LECTURE NOTES ON ALGEBRAIC LOGIC

T. MORASCHINI

Lecture notes for the Algebraic Logic course of the Master in Pure and Applied Logic of the University of Barcelona, 2021

1. Algebras and equations

We begin by reviewing some fundamentals of general algebraic systems.

Definition 1.1.

(i) A *type* is a map $\rho\colon \mathcal{F} \to \mathbb{N}$, where $\mathcal{F}$ is a set of function symbols. In this case, $\rho(f)$ is said to be the *arity* of the function symbol f, for every $f \in \mathcal{F}$. Function symbols of arity zero are called *constants*.

(ii) An *algebra* of type ρ is a pair $\boldsymbol{A} = \langle A; F \rangle$ where A is a nonempty set and $F = \{f^{\boldsymbol{A}} : f \in \mathcal{F}\}$ is a set of operations on A whose arity is determined by ρ, in the sense that each $f^{\boldsymbol{A}}$ has arity $\rho(f)$. The set A is called the *universe* of $\boldsymbol{A}$.

When $\mathcal{F} = \{f_1, \dots, f_n\}$, we shall write $\langle A; f_1^{\boldsymbol{A}}, \dots, f_n^{\boldsymbol{A}} \rangle$ instead of $\langle A; F \rangle$. In this case, we often drop the superscripts, and write simply $\langle A; f_1, \dots, f_n \rangle$.

Classical examples of algebras are groups and rings. For instance, the type of groups ρ_G consists of a binary symbol $+$, a unary symbol $-$, and a constant symbol 0. Then a group is an algebra $\langle G; +, -, 0 \rangle$ of type ρ_G in which $+$ is associative, 0 is a neutral element for $+$, and $-$ produces inverses.

Lattices, Heyting algebras, and modal algebras are also algebras in the above sense. For instance, the type of lattices ρ_L consists of two binary symbols $\wedge$ and $\vee$ and a lattice is an algebra $\langle A; \wedge, \vee \rangle$ of type ρ_L that satisfies the idempotent, commutative, associative, and absorption laws. Similarly, the type of Heyting algebras ρ_H consists of three binary operations symbols $\wedge, \vee$, and $\to$ and of two constant symbols 0 and 1. Then a Heyting algebra is an algebra $\langle A; \wedge, \vee, \to, 0, 1 \rangle$ such that $\langle A; \wedge, \vee, 0, 1 \rangle$ is a bounded lattice and, for every $a, b, c \in A$,

$$a \wedge b \leqslant c \iff a \leqslant b \to c. \qquad (\textit{residuation law})$$

Boolean algebras can be viewed as the Heyting algebras that satisfy the following equational version of the *excluded middle law*:

$$x \vee (x \to 0) \approx 1.$$

In this case, the complement operation $\neg x$ can be defined as $x \to 0$.

Perhaps less obviously, even algebraic structures whose operations are apparently *external* can be viewed as algebras in the sense of the above definition. For instance, modules over a ring $\boldsymbol{R}$ can be viewed as algebras whose type ρ_R extends that of groups with the unary symbols $\{\lambda_r : r \in R\}$. From this point of view, a module over $\boldsymbol{R}$ is an

Date: February 25, 2021.

algebra $\langle G; +, -, 0, \{\lambda_r : r \in R\}\rangle$ of type ρ_R such that $\langle G; +, -, 0\rangle$ is an abelian group and, for every $r, s \in R$ and $a, c \in G$,

$$\begin{aligned}
\lambda_r(a+c) &= \lambda_r(a) + \lambda_r(c)\\
\lambda_{r+s}(a) &= \lambda_r(a) + \lambda_s(a)\\
\lambda_r(\lambda_s(a)) &= \lambda_{r\cdot s}(a)\\
\lambda_1(a) &= a.
\end{aligned}$$

Given a type $\rho\colon \mathcal{F} \to \mathbb{N}$ and a set of variables X disjoint from $\mathcal{F}$, the set of *terms of type ρ over X* is the least set $T_\rho(X)$ such that

(i) $X \subseteq T_\rho(X)$;
(ii) if $c \in \mathcal{F}$ is a constant, then $c \in T_\rho(X)$; and
(iii) if $\varphi_1, \ldots, \varphi_{\rho(f)} \in T_\rho(X)$ and $f \in \mathcal{F}$, then $f\varphi_1 \ldots \varphi_{\rho(f)} \in T_\rho(X)$.

For the sake of readability, we shall often write $f(\varphi_1, \ldots, \varphi_{\rho(f)})$ instead of $f\varphi_1 \ldots \varphi_{\rho(f)}$. Similarly, if f is a binary operation $+$, we often write $\varphi_1 + \varphi_2$ instead of $f(\varphi_1, \varphi_2)$.

Given a term $\varphi \in T_\rho(X)$, we write $\varphi(x_1, \ldots, x_n)$ to indicate that the variables occurring in φ are among $x_1, \ldots, x_n$. Furthermore, given an algebra $\boldsymbol{A}$ of type ρ and elements $a_1, \ldots, a_n \in A$, we define an element

$$\varphi^{\boldsymbol{A}}(a_1, \ldots, a_n)$$

of A, by recursion on the construction of φ, as follows:

(i) if φ is a variable x_i, then $\varphi^{\boldsymbol{A}}(a_1, \ldots, a_n) := a_i$;
(ii) if φ is a constant c, then $c^{\boldsymbol{A}}$ is the interpretation of c in $\boldsymbol{A}$;
(iii) if $\varphi = f(\psi_1, \ldots, \psi_m)$, then

$$\varphi^{\boldsymbol{A}}(a_1, \ldots, a_n) := f^{\boldsymbol{A}}(\psi_1^{\boldsymbol{A}}(a_1, \ldots, a_n), \ldots, \psi_m^{\boldsymbol{A}}(a_1, \ldots, a_n)).$$

An *equation of type ρ over X* is an expression of the form $\varphi \approx \psi$, where $\varphi, \psi \in T_\rho(X)$. Such an equation $\varphi \approx \psi$ is *valid* in an algebra $\boldsymbol{A}$ of type ρ, if

$$\varphi^{\boldsymbol{A}}(a_1, \ldots, a_n) = \psi^{\boldsymbol{A}}(a_1, \ldots, a_n), \text{ for every } a_1, \ldots, a_n \in A,$$

in which case we say that $\boldsymbol{A}$ *satisfies* $\varphi \approx \psi$.

For instance, groups are precisely the algebras of type ρ_G that satisfy the equations

$$x + (y + z) \approx (x + y) + z \quad x + 0 \approx x \quad 0 + x \approx x \quad x + -x \approx 0 \quad -x + x \approx 0.$$

Similarly, lattices are the algebras of type ρ_L that satisfy the equations

$$\begin{array}{llr}
x \wedge x \approx x & x \vee x \approx x & (\textit{idempotent laws})\\
x \wedge y \approx y \wedge x & x \vee y \approx y \vee x & (\textit{commutative laws})\\
x \wedge (y \wedge z) \approx (x \wedge y) \wedge z & x \vee (y \vee z) \approx (x \vee y) \vee z & (\textit{associative laws})\\
x \wedge (y \vee x) \approx x & x \vee (y \wedge x) \approx x. & (\textit{absorption laws})
\end{array}$$

2. Basic constructions

Algebras of the same type are called *similar* and can be compared by means of maps that preserve their structure.

Definition 2.1. Given two similar algebras $\boldsymbol{A}$ and $\boldsymbol{B}$, a *homomorphism* from $\boldsymbol{A}$ to $\boldsymbol{B}$ is a map $f\colon A \to B$ such that, for every n-ary operation g of the common type and $a_1, \dots, a_n \in A$,

$$f(g^{\boldsymbol{A}}(a_1, \dots, a_n)) = g^{\boldsymbol{B}}(f(a_1), \dots, f(a_n)).$$

An injective homomorphism is called an *embedding* and, if there exists an embedding from $\boldsymbol{A}$ to $\boldsymbol{B}$, we say that $\boldsymbol{A}$ *embeds* into $\boldsymbol{B}$. Lastly, a surjective embedding is called an *isomorphism*. Accordingly, $\boldsymbol{A}$ and $\boldsymbol{B}$ are said to be *isomorphic* if there exists an isomorphism between them, in which case we write $\boldsymbol{A} \cong \boldsymbol{B}$.

A simple induction on the construction of terms shows that, for every pair of algebras $\boldsymbol{A}$ and $\boldsymbol{B}$ of type ρ and every term $\varphi(x_1, \dots, x_n)$ of ρ, if f is a homomorphism from $\boldsymbol{A}$ to $\boldsymbol{B}$, then

$$f(\varphi^{\boldsymbol{A}}(a_1, \dots, a_n)) = \varphi^{\boldsymbol{B}}(f(a_1), \dots, f(a_n)),$$

for every $a_1, \dots, a_n \in A$. Therefore homomorphisms preserve not only basic operations, but also arbitrary terms.

In the particular case where $\boldsymbol{A}$ and $\boldsymbol{B}$ are lattices, a homomorphism from $\boldsymbol{A}$ to $\boldsymbol{B}$ is a map $f\colon A \to B$ such that, for every $a, c \in A$,

$$f(a \wedge^{\boldsymbol{A}} c) = f(a) \wedge^{\boldsymbol{B}} f(c) \text{ and } f(a \vee^{\boldsymbol{A}} c) = f(a) \vee^{\boldsymbol{B}} f(c).$$

For instance, the inclusion map from the lattice $\langle \mathbb{N}; \leqslant \rangle$ into the lattice $\langle \mathbb{Z}; \leqslant \rangle$ is an injective homomorphism, that is, an embedding. Similarly, given two sets $Y \subseteq X$, the inclusion map from the powerset lattice $\langle \mathcal{P}(Y); \subseteq \rangle$ to the powerset lattice $\langle \mathcal{P}(X); \subseteq \rangle$ is also an embedding. On the other hand, if $Y \subsetneq X$, the map

$$(-) \cap Y \colon \mathcal{P}(X) \to \mathcal{P}(Y)$$

that sends every $Z \subseteq X$ to $Z \cap Y$ is a noninjective homomorphism from $\langle \mathcal{P}(X); \subseteq \rangle$ to $\langle \mathcal{P}(Y); \subseteq \rangle$.

Definition 2.2. Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be algebras of the same type $\rho\colon \mathcal{F} \to \mathbb{N}$. Then $\boldsymbol{A}$ is said to be a *subalgebra* of $\boldsymbol{B}$ if $A \subseteq B$ and $f^{\boldsymbol{A}}$ is the restriction of $f^{\boldsymbol{B}}$ to A, for every $f \in \mathcal{F}$. In this case, we write $\boldsymbol{A} \leqslant \boldsymbol{B}$.

Given a class of algebras K, let

$$\mathbb{I}(\mathsf{K}) := \{\boldsymbol{A} : \boldsymbol{A} \cong \boldsymbol{B} \text{ for some } \boldsymbol{B} \in \mathsf{K}\}$$
$$\mathbb{S}(\mathsf{K}) := \{\boldsymbol{A} : \boldsymbol{A} \leqslant \boldsymbol{B} \text{ for some } \boldsymbol{B} \in \mathsf{K}\}.$$

When $\mathsf{K} = \{\boldsymbol{A}\}$, we write $\mathbb{I}(\boldsymbol{A})$ and $\mathbb{S}(\boldsymbol{A})$ as a shorthand for $\mathbb{I}(\{\boldsymbol{A}\})$ and $\mathbb{S}(\{\boldsymbol{A}\})$, respectively. The following observation is an immediate consequence of the definitions.

Proposition 2.3. *Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be algebras of the same type. Then $\boldsymbol{A} \in \mathbb{IS}(\boldsymbol{B})$ if and only if there exists an embedding $f\colon \boldsymbol{A} \to \boldsymbol{B}$. In this case, $\boldsymbol{A}$ is isomorphic to the unique subalgebra of $\boldsymbol{B}$ with universe $f[A]$.*

As we mentioned, homomorphisms can be used to compare similar algebras.

Definition 2.4. Given two similar algebras $\boldsymbol{A}$ and $\boldsymbol{B}$, we say that $\boldsymbol{A}$ is a *homomorphic image* of $\boldsymbol{B}$ if there exists a surjective homomorphism $f\colon \boldsymbol{B} \to \boldsymbol{A}$.

Accordingly, given a class of algebras K, we set

$$\mathbb{H}(\mathsf{K}) := \{\boldsymbol{A} : \boldsymbol{A} \text{ is a homomorphic image of some } \boldsymbol{B} \in \mathsf{K}\}.$$

As usual, when $\mathsf{K} = \{\boldsymbol{A}\}$, we write $\mathbb{H}(\boldsymbol{A})$ as a shorthand for $\mathbb{H}(\{\boldsymbol{A}\})$.

Observe that every (not necessarily surjective) homomorphism $f\colon \boldsymbol{A} \to \boldsymbol{B}$ induces a homomorphic image of $\boldsymbol{A}$.

Proposition 2.5. *If $f\colon \boldsymbol{A} \to \boldsymbol{B}$ is a homomorphism, then $f[A]$ is the universe of a subalgebra of $\boldsymbol{B}$ that, moreover, is a homomorphic image of $\boldsymbol{A}$.*

Proof. Observe that $f[A]$ is nonempty, because A is. Then consider an n-ary function symbol g of the common type of $\boldsymbol{A}$ and $\boldsymbol{B}$ and $b_1, \dots, b_n \in f[A]$. Clearly, there are $a_1, \dots, a_n \in A$ such that $f(a_i) = b_i$, for every $i \leqslant n$. Since f is a homomorphism from $\boldsymbol{A}$ to $\boldsymbol{B}$, we obtain

$$g^{\boldsymbol{B}}(b_1, \dots, b_n) = g^{\boldsymbol{B}}(f(a_1), \dots, g(a_n)) = f(g^{\boldsymbol{A}}(a_1, \dots, a_n)) \in f[A].$$

Hence, we conclude that $f[A]$ is the universe of a subalgebra $f[\boldsymbol{A}]$ of $\boldsymbol{B}$.

Furthermore, $f\colon \boldsymbol{A} \to f[\boldsymbol{A}]$ is a homomorphism, because for every basic n-ary function symbol g of the common type and $a_1, \dots, a_n \in A$,

$$f(g^{\boldsymbol{A}}(a_1, \dots, a_n)) = g^{\boldsymbol{B}}(f(a_1), \dots, f(a_n)) = g^{f[\boldsymbol{A}]}(f(a_1), \dots, f(a_n)),$$

where the first equality follows from the assumption that $f\colon \boldsymbol{A} \to \boldsymbol{B}$ is a homomorphism. Since the map $f\colon A \to f[A]$ is surjective, we conclude that $f[\boldsymbol{A}] \in \mathbb{H}(\boldsymbol{A})$. ⊠

In view of the above result, when $f\colon \boldsymbol{A} \to \boldsymbol{B}$ is a homomorphism, we denote by $f[\boldsymbol{A}]$ the unique subalgebra of $\boldsymbol{B}$ with universe $f[A]$.

For instance, let $f\colon \mathbb{Z} \to \mathbb{R}$ be the absolute value map, that is, the function defined by the rule

$$f(n) := \text{the absolute value of } n.$$

Observe that f is a nonsurjective homomorphism from the lattice of integers to that of reals. Furthermore, the homomorphic image $f[\langle \mathbb{Z}; \leqslant \rangle]$ of $\langle \mathbb{Z}; \leqslant \rangle$ is the lattice of natural numbers $\langle \mathbb{N}; \leqslant \rangle$, which, in turn, is a subalgebra of lattice of reals.

Notably, the homomorphic images of an algebra $\boldsymbol{A}$ can be "internalized" as special equivalence relations on $\boldsymbol{A}$ as follows.

Definition 2.6. A *congruence* of an algebra $\boldsymbol{A}$ is an equivalence relation θ on A such that, for every basic n-ary opearation f of $\boldsymbol{A}$ and $a_1, \dots, a_n, c_1, \dots, c_n \in A$,

$$\text{if } \langle a_1, c_1 \rangle, \dots, \langle a_n, c_n \rangle \in \theta, \text{ then } \langle f^{\boldsymbol{A}}(a_1, \dots, a_n), f^{\boldsymbol{A}}(c_1, \dots, c_n) \rangle \in \theta. \tag{1}$$

In this case, we often write $a \equiv_\theta c$ as a shorthand for $\langle a, c \rangle \in \theta$. The poset of congruences of $\boldsymbol{A}$ ordered under the inclusion relation will be denoted by $\mathsf{Con}(\boldsymbol{A})$.

A simple induction on the construction of terms shows that, for every congruence θ of $\boldsymbol{A}$ and every term $\varphi(x_1, \dots, x_n)$,

$$\text{if } \langle a_1, c_1 \rangle, \dots, \langle a_n, c_n \rangle \in \theta, \text{ then } \langle \varphi^{\boldsymbol{A}}(a_1, \dots, a_n), \varphi^{\boldsymbol{A}}(c_1, \dots, c_n) \rangle \in \theta,$$

for every $a_1, \dots, a_n \in A$. Therefore, congruences preserve not only basic operations, but also arbitrary terms. Furthermore, a simple argument shows that $\mathsf{Con}(\boldsymbol{A})$ is an inductive closure system and, therefore, an algebraic lattice whose maximum is the total relation $A \times A$ and whose minimum is the identity relation $\mathrm{id}_A := \{\langle a, a \rangle : a \in A\}$.

Example 2.7 (Boolean algebras)**.** Recall that a *filter* of a Boolean algebra $\boldsymbol{A}$ is a nonempty upset $F \subseteq A$ closed under binary meets. We denote by $\mathsf{Fi}(\boldsymbol{A})$ the poset of filters of $\boldsymbol{A}$ ordered under the inclusion relation. It is easy to see $\mathsf{Fi}(\boldsymbol{A})$ is an inductive closure system and, therefore, an algebraic lattice. Furthermore, the lattices $\mathsf{Fi}(\boldsymbol{A})$ and $\mathsf{Con}(\boldsymbol{A})$ are isomorphic via the inverse isomorphisms

$$\boldsymbol{\Omega}^{\boldsymbol{A}}(-)\colon \mathsf{Fi}(\boldsymbol{A}) \to \mathsf{Con}(\boldsymbol{A}) \text{ and } \boldsymbol{\tau}(-)\colon \mathsf{Con}(\boldsymbol{A}) \to \mathsf{Fi}(\boldsymbol{A})$$

defined by the rules

$$\begin{aligned}\boldsymbol{\Omega}^{\boldsymbol{A}}(F) &:= \{\langle a, c\rangle \in A \times A : a \to c, c \to a \in F\}\\ \boldsymbol{\tau}(\theta) &:= \{a \in A : \langle a, 1\rangle \in \theta\}.\end{aligned}$$

Because of this, every congruence θ of a Boolean algebra $\boldsymbol{A}$ is induced by some filter F, in the sense that $\theta = \boldsymbol{\Omega}^{\boldsymbol{A}}F$. This correspondence between filters and congruences generalizes straightforwardly to all Heyting algebras. ⊠

Example 2.8 (Modal algebras)**.** A *modal algebra* is an algebra $\boldsymbol{A} = \langle A; \wedge, \vee, \neg, \Box, 0, 1\rangle$ such that $\langle A; \wedge, \vee, \neg, 0, 1\rangle$ is a Boolean algebra and $\Box$ is a unary operation such that

$$\Box(a \wedge c) = \Box a \wedge \Box c \text{ and } \Box 1 = 1,$$

for every $a, c \in A$. An *open filter* of a modal algebra $\boldsymbol{A}$ is a filter of the Boolean reduct of $\boldsymbol{A}$ that, moreover, is closed under the operation $\Box$. The poset of open filters of $\boldsymbol{A}$ ordered under the inclusion relation will be denoted by $\mathsf{Op}(\boldsymbol{A})$. It forms an inductive closure system and, therefore, an algebraic lattice. Furthermore, the lattices $\mathsf{Op}(\boldsymbol{A})$ and $\mathsf{Con}(\boldsymbol{A})$ are isomorphic via the inverse isomorphisms described in Example 2.7. Because of this, every congruence of a modal algebra $\boldsymbol{A}$ has the form $\theta = \boldsymbol{\Omega}^{\boldsymbol{A}}F$, for some open filter F. ⊠

Example 2.9 (Groups)**.** Similarly, it is well known that the lattice of congruences of a group is isomorphic to that of its normal subgroups. Because of this, every congruence of a group is induced by some normal subgroup. ⊠

As we mentioned, there is a tight correspondence between the homomorphic images and the congruences of an algebra $\boldsymbol{A}$. On the one hand, every congruence θ of $\boldsymbol{A}$ gives rise to a homomorphic image $\boldsymbol{A}/\theta$ of $\boldsymbol{A}$. Let $\mathcal{F}$ be the set of function symbols of $\boldsymbol{A}$. Given $\theta \in \mathsf{Con}(\boldsymbol{A})$ and a basic n-ary function symbol $f \in \mathcal{F}$, let $f^{\boldsymbol{A}/\theta}$ be the n-ary operation on A/θ defined by the rule

$$f^{\boldsymbol{A}/\theta}(a_1/\theta, \dots, a_n/\theta) := f^{\boldsymbol{A}}(a_1, \dots, a_n)/\theta.$$

Notice that $f^{\boldsymbol{A}/\theta}$ is well-defined, by condition (1). As a consequence, the structure

$$\boldsymbol{A}/\theta := \langle A/\theta; \{f^{\boldsymbol{A}/\theta} : f \in \mathcal{F}\}\rangle$$

is a well-defined algebra of the type as $\boldsymbol{A}$. Furthermore, $\boldsymbol{A}/\theta \in \mathbb{H}(\boldsymbol{A})$, because the map $\pi_\theta\colon A \to A/\theta$, defined, for every $a \in A$, as $\pi_\theta(a) := a/\theta$, is a surjective homomorphism from $\boldsymbol{A}$ to $\boldsymbol{A}/\theta$. To prove this, consider $a_1, \dots, a_n \in A$. We have

$$\begin{aligned}\pi_\theta(f^{\boldsymbol{A}}(a_1, \dots, a_n)) &= f^{\boldsymbol{A}}(a_1, \dots, a_n)/\theta\\ &= f^{\boldsymbol{A}/\theta}(a_1/\theta, \dots, a_n/\theta)\\ &= f^{\boldsymbol{A}/\theta}(\pi_\theta(a_1), \dots, \pi_\theta(a_n)),\end{aligned}$$

where the second equality follows from the definition of the operation $f^{\boldsymbol{A}/\theta}$.

Corollary 2.10. *If θ is a congruence of an algebra $\boldsymbol{A}$, then $\boldsymbol{A}/\theta$ is a well-defined homomorphic image of $\boldsymbol{A}$.*

In view of the above result, every congruence θ of an algebra $\boldsymbol{A}$ induces a homomorphic image of $\boldsymbol{A}$, namely $\boldsymbol{A}/\theta$. The converse is also true, as we proceed to explain.

Definition 2.11. The *kernel* of a homomorphism $f\colon \boldsymbol{A} \to \boldsymbol{B}$ is the binary relation

$$\mathsf{Ker}(f) \coloneqq \{\langle a, c\rangle \in A \times A : f(a) = f(c)\}.$$

Proposition 2.12. *The kernel of a homomorphism $f\colon \boldsymbol{A} \to \boldsymbol{B}$ is a congruence of $\boldsymbol{A}$.*

Proof. It is obvious that $\mathsf{Ker}(f)$ is an equivalence relation on A. Therefore, to prove that $\mathsf{Ker}(f)$ is a congruence of $\boldsymbol{A}$, it suffices to show that it preserves the basic operations of $\boldsymbol{A}$. Consider a basic n-ary operation g of $\boldsymbol{A}$ and $a_1, \dots, a_n, c_1, \dots, c_n \in A$ such that $\langle a_1, c_1\rangle, \dots, \langle a_n, c_n\rangle \in \mathsf{Ker}(f)$. By the definition of $\mathsf{Ker}(f)$,

$$f(a_i) = f(c_i), \text{ for every } i \leqslant n.$$

It follows that $g^{\boldsymbol{B}}(f(a_1), \dots, f(a_n)) = g^{\boldsymbol{B}}(f(c_1), \dots, f(c_n))$. Since $f\colon \boldsymbol{A} \to \boldsymbol{B}$ is a homomorphism, this yields

$$f(g^{\boldsymbol{A}}(a_1, \dots, a_n)) = g^{\boldsymbol{B}}(f(a_1), \dots, f(a_n)) = g^{\boldsymbol{B}}(f(c_1), \dots, f(c_n)) = f(g^{\boldsymbol{A}}(c_1, \dots, c_n)).$$

Hence, we conclude that $\langle g^{\boldsymbol{A}}(a_1, \dots, a_n), g^{\boldsymbol{A}}(c_1, \dots, c_n)\rangle \in \mathsf{Ker}(f)$, as desired. ⊠

The behaviour of kernels is governed by the next principle.

Fundamental Homomorphism Theorem 2.13. *If $f\colon \boldsymbol{A} \to \boldsymbol{B}$ is a homomorphism with kernel θ, then there exists a unique embedding $g\colon \boldsymbol{A}/\theta \to \boldsymbol{B}$ such that $f = g \circ \pi_\theta$.*

Proof. We begin by proving the existence of g. Let $g\colon A/\theta \to B$ be the map defined as $g(a/\theta) \coloneqq f(a)$, for every $a \in A$. To show that g is well-defined, consider $a, c \in A$ such that $a/\theta = c/\theta$. Since $\theta = \mathsf{Ker}(f)$, this means that $f(a) = f(c)$, as desired. Furthermore, the definition of g guarantees that $f = g \circ \pi_\theta$.

Now, observe g is injective, because, for every $a, c \in A$ such that $g(a/\theta) = g(c/\theta)$, we have $f(a) = f(c)$, that is, $\langle a, c\rangle \in \mathsf{Ker}(f) = \theta$ and, therefore, $a/\theta = c/\theta$. Moreover, for every basic n-ary operation p of $\boldsymbol{A}$ and $a_1, \dots, a_n \in A$, we have

$$\begin{aligned}
g(p^{\boldsymbol{A}/\theta}(a_1/\theta, \dots, a_n/\theta)) &= g(p^{\boldsymbol{A}}(a_1, \dots, a_n)/\theta) \\
&= f(p^{\boldsymbol{A}}(a_1, \dots, a_n)) \\
&= p^{\boldsymbol{B}}(f(a_1), \dots, f(a_n)) \\
&= p^{\boldsymbol{B}}(g(a_1/\theta), \dots, g(a_n/\theta)).
\end{aligned}$$

The first equality above follows from the definition of $\boldsymbol{A}/\theta$, the second and the last from the definition of g, and the third from the assumption that $f\colon \boldsymbol{A} \to \boldsymbol{B}$ is a homomorphism. Hence, we conclude that $g\colon \boldsymbol{A}/\theta \to \boldsymbol{B}$ is a homomorphism and, therefore, an embedding, as desired.

The uniqueness of g follows from the fact that, if a map g^* satisfies the condition in the statement of the theorem, then, for every $a \in A$,

$$f(a) = g^* \circ \pi_\theta(a) = g^*(a/\theta),$$

that is, g^* coincides with g. ⊠

Corollary 2.14. *If $f\colon \boldsymbol{A} \to \boldsymbol{B}$ is a homomorphism, then $f[\boldsymbol{A}] \cong \boldsymbol{A}/\mathsf{Ker}(f)$. In particular, if f is surjective, $\boldsymbol{B} \cong \boldsymbol{A}/\mathsf{Ker}(f)$.*

Proof. In the proof of the Fundamental Homomorphism Theorem we showed that the map $g\colon A/\mathsf{Ker}(f) \to B$, defined by the rule $g(a/\mathsf{Ker}(f)) := f(a)$, is an embedding of $\boldsymbol{A}/\mathsf{Ker}(f)$ into $\boldsymbol{B}$. As g can be viewed as a surjective embedding of $\boldsymbol{A}/\mathsf{Ker}(f)$ into $f[\boldsymbol{A}]$, we conclude that $f[\boldsymbol{A}] \cong \boldsymbol{A}/\mathsf{Ker}(f)$. ⊠

At this stage, it should be clear that if θ is a congruence on an algebra $\boldsymbol{A}$, then $\pi_\theta\colon \boldsymbol{A} \to \boldsymbol{A}/\theta$ is a surjective homomorphism whose kernel is θ. Similarly, if $f\colon \boldsymbol{A} \to \boldsymbol{B}$ is a surjective homomorphism, then $\boldsymbol{A}/\mathsf{Ker}(f) \cong \boldsymbol{B}$, by Corollary 2.14. As a consequence, for every class of algebras K,

$$\mathbb{H}(\mathsf{K}) = \mathbb{I}\{\boldsymbol{A}/\theta : \boldsymbol{A} \in \mathsf{K} \text{ and } \theta \in \mathsf{Con}(\boldsymbol{A})\}. \tag{2}$$

Now, recall that the Cartesian product of a family of sets $\{A_i : i \in I\}$ is the set

$$\prod_{i\in I} A_i := \{f\colon I \to \bigcup_{i \in I} A_i \ :\ f(i) \in A_i, \text{for all } i \in I\}.$$

In particular, if I is empty, then $\prod_{i\in I} A_i$ is the singleton containing only the empty map.

Definition 2.15. The *direct product* of a family of similar algebras $\{\boldsymbol{A}_i : i \in I\}$ is the unique algebra of the common type whose universe is the Cartesian product $\prod_{i\in I} A_i$ and such that, for every basic n-ary operation symbol f and every $\vec{a}_1, \dots, \vec{a}_n \in \prod_{i\in I} A_i$,

$$f^{\prod_{i\in I} \boldsymbol{A}_i}(\vec{a}_1, \dots, \vec{a}_n)(i) = f^{\boldsymbol{A}_i}(\vec{a}_1(i), \dots, \vec{a}_n(i)), \text{ for every } i \in I.$$

We denote this algebra by $\prod_{i\in I} \boldsymbol{A}_i$.

In this case, for every $j \in I$, the projection map on the j-th component $p_j\colon \prod_{i\in I} A_i \to A_j$, defined by the rule $p_j(\vec{a}) := \vec{a}(j)$, is a surjective homomorphism from $\prod_{i\in I} \boldsymbol{A}_i$ to $\boldsymbol{A}_j$.

Given a class of similar algebras K, we set

$$\mathbb{P}(\mathsf{K}) := \{\boldsymbol{A} : \boldsymbol{A} \text{ is a direct product of a family } \{\boldsymbol{B}_i : i \in I\} \subseteq \mathsf{K}\}.$$

As usual, when $\mathsf{K} = \{\boldsymbol{A}\}$, we write $\mathbb{P}(\boldsymbol{A})$ as a shorthand for $\mathbb{P}(\{\boldsymbol{A}\})$.

Notice that up to isomorphism, there exists a unique one-element algebra of a given type. Because of this, one-element algebras are called *trivial*. Accordingly, when the set of indexes I is empty, the direct product $\prod_{i\in I} \boldsymbol{A}_i$ is the trivial algebra of the given type. It follows that $\mathbb{P}(\mathsf{K})$ contains always a trivial algebra, for every class of similar algebras K.

Example 2.16 (Powerset algebras)**.** Boolean algebras of the form $\langle \mathcal{P}(X); \cap, \cup, -, \emptyset, X\rangle$ are called *powerset Boolean algebras*. Let $\boldsymbol{B}$ be the two-element Boolean algebra and observe that $\mathbb{IP}(\boldsymbol{B})$ is the class of algebras isomorphic to some powerset Boolean algebra. To prove this, observe that every powerset Boolean algebra $\mathcal{P}(X)$ is isomorphic to a direct product of $\boldsymbol{B}$ via the *characteristic function* $f_X\colon \mathcal{P}(X) \to \prod_{x\in X} \boldsymbol{B}_x$, defined by the rule

$$f(Y)(x) := \begin{cases} 1 & \text{if } x \in Y \\ 0 & \text{if } x \notin Y, \end{cases}$$

where $Y \in \mathcal{P}(X)$ and $x \in X$. By the same token, every direct product $\prod_{i\in I} \boldsymbol{B}_i$ of $\boldsymbol{B}$ is isomorphic to the powerset Boolean algebra $\mathcal{P}(I)$ via the isomorphism f_I. ⊠

We close this section by reviewing the subdirect product construction.

Definition 2.17. A subalgebra $\boldsymbol{B}$ of a direct product $\prod_{i \in I} \boldsymbol{A}_i$ is said to be a *subdirect product* of $\{\boldsymbol{A}_i : i \in I\}$ if the projection map p_i is surjective, for every $i \in I$. Similarly, an embedding $f \colon \boldsymbol{B} \to \prod_{i \in I} \boldsymbol{A}_i$ is said to be *subdirect* when $f[\boldsymbol{B}]$ is a subdirect product of the family $\{\boldsymbol{A}_i : i \in I\}$.

Given a class of similar algebras K, we set

$$\mathbb{P}_{\text{SD}}(\mathsf{K}) := \{\boldsymbol{A} : \boldsymbol{A} \text{ is a subdirect direct product of a family } \{\boldsymbol{B}_i : i \in I\} \subseteq \mathsf{K}\}.$$

As usual, when $\mathsf{K} = \{\boldsymbol{A}\}$, we write $\mathbb{P}_{\text{SD}}(\boldsymbol{A})$ as a shorthand for $\mathbb{P}_{\text{SD}}(\{\boldsymbol{A}\})$. Clearly, $\mathbb{P}_{\text{SD}}(\mathsf{K}) \subseteq \mathbb{SP}(\mathsf{K})$. Furthermore, $\mathbb{P}_{\text{SD}}(\mathsf{K})$ contains always a trivial algebra.

Example 2.18 (Distributive lattices)**.** Let DL be the class of distributive lattices and $\boldsymbol{B}$ be the two-element distributive lattice. Birkhoff's Representation Theorem states that $\mathsf{DL} = \mathbb{IP}_{\text{SD}}(\boldsymbol{B})$. The inclusion $\mathbb{IP}_{\text{SD}}(\boldsymbol{B}) \subseteq \mathsf{DL}$ follows from the fact that DL is closed under $\mathbb{I}, \mathbb{S}$, and $\mathbb{P}$. For the other inclusion, consider a distributive lattice $\boldsymbol{A}$ and let I be the set of its prime filters. By Birkhoff's Representation Theorem, the map

$$\gamma \colon \boldsymbol{A} \to \prod_{F \in I} \boldsymbol{B}_F,$$

defined, for every $a \in A$ and $F \in I$, by the rule

$$\gamma(a)(F) := \begin{cases} 1 & \text{if } a \in F \\ 0 & \text{if } a \notin F, \end{cases}$$

is a well-defined subdirect embedding. ⊠

Example 2.19 (Boolean algebras)**.** Similarly, Stone's Representation Theorem states that the class of Boolean algebras coincides with $\mathbb{IP}_{\text{SD}}(\boldsymbol{B})$, where $\boldsymbol{B}$ the two-element Boolean algebra. ⊠

The next result provides a general recipe to construct subdirect products.

Proposition 2.20. *Let $\boldsymbol{A}$ be an algebra and $\{\theta_i : i \in I\} \subseteq \mathsf{Con}(\boldsymbol{A})$. Then the map*

$$f \colon \boldsymbol{A}/\bigcap_{i \in I} \theta_i \to \prod_{i \in I} \boldsymbol{A}/\theta_i,$$

defined, for every $a \in A$ and $j \in I$, as

$$f(a/\bigcap_{i \in I} \theta_i)(j) := a/\theta_j,$$

is a subdirect embedding.

Proof. For the sake of readability, set $\boldsymbol{B} := \boldsymbol{A}/\bigcap_{i \in I} \theta_i$. To prove that f is injective, consider $a, c \in A$ such that $\langle a, c \rangle \notin \bigcap_{i \in I} \theta_i$. Then there exists $j \in I$ such that $\langle a, c \rangle \notin \theta_j$ and, therefore,

$$f(a/\bigcap_{i \in I} \theta_i)(j) := a/\theta_j \neq c/\theta_j = f(c/\bigcap_{i \in I} \theta_i)(j).$$

It follows that $f(a/\bigcap_{i \in I} \theta_i) \neq f(c/\bigcap_{i \in I} \theta_i)$. Thus, f is injective. Moreover, by the definition of f, the composition $p_i \circ f \colon \boldsymbol{B} \to \boldsymbol{A}/\theta_i$ is surjective, for every $i \in I$.

It only remains to prove that f is a homomorphism. Consider an n-ary basic operation g and $a_1, \dots, a_n \in A$. For every $j \in I$, we have

$$\begin{aligned} f(g^{\boldsymbol{B}}(a_1/\bigcap_{i\in I}\theta_i, \dots, a_n/\bigcap_{i\in I}\theta_i))(j) &= f(g^{\boldsymbol{A}}(a_1, \dots, a_n)/\bigcap_{i\in I}\theta_i)(j) \\ &= g^{\boldsymbol{A}}(a_1, \dots, a_n)/\theta_j \\ &= g^{\boldsymbol{A}/\theta_j}(a_1/\theta_j, \dots, a_n/\theta_j) \\ &= g^{\boldsymbol{A}/\theta_j}(f(a_1/\bigcap_{i\in I}\theta_i)(j), \dots, f(a_n/\bigcap_{i\in I}\theta_i)(j)) \\ &= g^{\prod_{i\in I}\boldsymbol{A}/\theta_i}(f(a_1/\bigcap_{i\in I}\theta_i), \dots, f(a_n/\bigcap_{i\in I}\theta_i))(j). \end{aligned}$$

It follows that

$$f(g^{\boldsymbol{B}}(a_1/\bigcap_{i\in I}\theta_i, \dots, a_n/\bigcap_{i\in I}\theta_i)) = g^{\prod_{i\in I}\boldsymbol{A}/\theta_i}(f(a_1/\bigcap_{i\in I}\theta_i), \dots, f(a_n/\bigcap_{i\in I}\theta_i)). \qquad \boxtimes$$

3. Prevarieties

Definition 3.1. A class of similar algebras closed under $\mathbb{I}, \mathbb{S}$, and $\mathbb{P}$ is a said to be a *prevariety*.

Given a class of similar algebras K, the least prevariety extending K is $\mathbb{ISP}(\mathsf{K})$ and is called the prevariety *generated* by K. For instance, in view of Examples 2.18, the class of distributive lattices is the prevariety generated by the two-element distributive lattice. Similarly, the class of Boolean algebras is the prevariety generated by the two-element Boolean algebra (see Example 2.19, if necessary).

Our aim will be to prove that prevarieties are precisely the classes of algebras axiomatized by a certain kind of infinitary formulas. To this end, we rely on the following notational convention. When no confusion shall arise, given a sequence $\vec{a}$ and a set A, we write $\vec{a} \in A$ to indicate that the elements of the sequence $\vec{a}$ belong to A.

Definition 3.2. A *generalized quasi-equation* of type ρ is an expression Φ of the form

$$\Big(\mathop{\&}_{i\in I} \varphi_i(\vec{x}) \approx \psi_i(\vec{x}) \Big) \Longrightarrow \varepsilon(\vec{x}) \approx \delta(\vec{x}),$$

where $\{\varphi_i \approx \psi_i : i \in I\} \cup \{\varepsilon \approx \delta\}$ is a set of equations of type ρ. Then Φ is *valid* in an algebra $\boldsymbol{A}$ of type ρ when so is its universal closure, that is, for every $\vec{a} \in A$,

$$\text{if } (\varphi_i^{\boldsymbol{A}}(\vec{a}) = \psi_i^{\boldsymbol{A}}(\vec{a}), \text{ for all } i \in I), \text{ then } \varepsilon^{\boldsymbol{A}}(\vec{a}) = \delta^{\boldsymbol{A}}(\vec{a}).$$

In this case, we often say that $\boldsymbol{A}$ *satisfies* Φ and write $\boldsymbol{A} \vDash \Phi$.

Notice that, in the above definition, the set of indexes I can be arbitrarily large and that the same applies to the sequence of variables $\vec{x}$ that appear in the equations of Φ. This motivates the following.

Definition 3.3. A generalized quasi-equation is said to be

(i) a *quasi-equation* when the index set I is finite; and
(ii) an *equation* when the index set I is empty.

Remark 3.4. It might seem that we are using the term *equations* to refer to two distinct kinds of expressions, namely those of the form $\varepsilon \approx \delta$ and $\emptyset \Longrightarrow \varepsilon \approx \delta$. This is not a problem, however, because these expressions are synonyms, in the sense that an algebra satisfies $\varepsilon \approx \delta$ if and only if it satisfies $\emptyset \Longrightarrow \varepsilon \approx \delta$. Because of this, we will continue to denote equations by $\varepsilon \approx \delta$, while keeping in mind that they are special instances of generalized quasi-equations. ⊠

Definition 3.5. Let $\rho \colon \mathcal{F} \to \mathbb{N}$ be a type and X a set of variables disjoint from $\mathcal{F}$. The *term algebra* $\boldsymbol{T}_\rho(X)$ of type ρ over X is the unique algebra of type ρ whose universe is $T_\rho(X)$ and with basic n-ary operations f defined, for every $\varphi_1, \dots, \varphi_n \in T_\rho(X)$, as

$$f^{\boldsymbol{T}_\rho(X)}(\varphi_1, \dots, \varphi_n) := f(\varphi_1, \dots, \varphi_n).$$

When no confusion might arise, we drop the subscript and write $T(X)$ instead of $T_\rho(X)$. Term algebras have the following fundamental property.

Proposition 3.6. *Let $\boldsymbol{A}$ be an algebra of type ρ and X a set of variables. Every function $f \colon X \to A$ extends uniquely to a homomorphism $f^* \colon \boldsymbol{T}_\rho(X) \to \boldsymbol{A}$.*

Proof. The unique extension f^* is defined, for every $\varphi(x_{\alpha_1}, \dots x_{\alpha_n}) \in T_\rho(X)$, as

$$f^*(\varphi) = \varphi^{\boldsymbol{A}}(f(x_{\alpha_1}), \dots, f(x_{\alpha_n})).$$ ⊠

Exercise 3.7. Prove the above proposition. ⊠

Theorem 3.8. *A class of similar algebras is a prevariety if and only if it can be axiomatized by a class of generalized quasi-equations.*

Proof. The "if" part follows from the fact that the validity of generalized quasi-equations persists under the formation of isomorphic copies, subalgebras, and direct product. To prove the converse, consider a prevariety K and let Σ be the class of generalized quasi-equations valid in it. Let K^+ be the class of algebras in which the generalized quasi-equations in Σ are valid. Clearly, $\mathsf{K} \subseteq \mathsf{K}^+$. To prove the other inclusion, consider an algebra $\boldsymbol{A} \in \mathsf{K}^+$. Let also X be a set of variables for which there exists a surjective map $f \colon X \to A$. By Proposition 3.6, f extends to a surjective homomorphism $f^* \colon \boldsymbol{T}(X) \to \boldsymbol{A}$. Together with Corollary 2.14, this yields

$$\boldsymbol{A} \cong \boldsymbol{T}(X)/\mathsf{Ker}(f^*). \tag{3}$$

Now, consider an arbitrary pair $\langle \varphi, \psi \rangle \in (T(X) \times T(X)) \smallsetminus \mathsf{Ker}(f^*)$. Notice that the elements of $T(X) \times T(X)$ are ordered pairs of terms and, therefore, can be viewed as equations under the identification of $\langle \varepsilon, \delta \rangle$ with $\varepsilon \approx \delta$. In this way, $\mathsf{Ker}(f^*)$ becomes a set of equations in variables X. Bearing this in mind, consider the generalized quasi-equation

$$\Phi := \Big(\mathop{\&} \mathsf{Ker}(f^*) \Big) \Longrightarrow \varphi \approx \psi.$$

We will prove that Φ fails in $\boldsymbol{A}$. For the sake of readability we will denote by $\vec{x}$ the sequence of all variables in X. Observe that every element $\varepsilon \in T(X)$ is of the form $\varepsilon(\vec{x})$. Then consider the assignment $f \colon X \to A$. We will denote by $f(\vec{x})$ the sequence obtained by applying f component-wise to $\vec{x}$. For every pair $\langle \varepsilon, \delta \rangle \in \mathsf{Ker}(f^*)$, we have

$$\varepsilon^{\boldsymbol{A}}(f(\vec{x})) = \varepsilon^{\boldsymbol{A}}(f^*(\vec{x})) = f^*(\varepsilon(\vec{x})) = f^*(\delta(\vec{x})) = \delta^{\boldsymbol{A}}(f^*(\vec{x})) = \delta^{\boldsymbol{A}}(f(\vec{x})).$$

The equalities above can be justified as follows. The first and the last holds because f^* extends f, the second and the fourth because $f^* \colon \boldsymbol{T}(X) \to \boldsymbol{A}$ is a homomorphism, and the third because $\langle \varepsilon, \delta \rangle \in \mathsf{Ker}(f^*)$. On the other hand, since $\langle \varphi, \psi \rangle \notin \mathsf{Ker}(f^*)$, a similar argument shows

$$\varphi^{\boldsymbol{A}}(f(\vec{x})) \neq \psi^{\boldsymbol{A}}(f(\vec{x})).$$

Thus, $\boldsymbol{A}$ refutes Φ, as desired.

Since $\boldsymbol{A} \in \mathsf{K}^+$, this implies that there exists some algebra $\boldsymbol{C}_{\varphi,\psi} \in \mathsf{K}$ and an assignment $g_{\varphi,\psi} \colon X \to C_{\varphi,\psi}$ such that

$$\varepsilon^{\boldsymbol{C}_{\varphi,\psi}}(g_{\varphi,\psi}(\vec{x})) = \delta^{\boldsymbol{C}_{\varphi,\psi}}(g_{\varphi,\psi}(\vec{x})), \text{ for all } \langle \varepsilon, \delta \rangle \in \mathsf{Ker}(f^*), \text{ and } \varphi^{\boldsymbol{C}_{\varphi,\psi}}(g_{\varphi,\psi}(\vec{x})) \neq \psi^{\boldsymbol{C}_{\varphi,\psi}}(g_{\varphi,\psi}(\vec{x})).$$

Recall that $g_{\varphi,\psi}$ extends uniquely to a homomorphism $g^*_{\varphi,\psi} \colon \boldsymbol{T}(X) \to \boldsymbol{C}_{\varphi,\psi}$. Moreover, from the above display it follows

$$g^*_{\varphi,\psi}(\varepsilon) = g^*_{\varphi,\psi}(\delta), \text{ for all } \langle \varepsilon, \delta \rangle \in \mathsf{Ker}(f^*), \text{ and } g^*_{\varphi,\psi}(\varphi) \neq g^*_{\varphi,\psi}(\psi).$$

Consequently,

$$\mathsf{Ker}(f^*) \subseteq \mathsf{Ker}(g^*_{\varphi,\psi}) \text{ and } \langle \varphi, \psi \rangle \notin \mathsf{Ker}(g^*_{\varphi,\psi}).$$

It follows that

$$\mathsf{Ker}(f^*) = \bigcap \{\mathsf{Ker}(g^*_{\varphi,\psi}) : \langle \varphi, \psi \rangle \in (T(X) \times T(X)) \smallsetminus \mathsf{Ker}(f^*)\}.$$

By Proposition 2.20, this yields

$$\boldsymbol{T}(X)/\mathsf{Ker}(f^*) \in \mathbb{IP}_{\mathrm{SD}}(\{\boldsymbol{T}(X)/\mathsf{Ker}(g^*_{\varphi,\psi}) : \langle \varphi, \psi \rangle \in (T(X) \times T(X)) \smallsetminus \mathsf{Ker}(f^*)\}). \tag{4}$$

Moreover, from Corollary 2.14 and the fact that K is closed under $\mathbb{I}$ and $\mathbb{S}$ it follows that

$$\boldsymbol{T}(X)/\mathsf{Ker}(g^*_{\varphi,\psi}) \in \mathbb{IS}(\boldsymbol{C}_{\varphi,\psi}) \subseteq \mathsf{K},$$

for every $\langle \varphi, \psi \rangle \in (T(X) \times T(X)) \smallsetminus \mathsf{Ker}(f^*)$. Consequently, (4) simplifies to

$$\boldsymbol{T}(X)/\mathsf{Ker}(f^*) \in \mathbb{IP}_{\mathrm{SD}}(\mathsf{K}) \subseteq \mathsf{K},$$

where the last inclusion follows from the fact that K is a prevariety. Together with (3), this yields $\boldsymbol{A} \in \mathbb{I}(\mathsf{K}) \subseteq \mathsf{K}$. ⊠

Remark 3.9. In view of Theorem 3.8, prevarieties are classes of algebras axiomatized by classes of generalized quasi-equations. It is therefore natural to wonder whether there exists a prevariety that cannot be axiomatized by a set (as opposed to proper class) of generalized quasi-equations. It turns out that the answer to this question depends on the set theory we live in, as the nonexistence of such a prevariety is equivalent to *Vopěnka's Principle*. ⊠

Nonetheless, prevarieties axiomatizable by a set of generalized quasi-equations admit a relatively transparent description, as we proceed to explain. Given an infinite cardinal κ and a class of algebras K, let

$$\mathbb{U}_\kappa(\mathsf{K}) \coloneqq \{\boldsymbol{A} : \boldsymbol{B} \in \mathsf{K}, \text{ for all } \kappa\text{-generated } \boldsymbol{B} \leqslant \boldsymbol{A}\}.$$

Definition 3.10. Let κ be an infinite cardinal. A κ*-generalized quasi-variety* is a prevariety closed under $\mathbb{U}_\kappa$.

When $\kappa = \aleph_0$, we often say that K is simply a *generalized quasi-variety*. Given a class of similar algebras K, the least κ-generalized quasi-variety extending K is $\mathbb{U}_\kappa\mathbb{ISP}(\mathsf{K})$ and is called the κ-generalized quasi-variety *generated* by K.

Theorem 3.11. *Let κ be an infinite cardinal. A class of similar algebras is a κ-generalized quasi-variety if and only if it can be axiomatized by a set of generalized quasi-equations in which at most κ variables occur.*

Proof. The "if" part follows from the fact that the validity of generalized quasi-equations in $\leqslant \kappa$ variables persist under the $\mathbb{I}, \mathbb{S}, \mathbb{P}, \mathbb{U}_\kappa$. To prove the converse, consider a κ-generalized quasi-variety K. Then let X be a set of variables of cardinality κ and Σ the class of generalized quasi-equations written with variables in X. Since X is a set, so is Σ. It only remains to prove that K coincides with the class K^+ of algebras satisfying the generalized quasi-equations in Σ. Clearly, $\mathsf{K} \subseteq \mathsf{K}^+$. To prove the other inclusion, consider an algebra $\boldsymbol{A} \in \mathsf{K}^+$. We need to prove that $\boldsymbol{A} \in \mathsf{K}$. Since K is closed under $\mathbb{U}_\kappa$, it suffices to show that all the κ-generated subalgebras of $\boldsymbol{A}$ belong to K.

Accordingly, let $\boldsymbol{B}$ be a κ-generated subalgebra of A and $Y \subseteq B$ a set of generators for $\boldsymbol{B}$ of size $\leqslant \kappa$. There exists a surjective map $f \colon X \to Y$. By Proposition 3.6, f extends to a surjective homomorphism $f^* \colon \boldsymbol{T}(X) \to \boldsymbol{B}$. Now, we repeat the argument in the proof of Theorem 3.8, obtaining $\boldsymbol{B} \in \mathsf{K}$, as desired. ⊠

Corollary 3.12. *A prevariety can be axiomatized by a set of generalized quasi-equations if and only if it is a κ-generalized quasi-variety, for some infinite cardinal κ.*

*Exercise** 3.13. Let K be a class of similar algebras and κ an infinite cardinal. Prove that the prevariety and the κ-generalized quasi-variety generated by K are, respectively, $\mathbb{ISP}(\mathsf{K})$ and $\mathbb{U}_\kappa\mathbb{ISP}(\mathsf{K})$. ⊠

4. Ultraproducts

Let $\boldsymbol{A}$ be a Boolean algebra. A nonempty subset $F \subseteq A$ is said to be a *filter* of $\boldsymbol{A}$ if it is an upset closed under binary meets. A filter is said to be *proper* when it differs from A. Lastly, a proper filter U of $\boldsymbol{A}$ is said to be a *ultrafilter* of $\boldsymbol{A}$ if it is maximal among the proper filters of $\boldsymbol{A}$ or, equivalently, if

$$a \in U \text{ or } \neg a \in U, \text{ for every } a \in A.$$

While the following result holds in ZFC, it cannot be proved in ZF (although it is strictly weaker then the axiom of choice).

Ultrafilter Lemma 4.1. *Every proper filter on a Boolean algebra can be extended to a ultrafilter.*

Ultrafilters on powerset Boolean algebras $\mathcal{P}(X)$ are also called *ultrafilters on* X. In this section we will use them to define a product-like construction known as *ultraproduct*. To this end, let $\{\boldsymbol{A}_i : i \in I\}$ be a family of similar algebras. The *equalizer* of a pair of elements $\vec{a}, \vec{c} \in \prod_{i \in I} A_i$ is the set of indexes on which the sequences $\vec{a}$ and $\vec{c}$ agree, that is,

$$[\![\vec{a} = \vec{c}]\!] := \{i \in I : \vec{a}(i) = \vec{c}(i)\}.$$

Moreover, given an ultrafilter U on the index set I, let θ_U be the binary relation on the Cartesian product $\prod_{i \in I} A_i$ defined as

$$\theta_U := \{\langle \vec{a}, \vec{c} \rangle : [\![\vec{a} = \vec{c}]\!] \in U\}.$$

Proposition 4.2. *If $\{\boldsymbol{A}_i : i \in I\}$ is a family of similar algebras and U an ultrafilter on I, then θ_U is a congruence of $\prod_{i \in I} \boldsymbol{A}_i$.*

Proof. We begin by proving that θ_U is an equivalence relation on $\prod_{i \in I} A_i$. To this end, consider $\vec{a}, \vec{b}, \vec{c} \in \prod_{i \in I} A_i$. We have

$$[\![\vec{a} = \vec{a}]\!] = \{i \in I : \vec{a}(i) = \vec{a}(i)\} = I.$$

Observe that $I \in U$, since U is a nonempty upset of $\mathcal{P}(I)$. Together with the above display, this yields $[\![\vec{a} = \vec{a}]\!] \in U$ and, therefore, $\langle \vec{a}, \vec{a} \rangle \in \theta_U$. It follows that θ_U is reflexive. To prove that it is symmetric, suppose that $\langle \vec{a}, \vec{c} \rangle \in \theta_U$. Then $[\![\vec{a} = \vec{c}]\!] \in U$. Since $[\![\vec{a} = \vec{c}]\!] = [\![\vec{c} = \vec{a}]\!]$, this implies $[\![\vec{c} = \vec{a}]\!] \in U$ and, therefore, $\langle \vec{c}, \vec{a} \rangle \in \theta_U$. Lastly, to prove that θ_U is transitive, suppose that $\langle \vec{a}, \vec{b} \rangle, \langle \vec{b}, \vec{c} \rangle \in \theta_U$, that is, $[\![\vec{a} = \vec{b}]\!], [\![\vec{b} = \vec{c}]\!] \in U$. Since U is closed under binary meets,

$$[\![\vec{a} = \vec{b}]\!] \cap [\![\vec{b} = \vec{c}]\!] \in U$$

Clearly, $[\![\vec{a} = \vec{b}]\!] \cap [\![\vec{b} = \vec{c}]\!] \subseteq [\![\vec{a} = \vec{c}]\!]$. Since U is an upset of $\mathcal{P}(I)$, we obtain that $[\![\vec{a} = \vec{c}]\!] \in U$, whence $\langle \vec{a}, \vec{c} \rangle \in \theta_U$. We conclude that θ_U is an equivalence relation.

To prove that θ_U is a congruence, it only remains to show that it preserves the basic operations. Accordingly, let f be a basic n-ary operation and $\vec{a}_1, \dots, \vec{a}_n, \vec{c}_1, \dots, \vec{c}_n \in \prod_{i \in I} A_i$ such that

$$\langle \vec{a}_1, \vec{c}_1 \rangle, \dots, \langle \vec{a}_n, \vec{c}_n \rangle \in \theta_U.$$

By definition of θ_U, this amounts to $[\![\vec{a}_1 = \vec{c}_1]\!], \dots, [\![\vec{a}_n = \vec{c}_n]\!] \in U$. Since U is a filter, it is closed under finite meets, whence

$$[\![\vec{a}_1 = \vec{c}_1]\!] \cap \dots \cap [\![\vec{a}_n = \vec{c}_n]\!] \in U. \tag{5}$$

We will show that

$$[\![\vec{a}_1 = \vec{c}_1]\!] \cap \dots \cap [\![\vec{a}_n = \vec{c}_n]\!] \subseteq [\![f^{\prod_{i \in I} \boldsymbol{A}_i}(\vec{a}_1, \dots, \vec{a}_n) = f^{\prod_{i \in I} \boldsymbol{A}_i}(\vec{c}_1, \dots, \vec{c}_n)]\!]. \tag{6}$$

To this end, consider $j \in [\![\vec{a}_1 = \vec{c}_1]\!] \cap \dots \cap [\![\vec{a}_n = \vec{c}_n]\!]$. We have

$$\vec{a}_1(j) = \vec{c}_1(j), \dots, \vec{a}_n(j) = \vec{c}_n(j).$$

Consequently,

$$\begin{aligned} f^{\prod_{i \in I} \boldsymbol{A}_i}(\vec{a}_1, \dots, \vec{a})(j) &= f^{\boldsymbol{A}_j}(\vec{a}_1(j), \dots, \vec{a}_n(j)) \\ &= f^{\boldsymbol{A}_j}(\vec{c}_1(j), \dots, \vec{c}_n(j)) \\ &= f^{\prod_{i \in I} \boldsymbol{A}_i}(\vec{c}_1, \dots, \vec{c})(j), \end{aligned}$$

that is, $j \in [\![f^{\prod_{i \in I} \boldsymbol{A}_i}(\vec{a}_1, \dots, \vec{a}_n) = f^{\prod_{i \in I} \boldsymbol{A}_i}(\vec{c}_1, \dots, \vec{c}_n)]\!]$. This establishes (6). Since U is an upset of $\mathcal{P}(I)$, from (5) and (6) it follows

$$[\![f^{\prod_{i \in I} \boldsymbol{A}_i}(\vec{a}_1, \dots, \vec{a}_n) = f^{\prod_{i \in I} \boldsymbol{A}_i}(\vec{c}_1, \dots, \vec{c}_n)]\!] \in U.$$

Hence, we conclude that $\langle f^{\prod_{i \in I} \boldsymbol{A}_i}(\vec{a}_1, \dots, \vec{a}_n), f^{\prod_{i \in I} \boldsymbol{A}_i}(\vec{c}_1, \dots, \vec{c}_n) \rangle \in \theta_U$, as desired. ⊠

In view of the above result, we can make the following definition.

Definition 4.3. An *ultraproduct* of a family of similar algebras $\{\boldsymbol{A}_i : i \in I\}$ is an algebra of the form $\prod_{i \in I} \boldsymbol{A}_i / \theta_U$, for some ultrafilter U on I.

Given a class of similar algebras K, we set

$$\mathbb{P}_{\mathrm{U}}(\mathsf{K}) := \{\boldsymbol{A} : \boldsymbol{A} \text{ is an ultraproduct of a family } \{\boldsymbol{B}_i : i \in I\} \subseteq \mathsf{K}\}.$$

Notice that $\mathbb{P}_{\mathrm{U}}(\mathsf{K}) \subseteq \mathbb{HP}(\mathsf{K})$. Furthermore, as usual, when $\mathsf{K} = \{\boldsymbol{A}\}$, we write $\mathbb{P}_{\mathrm{U}}(\boldsymbol{A})$ as a shorthand for $\mathbb{P}_{\mathrm{U}}(\{\boldsymbol{A}\})$.

Exercise 4.4. Prove that if U is not free (that is, it is principal), then $\prod_{i\in I} \boldsymbol{A}_i/\theta_U$ is isomorphic to some $\boldsymbol{A}_i$. Conclude that if I is finite, then $\prod_{i\in I} \boldsymbol{A}_i/\theta_U$ belongs to $\mathbb{I}\{\boldsymbol{A}_i : i \in I\}$. Because of this, interesting ultraproducts arise from free ultrafilters only. ⊠

The importance of ultraproducts is tightly related to the following fundamental result.

Theorem 4.5 (Łos). *Let $\{\boldsymbol{A}_i : i \in I\}$ be a family of similar algebras, U an ultrafilter on I, and $\phi(x_1, \dots, x_n)$ a first-order formula. For every $\vec{a}_1, \dots, \vec{a}_n \in \prod_{i\in I} A_i$,*

$$\prod_{i\in I} \boldsymbol{A}_i/\theta_U \vDash \phi(\vec{a}_1/\theta_U, \dots, \vec{a}_n/\theta_U) \iff \{i \in I : \boldsymbol{A}_i \vDash \phi(\vec{a}_1(i), \dots, \vec{a}_n(i))\} \in U.$$

Corollary 4.6. *Let $\{\boldsymbol{A}_i : i \in I\}$ be a family of similar algebras, U an ultrafilter on I, and ϕ a first-order sentence. If ϕ is valid in all the $\boldsymbol{A}_i$, then it is valid in $\prod_{i\in I} \boldsymbol{A}_i/\theta_U$.*

In view Łos' Theorem, ultraproducts are instrumental to construct nonstandard models of first-order theories. For instance, let $\mathbb{N} = \langle \mathbb{N}; s, +, \cdot, 0\rangle$ be the standard model of Peano Arithmetic. If U is a free ultrafilter on $\mathbb{N}$, the ultraproduct $\prod_{n\in\mathbb{N}} \mathbb{N}_n/U$ is *elementarily equivalent* to $\mathbb{N}$, that is, it satisfies the same first-order sentences as $\mathbb{N}$. On the other hand, it is not hard to see that $\prod_{n\in\mathbb{N}} \mathbb{N}_n/U$ is uncountable and, therefore, contains many "infinite" (or nonstandard) natural numbers.

For the present purpose, however, we will not need the full strength of Łos Theorem and, therefore, we shall omit its proof. Instead, we shall focus on a particular embedding theorem for ultraproducts that depends on the following notion.

Definition 4.7. A *local subgraph* $\mathbb{X}$ of an algebra $\boldsymbol{A}$ is a finite subset $X \subseteq A$ endowed with the restriction of finitely many basic operations of $\boldsymbol{A}$ to X.

In this case, $\mathbb{X}$ is a finite *partial* algebra of finite type (even when the type of $\boldsymbol{A}$ is infinite).

Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be similar algebras and $\mathbb{X}$ a local subgraph of $\boldsymbol{A}$. A map $f\colon X \to B$ is said to be an *embedding* of $\mathbb{X}$ into $\boldsymbol{B}$ if it is injective and, for every basic n-ary operation g of the type of $\mathbb{X}$ and $a_1, \dots, a_n \in X$ such that $g^{\boldsymbol{A}}(a_1, \dots, a_n) \in X$,

$$f(g^{\boldsymbol{A}}(a_1, \dots, a_n)) = g^{\boldsymbol{B}}(f(a_1), \dots, f(a_n)).$$

Theorem 4.8. *Let $\mathsf{K} \cup \{\boldsymbol{A}\}$ be a class of similar algebras. If every local subgraph of $\boldsymbol{A}$ can be embedded into some member of K, then $\boldsymbol{A} \in \mathbb{ISP}_{\mathrm{U}}(\mathsf{K})$.*

Proof. Let I be the set of local subgraphs of $\boldsymbol{A}$. By assumption, for every $\mathbb{X} \in I$ there are an algebra $\boldsymbol{B}_{\mathbb{X}} \in \mathsf{K}$ and an embedding $h_{\mathbb{X}}\colon \mathbb{X} \to \boldsymbol{B}_{\mathbb{X}}$. We define a partial order $\sqsubseteq$ on I as follows:

$$\mathbb{X} \sqsubseteq \mathbb{Y} \iff X \subseteq Y \text{ and the type of } \mathbb{Y} \text{ extends that of } \mathbb{X}.$$

Then, for every $\mathbb{X} \in I$, define

$$J_{\mathbb{X}} := \{\mathbb{Y} \in I \colon \mathbb{X} \sqsubseteq \mathbb{Y}\}.$$

Moreover, let $\mathcal{F}$ be the filter of $\mathcal{P}(I)$ generated by $\{J_{\mathbb{X}} \colon \mathbb{X} \in I\}$. Recall that

$$\mathcal{F} = \{Y \subseteq I : J_{\mathbb{X}_1} \cap \dots \cap J_{\mathbb{X}_n} \subseteq Y, \text{ for some } \mathbb{X}_1, \dots, \mathbb{X}_n \in I\}.$$

We will prove that $\mathcal{F}$ is proper. To this end, consider $\mathbb{X}_1, \dots, \mathbb{X}_n \in I$. Then let $\mathbb{Y}$ be the local subgraph of $\boldsymbol{A}$ with universe $Y := X_1 \cup \dots \cup X_n$ and whose type in the union of the types of the various $\mathbb{X}_i$. Then

$$\mathbb{X}_i \sqsubseteq \mathbb{Y}, \text{ for every } i \leqslant n,$$

that is, $\mathbb{Y} \in J_{\mathbb{X}_1} \cap \cdots \cap J_{\mathbb{X}_n}$. It follows that $\emptyset \notin \mathcal{F}$ and, therefore, that $\mathcal{F}$ is proper. As $\mathcal{F}$ is a proper filter, by the Ultrafilter Lemma, it can be extended to an ultrafilter U on I.

Now, consider a map

$$f \colon A \to \prod_{\mathbb{X} \in I} B_{\mathbb{X}}$$

such that $f(a)(\mathbb{X}) = h_{\mathbb{X}}(a)$, for every $a \in A$ and $\mathbb{X} \in I$ such that $a \in \mathbb{X}$. Moreover, let

$$f^* \colon A \to \prod_{\mathbb{X} \in I} B_{\mathbb{X}} / \theta_U$$

be the map defined by the rule

$$f^*(a) := f(a)/\theta_U.$$

We will show f^* is an embedding of $\boldsymbol{A}$ into $\prod_{\mathbb{X} \in I} \boldsymbol{B}_{\mathbb{X}} / \theta_U$.

In order to prove that f^* is injective, consider a pair of distinct elements $a, c \in A$. Consider a local subgraph $\mathbb{Y}$ of $\boldsymbol{A}$ containing a and c. We will show that

$$J_{\mathbb{Y}} \subseteq \{\mathbb{X} \in I : f(a)(\mathbb{X}) \neq f(c)(\mathbb{X})\} \tag{7}$$

Consider $\mathbb{X} \in J_{\mathbb{Y}}$. Then $\mathbb{Y} \sqsubseteq \mathbb{X}$ and, therefore, $a, c \in Y \subseteq X$. Since $a, c \in X$, we have

$$f(a)(\mathbb{X}) = h_{\mathbb{X}}(a) \text{ and } f(c)(\mathbb{X}) = h_{\mathbb{X}}(c).$$

Furthermore, $h_{\mathbb{X}}(a) \neq h_{\mathbb{X}}(c)$, because $h_{\mathbb{X}}$ is injective and $a \neq c$. This yields $f(a)(\mathbb{X}) \neq f(c)(\mathbb{X})$, establishing (7).

Recall that the definition of U guarantees that $J_{\mathbb{Y}} \in \mathcal{F} \subseteq U$. Therefore, since U is an upset of $\mathcal{P}(I)$, we can apply (7) obtaining

$$I \smallsetminus [\![f(a) = f(c)]\!] = \{\mathbb{X} \in I : f(a)(\mathbb{X}) \neq f(c)(\mathbb{X})\} \in U.$$

Since U is a proper filter, this implies

$$[\![f(a) = f(c)]\!] \notin U$$

and, therefore,

$$f^*(a) = f(a)/\theta_U \neq f(c)/\theta_U = f^*(c).$$

Hence, we conclude that f^* is injective.

To prove that it is a homomorphism, consider a basic n-ary operation g and $a_1, \dots, a_n \in A$. Then consider a local subgraph $\mathbb{Y}$ of $\boldsymbol{A}$ whose universe contains $a_1, \dots, a_n, g^{\boldsymbol{A}}(a_1, \dots, a_n)$ and whose type contains g. We will prove that

$$J_{\mathbb{Y}} \subseteq [\![f(g^{\boldsymbol{A}}(a_1, \dots, a_n)) = g^{\prod_{\mathbb{X} \in I} \boldsymbol{B}_{\mathbb{X}}}(f(a_1), \dots, f(a_n))]\!]. \tag{8}$$

Consider $\mathbb{V} \in J_{\mathbb{Y}}$. Since $\mathbb{Y} \sqsubseteq \mathbb{V}$, the type of $\mathbb{V}$ contains g and $a_1, \dots, a_n, g^{\boldsymbol{A}}(a_1, \dots, a_n) \in V$. Since $a_1, \dots, a_n, g^{\boldsymbol{A}}(a_1, \dots, a_n) \in V$, we have

$$\begin{aligned} f(a_1)(\mathbb{V}) &= h_{\mathbb{V}}(a_1) \\ &\vdots \\ f(a_n)(\mathbb{V}) &= h_{\mathbb{V}}(a_n) \\ f(g^{\boldsymbol{A}}(a_1, \dots, a_n))(\mathbb{V}) &= h_{\mathbb{V}}(g^{\boldsymbol{A}}(a_1, \dots, a_n)). \end{aligned}$$

Furthermore, as the type of $\mathbb{V}$ contains g,

$$h_{\mathbb{V}}(g^{\boldsymbol{A}}(a_1, \dots, a_n)) = g^{\boldsymbol{B}_{\mathbb{V}}}(h_{\mathbb{V}}(a_1), \dots, h_{\mathbb{V}}(a_n)).$$

From the above displays it follows

$$f(g^{\boldsymbol{A}}(a_1,\dots,a_n))(\mathbb{V}) = g^{\boldsymbol{B}_{\mathbb{V}}}(f(a_1)(\mathbb{V}),\dots,f(a_n)(\mathbb{V})) = g^{\prod_{\mathbb{X}\in I}\boldsymbol{B}_{\mathbb{X}}}(f(a_1),\dots,f(a_n))(\mathbb{V}),$$

that is, $\mathbb{V} \in [\![f(g^{\boldsymbol{A}}(a_1,\dots,a_n)) = g^{\prod_{\mathbb{X}\in I}\boldsymbol{B}_{\mathbb{X}}}(f(a_1),\dots,f(a_n))]\!]$. This establishes (8). Lastly, as $J_{\mathbb{Y}} \in U$ and U is an upset of $\mathcal{P}(I)$, condition (8) implies

$$[\![f(g^{\boldsymbol{A}}(a_1,\dots,a_n)) = g^{\prod_{\mathbb{X}\in I}\boldsymbol{B}_{\mathbb{X}}}(f(a_1),\dots,f(a_n))]\!] \in U,$$

and, therefore,

$$\begin{aligned} f^*(g^{\boldsymbol{A}}(a_1,\dots,a_n)) &= f(g^{\boldsymbol{A}}(a_1,\dots,a_n))/\theta_U \\ &= g^{\prod_{\mathbb{X}\in I}\boldsymbol{B}_{\mathbb{X}}}(f(a_1),\dots,f(a_n))/\theta_U \\ &= g^{\prod_{\mathbb{X}\in I}\boldsymbol{B}_{\mathbb{X}}/\theta_U}(f(a_1)/\theta_U,\dots,f(a_n)/\theta_U) \\ &= g^{\prod_{\mathbb{X}\in I}\boldsymbol{B}_{\mathbb{X}}/\theta_U}(f^*(a_1),\dots,f^*(a_n)). \end{aligned}$$

Hence, we conclude that f^* is a homomorphism and, therefore, an embedding of $\boldsymbol{A}$ into $\prod_{\mathbb{Y}\in I}\boldsymbol{B}_{\mathbb{Y}}/\theta_U$. As a consequence,

$$\boldsymbol{A} \in \mathbb{ISP}_{\mathrm{U}}(\{\boldsymbol{B}_{\mathbb{X}} : \mathbb{X} \in I\}) \subseteq \mathbb{ISP}_{\mathrm{U}}(\mathsf{K}). \qquad \boxtimes$$

Corollary 4.9. *Every algebra embeds into an ultraproduct of its finitely generated subalgebras.*

Example 4.10 (Lattices)**.** Let Latt be the class of all lattices and $\mathsf{Latt}^{<\omega}$ that of finite lattices. We will show that $\mathsf{Latt} = \mathbb{ISP}_{\mathrm{U}}(\mathsf{Latt}^{<\omega})$. The inclusion from right to left follows from the fact that Latt is closed under $\mathbb{I}, \mathbb{S}$, and $\mathbb{P}_{\mathrm{U}}$. For the other inclusion, consider a lattice$\boldsymbol{A}$. We know that every local subgraph $\mathbb{X}$ of $\boldsymbol{A}$ can be embedded into the Dedekind-MacNeille completion of the subposet of $\boldsymbol{A}$ with universe X. As the Dedekind-MacNeille completion of a finite poset is finite, it follows that $\mathbb{X}$ can be emdebbed into a finite lattice. As a consequence, every local subgraph of $\boldsymbol{A}$ can be embedded into some finite lattice. By Theorem 4.8, this implies that $\boldsymbol{A} \in \mathbb{ISP}_{\mathrm{U}}(\mathsf{Latt}^{<\omega})$, as desired. $\boxtimes$

5. Quasi-varieties

At this stage it is natural to wonder whether it is possible to characterize classes of algebras axiomatized by quasi-equations (Definition 3.3) in terms of closure under certain class operators. The answer is affirmative, as we proceed to explain.

Definition 5.1. A prevariety closed under $\mathbb{P}_{\mathrm{U}}$ is a said to be a *quasi-variety*.

The aim of this section is to prove the following classical result.

Maltsev's Theorem 5.2. *A class of similar algebras is a quasi-variety if and only if it can be axiomatized by a set of quasi-equations.*

Proof. The "only if" part follows from the fact that the validity of quasi-equations is preserved by the class operators $\mathbb{I}, \mathbb{S}, \mathbb{P}$, and $\mathbb{P}_{\mathrm{U}}$. To prove the converse, consider a prevariety K closed under $\mathbb{P}_{\mathrm{U}}$. Moreover, let *Var* be a denumerable set of variables and Σ the set of quasi-equations, with variables in *Var*, valid in K. Let also K^+ be the class of algebras axiomatized by Σ. Our aim is to prove that $\mathsf{K} = \mathsf{K}^+$.

The inclusion $\mathsf{K} \subseteq \mathsf{K}^+$ is straightforward. To prove the other one, consider an algebra $\boldsymbol{A} \in \mathsf{K}^+$. In order to prove that $\boldsymbol{A} \in \mathsf{K}^+$, it suffices to show that every local subgraph of $\boldsymbol{A}$ embeds in some members of K. This is because, in this case, $\boldsymbol{A} \in \mathbb{ISP}_{\mathrm{U}}(\mathsf{K})$, by Theorem 4.8. Since K is closed under $\mathbb{I}, \mathbb{S}$, and $\mathbb{P}_{\mathrm{U}}$, this implies $\boldsymbol{A} \in \mathsf{K}$, as desired.

Then consider a local subgraph $\mathbb{X}$ of $\boldsymbol{A}$. By definition, $\mathbb{X}$ consists of a finite set $\{a_1, \dots, a_n\}$ endowed with the restriction of finitely many basic operations $f_1, \dots, f_m$ of $\boldsymbol{A}$ to X. Fix n distinct variables $x_1, \dots, x_n \in Var$, corresponding to the elements $a_1, \dots, a_n$ of X. The *positive* and *negative atomic diagrams* of $\mathbb{X}$ are, respectively,

$$\mathcal{D}^+(\mathbb{X}) \coloneqq \{f_i(x_{k_1}, \dots, x_{k_s}) \approx x_j \colon i \leqslant m \text{ and } k_1, \dots, k_s, j \leqslant n \text{ and } f_i^{\boldsymbol{A}}(a_{k_1}, \dots, a_{k_s}) = a_j\}$$
$$\mathcal{D}^-(\mathbb{X}) \coloneqq \{x_i \not\approx x_j \colon i, j \leqslant n \text{ and } a_i \neq a_j\}.$$

Observe that both $\mathcal{D}^+(\mathbb{X})$ and $\mathcal{D}^-(\mathbb{X})$ are finite sets. Then take an enumeration

$$\mathcal{D}^-(\mathbb{X}) = \{\varepsilon_1 \not\approx \delta_1, \dots, \varepsilon_t \not\approx \delta_t\}.$$

Moreover, for each $i \leqslant t$, consider the quasi-equation

$$\Phi_i \coloneqq \left(\mathop{\&} \mathcal{D}^+(\mathbb{X}) \right) \Longrightarrow \varepsilon_i \approx \delta_i.$$

As witnessed by the natural assignment

$$x_1 \longmapsto a_1, \dots, x_n \longmapsto a_n,$$

the quasi-equations $\Phi_1, \dots, \Phi_t$ fail in $\boldsymbol{A}$. Since they are written with variables in Var and $\boldsymbol{A}$ satisfies all the quasi-equations with variables in Var valid in K, this implies that each Φ_i fails in some $\boldsymbol{B}_i \in \mathsf{K}$ under an assignment

$$x_1 \longmapsto b_1^i, \dots, x_n \longmapsto b_n^i. \tag{9}$$

Now, consider the map $h \colon X \to (B_1 \times \cdots \times B_t)$, defined by the rule

$$a_1 \longmapsto \langle b_1^1, \dots, b_1^t \rangle, \dots, a_n \longmapsto \langle b_n^1, \dots, b_n^t \rangle.$$

We will prove that h is an embedding of $\mathbb{X}$ into $\boldsymbol{B}_1 \times \cdots \times \boldsymbol{B}_t$. To prove that h is injective, consider two distinct elements $a_p, a_q \in X$. Then the formula $x_p \not\approx x_q$ belongs to the negative atomic diagram of $\mathbb{X}$. Then there exists $i \leqslant t$ such that

$$\Phi_i = \left(\mathop{\&} \mathcal{D}^+(\mathbb{X}) \right) \Longrightarrow x_p \approx x_q.$$

Since Φ_i fails in $\boldsymbol{B}_i$ under the assignment in (9), we obtain $b_p^i \neq b_q^i$. As a consequence,

$$h(a_p)(i) = b_p^i \neq b_q^i = h(a_q)(i)$$

and, therefore, $h(a_p) \neq h(a_q)$. Hence, h is injective. To prove that it preserves the partial operations, consider a basic s-ary operation f_j in the type of $\mathbb{X}$ and $a_{k_1}, \dots, a_{k_s} \in X$ such that $f_j^{\boldsymbol{A}}(a_{k_1}, \dots, a_{k_s}) \in X$. Then there exists some $p \leqslant n$ such that $a_p = f_j^{\boldsymbol{A}}(a_{k_1}, \dots, a_{k_s})$. Moreover, the equation

$$f_j(x_{k_1}, \dots, x_{k_s}) \approx x_p$$

belongs to the positive atomic diagram $\mathcal{D}^+(\mathbb{X})$ of $\mathbb{X}$. As each quasi-equation Φ_i fails under the assignment in (9), the same assignment satisfies the antecedent of Φ_i, namely $\mathcal{D}^+(\mathbb{X})$. It follows that

$$f_j^{\boldsymbol{B}_i}(b_{k_1}^i, \dots, b_{k_s}^i) = b_p^i, \text{ for each } i \leqslant t.$$

As a consequence, for every $i \leqslant t$,

$$\begin{aligned} h(f_j^{\boldsymbol{A}}(a_{k_1}, \dots, a_{k_s}))(i) &= h(a_p)(i) \\ &= b_p^i \\ &= f_j^{\boldsymbol{B}_i}(b_{k_1}^i, \dots, b_{k_s}^i) \\ &= f_j^{\boldsymbol{B}_i}(h(a_{k_1})(i), \dots, h(a_{k_s})(i)) \\ &= f_j^{\boldsymbol{B}_1 \times \cdots \times \boldsymbol{B}_t}(h(a_{k_1}), \dots, h(a_{k_s}))(i). \end{aligned}$$

Thus, $h(f_j^{\boldsymbol{A}}(a_{k_1}, \dots, a_{k_s})) = f_j^{\boldsymbol{B}_1 \times \cdots \times \boldsymbol{B}_t}(h(a_{k_1}), \dots, h(a_{k_s}))$. We conclude that $h \colon \mathbb{X} \to (\boldsymbol{B}_1 \times \cdots \times \boldsymbol{B}_t)$ is an embedding. Since $\boldsymbol{B}_1, \dots, \boldsymbol{B}_t \in \mathsf{K}$ and K is closed under $\mathbb{P}$, the direct product $\boldsymbol{B}_1 \times \cdots \times \boldsymbol{B}_t$ belongs to K. Hence, $\mathbb{X}$ embeds into some member of K, as desired. ⊠

Exercise 5.3. Prove that if a quasi-equation Φ is valid in a class of similar algebras K, then it is also valid in $\mathbb{P}_{\mathrm{U}}(\mathsf{K})$. ⊠

Given a class of similar algebras K, the least quasi-variety extending K exists and will be denoted by $\mathbb{Q}(\mathsf{K})$ and called the quasi-variety *generated* by K.

Proposition 5.4 (Maltsev). *For every class of algebras* K*,*

$$\mathbb{Q}(\mathsf{K}) = \mathbb{ISPP}_{\mathrm{U}}(\mathsf{K}).$$

Proof. The inclusion $\mathbb{ISPP}_{\mathrm{U}}(\mathsf{K}) \subseteq \mathbb{Q}(\mathsf{K})$ is obvious. To prove the other, consider $\boldsymbol{A} \in \mathbb{Q}(\mathsf{K})$. By Maltsev's Theorem, $\mathbb{Q}(\mathsf{K})$ is the class of all algebras satisfying the quasi-equations valid in K. The proof of the hard part of Maltsev's Theorem show that $\boldsymbol{A} \in \mathbb{ISP}_{\mathrm{U}}\mathbb{P}(\mathsf{K})$. Therefore, it only remains to show that $\mathbb{P}_{\mathrm{U}}\mathbb{P}(\mathsf{K}) \subseteq \mathbb{ISPP}_{\mathrm{U}}(\mathsf{K})$. But this is an easy exercise on class operators (the details are sketched below).

Consider an algebra $\boldsymbol{B} \in \mathbb{P}_{\mathrm{U}}\mathbb{P}(\mathsf{K})$. There exists an index set I, an ultrafilter U on I, and a family of algebras $\{\boldsymbol{B}_j : j \in J_i\}$ for each $i \in I$ such that

$$\boldsymbol{B} = \Big(\prod_{i \in I}\Big(\prod_{j \in J_i}\boldsymbol{B}_j\Big)\Big)/\theta_U.$$

Let J be the set of all maps $f \colon I \to \bigcup_{i \in I} J_i$ such that $f(i) \in J_i$. Moreover, let

$$g \colon B \to \prod_{f \in J}\Big(\prod_{i \in I} B_{f}(i)\Big)$$

be the map defined by the rule $g(b)(f)(i) \coloneqq b(i)(f(i))$. It is not hard to check that the map

$$g^* \colon B \to \Big(\prod_{f \in J}\Big(\prod_{i \in I} B_{f}(i)\Big)\Big)/\theta_U$$

that sends an element $b \in B$ to $f(b)/\theta_U$ is an embedding, whence $\boldsymbol{B} \in \mathbb{ISPP}_{\mathrm{U}}(\mathsf{K})$. ⊠

Corollary 5.5. *If* K *be a finite set of finite similar algebras, then* $\mathbb{Q}(\mathsf{K}) = \mathbb{ISP}_{\mathrm{U}}(\mathsf{K})$*, that is, the quasi-variety and the prevariety generated by* K *coincide.*

Proof. Since K is a finite set of finite algebras, $\mathbb{P}_{\mathrm{U}}(\mathsf{K}) \subseteq \mathbb{I}(\mathsf{K})$. As a consequence, we obtain $\mathbb{ISP}(\mathsf{K}) = \mathbb{ISPP}_{\mathrm{U}}(\mathsf{K})$. Together with Proposition 5.4, this yields $\mathbb{Q}(\mathsf{K}) = \mathbb{ISP}(\mathsf{K})$. ⊠

Exercise 5.6. Prove that if K is a class of similar algebras, then $\mathbb{P}_{\mathrm{U}}\mathbb{P}(\mathsf{K}) \subseteq \mathbb{ISPP}_{\mathrm{U}}(\mathsf{K})$. Hint: use the sketch in the last part of Proposition 5.4. ⊠

Fix a denumerable set of variables X. The *quasi-equational theory* of a class of similar algebras K is the set of quasi-equations with variables in X valid in K.

Example 5.7 (Lattices)**.** Recall from Example 4.10 that $\mathsf{Latt} = \mathbb{ISP}_{\mathrm{U}}(\mathsf{Latt}^{<\omega})$. As a consequence, $\mathsf{Latt} = \mathbb{Q}(\mathsf{Latt}^{<\omega})$. Thus, a quasi-equation is valid in Latt if and only if it is valid in $\mathsf{Latt}^{<\omega}$. Since the class of lattices is finitely axiomatizable, this implies that the quasi-equational theory of Latt is decidable.

At this stage, it is natural to wonder whether Latt is also the prevariety generated by $\mathsf{Latt}^{<\omega}$. This is not the case, as we proceed to explain. First, consider a lattice $\boldsymbol{A}$ with precisely two congruences, namely id_A and $A \times A$. For instance, we can take $\boldsymbol{A}$ to the the poset of equivalence relations on an infinite set. Then suppose, with a view to contradiction, that $\mathsf{Latt} = \mathbb{ISP}(\mathsf{Latt}^{<\omega})$. Since $\mathbb{ISP}(\mathsf{Latt}^{<\omega}) = \mathbb{IP}_{\mathrm{SD}}\mathbb{S}(\mathsf{Latt}^{<\omega}) = \mathbb{IP}_{\mathrm{SD}}(\mathsf{Latt}^{<\omega})$, there exists a subdirect embedding $f \colon \boldsymbol{A} \to \prod_{i \in I} \boldsymbol{B}_i$ where $\{\boldsymbol{B}_i : i \in I\}$ is a family of finite lattices. For the sake of simplicity, we can assume that f is the identity map and that $\boldsymbol{A}$ is a subdirect product of $\{\boldsymbol{B}_i : i \in I\}$. Then consider consider $j \in I$. Since $\boldsymbol{A}$ is a subdirect product of $\{\boldsymbol{B}_i : i \in I\}$, the projection map $p_j \colon \boldsymbol{A} \to \boldsymbol{B}_j$ is surjective. As $\boldsymbol{A}$ is simple, $\mathsf{Ker}(p_j)$ is either id_A and $A \times A$. If it is id_A, then p_j is injective and, therefore, $\boldsymbol{B}_j$ is infinite, which is false. Then $\mathsf{Ker}(p_j) = A \times A$. But this implies that $\boldsymbol{B}_j$ is the trivial lattice. It follows that each $\boldsymbol{B}_i$ is trivial, whence so is $\prod_{i \in I} \boldsymbol{B}_i$. But this contradicts the assumption that $\boldsymbol{A}$ is an infinite subalgebra of $\prod_{i \in I} \boldsymbol{B}_i$. We conclude that $\mathbb{ISP}(\mathsf{Latt}^{<\omega}) \subsetneq \mathsf{Latt}$ and, therefore, that there are generalized quasi-equations that are valid in $\mathsf{Latt}^{<\omega}$, but not in Latt.

The contrasts with the case of distributive lattices. Indeed the class DL of distributive lattices is the prevariety generated by its finite members. Even more is true: DL is the prevariety generated by the two-element distributive lattice. Similarly, the class of Boolean algebras is the prevariety generated the two-element Boolean algebra (and, therefore, by finite Boolean algebras). ⊠

Given a set of variables X and a type ρ, we denote by $E_\rho(X)$ the set of equations of type ρ with variables in X.

Definition 5.8. Let K be a class of similar algebras and X a set of variables. We define a binary relation $\vDash_{\mathsf{K}}^{X} \subseteq \mathcal{P}(E_\rho(X)) \times E_\rho(X)$ as follows:

$$\begin{aligned}\Theta \vDash_{\mathsf{K}}^{X} \varepsilon \approx \delta \iff & \text{ for every } \boldsymbol{A} \in \mathsf{K} \text{ and every } \vec{a} \in A, \\ & \text{if } \varphi^{\boldsymbol{A}}(\vec{a}) = \psi^{\boldsymbol{A}}(\vec{a}) \text{ for all } \varphi \approx \psi \in \Theta, \text{ then } \varepsilon^{\boldsymbol{A}}(\vec{a}) = \delta^{\boldsymbol{A}}(\vec{a}).\end{aligned}$$

The relation $\vDash_{\mathsf{K}}^{X}$ is known as the *equational consequence relative to* K (with variables in X).

Notice that the equational consequence relative to K describes the validity of generalized quasi-equations in K, in the sense that

$$\Theta \vDash_{\mathsf{K}}^{X} \varepsilon \approx \delta \iff \mathsf{K} \vDash \mathop{\&} \Theta \Longrightarrow \varepsilon \approx \delta.$$

When the set X is understood, we drop the superscript and write $\vDash_{\mathsf{K}}$ instead of $\vDash_{\mathsf{K}}^{X}$. Moreover, when $\mathsf{K} = \{\boldsymbol{A}\}$ we write $\vDash_{\boldsymbol{A}}^{X}$ as a shorthand for $\vDash_{\{\boldsymbol{A}\}}^{X}$.

Example 5.9 (Lattices)**.** Since the class of distributive lattices DL is the prevariety generated by the two-element distributive lattice $\boldsymbol{B}$, the equational consequences relative to DL and to $\boldsymbol{B}$ coincide, for every set of variables X. On the other hand, as the class Latt of all lattices is not the prevariety generated by the class $\mathsf{Latt}^{<\omega}$ of finite lattices, the equational consequence relative to Latt and $\mathsf{Latt}^{<\omega}$ do not coincide in general. ⊠

Example 5.10 (Boolean algebras)**.** Recall that the class BA of Boolean algebras is the prevariety generated by the two-element Boolean algebras $\boldsymbol{B}$. Consequently, the equational consequences relative to BA and $\boldsymbol{B}$ coincide, for every set of variables X. Notice that these relative equational consequences are related to Classical Propositional Logic **CPC** by the following completeness theorem: for every set of variables X and $\Gamma \cup \{\varphi\} \subseteq T(X)$,

$$\begin{aligned}\Gamma \vdash_{\mathbf{CPC}} \varphi &\iff \{\gamma \approx 1 : \gamma \in \Gamma\} \vDash_{\mathsf{BA}} \varphi \approx 1\\ &\iff \{\gamma \approx 1 : \gamma \in \Gamma\} \vDash_{\boldsymbol{B}} \varphi \approx 1.\end{aligned}$$

The first of the above equivalences can be proved using the Lindenbaum-Tarski method, while the second follows from the Ultrafilter Lemma. ⊠

Notably, quasi-varieties coincide with the prevarieties whose relative equational consequence is finitary.

Theorem 5.11. *A K prevariety is a quasi-variety if and only if the equational consequence relative to K is finitary on every set of variables X, in the sense that*

$$\textit{if } \Theta \vDash_{\mathsf{K}}^{X} \varepsilon \approx \delta, \textit{ there exists a finite } \Sigma \subseteq \Theta \textit{ such that } \Sigma \vDash_{\mathsf{K}}^{X} \varepsilon \approx \delta.$$

The proof of the above result is based on an ultraproduct construction, whose standard application is a proof of the Compactness Theorem of first-order logic (semantically defined). Here we will use it to prove the compactness (a.k.a. finitarity) of relative equational consequences.

*Exercise** 5.12. Prove Theorem 5.11. To do this, use the following proof strategy:

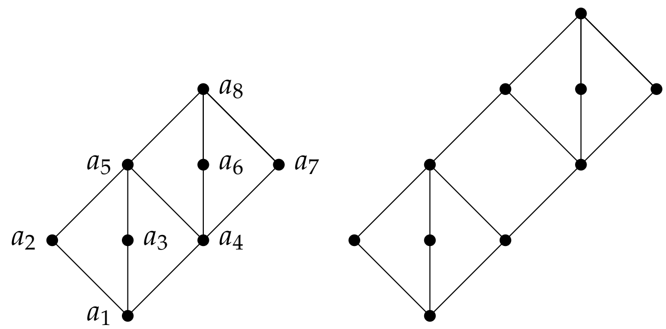
(i) First, suppose that the equational consequence relative to K is finitary (on every set of variables). Use this fact to show that every generalized quasi-equation valid in K can be finitized, i.e., transformed into a quasi-equation valid in K.
(ii) Then prove that ultraproducts preserve the validity of quasi-equations (Exercise 5.3). Use this fact to conclude that all the generalized q uasi-equations valid in K are also valid in $\mathbb{P}_{\mathrm{U}}(\mathsf{K})$.
(iii) Lastly, use Theorem 3.8 to conclude that $\mathbb{P}_{\mathrm{U}}(\mathsf{K})$ belongs to the prevariety generated by K. Since K is a prevariety, this yields $\mathbb{P}_{\mathrm{U}}(\mathsf{K}) \subseteq \mathsf{K}$, as desired.
(iv) The converse implication is the standard ultraproduct argument. Suppose that K is closed under $\mathbb{P}_{\mathrm{U}}$. Then consider $\Theta \cup \{\varepsilon \approx \delta\} \subseteq E(X)$ such that $\Sigma \nvDash_{\mathsf{K}} \varepsilon \approx \delta$, for every finite $\Sigma \subseteq \Theta$. Then, for every finite $\Sigma \subseteq \Theta$ there are $\boldsymbol{A}_\Sigma \in \mathsf{K}$ and an assignment $h_\Sigma \colon X \to \boldsymbol{A}_\Sigma$ that falsify the quasi-equation $\& \Sigma \Longrightarrow \varepsilon \approx \delta$.
(v) Let I be the set set of finite subsets of Θ. For each $\Sigma \in I$, define

$$J_\Sigma \coloneqq \{\Delta \in I : \Sigma \subseteq \Delta\}.$$

Prove that there exists a ultrafilter U on I extending the set $\{J_\Sigma : \Sigma \in I\}$.
(vi) Then consider the ultraproduct $\prod_{\Sigma \in I} \boldsymbol{A}_\Sigma / \theta_U$ and observe that it belongs to K by assumption. Let $h \colon X \to \prod_{\Sigma \in I} \boldsymbol{A}_\Sigma / \theta_U$ be the natural assignment induced by the various h_Σ. Prove that it rejects the generalized quasi-equation $\& \Theta \Longrightarrow \varepsilon \approx \delta$. Then conclude that $\Theta \nvDash_{\mathsf{K}} \varepsilon \approx \delta$ and, therefore, that $\vDash_{\mathsf{K}}^{X}$ is finitary. ⊠

6. Relative congruences

Quasi-varieties need not be closed under homomorphic images. For instance, consider the lattices $\boldsymbol{A}$ and $\boldsymbol{B}$ depicted below. We will show that the quasi-variety $\mathbb{Q}(\boldsymbol{B})$ is not closed under $\mathbb{H}$.



$\boldsymbol{A}$ $\boldsymbol{B}$

To this end, let $\mathcal{D}^+(\boldsymbol{A})$ be the positive atomic diagram of $\boldsymbol{A}$ written with the variables $x_1, \dots, x_8$ corresponding to the elements $a_1, \dots, a_8$ and consider the quasi-equation

$$\Phi = \mathop{\&} \mathcal{D}^+(\boldsymbol{A}) \Longrightarrow x_1 \approx x_8.$$

Notice that $\boldsymbol{B}$ validates Φ. To prove this, consider an assignment $f \colon \{x_1, \dots, x_8\} \to B$ that validates $\mathcal{D}^+(\boldsymbol{A})$ in $\boldsymbol{B}$. Using the definition of $\mathcal{D}^+(\boldsymbol{A})$, it is easy to see that the map $h \colon A \to B$ that sends a_i to $f(a_i)$ is a homomorphism from $\boldsymbol{A}$ to $\boldsymbol{B}$. Since $\boldsymbol{A}$ is simple, $\mathsf{Ker}(h)$ is either id_A or $A \times A$. Notice that there is no embedding of $\boldsymbol{A}$ into $\boldsymbol{B}$. Therefore, $\mathsf{Ker}(h)$ cannot be the identity relation. It follows that $\mathsf{Ker}(h) = A \times A$. In particular, $\langle a_1, a_8 \rangle \in \mathsf{Ker}(h)$ and, therefore, $f(x_1) = h(a_1) = f(a_8) = f(x_8)$. Hence, we conclude that $\boldsymbol{B} \vDash \Phi$, as desired. Moreover, Φ fails in $\boldsymbol{A}$, as witnessed by the assignment $x_i \longmapsto a_i$.

In brief, Φ holds in $\boldsymbol{B}$ but fails in $\boldsymbol{A}$. By Maltsev's Theorem, we conclude that $\boldsymbol{A}$ does not belong to the quasi-variety $\mathbb{Q}(\boldsymbol{B})$ generated by $\boldsymbol{B}$. On the other hand, $\boldsymbol{A}$ is a homomorphic image of $\boldsymbol{B}$ (obtained by glueing two pairs of elements of $\boldsymbol{B}$). Thus, the quasi-variety $\mathbb{Q}(\boldsymbol{B})$ is not closed under $\mathbb{H}$.

More in general, if K is a quasi-variety and θ a congruence of some $\boldsymbol{A} \in \mathsf{K}$, the algebra $\boldsymbol{A}/\theta$ need not belong to K. This makes the following concept attractive.

Definition 6.1. Let $\mathsf{K} \cup \{\boldsymbol{A}\}$ be a class of similar algebras. A congruence $\theta \in \mathsf{Con}(\boldsymbol{A})$ is said to be a K-*congruence* of $\boldsymbol{A}$ if $\boldsymbol{A}/\theta \in \mathsf{K}$. We denote the poset of K-congruences of $\boldsymbol{A}$, ordered under the inclusion relation, by $\mathsf{Con}_{\mathsf{K}}(\boldsymbol{A})$.

Notice that if K is closed under $\mathbb{H}$, then $\mathsf{Con}_{\mathsf{K}}(\boldsymbol{A}) = \mathsf{Con}(\boldsymbol{A})$, for all $\boldsymbol{A} \in \mathsf{K}$ (see (2, if necessary). Under some minimal assumptions, the converse is also true.

Proposition 6.2. *Let K be a class of similar algebras closed under $\mathbb{I}$. Then K is closed under $\mathbb{H}$ if and only if $\mathsf{Con}(\boldsymbol{A}) = \mathsf{Con}_{\mathsf{K}}(\boldsymbol{A})$, for all $\boldsymbol{A} \in \mathsf{K}$.*

Proof. First, suppose that $\mathsf{Con}(\boldsymbol{A}) = \mathsf{Con}_{\mathsf{K}}(\boldsymbol{A})$, for all $\boldsymbol{A} \in \mathsf{K}$. Then consider an algebra $\boldsymbol{B} \in \mathbb{H}(\mathsf{K})$. In view of (2), there exists $\boldsymbol{A} \in \mathsf{K}$ and a congruence θ of $\boldsymbol{A}$ such that $\boldsymbol{B} \cong \boldsymbol{A}/\theta$. Since $\theta \in \mathsf{Con}\boldsymbol{A} = \mathsf{Con}_{\mathsf{K}}\boldsymbol{A}$, then $\boldsymbol{A}/\theta \in \mathsf{K}$. As K is closed under the formation of isomorphic copies, we conclude that $\boldsymbol{B} \in \mathsf{K}$, as desired.

Conversely, suppose that K is closed under $\mathbb{H}$. Consider $\boldsymbol{A} \in \mathsf{K}$. Since $\boldsymbol{A}/\theta$ is a homomorphic image of $\boldsymbol{A}$, we obtain $\boldsymbol{A}/\theta \in \mathbb{H}(\boldsymbol{A}) \subseteq \mathbb{H}(\mathsf{K}) \subseteq \mathsf{K}$. Thus, $\theta \in \mathsf{Con}_{\mathsf{K}}\boldsymbol{A}$. ⊠

Relative congruences and subdirect products are related as follows.

Proposition 6.3. *Let K be a class of algebras of type ρ closed under $\mathbb{I}$. Then K is closed under $\mathbb{P}_{\text{SD}}$ if and only if $\mathsf{Con}_{\mathsf{K}}(\boldsymbol{A})$ is a closure system on $A \times A$, for all algebras $\boldsymbol{A}$ of type ρ.*

Proof. Suppose first that K is closed under $\mathbb{P}_{\text{SD}}$. Then let $\boldsymbol{A}$ be an algebra of type ρ. Since K is closed under $\mathbb{P}_{\text{SD}}$ it contains a trivial algebra (the subdirect product of the empty family). Therefore, as K is closed under $\mathbb{I}$ by assumption, it contains all trivial algebras and, in particular, $\boldsymbol{A}/(A \times A)$. Thus, $A \times A \in \mathsf{Con}_{\mathsf{K}}\boldsymbol{A}$. Then consider a nonempty family $\{\theta_i : i \in I\} \subseteq \mathsf{Con}_{\mathsf{K}}(\boldsymbol{A})$. By Proposition 2.20,

$$\boldsymbol{A}/\bigcap_{i \in I}\theta_i \in \mathbb{I}\mathbb{P}_{\text{SD}}(\{\boldsymbol{A}/\theta_i : i \in I\}).$$

Observe that $\{\boldsymbol{A}/\theta_i : i \in I\} \subseteq \mathsf{K}$, since the various θ_i are K-congruences of $\boldsymbol{A}$. Together with the above display and the assumption that K is closed under $\mathbb{I}$ and $\mathbb{P}_{\text{SD}}$, this yields $\boldsymbol{A}/\bigcap_{i \in I}\theta_i \in \mathsf{K}$, whence $\bigcap_{i \in I}\theta_i \in \mathsf{Con}_{\mathsf{K}}(\boldsymbol{A})$. It follows that $\mathsf{Con}_{\mathsf{K}}(\boldsymbol{A})$ is a closure system.

To prove the converse, suppose that $\mathsf{Con}_{\mathsf{K}}(\boldsymbol{A})$ is a closure system on $A \times A$, for all algebras $\boldsymbol{A}$ of type ρ. Then consider a subdirect product $\boldsymbol{A}$ of a family $\{\boldsymbol{B}_i : i \in I\} \subseteq \mathsf{K}$. Then the canonical projection $p_i \colon \boldsymbol{A} \to \boldsymbol{B}_i$ is a surjective homomorphism, for every $i \in I$. We will show that $\{\mathsf{Ker}(p_i) : i \in I\} \subseteq \mathsf{Con}_{\mathsf{K}}\boldsymbol{A}$. To this end, consider $i \in I$. Since $p_i \colon \boldsymbol{A} \to \boldsymbol{B}_i$ is surjective, we have $\boldsymbol{A}/\mathsf{Ker}(p_i) \cong \boldsymbol{B}_i$, by Corollary 2.14. Since $\boldsymbol{B}_i \in \mathsf{K}$ and K is closed under $\mathbb{I}$, this yields $\boldsymbol{A}/\mathsf{Ker}(p_i) \in \mathsf{K}$ and, therefore, $\mathsf{Ker}(p_i) \in \mathsf{Con}_{\mathsf{K}}\boldsymbol{A}$, as desired.

Now, recall that $\mathsf{Con}_{\mathsf{K}}\boldsymbol{A}$ is a closure system, by assumption. Therefore, from $\{\mathsf{Ker}(p_i) : i \in I\} \subseteq \mathsf{Con}_{\mathsf{K}}\boldsymbol{A}$ it follows $\bigcap_{i \in I}\mathsf{Ker}(p_i) \in \mathsf{Con}_{\mathsf{K}}\boldsymbol{A}$. We will prove that $\bigcap_{i \in I}\mathsf{Ker}(p_i)$ is the identity relation on A. Consider two distinct $a, c \in A$. Since $\boldsymbol{A}$ is a subdirect product of $\{\boldsymbol{B}_i : i \in I\}$, we have $\boldsymbol{A} \leqslant \prod_{i \in I}\boldsymbol{B}_i$. Then there must be some $j \in I$ such that $a(j) \neq c(j)$. As a consequence, $\langle a, c\rangle \notin \mathsf{Ker}(p_j)$, whence $\langle a, c\rangle \notin \bigcap_{i \in I}\mathsf{Ker}(p_i)$. We conclude that

$$\bigcap_{i \in I}\mathsf{Ker}(p_i) = \mathrm{id}_A.$$

Therefore, $\mathrm{id}_A \in \mathsf{Con}_{\mathsf{K}}\boldsymbol{A}$, whence $\boldsymbol{A}/\mathrm{id}_A \in \mathsf{K}$. Since K is closed under $\mathbb{I}$ and $\boldsymbol{A} \cong \boldsymbol{A}/\mathrm{id}_A$, we obtain $\boldsymbol{A} \in \mathsf{K}$, as desired. ⊠

Corollary 6.4. *If K is a prevariety, then $\mathsf{Con}_{\mathsf{K}}\boldsymbol{A}$ is a closure system, for every algebra $\boldsymbol{A}$ of the same type as K.*

In the case of quasi-varieties something more is true.

Proposition 6.5. *Let K be a quasi-variety. If $\boldsymbol{A}$ is an algebra of the same type as K, then $\mathsf{Con}_{\mathsf{K}}\boldsymbol{A}$ is an inductive closure system and, therefore, an algebraic lattice.*

Proof. In view of Corollary 6.4, it suffices to show that the union ϕ of a nonempty upward directed family $\{\theta_i : i \in I\} \subseteq \mathsf{Con}_{\mathsf{K}}(\boldsymbol{A})$ is still a K-congruence of $\boldsymbol{A}$. It is clear that ϕ is a congruence of $\boldsymbol{A}$. Therefore, we only detail a proof of the fact that $\boldsymbol{A}/\phi \in \mathsf{K}$.

In view of Maltsev's Theorem, it suffices to show that all quasi-equations valid in K are also valid in $\boldsymbol{A}/\phi$. Accordingly, consider a quasi-equation

$$\Phi = (\varphi_1 \approx \psi_1 \,\&\, \dots \,\&\, \varphi_n \approx \psi_n) \Longrightarrow \varepsilon \approx \delta$$

valid in K. Moreover, let $\vec{a} \in A$ be such that $\varphi_j^{\boldsymbol{A}/\phi}(\vec{a}/\phi) = \psi_j^{\boldsymbol{A}/\phi}(\vec{a}/\phi)$, for every $j \leqslant n$. Then consider $j \leqslant n$. Since

$$\varphi_j^{\boldsymbol{A}}(\vec{a})/\phi = \varphi_j^{\boldsymbol{A}/\phi}(\vec{a}/\phi) = \psi_j^{\boldsymbol{A}/\phi}(\vec{a}/\phi) = \psi_j^{\boldsymbol{A}}(\vec{a})/\phi$$

and $\phi = \bigcup_{i \in I} \theta_i$, there exists $i_j \in I$ such that $\langle \varphi_j^{\boldsymbol{A}}(\vec{a}), \psi_j^{\boldsymbol{A}}(\vec{a}) \rangle \in \theta_{i_j}$. Thus,

$$\langle \varphi_1^{\boldsymbol{A}}(\vec{a}), \psi_1^{\boldsymbol{A}}(\vec{a}) \rangle \in \theta_{i_1}, \dots, \langle \varphi_n^{\boldsymbol{A}}(\vec{a}), \psi_n^{\boldsymbol{A}}(\vec{a}) \rangle \in \theta_{i_n}.$$

Since the family $\{\theta_i : i \in I\}$ is upward directed, there exists $k \in I$ such that $\theta_{i_1}, \dots, \theta_{i_n} \subseteq \theta_k$. Therefore,

$$\langle \varphi_1^{\boldsymbol{A}}(\vec{a}), \psi_1^{\boldsymbol{A}}(\vec{a}) \rangle, \dots, \langle \varphi_n^{\boldsymbol{A}}(\vec{a}), \psi_n^{\boldsymbol{A}}(\vec{a}) \rangle \in \theta_k.$$

This implies

$$\varphi_j^{\boldsymbol{A}/\theta_k}(\vec{a}/\theta_k) = \varphi_j^{\boldsymbol{A}}(\vec{a})/\theta_k = \psi_j^{\boldsymbol{A}}(\vec{a})/\theta_k = \psi_j^{\boldsymbol{A}/\theta_k}(\vec{a}/\theta_k), \text{ for every } j \leqslant n.$$

Since $\boldsymbol{A}/\theta_k \in \mathsf{K}$, we know that this algebra validates the quasi-equation Φ. Together with the above display, this yields

$$\langle \varepsilon^{\boldsymbol{A}}(\vec{a}), \delta^{\boldsymbol{A}}(\vec{a}) \rangle \in \theta_k \subseteq \phi.$$

It follows that $\varepsilon^{\boldsymbol{A}/\phi}(\vec{a}/\phi) = \delta^{\boldsymbol{A}/\phi}(\vec{a}/\phi)$. Therefore, $\boldsymbol{A}/\phi \vDash \Phi$, as desired. ⊠

Corollary 6.6. *Let K be a quasi-variety and $\boldsymbol{A} \in \mathsf{K}$. Every K-congruence of $\boldsymbol{A}$ is the intersection of a family of K-congruences of $\boldsymbol{A}$ that are completely meet irreducible in $\mathsf{Con}_{\mathsf{K}}\boldsymbol{A}$.*

Proof. Every element of an algebraic lattice is a meet of a family of completely meet irreducible elements. Therefore, the result follows immediately from Proposition 6.5. ⊠

7. Subdirect decomposition

In this section, we shall present a general decomposition of for algebraic structures in terms of subdirect products. Because of this, it makes sense to isolate the building blocks of subdirect products, that is, the algebras that cannot be obtained as subdirect products of algebras other than themselves.

Definition 7.1. Let K be a quasi-variety. A member $\boldsymbol{A}$ of K is said to be *subdirectly irreducible relative to* K when for every subdirect emebedding $f \colon \boldsymbol{A} \to \prod_{i \in I} \boldsymbol{B}_i$ with $\{\boldsymbol{B}_i : i \in I\} \subseteq \mathsf{K}$, there exists some $i \in I$ such that the composition $p_i \circ f \colon \boldsymbol{A} \to \boldsymbol{B}_i$ is an isomorphism. The class of all subdirectly irreducible algebras relative to K will be denoted by $\mathsf{K}_{\mathrm{RSI}}$.

An algebra $\boldsymbol{A}$ is said to be *subdirectly irreducible* (in the absolute sense) when it is subdirectly irreducible relative to the quasi-variety of all algebras of its type.

The next result connects subdirect irreduciblity with congruence lattices.

Proposition 7.2. *Let K be a quasi-variety. An algebra $\boldsymbol{A} \in \mathsf{K}$ is subdirectly irreducible relative to K if and only if id_A is completely meet irreducible in $\mathsf{Con}_{\mathsf{K}}\boldsymbol{A}$.*

Proof. Suppose first that id_A is not completely meet irreducible. Then there exists a family $\{\theta_i : i \in I\} \subseteq \mathsf{Con}_{\mathsf{K}}\boldsymbol{A} \smallsetminus \{\mathrm{id}_A\}$. By Proposition 2.20, there exists a subdirect embedding $f \colon \boldsymbol{A} \to \prod_{i \in I} \boldsymbol{A}/\theta_i$. We will show that $p_i \circ f$ is not injective, for every $i \in I$. To this end, consider $i \in I$. We have $\mathsf{Ker}(p_i \circ f) = \theta_i$. Since θ_i is not the identity, we obtain $\mathsf{Ker}(p_i \circ f) \neq \mathrm{id}_A$, whence $p_i \circ f$ is not injective, as desired. It follows that $\boldsymbol{A}$ is not subdirectly irreducible relative to K.

To prove the other implication, suppose that id_A is completely meet irreducible in $\mathsf{Con}_{\mathsf{K}}\boldsymbol{A}$. Then consider a subdirect embedding $f\colon \boldsymbol{A} \to \prod_{i\in I} \boldsymbol{B}_i$ with $\{\boldsymbol{B}_i : i \in I\} \subseteq \mathsf{K}$. For each $i \in I$, we consider the congruence $\mathsf{Ker}(p_i \circ f)$ of $\boldsymbol{A}$. Since $p_i \circ f$ is sujective (because f is subdirect), we can apply Corollary 2.14, obtaining that $\boldsymbol{A}/\mathsf{Ker}(p_i \circ f) \cong \boldsymbol{B}_i$ and, therefore, $\boldsymbol{A}/\mathsf{Ker}(p_i \circ f) \in \mathsf{K}$. It follows that $\mathsf{Ker}(p_i \circ f) \in \mathsf{Con}_{\mathsf{K}}\boldsymbol{A}$. We will show that

$$\mathrm{id}_A = \bigcap_{i\in I} \mathsf{Ker}(p_i \circ f). \tag{10}$$

To this end, consider $a, c \in A$. We have

$$\begin{aligned}
\langle a, c\rangle \in \mathrm{id}_A &\iff a = c\\
&\iff f(a) = f(c)\\
&\iff f(a)(i) = f(c)(i), \text{ for every } i \in I\\
&\iff p_i \circ f(a) = p_i \circ f(c), \text{ for every } i \in I\\
&\iff \langle a, c\rangle \in \mathsf{Ker}(p_i \circ f), \text{ for every } i \in I\\
&\iff \langle a, c\rangle \in \bigcap_{i\in I} \mathsf{Ker}(p_i \circ f),
\end{aligned}$$

where the second equivalence follows from the injectivity of f. From (10) and the assumption that id_A is completely meet irreducible in $\mathsf{Con}_{\mathsf{K}}\boldsymbol{A}$ it follows that there exists $i \in I$ such that $\mathrm{id}_A = \mathsf{Ker}(p_i \circ f)$. It follows that the homomorphism $p_i \circ f\colon \boldsymbol{A} \to \boldsymbol{B}_i$ is injective. As it is also surjective, because f is subdirect, we conclude that it is an isomorphism. Hence, $\boldsymbol{A}$ is subdirectly irreducible relative to K. ⊠

Corollary 7.3. *An algebra $\boldsymbol{A}$ is subdirectly irreducible if and only if id_A is completely meet irreducible in $\mathsf{Con}\boldsymbol{A}$.*

Remark 7.4. Let K be a quasi-variety and $\boldsymbol{A} \in \mathsf{K}$. In view of Proposition 7.2, $\boldsymbol{A}$ is subdirectly irreducible relative to K precisely when there exists $\phi \in \mathsf{Con}_{\mathsf{K}}(\boldsymbol{A}) \smallsetminus \{\mathrm{id}_A\}$ such that every element of $\mathsf{Con}_{\mathsf{K}}(\boldsymbol{A}) \smallsetminus \{\mathrm{id}_A\}$ extends ϕ. In this case, ϕ is sometimes called the *relative monolith* of $\boldsymbol{A}$.

To prove the above equivalence, observe that if such a ϕ exists, then id_A is clearly completely meet irreducible in $\mathsf{Con}_{\mathsf{K}}(\boldsymbol{A})$ and, therefore, $\boldsymbol{A}$ is subdirectly irreducible relative to K, by Proposition 7.2. Conversely, suppose that $\boldsymbol{A}$ is subdirectly irreducible and, therefore, that id_A is completely meet irreducible in $\mathsf{Con}_{\mathsf{K}}(\boldsymbol{A})$. Then the K-congruence

$$\phi := \bigcap \{\theta \in \mathsf{Con}_{\mathsf{K}}(\boldsymbol{A}) : \theta \neq \mathrm{id}_A\}$$

is different from identity relation id_A. Furthermore, every element of $\mathsf{Con}_{\mathsf{K}}(\boldsymbol{A}) \smallsetminus \{\mathrm{id}_A\}$ extends ϕ, as desired. ⊠

A special kind of relative subdirectly irreducible algebras is the following.

Definition 7.5. Let K be a quasi-variety. An algebra $\boldsymbol{A} \in \mathsf{K}$ is *simple relative to* K if it has exactly two K-congruences.

In this case, $\mathsf{Con}_{\mathsf{K}}(\boldsymbol{A}) = \{\mathrm{id}_A, A \times A\}$ and $\mathrm{id}_A \neq A \times A$, whence $\boldsymbol{A}$ is nontrivial. Moreover, $\mathsf{Con}_{\mathsf{K}}(\boldsymbol{A})$ is the two-element chain with minimum id_A and, therefore, id_A is completely meet irreducible in $\mathsf{Con}_{\mathsf{K}}(\boldsymbol{A})$. Therefore, by Proposition 7.2, we obtain the following.

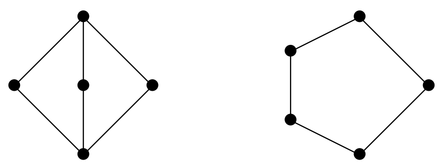
Corollary 7.6. *If an algebra $\boldsymbol{A}$ is simple relative to a quasi-variety K, it is also subdirectly irreducible relative to K.*

As for the case of subdirect irreducibility, the notion of a simple algebra admits an absolute variant. More precisely, an algebra $\boldsymbol{A}$ is *simple* (in the absolute sense) if $\mathsf{Con}(\boldsymbol{A})$ has precisely two elements.

Example 7.7 (Distributive lattices)**.** Recall that the class of distributive lattices DL coincides with $\mathbb{IP}_{\text{SD}}(\boldsymbol{B})$, where $\boldsymbol{B}$ is the two-element distributive lattice. It follows that the class of (relative) subdirectly irreducible members of DL is included into $\mathbb{I}(\boldsymbol{B})$. Furthermore, notice that $\mathsf{Con}\boldsymbol{B}$ is a two-element chain with maximum $B \times B$ and minimum id_B. Consequently, $\boldsymbol{B}$ is simple and, therefore, subdirectly irreducible. It follows that $\mathbb{I}(\boldsymbol{B})$ is the class of all (relative) subdirectly irreducible members of DL.

A similar argument shows that the (relative) subdirectly irreducible members of the class of Boolean algebras are the two-element chains. ⊠

Exercise 7.8. The following lattices are called, respectively, $\boldsymbol{M}_3$ and $\boldsymbol{N}_5$.



Their importance is related to two classical result. The first, due to Dedekind, states that a lattice is nonmodular if and only if $\boldsymbol{N}_5$ embeds into it. The second, due to Birkhoff, states that a lattice fails to be distributive precisely when $\boldsymbol{M}_3$ or $\boldsymbol{N}_5$ can be embedded into it.

Find the congruence lattices of $\boldsymbol{M}_3$ and $\boldsymbol{N}_5$ and use them to convince yourself that both $\boldsymbol{M}_3$ and $\boldsymbol{N}_5$ are subdirectly irreducible. Prove also that $\boldsymbol{M}_3$ is simple, but $\boldsymbol{N}_5$ is not. ⊠

Example 7.9 (Heyting algebras)**.** A filter on a Heyting algebra $\boldsymbol{A}$ is a nonempty upset closed under binary meets. The poset of all filters of $\boldsymbol{A}$ ordered under the inclusion relation will be denoted by $\mathsf{Fi}(\boldsymbol{A})$. Recall that $\mathsf{Fi}(\boldsymbol{A})$ is isomorphic to $\mathsf{Con}(\boldsymbol{A})$ via the map $\boldsymbol{\Omega}^{\boldsymbol{A}} \colon \mathsf{Fi}(\boldsymbol{A}) \to \mathsf{Con}(\boldsymbol{A})$, defined in Example 2.7. We shall prove that a Heyting algebra $\boldsymbol{A}$ is subdirectly irreducible if and only if there exists an element $a \in A \smallsetminus \{1\}$ such that $A = \{1\} \cup {\downarrow} a$.

Suppose first that $\boldsymbol{A}$ is subdirectly irreducible. In view of Remark 7.4, there exists the least congruence θ of $\boldsymbol{A}$ different from the identity. Since $\mathsf{Fi}(\boldsymbol{A}) \cong \mathsf{Con}(\boldsymbol{A})$, there exists also the least filter F of $\boldsymbol{A}$ different from the minimum $\{1\}$ of $\mathsf{Fi}(\boldsymbol{A})$. Since $F \neq \{1\}$, there exists some $a \in F \smallsetminus \{1\}$. As ${\uparrow} a$ is a filter different from $\{1\}$ and F is the least such, we obtain $F \subseteq {\uparrow} a$. On the other hand, as $a \in F$, the other inclusion holds, whence $F = {\uparrow} a$. Then consider an element $c \in A \smallsetminus \{1\}$. Since ${\uparrow} c$ is a filter different from $\{1\}$ and F is the least such, we obtain ${\uparrow} a = F \subseteq {\uparrow} c$, that is, $c \leqslant a$. Hence, $A = \{1\} \cup {\downarrow} a$, as desired.

To prove the converse, suppose that there exists an element $a \in A \smallsetminus \{1\}$ such that $A = \{1\} \cup {\downarrow} a$. In this case, ${\uparrow} a$ is the least filter of $\boldsymbol{A}$ different from the minimum filter $\{1\}$. Since $\mathsf{Fi}(\boldsymbol{A}) \cong \mathsf{Con}(\boldsymbol{A})$, there exists also the least congruence of $\boldsymbol{A}$ different from the identity relation. But, by Remark 7.4, this implies that $\boldsymbol{A}$ is subdirectly irreducible. ⊠

The last ingredient of the general representation theorem is the following.

Correspondence Theorem 7.10. *Let K be a quasi-variety and $\boldsymbol{A} \in \mathsf{K}$. Given a K-congruence θ of $\boldsymbol{A}$, the subposet ${\uparrow}\theta$ of $\mathsf{Con}_{\mathsf{K}}(\boldsymbol{A})$ is isomorphic to $\mathsf{Con}_{\mathsf{K}}(\boldsymbol{A}/\theta)$ under the map $f \colon {\uparrow}\theta \to \mathsf{Con}_{\mathsf{K}}(\boldsymbol{A}/\theta)$,* *defined by the rule*

$$f(\phi) := \{\langle a/\theta, c/\theta\rangle \in A/\theta \times A/\theta : \langle a, c\rangle \in \phi\}.$$

Proof. We claim that, for every $\phi \in \mathsf{Con}_{\mathsf{K}}(\boldsymbol{A})$ such that $\theta \subseteq \phi$ and $a, c \in A$,

$$\langle a, c\rangle \in \phi \Longleftrightarrow \langle a/\theta, c/\theta\rangle \in f(\phi).$$

The implication from left to right is an immediate consequence of the definition of f. To prove the other implication, suppose that $\langle a/\theta, c/\theta\rangle \in f(\phi)$. By definition of $f(\phi)$, there is a pair $\langle b, d\rangle \in \phi$ such that $a/\theta = b/\theta$ and $c/\theta = d/\theta$. Since ϕ is an equivalence relation extending θ, this implies $\langle a, c\rangle \in \phi$, as desired.

Then we turn to prove that f is well-defined. Consider $\phi \in \mathsf{Con}_{\mathsf{K}}(\boldsymbol{A})$ such that $\theta \subseteq \phi$. Since ϕ is an equivalence relation on A, the definition of f guarantees that $f(\phi)$ is an equivalence relation on A/θ. Then consider a basic n-ary operation g and $a_1, \dots, a_n, c_1, \dots, c_n \in A$ such that

$$\langle a_1/\theta, c_1/\theta\rangle, \dots, \langle a_n/\theta, c_n/\theta\rangle \in f(\phi).$$

By the claim, $\langle a_1, c_1\rangle, \dots, \langle a_n, c_n\rangle \in \phi$. Since ϕ is a congruence of $\boldsymbol{A}$, this yields

$$\langle g^{\boldsymbol{A}}(a_1, \dots, a_n), g^{\boldsymbol{A}}(c_1, \dots, c_n)\rangle \in \phi.$$

Hence,

$$\begin{aligned} g^{\boldsymbol{A}/\theta}(a_1/\theta, \dots, a_n/\theta) &= g^{\boldsymbol{A}}(a_1, \dots, a_n)/\theta \\ &\equiv_{f(\phi)} g^{\boldsymbol{A}}(c_1, \dots, c_n)/\theta \\ &= g^{\boldsymbol{A}/\theta}(c_1/\theta, \dots, c_n/\theta) \end{aligned}$$

and, therefore, $f(\phi)$ is a congruence of $\boldsymbol{A}/\theta$. The proof that $f(\phi)$ is also a K-congruence of $\boldsymbol{A}/\theta$ is left an an exercise. We conclude that f is well-defined.

Then consider $\phi, \eta \in \mathsf{Con}_{\mathsf{K}}(\boldsymbol{A})$ such that $\theta \subseteq \phi, \eta$. We have

$$\phi \subseteq \eta \Longleftrightarrow f(\phi) \subseteq f(\eta).$$

The implication from left to right is an immediate consequence of the definition of f. To prove the other implication, suppose that $f(\phi) \subseteq f(\eta)$ and consider a pair $\langle a, c\rangle \in \phi$. We have $\langle a/\theta, c/\theta\rangle \in f(\phi) \subseteq f(\eta)$. With an application of the claim, we obtain $\langle a, c\rangle \in \eta$, as desired. Hence, f is an order embedding.

To prove that it is surjective, consider a K-congruence ϕ of $\boldsymbol{A}/\theta$ and let

$$\eta := \{\langle a, c\rangle \in A \times A : \langle a/\theta, c/\theta\rangle \in \phi\}.$$

Since ϕ is an equivalence relation on A/θ, the definition of η guarantees that η is an equivalence relation on A. Then consider an n-ary operation g and $a_1, \dots, a_n, c_1, \dots, c_n \in A$ such that $\langle a_1, c_1\rangle, \dots, \langle a_n, c_n\rangle \in \eta$. We have $\langle a_1/\theta, c_1/\theta\rangle, \dots, \langle a_n/\theta, c_n/\theta\rangle \in \phi$. Since ϕ is a congruence on $\boldsymbol{A}/\theta$, this yields

$$g^{\boldsymbol{A}}(a_1, \dots, a_n)/\theta = g^{\boldsymbol{A}/\theta}(a_1/\theta, \dots, a_n/\theta) \equiv_{\phi} g^{\boldsymbol{A}/\theta}(c_1/\theta, \dots, c_n/\theta) = g^{\boldsymbol{A}}(c_1, \dots, c_n)/\theta$$

and, therefore, $\langle g^{\boldsymbol{A}}(a_1, \dots, a_n), g^{\boldsymbol{A}}(c_1, \dots, c_n)\rangle \in \eta$. We conclude that η is a congruence of $\boldsymbol{A}$. The proof that η is also a K-congruence of $\boldsymbol{A}$ is left as an exercise.

To prove that it extends θ, consider a pair $\langle a, c\rangle \in \theta$. Then $a/\theta = c/\theta$ and, since ϕ is reflexive, $\langle a/\theta, c/\theta\rangle \in \phi$. It follows that $\langle a, c\rangle \in \phi$, as desired. Thus, $\phi \in {\uparrow}\theta$. Furthermore, the definition of f implies that $f(\eta) = \phi$. Hence, f is surjective and, therefore, an isomorphism. ⊠

Exercise 7.11. Two parts of the above proof (both related to K-congruences) were left as an exercise. Complete the missing details. ⊠

The following representation theorem is as an application of lattice theory to general algebra, its main ingredient being the observation that every element of an algebraic lattice can be obtained as a meet of completely meet irreducible ones.

Subdirect Decomposition Theorem 7.12. *If K is a quasi-variety, then $\mathsf{K} = \mathbb{I}\mathbb{P}_{\text{SD}}(\mathsf{K}_{\text{RSI}})$.*

Proof. Consider an algebra $\boldsymbol{A} \in \mathsf{K}$. By Corollary 6.6, there exists a family $\{\theta_i : i \in I\} \subseteq \mathsf{Con}_{\mathsf{K}}(\boldsymbol{A})$ such that each θ_i is completely meet irreducible in $\mathsf{Con}_{\mathsf{K}}(\boldsymbol{A})$ and, moreover,

$$\mathrm{id}_A = \bigcap_{i \in I} \theta_i.$$

By Proposition 2.20, $\boldsymbol{A} \in \mathbb{I}\mathbb{P}_{\text{SD}}(\{\boldsymbol{A}/\theta_i : i \in I\})$.

Now, we know that each $\boldsymbol{A}/\theta_i$ belongs to K, because θ_i is a K-congruence of $\boldsymbol{A}$. Therefore, to conclude the proof, it only remains to show that $\boldsymbol{A}/\theta_i$ is subdirectly irreducible relative to K. By the Correspondence Theorem, $\mathsf{Con}_{\mathsf{K}}(\boldsymbol{A}/\theta_i)$ is isomorphic to the upset generated by θ_i in $\mathsf{Con}_{\mathsf{K}}\boldsymbol{A}$. Moreover, this isomorphism sends θ_i to id_{A/θ_i}. Therefore, from the assumption that θ_i is completely meet irreducible in $\mathsf{Con}_{\mathsf{K}}(\boldsymbol{A})$, it follows that id_{A/θ_i} is completely meet irreducible in $\mathsf{Con}_{\mathsf{K}}(\boldsymbol{A}/\theta_i)$. By Proposition 7.2, we conclude that $\boldsymbol{A}/\theta_i$ is subdirectly irreducible relative to K. ⊠

Corollary 7.13. *Every algebra is isomorphic to a subdirect product of subdirectly irreducible algebras.*

The Subdirect Decomposition Theorem was discovered by Birkhoff in the form of the above corollary. Its posterior formulation for quasi-varieties is due to Maltsev.

Example 7.14 (Subdirect decomposition)**.** Birkhoff's representation of distributive lattices as subdirect products of the two-element chain and Stone's representation of Boolean algebras as subdirect products of the two-element Boolean algebra are special instances of the Subdirect Decomposition Theorem. Another such application is the observation that every Heyting algebra is isomorphic to a subdirect products of subdirectly irreducible Heyting algebras, where the latter were described in Example 7.9. Lastly, an example from classical algebra is the following: every finite Abelian group is a direct product of cyclic groups of prime power order. The latter happen to be precisely the finite subdirectly irreducible Abelian groups. ⊠

Since relative subdirectly irreducible algebras form the building blocks of quasi-varieties, it is natural to wonder how do they arise. The next results provides an answer in terms of the generators of a quasi-variety.

Proposition 7.15. *If K is a class of similar algebras, then $\mathbb{Q}(\mathsf{K})_{\text{RSI}} \subseteq \mathbb{I}\mathbb{S}\mathbb{P}_{\text{U}}(\mathsf{K})$.*

Proof. Consider an algebra $\boldsymbol{A} \in \mathbb{Q}(\mathsf{K})_{\text{RSI}}$. Recall from Corollary 5.5 that $\mathbb{Q}(\mathsf{K}) = \mathbb{I}\mathbb{S}\mathbb{P}\mathbb{P}_{\text{U}}(\mathsf{K})$. Moreover, it is easy to see that $\mathbb{I}\mathbb{S}\mathbb{P}\mathbb{P}_{\text{U}}(\mathsf{K}) = \mathbb{I}\mathbb{P}_{\text{SD}}\mathbb{S}\mathbb{P}_{\text{U}}(\mathsf{K})$, whence $\mathbb{Q}(\mathsf{K}) = \mathbb{I}\mathbb{P}_{\text{SD}}\mathbb{S}\mathbb{P}_{\text{U}}(\mathsf{K})$. In particular, this implies $\boldsymbol{A} \in \mathbb{I}\mathbb{P}_{\text{SD}}\mathbb{S}\mathbb{P}_{\text{U}}(\mathsf{K})$. Accordingly, there exists a subdirect embedding $f\colon \boldsymbol{A} \to \prod_{i \in I} \boldsymbol{B}_i$ with $\{\boldsymbol{B}_i\colon i \in I\} \subseteq \mathbb{S}\mathbb{P}_{\text{U}}(\mathsf{K})$. As $\boldsymbol{A}$ is subdirectly irreducible relative to $\mathbb{Q}(\mathsf{K})$ and $\mathbb{S}\mathbb{P}_{\text{U}}(\mathsf{K}) \subseteq \mathbb{Q}(\mathsf{K})$, there exists $i \in I$ such that $p_i \circ f\colon \boldsymbol{A} \to \boldsymbol{B}_i$ is an isomorphism. Thus, $\boldsymbol{A} \in \mathbb{I}(\boldsymbol{B}) \subseteq \mathbb{I}\mathbb{S}\mathbb{P}_{\text{U}}(\mathsf{K})$. ⊠

Corollary 7.16. *If* K *is a finite set of finite similar algebras, then* $\mathbb{Q}(\mathsf{K})_{\mathsf{RSI}} \subseteq \mathbb{IS}(\mathsf{K})$*.*

Example 7.17 (Closure under $\mathbb{H}$)**.** Consider the lattices $\boldsymbol{A}$ and $\boldsymbol{B}$ defined at the beginning of Section 6. There, we prove that the quasi-variety $\mathbb{Q}(\boldsymbol{B})$ is not closed under $\mathbb{H}$. We are now in the position of offering a simpler proof that, moreover, can be easily adapted to other cases.

First, observe that $\boldsymbol{A} \in \mathbb{H}(\boldsymbol{B})$ and, therefore, it suffices to show that $\boldsymbol{A} \notin \mathbb{Q}(\boldsymbol{B})$. Suppose the contrary. Since $\boldsymbol{A}$ is simple and, therefore, subdirectly irreducible, it must be also subdirectly irreducible relative to $\mathbb{Q}(\boldsymbol{B})$. Therefore, $\boldsymbol{A} \in \mathbb{IS}(\boldsymbol{B})$, by Corollary 7.16. This contradicts the fact that there cannot be any embedding of $\boldsymbol{A}$ into $\boldsymbol{B}$ (look at their Hasse diagrams to convince you of this). ⊠

*Exercise** 7.18 (Modal algebras). Forthcoming. ⊠

8. Categories, functors, and natural transformations

Intuitively, a *category* is a class of mathematical structures (called *objects*) with structure preserving maps (called *arrows*) between them. The arrows of a category are endowed with a composition operation that abstracts the behaviour of the usual composition of functions and with identity arrows that abstract the behaviour of the usual identity functions.

While reading the following definition, you might wish to keep in mind the example of the category of topological spaces, whose objects are all the topological spaces and whose arrows are the continuous functions. In this category composition is the usual composition of functions and identity arrows are identity functions.

Definition 8.1. A category C consists of a class of objects Obj and a class Arr of arrows with two class functions

$$\mathsf{dom}\colon \mathsf{Arr} \to \mathsf{Obj} \text{ and } \mathsf{cdom}\colon \mathsf{Arr} \to \mathsf{Obj}$$

assigning a *domain* $A = \mathsf{dom}(f)$ and a *codomain* $B = \mathsf{cdom}(f)$ with each arrow f (in which case we write $f\colon A \to B$) in such a way that

(i) for each pair of arrows $f\colon A \to B$ and $g\colon B \to C$, there exists a special arrow $g \circ f\colon A \to C$, called the *composition* of f and g; and
(ii) for each object A there exists a special arrow $1_A\colon A \to A$, called the *identity* arrow on A, such that

- *composition is associative*: for every triple of arrows $f\colon A \to B$, $g\colon B \to C$, and $h\colon C \to D$,
$$h \circ (g \circ f) = (h \circ g) \circ f; \text{ and}$$
- *identity arrows can be cancelled*: for every pair of arrows $f\colon A \to B$ and $g\colon C \to A$,
$$f \circ 1_A = f \text{ and } 1_A \circ g = g.$$

The notion of a category is general enough to encompass most prominent collections of mathematical objects. We already pointed at the example of topological spaces with continuous maps. To mention a few more, every prevariety K can be viewed as a category whose objects are the members of K and whose arrows are the homomorphisms between them. In this category, which we also denote by K, composition and identity arrows are defined in the standard way. For the present purpose, the following example will play a fundamental role.

Example 8.2 (Bounded distributive lattices)**.** A *bounded distributive lattice* is an algebra $\boldsymbol{A} = \langle A; \wedge, \vee, 0, 1\rangle$ such that $\langle A; \wedge, \vee\rangle$ is a distributive lattice with maximum 1 and minimum 0. Bounded distributive lattices form a quasi-variety that we denote by BDL. Given two bounded distributive lattices $\boldsymbol{A}$ and $\boldsymbol{B}$, a homomorphism $f\colon \boldsymbol{A} \to \boldsymbol{B}$ is a map $f\colon A \to B$ such that for every $a, c \in A$,

$$f(a \wedge^{\boldsymbol{A}} c) = f(a) \wedge^{\boldsymbol{B}} f(c) \quad f(a \vee^{\boldsymbol{A}} c) = f(a) \vee^{\boldsymbol{B}} f(c) \quad f(0^{\boldsymbol{A}}) = 0^{\boldsymbol{B}} \quad f(1^{\boldsymbol{A}}) = 1^{\boldsymbol{B}}.$$

As we mentioned, BDL can be viewed as a category whose objects are bounded distributive lattices and whose arrows are homomorphisms between them. ⊠

Another useful example is the following.

Example 8.3 (Posets)**.** The class Pos of all (possibly empty) posets can be viewed as a category whose objects are posets and whose arrows are order preserving maps between them. In this category, composition and identity arrows are defined in the usual way. ⊠

At this stage, it is important to stress that the arrows of a category need not be functions (they just need to behave as such). For instance, given a category C, the *opposite category* C^{op} is the category obtained from C by inverting the direction of arrows. Formally speaking, the class of objects of C^{op} is that of C, the domain function of C^{op} is the codomain function of C, the codomain function of C^{op} is the domain function of C, the identity arrows of C^{op} are the same as those of C, and the composition $f \circ g$ in C^{op} of two arrows f and g is the arrow $g \circ f$ of C. Notice that $\mathsf{C}^{opop} = \mathsf{C}$.

Given a category C, we denote by $\mathsf{Obj}(\mathsf{C})$ and $\mathsf{Arr}(\mathsf{C})$, respectively, the classes of objects and arrows of C.

Definition 8.4. Let C and D be categories. A *covariant functor* $\mathcal{F}$ from C to D consists of two class functions

$$\mathcal{F}_{\mathsf{Obj}}\colon \mathsf{Obj}(\mathsf{C}) \to \mathsf{Obj}(\mathsf{D}) \;\text{ and }\; \mathcal{F}_{\mathsf{Arr}}\colon \mathsf{Arr}(\mathsf{C}) \to \mathsf{Arr}(\mathsf{D})$$

that satisfy the following requirements:

- *preservation of domains and codomains*: if $f\colon A \to B$ is an arrow of C, then the domain and the codomain of $\mathcal{F}_{\mathsf{Arr}}(f)$ are, respectively, $\mathcal{F}_{\mathsf{Obj}}(A)$ and $\mathcal{F}_{\mathsf{Obj}}(B)$;
- *preservation of identity arrows*: if A is an object of C, then $\mathcal{F}_{\mathsf{Arr}}(1_A) = 1_{\mathcal{F}_{\mathsf{Obj}}(A)}$;
- *preservation of composition*: for every pair of arrows $f\colon A \to B$ and $g\colon B \to C$ in C,

$$\mathcal{F}_{\mathsf{Arr}}(g \circ f) = \mathcal{F}_{\mathsf{Arr}}(g) \circ \mathcal{F}_{\mathsf{Arr}}(f).$$

In this case, we often drop the subscripts from $\mathcal{F}_{\mathsf{Obj}}$ and $\mathcal{F}_{\mathsf{Arr}}$ and write simply $\mathcal{F}$.

A *contravariant* functor from C to D is a covariant functor from C to D^{op}. Notice that, in this case, if $f\colon A \to B$ is an arrow in C, then $\mathcal{F}(f)\colon \mathcal{F}(B) \to \mathcal{F}(A)$ is an arrow in D. We write $\mathcal{F}\colon \mathsf{C} \to \mathsf{D}$ to indicate that $\mathcal{F}$ is a functor (covariant or contravariant) from C to D.

Example 8.5 (Prime spectra)**.** We shall define a contravariant functor $(-)_*\colon \mathsf{BDL} \to \mathsf{Pos}$. First, given a bounded distributive lattice $\boldsymbol{A}$, let $\boldsymbol{A}_*$ be the poset of its prime filters ordered under the inclusion relation (sometimes called the *prime spectrum* of $\boldsymbol{A}$). Clearly, $\boldsymbol{A}_*$ is an object of Pos. Then, given two bounded distributive lattices $\boldsymbol{A}$ and $\boldsymbol{B}$ and a homomorphism $f\colon \boldsymbol{A} \to \boldsymbol{B}$, let $f_*\colon \boldsymbol{B}_* \to \boldsymbol{A}_*$ be the map defined, for every $F \in \boldsymbol{B}_*$, as $f_*(F) := f^{-1}[F]$. Notice that, if $f_*\colon \boldsymbol{B}_* \to \boldsymbol{A}_*$ is indeed an arrow of Pos, then the application

$(-)_* \colon \mathsf{BDL} \to \mathsf{Pos}$ is a contravariant functor, because it reverses domains and codomains and preserves identity arrows and composition.

Therefore, it only remains to prove that if $f \colon \boldsymbol{A} \to \boldsymbol{B}$ is a homomorphism between bounded distributive lattices, then $f_* \colon \boldsymbol{B}_* \to \boldsymbol{A}_*$ is an arrow of Pos, that is, a well-defined order preserving map from $\boldsymbol{B}_*$ to $\boldsymbol{A}_*$. The fact that it is order preserving is obvious. To prove that it is well-defined, consider a prime filter F of $\boldsymbol{B}$. We need to show that $f_*(F)$ is a prime filter of $\boldsymbol{A}$.

We begin by showing that $f_*(F)$ is a filter of $\boldsymbol{A}$. As F is nonempty, so is $f_*(F) = f^{-1}[F]$. To prove that $f_*(F)$ is an upset of $\boldsymbol{A}$, consider $a, c \in A$ such that $a \in f_*(F)$ and $a \leqslant^{\boldsymbol{A}} c$. Since f is order preserving (being a lattice homomorphism), $f(a) \leqslant^{\boldsymbol{B}} f(c)$. Moreover, by assumption, $a \in f_*(F) = f^{-1}[F]$, whence $f(a) \in F$. Since F is an upset of $\boldsymbol{B}$, we obtain $f(c) \in F$ and, therefore, $c \in f^{-1}[F] = f_*(F)$. Thus, $f_*(F)$ is an upset of $\boldsymbol{A}$. Lastly, to prove that $f_*(F)$ is closed under binary meets, consider $a, c \in f_*(F)$. We have $f(a), f(c) \in F$ and, since F is a filter of $\boldsymbol{B}$, $f(a) \wedge^{\boldsymbol{B}} f(c) \in F$. Since f is a homomorphism, $f(a \wedge^{\boldsymbol{A}} c) = f(a) \wedge^{\boldsymbol{B}} f(c) \in F$, whence $a \wedge^{\boldsymbol{A}} c \in f^{-1}[F] = f_*(F)$. Hence, $f_*(F)$ is a filter of $\boldsymbol{A}$.

To prove that it is prime, it only remains to show that $f_*(F)$ is proper and that, for every $a, c \in A$, if $a \vee^{\boldsymbol{A}} c \in f_*(F)$, then $a \in f_*(F)$ or $c \in f_*(F)$. First, observe that $0^{\boldsymbol{B}} \notin F$ (because F proper) and $f(0^{\boldsymbol{A}}) = 0^{\boldsymbol{B}}$ (because f is a homomorphism), then $f(0^{\boldsymbol{A}}) = 0^{\boldsymbol{B}} \notin F$, whence $0^{\boldsymbol{A}} \notin f^{-1}[F] = f_*(F)$. We conclude that $f_*(F)$ is proper. Then consider $a, c \in A$ such that $a \vee^{\boldsymbol{A}} c \in f_*(F)$. Since f is a homomorphism,

$$f(a) \vee^{\boldsymbol{B}} f(c) = f(a \vee^{\boldsymbol{A}} c) \in F.$$

Since F is prime, $f(a) \in F$ or $f(c)$, that is, $a \in f_*(F)$ or $c \in f_*(F)$. ⊠

Example 8.6 (Upsets)**.** Recall that, given a poset $\mathbb{X}$, the collection $\mathsf{Up}(\mathbb{X})$ of its upsets can be viewed as a bounded distributive lattice

$$\mathbb{X}^* := \langle \mathsf{Up}(\mathbb{X}); \cap, \cup, \emptyset, X \rangle.$$

We will define a contravariant functor $(-)^* \colon \mathsf{Pos} \to \mathsf{BDL}$ exploiting this observation.

Clearly, if $\mathbb{X}$ is a poset, then $\mathbb{X}^*$ is an object of BDL. Moreover, given two posets $\mathbb{X}$ and $\mathbb{Y}$ and an order preserving map $f \colon \mathbb{X} \to \mathbb{Y}$, let $f^* \colon \mathbb{Y}^* \to \mathbb{X}^*$ be the map defined, for every $U \in \mathsf{Up}(\mathbb{Y})$, as $f^*(U) := f^{-1}[U]$. Clearly, f_* is well-defined, because inverse images of upsets under order preserving maps are still upsets. Moreover, as the inverse image map $f^{-1} \colon \mathcal{P}(Y) \to \mathcal{P}(X)$ preserves intersections and unions, f^* is a lattice homomorphism. As $f^{-1}(\emptyset) = \emptyset$ and $f^{-1}[Y] = X$, we conclude that $f^* \colon \mathbb{Y}^* \to \mathbb{X}^*$ is a well-defined homomorphism of bounded lattices and, therefore, an arrow of BDL. It follows that $(-)^* \colon \mathsf{Pos} \to \mathsf{BDL}$ is indeed a contravariant functor. ⊠

The following notion is useful to compare pair of functors with the same domain and codomain.

Definition 8.7. Let $\mathcal{F} \colon \mathsf{C} \to \mathsf{D}$ and $\mathcal{G} \colon \mathsf{C} \to \mathsf{D}$ be covariant functors. A *natural transformation η* from $\mathcal{F}$ to $\mathcal{G}$ is a is a collection $\{\eta_A : A \in \mathsf{Obj}(\mathsf{C})\}$ such that each $\eta_A \colon \mathcal{F}(A) \to \mathcal{G}(A)$ is an arrow in D and, for every arrow $f \colon B \to C$ in C,

$$\eta_C \circ \mathcal{F}(f) = \mathcal{G}(f) \circ \eta_B.$$

We write $\eta \colon \mathcal{F} \to \mathcal{G}$ to indicate that ε is a natural transformation from $\mathcal{F}$ to $\mathcal{G}$.

Notice that, for every category C, there exists a (covariant) *identity functor* $id_{\mathsf{C}} \colon \mathsf{C} \to \mathsf{C}$ that behaves as the identity on the objects and arrows of C. Furthermore, if $\mathcal{F} \colon \mathsf{C} \to \mathcal{D}$ and $\mathcal{G} \colon \mathsf{D} \to \mathsf{E}$ are functors, then the natural composition $\mathcal{G} \circ \mathcal{F}$ is a functor from C to E.

Example 8.8 (Canonical extensions)**.** Since the functors $(-)_* \colon \mathsf{BDL} \to \mathsf{Pos}$ and $(-)^* \colon \mathsf{Pos} \to \mathsf{BDL}$ are contravariant, their composition

$$(-)^* \circ (-)_* \colon \mathsf{BDL} \to \mathsf{BDL}$$

is covariant. We shall define a natural transformation $\varepsilon \colon id_{\mathsf{BDL}} \to (-)^* \circ (-)_*$ as follows. Observe that, for a bounded distributive lattice $\boldsymbol{A}$, the structure $(\boldsymbol{A}_*)^*$ is the bounded distributive lattice of upsets of the poset of prime filters of $\boldsymbol{A}$. By Birkhoff's representation theorem for distributive lattices, we know that the map $\varepsilon_{\boldsymbol{A}} \colon \boldsymbol{A} \to (\boldsymbol{A}_*)^*$, defined by the rule

$$\varepsilon_{\boldsymbol{A}}(a) \coloneqq \{F \in \boldsymbol{A}_* : a \in F\},$$

is a well-defined embedding of bounded distributive lattices and, therefore, an arrow in BDL. Notice that the pair $\langle \boldsymbol{A}, \varepsilon_{\boldsymbol{A}} \rangle$ is the *canonical extension* of $\boldsymbol{A}$. It is easy to prove that the collection $\varepsilon \coloneqq \{\varepsilon_{\boldsymbol{A}} : \boldsymbol{A} \in \mathsf{DBL}\}$ is indeed a natural transformation from id_{BDL} to $(-)^* \circ (-)_*$. ⊠

Another useful example of a natural transformation is the following.

Example 8.9. Again, since the functors $(-)_* \colon \mathsf{BDL} \to \mathsf{Pos}$ and $(-)^* \colon \mathsf{Pos} \to \mathsf{BDL}$ are contravariant, their composition

$$(-)_* \circ (-)^* \colon \mathsf{Pos} \to \mathsf{Pos}$$

is covariant. We shall define a natural transformation $\eta \colon id_{\mathsf{Pos}} \to (-)_* \circ (-)^*$ as follows. Recall that, for a poset $\mathbb{X}$, the structure $(\mathbb{X}^*)_*$ is the poset of prime filters of the distributive lattice of upsets of $\mathbb{X}$. Bearing this in mind, consider the map $\eta_{\mathbb{X}} \colon \mathbb{X} \to (\mathbb{X}^*)_*$, defined by the rule

$$\eta_{\mathbb{X}}(x) \coloneqq \{U \in \mathsf{Up}(\mathbb{X}) : x \in U\}.$$

It is easy to prove that the collection $\eta \coloneqq \{\eta_{\mathbb{X}} : \mathbb{X} \in \mathsf{Pos}\}$ is a natural transformation from id_{Pos} to $(-)_* \circ (-)^*$. ⊠

Remark 8.10. The categorically minded reader might wish to observe that $(-)_* \colon \mathsf{BDL} \to \mathsf{Pos}$ and $(-)^* \colon \mathsf{Pos} \to \mathsf{BDL}$ is a dual adjunction with unit η and counit ε. ⊠

*Exercise** 8.11. To be written: check that the examples are indeed natural transformations.

9. Discrete duality

Department of Philosophy, Faculty of Philosophy, University of Barcelona, Carrer de Montalegre 6, 08001, Barcelona, Spain

Email address: tommaso.moraschini@ub.edu