

LECTURE NOTES ON THE ALGEBRA OF LOGIC

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1. ALGEBRAS AND EQUATIONS

We begin by reviewing some fundamentals of general algebraic systems. For more information, the reader may consult [2, 8].

Definition 1.1.

- (i) A *type* is a map $\rho: \mathcal{F} \rightarrow \mathbb{N}$, where \mathcal{F} is a set of function symbols. In this case, $\rho(f)$ is said to be the *arity* of the function symbol f , for every $f \in \mathcal{F}$. Function symbols of arity zero are called *constants*.
- (ii) An *algebra* of type ρ is a pair $A = \langle A; F \rangle$ where A is a nonempty set and $F = \{f^A : f \in \mathcal{F}\}$ is a set of operations on A whose arity is determined by ρ , in the sense that each f^A has arity $\rho(f)$. The set A is called the *universe* of A .

When $\mathcal{F} = \{f_1, \dots, f_n\}$, we shall write $\langle A; f_1^A, \dots, f_n^A \rangle$ instead of $\langle A; F \rangle$. In this case, we often drop the superscripts, and write simply $\langle A; f_1, \dots, f_n \rangle$.

Classical examples of algebras are groups and rings. For instance, the type of groups ρ_G consists of a binary symbol $+$, a unary symbol $-$ and a constant symbol 0 . Then a group is an algebra $\langle G; +, -, 0 \rangle$ of type ρ_G in which $+$ is associative, 0 is a neutral element for $+$ and $-$ produces inverses.

Lattices, Heyting algebras and modal algebras are also algebras in the above sense. For instance, the type of lattices ρ_L consists of two binary symbols \wedge and \vee and a lattice is an algebra $\langle A; \wedge, \vee \rangle$ of type ρ_L that satisfies the idempotent, commutative, associative and absorption laws. Similarly, the type of Heyting algebras ρ_H consists of three binary operations symbols \wedge, \vee and \rightarrow and of two constant symbols 0 and 1 . Then a Heyting algebra is an algebra $\langle A; \wedge, \vee, \rightarrow, 0, 1 \rangle$ such that $\langle A; \wedge, \vee, 0, 1 \rangle$ is a bounded lattice and, for every $a, b, c \in A$,

$$a \wedge b \leq c \iff a \leq b \rightarrow c. \quad (\text{residuation law})$$

Boolean algebras can be viewed as the Heyting algebras that satisfy the following equational version of the *excluded middle law*:

$$x \vee (x \rightarrow 0) \approx 1.$$

In this case, the complement operation $\neg x$ can be defined as $x \rightarrow 0$.

Perhaps less obviously, even algebraic structures whose operations are apparently *external* can be viewed as algebras in the sense of the above definition. For instance, modules over a ring R can be viewed as algebras whose type ρ_R extends that of groups with the unary symbols $\{\lambda_r : r \in R\}$. From this point of view, a module over R is an

algebra $\langle G; +, -, 0, \{\lambda_r : r \in R\} \rangle$ of type ρ_R such that $\langle G; +, -, 0 \rangle$ is an abelian group and, for every $r, s \in R$ and $a, c \in G$,

$$\begin{aligned}\lambda_r(a + c) &= \lambda_r(a) + \lambda_r(c) \\ \lambda_{r+s}(a) &= \lambda_r(a) + \lambda_s(a) \\ \lambda_r(\lambda_s(a)) &= \lambda_{r \cdot s}(a) \\ \lambda_1(a) &= a.\end{aligned}$$

Given a type $\rho: \mathcal{F} \rightarrow \mathbb{N}$ and a set of variables X disjoint from \mathcal{F} , the set of *terms of type ρ over X* is the least set $T_\rho(X)$ such that

- (i) $X \subseteq T_\rho(X)$;
- (ii) if $c \in \mathcal{F}$ is a constant, then $c \in T_\rho(X)$; and
- (iii) if $\varphi_1, \dots, \varphi_{\rho(f)} \in T_\rho(X)$ and $f \in \mathcal{F}$, then $f\varphi_1 \dots \varphi_{\rho(f)} \in T_\rho(X)$.

For the sake of readability, we shall often write $f(\varphi_1, \dots, \varphi_{\rho(f)})$ instead of $f\varphi_1 \dots \varphi_{\rho(f)}$. Similarly, if f is a binary operation $+$, we often write $\varphi_1 + \varphi_2$ instead of $f(\varphi_1, \varphi_2)$.

Definition 1.2. Let $\rho: \mathcal{F} \rightarrow \mathbb{N}$ be a type and X a set of variables disjoint from \mathcal{F} . The *term algebra $T_\rho(X)$ of type ρ over X* is the unique algebra of type ρ whose universe is $T_\rho(X)$ and with basic n -ary operations f defined, for every $\varphi_1, \dots, \varphi_n \in T_\rho(X)$, as

$$f^{T_\rho(X)}(\varphi_1, \dots, \varphi_n) := f(\varphi_1, \dots, \varphi_n).$$

When no confusion might arise, we drop the subscript and write $T(X)$ instead of $T_\rho(X)$. Term algebras have the following fundamental property.

Proposition 1.3. Let A be an algebra of type ρ and X a set of variables. Every function $f: X \rightarrow A$ extends uniquely to a homomorphism $f^*: T_\rho(X) \rightarrow A$.

Proof. The unique extension f^* is defined, for every $\varphi(x_{\alpha_1}, \dots, x_{\alpha_n}) \in T_\rho(X)$, as

$$f^*(\varphi) = \varphi^A(f(x_{\alpha_1}), \dots, f(x_{\alpha_n})). \quad \square$$

Exercise 1.4. Prove the above proposition. \square

Given a term $\varphi \in T_\rho(X)$, we write $\varphi(x_1, \dots, x_n)$ to indicate that the variables occurring in φ are among x_1, \dots, x_n . Furthermore, given an algebra A of type ρ and elements $a_1, \dots, a_n \in A$, we define an element

$$\varphi^A(a_1, \dots, a_n)$$

of A , by recursion on the construction of φ , as follows:

- (i) if φ is a variable x_i , then $\varphi^A(a_1, \dots, a_n) := a_i$;
- (ii) if φ is a constant c , then c^A is the interpretation of c in A ;
- (iii) if $\varphi = f(\psi_1, \dots, \psi_m)$, then

$$\varphi^A(a_1, \dots, a_n) := f^A(\psi_1^A(a_1, \dots, a_n), \dots, \psi_m^A(a_1, \dots, a_n)).$$

An *equation of type ρ over X* is an expression of the form $\varphi \approx \psi$, where $\varphi, \psi \in T_\rho(X)$. We denote by $E_\rho(X)$ the set of equations of type ρ over X . Such an equation $\varphi \approx \psi$ is *valid* in an algebra A of type ρ , if

$$\varphi^A(a_1, \dots, a_n) = \psi^A(a_1, \dots, a_n), \text{ for every } a_1, \dots, a_n \in A,$$

in which case we say that A *satisfies* $\varphi \approx \psi$.

For instance, groups are precisely the algebras of type ρ_G that satisfy the equations

$$x + (y + z) \approx (x + y) + z \quad x + 0 \approx x \quad 0 + x \approx x \quad x + -x \approx 0 \quad -x + x \approx 0.$$

Similarly, lattices are the algebras of type ρ_L that satisfy the equations

$$\begin{array}{lll} x \wedge x \approx x & x \vee x \approx x & (\text{idempotent laws}) \\ x \wedge y \approx y \wedge x & x \vee y \approx y \vee x & (\text{commutative laws}) \\ x \wedge (y \wedge z) \approx (x \wedge y) \wedge z & x \vee (y \vee z) \approx (x \vee y) \vee z & (\text{associative laws}) \\ x \wedge (y \vee x) \approx x & x \vee (y \wedge x) \approx x & (\text{absorption laws}) \end{array}$$

From now on, we will work with a fixed denumerable set of variables

$$\text{Var} = \{x_n : n \in \mathbb{N}\}.$$

Accordingly, when we write $x, y, z \dots$ for variables, it should be understood that these are variables in Var .

2. BASIC CONSTRUCTIONS

Algebras of the same type are called *similar* and can be compared by means of maps that preserve their structure.

Definition 2.1. Given two similar algebras A and B , a *homomorphism* from A to B is a map $f: A \rightarrow B$ such that, for every n -ary operation g of the common type and $a_1, \dots, a_n \in A$,

$$f(g^A(a_1, \dots, a_n)) = g^B(f(a_1), \dots, f(a_n)).$$

An injective homomorphism is called an *embedding* and, if there exists an embedding from A to B , we say that A *embeds* into B . Lastly, a surjective embedding is called an *isomorphism*. Accordingly, A and B are said to be *isomorphic* if there exists an isomorphism between them, in which case we write $A \cong B$.

A simple induction on the construction of terms shows that, for every pair of algebras A and B of type ρ and every term $\varphi(x_1, \dots, x_n)$ of ρ , if f is a homomorphism from A to B , then

$$f(\varphi^A(a_1, \dots, a_n)) = \varphi^B(f(a_1), \dots, f(a_n)),$$

for every $a_1, \dots, a_n \in A$. Therefore homomorphisms preserve not only basic operations, but also arbitrary terms.

In the particular case where A and B are lattices, a homomorphism from A to B is a map $f: A \rightarrow B$ such that, for every $a, c \in A$,

$$f(a \wedge^A c) = f(a) \wedge^B f(c) \quad \text{and} \quad f(a \vee^A c) = f(a) \vee^B f(c).$$

For instance, the inclusion map from the lattice $\langle \mathbb{N}; \leq \rangle$ into the lattice $\langle \mathbb{Z}; \leq \rangle$ is an injective homomorphism, that is, an embedding. Similarly, given two sets $Y \subseteq X$, the inclusion map from the powerset lattice $\langle \mathcal{P}(Y); \subseteq \rangle$ to the powerset lattice $\langle \mathcal{P}(X); \subseteq \rangle$ is also an embedding. On the other hand, if $Y \subsetneq X$, the map

$$(-) \cap Y: \mathcal{P}(X) \rightarrow \mathcal{P}(Y)$$

that sends every $Z \subseteq X$ to $Z \cap Y$ is a noninjective homomorphism from $\langle \mathcal{P}(X); \subseteq \rangle$ to $\langle \mathcal{P}(Y); \subseteq \rangle$.

Definition 2.2. Let A and B be algebras of the same type $\rho: \mathcal{F} \rightarrow \mathbb{N}$. Then A is said to be a *subalgebra* of B if $A \subseteq B$ and f^A is the restriction of f^B to A , for every $f \in \mathcal{F}$. In this case, we write $A \leq B$.

Given a class of algebras K , let

$$\mathbb{I}(K) := \{A : A \cong B \text{ for some } B \in K\}$$

$$\mathbb{S}(K) := \{A : A \leq B \text{ for some } B \in K\}.$$

When $K = \{A\}$, we write $\mathbb{I}(A)$ and $\mathbb{S}(A)$ as a shorthand for $\mathbb{I}(\{A\})$ and $\mathbb{S}(\{A\})$, respectively. The following observation is an immediate consequence of the definitions.

Proposition 2.3. Let A and B be algebras of the same type. Then $A \in \mathbb{IS}(B)$ if and only if there exists an embedding $f: A \rightarrow B$. In this case, A is isomorphic to the unique subalgebra of B with universe $f[A]$.

As we mentioned, homomorphisms can be used to compare similar algebras.

Definition 2.4. Given two similar algebras A and B , we say that A is a *homomorphic image* of B if there exists a surjective homomorphism $f: B \rightarrow A$.

Accordingly, given a class of algebras K , we set

$$\mathbb{H}(K) := \{A : A \text{ is a homomorphic image of some } B \in K\}.$$

As usual, when $K = \{A\}$, we write $\mathbb{H}(A)$ as a shorthand for $\mathbb{H}(\{A\})$.

Observe that every (not necessarily surjective) homomorphism $f: A \rightarrow B$ induces a homomorphic image of A .

Proposition 2.5. If $f: A \rightarrow B$ is a homomorphism, then $f[A]$ is the universe of a subalgebra of B that, moreover, is a homomorphic image of A .

Proof. Observe that $f[A]$ is nonempty, because A is. Then consider an n -ary function symbol g of the common type of A and B and $b_1, \dots, b_n \in f[A]$. Clearly, there are $a_1, \dots, a_n \in A$ such that $f(a_i) = b_i$, for every $i \leq n$. Since f is a homomorphism from A to B , we obtain

$$g^B(b_1, \dots, b_n) = g^B(f(a_1), \dots, f(a_n)) = f(g^A(a_1, \dots, a_n)) \in f[A].$$

Hence, we conclude that $f[A]$ is the universe of a subalgebra $f[A]$ of B .

Furthermore, $f: A \rightarrow f[A]$ is a homomorphism, because for every basic n -ary function symbol g of the common type and $a_1, \dots, a_n \in A$,

$$f(g^A(a_1, \dots, a_n)) = g^B(f(a_1), \dots, f(a_n)) = g^{f[A]}(f(a_1), \dots, f(a_n)),$$

where the first equality follows from the assumption that $f: A \rightarrow B$ is a homomorphism. Since the map $f: A \rightarrow f[A]$ is surjective, we conclude that $f[A] \in \mathbb{H}(A)$. \square

In view of the above result, when $f: A \rightarrow B$ is a homomorphism, we denote by $f[A]$ the unique subalgebra of B with universe $f[A]$.

For instance, let $f: \mathbb{Z} \rightarrow \mathbb{R}$ be the absolute value map, that is, the function defined by the rule

$$f(n) := \text{the absolute value of } n.$$

Observe that f is a nonsurjective homomorphism from the lattice of integers to that of reals. Furthermore, the homomorphic image $f[\langle \mathbb{Z}; \leq \rangle]$ of $\langle \mathbb{Z}; \leq \rangle$ is the lattice of natural numbers $\langle \mathbb{N}; \leq \rangle$, which, in turn, is a subalgebra of lattice of reals.

Notably, the homomorphic images of an algebra A can be “internalized” as special equivalence relations on A as follows.

Definition 2.6. A *congruence* of an algebra A is an equivalence relation θ on A such that, for every basic n -ary operation f of A and $a_1, \dots, a_n, c_1, \dots, c_n \in A$,

$$\text{if } \langle a_1, c_1 \rangle, \dots, \langle a_n, c_n \rangle \in \theta, \text{ then } \langle f^A(a_1, \dots, a_n), f^A(c_1, \dots, c_n) \rangle \in \theta. \quad (1)$$

In this case, we often write $a \equiv_\theta c$ as a shorthand for $\langle a, c \rangle \in \theta$. The poset of congruences of A ordered under the inclusion relation will be denoted by $\text{Con}(A)$.

A simple induction on the construction of terms shows that, for every congruence θ of A and every term $\varphi(x_1, \dots, x_n)$,

$$\text{if } \langle a_1, c_1 \rangle, \dots, \langle a_n, c_n \rangle \in \theta, \text{ then } \langle \varphi^A(a_1, \dots, a_n), \varphi^A(c_1, \dots, c_n) \rangle \in \theta,$$

for every $a_1, \dots, a_n \in A$. Therefore, congruences preserve not only basic operations, but also arbitrary terms. Furthermore, a simple argument shows that $\text{Con}(A)$ is a complete (indeed algebraic) lattice whose maximum is the total relation $A \times A$ and whose minimum is the identity relation $\text{id}_A := \{ \langle a, a \rangle : a \in A \}$.

Example 2.7 (Heyting algebras). Recall that a *filter* of a Heyting algebra A is a nonempty upset $F \subseteq A$ closed under binary meets. We denote by $\text{Fi}(A)$ the poset of filters of A ordered under the inclusion relation. It is easy to see $\text{Fi}(A)$ is a complete lattice. Furthermore, the lattices $\text{Fi}(A)$ and $\text{Con}(A)$ are isomorphic via the inverse isomorphisms

$$\Omega^A(-): \text{Fi}(A) \rightarrow \text{Con}(A) \quad \text{and} \quad \tau^A(-): \text{Con}(A) \rightarrow \text{Fi}(A)$$

defined by the rules

$$\begin{aligned} \Omega^A(F) &:= \{ \langle a, c \rangle \in A \times A : a \rightarrow c, c \rightarrow a \in F \} \\ \tau^A(\theta) &:= \{ a \in A : \langle a, 1 \rangle \in \theta \}. \end{aligned}$$

Because of this, every congruence θ of a Heyting algebra A is induced by some filter F , in the sense that $\theta = \Omega^A F$. \square

Example 2.8 (Modal algebras). A *modal algebra* is an algebra $A = \langle A; \wedge, \vee, \neg, \Box, 0, 1 \rangle$ such that $\langle A; \wedge, \vee, \neg, 0, 1 \rangle$ is a Boolean algebra and \Box is a unary operation such that

$$\Box(a \wedge c) = \Box a \wedge \Box c \quad \text{and} \quad \Box 1 = 1,$$

for every $a, c \in A$. An *open filter* of a modal algebra A is a filter of the Boolean reduct of A that, moreover, is closed under the operation \Box . The poset of open filters of A ordered under the inclusion relation will be denoted by $\text{Op}(A)$. It forms a complete lattice. Furthermore, the lattices $\text{Op}(A)$ and $\text{Con}(A)$ are isomorphic via the inverse isomorphisms described in Example 2.7. Because of this, every congruence of a modal algebra A has the form $\theta = \Omega^A F$, for some open filter F . \square

Example 2.9 (Groups). Similarly, it is well known that the lattice of congruences of a group is isomorphic to that of its normal subgroups. Because of this, every congruence of a group is induced by some normal subgroup. \square

As we mentioned, there is a tight correspondence between the homomorphic images and the congruences of an algebra A . On the one hand, every congruence θ of A gives rise to a homomorphic image A/θ of A . Let \mathcal{F} be the set of function symbols of A . Given

$\theta \in \text{Con}(A)$ and a basic n -ary function symbol $f \in \mathcal{F}$, let $f^{A/\theta}$ be the n -ary operation on A/θ defined by the rule

$$f^{A/\theta}(a_1/\theta, \dots, a_n/\theta) := f^A(a_1, \dots, a_n)/\theta.$$

Notice that $f^{A/\theta}$ is well-defined, by condition (1). As a consequence, the structure

$$A/\theta := \langle A/\theta; \{f^{A/\theta} : f \in \mathcal{F}\} \rangle$$

is a well-defined algebra of the type as A . Furthermore, $A/\theta \in \mathbb{H}(A)$, because the map $\pi_\theta: A \rightarrow A/\theta$, defined, for every $a \in A$, as $\pi_\theta(a) := a/\theta$, is a surjective homomorphism from A to A/θ . To prove this, consider $a_1, \dots, a_n \in A$. We have

$$\begin{aligned} \pi_\theta(f^A(a_1, \dots, a_n)) &= f^A(a_1, \dots, a_n)/\theta \\ &= f^{A/\theta}(a_1/\theta, \dots, a_n/\theta) \\ &= f^{A/\theta}(\pi_\theta(a_1), \dots, \pi_\theta(a_n)), \end{aligned}$$

where the second equality follows from the definition of the operation $f^{A/\theta}$.

Corollary 2.10. *If θ is a congruence of an algebra A , then A/θ is a well-defined homomorphic image of A .*

In view of the above result, every congruence θ of an algebra A induces a homomorphic image of A , namely A/θ . The converse is also true, as we proceed to explain.

Definition 2.11. The *kernel* of a homomorphism $f: A \rightarrow B$ is the binary relation

$$\text{Ker}(f) := \{ \langle a, c \rangle \in A \times A : f(a) = f(c) \}.$$

Proposition 2.12. *The kernel of a homomorphism $f: A \rightarrow B$ is a congruence of A .*

Proof. It is obvious that $\text{Ker}(f)$ is an equivalence relation on A . Therefore, to prove that $\text{Ker}(f)$ is a congruence of A , it suffices to show that it preserves the basic operations of A . Consider a basic n -ary operation g of A and $a_1, \dots, a_n, c_1, \dots, c_n \in A$ such that $\langle a_1, c_1 \rangle, \dots, \langle a_n, c_n \rangle \in \text{Ker}(f)$. By the definition of $\text{Ker}(f)$,

$$f(a_i) = f(c_i), \text{ for every } i \leq n.$$

It follows that $g^B(f(a_1), \dots, f(a_n)) = g^B(f(c_1), \dots, f(c_n))$. Since $f: A \rightarrow B$ is a homomorphism, this yields

$$f(g^A(a_1, \dots, a_n)) = g^B(f(a_1), \dots, f(a_n)) = g^B(f(c_1), \dots, f(c_n)) = f(g^A(c_1, \dots, c_n)).$$

Hence, we conclude that $\langle g^A(a_1, \dots, a_n), g^A(c_1, \dots, c_n) \rangle \in \text{Ker}(f)$, as desired. \square

The behaviour of kernels is governed by the next principle.

Fundamental Homomorphism Theorem 2.13. *If $f: A \rightarrow B$ is a homomorphism with kernel θ , then there exists a unique embedding $g: A/\theta \rightarrow B$ such that $f = g \circ \pi_\theta$.*

Proof. We begin by proving the existence of g . Let $g: A/\theta \rightarrow B$ be the map defined as $g(a/\theta) := f(a)$, for every $a \in A$. To show that g is well-defined, consider $a, c \in A$ such that $a/\theta = c/\theta$. Since $\theta = \text{Ker}(f)$, this means that $f(a) = f(c)$, as desired. Furthermore, the definition of g guarantees that $f = g \circ \pi_\theta$.

Now, observe g is injective, because, for every $a, c \in A$ such that $g(a/\theta) = g(c/\theta)$, we have $f(a) = f(c)$, that is, $\langle a, c \rangle \in \text{Ker}(f) = \theta$ and, therefore, $a/\theta = c/\theta$. Moreover, for every basic n -ary operation p of A and $a_1, \dots, a_n \in A$, we have

$$\begin{aligned} g(p^{A/\theta}(a_1/\theta, \dots, a_n/\theta)) &= g(p^A(a_1, \dots, a_n)/\theta) \\ &= f(p^A(a_1, \dots, a_n)) \\ &= p^B(f(a_1), \dots, f(a_n)) \\ &= p^B(g(a_1/\theta), \dots, g(a_n/\theta)). \end{aligned}$$

The first equality above follows from the definition of A/θ , the second and the last from the definition of g and the third from the assumption that $f: A \rightarrow B$ is a homomorphism. Hence, we conclude that $g: A/\theta \rightarrow B$ is a homomorphism and, therefore, an embedding, as desired.

The uniqueness of g follows from the fact that, if a map g^* satisfies the condition in the statement of the theorem, then, for every $a \in A$,

$$f(a) = g^* \circ \pi_\theta(a) = g^*(a/\theta),$$

that is, g^* coincides with g . \square

Corollary 2.14. *If $f: A \rightarrow B$ is a homomorphism, then $f[A] \cong A/\text{Ker}(f)$. In particular, if f is surjective, $B \cong A/\text{Ker}(f)$.*

Proof. In the proof of the Fundamental Homomorphism Theorem we showed that the map $g: A/\text{Ker}(f) \rightarrow B$, defined by the rule $g(a/\text{Ker}(f)) := f(a)$, is an embedding of $A/\text{Ker}(f)$ into B . As g can be viewed as a surjective embedding of $A/\text{Ker}(f)$ into $f[A]$, we conclude that $f[A] \cong A/\text{Ker}(f)$. \square

At this stage, it should be clear that if θ is a congruence on an algebra A , then $\pi_\theta: A \rightarrow A/\theta$ is a surjective homomorphism whose kernel is θ . Similarly, if $f: A \rightarrow B$ is a surjective homomorphism, then $A/\text{Ker}(f) \cong B$, by Corollary 2.14. As a consequence, for every class of algebras K ,

$$\mathbb{H}(K) = \mathbb{I}\{A/\theta : A \in K \text{ and } \theta \in \text{Con}(A)\}. \quad (2)$$

Now, recall that the Cartesian product of a family of sets $\{A_i : i \in I\}$ is the set

$$\prod_{i \in I} A_i := \{f: I \rightarrow \bigcup_{i \in I} A_i : f(i) \in A_i, \text{ for all } i \in I\}.$$

In particular, if I is empty, then $\prod_{i \in I} A_i$ is the singleton containing only the empty map.

Definition 2.15. The *direct product* of a family of similar algebras $\{A_i : i \in I\}$ is the unique algebra of the common type whose universe is the Cartesian product $\prod_{i \in I} A_i$ and such that, for every basic n -ary operation symbol f and every $\vec{a}_1, \dots, \vec{a}_n \in \prod_{i \in I} A_i$,

$$f^{\prod_{i \in I} A_i}(\vec{a}_1, \dots, \vec{a}_n)(i) = f^{A_i}(\vec{a}_1(i), \dots, \vec{a}_n(i)), \text{ for every } i \in I.$$

We denote this algebra by $\prod_{i \in I} A_i$.

In this case, for every $j \in I$, the projection map on the j -th component $p_j: \prod_{i \in I} A_i \rightarrow A_j$, defined by the rule $p_j(\vec{a}) := \vec{a}(j)$, is a surjective homomorphism from $\prod_{i \in I} A_i$ to A_j .

Given a class of similar algebras K , we set

$$\mathbb{P}(K) := \{A : A \text{ is a direct product of a family } \{B_i : i \in I\} \subseteq K\}.$$

As usual, when $K = \{A\}$, we write $\mathbb{P}(A)$ as a shorthand for $\mathbb{P}(\{A\})$.

Notice that up to isomorphism, there exists a unique one-element algebra of a given type. Because of this, one-element algebras are called *trivial*. Accordingly, when the set of indexes I is empty, the direct product $\prod_{i \in I} A_i$ is the trivial algebra of the given type. It follows that $\mathbb{P}(K)$ contains always a trivial algebra, for every class of similar algebras K .

Example 2.16 (Powerset algebras). Boolean algebras of the form $\langle \mathcal{P}(X); \cap, \cup, -, \emptyset, X \rangle$ are called *powerset Boolean algebras*. Let B be the two-element Boolean algebra and observe that $\mathbb{IP}(B)$ is the class of algebras isomorphic to some powerset Boolean algebra. To prove this, observe that every powerset Boolean algebra $\mathcal{P}(X)$ is isomorphic to a direct product of B via the *characteristic function* $f_X: \mathcal{P}(X) \rightarrow \prod_{x \in X} B_x$, defined by the rule

$$f(Y)(x) := \begin{cases} 1 & \text{if } x \in Y \\ 0 & \text{if } x \notin Y, \end{cases}$$

where $Y \in \mathcal{P}(X)$ and $x \in X$. By the same token, every direct product $\prod_{i \in I} B_i$ of B is isomorphic to the powerset Boolean algebra $\mathcal{P}(I)$ via the isomorphism f_I . \square

We close this section by reviewing the subdirect product construction.

Definition 2.17. A subalgebra B of a direct product $\prod_{i \in I} A_i$ is said to be a *subdirect product* of $\{A_i : i \in I\}$ if the projection map p_i is surjective, for every $i \in I$. Similarly, an embedding $f: B \rightarrow \prod_{i \in I} A_i$ is said to be *subdirect* when $f[B]$ is a subdirect product of the family $\{A_i : i \in I\}$.

Given a class of similar algebras K , we set

$$\mathbb{P}_{\text{SD}}(K) := \{A : A \text{ is a subdirect direct product of a family } \{B_i : i \in I\} \subseteq K\}.$$

As usual, when $K = \{A\}$, we write $\mathbb{P}_{\text{SD}}(A)$ as a shorthand for $\mathbb{P}_{\text{SD}}(\{A\})$. Clearly, $\mathbb{P}_{\text{SD}}(K) \subseteq \mathbb{SP}(K)$. Furthermore, $\mathbb{P}_{\text{SD}}(K)$ contains always a trivial algebra.

Example 2.18 (Distributive lattices). Let DL be the class of distributive lattices and B be the two-element distributive lattice. Birkhoff's Representation Theorem states that $\text{DL} = \mathbb{IP}_{\text{SD}}(B)$. The inclusion $\mathbb{IP}_{\text{SD}}(B) \subseteq \text{DL}$ follows from the fact that DL is closed under \mathbb{I}, \mathbb{S} and \mathbb{P} . For the other inclusion, consider a distributive lattice A and let I be the set of its prime filters. By Birkhoff's Representation Theorem, the map

$$\gamma: A \rightarrow \prod_{F \in I} B_F,$$

defined, for every $a \in A$ and $F \in I$, by the rule

$$\gamma(a)(F) := \begin{cases} 1 & \text{if } a \in F \\ 0 & \text{if } a \notin F, \end{cases}$$

is a well-defined subdirect embedding. \square

Example 2.19 (Boolean algebras). Similarly, Stone's Representation Theorem states that the class of Boolean algebras coincides with $\mathbb{IP}_{\text{SD}}(B)$, where B the two-element Boolean algebra. \square

The next result provides a general recipe to construct subdirect products.

Proposition 2.20. *Let A be an algebra and $\{\theta_i : i \in I\} \subseteq \text{Con}(A)$. Then the map*

$$f: A / \bigcap_{i \in I} \theta_i \rightarrow \prod_{i \in I} A / \theta_i,$$

defined, for every $a \in A$ and $j \in I$, as

$$f(a / \bigcap_{i \in I} \theta_i)(j) := a / \theta_j,$$

is a subdirect embedding.

Proof. For the sake of readability, set $B := A / \bigcap_{i \in I} \theta_i$. To prove that f is injective, consider $a, c \in A$ such that $\langle a, c \rangle \notin \bigcap_{i \in I} \theta_i$. Then there exists $j \in I$ such that $\langle a, c \rangle \notin \theta_j$ and, therefore,

$$f(a / \bigcap_{i \in I} \theta_i)(j) := a / \theta_j \neq c / \theta_j = f(c / \bigcap_{i \in I} \theta_i)(j).$$

It follows that $f(a / \bigcap_{i \in I} \theta_i) \neq f(c / \bigcap_{i \in I} \theta_i)$. Thus, f is injective. Moreover, by the definition of f , the composition $p_i \circ f: B \rightarrow A / \theta_i$ is surjective, for every $i \in I$.

It only remains to prove that f is a homomorphism. Consider an n -ary basic operation g and $a_1, \dots, a_n \in A$. For every $j \in I$, we have

$$\begin{aligned} f(g^B(a_1 / \bigcap_{i \in I} \theta_i, \dots, a_n / \bigcap_{i \in I} \theta_i))(j) &= f(g^A(a_1, \dots, a_n) / \bigcap_{i \in I} \theta_i)(j) \\ &= g^A(a_1, \dots, a_n) / \theta_j \\ &= g^{A/\theta_j}(a_1 / \theta_j, \dots, a_n / \theta_j) \\ &= g^{A/\theta_j}(f(a_1 / \bigcap_{i \in I} \theta_i)(j), \dots, f(a_n / \bigcap_{i \in I} \theta_i)(j)) \\ &= g^{\prod_{i \in I} A/\theta_i}(f(a_1 / \bigcap_{i \in I} \theta_i), \dots, f(a_n / \bigcap_{i \in I} \theta_i))(j). \end{aligned}$$

It follows that

$$f(g^B(a_1 / \bigcap_{i \in I} \theta_i, \dots, a_n / \bigcap_{i \in I} \theta_i)) = g^{\prod_{i \in I} A/\theta_i}(f(a_1 / \bigcap_{i \in I} \theta_i), \dots, f(a_n / \bigcap_{i \in I} \theta_i)). \quad \square$$

3. PROPOSITIONAL LOGICS AND EQUATIONAL COMPLETENESS THEOREMS

For general information on propositional logics we refer the reader to [12, 15, 16, 17]. Recall that a *closure operator* on a set A is a map $C: \mathcal{P}(A) \rightarrow \mathcal{P}(A)$ such that, for every $X \subseteq Y \subseteq A$,

$$X \subseteq C(X) = C(C(X)) \quad \text{and} \quad C(X) \subseteq C(Y).$$

Given a closure operator C on A , a subset $X \subseteq A$ is said to be *closed* if $X = C(X)$. A *closure system* on A is a family $\mathcal{C} \subseteq \mathcal{P}(A)$ that contains A and such that $\bigcap \mathcal{F}$, for every nonempty $\mathcal{F} \subseteq \mathcal{C}$. Closure operators and systems on A are two faces of the same coin. More precisely, if the family of closed sets of a closure operator on A is a closure system on A . On the other hand, if \mathcal{C} is a closure system on A , then the map $C: \mathcal{P}(A) \rightarrow \mathcal{P}(A)$, defined by the rule

$$C(X) := \bigcap \{Y \in \mathcal{C} : X \subseteq Y\},$$

is a closure operator on A . These transformations between closure operators and systems on A are one inverse to the other.

Exercise 3.1. Prove that these transformations are well-defined and one inverse to the other. \square

Another way of presenting closure operators or systems is by means of the following concept.

Definition 3.2. A *consequence relation* on a set A is a binary relation $\vdash \subseteq \mathcal{P}(A) \times A$ such that, for every $X \cup Y \cup \{a\} \subseteq A$,

- (i) if $a \in X$, then $X \vdash a$; and
- (ii) if $X \vdash y$ for all $y \in Y$ and $Y \vdash a$, then $X \vdash a$.

Furthermore, \vdash is said to be *finitary* when, for every $X \cup \{a\} \subseteq A$,

if $X \vdash a$, there exists a finite $Y \subseteq X$ such that $Y \vdash a$.

Remark 3.3. The relation $X \vdash a$ should be read, intuitively, as “ X proves a ” or “ a follows from X ”. In this reading, the demand expressed by condition (i) is rather natural, while (ii) is an abstract of the Cut rule. \square

Formally speaking, a consequence relation on a set A is a binary relation $\vdash \subseteq \mathcal{P}(A) \times A$. However, to simplify the notation, we will often write $a_1, \dots, a_n \vdash c$ as a shorthand for $\{a_1, \dots, a_n\} \vdash c$. Similarly, we will use $X, a \vdash c$ as a shorthand for $X \cup \{a\} \vdash c$. Lastly, for every set of formulas $X \cup Y \cup \{a, c\}$, we write

- (i) $X \vdash Y$, when $X \vdash y$ for every $y \in Y$;
- (ii) $a \dashv\vdash c$, when $a \vdash c$ and $c \vdash a$; and
- (iii) $X \dashv\vdash Y$, when $X \vdash Y$ and $Y \vdash X$.

Definition 3.4. Let \vdash be a consequence relation on a set A . A *theory* of \vdash is a subset $X \subseteq A$ such that, for every $a \in A$, if $X \vdash a$, then $a \in X$. The set of theories of A will be denoted by $\mathcal{Th}(\vdash)$.

It is easy to see that $\mathcal{Th}(\vdash)$ is a closure system on A . Moreover, given a closure operator C on A , the following is a consequence relation on A :

$$\{(X, a) \in \mathcal{P}(A) \times A : a \in C(X)\}.$$

Together with the correspondence between closure systems and operators, these transformations induce a one-to-one correspondence between consequence relations, closure operators and closure systems on A .

Exercise 3.5. Prove these facts. \square

In the context of logic, the term algebra $T_\rho(\text{Var})$ is often called the *algebra of formulas* (of type ρ) and its elements are referred to as *formulas*. An *endomorphism* of an algebra A is a homomorphism whose domain and codomain is A . Endomorphisms of the algebra of formulas play a fundamental role in logic.

Definition 3.6. A *substitution* of type ρ is an endomorphism σ of $T_\rho(\text{Var})$.

When the type ρ is clear from the context, we will simply say that σ is a substitution.

In view of Proposition 1.3 and of the fact that Var is a set of generators for $T_\rho(\text{Var})$, every function $\sigma: \text{Var} \rightarrow T_\rho(\text{Var})$ can be uniquely extended to a substitution σ^+ of type ρ , namely the function defined by the rule

$$\varphi(x_1, \dots, x_n) \mapsto \varphi(\sigma(x_1), \dots, \sigma(x_n)).$$

Because of this, substitutions of type ρ can be presented by exhibiting functions $\sigma: \text{Var} \rightarrow T_\rho(\text{Var})$.

Definition 3.7. A logic of type ρ is a consequence relation \vdash on the set of formulas $T_\rho(\text{Var})$ that, moreover, is *substitution invariant* in the sense that for every substitution σ of type ρ and every set of formulas $\Gamma \cup \{\varphi\} \subseteq T_\rho(\text{Var})$,

$$\text{if } \Gamma \vdash \varphi, \text{ then } \sigma[\Gamma] \vdash \sigma(\varphi).$$

Remark 3.8. As mentioned above, $\Gamma \vdash \varphi$ should be read as “ Γ proves φ ” or “ φ follows from Γ ”. The requirement that \vdash is substitution invariant, instead, is intended to capture the idea that logical inferences are true only in virtue of their form (as opposed to their content). \square

Example 3.9 (Hilbert calculi). We work within a fixed, but arbitrary, type ρ . A *rule* is an expression of the form $\Gamma \triangleright \varphi$, where $\Gamma \cup \{\varphi\} \subseteq T_\rho(\text{Var})$. In this case, Γ is said to be the set of *premises* of the rule and φ the *conclusion*. When $\Gamma = \emptyset$, the rule $\Gamma \triangleright \varphi$ is sometimes called an *axiom*. A *Hilbert calculus* is a set of rules.

Every Hilbert calculus H induces a logic, as we proceed to explain. Consider a set of formulas $\Gamma \cup \{\varphi\} \subseteq T_\rho(\text{Var})$. A *proof of φ from Γ in H* is a well-ordered sequence $\langle \psi_\alpha : \alpha \leq \gamma \rangle$ of formulas $\psi_\alpha \in T_\rho(\text{Var})$ whose last element ψ_γ is φ and such that, for every $\alpha \leq \gamma$, either $\psi_\alpha \in \Gamma$ or there exist a substitution σ and a rule $\Delta \triangleright \delta$ in H such that the formulas in $\sigma[\Delta]$ occur in the initial segment $\langle \psi_\beta : \beta < \alpha \rangle$ and $\psi_\alpha = \sigma(\delta)$.

The logic \vdash_H induced by H is defined, for every $\Gamma \cup \{\varphi\} \subseteq T_\rho(\text{Var})$, as

$$\Gamma \vdash_H \varphi \iff \text{there exists a proof of } \varphi \text{ from } \Gamma \text{ in } H.$$

As expected, \vdash_H is a logic in the sense of Definition 3.7. Furthermore, it is the least logic \vdash such that $\Gamma \vdash \varphi$, for every rule $\Gamma \triangleright \varphi$ in H .

A logic \vdash is said to be *axiomatized* by a Hilbert calculus H when it coincides with \vdash_H . Notice that every logic \vdash is vacuously axiomatized by the Hilbert calculus

$$\{\Gamma \triangleright \varphi : \Gamma \vdash \varphi\}.$$

Because of this, axiomatizations in terms of Hilbert calculi H acquire special interest when H is finite or, at least, recursive. \square

When no confusion shall arise, given a sequence \vec{a} and a set A , we write $\vec{a} \in A$ to indicate that the elements of the sequence \vec{a} belong to A . The following concept is instrumental to exhibit further examples of logics.

Definition 3.10. Let K be a class of similar algebras. We define a binary relation $\models_K \subseteq \mathcal{P}(E_\rho(\text{Var})) \times E_\rho(\text{Var})$ as follows:

$$\Theta \models_K \varepsilon \approx \delta \iff \text{for every } A \in K \text{ and every } \vec{a} \in A,$$

$$\text{if } \varphi^A(\vec{a}) = \psi^A(\vec{a}) \text{ for all } \varphi \approx \psi \in \Theta, \text{ then } \varepsilon^A(\vec{a}) = \delta^A(\vec{a}).$$

The relation \models_K is known as the *equational consequence relative to K* .

Example 3.11 (Equationally defined logics). We work within a fixed, but arbitrary, type ρ . Given a set of equations $\tau(x)$ in a single variable x and a set of formulas $\Gamma \cup \{\varphi\} \subseteq T(\text{Var})$, we abbreviate

$$\{\varepsilon(\varphi) \approx \delta(\varphi) : \varepsilon \approx \delta \in \tau\} \text{ as } \tau(\varphi), \text{ and } \bigcup_{\gamma \in \Gamma} \tau(\gamma) \text{ as } \tau[\Gamma].$$

Given a class of algebras \mathbf{K} and a set of equations $\tau(x)$, we define a logic $\vdash_{\mathbf{K},\tau}$ as follows: for every $\Gamma \cup \{\varphi\} \subseteq T(\text{Var})$,

$$\Gamma \vdash_{\mathbf{K},\tau} \varphi \iff \tau[\Gamma] \models_{\mathbf{K}} \tau(\varphi).$$

It is easy to prove that $\vdash_{\mathbf{K},\tau}$ is indeed a logic in the sense of Definition 3.7. Notice that, in this case, \vdash is related to \mathbf{K} by a *completeness theorem* witnessed by the set of equations $\tau(x)$ that allows to translate formulas into equations and, therefore, to interpret $\vdash_{\mathbf{K},\tau}$ into $\models_{\mathbf{K}}$.

For instance, the completeness theorem of classical propositional logic **CPC** with respect to the class of Boolean algebras **BA** states precisely that **CPC** coincides with $\vdash_{\mathbf{BA},\tau}$ where $\tau = \{x \approx 1\}$. Similarly, the completeness theorem of intuitionistic propositional logic **IPC** with respect to the class of Heyting algebras **HA** states precisely that **IPC** coincides with $\vdash_{\mathbf{HA},\tau}$ where $\tau = \{x \approx 1\}$. Because of this, **CPC** and **IPC** can be defined as follows: for every set of formulas $\Gamma \cup \{\varphi\}$ of the appropriate type,

$$\begin{aligned} \Gamma \vdash_{\mathbf{CPC}} \varphi &\iff \tau[\Gamma] \models_{\mathbf{BA}} \tau(\varphi) \\ \Gamma \vdash_{\mathbf{IPC}} \varphi &\iff \tau[\Gamma] \models_{\mathbf{HA}} \tau(\varphi), \end{aligned}$$

where $\tau = \{x \approx 1\}$. □

The relation between logic and algebra is often explained in terms of the existence of equational completeness theorems. The following definition makes this concept precise. As we will see, however, equational completeness theorems alone are not sufficient to account for the relation between logic and algebra.

Definition 3.12. A logic \vdash is said to admit an *equational completeness theorem* if there are a set of equations $\tau(x)$ and a class \mathbf{K} of algebras such that for all $\Gamma \cup \{\varphi\} \subseteq T(\text{Var})$,

$$\Gamma \vdash \varphi \iff \tau[\Gamma] \models_{\mathbf{K}} \tau(\varphi).$$

In this case, \vdash coincides with $\vdash_{\mathbf{K},\tau}$ and \mathbf{K} is said to be a τ -*algebraic semantics* (or simply an *algebraic semantics*) for \vdash .

This notion was introduced in [5] and studied in depth in [6, 26, 28]. For instance, the classes of Boolean and Heyting algebras are, respectively, τ -algebraic semantics for **CPC** and **IPC** where $\tau = \{x \approx 1\}$.

Another familiar example of equational completeness theorem arises from the field of modal logic. Let Fr be the class of all Kripke frames. We can associate two distinct logics with Fr , see for instance [21, 22]. The *global consequence* \mathbf{K}_g of the modal system \mathbf{K} is the logic defined, for every set of modal formulas $\Gamma \cup \{\varphi\}$, as follows:

$$\begin{aligned} \Gamma \vdash_{\mathbf{K}_g} \varphi &\iff \text{for every } \langle W, R \rangle \in \text{Fr} \text{ and evaluation } v \text{ in } \langle W, R \rangle, \\ &\quad \text{if } w, v \Vdash \Gamma \text{ for all } w \in W, \text{ then } w, v \Vdash \varphi \text{ for all } w \in W. \end{aligned}$$

On the other hand, the *local consequence* \mathbf{K}_ℓ of the modal system \mathbf{K} is defined, for every set of modal formulas $\Gamma \cup \{\varphi\}$, as follows:

$$\begin{aligned} \Gamma \vdash_{\mathbf{K}_\ell} \varphi &\iff \text{for every } \langle W, R \rangle \in \text{Fr}, w \in W, \text{ and evaluation } v \text{ in } \langle W, R \rangle, \\ &\quad \text{if } w, v \Vdash \Gamma, \text{ then } w, v \Vdash \varphi. \end{aligned}$$

It is easy to see that \mathbf{K}_g and \mathbf{K}_ℓ are logics. Moreover, they are distinct, because

$$x \vdash_{\mathbf{K}_g} \Box x \text{ and } x \not\vdash_{\mathbf{K}_\ell} \Box x. \quad (3)$$

Exercise 3.13. Prove that \mathbf{K}_g and \mathbf{K}_ℓ are logics. Notice also that the modal system \mathbf{K} is not a logic itself, because it is not a consequence relation. Indeed, there are two ways to turn \mathbf{K} into a logic, namely, \mathbf{K}_g and \mathbf{K}_ℓ . \square

Exercise 3.14. Prove that \mathbf{K}_g and \mathbf{K}_ℓ have the same *theorems*, i.e., formulas provable from the empty set. Prove also that the set of their theorems is the modal system \mathbf{K} . This indicates that, even in the modal setting, logics should not be identified with their sets of theorems. \square

The global consequence \mathbf{K}_g is related to the class MA of modal algebras by the following equational completeness theorem.

Theorem 3.15. *For every set $\Gamma \cup \{\varphi\}$ of modal formulas,*

$$\Gamma \vdash_{\mathbf{K}_g} \varphi \iff \tau[\Gamma] \models_{\mathbf{MA}} \tau(\varphi),$$

where $\tau = \{x \approx 1\}$. Consequently, the class of modal algebras is a τ -algebraic semantics for \mathbf{K}_g .

In order to prove it, recall that a filter on a Boolean algebra A is said to be *proper* when it differs from A . Moreover, a proper filter U of A is said to be a *ultrafilter* of A if it is maximal among the proper filters of A or, equivalently, if

$$a \in U \text{ or } \neg a \in U, \text{ for every } a \in A.$$

While the following result holds in ZFC, it cannot be proved in ZF (although it is strictly weaker than the axiom of choice).

Ultrafilter Lemma 3.16. *Every proper filter on a Boolean algebra can be extended to a ultrafilter.*

We are now ready to prove Theorem 3.15.

Proof sketch. It suffices to prove that

$$\Gamma \not\vdash_{\mathbf{K}_g} \varphi \iff \tau[\Gamma] \not\models_{\mathbf{MA}} \tau(\varphi).$$

Suppose first that $\Gamma \not\vdash_{\mathbf{K}_g} \varphi$. Then there are a Kripke frame $\langle W, R \rangle$, an evaluation v in it and a world u such that

$$w, v \Vdash \Gamma \text{ for all } w \in W \text{ and } u, v \not\Vdash \varphi. \quad (4)$$

Then consider the complex algebra of $\langle W, R \rangle$, that is, the structure

$$A := \langle \mathcal{P}(W); \cap, \cup, -, \Box, \emptyset, W \rangle,$$

where $-$ is set theoretic complement and, for every $V \subseteq W$,

$$\Box V := \{w \in W : \text{if } \langle w, t \rangle \in R, \text{ then } t \in V\}.$$

It is easy to prove that A is a modal algebra. Then consider the unique homomorphism $f: T(\text{Var}) \rightarrow A$ such that

$$f(x) = \{w \in W : w, v \Vdash x\},$$

for every $x \in \text{Var}$. A simple induction of the construction of terms shows that, for every formula ψ ,

$$f(\psi) = \{w \in W : w, v \Vdash \psi\}.$$

Together with (4), this yields

$$f[\Gamma] \subseteq \{W\} \text{ and } f(\varphi) \neq W.$$

Hence, we conclude that $\tau[\Gamma] \not\models_{\mathbf{MA}} \tau(\varphi)$.

To prove the converse, suppose that $\tau[\Gamma] \not\vdash_{\mathbf{MA}} \tau(\varphi)$. Then there are a modal algebra \mathbf{A} and a homomorphism $f: T(\text{Var}) \rightarrow \mathbf{A}$ such that

$$f[\Gamma] \subseteq \{1\} \text{ and } f(\varphi) \neq 1.$$

Then consider the Kripke frame dual to \mathbf{A} , that is, the structure $\langle W, R \rangle$, where W is the set of ultrafilters of \mathbf{A} and R the binary relation on W defined as follows:

$$R := \{ \langle U, V \rangle \in W \times W : \{a \in A : \Box a \in U\} \subseteq V \}.$$

Let then $v: \text{Var} \rightarrow \mathcal{P}(W)$ be the evaluation in $\langle W, R \rangle$ defined by the rule

$$v(x) := \{U \in W : f(x) \in U\}.$$

An easy induction on the construction of terms shows that, for every formula ψ ,

$$\{U \in W : f(\psi) \in U\} = \{U \in W : U, v \Vdash \psi\}. \quad (5)$$

Now, since $f(\varphi) \neq 1$, the Ultrafilter Lemma guarantees the existence of an ultrafilter F such that $f(\varphi) \notin F$. Furthermore, as every ultrafilter contain 1, from $f[\Gamma] \subseteq \{1\}$ it follows that $f[\Gamma] \subseteq U$, for all $U \in W$. In short,

$$f[\Gamma] \subseteq U \text{ for all } U \in W \text{ and } f(\varphi) \notin F.$$

Together with (5), this yields

$$U, v \Vdash \Gamma \text{ for all } U \in W \text{ and } F, v \not\Vdash \varphi.$$

Hence, we conclude that $\Gamma \not\vdash_{\mathbf{K}_s} \varphi$. \(\square\)

At this stage, it is tempting to conjecture that the relation between logic and algebra can be explained in terms of equational completeness theorems only. As we anticipated, however, this is not the case. For instance, the relation between **CPC** and **BA** cannot be explained in terms of completeness theorems only, because the class of Heyting algebras **HA** is also an algebraic semantics for **CPC**. To explain why, it is convenient to recall the following classical result relating **CPC** and **IPC** [18].

Glivenko's Theorem 3.17. *For every set of formulas $\Gamma \cup \{\varphi\}$,*

$$\Gamma \vdash_{\mathbf{CPC}} \varphi \iff \{\neg\neg\gamma : \gamma \in \Gamma\} \vdash_{\mathbf{IPC}} \neg\neg\varphi.$$

As a consequence, we obtain the desired result.

Corollary 3.18. *The class of Heyting algebras is an algebraic semantics for **CPC**.*

Proof. For every set of formulas $\Gamma \cup \{\varphi\}$, we have

$$\begin{aligned} \Gamma \vdash_{\mathbf{CPC}} \varphi &\iff \{\neg\neg\gamma : \gamma \in \Gamma\} \vdash_{\mathbf{IPC}} \neg\neg\varphi \\ &\iff \{\neg\neg\gamma \approx 1 : \gamma \in \Gamma\} \models_{\mathbf{HA}} \neg\neg\varphi \approx 1. \end{aligned}$$

The first equivalent above is Glivenko's Theorem, while the second is a consequence of the completeness theorem of **IPC** with respect to **HA**. As a consequence, the class of Heyting algebras is a τ -algebraic semantics for **IPC**, where $\tau = \{\neg\neg x \approx 1\}$. \(\square\)

This means that the univocal relation between **CPC** and the class of Boolean algebras cannot be explained in terms of the existence of completeness theorems only. As we shall see, this relation arises from a deeper phenomenon, known as *algebraizability* [5, 15].

Exercise 3.19. One may wonder whether the fact that **CPC** has many distinct algebraic semantics cannot be amended by restricting our attention to τ -algebraic semantics where $\tau = \{x \approx 1\}$. This is not the case, as this exercise asks you to check. Let A be the three-element algebra $\langle \{0, 1, a\}; \wedge, \vee, \neg, 0, 1 \rangle$ where $\langle A; \wedge, \vee \rangle$ is the lattice with order $0 < a < 1$ and $\neg: A \rightarrow A$ is the map described by the rule

$$\neg 0 = \neg a = 1 \text{ and } \neg 1 = 0.$$

Clearly, A is not a Boolean algebra (as there is no three-element Boolean algebra). Prove that $\{A\}$ is τ -algebraic semantics for **CPC** where $\tau = \{x \approx 1\}$. Hint: use the fact that the two-element Boolean algebra is a homomorphic image of A . \boxtimes

Indeed the existence of equational completeness theorems between a logic and a class of algebras turns out to be a very weak relation, as shown in [6, 26]. For instance, while many interesting logics lack a natural equational completeness theorem, they still admit a nonstandard one. This is the case of \mathbf{K}_ℓ , as we proceed to explain.

A logic \vdash is said to be *protoalgebraic* if there exists a set $\Delta(x, y)$ of formulas such that $\emptyset \vdash \Delta(x, x)$ and $x, \Delta(x, y) \vdash y$. Notice that all logics \vdash with a binary connective \rightarrow such that $\emptyset \vdash x \rightarrow x$ and $x, x \rightarrow y \vdash y$ are protoalgebraic, as witnessed by the set $\Delta := \{x \rightarrow y\}$. Furthermore, a logic \vdash is said to be *nontrivial* if $x \not\vdash y$.

Theorem 3.20 (M. [26, Thm. 9.3]). *A nontrivial protoalgebraic logic \vdash has an algebraic semantics if and only if there are two distinct formulas φ and ψ that are logically equivalent in the sense that*

$$\delta(\varphi, \vec{z}) \dashv\vdash \delta(\psi, \vec{z}), \text{ for all } \delta(x, \vec{z}) \in T(\text{Var}).$$

As a consequence, we obtain the following.

Corollary 3.21. *The logic \mathbf{K}_ℓ has an algebraic semantics.*

Proof. Clearly, \mathbf{K}_ℓ is nontrivial and protoalgebraic. Furthermore, the formulas x and $x \wedge x$ are distinct, but logical equivalent in \mathbf{K}_ℓ . Therefore, \mathbf{K}_ℓ has an algebraic semantics in view of Theorem 3.20. \boxtimes

On the other hand, \mathbf{K}_ℓ lacks any natural equational completeness theorem.

Theorem 3.22 (M. [26, Cor. 9.7]). *No class of modal algebras is an algebraic semantics for \mathbf{K}_ℓ .*

Proof. We begin by proving that, for all $\varphi, \psi \in T(\text{Var})$,

$$\text{MA} \models \varphi \approx \psi \iff \varphi \dashv\vdash_{\mathbf{K}_\ell} \psi. \quad (6)$$

To this end, observe that

$$\begin{aligned} \text{MA} \models \varphi \approx \psi &\iff \text{MA} \models \varphi \leftrightarrow \psi \approx 1 \\ &\iff \emptyset \vdash_{\mathbf{K}_g} \varphi \leftrightarrow \psi \\ &\iff \emptyset \vdash_{\mathbf{K}_\ell} \varphi \leftrightarrow \psi \\ &\iff \varphi \dashv\vdash_{\mathbf{K}_\ell} \psi. \end{aligned}$$

The above equivalences are justified as follows. The first is an easy property of Boolean algebras, the second is a consequence of Theorem 3.15, the third holds because \mathbf{K}_g and \mathbf{K}_ℓ have the same theorems (see Exercise 3.14) and the last one because \mathbf{K}_ℓ has a standard deduction theorem.

Now, suppose, with a view to contradiction, that \mathbf{K}_ℓ has a τ -algebraic semantics $\mathbf{K} \subseteq \text{MA}$. This implies that there exists an equation $\varepsilon \approx \delta \in \tau$ such that $\text{MA} \not\models \varepsilon \approx \delta$. Thus, in view

of the above display, we can assume, by symmetry, that $\varepsilon \not\vdash_{\mathbf{K}_\ell} \delta$. This means that there are a Kripke frame $\mathbb{X} = \langle X, R \rangle$, an element $w \in X$ and a valuation v in \mathbb{X} such that $w, v \Vdash \varepsilon$ and $w, v \not\vdash \delta$.

Let $\mathbb{X}^+ = \langle X^+, R^+ \rangle$ be the Kripke frame obtained by adding a new point w^+ to \mathbb{X} and defining the relation R^+ as follows:

$$\langle p, q \rangle \in R^+ \iff p = w^+ \text{ or } \langle p, q \rangle \in R.$$

Let also v^+ be the unique evaluation in \mathbb{X}^+ such that for every $y \in \text{Var}$ and $q \in \mathbb{X}^+$:

$$q, v^+ \Vdash y \iff \text{either } (q \in X \text{ and } q, v \Vdash y) \text{ or } q = w^+.$$

From the definition of \mathbb{X}^+ and v^+ it follows that

$$q, v^+ \Vdash \varphi \iff q, v \Vdash \varphi$$

for all $\varphi \in T(\text{Var})$ and $q \in \mathbb{X}$. Consequently, as $w, v \Vdash \varepsilon$ and $w, v \not\vdash \delta$,

$$w^+, v^+ \Vdash \varepsilon \text{ and } w^+, v^+ \not\vdash \Box(\varepsilon \rightarrow \delta).$$

This implies

$$x \not\vdash_{\mathbf{K}_\ell} \Box(\varepsilon \rightarrow \delta).$$

On the other hand, clearly $\emptyset \vdash_{\mathbf{K}_\ell} \Box(\delta \rightarrow \delta)$. Consequently,

$$x, \Box(\delta \rightarrow \delta) \not\vdash_{\mathbf{K}_\ell} \Box(\varepsilon \rightarrow \delta). \quad (7)$$

Now, observe that, for every $\varphi, \psi \in T(\text{Var})$,

$$\varepsilon(x) \approx \delta(x), \varphi(\Box(\delta \rightarrow \delta)) \approx \psi(\Box(\delta \rightarrow \delta)) \models_K \varphi(\Box(\varepsilon \rightarrow \delta)) \approx \psi(\Box(\varepsilon \rightarrow \delta)).$$

Since $\varepsilon \approx \delta \in \tau(x)$, this implies

$$\tau(x), \tau(\Box(\delta \rightarrow \delta)) \models_K \tau(\Box(\varepsilon \rightarrow \delta)).$$

Since \mathbf{K} is a τ -algebraic semantics for \mathbf{K}_ℓ , this yields $x, \Box(\delta \rightarrow \delta) \vdash_{\mathbf{K}_\ell} \Box(\varepsilon \rightarrow \delta)$, a contradiction with (7). \square

While, in view of Theorem 3.22, most logics have an algebraic semantics, examples of logics lacking any algebraic semantics are known since [4].

Exercise 3.23. Prove that \mathbf{K}_ℓ has a standard deduction theorem, i.e., that for every set of formulas $\Gamma \cup \{\psi, \varphi\}$,

$$\Gamma, \psi \vdash_{\mathbf{K}_\ell} \varphi \iff \Gamma \vdash_{\mathbf{K}_\ell} \psi \rightarrow \varphi.$$

Prove that this is not the case for \mathbf{K}_g . \square

Exercise 3.24. Prove that no class of distributive lattices is an algebraic semantics for the $\langle \wedge, \vee \rangle$ -fragment $\mathbf{CPC}_{\wedge\vee}$ of \mathbf{CPC} . Hint: use the fact that every equation in a single variable holds in the class of distributive lattices.

Furthermore, prove that $\mathbf{CPC}_{\wedge\vee}$ has a nonstandard algebraic semantics. To this end, consider the three-element algebra $\mathbf{A} = \langle \{0^+, 0^-, 1\}; \wedge, \vee \rangle$ whose binary commutative operations are defined by the following tables

\wedge	0^-	0^+	1	\vee	0^-	0^+	1
0^-	0^+	0^+	0^+	0^-	0^+	0^+	1
0^+		0^-	0^+	0^+		0^-	1
1			1	1			1

and prove that $\{A\}$ is a τ -algebraic semantics for $\tau = \{x \approx x \wedge x\}$. Conclude that $\mathbf{CPC}_{\wedge\vee}$ is another example of logic that admits a nonstandard algebraic semantics, but lacks a standard one. \square

4. ULTRAPRODUCTS AND UNIVERSAL CLASSES

In order to understand the relation between logic and algebra, we need to take a short detour in universal algebra and the theory of quasi-varieties. We begin by reviewing a product-like construction known as *ultraproduct* [1, 9]. First, recall that ultrafilters on powerset Boolean algebras $\mathcal{P}(X)$ are also called *ultrafilters on X* . Then let $\{A_i : i \in I\}$ be a family of similar algebras. The *equalizer* $\llbracket \vec{a} = \vec{c} \rrbracket$ of a pair of elements $\vec{a}, \vec{c} \in \prod_{i \in I} A_i$ is the set of indexes on which the sequences \vec{a} and \vec{c} agree, that is,

$$\llbracket \vec{a} = \vec{c} \rrbracket := \{i \in I : \vec{a}(i) = \vec{c}(i)\}.$$

Moreover, given an ultrafilter U on the index set I , let θ_U be the binary relation on the Cartesian product $\prod_{i \in I} A_i$ defined as

$$\theta_U := \{\langle \vec{a}, \vec{c} \rangle : \llbracket \vec{a} = \vec{c} \rrbracket \in U\}.$$

Proposition 4.1. *If $\{A_i : i \in I\}$ is a family of similar algebras and U an ultrafilter on I , then θ_U is a congruence of $\prod_{i \in I} A_i$.*

Proof. We begin by proving that θ_U is an equivalence relation on $\prod_{i \in I} A_i$. To this end, consider $\vec{a}, \vec{b}, \vec{c} \in \prod_{i \in I} A_i$. We have

$$\llbracket \vec{a} = \vec{a} \rrbracket = \{i \in I : \vec{a}(i) = \vec{a}(i)\} = I.$$

Observe that $I \in U$, since U is a nonempty upset of $\mathcal{P}(I)$. Together with the above display, this yields $\llbracket \vec{a} = \vec{a} \rrbracket \in U$ and, therefore, $\langle \vec{a}, \vec{a} \rangle \in \theta_U$. It follows that θ_U is reflexive. To prove that it is symmetric, suppose that $\langle \vec{a}, \vec{c} \rangle \in \theta_U$. Then $\llbracket \vec{a} = \vec{c} \rrbracket \in U$. Since $\llbracket \vec{a} = \vec{c} \rrbracket = \llbracket \vec{c} = \vec{a} \rrbracket$, this implies $\llbracket \vec{c} = \vec{a} \rrbracket \in U$ and, therefore, $\langle \vec{c}, \vec{a} \rangle \in \theta_U$. Lastly, to prove that θ_U is transitive, suppose that $\langle \vec{a}, \vec{b} \rangle, \langle \vec{b}, \vec{c} \rangle \in \theta_U$, that is, $\llbracket \vec{a} = \vec{b} \rrbracket, \llbracket \vec{b} = \vec{c} \rrbracket \in U$. Since U is closed under binary meets,

$$\llbracket \vec{a} = \vec{b} \rrbracket \cap \llbracket \vec{b} = \vec{c} \rrbracket \in U$$

Clearly, $\llbracket \vec{a} = \vec{b} \rrbracket \cap \llbracket \vec{b} = \vec{c} \rrbracket \subseteq \llbracket \vec{a} = \vec{c} \rrbracket$. Since U is an upset of $\mathcal{P}(I)$, we obtain that $\llbracket \vec{a} = \vec{c} \rrbracket \in U$, whence $\langle \vec{a}, \vec{c} \rangle \in \theta_U$. We conclude that θ_U is an equivalence relation.

To prove that θ_U is a congruence, it only remains to show that it preserves the basic operations. Accordingly, let f be a basic n -ary operation and $\vec{a}_1, \dots, \vec{a}_n, \vec{c}_1, \dots, \vec{c}_n \in \prod_{i \in I} A_i$ such that

$$\langle \vec{a}_1, \vec{c}_1 \rangle, \dots, \langle \vec{a}_n, \vec{c}_n \rangle \in \theta_U.$$

By definition of θ_U , this amounts to $\llbracket \vec{a}_1 = \vec{c}_1 \rrbracket, \dots, \llbracket \vec{a}_n = \vec{c}_n \rrbracket \in U$. Since U is a filter, it is closed under finite meets, whence

$$\llbracket \vec{a}_1 = \vec{c}_1 \rrbracket \cap \dots \cap \llbracket \vec{a}_n = \vec{c}_n \rrbracket \in U. \quad (8)$$

We will show that

$$\llbracket \vec{a}_1 = \vec{c}_1 \rrbracket \cap \dots \cap \llbracket \vec{a}_n = \vec{c}_n \rrbracket \subseteq \llbracket f^{\prod_{i \in I} A_i}(\vec{a}_1, \dots, \vec{a}_n) = f^{\prod_{i \in I} A_i}(\vec{c}_1, \dots, \vec{c}_n) \rrbracket. \quad (9)$$

To this end, consider $j \in \llbracket \vec{a}_1 = \vec{c}_1 \rrbracket \cap \dots \cap \llbracket \vec{a}_n = \vec{c}_n \rrbracket$. We have

$$\vec{a}_1(j) = \vec{c}_1(j), \dots, \vec{a}_n(j) = \vec{c}_n(j).$$

Consequently,

$$\begin{aligned} f^{\prod_{i \in I} A_i}(\vec{a}_1, \dots, \vec{a}_n)(j) &= f^{A_j}(\vec{a}_1(j), \dots, \vec{a}_n(j)) \\ &= f^{A_j}(\vec{c}_1(j), \dots, \vec{c}_n(j)) \\ &= f^{\prod_{i \in I} A_i}(\vec{c}_1, \dots, \vec{c}_n)(j), \end{aligned}$$

that is, $j \in \llbracket f^{\prod_{i \in I} A_i}(\vec{a}_1, \dots, \vec{a}_n) = f^{\prod_{i \in I} A_i}(\vec{c}_1, \dots, \vec{c}_n) \rrbracket$. This establishes (9). Since U is an upset of $\mathcal{P}(I)$, from (8) and (9) it follows

$$\llbracket f^{\prod_{i \in I} A_i}(\vec{a}_1, \dots, \vec{a}_n) = f^{\prod_{i \in I} A_i}(\vec{c}_1, \dots, \vec{c}_n) \rrbracket \in U.$$

Hence, we conclude that $\langle f^{\prod_{i \in I} A_i}(\vec{a}_1, \dots, \vec{a}_n), f^{\prod_{i \in I} A_i}(\vec{c}_1, \dots, \vec{c}_n) \rangle \in \theta_U$, as desired. \square

In view of the above result, we can make the following definition.

Definition 4.2. An *ultraproduct* of a family of similar algebras $\{A_i : i \in I\}$ is an algebra of the form $\prod_{i \in I} A_i / \theta_U$, for some ultrafilter U on I .

Given a class of similar algebras \mathbf{K} , we set

$$\mathbb{P}_U(\mathbf{K}) := \{A : A \text{ is an ultraproduct of a family } \{B_i : i \in I\} \subseteq \mathbf{K}\}.$$

Notice that $\mathbb{P}_U(\mathbf{K}) \subseteq \mathbb{HP}(\mathbf{K})$. Furthermore, as usual, when $\mathbf{K} = \{A\}$, we write $\mathbb{P}_U(A)$ as a shorthand for $\mathbb{P}_U(\{A\})$.

Exercise 4.3. Prove that if U is not free (that is, it is principal), then $\prod_{i \in I} A_i / \theta_U$ is isomorphic to some A_i . Conclude that if I is finite, then $\prod_{i \in I} A_i / \theta_U$ belongs to $\mathbb{I}\{A_i : i \in I\}$. Because of this, interesting ultraproducts arise from free ultrafilters only. \square

Exercise 4.4. Prove that \mathbf{K} is a finite set of finite algebras, $\mathbb{P}_U(\mathbf{K}) \subseteq \mathbb{I}(\mathbf{K})$. \square

The importance of ultraproducts is largely due to the following result [8, Thm. V.2.9].

Łoś' Theorem 4.5. Let $\{A_i : i \in I\}$ be a family of similar algebras, U an ultrafilter on I and $\phi(x_1, \dots, x_n)$ a first order formula. For every $\vec{a}_1, \dots, \vec{a}_n \in \prod_{i \in I} A_i$,

$$\prod_{i \in I} A_i / \theta_U \models \phi(\vec{a}_1 / \theta_U, \dots, \vec{a}_n / \theta_U) \iff \{i \in I : A_i \models \phi(\vec{a}_1(i), \dots, \vec{a}_n(i))\} \in U.$$

Corollary 4.6. Let $\{A_i : i \in I\}$ be a family of similar algebras, U an ultrafilter on I and ϕ a sentence. If ϕ is valid in all the A_i , then it is valid in $\prod_{i \in I} A_i / \theta_U$.

In view of Łoś' Theorem, ultraproducts are instrumental to construct nonstandard models of first order theories. For instance, let $\mathbb{N} = \langle \mathbb{N}; s, +, \cdot, 0 \rangle$ be the standard model of Peano Arithmetic. If U is an ultrafilter on \mathbb{N} , the ultraproduct $\prod_{n \in \mathbb{N}} \mathbb{N}_n / U$ is *elementarily equivalent* to \mathbb{N} , that is, it satisfies the same sentences as \mathbb{N} . On the other hand, it is not hard to see that if U is free, $\prod_{n \in \mathbb{N}} \mathbb{N}_n / U$ is uncountable and, therefore, contains many “infinite” (or nonstandard) natural numbers.

For the present purpose, however, we will not need the full strength of Łoś' Theorem and, therefore, we shall omit its proof. Instead, we shall focus on a particular embedding theorem for ultraproducts that depends on the following notion.

Definition 4.7. A *local subgraph* \mathbb{X} of an algebra A is a finite subset $X \subseteq A$ endowed with the restriction of finitely many basic operations of A to X .

In this case, \mathbb{X} is a finite *partial* algebra of finite type (even when the type of A is infinite).

Let A and B be similar algebras and \mathbb{X} a local subgraph of A . A map $f: X \rightarrow B$ is said to be an *embedding* of \mathbb{X} into B if it is injective and, for every basic n -ary operation g of the type of \mathbb{X} and $a_1, \dots, a_n \in X$ such that $g^A(a_1, \dots, a_n) \in X$,

$$f(g^A(a_1, \dots, a_n)) = g^B(f(a_1), \dots, f(a_n)).$$

Theorem 4.8. *Let $K \cup \{A\}$ be a class of similar algebras. If every local subgraph of A can be embedded into some member of K , then $A \in \mathbb{ISP}_U(K)$.*

Proof. Let I be the set of local subgraphs of A . By assumption, for every $\mathbb{X} \in I$ there are an algebra $B_{\mathbb{X}} \in K$ and an embedding $h_{\mathbb{X}}: \mathbb{X} \rightarrow B_{\mathbb{X}}$. We define a partial order \sqsubseteq on I as follows:

$$\mathbb{X} \sqsubseteq \mathbb{Y} \iff X \subseteq Y \text{ and the type of } \mathbb{Y} \text{ extends that of } \mathbb{X}.$$

Then, for every $\mathbb{X} \in I$, define

$$J_{\mathbb{X}} := \{\mathbb{Y} \in I : \mathbb{X} \sqsubseteq \mathbb{Y}\}.$$

Moreover, let \mathcal{F} be the filter of $\mathcal{P}(I)$ generated by $\{J_{\mathbb{X}} : \mathbb{X} \in I\}$. Recall that

$$\mathcal{F} = \{Y \subseteq I : J_{\mathbb{X}_1} \cap \dots \cap J_{\mathbb{X}_n} \subseteq Y, \text{ for some } \mathbb{X}_1, \dots, \mathbb{X}_n \in I\}.$$

We will prove that \mathcal{F} is proper. To this end, consider $\mathbb{X}_1, \dots, \mathbb{X}_n \in I$. Then let \mathbb{Y} be the local subgraph of A with universe $Y := X_1 \cup \dots \cup X_n$ and whose type in the union of the types of the various \mathbb{X}_i . Then

$$\mathbb{X}_i \sqsubseteq \mathbb{Y}, \text{ for every } i \leq n,$$

that is, $\mathbb{Y} \in J_{\mathbb{X}_1} \cap \dots \cap J_{\mathbb{X}_n}$. It follows that $\emptyset \notin \mathcal{F}$ and, therefore, that \mathcal{F} is proper. As \mathcal{F} is a proper filter, by the Ultrafilter Lemma, it can be extended to an ultrafilter U on I .

Now, consider a map

$$f: A \rightarrow \prod_{\mathbb{X} \in I} B_{\mathbb{X}}$$

such that $f(a)(\mathbb{X}) = h_{\mathbb{X}}(a)$, for every $a \in A$ and $\mathbb{X} \in I$ such that $a \in \mathbb{X}$. Moreover, let

$$f^*: A \rightarrow \prod_{\mathbb{X} \in I} B_{\mathbb{X}} / \theta_U$$

be the map defined by the rule

$$f^*(a) := f(a) / \theta_U.$$

We will show f^* is an embedding of A into $\prod_{\mathbb{X} \in I} B_{\mathbb{X}} / \theta_U$.

In order to prove that f^* is injective, consider a pair of distinct elements $a, c \in A$. Consider a local subgraph \mathbb{Y} of A containing a and c . We will show that

$$J_{\mathbb{Y}} \subseteq \{\mathbb{X} \in I : f(a)(\mathbb{X}) \neq f(c)(\mathbb{X})\} \quad (10)$$

Consider $\mathbb{X} \in J_{\mathbb{Y}}$. Then $\mathbb{Y} \sqsubseteq \mathbb{X}$ and, therefore, $a, c \in Y \subseteq X$. Since $a, c \in X$, we have

$$f(a)(\mathbb{X}) = h_{\mathbb{X}}(a) \text{ and } f(c)(\mathbb{X}) = h_{\mathbb{X}}(c).$$

Furthermore, $h_{\mathbb{X}}(a) \neq h_{\mathbb{X}}(c)$, because $h_{\mathbb{X}}$ is injective and $a \neq c$. This yields $f(a)(\mathbb{X}) \neq f(c)(\mathbb{X})$, establishing (10).

Recall that the definition of U guarantees that $J_{\mathbb{Y}} \in \mathcal{F} \subseteq U$. Therefore, since U is an upset of $\mathcal{P}(I)$, we can apply (10) obtaining

$$I \setminus \llbracket f(a) = f(c) \rrbracket = \{\mathbb{X} \in I : f(a)(\mathbb{X}) \neq f(c)(\mathbb{X})\} \in U.$$

Since U is a proper filter, this implies

$$\llbracket f(a) = f(c) \rrbracket \notin U$$

and, therefore,

$$f^*(a) = f(a)/\theta_U \neq f(c)/\theta_U = f^*(c).$$

Hence, we conclude that f^* is injective.

To prove that it is a homomorphism, consider a basic n -ary operation g and $a_1, \dots, a_n \in A$. Then consider a local subgraph \mathbb{Y} of A whose universe contains $a_1, \dots, a_n, g^A(a_1, \dots, a_n)$ and whose type contains g . We will prove that

$$J_{\mathbb{Y}} \subseteq \llbracket f(g^A(a_1, \dots, a_n)) = g^{\prod_{\mathbb{X} \in I} B_{\mathbb{X}}}(f(a_1), \dots, f(a_n)) \rrbracket. \quad (11)$$

Consider $\mathbb{V} \in J_{\mathbb{Y}}$. Since $\mathbb{Y} \sqsubseteq \mathbb{V}$, the type of \mathbb{V} contains g and $a_1, \dots, a_n, g^A(a_1, \dots, a_n) \in V$. Since $a_1, \dots, a_n, g^A(a_1, \dots, a_n) \in V$, we have

$$\begin{aligned} f(a_1)(\mathbb{V}) &= h_{\mathbb{V}}(a_1) \\ &\vdots \\ f(a_n)(\mathbb{V}) &= h_{\mathbb{V}}(a_n) \\ f(g^A(a_1, \dots, a_n))(\mathbb{V}) &= h_{\mathbb{V}}(g^A(a_1, \dots, a_n)). \end{aligned}$$

Furthermore, as the type of \mathbb{V} contains g ,

$$h_{\mathbb{V}}(g^A(a_1, \dots, a_n)) = g^{B_{\mathbb{V}}}(h_{\mathbb{V}}(a_1), \dots, h_{\mathbb{V}}(a_n)).$$

From the above displays it follows

$$f(g^A(a_1, \dots, a_n))(\mathbb{V}) = g^{B_{\mathbb{V}}}(f(a_1)(\mathbb{V}), \dots, f(a_n)(\mathbb{V})) = g^{\prod_{\mathbb{X} \in I} B_{\mathbb{X}}}(f(a_1), \dots, f(a_n))(\mathbb{V}),$$

that is, $\mathbb{V} \in \llbracket f(g^A(a_1, \dots, a_n)) = g^{\prod_{\mathbb{X} \in I} B_{\mathbb{X}}}(f(a_1), \dots, f(a_n)) \rrbracket$. This establishes (11). Lastly, as $J_{\mathbb{Y}} \in U$ and U is an upset of $\mathcal{P}(I)$, condition (11) implies

$$\llbracket f(g^A(a_1, \dots, a_n)) = g^{\prod_{\mathbb{X} \in I} B_{\mathbb{X}}}(f(a_1), \dots, f(a_n)) \rrbracket \in U,$$

and, therefore,

$$\begin{aligned} f^*(g^A(a_1, \dots, a_n)) &= f(g^A(a_1, \dots, a_n))/\theta_U \\ &= g^{\prod_{\mathbb{X} \in I} B_{\mathbb{X}}}(f(a_1), \dots, f(a_n))/\theta_U \\ &= g^{\prod_{\mathbb{X} \in I} B_{\mathbb{X}}/\theta_U}(f(a_1)/\theta_U, \dots, f(a_n)/\theta_U) \\ &= g^{\prod_{\mathbb{X} \in I} B_{\mathbb{X}}/\theta_U}(f^*(a_1), \dots, f^*(a_n)). \end{aligned}$$

Hence, we conclude that f^* is a homomorphism and, therefore, an embedding of A into $\prod_{\mathbb{Y} \in I} B_{\mathbb{Y}}/\theta_U$. As a consequence,

$$A \in \text{ISP}_U(\{B_{\mathbb{X}} : \mathbb{X} \in I\}) \subseteq \text{ISP}_U(K). \quad \square$$

Corollary 4.9. *Every algebra embeds into an ultraproduct of its finitely generated subalgebras.*

Exercise 4.10. Consider the consequence relation \vdash on the set of sentences of a given algebraic language defined as follows:

$$\begin{aligned} \Gamma \vdash \varphi &\iff \text{for every algebra } A \text{ of the appropriate type,} \\ &\text{if } A \models \Gamma, \text{ then } A \models \varphi. \end{aligned}$$

Notice that, in view of the Completeness Theorem, \vdash is classical first order logic for algebraic languages. This exercise asks you to prove the Compactness Theorem for \vdash using the ultraproduct construction (the same proof works for relational languages too). To this end, consider a set of sentences $\Gamma \cup \{\varphi\}$ such that $\Delta \not\models \varphi$, for every finite $\Delta \subseteq \Gamma$. Then we can associate with every finite $\Delta \subseteq \Gamma$ an algebra A_Δ such that

$$A_\Delta \models \Delta \text{ and } A_\Delta \not\models \varphi.$$

Prove that there exists an ultraproduct B of the family $\{A_\Delta : \Delta \text{ is a finite subset of } \Gamma\}$ such that $B \models \Gamma$ and $B \not\models \varphi$. Then conclude that $\Gamma \not\models \varphi$, as desired. Hint: use Łoś' Theorem. \square

Theorem 4.8 is related to the following first order formulas.

Definition 4.11. A sentence is said to be *universal* if it is of the form $\forall x_1, \dots, x_n \varphi$ for some quantifier free formula φ . Accordingly, a class of similar algebras is said to be *universal* if it can be axiomatized by a set of universal sentences.

The following concept is instrumental for describing universal classes.

Definition 4.12. Let \mathbb{X} be a local subgraph of an algebra A and assume that the universe and the type of \mathbb{X} are, respectively, $\{a_1, \dots, a_n\}$ and f_1, \dots, f_t .

(i) The *positive atomic diagram* of \mathbb{X} is the set of equations

$$\mathcal{D}^+(\mathbb{X}) := \{f_i(x_{m_1}, \dots, x_{m_k}) \approx x_j : m_1, \dots, m_k, j \leq n \text{ and } i \leq t \text{ and } f_i^A(a_{m_1}, \dots, a_{m_k}) = a_j\}.$$

(ii) The *negative atomic diagram* of \mathbb{X} is the set of negated equations

$$\mathcal{D}^-(\mathbb{X}) := \{x_m \not\approx x_k : m, k \leq n \text{ and } a_m \neq a_k\}.$$

Theorem 4.13 (Łoś). *A class of similar algebras is universal if and only if it is closed under \mathbb{I} , \mathbb{S} and \mathbb{P}_U .*

Proof. The implication from left to right follows from the easy observation that the validity of universal sentences is preserved by \mathbb{I} , \mathbb{S} and \mathbb{P}_U . To prove the converse, consider a class K of similar algebras closed under \mathbb{I} , \mathbb{S} and \mathbb{P}_U . Moreover, let $\text{Th}_\forall(K)$ be the set of universal sentences valid in K and let K_\forall be the class of algebras satisfying $\text{Th}_\forall(K)$. In order to conclude the proof, it suffices to show that $K = K_\forall$.

The inclusion $K \subseteq K_\forall$ follows immediately from the fact that $K \models \text{Th}_\forall(K)$. To prove the other inclusion, consider $A \in K_\forall$. We will show that every local subgraph of A can be embedded into an element of K . To this end, consider a local subgraph \mathbb{X} of A with universe $\{a_1, \dots, a_n\}$ and take the sentence

$$\Phi := \exists x_1, \dots, x_n \left(\bigwedge \mathcal{D}^+(\mathbb{X}) \ \& \ \bigwedge \mathcal{D}^-(\mathbb{X}) \right).$$

Now, suppose, with a view to contradiction, that $K \models \neg\Phi$. Since $\neg\Phi$ is equivalent to a universal sentence and $A \models \text{Th}_\forall(K)$, we obtain $A \models \neg\Phi$. But this is false, as witnessed by the assignment $x_1 \mapsto a_1, \dots, x_n \mapsto a_n$ in A . Thus, we conclude that $K \not\models \neg\Phi$. Consequently, there exists an element $B \in K$ such that $B \models \Phi$. Let then b_1, \dots, b_n be the elements that witness the validity of the existential part of Φ in B . The map $f: X \rightarrow B$ defined by the rule $a_i \mapsto b_i$ is an embedding of \mathbb{X} into B , as desired.

Since every local subgraph of A can be embedded into a member of K , we can apply Theorem 4.8 obtaining that $A \in \mathbb{ISP}_U(K)$. As, by assumption, K is closed under \mathbb{I} , \mathbb{S} and \mathbb{P}_U , this yields $A \in K$. Hence, we conclude that $K_\forall \subseteq K$. \square

Given a class of similar algebras K , the least universal class extending K exists and will be denoted by $\mathbb{U}(K)$ and called the universal class *generated* by K .

Corollary 4.14. *If K is a class of similar algebras, $\mathbb{U}(K) = \mathbb{ISP}_U(K)$.*

Proof. From Theorem 4.13 and the fact that $\mathbb{U}(K)$ is a universal class it follows that it is closed under \mathbb{I} , \mathbb{S} and \mathbb{P}_U , whence $\mathbb{ISP}_U(K) \subseteq \mathbb{U}(K)$. To prove the reverse inclusion, observe that $\mathbb{U}(K)$ is the class of all algebras satisfying all the universal sentences valid in K . In the proof of Theorem 4.13, we showed that this guarantees the inclusion $\mathbb{U}(K) \subseteq \mathbb{ISP}_U(K)$. \square

Corollary 4.15. *Let $K \cup \{A\}$ be a class of similar algebras. If $A \in \mathbb{U}(K)$, then every local subgraph of A embeds into some member of K .*

Proof. This is established in the second part of the proof of Theorem 4.13. \square

The following provides an algebraic path to the strong finite model property in logic.

Definition 4.16. A class of similar algebras K is said to have the *finite embeddability property* (FEP, for short) if every local subgraph of a member of K can be embedded into a finite member of K .

Given a class of algebras K , we denote by $K^{<\omega}$ the class of its finite members.

Proposition 4.17. *A universal class K has the FEP if and only if $K = \mathbb{U}(K^{<\omega})$.*

Proof. Suppose first that K has the FEP. Then

$$K \subseteq \mathbb{ISP}_U(K^{<\omega}) = \mathbb{U}(K^{<\omega}).$$

The first inclusion in the above display follows from Theorem 4.8 and the second from Corollary 4.14. Furthermore, since K is a universal class, $\mathbb{U}(K^{<\omega}) \subseteq K$. Therefore, we conclude that $K = \mathbb{U}(K^{<\omega})$. To prove the converse, suppose that $K = \mathbb{U}(K^{<\omega})$. By Corollary 4.15, every local subgraph of a member of K can be embedded into some element of $K^{<\omega}$, that is, K has the FEP. \square

The *universal theory* $\text{Th}_\forall(K)$ of a class of algebras K is the set of universal sentences valid in K . $\text{Th}_\forall(K)$ is said to be *decidable* when so is the problem of determining whether a universal sentence is valid in K . Furthermore, we say that a class of algebras is *finitely axiomatizable* if it can be axiomatized by finitely many sentences.

Proposition 4.18. *Let K be a finitely axiomatizable class of algebras. If K has the FEP, then $\text{Th}_\forall(K)$ is decidable.*

Proof. In order to prove that $\text{Th}_\forall(K)$ is decidable it suffices to show that

- (i) the problem of determining whether a universal sentence belongs to $\text{Th}_\forall(K)$ is semidecidable; and
- (ii) the problem of determining whether a universal sentence does not belong to $\text{Th}_\forall(K)$ is semidecidable.

Condition (i) holds, because K is finitely axiomatizable. Therefore, it only remains to prove (ii). To this end, let Σ be a finite set of axioms for K and $\{f_1, \dots, f_n\}$ the function symbols that appear in Σ . Given a universal sentence Φ , we enumerate the finite models A_1, A_2, \dots of Σ in the type $\{f_1, \dots, f_n, g_1, \dots, g_m\}$, where g_1, \dots, g_m are the function symbols that occur in Φ . This can be done mechanically, because Σ is finite. Our algorithm

tests if Φ fails in some A_n . If this is the case, it stops and answers that Φ does not belong to $\text{Th}_\forall(K)$, otherwise it runs forever.

In order to establish (ii), it suffices to prove that the algorithm stops if and only if $\Phi \notin \text{Th}_\forall(K)$. First, if it stops, then Φ fails in some A_n . Let then B be an algebra of the type of K obtained by expanding A_n with an arbitrary interpretation of the missing function symbols. From $A_n \not\models \Phi$ it follows $B \not\models \Phi$. Moreover, since $A_n \models \Sigma$, we obtain $B \models \Sigma$ and, therefore, $B \in K$ (as Σ axiomatizes K). Hence, we conclude that $\Phi \notin \text{Th}_\forall(K)$.

To prove the converse, consider a universal sentence $\forall \vec{x} \varphi$ that fails in K . Then there exist $B \in K$ and $b_1, \dots, b_n \in B$ such that $B \not\models \varphi(b_1, \dots, b_n)$. Let \mathbb{X} be the local subgraph of B whose universe is $\{b_1, \dots, b_n\}$ and whose type consists of the function symbols occurring in φ . Since K has the FEP, there is an embedding $f: \mathbb{X} \rightarrow C$, for some $C \in K^{<\omega}$. It follows that $C \not\models \varphi(f(a_1), \dots, f(a_n))$, whence $\forall \vec{x} \varphi$ fails in C . Let C^- be the reduct of C to the language \mathcal{L} consisting of the function symbols occurring in $\Sigma \cup \{\varphi\}$. Clearly, $C^- \not\models \varphi(f(a_1), \dots, f(a_n))$ and, therefore, $\forall \vec{x} \varphi$ fails in C^- . As C^- is a finite model of Σ in the language \mathcal{L} , there must be some $n \in \mathbb{N}$ such that $C^- \cong A_n$. It follows that $\forall \vec{x} \varphi$ fails in A_n , whence the algorithm stops, as desired. \square

Example 4.19 (Lattices). We will prove that the class **Latt** of all lattices has the FEP. For consider a lattice A and let \mathbb{X} be one of its local subgraphs. Let also

$$B := \{a_1 \wedge^A \dots \wedge^A a_n : a_1, \dots, a_n \in X \text{ and } n \in \mathbb{N}\}.$$

Since the operation \wedge^A is idempotent, commutative and associative, the set B is finite. Furthermore, B can be viewed as a subposet of A . Let B^+ be the poset obtained extending $\langle B; \leq \rangle$ with a new top element. Since B^+ is a finite meet-semilattice with maximum, it is also a lattice. Furthermore, \mathbb{X} embeds into B^+ . Hence, **Latt** has the FEP. By Propositions 4.18 and 4.17, $\text{Th}_\forall(\text{Latt})$ is decidable [31] and $\text{Latt} = \mathbb{U}(\text{Latt}^{<\omega})$. On the other hand, the first order theory of distributive lattices (and, therefore, of any nontrivial equational class of lattices) is undecidable [20]. \square

Example 4.20 (Heyting algebras). We will prove that the class **HA** of Heyting algebras has the FEP. For consider a Heyting algebra A and let \mathbb{X} be one of its local subgraphs. Then let B be the bounded sublattice of A generated by \mathbb{X} . Notice that B is finite, because it is a finitely generated bounded distributive lattice. Since B is a finite distributive lattice, it can be viewed as a finite Heyting algebra B^+ . Furthermore, it is easy to see that the identity map is an embedding of \mathbb{X} into B^+ . Hence, we conclude that **HA** has the FEP. By Propositions 4.18 and 4.17, $\text{Th}_\forall(\text{HA})$ is decidable and $\text{HA} = \mathbb{U}(\text{HA}^{<\omega})$ [25]. On the other hand, the first order theory of nontrivial equational class of Heyting algebras other than that of Boolean algebras is known to be undecidable [7].

Notably, the fact that $\text{Th}_\forall(\text{HA})$ is decidable implies that **IPC** is also decidable, in the sense that, given a finite set of formulas $\{\gamma_1, \dots, \gamma_n, \varphi\}$, we can decide whether $\gamma_1, \dots, \gamma_n \vdash_{\text{IPC}} \varphi$. This is because, in view of the fact that **HA** is a $\{x \approx 1\}$ -algebraic semantics for **IPC**, we have

$$\begin{aligned} \gamma_1, \dots, \gamma_n \vdash_{\text{IPC}} \varphi &\iff \gamma_1 \approx 1, \dots, \gamma_n \approx 1 \models_{\text{HA}} \varphi \approx 1 \\ &\iff \text{HA} \models \forall \vec{x} ((\gamma_1 \approx 1 \ \& \ \dots \ \& \ \gamma_n \approx 1) \implies \varphi \approx 1). \end{aligned}$$

As $\forall \vec{x} ((\gamma_1 \approx 1 \ \& \ \dots \ \& \ \gamma_n \approx 1) \implies \varphi \approx 1)$ is a universal sentence, we can decide whether it holds in **HA** or not, because the universal theory of **HA** is decidable. \square

Exercise 4.21. The methods in [24] can be used to show that the class MA of modal algebras has the FEP. For let \mathbb{X} be a local subgraph of a model algebra A . Moreover, let B be the Boolean subalgebra of A generated by $X \cup \{\Box 0\}$. We consider the algebra B^+ obtained by endowing B with a unary operation \Box defined as follows:

$$\Box^{B^+} b := \bigvee^B \{a \in B : \Box^A a \in B \text{ and } a \leq b\}.$$

Prove that B^+ is a modal algebra and that \mathbb{X} embeds into B^+ . Then conclude that $\text{Th}_V(\text{MA})$ is decidable and $\text{MA} = \mathbb{U}(\text{MA}^{<\omega})$. Use these facts to infer that the logic \mathbf{K}_g is decidable. \square

5. QUASI-VARIETIES

For the theory of quasi-varieties, we refer the reader to [19, 23].

Definition 5.1. A class of similar algebras closed under $\mathbb{I}, \mathbb{S}, \mathbb{P}$ and \mathbb{P}_U is said to be a *quasi-variety*.

Examples of quasi-varieties include the classes of Boolean, Heyting and modal algebras, as well as the class of (bounded) distributive lattices and groups. Our aim will be to prove that quasi-varieties are precisely the classes of algebras axiomatized by the following kind of first order formulas.

Definition 5.2. A *quasi-equation* of type ρ is an expression Φ of the form

$$(\varphi_1 \approx \psi_1 \ \& \ \dots \ \& \ \varphi_n \approx \psi_n) \implies \varepsilon \approx \delta,$$

where $\{\varphi_1 \approx \psi_1, \dots, \varphi_n \approx \psi_n, \varepsilon \approx \delta\}$ is a set of equations of type ρ . Then Φ is *valid* in an algebra A of type ρ when so is its universal closure, that is, for every $\vec{a} \in A$,

$$\text{if } \varphi_1^A(\vec{a}) = \psi_1^A(\vec{a}), \dots, \varphi_n^A(\vec{a}) = \psi_n^A(\vec{a}), \text{ then } \varepsilon^A(\vec{a}) = \delta^A(\vec{a}).$$

In this case, we often say that A *satisfies* Φ and write $A \models \Phi$. Lastly, a quasi-equation is said to be an *equation* when its set of premises is empty.

Notably, for every class of algebras K and equations $\varphi_1 \approx \psi_1, \dots, \varphi_n \approx \psi_n, \varepsilon \approx \delta$,

$$K \models (\bigwedge_{i \leq n} \varphi_i \approx \psi_i) \implies \varepsilon \approx \delta \text{ iff } \{\varphi_1 \approx \psi_1, \dots, \varphi_n \approx \psi_n\} \models_K \varepsilon \approx \delta.$$

Remark 5.3. It might seem that we are using the term *equations* to refer to two distinct kinds of expressions, namely those of the form $\varepsilon \approx \delta$ and $\emptyset \implies \varepsilon \approx \delta$. This is not a problem, however, because these expressions are synonyms, in the sense that an algebra satisfies $\varepsilon \approx \delta$ if and only if it satisfies $\emptyset \implies \varepsilon \approx \delta$. Because of this, we will continue to denote equations by $\varepsilon \approx \delta$, while keeping in mind that they are special instances of quasi-equations. \square

The aim of this section is to prove the following classical result.

Maltsev's Theorem 5.4. A class of similar algebras is a quasi-variety if and only if it can be axiomatized by a set of quasi-equations.

Proof. The “only if” part follows from the fact that the validity of quasi-equations is preserved by the class operators $\mathbb{I}, \mathbb{S}, \mathbb{P}$ and \mathbb{P}_U . To prove the converse, consider a prevariety K closed under \mathbb{P}_U . Moreover, let Σ the set of quasi-equations valid in K . Let also K^+ be the class of algebras axiomatized by Σ . Our aim is to prove that $K = K^+$.

The inclusion $K \subseteq K^+$ is straightforward. To prove the other one, consider an algebra $A \in K^+$. In order to prove that $A \in K^+$, it suffices to show that every local subgraph of A embeds in some members of K . This is because, in this case, $A \in \mathbb{ISP}_U(K)$, by Theorem 4.8. Since K is closed under \mathbb{I} , \mathbb{S} and \mathbb{P}_U , this implies $A \in K$, as desired.

Then consider a local subgraph \mathbb{X} of A . By definition, \mathbb{X} consists of a finite set $\{a_1, \dots, a_n\}$ endowed with the restriction of finitely many basic operations f_1, \dots, f_m of A to X . Observe that both $\mathcal{D}^+(\mathbb{X})$ and $\mathcal{D}^-(\mathbb{X})$ are finite sets. Then take an enumeration

$$\mathcal{D}^-(\mathbb{X}) = \{\varepsilon_1 \not\approx \delta_1, \dots, \varepsilon_t \not\approx \delta_t\}.$$

Moreover, for each $i \leq t$, consider the quasi-equation

$$\Phi_i := \left(\& \mathcal{D}^+(\mathbb{X}) \right) \implies \varepsilon_i \approx \delta_i.$$

As witnessed by the natural assignment

$$x_1 \mapsto a_1, \dots, x_n \mapsto a_n,$$

the quasi-equations Φ_1, \dots, Φ_t fail in A . Since A satisfies all the quasi-equations valid in K , this implies that each Φ_i fails in some $B_i \in K$ under an assignment

$$x_1 \mapsto b_1^i, \dots, x_n \mapsto b_n^i. \quad (12)$$

Now, consider the map $h: X \rightarrow (B_1 \times \dots \times B_t)$, defined by the rule

$$a_1 \mapsto \langle b_1^1, \dots, b_1^t \rangle, \dots, a_n \mapsto \langle b_n^1, \dots, b_n^t \rangle.$$

We will prove that h is an embedding of \mathbb{X} into $B_1 \times \dots \times B_t$. To prove that h is injective, consider two distinct elements $a_p, a_q \in X$. Then the formula $x_p \not\approx x_q$ belongs to the negative atomic diagram of \mathbb{X} . Then there exists $i \leq t$ such that

$$\Phi_i = \left(\& \mathcal{D}^+(\mathbb{X}) \right) \implies x_p \approx x_q.$$

Since Φ_i fails in B_i under the assignment in (12), we obtain $b_p^i \neq b_q^i$. As a consequence,

$$h(a_p)(i) = b_p^i \neq b_q^i = h(a_q)(i)$$

and, therefore, $h(a_p) \neq h(a_q)$. Hence, h is injective. To prove that it preserves the partial operations, consider a basic s -ary operation f_j in the type of \mathbb{X} and $a_{k_1}, \dots, a_{k_s} \in X$ such that $f_j^A(a_{k_1}, \dots, a_{k_s}) \in X$. Then there exists some $p \leq n$ such that $a_p = f_j^A(a_{k_1}, \dots, a_{k_s})$. Moreover, the equation

$$f_j(x_{k_1}, \dots, x_{k_s}) \approx x_p$$

belongs to $\mathcal{D}^+(\mathbb{X})$. As each quasi-equation Φ_i fails under the assignment in (12), the same assignment satisfies the antecedent of Φ_i , namely $\mathcal{D}^+(\mathbb{X})$. It follows that

$$f_j^{B_i}(b_{k_1}^i, \dots, b_{k_s}^i) = b_p^i, \text{ for each } i \leq t.$$

As a consequence, for every $i \leq t$,

$$\begin{aligned}
 h(f_j^A(a_{k_1}, \dots, a_{k_s}))(i) &= h(a_p)(i) \\
 &= b_p^i \\
 &= f_j^{B_i}(b_{k_1}^i, \dots, b_{k_s}^i) \\
 &= f_j^{B_i}(h(a_{k_1})(i), \dots, h(a_{k_s})(i)) \\
 &= f_j^{B_1 \times \dots \times B_t}(h(a_{k_1}), \dots, h(a_{k_s}))(i).
 \end{aligned}$$

Thus, $h(f_j^A(a_{k_1}, \dots, a_{k_s})) = f_j^{B_1 \times \dots \times B_t}(h(a_{k_1}), \dots, h(a_{k_s}))$. We conclude that $h: \mathbb{X} \rightarrow (B_1 \times \dots \times B_t)$ is an embedding. Since $B_1, \dots, B_t \in \mathbf{K}$ and \mathbf{K} is closed under \mathbb{P} , the direct product $B_1 \times \dots \times B_t$ belongs to \mathbf{K} . Hence, \mathbb{X} embeds into some member of \mathbf{K} , as desired. \square

Exercise 5.5. In view of Łoś' Theorem quasi-equations persist in ultraproducts. If you are not familiar with the proof of Łoś' Theorem, offer a direct proof of this fact. \square

Given a class of similar algebras \mathbf{K} , the least quasi-variety extending \mathbf{K} exists and will be denoted by $\mathbb{Q}(\mathbf{K})$ and called the quasi-variety *generated* by \mathbf{K} .

Proposition 5.6 (Maltsev). *For every class of algebras \mathbf{K} ,*

$$\mathbb{Q}(\mathbf{K}) = \text{ISP}_{\mathbb{U}}(\mathbf{K}).$$

Proof. The inclusion $\text{ISP}_{\mathbb{U}}(\mathbf{K}) \subseteq \mathbb{Q}(\mathbf{K})$ is obvious. To prove the other, consider $A \in \mathbb{Q}(\mathbf{K})$. By Maltsev's Theorem, $\mathbb{Q}(\mathbf{K})$ is the class of all algebras satisfying the quasi-equations valid in \mathbf{K} . The proof of the hard part of Maltsev's Theorem show that $A \in \text{ISP}_{\mathbb{U}}(\mathbf{K})$. Therefore, it only remains to show that $\mathbb{P}_{\mathbb{U}}(\mathbf{K}) \subseteq \text{ISP}_{\mathbb{U}}(\mathbf{K})$. But this is an easy exercise on class operators (the details are sketched below).

Consider an algebra $B \in \mathbb{P}_{\mathbb{U}}(\mathbf{K})$. There exists an index set I , an ultrafilter U on I and a family of algebras $\{B_j : j \in J_i\}$ for each $i \in I$ such that

$$B = \left(\prod_{i \in I} \left(\prod_{j \in J_i} B_j \right) \right) / \theta_U.$$

Let J be the set of all maps $f: I \rightarrow \bigcup_{i \in I} J_i$ such that $f(i) \in J_i$. Moreover, let

$$g: B \rightarrow \prod_{f \in J} \left(\prod_{i \in I} B_f(i) \right)$$

be the map defined by the rule $g(b)(f)(i) := b(i)(f(i))$. It is not hard to check that the map

$$g^*: B \rightarrow \left(\prod_{f \in J} \left(\prod_{i \in I} B_f(i) \right) \right) / \theta_U$$

that sends an element $b \in B$ to $f(b)/\theta_U$ is an embedding, whence $B \in \text{ISP}_{\mathbb{U}}(\mathbf{K})$. \square

Corollary 5.7. *If \mathbf{K} be a finite set of finite similar algebras, then $\mathbb{Q}(\mathbf{K}) = \text{ISP}(\mathbf{K})$.*

Proof. Since \mathbf{K} is a finite set of finite algebras, $\mathbb{P}_{\mathbb{U}}(\mathbf{K}) \subseteq \text{I}(\mathbf{K})$ (see Exercise 4.4). As a consequence, we obtain $\text{ISP}(\mathbf{K}) = \text{ISP}_{\mathbb{U}}(\mathbf{K})$. Together with Proposition 5.6, this yields $\mathbb{Q}(\mathbf{K}) = \text{ISP}(\mathbf{K})$. \square

Exercise 5.8. Prove that if K is a class of similar algebras, then $\mathbb{P}_U \mathbb{P}(K) \subseteq \mathbb{ISP}_U(K)$. Hint: use the sketch in the last part of Proposition 5.6. \square

Example 5.9 (Quasi-varieties). In view of Examples 4.19 and 4.20, we know that

$$\text{Latt} = \mathbb{ISP}_U(\text{Latt}^{<\omega}) \quad \text{and} \quad \text{HA} = \mathbb{ISP}_U(\text{HA}^{<\omega}).$$

This implies that $\text{Latt} \subseteq \mathbb{Q}(\text{Latt}^{<\omega})$ and $\text{HA} \subseteq \mathbb{Q}(\text{HA}^{<\omega})$. Since both Latt and HA are closed under $\mathbb{I}, \mathbb{S}, \mathbb{P}$ and \mathbb{P}_U , this yields

$$\text{Latt} = \mathbb{Q}(\text{Latt}^{<\omega}) \quad \text{and} \quad \text{HA} = \mathbb{Q}(\text{HA}^{<\omega}).$$

Let DL be the class of distributive lattices and \mathbf{B} the two-element distributive lattice. In view of Example 2.18, we have $\text{DL} = \mathbb{IP}_{\text{SD}}(\mathbf{B})$. Clearly, $\mathbb{IP}_{\text{SD}}(\mathbf{B}) \subseteq \mathbb{ISP}(\mathbf{B}) \subseteq \mathbb{Q}(\mathbf{B})$. On the other hand, since DL is closed under $\mathbb{I}, \mathbb{S}, \mathbb{P}$ and \mathbb{P}_U , we obtain $\mathbb{Q}(\mathbf{B}) \subseteq \text{DL}$. Hence, $\text{DL} = \mathbb{Q}(\mathbf{B})$. Similarly, the class of Boolean algebras is the quasi-variety generated by the two-element Boolean algebra (see 2.19, if necessary). \square

Exercise 5.10. Prove that there is no finite Heyting algebra A such that $\text{HA} = \mathbb{Q}(A)$, cf. with $\text{HA} = \mathbb{Q}(\text{HA}^{<\omega})$. Prove that, however, there exists an infinite Heyting algebra A such that $\text{HA} = \mathbb{Q}(A)$, see for instance [27, 32]. \square

The next observation will be needed later on.

Theorem 5.11. *If K is a quasi-variety, the consequence relation \models_K is finitary.*

Proof. In view of Maltsev's Theorem, K can be axiomatized by a set Σ of quasi-equations. Formally speaking, this means that K is the class of models of the set of sentences

$$\Sigma_{\forall} := \{\forall \vec{x} \varphi : \varphi(\vec{x}) \in \Sigma\}.$$

Bearing this in mind, let $\Theta \cup \{\varphi \approx \psi\} \subseteq E(\text{Var})$ be such that $\Theta \models_K \varphi \approx \psi$. Moreover, let $\{a_n : n \in \mathbb{N}\}$ be a set of new constants. Given a formula $\varphi(x_1, \dots, x_n)$, we write

$$\varphi(\vec{a}) \text{ as a shorthand for } \varphi(a_1, \dots, a_n).$$

Since Σ_{\forall} axiomatizes K , the fact that $\Theta \models_K \varphi \approx \psi$ is equivalent to the demand that

$$\Sigma_{\forall} \cup \{\varepsilon(\vec{a}) \approx \delta(\vec{a}) : \varepsilon \approx \delta \in \Theta\} \vdash \varphi(\vec{a}) \approx \psi(\vec{a}),$$

where \vdash is the derivability symbol of classical first order logic. Consequently, by the Compactness Theorem, there exists a finite $\Delta \subseteq \Theta$ such that

$$\Sigma_{\forall} \cup \{\varepsilon(\vec{a}) \approx \delta(\vec{a}) : \varepsilon \approx \delta \in \Delta\} \vdash \varphi(\vec{a}) \approx \psi(\vec{a}).$$

Since Σ_{\forall} axiomatizes K , this yields $\Delta \models_K \varphi \approx \psi$, as desired. \square

Observe that, for every class K of similar algebras, $\mathbb{I}(K)$ and $\mathbb{P}_U(K) \subseteq \mathbb{HIP}(K)$. Because of this, the following classes of algebras [2, 8] are special cases of quasi-varieties.

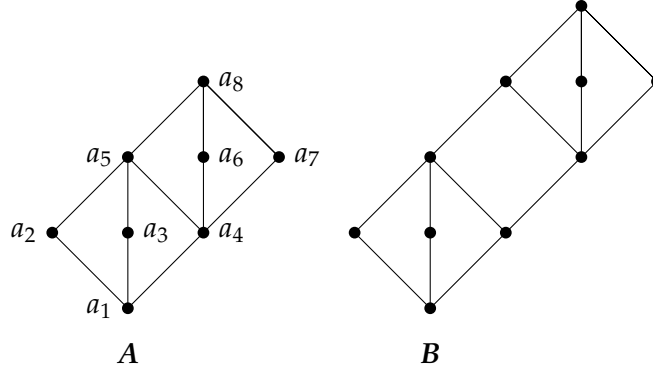
Definition 5.12. A class of similar algebras closed under \mathbb{H}, \mathbb{S} and \mathbb{P} is said to be a *variety*.

Notably, varieties coincide with equational classes of algebras [2, Thm. 4.41].

Birkhoff's Theorem 5.13. *A class of similar algebras is a variety if and only if it can be axiomatized by a set of equations.*

As a consequence, examples of varieties include the classes of (distributive) lattices, Heyting, Boolean and modal algebras. While every variety is a quasi-variety, the converse is not true in general, as we proceed to explain.

Example 5.14 (Lattices). Consider the lattices A and B depicted below. We will show that the quasi-variety $\mathbb{Q}(B)$ is not closed under \mathbb{H} and, therefore, is not a variety.



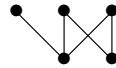
To this end, let $\mathcal{D}^+(A)$ be the positive atomic diagram of A written with the variables x_1, \dots, x_8 corresponding to the elements a_1, \dots, a_8 and consider the quasi-equation

$$\Phi = \& \mathcal{D}^+(A) \implies x_1 \approx x_8.$$

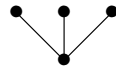
Notice that B validates Φ . To prove this, consider an assignment $f: \{x_1, \dots, x_8\} \rightarrow B$ that validates $\mathcal{D}^+(A)$ in B . Using the definition of $\mathcal{D}^+(A)$, it is easy to see that the map $h: A \rightarrow B$ that sends a_i to $f(a_i)$ is a homomorphism from A to B . Since $\text{Con}(A) = \{\text{id}_A, A \times A\}$, the kernel $\text{Ker}(h)$ is either id_A or $A \times A$. Notice that there is no embedding of A into B . Therefore, $\text{Ker}(h)$ cannot be the identity relation. It follows that $\text{Ker}(h) = A \times A$. In particular, $\langle a_1, a_8 \rangle \in \text{Ker}(h)$ and, therefore, $f(x_1) = h(a_1) = f(a_8) = f(x_8)$. Hence, we conclude that $B \models \Phi$, as desired. Moreover, Φ fails in A , as witnessed by the assignment $x_i \mapsto a_i$.

In brief, Φ holds in B but fails in A . By Maltsev's Theorem, we conclude that A does not belong to the quasi-variety $\mathbb{Q}(B)$ generated by B . On the other hand, A is a homomorphic image of B (obtained by glueing two pairs of elements of B). Thus, the quasi-variety $\mathbb{Q}(B)$ is not closed under \mathbb{H} . \square

Example 5.15 (Heyting algebras). We assume the reader is familiar with *Esakia duality* for Heyting algebras [13, 14]. Let A be the finite Heyting algebra whose dual Esakia space is the following poset A_* endowed with the discrete topology.



Our aim is to show that $\mathbb{Q}(A)$ is not closed under \mathbb{H} and, therefore, fails to be a variety. To this end, consider the finite Heyting algebra B whose dual Esakia space is the the rooted poset B_* (endowed with the discrete topology) depicted below.



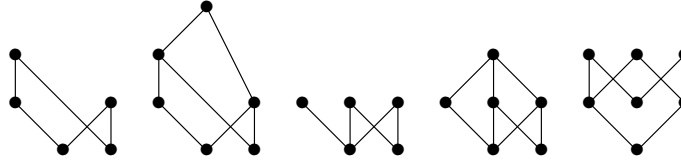
Notice that, as B_* is rooted and finite, the algebra B is subdirectly irreducible. Moreover, observe that, as B_* is an upset of A_* , by Esakia duality we obtain $B \in \mathbb{H}(A)$. Therefore, it

suffices to show that $B \notin \mathbb{Q}(A)$. Suppose the contrary, with a view to contradiction. By Corollary 5.7,

$$B \in \mathbb{ISP}(A) \subseteq \mathbb{IP}_{\text{SD}}\mathbb{S}(A).$$

As B is subdirectly irreducible, we conclude that $B \in \mathbb{IS}(A)$. By Esakia duality, this means that B_* is a p-morphic image of A_* . But a quick inspection of the posets A_* and B_* shows that this is impossible, a contradiction. Hence, we conclude that $\mathbb{Q}(A)$ does not contain B and, therefore, is not closed under \mathbb{H} . \square

Exercise 5.16. Let A_1, \dots, A_5 be the Heyting algebras whose Esakia duals are the five posets depicted below.



A variety K is said to be *primitive* whenever if $M \subseteq K$ is a quasi-variety, then M is also variety. Prove that if a variety K of Heyting algebras is primitive, then it does not contain any of the various A_i .

Notably, the converse is also true: it turns out that a variety K of Heyting algebras is primitive if and only if it omits A_1, \dots, A_5 [10, 11], see also [3] for a duality theoretic proof. A similar result has been obtained in the setting of modal logic [29]. These results admit a logical interpretation in terms of structural completeness [30]. \square

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