# LECTURE NOTES ON ALGEBRAIC LOGIC

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## 1. Algebras and equations

We begin by reviewing some fundamentals of general algebraic systems.

**Definition 1.1.**

(i) A *type* is a map $\rho\colon \mathcal{F} \to \mathbb{N}$, where $\mathcal{F}$ is a set of function symbols. In this case, $\rho(f)$ is said to be the *arity* of the function symbol $f$, for every $f \in \mathcal{F}$. Function symbols of arity zero are called *constants*.

(ii) An *algebra* of type $\rho$ is a pair $\boldsymbol{A} = \langle A; F\rangle$ where $A$ is a nonempty set and $F = \{f^{\boldsymbol{A}} : f \in \mathcal{F}\}$ is a set of operations on $A$ whose arity is determined by $\rho$, in the sense that each $f^{\boldsymbol{A}}$ has arity $\rho(f)$. The set $A$ is called the *universe* of $\boldsymbol{A}$.

When $\mathcal{F} = \{f_1, \dots, f_n\}$, we shall write $\langle A; f_1^{\boldsymbol{A}}, \dots, f_n^{\boldsymbol{A}}\rangle$ instead of $\langle A; F\rangle$. In this case, we often drop the superscripts, and write simply $\langle A; f_1, \dots, f_n\rangle$.

Classical examples of algebras are groups and rings. For instance, the type of groups $\rho_G$ consists of a binary symbol $+$, a unary symbol $-$, and a constant symbol 0. Then a group is an algebra $\langle G; +, -, 0\rangle$ of type $\rho_G$ in which $+$ is associative, 0 is a neutral element for $+$, and $-$ produces inverses.

Lattices, Heyting algebras, and modal algebras are also algebras in the above sense. For instance, the type of lattices $\rho_L$ consists of two binary symbols $\wedge$ and $\vee$ and a lattice is an algebra $\langle A; \wedge, \vee\rangle$ of type $\rho_L$ that satisfies the idempotent, commutative, associative, and absorption laws. Similarly, the type of Heyting algebras $\rho_H$ consists of three binary operations symbols $\wedge, \vee$, and $\to$ and of two constant symbols 0 and 1. Then a Heyting algebra is an algebra $\langle A; \wedge, \vee, \to, 0, 1\rangle$ such that $\langle A; \wedge, \vee, 0, 1\rangle$ is a bounded lattice and, for every $a, b, c \in A$,

$$a \wedge b \leqslant c \iff a \leqslant b \to c. \qquad (\textit{residuation law})$$

Boolean algebras can be viewed as the Heyting algebras that satisfy the following equational version of the *excluded middle law*:

$$x \vee (x \to 0) \approx 1.$$

In this case, the complement operation $\neg x$ can be defined as $x \to 0$.

Perhaps less obviously, even algebraic structures whose operations are apparently *external* can be viewed as algebras in the sense of the above definition. For instance, modules over a ring $\boldsymbol{R}$ can be viewed as algebras whose type $\rho_R$ extends that of groups with the unary symbols $\{\lambda_r : r \in R\}$. From this point of view, a module over $\boldsymbol{R}$ is an

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algebra $\langle G; +, -, 0, \{\lambda_r : r \in R\}\rangle$ of type $\rho_R$ such that $\langle G; +, -, 0\rangle$ is an abelian group and, for every $r, s \in R$ and $a, c \in G$,

$$\begin{aligned}
\lambda_r(a + c) &= \lambda_r(a) + \lambda_r(c)\\
\lambda_{r+s}(a) &= \lambda_r(a) + \lambda_s(a)\\
\lambda_r(\lambda_s(a)) &= \lambda_{r\cdot s}(a)\\
\lambda_1(a) &= a.
\end{aligned}$$

Given a type $\rho\colon \mathcal{F} \to \mathbb{N}$ and a set of variables $X$ disjoint from $\mathcal{F}$, the set of *terms of type $\rho$ over $X$* is the least set $T_\rho(X)$ such that

(i) $X \subseteq T_\rho(X)$;
(ii) if $c \in \mathcal{F}$ is a constant, then $c \in T_\rho(X)$; and
(iii) if $\varphi_1, \dots, \varphi_{\rho(f)} \in T_\rho(X)$ and $f \in \mathcal{F}$, then $f\varphi_1 \dots \varphi_{\rho(f)} \in T_\rho(X)$.

For the sake of readability, we shall often write $f(\varphi_1, \dots, \varphi_{\rho(f)})$ instead of $f\varphi_1 \dots \varphi_{\rho(f)}$. Similarly, if $f$ is a binary operation $+$, we often write $\varphi_1 + \varphi_2$ instead of $f(\varphi_1, \varphi_2)$.

Given a term $\varphi \in T_\rho(X)$, we write $\varphi(x_1, \dots, x_n)$ to indicate that the variables occurring in $\varphi$ are among $x_1, \dots, x_n$. Furthermore, given an algebra $\boldsymbol{A}$ of type $\rho$ and elements $a_1, \dots, a_n \in A$, we define an element

$$\varphi^{\boldsymbol{A}}(a_1, \dots, a_n)$$

of $A$, by recursion on the construction of $\varphi$, as follows:

(i) if $\varphi$ is a variable $x_i$, then $\varphi^{\boldsymbol{A}}(a_1, \dots, a_n) \coloneqq a_i$;
(ii) if $\varphi$ is a constant $c$, then $c^{\boldsymbol{A}}$ is the interpretation of $c$ in $\boldsymbol{A}$;
(iii) if $\varphi = f(\psi_1, \dots, \psi_m)$, then

$$\varphi^{\boldsymbol{A}}(a_1, \dots, a_n) \coloneqq f^{\boldsymbol{A}}(\psi_1^{\boldsymbol{A}}(a_1, \dots, a_n), \dots, \psi_m^{\boldsymbol{A}}(a_1, \dots, a_n)).$$

An *equation of type $\rho$ over $X$* is an expression of the form $\varphi \approx \psi$, where $\varphi, \psi \in T_\rho(X)$. Such an equation $\varphi \approx \psi$ is *valid* in an algebra $\boldsymbol{A}$ of type $\rho$, if

$$\varphi^{\boldsymbol{A}}(a_1, \dots, a_n) = \psi^{\boldsymbol{A}}(a_1, \dots, a_n), \text{ for every } a_1, \dots, a_n \in A,$$

in which case we say that $\boldsymbol{A}$ *satisfies* $\varphi \approx \psi$.

For instance, groups are precisely the algebras of type $\rho_G$ that satisfy the equations

$$x + (y + z) \approx (x + y) + z \quad x + 0 \approx x \quad 0 + x \approx x \quad x + -x \approx 0 \quad -x + x \approx 0.$$

Similarly, lattices are the algebras of type $\rho_L$ that satisfy the equations

$$\begin{array}{llr}
x \wedge x \approx x & x \vee x \approx x & (\textit{idempotent laws})\\
x \wedge y \approx y \wedge x & x \vee y \approx y \vee x & (\textit{commutative laws})\\
x \wedge (y \wedge z) \approx (x \wedge y) \wedge z & x \vee (y \vee z) \approx (x \vee y) \vee z & (\textit{associative laws})\\
x \wedge (y \vee x) \approx x & x \vee (y \wedge x) \approx x. & (\textit{absorption laws})
\end{array}$$

## 2. Basic constructions

Algebras of the same type are called *similar* and can be compared by means of maps that preserve their structure.

**Definition 2.1.** Given two similar algebras $\boldsymbol{A}$ and $\boldsymbol{B}$, a *homomorphism* from $\boldsymbol{A}$ to $\boldsymbol{B}$ is a map $f\colon A \to B$ such that, for every $n$-ary operation $g$ of the common type and $a_1, \dots, a_n \in A$,

$$f(g^{\boldsymbol{A}}(a_1, \dots, a_n)) = g^{\boldsymbol{B}}(f(a_1), \dots, f(a_n)).$$

An injective homomorphism is called an *embedding* and, if there exists an embedding from $\boldsymbol{A}$ to $\boldsymbol{B}$, we say that $\boldsymbol{A}$ *embeds* into $\boldsymbol{B}$. Lastly, a surjective embedding is called an *isomorphism*. Accordingly, $\boldsymbol{A}$ and $\boldsymbol{B}$ are said to be *isomorphic* if there exists an isomorphism between them, in which case we write $\boldsymbol{A} \cong \boldsymbol{B}$.

A simple induction on the construction of terms shows that, for every pair of algebras $\boldsymbol{A}$ and $\boldsymbol{B}$ of type $\rho$ and every term $\varphi(x_1, \dots, x_n)$ of $\rho$, if $f$ is a homomorphism from $\boldsymbol{A}$ to $\boldsymbol{B}$, then

$$f(\varphi^{\boldsymbol{A}}(a_1, \dots, a_n)) = \varphi^{\boldsymbol{B}}(f(a_1), \dots, f(a_n)),$$

for every $a_1, \dots, a_n \in A$. Therefore homomorphisms preserve not only basic operations, but also arbitrary terms.

In the particular case where $\boldsymbol{A}$ and $\boldsymbol{B}$ are lattices, a homomorphism from $\boldsymbol{A}$ to $\boldsymbol{B}$ is a map $f\colon A \to B$ such that, for every $a, c \in A$,

$$f(a \wedge^{\boldsymbol{A}} c) = f(a) \wedge^{\boldsymbol{B}} f(c) \text{ and } f(a \vee^{\boldsymbol{A}} c) = f(a) \vee^{\boldsymbol{B}} f(c).$$

For instance, the inclusion map from the lattice $\langle \mathbb{N}; \leqslant \rangle$ into the lattice $\langle \mathbb{Z}; \leqslant \rangle$ is an injective homomorphism, that is, an embedding. Similarly, given two sets $Y \subseteq X$, the inclusion map from the powerset lattice $\langle \mathcal{P}(Y); \subseteq \rangle$ to the powerset lattice $\langle \mathcal{P}(X); \subseteq \rangle$ is also an embedding. On the other hand, if $Y \subsetneq X$, the map

$$(-) \cap Y\colon \mathcal{P}(X) \to \mathcal{P}(Y)$$

that sends every $Z \subseteq X$ to $Z \cap Y$ is a noninjective homomorphism from $\langle \mathcal{P}(X); \subseteq \rangle$ to $\langle \mathcal{P}(Y); \subseteq \rangle$.

**Definition 2.2.** Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be algebras of the same type $\rho\colon \mathcal{F} \to \mathbb{N}$. Then $\boldsymbol{A}$ is said to be a *subalgebra* of $\boldsymbol{B}$ if $A \subseteq B$ and $f^{\boldsymbol{A}}$ is the restriction of $f^{\boldsymbol{B}}$ to $A$, for every $f \in \mathcal{F}$. In this case, we write $\boldsymbol{A} \leqslant \boldsymbol{B}$.

Given a class of algebras $\mathsf{K}$, let

$$\mathbb{I}(\mathsf{K}) := \{\boldsymbol{A} : \boldsymbol{A} \cong \boldsymbol{B} \text{ for some } \boldsymbol{B} \in \mathsf{K}\}$$
$$\mathbb{S}(\mathsf{K}) := \{\boldsymbol{A} : \boldsymbol{A} \leqslant \boldsymbol{B} \text{ for some } \boldsymbol{B} \in \mathsf{K}\}.$$

When $\mathsf{K} = \{\boldsymbol{A}\}$, we write $\mathbb{I}(\boldsymbol{A})$ and $\mathbb{S}(\boldsymbol{A})$ as a shorthand for $\mathbb{I}(\{\boldsymbol{A}\})$ and $\mathbb{S}(\{\boldsymbol{A}\})$, respectively. The following observation is an immediate consequence of the definitions.

**Proposition 2.3.** *Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be algebras of the same type. Then $\boldsymbol{A} \in \mathbb{IS}(\boldsymbol{B})$ if and only if there exists an embedding $f\colon \boldsymbol{A} \to \boldsymbol{B}$. In this case, $\boldsymbol{A}$ is isomorphic to the unique subalgebra of $\boldsymbol{B}$ with universe $f[A]$.*

As we mentioned, homomorphisms can be used to compare similar algebras.

**Definition 2.4.** Given two similar algebras $\boldsymbol{A}$ and $\boldsymbol{B}$, we say that $\boldsymbol{A}$ is a *homomorphic image* of $\boldsymbol{B}$ if there exists a surjective homomorphism $f\colon \boldsymbol{B} \to \boldsymbol{A}$.

Accordingly, given a class of algebras $\mathsf{K}$, we set

$$\mathbb{H}(\mathsf{K}) := \{\boldsymbol{A} : \boldsymbol{A} \text{ is a homomorphic image of some } \boldsymbol{B} \in \mathsf{K}\}.$$

As usual, when $\mathsf{K} = \{\boldsymbol{A}\}$, we write $\mathbb{H}(\boldsymbol{A})$ as a shorthand for $\mathbb{H}(\{\boldsymbol{A}\})$.

Observe that every (not necessarily surjective) homomorphism $f \colon \boldsymbol{A} \to \boldsymbol{B}$ induces a homomorphic image of $\boldsymbol{A}$.

**Proposition 2.5.** *If $f \colon \boldsymbol{A} \to \boldsymbol{B}$ is a homomorphism, then $f[A]$ is the universe of a subalgebra of $\boldsymbol{B}$ that, moreover, is a homomorphic image of $\boldsymbol{A}$.*

*Proof.* Observe that $f[A]$ is nonempty, because $A$ is. Then consider an $n$-ary function symbol $g$ of the common type of $\boldsymbol{A}$ and $\boldsymbol{B}$ and $b_1, \dots, b_n \in f[A]$. Clearly, there are $a_1, \dots, a_n \in A$ such that $f(a_i) = b_i$, for every $i \leqslant n$. Since $f$ is a homomorphism from $\boldsymbol{A}$ to $\boldsymbol{B}$, we obtain

$$g^{\boldsymbol{B}}(b_1, \dots, b_n) = g^{\boldsymbol{B}}(f(a_1), \dots, g(a_n)) = f(g^{\boldsymbol{A}}(a_1, \dots, a_n)) \in f[A].$$

Hence, we conclude that $f[A]$ is the universe of a subalgebra $f[\boldsymbol{A}]$ of $\boldsymbol{B}$.

Furthermore, $f \colon \boldsymbol{A} \to f[\boldsymbol{A}]$ is a homomorphism, because for every basic $n$-ary function symbol $g$ of the common type and $a_1, \dots, a_n \in A$,

$$f(g^{\boldsymbol{A}}(a_1, \dots, a_n)) = g^{\boldsymbol{B}}(f(a_1), \dots, f(a_n)) = g^{f[\boldsymbol{A}]}(f(a_1), \dots, f(a_n)),$$

where the first equality follows from the assumption that $f \colon \boldsymbol{A} \to \boldsymbol{B}$ is a homomorphism. Since the map $f \colon A \to f[A]$ is surjective, we conclude that $f[\boldsymbol{A}] \in \mathbb{H}(\boldsymbol{A})$. ⊠

In view of the above result, when $f \colon \boldsymbol{A} \to \boldsymbol{B}$ is a homomorphism, we denote by $f[\boldsymbol{A}]$ the unique subalgebra of $\boldsymbol{B}$ with universe $f[A]$.

For instance, let $f \colon \mathbb{Z} \to \mathbb{R}$ be the absolute value map, that is, the function defined by the rule

$$f(n) := \text{the absolute value of } n.$$

Observe that $f$ is a nonsurjective homomorphism from the lattice of integers to that of reals. Furthermore, the homomorphic image $f[\langle \mathbb{Z}; \leqslant \rangle]$ of $\langle \mathbb{Z}; \leqslant \rangle$ is the lattice of natural numbers $\langle \mathbb{N}; \leqslant \rangle$, which, in turn, is a subalgebra of lattice of reals.

Notably, the homomorphic images of an algebra $\boldsymbol{A}$ can be "internalized" as special equivalence relations on $\boldsymbol{A}$ as follows.

**Definition 2.6.** A *congruence* of an algebra $\boldsymbol{A}$ is an equivalence relation $\theta$ on $A$ such that, for every basic $n$-ary opearation $f$ of $\boldsymbol{A}$ and $a_1, \dots, a_n, c_1, \dots, c_n \in A$,

$$\text{if } \langle a_1, c_1 \rangle, \dots, \langle a_n, c_n \rangle \in \theta, \text{ then } \langle f^{\boldsymbol{A}}(a_1, \dots, a_n), f^{\boldsymbol{A}}(c_1, \dots, c_n) \rangle \in \theta. \tag{1}$$

In this case, we often write $a \equiv_\theta c$ as a shorthand for $\langle a, c \rangle \in \theta$. The poset of congruences of $\boldsymbol{A}$ ordered under the inclusion relation will be denoted by $\mathsf{Con}(\boldsymbol{A})$.

A simple induction on the construction of terms shows that, for every congruence $\theta$ of $\boldsymbol{A}$ and every term $\varphi(x_1, \dots, x_n)$,

$$\text{if } \langle a_1, c_1 \rangle, \dots, \langle a_n, c_n \rangle \in \theta, \text{ then } \langle \varphi^{\boldsymbol{A}}(a_1, \dots, a_n), \varphi^{\boldsymbol{A}}(c_1, \dots, c_n) \rangle \in \theta,$$

for every $a_1, \dots, a_n \in A$. Therefore, congruences preserve not only basic operations, but also arbitrary terms. Furthermore, a simple argument shows that $\mathsf{Con}(\boldsymbol{A})$ is an inductive closure system and, therefore, an algebraic lattice whose maximum is the total relation $A \times A$ and whose minimum is the identity relation $\mathrm{id}_A := \{\langle a, a \rangle : a \in A\}$.

**Example 2.7** (Boolean algebras)**.** Recall that a *filter* of a Boolean algebra $\boldsymbol{A}$ is a nonempty upset $F \subseteq A$ closed under binary meets. We denote by $\mathsf{Fi}(\boldsymbol{A})$ the poset of filters of $\boldsymbol{A}$ ordered under the inclusion relation. It is easy to see $\mathsf{Fi}(\boldsymbol{A})$ is an inductive closure system and, therefore, an algebraic lattice. Furthermore, the lattices $\mathsf{Fi}(\boldsymbol{A})$ and $\mathsf{Con}(\boldsymbol{A})$ are isomorphic via the inverse isomorphisms

$$\boldsymbol{\Omega}^{\boldsymbol{A}}(-)\colon \mathsf{Fi}(\boldsymbol{A}) \to \mathsf{Con}(\boldsymbol{A}) \text{ and } \boldsymbol{\tau}(-)\colon \mathsf{Con}(\boldsymbol{A}) \to \mathsf{Fi}(\boldsymbol{A})$$

defined by the rules

$$\begin{aligned}\boldsymbol{\Omega}^{\boldsymbol{A}}(F) &:= \{\langle a,c\rangle \in A \times A : a \to c, c \to a \in F\}\\ \boldsymbol{\tau}(\theta) &:= \{a \in A : \langle a,1\rangle \in \theta\}.\end{aligned}$$

Because of this, every congruence $\theta$ of a Boolean algebra $\boldsymbol{A}$ is induced by some filter $F$, in the sense that $\theta = \boldsymbol{\Omega}^{\boldsymbol{A}}F$. This correspondence between filters and congruences generalizes straightforwardly to all Heyting algebras. ⊠

**Example 2.8** (Modal algebras)**.** A *modal algebra* is an algebra $\boldsymbol{A} = \langle A; \wedge, \vee, \neg, \Box, 0, 1\rangle$ such that $\langle A; \wedge, \vee, \neg, 0, 1\rangle$ is a Boolean algebra and $\Box$ is a unary operation such that

$$\Box(a \wedge c) = \Box a \wedge \Box c \text{ and } \Box 1 = 1,$$

for every $a, c \in A$. An *open filter* of a modal algebra $\boldsymbol{A}$ is a filter of the Boolean reduct of $\boldsymbol{A}$ that, moreover, is closed under the operation $\Box$. The poset of open filters of $\boldsymbol{A}$ ordered under the inclusion relation will be denoted by $\mathsf{Op}(\boldsymbol{A})$. It forms an inductive closure system and, therefore, an algebraic lattice. Furthermore, the lattices $\mathsf{Op}(\boldsymbol{A})$ and $\mathsf{Con}(\boldsymbol{A})$ are isomorphic via the inverse isomorphisms described in Example 2.7. Because of this, every congruence of a modal algebra $\boldsymbol{A}$ has the form $\theta = \boldsymbol{\Omega}^{\boldsymbol{A}}F$, for some open filter $F$. ⊠

**Example 2.9** (Groups)**.** Similarly, it is well known that the lattice of congruences of a group is isomorphic to that of its normal subgroups. Because of this, every congruence of a group is induced by some normal subgroup. ⊠

As we mentioned, there is a tight correspondence between the homomorphic images and the congruences of an algebra $\boldsymbol{A}$. On the one hand, every congruence $\theta$ of $\boldsymbol{A}$ gives rise to a homomorphic image $\boldsymbol{A}/\theta$ of $\boldsymbol{A}$. Let $\mathcal{F}$ be the set of function symbols of $\boldsymbol{A}$. Given $\theta \in \mathsf{Con}(\boldsymbol{A})$ and a basic $n$-ary function symbol $f \in \mathcal{F}$, let $f^{\boldsymbol{A}/\theta}$ be the $n$-ary operation on $A/\theta$ defined by the rule

$$f^{\boldsymbol{A}/\theta}(a_1/\theta, \dots, a_n/\theta) := f^{\boldsymbol{A}}(a_1, \dots, a_n)/\theta.$$

Notice that $f^{\boldsymbol{A}/\theta}$ is well-defined, by condition (1). As a consequence, the structure

$$\boldsymbol{A}/\theta := \langle A/\theta; \{f^{\boldsymbol{A}/\theta} : f \in \mathcal{F}\}\rangle$$

is a well-defined algebra of the type as $\boldsymbol{A}$. Furthermore, $\boldsymbol{A}/\theta \in \mathbb{H}(\boldsymbol{A})$, because the map $\pi_\theta\colon A \to A/\theta$, defined, for every $a \in A$, as $\pi_\theta(a) := a/\theta$, is a surjective homomorphism from $\boldsymbol{A}$ to $\boldsymbol{A}/\theta$. To prove this, consider $a_1, \dots, a_n \in A$. We have

$$\begin{aligned}\pi_\theta(f^{\boldsymbol{A}}(a_1, \dots, a_n)) &= f^{\boldsymbol{A}}(a_1, \dots, a_n)/\theta\\ &= f^{\boldsymbol{A}/\theta}(a_1/\theta, \dots, a_n/\theta)\\ &= f^{\boldsymbol{A}/\theta}(\pi_\theta(a_1), \dots, \pi_\theta(a_n)),\end{aligned}$$

where the second equality follows from the definition of the operation $f^{\boldsymbol{A}/\theta}$.

**Corollary 2.10.** *If $\theta$ is a congruence of an algebra $\boldsymbol{A}$, then $\boldsymbol{A}/\theta$ is a well-defined homomorphic image of $\boldsymbol{A}$.*

In view of the above result, every congruence $\theta$ of an algebra $\boldsymbol{A}$ induces a homomorphic image of $\boldsymbol{A}$, namely $\boldsymbol{A}/\theta$. The converse is also true, as we proceed to explain.

**Definition 2.11.** The *kernel* of a homomorphism $f\colon \boldsymbol{A} \to \boldsymbol{B}$ is the binary relation

$$\mathsf{Ker}(f) := \{\langle a, c\rangle \in A \times A : f(a) = f(c)\}.$$

**Proposition 2.12.** *The kernel of a homomorphism $f\colon \boldsymbol{A} \to \boldsymbol{B}$ is a congruence of $\boldsymbol{A}$.*

*Proof.* It is obvious that $\mathsf{Ker}(f)$ is an equivalence relation on $A$. Therefore, to prove that $\mathsf{Ker}(f)$ is a congruence of $\boldsymbol{A}$, it suffices to show that it preserves the basic operations of $\boldsymbol{A}$. Consider a basic $n$-ary operation $g$ of $\boldsymbol{A}$ and $a_1, \dots, a_n, c_1, \dots, c_n \in A$ such that $\langle a_1, c_1\rangle, \dots, \langle a_n, c_n\rangle \in \mathsf{Ker}(f)$. By the definition of $\mathsf{Ker}(f)$,

$$f(a_i) = f(c_i), \text{ for every } i \leqslant n.$$

It follows that $g^{\boldsymbol{B}}(f(a_1), \dots, f(a_n)) = g^{\boldsymbol{B}}(f(c_1), \dots, f(c_n))$. Since $f\colon \boldsymbol{A} \to \boldsymbol{B}$ is a homomorphism, this yields

$$f(g^{\boldsymbol{A}}(a_1, \dots, a_n)) = g^{\boldsymbol{B}}(f(a_1), \dots, f(a_n)) = g^{\boldsymbol{B}}(f(c_1), \dots, f(c_n)) = f(g^{\boldsymbol{A}}(c_1, \dots, c_n)).$$

Hence, we conclude that $\langle g^{\boldsymbol{A}}(a_1, \dots, a_n), g^{\boldsymbol{A}}(c_1, \dots, c_n)\rangle \in \mathsf{Ker}(f)$, as desired. ⊠

The behaviour of kernels is governed by the next principle.

**Fundamental Homomorphism Theorem 2.13.** *If $f\colon \boldsymbol{A} \to \boldsymbol{B}$ is a homomorphism with kernel $\theta$, then there exists a unique embedding $g\colon \boldsymbol{A}/\theta \to \boldsymbol{B}$ such that $f = g \circ \pi_\theta$.*

*Proof.* We begin by proving the existence of $g$. Let $g\colon A/\theta \to B$ be the map defined as $g(a/\theta) := f(a)$, for every $a \in A$. To show that $g$ is well-defined, consider $a, c \in A$ such that $a/\theta = c/\theta$. Since $\theta = \mathsf{Ker}(f)$, this means that $f(a) = f(c)$, as desired. Furthermore, the definition of $g$ guarantees that $f = g \circ \pi_\theta$.

Now, observe $g$ is injective, because, for every $a, c \in A$ such that $g(a/\theta) = g(c/\theta)$, we have $f(a) = f(c)$, that is, $\langle a, c\rangle \in \mathsf{Ker}(f) = \theta$ and, therefore, $a/\theta = c/\theta$. Moreover, for every basic $n$-ary operation $p$ of $\boldsymbol{A}$ and $a_1, \dots, a_n \in A$, we have

$$\begin{aligned}
g(p^{\boldsymbol{A}/\theta}(a_1/\theta, \dots, a_n/\theta)) &= g(p^{\boldsymbol{A}}(a_1, \dots, a_n)/\theta)\\
&= f(p^{\boldsymbol{A}}(a_1, \dots, a_n))\\
&= p^{\boldsymbol{B}}(f(a_1), \dots, f(a_n))\\
&= p^{\boldsymbol{B}}(g(a_1/\theta), \dots, g(a_n/\theta)).
\end{aligned}$$

The first equality above follows from the definition of $\boldsymbol{A}/\theta$, the second and the last from the definition of $g$, and the third from the assumption that $f\colon \boldsymbol{A} \to \boldsymbol{B}$ is a homomorphism. Hence, we conclude that $g\colon \boldsymbol{A}/\theta \to \boldsymbol{B}$ is a homomorphism and, therefore, an embedding, as desired.

The uniqueness of $g$ follows from the fact that, if a map $g^*$ satisfies the condition in the statement of the theorem, then, for every $a \in A$,

$$f(a) = g^* \circ \pi_\theta(a) = g^*(a/\theta),$$

that is, $g^*$ coincides with $g$. ⊠

**Corollary 2.14.** *If $f\colon \boldsymbol{A} \to \boldsymbol{B}$ is a homomorphism, then $f[\boldsymbol{A}] \cong \boldsymbol{A}/\mathsf{Ker}(f)$. In particular, if $f$ is surjective, $\boldsymbol{B} \cong \boldsymbol{A}/\mathsf{Ker}(f)$.*

*Proof.* In the proof of the Fundamental Homomorphism Theorem we showed that the map $g\colon A/\mathsf{Ker}(f) \to B$, defined by the rule $g(a/\mathsf{Ker}(f)) := f(a)$, is an embedding of $\boldsymbol{A}/\mathsf{Ker}(f)$ into $\boldsymbol{B}$. As $g$ can be viewed as a surjective embedding of $\boldsymbol{A}/\mathsf{Ker}(f)$ into $f[\boldsymbol{A}]$, we conclude that $f[\boldsymbol{A}] \cong \boldsymbol{A}/\mathsf{Ker}(f)$. ⊠

At this stage, it should be clear that if $\theta$ is a congruence on an algebra $\boldsymbol{A}$, then $\pi_\theta\colon \boldsymbol{A} \to \boldsymbol{A}/\theta$ is a surjective homomorphism whose kernel is $\theta$. Similarly, if $f\colon \boldsymbol{A} \to \boldsymbol{B}$ is a surjective homomorphism, then $\boldsymbol{A}/\mathsf{Ker}(f) \cong \boldsymbol{B}$, by Corollary 2.14. As a consequence, for every class of algebras $\mathsf{K}$,

$$\mathbb{H}(\mathsf{K}) = \mathbb{I}\{\boldsymbol{A}/\theta : \boldsymbol{A} \in \mathsf{K} \text{ and } \theta \in \mathsf{Con}(\boldsymbol{A})\}. \tag{2}$$

Now, recall that the Cartesian product of a family of sets $\{A_i : i \in I\}$ is the set

$$\prod_{i\in I} A_i := \{f\colon I \to \bigcup_{i\in I} A_i \ : \ f(i) \in A_i, \text{ for all } i \in I\}.$$

In particular, if $I$ is empty, then $\prod_{i\in I} A_i$ is the singleton containing only the empty map.

**Definition 2.15.** The *direct product* of a family of similar algebras $\{\boldsymbol{A}_i : i \in I\}$ is the unique algebra of the common type whose universe is the Cartesian product $\prod_{i\in I} A_i$ and such that, for every basic $n$-ary operation symbol $f$ and every $\vec{a}_1, \dots, \vec{a}_n \in \prod_{i\in I} A_i$,

$$f^{\prod_{i\in I} \boldsymbol{A}_i}(\vec{a}_1, \dots, \vec{a}_n)(i) = f^{\boldsymbol{A}_i}(\vec{a}_1(i), \dots, \vec{a}_n(i)), \text{ for every } i \in I.$$

We denote this algebra by $\prod_{i\in I} \boldsymbol{A}_i$.

In this case, for every $j \in I$, the projection map on the $j$-th component $p_j\colon \prod_{i\in I} A_i \to A_j$, defined by the rule $p_j(\vec{a}) := \vec{a}(j)$, is a surjective homomorphism from $\prod_{i\in I} \boldsymbol{A}_i$ to $\boldsymbol{A}_j$.

Given a class of similar algebras $\mathsf{K}$, we set

$$\mathbb{P}(\mathsf{K}) := \{\boldsymbol{A} : \boldsymbol{A} \text{ is a direct product of a family } \{\boldsymbol{B}_i : i \in I\} \subseteq \mathsf{K}\}.$$

As usual, when $\mathsf{K} = \{\boldsymbol{A}\}$, we write $\mathbb{P}(\boldsymbol{A})$ as a shorthand for $\mathbb{P}(\{\boldsymbol{A}\})$.

Notice that up to isomorphism, there exists a unique one-element algebra of a given type. Because of this, one-element algebras are called *trivial*. Accordingly, when the set of indexes $I$ is empty, the direct product $\prod_{i\in I} \boldsymbol{A}_i$ is the trivial algebra of the given type. It follows that $\mathbb{P}(\mathsf{K})$ contains always a trivial algebra, for every class of similar algebras $\mathsf{K}$.

**Example 2.16** (Powerset algebras)**.** Boolean algebras of the form $\langle \mathcal{P}(X); \cap, \cup, -, \emptyset, X\rangle$ are called *powerset Boolean algebras*. Let $\boldsymbol{B}$ be the two-element Boolean algebra and observe that $\mathbb{IP}(\boldsymbol{B})$ is the class of algebras isomorphic to some powerset Boolean algebra. To prove this, observe that every powerset Boolean algebra $\mathcal{P}(X)$ is isomorphic to a direct product of $\boldsymbol{B}$ via the *characteristic function* $f_X\colon \mathcal{P}(X) \to \prod_{x\in X} \boldsymbol{B}_x$, defined by the rule

$$f(Y)(x) := \begin{cases} 1 & \text{if } x \in Y \\ 0 & \text{if } x \notin Y, \end{cases}$$

where $Y \in \mathcal{P}(X)$ and $x \in X$. By the same token, every direct product $\prod_{i\in I} \boldsymbol{B}_i$ of $\boldsymbol{B}$ is isomorphic to the powerset Boolean algebra $\mathcal{P}(I)$ via the isomorphism $f_I$. ⊠

We close this section by reviewing the subdirect product construction.

**Definition 2.17.** A subalgebra $\boldsymbol{B}$ of a direct product $\prod_{i\in I} \boldsymbol{A}_i$ is said to be a *subdirect product* of $\{\boldsymbol{A}_i : i \in I\}$ if the projection map $p_i$ is surjective, for every $i \in I$. Similarly, an embedding $f\colon \boldsymbol{B} \to \prod_{i\in I} \boldsymbol{A}_i$ is said to be *subdirect* when $f[\boldsymbol{B}]$ is a subdirect product of the family $\{\boldsymbol{A}_i : i \in I\}$.

Given a class of similar algebras $\mathsf{K}$, we set

$$\mathbb{P}_{\text{SD}}(\mathsf{K}) := \{\boldsymbol{A} : \boldsymbol{A} \text{ is a subdirect direct product of a family } \{\boldsymbol{B}_i : i \in I\} \subseteq \mathsf{K}\}.$$

As usual, when $\mathsf{K} = \{\boldsymbol{A}\}$, we write $\mathbb{P}_{\text{SD}}(\boldsymbol{A})$ as a shorthand for $\mathbb{P}_{\text{SD}}(\{\boldsymbol{A}\})$. Clearly, $\mathbb{P}_{\text{SD}}(\mathsf{K}) \subseteq \mathbb{SP}(\mathsf{K})$. Furthermore, $\mathbb{P}_{\text{SD}}(\mathsf{K})$ contains always a trivial algebra.

**Example 2.18** (Distributive lattices)**.** Let $\mathsf{DL}$ be the class of distributive lattices and $\boldsymbol{B}$ be the two-element distributive lattice. Birkhoff's Representation Theorem states that $\mathsf{DL} = \mathbb{IP}_{\text{SD}}(\boldsymbol{B})$. The inclusion $\mathbb{IP}_{\text{SD}}(\boldsymbol{B}) \subseteq \mathsf{DL}$ follows from the fact that $\mathsf{DL}$ is closed under $\mathbb{I}, \mathbb{S}$, and $\mathbb{P}$. For the other inclusion, consider a distributive lattice $\boldsymbol{A}$ and let $I$ be the set of its prime filters. By Birkhoff's Representation Theorem, the map

$$\gamma\colon \boldsymbol{A} \to \prod_{F\in I} \boldsymbol{B}_F,$$

defined, for every $a \in A$ and $F \in I$, by the rule

$$\gamma(a)(F) := \begin{cases} 1 & \text{if } a \in F \\ 0 & \text{if } a \notin F, \end{cases}$$

is a well-defined subdirect embedding. ⊠

**Example 2.19** (Boolean algebras)**.** Similarly, Stone's Representation Theorem states that the class of Boolean algebras coincides with $\mathbb{IP}_{\text{SD}}(\boldsymbol{B})$, where $\boldsymbol{B}$ the two-element Boolean algebra. ⊠

The next result provides a general recipe to construct subdirect products.

**Proposition 2.20.** *Let $\boldsymbol{A}$ be an algebra and $\{\theta_i : i \in I\} \subseteq \mathsf{Con}(\boldsymbol{A})$. Then the map*

$$f\colon \boldsymbol{A}/\bigcap_{i\in I}\theta_i \to \prod_{i\in I} \boldsymbol{A}/\theta_i,$$

*defined, for every $a \in A$ and $j \in I$, as*

$$f(a/\bigcap_{i\in I}\theta_i)(j) := a/\theta_j,$$

*is a subdirect embedding.*

*Proof.* For the sake of readability, set $\boldsymbol{B} := \boldsymbol{A}/\bigcap_{i\in I}\theta_i$. To prove that $f$ is injective, consider $a, c \in A$ such that $\langle a, c\rangle \notin \bigcap_{i\in I}\theta_i$. Then there exists $j \in I$ such that $\langle a, c\rangle \notin \theta_j$ and, therefore,

$$f(a/\bigcap_{i\in I}\theta_i)(j) := a/\theta_j \neq c/\theta_j = f(c/\bigcap_{i\in I}\theta_i)(j).$$

It follows that $f(a/\bigcap_{i\in I}\theta_i) \neq f(c/\bigcap_{i\in I}\theta_i)$. Thus, $f$ is injective. Moreover, by the definition of $f$, the composition $p_i \circ f\colon \boldsymbol{B} \to \boldsymbol{A}/\theta_i$ is surjective, for every $i \in I$.

It only remains to prove that $f$ is a homomorphism. Consider an $n$-ary basic operation $g$ and $a_1, \dots, a_n \in A$. For every $j \in I$, we have

$$\begin{aligned}
f(g^{\boldsymbol{B}}(a_1/\bigcap_{i\in I}\theta_i, \dots, a_n/\bigcap_{i\in I}\theta_i))(j) &= f(g^{\boldsymbol{A}}(a_1, \dots, a_n)/\bigcap_{i\in I}\theta_i)(j) \\
&= g^{\boldsymbol{A}}(a_1, \dots, a_n)/\theta_j \\
&= g^{\boldsymbol{A}/\theta_j}(a_1/\theta_j, \dots, a_n/\theta_j) \\
&= g^{\boldsymbol{A}/\theta_j}(f(a_1/\bigcap_{i\in I}\theta_i)(j), \dots, f(a_n/\bigcap_{i\in I}\theta_i)(j)) \\
&= g^{\prod_{i\in I}\boldsymbol{A}/\theta_i}(f(a_1/\bigcap_{i\in I}\theta_i), \dots, f(a_n/\bigcap_{i\in I}\theta_i))(j).
\end{aligned}$$

It follows that

$$f(g^{\boldsymbol{B}}(a_1/\bigcap_{i\in I}\theta_i, \dots, a_n/\bigcap_{i\in I}\theta_i)) = g^{\prod_{i\in I}\boldsymbol{A}/\theta_i}(f(a_1/\bigcap_{i\in I}\theta_i), \dots, f(a_n/\bigcap_{i\in I}\theta_i)). \qquad \boxtimes$$

## 3. Prevarieties

**Definition 3.1.** A class of similar algebras closed $\mathbb{I}, \mathbb{S}$, and $\mathbb{P}$ is a said to be a *prevariety*.

Given a class of similar algebras $\mathsf{K}$, the least prevariety extending $\mathsf{K}$ is $\mathbb{ISP}(\mathsf{K})$ and is called the prevariety *generated* by $\mathsf{K}$. Our aim will be to prove that prevarieties are precisely the classes of algebras axiomatized by a certain kind of infinitary formulas.

When no confusion shall arise, given a sequence $\vec{a}$ and a set $A$, we write $\vec{a} \in A$ to indicate that the elements of the sequence $\vec{a}$ belong to $A$.

**Definition 3.2.** A *generalized quasi-equation* of type $\rho$ is an expression $\Phi$ of the form

$$\Big(\mathop{\&}_{i\in I} \varphi_i(\vec{x}) \approx \psi_i(\vec{x})\Big) \Longrightarrow \varepsilon(\vec{x}) \approx \delta(\vec{x}),$$

where $\{\varphi_i \approx \psi_i : i \in I\} \cup \{\varepsilon \approx \delta\}$ is a set of equations of type $\rho$. Then $\Phi$ is *valid* in an algebra $\boldsymbol{A}$ of type $\rho$ when so is its universal closure, that is, for every $\vec{a} \in A$,

$$\text{if } (\varphi_i^{\boldsymbol{A}}(\vec{a}) = \psi_i^{\boldsymbol{A}}(\vec{a}), \text{ for all } i \in I), \text{ then } \varepsilon^{\boldsymbol{A}}(\vec{a}) = \delta^{\boldsymbol{A}}(\vec{a}).$$

In this case, we often say that $\boldsymbol{A}$ *satisfies* $\Phi$.

Notice that, in the above definition, the set of indexes $I$ can be arbitrarily large and that the same applies to the sequence of variables $\vec{x}$ that appear in the equations of $\Phi$. This motivates the following.

**Definition 3.3.** A generalized quasi-equation is said to be

(i) a *quasi-equation* when the index set $I$ is finite; and
(ii) an *equation* when the index set $I$ is empty.

*Remark* 3.4. It might seem that we are using the term *equations* to refer to two distinct kinds of expressions, namely those of the form $\varepsilon \approx \delta$ and $\emptyset \Longrightarrow \varepsilon \approx \delta$. This is not a problem, however, because these expressions are synonyms, in the sense that an algebra satisfies $\varepsilon \approx \delta$ if and only if it satisfies $\emptyset \Longrightarrow \varepsilon \approx \delta$. Because of this, we will continue to denote equations by $\varepsilon \approx \delta$, while keeping in mind that they are special instances of generalized quasi-equations. $\boxtimes$

**Definition 3.5.** Let $\rho\colon \mathcal{F} \to \mathbb{N}$ be a type and $X$ a set of variables disjoint from $\mathcal{F}$. The *term algebra* $\boldsymbol{T}_\rho(X)$ of type $\rho$ over $X$ is the unique algebra of type $\rho$ whose universe is $T_\rho(X)$ and with basic $n$-ary operations $f$ defined, for every $\varphi_1, \dots, \varphi_n \in T_\rho(X)$, as

$$f^{\boldsymbol{T}_\rho(X)}(\varphi_1, \dots, \varphi_n) := f(\varphi_1, \dots, \varphi_n).$$

Term algebras have the following fundamental property.

**Proposition 3.6.** *Let $\boldsymbol{A}$ be an algebra of type $\rho$ and $X$ a set of variables. Every function $f\colon X \to A$ extends uniquely to a homomorphism $f^*\colon \boldsymbol{T}_\rho(X) \to \boldsymbol{A}$.*

*Proof.* The unique extension $f^*$ is defined, for every $\varphi(x_{\alpha_1}, \dots x_{\alpha_n}) \in T_\rho(X)$, as

$$f^*(\varphi) = \varphi^{\boldsymbol{A}}(f(x_{\alpha_1}), \dots, f(x_{\alpha_n})). \qquad \boxtimes$$

*Exercise* 3.7. Prove the above proposition. $\boxtimes$

**Theorem 3.8.** *A class of similar algebras is a prevariety if and only if it can be axiomatized by a class of generalized quasi-equations.*

*Proof.* The "if" part follows from the fact that the validity of generalized quasi-equations persists under the formation of isomorphic copies, subalgebras, and direct product. To prove the converse, consider a prevariety $\mathsf{K}$ and let $\Sigma$ be the class of generalized quasi-equations valid in it. Let $\mathsf{K}^+$ be the class of algebras in which the generalized quasi-equations in $\Sigma$ are valid. Clearly, $\mathsf{K} \subseteq \mathsf{K}^+$. To prove the other inclusion, consider an algebra $\boldsymbol{A} \in \mathsf{K}^+$. Let also $X$ be a set of variables for which there exists a surjective map $f\colon X \to A$. By Proposition 3.6, $f$ extends to a surjective homomorphism $f^*\colon \boldsymbol{T}_\rho(X) \to \boldsymbol{A}$. Together with Corollary 2.14, this yields

$$\boldsymbol{A} \cong \boldsymbol{T}_\rho(X)/\mathsf{Ker}(f^*). \tag{3}$$

Now, consider an arbitrary pair $\langle \varphi, \psi \rangle \in (T_\rho(X) \times T_\rho(X)) \smallsetminus \mathsf{Ker}(f^*)$. Notice that the elements of $T_\rho(X) \times T_\rho(X)$ are ordered pairs of terms and, therefore, can be viewed as equations under the identification of $\langle \varepsilon, \delta \rangle$ with $\varepsilon \approx \delta$. In this way, $\mathsf{Ker}(f^*)$ becomes a set of equations in variables $X$. Bearing this in mind, consider the generalized quasi-equation

$$\Phi := \Big( \mathop{\&} \mathsf{Ker}(f^*) \Big) \Longrightarrow \varphi \approx \psi.$$

We will prove that $\Phi$ fails in $\boldsymbol{A}$. For the sake of readability we will denote by $\vec{x}$ the sequence of all variables in $X$. Observe that every element $\varepsilon \in T_\rho(X)$ is of the form $\varepsilon(\vec{x})$. Then consider the assignment $f\colon X \to A$. We will denote by $f(\vec{x})$ the sequence obtained by applying $f$ component-wise to $\vec{x}$. For every pair $\langle \varepsilon, \delta \rangle \in \mathsf{Ker}(f^*)$, we have

$$\varepsilon^{\boldsymbol{A}}(f(\vec{x})) = \varepsilon^{\boldsymbol{A}}(f^*(\vec{x})) = f^*(\varepsilon(\vec{x})) = f^*(\delta(\vec{x})) = \delta^{\boldsymbol{A}}(f^*(\vec{x})) = \delta^{\boldsymbol{A}}(f(\vec{x})).$$

The equalities above can be justified as follows. The first and the last holds because $f^*$ extends $f$, the second and the fourth because $f^*\colon \boldsymbol{T}_\rho(X) \to \boldsymbol{A}$ is a homomorphism, and the third because $\langle \varepsilon, \delta \rangle \in \mathsf{Ker}(f^*)$. On the other hand, since $\langle \varphi, \psi \rangle \notin \mathsf{Ker}(f^*)$, a similar argument shows

$$\varphi^{\boldsymbol{A}}(f(\vec{x})) \neq \psi^{\boldsymbol{A}}(f(\vec{x})).$$

Thus, $\boldsymbol{A}$ refutes $\Phi$, as desired.

Since $\boldsymbol{A} \in \mathsf{K}^{+}$, this implies that there exists some algebra $\boldsymbol{C}_{\varphi,\psi} \in \mathsf{K}$ and an assignment $g_{\varphi,\psi} \colon X \to C_{\varphi,\psi}$ such that

$$\varepsilon^{\boldsymbol{C}_{\varphi,\psi}}(g_{\varphi,\psi}(\vec{x})) = \delta^{\boldsymbol{C}_{\varphi,\psi}}(g_{\varphi,\psi}(\vec{x})), \text{ for all } \langle \varepsilon, \delta \rangle \in \mathsf{Ker}(f^*), \text{ and } \varphi^{\boldsymbol{C}_{\varphi,\psi}}(g_{\varphi,\psi}(\vec{x})) \neq \psi^{\boldsymbol{C}_{\varphi,\psi}}(g_{\varphi,\psi}(\vec{x})).$$

Recall that $g_{\varphi,\psi}$ extends uniquely to a homomorphism $g^*_{\varphi,\psi} \colon \boldsymbol{T}_\rho(X) \to \boldsymbol{C}_{\varphi,\psi}$. Moreover, from the above display it follows

$$g^*_{\varphi,\psi}(\varepsilon) = g^*_{\varphi,\psi}(\delta), \text{ for all } \langle \varepsilon, \delta \rangle \in \mathsf{Ker}(f^*), \text{ and } g^*_{\varphi,\psi}(\varphi) \neq g^*_{\varphi,\psi}(\psi).$$

Consequently,

$$\mathsf{Ker}(f^*) \subseteq \mathsf{Ker}(g^*_{\varphi,\psi}) \text{ and } \langle \varphi, \psi \rangle \notin \mathsf{Ker}(g^*_{\varphi,\psi}).$$

It follows that

$$\mathsf{Ker}(f^*) = \bigcap \{\mathsf{Ker}(g^*_{\varphi,\psi}) : \langle \varphi, \psi \rangle \in (T_\rho(X) \times T_\rho(X)) \smallsetminus \mathsf{Ker}(f^*)\}.$$

By Proposition 2.20, this yields

$$\boldsymbol{T}_\rho(X)/\mathsf{Ker}(f^*) \in \mathbb{IP}_{\mathsf{SD}}(\{\boldsymbol{T}_\rho(X)/\mathsf{Ker}(g^*_{\varphi,\psi}) : \langle \varphi, \psi \rangle \in (T_\rho(X) \times T_\rho(X)) \smallsetminus \mathsf{Ker}(f^*)\}). \quad (4)$$

Moreover, from Corollary 2.14 and the fact that $\mathsf{K}$ is closed under $\mathbb{I}$ and $\mathbb{S}$ it follows that

$$\boldsymbol{T}_\rho(X)/\mathsf{Ker}(g^*_{\varphi,\psi}) \in \mathbb{IS}(\boldsymbol{C}_{\varphi,\psi}) \subseteq \mathsf{K},$$

for every $\langle \varphi, \psi \rangle \in (T_\rho(X) \times T_\rho(X)) \smallsetminus \mathsf{Ker}(f^*)$. Consequently, (4) simplifies to

$$\boldsymbol{T}_\rho(X)/\mathsf{Ker}(f^*) \in \mathbb{IP}_{\mathsf{SD}}(\mathsf{K}) \subseteq \mathsf{K},$$

where the last inclusion follows from the fact that $\mathsf{K}$ is a prevariety. Together with (3), this yields $\boldsymbol{A} \in \mathbb{I}(\mathsf{K}) \subseteq \mathsf{K}$. ⊠

*Remark* 3.9. In view of Theorem 3.8, prevarieties are classes of algebras axiomatized by classes of generalized quasi-equations. It is therefore natural to wonder whether there exists a prevariety that cannot be axiomatized by a set (as opposed to proper class) of generalized quasi-equations. It turns out that the answer to this question depends on the set theory we live in, as the nonexistence of such a prevariety is equivalent to *Vopěnka's Principle*. ⊠

Nonetheless, prevarieties axiomatizable by a set of generalized quasi-equations admit a relatively transparent description, as we proceed to explain. Given an infinite cardinal $\kappa$ and a class of algebras $\mathsf{K}$, let

$$\mathbb{U}_\kappa(\mathsf{K}) := \{\boldsymbol{A} : \boldsymbol{B} \in \mathsf{K}, \text{ for all } \kappa\text{-generated } \boldsymbol{B} \leqslant \boldsymbol{A}\}.$$

**Definition 3.10.** Let $\kappa$ be an infinite cardinal. A *$\kappa$-generalized quasi-variety* is a prevariety closed under $\mathbb{U}_\kappa$.

When $\kappa = \aleph_0$, we often say that $\mathsf{K}$ is simply a *generalized quasi-variety*. Given a class of similar algebras $\mathsf{K}$, the least $\kappa$-generalized quasi-variety extending $\mathsf{K}$ is $\mathbb{U}_\kappa\mathbb{ISP}(\mathsf{K})$ and is called the $\kappa$-generalized quasi-variety *generated* by $\mathsf{K}$.

**Theorem 3.11.** *Let $\kappa$ be an infinite cardinal. A class of similar algebras is a $\kappa$-generalized quasi-variety if and only if it can be axiomatized by a set of generalized quasi-equations in which at most $\kappa$ variables occur.*

*Proof.* The "if" part follows from the fact that the validity of generalized quasi-equations in $\leqslant \kappa$ variables persist under the $\mathbb{I}, \mathbb{S}, \mathbb{P}, \mathbb{U}_\kappa$. To prove the converse, consider a $\kappa$-generalized quasi-variety $\mathsf{K}$. Then let $X$ be a set of variables of cardinality $\kappa$ and $\Sigma$ the class of generalized quasi-equations written with variables in $X$. Since $X$ is a set, so is $\Sigma$. It only remains to prove that $\mathsf{K}$ coincides with the class $\mathsf{K}^+$ of algebras satisfying the generalized quasi-equations in $\Sigma$. Clearly, $\mathsf{K} \subseteq \mathsf{K}^+$. To prove the other inclusion, consider an algebra $\boldsymbol{A} \in \mathsf{K}^+$. We need to prove that $\boldsymbol{A} \in \mathsf{K}$. Since $\mathsf{K}$ is closed under $\mathbb{U}_\kappa$, it suffices to show that all the $\kappa$-generated subalgebras of $\boldsymbol{A}$ belong to $\mathsf{K}$.

Accordingly, let $\boldsymbol{B}$ be a $\kappa$-generated subalgebra of $\boldsymbol{A}$ and $Y \subseteq B$ a set of generators for $\boldsymbol{B}$ of size $\leqslant \kappa$. There exists a surjective map $f \colon X \to Y$. By Proposition 3.6, $f$ extends to a surjective homomorphism $f^* \colon \boldsymbol{T}_\rho(X) \to \boldsymbol{B}$. Now, we repeat the argument in the proof of Theorem 3.8, obtaining $\boldsymbol{B} \in \mathsf{K}$, as desired. ⊠

**Corollary 3.12.** *A prevariety can be axiomatized by a set of generalized quasi-equations if and only if it is a $\kappa$-generalized quasi-variety, for some infinite cardinal $\kappa$.*

*Exercise** 3.13. Let $\mathsf{K}$ be a class of similar algebras and $\kappa$ an infinite cardinal. Prove that the prevariety and the $\kappa$-generalized quasi-variety generated by $\mathsf{K}$ are, respectively, $\mathbb{ISP}(\mathsf{K})$ and $\mathbb{U}_\kappa\mathbb{ISP}(\mathsf{K})$. ⊠

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