(p_i \circ f) = \theta_i$. Since θ_i is not the identity, we obtain $\mathsf{Ker}(p_i \circ f) \neq \mathrm{id}_A$, whence $p_i \circ f$ is not injective, as desired. It follows that $\boldsymbol{A}$ is not subdirectly irreducible relative to K.

To prove the other implication, suppose that id_A is completely meet irreducible in $\mathsf{Con}_{\mathsf{K}}\boldsymbol{A}$. Then consider a subdirect embedding $f\colon \boldsymbol{A} \to \prod_{i \in I} \boldsymbol{B}_i$ with $\{\boldsymbol{B}_i : i \in I\} \subseteq \mathsf{K}$. For each $i \in I$, we consider the congruence $\mathsf{Ker}(p_i \circ f)$ of $\boldsymbol{A}$. Since $p_i \circ f$ is sujective (because f is subdirect), we can apply Corollary 2.14, obtaining that $\boldsymbol{A}/\mathsf{Ker}(p_i \circ f) \cong \boldsymbol{B}_i$ and, therefore, $\boldsymbol{A}/\mathsf{Ker}(p_i \circ f) \in \mathsf{K}$. It follows that $\mathsf{Ker}(p_i \circ f) \in \mathsf{Con}_{\mathsf{K}}\boldsymbol{A}$. We will show that

$$\mathrm{id}_A = \bigcap_{i \in I} \mathsf{Ker}(p_i \circ f). \tag{13}$$

To this end, consider $a, c \in A$. We have

$$\begin{aligned}
\langle a, c \rangle \in \mathrm{id}_A &\iff a = c \\
&\iff f(a) = f(c) \\
&\iff f(a)(i) = f(c)(i), \text{ for every } i \in I \\
&\iff p_i \circ f(a) = p_i \circ f(c), \text{ for every } i \in I \\
&\iff \langle a, c \rangle \in \mathsf{Ker}(p_i \circ f), \text{ for every } i \in I \\
&\iff \langle a, c \rangle \in \bigcap_{i \in I} \mathsf{Ker}(p_i \circ f),
\end{aligned}$$

where the second equivalence follows from the injectivity of f. From (13) and the assumption that id_A is completely meet irreducible in $\mathsf{Con}_{\mathsf{K}}\boldsymbol{A}$ it follows that there exists $i \in I$ such that $\mathrm{id}_A = \mathsf{Ker}(p_i \circ f)$. It follows that the homomorphism $p_i \circ f \colon \boldsymbol{A} \to \boldsymbol{B}_i$ is injective. As it is also surjective, because f is subdirect, we conclude that it is an isomorphism. Hence, $\boldsymbol{A}$ is subdirectly irreducible relative to K. ⊠

Corollary 7.3. *An algebra $\boldsymbol{A}$ is subdirectly irreducible if and only if id_A is completely meet irreducible in $\mathsf{Con}\boldsymbol{A}$.*

Remark 7.4. Let K be a quasi-variety and $\boldsymbol{A} \in \mathsf{K}$. In view of Proposition 7.2, $\boldsymbol{A}$ is subdirectly irreducible relative to K precisely when there exists $\phi \in \mathsf{Con}_{\mathsf{K}}(\boldsymbol{A}) \smallsetminus \{\mathrm{id}_A\}$ such that every element of $\mathsf{Con}_{\mathsf{K}}(\boldsymbol{A}) \smallsetminus \{\mathrm{id}_A\}$ extends ϕ. In this case, ϕ is sometimes called the *relative monolith* of $\boldsymbol{A}$.

To prove the above equivalence, observe that if such a ϕ exists, then id_A is clearly completely meet irreducible in $\mathsf{Con}_{\mathsf{K}}(\boldsymbol{A})$ and, therefore, $\boldsymbol{A}$ is subdirectly irreducible relative to K, by Proposition 7.2. Conversely, suppose that $\boldsymbol{A}$ is subdirectly irreducible and, therefore, that id_A is completely meet irreducible in $\mathsf{Con}_{\mathsf{K}}(\boldsymbol{A})$. Then the K-congruence

$$\phi := \bigcap\{\theta \in \mathsf{Con}_{\mathsf{K}}(\boldsymbol{A}) : \theta \neq \mathrm{id}_A\}$$

is different from identity relation id_A. Furthermore, every element of $\mathsf{Con}_{\mathsf{K}}(\boldsymbol{A}) \smallsetminus \{\mathrm{id}_A\}$ extends ϕ, as desired. ⊠

A special kind of relative subdirectly irreducible algebras is the following.

Definition 7.5. Let K be a quasi-variety. An algebra $\boldsymbol{A} \in \mathsf{K}$ is *simple relative to* K if it has exactly two K-congruences.

In this case, $\mathsf{Con}_{\mathsf{K}}(\boldsymbol{A}) = \{\mathrm{id}_A, A \times A\}$ and $\mathrm{id}_A \neq A \times A$, whence $\boldsymbol{A}$ is nontrivial. Moreover, $\mathsf{Con}_{\mathsf{K}}(\boldsymbol{A})$ is the two-element chain with minimum id_A and, therefore, id_A is completely meet irreducible in $\mathsf{Con}_{\mathsf{K}}(\boldsymbol{A})$. Therefore, by Proposition 7.2, we obtain the following.

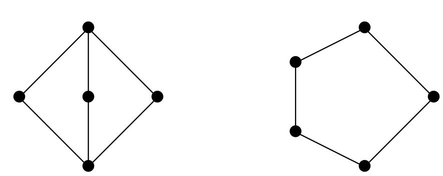
Corollary 7.6. *If an algebra $\boldsymbol{A}$ is simple relative to a quasi-variety K, it is also subdirectly irreducible relative to K.*

As for the case of subdirect irreducibility, the notion of a simple algebra admits an absolute variant. More precisely, an algebra $\boldsymbol{A}$ is *simple* (in the absolute sense) if $\mathsf{Con}(\boldsymbol{A})$ has precisely two elements.

Example 7.7 (Distributive lattices)**.** Recall that the class of distributive lattices DL coincides with $\mathbb{IP}_{\mathrm{SD}}(\boldsymbol{B})$, where $\boldsymbol{B}$ is the two-element distributive lattice. It follows that the class of (relative) subdirectly irreducible members of DL is included into $\mathbb{I}(\boldsymbol{B})$. Furthermore, notice that $\mathsf{Con}\boldsymbol{B}$ is a two-element chain with maximum $B \times B$ and minimum id_B. Consequently, $\boldsymbol{B}$ is simple and, therefore, subdirectly irreducible. It follows that $\mathbb{I}(\boldsymbol{B})$ is the class of all (relative) subdirectly irreducible members of DL.

A similar argument shows that the (relative) subdirectly irreducible members of the class of Boolean algebras are the two-element chains. ⊠

Exercise 7.8. The following lattices are called, respectively, $\boldsymbol{M}_3$ and $\boldsymbol{N}_5$.



Their importance is related to two classical result. The first, due to Dedekind, states that a lattice is nonmodular if and only if $\boldsymbol{N}_5$ embeds into it. The second, due to Birkhoff, states that a lattice fails to be distributive precisely when $\boldsymbol{M}_3$ or $\boldsymbol{N}_5$ can be embedded into it.

Find the congruence lattices of $\boldsymbol{M}_3$ and $\boldsymbol{N}_5$ and use them to convince yourself that both $\boldsymbol{M}_3$ and $\boldsymbol{N}_5$ are subdirectly irreducible. Prove also that $\boldsymbol{M}_3$ is simple, but $\boldsymbol{N}_5$ is not. $\boxtimes$

Example 7.9 (Heyting algebras)**.** A filter on a Heyting algebra $\boldsymbol{A}$ is a nonempty upset closed under binary meets. The poset of all filters of $\boldsymbol{A}$ ordered under the inclusion relation will be denoted by $\mathsf{Fi}(\boldsymbol{A})$. Recall that $\mathsf{Fi}(\boldsymbol{A})$ is isomorphic to $\mathsf{Con}(\boldsymbol{A})$ via the map $\boldsymbol{\Omega}^{\boldsymbol{A}} \colon \mathsf{Fi}(\boldsymbol{A}) \to \mathsf{Con}(\boldsymbol{A})$, defined in Example **??**. We shall prove that a Heyting algebra $\boldsymbol{A}$ is subdirectly irreducible if and only if there exists an element $a \in A \smallsetminus \{1\}$ such that $A = \{1\} \cup {\downarrow}a$.

Suppose first that $\boldsymbol{A}$ is subdirectly irreducible. In view of Remark 7.4, there exists the least congruence θ of $\boldsymbol{A}$ different from the identity. Since $\mathsf{Fi}(\boldsymbol{A}) \cong \mathsf{Con}(\boldsymbol{A})$, there exists also the least filter F of $\boldsymbol{A}$ different from the minimum $\{1\}$ of $\mathsf{Fi}(\boldsymbol{A})$. Since $F \neq \{1\}$, there exists some $a \in F \smallsetminus \{1\}$. As ${\uparrow}a$ is a filter different from $\{1\}$ and F is the least such, we obtain $F \subseteq {\uparrow}a$. On the other hand, as $a \in F$, the other inclusion holds, whence $F = {\uparrow}a$. Then consider an element $c \in A \smallsetminus \{1\}$. Since ${\uparrow}c$ is a filter different from $\{1\}$ and F is the least such, we obtain ${\uparrow}a = F \subseteq {\uparrow}c$, that is, $c \leqslant a$. Hence, $A = \{1\} \cup {\downarrow}a$, as desired.

To prove the converse, suppose that there exists an element $a \in A \smallsetminus \{1\}$ such that $A = \{1\} \cup {\downarrow}a$. In this case, ${\uparrow}a$ is the least filter of $\boldsymbol{A}$ different from the minimum filter $\{1\}$. Since $\mathsf{Fi}(\boldsymbol{A}) \cong \mathsf{Con}(\boldsymbol{A})$, there exists also the least congruence of $\boldsymbol{A}$ different from the identity relation. But, by Remark 7.4, this implies that $\boldsymbol{A}$ is subdirectly irreducible. $\boxtimes$

The last ingredient of the general representation theorem is the following.

Correspondence Theorem 7.10. *Let* K *be a quasi-variety and* $\boldsymbol{A} \in \mathsf{K}$. *Given a* K*-congruence* θ *of* $\boldsymbol{A}$, *the subposet* ${\uparrow}\theta$ *of* $\mathsf{Con}_{\mathsf{K}}(\boldsymbol{A})$ *is isomorphic to* $\mathsf{Con}_{\mathsf{K}}(\boldsymbol{A}/\theta)$ *under the map* $f \colon {\uparrow}\theta \to \mathsf{Con}_{\mathsf{K}}(\boldsymbol{A}/\theta)$, *defined by the rule*

$$f(\phi) := \{\langle a/\theta, c/\theta\rangle \in A/\theta \times A/\theta : \langle a, c\rangle \in \phi\}.$$

Proof. We claim that, for every $\phi \in \mathsf{Con}_{\mathsf{K}}(\boldsymbol{A})$ such that $\theta \subseteq \phi$ and $a, c \in A$,

$$\langle a, c\rangle \in \phi \Longleftrightarrow \langle a/\theta, c/\theta\rangle \in f(\phi).$$

The implication from left to right is an immediate consequence of the definition of f. To prove the other implication, suppose that $\langle a/\theta, c/\theta\rangle \in f(\phi)$. By definition of $f(\phi)$, there is a pair $\langle b, d\rangle \in \phi$ such that $a/\theta = b/\theta$ and $c/\theta = d/\theta$. Since ϕ is an equivalence relation extending θ, this implies $\langle a, c\rangle \in \phi$, as desired.

Then we turn to prove that f is well-defined. Consider $\phi \in \mathsf{Con}_{\mathsf{K}}(\boldsymbol{A})$ such that $\theta \subseteq \phi$. Since ϕ is an equivalence relation on A, the definition of f guarantees that $f(\phi)$ is an equivalence relation on A/θ. Then consider a basic n-ary operation g and $a_1, \dots, a_n, c_1, \dots, c_n \in A$ such that

$$\langle a_1/\theta, c_1/\theta\rangle, \dots, \langle a_n/\theta, c_n/\theta\rangle \in f(\phi).$$

By the claim, $\langle a_1, c_1\rangle, \dots, \langle a_n, c_n\rangle \in \phi$. Since ϕ is a congruence of $\boldsymbol{A}$, this yields

$$\langle g^{\boldsymbol{A}}(a_1, \dots, a_n), g^{\boldsymbol{A}}(c_1, \dots, c_n)\rangle \in \phi.$$

Hence,

$$\begin{aligned} g^{A/\theta}(a_1/\theta, \dots, a_n/\theta) &= g^{A}(a_1, \dots, a_n)/\theta \\ &\equiv_{f(\phi)} g^{A}(c_1, \dots, c_n)/\theta \\ &= g^{A/\theta}(c_1/\theta, \dots, c_n/\theta) \end{aligned}$$

and, therefore, $f(\phi)$ is a congruence of A/θ. The proof that $f(\phi)$ is also a K-congruence of A/θ is left an an exercise. We conclude that f is well-defined.

Then consider $\phi, \eta \in \mathsf{Con}_{\mathsf{K}}(A)$ such that $\theta \subseteq \phi, \eta$. We have

$$\phi \subseteq \eta \iff f(\phi) \subseteq f(\eta).$$

The implication from left to right is an immediate consequence of the definition of f. To prove the other implication, suppose that $f(\phi) \subseteq f(\eta)$ and consider a pair $\langle a, c \rangle \in \phi$. We have $\langle a/\theta, c/\theta \rangle \in f(\phi) \subseteq f(\eta)$. With an application of the claim, we obtain $\langle a, c \rangle \in \eta$, as desired. Hence, f is an order embedding.

To prove that it is surjective, consider a K-congruence ϕ of A/θ and let

$$\eta := \{\langle a, c \rangle \in A \times A : \langle a/\theta, c/\theta \rangle \in \phi\}.$$

Since ϕ is an equivalence relation on A/θ, the definition of η guarantees that η is an equivalence relation on A. Then consider an n-ary operation g and $a_1, \dots, a_n, c_1, \dots, c_n \in A$ such that $\langle a_1, c_1 \rangle, \dots, \langle a_n, c_n \rangle \in \eta$. We have $\langle a_1/\theta, c_1/\theta \rangle, \dots, \langle a_n/\theta, c_n/\theta \rangle \in \phi$. Since ϕ is a congruence on A/θ, this yields

$$g^{A}(a_1, \dots, a_n)/\theta = g^{A/\theta}(a_1/\theta, \dots, a_n/\theta) \equiv_{\phi} g^{A/\theta}(c_1/\theta, \dots, c_n/\theta) = g^{A}(c_1, \dots, c_n)/\theta$$

and, therefore, $\langle g^{A}(a_1, \dots, a_n), g^{A}(c_1, \dots, c_n) \rangle \in \eta$. We conclude that η is a congruence of A. The proof that η is also a K-congruence of A is left as an exercise.

To prove that it extends θ, consider a pair $\langle a, c \rangle \in \theta$. Then $a/\theta = c/\theta$ and, since ϕ is reflexive, $\langle a/\theta, c/\theta \rangle \in \phi$. It follows that $\langle a, c \rangle \in \phi$, as desired. Thus, $\phi \in {\uparrow}\theta$. Furthermore, the definition of f implies that $f(\eta) = \phi$. Hence, f is surjective and, therefore, an isomorphism. ⊠

Exercise 7.11. Two parts of the above proof (both related to K-congruences) were left as an exercise. Complete the missing details. ⊠

The following representation theorem is as an application of lattice theory to general algebra, its main ingredient being the observation that every element of an algebraic lattice can be obtained as a meet of completely meet irreducible ones.

Subdirect Decomposition Theorem 7.12. *If K is a quasi-variety, then $\mathsf{K} = \mathbb{IP}_{\mathsf{SD}}(\mathsf{K}_{\mathsf{RSI}})$.*

Proof. Consider an algebra $A \in \mathsf{K}$. By Corollary 6.14, there exists a family $\{\theta_i : i \in I\} \subseteq \mathsf{Con}_{\mathsf{K}}(A)$ such that each θ_i is completely meet irreducible in $\mathsf{Con}_{\mathsf{K}}(A)$ and, moreover,

$$\mathrm{id}_A = \bigcap_{i \in I} \theta_i.$$

By Proposition 2.20, $A \in \mathbb{IP}_{\mathsf{SD}}(\{A/\theta_i : i \in I\})$.

Now, we know that each A/θ_i belongs to K, because θ_i is a K-congruence of A. Therefore, to conclude the proof, it only remains to show that A/θ_i is subdirectly irreducible relative to K. By the Correspondence Theorem, $\mathsf{Con}_{\mathsf{K}}(A/\theta_i)$ is isomorphic to the upset generated by θ_i in $\mathsf{Con}_{\mathsf{K}}A$. Moreover, this isomorphism sends θ_i to id_{A/θ_i}. Therefore, from the assumption that θ_i is completely meet irreducible in $\mathsf{Con}_{\mathsf{K}}(\boldsymbol{A})$, it follows that id_{A/θ_i} is completely meet irreducible in $\mathsf{Con}_{\mathsf{K}}(\boldsymbol{A}/\theta_i)$. By Proposition 7.2, we conclude that $\boldsymbol{A}/\theta_i$ is subdirectly irreducible relative to K. ⊠

Corollary 7.13. *Every algebra is isomorphic to a subdirect product of subdirectly irreducible algebras.*

The Subdirect Decomposition Theorem was discovered by Birkhoff in the form of the above corollary. Its posterior formulation for quasi-varieties is due to Maltsev.

Example 7.14 (Subdirect decomposition)**.** Birkhoff's representation of distributive lattices as subdirect products of the two-element chain and Stone's representation of Boolean algebras as subdirect products of the two-element Boolean algebra are special instances of the Subdirect Decomposition Theorem. Another such application is the observation that every Heyting algebra is isomorphic to a subdirect products of subdirectly irreducible Heyting algebras, where the latter were described in Example 7.9. Lastly, an example from classical algebra is the following: every finite Abelian group is a direct product of cyclic groups of prime power order. The latter happen to be precisely the finite subdirectly irreducible Abelian groups. ⊠

Since relative subdirectly irreducible algebras form the building blocks of quasi-varieties, it is natural to wonder how do they arise. The next results provides an answer in terms of the generators of a quasi-variety.

Proposition 7.15. *If* K *is a class of similar algebras, then* $\mathbb{Q}(\mathsf{K})_{\mathrm{RSI}} \subseteq \mathbb{ISP}_{\mathrm{U}}(\mathsf{K})$*.*

Proof. Consider an algebra $\boldsymbol{A} \in \mathbb{Q}(\mathsf{K})_{\mathrm{RSI}}$. Recall from Corollary 5.7 that $\mathbb{Q}(\mathsf{K}) = \mathbb{ISPP}_{\mathrm{U}}(\mathsf{K})$. Moreover, it is easy to see that $\mathbb{ISPP}_{\mathrm{U}}(\mathsf{K}) = \mathbb{IP}_{\mathrm{SD}}\mathbb{SP}_{\mathrm{U}}(\mathsf{K})$, whence $\mathbb{Q}(\mathsf{K}) = \mathbb{IP}_{\mathrm{SD}}\mathbb{SP}_{\mathrm{U}}(\mathsf{K})$. In particular, this implies $\boldsymbol{A} \in \mathbb{IP}_{\mathrm{SD}}\mathbb{SP}_{\mathrm{U}}(\mathsf{K})$. Accordingly, there exists a subdirect embedding $f\colon \boldsymbol{A} \to \prod_{i\in I} \boldsymbol{B}_i$ with $\{\boldsymbol{B}_i \colon i \in I\} \subseteq \mathbb{SP}_{\mathrm{U}}(\mathsf{K})$. As $\boldsymbol{A}$ is subdirectly irreducible relative to $\mathbb{Q}(\mathsf{K})$ and $\mathbb{SP}_{\mathrm{U}}(\mathsf{K}) \subseteq \mathbb{Q}(\mathsf{K})$, there exists $i \in I$ such that $p_i \circ f\colon \boldsymbol{A} \to \boldsymbol{B}_i$ is an isomorphism. Thus, $\boldsymbol{A} \in \mathbb{I}(\boldsymbol{B}) \subseteq \mathbb{ISP}_{\mathrm{U}}(\mathsf{K})$. ⊠

Corollary 7.16. *If* K *is a finite set of finite similar algebras, then* $\mathbb{Q}(\mathsf{K})_{\mathrm{RSI}} \subseteq \mathbb{IS}(\mathsf{K})$*.*

Example 7.17 (Closure under $\mathbb{H}$)**.** Consider the lattices $\boldsymbol{A}$ and $\boldsymbol{B}$ defined at the beginning of Section 6. There, we prove that the quasi-variety $\mathbb{Q}(\boldsymbol{B})$ is not closed under $\mathbb{H}$. We are now in the position of offering a simpler proof that, moreover, can be easily adapted to other cases.

First, observe that $\boldsymbol{A} \in \mathbb{H}(\boldsymbol{B})$ and, therefore, it suffices to show that $\boldsymbol{A} \notin \mathbb{Q}(\boldsymbol{B})$. Suppose the contrary. Since $\boldsymbol{A}$ is simple and, therefore, subdirectly irreducible, it must be also subdirectly irreducible relative to $\mathbb{Q}(\boldsymbol{B})$. Therefore, $\boldsymbol{A} \in \mathbb{IS}(\boldsymbol{B})$, by Corollary 7.16. This contradicts the fact that there cannot be any embedding of $\boldsymbol{A}$ into $\boldsymbol{B}$ (look at their Hasse diagrams to convince you of this). ⊠

*Exercise** 7.18 (Modal algebras). Modal algebras and open filters were defined in Example 2.8, where it is also explained that the lattice $\mathsf{Op}(\boldsymbol{A})$ of open filters of a modal algebra $\boldsymbol{A}$ is isomorphic to $\mathsf{Con}(\boldsymbol{A})$ under the maps $\boldsymbol{\Omega}^{\boldsymbol{A}}$ and $\boldsymbol{\tau}^{\boldsymbol{A}}$. Given a modal algebra $\boldsymbol{A}$, an element $a \in A$, and a natural number $n \in \mathbb{N}$, we define recursively an element $\Box^n a$ of $\boldsymbol{A}$ as follows:

$$\Box^0 a := a \text{ and } \Box^{k+1} a := \Box^{\boldsymbol{A}} \Box^k a.$$

Similarly, we define an element $\boxplus^n a$ as follows:

$$\boxplus^0 a := a \text{ and } \boxplus^{k+1} a := \Box^{k+1} a \wedge \boxplus^k a.$$

Thus, for instance, $\boxplus^2 a = \Box\Box a \wedge \Box a \wedge a$, where $\Box$ binds stronger than $\wedge$.

Prove that a modal algebra $\boldsymbol{A}$ is subdirectly irreducible if and only if there exists an element $a \in A \smallsetminus \{1\}$ such that for every $c \in A \smallsetminus \{1\}$ there exists some $n \in \mathbb{N}$ such that $\boxplus^n c \leqslant a$. To do this, take inspiration from the proof of the characterization of subdirectly irreducible Heyting algebras given in Example 7.9. While doing so, be careful to the fact that arbitrary filters of $\boldsymbol{A}$ need not be open! Indeed, if F is an open filter of $\boldsymbol{A}$ and $a \in F$, then $\boxplus^n a \in F$, for every $n \in \mathbb{N}$. More precisely, you will need to use (and prove) the fact that the open filter of $\boldsymbol{A}$ generated by a subset $X \subseteq A$ is

$$\{1\} \cup \{a \in A : \boxplus^m (c_1 \wedge \cdots \wedge c_n) \leqslant a : m, n \in \mathbb{N}, 1 \leqslant n, \text{ and } c_1, \ldots, c_n \in X\}. \qquad \boxtimes$$

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