LECTURE NOTES ON ALGEBRAIC LOGIC

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1. Algebras and equations

We begin by reviewing some fundamentals of general algebraic systems.

Definition 1.1.

(i) A *type* is a map $\rho\colon \mathcal{F} \to \mathbb{N}$, where $\mathcal{F}$ is a set of function symbols. In this case, $\rho(f)$ is said to be the *arity* of the function symbol f, for every $f \in \mathcal{F}$. Function symbols of arity zero are called *constants*.

(ii) An *algebra* of type ρ is a pair $\boldsymbol{A} = \langle A; F\rangle$ where A is a nonempty set and $F = \{f^{\boldsymbol{A}} : f \in \mathcal{F}\}$ is a set of operations on A whose arity is determined by ρ, in the sense that each $f^{\boldsymbol{A}}$ has arity $\rho(f)$. The set A is called the *universe* of $\boldsymbol{A}$.

When $\mathcal{F} = \{f_1, \dots, f_n\}$, we shall write $\langle A; f_1^{\boldsymbol{A}}, \dots, f_n^{\boldsymbol{A}}\rangle$ instead of $\langle A; F\rangle$. In this case, we often drop the superscripts, and write simply $\langle A; f_1, \dots, f_n\rangle$.

Classical examples of algebras are groups and rings. For instance, the type of groups ρ_G consists of a binary symbol $+$, a unary symbol $-$, and a constant symbol 0. Then a group is an algebra $\langle G; +, -, 0\rangle$ of type ρ_G in which $+$ is associative, 0 is a neutral element for $+$, and $-$ produces inverses.

Lattices, Heyting algebras, and modal algebras are also algebras in the above sense. For instance, the type of lattices ρ_L consists of two binary symbols $\wedge$ and $\vee$ and a lattice is an algebra $\langle A; \wedge, \vee\rangle$ of type ρ_L that satisfies the idempotent, commutative, associative, and absorption laws. Similarly, the type of Heyting algebras ρ_H consists of three binary operations symbols $\wedge, \vee$, and $\to$ and of two constant symbols 0 and 1. Then a Heyting algebra is an algebra $\langle A; \wedge, \vee, \to, 0, 1\rangle$ such that $\langle A; \wedge, \vee, 0, 1\rangle$ is a bounded lattice and, for every $a, b, c \in A$,

$$a \wedge b \leqslant c \iff a \leqslant b \to c. \qquad \text{(\textit{residuation law})}$$

Boolean algebras can be viewed as the Heyting algebras that satisfy the following equational version of the *excluded middle law*:

$$x \vee (x \to 0) \approx 1.$$

In this case, the complement operation $\neg x$ can be defined as $x \to 0$.

Perhaps less obviously, even algebraic structures whose operations are apparently *external* can be viewed as algebras in the sense of the above definition. For instance, modules over a ring $\boldsymbol{R}$ can be viewed as algebras whose type ρ_R extends that of groups with the unary symbols $\{\lambda_r : r \in R\}$. From this point of view, a module over $\boldsymbol{R}$ is an

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algebra $\langle G; +, -, 0, \{\lambda_r : r \in R\}\rangle$ of type ρ_R such that $\langle G; +, -, 0\rangle$ is an abelian group and, for every $r, s \in R$ and $a, c \in G$,

$$\begin{aligned}
\lambda_r(a+c) &= \lambda_r(a) + \lambda_r(c)\\
\lambda_{r+s}(a) &= \lambda_r(a) + \lambda_s(a)\\
\lambda_r(\lambda_s(a)) &= \lambda_{r\cdot s}(a)\\
\lambda_1(a) &= a.
\end{aligned}$$

Given a type $\rho\colon \mathcal{F} \to \mathbb{N}$ and a set of variables X disjoint from $\mathcal{F}$, the set of *terms of type ρ over X* is the least set $T_\rho(X)$ such that

(i) $X \subseteq T_\rho(X)$;
(ii) if $c \in \mathcal{F}$ is a constant, then $c \in T_\rho(X)$; and
(iii) if $\varphi_1, \dots, \varphi_{\rho(f)} \in T_\rho(X)$ and $f \in \mathcal{F}$, then $f\varphi_1 \dots \varphi_{\rho(f)} \in T_\rho(X)$.

For the sake of readability, we shall often write $f(\varphi_1, \dots, \varphi_{\rho(f)})$ instead of $f\varphi_1 \dots \varphi_{\rho(f)}$. Similarly, if f is a binary operation $+$, we often write $\varphi_1 + \varphi_2$ instead of $f(\varphi_1, \varphi_2)$.

Given a term $\varphi \in T_\rho(X)$, we write $\varphi(x_1, \dots, x_n)$ to indicate that the variables occurring in φ are among $x_1, \dots, x_n$. Furthermore, given an algebra $\boldsymbol{A}$ of type ρ and elements $a_1, \dots, a_n \in A$, we define an element

$$\varphi^{\boldsymbol{A}}(a_1, \dots, a_n)$$

of A, by recursion on the construction of φ, as follows:

(i) if φ is a variable x_i, then $\varphi^{\boldsymbol{A}}(a_1, \dots, a_n) := a_i$;
(ii) if φ is a constant c, then $c^{\boldsymbol{A}}$ is the interpretation of c in $\boldsymbol{A}$;
(iii) if $\varphi = f(\psi_1, \dots, \psi_m)$, then

$$\varphi^{\boldsymbol{A}}(a_1, \dots, a_n) := f^{\boldsymbol{A}}(\psi_1^{\boldsymbol{A}}(a_1, \dots, a_n), \dots, \psi_m^{\boldsymbol{A}}(a_1, \dots, a_n)).$$

An *equation of type ρ over X* is an expression of the form $\varphi \approx \psi$, where $\varphi, \psi \in T_\rho(X)$. Such an equation $\varphi \approx \psi$ is *valid* in an algebra $\boldsymbol{A}$ of type ρ, if

$$\varphi^{\boldsymbol{A}}(a_1, \dots, a_n) = \psi^{\boldsymbol{A}}(a_1, \dots, a_n), \text{ for every } a_1, \dots, a_n \in A,$$

in which case we say that $\boldsymbol{A}$ *satisfies* $\varphi \approx \psi$.

For instance, groups are precisely the algebras of type ρ_G that satisfy the equations

$$x + (y+z) \approx (x+y) + z \quad x + 0 \approx x \quad 0 + x \approx x \quad x + -x \approx 0 \quad -x + x \approx 0.$$

Similarly, lattices are the algebras of type ρ_L that satisfy the equations

$$\begin{array}{ccr}
x \wedge x \approx x & x \vee x \approx x & (\textit{idempotent laws})\\
x \wedge y \approx y \wedge x & x \vee y \approx y \vee x & (\textit{commutative laws})\\
x \wedge (y \wedge z) \approx (x \wedge y) \wedge z & x \vee (y \vee z) \approx (x \vee y) \vee z & (\textit{associative laws})\\
x \wedge (y \vee x) \approx x & x \vee (y \wedge x) \approx x. & (\textit{absorption laws})
\end{array}$$

2. Basic constructions

Algebras of the same type are called *similar* and can be compared by means of maps that preserve their structure.

Definition 2.1. Given two similar algebras $\boldsymbol{A}$ and $\boldsymbol{B}$, a *homomorphism* from $\boldsymbol{A}$ to $\boldsymbol{B}$ is a map $f\colon A \to B$ such that, for every n-ary operation g of the common type and $a_1, \dots, a_n \in A$,

$$f(g^{\boldsymbol{A}}(a_1, \dots, a_n)) = g^{\boldsymbol{B}}(f(a_1), \dots, f(a_n)).$$

An injective homomorphism is called an *embedding* and, if there exists an embedding from $\boldsymbol{A}$ to $\boldsymbol{B}$, we say that $\boldsymbol{A}$ *embeds* into $\boldsymbol{B}$. Lastly, a surjective embedding is called an *isomorphism*. Accordingly, $\boldsymbol{A}$ and $\boldsymbol{B}$ are said to be *isomorphic* if there exists an isomorphism between them, in which case we write $\boldsymbol{A} \cong \boldsymbol{B}$.

A simple induction on the construction of terms shows that, for every pair of algebras $\boldsymbol{A}$ and $\boldsymbol{B}$ of type ρ and every term $\varphi(x_1, \dots, x_n)$ of ρ, if f is a homomorphism from $\boldsymbol{A}$ to $\boldsymbol{B}$, then

$$f(\varphi^{\boldsymbol{A}}(a_1, \dots, a_n)) = \varphi^{\boldsymbol{B}}(f(a_1), \dots, f(a_n)),$$

for every $a_1, \dots, a_n \in A$. Therefore homomorphisms preserve not only basic operations, but also arbitrary terms.

In the particular case where $\boldsymbol{A}$ and $\boldsymbol{B}$ are lattices, a homomorphism from $\boldsymbol{A}$ to $\boldsymbol{B}$ is a map $f\colon A \to B$ such that, for every $a, c \in A$,

$$f(a \wedge^{\boldsymbol{A}} c) = f(a) \wedge^{\boldsymbol{B}} f(c) \text{ and } f(a \vee^{\boldsymbol{A}} c) = f(a) \vee^{\boldsymbol{B}} f(c).$$

For instance, the inclusion map from the lattice $\langle \mathbb{N}; \leqslant \rangle$ into the lattice $\langle \mathbb{Z}; \leqslant \rangle$ is an injective homomorphism, that is, an embedding. Similarly, given two sets $Y \subseteq X$, the inclusion map from the powerset lattice $\langle \mathcal{P}(Y); \subseteq \rangle$ to the powerset lattice $\langle \mathcal{P}(X); \subseteq \rangle$ is also an embedding. On the other hand, if $Y \subsetneq X$, the map

$$(-) \cap Y\colon \mathcal{P}(X) \to \mathcal{P}(Y)$$

that sends every $Z \subseteq X$ to $Z \cap Y$ is a noninjective homomorphism from $\langle \mathcal{P}(X); \subseteq \rangle$ to $\langle \mathcal{P}(Y); \subseteq \rangle$.

Definition 2.2. Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be algebras of the same type $\rho\colon \mathcal{F} \to \mathbb{N}$. Then $\boldsymbol{A}$ is said to be a *subalgebra* of $\boldsymbol{B}$ if $A \subseteq B$ and $f^{\boldsymbol{A}}$ is the restriction of $f^{\boldsymbol{B}}$ to A, for every $f \in \mathcal{F}$. In this case, we write $\boldsymbol{A} \leqslant \boldsymbol{B}$.

Given a class of algebras K, let

$$\mathbb{I}(\mathsf{K}) := \{\boldsymbol{A} : \boldsymbol{A} \cong \boldsymbol{B} \text{ for some } \boldsymbol{B} \in \mathsf{K}\}$$
$$\mathbb{S}(\mathsf{K}) := \{\boldsymbol{A} : \boldsymbol{A} \leqslant \boldsymbol{B} \text{ for some } \boldsymbol{B} \in \mathsf{K}\}.$$

When $\mathsf{K} = \{\boldsymbol{A}\}$, we write $\mathbb{I}(\boldsymbol{A})$ and $\mathbb{S}(\boldsymbol{A})$ as a shorthand for $\mathbb{I}(\{\boldsymbol{A}\})$ and $\mathbb{S}(\{\boldsymbol{A}\})$, respectively. The following observation is an immediate consequence of the definitions.

Proposition 2.3. *Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be algebras of the same type. Then $\boldsymbol{A} \in \mathbb{IS}(\boldsymbol{B})$ if and only if there exists an embedding $f\colon \boldsymbol{A} \to \boldsymbol{B}$. In this case, $\boldsymbol{A}$ is isomorphic to the unique subalgebra of $\boldsymbol{B}$ with universe $f[A]$.*

As we mentioned, homomorphisms can be used to compare similar algebras.

Definition 2.4. Given two similar algebras $\boldsymbol{A}$ and $\boldsymbol{B}$, we say that $\boldsymbol{A}$ is a *homomorphic image* of $\boldsymbol{B}$ if there exists a surjective homomorphism $f\colon \boldsymbol{B} \to \boldsymbol{A}$.

Accordingly, given a class of algebras K, we set

$$\mathbb{H}(\mathsf{K}) := \{\boldsymbol{A} : \boldsymbol{A} \text{ is a homomorphic image of some } \boldsymbol{B} \in \mathsf{K}\}.$$

As usual, when $\mathsf{K} = \{\boldsymbol{A}\}$, we write $\mathbb{H}(\boldsymbol{A})$ as a shorthand for $\mathbb{H}(\{\boldsymbol{A}\})$.

Observe that every (not necessarily surjective) homomorphism $f\colon \boldsymbol{A} \to \boldsymbol{B}$ induces a homomorphic image of $\boldsymbol{A}$.

Proposition 2.5. *If $f\colon \boldsymbol{A} \to \boldsymbol{B}$ is a homomorphism, then $f[A]$ is the universe of a subalgebra of $\boldsymbol{B}$ that, moreover, is a homomorphic image of $\boldsymbol{A}$.*

Proof. Observe that $f[A]$ is nonempty, because A is. Then consider an n-ary function symbol g of the common type of $\boldsymbol{A}$ and $\boldsymbol{B}$ and $b_1, \dots, b_n \in f[A]$. Clearly, there are $a_1, \dots, a_n \in A$ such that $f(a_i) = b_i$, for every $i \leqslant n$. Since f is a homomorphism from $\boldsymbol{A}$ to $\boldsymbol{B}$, we obtain

$$g^{\boldsymbol{B}}(b_1, \dots, b_n) = g^{\boldsymbol{B}}(f(a_1), \dots, g(a_n)) = f(g^{\boldsymbol{A}}(a_1, \dots, a_n)) \in f[A].$$

Hence, we conclude that $f[A]$ is the universe of a subalgebra $f[\boldsymbol{A}]$ of $\boldsymbol{B}$.

Furthermore, $f\colon \boldsymbol{A} \to f[\boldsymbol{A}]$ is a homomorphism, because for every basic n-ary function symbol g of the common type and $a_1, \dots, a_n \in A$,

$$f(g^{\boldsymbol{A}}(a_1, \dots, a_n)) = g^{\boldsymbol{B}}(f(a_1), \dots, f(a_n)) = g^{f[\boldsymbol{A}]}(f(a_1), \dots, f(a_n)),$$

where the first equality follows from the assumption that $f\colon \boldsymbol{A} \to \boldsymbol{B}$ is a homomorphism. Since the map $f\colon A \to f[A]$ is surjective, we conclude that $f[\boldsymbol{A}] \in \mathbb{H}(\boldsymbol{A})$. ⊠

In view of the above result, when $f\colon \boldsymbol{A} \to \boldsymbol{B}$ is a homomorphism, we denote by $f[\boldsymbol{A}]$ the unique subalgebra of $\boldsymbol{B}$ with universe $f[A]$.

For instance, let $f\colon \mathbb{Z} \to \mathbb{R}$ be the absolute value map, that is, the function defined by the rule

$$f(n) := \text{the absolute value of } n.$$

Observe that f is a nonsurjective homomorphism from the lattice of integers to that of reals. Furthermore, the homomorphic image $f[\langle \mathbb{Z}; \leqslant \rangle]$ of $\langle \mathbb{Z}; \leqslant \rangle$ is the lattice of natural numbers $\langle \mathbb{N}; \leqslant \rangle$, which, in turn, is a subalgebra of lattice of reals.

Notably, the homomorphic images of an algebra $\boldsymbol{A}$ can be "internalized" as special equivalence relations on $\boldsymbol{A}$ as follows.

Definition 2.6. A *congruence* of an algebra $\boldsymbol{A}$ is an equivalence relation θ on A such that, for every basic n-ary opearation f of $\boldsymbol{A}$ and $a_1, \dots, a_n, c_1, \dots, c_n \in A$,

$$\text{if } \langle a_1, c_1 \rangle, \dots, \langle a_n, c_n \rangle \in \theta, \text{ then } \langle f^{\boldsymbol{A}}(a_1, \dots, a_n), f^{\boldsymbol{A}}(c_1, \dots, c_n) \rangle \in \theta. \tag{1}$$

In this case, we often write $a \equiv_\theta c$ as a shorthand for $\langle a, c \rangle \in \theta$. The poset of congruences of $\boldsymbol{A}$ ordered under the inclusion relation will be denoted by $\mathsf{Con}(\boldsymbol{A})$.

A simple induction on the construction of terms shows that, for every congruence θ of $\boldsymbol{A}$ and every term $\varphi(x_1, \dots, x_n)$,

$$\text{if } \langle a_1, c_1 \rangle, \dots, \langle a_n, c_n \rangle \in \theta, \text{ then } \langle \varphi^{\boldsymbol{A}}(a_1, \dots, a_n), \varphi^{\boldsymbol{A}}(c_1, \dots, c_n) \rangle \in \theta,$$

for every $a_1, \dots, a_n \in A$. Therefore, congruences preserve not only basic operations, but also arbitrary terms. Furthermore, a simple argument shows that $\mathsf{Con}(\boldsymbol{A})$ is an inductive closure system and, therefore, an algebraic lattice whose maximum is the total relation $A \times A$ and whose minimum is the identity relation $\mathrm{id}_A := \{\langle a, a \rangle : a \in A\}$.

Example 2.7 (Boolean algebras)**.** Recall that a *filter* of a Boolean algebra $\boldsymbol{A}$ is a nonempty upset $F \subseteq A$ closed under binary meets. We denote by $\mathsf{Fi}(\boldsymbol{A})$ the poset of filters of $\boldsymbol{A}$ ordered under the inclusion relation. It is easy to see $\mathsf{Fi}(\boldsymbol{A})$ is an inductive closure system and, therefore, an algebraic lattice. Furthermore, the lattices $\mathsf{Fi}(\boldsymbol{A})$ and $\mathsf{Con}(\boldsymbol{A})$ are isomorphic via the inverse isomorphisms

$$\boldsymbol{\Omega}^{\boldsymbol{A}}(-)\colon \mathsf{Fi}(\boldsymbol{A}) \to \mathsf{Con}(\boldsymbol{A}) \text{ and } \boldsymbol{\tau}(-)\colon \mathsf{Con}(\boldsymbol{A}) \to \mathsf{Fi}(\boldsymbol{A})$$

defined by the rules

$$\begin{aligned}\boldsymbol{\Omega}^{\boldsymbol{A}}(F) &:= \{\langle a, c\rangle \in A \times A : a \to c, c \to a \in F\}\\ \boldsymbol{\tau}(\theta) &:= \{a \in A : \langle a, 1\rangle \in \theta\}.\end{aligned}$$

Because of this, every congruence θ of a Boolean algebra $\boldsymbol{A}$ is induced by some filter F, in the sense that $\theta = \boldsymbol{\Omega}^{\boldsymbol{A}}F$. This correspondence between filters and congruences generalizes straightforwardly to all Heyting algebras. ⊠

Example 2.8 (Modal algebras)**.** A *modal algebra* is an algebra $\boldsymbol{A} = \langle A; \wedge, \vee, \neg, \Box, 0, 1\rangle$ such that $\langle A; \wedge, \vee, \neg, 0, 1\rangle$ is a Boolean algebra and $\Box$ is a unary operation such that

$$\Box(a \wedge c) = \Box a \wedge \Box c \text{ and } \Box 1 = 1,$$

for every $a, c \in A$. An *open filter* of a modal algebra $\boldsymbol{A}$ is a filter of the Boolean reduct of $\boldsymbol{A}$ that, moreover, is closed under the operation $\Box$. The poset of open filters of $\boldsymbol{A}$ ordered under the inclusion relation will be denoted by $\mathsf{Op}(\boldsymbol{A})$. It forms an inductive closure system and, therefore, an algebraic lattice. Furthermore, the lattices $\mathsf{Op}(\boldsymbol{A})$ and $\mathsf{Con}(\boldsymbol{A})$ are isomorphic via the inverse isomorphisms described in Example 2.7. Because of this, every congruence of a modal algebra $\boldsymbol{A}$ has the form $\theta = \boldsymbol{\Omega}^{\boldsymbol{A}}F$, for some open filter F. ⊠

Example 2.9 (Groups)**.** Similarly, it is well known that the lattice of congruences of a group is isomorphic to that of its normal subgroups. Because of this, every congruence of a group is induced by some normal subgroup. ⊠

As we mentioned, there is a tight correspondence between the homomorphic images and the congruences of an algebra $\boldsymbol{A}$. On the one hand, every congruence θ of $\boldsymbol{A}$ gives rise to a homomorphic image $\boldsymbol{A}/\theta$ of $\boldsymbol{A}$. Let $\mathcal{F}$ be the set of function symbols of $\boldsymbol{A}$. Given $\theta \in \mathsf{Con}(\boldsymbol{A})$ and a basic n-ary function symbol $f \in \mathcal{F}$, let $f^{\boldsymbol{A}/\theta}$ be the n-ary operation on A/θ defined by the rule

$$f^{\boldsymbol{A}/\theta}(a_1/\theta, \dots, a_n/\theta) := f^{\boldsymbol{A}}(a_1, \dots, a_n)/\theta.$$

Notice that $f^{\boldsymbol{A}/\theta}$ is well-defined, by condition (1). As a consequence, the structure

$$\boldsymbol{A}/\theta := \langle A/\theta; \{f^{\boldsymbol{A}/\theta} : f \in \mathcal{F}\}\rangle$$

is a well-defined algebra of the type as $\boldsymbol{A}$. Furthermore, $\boldsymbol{A}/\theta \in \mathbb{H}(\boldsymbol{A})$, because the map $\pi_\theta\colon A \to A/\theta$, defined, for every $a \in A$, as $\pi_\theta(a) := a/\theta$, is a surjective homomorphism from $\boldsymbol{A}$ to $\boldsymbol{A}/\theta$. To prove this, consider $a_1, \dots, a_n \in A$. We have

$$\begin{aligned}\pi_\theta(f^{\boldsymbol{A}}(a_1, \dots, a_n)) &= f^{\boldsymbol{A}}(a_1, \dots, a_n)/\theta\\ &= f^{\boldsymbol{A}/\theta}(a_1/\theta, \dots, a_n/\theta)\\ &= f^{\boldsymbol{A}/\theta}(\pi_\theta(a_1), \dots, \pi_\theta(a_n)),\end{aligned}$$

where the second equality follows from the definition of the operation $f^{\boldsymbol{A}/\theta}$.

Corollary 2.10. *If θ is a congruence of an algebra $\boldsymbol{A}$, then $\boldsymbol{A}/\theta$ is a well-defined homomorphic image of $\boldsymbol{A}$.*

In view of the above result, every congruence θ of an algebra $\boldsymbol{A}$ induces a homomorphic image of $\boldsymbol{A}$, namely $\boldsymbol{A}/\theta$. The converse is also true, as we proceed to explain.

Definition 2.11. The *kernel* of a homomorphism $f\colon \boldsymbol{A} \to \boldsymbol{B}$ is the binary relation

$$\mathsf{Ker}(f) \coloneqq \{\langle a, c\rangle \in A \times A : f(a) = f(c)\}.$$

Proposition 2.12. *The kernel of a homomorphism $f\colon \boldsymbol{A} \to \boldsymbol{B}$ is a congruence of $\boldsymbol{A}$.*

Proof. It is obvious that $\mathsf{Ker}(f)$ is an equivalence relation on A. Therefore, to prove that $\mathsf{Ker}(f)$ is a congruence of $\boldsymbol{A}$, it suffices to show that it preserves the basic operations of $\boldsymbol{A}$. Consider a basic n-ary operation g of $\boldsymbol{A}$ and $a_1, \dots, a_n, c_1, \dots, c_n \in A$ such that $\langle a_1, c_1\rangle, \dots, \langle a_n, c_n\rangle \in \mathsf{Ker}(f)$. By the definition of $\mathsf{Ker}(f)$,

$$f(a_i) = f(c_i), \text{ for every } i \leqslant n.$$

It follows that $g^{\boldsymbol{B}}(f(a_1), \dots, f(a_n)) = g^{\boldsymbol{B}}(f(c_1), \dots, f(c_n))$. Since $f\colon \boldsymbol{A} \to \boldsymbol{B}$ is a homomorphism, this yields

$$f(g^{\boldsymbol{A}}(a_1, \dots, a_n)) = g^{\boldsymbol{B}}(f(a_1), \dots, f(a_n)) = g^{\boldsymbol{B}}(f(c_1), \dots, f(c_n)) = f(g^{\boldsymbol{A}}(c_1, \dots, c_n)).$$

Hence, we conclude that $\langle g^{\boldsymbol{A}}(a_1, \dots, a_n), g^{\boldsymbol{A}}(c_1, \dots, c_n)\rangle \in \mathsf{Ker}(f)$, as desired. ⊠

The behaviour of kernels is governed by the next principle.

Fundamental Homomorphism Theorem 2.13. *If $f\colon \boldsymbol{A} \to \boldsymbol{B}$ is a homomorphism with kernel θ, then there exists a unique embedding $g\colon \boldsymbol{A}/\theta \to \boldsymbol{B}$ such that $f = g \circ \pi_\theta$.*

Proof. We begin by proving the existence of g. Let $g\colon A/\theta \to B$ be the map defined as $g(a/\theta) \coloneqq f(a)$, for every $a \in A$. To show that g is well-defined, consider $a, c \in A$ such that $a/\theta = c/\theta$. Since $\theta = \mathsf{Ker}(f)$, this means that $f(a) = f(c)$, as desired. Furthermore, the definition of g guarantees that $f = g \circ \pi_\theta$.

Now, observe g is injective, because, for every $a, c \in A$ such that $g(a/\theta) = g(c/\theta)$, we have $f(a) = f(c)$, that is, $\langle a, c\rangle \in \mathsf{Ker}(f) = \theta$ and, therefore, $a/\theta = c/\theta$. Moreover, for every basic n-ary operation p of $\boldsymbol{A}$ and $a_1, \dots, a_n \in A$, we have

$$\begin{aligned}
g(p^{\boldsymbol{A}/\theta}(a_1/\theta, \dots, a_n/\theta)) &= g(p^{\boldsymbol{A}}(a_1, \dots, a_n)/\theta)\\
&= f(p^{\boldsymbol{A}}(a_1, \dots, a_n))\\
&= p^{\boldsymbol{B}}(f(a_1), \dots, f(a_n))\\
&= p^{\boldsymbol{B}}(g(a_1/\theta), \dots, g(a_n/\theta)).
\end{aligned}$$

The first equality above follows from the definition of $\boldsymbol{A}/\theta$, the second and the last from the definition of g, and the third from the assumption that $f\colon \boldsymbol{A} \to \boldsymbol{B}$ is a homomorphism. Hence, we conclude that $g\colon \boldsymbol{A}/\theta \to \boldsymbol{B}$ is a homomorphism and, therefore, an embedding, as desired.

The uniqueness of g follows from the fact that, if a map g^* satisfies the condition in the statement of the theorem, then, for every $a \in A$,

$$f(a) = g^* \circ \pi_\theta(a) = g^*(a/\theta),$$

that is, g^* coincides with g. ⊠

Corollary 2.14. *If* $f\colon \boldsymbol{A} \to \boldsymbol{B}$ *is a homomorphism, then* $f[\boldsymbol{A}] \cong \boldsymbol{A}/\mathsf{Ker}(f)$. *In particular, if* f *is surjective,* $\boldsymbol{B} \cong \boldsymbol{A}/\mathsf{Ker}(f)$.

Proof. In the proof of the Fundamental Homomorphism Theorem we showed that the map $g\colon A/\mathsf{Ker}(f) \to B$, defined by the rule $g(a/\mathsf{Ker}(f)) := f(a)$, is an embedding of $\boldsymbol{A}/\mathsf{Ker}(f)$ into $\boldsymbol{B}$. As g can be viewed as a surjective embedding of $\boldsymbol{A}/\mathsf{Ker}(f)$ into $f[\boldsymbol{A}]$, we conclude that $f[\boldsymbol{A}] \cong \boldsymbol{A}/\mathsf{Ker}(f)$. ⊠

At this stage, it should be clear that if θ is a congruence on an algebra $\boldsymbol{A}$, then $\pi_\theta\colon \boldsymbol{A} \to \boldsymbol{A}/\theta$ is a surjective homomorphism whose kernel is θ. Similarly, if $f\colon \boldsymbol{A} \to \boldsymbol{B}$ is a surjective homomorphism, then $\boldsymbol{A}/\mathsf{Ker}(f) \cong \boldsymbol{B}$, by Corollary 2.14. As a consequence, for every class of algebras K,

$$\mathbb{H}(\mathsf{K}) = \mathbb{I}\{\boldsymbol{A}/\theta : \boldsymbol{A} \in \mathsf{K} \text{ and } \theta \in \mathsf{Con}(\boldsymbol{A})\}. \tag{2}$$

Now, recall that the Cartesian product of a family of sets $\{A_i : i \in I\}$ is the set

$$\prod_{i \in I} A_i := \{f\colon I \to \bigcup_{i \in I} A_i \,:\, f(i) \in A_i, \text{ for all } i \in I\}.$$

In particular, if I is empty, then $\prod_{i \in I} A_i$ is the singleton containing only the empty map.

Definition 2.15. The *direct product* of a family of similar algebras $\{\boldsymbol{A}_i : i \in I\}$ is the unique algebra of the common type whose universe is the Cartesian product $\prod_{i \in I} A_i$ and such that, for every basic n-ary operation symbol f and every $\vec{a}_1, \ldots, \vec{a}_n \in \prod_{i \in I} A_i$,

$$f^{\prod_{i \in I} \boldsymbol{A}_i}(\vec{a}_1, \ldots, \vec{a}_n)(i) = f^{\boldsymbol{A}_i}(\vec{a}_1(i), \ldots, \vec{a}_n(i)), \text{ for every } i \in I.$$

We denote this algebra by $\prod_{i \in I} \boldsymbol{A}_i$.

In this case, for every $j \in I$, the projection map on the j-th component $p_j\colon \prod_{i \in I} A_i \to A_j$, defined by the rule $p_j(\vec{a}) := \vec{a}(j)$, is a surjective homomorphism from $\prod_{i \in I} \boldsymbol{A}_i$ to $\boldsymbol{A}_j$.

Given a class of similar algebras K, we set

$$\mathbb{P}(\mathsf{K}) := \{\boldsymbol{A} : \boldsymbol{A} \text{ is a direct product of a family } \{\boldsymbol{B}_i : i \in I\} \subseteq \mathsf{K}\}.$$

As usual, when $\mathsf{K} = \{\boldsymbol{A}\}$, we write $\mathbb{P}(\boldsymbol{A})$ as a shorthand for $\mathbb{P}(\{\boldsymbol{A}\})$.

Notice that up to isomorphism, there exists a unique one-element algebra of a given type. Because of this, one-element algebras are called *trivial*. Accordingly, when the set of indexes I is empty, the direct product $\prod_{i \in I} \boldsymbol{A}_i$ is the trivial algebra of the given type. It follows that $\mathbb{P}(\mathsf{K})$ contains always a trivial algebra, for every class of similar algebras K.

Example 2.16 (Powerset algebras)**.** Boolean algebras of the form $\langle \mathcal{P}(X); \cap, \cup, -, \emptyset, X\rangle$ are called *powerset Boolean algebras*. Let $\boldsymbol{B}$ be the two-element Boolean algebra and observe that $\mathbb{IP}(\boldsymbol{B})$ is the class of algebras isomorphic to some powerset Boolean algebra. To prove this, observe that every powerset Boolean algebra $\mathcal{P}(X)$ is isomorphic to a direct product of $\boldsymbol{B}$ via the *characteristic function* $f_X\colon \mathcal{P}(X) \to \prod_{x \in X} \boldsymbol{B}_x$, defined by the rule

$$f(Y)(x) := \begin{cases} 1 & \text{if } x \in Y \\ 0 & \text{if } x \notin Y, \end{cases}$$

where $Y \in \mathcal{P}(X)$ and $x \in X$. By the same token, every direct product $\prod_{i \in I} \boldsymbol{B}_i$ of $\boldsymbol{B}$ is isomorphic to the powerset Boolean algebra $\mathcal{P}(I)$ via the isomorphism f_I. ⊠

We close this section by reviewing the subdirect product construction.

Definition 2.17. A subalgebra $\boldsymbol{B}$ of a direct product $\prod_{i \in I} \boldsymbol{A}_i$ is said to be a *subdirect product* of $\{\boldsymbol{A}_i : i \in I\}$ if the projection map p_i is surjective, for every $i \in I$. Similarly, an embedding $f \colon \boldsymbol{B} \to \prod_{i \in I} \boldsymbol{A}_i$ is said to be *subdirect* when $f[\boldsymbol{B}]$ is a subdirect product of the family $\{\boldsymbol{A}_i : i \in I\}$.

Given a class of similar algebras K, we set

$$\mathbb{P}_{\text{SD}}(\mathsf{K}) := \{\boldsymbol{A} : \boldsymbol{A} \text{ is a subdirect direct product of a family } \{\boldsymbol{B}_i : i \in I\} \subseteq \mathsf{K}\}.$$

As usual, when $\mathsf{K} = \{\boldsymbol{A}\}$, we write $\mathbb{P}_{\text{SD}}(\boldsymbol{A})$ as a shorthand for $\mathbb{P}_{\text{SD}}(\{\boldsymbol{A}\})$. Clearly, $\mathbb{P}_{\text{SD}}(\mathsf{K}) \subseteq \mathbb{SP}(\mathsf{K})$. Furthermore, $\mathbb{P}_{\text{SD}}(\mathsf{K})$ contains always a trivial algebra.

Example 2.18 (Distributive lattices)**.** Let DL be the class of distributive lattices and $\boldsymbol{B}$ be the two-element distributive lattice. Birkhoff's Representation Theorem states that $\mathsf{DL} = \mathbb{IP}_{\text{SD}}(\boldsymbol{B})$. The inclusion $\mathbb{IP}_{\text{SD}}(\boldsymbol{B}) \subseteq \mathsf{DL}$ follows from the fact that DL is closed under $\mathbb{I}, \mathbb{S}$, and $\mathbb{P}$. For the other inclusion, consider a distributive lattice $\boldsymbol{A}$ and let I be the set of its prime filters. By Birkhoff's Representation Theorem, the map

$$\gamma \colon \boldsymbol{A} \to \prod_{F \in I} \boldsymbol{B}_F,$$

defined, for every $a \in A$ and $F \in I$, by the rule

$$\gamma(a)(F) := \begin{cases} 1 & \text{if } a \in F \\ 0 & \text{if } a \notin F, \end{cases}$$

is a well-defined subdirect embedding. ⊠

Example 2.19 (Boolean algebras)**.** Similarly, Stone's Representation Theorem states that the class of Boolean algebras coincides with $\mathbb{IP}_{\text{SD}}(\boldsymbol{B})$, where $\boldsymbol{B}$ the two-element Boolean algebra. ⊠

The next result provides a general recipe to construct subdirect products.

Proposition 2.20. *Let $\boldsymbol{A}$ be an algebra and $\{\theta_i : i \in I\} \subseteq \mathsf{Con}(\boldsymbol{A})$. Then the map*

$$f \colon \boldsymbol{A} / \bigcap_{i \in I} \theta_i \to \prod_{i \in I} \boldsymbol{A}/\theta_i,$$

defined, for every $a \in A$ and $j \in I$, as

$$f(a / \bigcap_{i \in I} \theta_i)(j) := a/\theta_j,$$

is a subdirect embedding.

Proof. For the sake of readability, set $\boldsymbol{B} := \boldsymbol{A} / \bigcap_{i \in I} \theta_i$. To prove that f is injective, consider $a, c \in A$ such that $\langle a, c \rangle \notin \bigcap_{i \in I} \theta_i$. Then there exists $j \in I$ such that $\langle a, c \rangle \notin \theta_j$ and, therefore,

$$f(a / \bigcap_{i \in I} \theta_i)(j) := a/\theta_j \neq c/\theta_j = f(c / \bigcap_{i \in I} \theta_i)(j).$$

It follows that $f(a / \bigcap_{i \in I} \theta_i) \neq f(c / \bigcap_{i \in I} \theta_i)$. Thus, f is injective. Moreover, by the definition of f, the composition $p_i \circ f \colon \boldsymbol{B} \to \boldsymbol{A}/\theta_i$ is surjective, for every $i \in I$.

It only remains to prove that f is a homomorphism. Consider an n-ary basic operation g and $a_1, \dots, a_n \in A$. For every $j \in I$, we have

$$\begin{aligned} f(g^{\boldsymbol{B}}(a_1/\bigcap_{i\in I}\theta_i, \dots, a_n/\bigcap_{i\in I}\theta_i))(j) &= f(g^{\boldsymbol{A}}(a_1, \dots, a_n)/\bigcap_{i\in I}\theta_i)(j) \\ &= g^{\boldsymbol{A}}(a_1, \dots, a_n)/\theta_j \\ &= g^{\boldsymbol{A}/\theta_j}(a_1/\theta_j, \dots, a_n/\theta_j) \\ &= g^{\boldsymbol{A}/\theta_j}(f(a_1/\bigcap_{i\in I}\theta_i)(j), \dots, f(a_n/\bigcap_{i\in I}\theta_i)(j)) \\ &= g^{\prod_{i\in I}\boldsymbol{A}/\theta_i}(f(a_1/\bigcap_{i\in I}\theta_i), \dots, f(a_n/\bigcap_{i\in I}\theta_i))(j). \end{aligned}$$

It follows that

$$f(g^{\boldsymbol{B}}(a_1/\bigcap_{i\in I}\theta_i, \dots, a_n/\bigcap_{i\in I}\theta_i)) = g^{\prod_{i\in I}\boldsymbol{A}/\theta_i}(f(a_1/\bigcap_{i\in I}\theta_i), \dots, f(a_n/\bigcap_{i\in I}\theta_i)). \qquad \boxtimes$$

3. PREVARIETIES

Definition 3.1. A class of similar algebras closed $\mathbb{I}, \mathbb{S}$, and $\mathbb{P}$ is a said to be a *prevariety*.

Given a class of similar algebras K, the least prevariety extending K is $\mathbb{ISP}(\mathsf{K})$ and is called the prevariety *generated* by K. For instance, in view of Examples 2.18, the class of distributive lattices is the prevariety generated by the two-element distributive lattice. Similarly, the class of Boolean algebras is the prevariety generated by the two-element Boolean algebra (see Example 2.19, if necessary).

Our aim will be to prove that prevarieties are precisely the classes of algebras axiomatized by a certain kind of infinitary formulas. To this end, we rely on the following notational convention. When no confusion shall arise, given a sequence $\vec{a}$ and a set A, we write $\vec{a} \in A$ to indicate that the elements of the sequence $\vec{a}$ belong to A.

Definition 3.2. A *generalized quasi-equation* of type ρ is an expression Φ of the form

$$\Big(\mathop{\&}_{i\in I} \varphi_i(\vec{x}) \approx \psi_i(\vec{x}) \Big) \Longrightarrow \varepsilon(\vec{x}) \approx \delta(\vec{x}),$$

where $\{\varphi_i \approx \psi_i : i \in I\} \cup \{\varepsilon \approx \delta\}$ is a set of equations of type ρ. Then Φ is *valid* in an algebra $\boldsymbol{A}$ of type ρ when so is its universal closure, that is, for every $\vec{a} \in A$,

$$\text{if } (\varphi_i^{\boldsymbol{A}}(\vec{a}) = \psi_i^{\boldsymbol{A}}(\vec{a}), \text{ for all } i \in I), \text{ then } \varepsilon^{\boldsymbol{A}}(\vec{a}) = \delta^{\boldsymbol{A}}(\vec{a}).$$

In this case, we often say that $\boldsymbol{A}$ *satisfies* Φ.

Notice that, in the above definition, the set of indexes I can be arbitrarily large and that the same applies to the sequence of variables $\vec{x}$ that appear in the equations of Φ. This motivates the following.

Definition 3.3. A generalized quasi-equation is said to be

(i) a *quasi-equation* when the index set I is finite; and
(ii) an *equation* when the index set I is empty.

Remark 3.4. It might seem that we are using the term *equations* to refer to two distinct kinds of expressions, namely those of the form $\varepsilon \approx \delta$ and $\emptyset \Longrightarrow \varepsilon \approx \delta$. This is not a problem, however, because these expressions are synonyms, in the sense that an algebra satisfies $\varepsilon \approx \delta$ if and only if it satisfies $\emptyset \Longrightarrow \varepsilon \approx \delta$. Because of this, we will continue to denote equations by $\varepsilon \approx \delta$, while keeping in mind that they are special instances of generalized quasi-equations. ⊠

Definition 3.5. Let $\rho\colon \mathcal{F} \to \mathbb{N}$ be a type and X a set of variables disjoint from $\mathcal{F}$. The *term algebra* $\boldsymbol{T}_\rho(X)$ of type ρ over X is the unique algebra of type ρ whose universe is $T_\rho(X)$ and with basic n-ary operations f defined, for every $\varphi_1, \dots, \varphi_n \in T_\rho(X)$, as

$$f^{\boldsymbol{T}_\rho(X)}(\varphi_1, \dots, \varphi_n) := f(\varphi_1, \dots, \varphi_n).$$

Term algebras have the following fundamental property.

Proposition 3.6. *Let $\boldsymbol{A}$ be an algebra of type ρ and X a set of variables. Every function $f\colon X \to A$ extends uniquely to a homomorphism $f^*\colon \boldsymbol{T}_\rho(X) \to \boldsymbol{A}$.*

Proof. The unique extension f^* is defined, for every $\varphi(x_{\alpha_1}, \dots x_{\alpha_n}) \in T_\rho(X)$, as

$$f^*(\varphi) = \varphi^{\boldsymbol{A}}(f(x_{\alpha_1}), \dots, f(x_{\alpha_n})).$$ ⊠

Exercise 3.7. Prove the above proposition. ⊠

Theorem 3.8. *A class of similar algebras is a prevariety if and only if it can be axiomatized by a class of generalized quasi-equations.*

Proof. The "if" part follows from the fact that the validity of generalized quasi-equations persists under the formation of isomorphic copies, subalgebras, and direct product. To prove the converse, consider a prevariety K and let Σ be the class of generalized quasi-equations valid in it. Let K^+ be the class of algebras in which the generalized quasi-equations in Σ are valid. Clearly, $\mathsf{K} \subseteq \mathsf{K}^+$. To prove the other inclusion, consider an algebra $\boldsymbol{A} \in \mathsf{K}^+$. Let also X be a set of variables for which there exists a surjective map $f\colon X \to A$. By Proposition 3.6, f extends to a surjective homomorphism $f^*\colon \boldsymbol{T}_\rho(X) \to \boldsymbol{A}$. Together with Corollary 2.14, this yields

$$\boldsymbol{A} \cong \boldsymbol{T}_\rho(X)/\mathsf{Ker}(f^*). \tag{3}$$

Now, consider an arbitrary pair $\langle \varphi, \psi \rangle \in (T_\rho(X) \times T_\rho(X)) \smallsetminus \mathsf{Ker}(f^*)$. Notice that the elements of $T_\rho(X) \times T_\rho(X)$ are ordered pairs of terms and, therefore, can be viewed as equations under the identification of $\langle \varepsilon, \delta \rangle$ with $\varepsilon \approx \delta$. In this way, $\mathsf{Ker}(f^*)$ becomes a set of equations in variables X. Bearing this in mind, consider the generalized quasi-equation

$$\Phi := \Big(\mathop{\&} \mathsf{Ker}(f^*) \Big) \Longrightarrow \varphi \approx \psi.$$

We will prove that Φ fails in $\boldsymbol{A}$. For the sake of readability we will denote by $\vec{x}$ the sequence of all variables in X. Observe that every element $\varepsilon \in T_\rho(X)$ is of the form $\varepsilon(\vec{x})$. Then consider the assignment $f\colon X \to A$. We will denote by $f(\vec{x})$ the sequence obtained by applying f component-wise to $\vec{x}$. For every pair $\langle \varepsilon, \delta \rangle \in \mathsf{Ker}(f^*)$, we have

$$\varepsilon^{\boldsymbol{A}}(f(\vec{x})) = \varepsilon^{\boldsymbol{A}}(f^*(\vec{x})) = f^*(\varepsilon(\vec{x})) = f^*(\delta(\vec{x})) = \delta^{\boldsymbol{A}}(f^*(\vec{x})) = \delta^{\boldsymbol{A}}(f(\vec{x})).$$

The equalities above can be justified as follows. The first and the last holds because f^* extends f, the second and the fourth because $f^*\colon \boldsymbol{T}_\rho(X) \to \boldsymbol{A}$ is a homomorphism, and the third because $\langle \varepsilon, \delta \rangle \in \mathsf{Ker}(f^*)$. On the other hand, since $\langle \varphi, \psi \rangle \notin \mathsf{Ker}(f^*)$, a similar argument shows

$$\varphi^{\boldsymbol{A}}(f(\vec{x})) \neq \psi^{\boldsymbol{A}}(f(\vec{x})).$$

Thus, $\boldsymbol{A}$ refutes Φ, as desired.

Since $\boldsymbol{A} \in \mathsf{K}^+$, this implies that there exists some algebra $\boldsymbol{C}_{\varphi,\psi} \in \mathsf{K}$ and an assignment $g_{\varphi,\psi} \colon X \to C_{\varphi,\psi}$ such that

$$\varepsilon^{\boldsymbol{C}_{\varphi,\psi}}(g_{\varphi,\psi}(\vec{x})) = \delta^{\boldsymbol{C}_{\varphi,\psi}}(g_{\varphi,\psi}(\vec{x})), \text{ for all } \langle \varepsilon, \delta \rangle \in \mathsf{Ker}(f^*), \text{ and } \varphi^{\boldsymbol{C}_{\varphi,\psi}}(g_{\varphi,\psi}(\vec{x})) \neq \psi^{\boldsymbol{C}_{\varphi,\psi}}(g_{\varphi,\psi}(\vec{x})).$$

Recall that $g_{\varphi,\psi}$ extends uniquely to a homomorphism $g^*_{\varphi,\psi} \colon \boldsymbol{T}_\rho(X) \to \boldsymbol{C}_{\varphi,\psi}$. Moreover, from the above display it follows

$$g^*_{\varphi,\psi}(\varepsilon) = g^*_{\varphi,\psi}(\delta), \text{ for all } \langle \varepsilon, \delta \rangle \in \mathsf{Ker}(f^*), \text{ and } g^*_{\varphi,\psi}(\varphi) \neq g^*_{\varphi,\psi}(\psi).$$

Consequently,

$$\mathsf{Ker}(f^*) \subseteq \mathsf{Ker}(g^*_{\varphi,\psi}) \text{ and } \langle \varphi, \psi \rangle \notin \mathsf{Ker}(g^*_{\varphi,\psi}).$$

It follows that

$$\mathsf{Ker}(f^*) = \bigcap \{\mathsf{Ker}(g^*_{\varphi,\psi}) : \langle \varphi, \psi \rangle \in (T_\rho(X) \times T_\rho(X)) \smallsetminus \mathsf{Ker}(f^*)\}.$$

By Proposition 2.20, this yields

$$\boldsymbol{T}_\rho(X)/\mathsf{Ker}(f^*) \in \mathbb{IP}_{\mathrm{SD}}(\{\boldsymbol{T}_\rho(X)/\mathsf{Ker}(g^*_{\varphi,\psi}) : \langle \varphi, \psi \rangle \in (T_\rho(X) \times T_\rho(X)) \smallsetminus \mathsf{Ker}(f^*)\}). \quad (4)$$

Moreover, from Corollary 2.14 and the fact that K is closed under $\mathbb{I}$ and $\mathbb{S}$ it follows that

$$\boldsymbol{T}_\rho(X)/\mathsf{Ker}(g^*_{\varphi,\psi}) \in \mathbb{IS}(\boldsymbol{C}_{\varphi,\psi}) \subseteq \mathsf{K},$$

for every $\langle \varphi, \psi \rangle \in (T_\rho(X) \times T_\rho(X)) \smallsetminus \mathsf{Ker}(f^*)$. Consequently, (4) simplifies to

$$\boldsymbol{T}_\rho(X)/\mathsf{Ker}(f^*) \in \mathbb{IP}_{\mathrm{SD}}(\mathsf{K}) \subseteq \mathsf{K},$$

where the last inclusion follows from the fact that K is a prevariety. Together with (3), this yields $\boldsymbol{A} \in \mathbb{I}(\mathsf{K}) \subseteq \mathsf{K}$. ⊠

Remark 3.9. In view of Theorem 3.8, prevarieties are classes of algebras axiomatized by classes of generalized quasi-equations. It is therefore natural to wonder whether there exists a prevariety that cannot be axiomatized by a set (as opposed to proper class) of generalized quasi-equations. It turns out that the answer to this question depends on the set theory we live in, as the nonexistence of such a prevariety is equivalent to *Vopěnka's Principle*. ⊠

Nonetheless, prevarieties axiomatizable by a set of generalized quasi-equations admit a relatively transparent description, as we proceed to explain. Given an infinite cardinal κ and a class of algebras K, let

$$\mathbb{U}_\kappa(\mathsf{K}) := \{\boldsymbol{A} : \boldsymbol{B} \in \mathsf{K}, \text{ for all } \kappa\text{-generated } \boldsymbol{B} \leqslant \boldsymbol{A}\}.$$

Definition 3.10. Let κ be an infinite cardinal. A *κ-generalized quasi-variety* is a prevariety closed under $\mathbb{U}_\kappa$.

When $\kappa = \aleph_0$, we often say that K is simply a *generalized quasi-variety*. Given a class of similar algebras K, the least κ-generalized quasi-variety extending K is $\mathbb{U}_\kappa\mathbb{ISP}(\mathsf{K})$ and is called the κ-generalized quasi-variety *generated* by K.

Theorem 3.11. *Let κ be an infinite cardinal. A class of similar algebras is a κ-generalized quasi-variety if and only if it can be axiomatized by a set of generalized quasi-equations in which at most κ variables occur.*

Proof. The "if" part follows from the fact that the validity of generalized quasi-equations in $\leqslant \kappa$ variables persist under the $\mathbb{I}, \mathbb{S}, \mathbb{P}, \mathbb{U}_\kappa$. To prove the converse, consider a κ-generalized quasi-variety K. Then let X be a set of variables of cardinality κ and Σ the class of generalized quasi-equations written with variables in X. Since X is a set, so is Σ. It only remains to prove that K coincides with the class K^+ of algebras satisfying the generalized quasi-equations in Σ. Clearly, $\mathsf{K} \subseteq \mathsf{K}^+$. To prove the other inclusion, consider an algebra $\boldsymbol{A} \in \mathsf{K}^+$. We need to prove that $\boldsymbol{A} \in \mathsf{K}$. Since K is closed under $\mathbb{U}_\kappa$, it suffices to show that all the κ-generated subalgebras of $\boldsymbol{A}$ belong to K.

Accordingly, let $\boldsymbol{B}$ be a κ-generated subalgebra of A and $Y \subseteq B$ a set of generators for $\boldsymbol{B}$ of size $\leqslant \kappa$. There exists a surjective map $f\colon X \to Y$. By Proposition 3.6, f extends to a surjective homomorphism $f^*\colon \boldsymbol{T}_\rho(X) \to \boldsymbol{B}$. Now, we repeat the argument in the proof of Theorem 3.8, obtaining $\boldsymbol{B} \in \mathsf{K}$, as desired. ⊠

Corollary 3.12. *A prevariety can be axiomatized by a set of generalized quasi-equations if and only if it is a κ-generalized quasi-variety, for some infinite cardinal κ.*

*Exercise** 3.13. Let K be a class of similar algebras and κ an infinite cardinal. Prove that the prevariety and the κ-generalized quasi-variety generated by K are, respectively, $\mathbb{ISP}(\mathsf{K})$ and $\mathbb{U}_\kappa\mathbb{ISP}(\mathsf{K})$. ⊠

4. ULTRAPRODUCTS

Let $\boldsymbol{A}$ be a Boolean algebra. A nonempty subset $F \subseteq A$ is said to be a *filter* of $\boldsymbol{A}$ if it is an upset closed under binary meets. A filter is said to be *proper* when it differs from A. Lastly, a proper filter U of $\boldsymbol{A}$ is said to be a *ultrafilter* of $\boldsymbol{A}$ if it is maximal among the proper filters of $\boldsymbol{A}$ or, equivalently, if

$$a \in U \text{ or } \neg a \in U, \text{ for every } a \in A.$$

While the following result holds in ZFC, it cannot be proved in ZF (although it is strictly weaker then the axiom of choice).

Ultrafilter Lemma 4.1. *Every proper filter on a Boolean algebra can be extended to a ultrafilter.*

Ultrafilters on powerset Boolean algebras $\mathcal{P}(X)$ are also called *ultrafilters on X*. In this section we will use them to define a product-like construction known as *ultraproduct*. To this end, let $\{\boldsymbol{A}_i : i \in I\}$ be a family of similar algebras. The *equalizer* of a pair of elements $\vec{a}, \vec{c} \in \prod_{i \in I} A_i$ is the set of indexes on which the sequences $\vec{a}$ and $\vec{c}$ agree, that is,

$$[\![\vec{a} = \vec{c}]\!] \coloneqq \{i \in I : \vec{a}(i) = \vec{c}(i)\}.$$

Moreover, given an ultrafilter U on the index set I, let θ_U be the binary relation on the Cartesian product $\prod_{i \in I} A_i$ defined as

$$\theta_U \coloneqq \{\langle \vec{a}, \vec{c} \rangle : [\![\vec{a} = \vec{c}]\!] \in U\}.$$

Proposition 4.2. *If $\{\boldsymbol{A}_i : i \in I\}$ is a family of similar algebras and U an ultrafilter on I, then θ_U is a congruence of $\prod_{i \in I} \boldsymbol{A}_i$.*

Proof. We begin by proving that θ_U is an equivalence relation on $\prod_{i \in I} A_i$. To this end, consider $\vec{a}, \vec{b}, \vec{c} \in \prod_{i \in I} A_i$. We have

$$[\![\vec{a} = \vec{a}]\!] = \{i \in I : \vec{a}(i) = \vec{a}(i)\} = I.$$

Observe that $I \in U$, since U is a nonempty upset of $\mathcal{P}(I)$. Together with the above display, this yields $[\![\vec{a} = \vec{a}]\!] \in U$ and, therefore, $\langle \vec{a}, \vec{a} \rangle \in \theta_U$. It follows that θ_U is reflexive. To prove that it is symmetric, suppose that $\langle \vec{a}, \vec{c} \rangle \in \theta_U$. Then $[\![\vec{a} = \vec{c}]\!] \in U$. Since $[\![\vec{a} = \vec{c}]\!] = [\![\vec{c} = \vec{a}]\!]$, this implies $[\![\vec{c} = \vec{a}]\!] \in U$ and, therefore, $\langle \vec{c}, \vec{a} \rangle \in \theta_U$. Lastly, to prove that θ_U is transitive, suppose that $\langle \vec{a}, \vec{b} \rangle, \langle \vec{b}, \vec{c} \rangle \in \theta_U$, that is, $[\![\vec{a} = \vec{b}]\!], [\![\vec{b} = \vec{c}]\!] \in U$. Since U is closed under binary meets,

$$[\![\vec{a} = \vec{b}]\!] \cap [\![\vec{b} = \vec{c}]\!] \in U$$

Clearly, $[\![\vec{a} = \vec{b}]\!] \cap [\![\vec{b} = \vec{c}]\!] \subseteq [\![\vec{a} = \vec{c}]\!]$. Since U is an upset of $\mathcal{P}(I)$, we obtain that $[\![\vec{a} = \vec{c}]\!] \in U$, whence $\langle \vec{a}, \vec{c} \rangle \in \theta_U$. We conclude that θ_U is an equivalence relation.

To prove that θ_U is a congruence, it only remains to show that it preserves the basic operations. Accordingly, let f be a basic n-ary operation and $\vec{a}_1, \dots, \vec{a}_n, \vec{c}_1, \dots, \vec{c}_n \in \prod_{i \in I} A_i$ such that

$$\langle \vec{a}_1, \vec{c}_1 \rangle, \dots, \langle \vec{a}_n, \vec{c}_n \rangle \in \theta_U.$$

By definition of θ_U, this amounts to $[\![\vec{a}_1 = \vec{c}_1]\!], \dots, [\![\vec{a}_n = \vec{c}_n]\!] \in U$. Since U is a filter, it is closed under finite meets, whence

$$[\![\vec{a}_1 = \vec{c}_1]\!] \cap \dots \cap [\![\vec{a}_n = \vec{c}_n]\!] \in U. \tag{5}$$

We will show that

$$[\![\vec{a}_1 = \vec{c}_1]\!] \cap \dots \cap [\![\vec{a}_n = \vec{c}_n]\!] \subseteq [\![f^{\prod_{i \in I} \boldsymbol{A}_i}(\vec{a}_1, \dots, \vec{a}_n) = f^{\prod_{i \in I} \boldsymbol{A}_i}(\vec{c}_1, \dots, \vec{c}_n)]\!]. \tag{6}$$

To this end, consider $j \in [\![\vec{a}_1 = \vec{c}_1]\!] \cap \dots \cap [\![\vec{a}_n = \vec{c}_n]\!]$. We have

$$\vec{a}_1(j) = \vec{c}_1(j), \dots, \vec{a}_n(j) = \vec{c}_n(j).$$

Consequently,

$$\begin{aligned} f^{\prod_{i \in I} \boldsymbol{A}_i}(\vec{a}_1, \dots, \vec{a})(j) &= f^{\boldsymbol{A}_j}(\vec{a}_1(j), \dots, \vec{a}_n(j)) \\ &= f^{\boldsymbol{A}_j}(\vec{c}_1(j), \dots, \vec{c}_n(j)) \\ &= f^{\prod_{i \in I} \boldsymbol{A}_i}(\vec{c}_1, \dots, \vec{c})(j), \end{aligned}$$

that is, $j \in [\![f^{\prod_{i \in I} \boldsymbol{A}_i}(\vec{a}_1, \dots, \vec{a}_n) = f^{\prod_{i \in I} \boldsymbol{A}_i}(\vec{c}_1, \dots, \vec{c}_n)]\!]$. This establishes (6). Since U is an upset of $\mathcal{P}(I)$, from (5) and (6) it follows

$$[\![f^{\prod_{i \in I} \boldsymbol{A}_i}(\vec{a}_1, \dots, \vec{a}_n) = f^{\prod_{i \in I} \boldsymbol{A}_i}(\vec{c}_1, \dots, \vec{c}_n)]\!] \in U.$$

Hence, we conclude that $\langle f^{\prod_{i \in I} \boldsymbol{A}_i}(\vec{a}_1, \dots, \vec{a}_n), f^{\prod_{i \in I} \boldsymbol{A}_i}(\vec{c}_1, \dots, \vec{c}_n) \rangle \in \theta_U$, as desired. ⊠

In view of the above result, we can make the following definition.

Definition 4.3. An *ultraproduct* of a family of similar algebras $\{\boldsymbol{A}_i : i \in I\}$ is an algebra of the form $\prod_{i \in I} \boldsymbol{A}_i / \theta_U$, for some ultrafilter U on I.

Given a class of similar algebras K, we set

$$\mathbb{P}_{\text{U}}(\mathsf{K}) := \{\boldsymbol{A} : \boldsymbol{A} \text{ is an ultraproduct of a family } \{\boldsymbol{B}_i : i \in I\} \subseteq \mathsf{K}\}.$$

Notice that $\mathbb{P}_{\text{U}}(\mathsf{K}) \subseteq \mathbb{HP}(\mathsf{K})$. Furthermore, as usual, when $\mathsf{K} = \{\boldsymbol{A}\}$, we write $\mathbb{P}_{\text{U}}(\boldsymbol{A})$ as a shorthand for $\mathbb{P}_{\text{U}}(\{\boldsymbol{A}\})$.

Exercise 4.4. Prove that if U is not free (that is, it is principal), then $\prod_{i\in I} \boldsymbol{A}_i/\theta_U$ is isomorphic to some $\boldsymbol{A}_i$. Conclude that if I is finite, then $\prod_{i\in I} \boldsymbol{A}_i/\theta_U$ belongs to $\mathbb{I}\{\boldsymbol{A}_i : i \in I\}$. Because of this, interesting ultraproducts arise from free ultrafilters only. ⊠

The importance of ultraproducts is tightly related to the following fundamental result.

Theorem 4.5 (Łos). *Let $\{\boldsymbol{A}_i : i \in I\}$ be a family of similar algebras, U an ultrafilter on I, and $\phi(x_1, \dots, x_n)$ a first-order formula. For every $\vec{a}_1, \dots, \vec{a}_n \in \prod_{i\in I} A_i$,*

$$\prod_{i\in I} \boldsymbol{A}_i/\theta_U \vDash \phi(\vec{a}_1/\theta_U, \dots, \vec{a}_n/\theta_U) \iff \{i \in I : \boldsymbol{A}_i \vDash \phi(\vec{a}_1(i), \dots, \vec{a}_n(i))\} \in U.$$

Corollary 4.6. *Let $\{\boldsymbol{A}_i : i \in I\}$ be a family of similar algebras, U an ultrafilter on I, and ϕ a first-order sentence. If ϕ is valid in all the $\boldsymbol{A}_i$, then it is valid in $\prod_{i\in I} \boldsymbol{A}_i/\theta_U$.*

In view Łos' Theorem, ultraproducts are instrumental to construct nonstandard models of first-order theories. For instance, let $\mathbb{N} = \langle \mathbb{N}; s, +, \cdot, 0\rangle$ be the standard model of Peano Arithmetic. If U is a free ultrafilter on $\mathbb{N}$, the ultraproduct $\prod_{n\in\mathbb{N}} \mathbb{N}_n/U$ is *elementarily equivalent* to $\mathbb{N}$, that is, it satisfies the same first-order sentences as $\mathbb{N}$. On the other hand, it is not hard to see that $\prod_{n\in\mathbb{N}} \mathbb{N}_n/U$ is uncountable and, therefore, contains many "infinite" (or nonstandard) natural numbers.

For the present purpose, however, we will not need the full strength of Łos Theorem and, therefore, we shall omit its proof. Instead, we shall focus on a particular embedding theorem for ultraproducts that depends on the following notion.

Definition 4.7. A *local subgraph* $\mathbb{X}$ of an algebra $\boldsymbol{A}$ is a finite subset $X \subseteq A$ endowed with the restriction of finitely many basic operations of $\boldsymbol{A}$ to X.

In this case, $\mathbb{X}$ is a finite *partial* algebra of finite type (even when the type of $\boldsymbol{A}$ is infinite).

Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be similar algebras and $\mathbb{X}$ a local subgraph of $\boldsymbol{A}$. A map $f\colon X \to B$ is said to be an *embedding* of $\mathbb{X}$ into $\boldsymbol{B}$ if it is injective and, for every basic n-ary operation g of the type of $\mathbb{X}$ and $a_1, \dots, a_n \in X$ such that $g^{\boldsymbol{A}}(a_1, \dots, a_n) \in X$,

$$f(g^{\boldsymbol{A}}(a_1, \dots, a_n)) = g^{\boldsymbol{B}}(f(a_1), \dots, f(a_n)).$$

Theorem 4.8. *Let $\mathsf{K} \cup \{\boldsymbol{A}\}$ be a class of similar algebras. If every local subgraph of $\boldsymbol{A}$ can be embedded into some member of K, then $\boldsymbol{A} \in \mathbb{ISP}_{\mathrm{U}}(\mathsf{K})$.*

Proof. Let I be the set of local subgraphs of $\boldsymbol{A}$. By assumption, for every $\mathbb{X} \in I$ there are an algebra $\boldsymbol{B}_{\mathbb{X}} \in \mathsf{K}$ and an embedding $h_{\mathbb{X}}\colon \mathbb{X} \to \boldsymbol{B}_{\mathbb{X}}$. We define a partial order $\sqsubseteq$ on I as follows:

$$\mathbb{X} \sqsubseteq \mathbb{Y} \iff X \subseteq Y \text{ and the type of } \mathbb{Y} \text{ extends that of } \mathbb{X}.$$

Then, for every $\mathbb{X} \in I$, define

$$J_{\mathbb{X}} := \{\mathbb{Y} \in I \colon \mathbb{X} \sqsubseteq \mathbb{Y}\}.$$

Moreover, let $\mathcal{F}$ be the filter of $\mathcal{P}(I)$ generated by $\{J_{\mathbb{X}} \colon \mathbb{X} \in I\}$. Recall that

$$\mathcal{F} = \{Y \subseteq I : J_{\mathbb{X}_1} \cap \dots \cap J_{\mathbb{X}_n} \subseteq Y, \text{ for some } \mathbb{X}_1, \dots, \mathbb{X}_n \in I\}.$$

We will prove that $\mathcal{F}$ is proper. To this end, consider $\mathbb{X}_1, \dots, \mathbb{X}_n \in I$. Then let $\mathbb{Y}$ be the local subgraph of $\boldsymbol{A}$ with universe $Y := X_1 \cup \dots \cup X_n$ and whose type in the union of the types of the various $\mathbb{X}_i$. Then

$$\mathbb{X}_i \sqsubseteq \mathbb{Y}, \text{ for every } i \leqslant n,$$

that is, $\mathbb{Y} \in J_{\mathbb{X}_1} \cap \cdots \cap J_{\mathbb{X}_n}$. It follows that $\emptyset \notin \mathcal{F}$ and, therefore, that $\mathcal{F}$ is proper. As $\mathcal{F}$ is a proper filter, by the Ultrafilter Lemma, it can be extended to an ultrafilter U on I.

Now, consider a map

$$f\colon A \to \prod_{\mathbb{X} \in I} B_{\mathbb{X}}$$

such that $f(a)(\mathbb{X}) = h_{\mathbb{X}}(a)$, for every $a \in A$ and $\mathbb{X} \in I$ such that $a \in \mathbb{X}$. Moreover, let

$$f^*\colon A \to \prod_{\mathbb{X} \in I} B_{\mathbb{X}} / \theta_U$$

be the map defined by the rule

$$f^*(a) := f(a)/\theta_U.$$

We will show f^* is an embedding of $\boldsymbol{A}$ into $\prod_{\mathbb{X} \in I} \boldsymbol{B}_{\mathbb{X}}/\theta_U$.

In order to prove that f^* is injective, consider a pair of distinct elements $a, c \in A$. Consider a local subgraph $\mathbb{Y}$ of $\boldsymbol{A}$ containing a and c. We will show that

$$J_{\mathbb{Y}} \subseteq \{\mathbb{X} \in I : f(a)(\mathbb{X}) \neq f(c)(\mathbb{X})\} \tag{7}$$

Consider $\mathbb{X} \in J_{\mathbb{Y}}$. Then $\mathbb{Y} \sqsubseteq \mathbb{X}$ and, therefore, $a, c \in Y \subseteq X$. Since $a, c \in X$, we have

$$f(a)(\mathbb{X}) = h_{\mathbb{X}}(a) \ \text{ and } \ f(c)(\mathbb{X}) = h_{\mathbb{X}}(c).$$

Furthermore, $h_{\mathbb{X}}(a) \neq h_{\mathbb{X}}(c)$, because $h_{\mathbb{X}}$ is injective and $a \neq c$. This yields $f(a)(\mathbb{X}) \neq f(c)(\mathbb{X})$, establishing (7).

Recall that the definition of U guarantees that $J_{\mathbb{Y}} \in \mathcal{F} \subseteq U$. Therefore, since U is an upset of $\mathcal{P}(I)$, we can apply (7) obtaining

$$I \smallsetminus [\![f(a) = f(c)]\!] = \{\mathbb{X} \in I : f(a)(\mathbb{X}) \neq f(c)(\mathbb{X})\} \in U.$$

Since U is a proper filter, this implies

$$[\![f(a) = f(c)]\!] \notin U$$

and, therefore,

$$f^*(a) = f(a)/\theta_U \neq f(c)/\theta_U = f^*(c).$$

Hence, we conclude that f^* is injective.

To prove that it is a homomorphism, consider a basic n-ary operation g and $a_1, \dots, a_n \in A$. Then consider a local subgraph $\mathbb{Y}$ of $\boldsymbol{A}$ whose universe contains $a_1, \dots, a_n, g^{\boldsymbol{A}}(a_1, \dots, a_n)$ and whose type contains g. We will prove that

$$J_{\mathbb{Y}} \subseteq [\![f(g^{\boldsymbol{A}}(a_1, \dots, a_n)) = g^{\prod_{\mathbb{X} \in I} \boldsymbol{B}_{\mathbb{X}}}(f(a_1), \dots, f(a_n))]\!]. \tag{8}$$

Consider $\mathbb{V} \in J_{\mathbb{Y}}$. Since $\mathbb{Y} \sqsubseteq \mathbb{V}$, the type of $\mathbb{V}$ contains g and $a_1, \dots, a_n, g^{\boldsymbol{A}}(a_1, \dots, a_n) \in V$. Since $a_1, \dots, a_n, g^{\boldsymbol{A}}(a_1, \dots, a_n) \in V$, we have

$$\begin{aligned} f(a_1)(\mathbb{V}) &= h_{\mathbb{V}}(a_1) \\ &\vdots \\ f(a_n)(\mathbb{V}) &= h_{\mathbb{V}}(a_n) \\ f(g^{\boldsymbol{A}}(a_1, \dots, a_n))(\mathbb{V}) &= h_{\mathbb{V}}(g^{\boldsymbol{A}}(a_1, \dots, a_n)). \end{aligned}$$

Furthermore, as the type of $\mathbb{V}$ contains g,

$$h_{\mathbb{V}}(g^{\boldsymbol{A}}(a_1, \dots, a_n)) = g^{\boldsymbol{B}_{\mathbb{V}}}(h_{\mathbb{V}}(a_1), \dots, h_{\mathbb{V}}(a_n)).$$

From the above displays it follows

$$f(g^{\boldsymbol{A}}(a_1,\dots,a_n))(\mathbb{V}) = g^{\boldsymbol{B}_{\mathbb{V}}}(f(a_1)(\mathbb{V}),\dots,f(a_n)(\mathbb{V})) = g^{\prod_{\mathbb{X}\in I}\boldsymbol{B}_{\mathbb{X}}}(f(a_1),\dots,f(a_n))(\mathbb{V}),$$

that is, $\mathbb{V} \in [\![f(g^{\boldsymbol{A}}(a_1,\dots,a_n)) = g^{\prod_{\mathbb{X}\in I}\boldsymbol{B}_{\mathbb{X}}}(f(a_1),\dots,f(a_n))]\!]$. This establishes (8). Lastly, as $J_{\mathbb{Y}} \in U$ and U is an upset of $\mathcal{P}(I)$, condition (8) implies

$$[\![f(g^{\boldsymbol{A}}(a_1,\dots,a_n)) = g^{\prod_{\mathbb{X}\in I}\boldsymbol{B}_{\mathbb{X}}}(f(a_1),\dots,f(a_n))]\!] \in U,$$

and, therefore,

$$\begin{aligned} f^*(g^{\boldsymbol{A}}(a_1,\dots,a_n)) &= f(g^{\boldsymbol{A}}(a_1,\dots,a_n))/\theta_U \\ &= g^{\prod_{\mathbb{X}\in I}\boldsymbol{B}_{\mathbb{X}}}(f(a_1),\dots,f(a_n))/\theta_U \\ &= g^{\prod_{\mathbb{X}\in I}\boldsymbol{B}_{\mathbb{X}}/\theta_U}(f(a_1)/\theta_U,\dots,f(a_n)/\theta_U) \\ &= g^{\prod_{\mathbb{X}\in I}\boldsymbol{B}_{\mathbb{X}}/\theta_U}(f^*(a_1),\dots,f^*(a_n)). \end{aligned}$$

Hence, we conclude that f^* is a homomorphism and, therefore, an embedding of $\boldsymbol{A}$ into $\prod_{\mathbb{Y}\in I}\boldsymbol{B}_{\mathbb{Y}}/\theta_U$. As a consequence,

$$\boldsymbol{A} \in \mathbb{ISP}_{\mathrm{U}}(\{\boldsymbol{B}_{\mathbb{X}} : \mathbb{X} \in I\}) \subseteq \mathbb{ISP}_{\mathrm{U}}(\mathsf{K}). \qquad \boxtimes$$

Corollary 4.9. *Every algebra embeds into an ultraproduct of its finitely generated subalgebras.*

Example 4.10 (Lattices)**.** Let Latt be the class of all lattices and $\mathsf{Latt}^{<\omega}$ that of finite lattices. We will show that $\mathsf{Latt} = \mathbb{ISP}_{\mathrm{U}}(\mathsf{Latt}^{<\omega})$. The inclusion from right to left follows from the fact that Latt is closed under $\mathbb{I}, \mathbb{S}$, and $\mathbb{P}_{\mathrm{U}}$. For the other inclusion, consider a lattice$\boldsymbol{A}$. We know that every local subgraph $\mathbb{X}$ of $\boldsymbol{A}$ can be embedded into the Dedekind-MacNeille completion of the subposet of $\boldsymbol{A}$ with universe X. As the Dedekind-MacNeille completion of a finite poset is finite, it follows that $\mathbb{X}$ can be emdebbed into a finite lattice. As a consequence, every local subgraph of $\boldsymbol{A}$ can be embedded into some finite lattice. By Theorem 4.8, this implies that $\boldsymbol{A} \in \mathbb{ISP}_{\mathrm{U}}(\mathsf{Latt}^{<\omega})$, as desired. $\boxtimes$

5. Quasi-varieties

At this stage it is natural to wonder whether it is possible to characterize classes of algebras axiomatized by quasi-equations (Definition 3.3) in terms of closure under certain class operators. The answer is affirmative, as we proceed to explain.

Definition 5.1. A prevariety closed under $\mathbb{P}_{\mathrm{U}}$ is a said to be a *quasi-variety*.

The aim of this section is to prove the following classical result.

Maltsev's Theorem 5.2. *A class of similar algebras is a quasi-variety if and only if it can be axiomatized by a set of quasi-equations.*

Proof. The "only if" part follows from the fact that the validity of quasi-equations is preserved by the class operators $\mathbb{I}, \mathbb{S}, \mathbb{P}$, and $\mathbb{P}_{\mathrm{U}}$. To prove the converse, consider a prevariety K closed under $\mathbb{P}_{\mathrm{U}}$. Moreover, let *Var* be a denumerable set of variables and Σ the set of quasi-equations, with variables among *Var*, valid in K. Let also K^+ be the class of algebras axiomatized by Σ. Our aim is to prove that $\mathsf{K} = \mathsf{K}^+$.

The inclusion $\mathsf{K} \subseteq \mathsf{K}^+$ is straightforward. To prove the other one, consider an algebra $\boldsymbol{A} \in \mathsf{K}^+$. In order to prove that $\boldsymbol{A} \in \mathsf{K}^+$, it suffices to show that every local subgraph of $\boldsymbol{A}$ embeds in some members of K. This is because, in this case, $\boldsymbol{A} \in \mathbb{ISP}_{\mathrm{U}}(\mathsf{K})$, by Theorem 4.8. Since K is closed under $\mathbb{I}, \mathbb{S}$, and $\mathbb{P}_{\mathrm{U}}$, this implies $\boldsymbol{A} \in \mathsf{K}$, as desired.

Then consider a local subgraph $\mathbb{X}$ of $\boldsymbol{A}$. By definition, $\mathbb{X}$ consists of a finite set $\{a_1, \ldots, a_n\}$ endowed with the restriction of finitely many basic operations $f_1, \ldots, f_m$ of $\boldsymbol{A}$ to X. Fix n distinct variables $x_1, \ldots, x_n \in Var$, corresponding to the elements $a_1, \ldots, a_n$ of X. The *positive* and *negative atomic diagrams* of $\mathbb{X}$ are, respectively,

$$\mathcal{D}^+(\mathbb{X}) := \{f_i(x_{k_1}, \ldots, x_{k_s}) \approx x_j \colon i \leqslant m \text{ and } k_1, \ldots, k_s, j \leqslant n \text{ and } f_i^{\boldsymbol{A}}(a_{k_1}, \ldots, a_{k_s}) = a_j\}$$
$$\mathcal{D}^-(\mathbb{X}) := \{x_i \not\approx x_j \colon i, j \leqslant n \text{ and } a_i \neq a_j\}.$$

Observe that both $\mathcal{D}^+(\mathbb{X})$ and $\mathcal{D}^-(\mathbb{X})$ are finite sets. Then take an enumeration

$$\mathcal{D}^-(\mathbb{X}) = \{\varepsilon_1 \not\approx \delta_1, \ldots, \varepsilon_t \not\approx \delta_t\}.$$

Moreover, for each $i \leqslant t$, consider the quasi-equation

$$\Phi_i := \Big(\mathop{\&} \mathcal{D}^+(\mathbb{X}) \Big) \Longrightarrow \varepsilon_i \approx \delta_i.$$

As witnessed by the natural assignment

$$x_1 \longmapsto a_1, \ldots, x_n \longmapsto a_n,$$

the quasi-equations $\Phi_1, \ldots, \Phi_t$ fail in $\boldsymbol{A}$. Since they are written with variables among Var and $\boldsymbol{A}$ satisfies all the quasi-equations with variables among Var valid in K, this implies that each Φ_i fails in some $\boldsymbol{B}_i \in \mathsf{K}$ under an assignment

$$x_1 \longmapsto b_1^i, \ldots, x_n \longmapsto b_n^i. \tag{9}$$

Now, consider the map $h \colon X \to (B_1 \times \cdots \times B_t)$, defined by the rule

$$a_1 \longmapsto \langle b_1^1, \ldots, b_1^t \rangle, \ldots, a_n \longmapsto \langle b_n^1, \ldots, b_n^t \rangle.$$

We will prove that h is an embedding of $\mathbb{X}$ into $\boldsymbol{B}_1 \times \cdots \times \boldsymbol{B}_t$. To prove that h is injective, consider two distinct elements $a_p, a_q \in X$. Then the formula $x_p \not\approx x_q$ belongs to the negative atomic diagram of $\mathbb{X}$. Then there exists $i \leqslant t$ such that

$$\Phi_i = \Big(\mathop{\&} \mathcal{D}^+(\mathbb{X}) \Big) \Longrightarrow x_p \approx x_q.$$

Since Φ_i fails in $\boldsymbol{B}_i$ under the assignment in (9), we obtain $b_p^i \neq b_q^i$. As a consequence,

$$h(a_p)(i) = b_p^i \neq b_q^i = h(a_q)(i)$$

and, therefore, $h(a_p) \neq h(a_q)$. Hence, h is injective. To prove that it preserves the partial operations, consider a basic s-ary operation f_j in the type of $\mathbb{X}$ and $a_{k_1}, \ldots, a_{k_s} \in X$ such that $f_j^{\boldsymbol{A}}(a_{k_1}, \ldots, a_{k_s}) \in X$. Then there exists some $p \leqslant n$ such that $a_p = f_j^{\boldsymbol{A}}(a_{k_1}, \ldots, a_{k_s})$. Moreover, the equation

$$f_j(x_{k_1}, \ldots, x_{k_s}) \approx x_p$$

belongs to the positive atomic diagram $\mathcal{D}^+(\mathbb{X})$ of $\mathbb{X}$. As each quasi-equation Φ_i fails under the assignment in (9), the same assignment satisfies the antecedent of Φ_i, namely $\mathcal{D}^+(\mathbb{X})$. It follows that

$$f_j^{\boldsymbol{B}_i}(b_{k_1}^i, \ldots, b_{k_s}^i) = b_p^i, \text{ for each } i \leqslant t.$$

As a consequence, for every $i \leqslant t$,

$$\begin{aligned} h(f_j^{\boldsymbol{A}}(a_{k_1}, \dots, a_{k_s}))(i) &= h(a_p)(i) \\ &= b_p^i \\ &= f_j^{\boldsymbol{B}_i}(b_{k_1}^i, \dots, b_{k_s}^i) \\ &= f_j^{\boldsymbol{B}_i}(h(a_{k_1})(i), \dots, h(a_{k_s})(i)) \\ &= f_j^{\boldsymbol{B}_1 \times \cdots \times \boldsymbol{B}_t}(h(a_{k_1}), \dots, h(a_{k_s}))(i). \end{aligned}$$

Thus, $h(f_j^{\boldsymbol{A}}(a_{k_1}, \dots, a_{k_s})) = f_j^{\boldsymbol{B}_1 \times \cdots \times \boldsymbol{B}_t}(h(a_{k_1}), \dots, h(a_{k_s}))$. We conclude that $h \colon \mathbb{X} \to (\boldsymbol{B}_1 \times \cdots \times \boldsymbol{B}_t)$ is an embedding. Since $\boldsymbol{B}_1, \dots, \boldsymbol{B}_t \in \mathsf{K}$ and K is closed under $\mathbb{P}$, the direct product $\boldsymbol{B}_1 \times \cdots \times \boldsymbol{B}_t$ belongs to K. Hence, $\mathbb{X}$ embeds into some member of K, as desired. ⊠

Exercise 5.3. Prove that if a quasi-equation Φ is valid in a class of similar algebras K, then it is also valid in $\mathbb{P}_{\mathrm{U}}(\mathsf{K})$. ⊠

Given a class of similar algebras K, the least quasi-variety extending K exists and will be denoted by $\mathbb{Q}(\mathsf{K})$ and called the quasi-variety *generated* by K.

Proposition 5.4 (Maltsev)**.** *For every class of algebras* K*,*

$$\mathbb{Q}(\mathsf{K}) = \mathbb{ISPP}_{\mathrm{U}}(\mathsf{K}).$$

Proof. The inclusion $\mathbb{ISPP}_{\mathrm{U}}(\mathsf{K}) \subseteq \mathbb{Q}(\mathsf{K})$ is obvious. To prove the other, consider $\boldsymbol{A} \in \mathbb{Q}(\mathsf{K})$. By Maltsev's Theorem, $\mathbb{Q}(\mathsf{K})$ is the class of all algebras satisfying the quasi-equations valid in K. The proof of the hard part of Maltsev's Theorem show that $\boldsymbol{A} \in \mathbb{ISP}_{\mathrm{U}}\mathbb{P}(\mathsf{K})$. Therefore, it only remains to show that $\mathbb{P}_{\mathrm{U}}\mathbb{P}(\mathsf{K}) \subseteq \mathbb{ISPP}_{\mathrm{U}}(\mathsf{K})$. But this is an easy exercise on class operators (the details are sketched below).

Consider an algebra $\boldsymbol{B} \in \mathbb{P}_{\mathrm{U}}\mathbb{P}(\mathsf{K})$. There exists an index set I, an ultrafilter U on I, and a family of algebras $\{\boldsymbol{B}_j : j \in J_i\}$ for each $i \in I$ such that

$$\boldsymbol{B} = \Big(\prod_{i \in I} \Big(\prod_{j \in J_i} \boldsymbol{B}_j \Big) \Big) / \theta_U.$$

Let J be the set of all maps $f \colon I \to \bigcup_{i \in I} J_i$ such that $f(i) \in J_i$. Moreover, let

$$g \colon B \to \prod_{f \in J} \Big(\prod_{i \in I} B_f(i) \Big)$$

be the map defined by the rule $g(b)(f)(i) \coloneqq b(i)(f(i))$. It is not hard to check that the map

$$g^* \colon B \to \Big(\prod_{f \in J} \Big(\prod_{i \in I} B_f(i) \Big) \Big) / \theta_U$$

that sends an element $b \in B$ to $f(b)/\theta_U$ is an embedding, whence $\boldsymbol{B} \in \mathbb{ISPP}_{\mathrm{U}}(\mathsf{K})$. ⊠

Corollary 5.5. *If* K *be a finite set of finite similar algebras, then* $\mathbb{Q}(\mathsf{K}) = \mathbb{ISP}_{\mathrm{U}}(\mathsf{K})$*, that is, the quasi-variety and the prevariety generated by* K *coincide.*

Proof. Since K is a finite set of finite algebras, $\mathbb{P}_{\mathrm{U}}(\mathsf{K}) \subseteq \mathbb{I}(\mathsf{K})$. As a consequence, we obtain $\mathbb{ISP}(\mathsf{K}) = \mathbb{ISPP}_{\mathrm{U}}(\mathsf{K})$. Together with Proposition 5.4, this yields $\mathbb{Q}(\mathsf{K}) = \mathbb{ISP}(\mathsf{K})$. ⊠

Exercise 5.6. Prove that if K is a class of similar algebras, then $\mathbb{P}_{\mathrm{U}}\mathbb{P}(\mathsf{K}) \subseteq \mathbb{I}\mathbb{S}\mathbb{P}\mathbb{P}_{\mathrm{U}}(\mathsf{K})$. Hint: use the sketch in the last part of Proposition 5.4. ⊠

Fix a denumerable set of variables X. The *quasi-equational theory* of a class of similar algebras K is the set of quasi-equations with variables in X valid in K.

Example 5.7 (Lattices)**.** Recall from Example 4.10 that $\mathsf{Latt} = \mathbb{I}\mathbb{S}\mathbb{P}_{\mathrm{U}}(\mathsf{Latt}^{<\omega})$. As a consequence, $\mathsf{Latt} = \mathbb{Q}(\mathsf{Latt}^{<\omega})$. Thus, a quasi-equation is valid in Latt if and only if it is valid in $\mathsf{Latt}^{<\omega}$. Since the class of lattices is finitely axiomatizable, this implies that the quasi-equational theory of Latt is decidable.

At this stage, it is natural to wonder whether Latt is also the prevariety generated by $\mathsf{Latt}^{<\omega}$. This is not the case, as we proceed to explain. First, consider a lattice $\boldsymbol{A}$ with precisely two congruences, namely id_A and $A \times A$. For instance, we can take $\boldsymbol{A}$ to the the poset of equivalence relations on an infinite set. Then suppose, with a view to contradiction, that $\mathsf{Latt} = \mathbb{I}\mathbb{S}\mathbb{P}(\mathsf{Latt}^{<\omega})$. Since $\mathbb{I}\mathbb{S}\mathbb{P}(\mathsf{Latt}^{<\omega}) = \mathbb{I}\mathbb{P}_{\mathrm{SD}}\mathbb{S}(\mathsf{Latt}^{<\omega}) = \mathbb{I}\mathbb{P}_{\mathrm{SD}}(\mathsf{Latt}^{<\omega})$, there exists a subdirect embedding $f \colon \boldsymbol{A} \to \prod_{i \in I} \boldsymbol{B}_i$ where $\{\boldsymbol{B}_i : i \in I\}$ is a family of finite lattices. For the sake of simplicity, we can assume that f is the identity map and that $\boldsymbol{A}$ is a subdirect product of $\{\boldsymbol{B}_i : i \in I\}$. Then consider consider $j \in I$. Since $\boldsymbol{A}$ is a subdirect product of $\{\boldsymbol{B}_i : i \in I\}$, the projection map $p_j \colon \boldsymbol{A} \to \boldsymbol{B}_j$ is surjective. As $\boldsymbol{A}$ is simple, $\mathsf{Ker}(p_j)$ is either id_A and $A \times A$. If it is id_A, then p_j is injective and, therefore, $\boldsymbol{B}_j$ is infinite, which is false. Then $\mathsf{Ker}(p_j) = A \times A$. But this implies that $\boldsymbol{B}_j$ is the trivial lattice. It follows that each $\boldsymbol{B}_i$ is trivial, whence so is $\prod_{i \in I} \boldsymbol{B}_i$. But this contradicts the assumption that $\boldsymbol{A}$ is an infinite subalgebra of $\prod_{i \in I} \boldsymbol{B}_i$. We conclude that $\mathbb{I}\mathbb{S}\mathbb{P}(\mathsf{Latt}^{<\omega}) \subsetneq \mathsf{Latt}$ and, therefore, that there are generalized quasi-equations that are valid in $\mathsf{Latt}^{<\omega}$, but not in Latt.

The contrasts with the case of distributive lattices. Indeed the class DL of distributive lattices is the prevariety generated by its finite members. Even more is true: DL is the prevariety generated by the two-element distributive lattice. Similarly, the class of Boolean algebras is the prevariety generated the two-element Boolean algebra (and, therefore, by finite Boolean algebras). ⊠

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