DEGREES OF INCOMPLETENESS OF IMPLICATIVE LOGICS: THE TRICHOTOMY THEOREM

DAMIANO FORNASIERE AND TOMMASO MORASCHINI

ABSTRACT. The degree of incompleteness of a logic L is the cardinality of the set of logics with the same Kripke frames as L. Blok's celebrated Dichotomy Theorem states that every normal modal logic has degree of incompleteness either 1 or $2^{\aleph_0}$. While characterising the degrees of incompleteness of all intermediate logics remains an outstanding open problem, we answer it in a reduced signature. To this end, we call *implicative logics* the axiomatic extensions of the implicative fragment of the intuitionistic propositional calculus. Our main result is a Trichotomy Theorem stating that the degree of incompleteness of an implicative logic is either 1, ω, or $2^{\aleph_0}$. A concrete description of the implicative logics with each of these degrees is also obtained.

1. INTRODUCTION

Although Kripke semantics plays a fundamental role in the study of intuitionistic and modal logics, examples of Kripke *incomplete logics* are known since the 70s (see, e.g., [17, Ch. 6]). These are logics for which Kripke semantics is not expressive enough, in the sense that they are not complete with respect to any class of Kripke frames. In order to understand the limitations of Kripke semantics imposed by the existence of Kripke incomplete logics, Fine associated a *degree of incompleteness* with every normal modal logic [24]. More precisely, the degree of incompleteness of a normal modal logic L is the number of normal modal logics L' that are "indistinguisheable" from L from the point of view of Kripke semantics (formally, L and L' are validated exactly by the same Kripke frames).

As there exists exactly a continuum of normal modal logics, and every normal modal logic is indistinguisheable from itself, the degree of incompleteness of a normal logic could, in principle, be any cardinal between 1 and the continuum $2^{\aleph_0}$. However, *Blok's Dichotomy Theorem* reduces dramatically the realm of possibilities by stating that the sole possible values for the degree of incompleteness of a normal modal logic are the extrema 1 and $2^{\aleph_0}$ [8, 9] (see also [36, 45]). Notably, the same dichotomy holds if Kripke semantics is replaced by neighborhood semantics, as shown by Chagrova [18].

A notable consequence of Blok Dichotomy Theorem is that the most familiar normal modal logics (such as $\mathsf{K4}$ or $\mathsf{S4}$) have degree of incompleteness $2^{\aleph_0}$: this is because the normal modal logics with degree of incompleteness 1 are rather artificial and coincide with the join-splitting ones (see, e.g., [17, Thm. 10.59]) or, equivalently, with those axiomatized by the Jankov formulas of finite rooted cycle-free Kripke frame (see, e.g., [17, Thm. 10.53]).

Date: October 22, 2025.

The first example of a Kripke incomplete superintuitionistic logic was discovered by Shehtman [47] and a continuum of such logics was exhibited by Litak in [35, 37]. This raises the question of understanding the degrees of incompleteness in the context of superintuitionistic logics as well. More precisely, the degree of incompleteness of a superintuitionistic logic L is the number of superintuitionistic logics that are validated exactly by the same intuitionistic Kripke frames as L. However, to this day, the problem of describing the degrees of incompleteness of superintuitionistic logics remains equally fascinating, elusive, and open.

In this paper, we answer this problem in a reduced signature. To this end, we call *implicative logics* the axiomatic extensions of the implicative fragment $\mathsf{IPC}_{\to}$ of the intuitionistic propositional calculus IPC. Our main result is a Trichotomy Theorem stating that the degree of incompleteness of an implicative logic is either 1, $\aleph_0$, or $2^{\aleph_0}$ (Theorem 5.4). More precisely, besides $\mathsf{IPC}_{\to}$, the sole implicative logics L with degree of incompleteness 1 are those for which there exists $n \in \mathbb{N}$ such that L is the implicative logic of all the Kripke frames whose chains have size $\leqslant n$. Furthermore, the implicative logics with degree of incompleteness $\aleph_0$ are those complete with respect to certain Hilbert algebras whose order reducts are depicted in Figure 1. All the remaining implicative logics have degree of incompleteness $2^{\aleph_0}$. As a consequence, the implicative fragments of most familiar superintuitionistic logics have degree of incompleteness $2^{\aleph_0}$.

To our knowledge this Trichotomy Theorem is the first result on degrees of incompleteness related to IPC, and we hope it will stimulate research on the degrees of incompleteness of superintuitionistic logics. Furthermore, it answers a problem raised by Litak in [36, p. 409], namely, "if there is any nontrivial completeness notion for which the Blok Dichotomy does not hold". While this problem was answered in [7] by introducing the notion of a *degree of the finite model property* for intuitionistic and modal logics, our Thrichotomy Theorem shows that the usual notion of a degree of incompleteness suffices to produce exactly three degrees in the context of implicative logics.

2. Hilbert algebras

2.1. **Basic concepts.** A structure $\boldsymbol{A} = \langle A; \wedge, \vee, \to, 0, 1\rangle$ is said to be a *Heyting algebra* when $\langle A; \wedge, \vee, 0, 1\rangle$ is a bounded distributive lattice satisfying the *residuation law*, *i.e.*, for every $a, b, c \in A$ it holds

$$a \wedge b \leqslant c \text{ if and only if } a \leqslant b \to c,$$

where $\leqslant$ is the lattice order of $\boldsymbol{A}$. In this case, the order relation $\leqslant$ can also be defined as

$$a \leqslant b \text{ if and only if } a \to b = 1. \tag{1}$$

We denote the class of Heyting algebras by HA [2, 17, 23, 44].

Remark 2.1. Each Heyting algebra is uniquely determined by its universe and order, in the sense that, if $\boldsymbol{A}$ and $\boldsymbol{B}$ are Heyting algebras with the same universe and order, then $\boldsymbol{A} = \boldsymbol{B}$ (see, e.g., [17, Cor. 7.12]). ⊠

The operation $\to$ of a Heyting algebra $\boldsymbol{A} = \langle A; \wedge, \vee, \to, 0, 1\rangle$ is often called *implication*. Because of this, the algebra $\boldsymbol{A}^{-} \coloneqq \langle A; \to\rangle$ is called the *implicative reduct* of $\boldsymbol{A}$. The next concept will play a fundamental role in this paper [22, 26, 27].

Definition 2.2. An algebra $\langle B; \to\rangle$ is said to be an *implicative subreduct* of a Heyting algebra $\boldsymbol{A}$ when it is a subalgebra of $\boldsymbol{A}^{-}$. The implicative subreducts of Heyting algebras will be called *Hilbert algebras*. We denote the class of Hilbert algebras by Hil.

Equivalently, Hilbert algebras can be defined as the algebras $\langle A; \to\rangle$ satisfying the following equations [22, Def. 1', pg. 8]:

$$
\begin{array}{ll}
(x \to x) \to x \approx x; & x \to (y \to z) \approx (x \to y) \to (x \to z);\\
x \to x \approx y \to y; & (x \to y) \to ((y \to x) \to x) \approx (y \to x) \to ((x \to y) \to y).
\end{array}
$$

Every Hilbert algebra $\boldsymbol{A}$ can be endowed with a partial order, as we proceed to illustrate. Since $a \to a = b \to b$ for every $a, b \in A$, we can expand $\boldsymbol{A}$ with a term-definable constant $1 \coloneqq x \to x$. Then the binary relation $\leqslant$ on A defined by the rule in (1) is a partial order with maximum 1. Obviously, the order of a Heyting algebra coincides with that of its implicative reduct.

Example 2.3. Let $\mathbb{X}$ be a poset. Then the structure $\langle \mathsf{Up}(\mathbb{X}); \cap, \cup, \to, \emptyset, X\rangle$, where

$$U \to V \coloneqq \{x \in X : U \cap {\uparrow} x \subseteq V\},$$

is a Heyting algebra which we also denote by $\mathsf{Up}(\mathbb{X})$. Consequently, its implicative reduct $\mathsf{Up}(\mathbb{X})^{-}$ is a Hilbert algebra. ⊠

Example 2.4. With every poset $\mathbb{X} = \langle X; \leqslant\rangle$ with maximum 1 we associate a binary operation $\to$ on X defined by the rule

$$x \to y \coloneqq \begin{cases} 1 & \text{if } x \leqslant y;\\ y & \text{otherwise.}\end{cases}$$

Then $\mathsf{H}(\mathbb{X}) \coloneqq \langle X; \to\rangle$ is a Hilbert algebra with underlying partial order $\leqslant$ [22, Example 2, pg. 11] (see also [11, 13]). ⊠

In contrast to the case of Heyting algebras, Hilbert algebras are not determined by their universes and underlying partial orders. For instance, let $\boldsymbol{A}$ be the four-element Boolean algebra and $\langle A; \leqslant\rangle$ its underlying poset. While the Hilbert algebras $\boldsymbol{A}^{-}$ and $\mathsf{H}(A; \leqslant)$ have the same universe and underlying partial order (that is, A and $\leqslant$), they are different because for every element $a \in A - \{0, 1\}$ we have that

$$a \to^{\boldsymbol{A}^{-}} 0 \neq 0 \;\text{ and }\; a \to^{\mathsf{H}(A;\leqslant)} 0 = 0.$$

Given a poset $\mathbb{X} = \langle X; \leqslant\rangle$ and a set $Y \subseteq X$, we define:

$$
\begin{aligned}
{\uparrow}^{\mathbb{X}} Y &\coloneqq \{x \in X : y \leqslant x \text{ for some } y \in Y\};\\
{\downarrow}^{\mathbb{X}} Y &\coloneqq \{x \in X : x \leqslant y \text{ for some } y \in Y\}.
\end{aligned}
$$

To simplify notation, we will omit the superscripts from $\uparrow^{\mathbb{X}}$ and $\downarrow^{\mathbb{X}}$ when no confusion shall arise. The subset Y is said to be an *upset* (resp. *downset*) of $\mathbb{X}$ if $Y = {\uparrow}Y$ (resp. $Y = {\downarrow}Y$). When $Y = \{y\}$, we will write ${\uparrow}y$ and ${\downarrow}y$ instead of ${\uparrow}\{y\}$ and ${\downarrow}\{y\}$. We denote the set of upsets (resp. downsets) of $\mathbb{X}$ by $\mathsf{Up}(\mathbb{X})$ (resp. $\mathsf{Down}(\mathbb{X})$). Given a Hilbert algebra $\boldsymbol{A}$ with underlying poset $\langle A; \leqslant\rangle$ and $Y \subseteq A$, we will write ${\uparrow}Y$ and ${\downarrow}Y$ for $\uparrow^{\langle A;\leqslant\rangle}Y$ and $\downarrow^{\langle A;\leqslant\rangle}Y$, respectively.

Proposition 2.5. *Let $f\colon \boldsymbol{A} \to \boldsymbol{B}$ be an embedding of Hilbert algebras. Then the following conditions hold:*

(i) *f is an order embedding that preserves maxima;*
(ii) *$f[{\uparrow}a] \subseteq {\uparrow}f(a)$, for every $a \in A$;*
(iii) *Assume that both $\boldsymbol{A}$ and $\boldsymbol{B}$ have second largest elements, say a and b respectively. Then $f^{-1}[\{b\}] \subseteq \{a\}$.*

Proof. Item (i) is an immediate consequence of (1) and the fact that, in Hilbert algebras, the element 1 is term-definable as $x \to x$. Lastly, (ii) and (iii) are both consequences of the fact that f is an order embedding that preserves maxima. ⊠

The next observation is an immediate consequence of Proposition 2.5(i) and of the definition of Hilbert algebras of the form $\mathsf{H}(\mathbb{X})$.

Proposition 2.6. *Let $f\colon \mathbb{X} \to \mathbb{Y}$ be a map between posets with maxima. Then f is an order embedding of $\mathbb{X}$ into $\mathbb{Y}$ that preserves maxima if and only if it is an Hilbert algebra embedding of $\mathsf{H}(\mathbb{X})$ into $\mathsf{H}(\mathbb{Y})$.*

We will also make use of an observation connecting embeddings and Hilbert algebras of the form $\mathsf{H}(\mathbb{X})$ and $\mathsf{Up}(\mathbb{X})$. In order to formulate it, we fix the following notation.

Definition 2.7. For every positive integer n let $\mathbb{C}_n$ be the n-element chain viewed as a poset.

Notice that the Heyting algebras of the form $\mathsf{Up}(\mathbb{C}_n)$ are precisely the finite nontrivial linearly ordered ones. This is because the $\mathsf{Up}(\mathbb{C}_n)$ is order isomorphic to $\mathbb{C}_{n+1}$ and each Heyting algebra is uniquely determined by its universe and order (Remark 2.1).

Proposition 2.8. *Let $\boldsymbol{A}$ be a finite nontrivial Hilbert algebra and $\mathbb{X}$ a poset. If $f\colon \boldsymbol{A} \to \mathsf{H}(\mathbb{X})$ is a Hilbert algebra embedding and $\boldsymbol{A}$ is the implicative reduct of a Heyting algebra, there exists some $n \in \mathbb{Z}^+$ such that $\boldsymbol{A} \cong \mathsf{Up}(\mathbb{C}_n)^-$.*

Proof. As $f\colon \boldsymbol{A} \to \mathsf{H}(\mathbb{X})$ is an embedding of Hilbert algebras, there exists a subposet $\mathbb{Y}$ of $\mathbb{X}$ such that $\boldsymbol{A} \cong \mathsf{H}(\mathbb{Y})$. Therefore, we may assume that $\boldsymbol{A} = \mathsf{H}(\mathbb{Y})$. Furthermore, by assumption $\boldsymbol{A}$ is the implicative reduct of some Heyting algebra $\boldsymbol{A}^+$, which has the same order $\leqslant$ as $\boldsymbol{A}$. Notice that $\boldsymbol{A}^+$ is finite and nontrivial, as so is $\boldsymbol{A}$ by assumption. Therefore, if $\boldsymbol{A}^+$ is a chain, then $\boldsymbol{A}^+ \cong \mathsf{Up}(\mathbb{C}_n)$ for some $n \in \mathbb{Z}^+$ and we are done. Consequently, it only remains to prove that for every $a, b \in A$ either $a \leqslant b$ or $b \leqslant a$.

To this end, consider $a, b \in A$ such that $a \nleqslant b$. We need to prove that $b \leqslant a$. Applying in succession the fact that $\boldsymbol{A}^+$ is a Heyting algebra, the assumption that $\boldsymbol{A}$ is the implicative reduct of $\boldsymbol{A}^+$, and the fact that $a \nleqslant b$ and $\boldsymbol{A} = \mathsf{H}(\mathbb{Y})$, we obtain

$$a \to^{\boldsymbol{A}^+} (a \wedge^{\boldsymbol{A}^+} b) = a \to^{\boldsymbol{A}^+} b = a \to^{\boldsymbol{A}} b = b.$$

On the other hand, applying in succession the fact $\boldsymbol{A}$ is the implicative reduct of $\boldsymbol{A}^+$ and the assumptions that $\boldsymbol{A} = \mathsf{H}(\mathbb{Y})$ and that $a \nleqslant a \wedge^{\boldsymbol{A}^+} b$ (the latter because $a \nleqslant b$), we obtain

$$a \to^{\boldsymbol{A}^+} (a \wedge^{\boldsymbol{A}^+} b) = a \to^{\boldsymbol{A}} (a \wedge^{\boldsymbol{A}^+} b) = a \wedge^{\boldsymbol{A}^+} b.$$

The two displays above imply that $b = a \wedge^{\boldsymbol{A}^+} b$. Hence, we conclude that $b \leqslant a$. ⊠

A subset F of the universe of a Hilbert algebra $\boldsymbol{A}$ is an *implicative filter* if it contains 1 and is closed under *modus ponens*, in the sense that for every $a, b \in A$,

$$a, a \to b \in F \text{ implies } b \in F.$$

When ordered under the inclusion relation, the set of implicative filters of $\boldsymbol{A}$ forms a lattice. The importance of implicative filters is due to the next observation [22, Sec. I.2].

Proposition 2.9. *The congruence lattice of a Hilbert algebra is isomorphic to the lattice of its implicative filters via the map that associates the coset $1/\theta$ with every congruence θ.*

An implicative filter F of a Hilbert algebra $\boldsymbol{A}$ is said to be *meet irreducible* when it is proper and for every pair of implicative filters G and H,

$$F = G \cap H \text{ implies that either } F = G \text{ or } F = H.$$

When ordered under the inclusion relation, the meet irreducible implicative filters of $\boldsymbol{A}$ form a poset that we denote by $\boldsymbol{A}_*$. The following concept is instrumental to describe the meet irreducible implicative filters of a finite Hilbert algebra. An element $a < 1$ of a Hilbert algebra $\boldsymbol{A}$ is called *irreducible* when for every $b \in A$ it holds that either $b \to a = 1$ or $b \to a = a$.

Proposition 2.10 ([14, Lems. 13 & 16])**.** *Let $\boldsymbol{A}$ be a finite Hilbert algebra. Then*

$$\boldsymbol{A}_* = \{({\downarrow}a)^c : a \in A \textit{ and } a \textit{ is irreducible}\}.$$

Remark 2.11. If $\boldsymbol{A}$ is a Heyting algebra, the following equalities hold:

$$\text{implicative filters of } \boldsymbol{A}^- = \text{lattice filters of } \boldsymbol{A};$$
$$\text{meet irreducible implicative filters of } \boldsymbol{A}^- = \text{prime filters of } \boldsymbol{A};$$
$$\text{irreducible elements of } \boldsymbol{A}^- = \text{meet irreducible elements of } \boldsymbol{A},$$

where an element a of $\boldsymbol{A}$ is called *meet irreducible* when it differs from 1 and cannot be obtained as the meet of two elements other than a. ⊠

Given a Hilbert algebra $\boldsymbol{A}$, let $\epsilon_{\boldsymbol{A}} \colon \boldsymbol{A} \to \mathsf{Up}(\boldsymbol{A}_*)$ be the map defined by the rule

$$\epsilon_{\boldsymbol{A}}(a) \coloneqq \{F \in \boldsymbol{A}_* : a \in F\}.$$

Theorem 2.12 ([22, Thm. 12]). *If $\boldsymbol{A}$ is a Hilbert algebra, then the map $\epsilon_{\boldsymbol{A}}\colon \boldsymbol{A} \to \mathsf{Up}(\boldsymbol{A}_*)^-$ is a Hilbert algebra embedding.*

2.2. **Varieties.** An algebra is said to be *subdirectly irreducible* (SI for short) when it has a least nonidentity congruence (see, e.g., [4, Thm. 3.23]). Given a class K of algebras, we denote the class of its SI members by K_{SI}. We recall that a poset $\mathbb{X}$ is said to be *rooted* when it has a minimum, sometimes called the *root* of $\mathbb{X}$. The SI Heyting and Hilbert algebras can be described as follows (see, e.g., [5, Thm. 2.9] and [11, Lem. 4]).

Theorem 2.13. *Let $\boldsymbol{A}$ be a Heyting or a Hilbert algebra. Then $\boldsymbol{A}$ is SI if and only if it has has a second largest element. Moreover, when $\boldsymbol{A}$ is finite, this is equivalent to the demand that $\boldsymbol{A}_*$ is rooted (in which case the root of $\boldsymbol{A}_*$ is the singleton $\{1\}$).*

We denote the class operators of closure under isomorphic copies, homomorphic images, subalgebras, direct products, and ultraproducts by $\mathbb{I}, \mathbb{H}, \mathbb{S}$, $\mathbb{P}$, and $\mathbb{P}_{\text{u}}$, respectively. If $\mathbb{O}$ is one of these class operators and $\boldsymbol{A}$ an algebra, we write $\mathbb{O}(\boldsymbol{A})$ as a shorthand for $\mathbb{O}(\{\boldsymbol{A}\})$. A class of similar algebras is said to be a *variety* when it is closed under $\mathbb{H}$, $\mathbb{S}$, and $\mathbb{P}$ or, equivalently, when it can be axiomatised by a set of equations (see, e.g., [12, Thm. II.11.9]). We denote the smallest variety containing a class K of similar algebra by $\mathbb{V}(\mathsf{K})$. Examples of varieties include the class HA of Heyting algebras and the class Hil of Hilbert algebras.

We will make extensive use of the following observation, whose origin can be traced back to *Prucnal's trick* [40] (see also [32, Lem. 5.1] and [41]).

Proposition 2.14. *For every $\boldsymbol{A}, \boldsymbol{B} \in \mathsf{Hil}$ such that $\boldsymbol{A}$ is finite and SI,*

$$\boldsymbol{A} \in \mathbb{HS}(\boldsymbol{B}) \quad \textit{if and only if} \quad \boldsymbol{A} \in \mathbb{IS}(\boldsymbol{B}).$$

Proof. It suffices to prove the implication from left to right. Let $\boldsymbol{A} \in \mathbb{HS}(\boldsymbol{B})$. By [40] the implicative fragment of the intuitionistic propositional calculus is hereditarily strucutrally complete, a property that in this context amounts to the demand that $\mathbb{V}(\mathsf{K}) = \mathbb{ISPP}_{\text{u}}(\mathsf{K})$ for every $\mathsf{K} \subseteq \mathsf{Hil}$ (see [3, Prop. 2.4] and [42, Thm. 6.12]). Consequently, $\mathbb{V}(\boldsymbol{B}) = \mathbb{ISPP}_{\text{u}}(\boldsymbol{B})$. Together with $\boldsymbol{A} \in \mathbb{HS}(\boldsymbol{B}) \subseteq \mathbb{V}(\boldsymbol{B})$, this yields $\boldsymbol{A} \in \mathbb{ISPP}_{\text{u}}(\boldsymbol{B})$. As $\boldsymbol{A}$ is SI, we obtain $\boldsymbol{A} \in \mathbb{ISP}_{\text{u}}(\boldsymbol{B})$ by [19, Thm. 1.5]. Lastly, since $\boldsymbol{A}$ is finite and of finite type, we conclude that $\boldsymbol{A} \in \mathbb{IS}(\boldsymbol{B})$ (see, e.g., [12, Thm. V.2.9]). ⊠

A variety is *locally finite* when its finitely generated members are finite. While it is well known that HA is not locally finite [39, 46] (see also [17, Example 7.66]), the opposite is true for Hil. More precisely, we have the following.

Diego's Theorem 2.15. *Let $\boldsymbol{A}$ be a Heyting algebra and $B \subseteq A$ finite. The smallest subset of A containing B and closed under $\wedge$ and $\to$ is finite.*

Proof. This was established in [22, Thm. 18] under the assumption that B is closed under $\to$ only. The easy adaptation to the case where B is also closed under meets can be found in [38]. ⊠

Since Hilbert algebras are implicative subreducts of Heyting algebras, we deduce the following.

Corollary 2.16 ([22, Thm. 18])**.** *The variety of Hilbert algebras is locally finite.*

Notably, locally finite varieties are determined by their finite SI members, in the sense that two locally finite varieties V and W coincide if and only if V_{SI} and W_{SI} have the same finite members (see, e.g., [12, Thm. II.8.6]). As a consequence, we obtain the following.

Proposition 2.17. *Let* V *be a variety of Hilbert algebras. If* $\mathsf{Up}(\mathbb{X})^- \in \mathsf{V}$ *for every finite rooted poset* $\mathbb{X}$*, then* $\mathsf{V} = \mathsf{Hil}$*.*

Proof. As Hil and V are locally finite by Corollary 2.16, it suffices to show that Hil_{SI} and V_{SI} have the same finite members. Clearly, $\mathsf{V}_{\text{SI}} \subseteq \mathsf{Hil}_{\text{SI}}$. Next we prove that every finite SI Hilbert algebra belongs to V. To this end, let $\boldsymbol{A}$ be a finite SI Hilbert algebra. Since $\boldsymbol{A}$ is SI, the poset $\boldsymbol{A}_*$ is rooted by Theorem 2.13. Moreover, $\boldsymbol{A}_*$ is finite because so is $\boldsymbol{A}$. By the assumption this implies $\mathsf{Up}(\boldsymbol{A}_*)^- \in \mathsf{V}$. As $\boldsymbol{A}$ embeds into $\mathsf{Up}(\boldsymbol{A}_*)^-$ by Theorem 2.12 and V is closed under $\mathbb{I}$ and $\mathbb{S}$, this yields $\boldsymbol{A} \in \mathsf{V}$. ⊠

Another useful consequence of Diego's Theorem 2.15 is the following.

Corollary 2.18. *Let* $\boldsymbol{A}$ *be a Heyting algebra,* $B \subseteq A$ *finite, and* C *the smallest subset of* A *containing* B *and closed under* $\wedge$ *and* $\to$*. The Hilbert algebra* $\langle C; \to \rangle$ *isomorphic to* $\mathsf{Up}(\mathbb{X})^-$ *for some finite poset* $\mathbb{X}$*.*

Proof. It is well known that if $\boldsymbol{A} = \langle A; \wedge, \vee, \to, 0, 1 \rangle$ is a Heyting algebra and C a finite subset of A closed under $\wedge$ and $\to$, there exists a finite poset $\mathbb{X}$ and a bijection $f \colon C \to \mathsf{Up}(\mathbb{X})$ that preserves $\wedge$ and $\to$ (see, e.g., [32, p. 110]). Together with Diego's Theorem 2.15, this yields the desired conclusion. ⊠

Let V be a variety. A class K is a *subvariety* of V if it is a variety and $\mathsf{K} \subseteq \mathsf{V}$. When ordered under the inclusion relation, the collection of subvarieties of V forms a complete lattice that we denote by $\Lambda(\mathsf{V})$. If V is locally finite and *congruence distributive* (that is, the congruence lattices of its members are distributive), the lattice $\Lambda(\mathsf{V})$ admits a transparent description, as we proceed to recall.

Given two algebras $\boldsymbol{A}$ and $\boldsymbol{B}$, we write $\boldsymbol{A} \sqsubseteq \boldsymbol{B}$ as a shorthand for $\boldsymbol{A} \in \mathbb{IS}(\boldsymbol{B})$ (or, equivalently, for "there exists an embedding from $\boldsymbol{A}$ to $\boldsymbol{B}$"). We will also write $\boldsymbol{A} \sqsubset \boldsymbol{B}$ when there exists a nonsurjective embedding from $\boldsymbol{A}$ to $\boldsymbol{B}$. Moreover, given a class K of algebras, let $\mathsf{Fin}(\mathsf{K}_{\text{SI}})$ be the preordered class whose universe is the class of finite members of K_{SI} and whose preorder $\ll$ is defined as follows:

$$\boldsymbol{A} \ll \boldsymbol{B} \;\text{ if and only if }\; \boldsymbol{A} \in \mathbb{HS}(\boldsymbol{B}).$$

The notion of a downset extends naturally to preordered classes and, in particular, to $\mathsf{Fin}(\mathsf{K}_{\text{SI}})$. When V is a locally finite congruence distributive variety, the map

$$\kappa \colon \Lambda(\mathsf{V}) \to \mathsf{Down}(\mathsf{Fin}(\mathsf{V}_{\text{SI}})) \;\text{ defined as }\; \kappa(\mathsf{W}) \coloneqq \mathsf{W} \cap \mathsf{Fin}(\mathsf{V}_{\text{SI}})$$

is a lattice isomorphism whose inverse is defined by the rule $\mathsf{K} \longmapsto \mathbb{V}(\mathsf{K})$ [20, Thm. 3.3].[1] In the case of Hilbert algebras, this specialises as follows.

Theorem 2.19. *The following conditions hold:*

(i) *The preorder relation of* $\mathsf{Fin}(\mathsf{Hil}_{\text{si}})$ *is* $\sqsubseteq$*;*

(ii) *The lattices* $\Lambda(\mathsf{Hil})$ *and* $\mathsf{Down}(\mathsf{Fin}(\mathsf{Hil}_{\text{si}}))$ *are isomorphic under the map* κ*.*

Proof. The first part of the statement holds by Proposition 2.14, while the second holds because Hil is locally finite by Corollary 2.16 and congruence distributive [22, Thm. 6]. ⊠

If $\boldsymbol{A}$ is a Hilbert algebra and φ a formula in the appropriate language, we write $\boldsymbol{A} \vDash \varphi$ to indicate that $\boldsymbol{A}$ validates the equation $\varphi \approx 1$. Similarly, given a class K of Hilbert algebras, we write $\mathsf{K} \vDash \varphi$ when $\boldsymbol{A} \vDash \varphi$ for every $\boldsymbol{A} \in \mathsf{K}$. Lastly, we say that K is *axiomatised* by a set Σ of formulas when $\mathsf{K} = \{\boldsymbol{A} \in \mathsf{Hil} : \boldsymbol{A} \vDash \Sigma\}$.

With every finite SI Hilbert algebra $\boldsymbol{A}$ we can associate a formula $\mathcal{J}(\boldsymbol{A})$, called the *Jankov formula* of $\boldsymbol{A}$, in a similar manner as the one detailed for the case of Heyting algebras in [29, 30, 31] so that the following holds.

Jankov's Lemma 2.20. *If* $\boldsymbol{A}$ *and* $\boldsymbol{B}$ *are Hilbert algebras with* $\boldsymbol{A}$ *finite and SI, then*

$$\boldsymbol{B} \vDash \mathcal{J}(\boldsymbol{A}) \quad \textit{if and only if} \quad \boldsymbol{A} \not\sqsubseteq \boldsymbol{B}.$$

Consequently, $\mathcal{J}(\boldsymbol{A})$ *axiomatises the largest subvariety of* Hil *that omits* $\boldsymbol{A}$*.*

Proof. From [10, Cor. 3.2] it follows that the statement holds once $\boldsymbol{A} \not\sqsubseteq \boldsymbol{B}$ is replaced by $\boldsymbol{A} \notin \mathbb{HS}(\boldsymbol{B})$, while Proposition 2.14 guarantees that such a replacement is harmless. ⊠

If $\mathbb{X}$ is a finite rooted poset, the Hilbert algebra $\mathsf{Up}(\mathbb{X})^{-}$ is finite and SI (the latter by Theorem 2.13). We will denote the associated Jankov formula by $\mathcal{J}(\mathbb{X})$.

Corollary 2.21. *If* $\boldsymbol{A}$ *is a finite SI Hilbert algebra and* $\mathsf{K} \subseteq \mathsf{Hil}$*, then*

$$\boldsymbol{A} \in \mathbb{V}(\mathsf{K}) \quad \textit{if and only if} \quad \boldsymbol{A} \in \mathbb{IS}(\mathsf{K}).$$

Proof. The implication from right to left is straightforward. To prove the other, we reason by contraposition. Suppose that $\boldsymbol{A} \notin \mathbb{IS}(\mathsf{K})$. By Jankov's Lemma 2.20 this implies $\mathsf{K} \vDash \mathcal{J}(\boldsymbol{A})$ and, therefore, $\mathbb{V}(\mathsf{K}) \vDash \mathcal{J}(\boldsymbol{A})$. Since $\boldsymbol{A} \nvDash \mathcal{J}(\boldsymbol{A})$ by Jankov's Lemma 2.20, we conclude that $\boldsymbol{A} \notin \mathbb{V}(\mathsf{K})$. ⊠

The class of SI homomorphic images of the members of a class K of algebras will be denoted by $\mathbb{H}_{\text{SI}}(\mathsf{K})$. We will make use of the following fact.

Theorem 2.22. *The following conditions hold true for every class* $\mathsf{K} \cup \{\boldsymbol{A}\}$ *of similar algebras:*

(i) $\boldsymbol{A} \in \mathbb{ISPH}_{\text{SI}}(\boldsymbol{A})$*;*

(ii) *If every finitely generated subalgebra of* $\boldsymbol{A}$ *belongs to* $\mathbb{IS}(\mathsf{K})$*, then* $\boldsymbol{A} \in \mathbb{ISP}_{\text{u}}(\mathsf{K})$*.*

[1] This is a direct consequence of the fact that every finite SI member of a locally finite congruence distributive variety is a splitting algebra [21, Cor. 3.8].

Proof. Condition (i) is a consequence of the Subdirect Decomposition Theorem see, e.g., [12, Thm. II.8.4], while for (ii) see, e.g., [12, Thm. V.2.14]. ⊠

3. Varieties of bounded depth

The concept of depth for Heyting algebras originates in [28]. This concept can be adapted to Hilbert algebras as follows.

Definition 3.1. Given $n \in \mathbb{N}$, we say that a poset has *depth* $\leqslant n$ when it does not contain any $(n+1)$-element chain.

Definition 3.2. Given $n \in \mathbb{N}$, we say that a Hilbert algebra $\boldsymbol{A}$ has *depth* $\leqslant n$ when so does the poset $\boldsymbol{A}_*$ of its meet irreducible implicative filters. Then let

$$\mathsf{D}_n \coloneqq \{\boldsymbol{A} \in \mathsf{Hil} : \boldsymbol{A} \text{ has depth} \leqslant n\}.$$

Notice that D_0 is the trivial variety. To see this, observe that every trivial Hilbert algebra has depth zero. On the other hand, every Hilbert algebra of depth zero is trivial because it embeds into the trivial algebra $\mathsf{Up}(\emptyset)^-$ by Theorem 2.12. This fact will be used repeatedly in the paper.

The following observation on Heyting algebra is folklore.

Proposition 3.3. *Let $n \in \mathbb{N}$. A poset $\mathbb{X}$ has depth $\leqslant n$ if and only if $\mathsf{Up}(\mathbb{C}_n)$ does not embed into $\mathsf{Up}(\mathbb{X})$.*

The next result is also well known in the context of Heyting and modal algebras (see, e.g., [17, p. 43]).

Proposition 3.4. *The class D_n is a variety for every $n \in \mathbb{N}$.*

Proof. In [34] it is shown that if K is a variety with "equationally definable principal congruences" (EDPC for short, [33]), then

$$\mathsf{K}_n \coloneqq \{\boldsymbol{A} \in \mathsf{K} : \text{the poset of meet irreducible congruences of } \boldsymbol{A} \text{ has depth} \leqslant n\}$$

is a variety. Since Hil has EDPC [10, p. 203], it follows that Hil_n is a variety. As Proposition 2.9 guarantees that $\mathsf{Hil}_n = \mathsf{D}_n$, we conclude that D_n is a variety. ⊠

Recall that $\mathbb{C}_n$ is the n-element chain, viewed as a poset. Our aim is to prove the following.

Theorem 3.5. *Let $n \in \mathbb{N}$. The variety D_n is axiomatised by the Jankov formula $\mathcal{J}(\mathbb{C}_{n+1})$.*

To this end, we rely on the next construction. Given a Heyting algebra $\boldsymbol{A}$, we denote by $\boldsymbol{A}_\bot$ the Heyting algebra obtained by adding a new minimum $\bot$ to $\boldsymbol{A}$ and defining the implication as follows: for every $a, b \in A_\bot$,

$$a \to b \coloneqq \begin{cases} a \to^{\boldsymbol{A}} b & \text{if } a, b \in A; \\ 1 & \text{if } a = \bot; \\ \bot & \text{if } b = \bot < a. \end{cases}$$

We will rely on the next simple observation.

Lemma 3.6. *Every map $f\colon \boldsymbol{A} \to \boldsymbol{B}$ between Heyting algebras that preserves $\wedge, \vee$, and $\to$ can be extended to a Heyting algebra homomorphism $f^+\colon \boldsymbol{A}_\bot \to \boldsymbol{B}_\bot$ by stipulating that $f^+(\bot) = \bot$.*

We are now ready to prove Theorem 3.5.

Proof. Let K_n be the variety of Hilbert algebras axiomatised by $\mathcal{J}(\mathbb{C}_{n+1})$. We need to prove that $\mathsf{D}_n = \mathsf{K}_n$. As D_n is also a variety by Proposition 3.4 and both D_n and K_n are locally finite by Corollary 2.16, it suffices to show that D_n and K_n have the same finite members, *i.e.*, that for every finite Hilbert algebra $\boldsymbol{A}$,

$$\boldsymbol{A} \text{ has depth} \leqslant n \text{ if and only if } \boldsymbol{A} \vDash \mathcal{J}(\mathbb{C}_{n+1}). \tag{2}$$

Consider a finite Hilbert algebra $\boldsymbol{A}$. To prove the implication from left to right in the above display, we reason by contraposition. Suppose that $\boldsymbol{A} \nvDash \mathcal{J}(\mathbb{C}_{n+1})$. Then Jankov's Lemma 2.20 guarantees that $\mathsf{Up}(\mathbb{C}_{n+1})^- \sqsubseteq \boldsymbol{A}$. By Theorem 2.12 we also have $\boldsymbol{A} \sqsubseteq \mathsf{Up}(\boldsymbol{A}_*)^-$. Consequently, there exists an embedding $f\colon \mathsf{Up}(\mathbb{C}_{n+1})^- \to \mathsf{Up}(\boldsymbol{A}_*)^-$. Since $\mathsf{Up}(\mathbb{C}_{n+1})$ is a chain, the map f preserves not only $\to$, but also the operations $\wedge$ and $\vee$ of the Heyting algebras $\mathsf{Up}(\mathbb{C}_{n+1})$ and $\mathsf{Up}(\boldsymbol{A}_*)$. By Lemma 3.6 the extension $f^+\colon \mathsf{Up}(\mathbb{C}_{n+1})_\bot \to \mathsf{Up}(\boldsymbol{A}_*)_\bot$ is a Heyting algebra homomorphism. Furthermore, it is injective because so is f.

Observe that $\mathsf{Up}(\mathbb{C}_{n+1})_\bot \cong \mathsf{Up}(\mathbb{C}_{n+2})$ and $\mathsf{Up}(\boldsymbol{A}_*)_\bot \cong \mathsf{Up}(\boldsymbol{A}_*^\top)$, where $\boldsymbol{A}_*^\top$ is the poset obtained by adding a new maximum to the poset $\boldsymbol{A}_*$. Therefore, $\mathsf{Up}(\mathbb{C}_{n+2})$ embeds into $\mathsf{Up}(\boldsymbol{A}_*^\top)$. As a consequence, $\boldsymbol{A}_*^\top$ contains an $(n+2)$-element chain by Proposition 3.3. By the definition of $\boldsymbol{A}_*^\top$, this means that $\boldsymbol{A}_*$ contains an $(n+1)$-element chain. Hence, $\boldsymbol{A}_*$ does not have depth $\leqslant n$ as desired.

Next we prove the implication from right to left in (2). Also in this case, we reason by contraposition. Suppose that $\boldsymbol{A}$ does not have depth $\leqslant n$. Therefore, $\boldsymbol{A}_*$ contains a chain

$$F_1 \subsetneq F_2 \subsetneq \cdots \subsetneq F_{n+1}.$$

Since $\boldsymbol{A}$ is finite, we can apply Lemma 2.10 obtaining that for each F_i there exists an irreducible element a_i of $\boldsymbol{A}$ such that $F_i = ({\downarrow} a_i)^c$. Together with the display above, this yields

$$({\downarrow} a_1)^c \subsetneq ({\downarrow} a_2)^c \subsetneq \cdots \subsetneq ({\downarrow} a_{n+1})^c.$$

Consequently, $a_{n+1} < \cdots < a_2 < a_1$. Notice that the irreducibility of a_1 ensures that $a_1 < 1$. Furthermore, as the equations $x \to 1 \approx 1$ and $1 \to x \approx x$ hold in every Hilbert algebra, the irreducibility of $a_1, \dots, a_{n+1}$ guarantees that $\{a_1, \dots, a_{n+1}, 1\}$ is the universe of a subalgebra of $\boldsymbol{A}$ isomorphic to $\mathsf{Up}(\mathbb{C}_{n+1})$. Consequently, $\mathsf{Up}(\mathbb{C}_{n+1}) \sqsubseteq \boldsymbol{A}$. By Jankov's Lemma 2.20 this implies that $\boldsymbol{A} \nvDash \mathcal{J}(\mathbb{C}_{n+1})$. ⊠

Lastly, we will make use of the following observation.

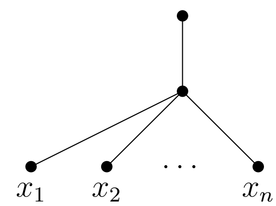

FIGURE 1. The poset $\mathbb{B}_n$.

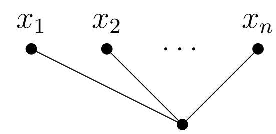

FIGURE 2. The poset $\mathbb{F}_n$.

Proposition 3.7. *For every $n \in \mathbb{N}$ we have*

$$\mathsf{D}_n = \mathbb{V}(\{\mathsf{Up}(\mathbb{X})^- : \mathbb{X} \textit{ is a finite rooted poset and } \mathsf{Up}(\mathbb{X})^- \textit{ has depth } \leqslant n\}).$$

Proof. The inclusion from right to left holds by the definition of D_n. To prove the other inclusion, it suffices to show that the finite SI members of D_n belong to the variety

$$\mathsf{K} \coloneqq \mathbb{V}(\{\mathsf{Up}(\mathbb{X})^- : \mathbb{X} \text{ is a finite rooted poset and } \mathsf{Up}(\mathbb{X})^- \text{ has depth } \leqslant n\}).$$

To this end, consider $\boldsymbol{A} \in \mathsf{D}_n$ finite and SI. Since $\boldsymbol{A} \in \mathsf{D}_n$, the poset $\boldsymbol{A}_*$ has depth $\leqslant n$. Moreover, it is rooted by Theorem 2.13 and finite because so is $\boldsymbol{A}$. In addition, $\mathsf{Up}(\boldsymbol{A}_*)^-$ has depth $\leqslant n$ because the implicative filters of $\mathsf{Up}(\boldsymbol{A}_*)^-$ coincide with the prime filters of $\mathsf{Up}(\boldsymbol{A}_*)$ by Remark 2.11 and the latter form a poset isomorphic to $\boldsymbol{A}_*$ (see, e.g., [25, Thm. 1.21]), which has depth $\leqslant n$. Therefore, $\mathsf{Up}(\boldsymbol{A}_*)^- \in \mathsf{K}$. As $\boldsymbol{A} \sqsubseteq \mathsf{Up}(\boldsymbol{A}_*)^-$ by Theorem 2.12, we conclude that $\boldsymbol{A} \in \mathsf{K}$. ⊠

4. A sequence of varieties

The following varieties will play a fundamental role in the paper (see Example 2.4 if necessary).

Definition 4.1. For every positive integer n let $\boldsymbol{B}_n \coloneqq \mathsf{H}(\mathbb{B}_n)$, where $\mathbb{B}_n$ is the poset depicted in Figure 1. Furthermore, let

$$\mathsf{B}_n \coloneqq \mathbb{V}(\boldsymbol{B}_n) \text{ and } \mathsf{B}_\omega \coloneqq \mathbb{V}(\{\boldsymbol{B}_n : n \in \mathbb{Z}^+\}).$$

We will also make use of the following posets.

Definition 4.2. For every positive integer n let $\mathbb{F}_n$ be the poset depicted in Figure 2.

The aim of this section is to establish the next result.

Theorem 4.3. *The variety B_ω is axiomatised by the Jankov formulas $\mathcal{J}(\mathbb{F}_2)$ and $\mathcal{J}(\mathbb{C}_3)$. Furthermore, its subvarieties are precisely $\mathsf{D}_0, \mathsf{D}_1, \mathsf{B}_1, \mathsf{B}_2, \dots, \mathsf{B}_\omega$.*

The proof of Theorem 4.3 hinges upon a series of observations, the first of which holds by a straightforward inspection.

Lemma 4.4. *For every* $\boldsymbol{A} \in \mathsf{Hil}_{\mathrm{SI}}$ *and* $n \in \mathbb{Z}^+$ *have*

$$\boldsymbol{A} \sqsubseteq \boldsymbol{B}_n \ \text{ if and only if } \ \boldsymbol{A} \in \mathbb{I}(\{\mathsf{Up}(\mathbb{C}_1)^-, \boldsymbol{B}_1, \dots, \boldsymbol{B}_n\}).$$

Since $\mathsf{B}_n = \mathbb{V}(\boldsymbol{B}_n)$ and $\mathsf{B}_\omega = \mathbb{V}(\{\boldsymbol{B}_m : m \in \mathbb{Z}^+\})$, from Corollary 2.21 and Lemma 4.4 we deduce the following.

Corollary 4.5. *For every* $n \in \mathbb{Z}^+$ *we have*

$$\mathsf{Fin}((\mathsf{B}_n)_{\mathrm{SI}}) = \mathbb{I}\{\mathsf{Up}(\mathbb{C}_1)^-, \boldsymbol{B}_1, \dots, \boldsymbol{B}_n\};$$
$$\mathsf{Fin}((\mathsf{B}_\omega)_{\mathrm{SI}}) = \mathbb{I}(\{\mathsf{Up}(\mathbb{C}_1)^-\} \cup \{\boldsymbol{B}_m : m \in \mathbb{Z}^+\}).$$

Given a Hilbert algebra $\boldsymbol{A}$ with a second largest element a, for every $b, c \in A$ we have

$$b \to c = a \ \text{ if and only if } \ (c = a \ \text{ and } \ b \nleq c). \tag{3}$$

Therefore, $A - \{a\}$ is a subuniverse of $\boldsymbol{A}$. We denote the corresponding subalgebra of $\boldsymbol{A}$ by $\boldsymbol{A}_\times$. In addition, given a Hilbert algebra $\boldsymbol{A}$, we denote by $\boldsymbol{A}^\top$ the Hilbert algebra obtained by adding a new element $\top$ to $\boldsymbol{A}$ and defining the implication as follows: for every $a, b \in A_\top$,

$$a \to^{\boldsymbol{A}^\top} b \coloneqq \begin{cases} a \to^{\boldsymbol{A}} b & \text{if } a, b \in A \text{ and } a \to^{\boldsymbol{A}} b \neq 1^{\boldsymbol{A}}; \\ \top & \text{if } b = \top \text{ or } (a, b \in A \text{ and } a \to^{\boldsymbol{A}} b = 1^{\boldsymbol{A}}); \\ b & \text{if } a = \top. \end{cases} \tag{4}$$

Notice that the poset underlying $\boldsymbol{A}^\top$ is the one obtained by adding a new maximum $\top$ to the poset underlying $\boldsymbol{A}$. We will rely on the next simple observation.

Proposition 4.6. *For every Hilbert algebra* $\boldsymbol{A}$ *we have* $\boldsymbol{A} \cong (\boldsymbol{A}^\top)_\times$.

Corollary 4.7. *For every* $\boldsymbol{A}, \boldsymbol{B} \in \mathsf{Hil}_{\mathrm{si}}$ *such that* $\boldsymbol{A} \sqsubseteq \boldsymbol{B}$ *we have* $\boldsymbol{A}_\times \sqsubseteq \boldsymbol{B}_\times$ *and* $\boldsymbol{A}^\top \sqsubseteq \boldsymbol{B}^\top$.

Proof. Let $f \colon \boldsymbol{A} \to \boldsymbol{B}$ be an embedding. Clearly, the unique map $f^\top \colon \boldsymbol{A}^\top \to \boldsymbol{B}^\top$ that extends f and preserves $\top$ is also an embedding. Then let $f_\times$ be the restriction of f to $\boldsymbol{A}_\times$. We will prove that $f_\times$ is an embedding of $\boldsymbol{A}_\times$ into $\boldsymbol{B}_\times$. Clearly, $f_\times$ is an embedding of $\boldsymbol{A}_\times$ into $\boldsymbol{B}$. Moreover, $f[A_\times] \subseteq B_\times$ by Proposition 2.5(iii). Thus, $f_\times$ is an embedding of $\boldsymbol{A}_\times$ into $\boldsymbol{B}_\times$. ⊠

We will also make use of the following class of Hilbert algebras (see, e.g., [43]).

Definition 4.8. The implicative subreducts of Boolean algebras are called *Tarski algebras*.

Notice that every Tarski algebra is a Hilbert algebra and, therefore, possesses an underlying order defined by the rule in (1).

Proposition 4.9. *The following conditions hold:*

(i) *An algebra* $\langle A; \to \rangle$ *is a Tarski algebra if and only if there exists a join semilattice* $\langle A; \leqslant \rangle$ *whose principal upsets are Boolean lattices and for every* $a, b \in A$,

$$a \to b = \textit{the complement of } a \vee b \textit{ in the Boolean lattice corresponding to } {\uparrow} b.$$

In this case, $\leqslant$ *is the order underlying the Hilbert algebra* $\langle A; \to \rangle$*;*

(ii) *Tarski algebras form a variety that coincides with* D_1*. Up to isomorphism, the only SI Tarski algebra is* $\mathsf{Up}(\mathbb{C}_1)^-$.

Proof. For (i), see [1, Thm. 18] and the paragraph preceding it. Next we prove the first half of (ii). To this end, we denote the powerset Boolean algebra induced by a set X by $\mathscr{P}(X)$. Observe that $\mathscr{P}(X)^-$ is a Tarski algebra. We will prove that $\mathscr{P}(X)^- \in \mathsf{D}_1$. First, recall that $(\mathscr{P}(X)^-)_*$ coincides with the poset of prime filters of the Boolean algebra $\mathscr{P}(X)$ by Remark 2.11. As the prime filters of a Boolean algebra are precisely its ultrafilters (i.e., the maximal proper filters), the poset $(\mathscr{P}(X)^-)_*$ is ordered under the identity relation. Consequently, it has depth $\leqslant 1$ and, therefore, so does $\mathscr{P}(X)^-$. Hence, we conclude that $\mathscr{P}(X)^- \in \mathsf{D}_1$ as desired.

Now, consider a Tarski algebra $\boldsymbol{A}$. Then $\boldsymbol{A} \sqsubseteq \mathscr{P}(X)^-$ for some set X (see, e.g., [16, Thm. 4]). Since $\mathscr{P}(X)^- \in \mathsf{D}_1$, we obtain $\boldsymbol{A} \in \mathbb{IS}(\mathsf{D}_1)$. As D_1 is a variety by Proposition 3.4 and varieties are closed under $\mathbb{I}$ and $\mathbb{S}$, we conclude that $\boldsymbol{A} \in \mathsf{D}_1$. Conversely, suppose $\boldsymbol{A} \in \mathsf{D}_1$. As $\boldsymbol{A}$ has depth $\leqslant 1$, the poset $\boldsymbol{A}_*$ is ordered under the identity relation. Consequently, $\mathsf{Up}(\boldsymbol{A}_*) = \mathscr{P}(\boldsymbol{A}_*)$, whence $\mathsf{Up}(\boldsymbol{A}_*)^- = \mathscr{P}(\boldsymbol{A}_*)^-$. Together with Theorem 2.12, this yields $\boldsymbol{A} \sqsubseteq \mathscr{P}(\boldsymbol{A}_*)^-$. Hence, $\boldsymbol{A}$ is a subreduct of a Boolean algebra and, therefore, it is a Tarski algebra. This concludes the proof of the first half of (ii).

To prove the second half of (ii), observe that $\mathsf{Up}(\mathbb{C}_1)^-$ is an SI Tarski algebra. Then consider an SI Tarski algebra $\boldsymbol{A}$. From Theorem 2.13 it follows that $\boldsymbol{A}$ has a second largest element a. Therefore, both $\{1\}$ and $\{1, a\}$ are implicative filters of $\boldsymbol{A}$ and every other implicative filter of $\boldsymbol{A}$ extends $\{1, a\}$. Consequently, the implicative filter $\{1\}$ is meet irreducible and, therefore, the minimum of $\boldsymbol{A}_*$. As $\boldsymbol{A}_*$ is a Tarski algebra, it has depth $\leqslant 1$ by (i). Therefore, $\boldsymbol{A}_*$ is a rooted poset of depth $\leqslant 1$, whence it is isomorphic to $\mathbb{C}_1$. Together with Theorem 2.12, this yields $\boldsymbol{A} \sqsubseteq \mathsf{Up}(\mathbb{C}_1)^-$. Since $\boldsymbol{A}$ is nontrivial (because it is SI) and $\mathsf{Up}(\mathbb{C}_1)^-$ is a two-element algebra, we conclude that $\boldsymbol{A} \cong \mathsf{Up}(\mathbb{C}_1)^-$. ⊠

Proposition 4.10. *The following conditions hold for every* $n, m \in \mathbb{Z}^+$*:*

(i) *The algebra* $(\mathsf{Up}(\mathbb{F}_n)^-)_\times$ *is a Tarski algebra;*

(ii) *If* $n \leqslant m$*, then* $\mathsf{Up}(\mathbb{F}_n) \sqsubseteq \mathsf{Up}(\mathbb{F}_m)$*. In particular,* $\mathsf{Up}(\mathbb{F}_n)^- \sqsubseteq \mathsf{Up}(\mathbb{F}_m)^-$.

Proof. Condition (i) holds because $(\mathsf{Up}(\mathbb{F}_n)^-)_\times$ is the implicative reduct of a Boolean algebra, while (ii) is straightforward. ⊠

Lastly, we rely on the next observation.

Lemma 4.11. *Let* $\boldsymbol{A} \in \mathsf{Hil}$ *and* $n \in \mathbb{Z}^+$*. Then* $\boldsymbol{A} \cong \boldsymbol{B}_n$ *if and only if the posets underlying* $\boldsymbol{A}$ *and* $\boldsymbol{B}_n$ *are isomorphic.*

Proof. It suffices to prove the implication from right to left. Let $\boldsymbol{A}$ be a Hilbert algebra whose underlying poset is isomorphic to the one underlying $\boldsymbol{B}_n$. As $\boldsymbol{B}_n = \mathsf{H}(\mathbb{B}_n)$ by definition, the poset underlying $\boldsymbol{B}_n$ is $\mathbb{B}_n$. Consequently, we may assume that the poset underlying $\boldsymbol{A}$ is $\mathbb{B}_n$ as well. We need to prove that $\boldsymbol{A} \cong \boldsymbol{B}_n$. As $\boldsymbol{A}$ and $\boldsymbol{B}_n$ have the same universe (because they have the samer underlying poset), it suffices to show that $a \to^{\boldsymbol{A}} b = a \to^{\boldsymbol{B}_n} b$ for every $a, b \in A$.

To this end, we recall that the following conditions hold for every Hilbert algebra $\boldsymbol{C}$ and $a, b \in C$:

(i) if $a \leqslant b$, then $a \to^{\boldsymbol{C}} b = 1$;
(ii) $1 \to^{\boldsymbol{C}} a = a$;
(iii) if a is the second largest element of $\boldsymbol{C}$ and $b < a$, then $a \to^{\boldsymbol{C}} b = b$.

Then consider $a, b \in A$. We denote the order underlying $\boldsymbol{A}$ and $\boldsymbol{B}_n$ by $\leqslant$. In view of the above conditions, if either $a \leqslant b$ or $a = 1$ or a is the second largest element of $\boldsymbol{A}$ (equiv. of $\boldsymbol{B}_n$), then $a \to^{\boldsymbol{A}} b = a \to^{\boldsymbol{B}_n} b$ as desired. Notice that $\boldsymbol{A}$ and $\boldsymbol{B}_n$ have a second largest element c because the poset underlying them is $\mathbb{B}_n$. Therefore, we may assume that $a \nleqslant b$ and that $a < c$. As $\mathbb{B}_n$ is the poset underlying $\boldsymbol{A}$ and $\boldsymbol{B}_n$ and c the second largest element of $\mathbb{B}_n$, this guarantees that a and b are minimal incomparable elements of $\mathbb{B}_n$. Therefore, from the definition of $\boldsymbol{B}_n$ it follows that $a \to^{\boldsymbol{B}_n} b = b$.

To conclude the proof, it will be enough to show that $a \to^{\boldsymbol{A}} b = b$. As $\boldsymbol{A}$ is a Hilbert algebra, we have $b \leqslant a \to^{\boldsymbol{A}} b$. Since the poset underlying $\boldsymbol{A}$ is $\mathbb{B}_n$, this implies that

$$\text{either} \quad a \to^{\boldsymbol{A}} b = b \quad \text{or} \quad c \leqslant a \to^{\boldsymbol{A}} b,$$

where c is the second largest element of $\mathbb{B}_n$. Suppose, with a view to contradiction, that $a \to^{\boldsymbol{A}} b \neq b$. Then $c \leqslant a \to^{\boldsymbol{A}} b$. Since a and b are minimal elements of $\mathbb{B}_n$, we obtain $a \leqslant c \leqslant a \to^{\boldsymbol{A}} b$. As $\boldsymbol{A}$ is a Hilbert algebra, this implies $a \leqslant b$, a contradiction. Hence, we conclude that $a \to^{\boldsymbol{A}} b = b$.+ ⊠

We are now ready to conclude the proof of Theorem 4.3.

Proof. We begin by proving that B_ω is axiomatised by $\mathcal{J}(\mathbb{F}_2)$ and $\mathcal{J}(\mathbb{C}_3)$, that is,

$$\mathsf{B}_\omega = \{\boldsymbol{A} \in \mathsf{Hil} : \boldsymbol{A} \vDash \mathcal{J}(\mathbb{F}_2) \text{ and } \boldsymbol{A} \vDash \mathcal{J}(\mathbb{C}_3)\}. \tag{5}$$

To prove the inclusion from left to right, observe that $\mathsf{Up}(\mathbb{F}_2)^-$ and $\mathsf{Up}(\mathbb{C}_3)^-$ are finite and SI because $\mathbb{F}_2$ and $\mathbb{C}_3$ are finite rooted posets. Therefore, from Corollary 2.21 and the definition of B_ω it follows that

$$\mathsf{Up}(\mathbb{F}_2)^- \in \mathsf{B}_\omega \;\text{ if and only if }\; \mathsf{Up}(\mathbb{F}_2)^- \sqsubseteq \mathsf{H}(\mathbb{B}_n) \text{ for some } n \in \mathbb{Z}^+;$$
$$\mathsf{Up}(\mathbb{C}_3)^- \in \mathsf{B}_\omega \;\text{ if and only if }\; \mathsf{Up}(\mathbb{C}_3)^- \sqsubseteq \mathsf{H}(\mathbb{B}_n) \text{ for some } n \in \mathbb{Z}^+.$$

Notice that the posets underlying $\mathsf{Up}(\mathbb{F}_2)^-$ and $\mathsf{Up}(\mathbb{C}_3)^-$ possess a four-element chain. On the other hand, the poset $\mathbb{B}_n$ underlying $\mathsf{H}(\mathbb{B}_n)$ lacks such a chain for each $n \in \mathbb{Z}^+$. As the existence of such a chain is preserved under extensions, the right hand sides of the pair of above equivalences in the above display above fail, whence $\mathsf{Up}(\mathbb{F}_2)^-, \mathsf{Up}(\mathbb{C}_3)^- \notin \mathsf{B}_\omega$. By

Jankov's Lemma 2.20 we conclude that $\mathcal{J}(\mathbb{F}_2)$ and $\mathcal{J}(\mathbb{C}_3)$ are valid in B_ω. This establishes the inclusion from left to right in (5).

Next we prove the inclusion from right to left in (5). Let V be the class of algebras in the right hand side of (5), that is, the variety of Hilbert algebras axiomatised by $\mathcal{J}(\mathbb{F}_2)$ and $\mathcal{J}(\mathbb{C}_3)$. In view of Theorem 2.19(ii), in order to prove that $\mathsf{V} \subseteq \mathsf{B}_\omega$, is suffices to show that $\boldsymbol{A} \in \mathsf{B}_\omega$ for every $\boldsymbol{A} \in \mathsf{Fin}(\mathsf{V}_{\text{SI}})$.

Then let $\boldsymbol{A} \in \mathsf{Fin}(\mathsf{V}_{\text{SI}})$. Since $\boldsymbol{A}$ is SI, it has a second largest element a by Theorem 2.13. If $A - \{a, 1\}$ has $\leqslant 1$ elements, then $\boldsymbol{A}$ is either a two-element or a three-element SI Hilbert algebra. Up to isomorphism, the only two-element Hilbert algebra is $\mathsf{Up}(\mathbb{C}_1)^-$ and the only SI three-element one $\boldsymbol{B}_1$. As both $\boldsymbol{B}_1$ and $\mathsf{Up}(\mathbb{C}_1)^-$ belong to B_ω (the first by the definition of B_ω and the second because $\mathsf{Up}(\mathbb{C}_1)^- \sqsubseteq \boldsymbol{B}_1 \in \mathsf{B}_\omega$), we conclude that $\boldsymbol{A} \in \mathsf{B}_\omega$ as desired.

Then we consider the case where $A - \{a, 1\}$ has $\geqslant 2$ elements. Since $\boldsymbol{A}$ is finite, then $A - \{a, 1\}$ has size n for some $n \geqslant 2$. We will rely on the following observation.

Claim 4.12. *The poset underlying $\boldsymbol{A}$ is isomorphic to $\mathbb{B}_n$.*

Proof of the Claim. Observe that the poset $\boldsymbol{A}_*$ is finite because so is $\boldsymbol{A}$. As $\boldsymbol{A}$ is SI, $\boldsymbol{A}_*$ is rooted by Theorem 2.13. Furthermore, from the assumption that $\boldsymbol{A} \vDash \mathcal{J}(\mathbb{C}_3)$ and Theorem 3.5 it follows that $\boldsymbol{A}_*$ has depth $\leqslant 2$. Lastly, we will prove that $\boldsymbol{A}_*$ is nontrivial. Suppose the contrary, with a view to contradiction. Then $\boldsymbol{A}_*$ is the trivial poset and $\mathsf{Up}(\boldsymbol{A}_*)^- \cong \mathsf{Up}(\mathbb{C}_1)^-$. Therefore, $\boldsymbol{A}$ embeds into the two-element algebra $\mathsf{Up}(\mathbb{C}_1)^-$ by Theorem 2.12. Since $\boldsymbol{A}$ is nontrivial (because it is SI), this implies that $\boldsymbol{A} \cong \mathsf{Up}(\mathbb{C}_1)^-$, a contradiction with the assumption that $A - \{a, 1\}$ is nonempty. Consequently, $\boldsymbol{A}_*$ is a nontrivial finite rooted poset of depth $\leqslant 2$. Therefore, we may assume that $\boldsymbol{A}_* = \mathbb{F}_m$ for some $m \in \mathbb{Z}^+$.

Next we prove the statement of the claim. Because of the simple structure of $\mathbb{B}_n$ and because a is the second largest element of $\boldsymbol{A}$ and $A - \{a, 1\}$ has size n, it suffices to prove that the elements of $A - \{a, 1\}$ are all incomparable. Suppose the contrary, with a view to contradiction. Then there exist $b, c \in A$ such that $b, c \in A$ and $b < c < a$. From $b, c < a$ and the assumption that a is the second largest element of $\boldsymbol{A}$ it follows that $b, c \in A_\times$. Furthermore, from $\boldsymbol{A}_* = \mathbb{F}_m$ and Theorem 2.12 we obtain that $\boldsymbol{A} \sqsubseteq (\mathsf{Up}(\mathbb{F}_m))^-$. As $\boldsymbol{A}$ and $\mathsf{Up}(\mathbb{F}_m)^-$ are both SI, we can apply Proposition 4.7, obtaining $\boldsymbol{A}_\times \sqsubseteq (\mathsf{Up}(\mathbb{F}_m)^-)_\times$. Together with Proposition 4.10(i), this implies that $\boldsymbol{A}_\times$ is a Tarski algebra. Thus, we can apply Proposition 4.9(i), obtaining that the poset underlying $\boldsymbol{A}_\times$ is a join semilattice whose principal upsets are Boolean lattices. In particular, the subposet $\uparrow^{\boldsymbol{A}_\times} b$ of $\boldsymbol{A}_\times$ is a Boolean lattice. Since $b < c$, we have $c \in \uparrow^{\boldsymbol{A}_\times} b$. Let then c^* be the complement of c in the Boolean lattice $\uparrow^{\boldsymbol{A}_\times} b$. Notice that the elements $b, c, c^*, 1$ are all distinct because $b < c < a < 1$ and $\uparrow^{\boldsymbol{A}_\times} b$ is a Boolean lattice. Furthermore, $\{b, c, c^*, 1\}$ is the universe of a subalgebra $\boldsymbol{B}$ of $\boldsymbol{A}_\times$ by Proposition 4.9(i). Furthermore, $\boldsymbol{B}$ is the implicative reduct of the four-element Boolean algebra. As $(\mathsf{Up}(\mathbb{F}_2)^-)_\times$ is also the implicative reduct of the four-element Boolean algebra, we conclude that $(\mathsf{Up}(\mathbb{F}_2)^-)_\times \cong \boldsymbol{B} \sqsubseteq \boldsymbol{A}_\times$. By Proposition

4.6 and Corollary 4.7 we obtain that

$$\mathsf{Up}(\mathbb{F}_2)^- \cong ((\mathsf{Up}(\mathbb{F}_2)^-)_\times)^\top \sqsubseteq (\boldsymbol{A}_\times)^\top \cong \boldsymbol{A}.$$

Thus, $\mathsf{Up}(\mathbb{F}_2)^- \sqsubseteq \boldsymbol{A}$. By Jankov's Lemma 2.20 this implies $\boldsymbol{A} \nvDash \mathcal{J}(\mathbb{F}_2)$, a contradiction with the assumption that $\boldsymbol{A} \in \mathsf{V}$. ⊠

From Lemma 4.11 and Claim 4.12 it follows that $\boldsymbol{A} \cong \boldsymbol{B}_n$. By the definition of B_ω we conclude that $\boldsymbol{A} \in \mathsf{B}_n$. This establishes the inclusion from right to left in (5). Thus, we conclude that B_ω is axiomatised by $\mathcal{J}(\mathbb{F}_2)$ and $\mathcal{J}(\mathbb{C}_3)$.

It only remains to prove that the subvarieties of B_ω are $\mathsf{D}_0, \mathsf{D}_1, \mathsf{B}_1, \mathsf{B}_2, \dots, \mathsf{B}_\omega$. We begin by showing that every subvariety of B_ω is of this form. To this end, let V be a subvariety of B_ω. By Theorem 2.19(ii) we have that $\kappa(\mathsf{V})$ is a downset of $\mathsf{Fin}((\mathsf{B}_\omega)_{\mathrm{SI}})$. Moreover, by Corollary 4.5 poset associated with the preordered class $\mathsf{Fin}((\mathsf{B}_\omega)_{\mathrm{SI}})$ is the chain

$$\mathsf{Up}(\mathbb{C}_1)^- \sqsubset \boldsymbol{B}_1 \sqsubset \boldsymbol{B}_2 \sqsubset \dots.$$

Consequently, $\kappa(\mathsf{V})$ is either $\emptyset$ or $\mathbb{I}(\mathsf{Up}(\mathbb{C}_1)^-)$ or $\mathsf{Fin}((\mathsf{B}_n)_{\mathrm{SI}})$ or $\mathsf{Fin}((\mathsf{B}_\omega)_{\mathrm{SI}})$ for some $n \in \mathbb{Z}^+$. If $\kappa(\mathsf{V}) = \emptyset$, then V is the trivial variety D_0. If $\kappa(\mathsf{V})$ is either $\mathsf{Fin}((\mathsf{B}_n)_{\mathrm{SI}})$ or $\mathsf{Fin}((\mathsf{B}_\omega)_{\mathrm{SI}})$, then V is either B_n or B_ω by Theorem 2.19(ii). Lastly, we consider the case where $\kappa(\mathsf{V}) = \mathbb{I}(\mathsf{Up}(\mathbb{C}_1)^-)$. By Proposition 4.9(ii) this implies $\kappa(\mathsf{V}) = \kappa(\mathsf{D}_1)$. Hence, we conclude that $\mathsf{V} = \mathsf{D}_1$ by Theorem 2.19(ii). Hence, every subvariety of B_ω appears in the sequence $\mathsf{D}_0, \mathsf{D}_1, \mathsf{B}_1, \mathsf{B}_2, \dots, \mathsf{B}_\omega$.

Therefore, it only remains to prove that $\mathsf{D}_0, \mathsf{D}_1, \mathsf{B}_1, \mathsf{B}_2, \dots, \mathsf{B}_\omega$ are subvarieties of B_ω. By the definition of B_ω this is clear for $\mathsf{B}_1, \mathsf{B}_2, \dots, \mathsf{B}_\omega$. In order to prove that $\mathsf{D}_1 \subseteq \mathsf{B}_\omega$, recall from Proposition 4.9(ii) that $\kappa(\mathsf{D}_1) = \mathbb{I}(\mathsf{Up}(\mathbb{C}_1)^-)$. By Corollary 4.5 this implies $\kappa(\mathsf{D}_1) \subseteq \kappa(\mathsf{B}_\omega)$, which amounts to $\mathsf{D}_1 \subseteq \mathsf{B}_\omega$ by Theorem 2.19(ii). Lastly, as D_0 is the trivial variety, we have $\mathsf{D}_0 \subseteq \mathsf{V}$. ⊠

5. Degrees of incompleteness

Let IPC be the *intuitionistic propositional calculus* (see, e.g., [17]). A formula of IPC is said to be *implicative* when it contains no connective other than $\to$.

Definition 5.1. We introduce the set

$$\mathsf{IPC}_\to \coloneqq \{\varphi \in \mathsf{IPC} : \varphi \text{ is an implicative formula}\}.$$

Notably, $\mathsf{IPC}_\to$ coincides with the set of implicative formulas φ such that $\mathsf{Hil} \vDash \varphi$ [26, Thm. 1].

Definition 5.2. An *implicative logic* is a set of implicative formulas containing $\mathsf{IPC}_\to$ that, moreover, is closed under modus ponens and uniform substitutions.

When ordered under inclusion, the set of implicative logics forms a complete lattice $\mathrm{Ext}(\mathsf{IPC}_\to)$ which is dually isomorphic to the lattice $\Lambda(\mathsf{Hil})$ of varieties of Hilbert algebras.

This dual isomorphism is witnessed by the maps $\mathsf{Var}(-)$ and $\mathsf{Log}(-)$ defined for every $\mathsf{L} \in \mathrm{Ext}(\mathsf{IPC}_{\to})$ and $\mathsf{V} \in \Lambda(\mathsf{Hil})$ as

$$\mathsf{Var}(\mathsf{L}) \coloneqq \{\boldsymbol{A} \in \mathsf{Hil} : \boldsymbol{A} \vDash \mathsf{L}\};$$
$$\mathsf{Log}(\mathsf{V}) \coloneqq \{\varphi : \varphi \text{ is an implicative formula such that } \mathsf{V} \vDash \varphi\}.$$

Consequently, $\mathsf{Var}(\mathsf{L})$ is a variety of Hilbert algebras and $\mathsf{Log}(\mathsf{V})$ an implicative logic.

Given a poset $\mathbb{X}$ and an implicative formula φ, we write $\mathbb{X} \Vdash \varphi$ if $\mathsf{Up}(\mathbb{X})^{-} \vDash \varphi$. In this case, we say that $\mathbb{X}$ *validates* φ. Similarly, given a set Γ of implicative formulas, we write $X \Vdash \Gamma$ if $\mathbb{X} \Vdash \varphi$ for every $\varphi \in \Gamma$ and say that $\mathbb{X}$ *validates* Γ.

The notion of a degree of incompleteness originates in the setting of modal logic [24], but can be adapted to the context of implicative logics in a straightforward manner.

Definition 5.3. The *span* of an implicative logic L is the set

$$\mathsf{span}(\mathsf{L}) := \{\mathsf{L}' \in \mathrm{Ext}(\mathsf{IPC}_{\to}) : \mathbb{X} \Vdash \mathsf{L} \text{ if and only if } \mathbb{X} \Vdash \mathsf{L}', \text{ for every poset } \mathbb{X}\}.$$

Furthermore, the *degree of incompleteness* of L is $\mathsf{deg}(\mathsf{L}) := |\mathsf{span}(\mathsf{L})|$.

Our main result is the following characterisation of the degrees of incompleteness of implicative logics.

Trichotomy Theorem 5.4. *The following conditions hold for an implicative logic* L*:*

(i) $\mathsf{deg}(\mathsf{L}) = 1$ *if and only if* $\mathsf{L} = \mathsf{IPC}_{\to}$ *or* $\mathsf{L} = \mathsf{Log}(\mathsf{D}_n)$ *for some* $n \in \mathbb{N}$*;*
(ii) $\mathsf{deg}(\mathsf{L}) = \aleph_0$ *if and only if* $\mathsf{L} = \mathsf{Log}(\mathsf{B}_\omega)$ *or* $\mathsf{L} = \mathsf{Log}(\mathsf{B}_n)$ *for some* $n \in \mathbb{Z}^+$*;*
(iii) $\mathsf{deg}(\mathsf{L}) = 2^{\aleph_0}$ *otherwise.*

In order to prove the Trichotomy Theorem 5.4, it is convenient to rephrase the notion of a span in purely algebraic terms.

Definition 5.5. The *span* of a variety V of Hilbert algebras is the

$$\mathsf{span}(\mathsf{V}) \coloneqq \{\mathsf{W} \in \Lambda(\mathsf{Hil}) : \mathsf{Up}(\mathbb{X})^{-} \in \mathsf{W} \text{ if and only if } \mathsf{Up}(\mathbb{X})^{-} \in \mathsf{V}, \text{ for every poset } \mathbb{X}\}.$$

Furthermore, the *degree of incompleteness* of V is $\mathsf{deg}(\mathsf{V}) := |\mathsf{span}(\mathsf{V})|$.

The degrees of incompleteness of implicative logics can be studied through those of varieties of Hilbert algebras, as we proceed to illustrate.

Proposition 5.6. *For every implicative logic* L *we have* $\mathsf{deg}(\mathsf{L}) = \mathsf{deg}(\mathsf{Var}(\mathsf{L}))$*.*

Proof. Since the map $\mathsf{Var}(-) : \mathrm{Ext}(\mathsf{IPC}_{\to}) \to \Lambda(\mathsf{Hil})$ is a bijection, it suffices to show that for every implicative logic L',

$$\mathsf{L}' \in \mathsf{span}(\mathsf{L}) \ \text{ if and only if } \ \mathsf{Var}(\mathsf{L}') \in \mathsf{span}(\mathsf{Var}(\mathsf{L})).$$

This is an immediate consequence of the fact for every poset $\mathbb{X}$ and implicative logic L'' we have

$$\mathbb{X} \Vdash \mathsf{L}'' \iff \mathsf{Up}(\mathbb{X})^{-} \vDash \mathsf{L}'' \iff \mathsf{Up}(\mathbb{X})^{-} \in \mathsf{Var}(\mathsf{L}'').$$

The above equivalences are justified as follows: the first holds by the definition of $\Vdash$ and the second by the definition of $\mathsf{Var}(\mathsf{L}'')$. $\boxtimes$

The next observation simplifies the task of determining the degree of incompleteness of a variety of Hilbert algebras.

Proposition 5.7. *For every variety* V *of Hilbert algebras,*

$$\begin{aligned}\mathsf{span}(\mathsf{V}) = \{\mathsf{W} \in \Lambda(\mathsf{Hil}) :\ & \mathsf{Up}(\mathbb{X})^- \in \mathsf{W} \textit{ if and only if } \mathsf{Up}(\mathbb{X})^- \in \mathsf{V},\\ & \textit{for every finite rooted poset } \mathbb{X}\}.\end{aligned}$$

Proof. As the inclusion from left to right is straightforward, we only detail the reverse inclusion. Consider a variety W of Hilbert algebras such that V and W contain exactly the same algebras of the form $\mathsf{Up}(\mathbb{X})^-$, where $\mathbb{X}$ is a finite rooted poset. We need to show that V and W contain also the same algebras of the form $\mathsf{Up}(\mathbb{X})^-$, where $\mathbb{X}$ is an arbitrary poset. By symmetry it suffices to show that $\mathsf{Up}(\mathbb{X})^- \in \mathsf{V}$, for every poset $\mathbb{X}$ such that $\mathsf{Up}(\mathbb{X})^- \in \mathsf{W}$. Accordingly, let $\mathbb{X}$ be a poset such that $\mathsf{Up}(\mathbb{X})^- \in \mathsf{W}$. We need to prove that $\mathsf{Up}(\mathbb{X})^- \in \mathsf{V}$.

Claim 5.8. *There exists a family* $\{\mathbb{Y}_i : i \in I\}$ *of finite posets such that each* $\mathsf{Up}(\mathbb{Y}_i)^-$ *belongs to* W *and* $\mathsf{Up}(\mathbb{X})^- \in \mathbb{ISP}_{\mathrm{u}}(\{\mathsf{Up}(\mathbb{Y}_i)^- : i \in I\})$.

Proof of the Claim. In view of Theorem 2.22(ii), it suffices to show that every finitely generated subalgebra of $\mathsf{Up}(\mathbb{X})^-$ embeds into an algebra of the form $\mathsf{Up}(\mathbb{Y})^-$, where $\mathbb{Y}$ is a finite rooted poset and $\mathsf{Up}(\mathbb{Y})^- \in \mathsf{W}$. Then let $\boldsymbol{A}$ be a subalgebra of $\mathsf{Up}(\mathbb{X})^-$ generated by a finite set Z. Let B be the least subset of the Heyting algebra $\mathsf{Up}(\mathbb{X})$ containing Z and closed under $\wedge$ and $\to$. By Corollary 2.18 the Hilbert algebra $\boldsymbol{B} = \langle B; \to\rangle$ is isomorphic to $\mathsf{Up}(\mathbb{Y})^-$ for some finite poset $\mathbb{Y}$. Furthermore, $\mathsf{Up}(\mathbb{Y})^- \in \mathsf{W}$ because $\boldsymbol{B} \in \mathbb{S}(\mathsf{Up}(\mathbb{X})^-) \subseteq \mathbb{S}(\mathsf{W}) \subseteq \mathsf{W}$. Lastly, the inclusion map is an embedding of $\boldsymbol{A}$ into $\boldsymbol{B}$ because Z generates $\boldsymbol{A}$ and is contained in B. As $\boldsymbol{B} \cong \mathsf{Up}(\mathbb{Y})^-$, we conclude that $\boldsymbol{A}$ embeds into $\mathsf{Up}(\mathbb{Y})^-$ as well. ⊠

Let $\{\mathbb{Y}_i : i \in I\}$ be the family of finite rooted posets given by Claim 5.8. It only remains to prove that each $\mathsf{Up}(\mathbb{Y}_i)^-$ belongs to V. For in this case, Claim 5.8 guarantees that $\mathsf{Up}(\mathbb{X})^- \in \mathbb{ISP}_{\mathrm{u}}(\{\mathsf{Up}(\mathbb{Y}_i)^- : i \in I\}) \subseteq \mathbb{ISP}_{\mathrm{u}}(\mathsf{V}) \subseteq \mathsf{V}$ and we are done. To this end, consider $i \in I$. It is well known that the SI homomorphic images of the finite Heyting algebra $\mathsf{Up}(\mathbb{Y}_i)$ are, up to isomorphism, the algebras of the form $\mathsf{Up}(\uparrow y)$ for $y \in \mathbb{Y}_i$ (see, e.g., [6, Lem. 2.9(ii)]). By Theorem 2.22(i) this implies that $\mathsf{Up}(\mathbb{Y}_i) \in \mathbb{ISP}(\{\mathsf{Up}(\uparrow y) : y \in \mathbb{Y}_i\})$. It follows that

$$\mathsf{Up}(\mathbb{Y}_i)^- \in \mathbb{ISP}(\{\mathsf{Up}(\uparrow y)^- : y \in \mathbb{Y}_i\}).$$

Now, consider $y \in \mathbb{Y}_i$. Since $\mathsf{Up}(\uparrow y) \in \mathbb{H}(\mathsf{Up}(\mathbb{Y}_i))$, we have $\mathsf{Up}(\uparrow y)^- \in \mathbb{H}(\mathsf{Up}(\mathbb{Y}_i)^-)$. As W is closed under $\mathbb{H}$ and by Claim 5.8 we have $\mathsf{Up}(\mathbb{Y}_i)^- \in \mathsf{W}$, this implies $\mathsf{Up}(\uparrow y)^- \in \mathsf{W}$. Since W and V have the same members of the form $\mathsf{Up}(\mathbb{P})^-$ for finite rooted posets $\mathbb{P}$, we conclude that $\mathsf{Up}(\uparrow y)^- \in \mathsf{V}$ too. Together with the above display, this yields $\mathsf{Up}(\mathbb{Y}_i)^- \in \mathbb{ISP}(\mathsf{V}) \subseteq \mathsf{V}$. ⊠

The following concept will play a fundamental role in the description of the spans of varieties of Hilbert algebras.

Definition 5.9. Given a variety V of Hilbert algebras, let

$$\begin{aligned}\mathsf{span}^*(\mathsf{V}) &:= \{D \in \mathsf{Down}(\mathsf{Fin}(\mathsf{Hil}_{\mathrm{SI}})) : \mathsf{Up}(\mathbb{X})^- \in D \text{ if and only if } \mathsf{Up}(\mathbb{X})^- \in \mathsf{V},\\ &\qquad \text{for every finite rooted poset } \mathbb{X}\};\\ \mathsf{deg}^*(\mathsf{V}) &:= |\mathsf{span}^*(\mathsf{V})|.\end{aligned}$$

Proposition 5.10. *For every variety* V *of Hilbert algebras we have* $\mathsf{deg}(\mathsf{V}) = \mathsf{deg}^*(\mathsf{V})$*.*

Proof. Recall from Theorem 2.19 that the map $\kappa\colon \Lambda(\mathsf{Hil}) \to \mathsf{Down}(\mathsf{Fin}(\mathsf{Hil}_{\mathrm{SI}}))$ is a bijection. Therefore, it suffices to prove that for every pair V and W of varieties of Hilbert algebras,

$$\mathsf{W} \in \mathsf{span}(\mathsf{V}) \ \text{ if and only if } \ \kappa(\mathsf{W}) \in \mathsf{span}^*(\mathsf{V}).$$

But this is a consequence of the following series of equivalences:

$$\begin{aligned}\mathsf{W} \in \mathsf{span}(\mathsf{V}) \iff& \mathsf{Up}(\mathbb{X})^- \in \mathsf{W} \text{ if and only if } \mathsf{Up}(\mathbb{X})^- \in \mathsf{V},\\ & \text{for every finite rooted poset } \mathbb{X};\\ \iff& \mathsf{Up}(\mathbb{X})^- \in \kappa(\mathsf{W}) \text{ if and only if } \mathsf{Up}(\mathbb{X})^- \in \mathsf{V},\\ & \text{for every finite rooted poset } \mathbb{X};\\ \iff& \kappa(\mathsf{W}) \in \mathsf{span}^*(\mathsf{V}).\end{aligned}$$

The first equivalence above holds by Proposition 5.7 and the second because $\mathsf{Up}(\mathbb{X})^-$ is finite and SI for every finite rooted poset $\mathbb{X}$ and because $\kappa(\mathsf{W})$ is the class of finite SI members of W. The last equivalence holds by the definitions of κ and $\mathsf{span}^*(\mathsf{V})$. ⊠

As a consequence of Propositions 5.6 and 5.10 we deduce the following.

Corollary 5.11. *For every implicative logic* L *we have* $\mathsf{deg}(\mathsf{L}) = \mathsf{deg}^*(\mathsf{Var}(\mathsf{L}))$*.*

In view of the above corollary and the fact that the map $\mathsf{Var}(-)\colon \mathrm{Ext}(\mathsf{IPC}_{\to}) \to \Lambda(\mathsf{Hil})$ is a dual lattice isomorphism, the Trichotomy Theorem 5.4 can be rephrased in purely algebraic terms as follows.

Theorem 5.12. *The following conditions hold for a variety of Hilbert algebras* V*:*

(i) $\mathsf{deg}^*(\mathsf{V}) = 1$ *if and only if* $\mathsf{V} = \mathsf{Hil}$ *or* $\mathsf{V} = \mathsf{D}_n$ *for some* $n \in \mathbb{N}$*;*
(ii) $\mathsf{deg}^*(\mathsf{V}) = \aleph_0$ *if and only if* $\mathsf{V} = \mathsf{B}_\omega$ *or* $\mathsf{V} = \mathsf{B}_n$ *for some* $n \in \mathbb{Z}^+$*;*
(iii) $\mathsf{deg}^*(\mathsf{V}) = 2^{\aleph_0}$ *otherwise.*

The rest of the paper is devoted to proving Theorem 5.12.

6. Varieties with degree of incompleteness 1

In this section we will prove condition (i) of Theorem 5.12, namely, the following result.

Theorem 6.1. *Let* V *be a variety of Hilbert algebras. Then* $\mathsf{deg}^*(\mathsf{V}) = 1$ *if and only if* $\mathsf{V} = \mathsf{Hil}$ *or* $\mathsf{V} = \mathsf{D}_n$ *for some* $n \in \mathbb{N}$*.*

To this end, we will make use of the following construction.

Definition 6.2. Given a Heyting algebra $\boldsymbol{A}$, we let

$$\mathsf{M}(\boldsymbol{A}) \coloneqq \{a \in A : a = 1 \text{ or } a \text{ is meet irreducible}\}.$$

From Remark 2.11 it follows that $\mathsf{M}(\boldsymbol{A})$ coincides with the set of irreducible elements of $\boldsymbol{A}^-$ together with the maximum 1. By the definition of an irreducible element of a Hilbert algebra and the fact that the equations $x \to 1 \approx 1$ and $1 \to x \approx x$ hold in every Hilbert algebra we obtain the following.

Proposition 6.3. *Let $\boldsymbol{A}$ be a Heyting algebra. Then $\langle \mathsf{M}(\boldsymbol{A}); \to \rangle$ is an implicative subreduct of $\boldsymbol{A}$. Furthermore, $\langle \mathsf{M}(\boldsymbol{A}); \to \rangle = \mathsf{H}(\mathbb{X})$, where $\mathbb{X}$ is the subposet of $\boldsymbol{A}$ with universe $\mathsf{M}(\boldsymbol{A})$.*

As no confusion shall arise, we will often denote the Hilbert algebra $\langle \mathsf{M}(\boldsymbol{A}); \to \rangle$ by $\mathsf{M}(\boldsymbol{A})$. The next observation identifies a fundamental property of $\mathsf{M}(\boldsymbol{A})$, which is a special case of [15, Prop. 6.6].

Embedding Lemma 6.4. *For every pair $\boldsymbol{A}$ and $\boldsymbol{B}$ of Heyting algebras with $\boldsymbol{A}$ finite,*

$$\mathsf{M}(\boldsymbol{A}) \sqsubseteq \boldsymbol{B}^- \textit{ implies } \boldsymbol{A}^- \sqsubseteq \boldsymbol{B}^-.$$

Proof. Suppose that $\mathsf{M}(\boldsymbol{A}) \sqsubseteq \boldsymbol{B}^-$. Then there exists an embedding $h\colon \mathsf{M}(\boldsymbol{A}) \to \boldsymbol{B}^-$ of Hilbert algebras. As $\boldsymbol{A}$ is a finite Heyting algebra, every element of $\boldsymbol{A}$ is a meet of a finite subset of $\mathsf{M}(\boldsymbol{A})$. Therefore, $\langle A; \wedge^{\boldsymbol{A}}, \to^{\boldsymbol{A}} \rangle$ with the inclusion map $i\colon \mathsf{M}(\boldsymbol{A}) \to \langle A; \wedge^{\boldsymbol{A}}, \to^{\boldsymbol{A}} \rangle$ forms an implicative semilattice envelope of $\mathsf{M}(\boldsymbol{A})$ in the sense of [15, Sec. 6]. Since $\boldsymbol{B}$ is a Heyting algebra and h an embedding of Hilbert algebras, we can apply [15, Prop. 6.6], obtaining an embedding $f\colon \boldsymbol{A}^- \to \boldsymbol{B}^-$. Hence, we conclude that $\boldsymbol{A}^- \sqsubseteq \boldsymbol{B}^-$. ⊠

Given a poset $\mathbb{X}$, we write $\mathsf{M}(\mathbb{X})$ as a shorthand for $\mathsf{M}(\mathsf{Up}(\mathbb{X}))$. As a consequence of the Embedding Lemma 6.4, we obtain the following.

Corollary 6.5. *For every pair of poset $\mathbb{X}$ and $\mathbb{Y}$ with $\mathbb{X}$ finite,*

$$\mathsf{M}(\mathbb{X}) \sqsubseteq \mathsf{Up}(\mathbb{Y})^- \textit{ implies } \mathsf{Up}(\mathbb{X})^- \sqsubseteq \mathsf{Up}(\mathbb{Y})^-.$$

The proof of Theorem 6.1 relies also on the next technical observation. We say that a Heyting or Hilbert algebra $\boldsymbol{A}$ is *linear* when it is linerarly ordered and *nonlinear* otherwise. Notice that if $\boldsymbol{A}$ is an SI Heyting algebra, then the Hilbert algebra $\boldsymbol{A}^-$ is also SI by Theorem 2.13.

Lemma 6.6. *For every nonlinear $\boldsymbol{A} \in \mathsf{Fin}(\mathsf{HA}_{\text{SI}})$ there exists a proper subalgebra $\boldsymbol{B}$ of $\boldsymbol{A}^-$ such that $\boldsymbol{B} \in \mathsf{Fin}(\mathsf{Hil}_{\text{SI}})$ and for every finite rooted poset $\mathbb{X}$ it holds*

$$\boldsymbol{B} \sqsubseteq \mathsf{Up}(\mathbb{X})^- \quad \textit{if and only if} \quad \boldsymbol{A}^- \sqsubseteq \mathsf{Up}(\mathbb{X})^-.$$

Proof. We call an element a of $\boldsymbol{A}$ *meet reducible* if there exist $b, c > a$ such that $a = b \wedge c$.

Claim 6.7. *There exists the least meet reducible element of $\boldsymbol{A}$.*

Proof of the Claim. As $\boldsymbol{A}$ is nonlinear, it contains two incomparable elements b and c. Then $b \wedge c < b, c$. Thus, $b \wedge c$ is meet reducible in $\boldsymbol{A}$. It follows that $\boldsymbol{A}$ contains at least one meet reducible element. Furthermore, $\boldsymbol{A}$ contains only finitely many meet reducible elements because it is finite.

Then consider an enumeration $\{a_1, \dots, a_n\}$ of the meet reducible elements of $\boldsymbol{A}$ and let

$$a \coloneqq a_1 \wedge \cdots \wedge a_n.$$

If $a \in \{a_1, \dots, a_n\}$, then a is the least meet reducible element of $\boldsymbol{A}$ and we are done. We will show that the case where $a \notin \{a_1, \dots, a_n\}$ never happens. For suppose that $a \notin \{a_1, \dots, a_n\}$. By the definition of a, this implies $a < a_1, \dots, a_n$. Together with the definition of a, this yields that there exist $b, c > a$ such that $a = b \wedge c$. Thus, a is meet reducible. But this implies $a \in \{a_1, \dots, a_n\}$, a contradiction. ⊠

Let a be least meet reducible element of $\boldsymbol{A}$ (which exists by Claim 6.7). We rely on the next observation.

Claim 6.8. *The set $B \coloneqq A - \{a\}$ is the universe of a subalgebra of $\boldsymbol{A}^-$.*

Proof of the Claim. We begin by showing that

$$A = {\uparrow}a \cup {\downarrow}a \;\; \text{and} \;\; {\downarrow}a \text{ is a chain.}$$

To prove the left hand side of the above display, it suffices to show that the inclusion $A \subseteq {\uparrow}a \cup {\downarrow}a$ holds. To this end, consider $b \in A$. If b is comparable with a, then $b \in {\uparrow}a \cup {\downarrow}a$ and we are done. We will show that the case where b is incomparable with a never happens. For suppose that $a \nleqslant b$ and $b \nleqslant a$, with a view to contradiction. Then $a \wedge b$ would be a meet reducible element of $\boldsymbol{A}$ strictly smaller than a, contradicting the assumption that a is the least meet irreducible element of $\boldsymbol{A}$.

A similar argument can be used to show that ${\downarrow}a$ is a chain, for if ${\downarrow}a$ contains two incomparable elements b and c, then $b \wedge c$ would be a meet reducible element of $\boldsymbol{A}$ strictly smaller than a, contradicting the assumption that a is the least meet irreducible element of $\boldsymbol{A}$. This establishes the above display.

Next we prove the statement of the claim. Consider $b, c \in B$. We need to show that $b \to c \in B$. From the left hand side of the above display and the assumption that $c \in B = A - \{a\}$ it follows that either $a < c$ or $c < a$. Suppose first that $a < c$. As $c \leqslant b \to c$, this implies $a < c \leqslant b \to c$ and, therefore, $a \neq b \to c$, whence $b \to c \in A - \{a\} = B$ as desired. Then we consider the case where $c < a$. Since ${\downarrow}a$ is a chain by the right hand side of the above display, from $c < a$ it follows that c is meet irreducible. Consequently, it is irreducible in $\boldsymbol{A}^-$ by Remark 2.11. In turn, this guarantees that $b \to c \in \{c, 1\}$. As both c and 1 differ from a (the first by the assumption that $c \in B = A - \{a\}$ and the second because a is meet reducible and 1 is not), we conclude that $b \to c \neq a$, whence $b \to c \in A - \{a\} = B$. ⊠

Now, let $\boldsymbol{B}$ be the subalgebra of $\boldsymbol{A}^-$ with universe $B = A - \{a\}$ given by Claim 6.8 and recall that $\boldsymbol{A}$ is SI by assumption. Therefore, $\boldsymbol{A}$ has a second largest element b by Theorem

2.13. Observe that b is meet irreducible and, therefore, it differs from a. Similarly, 1 is not meet reducible and, therefore, $a \neq 1$. Consequently, $b, 1 \in A - \{a\} = B$ and, therefore, b is also the second largest element of $\boldsymbol{B}$. By Theorem 2.13 we conclude that $\boldsymbol{B}$ is also SI. Furthermore, it is finite because so is $\boldsymbol{A}$. Thus, $\boldsymbol{B} \in \mathsf{Fin}(\mathsf{Hil}_{\text{SI}})$.

It only remains to prove that for every finite rooted poset $\mathbb{X}$,

$$\boldsymbol{B} \sqsubseteq \mathsf{Up}(\mathbb{X})^- \text{ if and only if } \boldsymbol{A}^- \sqsubseteq \mathsf{Up}(\mathbb{X})^-.$$

The implication from right to left holds because $\boldsymbol{B}$ is a subalgebra of $\boldsymbol{A}^-$ by definition. To prove the other implication, let $\mathbb{X}$ be a finite rooted poset such that $\boldsymbol{B} \sqsubseteq \mathsf{Up}(\mathbb{X})^-$. As the element a is meet reducible and $B = A - \{a\}$, we have that $\mathsf{M}(\boldsymbol{A})$ is a subalgebra of $\boldsymbol{B}$. Together with $\boldsymbol{B} \sqsubseteq \mathsf{Up}(\mathbb{X})^-$, this implies $\mathsf{M}(\boldsymbol{A}) \sqsubseteq \mathsf{Up}(\mathbb{X})^-$. By the Embedding Lemma 6.4 we conclude that $\boldsymbol{A}^- \sqsubseteq \mathsf{Up}(\mathbb{X})^-$. ⊠

In order to prove Theorem 6.1, we fix some notation.

Definition 6.9. Given a class $\mathsf{K} \subseteq \mathsf{Fin}(\mathsf{Hil}_{\text{SI}})$, let

$$\begin{aligned}
\mathsf{K}^c &\coloneqq \{\boldsymbol{A} \in \mathsf{Fin}(\mathsf{Hil}_{\text{SI}}) : \boldsymbol{A} \notin \mathsf{K}\};\\
\uparrow\mathsf{K} &\coloneqq \{\boldsymbol{A} \in \mathsf{Fin}(\mathsf{Hil}_{\text{SI}}) : \text{there exists } \boldsymbol{B} \in \mathsf{K} \text{ such that } \boldsymbol{B} \sqsubseteq \boldsymbol{A}\};\\
\downarrow\mathsf{K} &\coloneqq \{\boldsymbol{A} \in \mathsf{Fin}(\mathsf{Hil}_{\text{SI}}) : \text{there exists } \boldsymbol{B} \in \mathsf{K} \text{ such that } \boldsymbol{A} \sqsubseteq \boldsymbol{B}\}.
\end{aligned}$$

If $\mathsf{K} = \{\boldsymbol{A}\}$ for some $\boldsymbol{A} \in \mathsf{Fin}(\mathsf{Hil}_{\text{SI}})$, we will write $\uparrow\boldsymbol{A}$ (resp. $\downarrow\boldsymbol{A}$) instead of $\uparrow\{\boldsymbol{A}\}$ (resp. $\downarrow\{\boldsymbol{A}\}$).

We are now ready to prove Theorem 6.1.

Proof. We will prove the implication from left to right by contraposition. Accordingly, suppose that V is different from Hil and from each D_n.

Claim 6.10. *There exusts a nonlinear $\boldsymbol{A} \in \mathsf{Fin}(\mathsf{HA}_{\text{SI}})$ such that $\boldsymbol{A}^- \notin \mathsf{V}$ and for every finite rooted poset $\mathbb{X}$,*

$$\mathsf{Up}(\mathbb{X})^- \sqsubset \boldsymbol{A}^- \textit{ implies } \mathsf{Up}(\mathbb{X})^- \in \mathsf{V}.$$

Proof of the Claim. Throughout the proof, we will use repeatedly the fact the Heyting algebras of the form $\mathsf{Up}(\mathbb{C}_n)$ are precisely the finite nontrivial linear ones.

One of the following conditions holds:

(i) $\mathsf{Up}(\mathbb{C}_n)^- \in \mathsf{V}$ for every $n \geqslant 1$;
(ii) $\mathsf{Up}(\mathbb{C}_n)^- \notin \mathsf{V}$ for some $n \geqslant 1$.

First we consider case (i). Since $\mathsf{V} \neq \mathsf{Hil}$ by assumption, we can apply Proposition 2.17 obtaining some finite rooted poset $\mathbb{X}$ for which the implicative reduct $\boldsymbol{A}^-$ of the Heyting algebra $\boldsymbol{A} \coloneqq \mathsf{Up}(\mathbb{X})$ does not belong to V. In addition, $\boldsymbol{A}$ is finite and SI (the latter by Theorem 2.13). Thus, $\boldsymbol{A} \in \mathsf{Fin}(\mathsf{HA}_{\text{SI}})$. We will prove that $\boldsymbol{A}$ is nonlinear. For suppose the contrary. Then $\boldsymbol{A}^- \cong \mathsf{Up}(\mathbb{C}_n)^-$ for some $n \in \mathbb{Z}^+$ because $\boldsymbol{A}$ is a finite Heyting algebra which, moreover, is nontrivial (the latter holds since $\boldsymbol{A}$ is SI). From (i) and $\boldsymbol{A}^- \cong \mathsf{Up}(\mathbb{C}_n)^-$ it follows that $\boldsymbol{A}^- \in \mathsf{V}$, a contradiction. Hence, $\boldsymbol{A}$ is nonlinear as desired.

Therefore, the set

$$Y \coloneqq \{\boldsymbol{A} \in \mathsf{Fin}(\mathsf{HA}_{\text{SI}}) : \boldsymbol{A}^- \notin \mathsf{V} \text{ and } \boldsymbol{A} \text{ is nonlinear}\}$$

is nonempty. Then let $\boldsymbol{A}$ be an element of Y of minimal size. To establish the claim, it only remains to prove that for every finite rooted poset $\mathbb{X}$ such that $\mathsf{Up}(\mathbb{X})^- \sqsubset \boldsymbol{A}^-$ it holds $\mathsf{Up}(\mathbb{X})^- \in \mathsf{V}$. Suppose, on the contrary, that there exists a finite rooted poset $\mathbb{X}$ such that $\mathsf{Up}(\mathbb{X})^- \sqsubset \boldsymbol{A}^-$ and $\mathsf{Up}(\mathbb{X})^- \notin \mathsf{V}$. Together with the fact that $\boldsymbol{A}$ is finite and $\mathsf{Up}(\mathbb{X})^- \sqsubset \boldsymbol{A}^-$, the minimality of the size of $\boldsymbol{A}$ guarantees that $\mathsf{Up}(\mathbb{X}) \notin Y$. On the other hand, $\mathsf{Up}(\mathbb{X}) \in \mathsf{Fin}(\mathsf{HA}_{\text{SI}})$ by Theorem 2.13 and the assumption that $\mathbb{X}$ is finite and rooted. In addition, from (i) and $\mathsf{Up}(\mathbb{X})^- \notin \mathsf{V}$ it follows that $\mathsf{Up}(\mathbb{X})^-$ is nonlinear. Therefore, $\mathsf{Up}(\mathbb{X}) \in Y$, a contradiction.

Then we consider case (ii). Let n be the least positive integer m such that $\mathsf{Up}(\mathbb{C}_m)^- \notin \mathsf{V}$, which exists by (ii). By Jankov's Lemma 2.20 we have $\mathsf{V} \vDash \mathcal{J}(\mathbb{C}_n)$. Together with Theorem 3.5, this implies $\mathsf{V} \subseteq \mathsf{D}_{n-1}$. On the other hand, $\mathsf{V} \neq \mathsf{D}_{n-1}$ by assumption. Therefore, $\mathsf{V} \subsetneq \mathsf{D}_{n-1}$. By Proposition 3.7 there exists a finite rooted poset $\mathbb{X}$ such that the implicative reduct $\boldsymbol{A}^-$ of the Heyting algebra $\boldsymbol{A} \coloneqq \mathsf{Up}(\mathbb{X})$ belongs to $\mathsf{D}_{n-1} - \mathsf{V}$. Moreover, $\boldsymbol{A}$ must be nonlinear, otherwise it would be isomorphic to $\mathsf{Up}(\mathbb{C}_m)$ for some $m < n$ (the latter because $\boldsymbol{A} \in \mathsf{D}_{n-1}$), thus contradicting the minimality of n. Lastly, $\boldsymbol{A}$ is finite and SI because $\mathbb{X}$ is finite and rooted (see Theorem 2.13).

Therefore, the set

$$Y \coloneqq \{\boldsymbol{A} \in \mathsf{Fin}(\mathsf{HA}_{\text{SI}}) : \boldsymbol{A}^- \in \mathsf{D}_{n-1} - \mathsf{V} \text{ and } \boldsymbol{A} \text{ is nonlinear}\}$$

is nonempty. Then let $\boldsymbol{A}$ be an element of Y of minimal size. It only remains to prove that for every finite rooted poset $\mathbb{X}$ such that $\mathsf{Up}(\mathbb{X})^- \sqsubset \boldsymbol{A}^-$ it holds $\mathsf{Up}(\mathbb{X})^- \in \mathsf{V}$. Suppose, with a view to contradiction, that there exists a finite rooted poset $\mathbb{X}$ such that $\mathsf{Up}(\mathbb{X})^- \sqsubset \boldsymbol{A}^-$ and $\mathsf{Up}(\mathbb{X})^- \notin \mathsf{V}$. As in the previous case, we have that $\mathsf{Up}(\mathbb{X}) \notin Y$ and $\mathsf{Up}(\mathbb{X}) \in \mathsf{Fin}(\mathsf{HA}_{\text{SI}})$. Furthermore, from $\boldsymbol{A}^- \in \mathsf{D}_{n-1}$ and $\mathsf{Up}(\mathbb{X})^- \sqsubset \boldsymbol{A}^-$ it follows that $\mathsf{Up}(\mathbb{X})^- \in \mathsf{D}_{n-1}$. Together with the assumption that $\mathsf{Up}(\mathbb{X})^- \notin \mathsf{V}$, this implies $\mathsf{Up}(\mathbb{X})^- \in \mathsf{D}_{n-1} - \mathsf{V}$. As $\mathsf{Up}(\mathbb{X}) \notin Y$, it follows that $\mathsf{Up}(\mathbb{X})$ is linear. Moreover, $\mathsf{Up}(\mathbb{X})$ is nontrivial because $\mathbb{X}$ is nonempty. Therefore, we may assume that $\mathbb{X} = \mathbb{C}_m$ for some $m \in \mathbb{Z}^+$. Since $\mathsf{Up}(\mathbb{X})^- = \mathsf{Up}(\mathbb{C}_m)^-$ and $(\mathsf{Up}(\mathbb{C}_m)^-)_* \cong \mathbb{C}_m$, the poset $(\mathsf{Up}(\mathbb{X})^-)_*$ possesses an m-element chain. On the other hand, the chains of $(\mathsf{Up}(\mathbb{X})^-)_*$ have size $\leqslant n-1$ because $\mathsf{Up}(\mathbb{X})^- \in \mathsf{D}_{n-1}$. Consequently, $m \leqslant n-1 < n$. Together with $\mathsf{Up}(\mathbb{C}_m)^- = \mathsf{Up}(\mathbb{X})^- \notin \mathsf{V}$, this contradicts the minimality of n. ⊠

Let $\boldsymbol{A}$ be the finite nonlinear SI Heyting algebra given by Claim 6.10. By Lemma 6.6 there exists a proper subalgebra $\boldsymbol{B}$ of $\boldsymbol{A}^-$ with $\boldsymbol{B} \in \mathsf{Fin}(\mathsf{Hil}_{\text{SI}})$ such that for every finite rooted poset $\mathbb{X}$,

$$\boldsymbol{B} \sqsubseteq \mathsf{Up}(\mathbb{X})^- \text{ if and only if } \boldsymbol{A}^- \sqsubseteq \mathsf{Up}(\mathbb{X})^-. \tag{6}$$

Then consider the following dwonsets of $\mathsf{Fin}(\mathsf{Hil}_{\text{SI}})$:

$$D_1 \coloneqq \kappa(\mathsf{V}) \cup {\downarrow}\boldsymbol{B} \text{ and } D_2 \coloneqq \kappa(\mathsf{V}) - {\uparrow}\boldsymbol{B}.$$

To conclude the proof, it will be enough to show that D_1 and D_2 are two distinct members of $\mathsf{span}^*(\mathsf{V})$, for this would imply $\mathsf{deg}^*(\mathsf{V}) \geqslant 2$.

As $\boldsymbol{B} \in D_1 - D_2$, we get $D_1 \neq D_2$. Next we prove that $D_1, D_2 \in \mathsf{span}^*(\mathsf{V})$. Observe that D_1 and D_2 are downsets of $\mathsf{Fin}(\mathsf{Hil}_{\text{SI}})$ because so is $\kappa(\mathsf{V})$ by Theorem 2.19. Therefore, it only remains to show that for every finite rooted poset $\mathbb{X}$,

$$\mathsf{Up}(\mathbb{X})^- \in D_1 \text{ if and only if } \mathsf{Up}(\mathbb{X})^- \in \mathsf{V}; \tag{7}$$

$$\mathsf{Up}(\mathbb{X})^- \in D_2 \text{ if and only if } \mathsf{Up}(\mathbb{X})^- \in \mathsf{V}. \tag{8}$$

Since $\mathbb{X}$ is a finite rooted poset, the Hilbert algebra $\mathsf{Up}(\mathbb{X})^-$ is finite and SI by Theorem 2.13. By the definition of $\kappa(\mathsf{V})$ this yields

$$\mathsf{Up}(\mathbb{X})^- \in \mathsf{V} \text{ if and only if } \mathsf{Up}(\mathbb{X})^- \in \kappa(\mathsf{V}).$$

Furthermore, the definition of D_1 and D_2 guarantees that $\kappa(\mathsf{V}) \subseteq D_1$ and $D_2 \subseteq \kappa(\mathsf{V})$. Together with the above display, this implies that the right to left implication in (7) and the left to right implication in (8) hold.

To prove the left to right implication in (7), suppose that $\mathsf{Up}(\mathbb{X})^- \in D_1$. By the definition of D_1 we have $\mathsf{Up}(\mathbb{X})^- \in \kappa(\mathsf{V}) \cup {\downarrow}\boldsymbol{B}$. If $\mathsf{Up}(\mathbb{X})^- \in \kappa(\mathsf{V})$, then $\mathsf{Up}(\mathbb{X})^- \in \mathsf{V}$ because $\kappa(\mathsf{V}) \subseteq \mathsf{V}$ and we are done. Then we consider the case where $\mathsf{Up}(\mathbb{X})^- \in {\downarrow}\boldsymbol{B}$, that is, $\mathsf{Up}(\mathbb{X})^- \sqsubseteq \boldsymbol{B}$. As $\boldsymbol{B}$ is a proper subalgebra of $\boldsymbol{A}^-$, we obtain $\mathsf{Up}(\mathbb{X})^- \sqsubset \boldsymbol{A}^-$. But this implies that $\mathsf{Up}(\mathbb{X})^- \in \mathsf{V}$ by Claim 6.10.

It only remains to prove the implication from right to left in (8). Suppose that $\mathsf{Up}(\mathbb{X})^- \in \mathsf{V}$. By the above display we have $\mathsf{Up}(\mathbb{X})^- \in \kappa(\mathsf{V})$. We will prove that $\mathsf{Up}(\mathbb{X})^- \notin {\uparrow}\boldsymbol{B}$. For suppose, on the contrary, that $\boldsymbol{B} \sqsubseteq \mathsf{Up}(\mathbb{X})^-$. Then $\boldsymbol{A}^- \sqsubseteq \mathsf{Up}(\mathbb{X})^-$ by (6). Recall that $\boldsymbol{A}$ is finite and SI. By Theorem 2.13 this implies that $\boldsymbol{A}^-$ is also finite and SI. Thus, $\boldsymbol{A}^- \in \mathsf{Fin}(\mathsf{Hil}_{\text{SI}})$. Therefore, from $\boldsymbol{A}^- \sqsubseteq \mathsf{Up}(\mathbb{X})^-$ and $\mathsf{Up}(\mathbb{X})^- \in \kappa(\mathsf{V})$ and the fact that $\kappa(\mathsf{V})$ is a downset of $\mathsf{Fin}(\mathsf{Hil}_{\text{SI}})$ it follows that $\boldsymbol{A}^- \in \kappa(\mathsf{V}) \subseteq \mathsf{V}$. But this contradicts the fact that $\boldsymbol{A}^- \notin \mathsf{V}$, which holds by Claim 6.10. Thus, $\mathsf{Up}(\mathbb{X})^- \notin {\uparrow}\boldsymbol{B}$. Together with $\mathsf{Up}(\mathbb{X})^- \in \kappa(\mathsf{V})$, this implies $\mathsf{Up}(\mathbb{X})^- \in \kappa(\mathsf{V}) - {\uparrow}\boldsymbol{B} = D_2$ and concludes the proof of the implication form left to right in the statement.

To prove the implication from right to left in the statement, suppose that V is either Hil or some D_n. We need to show that $\mathsf{deg}^*(\mathsf{V}) = 1$. By Proposition 5.10 it suffices to show that $\mathsf{deg}(\mathsf{V}) = 1$. We will do this by establishing that $\mathsf{span}(\mathsf{V}) = \{\mathsf{V}\}$. As $\mathsf{V} \in \mathsf{span}(\mathsf{V})$ always holds, it suffices to prove $\mathsf{span}(\mathsf{V}) \subseteq \{\mathsf{V}\}$.

We begin with the case where $\mathsf{V} = \mathsf{Hil}$. Consider $\mathsf{W} \in \mathsf{span}(\mathsf{Hil})$. As $\mathsf{Up}(\mathbb{X})^- \in \mathsf{Hil}$ for every finite rooted poset $\mathbb{X}$, we obtain that $\mathsf{Up}(\mathbb{X})^- \in \mathsf{W}$ for every finite rooted poset $\mathbb{X}$ as well. By Proposition 2.17 we conclude that $\mathsf{W} = \mathsf{Hil}$ as desired. Then we consider the case where $\mathsf{V} = \mathsf{D}_n$ for some $n \in \mathbb{N}$. Consider $\mathsf{W} \in \mathsf{span}(\mathsf{D}_n)$. By the definition of D_n we know that $\mathsf{Up}(\mathbb{X})^- \in \mathsf{D}_n$ for every finite rooted poset $\mathbb{X}$ such that $\mathsf{Up}(\mathbb{X})^-$ has depth $\leqslant n$. Together with $\mathsf{W} \in \mathsf{span}(\mathsf{D}_n)$, this implies that $\mathsf{Up}(\mathbb{X})^- \in \mathsf{W}$ for every finite rooted poset $\mathbb{X}$ such that $\mathsf{Up}(\mathbb{X})^-$ has depth $\leqslant n$. By Proposition 3.7 this yields $\mathsf{D}_n \subseteq \mathsf{W}$. On the other hand, $\mathsf{Up}(\mathbb{C}_{n+1})^- \notin \mathsf{D}_n$ by the definition of D_n. As $\mathsf{W} \in \mathsf{span}(\mathsf{D}_n)$, this implies $\mathsf{Up}(\mathbb{C}_{n+1})^- \notin \mathsf{W}$. By Jankov's Lemma 2.20 we obtain $\mathsf{W} \vDash \mathcal{J}(\mathbb{C}_{n+1})$. Together with Theorem 3.5, this yields $\mathsf{W} \subseteq \mathsf{D}_n$. Hence, we conclude that $\mathsf{W} = \mathsf{D}_n$. ⊠

7. Varieties with degree of incompleteness $\aleph_0$

Recall that for every $n \in \mathbb{Z}^+$ we have $\boldsymbol{B}_n = \mathsf{H}(\mathbb{B}_n)$, where $\mathbb{B}_n$ is the poset depicted in Figure 1. Furthermore,

$$\mathsf{B}_n = \mathbb{V}(\boldsymbol{B}_n) \text{ and } \mathsf{B}_\omega = \mathbb{V}(\{\boldsymbol{B}_m : m \in \mathbb{Z}^+\})$$

by definition. The aim of this section is to prove the implication from right to left of (ii) of Theorem 5.12, namely, the following result.

Proposition 7.1. *For every positive integer n,*

$$\mathsf{deg}^*(\mathsf{B}_n) = \mathsf{deg}^*(\mathsf{B}_\omega) = \aleph_0.$$

Proof. We detail only the proof that $\mathsf{deg}^*(\mathsf{B}_n) = \aleph_0$ as that of $\mathsf{deg}^*(\mathsf{B}_\omega) = \aleph_0$ is analogous. Recall from Proposition 5.10 that $\mathsf{deg}^*(\mathsf{B}_n) = \mathsf{deg}(\mathsf{B}_n)$. Therefore, it suffices to show that

$$\mathsf{span}(\mathsf{B}_n) = \{\mathsf{B}_1, \mathsf{B}_2, \dots, \mathsf{B}_\omega\}, \tag{9}$$

for this would imply $\mathsf{deg}(\mathsf{B}_n) = \aleph_0$, as the varieties $\mathsf{B}_1, \mathsf{B}_2, \dots, \mathsf{B}_\omega$ are all different by Corollary 4.5.

In order to prove the inclusion from left to right in (9), consider $\mathsf{V} \in \mathsf{span}(\mathsf{B}_n)$. Recall from Theorem 4.3 that B_ω is axiomatised by $\mathcal{J}(\mathbb{F}_2)$ and $\mathcal{J}(\mathbb{C}_3)$. By Jankov's Lemma 2.20 this implies $\mathsf{Up}(\mathbb{F}_2)^-$ and $\mathsf{Up}(\mathbb{C}_3)^-$ do not belong to B_ω. As $\mathsf{B}_n \subseteq \mathsf{B}_\omega$, we also have $\mathsf{Up}(\mathbb{F}_2)^-, \mathsf{Up}(\mathbb{C}_3)^- \notin \mathsf{B}_n$. Since $\mathsf{V} \in \mathsf{span}(\mathsf{B}_n)$, this implies that $\mathsf{Up}(\mathbb{F}_2)^-, \mathsf{Up}(\mathbb{C}_3)^- \notin \mathsf{V}$. From Jankov's Lemma 2.20 it follows that V validates $\mathcal{J}(\mathbb{F}_2)$ and $\mathcal{J}(\mathbb{C}_3)$. As these formulas axiomatise B_ω, we conclude that $\mathsf{V} \subseteq \mathsf{B}_\omega$. In view of Theorem 4.3, this implies

$$\mathsf{V} \in \{\mathsf{D}_0, \mathsf{D}_1, \mathsf{B}_1, \mathsf{B}_2, \dots, \mathsf{B}_\omega\}.$$

Therefore, in order to show that V belongs to the right hand side of the equality in (9), it suffices to prove that V is neither D_0 nor D_1. Recall from Corollary 4.5 that $\mathsf{Up}(\mathbb{C}_1)^-$ and $\mathsf{Up}(\mathbb{C}_2)^- \cong \boldsymbol{B}_1$ belong to B_n. Since $\mathsf{V} \in \mathsf{span}(\mathsf{B}_n)$, this implies $\mathsf{Up}(\mathbb{C}_1)^-, \mathsf{Up}(\mathbb{C}_2)^- \in \mathsf{V}$ as well. Therefore, V cannot be the trivial variety D_0. On the other hand, as V contains the three-element SI algebra $\mathsf{Up}(\mathbb{C}_2)^-$, we can apply Proposition 4.9(ii) obtaining that $\mathsf{V} \neq \mathsf{D}_1$ as desired.

Next we prove the inclusion from right to left in (9). We will detail only the proof that each B_m belongs to $\mathsf{span}(\mathsf{B}_n)$ as the proof that $\mathsf{B}_\omega \in \mathsf{span}(\mathsf{B}_n)$ is analogous. Let $m \in \mathbb{Z}^+$. By Corollary 4.5 we have

$$\mathsf{Fin}((\mathsf{B}_n)_{\mathrm{SI}}) = \mathbb{I}\{\mathsf{Up}(\mathbb{C}_1)^-, \boldsymbol{B}_1, \dots, \boldsymbol{B}_n\};$$
$$\mathsf{Fin}((\mathsf{B}_m)_{\mathrm{SI}}) = \mathbb{I}\{\mathsf{Up}(\mathbb{C}_1)^-, \boldsymbol{B}_1, \dots, \boldsymbol{B}_m\}.$$

Now, of the algebras $\mathsf{Up}(\mathbb{C}_1)^-, \boldsymbol{B}_1, \boldsymbol{B}_2, \dots$ only $\mathsf{Up}(\mathbb{C}_1)^-$ and $\boldsymbol{B}_1$ are of the form $\mathsf{Up}(\mathbb{X})^-$ for a finite rooted poset $\mathbb{X}$. This is because $\mathsf{Up}(\mathbb{C}_1)^-$ and $\boldsymbol{B}_1 \cong \mathsf{Up}(\mathbb{C}_2)^-$ are obviously of this form, while $\boldsymbol{B}_2, \dots, \boldsymbol{B}_n$ are not implicative reducts of Heyting algebras, since the posets underlying them fail to be lattices (see Figure 1, if necessary). Consequently, $\mathsf{Fin}((\mathsf{B}_n)_{\mathrm{SI}})$ and $\mathsf{Fin}((\mathsf{B}_m)_{\mathrm{SI}})$ have the same members of the form $\mathsf{Up}(\mathbb{X})^-$ for a finite rooted poset $\mathbb{X}$. As every Hilbert algebra of the form $\mathsf{Up}(\mathbb{X})^-$ for a finite rooted poset $\mathbb{X}$ is finite and SI (see Theorem 2.13), this implies that B_n and B_m have the same members of the form $\mathsf{Up}(\mathbb{X})^-$ for a finite rooted poset $\mathbb{X}$. By Proposition 5.7 we conclude that $\mathsf{B}_m \in \mathsf{span}(\mathsf{B}_n)$. ⊠

8. Varieties with degree of incompleteness $2^{\aleph_0}$

The aim of this section is to prove the following result.

Theorem 8.1. *Let V be a variety of Hilbert algebras. If $\mathsf{V} \neq \mathsf{Hil}$, $\mathsf{V} \neq \mathsf{D}_n$ for every $n \in \mathbb{N}$, and $\mathsf{V} \nsubseteq \mathsf{B}_\omega$, then $\mathsf{deg}^*(\mathsf{V}) = 2^{\aleph_0}$.*

This will conclude the proof of Theorem 5.12, as we proceed to explain. Recall that condition (i) of Theorem 5.12 holds by Theorem 6.1. Moreover, the implication from right to left of condition (ii) of Theorem 5.12 holds by Proposition 7.1. To prove the implication from left to right of the latter condition, consider a variety V of Hilbert algebras with $\mathsf{deg}^*(\mathsf{V}) = \aleph_0$. By Theorem 8.1 either $\mathsf{V} = \mathsf{Hil}$, $\mathsf{V} = \mathsf{D}_n$ for some $n \in \mathbb{N}$, or $\mathsf{V} \subseteq \mathsf{B}_\omega$. As $\mathsf{deg}^*(\mathsf{V}) \neq 1$ by assumption, we can apply Theorem 5.12(i), obtaining $\mathsf{V} \subseteq \mathsf{B}_\omega$ and $\mathsf{V} \neq \mathsf{D}_n$ for each $n \in \mathbb{N}$. Together with Theorem 4.3, this implies $\mathsf{V} \in \{\mathsf{B}_1, \mathsf{B}_2, \dots, \mathsf{B}_\omega\}$ as desired. Therefore, it only remains to prove condition (iii) of Theorem 5.12. Consider a variety V of Hilbert algebras different from Hil, different from any D_n and B_n, and different from B_ω. By Theorem 4.3 we have $\mathsf{V} \nsubseteq \mathsf{B}_\omega$. Since $\mathsf{V} \neq \mathsf{Hil}$ and $\mathsf{V} \neq \mathsf{D}_n$ for each $n \in \mathbb{N}$, we can apply Theorem 8.1, obtaining $\mathsf{deg}^*(\mathsf{V}) = 2^{\aleph_0}$ as desired. Notice that, by concluding the proof of Theorem 5.12, this also establishes the Trichotomy Theorem 5.4.

The rest of this section is devoted to the proof of Theorem 8.1. We begin with a technical observation.

Lemma 8.2. *Let V be a variety of Hilbert algebras such that $\mathsf{deg}^*(\mathsf{V}) \neq 1$. Then there exists a finite rooted poset $\mathbb{X}$ satisfying the following conditions:*

(i) $\mathsf{Up}(\mathbb{X})^- \notin \mathsf{V}$ *and for every finite rooted poset $\mathbb{Y}$,*
$$\mathsf{Up}(\mathbb{Y})^- \sqsubset \mathsf{Up}(\mathbb{X})^- \textit{ implies } \mathsf{Up}(\mathbb{Y})^- \in \mathsf{V};$$
(ii) $\mathbb{X}$ *is not a chain.*

Proof. Let
$$P := \{\mathbb{X} : \mathbb{X} \text{ is a finite rooted poset such that } \mathsf{Up}(\mathbb{X})^- \notin \mathsf{V}\};$$
$$\mathsf{K} := \{\mathsf{Up}(\mathbb{X})^- : \mathbb{X} \in P\}.$$
Observe that $\mathsf{K} \subseteq \mathsf{Fin}(\mathsf{Hil}_{\mathrm{SI}})$ by Theorem 2.13. Then $({\uparrow}\mathsf{K})^c$ is a downset of $\mathsf{Fin}(\mathsf{Hil}_{\mathrm{SI}})$. Furthermore, the definition of K, the fact that $\mathsf{Up}(\mathbb{X})^-$ is a finite SI Hilbert algebra for every finite rooted poset $\mathbb{X}$ (see Theorem 2.13), and the assumption that V is a variety guarantee that for every finite rooted poset $\mathbb{X}$,
$$\mathsf{Up}(\mathbb{X})^- \in \mathsf{V} \text{ if and only if } \mathsf{Up}(\mathbb{X})^- \in ({\uparrow}\mathsf{K})^c.$$

Therefore, $(\uparrow\mathsf{K})^c \in \mathsf{span}^*(\mathsf{V})$. In turn, this implies

$$\mathsf{deg}^*(\mathsf{V}) = \mathsf{deg}^*(\kappa^{-1}(\uparrow\mathsf{K})^c).$$

Now, from the fact that the poset associated with the preordered class $\mathsf{Fin}(\mathsf{Hil}_{\mathrm{SI}})$ does not have infinite descending chains (because $\mathsf{Fin}(\mathsf{Hil}_{\mathrm{SI}})$ is a class of finite algebras preordered under the embeddability relation), and from the definition of K it follows that $(\uparrow\mathsf{K})^c$ is the largest downset of $\mathsf{Fin}(\mathsf{Hil}_{\mathrm{SI}})$ that omits every member of the class

$$\{\mathsf{Up}(\mathbb{X})^- : \mathbb{X} \text{ is a minimal element of } \langle P; \sqsubseteq\rangle\}.$$

By Theorem 2.19(ii) this means that $\kappa^{-1}((\uparrow\mathsf{K})^c)$ is the largest variety of Hilbert algebras omitting the algebras in the class in the above display. Thus, the last part of Jankov's Lemma 2.20 implies that $\kappa^{-1}((\uparrow\mathsf{K})^c)$ is axiomatised by

$$\Sigma \coloneqq \{\mathcal{J}(\mathbb{X}) : \mathbb{X} \text{ is a minimal element of } \langle P; \sqsubseteq\rangle\}.$$

We will prove that there exists a minimal element $\mathbb{X}$ of $\langle P; \sqsubseteq\rangle$ that is not a chain. Suppose the contrary, with a view to contradiction. We have two cases: either $\langle P; \sqsubseteq\rangle$ lacks minimal elements or not. Suppose first that P lacks minimal elements. Then $P = \emptyset$ and, consequently, $\Sigma = \emptyset$. Thus, $\kappa^{-1}((\uparrow\mathsf{K})^c)$ is the variety of Hilbert algebras axiomatised by $\emptyset$, that is, Hil. Then $\mathsf{deg}^*(\kappa^{-1}((\uparrow\mathsf{K})^c)) = 1$ by Theorem 6.1. Since $\mathsf{deg}^*(\mathsf{V}) = \mathsf{deg}^*(\kappa^{-1}((\uparrow\mathsf{K})^c))$, this implies $\mathsf{deg}^*(\mathsf{V}) = 1$, a contradiction with the assumptions. Then we consider the case where $\langle P; \sqsubseteq\rangle$ has minimal elements. Since every minimal element of $\langle P; \sqsubseteq\rangle$ is a chain by assumption and the minimal elements of $\langle P; \sqsubseteq\rangle$ are nonempty (because they must be rooted), there exists a nonempty $I \subseteq \mathbb{Z}^+$ such that the variety $\kappa^{-1}((\uparrow\mathsf{K})^c)$ is axiomatised by the Jankov formulas in $\{\mathcal{J}(\mathbb{C}_n) : n \in I\}$. Since $I \neq \emptyset$, we may consider $m \coloneqq \min(I)$. As Jankov's Lemma 2.20 guarantees that every variety validating $\mathcal{J}(\mathbb{C}_m)$ validates $\mathcal{J}(\mathbb{C}_k)$ for every $k \geqslant m$, we obtain that $\kappa^{-1}((\uparrow\mathsf{K})^c)$ is axiomatised by $\mathcal{J}(\mathbb{C}_m)$. Therefore, $\kappa^{-1}((\uparrow\mathsf{K})^c) = \mathsf{D}_{m-1}$ by Theorem 3.5. Hence, we conclude that $\mathsf{deg}^*(\kappa^{-1}((\uparrow\mathsf{K})^c)) = 1$ by Theorem 6.1. But this implies $\mathsf{deg}^*(\mathsf{V}) = 1$, a contradiction with the assumptions. Hence, we conclude that there exists a minimal element $\mathbb{X}$ of $\langle P; \sqsubseteq\rangle$ that is not a chain.

Since $\mathbb{X}$ is a minimal element of $\langle P; \sqsubseteq\rangle$, it satisfies (i) in the statement, while (ii) holds because $\mathbb{X}$ is not a chain. ⊠

We will make use of the following construction.

Definition 8.3. Given a rooted poset $\mathbb{X}$, let $\mathsf{d}(\mathbb{X})$ be the poset obtained by adding a new maximum to the order dual of $\mathbb{X}$.

Notice that if $\mathbb{X}$ is a finite rooted poset, then $\mathsf{d}(\mathbb{X})$ is a finite poset with a maximum and a second largest element. Therefore, the Hilbert algebra $\mathsf{H}(\mathsf{d}(\mathbb{X}))$ is finite and SI by Theorem 2.13.

Recall from Proposition 6.3 that for every poset $\mathbb{X}$ the set of elements of $\mathsf{Up}(\mathbb{X})$ that are either meet irreducible or equal to X is the universe of an implicative subreduct of

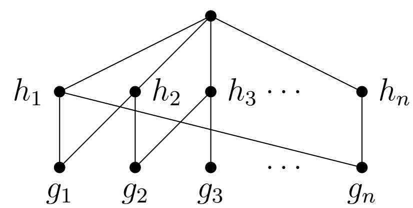

FIGURE 3. The poset $\mathbb{G}_n$.

$\mathsf{Up}(\mathbb{X})$ denoted by $\mathsf{M}(\mathbb{X})$. Furthermore, $\mathsf{M}(\mathbb{X})$ is an algebra of the form $\mathsf{H}(\mathbb{Y})$, where $\mathbb{Y}$ is the subposet of $\mathsf{Up}(\mathbb{X})$ corresponding to the universe of $\mathsf{M}(\mathbb{X})$.

Proposition 8.4. *Let $\mathbb{X}$ be a finite rooted poset. Then $\mathsf{H}(\mathsf{d}(\mathbb{X})) \cong \mathsf{M}(\mathbb{X})$.*

Proof. Let $\mathbb{Y}$ be the subposet of $\mathsf{Up}(\mathbb{X})$ corresponding to the universe of $\mathsf{M}(\mathbb{X})$. Since $\mathsf{M}(\mathbb{X}) = \mathsf{H}(\mathbb{Y})$, it suffices to show that the posets $\mathsf{d}(\mathbb{X})$ and $\mathbb{Y}$ are isomorphic. To this end, observe that set of the meet irreducible elements of $\mathsf{Up}(\mathbb{X})$ is $\{X - {\downarrow}x : x \in X\}$. Therefore, $\mathbb{Y}$ is the poset that has universe $\{X\} \cup \{X - {\downarrow}x : x \in X\}$ and is ordered under the inclusion relation. Clearly, this poset is isomorphic to $\mathsf{d}(\mathbb{X})$. ⊠

We will also make use of the following structures.

Definition 8.5. For every integer $n \geqslant 2$ we denote the poset depicted in Figure 3 by $\mathbb{G}_n$.

Notice that each $\mathbb{G}_n$ is a join semilattice whose principal upsets are Boolean lattices. In view of Proposition 4.9(i), there exists a unique Tarski algebra $\boldsymbol{G}_n$ whose underlying poset is $\mathbb{G}_n$. Recall that, given a Hilbert algebra $\boldsymbol{A}$, the notation $\boldsymbol{A}^\top$ refers to the Hilbert algebra obtained by adding a new maximum $\top$ to $\boldsymbol{A}$, and defining the implication as explain in (4).

Definition 8.6. For every integer $n \geqslant 2$ let $\boldsymbol{H}_n$ be the Hilbert algebra $\boldsymbol{G}_n^\top$.

The following result holds by a straightforward inspection.

Proposition 8.7. *Let $n, m \geqslant 2$ be distinct integers. Then $\mathbb{G}_n$ does not order embed into $\mathbb{G}_m$.*

Proposition 8.8. *Let $n, m \geqslant 2$ be distinct integers. Then $\boldsymbol{H}_n$ does not embed into $\boldsymbol{H}_m$.*

Proof. Suppose the contrary, with a view to contradiction. Then the poset underlying $\boldsymbol{H}_n$ order embeds into the one underlying $\boldsymbol{H}_m$. This means that $\mathbb{G}_n$ order embeds into $\mathbb{G}_m$, a contradiction with Proposition 8.8. ⊠

Notice each $\boldsymbol{H}_n$ is a Hilbert algebra with a second largest element and, therefore, it is SI by Theorem 2.13. Furthermore, it is finite by definition. Therefore, $\boldsymbol{H}_n \in \mathsf{Fin}(\mathsf{Hil}_{\mathrm{SI}})$. Consequently, from Proposition 8.8 we deduce the following.

Corollary 8.9. *Let $n, m \geqslant 2$ be distinct integers. Then $\boldsymbol{H}_n$ and $\boldsymbol{H}_m$ are incomparable in $\mathsf{Fin}(\mathsf{Hil}_{\mathrm{SI}})$.*

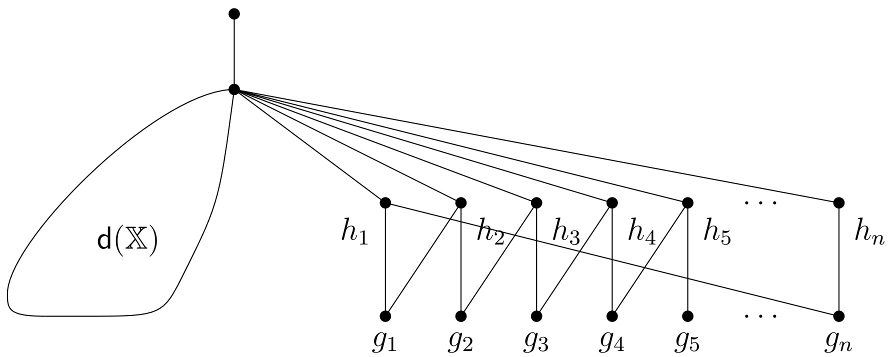

FIGURE 4. The poset $\mathbb{Y}_n$.

Recall that $\mathbb{F}_n$ is the poset depicted in Figure 2.

Proposition 8.10. *Let* $\boldsymbol{A} \in \mathsf{Fin}(\mathsf{HA}_{\text{SI}})$ *and assume that* $\boldsymbol{A}^- \sqsubseteq \boldsymbol{H}_n$ *for some integer* $n \geqslant 2$. *Then*

$$\boldsymbol{A} \in \mathbb{I}\{\mathsf{Up}(\mathbb{C}_1), \mathsf{Up}(\mathbb{C}_2), \mathsf{Up}(\mathbb{F}_2)\}.$$

Proof. We may assume that $\boldsymbol{A}^-$ is a subalgebra of $\boldsymbol{H}_n$. Then the poset $\langle A; \leqslant \rangle$ underlying $\boldsymbol{A}$ is a subposet of the poset $\langle H_n; \leqslant \rangle$ underlying $\boldsymbol{H}_n$, where the latter is the poset obtained by adding a new maximum to $\mathbb{G}_n$. Furthermore, $\langle A; \leqslant \rangle$ is a nontrivial lattice because $\boldsymbol{A}$ is a SI Heyting algebra by assumption. Now, the only nontrivial subposets of $\langle H_n; \leqslant \rangle$ that are lattices are those that are isomorphic to the poset underlying the four-element Boolean algebra $\boldsymbol{B}_4$ or to some of the posets $\mathsf{Up}(\mathbb{C}_1), \mathsf{Up}(\mathbb{C}_2), \mathsf{Up}(\mathbb{C}_3)$, and $\mathsf{Up}(\mathbb{F}_2)$. As Heyting algebras are uniquely determined by their underlying posets, this implies

$$\boldsymbol{A} \in \mathbb{I}\{\boldsymbol{B}_4, \mathsf{Up}(\mathbb{C}_1), \mathsf{Up}(\mathbb{C}_2), \mathsf{Up}(\mathbb{C}_3), \mathsf{Up}(\mathbb{F}_2)\}.$$

A simple inspection shows that $\mathsf{Up}(\mathbb{C}_3)^-$ does not embed into $\boldsymbol{H}_n$. Since $\boldsymbol{A}^-$ is a subalgebra of $\boldsymbol{H}_n$, this yields that $\boldsymbol{A}$ is not isomorphic to $\mathsf{Up}(\mathbb{C}_3)$. In addition, $\boldsymbol{A}$ has a second largest element because it is SI (see Theorem 2.13). Consequently, $\boldsymbol{A}$ is not isomorphic to $\boldsymbol{B}_4$. Together with the above display, this yields $\boldsymbol{A} \in \mathbb{I}\{\mathsf{Up}(\mathbb{C}_1), \mathsf{Up}(\mathbb{C}_2), \mathsf{Up}(\mathbb{F}_2)\}$. ⊠

The proof of Theorem 8.1 is split in two halves. The first is the next result.

Proposition 8.11. *Let* V *be a variety of Hilbert algebras. If* $\mathsf{Up}(\mathbb{C}_3)^- \in \mathsf{V}$ *and* $\mathsf{deg}^*(\mathsf{V}) \neq 1$, *then* $\mathsf{deg}^*(\mathsf{V}) = 2^{\aleph_0}$.

Proof. Assume that $\mathsf{Up}(\mathbb{C}_3)^- \in \mathsf{V}$ and $\mathsf{deg}^*(\mathsf{V}) \neq 1$. Then let $\mathbb{X}$ be the finite rooted poset given by Lemma 8.2. Recall that the poset $\mathsf{d}(\mathbb{X})$ has a second largest element. Then for each integer $n \geqslant |X|$ let $\mathbb{Y}_n$ be the finite poset obtained by taking the disjoint union of the posets $\mathsf{d}(\mathbb{X})$ and $\mathbb{G}_n$ and gluing the the second largest element of $\mathsf{d}(\mathbb{X})$ with the maximum of $\mathbb{G}_n$. A pictorial rendering of $\mathbb{Y}_n$ is given in Figure 4. Furthermore, let $\boldsymbol{A}_n \coloneqq \mathsf{H}(\mathbb{Y}_n)$. Observe that $\boldsymbol{A}_n$ is finite and SI, the latter because it has a second largest element (see Theorem 2.13). Therefore, $\boldsymbol{A}_n \in \mathsf{Fin}(\mathsf{Hil}_{\text{SI}})$ for each $n \geqslant |X|$.

Claim 8.12. *The members of* $\{\boldsymbol{A}_n : n \geqslant |X|\}$ *are all incomparable in* $\mathsf{Fin}(\mathsf{Hil}_{\text{SI}})$.

Proof of the Claim. Consider two integers $m > n \geqslant |X|$. We need to prove that neither $\boldsymbol{A}_m$ embeds into $\boldsymbol{A}_n$ nor $\boldsymbol{A}_n$ embeds into $\boldsymbol{A}_m$. On the one hand, $\boldsymbol{A}_m$ cannot embed into $\boldsymbol{A}_n$ on cardinality grounds because $n < m$ implies $|A_n| < |A_m|$. Next we prove that $\boldsymbol{A}_n$ does not embed into $\boldsymbol{A}_m$. Suppose, with a view to contradiction, that there exists an embedding $f\colon \boldsymbol{A}_n \to \boldsymbol{A}_m$. We will use repeatedly the fact that the posets underlying $\boldsymbol{A}_n$ and $\boldsymbol{A}_m$ are $\mathbb{Y}_n$ and $\mathbb{Y}_m$, respectively. Since f is an embedding, we have $f(1) = 1$ and $f(k^n) \leqslant k^m$, where k^n and k^m are the second largest elements of $\mathbb{Y}_n$ and $\mathbb{Y}_m$, respectively. Therefore, $f[A_n - \{1, k^n\}] \subseteq A_m - \{1, k^m\}$. In particular, f restricts to an order embedding $f^*\colon (G_n - \{k^n\}) \to (A_m - \{1, k^m\})$. Now, the subposet of $\mathbb{Y}_m$ with universe $(A_m - \{1, k^m\})$ is the disjoint union of the poset $G_m - \{k^m\}$ with the poset $\mathbb{X}^*$ obtained by removing the maximum from the order dual of $\mathbb{X}$. Since every pair of elements of $G_n - \{k^n\}$ is connected by a zig-zag (see Figure 3), this implies that f^* can be viewed as an order embedding of $G_n - \{k^n\}$ into either $G_m - \{k^m\}$ or $\mathbb{X}^*$. As $|X| \leqslant n$, the definition of $\mathbb{G}_n$ guarantees that $|X^*| = |X| - 1 < n \leqslant |G_n - \{k^n\}|$. Hence, we conclude that f^* order embeds $G_n - \{k^n\}$ into $G_m - \{k^m\}$. By the definition of $\mathbb{G}_n$ into $\mathbb{G}_m$ this guarantees that $\mathbb{G}_n$ order embeds into $\mathbb{G}_m$, a contradiction with Proposition 8.7. ⊠

We will prove that $\mathsf{H}(\mathsf{d}(\mathbb{X})) \sqsubseteq \boldsymbol{A}_n$ for every $n \geqslant |X|$. First, recall that $\mathsf{d}(\mathbb{X})$ is a subposet of $\mathbb{Y}_n$ containing the maximum of $\mathbb{Y}_n$ by the definition of $\mathbb{Y}_n$. Together with $\boldsymbol{A} = \mathsf{H}(\mathbb{Y}_n)$ and Proposition 2.6, this ensures $\mathsf{H}(\mathsf{d}(\mathbb{X})) \sqsubseteq \boldsymbol{A}_n$.

Next we prove that

$$\begin{gathered}\text{for every finite rooted poset } \mathbb{Y} \text{ and } n \geqslant |\mathbb{X}|,\\ \text{if } \boldsymbol{A}_n \sqsubseteq \mathsf{Up}(\mathbb{Y})^-, \text{ then } \mathsf{Up}(\mathbb{Y})^- \notin \mathsf{V}.\end{gathered} \tag{10}$$

Consider a finite rooted poset $\mathbb{Y}$ and $n \geqslant |X|$ such that $\boldsymbol{A}_n \sqsubseteq \mathsf{Up}(\mathbb{Y})^-$. Since $\mathsf{H}(\mathsf{d}(\mathbb{X})) \sqsubseteq \boldsymbol{A}_n$ and $\mathsf{H}(\mathsf{d}(\mathbb{X})) \cong \mathsf{M}(\mathbb{X})$ (the latter by Proposition 8.4), we have $\mathsf{M}(\mathbb{X}) \sqsubseteq \boldsymbol{A}_n$. Together with $\boldsymbol{A}_n \sqsubseteq \mathsf{Up}(\mathbb{Y})^-$, this yields $\mathsf{M}(\mathbb{X}) \sqsubseteq \mathsf{Up}(\mathbb{Y})^-$. Consequently, the Embedding Lemma 6.4 ensures $\mathsf{Up}(\mathbb{X})^- \sqsubseteq \mathsf{Up}(\mathbb{Y})^-$. As $\mathbb{X}$ satisfies condition (i) of Lemma 8.2, we have $\mathsf{Up}(\mathbb{X})^- \notin \mathsf{V}$. Since $\mathsf{Up}(\mathbb{X})^- \sqsubseteq \mathsf{Up}(\mathbb{Y})^-$, we conclude that $\mathsf{Up}(\mathbb{Y})^- \notin \mathsf{V}$. This establishes (10).

Now, for every $K \subseteq \{n \in \mathbb{Z}^+ : n \geqslant |\mathbb{X}|\}$ we define a downset D_K of $\mathsf{Fin}(\mathsf{Hil}_{\text{SI}})$ as the complement in $\mathsf{Fin}(\mathsf{Hil}_{\text{SI}})$ of the upset

$$\uparrow \left(\{\boldsymbol{A}_n : n \in K\} \cup \{\mathsf{Up}(\mathbb{Y})^- : \mathbb{Y} \text{ is a finite rooted poset and } \mathsf{Up}(\mathbb{Y})^- \notin \mathsf{V}\}\right).$$

To prove that $\mathsf{deg}^*(\mathsf{V}) = 2^{\aleph_0}$, it suffices to show that $D_K \in \mathsf{span}^*(\mathsf{V})$ for every K, and that $D_{K_1} \neq D_{K_2}$ whenever $K_1 \neq K_2$.

Claim 8.13. *For every* $K \subseteq \{n \in \mathbb{Z}^+ : n \geqslant |X|\}$ *we have* $D_K \in \mathsf{span}^*(\mathsf{V})$.

Proof of the Claim. Let $K \subseteq \{n \in \mathbb{Z}^+ : n \geqslant |X|\}$ and some $\mathsf{Up}(\mathbb{P})^-$. In order to prove that $D_K \in \mathsf{span}^*(\mathsf{V})$, it suffices to show that for every finite rooted poset $\mathbb{P}$,

$$\mathsf{Up}(\mathbb{P})^- \in D_K \text{ if and only if } \mathsf{Up}(\mathbb{P})^- \in \mathsf{V}.$$

Consider a finite rooted poset $\mathbb{P}$. The definition of D_K ensures that $\mathsf{Up}(\mathbb{P})^- \in D_K$ if and only if $\boldsymbol{A}_n \not\sqsubseteq \mathsf{Up}(\mathbb{P})^-$ and $\mathsf{Up}(\mathbb{Y})^- \not\sqsubseteq \mathsf{Up}(\mathbb{P})^-$ for every $n \in K$ and finite rooted poset $\mathbb{Y}$ such that $\mathsf{Up}(\mathbb{Y})^- \notin \mathsf{V}$. Consequently, from $\mathsf{Up}(\mathbb{P})^- \in D_K$ it follows that $\mathsf{Up}(\mathbb{P})^- \in \mathsf{V}$ (otherwise we would obtain a contradiction taking $\mathbb{Y} = \mathbb{P}$). Therefore, it only remains to prove the implication from right to left in the above display. To this end, we reason by contraposition. Suppose $\mathsf{Up}(\mathbb{P})^- \notin D_K$. Then either $\boldsymbol{A}_n \sqsubseteq \mathsf{Up}(\mathbb{P})^-$ for some $n \in K$ or $\mathsf{Up}(\mathbb{Y})^- \sqsubseteq \mathsf{Up}(\mathbb{P})^-$ for some finite rooted poset $\mathbb{Y}$ such that $\mathsf{Up}(\mathbb{Y})^- \notin \mathsf{V}$. In the former case, from (10) it follows that $\mathsf{Up}(\mathbb{P})^- \notin \mathsf{V}$ as desired. In the latter case, we obtain $\mathsf{Up}(\mathbb{P})^- \notin \mathsf{V}$ as well because $\mathsf{Up}(\mathbb{Y})^- \notin \mathsf{V}$. ⊠

To conclude the proof, it only remains to show that $D_{K_1} \neq D_{K_2}$ whenever $K_1 \neq K_2$. To this end, let K_1 and K_2 be distinct subsets of $\{n \in \mathbb{Z}^+ : n \geqslant |X|\}$. By symmetry we may assume that there exists $k \in K_1 - K_2$. Since $\boldsymbol{A}_k \sqsubseteq \boldsymbol{A}_k$ and $k \in K_1$, the definition D_{K_1} guarantees $\boldsymbol{A}_k \notin D_{K_1}$. Therefore, to conclude that $D_{K_1} \neq D_{K_2}$, it suffices to show that $\boldsymbol{A}_k \in D_{K_2}$. By the definition of D_{K_2} the latter amounts to the demand that $\boldsymbol{A}_n \not\sqsubseteq \boldsymbol{A}_k$ and $\mathsf{Up}(\mathbb{Y})^- \not\sqsubseteq \boldsymbol{A}_k$ for every $n \in K_2$ and finite rooted poset $\mathbb{Y}$ such that $\mathsf{Up}(\mathbb{Y})^- \notin \mathsf{V}$. From Claim 8.12 and $k \notin K_2$ it follows that $\boldsymbol{A}_n \not\sqsubseteq \boldsymbol{A}_k$ for every $n \in K_2$. Therefore, it only remains to show that $\mathsf{Up}(\mathbb{Y})^- \not\sqsubseteq \boldsymbol{A}_k$ for every finite rooted poset $\mathbb{Y}$ such that $\mathsf{Up}(\mathbb{Y})^- \notin \mathsf{V}$.

Suppose, with a view to contradiction, that there exists a finite rooted poset $\mathbb{Y}$ such that $\mathsf{Up}(\mathbb{Y})^- \sqsubseteq \boldsymbol{A}_k$ and $\mathsf{Up}(\mathbb{Y})^- \notin \mathsf{V}$. As $\boldsymbol{A}_k = \mathsf{H}(\mathbb{Y}_k)$ by definition and $\mathsf{Up}(\mathbb{Y})^-$ is the implicative reduct of a Heyting algebra, from Proposition 2.8 it follows that there exists $m \in \mathbb{Z}^+$ such that $\mathbb{Y} \cong \mathbb{C}_m$. For the sake of readability, we will assume that $\mathbb{Y} = \mathbb{C}_m$. So, $\mathsf{Up}(\mathbb{C}_m) \sqsubseteq \boldsymbol{A}_k$ and $\mathsf{Up}(\mathbb{C}_m) \notin \mathsf{V}$.

Recall that $\mathsf{Up}(\mathbb{C}_3)^- \in \mathsf{V}$ by assumption. Together with $\mathsf{Up}(\mathbb{C}_m)^- \notin \mathsf{V}$, this yields $m > 3$. Moreover, recall that $\mathsf{Up}(\mathbb{C}_m)^- \sqsubseteq \boldsymbol{A}_k = \mathsf{H}(\mathbb{Y}_k)$. Consequently, there exists an embedding $f\colon \mathsf{Up}(\mathbb{C}_m)^- \to \mathsf{H}(\mathbb{Y}_k)$. Since $\mathsf{Up}(\mathbb{C}_m)^-$ is chain of $\geqslant 5$ elements (because $m > 3$), the definition of $\mathbb{Y}_k$ guarantees that $f[\mathsf{Up}(\mathbb{C}_m)^-] \subseteq \mathsf{d}(\mathbb{X})$ (see Figure 4 if necessary). Thus, f can be viewed as an order embedding $f\colon \mathsf{Up}(\mathbb{C}_m)^- \to \mathsf{d}(\mathbb{X})$. Since $\mathsf{Up}(\mathbb{C}_m)^-$ is an $(m+1)$-element chain and $\mathsf{d}(\mathbb{X})$ is the poset obtained adding a new maximum to the order dual of $\mathbb{X}$, this implies that $\mathbb{X}$ contains an m-element chain. In turn, this yields $\mathsf{Up}(\mathbb{C}_m) \sqsubseteq \mathsf{Up}(\mathbb{X})$ by Proposition 3.3 and, therefore, $\mathsf{Up}(\mathbb{C}_m)^- \sqsubseteq \mathsf{Up}(\mathbb{X})^-$. As $\mathsf{Up}(\mathbb{X})^-$ is not a chain (because $\mathbb{X}$ satisfies condition (ii) of Lemma 8.2 by assumption) while $\mathsf{Up}(\mathbb{C}_m)^-$ is so, we get $\mathsf{Up}(\mathbb{C}_m)^- \sqsubset \mathsf{Up}(\mathbb{X})^-$. Therefore, the fact that $\mathbb{X}$ satisfies condition (i) of Lemma 8.2 ensures $\mathsf{Up}(\mathbb{C}_m)^- \in \mathsf{V}$, a contradiction. ⊠

To establish Theorem 8.1, we need to prove one last result.

Proposition 8.14. *Let* V *be a variety of Hilbert algebras. If* $\mathsf{Up}(\mathbb{F}_2)^- \in \mathsf{V}$ *and* $\mathsf{deg}^*(\mathsf{V}) \neq 1$, *then* $\mathsf{deg}^*(\mathsf{V}) = 2^{\aleph_0}$.

Proof. Suppose that $\mathsf{Up}(\mathbb{F}_2)^- \in \mathsf{V}$ and $\mathsf{deg}^*(\mathsf{V}) \neq 1$. We need to prove that $\mathsf{deg}^*(\mathsf{V}) = 2^{\aleph_0}$. To this end, we may also assume that $\mathsf{Up}(\mathbb{C}_3)^- \notin \mathsf{V}$, otherwise we are done by Proposition 8.14. Lastly, let $\mathbb{X}$ be the finite rooted poset given by Lemma 8.2.

For every $K \subseteq \{k \in \mathbb{Z}^+ : 2^{|X|} \leqslant 2k+2\}$ we consider the following downset of $\mathsf{Fin}(\mathsf{Hil}_{\mathrm{SI}})$:

$$D_K \coloneqq {\downarrow}\left(\{\mathsf{Up}(\mathbb{Y})^- : \mathbb{Y} \text{ is a finite rooted poset such that } \mathsf{Up}(\mathbb{Y})^- \in \mathsf{V}\} \cup \{\boldsymbol{H}_k : k \in K\}\right).$$

To prove that $\mathsf{deg}^*(\mathsf{V}) = 2^{\aleph_0}$, it suffices to show that $D_K \in \mathsf{span}^*(\mathsf{V})$ for every $K \subseteq \{k \in \mathbb{Z}^+ : 2^{|X|} \leqslant 2k+2\}$, and that $D_{K_1} \neq D_{K_2}$ whenever $K_1 \neq K_2$.

Claim 8.15. *For every $K \subseteq \{k \in \mathbb{Z}^+ : 2^{|X|} \leqslant 2k+2\}$ we have $D_K \in \mathsf{span}^*(\mathsf{V})$.*

Proof of the Claim. Let $K \subseteq \{k \in \mathbb{Z}^+ : 2^{|X|} \leqslant 2k+2\}$. In order to prove that $D_K \in \mathsf{span}^*(\mathsf{V})$, it suffices to show that for every finite rooted poset $\mathbb{Y}$,

$$\mathsf{Up}(\mathbb{Y})^- \in D_K \text{ if and only if } \mathsf{Up}(\mathbb{Y})^- \in \mathsf{V}.$$

The implication from right to left holds by definition of D_K. To prove the other implication, let $\mathbb{Y}$ be a finite rooted poset such that $\mathsf{Up}(\mathbb{Y})^- \in D_K$. By the definition of D_K we have that either $\mathsf{Up}(\mathbb{Y})^- \sqsubseteq \mathsf{Up}(\mathbb{P})^-$ for some finite rooted poset $\mathbb{P}$ such that $\mathsf{Up}(\mathbb{P})^- \in \mathsf{V}$ or $\mathsf{Up}(\mathbb{Y})^- \sqsubseteq \boldsymbol{H}_k$ for some $k \in K$. The former case immediately implies $\mathsf{Up}(\mathbb{Y})^- \in \mathsf{V}$ as desired. In the latter case, Proposition 8.10 ensures that

$$\mathsf{Up}(\mathbb{Y})^- \in \mathbb{I}\{\mathsf{Up}(\mathbb{C}_1)^-, \mathsf{Up}(\mathbb{C}_2)^-, \mathsf{Up}(\mathbb{F}_2)^-\}.$$

As $\mathsf{Up}(\mathbb{F}_2)^- \in \mathsf{V}$ by assumption and both $\mathsf{Up}(\mathbb{C}_1)^-$ and $\mathsf{Up}(\mathbb{C}_2)^-$ embed into $\mathsf{Up}(\mathbb{F}_2)^-$, we obtain $\mathsf{Up}(\mathbb{C}_1)^-, \mathsf{Up}(\mathbb{C}_2)^-, \mathsf{Up}(\mathbb{F}_2)^- \in \mathsf{V}$. Together with the above display, this yields $\mathsf{Up}(\mathbb{Y})^- \in \mathsf{V}$. ⊠

To conclude the proof, it only remains to prove that $D_{K_1} \neq D_{K_2}$ whenever $K_1 \neq K_2$. To this end, let K_1 and K_2 be distinct subsets of $\{k \in \mathbb{Z}^+ : 2^{|X|} \leqslant 2k+2\}$. By symmetry we may assume that there exists $k \in K_1 - K_2$. As $k \in K_1$, we definition of D_{K_1} guarantees that $\boldsymbol{H}_k \in D_{K_1}$. Therefore, to conclude that $D_{K_1} \neq D_{K_2}$, it suffices to show that $\boldsymbol{H}_k \notin D_{K_2}$. By the definition of D_{K_2} the latter amounts to the demand that $\boldsymbol{H}_k \not\sqsubseteq \boldsymbol{H}_n$ and $\boldsymbol{H}_k \not\sqsubseteq \mathsf{Up}(\mathbb{Y})^-$ for every $n \in K_2$ and finite rooted poset $\mathbb{Y}$ such that $\mathsf{Up}(\mathbb{Y})^- \in \mathsf{V}$. From Corollary 8.9 and $k \notin K_2$ it follows that $\boldsymbol{H}_k \not\sqsubseteq \boldsymbol{H}_n$ for every $n \in K_2$. Therefore, it only remains to show that $\boldsymbol{H}_k \not\sqsubseteq \mathsf{Up}(\mathbb{Y})^-$ for every finite rooted poset $\mathbb{Y}$ such that $\mathsf{Up}(\mathbb{Y})^- \in \mathsf{V}$.

Suppose, with a view to contradiction, that there exists a finite rooted poset $\mathbb{Y}$ such that $\boldsymbol{H}_k \sqsubseteq \mathsf{Up}(\mathbb{Y})^-$ and $\mathsf{Up}(\mathbb{Y})^- \in \mathsf{V}$.

Claim 8.16. *There exist $n, m \geqslant 2$ such that $\mathbb{X} \cong \mathbb{F}_n$ and $\mathbb{Y} \cong \mathbb{F}_m$.*

Proof of the Claim. Recall that $\mathsf{Up}(\mathbb{C}_3)^- \notin \mathsf{V}$ by assumption. As $\mathsf{Up}(\mathbb{Y})^- \in \mathsf{V}$ and $\mathbb{X}$ satisfies condition (i) of Lemma 8.2, this yields that $\mathsf{Up}(\mathbb{C}_3)^-$ does not embed in $\mathsf{Up}(\mathbb{X})^-$ or $\mathsf{Up}(\mathbb{Y})^-$. Consequently, $\mathbb{X}$ and $\mathbb{Y}$ are posets of depth $\leqslant 2$ by Proposition 3.3. As they are finite and rooted, this means that $\mathbb{X}$ and $\mathbb{Y}$ are isomorphic some of the following posets: $\mathbb{C}_1$, $\mathbb{C}_2$, or $\mathbb{F}_m$ for some $m \geqslant 2$. Since $\mathbb{X}$ is not a chain by condition (ii) of Lemma 8.2, we obtain $\mathbb{X} \cong \mathbb{F}_n$ for some $n \geqslant 2$. Moreover, from the assumption that $\boldsymbol{H}_k \sqsubseteq \mathsf{Up}(\mathbb{Y})^-$ and the fact that $\boldsymbol{H}_k$ does not embed into $\mathsf{Up}(\mathbb{C}_1)$ and $\mathsf{Up}(\mathbb{C}_2)$ on cardinality grounds (see Figure 3, if necessary), we conclude that $\mathbb{Y} \cong \mathbb{F}_m$ for some $m \geqslant 2$. ⊠

In view of Claim 8.16, we may assume that $\mathbb{X} = \mathbb{F}_n$ and $\mathbb{Y} = \mathbb{F}_m$ for some $n, m \geqslant 2$. Therefore, $|X| = n+1$ and $|Y| = m+1$. Recall from the definition of $\boldsymbol{H}_k$ that $|H_k| = 2k+2$. So, from the assumption that $\boldsymbol{H}_k \sqsubseteq \mathsf{Up}(\mathbb{Y})^-$ it follows that $2k+2 \leqslant 2^{|Y|} = 2^{m+1}$. Moreover, recall that $2^{|X|} \leqslant 2k+2$ by assumption. As $|X| = n+1$, this amounts to $2^{n+1} \leqslant 2k+2$. Consequently, $2^{n+1} \leqslant 2k+2 \leqslant 2^{m+1}$, whence $n \leqslant m$. Thus, Proposition 4.10(ii) yields $\mathsf{Up}(\mathbb{X})^- = \mathsf{Up}(\mathbb{F}_n)^- \sqsubseteq \mathsf{Up}(\mathbb{F}_m)^- = \mathsf{Up}(\mathbb{Y})^-$. Since $\mathsf{Up}(\mathbb{Y})^- \in \mathsf{V}$ by assumption, we conclude that $\mathsf{Up}(\mathbb{X})^- \in \mathsf{V}$, a contradiction with the fact that $\mathbb{X}$ satisfies condition (ii) of Lemma 8.2. ⊠

We are now ready to prove Theorem 8.1.

Proof. Let V be a variety of Hilbert algebras such that $\mathsf{V} \neq \mathsf{Hil}$, $\mathsf{V} \neq \mathsf{D}_n$ for every $n \in \mathbb{N}$, and $\mathsf{V} \not\subseteq \mathsf{B}_\omega$. Since $\mathsf{V} \neq \mathsf{Hil}$ and $\mathsf{V} \neq \mathsf{D}_n$ for every $n \in \mathbb{N}$, from Theorem 6.1 it follows that $\mathsf{deg}^*(\mathsf{V}) \neq 1$. There are two cases: either $\mathsf{Up}(\mathbb{C}_3)^- \in \mathsf{V}$ or not. In the former case, we have $\mathsf{deg}^*(\mathsf{V}) \neq 1$ and $\mathsf{Up}(\mathbb{C}_3)^- \in \mathsf{V}$. Therefore, $\mathsf{deg}^*(\mathsf{V}) = 2^{\aleph_0}$ by Proposition 8.14. Next we consider the case where $\mathsf{Up}(\mathbb{C}_3)^- \notin \mathsf{V}$. By Jankov's Lemma 2.20 we have $\mathsf{V} \vDash \mathcal{J}(\mathbb{C}_3)$. Together with the assumption that $\mathsf{V} \not\subseteq \mathsf{B}_\omega$ and the first half of Theorem 4.3, this yields $\mathsf{V} \nvDash \mathcal{J}(\mathbb{F}_2)$. Therefore, $\mathsf{Up}(\mathbb{F}_2)^- \in \mathsf{V}$ by Jankov's Lemma 2.20. Since $\mathsf{deg}^*(\mathsf{V}) \neq 1$ by assumption, we can apply Proposition 8.11, obtaining $\mathsf{deg}^*(\mathsf{V}) = 2^{\aleph_0}$. ⊠

Remark 8.17. We close this paper by observing that the problem of describing the degrees of incompleteness of the extensions of the $\langle \wedge, \to \rangle$-fragment of IPC is not interesting, for each of these logics has degree of incompleteness 1. To prove this, it suffices to show that each extension L of this fragment is Kripke complete. Now, from Corollary 2.18 it follows that L is complete with respect to a class of $\langle \wedge, \to \rangle$-reducts of algebras of the form $\mathsf{Up}(\mathbb{X})$, where $\mathbb{X}$ is a finite poset. Hence, L is Kripke complete and, in fact, complete with respect to a class of finite Kripke frames. ⊠

Acknowledgements. The authors are very grateful to Alex Citkin for suggesting the current proof of the Embedding Lemma 6.4. The first author was supported by the FPI scholarship PRE2020-093696, associated with the I+D+i research project PID2019-110843GA-100 I+D *La geometría de las lógicas no clásicas* funded by the Ministry of Science and Innovation of Spain. The second author was supported by the proyecto PID2022-141529NB-C21 de investigacióon financiado por MICIU/AEI/10.13039/501100011033 y por FEDER, UE. He was also supported by the Research Group in Mathematical Logic, 2021SGR00348 funded by the Agency for Management of University and Research Grants of the Government of Catalonia. All the authors were supported by the MSCA-RISE-Marie Skłodowska-Curie Research and Innovation Staff Exchange (RISE) project MOSAIC 101007627 funded by Horizon 2020 of the European Union.

References

[1] J. C. Abbott. Implicational algebras. *Bulletin mathématique de la Société des Sciences Mathématiques de la République Socialiste de Roumanie*, 11(59):3–23 (1968), 1967.

[2] R. Balbes and P. Dwinger. *Distributive lattices*. University of Missouri Press, Columbia, Mo., 1974.

[3] C. Bergman. Structural completeness in algebra and logic. In *Algebraic logic (Budapest, 1988)*, volume 54 of *Colloquia Mathematica Societatis János Bolyai*, 59–73. North-Holland, Amsterdam, 1991.

[4] C. Bergman. *Universal Algebra: Fundamentals and Selected Topics*. Chapman & Hall Pure and Applied Mathematics. Chapman and Hall/CRC, 2011.

[5] G. Bezhanishvili and N. Bezhanishvili. Profinite Heyting algebras. *Order*, 25(3):211–227, 2008.

[6] G. Bezhanishvili, N. Bezhanishvili, T. Moraschini, and M. Stronkowski. Profiniteness and representability of spectra of Heyting algebras. *Advances in Mathematics*, 391, 2021.

[7] G. Bezhanishvili, N. Bezhanishvili, and T. Moraschini. Degrees of the finite model property: The Antidichotomy Theorem. *Journal of Mathematical Logic*, 2025.

[8] W. J. Blok. On the degree of incompleteness of modal logics. *Bulletin of the Section of Logic*, 7:167–175, 1978.

[9] W. J. Blok. On the degree of incompleteness of modal logics and the covering relation in the lattice of modal logics. Technical Report 78–07, Department of Mathematics, University of Amsterdam, 1978.

[10] W. J. Blok and D. Pigozzi. On the structure of varieties with equationally definable principal congruences I. *Algebra Universalis*, 15:195–227, 1982.

[11] R. A. Bull. Some Results for Implicational Calculi. *The Journal of Symbolic Logic*, 29(1):33–39, 1964.

[12] S. Burris and H. P. Sankappanavar. *A course in Universal Algebra*. Available online https://www.math.uwaterloo.ca/~snburris/htdocs/ualg.html, the millennium edition, 2012.

[13] J. L. Castiglioni, S. A. Celani, and H. J. San Martín. On Hilbert algebras generated by the order. *Archive for Mathematical Logic*, 61:155–172, 2022.

[14] S. Celani and L. Cabrer. Duality for finite Hilbert algebras. *Discrete Mathematics*, 305(1-3):74–99, 2005.

[15] S. Celani and R. Jansana. On the free implicative semilattice extension of a Hilbert algebra. *Mathematical Logic Quarterly*, 68(3):188–207,2012.

[16] S. A. Celani and L. M. Cabrer. Topological duality for tarski algebras. *Algebra Universalis*, 58:73–94, February 2008.

[17] A. Chagrov and M. Zakharyaschev. *Modal Logic*, volume 35 of *Oxford Logic Guides*. Oxford University Press, 1997.

[18] L. Chagrova. On the degree of neighborhood incompleteness of normal modal logics. In *Advances in modal logic, Vol. 1 (Berlin, 1996)*, CSLI Lecture Notes, pages 63–72. 1998.

[19] J. Czelakowski and W. Dziobiak. Congruence distributive quasivarieties whose finitely subdirectly irreducible members form a universal class. *Algebra Universalis*, 27(1):128–149, 1990.

[20] B. A. Davey. On the lattice of subvarieties. *Houston Journal of Mathematics*, 5(2):183–192, 1979.

[21] A. Day. Splitting algebras and a weak notion of projectivity. *Algebra Universalis*, 5:153–162, 1975.

[22] A. Diego. *Sobre álgebras de Hilbert*, volume 12 of *Notas de Lógica Matemática*. Universidad Nacional del Sur, Bahía Blanca (Argentina), 1965.

[23] L. Esakia. *Heyting Algebras. Duality Theory*. Springer, English translation of the original 1985 book. 2019.

[24] K. Fine. An Incomplete Logic Containing S4. *Theoria*, 40:23–29, 1974.

[25] M. Gehrke and S. van Gool. *Topological Duality for Distributive Lattices: Theory and Applications*. Cambridge Tracts in Theoretical Computer Science, Cambridge University Press, 2024.

[26] L. Henkin. An algebraic characterization of quantifiers. *Fundamenta Mathematicae*, 37:63–74, 1950.

[27] A. Horn. The separation theorem of intuitionist propositional logic. *The Journal of Symbolic Logic*, 27:391–399, 1962.

[28] T. Hosoi. On intermediate logics i. *Journal of the Faculty of Science, University of Tokyo*, 14:293–312, 1967.

[29] V. A. Jankov. On the relation between deducibility in intuitionistic propositional calculus and finite implicative structures. *Doklady Akademii Nauk SSSR*, 151:1293–1294, 1963.

[30] V. A. Jankov. The construction of a sequence of strongly independent superintuitionistic propositional calculi. *Soviet Mathematics Doklady*, 9:806–807, 1968.

[31] V. A. Jankov. Conjunctively irresolvable formulae in propositional calculi. *Izvestiya Akademii Nauk SSSR. Seriya Matematicheskaya*, 33:18–38, 1969.

[32] P. Köhler. Brouwerian semilattices. *Transactions of the American Mathematical Society*, 268:103–126, 1981.

[33] P. Köhler and D. Pigozzi. Varieties with equationally definable principal congruences. *Algebra Universalis*, 11:213–219, 1980.

[34] M. Kurtzhals. Definability in logic and algebra. Manuscript, 1015.

[35] T. Litak. A continuum of incomplete intermediate logics. *Reports on Mathematical Logic*, 36:131–141, 2002.

[36] T. Litak. Stability of the Blok theorem. *Algebra Universalis*, 58(4):385–411, 2008.

[37] T. Litak. A continuum of incomplete intermediate logics (corrected version). Available online on the ArXiv, 2018.

[38] C. G. McKay. The decidability of certain intermediate propositional logics. *The Journal of Symbolic Logic*, 33:258–264, 1968.

[39] I. Nishimura. On formulas of one variable in intuitionistic propositional calculus. *The Journal of Symbolic Logic*, 25:327–331 (1962), 1960.

[40] T. Prucnal. On the structural completeness of some pure implicational propositional calculi. *Studia Logica*, 30:45–52, 1972.

[41] T. Prucnal. Structural completeness of some fragments of intermediate logics. *Bull. Sect. Logic*, 12(1):41–44, 1983.

[42] J. G. Raftery. Admissible Rules and the Leibniz Hierarchy. *Notre Dame Journal of Formal Logic*, 57(4):569–606, 2016.

[43] H. Rasiowa. *An algebraic approach to non-classical logics*, volume 78 of *Studies in Logic and the Foundations of Mathematics*. North-Holland, Amsterdam, 1974.

[44] H. Rasiowa and R. Sikorski. *The mathematics of metamathematics*. Monografie Matematyczne, Tom 41. Państwowe Wydawnictwo Naukowe, Warsaw, 1963.

[45] W. Rautenberg, M. Zakharyaschev, and F. Wolter. Willem Blok and modal logic. *Studia Logica*, 83(1-3):15–30, 2006.

[46] L. Rieger. On the lattice theory of Brouwerian propositional logic. *Acta Facultatis Rerum Naturalium Universitatis Carolinae*, 1949(189):40, Prague, 1949.

[47] V.B. Shehtman. On Incomplete Propositional Logics. *Soviet Mathematics Doklady*, 18:985–989, 1977.

Damiano Fornasiere: Departament de Filosofia, Facultat de Filosofia, Universitat de Barcelona (UB), Carrer Montalegre, 6, 08001 Barcelona, Spain

Email address: damiano.fornasiere@ub.edu

Tommaso Moraschini: Departament de Filosofia, Facultat de Filosofia, Universitat de Barcelona (UB), Carrer Montalegre, 6, 08001 Barcelona, Spain

Email address: tommaso.moraschini@ub.edu