THE THEORY OF IMPLICIT OPERATIONS

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Abstract

A family of partial functions of a class of algebras K is said to be an *implicit operation* of K when it is defined by a first order formula and it is preserved by homomorphisms. In this work, we develop the theory of implicit operations from an algebraic standpoint.

Notably, the implicit operations of an elementary class K are exactly the partial functions on K definable by existential positive formulas. For instance, "taking inverses" is an implicit operation of the class of monoids, defined by the conjunction of the equations $xy \approx 1$ and $yx \approx 1$, saying that y is the inverse of x. As this example suggests, implicit operations need not be definable by the terms of the corresponding class of algebras. In fact, the demand that every implicit operation of a universal class K be interpolated by a family of terms is equivalent to the demand that K has the strong epimorphism surjectivity property.

However, implicit operations are always interpolated by families of implicit operations of a simpler kind, namely, those defined by *pp formulas*, i.e., formulas of the form $\exists \vec{x} \varphi$, where φ is a conjunction of equations. Motivated by this, we establish an *existential elimination theorem* stating that, when K is a quasivariety with the amalgamation property, every implicit operation of K is interpolated by a family of implicit operations defined by conjunctions of equations (i.e., by pp formulas without existential quantifiers). We also provide a series of methods to test whether a concrete class of algebras has the strong epimorphism surjectivity property or, equivalently, to test whether interpolation can be carried out using terms only.

As the implicit operations of a class of algebras K need not belong to the language of K, it is natural to wonder whether K can be expanded with its implicit operations. The main obstacle is that, in general, implicit operations need not be total. Accordingly, we say that an implicit operation of K is *extendable* when every member of K can be extended to one in which the operation is total. For instance, the operation of "taking inverses" is not extendable in the class of monoids, but it becomes so in the class of cancellative commutative monoids because every such monoid embeds into an Abelian group.

When expanding a class of algebras K with its pp definable extendable implicit operations produces a class M in which every implicit operations is interpolated by a family of terms, we say that M is a *Beth companion* of K. In the context of quasivarieties, Beth companions are essentially unique, in the sense that all the Beth companions of a quasivariety are term equivalent. However, Beth companions need not exist in general: while Abelian groups are the Beth companion of cancellative commutative monoids, the class of all (commutative) monoids lacks a Beth companion. A series of tools to describe the Beth companion of a concrete class of algebras is also exhibited, drawing connections with absolutely closed, injective, and saturated algebras.

The appeal of Beth companions depends largely on whether the structure theory of a class is improved by moving to its Beth companion. We show that this is indeed the case by proving that, under minimal assumptions, every Beth companion of a relatively congruence distributive

quasivariety whose class of relatively finitely subdirectly irreducible members is closed under nontrivial subalgebras is an arithmetical variety with the congruence extension property. This theorem applies, for instance, to the quasivariety of reduced commutative rings of characteristic zero (which lacks the properties given by the theorem) and its Beth companion, namely, the variety of meadows of characteristic zero. As a corollary, we obtain that the Beth companion of a relatively filtral quasivariety must be a discriminator variety.

A variety of examples, together with their Beth companions, are discussed (see Table 1). These include both examples from classical algebra such as semigroups and monoids (both with and without commutativity and cancellativity), Abelian ℓ-groups, torsion-free Abelian groups, and reduced commutative rings, as well as examples with a logical motivation such as (bounded) distributive lattices, Hilbert algebras, Heyting algebras, and MV-algebras.

Class	Beth companion	Location
universal classes with the strong epimorphism surjectivity property	themselves	Thm. 11.9
relatively filtral quasivarieties	discriminator varieties	Cor. 12.11
monoids	no Beth companion	Thm. 14.1
semigroups	no Beth companion	Rem. 14.8
commutative monoids	no Beth companion	Thm. 14.1
commutative semigroups	no Beth companion	Rem. 14.8
cancellative commutative monoids	Abelian groups	Thm. 11.9
cancellative commutative semigroups	Abelian groups	Rem. 14.8
torsion-free Abelian groups	Abelian groups with division	Thm. 13.10
Abelian ℓ-groups	Abelian ℓ-groups with division	Thm. 13.16
reduced commutative rings of characteristic zero	meadows of characteristic zero	Exa. 12.12
distributive lattices	relatively complemented distributive lattices	Thm. 11.9
bounded distributive lattices	Boolean algebras	Thm. 11.9
Hilbert algebras	implicative semilattices	Thm. 11.9
pseudocomplemented distributive lattices	Heyting algebras of depth $\leqslant 2$	Thm. 11.9
varieties generated by a linearly ordered Heyting algebra $\boldsymbol{A}$	no Beth companion if $5 \leqslant |A| < \omega$ and $\mathbb{V}(\boldsymbol{A})$ otherwise	Thm. 14.17
MV-algebras	MV-algebras with division	Thm. 13.21
varieties generated by an MV-algebra of the form $Ł_n$	varieties generated by the expansion of $Ł_n$ with a constant for $\frac{1}{n}$	Thm. 13.27

TABLE 1. Some classes of algebras and their Beth companions.

Contents

1. Formulas, compactness, and preservation theorems

Throughout this work, we will assume familiarity with the notions of an algebra and a homomorphism from universal algebra (see, e.g., [11, 21]), with simple constructions such as direct products or subalgebras, as well as with first order formulas and their interpretation in mathematical structures (see, e.g., [35, 63]). In doing so, we will restrict our attention to algebraic languages. Furthermore, classes of algebras K will be always assumed to be classes of *similar* algebras, that is, algebras with a common language. Given a pair of algebras $\boldsymbol{A}$ and $\boldsymbol{B}$, we write $\boldsymbol{A} \leqslant \boldsymbol{B}$ to indicate that $\boldsymbol{A}$ is a subalgebra of $\boldsymbol{B}$. We denote the direct product of a family $\{\boldsymbol{A}_i : i \in I\}$ of similar algebras by $\prod_{i \in I} \boldsymbol{A}_i$ and the projection maps by $p_j \colon \prod_{i \in I} \boldsymbol{A}_i \to \boldsymbol{A}_j$ for every $j \in I$. We also write $\boldsymbol{A}_1 \times \cdots \times \boldsymbol{A}_n$ for the product of a finite family $\{\boldsymbol{A}_1, \ldots, \boldsymbol{A}_n\}$.

By a *formula* we always mean a first order formula. Given a formula φ, we write $\varphi(x_1, \ldots, x_n)$ to indicate that the free variables of φ are among $x_1, \ldots, x_n$. We denote the conjunction, disjunction, and implication of a pair of formulas φ and ψ by $\varphi \sqcap \psi$, $\varphi \sqcup \psi$, and $\varphi \to \psi$, respectively. Moreover, we denote the negation of a formula φ by $\neg\varphi$. When φ is an equation $t_1 \approx t_2$, we often write $t_1 \not\approx t_2$ as a shorthand for $\neg(t_1 \approx t_2)$. Given an algebra $\boldsymbol{A}$, a formula $\varphi(x_1, \ldots, x_n)$, and $a_1, \ldots, a_n \in A$, we write $\boldsymbol{A} \vDash \varphi(a_1, \ldots, a_n)$ to indicate that φ holds in $\boldsymbol{A}$ of the elements $a_1, \ldots, a_n$. When $\boldsymbol{A} \vDash \varphi(a_1, \ldots, a_n)$ for all $a_1, \ldots, a_n \in A$, we say that φ is *valid* in $\boldsymbol{A}$ and write $\boldsymbol{A} \vDash \varphi$. Similarly, if Φ is a set of formulas, we write $\boldsymbol{A} \vDash \Phi$, and say that $\boldsymbol{A}$ is *a model of* Φ, to indicate that $\boldsymbol{A} \vDash \varphi$ for each $\varphi \in \Phi$. This notion extends to classes of algebras K as follows: we say that a formula φ is *valid* in K and write $\mathsf{K} \vDash \varphi$ when $\boldsymbol{A} \vDash \varphi$ for each $\boldsymbol{A} \in \mathsf{K}$. Similarly, we write $\mathsf{K} \vDash \Phi$ when $\mathsf{K} \vDash \varphi$ for each $\varphi \in \Phi$. We always allow two

special formulas $\top$ and $\bot$ without free variables and assume that $\top$ is valid in every algebra, while $\bot$ is not valid in any algebra.

A pair of formulas $\varphi(x_1, \ldots, x_n)$ and $\psi(x_1, \ldots, x_n)$ is said to be *equivalent* in a class of algebras K when for all $\boldsymbol{A} \in \mathsf{K}$ and $a_1, \ldots, a_n \in A$,

$$\boldsymbol{A} \vDash \varphi(a_1, \ldots, a_n) \iff \boldsymbol{A} \vDash \psi(a_1, \ldots, a_n).$$

When φ and ψ are equivalent in every class of algebras, we simply say that they are *equivalent*.

A formula is said to be *existential positive* when it is of the form

$$\exists x_1, \ldots, x_n \varphi, \tag{1}$$

where φ is built from equations, $\top$, and $\bot$ using only $\sqcap$ and $\sqcup$. An existential positive formula of the form in (1) is called a *primitive positive formula* (*pp formula* for short) when φ is a finite conjunction of equations.[1] Since existential quantifiers and conjunctions distribute over disjunctions up to equivalence, every existential positive formula is equivalent to a disjunction of pp formulas. Lastly, a formula is said to be *universal* when it is of the form $\forall x_1, \ldots, x_n \varphi$, where φ is a quantifier-free formula.

Let K be a class of algebras. A formula $\varphi(x_1, \ldots, x_n)$ is said to be preserved by

(i) *homomorphisms* in K when for every homomorphism $h \colon \boldsymbol{A} \to \boldsymbol{B}$ with $\boldsymbol{A}, \boldsymbol{B} \in \mathsf{K}$ and $a_1, \ldots, a_n \in A$,

$$\text{if } \boldsymbol{A} \vDash \varphi(a_1, \ldots, a_n), \text{ then } \boldsymbol{B} \vDash \varphi(h(a_1), \ldots, h(a_n));$$

(ii) *direct products* in K when for all $\{\boldsymbol{A}_i : i \in I\} \subseteq \mathsf{K}$ and $a_1, \ldots, a_n \in \prod_{i \in I} A_i$,

$$\text{if } \boldsymbol{A}_i \vDash \varphi(p_i(a_1), \ldots, p_i(a_n)) \text{ for each } i \in I, \text{ then } \prod_{i \in I} \boldsymbol{A}_i \vDash \varphi(a_1, \ldots, a_n);$$

(iii) *subalgebras* in K when for all $\boldsymbol{A} \leqslant \boldsymbol{B} \in \mathsf{K}$ and $a_1, \ldots, a_n \in A$,

$$\text{if } \boldsymbol{B} \vDash \varphi(a_1, \ldots, a_n), \text{ then } \boldsymbol{A} \vDash \varphi(a_1, \ldots, a_n).$$

A class of algebras K is said to be *elementary* when it can be axiomatized by a set of formulas Φ, i.e., K is the class of models of Φ. We rely on the following preservation theorem for elementary classes.

Theorem 1.1. *Let K be an elementary class. A formula $\varphi(x_1, \ldots, x_n)$ is preserved by*

(i) *homomorphisms in K if and only if it is equivalent in K to an existential positive formula;*
(ii) *homomorphisms and direct products in K if it is a pp formula;*
(iii) *subalgebras in K if and only if it is equivalent in K to a universal formula.*

Proof. (i): This fact is known as the *homomorphism preservation Theorem* and is due to Łoś, Lyndon, and Tarski [96, 86, 102]. Since we have not been able to find this result relativized to elementary classes explicitly stated in the literature, we provide a complete proof here. The argument requires some basic notions from the model theory of structures in languages containing both function and relation symbols that can be found in any standard book on model theory such as [63].

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[1]We admit the empty conjunction, which is defined to be $\top$.

For every $\boldsymbol{A} \in \mathsf{K}$ let $R^{\boldsymbol{A}}$ be the n-ary relation on A defined by $\langle a_1, \ldots, a_n \rangle \in R^{\boldsymbol{A}}$ if and only if $\boldsymbol{A} \vDash \varphi(a_1, \ldots, a_n)$. We denote by $\boldsymbol{A}^*$ the structure obtained by equipping the algebra $\boldsymbol{A}$ with $R^{\boldsymbol{A}}$. Then $\mathsf{K}^* = \{\boldsymbol{A}^* : \boldsymbol{A} \in \mathsf{K}\}$ is an elementary class in the language of K expanded with an n-ary relation symbol R. Indeed, since K is an elementary class, an axiomatization of K^* is obtained by adding to the axiomatization of K the first order formula $R(x_1, \ldots, x_n) \leftrightarrow \varphi(x_1, \ldots, x_n)$. Since K^* is elementary, [26, Thm. 6.2(3)] yields that the following conditions are equivalent:

(a) for every homomorphism $h \colon \boldsymbol{A} \to \boldsymbol{B}$ between members of K and $a_1, \ldots, a_n \in A$ we have that $\langle a_1, \ldots, a_n \rangle \in R^{\boldsymbol{A}}$ implies $\langle h(a_1), \ldots, h(a_n) \rangle \in R^{\boldsymbol{B}}$;

(b) there exists an existential positive formula ψ in the language of K such that for all $\boldsymbol{A} \in \mathsf{K}$ and $a_1, \ldots, a_n \in A$ we have $\langle a_1, \ldots, a_n \rangle \in R^{\boldsymbol{A}}$ if and only if $\boldsymbol{A} \vDash \psi(a_1, \ldots, a_n)$.

It follows immediately from the definition of $R^{\boldsymbol{A}}$ and $R^{\boldsymbol{B}}$ that condition (a) is equivalent to φ being preserved by homomorphisms in K, while condition (b) says that φ is equivalent to an existential positive formula in K. We then conclude that φ is preserved by homomorphisms in K if and only if it is equivalent in K to an existential positive formula, as desired.

(ii): Suppose that φ is a pp formula. As every pp formula is an existential positive formula, (i) implies that φ is preserved by homomorphisms in K. We show that φ is also preserved by direct products as well. Since $\varphi(x_1, \ldots, x_n)$ is a pp formula, it is of the form

$$\exists z_1, \ldots, z_m \psi(x_1, \ldots, x_n, z_1, \ldots, z_m),$$

where ψ is a finite conjunction of equations. Consider $\{\boldsymbol{A}_i : i \in I\} \subseteq \mathsf{K}$ and $a_1, \ldots, a_n \in \prod_{i \in I} A_i$ such that $\boldsymbol{A}_i \vDash \varphi(p_i(a_1), \ldots, p_i(a_n))$ for each $i \in I$. Our goal is to show that $\prod_{i \in I} \boldsymbol{A}_i \vDash \varphi(a_1, \ldots, a_n)$. For every $i \in I$, from $\boldsymbol{A}_i \vDash \varphi(p_i(a_1), \ldots, p_i(a_n))$ it follows that there exists $\langle b_1^i, \ldots, b_m^i \rangle \in A_i^m$ such that

$$\boldsymbol{A}_i \vDash \psi(p_i(a_1), \ldots, p_i(a_n), b_1^i, \ldots, b_m^i).$$

It is straightforward to verify that conjunctions of equations and $\top$ are preserved by direct products. Therefore, letting $b_1 = \langle b_1^i : i \in I \rangle, \ldots, b_m = \langle b_m^i : i \in I \rangle$, we obtain

$$\prod_{i \in I} \boldsymbol{A}_i \vDash \psi(a_1, \ldots, a_n, b_1, \ldots, b_m).$$

Hence, we conclude that $\prod_{i \in I} \boldsymbol{A}_i \vDash \varphi(a_1, \ldots, a_n)$.

(iii): See, e.g., [63, Thm. 6.5.4] and the subsequent paragraph. $\boxtimes$

We recall the Compactness Theorem of first order logic (see, e.g., [63, Thm. 6.1.1]).

Compactness Theorem 1.2. *A set of formulas Φ has a model if every finite subset of Φ has a model.*

For the present purpose, it is convenient to phrase the Compactness Theorem in terms of infinite conjunctions and disjunctions as well. To this end, we denote the conjunction and the disjunction of a (possibly infinite) set of formulas Φ, respectively, by

$$\bigsqcap \Phi \quad \text{and} \quad \bigsqcup \Phi.$$

When $\Phi = \emptyset$, we assume that $\prod \Phi = \top$ and $\bigsqcup \Phi = \bot$. Given a pair of sets of formulas Φ and Ψ with free variables among $\langle x_i : i \in I \rangle$, an algebra $\boldsymbol{A}$, and a sequence $\vec{a} = \langle a_i : i \in I \rangle$ of elements of A, we write

$$\boldsymbol{A} \vDash \left(\prod \Phi \to \bigsqcup \Psi \right)(\vec{a})$$

to indicate that if $\boldsymbol{A} \vDash \varphi(\vec{a})$ for each $\varphi \in \Phi$, there exists $\psi \in \Psi$ such that $\boldsymbol{A} \vDash \psi(\vec{a})$. When the above display holds for every $\vec{a}$, we write $\boldsymbol{A} \vDash \prod \Phi \to \bigsqcup \Psi$. Similarly, given K a class of algebras, we write $\mathsf{K} \vDash \prod \Phi \to \bigsqcup \Psi$ to indicate that $\boldsymbol{A} \vDash \prod \Phi \to \bigsqcup \Psi$ for each $\boldsymbol{A} \in \mathsf{K}$.

A standard argument shows that the Compactness Theorem 1.2 is equivalent to the following.

Compactness Theorem 1.3. *Let K be an elementary class. For each pair of sets of formulas Φ and Ψ,*

$$\text{if } \mathsf{K} \vDash \prod \Phi \to \bigsqcup \Psi, \text{ then } \mathsf{K} \vDash \prod \Phi' \to \bigsqcup \Psi' \text{ for some finite } \Phi' \subseteq \Phi \text{ and } \Psi' \subseteq \Psi.$$

We will make use of the following version of the Compactness Theorem for pp formulas.

Corollary 1.4. *Let K be an elementary class closed under direct products. For each pair of sets of pp formulas Φ and Ψ with $\Psi \neq \emptyset$,*

$$\text{if } \mathsf{K} \vDash \prod \Phi \to \bigsqcup \Psi, \text{ then } \mathsf{K} \vDash \prod \Phi' \to \psi \text{ for some finite } \Phi' \subseteq \Phi \text{ and } \psi \in \Psi.$$

Proof. Suppose that $\mathsf{K} \vDash \prod \Phi \to \bigsqcup \Psi$. By the Compactness Theorem 1.3 there exist finite $\Phi' \subseteq \Phi$ and $\Psi' \subseteq \Psi$ such that

$$\mathsf{K} \vDash \prod \Phi' \to \bigsqcup \Psi'. \tag{2}$$

As $\Psi \neq \emptyset$, we may assume that $\Psi' \neq \emptyset$. Then consider an enumeration $\Psi' = \{\psi_1, \dots, \psi_n\}$. To conclude the proof, it suffices to show that $\mathsf{K} \vDash \prod \Phi' \to \psi_i$ for some $i \leqslant n$. Suppose the contrary, with a view to contradiction. Then let $x_1, \dots, x_m$ be the free variables of $\prod \Phi' \to \bigsqcup \Psi'$. For each $i \leqslant n$ there exist $\boldsymbol{A}_i \in \mathsf{K}$ and $a_1^i, \dots, a_m^i \in A_i$ such that

$$\boldsymbol{A}_i \vDash \prod \Phi'(a_1^i, \dots, a_m^i) \text{ and } \boldsymbol{A}_i \nvDash \psi_i(a_1^i, \dots, a_m^i). \tag{3}$$

Then consider the elements $a_1 = \langle a_1^i : i \leqslant n \rangle, \dots, a_m = \langle a_m^i : i \leqslant n \rangle$ of $A_1 \times \cdots \times A_n$. From the left hand side of the above display, the assumption that Φ' is a set of pp formulas, and Theorem 1.1(ii) it follows that

$$\boldsymbol{A}_1 \times \cdots \times \boldsymbol{A}_n \vDash \prod \Phi'(a_1, \dots, a_m).$$

As $\boldsymbol{A}_1, \dots, \boldsymbol{A}_n \in \mathsf{K}$ and K is closed under direct products by assumption, we obtain $\boldsymbol{A}_1 \times \cdots \times \boldsymbol{A}_n \in \mathsf{K}$. Together with (2) and the above display, this yields

$$\boldsymbol{A}_1 \times \cdots \times \boldsymbol{A}_n \vDash \psi_i(a_1, \dots, a_m)$$

for some $i \leqslant n$. From the above display, Theorem 1.1(ii) and the assumption that ψ_i is a pp formula it follows that $\boldsymbol{A}_i \vDash \psi_i(p_i(a_1), \dots, p_i(a_m))$. Together with the definition of $a_1, \dots, a_m$, this amounts to $\boldsymbol{A}_i \vDash \psi_i(a_1^i, \dots, a_m^i)$, a contradiction with the right hand side of (3). $\boxtimes$

The following may be regarded as a converse to Theorem 1.1(ii) under some additional assumptions.

Theorem 1.5. *Let* K *be an elementary class closed under direct products and* φ *a formula that is preserved by homomorphisms and direct products in* K*. Then* φ *is equivalent in* K *to a pp formula.*

Proof. As φ is preserved by homomorphisms in K, Theorem 1.1(i) yields an existential positive formula ψ that is equivalent in K to φ. Since ψ is an existential positive formula, it is equivalent to a finite disjunction $\psi_1 \sqcup \cdots \sqcup \psi_m$ of pp formulas ψ_i. It is then sufficient to show that there exists $i \leqslant m$ such that $\mathsf{K} \vDash \psi \leftrightarrow \psi_i$. Since $\mathsf{K} \vDash \psi \to (\psi_1 \sqcup \cdots \sqcup \psi_m)$, by Corollary 1.4 there exists $i \leqslant m$ such that $\mathsf{K} \vDash \psi \to \psi_i$. Because ψ is equivalent to $\psi_1 \sqcup \cdots \sqcup \psi_m$, we have $\mathsf{K} \vDash \psi_i \to \psi$. Thus, the formula ψ, and hence also φ, is equivalent in K to ψ_i, which is a pp formula. ⊠

We close this introductory section by recalling a fundamental theorem on ultraproducts known as Łoś' Theorem (see, e.g., [21, Thm. V.2.9]). To this end, given a family of algebras $\{\boldsymbol{A}_i : i \in I\}$, a formula $\varphi(x_1, \ldots, x_n)$, and $a_1, \ldots, a_n \in \prod_{i \in I} A_i$, let

$$[\![\varphi(a_1, \ldots, a_n)]\!] = \{i \in I : \boldsymbol{A}_i \vDash \varphi(p_i(a_1), \ldots, p_i(a_n))\}.$$

Łoś' Theorem 1.6. *Let* $\{\boldsymbol{A}_i : i \in I\}$ *be a family of algebras and* U *an ultrafilter on* I*. For every formula* $\varphi(x_1, \ldots, x_n)$ *and* $a_1, \ldots, a_n \in \prod_{i \in I} \boldsymbol{A}_i$ *we have*

$$\prod_{i \in I} \boldsymbol{A}_i/U \vDash \varphi(a_1/U, \ldots, a_n/U) \iff [\![\varphi(a_1, \ldots, a_n)]\!] \in U.$$

We denote by $\mathbb{P}_{\mathrm{u}}$ the class operator of closure under ultraproducts. As a consequence of Łoś' Theorem, every elementary class is closed under $\mathbb{P}_{\mathrm{u}}$.

2. Universal algebra

This section reviews the main tools of general algebraic nature that will be employed in this monograph. The reader need not read it in its entirety before proceeding with the subsequent sections and can come back to it each time they encounter a new notion.

We denote the class operators of closure under isomorphic copies, subalgebras, homomorphic images, direct products, and ultraproducts by $\mathbb{I}, \mathbb{S}, \mathbb{H}, \mathbb{P}$, and $\mathbb{P}_{\mathrm{u}}$, respectively. A class of algebras is said to be:

   (i) a *variety* when it is closed under $\mathbb{H}, \mathbb{S}$, and $\mathbb{P}$;
  (ii) a *quasivariety* when it is closed under $\mathbb{I}, \mathbb{S}, \mathbb{P}$, and $\mathbb{P}_{\mathrm{u}}$;
 (iii) a *universal class* when it is closed under $\mathbb{I}, \mathbb{S}$, and $\mathbb{P}_{\mathrm{u}}$.

While every variety is a quasivariety and every quasivariety is a universal class, the converses are not true in general. We call *proper* the quasivarieties that are not varieties and the universal classes that are not quasivarieties. Examples of a proper quasivariety and a proper universal class are the classes of cancellative commutative monoids (Example 8.8) and of fields (Example 3.16), respectively.

The next theorem provides an alternative characterization of the above mentioned classes in terms of axiomatizability by certain types of formulas (see, e.g., [21, Thms. II.11.9 &

V.2.25 & V.2.20]). We recall that a formula is called a *quasiequation* when it is of the form

$$\bigsqcap \Phi \to \varphi,$$

where $\Phi \cup \{\varphi\}$ is a finite set of equations. When $\Phi = \emptyset$, the above quasiequation is equivalent to the equation φ. Consequently, every equation is equivalent to a quasiequation.

Theorem 2.1. *The following conditions hold for a class of algebras* K:

  (i) K *is a variety if and only if it can be axiomatized by a set of equations;*
 (ii) K *is a quasivariety if and only if it can be axiomatized by a set of quasiequations;*
(iii) K *is a universal class if and only if it can be axiomatized by a set of universal formulas.*

We denote the least variety, the least quasivariety, and the least universal class containing a class of algebras K by $\mathbb{V}(\mathsf{K})$, $\mathbb{Q}(\mathsf{K})$, and $\mathbb{U}(\mathsf{K})$, respectively. A variety (resp. quasivariety) K is *finitely generated* when $\mathsf{K} = \mathbb{V}(\mathsf{M})$ (resp. $\mathsf{K} = \mathbb{Q}(\mathsf{M})$) for a finite set M of finite algebras. The following characterizes $\mathbb{V}(\mathsf{K})$, $\mathbb{Q}(\mathsf{K})$, and $\mathbb{U}(\mathsf{K})$ in terms of the class operators (see, e.g., [21, Thms. II.9.5 & V.2.25 & V.2.20]).

Theorem 2.2. *For every class of algebras* K,

$$\mathbb{V}(\mathsf{K}) = \mathbb{HSP}(\mathsf{K}), \quad \mathbb{Q}(\mathsf{K}) = \mathbb{ISPP}_{\mathrm{u}}(\mathsf{K}), \quad and \quad \mathbb{U}(\mathsf{K}) = \mathbb{ISP}_{\mathrm{u}}(\mathsf{K}).$$

The following is a straightforward consequence of Theorem 2.1.

Corollary 2.3. *The following conditions hold for a class of algebras* K:

  (i) $\mathbb{V}(\mathsf{K})$ *is the class of models of all the equations valid in* K;
 (ii) $\mathbb{Q}(\mathsf{K})$ *is the class of models of all the quasiequations valid in* K;
(iii) $\mathbb{U}(\mathsf{K})$ *is the class of models of all the universal formulas valid in* K.

We will make use of the following closure property of universal classes (see, e.g., [35, Thm. 3.2.3]).

Proposition 2.4. *Universal classes are closed under the formation of unions of chains of algebras.*

As quasivarieties need not be closed under $\mathbb{H}$, the following concept is often useful. Let K be a quasivariety and $\boldsymbol{A}$ and algebra (not necessarily in K). A congruence θ of $\boldsymbol{A}$ is said to be a K-*congruence* when $\boldsymbol{A}/\theta \in \mathsf{K}$. Owing to the fact that K is closed under $\mathbb{I}$ and $\mathbb{S}$, the Homomorphism Theorem [21, Thm. II.6.12] yields that the *kernel*

$$\mathsf{Ker}(h) = \{\langle a, b \rangle \in A \times A : h(a) = h(b)\}$$

of every homomorphism $h \colon \boldsymbol{A} \to \boldsymbol{B}$ with $\boldsymbol{B} \in \mathsf{K}$ is a K-congruence of $\boldsymbol{A}$ such that $\boldsymbol{A}/\mathsf{Ker}(h) \cong h[\boldsymbol{A}]$, where $h[\boldsymbol{A}]$ denotes the subalgebra of $\boldsymbol{B}$ with universe $h[A]$. When ordered under the inclusion relation, the set of K-congruences of $\boldsymbol{A}$ forms an algebraic lattice $\mathsf{Con}_{\mathsf{K}}(\boldsymbol{A})$ in which meets are intersections (see, e.g., [58, Prop. 1.4.7 & Cor. 1.4.11]). Given $X \subseteq A \times A$, we denote the least congruence of $\boldsymbol{A}$ containing X by $\mathsf{Cg}^{\boldsymbol{A}}(X)$ and the least K-congruence of $\boldsymbol{A}$ containing X by $\mathsf{Cg}_{\mathsf{K}}^{\boldsymbol{A}}(X)$. We will rely on the following observation.

Proposition 2.5. *A quasivariety* K *is a variety if and only if* $\mathsf{Con}(\boldsymbol{A}) = \mathsf{Con}_{\mathsf{K}}(\boldsymbol{A})$ *for every* $\boldsymbol{A} \in \mathsf{K}$.

Proof. First, suppose that K is a variety. If $\boldsymbol{A} \in \mathsf{K}$ and $\theta \in \mathsf{Con}(\boldsymbol{A})$, then $\boldsymbol{A}/\theta \in \mathbb{H}(\boldsymbol{A}) \subseteq \mathsf{K}$. Therefore, $\mathsf{Con}(\boldsymbol{A}) = \mathsf{Con}_{\mathsf{K}}(\boldsymbol{A})$ for every $\boldsymbol{A} \in \mathsf{K}$. Assume now that $\mathsf{Con}(\boldsymbol{A}) = \mathsf{Con}_{\mathsf{K}}(\boldsymbol{A})$ for every $\boldsymbol{A} \in \mathsf{K}$. Since K is a quasivariety, it suffices to prove that K is closed under homomorphic images. Consider $\boldsymbol{B} \in \mathbb{H}(\mathsf{K})$. Then there exists a surjective homomorphism $h \colon \boldsymbol{A} \to \boldsymbol{B}$ with $\boldsymbol{A} \in \mathsf{K}$. Since $\mathsf{Ker}(h) \in \mathsf{Con}(\boldsymbol{A})$, the assumption yields $\mathsf{Ker}(h) \in \mathsf{Con}_{\mathsf{K}}(\boldsymbol{A})$. Thus, $\boldsymbol{A}/\mathsf{Ker}(h) \in \mathsf{K}$. As $\boldsymbol{B} \cong \boldsymbol{A}/\mathsf{Ker}(h)$ and K is closed under $\mathbb{I}$, it follows that $\boldsymbol{B} \in \mathsf{K}$. ⊠

The following gives a necessary and sufficient condition for a homomorphism to factor through a quotient (see, e.g., [59, p. 62]).

Proposition 2.6. *Let* $h \colon \boldsymbol{A} \to \boldsymbol{B}$ *be a homomorphism,* $\theta \in \mathsf{Con}(\boldsymbol{A})$, *and* $f \colon \boldsymbol{A} \to \boldsymbol{A}/\theta$ *the canonical surjection. Then* $\theta \subseteq \mathsf{Ker}(h)$ *if and only if there exists a homomorphism* $g \colon \boldsymbol{A}/\theta \to \boldsymbol{B}$ *such that* $g \circ f = h$.

$$\begin{array}{ccc} \boldsymbol{A} & \xrightarrow{\ h\ } & \boldsymbol{B} \\ f \downarrow & \nearrow \; g & \\ \boldsymbol{A}/\theta & & \end{array}$$

An algebra $\boldsymbol{A}$ is a *subdirect product* of a family $\{\boldsymbol{B}_i : i \in I\}$ when $\boldsymbol{A} \leqslant \prod_{i \in I} \boldsymbol{B}_i$ and for every $i \in I$ the projection map $p_i \colon \boldsymbol{A} \to \boldsymbol{B}_i$ is surjective. Similarly, an embedding $h \colon \boldsymbol{A} \to \prod_{i \in I} \boldsymbol{B}_i$ is called *subdirect* when $h[\boldsymbol{A}] \leqslant \prod_{i \in I} \boldsymbol{B}_i$ is a subdirect product. The next result simplifies the task of constructing subdirect embeddings (see, e.g., [21, Lem. II.8.2]).

Proposition 2.7. *Let* $\boldsymbol{A}$ *be an algebra and* $X \subseteq \mathsf{Con}(\boldsymbol{A})$. *Then the map*

$$h \colon \boldsymbol{A}/\bigcap X \to \prod_{\theta \in X} \boldsymbol{A}/\theta$$

defined by the rule $h(a/\bigcap X) = \langle a/\theta : \theta \in X \rangle$ *is a subdirect embedding.*

Notably, every congruence can be viewed as a subdirect product.

Proposition 2.8. *Let* K *be a quasivariety and* $\boldsymbol{A} \in \mathsf{K}$. *Every congruence* θ *of* $\boldsymbol{A}$ *is the universe of an algebra* $\theta^* \in \mathsf{K}$ *such that* $\theta^* \leqslant \boldsymbol{A} \times \boldsymbol{A}$ *is a subdirect product.*

Proof. The fact that θ is the universe of a subalgebra θ^* of $\boldsymbol{A} \times \boldsymbol{A}$ is an immediate consequence of the definition of a congruence of $\boldsymbol{A}$. As $\boldsymbol{A} \in \mathsf{K}$ and K is closed under subalgebras and direct products (because it is a quasivariety), we obtain $\theta^* \in \mathsf{K}$. To conclude that $\theta^* \leqslant \boldsymbol{A} \times \boldsymbol{A}$ is a subdirect product, it suffices to show that the projection maps $p_1, p_2 \colon \theta^* \to \boldsymbol{A}$ are surjective. Consider $a \in A$. As θ is a reflexive relation on A, we have $\langle a, a \rangle \in \theta$. Consequently, $p_1(\langle a, a \rangle) = p_2(\langle a, a \rangle) = a$. ⊠

Let K be a quasivariety. An algebra $\boldsymbol{A} \in \mathsf{K}$ is said to be *subdirectly irreducible relative to* K when for every subdirect embedding $h \colon \boldsymbol{A} \to \prod_{i \in I} \boldsymbol{B}_i$ with $\{\boldsymbol{B}_i : i \in I\} \subseteq \mathsf{K}$ there exists $i \in I$ such that $p_i \circ h \colon \boldsymbol{A} \to \boldsymbol{B}_i$ is an isomorphism. In case this happens whenever

the index set I is finite, we say that $\boldsymbol{A}$ is *finitely subdirectly irreducible relative to* K.[2] The classes of algebras that are subdirectly irreducible relatively to K and finitely subdirectly irreducible relative to K will be denoted by $\mathsf{K}_{\mathrm{RSI}}$ and $\mathsf{K}_{\mathrm{RFSI}}$, respectively. When K is a variety, the requirement that $\{\boldsymbol{B}_i : i \in I\}$ is a subset of K in the above definitions can be harmlessly dropped and we simply say that $\boldsymbol{A}$ is *subdirectly irreducible* or *finitely subdirectly irreducible* (i.e., we drop the "relative to K"). In this case, we also write K_{SI} and $\mathsf{K}_{\mathrm{FSI}}$ instead of $\mathsf{K}_{\mathrm{RSI}}$ and $\mathsf{K}_{\mathrm{RFSI}}$.

The importance of subdirect embeddings and of algebras that are subdirectly irreducible relative to K derives from the following representation theorem (see, e.g., [58, Thm. 3.1.1]).

Subdirect Decomposition Theorem 2.9. *Let* K *be a quasivariety. For every* $\boldsymbol{A} \in \mathsf{K}$ *there exists a subdirect embedding* $f \colon \boldsymbol{A} \to \prod_{i \in I} \boldsymbol{B}_i$ *with* $\{\boldsymbol{B}_i : i \in I\} \subseteq \mathsf{K}_{\mathrm{RSI}}$.

For instance, an Abelian group is subdirectly irreducible precisely when it is either cyclic of prime-power order or quasicyclic (see, e.g., [11, Thm. 3.29]). Therefore, every Abelian group can be represented as a subdirect product of Abelian groups of this form.

Notably, algebras that are (finitely) subdirectly irreducible relative to K can be recognized by looking at the structure of their lattices of K-congruences. More precisely, we recall that an element a of a lattice $\boldsymbol{A}$ is said to be:

(i) *completely meet irreducible* when $a \in X$ for every $X \subseteq A$ such that $a = \bigwedge X$;
(ii) *meet irreducible* when $a \in X$ for every finite $X \subseteq A$ such that $a = \bigwedge X$.

Notice that every completely meet irreducible element is meet irreducible and that the maximum of a lattice is never meet irreducible because it coincides with $\bigwedge \emptyset$. Given a quasivariety K and $\boldsymbol{A} \in \mathsf{K}$, let

$$\mathsf{Irr}_{\mathsf{K}}^{\infty}(\boldsymbol{A}) = \text{the set of completely meet irreducible elements of } \mathsf{Con}_{\mathsf{K}}(\boldsymbol{A});$$

$$\mathsf{Irr}_{\mathsf{K}}(\boldsymbol{A}) = \text{the set of meet irreducible elements of } \mathsf{Con}_{\mathsf{K}}(\boldsymbol{A}).$$

Furthermore, we denote the identity relation on $\boldsymbol{A}$ by id_A. The following is a consequence of [58, Cor. 1.4.8] and the Correspondence Theorem [21, Thm. II.6.20].

Proposition 2.10. *Let* $\boldsymbol{A}$ *be a member of a quasivariety* K. *For every* $\theta \in \mathsf{Con}(\boldsymbol{A})$ *we have*

$$\boldsymbol{A}/\theta \in \mathsf{K}_{\mathrm{RSI}} \quad \text{if and only if} \quad \theta \in \mathsf{Irr}_{\mathsf{K}}^{\infty}(\boldsymbol{A});$$
$$\boldsymbol{A}/\theta \in \mathsf{K}_{\mathrm{RFSI}} \quad \text{if and only if} \quad \theta \in \mathsf{Irr}_{\mathsf{K}}(\boldsymbol{A}).$$

Therefore, $\boldsymbol{A} \in \mathsf{K}_{\mathrm{RSI}}$ *(resp.* $\boldsymbol{A} \in \mathsf{K}_{\mathrm{RFSI}}$*) if and only if* $\mathrm{id}_A \in \mathsf{Irr}_{\mathsf{K}}^{\infty}(\boldsymbol{A})$ *(resp.* $\mathrm{id}_A \in \mathsf{Irr}_{\mathsf{K}}(\boldsymbol{A})$*).*

As a consequence, a member $\boldsymbol{A}$ of a quasivariety K is relatively subdirectly irreducible precisely when it has a least nonidentity K-congruence, called the *monolith* of $\boldsymbol{A}$. When it exists, the monolith of $\boldsymbol{A}$ is always the K-congruence of $\boldsymbol{A}$ generated by a pair of distinct elements $a, b \in A$, which we denote by $\mathsf{Cg}_{\mathsf{K}}^{\boldsymbol{A}}(a, b)$.

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[2]We adopt the convention that the direct product of an empty family of algebras is the trivial algebra in the language under consideration. Consequently, we do not regard the trivial algebra as relatively (finitely) subdirectly irreducible.

Given two binary relations R_1 and R_2 on a set A, we let

$$R_1 \circ R_2 = \{\langle a, b \rangle \in A \times A : \text{there exists } c \in A \text{ s.t. } \langle a, c \rangle \in R_1 \text{ and } \langle c, b \rangle \in R_2\}.$$

A variety K is said to be *congruence permutable* when for all $\boldsymbol{A} \in \mathsf{K}$ and $\theta, \phi \in \mathsf{Con}(\boldsymbol{A})$ we have $\theta \circ \phi = \phi \circ \theta$. A quasivariety K is said to be *relatively congruence distributive* when $\mathsf{Con}_{\mathsf{K}}(\boldsymbol{A})$ is a distributive lattice for every $\boldsymbol{A} \in \mathsf{K}$. If K is a variety, we simply say that K is *congruence distributive*. Notably, every variety whose members have a group (resp. lattice) structure is congruence permutable (resp. distributive) (see, e.g., [21, p. 79]). A variety that is both congruence distributive and congruence permutable is called *arithmetical*.

Remark 2.11. Contrarily to the case of congruence distributivity, congruence permutability is usually understood as a property of varieties only (as opposed to arbitrary quasivarieties). The reason is that an algebra is congruence permutable if and only if $\theta \circ \phi$ is the join of θ and ϕ in $\mathsf{Con}(\boldsymbol{A})$ for all $\theta, \phi \in \mathsf{Con}(\boldsymbol{A})$. However, the sole quasivarieties K such that $\theta \circ \phi$ is the join of θ and ϕ in $\mathsf{Con}_{\mathsf{K}}(\boldsymbol{A})$ for all $\boldsymbol{A} \in \mathsf{K}$ and $\theta, \phi \in \mathsf{Con}_{\mathsf{K}}(\boldsymbol{A})$ are those that are varieties (see [28]). ⊠

The following is a generalization of Jónsson's Theorem to the setting of finitely subdirectly irreducible algebras, which can be obtained as a straightforward consequence of [40, Thm. 1.7].

Jónsson's Theorem 2.12. *Let K be a class of algebras such that $\mathbb{V}(\mathsf{K})$ is congruence distributive. Then $\mathbb{V}(\mathsf{K})_{\mathrm{FSI}} \subseteq \mathbb{HSP}_{\mathrm{u}}(\mathsf{K})$.*

We will also utilize the following analogous statement for quasivarieties (see [40, Thm. 1.5]).

Theorem 2.13. *Let K be a class of algebras. Then $\mathbb{Q}(\mathsf{K})_{\mathrm{RFSI}} \subseteq \mathbb{ISP}_{\mathrm{u}}(\mathsf{K})$.*

When the class K in the above result is a finite set of finite algebras, the class operator $\mathbb{P}_{\mathrm{u}}$ becomes superfluous because of the following observation (see, e.g., [11, Thm. 5.6(2)]).

Proposition 2.14. *If K is a finite set of finite algebras, then $\mathbb{P}_{\mathrm{u}}(\mathsf{K}) \subseteq \mathbb{I}(\mathsf{K})$.*

Lastly, given an algebra $\boldsymbol{A}$ and a set $X \subseteq A$, we denote the least subuniverse of $\boldsymbol{A}$ containing X by $\mathsf{Sg}^{\boldsymbol{A}}(X)$. When $\mathsf{Sg}^{\boldsymbol{A}}(X) \neq \emptyset$, the subalgebra of $\boldsymbol{A}$ with universe $\mathsf{Sg}^{\boldsymbol{A}}(X)$ will also be denoted by $\mathsf{Sg}^{\boldsymbol{A}}(X)$. When $X = \{a_1, \dots, a_n\}$ is finite, we write $\mathsf{Sg}^{\boldsymbol{A}}(a_1, \dots, a_n)$ in place of $\mathsf{Sg}^{\boldsymbol{A}}(\{a_1, \dots, a_n\})$. If $A = \mathsf{Sg}^{\boldsymbol{A}}(X)$ for some finite $X \subseteq A$, we say that $\boldsymbol{A}$ is *finitely generated*. If every finitely generated subalgebra of $\boldsymbol{A}$ is finite, we call $\boldsymbol{A}$ *locally finite*. A class of algebras is *locally finite* when its members are. We denote the class of finitely generated members of a class of algebras K by K^{fg}.

The following is an immediate consequence of [21, Thm. V.2.14].

Proposition 2.15. *Let K be a universal class. Then $\mathsf{K} = \mathbb{U}(\mathsf{K}^{\mathrm{fg}})$.*

The Subdirect Decomposition Theorem 2.9 readily implies that $\mathsf{K} = \mathbb{Q}(\mathsf{K}_{\mathrm{RSI}})$ for every quasivariety K. It is well known that this result can be improved by restricting to the class $\mathsf{K}_{\mathrm{RSI}}^{\mathrm{fg}}$ of finitely generated members of $\mathsf{K}_{\mathrm{RSI}}$. As we were unable to find a reference in the literature, we provide a proof.

Proposition 2.16. *Let K be a quasivariety. Then $\mathsf{K} = \mathbb{Q}(\mathsf{K}_{\mathrm{RSI}}^{\mathrm{fg}})$.*

Proof. From [58, Prop. 2.1.18] it follows that $\mathsf{K} = \mathbb{Q}(\mathsf{K}^{\mathrm{fg}})$, where K^{fg} is the class of the finitely generated members of K. Then let $\boldsymbol{A} \in \mathsf{K}^{\mathrm{fg}}$. By the Subdirect Decomposition Theorem 2.9 there exists a subdirect embedding $f\colon \boldsymbol{A} \to \prod_{i \in I} \boldsymbol{A}_i$, where $\boldsymbol{A}_i \in \mathsf{K}_{\mathrm{RSI}}$ for every $i \in I$. As f is a subdirect embedding, each $\boldsymbol{A}_i$ is a homomorphic image of $\boldsymbol{A}$ and, therefore, finitely generated. So, $\boldsymbol{A}_i \in \mathsf{K}_{\mathrm{RSI}}^{\mathrm{fg}}$ for every $i \in I$. Since f is an embedding into a direct product of members of $\mathsf{K}_{\mathrm{RSI}}^{\mathrm{fg}}$, we obtain $\boldsymbol{A} \in \mathbb{ISP}(\mathsf{K}_{\mathrm{RSI}}^{\mathrm{fg}}) \subseteq \mathbb{Q}(\mathsf{K}_{\mathrm{RSI}}^{\mathrm{fg}})$. Thus, $\mathsf{K}^{\mathrm{fg}} \subseteq \mathbb{Q}(\mathsf{K}_{\mathrm{RSI}}^{\mathrm{fg}})$, and hence $\mathbb{Q}(\mathsf{K}^{\mathrm{fg}}) \subseteq \mathbb{Q}(\mathsf{K}_{\mathrm{RSI}}^{\mathrm{fg}})$. Together with $\mathsf{K} = \mathbb{Q}(\mathsf{K}^{\mathrm{fg}})$, this yields $\mathsf{K} \subseteq \mathbb{Q}(\mathsf{K}_{\mathrm{RSI}}^{\mathrm{fg}})$. Since K is a quasivariety containing $\mathsf{K}_{\mathrm{RSI}}^{\mathrm{fg}}$, we conclude that $\mathsf{K} = \mathbb{Q}(\mathsf{K}_{\mathrm{RSI}}^{\mathrm{fg}})$. $\boxtimes$

Let K be a quasivariety. For every pair of algebras $\boldsymbol{A}$ and $\boldsymbol{B}$, $\theta \in \mathsf{Con}_{\mathsf{K}}(\boldsymbol{A})$, and $\phi \in \mathsf{Con}_{\mathsf{K}}(\boldsymbol{B})$, the relation

$$\theta \times \phi = \{\langle \langle a_1, b_1 \rangle, \langle a_2, b_2 \rangle \rangle \in (A \times B)^2 : \langle a_1, a_2 \rangle \in \theta \text{ and } \langle b_1, b_2 \rangle \in \phi\}$$

is a K-congruence of the direct product $\boldsymbol{A} \times \boldsymbol{B}$. Given a pair of algebras $\boldsymbol{A} \leqslant \boldsymbol{B}$ and $\theta \in \mathsf{Con}_{\mathsf{K}}(\boldsymbol{B})$, we write $\theta{\restriction}_A$ as a shorthand for $\theta \cap (A \times A)$. Notice that $\theta{\restriction}_A$ is a K-congruence of $\boldsymbol{A}$. The next result is an effortless generalization to quasivarieties of [73, Thm. 1.2.20].

Theorem 2.17. *A quasivariety K is relatively congruence distributive if and only if for every subdirect product $\boldsymbol{A} \leqslant \boldsymbol{B} \times \boldsymbol{C}$ with $\boldsymbol{B}, \boldsymbol{C} \in \mathsf{K}$ and every $\theta \in \mathsf{Con}_{\mathsf{K}}(\boldsymbol{A})$ there exist $\phi \in \mathsf{Con}_{\mathsf{K}}(\boldsymbol{B})$ and $\eta \in \mathsf{Con}_{\mathsf{K}}(\boldsymbol{C})$ such that $\theta = (\phi \times \eta){\restriction}_A$.*

As a consequence, we deduce the following.

Corollary 2.18. *Let K be a relatively congruence distributive quasivariety, $\boldsymbol{A}$ an algebra, and $\theta \in \mathsf{Con}_{\mathsf{K}}(\boldsymbol{A})$ such that $\boldsymbol{A}/\theta$ is either trivial or a member of $\mathsf{K}_{\mathrm{RFSI}}$. Then for every $\boldsymbol{B} \in \mathsf{K}$ such that $\boldsymbol{A} \leqslant \boldsymbol{B} \times \boldsymbol{B}$ is a subdirect product there exists $\phi \in \mathsf{Con}_{\mathsf{K}}(\boldsymbol{B})$ with*

$$\theta \in \{(\phi \times B^2){\restriction}_A, (B^2 \times \phi){\restriction}_A\}.$$

Proof. Consider $\boldsymbol{B} \in \mathsf{K}$ such that $\boldsymbol{A} \leqslant \boldsymbol{B} \times \boldsymbol{B}$ is a subdirect product. We have two cases: either $\boldsymbol{A}/\theta$ is trivial or it belongs to $\mathsf{K}_{\mathrm{RFSI}}$. First, suppose that $\boldsymbol{A}/\theta$ is trivial. Then $\theta = A \times A$. As $A \subseteq B \times B$ by assumption, we obtain

$$\theta = A \times A = (B^2 \times B^2){\restriction}_A.$$

Since B^2 is a K-congruence of $\boldsymbol{B}$ (because K contains all the trivial algebras in the appropriate language), we obtain $B^2 \in \mathsf{Con}_{\mathsf{K}}(\boldsymbol{B})$. Hence, we are done taking $\phi = B^2$.

Next we consider the case where $\boldsymbol{A}/\theta \in \mathsf{K}_{\mathrm{RFSI}}$. As $\boldsymbol{A} \leqslant \boldsymbol{B} \times \boldsymbol{B}$ is a subdirect product and K is relatively congruence distributive by assumption, we can apply Theorem 2.17, obtaining some $\phi_1, \phi_2 \in \mathsf{Con}_{\mathsf{K}}(\boldsymbol{B})$ such that $\theta = (\phi_1 \times \phi_2){\restriction}_A$. Observe that $\phi_1 \times \phi_2 = (\phi_1 \times B^2) \cap (B^2 \times \phi_2)$. Therefore,

$$\theta = (\phi_1 \times \phi_2){\restriction}_A = ((\phi_1 \times B^2) \cap (B^2 \times \phi_2)){\restriction}_A = (\phi_1 \times B^2){\restriction}_A \cap (B^2 \times \phi_2){\restriction}_A. \qquad (4)$$

Observe that $\phi_1 \times B^2, B^2 \times \phi_2 \in \mathsf{Con}_{\mathsf{K}}(\boldsymbol{B} \times \boldsymbol{B})$ because $\phi_1, \phi_2 \in \mathsf{Con}_{\mathsf{K}}(\boldsymbol{B})$. Together with $\boldsymbol{A} \leqslant \boldsymbol{B} \times \boldsymbol{B}$, this yields

$$(\phi_1 \times B^2){\restriction}_A, (B^2 \times \phi_2){\restriction}_A \in \mathsf{Con}_{\mathsf{K}}(\boldsymbol{A}). \qquad (5)$$

Lastly, recall that $\boldsymbol{A}/\theta \in \mathsf{K}_{\mathrm{RFSI}}$ by assumption. Then $\theta \in \mathsf{Irr}_\mathsf{K}(\boldsymbol{A})$ by Proposition 2.10. Therefore, from (4) and (5) it follows that $\theta \in \{(\phi_1 \times B^2)\restriction_A, (B^2 \times \phi_2)\restriction_A\}$. Since $\phi_1, \phi_2 \in \mathsf{Con}_\mathsf{K}(\boldsymbol{B})$, we are done. $\boxtimes$

Let K be a class of algebras and X a nonempty set of variables. The set of all terms built over variables in X can be made into an algebra $\boldsymbol{T}(X)$ called the *term algebra over* X. The binary relation θ_K on $T(X)$ defined by $\langle t, s \rangle \in \theta_\mathsf{K}$ if and only if $\mathsf{K} \vDash t \approx s$ is a congruence of $\boldsymbol{T}(X)$. We call the quotient $\boldsymbol{F}_\mathsf{K}(X) = \boldsymbol{T}(X)/\theta_\mathsf{K}$ the K-*free algebra over* X. Free algebras have the following universal mapping property: for every map $f\colon X \to A$ with $\boldsymbol{A} \in \mathsf{K}$ there exists a unique homomorphism $h\colon \boldsymbol{F}_\mathsf{K}(X) \to \boldsymbol{A}$ such that $h(x/\theta_\mathsf{K}) = f(x)$ for every $x \in X$. To simplify the notation, we will often denote an element t/θ_K of $F_\mathsf{K}(X)$ simply by t. The following states that quasivarieties contain all free algebras (see, e.g., [21, Thm. II.10.12]).

Theorem 2.19. *Let K be a quasivariety and X a nonempty set. Then $\boldsymbol{F}_\mathsf{K}(X) \in \mathsf{K}$.*

A member $\boldsymbol{A}$ of quasivariety K is called *finitely presented* when there exist a finite set of variables X and a finite $Y \subseteq T(X) \times T(X)$ such that $\boldsymbol{A} \cong \boldsymbol{T}(X)/\mathsf{Cg}_\mathsf{K}^{\boldsymbol{T}(X)}(Y)$.

We call an algebra in a language $\mathscr{L}$ an $\mathscr{L}$-*algebra*. When $\mathscr{L}$ and $\mathscr{L}'$ are two languages such that $\mathscr{L} \subseteq \mathscr{L}'$ we say that $\mathscr{L}'$ is an *expansion* of $\mathscr{L}$. If $\mathscr{L}'$ is an expansion of $\mathscr{L}$, then for every $\mathscr{L}'$-algebra $\boldsymbol{A}$, we can consider its $\mathscr{L}$-*reduct* $\boldsymbol{A}\restriction_\mathscr{L}$ obtained from $\boldsymbol{A}$ by forgetting the interpretations of all function symbols that are not in $\mathscr{L}$. Given a class K of $\mathscr{L}'$-algebras, we denote the class of the $\mathscr{L}$-reducts of members of K by $\mathsf{K}\restriction_\mathscr{L}$ and we call the members of $\mathbb{S}(\mathsf{K}\restriction_\mathscr{L})$ the $\mathscr{L}$-*subreducts* of K. For instance, the monoid subreducts of Abelian groups are precisely the cancellative commutative monoids (see, e.g., [84, pp. 39–40]). We will often denote the language of a given class of algebras K by $\mathscr{L}_\mathsf{K}$, and we will refer to the terms of $\mathscr{L}_\mathsf{K}$ simply as the *terms of* K.

Particular cases of language expansions are those obtained by adding to the language names for the elements of a given algebra. More precisely, given an $\mathscr{L}$-algebra $\boldsymbol{A}$, we consider the language $\mathscr{L}_A$ obtained by adding to $\mathscr{L}$ a set of new constants $\{c_a : a \in A\}$ that is in bijection with the elements of $\boldsymbol{A}$. Given a function $h\colon A \to B$ between the universes of a pair of $\mathscr{L}$-algebras $\boldsymbol{A}$ and $\boldsymbol{B}$, we denote by $\boldsymbol{B}_{h[A]}$ the $\mathscr{L}_A$-algebra whose $\mathscr{L}$-reduct is $\boldsymbol{B}$ and in which each constant c_a is interpreted as $h(a)$. In particular, we denote by $\boldsymbol{A}_A$ the expansion of $\boldsymbol{A}$ to an $\mathscr{L}_A$-algebra induced by the identity map on A. We define the *diagram* $\mathsf{diag}(\boldsymbol{A})$ of an $\mathscr{L}$-algebra $\boldsymbol{A}$ to be the set of all variable-free $\mathscr{L}_A$-formulas that are equations and negated equations valid in $\boldsymbol{A}_A$. The following lemma connects the validity of diagrams with the existence of embeddings (see, e.g., [35, Prop. 2.1.8]).

Diagram Lemma 2.20. *Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be $\mathscr{L}$-algebras and $h\colon A \to B$ a function. Then $h\colon \boldsymbol{A} \to \boldsymbol{B}$ is an embedding if and only if $\boldsymbol{B}_{h[A]} \vDash \mathsf{diag}(\boldsymbol{A})$.*

3. Implicit operations

An n-ary *partial function* on a set X is a function $f\colon Y \to X$ for some $Y \subseteq X^n$. In this case, the set Y will be called the *domain* of f and will be denoted by $\mathsf{dom}(f)$. This notion can be extended to classes of algebras as follows. An n-ary *partial function* on a class of

algebras K is a sequence $\langle f^{\boldsymbol{A}} : \boldsymbol{A} \in \mathsf{K}\rangle$, where $f^{\boldsymbol{A}}$ is an n-ary partial function on A for each $\boldsymbol{A} \in \mathsf{K}$. By a *partial function* on K we mean an n-ary partial function on K for some $n \in \mathbb{N}$. When f is a partial function on K and $\boldsymbol{A} \in \mathsf{K}$, we denote the $\boldsymbol{A}$-component of f by $f^{\boldsymbol{A}}$.

Example 3.1 (Monoids). The operation of "taking inverses" can be viewed as a partial function on the variety of monoids Mon. More precisely, recall that the inverse of an element a of a monoid $\boldsymbol{A} = \langle A; \cdot, 1\rangle$ is unique when it exists, in which case it will be denoted by a^{-1}. Then let $f^{\boldsymbol{A}}$ be the unary partial function on A with domain

$$\mathsf{dom}(f^{\boldsymbol{A}}) = \{a \in A : a \text{ has an inverse in } \boldsymbol{A}\},$$

defined for each $a \in \mathsf{dom}(f^{\boldsymbol{A}})$ as $f^{\boldsymbol{A}}(a) = a^{-1}$. The sequence $\langle f^{\boldsymbol{A}} : \boldsymbol{A} \in \mathsf{Mon}\rangle$ is a unary partial function on Mon. $\boxtimes$

Definition 3.2. A formula $\varphi(x_1, \ldots, x_n, y)$ with $n \geqslant 1$ in the language of a class of algebras K is said to be *functional* in K when for all $\boldsymbol{A} \in \mathsf{K}$ and $a_1, \ldots, a_n \in A$ there exists at most one $b \in A$ such that $\boldsymbol{A} \vDash \varphi(a_1, \ldots, a_n, b)$. When $\mathsf{K} = \{\boldsymbol{A}\}$, we often say that φ is *functional* in $\boldsymbol{A}$.

In other words, φ is functional in a class of algebras K when

$$\mathsf{K} \vDash (\varphi(x_1, \ldots, x_n, y) \sqcap \varphi(x_1, \ldots, x_n, z)) \to y \approx z.$$

In this case, φ induces an n-ary partial function $\varphi^{\boldsymbol{A}}$ on each $\boldsymbol{A} \in \mathsf{K}$ with domain

$$\mathsf{dom}(\varphi^{\boldsymbol{A}}) = \{\langle a_1, \ldots, a_n\rangle \in A^n : \text{there exists } b \in A \text{ such that } \boldsymbol{A} \vDash \varphi(a_1, \ldots, a_n, b)\},$$

defined for all $\langle a_1, \ldots, a_n\rangle \in \mathsf{dom}(\varphi^{\boldsymbol{A}})$ as $\varphi^{\boldsymbol{A}}(a_1, \ldots, a_n) = b$, where b is the unique element of A such that $\boldsymbol{A} \vDash \varphi(a_1, \ldots, a_n, b)$. Consequently,

$$\varphi^{\mathsf{K}} = \langle \varphi^{\boldsymbol{A}} : \boldsymbol{A} \in \mathsf{K}\rangle$$

is an n-ary partial function on K.

Definition 3.3. A partial function f on a class of algebras K is said to be
(i) *defined by a formula* φ when φ is functional in K and $f = \varphi^{\mathsf{K}}$;
(ii) *implicit* when it is defined by some formula.

We remark that the arity of implicit partial functions is always positive because the definition of a functional formula $\varphi = \varphi(x_1, \ldots, x_n, y)$ requires n to be positive.

Example 3.4 (Monoids). We will prove that the partial function of "taking inverses" in monoids introduced in Example 3.1 is defined by the formula

$$\varphi(x, y) = (x \cdot y \approx 1) \sqcap (y \cdot x \approx 1).$$

First, observe that for each monoid $\boldsymbol{A}$ and $a, b \in A$ we have

$$\boldsymbol{A} \vDash \varphi(a, b) \iff a \cdot b = 1 = b \cdot a \iff b = a^{-1}.$$

As a consequence, for all $a, b, c \in A$ such that $\boldsymbol{A} \vDash \varphi(a, b) \sqcap \varphi(a, c)$ we have $b = a^{-1} = c$, whence φ is functional in Mon. Together with the above display, this shows that φ defines the partial function of "taking inverses" in monoids which, therefore, is implicit. $\boxtimes$

Definition 3.5. An *n*-ary partial function f on a class of algebras K is said to be

(i) an *operation* of K when for each homomorphism $h\colon \boldsymbol{A} \to \boldsymbol{B}$ with $\boldsymbol{A}, \boldsymbol{B} \in \mathsf{K}$ and $\langle a_1, \ldots, a_n \rangle \in \mathsf{dom}(f^{\boldsymbol{A}})$ we have $\langle h(a_1), \ldots, h(a_n) \rangle \in \mathsf{dom}(f^{\boldsymbol{B}})$ and

$$h(f^{\boldsymbol{A}}(a_1, \ldots, a_n)) = f^{\boldsymbol{B}}(h(a_1), \ldots, h(a_n));$$

(ii) an *implicit operation* of K when it is both implicit and an operation of K.

We denote the class of implicit operations of K by $\mathsf{imp}(\mathsf{K})$. When $\mathsf{K} = \{\boldsymbol{A}\}$, we often write $\mathsf{imp}(\boldsymbol{A})$ instead of $\mathsf{imp}(\mathsf{K})$.

Partial functions on a class K of algebras are defined as sequences $\langle f^{\boldsymbol{A}} : \boldsymbol{A} \in \mathsf{K} \rangle$ of partial functions indexed by K. Consequently, when K is a proper class, so is each of the partial functions on K. Nevertheless, since implicit partial functions can be identified with their defining formulas, we will always treat $\mathsf{imp}(\mathsf{K})$ as a set.

Example 3.6 (Monoids). We will prove the following.

Theorem 3.7. *Taking inverses is a unary implicit operation of the variety of monoids which, moreover, can be defined by the conjunction of equations*

$$\varphi = (x \cdot y \approx 1) \sqcap (y \cdot x \approx 1).$$

Proof. Recall from Example 3.4 that the partial function f on the variety of monoids of "taking inverses" is implicit and defined by φ. We will show that f is also an operation. To this end, consider a homomorphism $h\colon \boldsymbol{A} \to \boldsymbol{B}$ of monoids and $a \in \mathsf{dom}(f^{\boldsymbol{A}})$. Then $f^{\boldsymbol{A}}(a) = a^{-1}$. Since monoid homomorphisms preserve inverses, we obtain

$$h(f^{\boldsymbol{A}}(a)) = h(a^{-1}) = h(a)^{-1}.$$

Consequently, $h(a)$ has an inverse, whence $h(a) \in \mathsf{dom}(f^{\boldsymbol{B}})$ and $f^{\boldsymbol{B}}(h(a)) = h(a)^{-1}$. Together with the above display, this yields $h(f^{\boldsymbol{A}}(a)) = f^{\boldsymbol{B}}(h(a))$. $\boxtimes$

Example 3.8 (Term functions). Let K be a class of algebras. Every term of K can be viewed as an implicit operation, as we proceed to illustrate. Let $t(x_1, \ldots, x_n)$ be a term. For each $\boldsymbol{A} \in \mathsf{K}$, evaluating t on tuples of elements of A induces a function $t^{\boldsymbol{A}}\colon A^n \to A$. Then the sequence $t^{\mathsf{K}} = \langle t^{\boldsymbol{A}} : \boldsymbol{A} \in \mathsf{K} \rangle$ is an *n*-ary implicit operation of K defined by the equation $t(x_1, \ldots, x_n) \approx y$. We call t^{K} a *term function of* K. These implicit operations are always "total", in the sense that each $t^{\boldsymbol{A}}$ is a total function on A. $\boxtimes$

In elementary classes, implicit operations admit the following description.

Theorem 3.9. *Let f be a partial function on an elementary class K. Then f is an implicit operation of K if and only if it is defined by an existential positive formula.*

Proof. To prove the implication from left to right, suppose that f is an implicit operation on K. Then there exists a formula $\varphi(x_1, \ldots, x_n, y)$ that defines f. We will prove that φ is preserved by homomorphisms in K. To this end, consider a homomorphism $h\colon \boldsymbol{A} \to \boldsymbol{B}$ with $\boldsymbol{A}, \boldsymbol{B} \in \mathsf{K}$ and $a_1, \ldots, a_n, b \in A$ such that $\boldsymbol{A} \vDash \varphi(a_1, \ldots, a_n, b)$. As φ defines f, from $\boldsymbol{A} \vDash \varphi(a_1, \ldots, a_n, b)$ it follows that $\langle a_1, \ldots, a_n \rangle \in \mathsf{dom}(f^{\boldsymbol{A}})$ and $f^{\boldsymbol{A}}(a_1, \ldots, a_n) = b$. Together

with the assumption that f is an operation of K, this yields $\langle h(a_1), \ldots, h(a_n) \rangle \in \mathsf{dom}(f^{\boldsymbol{B}})$ and

$$h(b) = h(f^{\boldsymbol{A}}(a_1, \ldots, a_n)) = f^{\boldsymbol{B}}(h(a_1), \ldots, h(a_n)).$$

Since φ defines f, we conclude that $\boldsymbol{B} \vDash \varphi(h(a_1), \ldots, h(a_n), h(b))$. Hence, φ is preserved by homomorphisms in K. Therefore, we can apply Theorem 1.1(i), obtaining that φ is equivalent to an existential positive formula ψ in K. As φ defines f, so does ψ. Thus, we conclude that f is defined by an existential positive formula.

Then we proceed to prove the implication from right to left. Suppose that f is defined by an existential positive formula $\varphi(x_1, \ldots, x_n, y)$. To conclude that f is an implicit operation of K, it suffices to show that it is an operation of K. Consider a homomorphism $h \colon \boldsymbol{A} \to \boldsymbol{B}$ with $\boldsymbol{A}, \boldsymbol{B} \in \mathsf{K}$ and $a_1, \ldots, a_n \in A$ such that $\langle a_1, \ldots, a_n \rangle \in \mathsf{dom}(f^{\boldsymbol{A}})$. As φ defines f, we have $\boldsymbol{A} \vDash \varphi(a_1, \ldots, a_n, f^{\boldsymbol{A}}(a_1, \ldots, a_n))$. Since φ is an existential positive formula, we can apply Theorem 1.1(i), obtaining $\boldsymbol{B} \vDash \varphi(h(a_1), \ldots, h(a_n), h(f^{\boldsymbol{A}}(a_1, \ldots, a_n)))$. Therefore, $\langle h(a_1), \ldots, h(a_n) \rangle \in \mathsf{dom}(f^{\boldsymbol{B}})$ and $h(f^{\boldsymbol{A}}(a_1, \ldots, a_n)) = f^{\boldsymbol{B}}(h(a_1), \ldots, h(a_n))$ because φ defines f. Hence, we conclude that f is an operation of K. ⊠

Let $f, f_1, \ldots, f_n$ be m-ary partial functions on a set X. We write $f = f_1 \cup \cdots \cup f_n$ to indicate that f is the union of $f_1, \ldots, f_n$ when they are viewed as subsets of $X^m \times X$. This condition is equivalent to the requirements that $\mathsf{dom}(f) = \mathsf{dom}(f_1) \cup \cdots \cup \mathsf{dom}(f_n)$ and that $f_i(x) = f(x)$ for all i and $x \in \mathsf{dom}(f_i)$. As a consequence of Theorem 3.9, we obtain the following.

Corollary 3.10. *Let f be an implicit operation of an elementary class K. Then there exist some implicit operations $f_1, \ldots, f_n$ of K defined by pp formulas such that for each $\boldsymbol{A} \in \mathsf{K}$,*

$$f^{\boldsymbol{A}} = f_1^{\boldsymbol{A}} \cup \cdots \cup f_n^{\boldsymbol{A}}.$$

Proof. As K is an elementary class, we can apply Theorem 3.9, obtaining that f is defined by an existential positive formula φ. We may assume that $\varphi = \varphi_1 \sqcup \cdots \sqcup \varphi_n$ for some pp formulas $\varphi_1, \ldots, \varphi_n$. Since φ defines f, it is functional in K. Together with $\varphi = \varphi_1 \sqcup \cdots \sqcup \varphi_n$, this implies that each φ_i is functional in K and, therefore, defines a partial function f_i on K. As φ_i is a pp formula, from Theorem 3.9 it follows that f_i is an implicit operation of K. Now, recall that $\varphi = \varphi_1 \sqcup \cdots \sqcup \varphi_n$ defines f and φ_i defines f_i for each $i \leqslant n$. Thus, we conclude that $f^{\boldsymbol{A}} = f_1^{\boldsymbol{A}} \cup \cdots \cup f_n^{\boldsymbol{A}}$ for each $\boldsymbol{A} \in \mathsf{K}$. ⊠

In view of Corollary 3.10, implicit operations of elementary classes are obtained by gluing together implicit operations defined by pp formulas. This is the reason why the most fundamental implicit operations in mathematics are defined by pp formulas (as opposed to arbitrary existential positive formulas), as shown by the forthcoming examples. We denote by $\mathsf{imp}_{pp}(\mathsf{K})$ the set of implicit operations of a given class K that are defined by pp formulas and, when $\mathsf{K} = \{\boldsymbol{A}\}$, we often write $\mathsf{imp}_{pp}(\boldsymbol{A})$ instead of $\mathsf{imp}_{pp}(\mathsf{K})$.

Corollary 3.11. *Let K be a class of algebras and φ an existential positive formula functional in K. Then φ defines an implicit operation of $\mathbb{Q}(\mathsf{K})$.*

Proof. Suppose that $\varphi(\vec{x}, y)$ is an existential positive formula, where $\vec{x}$ is a finite sequence of variables. Since φ is existential positive, it is equivalent to a formula of the form

$$\bigsqcup_{i=1}^{m} \exists \vec{z} \psi_i(\vec{x}, y, \vec{z}),$$

where each $\psi_i(\vec{x}, y, \vec{z})$ is a finite conjunction of equations. We rely on the following observation.

Claim 3.12. *The formula φ is functional in a class of algebras M if and only if for all $i, j \leqslant m$ we have*

$$\mathsf{M} \vDash (\psi_i(\vec{x}, y, \vec{z}) \sqcap \psi_j(\vec{x}, y', \vec{z}')) \to y \approx y',$$

where y' is a fresh variable and $\vec{z}'$ a sequence of fresh variables of the same length as $\vec{z}$.

Proof of the Claim. The functionality of φ in M amounts to

$$\mathsf{M} \vDash (\varphi(\vec{x}, y) \sqcap \psi(\vec{x}, y')) \to y \approx y'.$$

In turn, this amounts to

$$\mathsf{M} \vDash \left(\left(\bigsqcup_{i=1}^{m} \exists \vec{z} \psi_i(\vec{x}, y, \vec{z}) \right) \sqcap \left(\bigsqcup_{i=1}^{m} \exists \vec{z}' \psi_i(\vec{x}, y', \vec{z}') \right) \right) \to y \approx y'.$$

The latter amounts to

$$\mathsf{M} \vDash \left(\bigsqcup_{i,j=1}^{m} \exists \vec{z}, \vec{z}' (\psi_i(\vec{x}, y, \vec{z}) \sqcap \psi_j(\vec{x}, y', \vec{z}')) \right) \to y \approx y',$$

which is in turn equivalent to the condition displayed in the statement. $\boxtimes$

Now, suppose that φ is functional in K. By Claim 3.12 we obtain that for all $i, j \leqslant m$,

$$\mathsf{K} \vDash (\psi_i(\vec{x}, y, \vec{z}) \sqcap \psi_j(\vec{x}, y', \vec{z}')) \to y \approx y'.$$

As the formula in the above display is a quasiequation for all $i, j \leqslant m$, Corollary 2.3(ii) implies that it is also valid in $\mathbb{Q}(\mathsf{K})$. Together with Claim 3.12, this guarantees that φ is functional in $\mathbb{Q}(\mathsf{K})$. $\boxtimes$

Given an n-ary partial function g and m-ary partial functions $f_1, \ldots, f_n$ on a class K, their *composition* $g(f_1, \ldots, f_n)$ is the m-ary partial function on K such that $\mathsf{dom}(g(f_1, \ldots, f_n)^{\boldsymbol{A}})$ is

$$\bigcap_{i \leqslant m} \mathsf{dom}(f_i^{\boldsymbol{A}}) \cap \{\langle a_1, \ldots, a_m \rangle \in A^m : \langle f_i^{\boldsymbol{A}}(a_1, \ldots, a_m) : i \leqslant n \rangle \in \mathsf{dom}(g^{\boldsymbol{A}})\}$$

for all $\boldsymbol{A} \in \mathsf{K}$ and

$$g(f_1, \ldots, f_n)^{\boldsymbol{A}}(a_1, \ldots, a_m) = g^{\boldsymbol{A}}(f_1^{\boldsymbol{A}}(a_1, \ldots, a_m), \ldots, f_n^{\boldsymbol{A}}(a_1, \ldots, a_m))$$

for all $\langle a_1, \ldots, a_m \rangle \in \mathsf{dom}(g(f_1, \ldots, f_n)^{\boldsymbol{A}})$.

Proposition 3.13. *Let K be a class of algebras. Then $\mathsf{imp}(\mathsf{K})$ and $\mathsf{imp}_{pp}(\mathsf{K})$ are closed under composition.*

Proof. Consider $g, f_1, \ldots, f_n \in \mathsf{imp}(\mathsf{K})$, where g is n-ary and each f_i is m-ary. Let $h = g(f_1, \ldots, f_n)$ be their composition, which is a partial m-ary function on K. To show that h is an operation of K, consider a homomorphism $k \colon \boldsymbol{A} \to \boldsymbol{B}$ with $\boldsymbol{A}, \boldsymbol{B} \in \mathsf{K}$ and $\langle a_1, \ldots, a_m \rangle \in \mathsf{dom}(h^{\boldsymbol{A}})$. It follows from the definition of h that $\langle a_1, \ldots, a_m \rangle \in \mathsf{dom}(f_i^{\boldsymbol{A}})$ for all i and $\langle f_1^{\boldsymbol{A}}(a_1, \ldots, a_m), \ldots, f_n^{\boldsymbol{A}}(a_1, \ldots, a_m) \rangle \in \mathsf{dom}(g^{\boldsymbol{A}})$. As $g, f_1, \ldots, f_n$ are operations of K, we have that

$$\langle k(a_1), \ldots, k(a_m) \rangle \in \mathsf{dom}(f_i^{\boldsymbol{B}})$$

for all i and

$$\langle f_1^{\boldsymbol{B}}(k(a_1), \ldots, k(a_m)), \ldots, f_n^{\boldsymbol{B}}(k(a_1), \ldots, k(a_m)) \rangle$$
$$= \langle k(f_1^{\boldsymbol{A}}(a_1, \ldots, a_m)), \ldots, k(f_n^{\boldsymbol{A}}(a_1, \ldots, a_m)) \rangle \in \mathsf{dom}(g^{\boldsymbol{B}}).$$

Then the definition of $\mathsf{dom}(h^{\boldsymbol{B}})$ implies $\langle k(a_1), \ldots, k(a_m) \rangle \in \mathsf{dom}(h^{\boldsymbol{B}})$. Using again the fact that $g, f_1, \ldots, f_n$ are operations of K yields

$$\begin{aligned}
k(h^{\boldsymbol{A}}(a_1, \ldots, a_m)) &= k(g^{\boldsymbol{A}}(f_1^{\boldsymbol{A}}(a_1, \ldots, a_m), \ldots, f_n^{\boldsymbol{A}}(a_1, \ldots, a_m))) \\
&= g^{\boldsymbol{B}}(k(f_1^{\boldsymbol{A}}(a_1, \ldots, a_m)), \ldots, k(f_n^{\boldsymbol{A}}(a_1, \ldots, a_m))) \\
&= g^{\boldsymbol{B}}(f_1^{\boldsymbol{B}}(k(a_1), \ldots, k(a_m)), \ldots, f_n^{\boldsymbol{B}}(k(a_1), \ldots, k(a_m))) \\
&= h^{\boldsymbol{B}}(k(a_1), \ldots, k(a_m)).
\end{aligned}$$

Thus, h is an operation of K. We now prove that h is defined by a formula. Since $g, f_1, \ldots, f_n \in \mathsf{imp}(\mathsf{K})$, there exist functional formulas $\psi, \varphi_1, \ldots, \varphi_n$ that define $g, f_1, \ldots, f_n$, respectively. Therefore, for all $\boldsymbol{A} \in \mathsf{K}$, $a_1, \ldots, a_m, b, b_1, \ldots, b_n, c \in A$, and $i \leqslant n$ we have

$$\langle a_1, \ldots, a_m \rangle \in \mathsf{dom}(f_i^{\boldsymbol{A}}) \text{ and } f_i^{\boldsymbol{A}}(a_1, \ldots, a_m) = b \iff \boldsymbol{A} \vDash \varphi_i(a_1, \ldots, a_m, b) \tag{6}$$

and

$$\langle b_1, \ldots, b_n \rangle \in \mathsf{dom}(g^{\boldsymbol{A}}) \text{ and } g^{\boldsymbol{A}}(b_1, \ldots, b_n) = c \iff \boldsymbol{A} \vDash \psi(b_1, \ldots, b_n, c). \tag{7}$$

Let

$$\chi(x_1, \ldots, x_m, y) = \exists z_1, \ldots, z_n \left(\psi(z_1, \ldots, z_n, y) \sqcap \prod_{i=1}^n \varphi_i(x_1, \ldots, x_m, z_i) \right). \tag{8}$$

We show that χ defines h. Consider $\boldsymbol{A} \in \mathsf{K}$ and $a_1, \ldots, a_m, c \in A$. By (8) we have $\boldsymbol{A} \vDash \chi(a_1, \ldots, a_m, c)$ if and only if there exist $b_1, \ldots, b_n \in A$ such that $\boldsymbol{A} \vDash \varphi_i(a_1, \ldots, a_m, b_i)$ for all $i \leqslant n$ and $\boldsymbol{A} \vDash \psi(b_1, \ldots, b_n, c)$. By (6) and (7), the latter condition is equivalent to

$$\langle a_1, \ldots, a_m \rangle \in \mathsf{dom}(f_i^{\boldsymbol{A}}) \text{ and } f_i^{\boldsymbol{A}}(a_1, \ldots, a_m) = b_i \text{ for every } i \leqslant n, \text{ and}$$
$$\langle b_1, \ldots, b_n \rangle \in \mathsf{dom}(g^{\boldsymbol{A}}) \text{ and } g^{\boldsymbol{A}}(b_1, \ldots, b_n) = c.$$

In turn, this amounts to $\langle a_1, \ldots, a_m \rangle \in \mathsf{dom}(h^{\boldsymbol{A}})$ and $h^{\boldsymbol{A}}(a_1, \ldots, a_m) = c$ by the definition of h. Therefore, χ defines h, and hence $h \in \mathsf{imp}(\mathsf{K})$.

Suppose now that in addition $g, f_1, \ldots, f_n \in \mathsf{imp}_{pp}(\mathsf{K})$. We can then assume that the formulas $\psi, \varphi_1, \ldots, \varphi_n$ that define $g, f_1, \ldots, f_n$ are pp formulas. Let χ be defined as in (8). Since ψ and each φ_i are pp formulas, they are of the form $\exists \vec{v} \psi'$ and $\exists \vec{u}_i \varphi_i'$, where $\vec{v}$ and $\vec{u}_i$ are finite sequences of variables and ψ' and φ_i' are finite conjunctions of equations. It is

straightforward to verify that χ is equivalent to the formula obtained by pulling out the existential quantifiers $\exists \vec{v}$ and $\exists \vec{u}_i$ from the conjunction in (8), and hence χ is equivalent to a pp formula. Thus, h is defined by a pp formula, and so $h \in \mathsf{imp}_{pp}(\mathsf{K})$. $\boxtimes$

Example 3.14 (Isbell's operations)**.** A fundamental example of implicit operations of the variety of monoids is due to Isbell [71][3]. More precisely, for each $n \geqslant 1$ let

$$\psi_n(z_1, \ldots, z_n, w_1, \ldots, w_n, x_1, \ldots, x_{2n+1}, y)$$

be the conjunction of the following equations in the language of monoids:

$$y \approx x_1 z_1$$
$$x_1 \approx w_1 x_2$$
$$x_{2i} z_i \approx x_{2i+1} z_{i+1} \text{ for } i = 1, \ldots, n-1$$
$$w_i x_{2i+1} \approx w_{i+1} x_{2(i+1)} \text{ for } i = 1, \ldots, n-1$$
$$x_{2n} z_n \approx x_{2n+1}$$
$$w_n x_{2n+1} \approx y.$$

Then let $\varphi_0(x, y) = x \approx y$ and for each $n \geqslant 1$,

$$\varphi_n(x_1, \ldots, x_{2n+1}, y) = \exists z_1, \ldots, z_n, w_1, \ldots, w_n \psi_n(z_1, \ldots, z_n, w_1, \ldots, w_n, x_1, \ldots, x_{2n+1}, y).$$

We refer to φ_n as to the n-th *Isbell's formula*. Notice that Isbell's formulas are pp formulas. It follows from [23, Lem. 4.4] that each Isbell's formula is functional in the variety of monoids. Whence, from Corollary 3.11 we deduce the following.

Theorem 3.15. *Every Isbell's formula defines an implicit operation of the variety of monoids.*

Isbell's formulas and the implicit operations they define, which we term *Isbell's operations*, will play a prominent role in the next sections (see Example 4.8, Isbell's Zigzag Theorem 4.9, and Proposition 14.4). $\boxtimes$

Example 3.16 (Reduced commutative rings)**.** Throughout this work, rings will be assumed to possess an identity element. Given an element a of a ring $\langle A; +, \cdot, -, 0, 1 \rangle$, we denote its multiplicative inverse (when it exists) by a^{-1}. By a *field* we understand a commutative ring $\boldsymbol{A}$ with $0 \neq 1$ such that a^{-1} exists for each $a \in A - \{0\}$. The class of fields will be denoted by Field.

A commutative ring $\boldsymbol{A}$ is said to be *reduced* when for each $a \in A$,

$$a \cdot a = 0 \text{ implies } a = 0$$

(see, e.g., [48]). The class of reduced commutative rings forms a quasivariety RCRing axiomatized relative to commutative rings by the quasiequation $x \cdot x \approx 0 \to x \approx 0$. We remark that this quasivariety is proper because the ring of integers $\mathbb{Z}$ is reduced, while its quotient $\mathbb{Z}_4$ is not. We rely on the next characterization of reduced commutative rings.

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[3]While these operations are traditionally considered in the variety of semigroups, it is straightforward to verify that all their properties relevant to our discussion continue to hold in the variety of monoids.

Theorem 3.17. *The class of reduced commutative rings coincides with the quasivariety generated by all fields.*

Proof. We will prove that

$$\mathsf{RCRing} = \mathbb{ISP}(\mathsf{Field}) = \mathbb{ISPP}_{\mathrm{u}}(\mathsf{Field}) = \mathbb{Q}(\mathsf{Field}).$$

The first equality above holds because a commutative ring is reduced if and only if it embeds into a direct product of fields (see, e.g., [91, Prop. 3.1]), the second because Field is an elementary class and, therefore, closed under $\mathbb{P}_{\mathrm{u}}$, and the third follows from Theorem 2.2.   ⊠

Let $\boldsymbol{A}$ be a field and $a \in A$. The *weak inverse* of a in $\boldsymbol{A}$ is the element

$$\mathsf{wi}(a) = \begin{cases} a^{-1} & \text{if } a \neq 0; \\ 0 & \text{if } a = 0. \end{cases} \tag{9}$$

We will prove the following.

Theorem 3.18. *There exists a unary implicit operation f of the quasivariety of reduced commutative rings such that $f^{\boldsymbol{A}}$ is total and $f^{\boldsymbol{A}}(a) = \mathsf{wi}(a)$ for all fields $\boldsymbol{A}$ and $a \in A$. Moreover, f can be defined by the conjunction of equations*

$$\varphi = (x^2 y \approx x) \sqcap (xy^2 \approx y).$$

Proof. We will prove that for every field $\boldsymbol{A}$ and $a, b \in A$ we have

$$\boldsymbol{A} \vDash \varphi(a, b) \iff b = \mathsf{wi}(a). \tag{10}$$

The implication from right to left is straightforward. To prove the reverse implication, suppose that $\boldsymbol{A} \vDash \varphi(a, b)$, i.e.,

$$a^2 b = a \ \text{ and } \ ab^2 = b.$$

We have two cases: either $a \neq 0$ or $a = 0$. First, suppose that $a \neq 0$. Then $\mathsf{wi}(a) = a^{-1}$. Therefore, from $a^2 b = a$ it follows that

$$b = a^{-2} a^2 b = a^{-2} a = a^{-1},$$

where a^{-2} abbreviates $(a^{-1})^2$. Whence $b = a^{-1} = \mathsf{wi}(a)$. Then we consider the case where $a = 0$. In this case, $\mathsf{wi}(a) = 0$. From $a = 0$ and $ab^2 = b$ it follows that $b = 0 = \mathsf{wi}(a)$. This establishes (10). Consequently, φ is functional in Field.

Recall from Theorem 3.17 that $\mathbb{Q}(\mathsf{Field}) = \mathsf{RCRing}$. As φ is a pp formula and is functional in Field, we can apply Corollary 3.11, obtaining that φ defines an implicit operation f of RCRing. Lastly, (10) ensures that $f^{\boldsymbol{A}}(a) = \mathsf{wi}(a)$ for every field $\boldsymbol{A}$ and $a \in A$.   ⊠

Example 3.19 (Distributive lattices)**.** Given a lattice $\boldsymbol{A}$ and $b, c \in A$, we let

$$[b, c] = \{a \in A : b \leqslant a \leqslant c\}.$$

Moreover, given $a, d \in A$, we say that d is a *complement* of a *relative* to the interval $[b, c]$ when

$$a \wedge d = b \ \text{ and } \ a \vee d = c.$$

In distributive lattices, relative complements are unique when they exist [16, Cor. IX.1]. Consequently, with every distributive lattice $\boldsymbol{A}$ we can associate a ternary partial function $f^{\boldsymbol{A}}$ on A with domain

$$\mathsf{dom}(f^{\boldsymbol{A}}) = \{\langle a, b, c\rangle \in A^3 : a \text{ has a complement relative to } [a \wedge b \wedge c, a \vee b \vee c] \text{ in } \boldsymbol{A}\},$$

defined for each $\langle a, b, c\rangle \in \mathsf{dom}(f^{\boldsymbol{A}})$ as

$$f^{\boldsymbol{A}}(a, b, c) = \text{the complement of } a \text{ relative to } [a \wedge b \wedge c, a \vee b \vee c] \text{ in } \boldsymbol{A}.$$

Let DL be the variety of distributive lattices. Then the sequence $f = \langle f^{\boldsymbol{A}} : \boldsymbol{A} \in \mathsf{DL}\rangle$ is a partial function on DL, which captures the idea of "taking relative complements".

This construction acquires special interest in the case of *bounded* distributive lattices. For let $\boldsymbol{A} = \langle A; \wedge, \vee, 0, 1\rangle$ be a bounded distributive lattice and $a, b \in A$. Then b is said to be a *complement* of a when

$$a \wedge b = 0 \quad \text{and} \quad a \vee b = 1$$

or, equivalently, when b is a complement of a relative to $[0, 1] = A$. With every bounded distributive lattice $\boldsymbol{A}$ we can associate a unary partial function $f^{\boldsymbol{A}}$ on A with domain

$$\mathsf{dom}(f^{\boldsymbol{A}}) = \{a \in A : a \text{ has a complement in } \boldsymbol{A}\},$$

defined for each $a \in \mathsf{dom}(f^{\boldsymbol{A}})$ as

$$f^{\boldsymbol{A}}(a) = \text{the complement of } a \text{ in } \boldsymbol{A}.$$

Let bDL be the variety of bounded distributive lattices. Then the sequence $f = \langle f^{\boldsymbol{A}} : \boldsymbol{A} \in \mathsf{bDL}\rangle$ is a partial function on bDL, which captures the idea of "taking complements".

Theorem 3.20. *The following conditions hold:*

(i) *taking relative complements is a ternary implicit operation of the variety of distributive lattices which, moreover, can be defined by the conjunction of equations*

$$\varphi = (x_1 \wedge y \approx x_1 \wedge x_2 \wedge x_3) \sqcap (x_1 \vee y \approx x_1 \vee x_2 \vee x_3);$$

(ii) *taking complements is a unary implicit operation of the variety of bounded distributive lattices which, moreover, can be defined by the conjunction of equations*

$$\psi = (x \wedge y \approx 0) \sqcap (x \vee y \approx 1).$$

Proof. (i): Observe that the partial function f on DL of "taking relative complements" can be defined by the conjunction of equations

$$\varphi(x_1, x_2, x_3, y) = (x_1 \wedge y \approx x_1 \wedge x_2 \wedge x_3) \sqcap (x_1 \vee y \approx x_1 \vee x_2 \vee x_3).$$

Therefore, from Theorem 3.9 it follows that f is an implicit operation of DL.

(ii): Analogous to the proof of (i).   ⊠

Example 3.21 (Absolute value). Let φ be the pp formula

$$\varphi(x, y) = \exists z_1, z_2, z_3, z_4((y \approx z_1^2 + z_2^2 + z_3^2 + z_4^2) \sqcap (x^2 \approx y^2))$$

in the language of rings. By Lagrange's four squares theorem any nonnegative integer can be written as the sum of four integer squares (see, e.g., [3, Thm. 11-3]). Therefore, for all

$a, b \in \mathbb{Z}$ we have that $\mathbb{Z} \vDash \varphi(a, b)$ if and only if $b \geqslant 0$ and $a^2 = b^2$, which happens exactly when $b = |a|$. In particular, φ is functional in $\mathbb{Z}$. Then Corollary 3.11 implies that φ defines an implicit operation f of the quasivariety K of rings generated by $\mathbb{Z}$ such that $f^{\mathbb{Z}}$ is the absolute value function.

While f is an implicit operation defined by a pp formula, it is interesting to observe that f cannot be defined by a conjunction of equations. Indeed, suppose, on the contrary, that f is defined on K by a conjunction of equations ψ. In the variety of commutative rings, each equation in variables x and y is equivalent to an equation of the form $p(x, y) \approx 0$, where $p(x, y)$ is a polynomial with integer coefficients. So, we can assume that

$$\psi = \prod_{i=1}^{n} p_i(x, y) \approx 0,$$

where each $p_i(x, y)$ is a polynomial with integer coefficients. Since ψ defines the absolute value function on $\mathbb{Z}$, we have that $\mathbb{Z} \vDash \psi(a, a)$ for every nonnegative integer a. Thus, for every i the polynomial $p_i(x, x)$ in a single variable x vanishes on every nonnegative integer. We recall that the only polynomial in a single variable with rational coefficients that has infinitely many roots is the zero polynomial (see, e.g., [4, Prop. 12.2.20]). Then $p_i(x, x)$ is the zero polynomial, and hence $p_i(-1, -1) = 0$ for every i. We conclude that $\mathbb{Z} \vDash \psi(-1, -1)$, which contradicts that $f^{\mathbb{Z}}(-1) = |-1| = 1$. Therefore, f cannot be defined by a conjunction of equations.

4. Existential elimination

The idea of interpolating a given family of functions by simpler ones plays a fundamental role in mathematics. For instance, a well-known theorem of Lagrange states that every finite set of pairs of real numbers can be interpolated by a polynomial with real coefficients (see, e.g., [101, Thm. 6.1]). In this section, we will establish a general interpolation theorem for the implicit operations of a quasivariety K.

More precisely, recall from Corollary 3.10 that every implicit operation of K can be obtained by gluing together finitely many implicit operations defined by pp formulas. The main result of this section states that, if K has the amalgamation property, the study of its implicit operations can be further simplified by observing that each implicit operation defined by a pp formula is interpolated by one defined by a conjunction of equations (Theorem 4.3). We term this phenomenon *existential elimination* because conjunctions of equations are obtained by removing existential quantifiers from pp formulas.

As we mentioned, the reason for existential elimination is the amalgamation property, whose definition we proceed to recall.

Definition 4.1. A class of algebras K is said to have the *amalgamation property* when for every pair of embeddings $h_1 \colon \boldsymbol{A} \to \boldsymbol{B}$ and $h_2 \colon \boldsymbol{A} \to \boldsymbol{C}$ with $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C} \in \mathsf{K}$ there exists a pair of embeddings $g_1 \colon \boldsymbol{B} \to \boldsymbol{D}$ and $g_2 \colon \boldsymbol{C} \to \boldsymbol{D}$ with $\boldsymbol{D} \in \mathsf{K}$ such that $g_1 \circ h_1 = g_2 \circ h_2$.
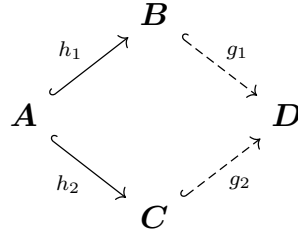
Furthermore, we will rely on the following notion of interpolation.

Definition 4.2. Let $\mathcal{F} \cup \{g\}$ be a family of n-ary implicit operations of a class of algebras K. We say that g is *interpolated* by $\mathcal{F}$ when for all $\boldsymbol{A} \in \mathsf{K}$ and $\langle a_1, \ldots, a_n \rangle \in \mathsf{dom}(g^{\boldsymbol{A}})$ there exists $f \in \mathcal{F}$ such that

$$\langle a_1, \ldots, a_n \rangle \in \mathsf{dom}(f^{\boldsymbol{A}}) \quad \text{and} \quad f^{\boldsymbol{A}}(a_1, \ldots, a_n) = g^{\boldsymbol{A}}(a_1, \ldots, a_n).$$

When $\mathcal{F} = \{f\}$, we often say that g *is interpolated by* f.

Given a class of algebras K, we denote by $\mathsf{imp}_{eq}(\mathsf{K})$ the set of implicit operations of K defined by a conjunction of equations. When $\mathsf{K} = \{\boldsymbol{A}\}$, we often write $\mathsf{imp}_{eq}(\boldsymbol{A})$ instead of $\mathsf{imp}_{eq}(\mathsf{K})$.[4] The aim of this section is to establish the following interpolation result.

Theorem 4.3. *The following conditions hold for a quasivariety K with the amalgamation property:*

   (i) *every member of $\mathsf{imp}_{pp}(\mathsf{K})$ can be interpolated by some member of $\mathsf{imp}_{eq}(\mathsf{K})$;*
  (ii) *every member of $\mathsf{imp}(\mathsf{K})$ can be interpolated by a finite subset of $\mathsf{imp}_{eq}(\mathsf{K})$.*

As the variety of monoids lacks the amalgamation property (see, e.g., [76, p. 100][5]), it falls outside the scope of Theorem 4.3. This is reflected by the fact that this variety possesses implicit operations defined by pp formulas that cannot be interpolated by any implicit operation defined by a conjunction of equations, an example being every n-th Isbell's operation for $n \geqslant 1$ (see Example 4.15). On the other hand, the variety of distributive lattices has the amalgamation property (see, e.g., [7, Thm. VII.8.4]) and, therefore, each of its implicit operations defined by pp formulas can be interpolated by one defined by a conjunction of equations.

The rest of this section is devoted to the proof of Theorem 4.3. The first ingredient of the proof is the following concept, introduced in [71] (see also [6]).

Definition 4.4. Let K be a class of algebras and $\boldsymbol{A} \leqslant \boldsymbol{B}$ a pair of $\mathscr{L}_{\mathsf{K}}$-algebras. The *dominion* of $\boldsymbol{A}$ in $\boldsymbol{B}$ relative to K is the set

$$\mathsf{d}_{\mathsf{K}}(\boldsymbol{A}, \boldsymbol{B}) = \{b \in B : \text{for each pair of homomorphisms } g, h \colon \boldsymbol{B} \to \boldsymbol{C} \text{ with } \boldsymbol{C} \in \mathsf{K},$$
$$\text{if } g{\restriction}_A = h{\restriction}_A, \text{ then } g(b) = h(b)\}.$$

---

[4]Although we will not rely on this fact, we will show in Example 4.16 that $\mathsf{imp}_{eq}(\mathsf{K})$ need not be closed under composition (cf. Proposition 3.13).

[5]As observed in [76, p. 108], the failure of the amalgamation property for the variety of monoids can be seen as a consequence of the corresponding result for the variety of semigroups proved in [75] (see also [37, Exa. 1]).

It is straightforward to verify that $\mathsf{d}_\mathsf{K}(\boldsymbol{A}, \boldsymbol{B})$ is the universe of a subalgebra of $\boldsymbol{B}$ that contains A. It follows immediately from its definition that $\mathsf{d}_\mathsf{K}(\boldsymbol{A}, \boldsymbol{B})$ is the intersection of all the equalizers of pairs of homomorphisms $g, h \colon \boldsymbol{B} \to \boldsymbol{C}$ with $\boldsymbol{C} \in \mathsf{K}$ that agree on A, where we recall that the equalizer of g and h is $\{b \in B : g(b) = h(b)\}$. Moreover, when K is closed under direct products, it turns out that $\mathsf{d}_\mathsf{K}(\boldsymbol{A}, \boldsymbol{B})$ is itself the equalizer of a pair of homomorphisms from $\boldsymbol{B}$ into an algebra of K that agree on A.

We rely on the following fact.

Proposition 4.5. *Let* K *be a class of algebras,* $\boldsymbol{A} \leqslant \boldsymbol{B} \in \mathsf{K}$*, and* $\boldsymbol{A}' \leqslant \boldsymbol{B}' \in \mathsf{K}$*. If* $g \colon \boldsymbol{B} \to \boldsymbol{B}'$ *is a homomorphism with* $g[A] \subseteq A'$*, then* $g[\mathsf{d}_\mathsf{K}(\boldsymbol{A}, \boldsymbol{B})] \subseteq \mathsf{d}_\mathsf{K}(\boldsymbol{A}', \boldsymbol{B}')$*.*

Proof. Let $b \in g[\mathsf{d}_\mathsf{K}(\boldsymbol{A}, \boldsymbol{B})]$. Then there exists $a \in \mathsf{d}_\mathsf{K}(\boldsymbol{A}, \boldsymbol{B})$ with $g(a) = b$. Suppose that $h_1, h_2 \colon \boldsymbol{B}' \to \boldsymbol{C}$ are homomorphisms with $\boldsymbol{C} \in \mathsf{K}$ and $h_1 {\restriction}_{A'} = h_2 {\restriction}_{A'}$. Since $g[A] \subseteq A'$, the homomorphisms $h_1 \circ g, h_2 \circ g \colon \boldsymbol{B} \to \boldsymbol{C}$ satisfy $(h_1 \circ g){\restriction}_A = (h_2 \circ g){\restriction}_A$. As $a \in \mathsf{d}_\mathsf{K}(\boldsymbol{A}, \boldsymbol{B})$, it follows that $h_1(g(a)) = h_2(g(a))$. Therefore, $b = g(a) \in \mathsf{d}_\mathsf{K}(\boldsymbol{A}', \boldsymbol{B}')$, as desired. $\boxtimes$

As an immediate consequence of the previous proposition we obtain the following result, where, for $\boldsymbol{A} \leqslant \boldsymbol{B}$ and $\theta \in \mathsf{Con}(\boldsymbol{B})$, we denote the subalgebra of $\boldsymbol{B}/\theta$ with universe $\{a/\theta : a \in A\}$ by $\boldsymbol{A}/\theta$.

Corollary 4.6. *The following conditions hold for every class* K *of algebras and* $\boldsymbol{A} \leqslant \boldsymbol{B} \in \mathsf{K}$*.*
  (i) *If* $\boldsymbol{A} \leqslant \boldsymbol{A}' \leqslant \boldsymbol{B}' \in \mathsf{K}$ *and* $\boldsymbol{B} \leqslant \boldsymbol{B}'$*, then* $\mathsf{d}_\mathsf{K}(\boldsymbol{A}, \boldsymbol{B}) \subseteq \mathsf{d}_\mathsf{K}(\boldsymbol{A}', \boldsymbol{B}')$*.*
  (ii) *If* $\theta \in \mathsf{Con}(\boldsymbol{B})$ *and* $\boldsymbol{B}/\theta \in \mathsf{K}$*, then* $b \in \mathsf{d}_\mathsf{K}(\boldsymbol{A}, \boldsymbol{B})$ *implies* $b/\theta \in \mathsf{d}_\mathsf{K}(\boldsymbol{A}/\theta, \boldsymbol{B}/\theta)$*.*

Proof. Both statements follow from Proposition 4.5: for (i) let $g \colon \boldsymbol{B} \to \boldsymbol{B}'$ be the inclusion map, and for (ii) let $g \colon \boldsymbol{B} \to \boldsymbol{B}/\theta$ be the canonical surjection. $\boxtimes$

In general, the task of describing dominions for concrete classes of algebras may be hard. However, in some cases a tangible description is within reach.

Example 4.7 (Distributive lattices). In the variety DL of distributive lattices dominions can be described as follows (see [105, Thm. 2.4]). For each $\boldsymbol{A} \leqslant \boldsymbol{B} \in \mathsf{DL}$ the dominion $\mathsf{d}_\mathsf{DL}(\boldsymbol{A}, \boldsymbol{B})$ is the least subset C of B containing A and closed under meets and joins such that for all $a, b, c \in C$ and $d \in B$,

$$\text{if } d \text{ is the complement of } a \text{ relative to } [b, c], \text{ then } d \in C. \qquad \boxtimes$$

Example 4.8 (Monoids). For each $n \in \mathbb{N}$ let $\varphi_n(x_1, \dots, x_{2n+1}, y)$ be the n-th Isbell's formula defined in Example 3.14. Dominions in the varieties of monoids and commutative monoids are described by the following classic result (see [68, Thm. 1.2]).

Isbell's Zigzag Theorem 4.9. *Let* K *be the variety of monoids or the variety of commutative monoids. For each* $\boldsymbol{A} \leqslant \boldsymbol{B} \in \mathsf{K}$ *and* $b \in B$ *we have*

$$b \in \mathsf{d}_\mathsf{K}(\boldsymbol{A}, \boldsymbol{B}) \iff \boldsymbol{B} \vDash \varphi_n(a_1, \dots, a_{2n+1}, b) \text{ for some } n \in \mathbb{N} \text{ and } a_1, \dots, a_{2n+1} \in A.$$

This theorem was originally stated for the variety of semigroups in [71]. Similar descriptions of dominions have been obtained for the varieties of commutative semigroups, rings, and commutative rings (see [69, 72]). $\boxtimes$

We will make use of the following description of dominions in terms of implicit operations (see [6, Thm. 1] and [23, Thm. 3.2]).

Theorem 4.10. *Let* K *be an elementary class. For every* $\boldsymbol{A} \leqslant \boldsymbol{B} \in \mathsf{K}$ *we have*

$$\mathsf{d}_{\mathsf{K}}(\boldsymbol{A}, \boldsymbol{B}) = \{b \in B : \text{there exist } f \in \mathsf{imp}_{pp}(\mathsf{K}) \text{ and } \langle a_1, \ldots, a_n \rangle \in \mathsf{dom}(f^{\boldsymbol{B}}) \cap A^n$$
$$\text{such that } f^{\boldsymbol{B}}(a_1, \ldots, a_n) = b\}.$$

As shown in the next result, the amalgamation property simplifies the task of describing dominions.

Proposition 4.11. *Let* K *be a class of algebras closed under finite direct products with the amalgamation property and* $\boldsymbol{A} \leqslant \boldsymbol{B} \leqslant \boldsymbol{C}$ *with* $\boldsymbol{B}, \boldsymbol{C} \in \mathsf{K}$. *Then*

$$\mathsf{d}_{\mathsf{K}}(\boldsymbol{A}, \boldsymbol{B}) = \mathsf{d}_{\mathsf{K}}(\boldsymbol{A}, \boldsymbol{C}) \cap B.$$

Proof. We first prove the inclusion from left to right. As $\boldsymbol{B} \leqslant \boldsymbol{C}$, from Corollary 4.6(i) it follows that $\mathsf{d}_{\mathsf{K}}(\boldsymbol{A}, \boldsymbol{B}) \subseteq \mathsf{d}_{\mathsf{K}}(\boldsymbol{A}, \boldsymbol{C})$. Since $\mathsf{d}_{\mathsf{K}}(\boldsymbol{A}, \boldsymbol{B}) \subseteq B$ by definition, we obtain that $\mathsf{d}_{\mathsf{K}}(\boldsymbol{A}, \boldsymbol{B}) \subseteq \mathsf{d}_{\mathsf{K}}(\boldsymbol{A}, \boldsymbol{C}) \cap B$.

Next we prove the inclusion from right to left. Suppose, with a view to contradiction, that this inclusion fails. Then there exists $b \in B$ such that

$$b \in \mathsf{d}_{\mathsf{K}}(\boldsymbol{A}, \boldsymbol{C}) \quad \text{and} \quad b \notin \mathsf{d}_{\mathsf{K}}(\boldsymbol{A}, \boldsymbol{B}). \tag{11}$$

As $b \in B$, the right hand side of the above display implies that there exists a pair of homomorphisms $f_1, f_2 \colon \boldsymbol{B} \to \boldsymbol{D}$ with $\boldsymbol{D} \in \mathsf{K}$ such that

$$f_1{\restriction}_A = f_2{\restriction}_A \quad \text{and} \quad f_1(b) \neq f_2(b). \tag{12}$$

We may assume that f_1 and f_2 are embeddings. Otherwise, we replace each f_i by the embedding $f_i^* \colon \boldsymbol{B} \to \boldsymbol{D} \times \boldsymbol{B}$ defined as $f_i^*(c) = \langle f_i(c), c \rangle$ for every $c \in B$. Observe that $\boldsymbol{D} \times \boldsymbol{B} \in \mathsf{K}$ because $\boldsymbol{B}, \boldsymbol{D} \in \mathsf{K}$ and K is closed under finite direct products by assumption. Furthermore, from (12) and the definition of f_1^* and f_2^* it follows that $f_1^*{\restriction}_A = f_2^*{\restriction}_A$ and $f_1^*(b) \neq f_2^*(b)$. Consequently, from now on we will assume that f_1 and f_2 are embeddings.

Recall from the assumptions that $\boldsymbol{B} \leqslant \boldsymbol{C}$. Then let $i \colon \boldsymbol{B} \to \boldsymbol{C}$ be the inclusion map, which is always an embedding. As $i \colon \boldsymbol{B} \to \boldsymbol{C}$ and $f_1 \colon \boldsymbol{B} \to \boldsymbol{D}$ are a pair of embeddings with $\boldsymbol{B}, \boldsymbol{C}, \boldsymbol{D} \in \mathsf{K}$, we can apply the assumption that K has the amalgamation property, obtaining a pair of embeddings $g_1 \colon \boldsymbol{C} \to \boldsymbol{E}$ and $g_2 \colon \boldsymbol{D} \to \boldsymbol{E}$ with $\boldsymbol{E} \in \mathsf{K}$ such that

$$g_1 \circ i = g_2 \circ f_1. \tag{13}$$

Since i, g_1, g_2, and f_1 are embeddings, so are the compositions $g_1 \circ i \colon \boldsymbol{B} \to \boldsymbol{E}$ and $g_2 \circ f_2 \colon \boldsymbol{B} \to \boldsymbol{E}$. Together with $\boldsymbol{E} \in \mathsf{K}$, this allows us to apply the amalgamation property of K, obtaining a pair of embeddings $h_1, h_2 \colon \boldsymbol{E} \to \boldsymbol{F}$ with $\boldsymbol{F} \in \mathsf{K}$ such that

$$h_1 \circ g_1 \circ i = h_2 \circ g_2 \circ f_2. \tag{14}$$
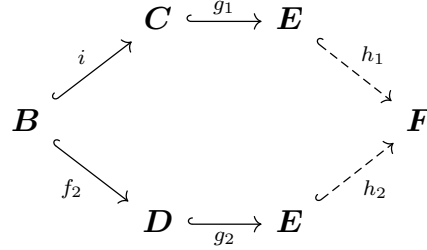
As i is the inclusion map from $\boldsymbol{B}$ to $\boldsymbol{C}$, from (14) and (13) it follows that for each $c \in B \subseteq C$,

$$h_1 \circ g_1(c) = h_2 \circ g_2 \circ f_2(c) \quad \text{and} \quad h_2 \circ g_1(c) = h_2 \circ g_2 \circ f_1(c). \tag{15}$$

By the left hand side of (12) we have $f_1(a) = f_2(a)$ for every $a \in A$. Together with (15) and $A \subseteq B$, this yields that for every $a \in A$,

$$h_1 \circ g_1(a) = h_2 \circ g_2 \circ f_2(a) = h_2 \circ g_2 \circ f_1(a) = h_2 \circ g_1(a).$$

Hence, $(h_1 \circ g_1){\restriction}_A = (h_2 \circ g_1){\restriction}_A$. On the other hand, recall that $f_1(b) \neq f_2(b)$ by the right hand side of (12). Since $h_2 \circ g_2 \colon \boldsymbol{C} \to \boldsymbol{F}$ is an embedding (because so are h_2 and g_2), we obtain $h_2 \circ g_2 \circ f_1(b) \neq h_2 \circ g_2 \circ f_2(b)$. Together with $b \in B$ and (15), this implies $h_1 \circ g_1(b) \neq h_2 \circ g_1(b)$. Since $h_2 \circ g_1 \colon \boldsymbol{C} \to \boldsymbol{F}$ is a homomorphism with $\boldsymbol{F} \in \mathsf{K}$ such that $(h_1 \circ g_1){\restriction}_A = (h_2 \circ g_1){\restriction}_A$, we conclude that $b \notin \mathsf{d}_\mathsf{K}(\boldsymbol{A}, \boldsymbol{C})$, a contradiction with the left hand side of (11). $\boxtimes$

The second ingredient of the proof of Theorem 4.3 is the following construction, which associates an algebra with every pp formula. We denote the set of variables occurring in a formula φ by $Var(\varphi)$. For instance, if $\varphi = \exists x(x + y \approx x)$, then $Var(\varphi) = \{x, y\}$. Moreover, we denote the term algebra with variables in $Var(\varphi)$ by $\boldsymbol{T}(Var(\varphi))$ and let

$$\ulcorner \varphi \urcorner = \{\langle t_1, t_2 \rangle : t_1 \approx t_2 \text{ is an equation occurring in } \varphi\}.$$

Observe that $\ulcorner \varphi \urcorner \subseteq T(Var(\varphi)) \times T(Var(\varphi))$.

Definition 4.12. Let K be a quasivariety. With every pp formula φ we associate the algebra

$$\boldsymbol{T}_\mathsf{K}(\varphi) = \boldsymbol{T}(Var(\varphi))/\theta(\varphi), \text{ where } \theta(\varphi) = \mathsf{Cg}_\mathsf{K}^{\boldsymbol{T}(Var(\varphi))}(\ulcorner \varphi \urcorner).$$

Recall that, when φ defines an implicit operation of K, we denote this operation by

$$\varphi^\mathsf{K} = \langle \varphi^{\boldsymbol{A}} : \boldsymbol{A} \in \mathsf{K} \rangle.$$

We rely on the next observation.

Proposition 4.13. *Let* $\varphi(x_1, \ldots, x_n, y)$ *be a pp formula that defines an implicit operation of a quasivariety* K. *Then the following conditions hold:*

(i) $\boldsymbol{T}_\mathsf{K}(\varphi)$ *is a finitely presented member of* K *such that*

$$\langle x_1/\theta(\varphi), \ldots, x_n/\theta(\varphi) \rangle \in \mathsf{dom}(\varphi^{\boldsymbol{T}_\mathsf{K}(\varphi)}) \quad \text{and} \quad \varphi^{\boldsymbol{T}_\mathsf{K}(\varphi)}(x_1/\theta(\varphi), \ldots, x_n/\theta(\varphi)) = y/\theta(\varphi);$$

(ii) *for all $\boldsymbol{A} \in \mathsf{K}$ and $\langle a_1, \ldots, a_n \rangle \in \mathsf{dom}(\varphi^{\boldsymbol{A}})$ there exists a homomorphism $h\colon \boldsymbol{T}_{\mathsf{K}}(\varphi) \to \boldsymbol{A}$ such that*

$$h(x_i/\theta(\varphi)) = a_i \text{ for each } i \leqslant n \text{ and } h(y/\theta(\varphi)) = \varphi^{\boldsymbol{A}}(a_1, \ldots, a_n).$$

Proof. Since φ is a pp formula, it is of the form $\exists z_1, \ldots, z_m \psi$, where ψ is a finite conjunction of equations. Therefore,

$$\varphi = \exists z_1, \ldots, z_m \prod_{i \leqslant k} t_i \approx s_i, \tag{16}$$

where t_i and s_i are terms in variables $x_1, \ldots, x_n, y, z_1, \ldots, z_m$.

(i): The algebra $\boldsymbol{T}_{\mathsf{K}}(\varphi)$ is a finitely presented member of K by definition. Therefore, it only remains to show that for each $i \leqslant k$,

$$\boldsymbol{T}_{\mathsf{K}}(\varphi) \vDash \varphi(x_1/\theta(\varphi), \ldots, x_n/\theta(\varphi), y/\theta(\varphi)).$$

In view of (16), it will be enough to prove

$$\boldsymbol{T}_{\mathsf{K}}(\varphi) \vDash \left(\prod_{i \leqslant k} t_i \approx s_i \right)(x_1/\theta(\varphi), \ldots, x_n/\theta(\varphi), y/\theta(\varphi), z_1/\theta(\varphi), \ldots, z_m/\theta(\varphi)).$$

Let $i \leqslant k$. From the definitions of $\ulcorner \varphi \urcorner$ and $\theta(\varphi)$ it follows that $\langle t_i, s_i \rangle \in \ulcorner \varphi \urcorner \subseteq \theta(\varphi)$. Consequently,

$$\begin{aligned}
t_i^{\boldsymbol{T}_{\mathsf{K}}(\varphi)} &(x_1/\theta(\varphi), \ldots, x_n/\theta(\varphi), y/\theta(\varphi), z_1/\theta(\varphi), \ldots, z_m/\theta(\varphi)) \\
&= t_i(x_1, \ldots, x_n, y, z_1, \ldots, z_m)/\theta(\varphi) \\
&= s_i(x_1, \ldots, x_n, y, z_1, \ldots, z_m)/\theta(\varphi) \\
&= s_i^{\boldsymbol{T}_{\mathsf{K}}(\varphi)}(x_1/\theta(\varphi), \ldots, x_n/\theta(\varphi), y/\theta(\varphi), z_1/\theta(\varphi), \ldots, z_m/\theta(\varphi)).
\end{aligned}$$

(ii): Let $\boldsymbol{A} \in \mathsf{K}$ and $\langle a_1, \ldots, a_n \rangle \in \mathsf{dom}(\varphi^{\boldsymbol{A}})$. Then $\boldsymbol{A} \vDash \varphi(a_1, \ldots, a_n, \varphi^{\boldsymbol{A}}(a_1, \ldots, a_n))$. In view of (16), there exist $b_1, \ldots, b_m \in A$ such that for each $i \leqslant k$,

$$t_i^{\boldsymbol{A}}(a_1, \ldots, a_n, \varphi^{\boldsymbol{A}}(a_1, \ldots, a_n), b_1, \ldots, b_m) = s_i^{\boldsymbol{A}}(a_1, \ldots, a_n, \varphi^{\boldsymbol{A}}(a_1, \ldots, a_n), b_1, \ldots, b_m). \tag{17}$$

Now, let $g\colon \boldsymbol{T}(Var(\varphi)) \to \boldsymbol{A}$ be the unique homomorphism such that

$$g(x_i) = a_i \text{ for each } i \leqslant n, \quad g(y) = \varphi^{\boldsymbol{A}}(a_1, \ldots, a_n), \quad \text{and} \quad g(z_j) = b_j \text{ for each } j \leqslant m. \tag{18}$$

From the above display and (17) it follows that $\ulcorner \varphi \urcorner \subseteq \mathsf{Ker}(g)$. As $\boldsymbol{A} \in \mathsf{K}$ by assumption, we also have $\mathsf{Ker}(g) \in \mathsf{Con}_{\mathsf{K}}(\boldsymbol{T}(Var(\varphi)))$. Consequently,

$$\theta(\varphi) = \mathrm{Cg}_{\mathsf{K}}^{\boldsymbol{T}(\varphi)}(\ulcorner \varphi \urcorner) \subseteq \mathsf{Ker}(g).$$

Since $\boldsymbol{T}_{\mathsf{K}}(\varphi) = \boldsymbol{T}(Var(\varphi))/\theta(\varphi)$, we can apply Proposition 2.6 to the above display, obtaining a homomorphism $h\colon \boldsymbol{T}_{\mathsf{K}}(\varphi) \to \boldsymbol{A}$ defined for every $t \in T(Var(\varphi))$ as $h(t/\theta(\varphi)) = g(t)$. Together with (18), this yields $h(x_i/\theta(\varphi)) = a_i$ for each $i \leqslant n$ and $h(y/\theta(\varphi)) = \varphi^{\boldsymbol{A}}(a_1, \ldots, a_n)$. $\boxtimes$

We are now ready to prove Theorem 4.3.

Proof. (i): Let f be an implicit operation of K defined by a pp formula $\varphi(x_1, \ldots, x_n, y)$. Consider the algebra

$$\boldsymbol{A} = \mathsf{Sg}^{\boldsymbol{T}_{\mathsf{K}}(\varphi)}(x_1/\theta(\varphi), \ldots, x_n/\theta(\varphi)).$$

By Proposition 4.13(i) we have

$$\langle x_1/\theta(\varphi), \ldots, x_n/\theta(\varphi) \rangle \in \mathsf{dom}(f^{\boldsymbol{T}_{\mathsf{K}}(\varphi)}) \cap A^n \quad \text{and} \quad f^{\boldsymbol{T}_{\mathsf{K}}(\varphi)}(x_1/\theta(\varphi), \ldots, x_n/\theta(\varphi)) = y/\theta(\varphi).$$

Together with Theorem 4.10, this yields $y/\theta(\varphi) \in \mathsf{d}_{\mathsf{K}}(\boldsymbol{A}, \boldsymbol{T}_{\mathsf{K}}(\varphi))$. Now, let

$$\boldsymbol{B} = \mathsf{Sg}^{\boldsymbol{T}_{\mathsf{K}}(\varphi)}(x_1/\theta(\varphi), \ldots, x_n/\theta(\varphi), y/\theta(\varphi)).$$

As $\boldsymbol{A} \leqslant \boldsymbol{B} \leqslant \boldsymbol{T}_{\mathsf{K}}(\varphi) \in \mathsf{K}$ and K is a quasivariety with the amalgamation property, we can apply Proposition 4.11 to $y/\theta(\varphi) \in \mathsf{d}_{\mathsf{K}}(\boldsymbol{A}, \boldsymbol{T}_{\mathsf{K}}(\varphi)) \cap B$, obtaining $y/\theta(\varphi) \in \mathsf{d}_{\mathsf{K}}(\boldsymbol{A}, \boldsymbol{B})$. By Theorem 4.10 there exist an m-ary $g \in \mathsf{imp}_{pp}(\mathsf{K})$ and $\langle a_1, \ldots, a_m \rangle \in \mathsf{dom}(g^{\boldsymbol{B}}) \cap A^m$ such that $g^{\boldsymbol{B}}(a_1, \ldots, a_m) = y/\theta(\varphi)$.

Since $g \in \mathsf{imp}_{pp}(\mathsf{K})$, there exists a formula $\exists z_1, \ldots, z_k \psi(x_1, \ldots, x_m, y, z_1, \ldots, z_k)$, where ψ is a conjunction of equations, defining g. Together with $\langle a_1, \ldots, a_m \rangle \in \mathsf{dom}(g^{\boldsymbol{B}})$ and $g^{\boldsymbol{B}}(a_1, \ldots, a_n) = y/\theta(\varphi)$, this guarantees the existence of $b_1, \ldots, b_k \in B$ such that

$$\boldsymbol{B} \vDash \psi(a_1, \ldots, a_m, y/\theta(\varphi), b_1, \ldots, b_k). \tag{19}$$

As $a_1, \ldots, a_m \in A$, $\boldsymbol{A} \leqslant \boldsymbol{B}$, and $\boldsymbol{A}$ is generated by $x_1/\theta(\varphi), \ldots, x_n/\theta(\varphi)$ by definition, for each $i \leqslant m$ there exists a term $t_i(x_1, \ldots, x_n)$ such that $a_i = t_i^{\boldsymbol{B}}(x_1/\theta(\varphi), \ldots, x_n/\theta(\varphi))$. Similarly, as $b_1, \ldots, b_k \in B$ and $\boldsymbol{B}$ is generated by $x_1/\theta(\varphi), \ldots, x_n/\theta(\varphi), y/\theta(\varphi)$ by definition, for each $j \leqslant k$ there exists a term $s_j(x_1, \ldots, x_n, y)$ such that $b_j = s_j^{\boldsymbol{B}}(x_1/\theta(\varphi), \ldots, x_n/\theta(\varphi), y/\theta(\varphi))$.

We consider the formula

$$\gamma = \psi(t_1(x_1, \ldots, x_n), \ldots, t_m(x_1, \ldots, x_n), y, s_1(x_1, \ldots, x_n, y), \ldots, s_k(x_1, \ldots, x_n, y)).$$

Notice that γ is a conjunction of equations because so is ψ. Then observe that the formula $\exists z_1, \ldots, z_k \psi(x_1, \ldots, x_m, y, z_1, \ldots, z_k)$ is functional in K because it defines g. Together with the definition of γ, this guarantees that γ is also functional in K. Hence, γ defines some $h \in \mathsf{imp}_{eq}(\mathsf{K})$ by Corollary 3.11. Therefore, to conclude the proof, it suffices to show that h interpolates f.

First, observe that from (19) and the definitions of γ and $t_1, \ldots, t_m, s_1, \ldots, s_k$ it follows that

$$\boldsymbol{B} \vDash \gamma(x_1/\theta(\varphi), \ldots, x_n/\theta(\varphi), y/\theta(\varphi)). \tag{20}$$

As γ defines h, this yields

$$\langle x_1/\theta(\varphi), \ldots, x_n/\theta(\varphi) \rangle \in \mathsf{dom}(h^{\boldsymbol{B}}) \quad \text{and} \quad h^{\boldsymbol{B}}(x_1/\theta(\varphi), \ldots, x_n/\theta(\varphi)) = y/\theta(\varphi).$$

We are now ready to prove that h interpolates f. To this end, consider $\boldsymbol{C} \in \mathsf{K}$ and $c_1, \ldots, c_n, d \in C$ such that $\langle c_1, \ldots, c_n \rangle \in \mathsf{dom}(f^{\boldsymbol{C}})$ and $f^{\boldsymbol{C}}(c_1, \ldots, c_n) = d$. As f is defined by the pp formula $\varphi(x_1, \ldots, x_n, y)$ by assumption, from Proposition 4.13(ii) it follows that there exists a homomorphism $e \colon \boldsymbol{T}_{\mathsf{K}}(\varphi) \to \boldsymbol{C}$ such that

$$e(x_i/\theta(\varphi)) = c_i \text{ for each } i \leqslant n \quad \text{and} \quad e(y/\theta(\varphi)) = d.$$

Since $\boldsymbol{B} \leqslant \boldsymbol{T}_{\mathsf{K}}(\varphi)$ and $x_1/\theta(\varphi), \ldots, x_n/\theta(\varphi), y/\theta(\varphi) \in B$ by the definition of $\boldsymbol{B}$, the above display still holds if we restrict e to a homomorphism $e \colon \boldsymbol{B} \to \boldsymbol{C}$. As h is an implicit

operation of K, it is preserved by homomorphism between members of K and, in particular, by e. Together with (20) and the above display, this yields

$$\langle c_1, \ldots, c_n \rangle = \langle e(x_1/\theta(\varphi)), \ldots, e(x_n/\theta(\varphi)) \rangle \in \mathsf{dom}(h^{\boldsymbol{C}})$$

and

$$d = e(y/\theta(\varphi)) = e(h^{\boldsymbol{B}}(x_1/\theta(\varphi), \ldots, x_n/\theta(\varphi))) = h^{\boldsymbol{C}}(e(x_1/\theta(\varphi)), \ldots, e(x_n/\theta(\varphi)))$$
$$= h^{\boldsymbol{C}}(c_1, \ldots, c_n).$$

Since $f^{\boldsymbol{C}}(c_1, \ldots, c_n) = d$, we conclude that h interpolates f.

(ii): Immediate consequence of (i) and Corollary 3.10.   ⊠

From Theorems 4.3(i) and 4.10 we deduce the following (for a similar observation, see [6, Thm. 1*, p. 475] and [5]).

Corollary 4.14. *Let K be a quasivariety with the amalgamation property. For every $\boldsymbol{A} \leqslant \boldsymbol{B} \in \mathsf{K}$ we have*

$$\mathsf{d}_{\mathsf{K}}(\boldsymbol{A}, \boldsymbol{B}) = \{b \in B : \text{there exist } f \in \mathsf{imp}_{eq}(\mathsf{K}) \text{ and } \langle a_1, \ldots, a_n \rangle \in \mathsf{dom}(f^{\boldsymbol{B}}) \cap A^n$$
$$\text{such that } f^{\boldsymbol{B}}(a_1, \ldots, a_n) = b\}.$$

We close this section with two examples. The first shows that Isbell's operations cannot be interpolated by any implicit operation defined by a conjunction of equations in the variety of commutative monoids, and the second shows that condition (i) of Theorem 4.3 cannot be improved by requiring the equality $\mathsf{imp}_{pp}(\mathsf{K}) = \mathsf{imp}_{eq}(\mathsf{K})$.

Example 4.15 (Isbell's operations)**.** For every positive n let f_n be the n-th Isbell's operation, viewed as an implicit operation of the variety Mon of monoids (see Example 3.14 and Theorem 3.15). Then $f_n \in \mathsf{imp}_{pp}(\mathsf{Mon})$. We will prove that f_n cannot be interpolated by any member of $\mathsf{imp}_{eq}(\mathsf{Mon})$. Suppose the contrary, with a view to contradiction. Then f_n is interpolated by some $g \in \mathsf{imp}_{eq}(\mathsf{Mon})$. Consider the commutative monoids $\mathbb{N} = \langle \mathbb{N}; \cdot, 1 \rangle$ and $\mathbb{Q} = \langle \mathbb{Q}; \cdot, 1 \rangle$. Moreover, let $a_1, \ldots, a_n, b_1, \ldots, b_n, c_1, \ldots, c_{2n+1}, d$ be the sequence of rationals defined as follows: for every $1 \leqslant i \leqslant n$ and $1 < j < 2n + 1$,

$$a_i = \frac{1}{2} = b_i, \quad c_j = 12, \quad c_1 = 6 = c_{2n+1}, \quad d = 3.$$

Using the formula ψ_n in Example 3.14, we have

$$\mathbb{Q} \vDash \psi_n(a_1, \ldots, a_n, b_1, \ldots, b_n, c_1, \ldots, c_{2n+1}, d).$$

By the definition of f_n this yields $\langle c_1, \ldots, c_{2n+1} \rangle \in \mathsf{dom}(f_n^{\mathbb{Q}})$ and $f_n^{\mathbb{Q}}(c_1, \ldots, c_{2n+1}) = d$. As g interpolates f_n, we obtain $\langle c_1, \ldots, c_{2n+1} \rangle \in \mathsf{dom}(g^{\mathbb{Q}})$ and $g^{\mathbb{Q}}(c_1, \ldots, c_{2n+1}) = d$. Let $\varphi(x_1, \ldots, x_{2n+1}, y)$ be the conjunction of equations defining g. Then $\mathbb{Q} \vDash \varphi(c_1, \ldots, c_{2n+1}, d)$. As φ is a universal formula and $c_1, \ldots, c_{2n+1}, d \in \mathbb{N}$, Theorem 1.1(iii) implies that $\mathbb{N} \vDash \varphi(c_1, \ldots, c_{2n+1}, d)$, and hence $\langle c_1, \ldots, c_{2n+1} \rangle \in \mathsf{dom}(g^{\mathbb{N}})$ and $g^{\mathbb{N}}(c_1, \ldots, c_{2n+1}) = d$. Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be the submonoids of $\mathbb{N}$ with universes $A = \{1\} \cup \{2m : m \in \mathbb{N}\}$ and $B = \{0, 1\}$, respectively.

Since $c_1, \ldots, c_{2n+1} \in A$ and $g \in \mathsf{imp}_{eq}(\mathsf{Mon})$, Theorem 4.10 yields $3 = d \in \mathsf{d}_{\mathsf{Mon}}(\boldsymbol{A}, \mathbb{N})$. Let $h, k \colon \mathbb{N} \to \boldsymbol{B}$ be given by

$$h(m) = \begin{cases} 1 & \text{if } m = 1, \\ 0 & \text{otherwise,} \end{cases} \qquad k(m) = \begin{cases} 1 & \text{if } m \text{ is odd,} \\ 0 & \text{otherwise.} \end{cases}$$

It is immediate to verify that h and k are homomorphisms such that $h{\restriction}_A = k{\restriction}_A$ and $h(3) \neq k(3)$, whence $3 \notin \mathsf{d}_{\mathsf{Mon}}(\boldsymbol{A}, \mathbb{N})$. As this is false, we conclude that f_n cannot be interpolated by any member of $\mathsf{imp}_{eq}(\mathsf{Mon})$.   ⊠

Example 4.16 (Cancellative commutative monoids)**.** An element a of a monoid $\boldsymbol{A} = \langle A; \cdot, 1 \rangle$ is said to be *cancellative* when for all $b, c \in A$,

$$(ab = ac \text{ implies } b = c) \quad \text{and} \quad (ba = ca \text{ implies } b = c).$$

When all the elements of $\boldsymbol{A}$ are cancellative, we say that $\boldsymbol{A}$ is *cancellative* (see, e.g., [38, 39]). The class of cancellative commutative monoids forms a quasivariety, denoted by CCMon, which is axiomatized relative to commutative monoids by the quasiequation $xy \approx xz \to y \approx z$.

We recall that CCMon has the amalgamation property (see, e.g., [76, pp. 100, 108]) and, therefore, falls within the scope of Theorem 4.3. On the other hand, we will show that $\mathsf{imp}_{pp}(\mathsf{CCMon}) \neq \mathsf{imp}_{eq}(\mathsf{CCMon})$.

Consider the pp formula

$$\varphi(x, y) = \exists z (xz \approx 1 \sqcap y \approx 1).$$

Notice that φ is functional in CCMon. Indeed, $\boldsymbol{A} \vDash \varphi(a, b)$ implies $b = 1$ for all $\boldsymbol{A} \in \mathsf{CCMon}$ and $a, b \in A$. Since φ is a pp formula, we can apply Corollary 3.11, obtaining that it defines a unary $f \in \mathsf{imp}_{pp}(\mathsf{CCMon})$. Moreover, if $\boldsymbol{A} \in \mathsf{CCMon}$, then $\mathsf{dom}(f^{\boldsymbol{A}})$ consists of the invertible elements of $\boldsymbol{A}$. We show that $f \notin \mathsf{imp}_{eq}(\mathsf{CCMon})$. Suppose the contrary, with a view to contradiction. Then consider the cancellative commutative monoids $\mathbb{N} = \langle \mathbb{N}; \cdot, 1 \rangle$ and $\mathbb{Q} = \langle \mathbb{Q}; \cdot, 1 \rangle$. We have

$$2 \in \mathsf{dom}(f^{\mathbb{Q}}) \quad \text{and} \quad f^{\mathbb{Q}}(2) = 1.$$

Since f is defined by a conjunction of equations, from the above display and $1, 2 \in \mathbb{N} \leqslant \mathbb{Q}$ it follows that $2 \in \mathsf{dom}(f^{\mathbb{N}})$, a contradiction with the fact that 2 is not invertible in $\mathbb{N}$. Hence, we conclude that $\mathsf{imp}_{pp}(\mathsf{CCMon}) \neq \mathsf{imp}_{eq}(\mathsf{CCMon})$.

We conclude this example by showing that $\mathsf{imp}_{eq}(\mathsf{CCMon})$ is not closed under composition. Observe that f coincides with the composition $h \circ g$, where $h \in \mathsf{imp}_{eq}(\mathsf{CCMon})$ is the unary implicit operation defined by the equation $y \approx 1$ and $g \in \mathsf{imp}_{eq}(\mathsf{CCMon})$ is the implicit operation of "taking inverses" in monoids (see Example 3.4) restricted to CCMon. Since $h \circ g = f \notin \mathsf{imp}_{eq}(\mathsf{CCMon})$, this shows that $\mathsf{imp}_{eq}(\mathsf{CCMon})$ is not closed under composition.   ⊠

5. The strong Beth definability property

As the implicit operations of a class of algebras need not be term functions, it is natural to wonder whether they can at least be interpolated by a set of terms, which can be thought of as a way of rendering them "explicit". This idea is reminiscent of the *Beth Definability Theorem* of first order logic (see, e.g., [63, pp. 301–302]), a fundamental result stating that

every implicit definition can be turned explicit (in the setting of first order theories). However, the notions of implicit and explicit definability typical of first order logic differ from ours. For instance, an explicit definition in first order logic is simply a definition given by a formula. As our implicit operations are defined by a formula by definition, they are already explicitly definable in the sense of first order logic. As a consequence, the Beth Definability Theorem cannot be applied to our implicit operations in a nontrivial way and, in particular, it does not guarantee they can be interpolated by a set of terms, that is, made explicit in our sense.

The next definition formalizes the idea of interpolating implicit operations by sets of terms and is a particular instance of the notion of interpolation introduced in Definition 4.2.

Definition 5.1. Let f be an n-ary implicit operation of a class of algebras K. We say that f is *interpolated* by a set $\{t_i : i \in I\}$ of n-ary terms of K when it is interpolated by $\{t_i^{\mathsf{K}} : i \in I\}$. This means that for all $\boldsymbol{A} \in \mathsf{K}$ and $\langle a_1, \ldots, a_n \rangle \in \mathsf{dom}(f^{\boldsymbol{A}})$ there exists $i \in I$ such that

$$f^{\boldsymbol{A}}(a_1, \ldots, a_n) = t_i^{\boldsymbol{A}}(a_1, \ldots, a_n).$$

Intuitively, the partial function f is made "explicit" by the terms in $\{t_i : i \in I\}$.

Notice that f is interpolated by a set of terms if and only if $f^{\boldsymbol{A}}(a_1, \ldots, a_n) \in \mathsf{Sg}^{\boldsymbol{A}}(a_1, \ldots, a_n)$ for all $\boldsymbol{A} \in \mathsf{K}$ and $\langle a_1, \ldots, a_n \rangle \in \mathsf{dom}(f^{\boldsymbol{A}})$. When f is defined by a formula φ, the demand that f be interpolated by the set of terms $\{t_i : i \in I\}$ can be rendered as follows:

$$\mathsf{K} \vDash \varphi(x_1, \ldots, x_n, y) \to \bigsqcup_{i \in I} t_i(x_1, \ldots, x_n) \approx y. \tag{21}$$

As a consequence of the Compactness Theorem 1.3, we obtain the following.

Proposition 5.2. *An implicit partial function on an elementary class can be interpolated by a set of terms if and only if it can be interpolated by a finite set of terms.*

Proof. Let f be a partial function on an elementary class K. Assume that f is defined by a formula $\varphi(x_1, \ldots, x_n, y)$ and that it can be interpolated by a set of terms $\{t_i : i \in I\}$. Then condition (21) holds. From the Compactness Theorem 1.3 it follows that there exists a finite $T \subseteq \{t_i : i \in I\}$ such that

$$\mathsf{K} \vDash \varphi(x_1, \ldots, x_n, y) \to \bigsqcup_{t \in T} t(x_1, \ldots, x_n) \approx y.$$

As f is defined by φ, this means that f is interpolated by the terms in T. ⊠

When viewed as an implicit operation on a class of algebras K, every term function $\langle t^{\boldsymbol{A}} : \boldsymbol{A} \in \mathsf{K} \rangle$ of K is interpolated by a single term, namely, t (see Example 3.8). However, not all implicit operations can be interpolated by terms. For instance, there is no set of terms interpolating the implicit operation of "taking inverses" in the variety of monoids as we will see in Example 6.8. It is therefore sensible to isolate the cases in which interpolation is always possible, something that indicates a good balance between the expressivity of the language (measured by what can be said in terms of implicit operations) and its actual richness (measured by what can be interpolated, or made explicit, by terms).

Definition 5.3. A class of algebras is said to have the *strong Beth definability property* when each of its implicit operations can be interpolated by a set of terms.

The reason why we termed our Beth definability property "strong" is to distinguish it from other weaker definability properties considered in the literature (see [17, 81]). Although we will not rely on this fact, we remark that, when a quasivariety is the equivalent algebraic semantics of a propositional logic in the sense of [19], the strong Beth definability property is the algebraic counterpart of the so called *projective Beth property* investigated in [9, p. 76] (see also [65, Sec. 2.2.3] and [88, 89, 90]).

In the context of elementary classes, the strong Beth definability property can be equivalently formulated by restricting our attention to implicit operations defined by pp formulas and interpolation by a finite set of terms. More precisely, we have the following.

Proposition 5.4. *The following conditions are equivalent for an elementary class* K*:*

(i) K *has the strong Beth definability property;*

(ii) *each implicit operation of* K *defined by a pp formula can be interpolated by a finite set of terms.*

Proof. The implication (i)⇒(ii) is an immediate consequence of Proposition 5.2. To prove (ii)⇒(i) suppose that each implicit operation of K defined by a pp formula can be interpolated by a finite set of terms. Then let f be an implicit operation of K. By Corollary 3.10 there exist some implicit operations $f_1, \ldots, f_n$ of K defined by pp formulas such that for each $\boldsymbol{A} \in \mathsf{K}$,

$$f^{\boldsymbol{A}} = f_1^{\boldsymbol{A}} \cup \cdots \cup f_n^{\boldsymbol{A}}.$$

By assumption each f_i is interpolated by a finite set of terms T_i. In view of the above display, we conclude that f is interpolated by the terms in $T_1 \cup \cdots \cup T_n$.   ⊠

Rephrasing condition (ii) of Proposition 5.4 in terms of the validity of certain formulas in K yields the following.

Corollary 5.5. *An elementary class* K *has the strong Beth definability property if and only if for each pp formula* $\varphi(x_1, \ldots, x_n, y)$ *such that*

$$\mathsf{K} \vDash (\varphi(x_1, \ldots, x_n, y) \sqcap \varphi(x_1, \ldots, x_n, z)) \to y \approx z$$

there exist terms $t_1(x_1, \ldots, x_n), \ldots, t_m(x_1, \ldots, x_n)$ *such that*

$$\mathsf{K} \vDash \varphi(x_1, \ldots, x_n, y) \to \bigsqcup_{i \leqslant m} t_i(x_1, \ldots, x_n) \approx y.$$

For elementary classes closed under direct products the equivalence in Proposition 5.4 can be refined as follows.

Proposition 5.6. *The following conditions are equivalent for an elementary class* K *closed under direct products:*

(i) K *has the strong Beth definability property;*

(ii) *each implicit operation of* K *defined by a pp formula can be interpolated by a single term.*

Proof. (i)⇒(ii): Suppose that K has the strong Beth definability property. Then consider an implicit operation f of K that can be defined by a pp formula $\varphi(x_1, \ldots, x_n, y)$. From the strong Beth definability property it follows that f can be interpolated by a set of terms $\{t_i : i \in I\}$. As f is defined by φ, this amounts to

$$\mathsf{K} \vDash \varphi(x_1, \ldots, x_n, y) \to \bigsqcup_{i \in I} t_i(x_1, \ldots, x_n) \approx y.$$

Since φ is a pp formula and K an elementary class closed under $\mathbb{P}$ by assumption, we can apply Corollary 1.4, obtaining that there exists $i \in I$ such that

$$\mathsf{K} \vDash \varphi(x_1, \ldots, x_n, y) \to t_i(x_1, \ldots, x_n) \approx y.$$

As φ defines f, we conclude that f is interpolated by t_i.

(ii)⇒(i): Immediate from the implication (ii)⇒(i) of Proposition 5.4.   ⊠
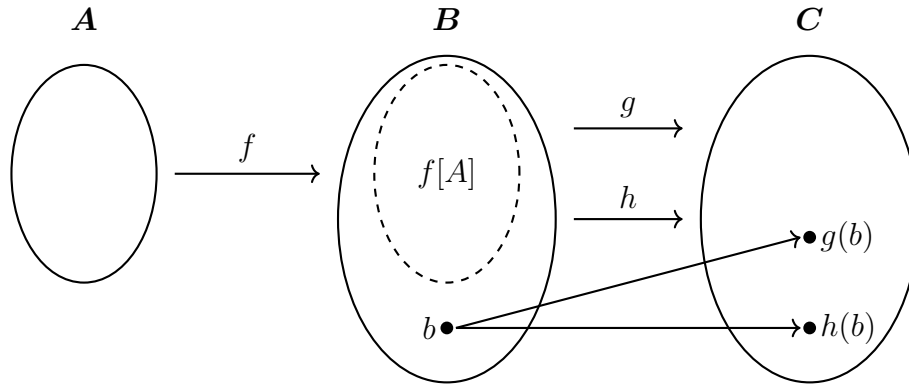
6. The strong epimorphism surjectivity property

The strong Beth definability property admits a purely algebraic formulation, as we proceed to illustrate. Let K be a class of algebras. A homomorphism $f \colon \boldsymbol{A} \to \boldsymbol{B}$ with $\boldsymbol{A}, \boldsymbol{B} \in \mathsf{K}$ is said to be a K-*epimorphism* when it is right cancellable, that is, when for every pair of homomorphisms $g, h \colon \boldsymbol{B} \to \boldsymbol{C}$ with $\boldsymbol{C} \in \mathsf{K}$,

$$g \circ f = h \circ f \quad \text{implies} \quad g = h.$$

While every surjective homomorphism between members of K is a K-epimorphism, the converse need not hold in general. For instance, the inclusion map of $\langle \mathbb{Z}; \cdot, 1 \rangle$ into $\langle \mathbb{Q}; \cdot, 1 \rangle$ is a nonsurjective epimorphism in the variety of monoids. Consequently, a class of algebras K is said to have the *epimorphism surjectivity property* when every K-epimorphism is surjective.

We will show that, in the setting of universal classes, the strong Beth definability property is equivalent to the following strengthening of the epimorphism surjectivity property (see Theorem 6.5).

Definition 6.1. A class of algebras K has the *strong epimorphism surjectivity property* when for every homomorphism $f \colon \boldsymbol{A} \to \boldsymbol{B}$ with $\boldsymbol{A}, \boldsymbol{B} \in \mathsf{K}$ and $b \in B - f[A]$ there exists a pair of homomorphisms $g, h \colon \boldsymbol{B} \to \boldsymbol{C}$ with $\boldsymbol{C} \in \mathsf{K}$ such that $g \circ f = h \circ f$ and $g(b) \neq h(b)$.

Remark 6.2. When K is closed under $\mathbb{I}$ and $\mathbb{S}$, we may assume that $\boldsymbol{A} \leqslant \boldsymbol{B}$ and that the map $f \colon \boldsymbol{A} \to \boldsymbol{B}$ in the above definition is an inclusion map. In this case, the strong epimorphism surjectivity property simplifies to the demand that for all $\boldsymbol{A} \leqslant \boldsymbol{B} \in \mathsf{K}$ and $b \in B - A$ there exists a pair of homomorphisms $g, h \colon \boldsymbol{B} \to \boldsymbol{C}$ with $\boldsymbol{C} \in \mathsf{K}$ such that $g{\restriction}_A = h{\restriction}_A$ and $g(b) \neq h(b)$. Moreover, when K is a quasivariety, the Subdirect Decomposition Theorem 2.9 allows us to assume $\boldsymbol{C} \in \mathsf{K}_{\mathrm{RSI}}$. $\boxtimes$

Example 6.3 (Abelian groups)**.** We view groups as algebras $\boldsymbol{A} = \langle A; \cdot, ^{-1}, 1 \rangle$, i.e., we assume that multiplication, taking inverses, and the neutral element are all basic operations. While it is known that every variety of Abelian groups has the strong epimorphism surjectivity property, we provide a short proof for the sake of completeness. For consider a variety of Abelian groups V, $\boldsymbol{A} \leqslant \boldsymbol{B} \in \mathsf{V}$, and $b \in B - A$. Since congruences in Abelian groups correspond to subgroups, there exists a congruence θ of $\boldsymbol{B}$ associated with the subgroup $\boldsymbol{A}$. Clearly, the canonical surjection $g \colon \boldsymbol{B} \to \boldsymbol{B}/\theta$ is a homomorphism such that $g^{-1}(1/\theta) = A$ and $\boldsymbol{B}/\theta \in \mathsf{V}$. Then let $h \colon \boldsymbol{B} \to \boldsymbol{B}/\theta$ be the homomorphism that sends every element of B to $1/\theta$. From $g^{-1}(1/\theta) = A$ and the definition of h it follows that $g(a) = 1/\theta = h(a)$ for each $a \in A$, whence $g{\restriction}_A = h{\restriction}_A$. On the other hand, from $b \notin A = g^{-1}(1/\theta)$ it follows that $g(b) \neq 1/\theta$, while $h(b) = 1/\theta$ by the definition of h. Hence, $g(b) \neq h(b)$. We conclude that V has the strong epimorphism surjectivity property. $\boxtimes$

While every class with the strong epimorphism surjectivity property has the epimorphism surjectivity property, the converse need not hold in general. For instance, it is known that only 16 varieties of Heyting algebras have the strong epimorphism surjectivity property [90, Thm. 8.1]. On the other hand, there exists a continuum of varieties of Heyting algebras with the nonstrong version of this property [14, p. 199]. However, the two properties coincide in quasivarieties with the amalgamation property (see Theorem 7.14).

Remark 6.4. In the context of quasivarieties, the strong epimorphism surjectivity property admits a purely categorical formulation. For observe that each quasivariety K can be viewed as a category whose objects are the members of K and whose arrows are the homomorphisms between them. As quasivarieties contain free algebras (see Theorem 2.19), monomorphisms coincide with embeddings in quasivarieties (see, e.g., [1, Prop. 8.29]). A monomorphism is said to be *regular* when it is an equalizer. It turns out that a quasivariety K has the strong epimorphism surjectivity property if and only if all monomorphisms are regular in K (see, e.g., [76, Prop. 6.1]). $\boxtimes$

For the present purpose, the interest of the strong epimorphism surjectivity property comes from the fact that it is the algebraic counterpart of the strong Beth definability property. More precisely, we will prove the following theorem which generalizes the correspondences between the strong epimorphism surjectivity property and definability properties established in [6, Thm. 4], [88, Thm. 3.6], [89, Thm. 3.1], and [64, Thm. 5].

Theorem 6.5. *A universal class has the strong epimorphism surjectivity property if and only if it has the strong Beth definability property.*

As an immediate consequence of Remark 6.2, we obtain an alternative formulation of the strong epimorphism surjectivity property in terms of dominions.

Proposition 6.6. *Let* K *be a class of algebras closed under* $\mathbb{I}$ *and* $\mathbb{S}$*. Then* K *has the strong epimorphism surjectivity property if and only if* $\mathsf{d}_K(\boldsymbol{A}, \boldsymbol{B}) = A$ *for every* $\boldsymbol{A} \leqslant \boldsymbol{B} \in$ K.

From Theorem 4.10 and Proposition 6.6 we deduce the following.

Corollary 6.7. *A universal class* K *has the strong epimorphism surjectivity property if and only if for all* $\boldsymbol{A} \leqslant \boldsymbol{B} \in$ K, $f \in \mathsf{imp}_{pp}(K)$, *and* $\langle a_1, \ldots, a_n \rangle \in \mathsf{dom}(f^{\boldsymbol{B}}) \cap A^n$ *we have* $f^{\boldsymbol{B}}(a_1, \ldots, a_n) \in A$.

We are now ready to prove Theorem 6.5.

Proof. Let K be a universal class. To prove the implication from left to right, we reason by contraposition. Suppose that K lacks the strong Beth definability property. By Propositions 5.2 and 5.4 there exists $f \in \mathsf{imp}_{pp}(K)$ that cannot be interpolated by any set of terms. Therefore, there exists $\boldsymbol{B} \in$ K and $\langle a_1, \ldots, a_n \rangle \in \mathsf{dom}(f^{\boldsymbol{B}})$ for which there exists no term t such that $f^{\boldsymbol{B}}(a_1, \ldots, a_n) = t^{\boldsymbol{B}}(a_1, \ldots, a_n)$. Then $f^{\boldsymbol{B}}(a_1, \ldots, a_n) \notin \mathsf{Sg}^{\boldsymbol{B}}(a_1, \ldots, a_n)$. Let $\boldsymbol{A} = \mathsf{Sg}^{\boldsymbol{B}}(a_1, \ldots, a_n)$. Since $\langle a_1, \ldots, a_n \rangle \in \mathsf{dom}(f^{\boldsymbol{B}}) \cap A^n$ and $f^{\boldsymbol{B}}(a_1, \ldots a_n) \notin A$, Corollary 6.7 implies that K lacks the strong epimorphism surjectivity property.

Next we prove the implication from right to left. Suppose that K has the strong Beth definability property. In order to prove that K has the strong epimorphism surjectivity property, it suffices to show that $\mathsf{d}_K(\boldsymbol{A}, \boldsymbol{B}) = A$ for every $\boldsymbol{A} \leqslant \boldsymbol{B} \in$ K (see Proposition 6.6). To this end, consider $\boldsymbol{A} \leqslant \boldsymbol{B} \in$ K and $b \in \mathsf{d}_K(\boldsymbol{A}, \boldsymbol{B})$. By Theorem 4.10 there exist $f \in \mathsf{imp}_{pp}(K)$ and $\langle a_1, \ldots, a_n \rangle \in \mathsf{dom}(f^{\boldsymbol{B}}) \cap A^n$ such that $b = f^{\boldsymbol{B}}(a_1, \ldots, a_n)$. Since K has the strong Beth definability property, there exists a term t such that $f^{\boldsymbol{B}}(a_1, \ldots, a_n) = t^{\boldsymbol{B}}(a_1, \ldots, a_n)$. Therefore,

$$b = f^{\boldsymbol{B}}(a_1, \ldots, a_n) = t^{\boldsymbol{B}}(a_1, \ldots, a_n) = t^{\boldsymbol{A}}(a_1, \ldots, a_n) \in A.$$

This shows that $\mathsf{d}_K(\boldsymbol{A}, \boldsymbol{B}) \subseteq A$. As the reverse inclusion always holds, we conclude that $\mathsf{d}_K(\boldsymbol{A}, \boldsymbol{B}) = A$. ⊠

We close this section with a series of examples of classes of algebras with and without the strong epimorphism surjectivity property.

Example 6.8 (Monoids)**.** As we mentioned, the inclusion map of $\mathbb{Z} = \langle \mathbb{Z}; \cdot, 1 \rangle$ into $\mathbb{Q} = \langle \mathbb{Q}; \cdot, 1 \rangle$ is a nonsurjective epimorphism in the variety of monoids Mon, whence Mon lacks the epimorphism surjectivity property, and consequently its strong version as well.

This can also be viewed through the lens of Corollary 6.7. For let f be the implicit operation of "taking inverses" in monoids and recall that it can be defined by a pp formula (see Theorem 3.7). Clearly, $\mathbb{Z} \leqslant \mathbb{Q}$ and $2 \in \mathsf{dom}(f^{\mathbb{Q}}) \cap \mathbb{Z}$. Moreover,

$$f^{\mathbb{Q}}(2) = 2^{-1} = \frac{1}{2} \notin \mathbb{Z}.$$

From Corollary 6.7 it follows that Mon lacks the strong epimorphism surjectivity property. The same proof yields the same conclusion for the variety of commutative monoids. In view of Theorem 6.5, both varieties lack the strong Beth definability property. ⊠

Example 6.9 (Reduced commutative rings)**.** Essentially the same argument shows that the quasivariety RCRing of reduced commutative rings lacks the strong epimorphism surjectivity property. More precisely, let $\mathbb{Z}$ and $\mathbb{Q}$ be the reduced commutative rings of the integers and the rationals, respectively. Moreover, let f be the implicit operation of RCRing given by Theorem 3.18. As $\mathbb{Q}$ is a field, Theorem 3.18 guarantees that $f^{\mathbb{Q}}$ is the operation of "taking weak inverses" in $\mathbb{Q}$. Therefore, we can replicate the argument detailed in Example 6.8, yielding that RCRing lacks the strong epimorphism surjectivity property, and hence also the strong Beth definability property.   ⊠

Example 6.10 (Distributive lattices)**.** We will show that the variety of distributive lattices DL lacks the strong epimorphism surjectivity property. For let f be the implicit operation of "taking relative complements" in distributive lattices and recall that it can be defined by a pp formula (see Theorem 3.20). Moreover, let $\boldsymbol{B}$ be the four-element Boolean lattice with universe $\{0, a, b, 1\}$, where 0 and 1 are the minimum and the maximum of $\boldsymbol{B}$, respectively. Lastly, let $\boldsymbol{A}$ be the subalgebra of $\boldsymbol{B}$ with universe $\{0, a, 1\}$. Since b is the complement of a relative to $[0, 1]$ in $\boldsymbol{B}$, we have $\langle a, 0, 1 \rangle \in \mathsf{dom}(f^{\boldsymbol{B}}) \cap A^3$ and

$$f^{\boldsymbol{B}}(a, 0, 1) = b \notin A.$$

Hence, we can apply Corollary 6.7, obtaining that DL lacks the strong epimorphism surjectivity property, and thus also the strong Beth definability property.   ⊠

The survey [76] contains plenty of examples of classes of algebras with and without the epimorphism surjectivity property and the *intersection property of amalgamation* (IPA, for short), which is equivalent to the strong epimorphism surjectivity property in varieties (see [76, Prop. 4.5]). Among these examples, we count the following. A *semigroup with zero* is an algebra $\boldsymbol{A} = \langle A; \cdot, 0 \rangle$, where $\langle A; \cdot \rangle$ is a semigroup and $a0 = 0a = 0$ for every $a \in A$. For every $n \geqslant 4$ the variety of semigroups with zero satisfying the equation $x^n \approx 0$ has the epimorphism surjectivity property, but not its strong version (see [76, p. 89]). Moreover, the varieties of semilattices and lattices have both the strong epimorphism surjectivity property (see [76, pp. 99, 102]).

7. Tangible epimorphism surjectivity

The aim of this section is to facilitate the task of determining whether a given class of algebras has the strong epimorphism surjectivity property. On the one hand, we will show that this problem can often be settled by considering only finitely generated or finitely presented algebras.

Theorem 7.1. *The following conditions are equivalent for a universal class* K*:*

  (i) K *has the strong epimorphism surjectivity property;*
  (ii) *for each* $\boldsymbol{A} \leqslant \boldsymbol{B} \in \mathsf{K}$ *with* $\boldsymbol{A}$ *and* $\boldsymbol{B}$ *finitely generated we have* $\mathsf{d}_{\mathsf{K}}(\boldsymbol{A}, \boldsymbol{B}) = A$.

In addition, when K *is a quasivariety, we may assume that* $\boldsymbol{B}$ *is finitely presented.*

On the other hand, we will provide a criterion for the validity of the strong epimorphism surjectivity property which applies to a large class of quasivarieties. More precisely, a term t

of arity $\geqslant 3$ is a *near unanimity term* (see, e.g., [73, Sec. 1.2.3]) for a class of algebras K when

$$\mathsf{K} \vDash x \approx t(y, x, \ldots, x) \approx t(x, y, x, \ldots, x) \approx \cdots \approx t(x, \ldots, x, y).$$

Intuitively, the term t returns x when its arguments are almost unanimously x. Notably, each variety with a near unanimity term is congruence distributive (see [92, Thm. 2]), although the converse need not hold in general (see [92, Lem. 3]). Ternary near unanimity terms play a prominent role in algebra and are known as *majority terms* (see, e.g., [21, Def. II.12.8]). As $t(x, y, z) = (x \wedge y) \vee (x \wedge z) \vee (y \wedge z)$ is a majority term for every class of algebras with a lattice reduct, we have the following.

Theorem 7.2. *Let K be a class of algebras with a lattice reduct. Then $\mathbb{V}(\mathsf{K})$ has a majority term and is congruence distributive.*

We will show that, in the presence of a near unanimity term, the task of determining whether a quasivariety has the strong epimorphism surjectivity property can be simplified as follows.

Theorem 7.3. *The following conditions are equivalent for a quasivariety K with a near unanimity term of arity n:*

(i) *K has the strong epimorphism surjectivity property;*
(ii) *for each finitely generated $\boldsymbol{A} \leqslant \boldsymbol{B}_1 \times \cdots \times \boldsymbol{B}_{n-1}$ with $\boldsymbol{B}_1, \ldots, \boldsymbol{B}_{n-1} \in \mathsf{K}_{\mathrm{RFSI}}$ finitely generated we have $\mathsf{d}_{\mathsf{K}}(\boldsymbol{A}, \boldsymbol{B}_1 \times \cdots \times \boldsymbol{B}_{n-1}) = A.$*

Before proving these results, let us provide an example of how to apply them in practice.

Example 7.4 (Relatively complemented distributive lattices)**.** An algebra $\langle A; \wedge, \vee, r \rangle$ is a *relatively complemented distributive lattice* when $\langle A; \wedge, \vee \rangle$ is a distributive lattice and r a ternary operation such that $r(a, b, c)$ is the complement of a relative to the interval $[a \wedge b \wedge c, a \vee b \vee c]$ for all $a, b, c \in A$. Notice that if $b \leqslant a \leqslant c$, then $r(a, b, c)$ is the complement of a relative to $[b, c]$. The class of relatively complemented distributive lattices forms a variety, which we denote by RCDL. We will prove the following.

Theorem 7.5. *The varieties of relatively complemented distributive lattices and of Boolean algebras have the strong epimorphism surjectivity property.*

Proof. We will detail the proof for RCDL only, as the case of Boolean algebras is analogous and well known (see, e.g., [76, p. 103]). Let $\boldsymbol{D}_2$ be the unique relatively complemented distributive lattice with universe $\{0, 1\}$ and $0 < 1$. We begin with the following observation.

Claim 7.6. *We have $\mathsf{RCDL}_{\mathrm{FSI}} = \mathbb{I}(\boldsymbol{D}_2)$.*

Proof of the Claim. Since every two-element algebra is finitely subdirectly irreducible, we obtain $\mathbb{I}(\boldsymbol{D}_2) \subseteq \mathsf{RCDL}_{\mathrm{FSI}}$. To prove the other inclusion, consider $\boldsymbol{A} \in \mathsf{RCDL}_{\mathrm{FSI}}$. Let $\boldsymbol{A}^-$ be the lattice reduct of $\boldsymbol{A}$. To conclude the proof, it will be enough to show that $\mathsf{Con}(\boldsymbol{A}) = \mathsf{Con}(\boldsymbol{A}^-)$. For suppose that this is the case. Then, as $\boldsymbol{A}$ is finitely subdirectly irreducible, we can apply Proposition 2.10, obtaining that id_A is meet irreducible in $\mathsf{Con}(\boldsymbol{A})$. Since $\mathsf{Con}(\boldsymbol{A}) = \mathsf{Con}(\boldsymbol{A}^-)$, this yields that id_A is also meet irreducible in $\mathsf{Con}(\boldsymbol{A}^-)$. Consequently, Proposition 2.10

guarantees that $\boldsymbol{A}^-$ is a finitely subdirectly irreducible distributive lattice. Up to isomorphism, the only such lattice is the lattice reduct of $\boldsymbol{D}_2$. Hence, we conclude that $\boldsymbol{A} \cong \boldsymbol{D}_2$, as desired.

Therefore, it only remains to show that $\mathsf{Con}(\boldsymbol{A}) = \mathsf{Con}(\boldsymbol{A}^-)$. As $\boldsymbol{A}^-$ is a reduct of $\boldsymbol{A}$, we have $\mathsf{Con}(\boldsymbol{A}) \subseteq \mathsf{Con}(\boldsymbol{A}^-)$. Then we proceed to prove the other inclusion. Consider $\theta \in \mathsf{Con}(\boldsymbol{A}^-)$. To prove that $\theta \in \mathsf{Con}(\boldsymbol{A})$, let $\langle a_1, a_2 \rangle, \langle b_1, b_2 \rangle, \langle c_1, c_2 \rangle \in \theta$. We need to show that $\langle r^{\boldsymbol{A}}(a_1, b_1, c_1), r^{\boldsymbol{A}}(a_2, b_2, c_2) \rangle \in \theta$. To this end, recall from Theorem 3.20 that "taking relative complements" is a ternary implicit operation f of DL. Moreover, consider the canonical surjection $\pi_\theta \colon \boldsymbol{A}^- \to \boldsymbol{A}^-/\theta$.

We will prove that for each $i \leqslant 2$,

$$\langle a_i/\theta, b_i/\theta, c_i/\theta \rangle \in \mathsf{dom}(f^{\boldsymbol{A}^-/\theta}) \text{ and } f^{\boldsymbol{A}^-/\theta}(a_i/\theta, b_i/\theta, c_i/\theta) = r^{\boldsymbol{A}}(a_i, b_i, c_i)/\theta. \quad (22)$$

First, observe that $\langle a_i, b_i, c_i \rangle \in \mathsf{dom}(f^{\boldsymbol{A}^-})$ and $f^{\boldsymbol{A}^-}(a_i, b_i, c_i) = r^{\boldsymbol{A}}(a_i, b_i, c_i)$ because $\boldsymbol{A}^-$ is the reduct of the relatively complemented distributive lattice $\boldsymbol{A}$ and f is the implicit operation of "taking relative complements". Since f is an operation of DL, it is preserved by the homomorphism $\pi \colon \boldsymbol{A}^- \to \boldsymbol{A}^-/\theta$. Therefore, from $\langle a_i, b_i, c_i \rangle \in \mathsf{dom}(f^{\boldsymbol{A}^-})$ and $f^{\boldsymbol{A}^-}(a_i, b_i, c_i) = r^{\boldsymbol{A}}(a_i, b_i, c_i)$ it follows that $\langle a_i/\theta, b_i/\theta, c_i/\theta \rangle = \langle \pi(a_i), \pi(b_i), \pi(c_i) \rangle \in \mathsf{dom}(f^{\boldsymbol{A}^-/\theta})$ and

$$\begin{aligned} f^{\boldsymbol{A}^-/\theta}(a_i/\theta, b_i/\theta, c_i/\theta) &= f^{\boldsymbol{A}^-/\theta}(\pi(a_i), \pi(b_i), \pi(c_i)) = \pi(f^{\boldsymbol{A}^-}(a_i, b_i, c_i)) \\ &= \pi(r^{\boldsymbol{A}}(a_i, b_i, c_i)) = r^{\boldsymbol{A}}(a_i, b_i, c_i)/\theta. \end{aligned}$$

This establishes (22). As $a_1/\theta = a_2/\theta$, $b_1/\theta = b_2/\theta$, and $c_1/\theta = c_2/\theta$ by assumption, this implies

$$r^{\boldsymbol{A}}(a_1, b_1, c_1)/\theta = f^{\boldsymbol{A}^-/\theta}(a_1/\theta, b_1/\theta, c_1/\theta) = f^{\boldsymbol{A}^-/\theta}(a_2/\theta, b_2/\theta, c_2/\theta) = r^{\boldsymbol{A}}(a_2, b_2, c_2)/\theta,$$

that is, $\langle r^{\boldsymbol{A}}(a_1, b_1, c_1), r^{\boldsymbol{A}}(a_2, b_2, c_2) \rangle \in \theta$, as desired. $\boxtimes$

Now, we prove that RCDL has the strong epimorphism surjectivity property. As RCDL has a lattice reduct, it possesses a majority term. Therefore, we can apply Theorem 7.3, obtaining that RCDL has the strong epimorphism surjectivity property if and only if for each finitely generated $\boldsymbol{A} \leqslant \boldsymbol{B} \times \boldsymbol{C}$ with $\boldsymbol{B}, \boldsymbol{C} \in \mathsf{RCDL}_{\mathrm{FSI}}$ finitely generated we have $\mathsf{d}_{\mathsf{K}}(\boldsymbol{A}, \boldsymbol{B} \times \boldsymbol{C}) = A$. Together with Claim 7.6 the latter specializes to the following: for each $\boldsymbol{A} \leqslant \boldsymbol{D}_2 \times \boldsymbol{D}_2$ we have $\mathsf{d}_{\mathsf{K}}(\boldsymbol{A}, \boldsymbol{D}_2 \times \boldsymbol{D}_2) = A$.

To prove this, consider $\boldsymbol{A} \leqslant \boldsymbol{D}_2 \times \boldsymbol{D}_2$. By inspection for each $b \in (D_2 \times D_2) - A$ one can find an endomorphism h of $\boldsymbol{D}_2 \times \boldsymbol{D}_2$ such that $id\restriction_A = h\restriction_A$ and $id(b) \neq g(b)$, where id is the identity map on $\boldsymbol{D}_2 \times \boldsymbol{D}_2$. Hence, we conclude that $\mathsf{d}_{\mathsf{K}}(\boldsymbol{A}, \boldsymbol{D}_2 \times \boldsymbol{D}_2) = A$. $\boxtimes$

We are now ready to prove Theorem 7.1.

Proof. The implication (i)$\Rightarrow$(ii) holds by Proposition 6.6. To prove the implication (ii)$\Rightarrow$(i), we reason by contraposition. Suppose that K lacks the strong epimorphism surjectivity property. In view of Corollary 6.7, there exist $\boldsymbol{C} \leqslant \boldsymbol{D} \in \mathsf{K}$, an implicit operation f of K defined by a pp formula $\varphi(x_1, \ldots, x_n, y)$, and

$$\langle c_1, \ldots, c_n \rangle \in \mathsf{dom}(f^{\boldsymbol{D}}) \cap C^n \text{ with } f^{\boldsymbol{D}}(c_1, \ldots, c_n) \notin C. \quad (23)$$

As φ defines an implicit operation of K, it also defines an implicit operation g of $\mathbb{Q}(\mathsf{K})$ by Corollary 3.11. As both f and g are defined by φ and $\boldsymbol{D} \in \mathsf{K} \subseteq \mathbb{Q}(\mathsf{K})$, we have $f^{\boldsymbol{D}} = g^{\boldsymbol{D}}$. We will make use of this observation without further notice.

Claim 7.7. *There exist a finitely presented member $\boldsymbol{B}$ of $\mathbb{Q}(\mathsf{K})$ and $\boldsymbol{A} \leqslant \boldsymbol{B}$ finitely generated with elements $a_1, \ldots, a_n \in A$ and a homomorphism $h \colon \boldsymbol{B} \to \boldsymbol{C}$ satisfying the following conditions:*

(i) $\langle a_1, \ldots, a_n \rangle \in \mathsf{dom}(g^{\boldsymbol{B}}) \cap A^n$ and $g^{\boldsymbol{B}}(a_1, \ldots, a_n) \notin A$;
(ii) $h(a_1) = c_1, \ldots, h(a_n) = c_n$, $h(g^{\boldsymbol{B}}(a_1, \ldots, a_n)) = f^{\boldsymbol{D}}(c_1, \ldots, c_n)$, and $h[A] \subseteq C$.

Proof of the Claim. Define $\boldsymbol{B} = \boldsymbol{T}_{\mathbb{Q}(\mathsf{K})}(\varphi)$, $a_1 = x_1/\theta(\varphi), \ldots, a_n = x_n/\theta(\varphi)$, and $\boldsymbol{A} = \mathsf{Sg}^{\boldsymbol{B}}(a_1, \ldots, a_n)$ (for the definition of $\boldsymbol{T}_{\mathbb{Q}(\mathsf{K})}(\varphi)$ see Definition 4.12). Clearly, $\boldsymbol{A}$ is a finitely generated subalgebra of $\boldsymbol{B}$. Moreover, from Proposition 4.13(i) it follows that $\boldsymbol{B}$ is finitely presented in $\mathbb{Q}(\mathsf{K})$ and that

$$\langle a_1, \ldots, a_n \rangle \in \mathsf{dom}(g^{\boldsymbol{B}}) \cap A^n \quad \text{and} \quad g^{\boldsymbol{B}}(a_1, \ldots, a_n) = y/\theta(\varphi). \tag{24}$$

Now, recall from (23) that $\langle c_1, \ldots, c_n \rangle \in \mathsf{dom}(f^{\boldsymbol{D}}) = \mathsf{dom}(g^{\boldsymbol{D}})$. Together with $\boldsymbol{D} \in \mathsf{K} \subseteq \mathbb{Q}(\mathsf{K})$ and Proposition 4.13(ii), this guarantees the existence of a homomorphism $h \colon \boldsymbol{B} \to \boldsymbol{D}$ such that

$$h(a_1) = c_1, \ldots, h(a_n) = c_n, \text{ and } h(y/\theta(\varphi)) = g^{\boldsymbol{D}}(c_1, \ldots, c_n) = f^{\boldsymbol{D}}(c_1, \ldots, c_n).$$

From the right hand sides of the above display and (24) it follows that $h(g^{\boldsymbol{B}}(a_1, \ldots, a_n)) = f^{\boldsymbol{D}}(c_1, \ldots, c_n)$.

Therefore, it only remains to prove that $g^{\boldsymbol{B}}(a_1, \ldots, a_n) \notin A$ and $h[A] \subseteq C$. To this end, recall that $\boldsymbol{A}$ is generated by $a_1, \ldots, a_n$ and that $c_1, \ldots, c_n$ belong to the subalgebra $\boldsymbol{C}$ of $\boldsymbol{D}$. Therefore, from the left hand side of the above display it follows that $h[A] \subseteq C$. Moreover, from the right hand side of (23) and $h(g^{\boldsymbol{B}}(a_1, \ldots, a_n)) = f^{\boldsymbol{D}}(c_1, \ldots, c_n)$ we obtain $h(g^{\boldsymbol{B}}(a_1, \ldots, a_n)) \notin C$. Together with $h[A] \subseteq C$, this yields $g^{\boldsymbol{B}}(a_1, \ldots, a_n) \notin A$. $\boxtimes$

Let $\boldsymbol{A}$, $\boldsymbol{B}$, and $a_1, \ldots, a_n$ be as in Claim 7.7. We have two cases: either K is a quasivariety or not. We begin with the case where K is a quasivariety. Then $\mathsf{K} = \mathbb{Q}(\mathsf{K})$. Therefore, $\boldsymbol{B}$ is a finitely presented member of K and $\boldsymbol{A} \leqslant \boldsymbol{B}$ finitely generated. From Claim 7.7(i) and Theorem 4.10 it follows that $g^{\boldsymbol{B}}(a_1, \ldots, a_n) \in \mathsf{d}_{\mathsf{K}}(\boldsymbol{A}, \boldsymbol{B}) - A$, whence $\mathsf{d}_{\mathsf{K}}(\boldsymbol{A}, \boldsymbol{B}) \neq A$, as desired.

It only remains to consider the case where K is not a quasivariety. Then let $h \colon \boldsymbol{B} \to \boldsymbol{D}$ be the homomorphism given by Claim 7.7. Define $\boldsymbol{A}' = h[\boldsymbol{A}]$ and $\boldsymbol{B}' = h[\boldsymbol{B}]$. As $\boldsymbol{D} \in \mathsf{K}$ and K is a universal class by assumption, from $\boldsymbol{A}' \leqslant \boldsymbol{B}' \leqslant \boldsymbol{D}$ it follows that $\boldsymbol{A}', \boldsymbol{B}' \in \mathsf{K}$. Furthermore, since $\boldsymbol{A}$ and $\boldsymbol{B}$ are finitely generated by Claim 7.7, the algebras $\boldsymbol{A}'$ and $\boldsymbol{B}'$ are also finitely generated. By condition (i) of the same claim we have $\boldsymbol{B} \vDash \varphi(a_1, \ldots, a_n, g^{\boldsymbol{B}}(a_1, \ldots, a_n))$. As $h \colon \boldsymbol{B} \to \boldsymbol{B}'$ is a homomorphism and φ a pp formula, we can apply Theorem 1.1(ii), obtaining $\boldsymbol{B}' \vDash \varphi(h(a_1), \ldots, h(a_n), h(g^{\boldsymbol{B}}(a_1, \ldots, a_n)))$. By Claim 7.7(ii) this amounts to $\boldsymbol{B}' \vDash \varphi(c_1, \ldots, c_n, f^{\boldsymbol{D}}(c_1, \ldots, c_n))$, that is,

$$\langle c_1, \ldots, c_n \rangle \in \mathsf{dom}(f^{\boldsymbol{B}'}) \quad \text{and} \quad f^{\boldsymbol{B}'}(c_1, \ldots, c_n) = f^{\boldsymbol{D}}(c_1, \ldots, c_n).$$

Recall from (23) that $f^{\boldsymbol{D}}(c_1, \ldots, c_n) \notin C$. Together with the right hand side of the above display and the fact that $A' = h[A] \subseteq C$ (see condition (ii) of Claim 7.7), this yields $f^{\boldsymbol{B}'}(c_1, \ldots, c_n) \notin A'$. On the other hand, from $a_1, \ldots, a_n \in A$ and $h(a_i) = c_i$ for each $i \leqslant n$ (see Claim 7.7(ii)) it follows that $c_1, \ldots, c_n \in h[A] = A'$. Hence, the left hand side of the above display can be improved to $\langle c_1, \ldots, c_n \rangle \in \mathsf{dom}(f^{\boldsymbol{B}'}) \cap (A')^n$. As a consequence, we can apply Theorem 4.10, obtaining $\varphi^{\boldsymbol{B}'}(c_1, \ldots, c_n) \in \mathsf{d}_{\mathsf{K}}(\boldsymbol{A}', \boldsymbol{B}') - A'$, whence $\mathsf{d}_{\mathsf{K}}(\boldsymbol{A}', \boldsymbol{B}') \neq A'$.   ⊠

Now, we proceed to prove Theorem 7.3. We will make use of the next concept from [29, Def. 4.4].

Definition 7.8. Let K be a quasivariety, $\boldsymbol{A} \in \mathsf{K}$, and $\theta \in \mathsf{Con}_{\mathsf{K}}(\boldsymbol{A})$. Given a positive integer n, we say that θ is *n-irreducible* in $\mathsf{Con}_{\mathsf{K}}(\boldsymbol{A})$ when $\theta = \theta_1 \cap \cdots \cap \theta_n$ with $\theta_1, \ldots, \theta_n \in \mathsf{Con}_{\mathsf{K}}(\boldsymbol{A})$ implies $\theta = \theta_1 \cap \cdots \cap \theta_{i-1} \cap \theta_{i+1} \cap \cdots \cap \theta_n$ for some $i \leqslant n$. When K is clear from the context, we will simply say that θ is *n-irreducible*.

Notice that the only 1-irreducible K-congruence of $\boldsymbol{A}$ is $A \times A$. Moreover, a K-congruence θ of $\boldsymbol{A}$ is 2-irreducible if and only if either $\theta \in \mathsf{Irr}_{\mathsf{K}}(\boldsymbol{A})$ or $\theta = A \times A$. We rely on the following observation (see [29, Prop. 4.5]).

Proposition 7.9. *Let K be a quasivariety, $\boldsymbol{A} \in \mathsf{K}$, and $\theta \in \mathsf{Con}_{\mathsf{K}}(\boldsymbol{A})$ n-irreducible. Then there exist $\phi_1, \ldots, \phi_{n-1} \in \mathsf{Irr}_{\mathsf{K}}(\boldsymbol{A})$ such that $\theta = \phi_1 \cap \cdots \cap \phi_{n-1}$.*

We recall that, for $\boldsymbol{A} \leqslant \boldsymbol{B}$ and $\phi \in \mathsf{Con}(\boldsymbol{B})$, we denote by $\boldsymbol{A}/\phi$ the subalgebra of $\boldsymbol{B}/\phi$ with universe $\{a/\phi \in B/\phi : a \in A\}$. We will need the following easy consequence of Zorn's Lemma (see the proof of [29, Prop. 3.7]).

Proposition 7.10. *Let K be a quasivariety, $\boldsymbol{A} \leqslant \boldsymbol{B} \in \mathsf{K}$, and $b \in B - A$. There exists $\phi \in \mathsf{Con}_{\mathsf{K}}(\boldsymbol{B})$ such that $b/\phi \notin A/\phi$ and for each $\theta \in \mathsf{Con}_{\mathsf{K}}(\boldsymbol{B}/\phi) - \{\mathrm{id}_{B/\phi}\}$ there exists $a \in A$ such that $\langle a/\phi, b/\phi \rangle \in \theta$.*

We are now ready to prove Theorem 7.3. We follow a strategy similar to the one used to establish an analogous result [29, Thm. 4.3] in the setting of epimorphisms between finitely generated algebras.

Proof. As the implication (i)⇒(ii) is straightforward, we only detail the implication (ii)⇒(i). To this end, we reason by contraposition. Suppose that K lacks the strong epimorphism surjectivity property. By Theorem 7.1 there exist $\boldsymbol{A} \leqslant \boldsymbol{B} \in \mathsf{K}$ with $\boldsymbol{A}$ and $\boldsymbol{B}$ finitely generated and some $b \in \mathsf{d}_{\mathsf{K}}(\boldsymbol{A}, \boldsymbol{B}) - A$.

Claim 7.11. *We may assume that for each $\theta \in \mathsf{Con}_{\mathsf{K}}(\boldsymbol{B}) - \{\mathrm{id}_B\}$ there exists $a \in A$ such that $\langle a, b \rangle \in \theta$.*

Proof of the Claim. As $\boldsymbol{A} \leqslant \boldsymbol{B} \in \mathsf{K}$ and $b \in \mathsf{d}_{\mathsf{K}}(\boldsymbol{A}, \boldsymbol{B}) - A \subseteq B - A$, we can apply Proposition 7.10, obtaining $\phi \in \mathsf{Con}_{\mathsf{K}}(\boldsymbol{B})$ satisfying the following requirements: $b/\phi \in B/\phi - A/\phi$ and for each $\theta \in \mathsf{Con}_{\mathsf{K}}(\boldsymbol{B}/\phi) - \{\mathrm{id}_{B/\phi}\}$ there exists $a \in A$ such that $\langle a/\phi, b/\phi \rangle \in \theta$.

Clearly, $\boldsymbol{A}/\phi \leqslant \boldsymbol{B}/\phi$ is a proper subalgebra. Moreover, $\boldsymbol{A}/\phi$ and $\boldsymbol{B}/\phi$ are finitely generated members of K because so are $\boldsymbol{A}$ and $\boldsymbol{B}$ by assumption and $\phi \in \mathsf{Con}_{\mathsf{K}}(\boldsymbol{B})$. Corollary 4.6(ii) implies that $b/\phi \in \mathsf{d}_{\mathsf{K}}(\boldsymbol{A}/\phi, \boldsymbol{B}/\phi)$. As $b/\phi \notin A/\phi$, we obtain $b/\phi \in \mathsf{d}_{\mathsf{K}}(\boldsymbol{A}/\phi, \boldsymbol{B}/\phi) - A/\phi$.

Therefore, we may assume that $\phi = \mathrm{id}_B$ (otherwise we replace $\boldsymbol{A}$ and $\boldsymbol{B}$ by $\boldsymbol{A}/\phi$ and $\boldsymbol{B}/\phi$, respectively).

When coupled with the assumption that $\phi = \mathrm{id}_B$, the fact that for each $\theta \in \mathsf{Con}_\mathsf{K}(\boldsymbol{B}/\phi) - \{\mathrm{id}_{B/\phi}\}$ there exists $a \in A$ such that $\langle a/\phi, b/\phi \rangle \in \theta$ implies that for each $\theta \in \mathsf{Con}_\mathsf{K}(\boldsymbol{B}) - \{\mathrm{id}_B\}$ there exists $a \in A$ such that $\langle a, b \rangle \in \theta$. $\boxtimes$

We will rely on the next observation.

Claim 7.12. *The congruence* id_B *is n-irreducible in* $\mathsf{Con}_\mathsf{K}(\boldsymbol{B})$.

Proof of the Claim. Let $\theta_1, \ldots, \theta_n \in \mathsf{Con}_\mathsf{K}(\boldsymbol{B})$ be such that $\mathrm{id}_B = \theta_1 \cap \cdots \cap \theta_n$. Let also $\phi_i = \theta_1 \cap \cdots \cap \theta_{i-1} \cap \theta_{i+1} \cdots \cap \theta_n$ for each $i \leqslant n$. We will show that $\phi_i = \mathrm{id}_B$ for some $i \leqslant n$. Suppose the contrary, with a view to contradiction. Claim 7.11 yields $a_1, \ldots, a_n \in A$ such that $\langle a_i, b \rangle \in \phi_i$ for every $i \leqslant n$. By assumption K has a near unanimity term $t(x_1, \ldots, x_n)$. We will prove that

$$\langle t^{\boldsymbol{B}}(a_1, \ldots, a_n), b \rangle \in \theta_j$$

for every $j \leqslant n$. To this end, consider $j \leqslant n$. As $\langle a_i, b \rangle \in \phi_i \subseteq \theta_j$ for every $i \leqslant n$ such that $i \neq j$, we obtain $\langle t^{\boldsymbol{B}}(a_1, \ldots, a_n), t^{\boldsymbol{B}}(b, \ldots, b, a_j, b, \ldots, b) \rangle \in \theta_j$. Furthermore, since t is a near unanimity term, we have $t^{\boldsymbol{B}}(b, \ldots, b, a_j, b, \ldots, b) = b$. Hence, $\langle t^{\boldsymbol{B}}(a_1, \ldots, a_n), b \rangle \in \theta_j$. This establishes the above display. Together with the assumption that $\mathrm{id}_B = \theta_1 \cap \cdots \cap \theta_n$, this implies $b = t^{\boldsymbol{B}}(a_1, \ldots, a_n)$. As $a_1, \ldots, a_n \in A$ and $\boldsymbol{A} \leqslant \boldsymbol{B}$, we conclude that $b \in A$, which is false. Hence, id_B is n-irreducible. $\boxtimes$

In view of Claim 7.12 and Proposition 7.9, there exist $\theta_1, \ldots, \theta_{n-1} \in \mathsf{Irr}_\mathsf{K}(\boldsymbol{B})$ such that $\mathrm{id}_B = \theta_1 \cap \cdots \cap \theta_{n-1}$. Therefore, we can apply Proposition 2.7 obtaining a subdirect embedding $h \colon \boldsymbol{B} \to \boldsymbol{B}/\theta_1 \times \cdots \times \boldsymbol{B}/\theta_{n-1}$. Let $\boldsymbol{B}_i = \boldsymbol{B}/\theta_i$ for each $i \leqslant n-1$. By replacing $\boldsymbol{A}$ and $\boldsymbol{B}$ by their isomorphic images $h[\boldsymbol{A}]$ and $h[\boldsymbol{B}]$, respectively, we may assume that $\boldsymbol{A} \leqslant \boldsymbol{B} \leqslant \boldsymbol{B}_1 \times \cdots \times \boldsymbol{B}_{n-1}$. Notice that each $\boldsymbol{B}_i = \boldsymbol{B}/\theta$ is finitely generated because so is $\boldsymbol{B}$. Furthermore, from Proposition 2.10 and $\theta_i \in \mathsf{Irr}_\mathsf{K}(\boldsymbol{B})$ it follows that $\boldsymbol{B}_i \in \mathsf{K}_{\mathrm{RFSI}}$. Lastly, as $b \in \mathsf{d}_\mathsf{K}(\boldsymbol{A}, \boldsymbol{B}) - A$ and $\boldsymbol{B} \leqslant \boldsymbol{B}_1 \times \cdots \times \boldsymbol{B}_{n-1}$, Corollary 4.6(ii) allows us to conclude that $b \in \mathsf{d}_\mathsf{K}(\boldsymbol{A}, \boldsymbol{B}_1 \times \cdots \times \boldsymbol{B}_{n-1}) - A$. Hence, $\mathsf{d}_\mathsf{K}(\boldsymbol{A}, \boldsymbol{B}_1 \times \cdots \times \boldsymbol{B}_{n-1}) \neq A$. $\boxtimes$

The literature on epimorphisms contains two variants of Theorem 7.3 in which the class K is required to be an arithmetical variety with the property that the class of its finitely subdirectly irreducible members is closed under ultraproducts and nontrivial subalgebras [23, Thm. 6.8] or only a congruence permutable variety [29, Thm. 5.3]. The first variant deals with the demand that all K-epimorphisms be surjective, while the second with the weaker demand that all K-epimorphisms between finitely generated algebras be surjective called the *weak epimorphism surjectivity property*. In both cases, the conclusion is that failures of the relevant property are witnessed by counterexamples of the form $\boldsymbol{A} \leqslant \boldsymbol{B}$ where $\boldsymbol{B}$ is a finitely subdirectly irreducible member of K. The possibility of obtaining similar results for the *strong* epimorphism surjectivity property is prevented by the following example. However, we will show in Corollary 7.16 that, under the amalgamation property, the above mentioned result for congruence permutable varieties becomes available in the context of the strong epimorphism surjectivity property as well.

Example 7.13 (Heyting algebras)**.** A *Heyting algebra* is an algebra $\langle A; \wedge, \vee, \to, 0, 1 \rangle$ which comprises a bounded distributive lattice $\langle A; \wedge, \vee, 0, 1 \rangle$ and a binary operation $\to$ (called *implication*) such that for all $a, b, c \in A$ we have

$$a \wedge b \leqslant c \iff a \leqslant b \to c.$$

This means that $b \to c$ is the largest element $d \in A$ such that $d \wedge (b \to c) \leqslant c$ (see [7, p. 173]).

As a consequence, Heyting algebras are uniquely determined by their lattice reduct. In particular, every finite distributive lattice $\boldsymbol{A}$ can be expanded uniquely to a Heyting algebra by letting 0 and 1 be the minimum and maximum of $\boldsymbol{A}$, respectively, and defining

$$a \to b = \max\{c \in A : a \wedge c \leqslant b\} \text{ for all } a, b \in A.$$

From a logical standpoint, the importance of Heyting algebras derives from the fact that they algebraize the intuitionistic propositional logic (see, e.g., [100, Ch. IX]).

Let $\boldsymbol{C}$ be the five-element chain, viewed as a Heyting algebra. Then $\mathbb{V}(\boldsymbol{C})$ is an arithmetical variety whose class of finitely subdirectly irreducible members is closed under nontrivial subalgebras and ultraproducts (see, e.g., [21, p. 80] and [40, p. 2 & Thm. 2.3]).

While it is known that $\mathbb{V}(\boldsymbol{C})$ lacks the strong epimorphism surjectivity property (see [88, Thm. 4.2]), it is impossible to find counterexamples to this property of the form $\boldsymbol{A} \leqslant \boldsymbol{B}$, where $\boldsymbol{B}$ is a finitely subdirectly irreducible member of $\mathbb{V}(\boldsymbol{C})$, for in this situation we always have $\mathsf{d}_{\mathbb{V}(\mathsf{C})}(\boldsymbol{A}, \boldsymbol{B}) = A$. ⊠

The next result is well known (see, e.g., [14, Thm. 1.3]). We provide a novel and short proof using the characterization of dominions in the presence of the amalgamation property established in Corollary 4.14.

Theorem 7.14. *Let* K *be a quasivariety with the amalgamation property. Then* K *has the strong epimorphism surjectivity property if and only if it has the weak epimorphism surjectivity property.*

Proof. The implication from left to right is straightforward. So, let us assume that K has the weak epimorphism surjectivity property. We will show that K has the strong epimorphism surjectivity property using Proposition 6.6. To this end, consider $\boldsymbol{A} \leqslant \boldsymbol{B} \in \mathsf{K}$ and $b \in \mathsf{dom}_{\mathsf{K}}(\boldsymbol{A}, \boldsymbol{B})$. Since K has the amalgamation property, by Corollary 4.14 there exist $f \in \mathsf{imp}_{eq}(\mathsf{K})$ and $\langle a_1, \dots, a_n \rangle \in \mathsf{dom}(f^{\boldsymbol{B}}) \cap A^n$ such that $f^{\boldsymbol{B}}(a_1, \dots, a_n) = b$. Let $\boldsymbol{A}' = \mathsf{Sg}^{\boldsymbol{A}}(a_1, \dots, a_n)$ and $\boldsymbol{B}' = \mathsf{Sg}^{\boldsymbol{B}}(a_1, \dots, a_n, b)$. Since f is defined by a conjunction of equations, $\langle a_1, \dots, a_n \rangle \in \mathsf{dom}(f^{\boldsymbol{B}'}) \cap (A')^n$ and $f^{\boldsymbol{B}'}(a_1, \dots, a_n) = b$. So, Corollary 4.14 implies that $b \in \mathsf{d}_{\mathsf{K}}(\boldsymbol{A}', \boldsymbol{B}')$. As $\boldsymbol{B}' = \mathsf{Sg}^{\boldsymbol{B}}(a_1, \dots, a_n, b)$ and $a_1, \dots, a_n, b \in \mathsf{d}_{\mathsf{K}}(\boldsymbol{A}', \boldsymbol{B}')$, we obtain $\mathsf{d}_{\mathsf{K}}(\boldsymbol{A}', \boldsymbol{B}') = B'$. Therefore, the inclusion map $\boldsymbol{A}' \to \boldsymbol{B}'$ is an epimorphism. Since $\boldsymbol{A}', \boldsymbol{B}'$ are finitely generated members of K and K has the weak epimorphism surjectivity property, it follows that $\boldsymbol{A}' = \boldsymbol{B}'$, and hence $b \in B' = A' \subseteq A$. We have shown that $\mathsf{dom}_{\mathsf{K}}(\boldsymbol{A}, \boldsymbol{B}) = A$. Thus, K has the strong epimorphism surjectivity property. ⊠

Given a class of algebras K closed under subalgebras and $\boldsymbol{B} \in \mathsf{K}$, we say that a subalgebra $\boldsymbol{A}$ of $\boldsymbol{B}$ is K-*epic* when the inclusion map $\boldsymbol{A} \to \boldsymbol{B}$ is a K-epimorphism. In this case, K has the epimorphism surjectivity property if and only if every $\boldsymbol{A} \in \mathsf{K}$ lacks proper K-epic subalgebras.

We rely on the following result, which is an immediate consequence of [29, Thm. 5.3] and the proof of [93, Thm. 5.4].

Theorem 7.15. *Let* K *be a congruence permutable variety. Then* K *has the weak epimorphism surjectivity property if and only if the finitely generated members of* $\mathsf{K}_{\mathrm{FSI}}$ *lack proper* K-*epic finitely generated subalgebras.*

As mentioned above, the amalgamation property allows us to obtain a result similar to [23, Thm. 6.8] and [29, Thm. 5.3] for the strong epimorphism surjectivity property in congruence permutable varieties.

Corollary 7.16. *Let* K *be a congruence permutable variety with the amalgamation property. Then* K *has the strong epimorphism surjectivity property if and only if every finitely generated* $\boldsymbol{B} \in \mathsf{K}_{\mathrm{FSI}}$ *lacks proper* K-*epic subalgebras.*

Proof. The implication from left to right is straightforward. On the other hand, if every finitely generated $\boldsymbol{B} \in \mathsf{K}_{\mathrm{FSI}}$ lacks proper K-epic subalgebras, then Theorem 7.15 guarantees that K has the weak epimorphism surjectivity property, which by Theorem 7.14 implies that K has the strong epimorphism surjectivity property as well. $\boxtimes$

The *join* of a family of varieties $\mathsf{K}_1, \ldots, \mathsf{K}_n$ is the least variety containing them, namely, $\mathbb{V}(\mathsf{K}_1 \cup \cdots \cup \mathsf{K}_n)$. While the weak and the strong epimorphism surjectivity properties need not be preserved by joins of varieties, in special cases they are, as we proceed to show.

Theorem 7.17. *Let* K *be an arithmetical variety. If* K *is the join of finitely many varieties with the weak epimorphism surjectivity property, then it has the weak epimorphism surjectivity property.*

Proof. Assume that $\mathsf{K} = \mathbb{V}(\mathsf{K}_1 \cup \cdots \cup \mathsf{K}_n)$, where each K_i is a variety with the weak epimorphism surjectivity property. Suppose, with a view to contradiction, that K lacks this property. By Theorem 7.15 this implies that there exists a finitely generated $\boldsymbol{B} \in \mathsf{K}_{\mathrm{FSI}}$ with a finitely generated subalgebra $\boldsymbol{A} \leqslant \boldsymbol{B}$ that is proper and K-epic. Applying Jónsson's Theorem 2.12, we obtain that $\boldsymbol{B} \in \mathbb{HSP}_{\mathrm{u}}(\mathsf{K}_1 \cup \cdots \cup \mathsf{K}_n)$. By [11, Thm. 5.6] we have $\mathbb{P}_{\mathrm{u}}(\mathsf{K}_1 \cup \cdots \cup \mathsf{K}_n) = \mathbb{P}_{\mathrm{u}}(\mathsf{K}_1) \cup \cdots \cup \mathbb{P}_{\mathrm{u}}(\mathsf{K}_n)$. Therefore, $\boldsymbol{B} \in \mathbb{HSP}_{\mathrm{u}}(\mathsf{K}_1) \cup \cdots \cup \mathbb{HSP}_{\mathrm{u}}(\mathsf{K}_n) \subseteq \mathsf{K}_1 \cup \cdots \cup \mathsf{K}_n$. Then $\boldsymbol{B} \in \mathsf{K}_i$ for some $i \leqslant n$. But K_i has the weak epimorphism surjectivity property by assumption, whence $\boldsymbol{A} \leqslant \boldsymbol{B}$ cannot be a K_i-epic subalgebra. As $\mathsf{K}_i \subseteq \mathsf{K}$, in particular it follows that $\boldsymbol{A} \leqslant \boldsymbol{B}$ cannot be a K-epic subalgebra either. But this contradicts the assumption and thus completes the proof. $\boxtimes$

The following is an immediate consequence of Theorems 7.14 and 7.17.

Corollary 7.18. *Let* K *be an arithmetical variety with the amalgamation property. If* K *is the join of finitely many varieties with the weak epimorphism surjectivity property, then it has the strong epimorphism surjectivity property.*

8. Extendable implicit operations

The implicit operations f of a class of algebras K behave well with respect to extensions, in the sense that if $\boldsymbol{A} \leqslant \boldsymbol{B}$ and $\boldsymbol{A}, \boldsymbol{B} \in \mathsf{K}$, then $f^{\boldsymbol{B}}$ extends $f^{\boldsymbol{A}}$. More precisely, we have the following.

Proposition 8.1. *Let f be an implicit operation of a class of algebras K. For all $\boldsymbol{A}, \boldsymbol{B} \in \mathsf{K}$ with $\boldsymbol{A} \leqslant \boldsymbol{B}$ the partial function $f^{\boldsymbol{B}}$ extends $f^{\boldsymbol{A}}$, in the sense that for all $\langle a_1, \ldots, a_n \rangle \in \mathsf{dom}(f^{\boldsymbol{A}})$ we have*

$$\langle a_1, \ldots, a_n \rangle \in \mathsf{dom}(f^{\boldsymbol{B}}) \quad and \quad f^{\boldsymbol{A}}(a_1, \ldots, a_n) = f^{\boldsymbol{B}}(a_1, \ldots, a_n).$$

Proof. Let $i \colon \boldsymbol{A} \to \boldsymbol{B}$ be the inclusion map and consider $\langle a_1, \ldots, a_n \rangle \in \mathsf{dom}(f^{\boldsymbol{A}})$. Since i is a homomorphism, $\boldsymbol{A}, \boldsymbol{B} \in \mathsf{K}$, and f an implicit operation of K, we obtain

$$\langle i(a_1), \ldots, i(a_n) \rangle \in \mathsf{dom}(f^{\boldsymbol{B}}) \quad \text{and} \quad i(f^{\boldsymbol{A}}(a_1, \ldots, a_n)) = f^{\boldsymbol{B}}(i(a_1), \ldots, i(a_n)).$$

As i is the inclusion map, this yields the desired conclusion. $\boxtimes$

Let f be an implicit operation of a class of algebras K. While Proposition 8.1 guarantees that $f^{\boldsymbol{B}}$ extends $f^{\boldsymbol{A}}$ whenever $\boldsymbol{A}, \boldsymbol{B} \in \mathsf{K}$ and $\boldsymbol{A} \leqslant \boldsymbol{B}$, there is no reason to expect that we can extend $f^{\boldsymbol{A}}$ to a total function in this way. More precisely, there may be no extension $\boldsymbol{B}$ of $\boldsymbol{A}$ in K for which $f^{\boldsymbol{B}}$ is a total function. This makes the following definition attractive.

Definition 8.2. Let M and K be classes of algebras with $\mathsf{M} \subseteq \mathsf{K}$. An n-ary implicit operation f of K is said to be *extendable relative to* M when for all $\boldsymbol{A} \in \mathsf{M}$ and $a_1, \ldots, a_n \in A$ there exists $\boldsymbol{B} \in \mathsf{K}$ such that

$$\boldsymbol{A} \leqslant \boldsymbol{B} \quad \text{and} \quad \langle a_1, \ldots, a_n \rangle \in \mathsf{dom}(f^{\boldsymbol{B}}).$$

The set of implicit operations of K that are extendable relative to M will be denoted by $\mathsf{ext}(\mathsf{M}, \mathsf{K})$. We also let

$$\mathsf{ext}_{pp}(\mathsf{M}, \mathsf{K}) = \mathsf{ext}(\mathsf{M}, \mathsf{K}) \cap \mathsf{imp}_{pp}(\mathsf{K}) \quad \text{and} \quad \mathsf{ext}_{eq}(\mathsf{M}, \mathsf{K}) = \mathsf{ext}(\mathsf{M}, \mathsf{K}) \cap \mathsf{imp}_{eq}(\mathsf{K}).$$

When $\mathsf{M} = \mathsf{K}$, we write $\mathsf{ext}(\mathsf{K}), \mathsf{ext}_{pp}(\mathsf{K})$, and $\mathsf{ext}_{eq}(\mathsf{K})$ instead of $\mathsf{ext}(\mathsf{K}, \mathsf{K}), \mathsf{ext}_{pp}(\mathsf{K}, \mathsf{K})$, and $\mathsf{ext}_{eq}(\mathsf{K}, \mathsf{K})$. Moreover, when an implicit operation is in $\mathsf{ext}(\mathsf{K})$, we simply say it is *extendable*.

Remark 8.3. Let $\mathsf{M}_1, \mathsf{M}_2, \mathsf{K}_1, \mathsf{K}_2$ be classes of algebras with $\mathsf{M}_1 \subseteq \mathsf{M}_2 \subseteq \mathsf{K}_2$ and $\mathsf{K}_1 \subseteq \mathsf{K}_2$. Then the definition of an extendable implicit operation immediately yields that $\mathsf{ext}(\mathsf{M}_2, \mathsf{K}_1) \subseteq \mathsf{ext}(\mathsf{M}_1, \mathsf{K}_2)$. In particular, if M and K are classes of algebras such that $\mathsf{M} \subseteq \mathsf{K}$, then $\mathsf{ext}(\mathsf{K}) \subseteq \mathsf{ext}(\mathsf{M}, \mathsf{K})$. $\boxtimes$

The relation between extendable implicit operations and the idea of "extending partial functions to total ones" is made precise by the next result.

Theorem 8.4. *Let K be a universal class and $\boldsymbol{A} \in \mathsf{K}$. Then there exists $\boldsymbol{B} \in \mathsf{K}$ with $\boldsymbol{A} \leqslant \boldsymbol{B}$ such that $f^{\boldsymbol{B}}$ is total for each $f \in \mathsf{ext}(\mathsf{K})$. When, in addition, K is a quasivariety and $\boldsymbol{A} \in \mathsf{K}_{\mathrm{RSI}}$, the algebra $\boldsymbol{B}$ can be chosen in $\mathsf{K}_{\mathrm{RSI}}$.*

The proof of Theorem 8.4 hinges on the following observation.

Proposition 8.5. *Let* K *be a quasivariety,* $\boldsymbol{A} \in \mathsf{K}_{\mathrm{RSI}}$, *and* $\boldsymbol{B} \in \mathsf{K}$ *with* $\boldsymbol{A} \leqslant \boldsymbol{B}$. *Then there exist* $\boldsymbol{C} \in \mathsf{K}_{\mathrm{RSI}}$ *with* $\boldsymbol{A} \leqslant \boldsymbol{C}$ *and a surjective homomorphism* $h \colon \boldsymbol{B} \to \boldsymbol{C}$.

Proof. By the Subdirect Decomposition Theorem 2.9 there exists a subdirect embedding $g \colon \boldsymbol{B} \to \prod_{i \in I} \boldsymbol{B}_i$ for some family $\{\boldsymbol{B}_i : i \in I\} \subseteq \mathsf{K}_{\mathrm{RSI}}$. From $\boldsymbol{A} \leqslant \boldsymbol{B}$ it follows that $g \colon \boldsymbol{A} \to \prod_{i \in I} p_i[g[\boldsymbol{A}]]$ is also a subdirect embedding. As $\boldsymbol{A} \in \mathsf{K}_{\mathrm{RSI}}$, there exists $j \in I$ such that $p_j \circ g \colon \boldsymbol{A} \to p_j[g[\boldsymbol{A}]]$ is an isomorphism. Together with $p_j[g[\boldsymbol{A}]] \leqslant \boldsymbol{B}_j$, this yields that $p_j \circ g \colon \boldsymbol{A} \to \boldsymbol{B}_j$ is an embedding. Since $\mathsf{K}_{\mathrm{RSI}}$ is closed under $\mathbb{I}$, there exist $\boldsymbol{C} \in \mathsf{K}_{\mathrm{RSI}}$ isomorphic to $\boldsymbol{B}_j$ such that $\boldsymbol{A} \leqslant \boldsymbol{C}$ and a surjective homomorphism $h \colon \boldsymbol{B} \to \boldsymbol{C}$ (the latter is obtained by composing $p_j \circ g \colon \boldsymbol{B} \to \boldsymbol{B}_j$ with the isomorphism between $\boldsymbol{B}_j$ and $\boldsymbol{C}$). $\boxtimes$

We are now ready to prove Theorem 8.4.

Proof. We begin with the following observation.

Claim 8.6. *Let* $\boldsymbol{A} \in \mathsf{K}$. *Then there exists* $\boldsymbol{B} \in \mathsf{K}$ *with* $\boldsymbol{A} \leqslant \boldsymbol{B}$ *such that* $A^n \subseteq \mathsf{dom}(f^{\boldsymbol{B}})$ *for each* n-*ary* $f \in \mathsf{ext}(\mathsf{K})$.

Proof of the Claim. Recall from Theorem 3.9 that each implicit operation f of K is defined by an existential positive formula φ_f. Then consider the following set of formulas in the language of K expanded with fresh constants $\{c_a : a \in A\}$ for the elements of A:

$$\Sigma = \{\exists y \varphi_f(c_{a_1}, \ldots, c_{a_n}, y) : n \in \mathbb{N},\ a_1, \ldots, a_n \in A,\ \text{and}\ f \in \mathsf{ext}(\mathsf{K})\ \text{is}\ n\text{-ary}\}.$$

Moreover, let Γ be a set of axioms for K (which is an elementary class by assumption) and define

$$\Delta = \mathsf{diag}(\boldsymbol{A}) \cup \Sigma \cup \Gamma.$$

We will prove that Δ has a model.

By the Compactness Theorem 1.2 it suffices to show that so does each finite subset of Δ. To this end, consider $a_1, \ldots, a_k \in A$ and $f_1, \ldots, f_m \in \mathsf{ext}(\mathsf{K})$ such that f_i has arity n_i for each $i \leqslant m$. Moreover, for each $i \leqslant m$ let $a_1^i, \ldots, a_{n_i}^i \in \{a_1, \ldots, a_k\}$. We need to prove that the following set has a model:

$$\mathsf{diag}(\mathsf{Sg}^{\boldsymbol{A}}(a_1, \ldots, a_k)) \cup \{\exists y \varphi_{f_i}(c_{a_1^i}, \ldots, c_{a_{n_i}^i}, y) : 1 \leqslant i \leqslant m\} \cup \Gamma. \tag{25}$$

To this end, we shall define a sequence $\boldsymbol{A}_0 \leqslant \boldsymbol{A}_1 \leqslant \cdots \leqslant \boldsymbol{A}_m$ of members of K. First, let $\boldsymbol{A}_0 = \mathsf{Sg}^{\boldsymbol{A}}(a_1, \ldots, a_k)$. Clearly, $\boldsymbol{A}_0 \in \mathsf{K}$ because $\boldsymbol{A}_0 \leqslant \boldsymbol{A} \in \mathsf{K}$ and K is a universal class by assumption. Then suppose that the sequence $\boldsymbol{A}_0 \leqslant \cdots \leqslant \boldsymbol{A}_i$ has already been defined for $i < m$. Since $\boldsymbol{A}_0 \leqslant \boldsymbol{A}_i$ we have $a_1^{i+1}, \ldots, a_{n_{i+1}}^{i+1} \in \{a_1, \ldots, a_k\} \subseteq A_0 \subseteq A_i$. As $f_{i+1} \in \mathsf{ext}(\mathsf{K})$ is n_{i+1}-ary and $\boldsymbol{A}_i \in \mathsf{K}$, there exists $\boldsymbol{A}_{i+1} \in \mathsf{K}$ such that $\langle a_1^{i+1}, \ldots, a_{n_{i+1}}^{i+1} \rangle \in \mathsf{dom}(f_{i+1}^{\boldsymbol{A}_{i+1}})$. Clearly, $\boldsymbol{A}_0 \leqslant \cdots \leqslant \boldsymbol{A}_{i+1}$ is still a sequence of members of K. This concludes the definition of $\boldsymbol{A}_0 \leqslant \boldsymbol{A}_1 \leqslant \cdots \leqslant \boldsymbol{A}_m$.

Observe that $\mathsf{Sg}^{\boldsymbol{A}}(a_1, \ldots, a_k) = \boldsymbol{A}_0 \leqslant \boldsymbol{A}_m$. Then let $\boldsymbol{A}_m^+$ be the expansion of $\boldsymbol{A}_m$ with constants in $\{c_a : a \in \mathsf{Sg}^{\boldsymbol{A}}(a_1, \ldots, a_k)\}$ in which each c_a is interpreted as a. We will prove that $\boldsymbol{A}_m^+$ is a model of the set of formulas in (25). From $\mathsf{Sg}^{\boldsymbol{A}}(a_1, \ldots, a_k) = \boldsymbol{A}_0 \leqslant \boldsymbol{A}_m$ and the Diagram Lemma 2.20 it follows that $\boldsymbol{A}_m^+$ is a model of $\mathsf{diag}(\mathsf{Sg}^{\boldsymbol{A}}(a_1, \ldots, a_k))$. Furthermore, $\boldsymbol{A}_m^+ \vDash \Gamma$ because $\boldsymbol{A}_m \in \mathsf{K}$ and Γ axiomatizes K. Therefore, it only remains to show that

$\boldsymbol{A}_m^+ \vDash \exists y \varphi_{f_i}(c_{a_1^i}, \ldots, c_{a_{n_i}^i}, y)$ for each $i \leqslant m$. As each c_a is interpreted as $a \in A_0 \subseteq A_m$ in $\boldsymbol{A}_m^+$, this amounts to

$$\boldsymbol{A}_m \vDash \exists y \varphi_{f_i}(a_1^i, \ldots, a_{n_i}^i, y) \text{ for each } i \leqslant m.$$

Consider $i \leqslant m$. The construction of $\boldsymbol{A}_i$ guarantees that $\langle a_1^i, \ldots, a_{n_i}^i \rangle \in \mathsf{dom}(f_i^{\boldsymbol{A}_i})$. As f_i is defined by φ_{f_i}, this yields $\boldsymbol{A}_i \vDash \varphi_{f_i}(a_1^i, \ldots, a_{n_i}^i, f_i^{\boldsymbol{A}_i}(a_1^i, \ldots, a_{n_i}^i))$. Since $\boldsymbol{A}_i \leqslant \boldsymbol{A}_m$ and φ_{f_i} is an existential positive formula, we can apply Theorem 1.1(i), obtaining $\boldsymbol{A}_m \vDash \varphi_{f_i}(a_1^i, \ldots, a_{n_i}^i, f_i^{\boldsymbol{A}_i}(a_1^i, \ldots, a_{n_i}^i))$, whence $\boldsymbol{A}_m \vDash \exists y \varphi_{f_i}(a_1^i, \ldots, a_{n_i}^i, y)$. Thus, we conclude that $\boldsymbol{A}_m^+$ is a model of the set of formulas in (25).

As we mentioned, from the fact that the set of formulas in (25) has a model it follows that Δ also has a model $\boldsymbol{B}^+$. Let $\boldsymbol{B}$ be the $\mathscr{L}_{\mathsf{K}}$-reduct of $\boldsymbol{B}^+$. Since $\boldsymbol{B}^+$ is a model of Γ, so is $\boldsymbol{B}$. Together with the assumption that Γ axiomatizes K, this yields $\boldsymbol{B} \in \mathsf{K}$. Furthermore, as $\boldsymbol{B}^+$ is a model of $\mathsf{diag}(\boldsymbol{A})$, we can apply the Diagram Lemma 2.20, obtaining that $\boldsymbol{A}$ embeds into $\boldsymbol{B}$ via the map that sends a to the interpretation of c_a in $\boldsymbol{B}^+$. Since K is an elementary class, it is closed under $\mathbb{I}$. Therefore, we may assume that $\boldsymbol{A} \leqslant \boldsymbol{B}$ and that c_a is interpreted as a in $\boldsymbol{B}^+$.

To conclude the proof of the claim it only remains to show that $A^n \subseteq \mathsf{dom}(f^{\boldsymbol{B}})$ for each n-ary $f \in \mathsf{ext}(\mathsf{K})$. To this end, consider an n-ary $f \in \mathsf{ext}(\mathsf{K})$ and $a_1, \ldots, a_n \in A$. As $\boldsymbol{B}^+$ is a model of Σ, we obtain $\boldsymbol{B} \vDash \exists y \varphi_f(a_1, \ldots, a_n, y)$. Since φ_f defines f, this amounts to $\langle a_1, \ldots, a_n \rangle \in \mathsf{dom}(f^{\boldsymbol{B}})$. ⊠

Now, we proceed to prove the first part of the statement of Theorem 8.4. Consider $\boldsymbol{A} \in \mathsf{K}$. We will define a sequence $\{\boldsymbol{A}_i : i \in \mathbb{N}\}$ of members of K. First, let $\boldsymbol{A}_0 = \boldsymbol{A}$. Then suppose $\boldsymbol{A}_i \in \mathsf{K}$ has already been defined. By Claim 8.6 there exists $\boldsymbol{A}_{i+1} \in \mathsf{K}$ with $\boldsymbol{A}_i \leqslant \boldsymbol{A}_{i+1}$ and $A_i^n \subseteq \mathsf{dom}(f^{\boldsymbol{A}_{i+1}})$ for each n-ary $f \in \mathsf{ext}(\mathsf{K})$. By definition the sequence $\{\boldsymbol{A}_i : i \in \mathbb{N}\}$ constructed in this way is such that

$$\boldsymbol{A} = \boldsymbol{A}_0 \leqslant \boldsymbol{A}_1 \leqslant \boldsymbol{A}_2 \leqslant \cdots$$

Now, as K is a universal class, it is closed under unions of chains of algebras by Proposition 2.4. Therefore, the union $\boldsymbol{B}$ of the chain in the above display belongs to K. Furthermore, $\boldsymbol{A} = \boldsymbol{A}_0 \leqslant \boldsymbol{B}$. To conclude the proof of the first part of the statement, it only remains to show that $f^{\boldsymbol{B}}$ is total for each $f \in \mathsf{ext}(\mathsf{K})$. To this end, let $f \in \mathsf{ext}(\mathsf{K})$ be n-ary and $b_1, \ldots, b_n \in B$. As $\boldsymbol{B}$ is the union of the chain in the above display, there exists $i \in \mathbb{N}$ such that $b_1, \ldots, b_n \in A_i$. By the definition of $\boldsymbol{A}_{i+1}$ we have $\langle b_1, \ldots, b_n \rangle \in A_i^n \subseteq \mathsf{dom}(f^{\boldsymbol{A}_{i+1}})$. As $\boldsymbol{A}_{i+1} \leqslant \boldsymbol{B}$, we can apply Proposition 8.1, obtaining $\langle b_1, \ldots, b_n \rangle \in \mathsf{dom}(f^{\boldsymbol{B}})$. Hence, $f^{\boldsymbol{B}}$ is a total operation, as desired.

To prove the second part of the statement of Theorem 8.4, suppose that K is a quasivariety and consider $\boldsymbol{A} \in \mathsf{K}_{\mathrm{RSI}}$. In view of the first part of the statement of Theorem 8.4, there exists $\boldsymbol{B} \in \mathsf{K}$ with $\boldsymbol{A} \leqslant \boldsymbol{B}$ such that $f^{\boldsymbol{B}}$ is total for each $f \in \mathsf{ext}(\mathsf{K})$. By Proposition 8.5 there also exist $\boldsymbol{C} \in \mathsf{K}_{\mathrm{RSI}}$ with $\boldsymbol{A} \leqslant \boldsymbol{C}$ and a surjective homomorphism $h \colon \boldsymbol{B} \to \boldsymbol{C}$. To conclude the proof, it only remains to show that $f^{\boldsymbol{C}}$ is total for each $f \in \mathsf{ext}(\mathsf{K})$. To this end, consider an n-ary $f \in \mathsf{ext}(\mathsf{K})$ and $c_1, \ldots, c_n \in C$. Since $h \colon \boldsymbol{B} \to \boldsymbol{C}$ is surjective, there exist $b_1, \ldots, b_n \in B$ such that $h(b_j) = c_j$ for each $j \leqslant n$. Recall that $f^{\boldsymbol{B}}$ is total because

$f \in \mathsf{ext}(\mathsf{K}^{\mathrm{fg}}, \mathsf{K})$. Therefore, $\langle b_1, \ldots, b_n \rangle \in \mathsf{dom}(f^{\boldsymbol{B}})$. As f is an implicit operation and h a homomorphism, we conclude that $\langle c_1, \ldots, c_n \rangle = \langle h(b_1), \ldots, h(b_n) \rangle \in \mathsf{dom}(f^{\boldsymbol{C}})$. ⊠

The following is a consequence of Theorem 8.4.

Corollary 8.7. *Given a universal class* K*, the classes* $\mathsf{ext}(\mathsf{K})$ *and* $\mathsf{ext}_{pp}(\mathsf{K})$ *are closed under composition.*

Proof. Consider an n-ary $g \in \mathsf{ext}(\mathsf{K})$ and m-ary $f_1, \ldots, f_n \in \mathsf{ext}(\mathsf{K})$. Let $\boldsymbol{A} \in \mathsf{K}$. By Theorem 8.4 there exists $\boldsymbol{B} \in \mathsf{K}$ with $\boldsymbol{A} \leqslant \boldsymbol{B}$ such that $g^{\boldsymbol{B}}, f_1^{\boldsymbol{B}}, \ldots, f_n^{\boldsymbol{B}}$ are total. It then follows from the definition of composition that $g(f_1, \ldots, _n)^{\boldsymbol{B}}$ is also total. Therefore, $\mathsf{ext}(\mathsf{K})$ is closed under composition. As $\mathsf{imp}_{pp}(\mathsf{K})$ is closed under composition by Proposition 3.13, we obtain that $\mathsf{ext}_{pp}(\mathsf{K})$ is also closed under composition because $\mathsf{ext}_{pp}(\mathsf{K}) = \mathsf{ext}(\mathsf{K}) \cap \mathsf{imp}_{pp}(\mathsf{K})$. ⊠

Example 8.8 (Cancellative commutative monoids)**.** Recall from Example 4.16 that the class of cancellative commutative monoids forms a quasivariety, which we denote by CCMon. The importance of cancellative commutative monoids is due to the following well-known result (see, e.g., [84, pp. 39–40]).

Theorem 8.9. *The quasivariety of cancellative commutative monoids is the class of monoid subreducts of Abelian groups.*

Recall from Theorem 3.7 that "taking inverses" is an implicit operation of the variety of monoids, definable by the conjunction of equations $\varphi = (x \cdot y \approx 1) \sqcap (y \cdot x \approx 1)$. Clearly, its restriction to CCMon is an implicit operation of CCMon, which is defined by the equation $x \cdot y \approx 1$. We will prove the following.

Theorem 8.10. *Taking inverses is a unary extendable implicit operation of the quasivariety of cancellative commutative monoids, which, moreover, can be defined by the equation* $x \cdot y \approx 1$*.*

Proof. It suffices to prove that the implicit operation f of "taking inverses" in CCMon is extendable. To this end, consider $\boldsymbol{A} \in \mathsf{CCMon}$ and $a \in A$. In view of Theorem 8.9, $\boldsymbol{A}$ is a subreduct of an Abelian group $\boldsymbol{B}$. Let $\boldsymbol{C}$ be the monoid reduct of $\boldsymbol{B}$. Since $\boldsymbol{B}$ is an Abelian group, $\boldsymbol{C}$ is a cancellative commutative monoid by Theorem 8.9. Therefore, $\boldsymbol{C} \in \mathsf{CCMon}$. Furthermore, $a \in A \subseteq C$ has an inverse in $\boldsymbol{C}$ because $\boldsymbol{C}$ is the reduct of a group. Therefore, $a \in \mathsf{dom}(f^{\boldsymbol{C}})$. Hence, we conclude that f is extendable. ⊠

On the other hand, the implicit operation f of "taking inverses" in the variety of all monoids is not extendable. For suppose the contrary, with a view to contradiction. By Theorem 8.4 this implies that for each monoid $\boldsymbol{A}$ there exists a monoid $\boldsymbol{B}$ such that $f^{\boldsymbol{B}}$ is total, that is, such that $\boldsymbol{B}$ is the reduct of a group. As a consequence, we obtain that every monoid embeds into the monoid reduct of a group. But this is false because monoid subreducts of groups need to be cancellative and noncancellative monoids exist (e.g., full transformation monoids). We conclude that f is not extendable. An analogous argument shows that the restriction of f to the variety of commutative monoids is also not extendable. ⊠

The next results simplify the task of proving that an implicit operation is extendable.

Proposition 8.11. *Let* K *be an elementary class and* $\mathsf{M} \subseteq \mathsf{K}$. *The following conditions hold:*

(i) *if* $\mathsf{K} \subseteq \mathbb{U}(\mathsf{M})$, *then* $\mathsf{ext}(\mathsf{K}) = \mathsf{ext}(\mathsf{M}, \mathsf{K})$;

(ii) *if* $\mathbb{P}(\mathsf{K}) \subseteq \mathsf{K} \subseteq \mathbb{Q}(\mathsf{M})$, *then* $\mathsf{ext}_{pp}(\mathsf{K}) = \mathsf{ext}_{pp}(\mathsf{M}, \mathsf{K})$.

Proof. We begin with the following observation.

Claim 8.12. *We have* $\mathsf{ext}(\mathsf{M}, \mathsf{K}) \subseteq \mathsf{ext}(\mathbb{P}_{\mathrm{u}}(\mathsf{M}), \mathsf{K})$.

Proof of the Claim. Consider $f \in \mathsf{ext}(\mathsf{M}, \mathsf{K})$, $\boldsymbol{A} \in \mathbb{P}_{\mathrm{u}}(\mathsf{M})$ and $a_1, \ldots, a_n \in A$. Then there exist $\{\boldsymbol{A}_i : i \in I\} \subseteq \mathsf{M}$ and an ultrafilter U on I such that $\boldsymbol{A} = \prod_{i \in I} \boldsymbol{A}_i / U$. Moreover, there exist $a_1^*, \ldots, a_n^* \in \prod_{i \in I} A_i$ such that $a_i = a_i^*/U$ for each $i \leqslant n$. Lastly, since f is an implicit operation of K, it is defined by a formula φ.

As $\{\boldsymbol{A}_i : i \in I\} \subseteq \mathsf{M}$, from the assumptions it follows that for each $i \in I$ there exists $\boldsymbol{B}_i \in \mathsf{K}$ with $\boldsymbol{A}_i \leqslant \boldsymbol{B}_i$ such that $\langle a_1^*(i), \ldots, a_n^*(i) \rangle \in \mathsf{dom}(f^{\boldsymbol{B}_i})$. Since φ defines f, this yields $\boldsymbol{B}_i \vDash \exists y \varphi(a_1^*(i), \ldots, a_n^*(i), y)$ for each $i \in I$. Let $\boldsymbol{B} = \prod_{i \in I} \boldsymbol{B}_i / U$. By Łoś' Theorem 1.6 we have

$$\boldsymbol{B} \vDash \exists y \varphi(a_1^*/U, \ldots, a_n^*/U, y).$$

Observe that $\boldsymbol{B} \in \mathsf{K}$ because $\{\boldsymbol{B}_i : i \in I\} \subseteq \mathsf{K}$ and K is an elementary class by assumption and, therefore, closed under $\mathbb{P}_{\mathrm{u}}$. Together with the fact that φ defines f and the above display, this yields $\langle a_1^*/U, \ldots, a_n^*/U \rangle \in \mathsf{dom}(f^{\boldsymbol{B}})$. Lastly, recall that $\boldsymbol{A}_i \leqslant \boldsymbol{B}_i$ for each $i \in I$. As a consequence, the map $h \colon \boldsymbol{A} \to \boldsymbol{B}$ defined by the rule $h(a/U) = a/U$ is an embedding and

$$\langle h(a_1), \ldots, h(a_n) \rangle = \langle h(a_1^*/U), \ldots, h(a_n^*/U) \rangle = \langle a_1^*/U, \ldots, a_n^*/U \rangle \in \mathsf{dom}(f^{\boldsymbol{B}}).$$

As K is closed under $\mathbb{I}$ (because it is an elementary class), we may assume that $h \colon \boldsymbol{A} \to \boldsymbol{B}$ is the inclusion map. Therefore, we obtain that $\boldsymbol{A} \leqslant \boldsymbol{B} \in \mathsf{K}$ and $\langle a_1, \ldots, a_n \rangle \in \mathsf{dom}(f^{\boldsymbol{B}})$, as desired. ⊠

(i): Suppose that $\mathsf{K} \subseteq \mathbb{U}(\mathsf{M})$. The inclusion from left to right follows from Remark 8.3. To prove the other inclusion consider $f \in \mathsf{ext}(\mathsf{M}, \mathsf{K})$, $\boldsymbol{A} \in \mathsf{K}$, and $a_1, \ldots, a_n \in A$. By Theorem 2.2 we have $\mathbb{U}(\mathsf{M}) = \mathbb{ISP}_{\mathrm{u}}(\mathsf{M})$. Together with $\boldsymbol{A} \in \mathsf{K} \subseteq \mathbb{U}(\mathsf{M})$, this yields $\boldsymbol{A} \in \mathbb{ISP}_{\mathrm{u}}(\mathsf{M})$. Therefore, there exist $\boldsymbol{B} \in \mathbb{P}_{\mathrm{u}}(\mathsf{M})$ and an embedding $h \colon \boldsymbol{A} \to \boldsymbol{B}$. By Claim 8.12 there exists also $\boldsymbol{C} \in \mathsf{K}$ such that $\boldsymbol{B} \leqslant \boldsymbol{C}$ and $\langle h(a_1), \ldots, h(a_n) \rangle \in \mathsf{dom}(f^{\boldsymbol{C}})$.

Since $\boldsymbol{C} \in \mathsf{K}$ and K is closed under $\mathbb{I}$, we may assume that $h \colon \boldsymbol{A} \to \boldsymbol{C}$ is the inclusion map. Consequently, we obtain that $\boldsymbol{A} \leqslant \boldsymbol{B} \leqslant \boldsymbol{C} \in \mathsf{K}$ and $\langle a_1, \ldots, a_n \rangle \in \mathsf{dom}(f^{\boldsymbol{C}})$. Hence, we conclude that f is extendable.

(ii): Suppose that $\mathbb{P}(\mathsf{K}) \subseteq \mathsf{K} \subseteq \mathbb{Q}(\mathsf{M})$. The inclusion from left to right follows from Remark 8.3. To prove the other inclusion consider $f \in \mathsf{ext}_{pp}(\mathsf{M}, \mathsf{K})$, $\boldsymbol{A} \in \mathsf{K}$, and $a_1, \ldots, a_n \in A$. Assume that f is defined by a pp formula φ. By Theorem 2.2 we have $\mathbb{Q}(\mathsf{M}) = \mathbb{ISPP}_{\mathrm{u}}(\mathsf{M})$. Together with $\boldsymbol{A} \in \mathsf{K} \subseteq \mathbb{Q}(\mathsf{M})$, this yields $\boldsymbol{A} \in \mathbb{ISPP}_{\mathrm{u}}(\mathsf{M})$. Therefore, there exist $\{\boldsymbol{B}_i : i \in I\} \subseteq \mathbb{P}_{\mathrm{u}}(\mathsf{M})$ and an embedding $h \colon \boldsymbol{A} \to \prod_{i \in I} \boldsymbol{B}_i$. By Claim 8.12 there exists also $\{\boldsymbol{C}_i : i \in I\} \subseteq \mathsf{K}$ such that $\boldsymbol{B}_i \leqslant \boldsymbol{C}_i$ and $\langle p_i(h(a_1)), \ldots, p_i(h(a_n)) \rangle \in \mathsf{dom}(f^{\boldsymbol{C}_i})$ for each $i \in I$.

Let $\boldsymbol{C} = \prod_{i \in I} \boldsymbol{C}_i$. From $\{\boldsymbol{C}_i : i \in I\} \subseteq \mathsf{K}$ and the assumption that $\mathbb{P}(\mathsf{K}) \subseteq \mathsf{K}$ it follows that $\boldsymbol{C} \in \mathsf{K}$. Furthermore, $\prod_{i \in I} \boldsymbol{B}_i \leqslant \boldsymbol{C}$ because $\boldsymbol{B}_i \leqslant \boldsymbol{C}_i$ for each $i \in I$. Therefore, $h \colon \boldsymbol{A} \to \prod_{i \in I} \boldsymbol{B}_i$ can be viewed as an embedding $h \colon \boldsymbol{A} \to \boldsymbol{C}$. Lastly, recall that

$\langle p_i(h(a_1)), \ldots, p_i(h(a_n)) \rangle \in \mathsf{dom}(f^{\boldsymbol{C}_i})$ for each $i \in I$. As φ defines f, this amounts to $\boldsymbol{C}_i \vDash \exists y \varphi(p_i(h(a_1)), \ldots, p_i(h(a_n)), y)$ for each $i \in I$. As φ is a pp formula by assumption, we can apply Theorem 1.1(ii) to the definition of $\boldsymbol{C}$, obtaining $\boldsymbol{C} \vDash \exists y \varphi(h(a_1), \ldots, h(a_n), y)$. Since $\boldsymbol{C} \in \mathsf{K}$, this amounts to $\langle h(a_1), \ldots, h(a_n) \rangle \in \mathsf{dom}(f^{\boldsymbol{C}})$.

Since $\boldsymbol{C} \in \mathsf{K}$ and K is closed under $\mathbb{I}$, we may assume that $h \colon \boldsymbol{A} \to \boldsymbol{C}$ is the inclusion map. Consequently, we obtain that $\boldsymbol{A} \leqslant \prod_{i \in I} \boldsymbol{B}_i \leqslant \boldsymbol{C} \in \mathsf{K}$ and $\langle a_1, \ldots, a_n \rangle \in \mathsf{dom}(f^{\boldsymbol{C}})$. Hence, we conclude that f is extendable. $\boxtimes$

Recall that, given a class K of algebras, we denote the class of finitely generated members of K by K^{fg}.

Theorem 8.13. *The following conditions hold for a class K of algebras:*

(i) *if K is a universal class, then $\mathsf{ext}(\mathsf{K}) = \mathsf{ext}(\mathsf{K}^{\mathrm{fg}}, \mathsf{K})$;*
(ii) *if K is a quasivariety, then $\mathsf{ext}_{pp}(\mathsf{K}) = \mathsf{ext}_{pp}(\mathsf{K}^{\mathrm{fg}}_{\mathrm{RSI}}, \mathsf{K})$.*

Proof. (i): Let K be a universal class and recall from Proposition 2.15 that $\mathsf{K} = \mathbb{U}(\mathsf{K}^{\mathrm{fg}})$. Therefore, K is elementary and $\mathsf{K}^{\mathrm{fg}} \subseteq \mathsf{K} \subseteq \mathbb{U}(\mathsf{K}^{\mathrm{fg}})$. Consequently, we can apply Proposition 8.11(i) to the case where $\mathsf{M} = \mathsf{K}^{\mathrm{fg}}$, obtaining $\mathsf{ext}(\mathsf{K}) = \mathsf{ext}(\mathsf{K}^{\mathrm{fg}}, \mathsf{K})$.

(ii): The inclusion from left to right is straightforward. To prove the other inclusion, observe that Proposition 2.16 guarantees that $\mathsf{K} = \mathbb{Q}(\mathsf{K}^{\mathrm{fg}}_{\mathrm{RSI}})$. Moreover, $\mathbb{P}(\mathsf{K}) \subseteq \mathsf{K}$ because K is a quasivariety. Therefore, we can apply Proposition 8.11(ii), obtaining $\mathsf{ext}_{pp}(\mathsf{K}) = \mathsf{ext}_{pp}(\mathsf{K}^{\mathrm{fg}}_{\mathrm{RSI}}, \mathsf{K})$. $\boxtimes$

Corollary 8.14. *A pp formula $\varphi(x_1, \ldots, x_n, y)$ defines an extendable implicit operation of a quasivariety K if and only if for each $\boldsymbol{A} \in \mathsf{K}^{\mathrm{fg}}_{\mathrm{RSI}}$ there exists $\boldsymbol{B} \in \mathsf{K}$ with $\boldsymbol{A} \leqslant \boldsymbol{B}$ such that for all $a_1, \ldots, a_n \in B$ there exists a unique $b \in B$ such that $\boldsymbol{B} \vDash \varphi(a_1, \ldots, a_n, b)$. The equivalence still holds if we require $\boldsymbol{B}$ to be a member of $\mathsf{K}_{\mathrm{RSI}}$.*

Proof. The implication from left to right and the last part of the statement follow from Theorem 8.4. To prove the implication from right to left, assume that for each $\boldsymbol{A} \in \mathsf{K}^{\mathrm{fg}}_{\mathrm{RSI}}$ there exists $\boldsymbol{A}^* \in \mathsf{K}$ with $\boldsymbol{A} \leqslant \boldsymbol{A}^*$ such that for all $a_1, \ldots, a_n \in A^*$ there exists a unique $a \in A^*$ such that $\boldsymbol{A}^* \vDash \varphi(a_1, \ldots, a_n, b)$.

We begin by showing that φ defines an implicit operation of K. Let

$$\mathsf{M} = \{ \boldsymbol{A}^* : \boldsymbol{A} \in \mathsf{K}^{\mathrm{fg}}_{\mathrm{RSI}} \}.$$

Observe that φ is functional in M by assumption. As φ is a pp formula, we can apply Corollary 3.11, obtaining that φ is functional in $\mathbb{Q}(\mathsf{M})$ as well. Since $\mathsf{M} \subseteq \mathsf{K}$ and $\mathsf{K}^{\mathrm{fg}}_{\mathrm{RSI}} \subseteq \mathbb{S}(\mathsf{M})$, we have $\mathsf{K} = \mathbb{Q}(\mathsf{M})$ by Proposition 2.16. Hence, we conclude that φ defines an implicit operation f of K which, moreover is extendable by Theorem 8.13(ii). $\boxtimes$

So far, the only concrete example of an extendable implicit operation that we have encountered is that of "taking inverses" in the quasivariety of cancellative commutative monoids (see Example 8.8). We close this section with five additional examples related to filtral quasivarieties, reduced commutative rings, distributive lattices, Hilbert algebras, and pseudocomplemented distributive lattices.

Example 8.15 (Filtral quasivarieties)**.** A quasivariety K is said to be *relatively filtral* when for every subdirect product $\boldsymbol{A} \leqslant \prod_{i \in I} \boldsymbol{A}_i$ with $\{\boldsymbol{A}_i : i \in I\} \subseteq \mathsf{K}_{\mathrm{RSI}}$ and every $\theta \in \mathsf{Con}_{\mathsf{K}}(\boldsymbol{A})$ there exists a filter F on I such that

$$\theta = \{\langle a, b\rangle \in A \times A : [\![a \approx b]\!] \in F\}.$$

When K is a variety, we simply say that it is *filtral*. This notion originated in the context of varieties [87] and was extended to quasivarieties in [24]. Examples of filtral varieties include the variety of (bounded) distributive lattices (see, e.g., [10, Ex. 3]).

We recall that the *quaternary discriminator* function on a set A is the function $d_A \colon A^4 \to A$ defined for all $a, b, c, d \in A$ as

$$d_A(a, b, c, d) = \begin{cases} c & \text{if } a = b; \\ d & \text{otherwise.} \end{cases}$$

We will prove the following.

Theorem 8.16. *Let K be a relatively filtral quasivariety. Then there exists a quaternary $f \in \mathsf{ext}_{eq}(\mathsf{K})$ such that $f^{\boldsymbol{A}}$ is total and coincides with the quaternary discriminator function on A for each $\boldsymbol{A} \in \mathsf{K}_{\mathrm{RSI}}$.*

Proof. Consider a relatively filtral quasivariety K. From [24, Thm. 6.3] and the implication $(4) \Rightarrow (1)$ in [25, Thm. 4.1] it follows that there exists a conjunction of equations $\varphi(x_1, x_2, x_3, x_4, y)$ such that for all $\boldsymbol{A} \in \mathsf{K}_{\mathrm{RSI}}$ and $a, b, c, d, e \in A$,

$$\boldsymbol{A} \vDash \varphi(a, b, c, d, e) \iff d_A(a, b, c, d) = e.$$

Therefore, for all $\boldsymbol{A} \in \mathsf{K}_{\mathrm{RSI}}$ and $a, b, c, d \in A$ there exists a unique $e \in A$ such that $\boldsymbol{A} \vDash \varphi(a, b, c, d, e)$. Then Corollary 8.14 implies that φ defines an extendable implicit operation f of K. In view of the above display, $f^{\boldsymbol{A}}$ coincides with d_A for each $\boldsymbol{A} \in \mathsf{K}_{\mathrm{RSI}}$. ⊠

Example 8.17 (Reduced commutative rings)**.** In Example 3.16 we proved that there exists an implicit operation of the quasivariety RCRing of reduced commutative rings that coincides with the operation of "taking weak inverses" in fields. We now show that this operation is extendable.

Theorem 8.18. *There exists a unary $f \in \mathsf{ext}_{eq}(\mathsf{RCRing})$ such that $f^{\boldsymbol{A}}$ is total and coincides with the operation of taking weak inverses for each field $\boldsymbol{A}$.*

Proof. Let f be the implicit operation of RCRing given by Theorem 3.18. Then $f^{\boldsymbol{A}}$ is total and coincides with the operation of taking weak inverses for every field $\boldsymbol{A}$. Moreover, f is defined by a conjunction of equations, whence $f \in \mathsf{imp}_{eq}(\mathsf{RCRing})$. Therefore, it suffices to prove that f is extendable. To this end, recall from Theorem 3.17 that $\mathsf{RCRing} = \mathbb{Q}(\mathsf{Field})$, where Field is the class of fields. As $f^{\boldsymbol{A}}$ is total for each field $\boldsymbol{A}$, we can apply Proposition 8.11(ii) (taking $\mathsf{M} = \mathsf{Fields}$), obtaining that f is extendable. ⊠

Example 8.19 (Distributive lattices)**.** Recall from Example 3.19 that "taking relative complements" defines an implicit operation of the variety DL of distributive lattices and that "taking complements" defines an implicit operation of the variety bDL of bounded distributive lattices. We show that these operations are extendable.

Theorem 8.20. *The following conditions hold:*

(i) *the operation of taking relative complements in* DL *is a ternary member of* $\mathsf{ext}_{eq}(\mathsf{DL})$*;*

(ii) *the operation of taking complements in* bDL *is a unary member of* $\mathsf{ext}_{eq}(\mathsf{bDL})$*.*

Proof. We detail only the proof of (i), as the proof of (ii) is analogous. In view of Theorem 3.20, it suffices to show that the implicit operation f of "taking relative complements" of DL is extendable. Let $\boldsymbol{D}_2$ be the two-element lattice and observe that $f^{\boldsymbol{D}_2}$ it total. As $\boldsymbol{D}_2$ is (up to isomorphism) the only subdirectly irreducible member of DL (see, e.g., [11, Ex. 3.19]), we obtain that $f^{\boldsymbol{A}}$ it total for each $\boldsymbol{A} \in \mathsf{DL}_{\mathrm{SI}}$. Therefore, we can apply Theorem 8.13(ii), obtaining that f is extendable. ⊠

Example 8.21 (Hilbert algebras)**.** The implication subreducts of Heyting algebras are known as *Hilbert algebras* (see, e.g., [44]). Hilbert algebras form a variety (see [44, Thm. 3]) that we denote by Hilbert. Every Hilbert algebra $\langle A; \to \rangle$ possesses a term-definable constant $1 = x \to x$ and can be endowed with a partial order $\leqslant$ defined for all $a, b \in A$ as

$$a \leqslant b \iff a \to b = 1.$$

We denote the implication reduct of a Heyting algebra $\boldsymbol{A}$ by $\boldsymbol{A}_\to$. Notably, the order of $\boldsymbol{A}_\to$ coincides with the lattice order of $\boldsymbol{A}$. We will prove the following.

Theorem 8.22. *There exists a binary $f \in \mathsf{ext}_{eq}(\mathsf{Hilbert})$ such that $f^{\boldsymbol{A}_\to}$ is total and coincides with $\wedge^{\boldsymbol{A}}$ for each Heyting algebra $\boldsymbol{A}$.*

Proof. Consider the conjunction of equations

$$\varphi = (y \to x_1 \approx 1) \sqcap (y \to x_2 \approx 1) \sqcap (x_1 \to (x_2 \to y) \approx 1).$$

We begin with the following observation.

Claim 8.23. *For all Heyting algebras $\boldsymbol{A}$ and $a, b, c \in A$,*

$$\boldsymbol{A}_\to \vDash \varphi(a, b, c) \iff a \wedge b = c.$$

Proof of the Claim. Observe that

$$\boldsymbol{A} \vDash (c \to a \approx 1) \sqcap (c \to b \approx 1) \iff (c \leqslant a \text{ and } c \leqslant b) \iff c \leqslant a \wedge b.$$

As the equation $x \to (y \to z) \approx x \wedge y \to z$ holds in every Heyting algebra, we also have

$$\boldsymbol{A} \vDash a \to (b \to c) \approx 1 \iff a \wedge b \to c \approx 1 \iff a \wedge b \leqslant c.$$

From the definition of φ and the two displays above it follows that

$$\boldsymbol{A} \vDash \varphi(a, b, c) \iff a \wedge b = c.$$

Since φ is a formula in the language $\langle \to \rangle$, the demand that $\boldsymbol{A} \vDash \varphi(a, b, c)$ is equivalent to $\boldsymbol{A}_\to \vDash \varphi(a, b, c)$. Together with the above display, this yields the desired conclusion. ⊠

Let M be the class of implication reducts of Heyting algebras. Since the variety Hilbert is the class of implication subreducts of Heyting algebras, we have $\mathsf{Hilbert} = \mathbb{Q}(\mathsf{M})$. Moreover, observe that φ is functional in M by Claim 8.23. As φ is a conjunction of equations, we can apply Corollary 3.11, obtaining that φ defines an implicit operation f of Hilbert. In view of

Claim 8.23, it only remains to show that f is extendable. To this end, consider a Hilbert algebra $\boldsymbol{A}$ and $a, b \in A$. Then $\boldsymbol{A}$ is a subreduct of a Heyting algebra $\boldsymbol{B}$, that is, $\boldsymbol{A} \leqslant \boldsymbol{B}_{\rightarrow}$. Since $f^{\boldsymbol{B}_{\rightarrow}}$ is total by Claim 8.23, we conclude that f is extendable. $\boxtimes$

Example 8.24 (Pseudocomplemented distributive lattices)**.** A *pseudocomplemented distributive lattice* is an algebra $\langle A; \wedge, \vee, \neg, 0, 1 \rangle$ which comprises a bounded distributive lattice $\langle A; \wedge, \vee, 0, 1 \rangle$ and a unary operation $\neg$ such that for all $a, b \in A$ we have

$$a \leqslant \neg b \iff a \wedge b = 0$$

(see, e.g., [7, Sec. VIII]).

This means that $\neg b$ is the largest $a \in A$ such that $a \wedge b = 0$. Consequently, pseudocomplemented distributive lattices are uniquely determined by their lattice reduct.

Given a Heyting algebra $\boldsymbol{A}$ and $a \in A$, we define $\neg a = a \rightarrow 0$. The interest of pseudocomplemented distributive lattices derives from the fact that they coincide with the $\langle \wedge, \vee, \neg, 0, 1 \rangle$-subreducts of Heyting algebras (see, e.g., [19, Proof of Thm. 2.6]).

It is well known that the class of pseudocomplemented distributive lattices forms a locally finite variety, which we denote by PDL (see, e.g., [7, Thm. VIII.3.1] and [11, Thm. 4.55]). The finitely generated members of $\mathsf{PDL}_{\mathrm{SI}}$ are precisely the pseudocomplemented distributive lattices whose lattice reduct is a finite Boolean lattice adjoined with a new top element (see [83, Thm. 2]). Being a finite distributive lattice, every finitely generated member $\boldsymbol{A} = \langle A; \wedge, \vee, \neg, 0, 1 \rangle$ of $\mathsf{PDL}_{\mathrm{SI}}$ can be expanded with an implication $\rightarrow^{\boldsymbol{A}}$ such that $\langle A; \wedge, \vee, \rightarrow^{\boldsymbol{A}}, 0, 1 \rangle$ is a Heyting algebra. We will show that this expansion is witnessed by an extendable operation of PDL. More precisely, we will establish the following.

Theorem 8.25. *There exists a binary* $f \in \mathsf{ext}_{eq}(\mathsf{PDL})$ *such that* $f^{\boldsymbol{A}}$ *is total and coincides with* $\rightarrow^{\boldsymbol{A}}$ *for each finitely generated* $\boldsymbol{A} \in \mathsf{PDL}_{\mathrm{SI}}$*.*

Proof. Let $\varphi(x_1, x_2, y)$ be the conjunction of the following equations:

$$x_1 \wedge y \leqslant x_2; \tag{26}$$

$$\neg x_1 \vee x_2 \leqslant y; \tag{27}$$

$$\neg(\neg x_1 \vee x_2) = \neg y; \tag{28}$$

$$y \vee x_1 = \neg\neg y \vee x_1. \tag{29}$$

It will be enough to show that for all finitely generated $\boldsymbol{A} \in \mathsf{PDL}_{\mathrm{SI}}$ and $a, b, c \in A$,

$$\boldsymbol{A} \vDash \varphi(a, b, c) \iff a \rightarrow^{\boldsymbol{A}} b = c. \tag{30}$$

For suppose this is true. Then φ defines an extendable implicit operation f of PDL by Corollary 8.14. In addition, the above display guarantees that $f^{\boldsymbol{A}}$ is total and coincides with $\rightarrow^{\boldsymbol{A}}$ for each finitely generated $\boldsymbol{A} \in \mathsf{PDL}_{\mathrm{SI}}$, as desired.

We proceed to prove (30). Let $\boldsymbol{A} \in \mathsf{PDL}_{\mathrm{SI}}$ be finitely generated. Then the lattice reduct of $\boldsymbol{A}$ is a finite Boolean lattice $\boldsymbol{B}$ adjoined with a new top element. We denote the maximum of $\boldsymbol{B}$ by $\top$, while the minimum and the maximum of $\boldsymbol{A}$ are 0 and 1, respectively. We also

write a' for the complement of $a \in B$ in the Boolean lattice $\boldsymbol{B}$. For all $a, b \in A$ we have

$$a \to^{\boldsymbol{A}} b = \begin{cases} a' \vee b & \text{if } a \in B \text{ and } a \nleqslant b; \\ 1 & \text{if } a \leqslant b; \\ b & \text{if } a = 1. \end{cases} \tag{31}$$

It follows that

$$\neg a = \begin{cases} a' & \text{if } a \in B - \{0\}; \\ 1 & \text{if } a = 0; \\ 0 & \text{if } a = 1. \end{cases} \qquad\qquad \neg\neg a = \begin{cases} a & \text{if } a \in B - \{\top\}; \\ 1 & \text{if } a \in \{\top, 1\}. \end{cases} \tag{32}$$

Therefore,

$$\neg a \vee b = \begin{cases} a' \vee b & \text{if } a \in B - \{0\}; \\ 1 & \text{if } a = 0; \\ b & \text{if } a = 1. \end{cases} \tag{33}$$

To prove the equivalence in (30), consider $a, b, c \in A$. We begin with the implication from right to left. It suffices to verify that $\boldsymbol{A} \vDash \varphi(a, b, a \to^{\boldsymbol{A}} b)$, which amounts to check that (26)–(29) hold in $\boldsymbol{A}$ once evaluated in $a, b, a \to^{\boldsymbol{A}} b$. It is a straightforward consequence of the properties of implications in Heyting algebras that $a \wedge (a \to^{\boldsymbol{A}} b) \leqslant b$ and $\neg a \vee b \leqslant a \to^{\boldsymbol{A}} b$ (see, e.g., [7, Thm. IX.2.3(i, iv, v)]). So, (26) and (27) hold in $\boldsymbol{A}$. From [7, Thm. IX.2.3(ix, xi)] it follows that $\neg(\neg a \vee b) = \neg\neg a \wedge \neg b = \neg(a \to^{\boldsymbol{A}} b)$, and hence (28) holds. It only remains to verify (29), which states that $(a \to^{\boldsymbol{A}} b) \vee a = \neg\neg(a \to^{\boldsymbol{A}} b) \vee a$. First observe that the equation clearly holds when $a \in \{0, 1\}$. So, we can assume that $a \notin \{0, 1\}$. By (28), (33), and (32) we have

$$\neg\neg(a \to^{\boldsymbol{A}} b) = \neg\neg(\neg a \vee b) = \neg\neg(a' \vee b) = \begin{cases} a' \vee b & \text{if } a \nleqslant b; \\ 1 & \text{if } a \leqslant b, \end{cases}$$

which coincides with $a \to^{\boldsymbol{A}} b$ by (31) because $a \neq 1$. Thus, $(a \to^{\boldsymbol{A}} b) \vee a = \neg\neg(a \to^{\boldsymbol{A}} b) \vee a$, and hence (29) holds.

We proceed to prove the implication from left to right in (30). To this end, assume that $\boldsymbol{A} \vDash \varphi(a, b, c)$. We will show that $a \to^{\boldsymbol{A}} b = c$. From (26) it follows that $a \wedge c \leqslant b$, which yields $c \leqslant a \to^{\boldsymbol{A}} b$. It then remains to show that $a \to^{\boldsymbol{A}} b \leqslant c$. We consider different cases separately. If $a = 1$, then (27) implies that $a \to^{\boldsymbol{A}} b = b = \neg a \vee b \leqslant c$. So, we can assume that $a \neq 1$. Consider the case in which $a \leqslant b$. Then

$$c \vee a = \neg\neg c \vee a = \neg\neg(\neg a \vee b) \vee a = 1 \vee a = 1,$$

where the first and second equalities follow from (29) and (28), and the third from (32) because $\neg a \vee b \in \{\top, 1\}$ as $a \leqslant b$. We have thus obtained that $c \vee a = 1$. Since $\boldsymbol{A}$ has a second largest element $\top$, we have that 1 is join irreducible in $\boldsymbol{A}$. So, $c \vee a = 1$ and $a \neq 1$ imply $c = 1$, and hence $c = a \to^{\boldsymbol{A}} b$, because $a \leqslant b$. Finally, we can assume that $a \neq 1$ and $a \nleqslant b$. Then $a, b \neq 1$, and so $a, b \in B$. Thus, $\neg a \vee b \notin \{\top, 1\}$ because $a \nleqslant b$. Then (32) yields $\neg\neg(\neg a \vee b) = \neg a \vee b$. So, (28) implies that $\neg\neg c = \neg a \vee b$. Therefore, $c \leqslant \neg a \vee b \in B - \{\top\}$,

and hence $c \in B - \{\top\}$. Then $c = \neg\neg c = \neg a \vee b$. Since $a \not\leqslant b$ and $a \neq 1$, we have $a \in B - \{0\}$ and, therefore, $\neg a \vee b = a' \vee b = a \to^{\boldsymbol{A}} b$ by (33) and (31). Consequently, $c = \neg a \vee b = a \to^{\boldsymbol{A}} b$. This establishes that $c = a \to^{\boldsymbol{A}} b$ in all possible cases and concludes the proof.   ⊠

9. Adding implicit operations

A fundamental question which arises in relation to implicit operations is the following: is it possible to expand the language of a given class of algebras K with new function symbols for some of its implicit operations so that every implicit operation of K becomes interpolable by a set of terms in a class M of algebras in the expanded language? Obviously, the interest of this possibility depends on whether M meets some basic desiderata: in addition to the demand that every implicit operation of K can be interpolated by a set of terms of M, we shall demand from our theory that

(D1) every member of K extends to one of M;
(D2) when such a class M exists, it is unique.

A familiar example of this situation is given by

$$\mathsf{K} = \text{the quasivariety of cancellative commutative monoids;}$$

$$\mathsf{M} = \text{the variety of Abelian groups.}$$

In this case, M is obtained by adding the implicit operation of "taking inverses" to K. The fact that every implicit operation of K is interpolated by a set of terms of M is a consequence of M having the strong Beth definability property (see Example 6.3). Furthermore, (D1) holds because every cancellative commutative monoid extends to an Abelian group by Theorem 8.9. Lastly, (D2) will be a consequence of the general theory (Theorem 11.7). On the other hand, we will show that the variety of all commutative monoids lacks an expansion with the desired properties (Theorem 14.1).

In this section, we begin to set the stage for this theory by describing how to expand a class of algebras with a given family of implicit operations.

Definition 9.1. Let K be a class of algebras and $\mathcal{F} \subseteq \mathsf{imp}(\mathsf{K})$. An $\mathcal{F}$-*expansion* of $\mathscr{L}_{\mathsf{K}}$ is a language $\mathscr{L}_{\mathsf{K}} \cup \{g_f : f \in \mathcal{F}\}$, where g_f is a new function symbol of the same arity as that of f for each $f \in \mathcal{F}$. We will often denote an $\mathcal{F}$-expansion of $\mathscr{L}_{\mathsf{K}}$ by $\mathscr{L}_{\mathcal{F}}$. When $\mathcal{F} = \{f\}$ for some $f \in \mathsf{imp}(\mathsf{K})$, we drop the braces and just write $\mathscr{L}_f$ and call it an f-*expansion*.

Definition 9.2. Let K be a class of algebras, $\mathcal{F} \subseteq \mathsf{imp}(\mathsf{K})$, and $\mathscr{L}_{\mathcal{F}}$ an $\mathcal{F}$-expansion of $\mathscr{L}_{\mathsf{K}}$.

(i) For each $\boldsymbol{A} \in \mathsf{K}$ such that $f^{\boldsymbol{A}}$ is total for every $f \in \mathcal{F}$, let

$$\boldsymbol{A}[\mathscr{L}_{\mathcal{F}}] = \text{the unique } \mathscr{L}_{\mathcal{F}}\text{-algebra whose } \mathscr{L}_{\mathsf{K}}\text{-reduct is } \boldsymbol{A} \text{ and in which}$$

$$g_f \text{ is interpreted as } f^{\boldsymbol{A}} \text{ for each } f \in \mathcal{F}.$$

(ii) For each $\mathsf{M} \subseteq \mathsf{K}$ let

$$\mathsf{M}[\mathscr{L}_{\mathcal{F}}] = \{\boldsymbol{A}[\mathscr{L}_{\mathcal{F}}] : \boldsymbol{A} \in \mathsf{M} \text{ and } f^{\boldsymbol{A}} \text{ is total for each } f \in \mathcal{F}\}.$$

The class $\mathsf{K}[\mathscr{L}_{\mathcal{F}}]$ can be viewed as the natural expansion of K induced by the implicit operations in $\mathcal{F}$. The next result provides an alternative description of the class $\mathsf{K}[\mathscr{L}_{\mathcal{F}}]$.

Proposition 9.3. *Let K be a class of algebras, $\mathcal{F} \subseteq \mathsf{imp}(\mathsf{K})$, and $\mathscr{L}_\mathcal{F} = \mathscr{L}_\mathsf{K} \cup \{g_f : f \in \mathcal{F}\}$ an $\mathcal{F}$-expansion of $\mathscr{L}_\mathsf{K}$. Assume that each $f \in \mathcal{F}$ is defined by a formula φ_f. Then*

$$\mathsf{K}[\mathscr{L}_\mathcal{F}] = \{\boldsymbol{B} : \boldsymbol{B} \text{ is an } \mathscr{L}_\mathcal{F}\text{-algebra such that } \boldsymbol{B}\!\restriction_{\mathscr{L}_\mathsf{K}} \in \mathsf{K} \text{ and}$$
$$\boldsymbol{B} \vDash \varphi_f(x_1, \ldots, x_n, g_f(x_1, \ldots, x_n)) \text{ for each } n\text{-ary } f \in \mathcal{F}\}.$$

Proof. To prove the inclusion from left to right, consider $\boldsymbol{B} \in \mathsf{K}[\mathscr{L}_\mathcal{F}]$. The definition of $\mathsf{K}[\mathscr{L}_\mathcal{F}]$ guarantees that there exists $\boldsymbol{A} \in \mathsf{K}$ such that $f^{\boldsymbol{A}}$ is total for all $f \in \mathcal{F}$ and $\boldsymbol{B} = \boldsymbol{A}[\mathscr{L}_\mathcal{F}]$. In particular, $\boldsymbol{B}$ is an $\mathscr{L}_\mathcal{F}$-algebra. As $\boldsymbol{A}$ is the $\mathscr{L}_\mathsf{K}$-reduct of $\boldsymbol{A}[\mathscr{L}_\mathcal{F}]$ by the definition of $\boldsymbol{A}[\mathscr{L}_\mathcal{F}]$, this yields $\boldsymbol{B}\!\restriction_{\mathscr{L}_\mathsf{K}} = \boldsymbol{A} \in \mathsf{K}$. Then consider an n-ary $f \in \mathcal{F}$. Since f is defined by φ_f and $f^{\boldsymbol{A}}$ is total, for all $a_1, \ldots, a_n \in A$ we have

$$\langle a_1, \ldots, a_n \rangle \in \mathsf{dom}(f^{\boldsymbol{A}}) \quad \text{and} \quad \boldsymbol{A} \vDash \varphi_f(a_1, \ldots, a_n, f^{\boldsymbol{A}}(a_1, \ldots, a_n)).$$

By the definition of $\boldsymbol{A}[\mathscr{L}_\mathcal{F}]$ the operation g_f is interpreted in $\boldsymbol{A}[\mathscr{L}_\mathcal{F}]$ as $f^{\boldsymbol{A}}$. As $\boldsymbol{A}$ is the $\mathscr{L}_\mathsf{K}$-reduct of $\boldsymbol{A}[\mathscr{L}_\mathcal{F}]$, from the above display it follows that $\boldsymbol{A}[\mathscr{L}_\mathcal{F}] \vDash \varphi_f(x_1, \ldots, x_n, g_f(x_1, \ldots, x_n))$. Since $\boldsymbol{B} = \boldsymbol{A}[\mathscr{L}_\mathcal{F}]$, we conclude that $\boldsymbol{B} \vDash \varphi_f(x_1, \ldots, x_n, g_f(x_1, \ldots, x_n))$, as desired.

Then we proceed to prove the inclusion from right to left. Consider an $\mathscr{L}_\mathcal{F}$-algebra $\boldsymbol{B}$ such that $\boldsymbol{B}\!\restriction_{\mathscr{L}_\mathsf{K}} \in \mathsf{K}$ and $\boldsymbol{B} \vDash \varphi_f(x_1, \ldots, x_n, g_f(x_1, \ldots, x_n))$ for each n-ary $f \in \mathcal{F}$. For the sake of readability, let $\boldsymbol{A} = \boldsymbol{B}\!\restriction_{\mathscr{L}_\mathsf{K}}$ and observe that $\boldsymbol{A} \in \mathsf{K}$ by assumption. We will prove that the algebra $\boldsymbol{A}[\mathscr{L}_\mathcal{F}]$ is defined and coincides with $\boldsymbol{B}$, whence $\boldsymbol{B} \in \mathsf{K}[\mathscr{L}_\mathcal{F}]$, as desired. Since $\boldsymbol{A} = \boldsymbol{B}\!\restriction_{\mathscr{L}_\mathsf{K}}$ and $\boldsymbol{A} \in \mathsf{K}$, it suffices to show that for each $f \in \mathcal{F}$ the function $f^{\boldsymbol{A}}$ is total and coincides with the interpretation of g_f in $\boldsymbol{B}$. To this end, consider an n-ary $f \in \mathcal{F}$ and $a_1, \ldots, a_n \in A$. We need to prove that

$$\langle a_1, \ldots, a_n \rangle \in \mathsf{dom}(f^{\boldsymbol{A}}) \quad \text{and} \quad g_f^{\boldsymbol{B}}(a_1, \ldots, a_n) = f^{\boldsymbol{A}}(a_1, \ldots, a_n).$$

First, from the assumption that $\boldsymbol{B} \vDash \varphi_f(x_1, \ldots, x_n, g_f(x_1, \ldots, x_n))$ it follows that $\boldsymbol{B} \vDash \varphi_f(a_1, \ldots, a_n, g_f^{\boldsymbol{B}}(a_1, \ldots, a_n))$. As φ_f is a formula of $\mathscr{L}_\mathsf{K}$ and $\boldsymbol{A} = \boldsymbol{B}\!\restriction_{\mathscr{L}_\mathsf{K}}$, this amounts to $\boldsymbol{A} \vDash \varphi_f(a_1, \ldots, a_n, g_f^{\boldsymbol{B}}(a_1, \ldots, a_n))$. Since φ_f defines f, the above display holds. $\boxtimes$

As a consequence of Proposition 9.3, we obtain the following.

Corollary 9.4. *Let K be a class of algebras, $\mathcal{F} \subseteq \mathsf{imp}(\mathsf{K})$, and $\mathscr{L}_\mathcal{F}$ an $\mathcal{F}$-expansion of $\mathscr{L}_\mathsf{K}$. If K is an elementary class, then $\mathsf{K}[\mathscr{L}_\mathcal{F}]$ is an elementary class.*

A useful feature of $\mathcal{F}$-expansions is that a map between members of $\mathsf{K}[\mathscr{L}_\mathcal{F}]$ preserves the operations in $\mathcal{F}$ if it preserves the operations in $\mathscr{L}_\mathsf{K}$. This is made precise in the next proposition.

Proposition 9.5. *Let K be a class of algebras, $\mathcal{F} \subseteq \mathsf{imp}(\mathsf{K})$, and $\mathscr{L}_\mathcal{F}$ an $\mathcal{F}$-expansion of $\mathscr{L}_\mathsf{K}$. Every homomorphism $h\colon \boldsymbol{A}\!\restriction_{\mathscr{L}_\mathsf{K}} \to \boldsymbol{B}\!\restriction_{\mathscr{L}_\mathsf{K}}$ with $\boldsymbol{A}, \boldsymbol{B} \in \mathsf{K}[\mathscr{L}_\mathcal{F}]$ is also a homomorphism $h\colon \boldsymbol{A} \to \boldsymbol{B}$.*

Proof. As $\mathscr{L}_\mathcal{F}$ is an $\mathcal{F}$-expansion of $\mathscr{L}_\mathsf{K}$, it is of the form $\mathscr{L}_\mathsf{K} \cup \{g_f : f \in \mathcal{F}\}$. Then let $h\colon \boldsymbol{A}\!\restriction_{\mathscr{L}_\mathsf{K}} \to \boldsymbol{B}\!\restriction_{\mathscr{L}_\mathsf{K}}$ be a homomorphism with $\boldsymbol{A}, \boldsymbol{B} \in \mathsf{K}[\mathscr{L}_\mathcal{F}]$. It suffices to prove that h preserves g_f for each $f \in \mathcal{F}$. To this end, consider $f \in \mathcal{F}$. Since $\boldsymbol{A}, \boldsymbol{B} \in \mathsf{K}[\mathscr{L}_\mathcal{F}]$, we have $\boldsymbol{A} = \boldsymbol{A}\!\restriction_{\mathscr{L}_\mathsf{K}}[\mathscr{L}_\mathcal{F}]$ and $\boldsymbol{B} = \boldsymbol{B}\!\restriction_{\mathscr{L}_\mathsf{K}}[\mathscr{L}_\mathcal{F}]$. Therefore,

$$g_f^{\boldsymbol{A}} = f^{\boldsymbol{A}\restriction_{\mathscr{L}_\mathsf{K}}} \quad \text{and} \quad g_f^{\boldsymbol{B}} = f^{\boldsymbol{B}\restriction_{\mathscr{L}_\mathsf{K}}}.$$

As $h \colon \boldsymbol{A}{\restriction}_{\mathscr{L}_{\mathsf{K}}} \to \boldsymbol{B}{\restriction}_{\mathscr{L}_{\mathsf{K}}}$ is a homomorphism between members of K and f an implicit operation of K, we know that h preserves f. In view of the above display, we conclude that h preserves g_f. $\boxtimes$

In general, there is no guarantee that each member of K is a subreduct of a member of $\mathsf{K}[\mathscr{L}_{\mathcal{F}}]$ or, equivalently, that condition (D1) is met. In the setting of universal classes, this is the case exactly when the members of $\mathcal{F}$ are extendable.

Proposition 9.6. *Let K be a universal class, $\mathcal{F} \subseteq \mathsf{imp}(\mathsf{K})$, and $\mathscr{L}_{\mathcal{F}}$ an $\mathcal{F}$-expansion of $\mathscr{L}_{\mathsf{K}}$. Then K is the class of $\mathscr{L}_{\mathsf{K}}$-subreducts of $\mathsf{K}[\mathscr{L}_{\mathcal{F}}]$ if and only if $\mathcal{F} \subseteq \mathsf{ext}(\mathsf{K})$.*

Proof. We begin with the implication from left to right. Suppose that K is the class of $\mathscr{L}_{\mathsf{K}}$-subreducts of $\mathsf{K}[\mathscr{L}_{\mathcal{F}}]$ and consider an n-ary $f \in \mathcal{F}$. We need to prove that f is extendable. To this end, consider $\boldsymbol{A} \in \mathsf{K}$. By assumption $\boldsymbol{A}$ is a subreduct of some $\boldsymbol{C} \in \mathsf{K}[\mathscr{L}_{\mathcal{F}}]$. By the definition of $\mathsf{K}[\mathscr{L}_{\mathcal{F}}]$ there exists $\boldsymbol{B} \in \mathsf{K}$ in which $g^{\boldsymbol{B}}$ is total for each $g \in \mathcal{F}$ such that $\boldsymbol{C} = \boldsymbol{B}[\mathscr{L}_{\mathcal{F}}]$. As $\boldsymbol{A}$ is an $\mathscr{L}_{\mathsf{K}}$-subreduct of $\boldsymbol{C} = \boldsymbol{B}[\mathscr{L}_{\mathcal{F}}]$ and $\boldsymbol{B}$ is the $\mathscr{L}_{\mathsf{K}}$-reduct of $\boldsymbol{B}[\mathscr{L}_{\mathcal{F}}]$, we obtain $\boldsymbol{A} \leqslant \boldsymbol{B}$. Furthermore, $f^{\boldsymbol{B}}$ is total because $f \in \mathcal{F}$. Since $\boldsymbol{A} \leqslant \boldsymbol{B} \in \mathsf{K}$, we conclude that f is extendable.

Then we proceed to prove the implication from right to left. Suppose that $\mathcal{F} \subseteq \mathsf{ext}(\mathsf{K})$ and consider $\boldsymbol{A} \in \mathsf{K}$. As K is a universal class, we can apply Theorem 8.4, obtaining some $\boldsymbol{B} \in \mathsf{K}$ with $\boldsymbol{A} \leqslant \boldsymbol{B}$ such that $f^{\boldsymbol{B}}$ is total for each $f \in \mathcal{F}$. By the definition of $\mathsf{K}[\mathscr{L}_{\mathcal{F}}]$ we get $\boldsymbol{B}[\mathscr{L}_{\mathcal{F}}] \in \mathsf{K}[\mathscr{L}_{\mathcal{F}}]$. Since $\boldsymbol{B}$ is the $\mathscr{L}_{\mathsf{K}}$-reduct of $\boldsymbol{B}[\mathscr{L}_{\mathcal{F}}]$ and $\boldsymbol{A} \leqslant \boldsymbol{B}$, the algebra $\boldsymbol{A}$ is an $\mathscr{L}_{\mathsf{K}}$-subreduct of a member of $\mathsf{K}[\mathscr{L}_{\mathcal{F}}]$. $\boxtimes$

We close this section with some observations governing the behavior of $\mathsf{K}[\mathscr{L}_{\mathcal{F}}]$ with respect to class operators.

Proposition 9.7. *Let K be a class of algebras and $\mathcal{F} \subseteq \mathsf{imp}_{pp}(\mathsf{K})$. Moreover, let*

$$\mathsf{M} = \{ \boldsymbol{A} \in \mathsf{K} : f^{\boldsymbol{A}} \text{ is total for each } f \in \mathcal{F} \}.$$

Then for each $\mathbb{O} \in \{\mathbb{H}, \mathbb{P}, \mathbb{P}_{\mathrm{u}}\}$ we have

$$\mathbb{O}(\mathsf{M}) \cap \mathsf{K} \subseteq \mathsf{M}.$$

Proof. In order to prove that $\mathbb{O}(\mathsf{M}) \cap \mathsf{K} \subseteq \mathsf{M}$, consider an n-ary $f \in \mathcal{F}$, $\boldsymbol{A} \in \mathbb{O}(\mathsf{M}) \cap \mathsf{K}$, and $a_1, \ldots, a_n \in A$. Since $\mathcal{F} \subseteq \mathsf{imp}_{pp}(\mathsf{K})$ by assumption, there exists a pp formula $\varphi(x_1, \ldots, x_n, y)$ defining f. We need to show that $\langle a_1, \ldots, a_n \rangle \in \mathsf{dom}(f^{\boldsymbol{A}})$, which is equivalent to $\boldsymbol{A} \vDash \exists y \varphi(a_1, \ldots, a_n, y)$. The definition of M guarantees that

$$\mathsf{M} \vDash \exists y \varphi(x_1, \ldots, x_n, y). \tag{34}$$

We have three cases depending on whether $\mathbb{O}$ is $\mathbb{H}$, $\mathbb{P}$, or $\mathbb{P}_{\mathrm{u}}$. First consider the case where $\mathbb{O} = \mathbb{H}$. Then $\boldsymbol{A} \in \mathbb{H}(\mathsf{M})$ implies that there exist $\boldsymbol{B} \in \mathsf{M}$ and a surjective homomorphism $h \colon \boldsymbol{B} \to \boldsymbol{A}$. Let $b_1, \ldots, b_n \in B$ be such that $h(b_i) = a_i$ for each $i \leqslant n$. From (34) and $\boldsymbol{B} \in \mathsf{M}$ it follows that $\boldsymbol{B} \vDash \exists y \varphi(b_1, \ldots, b_n, y)$. Since φ is a pp formula by assumption, so is $\exists y \varphi$. Therefore, we can apply Theorem 1.1(ii) to obtain that $\boldsymbol{A} \vDash \exists y \varphi(h(b_1), \ldots, h(b_n), y)$. As $h(b_i) = a_i$ for each $i \leqslant n$, it follows that $\boldsymbol{A} \vDash \exists y \varphi(a_1, \ldots, a_n, y)$, as desired.

Then we consider the case where $\mathbb{O} = \mathbb{P}$. From $\boldsymbol{A} \in \mathbb{P}(\mathsf{M})$ it follows that $\boldsymbol{A} = \prod_{i \in I} \boldsymbol{A}_i$ for some family $\{\boldsymbol{A}_i : i \in I\} \subseteq \mathsf{M}$. In view of (34), we have

$$\boldsymbol{A}_i \vDash \exists y \varphi(p_i(a_1), \ldots, p_i(a_n), y) \text{ for every } i \in I.$$

Since φ is a pp formula by assumption, so is $\exists y \varphi$. Therefore, we can apply Theorem 1.1(ii) to the above display, obtaining $\boldsymbol{A} \vDash \exists y \varphi(a_1, \ldots, a_n, y)$, as desired.

Lastly, we consider the case where $\mathbb{O} = \mathbb{P}_u$. From $\boldsymbol{A} \in \mathbb{P}_u(\mathsf{M})$ it follows that $\boldsymbol{A} = \prod_{i \in I} \boldsymbol{A}_i/U$ for some family $\{\boldsymbol{A}_i : i \in I\} \subseteq \mathsf{M}$ and ultrafilter U on I. Since $\{\boldsymbol{A}_i : i \in I\} \subseteq \mathsf{M}$, we can apply Łoś' Theorem 1.6 to (34), obtaining $\boldsymbol{A} \vDash \exists y \varphi(x_1, \ldots, x_n, y)$. Thus, $\boldsymbol{A} \vDash \exists y \varphi(a_1, \ldots, a_n, y)$. This concludes the proof. $\boxtimes$

Proposition 9.8. *Let* K *be a class of algebras,* $\mathcal{F} \subseteq \mathsf{imp}_{pp}(\mathsf{K})$, *and* $\mathscr{L}_{\mathcal{F}}$ *an* $\mathcal{F}$-*expansion of* $\mathscr{L}_{\mathsf{K}}$. *Moreover, let*

$$\mathsf{M} = \{\boldsymbol{A} \in \mathsf{K} : f^{\boldsymbol{A}} \text{ is total for each } f \in \mathcal{F}\}.$$

The following conditions hold for all $\mathsf{N} \subseteq \mathsf{M}$ *and class operators* $\mathbb{O}$ *such that* $\mathbb{O}(\mathsf{K}) \subseteq \mathsf{K}$:

(i) *if* $\mathbb{O} \in \{\mathbb{I}, \mathbb{H}, \mathbb{P}, \mathbb{P}_u\}$, *then*

$$\mathbb{O}(\mathsf{N}[\mathscr{L}_{\mathcal{F}}]) = (\mathbb{O}(\mathsf{N}))[\mathscr{L}_{\mathcal{F}}];$$

(ii) *if* $\mathbb{O} = \mathbb{S}$ *and* $\mathcal{F} \subseteq \mathsf{imp}_{eq}(\mathsf{K})$, *then*

$$\mathbb{S}(\mathsf{N}[\mathscr{L}_{\mathcal{F}}]) \subseteq (\mathbb{S}(\mathsf{N}))[\mathscr{L}_{\mathcal{F}}].$$

Proof. As $\mathscr{L}_{\mathcal{F}}$ is an $\mathcal{F}$-expansion of $\mathscr{L}_{\mathsf{K}}$, it is of the form $\mathscr{L}_{\mathsf{K}} \cup \{g_f : f \in \mathcal{F}\}$. This fact will be used repeatedly without further notice.

(i): Assume that $\mathsf{N} \subseteq \mathsf{M}$ and let $\mathbb{O} \in \{\mathbb{I}, \mathbb{H}, \mathbb{P}, \mathbb{P}_u\}$ be such that $\mathbb{O}(\mathsf{K}) \subseteq \mathsf{K}$. We first establish the following.

Claim 9.9. *For every* $\boldsymbol{A} \in \mathsf{N} \cup \mathbb{O}(\mathsf{N})$ *the algebra* $\boldsymbol{A}[\mathscr{L}_{\mathcal{F}}]$ *is defined.*

Proof of the Claim. The definition of M guarantees that $\boldsymbol{A}[\mathscr{L}_{\mathcal{F}}]$ is defined for each $\boldsymbol{A} \in \mathsf{M}$. It then suffices to show that $\mathsf{N} \cup \mathbb{O}(\mathsf{N}) \subseteq \mathsf{M}$. As $\mathsf{N} \subseteq \mathsf{M}$ holds by assumption, it remains to prove that $\mathbb{O}(\mathsf{N}) \subseteq \mathsf{M}$. From $\mathsf{N} \subseteq \mathsf{M} \subseteq \mathsf{K}$ and $\mathbb{O}(\mathsf{K}) \subseteq \mathsf{K}$ it follows that $\mathbb{O}(\mathsf{N}) \subseteq \mathbb{O}(\mathsf{M})$ and $\mathbb{O}(\mathsf{N}) \subseteq \mathbb{O}(\mathsf{K}) \subseteq \mathsf{K}$. Therefore, $\mathbb{O}(\mathsf{N}) \subseteq \mathbb{O}(\mathsf{M}) \cap \mathsf{K}$. Since Proposition 9.7 yields $\mathbb{O}(\mathsf{M}) \cap \mathsf{K} \subseteq \mathsf{M}$, we conclude that $\mathbb{O}(\mathsf{N}) \subseteq \mathsf{M}$. $\boxtimes$

We have to consider four cases depending on whether $\mathbb{O}$ is $\mathbb{I}$, $\mathbb{H}$, $\mathbb{P}$, or $\mathbb{P}_u$. We will start with the cases where $\mathbb{O} = \mathbb{I}$ and $\mathbb{O} = \mathbb{H}$, which can be treated simultaneously. Suppose that $\mathbb{O} \in \{\mathbb{I}, \mathbb{H}\}$. To prove that $\mathbb{O}(\mathsf{N}[\mathscr{L}_{\mathcal{F}}]) \subseteq (\mathbb{O}(\mathsf{N}))[\mathscr{L}_{\mathcal{F}}]$ consider $\boldsymbol{A} \in \mathbb{O}(\mathsf{N}[\mathscr{L}_{\mathcal{F}}])$. Then there exist $\boldsymbol{B} \in \mathsf{N}[\mathscr{L}_{\mathcal{F}}]$ and a surjective homomorphism $h \colon \boldsymbol{B} \to \boldsymbol{A}$, which we can assume to be an isomorphism when $\mathbb{O} = \mathbb{I}$. As $\boldsymbol{B}{\restriction}_{\mathscr{L}_{\mathsf{K}}} \in \mathsf{N}$ and h is also a homomorphism $h \colon \boldsymbol{B}{\restriction}_{\mathscr{L}_{\mathsf{K}}} \to \boldsymbol{A}{\restriction}_{\mathscr{L}_{\mathsf{K}}}$, which is an isomorphism when $\mathbb{O} = \mathbb{I}$, we obtain $\boldsymbol{A}{\restriction}_{\mathscr{L}_{\mathsf{K}}} \in \mathbb{O}(\mathsf{N})$. Hence, $\boldsymbol{A}{\restriction}_{\mathscr{L}_{\mathsf{K}}}[\mathscr{L}_{\mathcal{F}}]$ is defined by Claim 9.9 and, therefore, $\boldsymbol{A}{\restriction}_{\mathscr{L}_{\mathsf{K}}}[\mathscr{L}_{\mathcal{F}}] \in (\mathbb{O}(\mathsf{N}))[\mathscr{L}_{\mathcal{F}}]$. To show that $\boldsymbol{A} \in (\mathbb{O}(\mathsf{N}))[\mathscr{L}_{\mathcal{F}}]$, it is then sufficient to prove that $\boldsymbol{A} = \boldsymbol{A}{\restriction}_{\mathscr{L}_{\mathsf{K}}}[\mathscr{L}_{\mathcal{F}}]$. Let $f \in \mathcal{F}$ be n-ary, $a_1, \ldots, a_n \in A$, and

$b_1, \ldots, b_n \in B$ be such that $h(b_i) = a_i$ for each $i \leqslant n$. We have

$$
\begin{aligned}
g_f^{\boldsymbol{A}}(a_1, \ldots, a_n) &= g_f^{\boldsymbol{A}}(h(a_1), \ldots, h(a_n)) = h(g_f^{\boldsymbol{B}}(b_1, \ldots, b_n)) \\
&= h(f^{\boldsymbol{B}\restriction_{\mathscr{L}_{\mathsf{K}}}}(b_1, \ldots, b_n)) = f^{\boldsymbol{A}\restriction_{\mathscr{L}_{\mathsf{K}}}}(h(b_1), \ldots, h(b_n)) \\
&= f^{\boldsymbol{A}\restriction_{\mathscr{L}_{\mathsf{K}}}}(a_1, \ldots, a_n),
\end{aligned}
$$

where the first and last equalities hold because $a_i = h(b_i)$ for each $i \leqslant n$, the second and fourth hold because $h \colon \boldsymbol{B} \to \boldsymbol{A}$ and $h \colon \boldsymbol{B}\restriction_{\mathscr{L}_{\mathsf{K}}} \to \boldsymbol{A}\restriction_{\mathscr{L}_{\mathsf{K}}}$ are homomorphisms and $f \in \mathsf{imp}(\mathsf{K})$, and the third holds because $g_f^{\boldsymbol{B}} = f^{\boldsymbol{B}\restriction_{\mathscr{L}_{\mathsf{K}}}}$ by the definition of $g_f^{\boldsymbol{B}}$. Together with the fact that $\boldsymbol{A}$ and $\boldsymbol{A}\restriction_{\mathscr{L}_{\mathsf{K}}}[\mathscr{L}_{\mathcal{F}}]$ have the same $\mathscr{L}_{\mathsf{K}}$-reduct, the above display yields $\boldsymbol{A} = \boldsymbol{A}\restriction_{\mathscr{L}_{\mathsf{K}}}[\mathscr{L}_{\mathcal{F}}]$, as desired.

For the reverse inclusion, consider $\boldsymbol{A}[\mathscr{L}_{\mathcal{F}}] \in \mathbb{O}(\mathsf{N})[\mathscr{L}_{\mathcal{F}}]$. Then there exist $\boldsymbol{B} \in \mathsf{N}$ and a surjective homomorphism $h \colon \boldsymbol{B} \to \boldsymbol{A}$, which is an isomorphism when $\mathbb{O} = \mathbb{I}$. Since $\boldsymbol{B} \in \mathsf{N}$, Claim 9.9 yields that $\boldsymbol{B}[\mathscr{L}_{\mathcal{F}}]$ is defined and belongs to $\mathsf{N}[\mathscr{L}_{\mathcal{F}}]$. By Proposition 9.5, $h \colon \boldsymbol{B}[\mathscr{L}_{\mathcal{F}}] \to \boldsymbol{A}[\mathscr{L}_{\mathcal{F}}]$ is a homomorphism. Thus, we conclude that $\boldsymbol{A}[\mathscr{L}_{\mathcal{F}}] \in \mathbb{O}(\mathsf{N}[\mathscr{L}_{\mathcal{F}}])$.

Then we consider the case where $\mathbb{O} = \mathbb{P}$. From the definitions of $\mathbb{P}(\mathsf{N}[\mathscr{L}_{\mathcal{F}}])$ and $(\mathbb{P}(\mathsf{N}))[\mathscr{L}_{\mathcal{F}}]$ it follows that

$$
\boldsymbol{A} \in \mathbb{P}(\mathsf{N}[\mathscr{L}_{\mathcal{F}}]) \iff \boldsymbol{A} = \prod_{i \in I}(\boldsymbol{A}_i[\mathscr{L}_{\mathcal{F}}]) \text{ for some } \{\boldsymbol{A}_i : i \in I\} \subseteq \mathsf{N};
$$

$$
\boldsymbol{A} \in (\mathbb{P}(\mathsf{N}))[\mathscr{L}_{\mathcal{F}}] \iff \boldsymbol{A} = \Big(\prod_{i \in I} \boldsymbol{A}_i \Big)[\mathscr{L}_{\mathcal{F}}] \text{ for some } \{\boldsymbol{A}_i : i \in I\} \subseteq \mathsf{N}.
$$

Therefore, to conclude that $\mathbb{P}(\mathsf{N}[\mathscr{L}_{\mathcal{F}}]) = (\mathbb{P}(\mathsf{N}))[\mathscr{L}_{\mathcal{F}}]$, it suffices to show that for every family $\{\boldsymbol{A}_i : i \in I\} \subseteq \mathsf{N}$,

$$
\prod_{i \in I}(\boldsymbol{A}_i[\mathscr{L}_{\mathcal{F}}]) = \Big(\prod_{i \in I} \boldsymbol{A}_i \Big)[\mathscr{L}_{\mathcal{F}}],
$$

where the algebras in the above display are defined by Claim 9.9. Observe that $\prod_{i \in I}(\boldsymbol{A}_i[\mathscr{L}_{\mathcal{F}}])$ and $(\prod_{i \in I} \boldsymbol{A}_i)[\mathscr{L}_{\mathcal{F}}]$ have the same $\mathscr{L}_{\mathsf{K}}$-reduct, namely, $\prod_{i \in I} \boldsymbol{A}_i$. It will then be enough to prove that for all n-ary $f \in \mathcal{F}$ and $a_1, \ldots, a_n, b \in \prod_{i \in I} A_i$,

$$
g_f^{\prod_{i \in I}(\boldsymbol{A}_i[\mathscr{L}_{\mathcal{F}}])}(a_1, \ldots, a_n) = b \iff g_f^{(\prod_{i \in I} \boldsymbol{A}_i)[\mathscr{L}_{\mathcal{F}}]}(a_1, \ldots, a_n) = b. \tag{35}
$$

To this end, recall from the assumptions that f is defined by a pp formula φ. Observe that $\prod_{i \in I} \boldsymbol{A}_i \in \mathbb{P}(\mathsf{K}) \subseteq \mathsf{K}$, and hence $f^{\prod_{i \in I} \boldsymbol{A}_i}$ is defined. We will prove that

$$
\begin{aligned}
g_f^{\prod_{i \in I}(\boldsymbol{A}_i[\mathscr{L}_{\mathcal{F}}])}(a_1, \ldots, a_n) = b &\iff g_f^{\boldsymbol{A}_i[\mathscr{L}_{\mathcal{F}}]}(p_i(a_1), \ldots, p_i(a_n)) = p_i(b) \text{ for every } i \in I \\
&\iff f^{\boldsymbol{A}_i}(p_i(a_1), \ldots, p_i(a_n)) = p_i(b) \text{ for every } i \in I \\
&\iff \boldsymbol{A}_i \vDash \varphi(p_i(a_1), \ldots, p_i(a_n), p_i(b)) \text{ for every } i \in I \\
&\iff \prod_{i \in I} \boldsymbol{A}_i \vDash \varphi(a_1, \ldots, a_n, b) \\
&\iff f^{\prod_{i \in I} \boldsymbol{A}_i}(a_1, \ldots, a_n) = b \\
&\iff g_f^{(\prod_{i \in I} \boldsymbol{A}_i)[\mathscr{L}_{\mathcal{F}}]}(a_1, \ldots, a_n) = b.
\end{aligned}
$$

The above equivalences are justified as follows. The first holds by the definition of a direct product, the second by the definition of $\boldsymbol{A}_i[\mathscr{L}_\mathcal{F}]$, the third and the fifth because f is defined by φ, the fourth follows from an application of Theorem 1.1(ii) made possible by the assumption that φ is a pp formula, and the last one holds by the definition of $(\prod_{i\in I}\boldsymbol{A}_i)[\mathscr{L}_\mathcal{F}]$. This establishes (35), thus concluding the proof that $\mathbb{P}(\mathsf{N}[\mathscr{L}_\mathcal{F}]) = (\mathbb{P}(\mathsf{N}))[\mathscr{L}_\mathcal{F}]$.

Lastly, we consider the case where $\mathbb{O} = \mathbb{P}_\mathrm{u}$. From the definitions of $\mathbb{P}_\mathrm{u}(\mathsf{N}[\mathscr{L}_\mathcal{F}])$ and $(\mathbb{P}_\mathrm{u}(\mathsf{N}))[\mathscr{L}_\mathcal{F}]$ it follows that

$$\boldsymbol{A} \in \mathbb{P}_\mathrm{u}(\mathsf{N}[\mathscr{L}_\mathcal{F}]) \iff \text{there exist } \{\boldsymbol{A}_i : i \in I\} \subseteq \mathsf{N} \text{ and an ultrafilter } U \text{ on } I$$
$$\text{such that } \boldsymbol{A} = \prod_{i\in I}(\boldsymbol{A}_i[\mathscr{L}_\mathcal{F}])/U;$$

$$\boldsymbol{A} \in (\mathbb{P}_\mathrm{u}(\mathsf{N}))[\mathscr{L}_\mathcal{F}] \iff \text{there exist } \{\boldsymbol{A}_i : i \in I\} \subseteq \mathsf{N} \text{ and an ultrafilter } U \text{ on } I$$
$$\text{such that } \boldsymbol{A} = \Big(\prod_{i\in I}\boldsymbol{A}_i/U\Big)[\mathscr{L}_\mathcal{F}].$$

Therefore, to conclude that $\mathbb{P}_\mathrm{u}(\mathsf{N}[\mathscr{L}_\mathcal{F}]) = (\mathbb{P}_\mathrm{u}(\mathsf{N}))[\mathscr{L}_\mathcal{F}]$, it suffices to show that for every family $\{\boldsymbol{A}_i : i \in I\} \subseteq \mathsf{N}$ and ultrafilter U on I,

$$\prod_{i\in I}(\boldsymbol{A}_i[\mathscr{L}_\mathcal{F}])/U = \Big(\prod_{i\in I}\boldsymbol{A}_i/U\Big)[\mathscr{L}_\mathcal{F}],$$

where the algebras in the above display are defined by Claim 9.9.

Similarly to the case $\mathbb{O} = \mathbb{P}$ it suffices to show that for all n-ary $f \in \mathcal{F}$ and $a_1, \ldots, a_n, b \in \prod_{i\in I}A_i/U$

$$g_f^{\prod_{i\in I}(\boldsymbol{A}_i[\mathscr{L}_\mathcal{F}])/U}(a_1/U, \ldots, a_n/U) = b/U \iff g_f^{(\prod_{i\in I}\boldsymbol{A}_i/U)[\mathscr{L}_\mathcal{F}]}(a_1/U, \ldots, a_n/U) = b/U. \quad (36)$$

To this end, recall from the assumptions that f is defined by a pp formula φ. Observe that $\prod_{i\in I}\boldsymbol{A}_i/U \in \mathbb{P}_\mathrm{u}(\mathsf{K}) \subseteq \mathsf{K}$, and hence $f^{\prod_{i\in I}\boldsymbol{A}_i/U}$ is defined. We will prove that

$$g_f^{\prod_{i\in I}(\boldsymbol{A}_i[\mathscr{L}_\mathcal{F}])/U}(a_1/U, \ldots, a_n/U) = b/U \iff g_f^{\prod_{i\in I}(\boldsymbol{A}_i[\mathscr{L}_\mathcal{F}])}(a_1, \ldots, a_n)/U = b/U$$

$$\iff [\![g_f^{\prod_{i\in I}(\boldsymbol{A}_i[\mathscr{L}_\mathcal{F}])}(a_1, \ldots, a_n) \approx b]\!] \in U$$

$$\iff \{i \in I : g_f^{\boldsymbol{A}_i[\mathscr{L}_\mathcal{F}]}(p_i(a_1), \ldots, p_i(a_n)) = p_i(b)\} \in U$$

$$\iff \{i \in I : f^{\boldsymbol{A}_i}(p_i(a_1), \ldots, p_i(a_n)) = p_i(b)\} \in U$$

$$\iff \{i \in I : \boldsymbol{A}_i \vDash \varphi(p_i(a_1), \ldots, p_i(a_n), p_i(b))\} \in U$$

$$\iff [\![\varphi(a_1, \ldots, a_n, b)]\!] \in U$$

$$\iff \prod_{i\in I}\boldsymbol{A}_i/U \vDash \varphi(a_1/U, \ldots, a_n/U, b/U)$$

$$\iff f^{\prod_{i\in I}\boldsymbol{A}_i/U}(a_1/U, \ldots, a_n/U) = b/U$$

$$\iff g_f^{(\prod_{i\in I}\boldsymbol{A}_i/U)[\mathscr{L}_\mathcal{F}]}(a_1/U, \ldots, a_n/U) = b/U.$$

The above equivalences are justified as follows. The first holds by the definition of a quotient algebra, the second by the definition of an ultraproduct, the third and the sixth are straightforward, the fourth holds by the definition of $\boldsymbol{A}_i[\mathscr{L}_\mathcal{F}]$, the fifth and the eighth

because φ defines f, the seventh follows from Łoś' Theorem 1.6, and the last one from the definition of $(\prod_{i \in I} \boldsymbol{A}_i/U)[\mathscr{L}_\mathcal{F}]$. This establishes (36), thus concluding the proof that $\mathbb{P}_\mathrm{u}(\mathsf{N}[\mathscr{L}_\mathcal{F}]) = (\mathbb{P}_\mathrm{u}(\mathsf{N}))[\mathscr{L}_\mathcal{F}]$.

(ii): Suppose that $\mathbb{O} = \mathbb{S}$ and that each $f \in \mathcal{F}$ is defined by a conjunction of equations φ_f. We need to prove that $\mathbb{S}(\mathsf{N}[\mathscr{L}_\mathcal{F}]) \subseteq (\mathbb{S}(\mathsf{N}))[\mathscr{L}_\mathcal{F}]$. Consider $\boldsymbol{A} \in \mathbb{S}(\mathsf{N}[\mathscr{L}_\mathcal{F}])$. Then there exists $\boldsymbol{B} \in \mathsf{N}$ such that $\boldsymbol{A} \leqslant \boldsymbol{B}[\mathscr{L}_\mathcal{F}]$. As $\boldsymbol{B}$ is the $\mathscr{L}_\mathsf{K}$-reduct of $\boldsymbol{B}[\mathscr{L}_\mathcal{F}]$ and $\boldsymbol{A} \leqslant \boldsymbol{B}[\mathscr{L}_\mathcal{F}]$, we obtain $\boldsymbol{A}{\restriction}_{\mathscr{L}_\mathsf{K}} \leqslant \boldsymbol{B} \in \mathsf{N}$. Therefore, $\boldsymbol{A}{\restriction}_{\mathscr{L}_\mathsf{K}} \in \mathbb{S}(\mathsf{N})$. We will prove that the algebra $\boldsymbol{A}{\restriction}_{\mathscr{L}_\mathsf{K}}[\mathscr{L}_\mathcal{F}]$ is defined and coincides with $\boldsymbol{A}$, whence $\boldsymbol{A} \in (\mathbb{S}(\mathsf{N}))[\mathscr{L}_\mathcal{F}]$, as desired.

Since K is closed under $\mathbb{S}$ by assumption and $\mathsf{N} \subseteq \mathsf{K}$, we obtain $\boldsymbol{A}{\restriction}_{\mathscr{L}_\mathsf{K}} \in \mathbb{S}(\mathsf{N}) \subseteq \mathsf{K}$. It then suffices to show that for each $f \in \mathcal{F}$ the function $f^{\boldsymbol{A}{\restriction}_{\mathscr{L}_\mathsf{K}}}$ is total and coincides with the interpretation of g_f in $\boldsymbol{A}$. To this end, consider an n-ary $f \in \mathcal{F}$ and $a_1, \ldots, a_n \in A$. We need to prove that

$$\langle a_1, \ldots, a_n \rangle \in \mathsf{dom}(f^{\boldsymbol{A}{\restriction}_{\mathscr{L}_\mathsf{K}}}) \quad \text{and} \quad g_f^{\boldsymbol{A}}(a_1, \ldots, a_n) = f^{\boldsymbol{A}{\restriction}_{\mathscr{L}_\mathsf{K}}}(a_1, \ldots, a_n). \tag{37}$$

Observe that

$$f^{\boldsymbol{B}}(a_1, \ldots, a_n) = g_f^{\boldsymbol{B}[\mathscr{L}_\mathsf{K}]}(a_1, \ldots, a_n) = g_f^{\boldsymbol{A}}(a_1, \ldots, a_n),$$

where the first equality holds by $a_1, \ldots, a_n \in A \subseteq B$ and the definition of $\boldsymbol{B}[\mathscr{L}_\mathsf{K}]$, and the second holds because $\boldsymbol{A} \leqslant \boldsymbol{B}[\mathscr{L}_\mathsf{K}]$. Since f is defined by a conjunction of equations φ_f by assumption, the above display yields

$$\boldsymbol{B} \vDash \varphi_f(a_1, \ldots, a_n, g_f^{\boldsymbol{A}}(a_1, \ldots, a_n)).$$

From $\boldsymbol{A} \leqslant \boldsymbol{B}[\mathscr{L}_\mathcal{F}]$ it follows that $\boldsymbol{A}{\restriction}_{\mathscr{L}_\mathsf{K}} \leqslant \boldsymbol{B}$ because $\boldsymbol{B}$ is the $\mathscr{L}_\mathsf{K}$-reduct of $\boldsymbol{B}[\mathscr{L}_\mathcal{F}]$. As φ_f is a conjunction of equations and $\boldsymbol{A}{\restriction}_{\mathscr{L}_\mathsf{K}} \leqslant \boldsymbol{B}$, we can apply Theorem 1.1(iii) to the above display obtaining

$$\boldsymbol{A}{\restriction}_{\mathscr{L}_\mathsf{K}} \vDash \varphi_f(a_1, \ldots, a_n, g_f^{\boldsymbol{A}}(a_1, \ldots, a_n)).$$

Since φ_f defines f and $\boldsymbol{A}{\restriction}_{\mathscr{L}_\mathsf{K}} \in \mathsf{K}$, we conclude that (37) holds. $\boxtimes$

Proposition 9.10. *Let K be a class of algebras, $\mathcal{F} \subseteq \mathsf{imp}_{pp}(\mathsf{K})$, and $\mathscr{L}_\mathcal{F}$ an $\mathcal{F}$-expansion of $\mathscr{L}_\mathsf{K}$. Moreover, let*

$$\mathsf{M} = \{\boldsymbol{A} \in \mathsf{K} : f^{\boldsymbol{A}} \text{ is total for each } f \in \mathcal{F}\}.$$

Then for all $\mathsf{N} \subseteq \mathsf{M}$ and class operator $\mathbb{O} \in \{\mathbb{U}, \mathbb{Q}, \mathbb{ISP}\}$,

$$\text{if } \mathsf{K} \text{ is closed under } \mathbb{O}, \text{ then } \mathbb{O}(\mathsf{N}[\mathscr{L}_\mathcal{F}]) = \mathbb{S}((\mathbb{O}(\mathsf{N}))[\mathscr{L}_\mathcal{F}]).$$

Proof. We will detail the case in which $\mathbb{O} = \mathbb{U}$, as the proof of the case in which $\mathbb{O} \in \{\mathbb{Q}, \mathbb{ISP}\}$ is analogous. Assume that K is closed under $\mathbb{U}$. Then it is closed under $\mathbb{I}$ and $\mathbb{P}_\mathrm{u}$ as well. Moreover, by assumption

$$\mathsf{N} \subseteq \mathsf{M} \tag{38}$$

Therefore, from Proposition 9.8 it follows that

$$\mathbb{IP}_\mathrm{u}(\mathsf{N}[\mathscr{L}_\mathcal{F}]) = (\mathbb{IP}_\mathrm{u}(\mathsf{N}))[\mathscr{L}_\mathcal{F}]. \tag{39}$$

We will show that

$$\mathbb{U}(\mathsf{N}[\mathscr{L}_\mathcal{F}]) = \mathbb{ISP}_\mathrm{u}(\mathsf{N}[\mathscr{L}_\mathcal{F}]) = \mathbb{SIP}_\mathrm{u}(\mathsf{N}[\mathscr{L}_\mathcal{F}]) = \mathbb{S}((\mathbb{IP}_\mathrm{u}(\mathsf{N}))[\mathscr{L}_\mathcal{F}]) \subseteq \mathbb{S}((\mathbb{U}(\mathsf{N}))[\mathscr{L}_\mathcal{F}]).$$

The first equality above holds by Theorem 2.2, the second because $\mathbb{IS} = \mathbb{SI}$, the third follows from (39), and the last inclusion holds because $\mathbb{IP}_{\mathrm{u}}(\mathsf{N}) \subseteq \mathbb{U}(\mathsf{N})$. This establishes the inclusion $\mathbb{U}(\mathsf{N}[\mathscr{L}_{\mathcal{F}}]) \subseteq \mathbb{S}((\mathbb{U}(\mathsf{N}))[\mathscr{L}_{\mathcal{F}}])$.

Therefore, it only remains to prove the reverse inclusion $\mathbb{S}((\mathbb{U}(\mathsf{N}))[\mathscr{L}_{\mathcal{F}}]) \subseteq \mathbb{U}(\mathsf{N}[\mathscr{L}_{\mathcal{F}}])$. Consider $\boldsymbol{A} \in \mathbb{S}((\mathbb{U}(\mathsf{N}))[\mathscr{L}_{\mathcal{F}}])$. Then there exists $\boldsymbol{B} \in \mathbb{U}(\mathsf{N})$ such that $\boldsymbol{B}[\mathscr{L}_{\mathcal{F}}]$ is defined and $\boldsymbol{A} \leqslant \boldsymbol{B}[\mathscr{L}_{\mathcal{F}}]$. From $\boldsymbol{B} \in \mathbb{U}(\mathsf{N})$ and Theorem 2.2 it follows that $\boldsymbol{B} \in \mathbb{ISP}_{\mathrm{u}}(\mathsf{N})$. Therefore, there exist a family $\{\boldsymbol{B}_i : i \in I\} \subseteq \mathsf{N}$, an ultrafilter U on I, and an embedding $h \colon \boldsymbol{B} \to \prod_{i \in I} \boldsymbol{B}_i / U$. In view of (38), we have

$$\{\boldsymbol{B}_i : i \in I\} \subseteq \mathsf{M} \tag{40}$$

Since the hypotheses of Proposition 9.8 are satisfied, we can apply Claim 9.9 to the assumptions that K is closed under $\mathbb{P}_{\mathrm{u}}$ and the above display, obtaining that the algebra $(\prod_{i \in I} \boldsymbol{B}_i / U)[\mathscr{L}_{\mathcal{F}}]$ is defined. Observe that $\boldsymbol{B}, \prod_{i \in I} \boldsymbol{B}_i / U \in \mathbb{ISP}_{\mathrm{u}}(\mathsf{N}) \subseteq \mathsf{K}$ because $\mathsf{N} \subseteq \mathsf{K}$ and K is closed under $\mathbb{U}$ by assumption. Consequently,

$$\boldsymbol{B}[\mathscr{L}_{\mathcal{F}}], \Big(\prod_{i \in I} \boldsymbol{B}_i / U\Big)[\mathscr{L}_{\mathcal{F}}] \in \mathsf{K}[\mathscr{L}_{\mathcal{F}}].$$

Since $h \colon \boldsymbol{B} \to \prod_{i \in I} \boldsymbol{B}_i / U$ is an embedding between members of K, from the above display and Proposition 9.5 it follows that h can be regarded as an embedding $h \colon \boldsymbol{B}[\mathscr{L}_{\mathcal{F}}] \to (\prod_{i \in I} \boldsymbol{B}_i / U)[\mathscr{L}_{\mathcal{F}}]$. Lastly, (40) and the assumption that K is closed under $\mathbb{P}_{\mathrm{u}}$ allow us to apply Proposition 9.8(i), obtaining

$$\prod_{i \in I} (\boldsymbol{B}_i[\mathscr{L}_{\mathcal{F}}]) / U \in (\mathbb{P}_{\mathrm{u}}(\{\boldsymbol{B}_i : i \in I\}))[\mathscr{L}_{\mathcal{F}}].$$

As the $\mathscr{L}_{\mathsf{K}}$-reduct of $\prod_{i \in I} (\boldsymbol{B}_i[\mathscr{L}_{\mathcal{F}}]) / U$ is $\prod_{i \in I} \boldsymbol{B}_i / U$, the above display yields

$$\prod_{i \in I} (\boldsymbol{B}_i[\mathscr{L}_{\mathcal{F}}]) / U = \Big(\prod_{i \in I} \boldsymbol{B}_i / U\Big)[\mathscr{L}_{\mathcal{F}}].$$

Therefore, the map $h \colon \boldsymbol{B}[\mathscr{L}_{\mathcal{F}}] \to \prod_{i \in I} (\boldsymbol{B}_i[\mathscr{L}_{\mathcal{F}}]) / U$ is an embedding. Since $\{\boldsymbol{B}_i : i \in I\} \subseteq \mathsf{N}$, it follows that $\boldsymbol{B}[\mathscr{L}_{\mathcal{F}}] \in \mathbb{ISP}_{\mathrm{u}}(\mathsf{N}[\mathscr{L}_{\mathcal{F}}])$. As $\boldsymbol{A} \leqslant \boldsymbol{B}[\mathscr{L}_{\mathcal{F}}]$ by assumption, we conclude that $\boldsymbol{A} \in \mathbb{SISP}_{\mathrm{u}}(\mathsf{N}[\mathscr{L}_{\mathcal{F}}]) \subseteq \mathbb{U}(\mathsf{N}[\mathscr{L}_{\mathcal{F}}])$. ⊠

10. Primitive positive expansions

As we mentioned, our aim is to expand the language of a given elementary class of algebras K by adding to it enough implicit operations so that every implicit operation of K becomes interpolable by a set of terms in a class M of algebras in the expanded language. In view of Corollary 3.10, the latter can be stated as the demand that implicit operations of K defined by pp formulas be interpolated by terms of M. Because of this, from now on we shall restrict our attention to implicit operations defined by pp formulas. Furthermore, we require the implicit operations under consideration to be extendable in order to guarantee the validity of condition (D1) (see Proposition 9.6).

However, even when the implicit operations in $\mathcal{F}$ are defined by pp formulas and extendable, the class $\mathsf{K}[\mathscr{L}_{\mathcal{F}}]$ may lack some desirable closure properties. More precisely, there is no guarantee that if K is a universal class or a quasivariety, then so is $\mathsf{K}[\mathscr{L}_{\mathcal{F}}]$. This problem is

easily overcome by closing $\mathsf{K}[\mathscr{L}_{\mathcal{F}}]$ under $\mathbb{S}$, leading to the core of this section, namely, the notion of a pp expansion.

Definition 10.1. Let K and M be a pair of classes of algebras. Then M is said to be a *primitive positive expansion* (*pp expansion* for short) of K when $\mathsf{M} = \mathbb{S}(\mathsf{K}[\mathscr{L}_{\mathcal{F}}])$ for some $\mathcal{F} \subseteq \mathsf{ext}_{pp}(\mathsf{K})$ and $\mathcal{F}$-expansion $\mathscr{L}_{\mathcal{F}}$ of $\mathscr{L}_{\mathsf{K}}$. In this case, we say that M is *induced* by $\mathcal{F}$ and $\mathscr{L}_{\mathcal{F}}$.

From Proposition 9.6 we deduce that pp expansions satisfy condition (D1).

Proposition 10.2. *Let M be a pp expansion of a universal class K. Then K is the class of $\mathscr{L}_{\mathsf{K}}$-subreducts of M.*

Proof. Assume that M is a pp expansion of K of the form $\mathbb{S}(\mathsf{K}[\mathscr{L}_{\mathcal{F}}])$. As K is a universal class, we can apply Proposition 9.6, obtaining that K is the class of $\mathscr{L}_{\mathsf{K}}$-subreducts of $\mathsf{K}[\mathscr{L}_{\mathcal{F}}]$. Hence, K is also the class of $\mathscr{L}_{\mathsf{K}}$-subreducts of $\mathbb{S}(\mathsf{K}[\mathscr{L}_{\mathcal{F}}]) = \mathsf{M}$. ⊠

As we announced, the following holds true.

Theorem 10.3. *Let K be a class of algebras. The following conditions hold for a pp expansion M of K induced by $\mathcal{F}$ and $\mathscr{L}_{\mathcal{F}}$:*

(i) *if K is a universal class, then M is a universal class;*
(ii) *if K is a quasivariety, then M is a quasivariety such that $\mathsf{M}_{\mathrm{RSI}} \subseteq \mathbb{S}(\mathsf{K}_{\mathrm{RSI}}[\mathscr{L}_{\mathcal{F}}])$;*
(iii) *if K is a variety and $\mathcal{F} \subseteq \mathsf{ext}_{eq}(\mathsf{K})$, then M is a variety.*

As shown in [27, Thm. 2.1], the hypothesis that $\mathcal{F} \subseteq \mathsf{ext}_{eq}(\mathsf{K})$ in Theorem 10.3(iii) cannot be dispensed with.

Besides the above theorem, the main result of this section consists of four observations which facilitate the task of detecting the pp expansions of a given class of algebras. On the one hand, we will establish the following description of pp expansions induced by implicit operations definable by conjunctions of equations (for a similar result, see [32, Lem. 2.1]).

Theorem 10.4. *Let K be a universal class axiomatized by a set of formulas Σ and M a pp expansion of K induced by $\mathcal{F} \subseteq \mathsf{ext}_{eq}(\mathsf{K})$ and $\mathscr{L}_{\mathcal{F}} = \mathscr{L}_{\mathsf{K}} \cup \{g_f : f \in \mathcal{F}\}$. Then $\mathsf{M} = \mathsf{K}[\mathscr{L}_{\mathcal{F}}]$ and M is axiomatized by*

$$\Sigma \cup \{\varphi_f(x_1, \ldots, x_n, g_f(x_1, \ldots, x_n)) : f \text{ is an } n\text{-ary member of } \mathcal{F}\},$$

where φ_f denotes the conjunction of equations defining $f \in \mathcal{F}$.

On the other hand, we will establish the next description of pp expansions in terms of the class operators of universal class and quasivariety generation.

Theorem 10.5. *Let K be a class of algebras, $\mathcal{F} \subseteq \mathsf{ext}_{pp}(\mathsf{K})$, and $\mathscr{L}_{\mathcal{F}}$ an $\mathcal{F}$-expansion of $\mathscr{L}_{\mathsf{K}}$. Moreover, let $\mathsf{N} \subseteq \mathsf{K}$ and assume that $f^{\boldsymbol{A}}$ is total for all $\boldsymbol{A} \in \mathsf{N}$ and $f \in \mathcal{F}$. Then for each $\mathbb{O} \in \{\mathbb{U}, \mathbb{Q}\}$ such that $\mathsf{K} = \mathbb{O}(\mathsf{N})$ the class $\mathbb{O}(\mathsf{N}[\mathscr{L}_{\mathcal{F}}])$ is a pp expansion of K that coincides with $\mathbb{S}(\mathsf{K}[\mathscr{L}_{\mathcal{F}}])$.*

We will then show that the relation "being a pp expansion of" is transitive.

Theorem 10.6. *Every pp expansion of a pp expansion of a class of algebras* K *is a pp expansion of* K.

Lastly, we will show that enlarging the set of implicit operations inducing a pp expansion of a class of algebras also yields a pp expansion of the same class.

Theorem 10.7. *Let* K *be a universal class and* $\mathcal{F} \subseteq \mathcal{G} \subseteq \mathsf{ext}_{pp}(\mathsf{K})$. *Let also* $\mathscr{L}_{\mathcal{F}}$ *be an* $\mathcal{F}$*-expansion of* $\mathscr{L}_{\mathsf{K}}$ *and* $\mathscr{L}_{\mathcal{G}}$ *a* $\mathcal{G}$*-expansion of* $\mathscr{L}_{\mathsf{K}}$ *such that* $\mathscr{L}_{\mathcal{F}} \subseteq \mathscr{L}_{\mathcal{G}}$. *Then* $\mathbb{S}(\mathsf{K}[\mathscr{L}_{\mathcal{G}}])$ *is a pp expansion of* $\mathbb{S}(\mathsf{K}[\mathscr{L}_{\mathcal{F}}])$.

Before proving Theorems 10.3, 10.4, 10.5, 10.6, and 10.7, let us illustrate how these results can be used to describe pp expansions of familiar classes of algebras.

Example 10.8 (Lazy pp expansions)**.** Every class of algebras K closed under $\mathbb{S}$ is a pp expansion of itself. For let $\mathcal{F} = \emptyset$. Then $\mathcal{F} = \emptyset \subseteq \mathsf{ext}_{pp}(\mathsf{K})$. Furthermore, let $\mathscr{L}_{\mathcal{F}} = \mathscr{L}_{\mathsf{K}}$ and observe that $\mathscr{L}_{\mathcal{F}}$ is an $\mathcal{F}$-expansion of $\mathscr{L}_{\mathsf{K}}$ because $\mathcal{F} = \emptyset$. Therefore, the class $\mathsf{K}[\mathscr{L}_{\mathcal{F}}]$ coincides with K. As K is closed under $\mathbb{S}$, we conclude that $\mathsf{K} = \mathbb{S}(\mathsf{K}[\mathscr{L}_{\mathcal{F}}])$, whence K is a pp expansion of itself. $\boxtimes$

Example 10.9 (Cancellative commutative monoids)**.** We will prove the following.

Theorem 10.10. *The variety of Abelian groups is a pp expansion of the quasivariety of cancellative commutative monoids.*

Proof. Let f be the unary implicit operation of the quasivariety of cancellative commutative monoids CCMon given by Theorem 8.10. Recall from the same theorem that f is extendable and defined by the equation $x \cdot y \approx 1$. Moreover, let $(\ )^{-1}$ be a unary function symbol and denote by t^{-1} the result of applying $(\ )^{-1}$ to a term t. Then the language $\mathscr{L}_f = \mathscr{L}_{\mathsf{CCMon}} \cup \{(\ )^{-1}\}$ is an f-expansion of $\mathscr{L}_{\mathsf{CCMon}}$, in which the role of g_f is played by $(\ )^{-1}$.

As CCMon is a universal class, Theorem 10.4 yields that $\mathsf{CCMon}[\mathscr{L}_f]$ is a pp expansion of CCMon axiomatized by the axioms for cancellative commutative monoids plus the equation $x \cdot x^{-1} \approx 1$.

Clearly, every member of $\mathsf{CCMon}[\mathscr{L}_f]$ is an Abelian group. On the other hand, every Abelian group can be obtained by adding the implicit operation f to its monoid reduct, which is a cancellative commutative monoid. Therefore, $\mathsf{CCMon}[\mathscr{L}_f]$ coincides with the variety of Abelian groups. $\boxtimes$

Example 10.11 (Distributive lattices)**.** Our aim is to establish the next result.

Theorem 10.12. *The following conditions hold:*
  (i) *the variety of relatively complemented distributive lattices is a pp expansion of the variety of distributive lattices;*
  (ii) *the variety of Boolean algebras is a pp expansion of the variety of bounded distributive lattices.*

Proof. We detail only the proof of (i), as the proof of (ii) is analogous. Let f be the ternary implicit operation of the variety of distributive lattices DL given by Theorem 8.20. Recall

from the same theorem that f is extendable and defined by the conjunction of equations

$$\varphi = (x_1 \wedge y \approx x_1 \wedge x_2 \wedge x_3) \sqcap (x_1 \vee y \approx x_1 \vee x_2 \vee x_3).$$

Moreover, let r be a ternary function symbol. Then the language $\mathscr{L}_f = \mathscr{L}_{\mathsf{DL}} \cup \{r\}$ is an f-expansion of $\mathscr{L}_{\mathsf{DL}}$, in which the role of g_f is played by r.

As DL is a universal class, we can apply Theorem 10.4, obtaining that $\mathsf{DL}[\mathscr{L}_f]$ is a pp expansion of DL axiomatized by the axioms for distributive lattices plus the equations

$$x_1 \wedge r(x_1, x_2, x_3) \approx x_1 \wedge x_2 \wedge x_3 \quad \text{and} \quad x_1 \vee r(x_1, x_2, x_3) \approx x_1 \vee x_2 \vee x_3.$$

As a consequence, $\mathsf{DL}[\mathscr{L}_f]$ coincides with the variety of relatively complemented distributive lattices (see Example 7.4). $\boxtimes$

Example 10.13 (Reduced commutative rings). A *meadow* is an algebra $\langle A; +, \cdot, -, (\ )^*, 0, 1 \rangle$ which comprises a commutative ring $\langle A; +, \cdot, -, 0, 1 \rangle$ and a unary operation $(\ )^*$ such that for each $a \in A$,

$$(a \cdot a^*) \cdot a = a \quad \text{and} \quad a = a^{**}$$

(see, e.g., [13]). As a consequence, the class of meadows forms a variety.

Prototypical examples of meadows arise by adding the operation of "taking weak inverses" to fields. More precisely, a *zero-totalized field* is an algebra $\langle A; +, \cdot, -, (\ )^*, 0, 1 \rangle$ comprising a field $\langle A; +, \cdot, -, 0, 1 \rangle$ and a unary operation $(\ )^*$ defined as $a^* = \mathsf{wi}(a)$ for each $a \in A$, where $\mathsf{wi}(a)$ is the weak inverse of a (see Example 3.16). Meadows were introduced in response to the desire to construct an equational theory that captures the essence of fields. The following representation theorem (see [12, Sec. 3.2]) shows that they fulfill this expectation.

Theorem 10.14. *An algebra is a meadow if and only if it is isomorphic to a subdirect product of zero-totalized fields.*

We will prove the following.

Theorem 10.15. *The variety of meadows is a pp expansion of the quasivariety of reduced commutative rings.*

Proof. Let f be the unary implicit operation of the quasivariety RCRing of reduced commutative rings given by Theorem 8.18 and recall from the same theorem that $f \in \mathsf{ext}_{pp}(\mathsf{RCRing})$. Moreover, let $(\ )^*$ be a unary function symbol. Then the language $\mathscr{L}_f = \mathscr{L}_{\mathsf{RCRing}} \cup \{(\ )^*\}$ is an f-expansion of $\mathscr{L}_{\mathsf{RCRing}}$, in which the role of g_f is played by $(\ )^*$.

From Theorem 8.18 it follows that for each field $\boldsymbol{A}$ the function $f^{\boldsymbol{A}}$ is total and the algebra $\boldsymbol{A}[\mathscr{L}_f]$ coincides with the zero-totalized field obtained by adding the operation of "taking weak inverses" to $\boldsymbol{A}$. Therefore, letting Field and Field^* be the classes of fields and of zero-totalized fields, respectively, we obtain $\mathsf{Field}[\mathscr{L}_f] = \mathsf{Field}^*$. In view of Theorems 3.17 and 10.14, we also have

$$\mathsf{RCRing} = \mathbb{Q}(\mathsf{Field}) \quad \text{and} \quad \mathsf{Meadow} = \mathbb{Q}(\mathsf{Field}^*),$$

where Meadow is the variety of meadows. Lastly, recall that $f \in \mathsf{ext}_{pp}(\mathsf{RCRing})$ and that $f^{\boldsymbol{A}}$ is total for each $\boldsymbol{A} \in \mathsf{Field}$. Together with the left hand side of the above display, this allows us to apply Theorem 10.5, obtaining that $\mathbb{Q}(\mathsf{Field}[\mathscr{L}_f])$ is a pp expansion of RCRing. By

applying in succession the equality $\mathsf{Field}[\mathscr{L}_f] = \mathsf{Field}^*$ and the right hand side of the above display, we obtain

$$\mathbb{Q}(\mathsf{Field}[\mathscr{L}_f]) = \mathbb{Q}(\mathsf{Field}^*) = \mathsf{Meadow}.$$

Hence, we conclude that Meadow is a pp expansion of RCRing. ⊠

Example 10.16 (Hilbert algebras). An *implicative semilattice* is an algebra $\langle A; \wedge, \rightarrow \rangle$ which comprises a semilattice $\langle A; \wedge \rangle$ and a binary operation $\rightarrow$ such that for all $a, b, c \in A$,

$$a \wedge b \leqslant c \iff a \leqslant b \rightarrow c,$$

where $\leqslant$ is the meet order of $\langle A; \wedge \rangle$. The class of implicative semilattices forms a variety (see, e.g., [77, pp. 105–106]) which coincides with the class of $\langle \wedge, \rightarrow \rangle$-subreducts of Heyting algebras (see [66, Thms. 5.1 & 9.1]). We will prove the following.

Theorem 10.17. *The variety of implicative semilattices is a pp expansion of the variety of Hilbert algebras.*

Proof. First, let

$$\mathsf{Heyting}_{\rightarrow} = \text{the class of } \langle \rightarrow \rangle \text{-reducts of Heyting algebras};$$
$$\mathsf{Heyting}_{\wedge, \rightarrow} = \text{the class of } \langle \wedge, \rightarrow \rangle \text{-reducts of Heyting algebras}.$$

As the variety $\mathsf{Hilbert}$ of Hilbert algebras is the class of $\langle \rightarrow \rangle$-subreducts of Heyting algebras and the variety ISL of implicative semilattices is the class of $\langle \wedge, \rightarrow \rangle$-subreducts of Heyting algebras, we have

$$\mathsf{Hilbert} = \mathbb{Q}(\mathsf{Heyting}_{\rightarrow}) \quad \text{and} \quad \mathsf{ISL} = \mathbb{Q}(\mathsf{Heyting}_{\wedge, \rightarrow}). \tag{41}$$

Now, let $f \in \mathsf{ext}_{pp}(\mathsf{Hilbert})$ be the binary implicit operation given by Theorem 8.22. Moreover, let $\wedge$ be a binary function symbol. Then the language $\mathscr{L}_f = \mathscr{L}_{\mathsf{Hilbert}} \cup \{\wedge\}$ is an f-expansion of $\mathscr{L}_{\mathsf{Hilbert}}$, in which the role of g_f is played by $\wedge$.

Recall from Theorem 8.22 that for every Heyting algebra $\boldsymbol{A}$ with implication reduct $\boldsymbol{A}_{\rightarrow}$ the operation $f^{\boldsymbol{A}_{\rightarrow}}$ is total and coincides with the meet operation of $\boldsymbol{A}$. Therefore, the algebra $\boldsymbol{A}_{\rightarrow}[\mathscr{L}_f]$ is defined and coincides with the $\langle \wedge, \rightarrow \rangle$-reduct of $\boldsymbol{A}$. Consequently,

$$\mathsf{Heyting}_{\rightarrow}[\mathscr{L}_f] = \mathsf{Heyting}_{\wedge, \rightarrow}.$$

Lastly, recall that $f \in \mathsf{ext}_{pp}(\mathsf{Hilbert})$ and that $f^{\boldsymbol{A}}$ is total for each $\boldsymbol{A} \in \mathsf{Heyting}_{\rightarrow}$. Together with the left hand side of (41), this allows us to apply Theorem 10.5, obtaining that $\mathbb{Q}(\mathsf{Heyting}_{\rightarrow}[\mathscr{L}_f])$ is a pp expansion of $\mathsf{Hilbert}$. By applying in succession the above display and the right hand side of (41), we obtain

$$\mathbb{Q}(\mathsf{Heyting}_{\rightarrow}[\mathscr{L}_f]) = \mathbb{Q}(\mathsf{Heyting}_{\wedge, \rightarrow}) = \mathsf{ISL}.$$

Hence, we conclude that ISL is a pp expansion of $\mathsf{Hilbert}$. ⊠

Example 10.18 (Pseudocomplemented distributive lattices). A Heyting algebra $\boldsymbol{A}$ is said to have *depth* $\leqslant 2$ when the chains in the poset of prime filters of $\boldsymbol{A}$ have size at most two. The class of all Heyting algebras of depth $\leqslant 2$ forms a variety (see, e.g., [14, Thm. 4.1] and the references therein). We will prove the following.

Theorem 10.19. *The variety of Heyting algebras of depth $\leqslant 2$ is a pp expansion of the variety of pseudocomplemented distributive lattices.*

Proof. Let PDL and $\mathsf{Heyting}_2$ be the varieties of pseudocomplemented distributive lattices and of Heyting algebras of depth $\leqslant 2$, respectively. We recall that the members of $(\mathsf{Heyting}_2)^{\mathrm{fg}}_{\mathrm{SI}}$ are precisely the Heyting algebras whose lattice reduct is a finite Boolean lattice adjoined with a new top element(see [83, Thm. 2]). It follows from the characterization of subdirectly irreducible Heyting algebras (see, e.g., [7, Thm. IX.4.5]) and the local finiteness of $\mathsf{Heyting}_2$ (see [79, 82]) that $\mathsf{PDL}^{\mathrm{fg}}_{\mathrm{SI}}$ is the class of $\langle \wedge, \vee, \neg, 0, 1 \rangle$-reducts of $(\mathsf{Heyting}_2)^{\mathrm{fg}}_{\mathrm{SI}}$. Lastly, from Theorem 2.16 it follows that

$$\mathsf{PDL} = \mathbb{Q}(\mathsf{PDL}^{\mathrm{fg}}_{\mathrm{SI}}) \quad \text{and} \quad \mathsf{Heyting}_2 = \mathbb{Q}((\mathsf{Heyting}_2)^{\mathrm{fg}}_{\mathrm{SI}}). \tag{42}$$

Now, let $f \in \mathsf{ext}_{pp}(\mathsf{PDL})$ be the binary implicit operation of PDL given by Theorem 8.25. Moreover, let $\rightarrow$ be a binary function symbol. Then the language $\mathscr{L}_f = \mathscr{L}_{\mathsf{PDL}} \cup \{\rightarrow\}$ is an f-expansion of $\mathscr{L}_{\mathsf{PDL}}$, in which the role of g_f is played by $\rightarrow$.

Recall from Theorem 8.25 that for each $\boldsymbol{A} \in \mathsf{PDL}^{\mathrm{fg}}_{\mathrm{SI}}$ the function $f^{\boldsymbol{A}}$ is total and coincides with the implication $\rightarrow$ of the unique Heyting algebra with the same lattice reduct as $\boldsymbol{A}$. As $\mathsf{PDL}^{\mathrm{fg}}_{\mathrm{SI}}$ is the class of $\langle \wedge, \vee, \neg, 0, 1 \rangle$-reducts of $(\mathsf{Heyting}_2)^{\mathrm{fg}}_{\mathrm{SI}}$, this yields

$$\mathsf{PDL}^{\mathrm{fg}}_{\mathrm{SI}}[\mathscr{L}_f] = (\mathsf{Heyting}_2)^{\mathrm{fg}}_{\mathrm{SI}}.$$

Finally, recall that $f \in \mathsf{ext}_{pp}(\mathsf{PDL})$ and that $f^{\boldsymbol{A}}$ is total for each $\boldsymbol{A} \in \mathsf{PDL}^{\mathrm{fg}}_{\mathrm{SI}}$. Together with the left hand side of (42), this allows us to apply Theorem 10.5, obtaining that $\mathbb{Q}(\mathsf{PDL}^{\mathrm{fg}}_{\mathrm{SI}}[\mathscr{L}_f])$ is a pp expansion of PDL. By applying in succession the above display and the right hand side of (42), we obtain

$$\mathbb{Q}(\mathsf{PDL}^{\mathrm{fg}}_{\mathrm{SI}}[\mathscr{L}_f]) = \mathbb{Q}((\mathsf{Heyting}_2)^{\mathrm{fg}}_{\mathrm{SI}}) = \mathsf{Heyting}_2.$$

Hence, we conclude that $\mathsf{Heyting}_2$ is a pp expansion of PDL. $\boxtimes$

The rest of this section is devoted to proving Theorems 10.3, 10.4, 10.5, 10.6, and 10.7. We postpone the proof of Theorem 10.3 and begin by proving Theorem 10.4.

Proof. Proposition 9.3 yields that

$$\mathsf{K}[\mathscr{L}_{\mathcal{F}}] = \{\boldsymbol{B} : \boldsymbol{B} \text{ is an } \mathscr{L}_{\mathcal{F}}\text{-algebra such that } \boldsymbol{B}{\restriction}_{\mathscr{L}_{\mathsf{K}}} \in \mathsf{K} \text{ and}$$
$$\boldsymbol{B} \vDash \varphi_f(x_1, \ldots, x_n, g_f(x_1, \ldots, x_n)) \text{ for each } n\text{-ary } f \in \mathcal{F}\}.$$

Since Σ axiomatizes K, for every $\mathscr{L}_{\mathcal{F}}$-algebra $\boldsymbol{B}$ we have that $\boldsymbol{B}{\restriction}_{\mathscr{L}_{\mathsf{K}}} \in \mathsf{K}$ if and only if $\boldsymbol{B} \vDash \Sigma$. Therefore, from the above display it follows that the set of formulas

$$\Sigma \cup \{\varphi_f(x_1, \ldots, x_n, g_f(x_1, \ldots, x_n)) : f \text{ is an } n\text{-ary member of } \mathcal{F}\}$$

axiomatizes $\mathsf{K}[\mathscr{L}_{\mathcal{F}}]$. To show that $\mathsf{M} = \mathsf{K}[\mathscr{L}_{\mathcal{F}}]$ it is sufficient to prove that $\mathsf{K}[\mathscr{L}_{\mathcal{F}}]$ is closed under $\mathbb{S}$. As K is a universal class, Theorem 2.1(iii) allows us to assume that Σ consists of universal formulas. Together with the fact that each $\varphi_f(x_1, \ldots, x_n, g_f(x_1, \ldots, x_n))$ is a conjunction of equations, this implies that $\mathsf{K}[\mathscr{L}_{\mathcal{F}}]$ can be axiomatized by a set of universal formulas. Therefore, $\mathsf{K}[\mathscr{L}_{\mathcal{F}}]$ is a universal class by Theorem 2.1(iii), and hence it is closed under $\mathbb{S}$. $\boxtimes$

We are now ready to prove Theorem 10.3.

Proof. We first prove that if K is a universal class or quasivariety, then so is M. It suffices to show that for each class operator $\mathbb{O} \in \{\mathbb{U}, \mathbb{Q}\}$ if K is closed under $\mathbb{O}$, then $\mathbb{S}(\mathsf{K}[\mathscr{L}_\mathcal{F}])$ is closed under $\mathbb{O}$ as well.

To this end, let $\mathbb{O} \in \{\mathbb{U}, \mathbb{Q}\}$ and assume that K is closed under $\mathbb{O}$. Furthermore, let

$$\mathsf{N} = \{\boldsymbol{A} \in \mathsf{K} : f^{\boldsymbol{A}} \text{ is total for each } f \in \mathcal{F}\}.$$

We begin with the next observation.

Claim 10.20. *We have* $\mathsf{K} = \mathbb{O}(\mathsf{N})$.

Proof of the Claim. As $\mathbb{O} \in \{\mathbb{U}, \mathbb{Q}\}$ and K is closed under $\mathbb{O}$, we known that K is a universal class. Therefore, we can apply Theorem 8.4, obtaining that for each $\boldsymbol{A} \in \mathsf{K}$ there exists $\boldsymbol{B} \in \mathsf{K}$ with $\boldsymbol{A} \leqslant \boldsymbol{B}$ such that $f^{\boldsymbol{B}}$ is total for each $f \in \mathsf{ext}(\mathsf{K})$. Since $\mathcal{F} \subseteq \mathsf{ext}_{pp}(\mathsf{K}) \subseteq \mathsf{ext}(\mathsf{K})$ by assumption, for each $\boldsymbol{A} \in \mathsf{K}$ there exists $\boldsymbol{B} \in \mathsf{K}$ with $\boldsymbol{A} \leqslant \boldsymbol{B}$ such that $f^{\boldsymbol{B}}$ is total for each $f \in \mathcal{F}$. Together with the definition of N, this yields $\mathsf{K} \subseteq \mathbb{S}(\mathsf{N})$. As $\mathbb{O} \in \{\mathbb{U}, \mathbb{Q}\}$, we obtain $\mathsf{K} \subseteq \mathbb{O}(\mathsf{N})$. On the other hand, from $\mathsf{N} \subseteq \mathsf{K}$ and the assumption that K is closed under $\mathbb{O}$ it follows that $\mathbb{O}(\mathsf{N}) \subseteq \mathsf{K}$, whence $\mathsf{K} = \mathbb{O}(\mathsf{N})$. ⊠

Since K is closed under $\mathbb{O}$ by assumption, the definition of N allows us to apply Proposition 9.10, obtaining $\mathbb{O}(\mathsf{N}[\mathscr{L}_\mathcal{F}]) = \mathbb{S}((\mathbb{O}(\mathsf{N}))[\mathscr{L}_\mathcal{F}])$. Together with Claim 10.20, this yields $\mathbb{O}(\mathsf{N}[\mathscr{L}_\mathcal{F}]) = \mathbb{S}(\mathsf{K}[\mathscr{L}_\mathcal{F}])$. Hence, $\mathbb{S}(\mathsf{K}[\mathscr{L}_\mathcal{F}])$ is closed under $\mathbb{O}$, as desired.

Now, we prove the last part of (ii). Assume that K is a quasivariety. We need to show that $\mathsf{M}_{\text{RSI}} \subseteq \mathbb{S}(\mathsf{K}_{\text{RSI}}[\mathscr{L}_\mathcal{F}])$. Let

$$\mathsf{N}_{\text{RSI}} = \{\boldsymbol{A} \in \mathsf{K}_{\text{RSI}} : f^{\boldsymbol{A}} \text{ is total for each } f \in \mathcal{F}\}.$$

We rely on the following observation.

Claim 10.21. *We have* $\mathsf{K} = \mathbb{ISP}(\mathsf{N}_{\text{RSI}})$.

Proof of the Claim. Since K is a quasivariety, it is closed under $\mathbb{I}, \mathbb{S}$, and $\mathbb{P}$. Therefore, it suffices to prove the inclusion $\mathsf{K} \subseteq \mathbb{ISP}(\mathsf{N}_{\text{RSI}})$. To this end, consider $\boldsymbol{A} \in \mathsf{K}$. In view of the Subdirect Decomposition Theorem 2.9 there exist a family $\{\boldsymbol{A}_i : i \in I\} \subseteq \mathsf{K}_{\text{RSI}}$ and an embedding $h \colon \boldsymbol{A} \to \prod_{i \in I} \boldsymbol{A}_i$. By Theorem 8.4 for each $i \in I$ there exists $\boldsymbol{B}_i \in \mathsf{K}_{\text{RSI}}$ with $\boldsymbol{A}_i \leqslant \boldsymbol{B}_i$ such that $f^{\boldsymbol{B}_i}$ is total for each $f \in \mathsf{ext}(\mathsf{K})$. As $\mathcal{F} \subseteq \mathsf{ext}(\mathsf{K})$ by assumption, this guarantees that $\{\boldsymbol{B}_i : i \in I\} \subseteq \mathsf{N}_{\text{RSI}}$. Furthermore, from $\boldsymbol{A}_i \leqslant \boldsymbol{B}_i$ for each $i \in I$ it follows that $\prod_{i \in I} \boldsymbol{A}_i \leqslant \prod_{i \in I} \boldsymbol{B}_i$. Therefore, h can be viewed as an embedding of $\boldsymbol{A}$ into $\prod_{i \in I} \boldsymbol{B}_i$. Thus, we conclude that $\boldsymbol{A} \in \mathbb{ISP}(\mathsf{N}_{\text{RSI}})$. ⊠

Since the quasivariety K is closed under $\mathbb{ISP}$, the definition of N_{RSI} allows us to apply Proposition 9.10, obtaining $\mathbb{ISP}(\mathsf{N}_{\text{RSI}}[\mathscr{L}_\mathcal{F}]) = \mathbb{S}((\mathbb{ISP}(\mathsf{N}_{\text{RSI}}))[\mathscr{L}_\mathcal{F}])$. By Claim 10.21 this amounts to $\mathbb{ISP}(\mathsf{N}_{\text{RSI}}[\mathscr{L}_\mathcal{F}]) = \mathbb{S}(\mathsf{K}[\mathscr{L}_\mathcal{F}])$. As $\mathsf{M} = \mathbb{S}(\mathsf{K}[\mathscr{L}_\mathcal{F}])$, we conclude that $\mathbb{ISP}(\mathsf{N}_{\text{RSI}}[\mathscr{L}_\mathcal{F}]) = \mathsf{M}$.

We are now ready to prove that $\mathsf{M}_{\text{RSI}} \subseteq \mathbb{S}(\mathsf{K}_{\text{RSI}}[\mathscr{L}_\mathcal{F}])$. Consider $\boldsymbol{A} \in \mathsf{M}_{\text{RSI}}$. Since $\mathsf{M} = \mathbb{ISP}(\mathsf{N}_{\text{RSI}}[\mathscr{L}_\mathcal{F}])$, there exist a family $\{\boldsymbol{A}_i : i \in I\} \subseteq \mathsf{N}_{\text{RSI}}$ and an embedding $h \colon \boldsymbol{A} \to$

$\prod_{i \in I}(\boldsymbol{A}_i[\mathscr{L}_\mathcal{F}])$. Observe that h can be viewed as a subdirect embedding $h \colon \boldsymbol{A} \to \prod_{i \in I} p_i[h[\boldsymbol{A}]]$ whose factors belong to M because for each $i \in I$ we have

$$p_i[h[\boldsymbol{A}]] \in \mathbb{S}(\boldsymbol{A}_i[\mathscr{L}_\mathcal{F}]) \subseteq \mathbb{S}(\mathsf{N}_{\mathrm{RSI}}[\mathscr{L}_\mathcal{F}]) \subseteq \mathbb{ISP}(\mathsf{N}_{\mathrm{RSI}}[\mathscr{L}_\mathcal{F}]) = \mathsf{M}.$$

As $\boldsymbol{A} \in \mathsf{M}_{\mathrm{RSI}}$, this implies that there exists $i \in I$ such that the map $p_i \circ h \colon \boldsymbol{A} \to p_i[h[\boldsymbol{A}]]$ is an isomorphism. Since $p_i[h[\boldsymbol{A}]] \leqslant \boldsymbol{A}_i[\mathscr{L}_\mathcal{F}]$ and $\boldsymbol{A}_i \in \mathsf{N}_{\mathrm{RSI}}$, we obtain $\boldsymbol{A} \in \mathbb{IS}(\mathsf{N}_{\mathrm{RSI}}[\mathscr{L}_\mathcal{F}])$. Lastly, applying in succession $\mathbb{IS} = \mathbb{SI}$, Proposition 9.8(i) for the case where $\mathbb{O} = \mathbb{I}$, the fact that $\mathsf{N}_{\mathrm{RSI}}$ is closed under $\mathbb{I}$, and the inclusion $\mathsf{N}_{\mathrm{RSI}} \subseteq \mathsf{K}_{\mathrm{RSI}}$, we conclude that

$$\boldsymbol{A} \in \mathbb{IS}(\mathsf{N}_{\mathrm{RSI}}[\mathscr{L}_\mathcal{F}]) = \mathbb{SI}(\mathsf{N}_{\mathrm{RSI}}[\mathscr{L}_\mathcal{F}]) = \mathbb{S}((\mathbb{I}(\mathsf{N}_{\mathrm{RSI}}))[\mathscr{L}_\mathcal{F}]) = \mathbb{S}(\mathsf{N}_{\mathrm{RSI}}[\mathscr{L}_\mathcal{F}]) \subseteq \mathbb{S}(\mathsf{K}_{\mathrm{RSI}}[\mathscr{L}_\mathcal{F}]),$$

as desired.

It remains to prove (iii). Assume that K is a variety and $\mathcal{F} \subseteq \mathsf{ext}_{eq}(\mathsf{K})$. By Theorem 2.1(i) there exists a set of equations Σ that axiomatizes K. Then Theorem 10.4 yields that M is axiomatized by

$$\Sigma \cup \{\varphi_f(x_1, \ldots, x_n, g_f(x_1, \ldots, x_n)) : f \text{ is an } n\text{-ary member of } \mathcal{F}\},$$

where φ_f is the conjunction of equations defining $f \in \mathcal{F}$. Consequently, M is axiomatized by a set of equations. Thus, M is a variety by Theorem 2.1(i). ⊠

Then we prove Theorem 10.5.

Proof. We detail only the case in which $\mathbb{O} = \mathbb{U}$, as the case in which $\mathbb{O} = \mathbb{Q}$ is handled analogously. Accordingly, assume that $\mathsf{K} = \mathbb{U}(\mathsf{N})$. As $\mathcal{F} \subseteq \mathsf{ext}_{pp}(\mathsf{K})$ and $\mathscr{L}_\mathcal{F}$ is an $\mathcal{F}$-expansion of $\mathscr{L}_\mathsf{K}$ by assumption, the class $\mathbb{S}(\mathsf{K}[\mathscr{L}_\mathcal{F}])$ is a pp expansion of K. Therefore, it only remains to show that $\mathbb{S}(\mathsf{K}[\mathscr{L}_\mathcal{F}]) = \mathbb{U}(\mathsf{N}[\mathscr{L}_\mathcal{F}])$. Since

$$\mathsf{N} \subseteq \{\boldsymbol{A} \in \mathsf{K} : f^{\boldsymbol{A}} \text{ is total for each } f \in \mathcal{F}\}$$

by assumption, we can apply Proposition 9.10, obtaining $\mathbb{U}(\mathsf{N}[\mathscr{L}_\mathcal{F}]) = \mathbb{S}((\mathbb{U}(\mathsf{N}))[\mathscr{L}_\mathcal{F}])$. As $\mathsf{K} = \mathbb{U}(\mathsf{N})$, this amounts to $\mathbb{U}(\mathsf{N}[\mathscr{L}_\mathcal{F}]) = \mathbb{S}(\mathsf{K}[\mathscr{L}_\mathcal{F}])$. ⊠

It only remains to prove Theorem 10.6. The proof hinges on the next observation, in which a *nonconstant term* is simply a term that is not a constant.

Proposition 10.22. *Let M be a pp expansion of a class of algebras K induced by $\mathcal{F}$ and $\mathscr{L}_\mathcal{F}$. Then the following conditions hold:*

(i) *for every constant c of M there exists a unary $f_c \in \mathsf{ext}_{eq}(\mathsf{K})$ such that $f_c^{\boldsymbol{A} \restriction \mathscr{L}_\mathsf{K}}$ is total and $c^{\boldsymbol{A}} = f_c^{\boldsymbol{A} \restriction \mathscr{L}_\mathsf{K}}(a)$ for all $\boldsymbol{A} \in \mathsf{K}[\mathscr{L}_\mathcal{F}]$ and $a \in A$;*

(ii) *for every nonconstant term t of M there exists $f_t \in \mathsf{ext}_{pp}(\mathsf{K})$ such that $t^{\boldsymbol{A}} = f_t^{\boldsymbol{A} \restriction \mathscr{L}_\mathsf{K}}$ for each $\boldsymbol{A} \in \mathsf{K}[\mathscr{L}_\mathcal{F}]$;*

(iii) *for every $f \in \mathsf{imp}_{pp}(\mathsf{M})$ there exists $f_* \in \mathsf{imp}_{pp}(\mathsf{K})$ such that $f^{\boldsymbol{A}} = f_*^{\boldsymbol{A} \restriction \mathscr{L}_\mathsf{K}}$ for each $\boldsymbol{A} \in \mathsf{K}[\mathscr{L}_\mathcal{F}]$. Furthermore, if $f \in \mathsf{ext}_{pp}(\mathsf{M})$, then f_* can be chosen in $\mathsf{ext}_{pp}(\mathsf{K})$.*

Proof. By assumption M is a pp expansion of K induced by $\mathcal{F}$ and $\mathscr{L}_\mathcal{F}$. Therefore, $\mathscr{L}_\mathcal{F}$ is an $\mathcal{F}$-expansion of $\mathscr{L}_\mathsf{K}$. Consequently, $\mathcal{F} \subseteq \mathsf{ext}_{pp}(\mathsf{K})$ and $\mathscr{L}_\mathcal{F}$ is of the form $\mathscr{L}_\mathsf{K} \cup \{g_f : f \in \mathcal{F}\}$. This fact will be used repeatedly in the proof.

(i): Let c be a constant of M. Then c belongs to $\mathscr{L}_{\mathcal{F}}$, which is an $\mathcal{F}$-expansion of $\mathscr{L}_{\mathsf{K}}$. As $\mathcal{F}$-expansions are obtained by adding only functions symbols of positive arity, it follows that $c \in \mathscr{L}_{\mathsf{K}}$. Then the equation $\varphi(x, y) = y \approx c$ defines a unary $f_c \in \mathsf{ext}_{eq}(\mathsf{K})$ with the desired properties.

(ii): We proceed by induction on the construction of the nonconstant term $t(x_1, \ldots, x_n)$. In the base case, $t = p(x_1, \ldots, x_n)$ for a basic operation p of M. Therefore, $p \in \mathscr{L}_{\mathcal{F}}$. As $\mathscr{L}_{\mathcal{F}} = \mathscr{L}_{\mathsf{K}} \cup \{g_f : f \in \mathcal{F}\}$, we have two cases: either $p \in \mathscr{L}_{\mathsf{K}}$ or there exists $h \in \mathcal{F} \subseteq \mathsf{ext}_{pp}(\mathsf{K})$ such that $p = g_h$. In the first case, we let $f_t = \langle p^{\boldsymbol{A}} : \boldsymbol{A} \in \mathsf{K} \rangle$ and, in the second, $f_t = h$. In both cases, $f_t \in \mathsf{ext}_{pp}(\mathsf{K})$ and $t^{\boldsymbol{A}} = f_t^{\boldsymbol{A} \restriction \mathscr{L}_{\mathsf{K}}}$ for each $\boldsymbol{A} \in \mathsf{K}[\mathscr{L}_{\mathcal{F}}]$.

In the inductive step, $t = p(t_1, \ldots, t_m)$ for some m-ary $p \in \mathscr{L}_{\mathcal{F}}$ and terms $t_i(x_1, \ldots, x_n)$ of M. By the inductive hypothesis there exist $f_p, f_{t_1}, \ldots, f_{t_m} \in \mathsf{ext}_{pp}(\mathsf{K})$ satisfying the condition in the statement for $p, t_1, \ldots, t_m$, respectively. Define f_t as the composition $f_p(f_{t_1}, \ldots, f_{t_m})$. As $f_p, f_{t_1}, \ldots, f_{t_m} \in \mathsf{ext}_{pp}(\mathsf{K})$, from Corollary 8.7 it follows that $f_t \in \mathsf{ext}_{pp}(\mathsf{K})$ as well. Then consider $\boldsymbol{A} \in \mathsf{K}[\mathscr{L}_{\mathcal{F}}]$ and $a_1, \ldots, a_n \in A$. By the inductive hypothesis $p^{\boldsymbol{A}} = f_p^{\boldsymbol{A} \restriction \mathscr{L}_{\mathsf{K}}}$ and $t_i^{\boldsymbol{A}} = f_{t_i}^{\boldsymbol{A} \restriction \mathscr{L}_{\mathsf{K}}}$ for each $i \leqslant m$. Therefore, for each $i \leqslant m$,

$$\mathsf{dom}(f_p^{\boldsymbol{A} \restriction \mathscr{L}_{\mathsf{K}}}) = \mathsf{dom}(p^{\boldsymbol{A}}) = A^m \quad \text{and} \quad \mathsf{dom}(f_{t_i}^{\boldsymbol{A} \restriction \mathscr{L}_{\mathsf{K}}}) = \mathsf{dom}(t_i^{\boldsymbol{A}}) = A^n.$$

Together with $f_t = f_p(f_{t_1}, \ldots, f_{t_m})$, this yields $\langle a_1, \ldots, a_n \rangle \in \mathsf{dom}(f_t^{\boldsymbol{A} \restriction \mathscr{L}_{\mathsf{K}}})$ and

$$
\begin{aligned}
f_t^{\boldsymbol{A} \restriction \mathscr{L}_{\mathsf{K}}}(a_1, \ldots, a_n) &= f_p^{\boldsymbol{A} \restriction \mathscr{L}_{\mathsf{K}}}(f_{t_1}^{\boldsymbol{A} \restriction \mathscr{L}_{\mathsf{K}}}(a_1, \ldots, a_n), \ldots, f_{t_m}^{\boldsymbol{A} \restriction \mathscr{L}_{\mathsf{K}}}(a_1, \ldots, a_n)) \\
&= p^{\boldsymbol{A}}(t_1^{\boldsymbol{A}}(a_1, \ldots, a_n), \ldots, t_m^{\boldsymbol{A}}(a_1, \ldots, a_n)) \\
&= t^{\boldsymbol{A}}(a_1, \ldots, a_n).
\end{aligned}
$$

Hence, we conclude that $t^{\boldsymbol{A}} = f_t^{\boldsymbol{A} \restriction \mathscr{L}_{\mathsf{K}}}$.

(iii): Let f be an n-ary implicit operation of M defined by a pp formula

$$\varphi(x_1, \ldots, x_n, y) = \exists z_1, \ldots, z_k \prod_{j \leqslant m} t_j \approx s_j, \tag{43}$$

where each $t_j \approx s_j$ is an equation of M in variables $x_1, \ldots, x_n, z_1, \ldots, z_k, y$. For each $j \leqslant m$ let $\varphi_{t_j}(x_1, \ldots, x_{n+k+1}, y)$ and $\varphi_{s_j}(x_1, \ldots, x_{n+k+1}, y)$ be pp formulas of K defining the implicit operations f_{t_j} and f_{s_j} of K given by (i) and (ii).[6] Moreover, for each $j \leqslant m$ define

$$\alpha_j = \varphi_{t_j}(x_1, \ldots, x_n, z_1, \ldots, z_k, y, v_j) \sqcap \varphi_{s_j}(x_1, \ldots, x_n, z_1, \ldots, z_k, y, v_j);$$

$$\psi = \exists z_1, \ldots, z_k, v_1, \ldots, v_m \prod_{j \leqslant m} \alpha_j.$$

Claim 10.23. *For all $\boldsymbol{A} \in \mathsf{K}[\mathscr{L}_{\mathcal{F}}]$ and $a_1, \ldots, a_n, b \in A$,*

$$\boldsymbol{A} \vDash \varphi(a_1, \ldots, a_n, b) \iff \boldsymbol{A}\restriction_{\mathscr{L}_{\mathsf{K}}} \vDash \psi(a_1, \ldots, a_n, b).$$

Proof of the Claim. As φ is the formula in (43), we have $\boldsymbol{A} \vDash \varphi(a_1, \ldots, a_n, b)$ if and only if there exist $c_1, \ldots, c_k \in A$ such that $t_j^{\boldsymbol{A}}(a_1, \ldots, a_n, c_1, \ldots, c_k, b) = s_j^{\boldsymbol{A}}(a_1, \ldots, a_n, c_1, \ldots, c_k, b)$

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[6]When t_j is a constant c, we let the pp formula φ_{t_j} be $y \approx c$ (see the proof of (i)) and think of f_{t_j} as a constant operation of arity $n + k + 1$. Similarly for s_j when it is a constant.

for each $j \leqslant m$. In view of (i) and (ii), the latter is equivalent to the demand that there exist $c_1, \ldots, c_k \in A$ such that for each $j \leqslant m$,

$$\langle a_1, \ldots, a_n, c_1, \ldots, c_k, b \rangle \in \mathsf{dom}(f_{t_j}^{\boldsymbol{A} \restriction_{\mathscr{L}_{\mathsf{K}}}}) \cap \mathsf{dom}(f_{s_j}^{\boldsymbol{A} \restriction_{\mathscr{L}_{\mathsf{K}}}})$$

and

$$f_{t_j}^{\boldsymbol{A} \restriction_{\mathscr{L}_{\mathsf{K}}}}(a_1, \ldots, a_n, c_1, \ldots, c_k, b) = f_{s_j}^{\boldsymbol{A} \restriction_{\mathscr{L}_{\mathsf{K}}}}(a_1, \ldots, a_n, c_1, \ldots, c_k, b).$$

Since the formulas φ_{t_j} and φ_{s_j} define f_{t_j} and f_{s_j}, respectively, the definition of α_j guarantees that this happens precisely when there exist $c_1, \ldots, c_k \in A$ such that

$$\boldsymbol{A} \restriction_{\mathscr{L}_{\mathsf{K}}} \vDash \exists v_1, \ldots, v_m \bigsqcap_{j \leqslant m} \alpha_j(a_1, \ldots, a_n, c_1, \ldots, c_k, b, v_j).$$

By the definition of ψ this amounts to the demand that $\boldsymbol{A} \restriction_{\mathscr{L}_{\mathsf{K}}} \vDash \psi(a_1, \ldots, a_n, b)$. $\boxtimes$

Recall that φ defines the implicit operation f of M. Therefore, φ is functional in M and, in particular, in $\mathsf{K}[\mathscr{L}_{\mathcal{F}}]$. Together with Claim 10.23, this yields that ψ is functional in $\{\boldsymbol{A} \restriction_{\mathscr{L}_{\mathsf{K}}} : \boldsymbol{A} \in \mathsf{K}[\mathscr{L}_{\mathcal{F}}]\}$. As ψ is equivalent to a pp formula by definition, we can apply Corollary 3.11, obtaining that ψ defines some $f_* \in \mathsf{imp}_{pp}(\mathbb{S}(\{\boldsymbol{A} \restriction_{\mathscr{L}_{\mathsf{K}}} : \boldsymbol{A} \in \mathsf{K}[\mathscr{L}_{\mathcal{F}}]\}))$. Then recall that M is a pp expansion of K induced by $\mathcal{F}$ and $\mathscr{L}_{\mathcal{F}}$ and, therefore, $\mathcal{F} \subseteq \mathsf{ext}_{pp}(\mathsf{K})$. Consequently, we can apply Proposition 9.6, obtaining that $\mathsf{K} = \mathbb{S}(\{\boldsymbol{A} \restriction_{\mathscr{L}_{\mathsf{K}}} : \boldsymbol{A} \in \mathsf{K}[\mathscr{L}_{\mathcal{F}}]\})$. Hence,

$$f_* \in \mathsf{imp}_{pp}(\mathbb{S}(\{\boldsymbol{A} \restriction_{\mathscr{L}_{\mathsf{K}}} : \boldsymbol{A} \in \mathsf{K}[\mathscr{L}_{\mathcal{F}}]\})) = \mathsf{imp}_{pp}(\mathsf{K}).$$

Since φ and ψ define f and f_*, respectively, from Claim 10.23 it follows that for all $\boldsymbol{A} \in \mathsf{K}[\mathscr{L}_{\mathcal{F}}]$ and $a_1, \ldots, a_n, b \in A$,

$$\langle a_1, \ldots, a_n \rangle \in \mathsf{dom}(f^{\boldsymbol{A}}) \text{ and } f^{\boldsymbol{A}}(a_1, \ldots, a_n) = b$$
$$\Longleftrightarrow \quad \boldsymbol{A} \vDash \varphi(a_1, \ldots, a_n, b)$$
$$\Longleftrightarrow \quad \boldsymbol{A} \restriction_{\mathscr{L}_{\mathsf{K}}} \vDash \psi(a_1, \ldots, a_n, b)$$
$$\Longleftrightarrow \quad \langle a_1, \ldots, a_n \rangle \in \mathsf{dom}(f_*^{\boldsymbol{A} \restriction_{\mathscr{L}_{\mathsf{K}}}}) \text{ and } f_*^{\boldsymbol{A} \restriction_{\mathscr{L}_{\mathsf{K}}}}(a_1, \ldots, a_n) = b.$$

Hence, we conclude that $f^{\boldsymbol{A}} = f_*^{\boldsymbol{A} \restriction_{\mathscr{L}_{\mathsf{K}}}}$. This concludes the proof of the first half of (iii).

Therefore, it only remains to prove that if $f \in \mathsf{ext}_{pp}(\mathsf{M})$, then $f_* \in \mathsf{ext}_{pp}(\mathsf{K})$. Accordingly, suppose that $f \in \mathsf{ext}_{pp}(\mathsf{M})$. Since we already proved that $f_* \in \mathsf{imp}_{pp}(\mathsf{K})$, it suffices to show that $f_* \in \mathsf{ext}(\mathsf{K})$. To this end, consider $\boldsymbol{A} \in \mathsf{K}$ and $a_1, \ldots, a_n \in A$. As K is the class of $\mathscr{L}_{\mathsf{K}}$-subreducts of M by Proposition 10.2, there exists $\boldsymbol{B} \in \mathsf{M}$ such that $\boldsymbol{A} \leqslant \boldsymbol{B} \restriction_{\mathscr{L}_{\mathsf{K}}}$. Since $f \in \mathsf{ext}(\mathsf{M})$, $\boldsymbol{B} \in \mathsf{M}$, and $a_1, \ldots, a_n \in A \subseteq B$, there exists $\boldsymbol{C} \in \mathsf{M}$ such that $\boldsymbol{B} \leqslant \boldsymbol{C}$ and $\langle a_1, \ldots, a_n \rangle \in \mathsf{dom}(f^{\boldsymbol{C}})$. From $\boldsymbol{C} \in \mathsf{M} = \mathbb{S}(\mathsf{K}[\mathscr{L}_{\mathcal{F}}])$ it follows that there also exists $\boldsymbol{D} \in \mathsf{K}[\mathscr{L}_{\mathcal{F}}]$ with $\boldsymbol{C} \leqslant \boldsymbol{D}$. Since $\boldsymbol{A} \leqslant \boldsymbol{B} \restriction_{\mathscr{L}_{\mathsf{K}}}$ and $\boldsymbol{B} \leqslant \boldsymbol{C} \leqslant \boldsymbol{D} \in \mathsf{M}$ and K is the class of $\mathscr{L}_{\mathsf{K}}$-subreducts of M, we have $\boldsymbol{A} \leqslant \boldsymbol{D} \restriction_{\mathscr{L}_{\mathsf{K}}} \in \mathsf{K}$. Moreover, by applying Proposition 8.1 to $\langle a_1, \ldots, a_n \rangle \in \mathsf{dom}(f^{\boldsymbol{C}})$ and $\boldsymbol{C} \leqslant \boldsymbol{D}$, we obtain $\langle a_1, \ldots, a_n \rangle \in \mathsf{dom}(f^{\boldsymbol{D}})$. Hence,

$$\boldsymbol{D} \in \mathsf{K}[\mathscr{L}_{\mathcal{F}}], \quad \langle a_1, \ldots, a_n \rangle \in \mathsf{dom}(f^{\boldsymbol{D}}), \quad \text{and} \quad \boldsymbol{A} \leqslant \boldsymbol{D} \restriction_{\mathscr{L}_{\mathsf{K}}} \in \mathsf{K}.$$

Together with the first half of condition (iii), the first two items in the above display imply $\langle a_1, \ldots, a_n \rangle \in \mathsf{dom}(f_*^{\boldsymbol{D} \restriction_{\mathscr{L}_{\mathsf{K}}}})$. As $\boldsymbol{A} \leqslant \boldsymbol{D} \restriction_{\mathscr{L}_{\mathsf{K}}} \in \mathsf{K}$, we conclude that $f_* \in \mathsf{ext}(\mathsf{K})$. $\boxtimes$

We are now ready to prove Theorem 10.6.

Proof. Let M_2 be a pp expansion of a pp expansion M_1 of K. We will prove that M_2 is also a pp expansion of K. First, as M_1 is a pp expansion of K, there exist $\mathcal{F}_1 \subseteq \mathsf{ext}_{pp}(K)$ and an $\mathcal{F}_1$-expansion $\mathscr{L}_{\mathcal{F}_1}$ of $\mathscr{L}_K$ such that $M_1 = \mathbb{S}(K[\mathscr{L}_{\mathcal{F}_1}])$. Similarly, as M_2 is a pp expansion of M_1, there exist $\mathcal{F}_2 \subseteq \mathsf{ext}_{pp}(M_1)$ and an $\mathcal{F}_2$-expansion $\mathscr{L}_{\mathcal{F}_2}$ of $\mathscr{L}_{\mathcal{F}_1}$ such that $M_2 = \mathbb{S}(M_1[\mathscr{L}_{\mathcal{F}_2}])$. Hence,

$$M_1 = \mathbb{S}(K[\mathscr{L}_{\mathcal{F}_1}]) \quad \text{and} \quad M_2 = \mathbb{S}(M_1[\mathscr{L}_{\mathcal{F}_2}]). \tag{44}$$

Since $\mathscr{L}_{\mathcal{F}_1}$ is an $\mathcal{F}_1$-expansion of $\mathscr{L}_K$ and $\mathscr{L}_{\mathcal{F}_2}$ an $\mathcal{F}_2$-expansion of $\mathscr{L}_{\mathcal{F}_1}$, we may assume that

$$\mathscr{L}_{\mathcal{F}_1} = \mathscr{L}_K \cup \{g_f : f \in \mathcal{F}_1\} \quad \text{and} \quad \mathscr{L}_{\mathcal{F}_2} = \mathscr{L}_{\mathcal{F}_1} \cup \{g_f : f \in \mathcal{F}_2\}.$$

Consequently,

$$\mathscr{L}_{\mathcal{F}_2} = \mathscr{L}_K \cup \{g_f : f \in \mathcal{F}_1\} \cup \{g_f : f \in \mathcal{F}_2\}. \tag{45}$$

As $\mathcal{F}_2 \subseteq \mathsf{ext}_{pp}(M_1)$ and M_1 is a pp expansion of K, Proposition 10.22(iii) guarantees that for every $f \in \mathcal{F}_2$ there exists $f_* \in \mathsf{ext}_{pp}(K)$ such that

$$f^{\boldsymbol{A}} = f_*^{\boldsymbol{A} \restriction \mathscr{L}_K} \text{ for each } \boldsymbol{A} \in K[\mathscr{L}_{\mathcal{F}_1}]. \tag{46}$$

Since $\mathcal{F}_1 \subseteq \mathsf{ext}_{pp}(K)$ by assumption, the set

$$\mathcal{F} = \mathcal{F}_1 \cup \{f_* : f \in \mathcal{F}_2\}$$

is a subset of $\mathsf{ext}_{pp}(K)$. Define $g_{f_*} = g_f$ for each $f \in \mathcal{F}_2$ and consider the following $\mathcal{F}$-expansion of $\mathscr{L}_K$:

$$\mathscr{L}_{\mathcal{F}} = \mathscr{L}_K \cup \{g_f : f \in \mathcal{F}\}.$$

Then the pair $\mathcal{F}$ and $\mathscr{L}_{\mathcal{F}}$ induces a pp expansion $\mathbb{S}(K[\mathscr{L}_{\mathcal{F}}])$ of K. To conclude the proof, it will be enough to show that $M_2 = \mathbb{S}(K[\mathscr{L}_{\mathcal{F}}])$, for in this case M_2 would also be a pp expansion of K.

First, observe that

$$\begin{aligned} \mathscr{L}_{\mathcal{F}_2} &= \mathscr{L}_K \cup \{g_f : f \in \mathcal{F}_1\} \cup \{g_f : f \in \mathcal{F}_2\} \\ &= \mathscr{L}_K \cup \{g_f : f \in \mathcal{F}_1\} \cup \{g_{f_*} : f \in \mathcal{F}_2\} \\ &= \mathscr{L}_K \cup \{g_f : f \in \mathcal{F}\} \\ &= \mathscr{L}_{\mathcal{F}}. \end{aligned}$$

The above equalities are justified as follows. The first is (45), the second holds by the definition of g_{f_*} for $f \in \mathcal{F}_2$, the third by the definition of $\mathcal{F}$, and the fourth by that of $\mathscr{L}_{\mathcal{F}}$. This establishes $\mathscr{L}_{\mathcal{F}_2} = \mathscr{L}_{\mathcal{F}}$. Since $\mathscr{L}_{\mathcal{F}_2}$ and $\mathscr{L}_{\mathcal{F}}$ are, respectively, the languages of M_2 and $\mathbb{S}(K[\mathscr{L}_{\mathcal{F}}])$, we conclude that these classes have the same language.

In view of the right hand side of (44), in order to prove that $M_2 = \mathbb{S}(K[\mathscr{L}_{\mathcal{F}}])$, it suffices to show that

$$M_1[\mathscr{L}_{\mathcal{F}_2}] \subseteq \mathbb{S}(K[\mathscr{L}_{\mathcal{F}}]) \quad \text{and} \quad K[\mathscr{L}_{\mathcal{F}}] \subseteq M_1[\mathscr{L}_{\mathcal{F}_2}]. \tag{47}$$

We split the proof of the above display in two claims.

Claim 10.24. *We have* $M_1[\mathscr{L}_{\mathcal{F}_2}] \subseteq \mathbb{S}(K[\mathscr{L}_{\mathcal{F}}])$.

Proof of the Claim. Consider $\boldsymbol{A} \in \mathsf{M}_1[\mathscr{L}_{\mathcal{F}_2}]$. By the definition of $\mathsf{M}_1[\mathscr{L}_{\mathcal{F}_2}]$ we have $\boldsymbol{A}{\restriction}_{\mathscr{L}_{\mathsf{M}_1}} \in \mathsf{M}_1$. As $\mathsf{M}_1 = \mathbb{S}(\mathsf{K}[\mathscr{L}_{\mathcal{F}_1}])$ by the left hand side of (44), there exists $\boldsymbol{B} \in \mathsf{K}[\mathscr{L}_{\mathcal{F}_1}]$ with $\boldsymbol{A}{\restriction}_{\mathscr{L}_{\mathsf{M}_1}} \leqslant \boldsymbol{B}$. Furthermore, $\boldsymbol{B}{\restriction}_{\mathscr{L}_{\mathsf{K}}} \in \mathsf{K}$ because $\boldsymbol{B} \in \mathsf{K}[\mathscr{L}_{\mathcal{F}_1}]$. Since $\mathbb{S}(\mathsf{K}[\mathscr{L}_{\mathcal{F}}])$ is a pp expansion of K and $\boldsymbol{B}{\restriction}_{\mathscr{L}_{\mathsf{K}}} \in \mathsf{K}$, we can apply Proposition 9.6, obtaining some $\boldsymbol{C} \in \mathsf{K}[\mathscr{L}_{\mathcal{F}}]$ such that $\boldsymbol{B}{\restriction}_{\mathscr{L}_{\mathsf{K}}} \leqslant \boldsymbol{C}{\restriction}_{\mathscr{L}_{\mathsf{K}}}$.

We will prove that $\boldsymbol{A} \leqslant \boldsymbol{C}$. Since $\boldsymbol{C} \in \mathsf{K}[\mathscr{L}_{\mathcal{F}}]$, this will yield $\boldsymbol{A} \in \mathbb{S}(\mathsf{K}[\mathscr{L}_{\mathcal{F}}])$, thus concluding the proof of the inclusion $\mathsf{M}_1[\mathscr{L}_{\mathcal{F}_2}] \subseteq \mathbb{S}(\mathsf{K}[\mathscr{L}_{\mathcal{F}}])$. First, as $\mathscr{L}_{\mathcal{F}} = \mathscr{L}_{\mathcal{F}_2} = \mathscr{L}_{\mathcal{F}_1} \cup \{g_f : f \in \mathcal{F}_2\}$, we have $\mathscr{L}_{\mathcal{F}_1} \subseteq \mathscr{L}_{\mathcal{F}}$. Therefore, from $\boldsymbol{C} \in \mathsf{K}[\mathscr{L}_{\mathcal{F}}]$ it follows that $\boldsymbol{C}{\restriction}_{\mathscr{L}_{\mathcal{F}_1}} \in \mathsf{K}[\mathscr{L}_{\mathcal{F}_1}]$. By applying Proposition 9.5 to $\boldsymbol{B}, \boldsymbol{C}{\restriction}_{\mathscr{L}_{\mathcal{F}_1}} \in \mathsf{K}[\mathscr{L}_{\mathcal{F}_1}]$ and $\boldsymbol{B}{\restriction}_{\mathscr{L}_{\mathsf{K}}} \leqslant \boldsymbol{C}{\restriction}_{\mathscr{L}_{\mathsf{K}}} = (\boldsymbol{C}{\restriction}_{\mathscr{L}_{\mathcal{F}_1}}){\restriction}_{\mathscr{L}_{\mathsf{K}}}$, we obtain $\boldsymbol{B} \leqslant \boldsymbol{C}{\restriction}_{\mathscr{L}_{\mathcal{F}_1}}$. Together with $\boldsymbol{A}{\restriction}_{\mathscr{L}_{\mathsf{M}_1}} \leqslant \boldsymbol{B}$ and $\mathscr{L}_{\mathsf{M}_1} = \mathscr{L}_{\mathcal{F}_1}$, this yields $\boldsymbol{A}{\restriction}_{\mathscr{L}_{\mathcal{F}_1}} \leqslant \boldsymbol{C}{\restriction}_{\mathscr{L}_{\mathcal{F}_1}}$.

Since the language of $\boldsymbol{A}$ and $\boldsymbol{C}$ is $\mathscr{L}_{\mathcal{F}} = \mathscr{L}_{\mathcal{F}_1} \cup \{g_f : f \in \mathcal{F}_2\}$, in order to prove that $\boldsymbol{A} \leqslant \boldsymbol{C}$, it only remains to show that for all n-ary $f \in \mathcal{F}_2$ and $a_1, \ldots, a_n \in A$,

$$g_f^{\boldsymbol{A}}(a_1, \ldots, a_n) = g_f^{\boldsymbol{C}}(a_1, \ldots, a_n). \tag{48}$$

To this end, consider an n-ary $f \in \mathcal{F}_2$ and $a_1, \ldots, a_n \in A$. We will show that

$$\begin{aligned}
g_f^{\boldsymbol{A}}(a_1, \ldots, a_n) &= f^{\boldsymbol{A}{\restriction}_{\mathscr{L}_{\mathsf{M}_1}}}(a_1, \ldots, a_n) \\
&= f^{\boldsymbol{B}}(a_1, \ldots, a_n) \\
&= f_*^{\boldsymbol{B}{\restriction}_{\mathscr{L}_{\mathsf{K}}}}(a_1, \ldots, a_n) \\
&= f_*^{\boldsymbol{C}{\restriction}_{\mathscr{L}_{\mathsf{K}}}}(a_1, \ldots, a_n) \\
&= g_{f_*}^{\boldsymbol{C}}(a_1, \ldots, a_n) \\
&= g_f^{\boldsymbol{C}}(a_1, \ldots, a_n).
\end{aligned}$$

The equalities above are justified as follows. To prove the first, recall that $\boldsymbol{A} \in \mathsf{M}_1[\mathscr{L}_{\mathcal{F}_2}]$ and $f \in \mathcal{F}_2$, whence $f^{\boldsymbol{A}{\restriction}_{\mathscr{L}_{\mathsf{M}_1}}}$ is a total function and $g_f^{\boldsymbol{A}}(a_1, \ldots, a_n) = f^{\boldsymbol{A}{\restriction}_{\mathscr{L}_{\mathsf{M}_1}}}(a_1, \ldots, a_n)$. To prove the second, observe that the assumption that $f \in \mathcal{F}_2 \subseteq \mathsf{imp}(\mathsf{M}_1)$, $\boldsymbol{A}{\restriction}_{\mathscr{L}_{\mathsf{M}_1}} \leqslant \boldsymbol{B}$, and $\boldsymbol{A}{\restriction}_{\mathscr{L}_{\mathsf{M}_1}}, \boldsymbol{B} \in \mathsf{M}_1$ allows us to apply Proposition 8.1 to the fact that $\langle a_1, \ldots, a_n \rangle \in \mathsf{dom}(f^{\boldsymbol{A}{\restriction}_{\mathscr{L}_{\mathsf{M}_1}}})$, obtaining $\langle a_1, \ldots, a_n \rangle \in \mathsf{dom}(f^{\boldsymbol{B}})$ and $f^{\boldsymbol{A}{\restriction}_{\mathscr{L}_{\mathsf{M}_1}}}(a_1, \ldots, a_n) = f^{\boldsymbol{B}}(a_1, \ldots, a_n)$. The third equality follows from $\boldsymbol{B} \in \mathsf{K}[\mathscr{L}_{\mathcal{F}_1}]$ and (46). To prove the fourth, observe that $f_* \in \mathsf{imp}(\mathsf{K})$, $\boldsymbol{B}{\restriction}_{\mathscr{L}_{\mathsf{K}}}, \boldsymbol{C}{\restriction}_{\mathscr{L}_{\mathsf{K}}} \in \mathsf{K}$, and $\boldsymbol{B}{\restriction}_{\mathscr{L}_{\mathsf{K}}} \leqslant \boldsymbol{C}{\restriction}_{\mathscr{L}_{\mathsf{K}}}$. Therefore, we can apply Proposition 8.1 to the fact that $\langle a_1, \ldots, a_n \rangle \in \mathsf{dom}(f_*^{\boldsymbol{B}{\restriction}_{\mathscr{L}_{\mathsf{K}}}})$, obtaining $\langle a_1, \ldots, a_n \rangle \in \mathsf{dom}(f_*^{\boldsymbol{C}{\restriction}_{\mathscr{L}_{\mathsf{K}}}})$ and $f_*^{\boldsymbol{B}{\restriction}_{\mathscr{L}_{\mathsf{K}}}}(a_1, \ldots, a_n) = f_*^{\boldsymbol{C}{\restriction}_{\mathscr{L}_{\mathsf{K}}}}(a_1, \ldots, a_n)$. The fifth holds because $\boldsymbol{C} \in \mathsf{K}[\mathscr{L}_{\mathcal{F}}]$ and $f_* \in \mathcal{F}$. Lastly, the sixth equality holds because $g_{f_*} = g_f$ by assumption. This concludes the proof of (48). Hence, we obtain $\boldsymbol{A} \leqslant \boldsymbol{C}$, as desired.   ⊠

Claim 10.25. *We have* $\mathsf{K}[\mathscr{L}_{\mathcal{F}}] \subseteq \mathsf{M}_1[\mathscr{L}_{\mathcal{F}_2}]$.

Proof of the Claim. Consider $\boldsymbol{A} \in \mathsf{K}[\mathscr{L}_{\mathcal{F}}]$. Then $\boldsymbol{A}{\restriction}_{\mathscr{L}_{\mathsf{K}}} \in \mathsf{K}$ and $f^{\boldsymbol{A}{\restriction}_{\mathscr{L}_{\mathsf{K}}}}$ is total for each $f \in \mathcal{F}$. As $\mathcal{F}_1 \subseteq \mathcal{F}$, this implies that the algebra $\boldsymbol{A}{\restriction}_{\mathscr{L}_{\mathsf{K}}}[\mathscr{L}_{\mathcal{F}_1}]$ is defined and belongs to $\mathsf{K}[\mathscr{L}_{\mathcal{F}_1}]$ and, therefore, to M_1 as well. We will prove that $f^{\boldsymbol{A}{\restriction}_{\mathscr{L}_{\mathsf{K}}}[\mathscr{L}_{\mathcal{F}_1}]}$ is total for each $f \in \mathcal{F}_2$. To this end, consider $f \in \mathcal{F}_2$. By first applying (46) to $\boldsymbol{A}{\restriction}_{\mathscr{L}_{\mathsf{K}}}[\mathscr{L}_{\mathcal{F}_1}] \in \mathsf{K}[\mathscr{L}_{\mathcal{F}_1}]$ and then observing that

$\boldsymbol{A}{\upharpoonright}_{\mathscr{L}_{\mathsf{K}}}$ is the $\mathscr{L}_{\mathsf{K}}$-reduct of $\boldsymbol{A}{\upharpoonright}_{\mathscr{L}_{\mathsf{K}}}[\mathscr{L}_{\mathcal{F}_1}]$, we obtain

$$f^{\boldsymbol{A}{\upharpoonright}_{\mathscr{L}_{\mathsf{K}}}[\mathscr{L}_{\mathcal{F}_1}]} = f_*^{(\boldsymbol{A}{\upharpoonright}_{\mathscr{L}_{\mathsf{K}}}[\mathscr{L}_{\mathcal{F}_1}]){\upharpoonright}_{\mathscr{L}_{\mathsf{K}}}} = f_*^{\boldsymbol{A}{\upharpoonright}_{\mathscr{L}_{\mathsf{K}}}}.$$

The function on the right hand side of the above display is total because $\boldsymbol{A} \in \mathsf{K}[\mathscr{L}_{\mathcal{F}}]$ and $f_* \in \mathcal{F}$ (the latter by $f \in \mathcal{F}_2$ and the definition of $\mathcal{F}$). Hence, we conclude that the left hand side of the above display is also total, as desired. Since $\boldsymbol{A}{\upharpoonright}_{\mathscr{L}_{\mathsf{K}}}[\mathscr{L}_{\mathcal{F}_1}] \in \mathsf{M}_1$ and $f^{\boldsymbol{A}{\upharpoonright}_{\mathscr{L}_{\mathsf{K}}}[\mathscr{L}_{\mathcal{F}_1}]}$ is total for each $f \in \mathcal{F}_2$, the algebra $(\boldsymbol{A}{\upharpoonright}_{\mathscr{L}_{\mathsf{K}}}[\mathscr{L}_{\mathcal{F}_1}])[\mathscr{L}_{\mathcal{F}_2}]$ is defined and belongs to $\mathsf{M}_1[\mathscr{L}_{\mathcal{F}_2}]$. Therefore, in order to conclude the proof, it suffices to show that

$$\boldsymbol{A} = (\boldsymbol{A}{\upharpoonright}_{\mathscr{L}_{\mathsf{K}}}[\mathscr{L}_{\mathcal{F}_1}])[\mathscr{L}_{\mathcal{F}_2}]. \tag{49}$$

To this end, recall that the language of these algebras is $\mathscr{L}_{\mathcal{F}_1} \cup \{g_f : f \in \mathcal{F}_2\}$ and their universe is A. Moreover, recall that $\boldsymbol{A} \in \mathsf{K}[\mathscr{L}_{\mathcal{F}}]$. Then $\boldsymbol{A} = \boldsymbol{A}{\upharpoonright}_{\mathscr{L}_{\mathsf{K}}}[\mathscr{L}_{\mathcal{F}}]$ and $\boldsymbol{A}{\upharpoonright}_{\mathscr{L}_{\mathsf{K}}} \in \mathsf{K}$. Together with $\mathcal{F}_1 \subseteq \mathcal{F}$, this yields that the $\mathscr{L}_{\mathcal{F}_1}$-reduct of $\boldsymbol{A}$ is $\boldsymbol{A}{\upharpoonright}_{\mathscr{L}_{\mathsf{K}}}[\mathscr{L}_{\mathcal{F}_1}]$. On the other hand, the $\mathscr{L}_{\mathcal{F}_1}$-reduct of $(\boldsymbol{A}{\upharpoonright}_{\mathscr{L}_{\mathsf{K}}}[\mathscr{L}_{\mathcal{F}_1}])[\mathscr{L}_{\mathcal{F}_2}]$ is also $\boldsymbol{A}{\upharpoonright}_{\mathscr{L}_{\mathsf{K}}}[\mathscr{L}_{\mathcal{F}_1}]$ by construction. Therefore, in order to prove the above display, it only remains to show that for all n-ary $f \in \mathcal{F}_2$ and $a_1, \dots, a_n \in A$,

$$g_f^{\boldsymbol{A}}(a_1, \dots, a_n) = g_f^{(\boldsymbol{A}{\upharpoonright}_{\mathscr{L}_{\mathsf{K}}}[\mathscr{L}_{\mathcal{F}_1}])[\mathscr{L}_{\mathcal{F}_2}]}(a_1, \dots, a_n). \tag{50}$$

Consider an n-ary $f \in \mathcal{F}_2$ and $a_1, \dots, a_n \in A$. We will prove that

$$\begin{aligned} g_f^{\boldsymbol{A}}(a_1, \dots, a_n) &= g_{f_*}^{\boldsymbol{A}}(a_1, \dots, a_n) \\ &= f_*^{\boldsymbol{A}{\upharpoonright}_{\mathscr{L}_{\mathsf{K}}}}(a_1, \dots, a_n) \\ &= f_*^{(\boldsymbol{A}{\upharpoonright}_{\mathscr{L}_{\mathsf{K}}}[\mathscr{L}_{\mathcal{F}_1}]){\upharpoonright}_{\mathscr{L}_{\mathsf{K}}}}(a_1, \dots, a_n) \\ &= f^{\boldsymbol{A}{\upharpoonright}_{\mathscr{L}_{\mathsf{K}}}[\mathscr{L}_{\mathcal{F}_1}]}(a_1, \dots, a_n) \\ &= g_f^{(\boldsymbol{A}{\upharpoonright}_{\mathscr{L}_{\mathsf{K}}}[\mathscr{L}_{\mathcal{F}_1}])[\mathscr{L}_{\mathcal{F}_2}]}(a_1, \dots, a_n). \end{aligned}$$

The above equalities are justified as follows. The first holds because $g_{f_*} = g_f$ for each $f \in \mathcal{F}_2$ by definition, the second because $\boldsymbol{A} \in \mathsf{K}[\mathscr{L}_{\mathcal{F}}]$ and $f_* \in \mathcal{F}$, the third because $\boldsymbol{A}{\upharpoonright}_{\mathscr{L}_{\mathsf{K}}} = (\boldsymbol{A}{\upharpoonright}_{\mathscr{L}_{\mathsf{K}}}[\mathscr{L}_{\mathcal{F}_1}]){\upharpoonright}_{\mathscr{L}_{\mathsf{K}}}$, the fourth follows from (46) and $\boldsymbol{A}{\upharpoonright}_{\mathscr{L}_{\mathsf{K}}}[\mathscr{L}_{\mathcal{F}_1}] \in \mathsf{K}[\mathscr{L}_{\mathcal{F}_1}]$, and the fifth from $f \in \mathcal{F}_2$ and the definition of $(\boldsymbol{A}{\upharpoonright}_{\mathscr{L}_{\mathsf{K}}}[\mathscr{L}_{\mathcal{F}_1}])[\mathscr{L}_{\mathcal{F}_2}]$. This concludes the proof of (50) and, therefore, of (49). ⊠

As (47) is an immediate consequence of Claims 10.24 and 10.25, we are done. ⊠

The proof of Theorem 10.7 hinges on the following result, which describes how to lift implicit operations to pp expansions.

Proposition 10.26. *Let* M *be a pp expansion of a class of algebras* K *and* $f \in \mathsf{imp}(\mathsf{K})$ *defined by a formula* φ *of* $\mathscr{L}_{\mathsf{K}}$. *Let also* $f_\bullet$ *be the partial function on* M *given by* $f_\bullet = \langle f^{\boldsymbol{A}{\upharpoonright}_{\mathscr{L}_{\mathsf{K}}}} : \boldsymbol{A} \in \mathsf{M} \rangle$. *Then the following conditions hold:*

  (i) $f_\bullet$ *is defined by* φ *and belongs to* $\mathsf{imp}(\mathsf{M})$;
  (ii) *if* $f \in \mathsf{imp}_{pp}(\mathsf{K})$, *then* $f_\bullet \in \mathsf{imp}_{pp}(\mathsf{M})$;
  (iii) *if* K *is a universal class and* $f \in \mathsf{ext}(\mathsf{K})$, *then* $f_\bullet \in \mathsf{ext}(\mathsf{M})$.

Proof. Throughout the proof we assume that f is n-ary.

(i): Observe that Proposition 10.2 implies that $\boldsymbol{A}{\restriction}_{\mathscr{L}_{\mathsf{K}}} \in \mathsf{K}$ for every $\boldsymbol{A} \in \mathsf{M}$. Hence, $f_{\bullet}$ is a well-defined partial function on M. We show that $f_{\bullet}$ is preserved by homomorphisms in M. Consider a homomorphism $h \colon \boldsymbol{A} \to \boldsymbol{B}$ with $\boldsymbol{A}, \boldsymbol{B} \in \mathsf{M}$ and let $a_1, \ldots, a_n \in A$ be such that $\langle a_1, \ldots, a_n \rangle \in \mathsf{dom}(f_{\bullet}^{\boldsymbol{A}})$. Since $f_{\bullet}^{\boldsymbol{A}} = f^{\boldsymbol{A}{\restriction}_{\mathscr{L}_{\mathsf{K}}}}$, we have

$$\langle a_1, \ldots, a_n \rangle \in \mathsf{dom}(f^{\boldsymbol{A}{\restriction}_{\mathscr{L}_{\mathsf{K}}}}).$$

As $h \colon \boldsymbol{A}{\restriction}_{\mathscr{L}_{\mathsf{K}}} \to \boldsymbol{B}{\restriction}_{\mathscr{L}_{\mathsf{K}}}$ is a homomorphism and f is an implicit operation of K, the above display implies that

$$\langle h(a_1), \ldots, h(a_n) \rangle \in \mathsf{dom}(f^{\boldsymbol{B}{\restriction}_{\mathscr{L}_{\mathsf{K}}}}) \quad \text{and} \quad h(f^{\boldsymbol{A}{\restriction}_{\mathscr{L}_{\mathsf{K}}}}(a_1, \ldots, a_n)) = f^{\boldsymbol{B}{\restriction}_{\mathscr{L}_{\mathsf{K}}}}(h(a_1), \ldots, h(a_n)).$$

Together with $f_{\bullet}^{\boldsymbol{A}} = f^{\boldsymbol{A}{\restriction}_{\mathscr{L}_{\mathsf{K}}}}$ and $f_{\bullet}^{\boldsymbol{B}} = f^{\boldsymbol{B}{\restriction}_{\mathscr{L}_{\mathsf{K}}}}$, this yields

$$\langle h(a_1), \ldots, h(a_n) \rangle \in \mathsf{dom}(f_{\bullet}^{\boldsymbol{B}}) \quad \text{and} \quad h(f_{\bullet}^{\boldsymbol{A}}(a_1, \ldots, a_n)) = f_{\bullet}^{\boldsymbol{B}}(h(a_1), \ldots, h(a_n)).$$

Hence, $f_{\bullet}$ is preserved by homomorphisms, as desired.

Lastly, we will prove that for all $\boldsymbol{A} \in \mathsf{M}$ and $a_1, \ldots, a_n, b \in A$,

$$\begin{aligned}
&\langle a_1, \ldots, a_n \rangle \in \mathsf{dom}(f_{\bullet}^{\boldsymbol{A}}) \text{ and } f_{\bullet}^{\boldsymbol{A}}(a_1, \ldots, a_n) = b \\
&\iff \quad \langle a_1, \ldots, a_n \rangle \in \mathsf{dom}(f^{\boldsymbol{A}{\restriction}_{\mathscr{L}_{\mathsf{K}}}}) \text{ and } f^{\boldsymbol{A}{\restriction}_{\mathscr{L}_{\mathsf{K}}}}(a_1, \ldots, a_n) = b \\
&\iff \quad \boldsymbol{A}{\restriction}_{\mathscr{L}_{\mathsf{K}}} \vDash \varphi(a_1, \ldots, a_n, b) \\
&\iff \quad \boldsymbol{A} \vDash \varphi(a_1, \ldots, a_n, b).
\end{aligned}$$

The first equivalence above holds by the definition of $f_{\bullet}$, the second because φ defines f, and the third because φ is an $\mathscr{L}_{\mathsf{K}}$-formula. In view of the above series of equivalences, φ defines $f_{\bullet}$. As $f_{\bullet}$ is preserved by homomorphisms, we conclude that $f_{\bullet} \in \mathsf{imp}(\mathsf{M})$.

(ii): Suppose that $f \in \mathsf{imp}_{pp}(\mathsf{K})$. Then we may assume that φ is a pp formula. It follows from (i) that $f_{\bullet}$ is defined by φ and belongs to $\mathsf{imp}(\mathsf{M})$, whence $f_{\bullet} \in \mathsf{imp}_{pp}(\mathsf{M})$.

(iii): Since M is a pp expansion of K, it is of the form $\mathbb{S}(\mathsf{K}[\mathscr{L}_{\mathcal{F}}])$ for some $\mathcal{F} \subseteq \mathsf{ext}_{pp}(\mathsf{K})$ and $\mathscr{L}_{\mathcal{F}}$. Suppose that $f \in \mathsf{ext}(\mathsf{K})$. To show that $f_{\bullet} \in \mathsf{ext}(\mathsf{M})$, consider $\boldsymbol{A} \in \mathsf{M}$ and $a_1, \ldots, a_n \in A$. Since $\mathsf{M} = \mathbb{S}(\mathsf{K}[\mathscr{L}_{\mathcal{F}}])$, there exists $\boldsymbol{B}[\mathscr{L}_{\mathcal{F}}] \in \mathsf{K}[\mathscr{L}_{\mathcal{F}}]$ such that $\boldsymbol{A} \leqslant \boldsymbol{B}[\mathscr{L}_{\mathcal{F}}]$. As $\boldsymbol{B} \in \mathsf{K}$, Theorem 8.4 yields $\boldsymbol{C} \in \mathsf{K}$ such that $\boldsymbol{B} \leqslant \boldsymbol{C}$ and $g^{\boldsymbol{C}}$ is total for every $g \in \mathsf{ext}(\mathsf{K})$. In particular,

$$\boldsymbol{C}[\mathscr{L}_{\mathcal{F}}] \text{ is defined and } \langle a_1, \ldots, a_n \rangle \in \mathsf{dom}(f^{\boldsymbol{C}}). \tag{51}$$

By Proposition 9.5 from $\boldsymbol{B} \leqslant \boldsymbol{C}$ it follows that $\boldsymbol{B}[\mathscr{L}_{\mathcal{F}}] \leqslant \boldsymbol{C}[\mathscr{L}_{\mathcal{F}}]$. Then $\boldsymbol{A} \leqslant \boldsymbol{B}[\mathscr{L}_{\mathcal{F}}] \leqslant \boldsymbol{C}[\mathscr{L}_{\mathcal{F}}]$ with $\boldsymbol{C}[\mathscr{L}_{\mathcal{F}}] \in \mathsf{K}[\mathscr{L}_{\mathcal{F}}] \subseteq \mathsf{M}$. Since $\boldsymbol{C}[\mathscr{L}_{\mathcal{F}}]{\restriction}_{\mathscr{L}_{\mathsf{K}}} = \boldsymbol{C}$, the definition of $f_{\bullet}$ yields $f_{\bullet}^{\boldsymbol{C}[\mathscr{L}_{\mathcal{F}}]} = f^{\boldsymbol{C}}$. From (51) it follows that $\langle a_1, \ldots, a_n \rangle \in \mathsf{dom}(f_{\bullet}^{\boldsymbol{C}[\mathscr{L}_{\mathcal{F}}]})$ and, therefore, $f_{\bullet} \in \mathsf{ext}(\mathsf{M})$. $\boxtimes$

We now proceed to prove Theorem 10.7.

Proof. Let $\mathsf{M} = \mathbb{S}(\mathsf{K}[\mathscr{L}_{\mathcal{F}}])$ and $\mathscr{L}_{\mathcal{G}} = \mathscr{L}_{\mathsf{K}} \cup \{g_f : f \in \mathcal{G}\}$. Define $\mathcal{G}_{\bullet} = \{f_{\bullet} : f \in \mathcal{G} - \mathcal{F}\}$, where $f_{\bullet} = \langle f^{\boldsymbol{A}{\restriction}_{\mathscr{L}_{\mathsf{K}}}} : \boldsymbol{A} \in \mathsf{M} \rangle$. Since $\mathcal{G} \subseteq \mathsf{ext}_{pp}(\mathsf{K})$, Proposition 10.26 implies that $\mathcal{G}_{\bullet} \subseteq \mathsf{ext}_{pp}(\mathsf{M})$. Consider the pp expansion $\mathbb{S}(\mathsf{M}[\mathscr{L}_{\mathcal{G}_{\bullet}}])$ of M induced by $\mathcal{G}_{\bullet}$ and $\mathscr{L}_{\mathcal{G}_{\bullet}}$, where $\mathscr{L}_{\mathcal{G}_{\bullet}} = \mathscr{L}_{\mathcal{G}}$ and

$g_f^{\boldsymbol{A}[\mathscr{L}_{\mathcal{G}_\bullet}]} = f_\bullet^{\boldsymbol{A}}$ for all $f \in \mathcal{G} - \mathcal{F}$ and $\boldsymbol{A}[\mathscr{L}_{\mathcal{G}_\bullet}] \in \mathsf{M}[\mathscr{L}_{\mathcal{G}_\bullet}]$. Since $\mathbb{S}(\mathsf{M}[\mathscr{L}_{\mathcal{G}_\bullet}])$ is a pp expansion of M, to prove that $\mathbb{S}(\mathsf{K}[\mathscr{L}_{\mathcal{G}}])$ is a pp expansion of M, it suffices to show that

$$\mathbb{S}(\mathsf{M}[\mathscr{L}_{\mathcal{G}_\bullet}]) = \mathbb{S}(\mathsf{K}[\mathscr{L}_{\mathcal{G}}]). \tag{52}$$

For the inclusion from left to right of (52) it is enough to show that $\mathsf{M}[\mathscr{L}_{\mathcal{G}_\bullet}] \subseteq \mathbb{S}(\mathsf{K}[\mathscr{L}_{\mathcal{G}}])$. To this end, consider $\boldsymbol{A} \in \mathsf{M}[\mathscr{L}_{\mathcal{G}_\bullet}]$. Then $\boldsymbol{A}{\restriction}_{\mathscr{L}_{\mathcal{F}}} \in \mathsf{M} = \mathbb{S}(\mathsf{K}[\mathscr{L}_{\mathcal{F}}])$. Consequently, there exists $\boldsymbol{B} \in \mathsf{K}[\mathscr{L}_{\mathcal{F}}]$ such that $\boldsymbol{A}{\restriction}_{\mathscr{L}_{\mathcal{F}}} \leqslant \boldsymbol{B}$. As $\mathcal{G} \subseteq \mathsf{ext}(\mathsf{K})$, Proposition 9.6 yields that K is the class of subreducts of $\mathsf{K}[\mathscr{L}_{\mathcal{G}}]$. Together with $\boldsymbol{B}{\restriction}_{\mathscr{L}_{\mathsf{K}}} \in \mathsf{K}$ (which holds because $\boldsymbol{B} \in \mathsf{K}[\mathscr{L}_{\mathcal{F}}]$), this entails that there exists $\boldsymbol{C}[\mathscr{L}_{\mathcal{G}}] \in \mathsf{K}[\mathscr{L}_{\mathcal{G}}]$ such that $\boldsymbol{B}{\restriction}_{\mathscr{L}_{\mathsf{K}}} \leqslant \boldsymbol{C}$.

We will prove that $\boldsymbol{A} \leqslant \boldsymbol{C}[\mathscr{L}_{\mathcal{G}}]$. First, from $\mathscr{L}_{\mathsf{K}} \subseteq \mathscr{L}_{\mathcal{F}}$, $\boldsymbol{A}{\restriction}_{\mathscr{L}_{\mathcal{F}}} \leqslant \boldsymbol{B}$ and $\boldsymbol{B}{\restriction}_{\mathscr{L}_{\mathsf{K}}} \leqslant \boldsymbol{C}$ it follows that $\boldsymbol{A}{\restriction}_{\mathscr{L}_{\mathsf{K}}} \leqslant \boldsymbol{C}$. Therefore, it only remains to show that $g_f^{\boldsymbol{A}}(a_1, \dots, a_n) = g_f^{\boldsymbol{C}[\mathscr{L}_{\mathcal{G}}]}(a_1, \dots, a_n)$ for all n-ary $f \in \mathcal{G}$ and $a_1, \dots, a_n \in A$. We have that

$$\begin{aligned}
g_f^{\boldsymbol{A}}(a_1, \dots, a_n) &= f_\bullet^{\boldsymbol{A}}(a_1, \dots, a_n) \\
&= f^{\boldsymbol{A}{\restriction}_{\mathscr{L}_{\mathsf{K}}}}(a_1, \dots, a_n) \\
&= f^{\boldsymbol{C}}(a_1, \dots, a_n) \\
&= g_f^{\boldsymbol{C}[\mathscr{L}_{\mathcal{G}}]}(a_1, \dots, a_n).
\end{aligned}$$

The first equality above holds because $\boldsymbol{A} \in \mathsf{M}[\mathscr{L}_{\mathcal{G}_\bullet}]$ and the second is a consequence of the definition of $f_\bullet$. The third equality follows from Proposition 8.1 because $\boldsymbol{A}{\restriction}_{\mathscr{L}_{\mathsf{K}}} \leqslant \boldsymbol{C}$, and the last from the interpretation of g_f in $\mathsf{K}[\mathscr{L}_{\mathcal{G}}]$. Therefore, $\boldsymbol{A} \leqslant \boldsymbol{C}[\mathscr{L}_{\mathcal{G}}] \in \mathsf{K}[\mathscr{L}_{\mathcal{G}}]$. We conclude that $\mathsf{M}[\mathscr{L}_{\mathcal{G}_\bullet}] \subseteq \mathbb{S}(\mathsf{K}[\mathscr{L}_{\mathcal{G}}])$, as desired.

We now prove the inclusion from right to left of (52). It suffices to show that $\mathsf{K}[\mathscr{L}_{\mathcal{G}}] \subseteq \mathsf{M}[\mathscr{L}_{\mathcal{G}_\bullet}]$. To this end, consider $\boldsymbol{A} \in \mathsf{K}[\mathscr{L}_{\mathcal{G}}]$. Then $f^{\boldsymbol{A}{\restriction}_{\mathscr{L}_{\mathsf{K}}}}$ is a total function for every $f \in \mathcal{G}$. In particular, we have that $f^{\boldsymbol{A}{\restriction}_{\mathscr{L}_{\mathsf{K}}}}$ is total for every $f \in \mathcal{F}$ because $\mathcal{F} \subseteq \mathcal{G}$. It follows that $\boldsymbol{A}{\restriction}_{\mathscr{L}_{\mathcal{F}}} \in \mathsf{K}[\mathscr{L}_{\mathcal{F}}] \subseteq \mathsf{M}$ and $f_\bullet^{\boldsymbol{A}{\restriction}_{\mathscr{L}_{\mathcal{F}}}}$ is total for every $f \in \mathcal{G} - \mathcal{F}$. Then $\boldsymbol{A}{\restriction}_{\mathscr{L}_{\mathcal{F}}}[\mathscr{L}_{\mathcal{G}_\bullet}]$ is defined and belongs to $\mathsf{M}[\mathscr{L}_{\mathcal{G}_\bullet}]$. So, it only remains to show that $\boldsymbol{A} = \boldsymbol{A}{\restriction}_{\mathscr{L}_{\mathcal{F}}}[\mathscr{L}_{\mathcal{G}_\bullet}]$. Clearly, $\boldsymbol{A}$ and $\boldsymbol{A}{\restriction}_{\mathscr{L}_{\mathcal{F}}}$ have the same $\mathscr{L}_{\mathcal{F}}$-reduct. Moreover, for every $f \in \mathcal{G} - \mathcal{F}$ we have that

$$g_f^{\boldsymbol{A}} = f^{\boldsymbol{A}{\restriction}_{\mathscr{L}_{\mathsf{K}}}} = f_\bullet^{\boldsymbol{A}{\restriction}_{\mathscr{L}_{\mathcal{F}}}} = g_f^{\boldsymbol{A}{\restriction}_{\mathscr{L}_{\mathcal{F}}}[\mathscr{L}_{\mathcal{G}}]},$$

where the first equality is a consequence of the interpretation of g_f in $\mathsf{K}[\mathscr{L}_{\mathcal{G}}]$, the second follows from the definition of $f_\bullet$ and the fact that $(\boldsymbol{A}{\restriction}_{\mathscr{L}_{\mathcal{F}}}){\restriction}_{\mathscr{L}_{\mathsf{K}}} = \boldsymbol{A}{\restriction}_{\mathscr{L}_{\mathsf{K}}}$, and the last one from the interpretation of g_f in $\mathsf{M}[\mathscr{L}_{\mathcal{G}_\bullet}]$. Thus, $\mathsf{K}[\mathscr{L}_{\mathcal{G}}] \subseteq \mathsf{M}[\mathscr{L}_{\mathcal{G}_\bullet}]$, as desired. $\boxtimes$

11. THE BETH COMPANION

Recall that the strong Beth definability property is the demand that every implicit operation be interpolated by a set of terms (see Definition 5.3). We shall now extend the idea of interpolation to accommodate for situations in which the implicit operation and the set of terms belong to different classes of algebras.

Definition 11.1. Let K and M be a pair of classes of algebras with $\mathscr{L}_{\mathsf{K}} \subseteq \mathscr{L}_{\mathsf{M}}$ such that the $\mathscr{L}_{\mathsf{K}}$-reducts of M belong to K. We say that an n-ary implicit operation f of K is

(i) *interpolated* in M by a set of n-ary terms $\{t_i : i \in I\}$ of M when for all $\boldsymbol{A} \in \mathsf{M}$ and $\langle a_1, \ldots, a_n \rangle \in \mathsf{dom}(f^{\boldsymbol{A} \restriction \mathscr{L}_\mathsf{K}})$ there exists $i \in I$ such that

$$f^{\boldsymbol{A} \restriction \mathscr{L}_\mathsf{K}}(a_1, \ldots, a_n) = t_i^{\boldsymbol{A}}(a_1, \ldots, a_n);$$

(ii) *interpolated* in M by a set of n-ary partial functions $\{g_i : i \in I\}$ of M when for all $\boldsymbol{A} \in \mathsf{M}$ and $\langle a_1, \ldots, a_n \rangle \in \mathsf{dom}(f^{\boldsymbol{A} \restriction \mathscr{L}_\mathsf{K}})$ there exists $i \in I$ such that

$$\langle a_1, \ldots, a_n \rangle \in \mathsf{dom}(g_i^{\boldsymbol{A}}) \quad \text{and} \quad f^{\boldsymbol{A} \restriction \mathscr{L}_\mathsf{K}}(a_1, \ldots, a_n) = g_i^{\boldsymbol{A}}(a_1, \ldots, a_n).$$

Remark 11.2. When $\mathsf{K} = \mathsf{M}$, part (i) of the above definition specializes to the familiar demand that the implicit operation f be interpolated by the set of terms $\{t_i : i \in I\}$. Notice that part (ii) subsumes part (i), for term functions can be viewed as implicit operations (see Example 3.8). However, we opted for splitting the definition in two halves for the sake of clarity. Lastly, we remark that the above definition applies to the situation where M is a pp expansion of K because in this case $\mathscr{L}_\mathsf{K} \subseteq \mathscr{L}_\mathsf{M}$ and the $\mathscr{L}_\mathsf{K}$-reducts of M belong to K (see Proposition 10.2). ⊠

Recall that the notion of a pp expansion was introduced to address the following question: is it possible to expand a given class of algebras K by introducing new function symbols for some of its implicit operations so that

(i) every implicit operation of K becomes interpolable by a set of terms in the resulting expansion M, and

(ii) the basic desiderata (D1) and (D2)[7] are met?

This idea is made precise by the following definition.

Definition 11.3. A pp expansion M of a class of algebras K is said to be a *Beth companion* of K when every implicit operation of K is interpolated in M by a set of terms of M.

The aim of this section is to prove a triplet of results on Beth companions. The first governs the interplay between pp expansions and Beth companions.

Theorem 11.4. *Let K be a universal class, M_1 a pp expansion of K, and M_2 a pp expansion of M_1. Then the following conditions hold:*

(i) *if M_1 is a Beth companion of K, then M_2 is a Beth companion of K;*

(ii) *if M_2 is a Beth companion of M_1, then M_2 is a Beth companion of K.*

In general, a quasivariety K need not possess a Beth companion (see Section 14). However, when a Beth companion of K exists, the above result yields a description of a concrete Beth companion of K.

Corollary 11.5. *Let K be a universal class, $\mathcal{F} = \mathsf{ext}_{pp}(\mathsf{K})$, and $\mathscr{L}_\mathcal{F}$ an $\mathcal{F}$-expansion of $\mathscr{L}_\mathsf{K}$. Then K has a Beth companion if and only if $\mathbb{S}(\mathsf{K}[\mathscr{L}_\mathcal{F}])$ is a Beth companion of K.*

The second result connects Beth companions with the strong Beth definability and the strong epimorphism surjectivity properties as follows.

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[7]See the first paragraph of Section 9.

Theorem 11.6. *The following conditions are equivalent for a pp expansion* M *of a universal class* K*:*

(i) M *is a Beth companion of* K*;*
(ii) M *has the strong Beth definability property;*
(iii) M *has the strong epimorphism surjectivity property;*
(iv) *every member of* $\mathsf{imp}_{pp}(\mathsf{K})$ *is interpolated in* M *by a set of terms of* M*.*

In addition, when K *is a quasivariety, we can add the following equivalent condition:*

(v) *every member of* $\mathsf{imp}_{pp}(\mathsf{K})$ *is interpolated in* M *by a single term of* M*.*

The last result in this section states that, in the setting of quasivarieties, Beth companions are essentially unique (when they exist). To make this precise, we adapt the notion of term equivalence (see, e.g., [11, p. 131]) to expresses that two pp expansions of K in possibly distinct languages are essentially indistinguishable. Let M_1 and M_2 be a pair of pp expansions of a class of algebras K. For $i = 1, 2$ let T_i be the set of terms of M_i with variables in $\{x_n : n \in \mathbb{N}\}$. Let $\rho \colon \mathscr{L}_{\mathsf{M}_2} \to T_1$ be a map that preserves the arities. For each $\mathscr{L}_{\mathsf{M}_1}$-algebra $\boldsymbol{A}$ let $\rho(\boldsymbol{A})$ be the $\mathscr{L}_{\mathsf{M}_2}$-algebra with universe A such that $f^{\rho(\boldsymbol{A})} = \rho(f)^{\boldsymbol{A}}$ for each function symbol f in $\mathscr{L}_{\mathsf{M}_2}$. Similarly, given an arity-preserving map $\tau \colon \mathscr{L}_{\mathsf{M}_1} \to T_2$ and an $\mathscr{L}_{\mathsf{M}_2}$-algebra $\boldsymbol{B}$, we define an $\mathscr{L}_{\mathsf{M}_1}$-algebra $\tau(\boldsymbol{B})$. We say that M_1 and M_2 are *faithfully term equivalent relative to* K if there exist arity-preserving maps $\tau \colon \mathscr{L}_{\mathsf{M}_1} \to T_2$ and $\rho \colon \mathscr{L}_{\mathsf{M}_2} \to T_1$ such that $\tau(f) = f(x_1, \ldots, x_n)$ and $\rho(f) = f(x_1, \ldots, x_n)$ for each n-ary function symbol f in $\mathscr{L}_{\mathsf{K}}$, and for all $\boldsymbol{A} \in \mathsf{M}_1$ and $\boldsymbol{B} \in \mathsf{M}_2$ we have

(i) $\rho(\boldsymbol{A}) \in \mathsf{M}_2$;
(ii) $\tau(\boldsymbol{B}) \in \mathsf{M}_1$;
(iii) $\tau\rho(\boldsymbol{A}) = \boldsymbol{A}$;
(iv) $\rho\tau(\boldsymbol{B}) = \boldsymbol{B}$.

In view of the following theorem, from now on, we will talk about *the* Beth companion of a quasivariety.

Theorem 11.7. *All the Beth companions of a quasivariety* K *are faithfully term equivalent relative to* K*.*

Before proving these results, we shall illustrate their applicability by describing Beth companions of familiar classes of algebras.

Example 11.8 (Beth companions)**.** Our aim is to prove the next result, which describes Beth companions of some familiar classes of algebras. Further examples will be given once sufficient portions of the theory of Beth companions will become available. The curious reader may consult Table 1, which summarizes compactly all the examples considered in this work.

Theorem 11.9. *The following conditions hold:*

(i) *the variety of Abelian groups is the Beth companion of the quasivariety of cancellative commutative monoids;*
(ii) *the variety of Boolean algebras is the Beth companion of the variety of bounded distributive lattices;*

(iii) *the variety of relatively complemented distributive lattices is the Beth companion of the variety of distributive lattices;*

(iv) *the variety of implicative semilattices is the Beth companion of the variety of Hilbert algebras;*

(v) *the variety of Heyting algebras of depth $\leqslant 2$ is the Beth companion of the variety of pseudocomplemented distributive lattices;*

(vi) *every universal class with the strong epimorphism surjectivity property is a Beth companion of itself.*

Proof. (i): By Theorem 10.10 the variety AG of Abelian groups is a pp expansion of the quasivariety CCMon of cancellative commutative monoids. Recall from Example 6.3 that AG has the strong epimorphism surjectivity property. Hence, we can apply Theorem 11.6, obtaining that AG is the Beth companion of CCMon.

(ii)–(v): Analogous to the proof of (i). For (ii), use Theorems 10.12(ii) and 7.5. For (iii), use Theorems 10.12(i) and 7.5. For (iv), use Theorem 10.17 and the fact that the variety of implicative semilattices has the strong epimorphism surjectivity property (see [42, Props. 81 & 82]). For (v), use Theorem 10.19 and the fact that the variety of Heyting algebras of depth $\leqslant 2$ has the strong epimorphism surjectivity property (see [90, Thm. 8.1(3)]).

(vi): Consider a universal class K with the strong epimorphism surjectivity property. In view of Example 10.8, the class K is a pp expansion of itself. Therefore, we can apply Theorem 11.6, obtaining that K is a Beth companion of itself.　　　　　⊠

Remark 11.10. The following shows that the hypothesis that K is a quasivariety in Theorem 11.7 cannot be replaced with the requirement that K is a universal class. Let $\boldsymbol{A} = \langle A; \wedge, \vee, 0, 1 \rangle$ be the two-element bounded lattice and $\mathsf{K} = \mathbb{U}(\boldsymbol{A})$. Since $\boldsymbol{A}$ is finite and has no proper subalgebras, from Theorem 2.2 and Proposition 2.14 it follows that $\mathsf{K} = \mathbb{U}(\boldsymbol{A}) = \mathbb{ISP}_{\mathrm{u}}(\boldsymbol{A}) = \mathbb{I}(\boldsymbol{A})$. Therefore, every member of K lacks proper subalgebras because it is isomorphic to $\boldsymbol{A}$. Therefore, for all $\boldsymbol{B} \leqslant \boldsymbol{C} \in \mathsf{K}$ we have $\mathsf{d}_{\mathsf{K}}(\boldsymbol{B}, \boldsymbol{C}) = \mathsf{d}_{\mathsf{K}}(\boldsymbol{C}, \boldsymbol{C}) = \boldsymbol{C}$. Together with Proposition 6.6, this yields that K has the strong epimorphism surjectivity property. Therefore, Theorem 11.9(vi) implies that K is a Beth companion of itself.

We will show that there exists another Beth companion M of K such that K and M are not faithfully term equivalent relative to K. Since K is a class of bounded distributive lattices, it follows from Theorem 3.20(ii) that taking complements yields a unary implicit operation f of K defined by a conjunction of equations. As $f^{\boldsymbol{B}}$ is total for every $\boldsymbol{B} \in \mathbb{I}(\boldsymbol{A}) = \mathsf{K}$, we have $f \in \mathsf{ext}_{eq}(\mathsf{K})$. Let $\mathscr{L}_f = \mathscr{L}_{\mathsf{K}} \cup \{\neg\}$ be an f-expansion of $\mathscr{L}_{\mathsf{K}}$ and $\mathsf{M} = \mathbb{S}(\mathsf{K}[\mathscr{L}_f])$ the pp expansion of K induced by f and $\mathscr{L}_f$. As every member of K lacks proper subalgebras, so does every member of $\mathsf{K}[\mathscr{L}_f]$, and hence $\mathsf{M} = \mathsf{K}[\mathscr{L}_f]$. Together with $\mathsf{K} = \mathbb{I}(\boldsymbol{A})$ and the fact that $\boldsymbol{A}[\mathscr{L}_f]$ is defined (because $f^{\boldsymbol{A}}$ is total), this yields $\mathsf{M} = \mathbb{I}(\boldsymbol{A}[\mathscr{L}_f])$. By arguing as above we obtain that M has the strong epimorphism surjectivity property. Therefore, Theorem 11.6 implies that M is a Beth companion of K.

It only remains to show that M and K are not faithfully term equivalent relative to K. Suppose the contrary, with a view to contradiction. Then let $\tau \colon \mathscr{L}_{\mathsf{K}} \to T_1$ and $\rho \colon \mathscr{L}_f \to T_2$ be the maps witnessing the faithful term equivalence, where T_1 and T_2 are the sets of terms

of $\mathscr{L}_f$ and $\mathscr{L}_K$, respectively, with variables in $\{x_n : n \in \mathbb{N}\}$. Recall that $K = \mathbb{I}(\boldsymbol{A})$ and $M = \mathbb{I}(\boldsymbol{A}[\mathscr{L}_f])$. Therefore, $\boldsymbol{A}[\mathscr{L}_f] \cong \rho(\boldsymbol{A})$ because $\rho(\boldsymbol{A}) \in M$. Let $h\colon \rho(\boldsymbol{A}) \to \boldsymbol{A}[\mathscr{L}_f]$ be an isomorphism. For every $a \in A$ we have

$$h(\rho(\neg)^{\boldsymbol{A}}(a)) = h(\neg^{\rho(\boldsymbol{A})}a) = \neg^{\boldsymbol{A}[\mathscr{L}_f]}h(a) = f^{\boldsymbol{A}}(h(a)).$$

Consequently, there exists a term $t(x)$ of K (namely, $\rho(\neg)$) such that $h(t^{\boldsymbol{A}}(a)) = f^{\boldsymbol{A}}(h(a))$ for every $a \in A$. Since $f^{\boldsymbol{A}}(h(a))$ is the complement of $h(a)$ in the two-element bounded lattice $\boldsymbol{A}$, we have $f^{\boldsymbol{A}}(h(a)) \neq h(a)$ for every $a \in A$. Therefore, for every $a \in A$ we obtain $h(t^{\boldsymbol{A}}(a)) \neq h(a)$, and hence $t^{\boldsymbol{A}}(a) \neq a$. From $A = \{0, 1\}$ it follows that $t^{\boldsymbol{A}}(1) = 0$ and $t^{\boldsymbol{A}}(0) = 1$. Since all the basic operations of $\boldsymbol{A}$ are order preserving in each component, $t^{\boldsymbol{A}}$ is order preserving in each component as well. Therefore, $t^{\boldsymbol{A}}(0) \leqslant t^{\boldsymbol{A}}(1)$, a contradiction with the fact that $t^{\boldsymbol{A}}(1) = 0$ and $t^{\boldsymbol{A}}(0) = 1$. Hence, K and M are not faithfully term equivalent relative to K. In fact, the same argument shows that K and M are not even term equivalent (see, e.g., [11, p. 131] for the definition of term equivalence). $\boxtimes$

Now, we turn our attention to proving Theorem 11.4.

Proof. Recall from the assumptions that M_1 is a pp expansion of K and that M_2 is a pp expansion of M_1. Therefore, M_2 is also a pp expansion of K by Theorem 10.6. This fact will be used repeatedly in the proof.

(i): Suppose that M_1 is a Beth companion of K. We will prove that so is M_2. Since M_2 is a pp expansion of K, it suffices to show that every implicit operation of K is interpolated in M_2 by a set of terms of M_2. Accordingly, consider an n-ary $f \in \mathsf{imp}(K)$. As M_1 is a Beth companion of K, there exists a family $\{t_i : i \in I\}$ of M_1 that interpolates f in M_1. Since M_2 is a pp expansion of M_1, we know that $\{t_i : i \in I\}$ is also a set of terms of M_2. We will prove that it interpolates f in M_2.

To this end, consider $\boldsymbol{A} \in M_2$ and $a_1, \ldots, a_n \in A$ such that $\langle a_1, \ldots, a_n \rangle \in \mathsf{dom}(f^{\boldsymbol{A}\restriction_{\mathscr{L}_K}})$. From $\mathscr{L}_K \subseteq \mathscr{L}_{M_1}$ it follows that $(\boldsymbol{A}\restriction_{\mathscr{L}_{M_1}})\restriction_{\mathscr{L}_K} = \boldsymbol{A}\restriction_{\mathscr{L}_K}$. Therefore,

$$f^{\boldsymbol{A}\restriction_{\mathscr{L}_K}} = f^{(\boldsymbol{A}\restriction_{\mathscr{L}_{M_1}})\restriction_{\mathscr{L}_K}} \quad \text{and} \quad \langle a_1, \ldots, a_n \rangle \in \mathsf{dom}(f^{(\boldsymbol{A}\restriction_{\mathscr{L}_{M_1}})\restriction_{\mathscr{L}_K}}).$$

We will prove that $\boldsymbol{A}\restriction_{\mathscr{L}_{M_1}} \in M_1$. Recall from the assumptions that K is a universal class. Therefore, so is its pp expansion M_1 by Theorem 10.3(i). Together with the assumption that M_2 is a pp expansion of M_1, this allows us to apply Proposition 9.6, obtaining $\boldsymbol{A}\restriction_{\mathscr{L}_{M_1}} \in M_1$, as desired. Together with the above display and the assumption that $\{t_i : i \in I\}$ interpolates f in M_1, this implies that there exists $i \in I$ such that $f^{\boldsymbol{A}\restriction_{\mathscr{L}_K}}(a_1, \ldots, a_n) = t_i^{\boldsymbol{A}\restriction_{\mathscr{L}_{M_1}}}(a_1, \ldots, a_n) = t_i^{\boldsymbol{A}}(a_1, \ldots, a_n)$.

(ii): Assume that M_2 is a Beth companion of M_1. We will prove that M_2 is a Beth companion of K as well. As in the previous case, it suffices to show that every implicit operation of K is interpolated in M_2 by a family of terms of M_2. Accordingly, consider an n-ary $f \in \mathsf{imp}(K)$. By Proposition 10.26 there exists $g \in \mathsf{imp}(M_1)$ such that $g^{\boldsymbol{A}} = f^{\boldsymbol{A}\restriction_{\mathscr{L}_K}}$ for each $\boldsymbol{A} \in M_1$. Observe that g is interpolated in M_2 by a family $\{t_i : i \in I\}$ of terms of M_2 because M_2 is a Beth companion of M_1. We will prove that the same family interpolates f in M_2. To this end, consider $\boldsymbol{A} \in M_2$ and $a_1, \ldots, a_n \in A$ such that $\langle a_1, \ldots, a_n \rangle \in \mathsf{dom}(f^{\boldsymbol{A}\restriction_{\mathscr{L}_K}})$. Since K is

a universal class, Theorem 10.3(i) implies that M_1 is a universal class as well. As $\boldsymbol{A} \in \mathsf{M}_2$ and M_2 is a pp expansion of M_1 we have $\boldsymbol{A}{\restriction}_{\mathscr{L}_{\mathsf{M}_1}} \in \mathsf{M}_1$ by Proposition 9.6. Furthermore, $(\boldsymbol{A}{\restriction}_{\mathscr{L}_{\mathsf{M}_1}}){\restriction}_{\mathscr{L}_{\mathsf{K}}} = \boldsymbol{A}{\restriction}_{\mathscr{L}_{\mathsf{K}}}$ because $\mathscr{L}_{\mathsf{K}} \subseteq \mathscr{L}_{\mathsf{M}_1}$. Therefore,

$$g^{\boldsymbol{A}{\restriction}_{\mathscr{L}_{\mathsf{M}_1}}} = f^{(\boldsymbol{A}{\restriction}_{\mathscr{L}_{\mathsf{M}_1}}){\restriction}_{\mathscr{L}_{\mathsf{K}}}} = f^{\boldsymbol{A}{\restriction}_{\mathscr{L}_{\mathsf{K}}}} \text{ and } \langle a_1, \ldots, a_n \rangle \in \mathsf{dom}(g^{\boldsymbol{A}{\restriction}_{\mathscr{L}_{\mathsf{M}_1}}}).$$

As $\{t_i : i \in I\}$ interpolates g in M_2, the above display guarantees the existence of some $i \in I$ such that $f^{\boldsymbol{A}{\restriction}_{\mathscr{L}_{\mathsf{K}}}}(a_1, \ldots, a_n) = g^{\boldsymbol{A}{\restriction}_{\mathscr{L}_{\mathsf{M}_1}}}(a_1, \ldots, a_n) = t_i^{\boldsymbol{A}}(a_1, \ldots, a_n)$. ⊠

Then we prove Corollary 11.5.

Proof. It suffices to prove the implication from left to right, as the other one is straightforward. Accordingly, let M be a Beth companion of K.

Claim 11.11. *The class $\mathbb{S}(\mathsf{K}[\mathscr{L}_{\mathcal{F}}])$ is a pp expansion of K which, moreover, is term equivalent to a Beth companion of K.*

Proof of the Claim. Since M is a pp expansion of K, we may assume that it is induced by some $\mathcal{G} \subseteq \mathsf{ext}_{pp}(\mathsf{K})$ and $\mathcal{G}$-expansion $\mathscr{L}_{\mathcal{G}}$ of $\mathscr{L}_{\mathsf{K}}$. As $\mathcal{F} = \mathsf{ext}_{pp}(\mathsf{K})$ by assumption, we have $\mathcal{G} \subseteq \mathsf{ext}_{pp}(\mathsf{K}) = \mathcal{F}$. Then there exists an $\mathcal{F}$-expansion $\mathscr{L}'_{\mathcal{F}}$ of the form $\mathscr{L}_{\mathcal{G}} \cup \{g_f : f \in \mathcal{F} - \mathcal{G}\}$. Since $\mathcal{G} \subseteq \mathcal{F}$ and $\mathscr{L}_{\mathcal{G}} \subseteq \mathscr{L}'_{\mathcal{F}}$, Theorem 10.7 implies that $\mathbb{S}(\mathsf{K}[\mathscr{L}'_{\mathcal{F}}])$ is a pp expansion of $\mathsf{M} = \mathbb{S}(\mathsf{K}[\mathscr{L}_{\mathcal{G}}])$. Together with the assumption that M is a Beth companion of K, this allows us to apply Theorem 11.4(i), obtaining that $\mathbb{S}(\mathsf{K}[\mathscr{L}'_{\mathcal{F}}])$ is a Beth companion of K as well. Lastly, the definition of $\mathscr{L}'_{\mathcal{F}}$ guarantees that the classes $\mathbb{S}(\mathsf{K}[\mathscr{L}_{\mathcal{F}}])$ and $\mathbb{S}(\mathsf{K}[\mathscr{L}'_{\mathcal{F}}])$ are term equivalent. ⊠

Recall that $\mathbb{S}(\mathsf{K}[\mathscr{L}_{\mathcal{F}}])$ is a pp expansion of K by assumption. To prove that it is also a Beth companion of K, consider some $f \in \mathsf{imp}(\mathsf{K})$. By Claim 11.11 the class $\mathbb{S}(\mathsf{K}[\mathscr{L}_{\mathcal{F}}])$ is term equivalent to a class in which f is interpolated by a set of terms. By the definition of term equivalence, this guarantees that f is also interpolated in $\mathbb{S}(\mathsf{K}[\mathscr{L}_{\mathcal{F}}])$ by a set of terms. ⊠

Next we prove Theorem 11.6.

Proof. Let M be a pp expansion of a universal class K of the form $\mathbb{S}(\mathsf{K}[\mathscr{L}_{\mathcal{F}}])$. Theorem 10.3(i) implies that M is a universal class. So, conditions (ii) and (iii) are equivalent by Theorem 6.5. Furthermore, the implication (i)⇒(iv) is straightforward.

(iv)⇒(ii): To prove that M has the strong Beth definability property, it suffices to show that every implicit operation of M defined by a pp formula is interpolated by a set of terms. For suppose that this is the case. As M is an elementary class, we can apply Propositions 5.2 and 5.4, obtaining that M has the strong Beth definability property, as desired. Then consider an n-ary $f \in \mathsf{imp}_{pp}(\mathsf{M})$. By Proposition 10.22(iii) there exists $g \in \mathsf{imp}_{pp}(\mathsf{K})$ such that

$$f^{\boldsymbol{A}} = g^{\boldsymbol{A}{\restriction}_{\mathscr{L}_{\mathsf{K}}}} \text{ for each } \boldsymbol{A} \in \mathsf{K}[\mathscr{L}_{\mathcal{F}}]. \tag{53}$$

Applying (iv) to $g \in \mathsf{imp}_{pp}(\mathsf{K})$ yields a set $\{t_i : i \in I\}$ of terms of M that interpolates g in M. We will show that $\{t_i : i \in I\}$ interpolates f as well. To this end, consider $\boldsymbol{A} \in \mathsf{M}$ and $a_1, \ldots, a_n \in A$ such that $\langle a_1, \ldots, a_n \rangle \in \mathsf{dom}(f^{\boldsymbol{A}})$. Since $\boldsymbol{A} \in \mathsf{M} = \mathbb{S}(\mathsf{K}[\mathscr{L}_{\mathcal{F}}])$, we have

$\boldsymbol{A} \leqslant \boldsymbol{B}$ for some $\boldsymbol{B} \in \mathsf{K}[\mathscr{L}_{\mathcal{F}}]$. As $f \in \mathsf{imp}(\mathsf{M})$ and $\boldsymbol{A}, \boldsymbol{B} \in \mathsf{M}$ are such that $\boldsymbol{A} \leqslant \boldsymbol{B}$, from Proposition 8.1 and $\langle a_1, \ldots, a_n \rangle \in \mathsf{dom}(f^{\boldsymbol{A}})$ it follows that

$$\langle a_1, \ldots, a_n \rangle \in \mathsf{dom}(f^{\boldsymbol{B}}) \quad \text{and} \quad f^{\boldsymbol{A}}(a_1, \ldots, a_n) = f^{\boldsymbol{B}}(a_1, \ldots, a_n).$$

Furthermore, by applying (53) to the assumption that $\boldsymbol{B} \in \mathsf{K}[\mathscr{L}_{\mathcal{F}}]$, we obtain $f^{\boldsymbol{B}} = g^{\boldsymbol{B} \restriction \mathscr{L}_{\mathsf{K}}}$. Together with the above display, this yields

$$\langle a_1, \ldots, a_n \rangle \in \mathsf{dom}(g^{\boldsymbol{B} \restriction \mathscr{L}_{\mathsf{K}}}) \quad \text{and} \quad f^{\boldsymbol{A}}(a_1, \ldots, a_n) = g^{\boldsymbol{B} \restriction \mathscr{L}_{\mathsf{K}}}(a_1, \ldots, a_n).$$

Since $\{t_i : i \in I\}$ interpolates g in M, from the left hand side of the above display and $\boldsymbol{B} \in \mathsf{K}[\mathscr{L}_{\mathcal{F}}] \subseteq \mathsf{M}$ it follows that there exists $i \in I$ such that $g^{\boldsymbol{B} \restriction \mathscr{L}_{\mathsf{K}}}(a_1, \ldots, a_n) = t_i^{\boldsymbol{B}}(a_1, \ldots, a_n)$. Together with the right hand side of the above display and the fact that $\boldsymbol{A} \leqslant \boldsymbol{B}$, this yields

$$f^{\boldsymbol{A}}(a_1, \ldots, a_n) = g^{\boldsymbol{B} \restriction \mathscr{L}_{\mathsf{K}}}(a_1, \ldots, a_n) = t_i^{\boldsymbol{B}}(a_1, \ldots, a_n) = t_i^{\boldsymbol{A}}(a_1, \ldots, a_n).$$

Hence, we conclude that $\{t_i : i \in I\}$ interpolates f, as desired.

(ii)$\Rightarrow$(i): Suppose that M has the strong Beth definability property and consider an n-ary $f \in \mathsf{imp}(\mathsf{K})$. We need to prove that f is interpolated in M by a set of terms of M. In view of Proposition 10.26, there exists $g \in \mathsf{imp}(\mathsf{M})$ such that $g^{\boldsymbol{A}} = f^{\boldsymbol{A} \restriction \mathscr{L}_{\mathsf{K}}}$ for each $\boldsymbol{A} \in \mathsf{M}$. As M has the strong Beth definability property, the implicit operation g is interpolated by a set of terms $\{t_i : i \in I\}$ of M. We will show that this family interpolates f in M as well. Consider $\boldsymbol{A} \in \mathsf{M}$ and $a_1, \ldots, a_n \in A$ such that $\langle a_1, \ldots, a_n \rangle \in \mathsf{dom}(f^{\boldsymbol{A} \restriction \mathscr{L}_{\mathsf{K}}})$. Together with $g^{\boldsymbol{A}} = f^{\boldsymbol{A} \restriction \mathscr{L}_{\mathsf{K}}}$, this yields

$$\langle a_1, \ldots, a_n \rangle \in \mathsf{dom}(g^{\boldsymbol{A}}) \quad \text{and} \quad f^{\boldsymbol{A} \restriction \mathscr{L}_{\mathsf{K}}}(a_1, \ldots, a_n) = g^{\boldsymbol{A}}(a_1, \ldots, a_n).$$

Since g is interpolated by $\{t_i : i \in I\}$, there exists $i \in I$ such that $g^{\boldsymbol{A}}(a_1, \ldots, a_n) = t_i^{\boldsymbol{A}}(a_1, \ldots, a_n)$. By the right hand side of the above display this amounts to $f^{\boldsymbol{A} \restriction \mathscr{L}_{\mathsf{K}}}(a_1, \ldots, a_n) = t_i^{\boldsymbol{A}}(a_1, \ldots, a_n)$. Hence, we conclude that f is interpolated by $\{t_i : i \in I\}$ in M.

Next we prove the last part of the statement. To this end, in the rest of the proof, we assume that K is a quasivariety.

(v)$\Rightarrow$(iv): Straightforward.

(ii)$\Rightarrow$(v): Consider $f \in \mathsf{imp}_{pp}(\mathsf{K})$. In view of Proposition 10.26, there exists $g \in \mathsf{imp}_{pp}(\mathsf{M})$ such that $g^{\boldsymbol{A}} = f^{\boldsymbol{A} \restriction \mathscr{L}_{\mathsf{K}}}$ for each $\boldsymbol{A} \in \mathsf{M}$. Observe that M is a quasivariety by Theorem 10.3(ii). Together with the assumption that M has the strong Beth definability property and $g \in \mathsf{imp}_{pp}(\mathsf{M})$, this allows us to apply Proposition 5.6, obtaining that g is interpolated by a single term t of M. An argument analogous to the one detailed in the proof of the implication (ii)$\Rightarrow$(i) shows that t interpolates f in M as well (the only difference is that, in this case, the role of the family $\{t_i : i \in I\}$ is taken over by the single term t). $\boxtimes$

It only remains to prove Theorem 11.7. The proof hinges on the next observation.

Remark 11.12. The strong connection provided by a faithful term equivalence guarantees the preservation of many important properties. Let M_1 and M_2 be two pp expansions of K that are faithfully term equivalent relative to K and suppose that the term equivalence is witnessed by $\tau \colon \mathscr{L}_{\mathsf{M}_1} \to T_2$ and $\rho \colon \mathscr{L}_{\mathsf{M}_2} \to T_1$. Then it is straightforward to verify that the following conditions hold:

(i) if $h\colon \boldsymbol{A} \to \boldsymbol{B}$ is a homomorphism between $\mathscr{L}_{\mathsf{M}_1}$-algebras, then $h\colon \rho(\boldsymbol{A}) \to \rho(\boldsymbol{B})$ is a homomorphism between $\mathscr{L}_{\mathsf{M}_2}$-algebras;

(ii) if $\boldsymbol{A}, \boldsymbol{B}$ are $\mathscr{L}_{\mathsf{M}_1}$-algebras such that $\boldsymbol{A} \leqslant \boldsymbol{B}$, then $\rho(\boldsymbol{A}) \leqslant \rho(\boldsymbol{B})$;

(iii) if $\boldsymbol{A} \in \mathsf{M}_1$, then $\boldsymbol{A}{\restriction}_{\mathscr{L}_{\mathsf{K}}} = \rho(\boldsymbol{A}){\restriction}_{\mathscr{L}_{\mathsf{K}}}$;

(iv) if M_1 and M_2 are quasivarieties, then $\mathsf{Con}_{\mathsf{M}_1}(\boldsymbol{A}) = \mathsf{Con}_{\mathsf{M}_2}(\rho(\boldsymbol{A}))$ for every $\boldsymbol{A} \in \mathsf{M}_1$;

(v) if M_1 is a universal class, quasivariety, or variety, then so is M_2;

(vi) M_1 is a Beth companion of K if and only if M_2 is a Beth companion of K.

Since the roles of ρ and τ are interchangeable, M_1 and M_2 may be swapped in the conditions above.   ⊠

We are now ready to prove Theorem 11.7.

Proof. Let M_1 and M_2 be a pair of Beth companions of a quasivariety K. We will prove that M_1 and M_2 are faithfully term equivalent relative to K. For $i = 1, 2$ let $\mathcal{F}_i \subseteq \mathsf{ext}_{pp}(\mathsf{K})$ be such that $\mathsf{M}_i = \mathbb{S}(\mathsf{K}[\mathscr{L}_{\mathcal{F}_i}])$. We may assume that

$$\mathscr{L}_{\mathcal{F}_1} = \mathscr{L}_{\mathsf{K}} \cup \{g_f : f \in \mathcal{F}_1\} \quad \text{and} \quad \mathscr{L}_{\mathcal{F}_2} = \mathscr{L}_{\mathsf{K}} \cup \{g_f : f \in \mathcal{F}_2\}.$$

Then let T_i be the set of terms of $\mathscr{L}_{\mathcal{F}_i}$ with variables in $\{x_n : n \in \mathbb{N}\}$. Since K is a quasivariety, Theorem 11.6 implies that for every n-ary $f \in \mathsf{imp}_{pp}(\mathsf{K})$ and $i = 1, 2$ there exists an n-ary term $F_i(f) \in T_i$ that interpolates f in M_i. Therefore, for every n-ary $f \in \mathsf{imp}_{pp}(\mathsf{K})$, $\boldsymbol{A} \in \mathsf{M}_i$ with $i = 1, 2$, and $a_1, \ldots, a_n \in A$,

$$\langle a_1, \ldots, a_n \rangle \in \mathsf{dom}(f^{\boldsymbol{A}{\restriction}_{\mathscr{L}_{\mathsf{K}}}}) \text{ implies } f^{\boldsymbol{A}{\restriction}_{\mathscr{L}_{\mathsf{K}}}}(a_1, \ldots, a_n) = F_i(f)^{\boldsymbol{A}}(a_1, \ldots, a_n). \tag{54}$$

Define $\tau\colon \mathscr{L}_{\mathcal{F}_1} \to T_2$ and $\rho\colon \mathscr{L}_{\mathcal{F}_2} \to T_1$ by setting $\tau(f) = f(x_1, \ldots, x_n)$ and $\rho(f) = f(x_1, \ldots, x_n)$ for every n-ary $f \in \mathscr{L}_{\mathsf{K}}$, and

$$\tau(g_{f_1}) = F_2(f_1) \quad \text{and} \quad \rho(g_{f_2}) = F_1(f_2)$$

for each $f_1 \in \mathcal{F}_1$ and $f_2 \in \mathcal{F}_2$. We proceed to show that τ and ρ witness that M_1 and M_2 are faithfully term equivalent relative to K. Let $\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2$ and let $\mathscr{L}_{\mathcal{F}}$ be an $\mathcal{F}$-expansion of $\mathscr{L}_{\mathsf{K}}$ such that $\mathscr{L}_{\mathcal{F}_1}, \mathscr{L}_{\mathcal{F}_2} \subseteq \mathscr{L}_{\mathcal{F}}$. Recall that $\mathcal{F} = \mathcal{F}_2 \cup \mathcal{F}_2 \subseteq \mathsf{ext}_{pp}(\mathsf{K})$ by assumption. Therefore, Theorem 10.7 implies that $\mathbb{S}(\mathsf{K}[\mathscr{L}_{\mathcal{F}}])$ is a pp expansion of both $\mathsf{M}_1 = \mathbb{S}(\mathsf{K}[\mathscr{L}_{\mathcal{F}_1}])$ and $\mathsf{M}_2 = \mathbb{S}(\mathsf{K}[\mathscr{L}_{\mathcal{F}_2}])$. To verify conditions (i) and (iii) in the definition of faithful term equivalence, consider $\boldsymbol{A} \in \mathsf{M}_1$. We need to prove that $\rho(\boldsymbol{A}) \in \mathsf{M}_2$ and $\tau(\rho(\boldsymbol{A})) = \boldsymbol{A}$. Since $\mathbb{S}(\mathsf{K}[\mathscr{L}_{\mathcal{F}}])$ is a pp expansion of M_1, Proposition 10.2 implies that $\boldsymbol{A}$ is an $\mathscr{L}_{\mathcal{F}_1}$-subreduct of $\mathbb{S}(\mathsf{K}[\mathscr{L}_{\mathcal{F}}])$. As every member of $\mathbb{S}(\mathsf{K}[\mathscr{L}_{\mathcal{F}}])$ is a subalgebra of a member of $\mathsf{K}[\mathscr{L}_{\mathcal{F}}]$, it follows that there exists $\boldsymbol{B} \in \mathsf{K}[\mathscr{L}_{\mathcal{F}}]$ such that $\boldsymbol{A} \leqslant \boldsymbol{B}{\restriction}_{\mathscr{L}_{\mathcal{F}_1}}$. Since M_2 is a universal class by Theorem 10.3(i) and $\mathbb{S}(\mathsf{K}[\mathscr{L}_{\mathcal{F}}])$ is also a pp expansion of M_2, Proposition 10.2 implies that $\boldsymbol{B}{\restriction}_{\mathscr{L}_{\mathcal{F}_2}} \in \mathsf{M}_2$.

We now show that $\rho(\boldsymbol{A}) \in \mathsf{M}_2$. Observe that M_2 is closed under subalgebras because it is a pp expansion of K. Hence, it is sufficient to prove $\rho(\boldsymbol{A}) \leqslant \boldsymbol{B}{\restriction}_{\mathscr{L}_{\mathcal{F}_2}}$ because $\boldsymbol{B}{\restriction}_{\mathscr{L}_{\mathcal{F}_2}} \in \mathsf{M}_2$. Since $\boldsymbol{A} \leqslant \boldsymbol{B}{\restriction}_{\mathscr{L}_{\mathcal{F}_1}}$, from Remark 11.12(iii) it follows that

$$\rho(\boldsymbol{A}){\restriction}_{\mathscr{L}_{\mathsf{K}}} = \boldsymbol{A}{\restriction}_{\mathscr{L}_{\mathsf{K}}} \leqslant (\boldsymbol{B}{\restriction}_{\mathscr{L}_{\mathcal{F}_1}}){\restriction}_{\mathscr{L}_{\mathsf{K}}} = \boldsymbol{B}{\restriction}_{\mathscr{L}_{\mathsf{K}}} = (\boldsymbol{B}{\restriction}_{\mathscr{L}_{\mathcal{F}_2}}){\restriction}_{\mathscr{L}_{\mathsf{K}}}. \tag{55}$$

Therefore, it only remains to show that $g_f^{\rho(\boldsymbol{A})}(a_1,\ldots,a_n) = g_f^{\boldsymbol{B}\restriction_{\mathscr{L}_{\mathcal{F}_2}}}(a_1,\ldots,a_n)$ for all n-ary $g_f \in \mathscr{L}_{\mathcal{F}_2} - \mathscr{L}_{\mathsf{K}}$ and $a_1,\ldots,a_n \in A$. We have

$$
\begin{aligned}
g_f^{\rho(\boldsymbol{A})}(a_1,\ldots,a_n) &= \rho(g_f)^{\boldsymbol{A}}(a_1,\ldots,a_n) \\
&= F_1(f)^{\boldsymbol{A}}(a_1,\ldots,a_n) \\
&= F_1(f)^{\boldsymbol{B}\restriction_{\mathscr{L}_{\mathcal{F}_1}}}(a_1,\ldots,a_n) \\
&= f^{(\boldsymbol{B}\restriction_{\mathscr{L}_{\mathcal{F}_1}})\restriction_{\mathscr{L}_{\mathsf{K}}}}(a_1,\ldots,a_n) \\
&= f^{(\boldsymbol{B}\restriction_{\mathscr{L}_{\mathcal{F}_2}})\restriction_{\mathscr{L}_{\mathsf{K}}}}(a_1,\ldots,a_n) \\
&= g_f^{\boldsymbol{B}\restriction_{\mathscr{L}_{\mathcal{F}_2}}}(a_1,\ldots,a_n),
\end{aligned}
$$

where the first two equalities hold by the definitions of $\rho(\boldsymbol{A})$ and $\rho(g_f)$, the third holds because $\boldsymbol{A} \leqslant \boldsymbol{B}\restriction_{\mathscr{L}_{\mathcal{F}_1}}$, the fifth because $(\boldsymbol{B}\restriction_{\mathscr{L}_{\mathcal{F}_1}})\restriction_{\mathscr{L}_{\mathsf{K}}} = (\boldsymbol{B}\restriction_{\mathscr{L}_{\mathcal{F}_2}})\restriction_{\mathscr{L}_{\mathsf{K}}}$, and the last because $\boldsymbol{B} \in \mathsf{K}[\mathscr{L}_{\mathcal{F}}]$ implies $\boldsymbol{B}\restriction_{\mathscr{L}_{\mathcal{F}_2}} \in \mathsf{K}[\mathscr{L}_{\mathcal{F}_2}]$. The fourth equality follows from (54) because $(\boldsymbol{B}\restriction_{\mathscr{L}_{\mathcal{F}_1}})\restriction_{\mathscr{L}_{\mathsf{K}}} = (\boldsymbol{B}\restriction_{\mathscr{L}_{\mathcal{F}_2}})\restriction_{\mathscr{L}_{\mathsf{K}}}$ and $\boldsymbol{B}\restriction_{\mathscr{L}_{\mathcal{F}_2}} \in \mathsf{K}[\mathscr{L}_{\mathcal{F}_2}]$ imply $\langle a_1,\ldots,a_n \rangle \in \mathsf{dom}(f^{(\boldsymbol{B}\restriction_{\mathscr{L}_{\mathcal{F}_1}})\restriction_{\mathscr{L}_{\mathsf{K}}}})$. We conclude that $\rho(\boldsymbol{A}) \leqslant \boldsymbol{B}\restriction_{\mathscr{L}_{\mathcal{F}_2}}$, and hence $\rho(\boldsymbol{A}) \in \mathsf{M}_2$.

Next we prove that $\tau(\rho(\boldsymbol{A})) = \boldsymbol{A}$. By Remark 11.12(iii) we obtain

$$
\tau(\rho(\boldsymbol{A}))\restriction_{\mathscr{L}_{\mathsf{K}}} = \rho(\boldsymbol{A})\restriction_{\mathscr{L}_{\mathsf{K}}} = \boldsymbol{A}\restriction_{\mathscr{L}_{\mathsf{K}}}.
$$

Therefore, it remains to show that $g_f^{\tau(\rho(\boldsymbol{A}))} = g_f^{\boldsymbol{A}}$ for every n-ary $g_f \in \mathscr{L}_{\mathcal{F}_1} - \mathscr{L}_K$. Let $a_1,\ldots,a_n \in A$. Then

$$
\begin{aligned}
g_f^{\tau(\rho(\boldsymbol{A}))}(a_1,\ldots,a_n) &= \tau(g_f)^{\rho(\boldsymbol{A})}(a_1,\ldots,a_n) \\
&= F_2(f)^{\rho(\boldsymbol{A})}(a_1,\ldots,a_n) \\
&= F_2(f)^{\boldsymbol{B}\restriction_{\mathscr{L}_{\mathcal{F}_2}}}(a_1,\ldots,a_n) \\
&= f^{(\boldsymbol{B}\restriction_{\mathscr{L}_{\mathcal{F}_2}})\restriction_{\mathscr{L}_{\mathsf{K}}}}(a_1,\ldots,a_n) \\
&= f^{(\boldsymbol{B}\restriction_{\mathscr{L}_{\mathcal{F}_1}})\restriction_{\mathscr{L}_{\mathsf{K}}}}(a_1,\ldots,a_n) \\
&= g_f^{\boldsymbol{B}\restriction_{\mathscr{L}_{\mathcal{F}_1}}}(a_1,\ldots,a_n) \\
&= g_f^{\boldsymbol{A}}(a_1,\ldots,a_n),
\end{aligned}
$$

where the first equality follows from the definition of $\tau(\boldsymbol{C})$ for an $\mathscr{L}_{\mathcal{F}_2}$-algebra $\boldsymbol{C}$, the second from the definition of $\tau(g_f)$, the third holds because $\rho(\boldsymbol{A}) \leqslant \boldsymbol{B}\restriction_{\mathscr{L}_{\mathcal{F}_2}}$ as we established above, the fifth because $(\boldsymbol{B}\restriction_{\mathscr{L}_{\mathcal{F}_2}})\restriction_{\mathscr{L}_{\mathsf{K}}} = (\boldsymbol{B}\restriction_{\mathscr{L}_{\mathcal{F}_1}})\restriction_{\mathscr{L}_{\mathsf{K}}}$, the sixth because $\boldsymbol{B} \in \mathsf{K}[\mathscr{L}_{\mathcal{F}}]$ implies $\boldsymbol{B}\restriction_{\mathscr{L}_{\mathcal{F}_1}} \in \mathsf{K}[\mathscr{L}_{\mathcal{F}_1}]$, and the last because $\boldsymbol{A} \leqslant \boldsymbol{B}\restriction_{\mathscr{L}_{\mathcal{F}_1}}$. The fourth equality follows from (54) because $(\boldsymbol{B}\restriction_{\mathscr{L}_{\mathcal{F}_2}})\restriction_{\mathscr{L}_{\mathsf{K}}} = (\boldsymbol{B}\restriction_{\mathscr{L}_{\mathcal{F}_1}})\restriction_{\mathscr{L}_{\mathsf{K}}}$ and $\boldsymbol{B}\restriction_{\mathscr{L}_{\mathcal{F}_1}} \in \mathsf{K}[\mathscr{L}_{\mathcal{F}_1}]$ imply $\langle a_1,\ldots,a_n \rangle \in \mathsf{dom}(f^{(\boldsymbol{B}\restriction_{\mathscr{L}_{\mathcal{F}_2}})\restriction_{\mathscr{L}_{\mathsf{K}}}})$. We conclude that $\tau(\rho(\boldsymbol{A})) = \boldsymbol{A}$.

Thus, conditions (i) and (iii) in the definition of faithful term equivalence hold. The proof that conditions (ii) and (iv) hold is analogous. Since $\tau(f) = f(x_1,\ldots,x_n)$ and

$\rho(f) = f(x_1, \ldots, x_n)$ for every n-ary $f \in \mathscr{L}_\mathsf{K}$, we conclude that M_1 and M_2 are faithfully term equivalent relative to K.   ⊠

12. Structure theory

In this section, we address the following question: what do we gain by moving from a quasivariety to its Beth companion? We will answer it by showing that not only does the Beth companion of a quasivariety K inherit a significant portion of the structure theory of K, but its structure theory is often much richer than that of K. To this end, it is convenient to restrict our attention to the pp expansions that respect the structure of congruence lattices, which we term *congruence preserving*. More precisely, recall that if M is a pp expansion of a quasivariety K, then M is a quasivariety and K the class of $\mathscr{L}_\mathsf{K}$-subreducts of M (see Theorem 10.3(ii) and Proposition 10.2).

Definition 12.1. A pp expansion M of a quasivariety K is said to be *congruence preserving* when $\mathsf{Con}_\mathsf{M}(\boldsymbol{A}) = \mathsf{Con}_\mathsf{K}(\boldsymbol{A}{\restriction}_{\mathscr{L}_\mathsf{K}})$ for every $\boldsymbol{A} \in \mathsf{M}$.

Remark 12.2. Let M be a pp expansion of a quasivariety K. We will prove that M is congruence preserving when $\mathsf{Con}_\mathsf{K}(\boldsymbol{A}{\restriction}_{\mathscr{L}_\mathsf{K}}) \subseteq \mathsf{Con}_\mathsf{M}(\boldsymbol{A})$ for every $\boldsymbol{A} \in \mathsf{M}$. To this end, it suffices to show that the reverse inclusion $\mathsf{Con}_\mathsf{M}(\boldsymbol{A}) \subseteq \mathsf{Con}_\mathsf{K}(\boldsymbol{A}{\restriction}_{\mathscr{L}_\mathsf{K}})$ always holds for each $\boldsymbol{A} \in \mathsf{M}$. Consider $\theta \in \mathsf{Con}_\mathsf{M}(\boldsymbol{A})$. The inclusion $\mathscr{L}_\mathsf{K} \subseteq \mathscr{L}_\mathsf{M}$ guarantees that θ is a congruence of $\boldsymbol{A}{\restriction}_{\mathscr{L}_\mathsf{K}}$. Moreover, since θ is an M-congruence of $\boldsymbol{A}$, we have $\boldsymbol{A}/\theta \in \mathsf{M}$. Thus, $(\boldsymbol{A}{\restriction}_{\mathscr{L}_\mathsf{K}})/\theta = (\boldsymbol{A}/\theta){\restriction}_{\mathscr{L}_\mathsf{K}} \in \mathsf{M}{\restriction}_{\mathscr{L}_\mathsf{K}} \subseteq \mathsf{K}$. Consequently, we conclude that $\theta \in \mathsf{Con}_\mathsf{K}(\boldsymbol{A}{\restriction}_{\mathscr{L}_\mathsf{K}})$.   ⊠

Although pp expansions of quasivarieties need not be congruence preserving in general (see [27, Thm. 2.1]), the next result explains why most concrete pp expansions are indeed congruence preserving. To this end, it is convenient to introduce the following concept.

Definition 12.3. A pp expansion M of a class of algebras K is said to be *equational* when it is faithfully term equivalent relative to K to a pp expansion of K induced by some $\mathcal{F} \subseteq \mathsf{ext}_{eq}(\mathsf{K})$ and $\mathscr{L}_\mathcal{F}$.

We also recall that a quasivariety K has the *relative congruence extension property* when for all $\boldsymbol{A} \leqslant \boldsymbol{B} \in \mathsf{K}$ and $\theta \in \mathsf{Con}_\mathsf{K}(\boldsymbol{A})$ there exists $\phi \in \mathsf{Con}_\mathsf{K}(\boldsymbol{B})$ such that $\theta = \phi{\restriction}_A$. When K is a variety, we simply say that K has the *congruence extension property*.[8]

Theorem 12.4. *Let M be a pp expansion of a quasivariety K and assume that one of the following conditions holds:*
  (i) M *is equational;*
  (ii) K *has the relative congruence extension property.*
Then M is congruence preserving.

We say that a Beth companion is *congruence preserving* when it is a congruence preserving pp expansion. Similarly, we call a Beth companion *equational* when it is an equational pp

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[8]Although we will not rely on this fact, we remark that the relative congruence extension property persists in pp expansions of quasivarieties (see [27, Prop. 3.18]).

expansion. It is an immediate consequence of Remark 11.12(vi) that a Beth companion of a class of algebras K is equational if and only if it is faithfully term equivalent relative to K to a Beth companion of K induced by some $\mathcal{F} \subseteq \mathsf{ext}_{eq}(\mathsf{K})$ and $\mathscr{L}_{\mathcal{F}}$. In view of Theorem 12.4, all the Beth companions considered so far are congruence preserving as we proceed to illustrate.

Example 12.5 (Congruence preserving Beth companions). Notice that all the Beth companions of varieties mentioned in Theorem 11.9 are equational (in fact, they are all induced by sets of implicit operations defined by conjunctions of equations). Therefore, Theorem 12.4 guarantees that they are congruence preserving. ⊠

As the following example shows, being equational is not a necessary condition for a Beth companion to be congruence preserving.

Example 12.6 (Congruence preserving and not equational Beth companion). For every $n \in \mathbb{Z}^+$ let $\boldsymbol{A}_n$ be the unique Heyting algebra whose lattice reduct is obtained by adding a new maximum 1 to the powerset lattice $\langle \mathcal{P}(\{1, \ldots, n\}); \cap, \cup \rangle$. Then for every $n \geqslant 3$ the variety $\mathbb{V}(\boldsymbol{A}_n)$ admits a Beth companion that is congruence preserving and not equational, as shown in [27, Thm. 3.3]. Notably, the variety $\mathbb{V}(\boldsymbol{A}_n)$ can be equivalently described as the variety of Heyting algebras of depth $\leqslant 2$ (see Example 10.18) satisfying the bounded width $\leqslant n$ axiom in the variables $x_1, \ldots, x_{n+1}$, namely, the equation

$$\bigvee_{i=1}^{n+1} \left(x_i \to \bigvee_{j \neq i} x_j \right) \approx 1.$$ ⊠

We have seen in Section 4 that the amalgamation property allows us to eliminate existentials in certain situations. An instance of this phenomenon is described in the following theorem.

Theorem 12.7. *Let K be a quasivariety with the amalgamation property. Then every pp expansion of K is equational, congruence preserving, and has the amalgamation property.*

The congruence preserving pp expansions of a quasivariety K inherit a substantial portion of the structure theory of K, namely, the one related to the structure of lattices of K-congruences. The next concepts are instrumental to make this idea precise. Let K be a quasivariety. A *congruence equation* is a formal equation in the binary symbols $\wedge, \vee$, and $\circ$. A congruence equation is *valid* in an algebra $\boldsymbol{A}$ relative to K when it becomes true whenever we interpret the variables of the equation as K-congruences of $\boldsymbol{A}$, and for arbitrary binary relations α and β on A, we interpret $\alpha \wedge \beta$, $\alpha \vee \beta$, and $\alpha \circ \beta$ as $\alpha \cap \beta$, $\mathsf{Cg}_{\mathsf{K}}^{\boldsymbol{A}}(\alpha \cup \beta)$, and $\alpha \circ \beta$, respectively. We say that a congruence equation is *valid* in K when it is valid relative to K in every member of K [99, 108] (see also [74]). For instance, a quasivariety K is relatively congruence distributive precisely when the congruence equation $(x \vee y) \wedge (x \vee z) \approx x \vee (y \wedge z)$ is valid in K. Similarly, a variety K is congruence permutable if and only if the congruence equation $x \circ y = y \circ x$ is valid in K.

Theorem 12.8. *Let M be a congruence preserving pp expansion of a quasivariety K. Then every congruence equation valid in K is valid in M.*

Proof. Immediate from the definitions of a congruence preserving pp expansion and of a congruence equation. ⊠

In addition, congruence preserving pp expansions preserve and reflect the property of "being relatively (finitely) subdirectly irreducible" (cf. Theorem 10.3(ii)) and preserve the property of "being a variety". We recall that the latter fails for arbitrary pp expansions (see [27, Thm. 2.1]). More precisely, we will prove the following.

Theorem 12.9. *Let* M *be a congruence preserving pp expansion of a quasivariety* K. *Then*

$$\mathsf{M}_{\mathrm{RFSI}} = \{ \boldsymbol{A} \in \mathsf{M} : \boldsymbol{A}{\restriction}_{\mathscr{L}_{\mathsf{K}}} \in \mathsf{K}_{\mathrm{RFSI}} \} \quad and \quad \mathsf{M}_{\mathrm{RSI}} = \{ \boldsymbol{A} \in \mathsf{M} : \boldsymbol{A}{\restriction}_{\mathscr{L}_{\mathsf{K}}} \in \mathsf{K}_{\mathrm{RSI}} \}.$$

Moreover, if K *is a variety, then* M *is a variety.*

As we mentioned, the aim of this section is to explain what we gain by moving from a quasivariety to its Beth companion. Our main result states that not only does every congruence preserving Beth companion of a quasivariety K inherit a significant portion of the structure theory of K (see Theorems 12.8 and 12.9), but it often gains remarkable properties in comparison to those of K. More precisely, we will establish the following.

Theorem 12.10. *Let* K *be a relatively congruence distributive quasivariety for which* $\mathsf{K}_{\mathrm{RFSI}}$ *is closed under nontrivial subalgebras. Then every congruence preserving Beth companion* M *of* K *is an arithmetical variety with the congruence extension property such that* $\mathsf{M}_{\mathrm{FSI}}$ *is closed under nontrivial subalgebras.*

In view of Theorem 12.10, under reasonable assumptions, the structure theory of a congruence preserving Beth companion M of a quasivariety K enhances that of K as follows: M turns out to be a variety (as opposed to a quasivariety) which, moreover, is arithmetical (as opposed to relatively congruence distributive only) and possesses the congruence extension property.

Let us illustrate the effect of Theorem 12.10 in the setting of filtral quasivarieties. A variety K is said to be *discriminator* when there exist a class of algebras M and a quaternary term t such that $\mathsf{K} = \mathbb{V}(\mathsf{M})$ and $t^{\boldsymbol{A}}$ is the quaternary discriminator function on A for every $\boldsymbol{A} \in \mathsf{M}$ [22, 107] (see also [21, Sec. IV.9]). Examples of discriminator varieties include the variety of Boolean algebras and for each $n \in \mathbb{N}$ the variety of rings satisfying the equation $x \approx x^n$ (see, e.g., [20, pp. 179–180]). The importance of discriminator varieties derives from the fact that they admit a general representation theorem in terms of Boolean products with subdirectly irreducible factors [21, Thm. IV.9.4] (see also [104]). While every discriminator variety is filtral (see, e.g., [10, p. 101]), the converse need not hold in general: for instance, while the variety of (bounded) distributive lattices is filtral, it is not a discriminator variety. However, its Beth companion, the variety of Boolean algebras, is a discriminator variety. From Theorem 12.10 we will infer that this is true in general.

Corollary 12.11. *Every Beth companion of a relatively filtral quasivariety is a discriminator variety.*

Before proving these results, let us illustrate the applicability of Theorem 12.10 with a more concrete example from ring theory.

Example 12.12 (Reduced commutative rings of characteristic zero). We recall that a ring (resp. meadow) $\boldsymbol{A}$ has *characteristic* $n \in \mathbb{Z}^+$ when n is the least $m \in \mathbb{Z}^+$ such that $m1 = 0$.

If $\boldsymbol{A}$ is trivial or has not characteristic n for any $n \in \mathbb{Z}^+$, we say that it has *characteristic zero*. In other words, $\boldsymbol{A}$ has characteristic zero if and only if it validates the quasiequation $1n \approx 0 \to x \approx y$ for every $n \in \mathbb{Z}^+$. Consequently, the class RCRing_0 of reduced commutative rings of characteristic zero forms a quasivariety.

We will make use of the following observations established in [30]. The quasivariety RCRing_0 is relatively congruence distributive and its relatively finitely subdirectly irreducible members are precisely the integral domains of characteristic zero, where an *integral domain* is a commutative ring validating the formulas $0 \not\approx 1$ and $xy \approx 0 \to (x \approx 0 \sqcup y \approx 0)$. Furthermore, the class Meadow_0 of meadows of characteristic zero is a congruence preserving Beth companion of RCRing_0.

As the class of integral domains of characteristic zero is closed under nontrivial subalgebras, from Theorem 12.10 it follows that Meadow_0 is an arithmetical variety with the congruence extension property. None of these facts holds for RCRing_0, which is a proper quasivariety without the relative congruence extension property (congruence permutability is a property of varieties only), as we proceed to illustrate.

First, observe that the ring of integers $\mathbb{Z}$ is an integral domain of characteristic zero, while its quotient induced by the ideal $p\mathbb{Z}$ has characteristic p. Consequently, RCRing_0 is a proper quasivariety. To show that RCRing_0 lacks the relative congruence extension property, consider the polynomial ring $\mathbb{Z}[x]$ and observe that it is an integral domain of characteristic zero and, therefore, belongs to RCRing_0. Then let θ be the congruence of $\mathbb{Z}[x]$ generated by the pair $\langle x, 0 \rangle$. As $\mathbb{Z}[x]/\theta \cong \mathbb{Z}$, we obtain that θ is an RCRing_0-congruence of $\mathbb{Z}[x]$. On the other hand, observe that $\mathbb{Z}[x]$ is a subalgebra of a field $\boldsymbol{A}$ of characteristic zero because it is an integral domain of characteristic zero (see, e.g., [4, Thm. 11.7.2]). Since $\boldsymbol{A}$ is a field, it does not possess a congruence extending θ. Hence, we conclude that RCRing_0 lacks the relative congruence extension property, as desired. $\boxtimes$

Next we prove Theorems 12.4, 12.7, 12.9, and 12.10 and Corollary 12.11. The proof of Theorem 12.4 hinges upon the next pair of observations.

Proposition 12.13. *Let* M *be a pp expansion of a quasivariety* K *induced by* $\mathcal{F} \subseteq \mathsf{ext}_{pp}(\mathsf{K})$ *and* $\mathscr{L}_{\mathcal{F}}$. *Then* $\mathsf{Con}_{\mathsf{M}}(\boldsymbol{A}) = \mathsf{Con}_{\mathsf{K}}(\boldsymbol{A}{\restriction}_{\mathscr{L}_{\mathsf{K}}})$ *for every* $\boldsymbol{A} \in \mathsf{K}[\mathscr{L}_{\mathcal{F}}]$.

Proof. Consider $\boldsymbol{A} \in \mathsf{K}[\mathscr{L}_{\mathcal{F}}]$. As the inclusion $\mathsf{Con}_{\mathsf{M}}(\boldsymbol{A}) \subseteq \mathsf{Con}_{\mathsf{K}}(\boldsymbol{A}{\restriction}_{\mathscr{L}_{\mathsf{K}}})$ always holds (see Remark 12.2), we detail only the proof of the reverse inclusion. Consider $\theta \in \mathsf{Con}_{\mathsf{K}}(\boldsymbol{A}{\restriction}_{\mathscr{L}_{\mathsf{K}}})$. Then $(\boldsymbol{A}{\restriction}_{\mathscr{L}_{\mathsf{K}}})/\theta \in \mathsf{K}$. Since $\mathcal{F} \subseteq \mathsf{ext}_{pp}(\mathsf{K})$ and K is a quasivariety, by Proposition 9.6 there exists $\boldsymbol{B} \in \mathsf{K}[\mathscr{L}_{\mathcal{F}}]$ such that $(\boldsymbol{A}{\restriction}_{\mathscr{L}_{\mathsf{K}}})/\theta \leqslant \boldsymbol{B}{\restriction}_{\mathscr{L}_{\mathsf{K}}}$. Therefore, the canonical surjection $h \colon \boldsymbol{A}{\restriction}_{\mathscr{L}_{\mathsf{K}}} \to \boldsymbol{A}{\restriction}_{\mathscr{L}_{\mathsf{K}}}/\theta$ can be viewed as a homomorphism $h \colon \boldsymbol{A}{\restriction}_{\mathscr{L}_{\mathsf{K}}} \to \boldsymbol{B}{\restriction}_{\mathscr{L}_{\mathsf{K}}}$. As $\boldsymbol{A}, \boldsymbol{B} \in \mathsf{K}[\mathscr{L}_{\mathcal{F}}]$, Proposition 9.5 guarantees that h is also a homomorphism from $\boldsymbol{A}$ to $\boldsymbol{B}$. Together with $\boldsymbol{B} \in \mathsf{M}$ and $\mathsf{Ker}(h) = \theta$, this yields that θ is an M-congruence of $\boldsymbol{A}$. $\boxtimes$

Proposition 12.14. *Let* M_1 *and* M_2 *be a pair of pp expansions of a quasivariety* K. *Assume that* M_1 *and* M_2 *are faithfully term equivalent relative to* K. *Then* M_1 *is congruence preserving if and only if so is* M_2.

Proof. Let T_1 and T_2 be the sets of terms, respectively of M_1 and M_2, with variables in $\{x_n : n \in \mathbb{N}\}$. Moreover, let $\tau \colon \mathscr{L}_{\mathsf{M}_1} \to T_2$ and $\rho \colon \mathscr{L}_{\mathsf{M}_2} \to T_1$ be the maps witnessing the fact that M_1 and M_2 are faithfully term equivalent relative to K.

By symmetry it suffices to show that if M_1 is congruence preserving, then so is M_2. Assume that M_1 is congruence preserving. Then for every $\boldsymbol{A} \in \mathsf{M}_2$, we have

$$\mathsf{Con}_{\mathsf{M}_2}(\boldsymbol{A}) = \mathsf{Con}_{\mathsf{M}_1}(\tau(\boldsymbol{A})) = \mathsf{Con}_{\mathsf{K}}(\tau(\boldsymbol{A})\!\restriction_{\mathscr{L}_{\mathsf{K}}}) = \mathsf{Con}_{\mathsf{K}}(\boldsymbol{A}\!\restriction_{\mathscr{L}_{\mathsf{K}}}),$$

where the first equality follows from Remark 11.12(iv), the second from the assumption that M_1 is congruence preserving, and the last from $\tau(\boldsymbol{A})\!\restriction_{\mathscr{L}_{\mathsf{K}}} = \boldsymbol{A}\!\restriction_{\mathscr{L}_{\mathsf{K}}}$ (which holds because τ and ρ witness a term equivalence that is faithful relative to K). $\boxtimes$

We are now ready to prove Theorem 12.4.

Proof. (i): Since M is equational, there exists a pp expansion N of K induced by some $\mathcal{F} \subseteq \mathsf{ext}_{eq}(\mathsf{K})$ and $\mathscr{L}_{\mathcal{F}}$ such that M is faithfully term equivalent to N relative to K. We can apply Theorem 10.4, obtaining $\mathsf{N} = \mathsf{K}[\mathscr{L}_{\mathcal{F}}]$. Therefore, $\mathsf{Con}_{\mathsf{N}}(\boldsymbol{A}) = \mathsf{Con}_{\mathsf{K}}(\boldsymbol{A}\!\restriction_{\mathscr{L}_{\mathsf{K}}})$ for every $\boldsymbol{A} \in \mathsf{N}$ by Proposition 12.13. Hence, we conclude that N is congruence preserving. From Proposition 12.14 it follows that M is congruence preserving as well.

(ii): In view of Remark 12.2, it suffices to show that $\mathsf{Con}_{\mathsf{K}}(\boldsymbol{A}\!\restriction_{\mathscr{L}_{\mathsf{K}}}) \subseteq \mathsf{Con}_{\mathsf{M}}(\boldsymbol{A})$ for each $\boldsymbol{A} \in \mathsf{M}$. To this end, consider $\boldsymbol{A} \in \mathsf{M}$ and $\theta \in \mathsf{Con}_{\mathsf{K}}(\boldsymbol{A}\!\restriction_{\mathscr{L}_{\mathsf{K}}})$. As M is a pp expansion of K, it is of the form $\mathbb{S}(\mathsf{K}[\mathscr{L}_{\mathcal{F}}])$. Since $\boldsymbol{A} \in \mathsf{M}$, this implies the existence of some $\boldsymbol{B} \in \mathsf{K}[\mathscr{L}_{\mathcal{F}}]$ such that $\boldsymbol{A} \leqslant \boldsymbol{B}$. Clearly, $\boldsymbol{A}\!\restriction_{\mathscr{L}_{\mathsf{K}}} \leqslant \boldsymbol{B}\!\restriction_{\mathscr{L}_{\mathsf{K}}} \in \mathsf{K}$. Therefore, from the assumption that K has the relative congruence extension property it follows that there exists $\phi \in \mathsf{Con}_{\mathsf{K}}(\boldsymbol{B}\!\restriction_{\mathscr{L}_{\mathsf{K}}})$ such that $\theta = \phi\!\restriction_A$. Since $\boldsymbol{B} \in \mathsf{K}[\mathscr{L}_{\mathcal{F}}]$, we can apply Proposition 12.13, obtaining $\phi \in \mathsf{Con}_{\mathsf{K}}(\boldsymbol{B}\!\restriction_{\mathscr{L}_{\mathsf{K}}}) = \mathsf{Con}_{\mathsf{M}}(\boldsymbol{B})$. Together with $\boldsymbol{A} \leqslant \boldsymbol{B}$, this yields $\phi\!\restriction_A \in \mathsf{Con}_{\mathsf{M}}(\boldsymbol{A})$. As $\theta = \phi\!\restriction_A$, we conclude that $\theta \in \mathsf{Con}_{\mathsf{M}}(\boldsymbol{A})$. $\boxtimes$

We now prove Theorem 12.7.

Proof. Let M be a pp expansion of K of the form $\mathbb{S}(\mathsf{K}[\mathscr{L}_{\mathcal{F}}])$. From the assumption that K has the amalgamation property and Theorem 4.3(i) it follows that every $f \in \mathcal{F} \subseteq \mathsf{ext}_{pp}(\mathsf{K})$ is interpolated by some $f_* \in \mathsf{imp}_{eq}(\mathsf{K})$.

Consider the set of implicit operations

$$\mathcal{F}_* = \{f_* : f \in \mathcal{F}\}$$

of K. We will prove that $\mathcal{F}_* \subseteq \mathsf{ext}_{eq}(\mathsf{K})$. As $\mathcal{F}_* \subseteq \mathsf{imp}_{eq}(\mathsf{K})$ by definition, it suffices to show that $\mathcal{F}_* \subseteq \mathsf{ext}(\mathsf{K})$. Consider an n-ary $f_* \in \mathcal{F}_*$, $\boldsymbol{A} \in \mathsf{K}$, and $a_1, \ldots, a_n \in A$. Since $f \in \mathcal{F} \subseteq \mathsf{ext}(\mathsf{K})$, there exists $\boldsymbol{B} \in \mathsf{K}$ with $\boldsymbol{A} \leqslant \boldsymbol{B}$ such that $\langle a_1, \ldots, a_n \rangle \in \mathsf{dom}(f^{\boldsymbol{B}})$. Consequently, $\langle a_1, \ldots, a_n \rangle \in \mathsf{dom}(f_*^{\boldsymbol{B}})$ because f_* interpolates f. Hence, f_* is extendable.

Together with Theorem 10.4, the inclusion $\mathcal{F}_* \subseteq \mathsf{ext}_{eq}(\mathsf{K})$ implies that $\mathsf{M}_* = \mathsf{K}[\mathscr{L}_{\mathcal{F}_*}]$ is a pp expansion of K. Therefore, to prove that M is an equational pp expansion of K, it only remains to show that the pp expansions M_* and M are faithfully term equivalent relative to K.

Since $\mathscr{L}_{\mathcal{F}}$ is an $\mathcal{F}$-expansion of $\mathscr{L}_{\mathsf{K}}$ and $\mathscr{L}_{\mathcal{F}_*}$ is an $\mathcal{F}_*$-expansion of $\mathscr{L}_{\mathsf{K}}$, we may assume that

$$\mathscr{L}_{\mathcal{F}} = \mathscr{L}_{\mathsf{K}} \cup \{g_f : f \in \mathcal{F}\} \ \text{ and } \ \mathscr{L}_{\mathcal{F}_*} = \mathscr{L}_{\mathsf{K}} \cup \{h_{f_*} : f_* \in \mathcal{F}_*\}.$$

Let T and T_* be the sets of terms, respectively of $\mathscr{L}_{\mathcal{F}}$ and $\mathscr{L}_{\mathcal{F}_*}$, with variables in $\{x_n : n \in \mathbb{N}\}$. We will prove that M and M_* are faithfully term equivalent relative to K as witnessed by the maps $\tau \colon \mathscr{L}_{\mathcal{F}} \to T_*$ and $\rho \colon \mathscr{L}_{\mathcal{F}_*} \to T$ defined for every $p \in \mathscr{L}_{\mathcal{F}}$ and $q \in \mathscr{L}_{\mathcal{F}_*}$ as

$$\tau(p) = \begin{cases} p(x_1, \ldots, x_n) & \text{if } p \in \mathscr{L}_{\mathsf{K}}; \\ h_{f_*}(x_1, \ldots, x_n) & \text{if } p = g_f \text{ for some } f \in \mathcal{F}; \end{cases}$$

$$\rho(q) = \begin{cases} q(x_1, \ldots, x_n) & \text{if } q \in \mathscr{L}_{\mathsf{K}}; \\ g_f(x_1, \ldots, x_n) & \text{if } q = h_{f_*} \text{ for some } f_* \in \mathcal{F}_*. \end{cases}$$

To this end, we will use without further notice the fact that $\mathsf{M} = \mathbb{S}(\mathsf{K}[\mathscr{L}_{\mathcal{F}}])$ and $\mathsf{M}_* = \mathbb{S}(\mathsf{K}[\mathscr{L}_{\mathcal{F}_*}]) = \mathsf{K}[\mathscr{L}_{\mathcal{F}_*}]$.

The definition of τ and ρ guarantees the validity of conditions (iii) and (iv) in the definition of faithful term equivalence. Therefore, it only remains to show that for all $\boldsymbol{A} \in \mathbb{S}(\mathsf{K}[\mathscr{L}_{\mathcal{F}}])$ and $\boldsymbol{B} \in \mathbb{S}(\mathsf{K}[\mathscr{L}_{\mathcal{F}_*}])$,

$$\rho(\boldsymbol{A}) \in \mathbb{S}(\mathsf{K}[\mathscr{L}_{\mathcal{F}_*}]) \ \text{ and } \ \tau(\boldsymbol{B}) \in \mathbb{S}(\mathsf{K}[\mathscr{L}_{\mathcal{F}}]). \tag{56}$$

We begin by showing that

$$\rho(\boldsymbol{A}) \in \mathsf{K}[\mathscr{L}_{\mathcal{F}_*}], \text{ for every } \boldsymbol{A} \in \mathsf{K}[\mathscr{L}_{\mathcal{F}}]. \tag{57}$$

Consider $\boldsymbol{A} \in \mathsf{K}[\mathscr{L}_{\mathcal{F}}]$. Then there exists $\boldsymbol{B} \in \mathsf{K}$ such that $\boldsymbol{A} = \boldsymbol{B}[\mathscr{L}_{\mathcal{F}}]$. We will prove that $f_*^{\boldsymbol{B}}$ is total for every $f_* \in \mathcal{F}_*$. Consider $f_* \in \mathcal{F}_*$. Then $f \in \mathcal{F}$ by the definition of $\mathcal{F}_*$. Since $\boldsymbol{B}[\mathscr{L}_{\mathcal{F}}]$ is well defined, the function $f^{\boldsymbol{B}}$ is total. As f_* interpolates f, this implies $f_*^{\boldsymbol{B}} = f^{\boldsymbol{B}}$, whence $f_*^{\boldsymbol{B}}$ is total as well. Consequently, the algebra $\boldsymbol{B}[\mathscr{L}_{\mathcal{F}_*}]$ is defined and belongs to $\mathsf{K}[\mathscr{L}_{\mathcal{F}_*}]$. Moreover, for every $f \in \mathcal{F}$ we have

$$g_f^{\boldsymbol{B}[\mathscr{L}_{\mathcal{F}}]} = f^{\boldsymbol{B}} = f_*^{\boldsymbol{B}} = h_{f_*}^{\boldsymbol{B}[\mathscr{L}_{\mathcal{F}_*}]}.$$

Together with the definition of ρ and the fact that $\boldsymbol{B}$ is the $\mathscr{L}_{\mathsf{K}}$-reduct of both $\boldsymbol{B}[\mathscr{L}_{\mathcal{F}}]$ and $\boldsymbol{B}[\mathscr{L}_{\mathcal{F}_*}]$, the above display guarantees that $\rho(\boldsymbol{A}) = \rho(\boldsymbol{B}[\mathscr{L}_{\mathcal{F}}]) = \boldsymbol{B}[\mathscr{L}_{\mathcal{F}_*}] \in \mathsf{K}[\mathscr{L}_{\mathcal{F}_*}]$, thus establishing (57).

Next we prove the left hand side of (56). Consider $\boldsymbol{A} \in \mathbb{S}(\mathsf{K}[\mathscr{L}_{\mathcal{F}}])$. Then $\boldsymbol{A} \leqslant \boldsymbol{A}'$ for some $\boldsymbol{A}' \in \mathsf{K}[\mathscr{L}_{\mathcal{F}}]$. By (57) we have $\rho(\boldsymbol{A}') \in \mathsf{K}[\mathscr{L}_{\mathcal{F}_*}]$. Moreover, from $\boldsymbol{A} \leqslant \boldsymbol{A}'$ it follows that $\rho(\boldsymbol{A}) \leqslant \rho(\boldsymbol{A}')$ (see Remark 11.12(ii)). Together with $\rho(\boldsymbol{A}') \in \mathsf{K}[\mathscr{L}_{\mathcal{F}_*}]$, this yields $\rho(\boldsymbol{A}) \in \mathbb{S}(\mathsf{K}[\mathscr{L}_{\mathcal{F}_*}])$, as desired.

It only remains to prove the right hand side of (56). Consider $\boldsymbol{B} \in \mathbb{S}(\mathsf{K}[\mathscr{L}_{\mathcal{F}_*}]) = \mathsf{K}[\mathscr{L}_{\mathcal{F}_*}]$. Then $\boldsymbol{B}{\restriction}_{\mathscr{L}_{\mathsf{K}}} \in \mathsf{K}$. As $\mathcal{F} \subseteq \mathsf{ext}(\mathsf{K})$ and K is a quasivariety by assumption, we can apply Theorem 8.4, obtaining some $\boldsymbol{C} \in \mathsf{K}$ with $\boldsymbol{B}{\restriction}_{\mathscr{L}_{\mathsf{K}}} \leqslant \boldsymbol{C}$ such that $f^{\boldsymbol{C}}$ is total for every $f \in \mathcal{F}$. Consequently, the algebra $\boldsymbol{C}[\mathscr{L}_{\mathcal{F}}]$ is defined and belongs to $\mathsf{K}[\mathscr{L}_{\mathcal{F}}]$. Therefore, $\rho(\boldsymbol{C}[\mathscr{L}_{\mathcal{F}}]) \in \mathsf{K}[\mathscr{L}_{\mathcal{F}_*}]$ by (57). Moreover,

$$\boldsymbol{B}{\restriction}_{\mathscr{L}_{\mathsf{K}}} \leqslant \boldsymbol{C} = \boldsymbol{C}[\mathscr{L}_{\mathcal{F}}]{\restriction}_{\mathscr{L}_{\mathsf{K}}} = \rho(\boldsymbol{C}[\mathscr{L}_{\mathcal{F}}]){\restriction}_{\mathscr{L}_{\mathsf{K}}},$$

where the last equality holds because ρ sends each n-ary $f \in \mathscr{L}_\mathsf{K}$ to $f(x_1, \ldots, x_n)$ by definition. Together with $\boldsymbol{B}, \rho(\boldsymbol{C}[\mathscr{L}_\mathcal{F}]) \in \mathsf{K}[\mathscr{L}_{\mathcal{F}_*}]$, the above display allows us to apply Proposition 9.5, obtaining $\boldsymbol{B} \leqslant \rho(\boldsymbol{C}[\mathscr{L}_\mathcal{F}])$ and, therefore, $\tau(\boldsymbol{B}) \leqslant \tau(\rho(\boldsymbol{C}[\mathscr{L}_\mathcal{F}]))$ (see Remark 11.12(ii)). Since the definition of τ and ρ ensures that $\boldsymbol{C}[\mathscr{L}_\mathcal{F}] = \tau(\rho(\boldsymbol{C}[\mathscr{L}_\mathcal{F}]))$, we obtain that $\tau(\boldsymbol{B}) \leqslant \boldsymbol{C}[\mathscr{L}_\mathcal{F}] \in \mathsf{K}[\mathscr{L}_\mathcal{F}]$, whence $\tau(\boldsymbol{B}) \in \mathbb{S}(\mathsf{K}[\mathscr{L}_\mathcal{F}])$. We conclude that M_* and M are faithfully term equivalent relative to K.

Since M_* is induced by $\mathcal{F}_* \subseteq \mathsf{ext}_{eq}(\mathsf{K})$ and $\mathscr{L}_{\mathcal{F}_*}$, we obtain that M is an equational pp expansion of K. Then Theorem 12.4 implies that M is congruence preserving.

It remains to prove that M has the amalgamation property. Consider a pair of embeddings $f \colon \boldsymbol{A} \to \boldsymbol{B}$ and $g \colon \boldsymbol{A} \to \boldsymbol{C}$ with $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C} \in \mathsf{M}$. As $\boldsymbol{B}, \boldsymbol{C} \in \mathsf{M} = \mathbb{S}(\mathsf{K}[\mathscr{L}_\mathcal{F}])$, there exist $\boldsymbol{B}', \boldsymbol{C}' \in \mathsf{K}[\mathscr{L}_\mathcal{F}]$ and a pair of embeddings $i_1 \colon \boldsymbol{B} \to \boldsymbol{B}'$ and $i_2 \colon \boldsymbol{C} \to \boldsymbol{C}'$. Let $f' \colon \boldsymbol{A} \to \boldsymbol{B}'$ and $g' \colon \boldsymbol{A} \to \boldsymbol{C}'$ be the embeddings defined as $f' = i_1 \circ f$ and $g' = i_2 \circ g$. Observe that f' and g' can be viewed as embeddings $f' \colon \boldsymbol{A}{\upharpoonright}_{\mathscr{L}_\mathsf{K}} \to \boldsymbol{B}'{\upharpoonright}_{\mathscr{L}_\mathsf{K}}$ and $g' \colon \boldsymbol{A}{\upharpoonright}_{\mathscr{L}_\mathsf{K}} \to \boldsymbol{C}'{\upharpoonright}_{\mathscr{L}_\mathsf{K}}$. Furthermore, from $\boldsymbol{A}, \boldsymbol{B}', \boldsymbol{C}' \in \mathsf{M}$ and Proposition 10.2 it follows that $\boldsymbol{A}{\upharpoonright}_{\mathscr{L}_\mathsf{K}}, \boldsymbol{B}'{\upharpoonright}_{\mathscr{L}_\mathsf{K}}, \boldsymbol{C}'{\upharpoonright}_{\mathscr{L}_\mathsf{K}} \in \mathsf{K}$. Therefore, the assumption that K has the amalgamation property guarantees the existence of some $\boldsymbol{D} \in \mathsf{K}$ and embeddings $h_f \colon \boldsymbol{B}'{\upharpoonright}_{\mathscr{L}_\mathsf{K}} \to \boldsymbol{D}$ and $h_g \colon \boldsymbol{C}'{\upharpoonright}_{\mathscr{L}_\mathsf{K}} \to \boldsymbol{D}$ such that $h_f \circ f' = h_g \circ g'$.

Recall that $\boldsymbol{D} \in \mathsf{K}$ and that $\mathsf{M} = \mathbb{S}(\mathsf{K}[\mathscr{L}_\mathcal{F}])$ is a pp expansion of K. Therefore, by Proposition 10.2 there exists $\boldsymbol{D}' \in \mathsf{K}[\mathscr{L}_\mathcal{F}]$ such that $\boldsymbol{D} \leqslant \boldsymbol{D}'{\upharpoonright}_{\mathscr{L}_\mathsf{K}}$. Let $h'_f \colon \boldsymbol{B}'{\upharpoonright}_{\mathscr{L}_\mathsf{K}} \to \boldsymbol{D}'{\upharpoonright}_{\mathscr{L}_\mathsf{K}}$ and $h'_g \colon \boldsymbol{C}'{\upharpoonright}_{\mathscr{L}_\mathsf{K}} \to \boldsymbol{D}'{\upharpoonright}_{\mathscr{L}_\mathsf{K}}$ be the embeddings obtained by composing h_f and h_g, respectively, with the inclusion map of $\boldsymbol{D}$ into $\boldsymbol{D}'{\upharpoonright}_{\mathscr{L}_\mathsf{K}}$. Since $\boldsymbol{B}', \boldsymbol{C}', \boldsymbol{D}' \in \mathsf{K}[\mathscr{L}_\mathcal{F}]$, Proposition 9.5 implies that $h'_f \colon \boldsymbol{B}' \to \boldsymbol{D}'$ and $h'_g \colon \boldsymbol{C}' \to \boldsymbol{D}'$ are also embeddings. Moreover, from $h_f \circ f' = h_g \circ g'$ it follows that $h'_f \circ f' = h'_g \circ g'$. Together with $f' = i_1 \circ f$, and $g' = i_2 \circ g$, this yields

$$h'_f \circ i_1 \circ f = h'_f \circ f' = h'_g \circ g' = h'_g \circ i_2 \circ g.$$

In view of the above display and of $\boldsymbol{D}' \in \mathsf{K}[\mathscr{L}_\mathcal{F}] \subseteq \mathsf{M}$, the pair of embeddings $h'_f \circ i_1 \colon \boldsymbol{B} \to \boldsymbol{D}'$ and $h'_g \circ i_2 \colon \boldsymbol{C} \to \boldsymbol{D}'$ establishes the amalgamation property for M.

$\boxtimes$

Next we prove Theorem 12.9.

Proof. We begin by proving that $\mathsf{M}_{\mathrm{RFSI}} = \{\boldsymbol{A} \in \mathsf{M} : \boldsymbol{A}{\restriction}_{\mathscr{L}_\mathsf{K}} \in \mathsf{K}_{\mathrm{RFSI}}\}$, namely, that for every $\boldsymbol{A} \in \mathsf{M}$,

$$\boldsymbol{A} \in \mathsf{M}_{\mathrm{RFSI}} \iff \boldsymbol{A}{\restriction}_{\mathscr{L}_\mathsf{K}} \in \mathsf{K}_{\mathrm{RFSI}}. \tag{58}$$

To this end, consider $\boldsymbol{A} \in \mathsf{M}$ and observe that $\boldsymbol{A}{\restriction}_{\mathscr{L}_\mathsf{K}} \in \mathsf{K}$ by Proposition 10.2. From Proposition 2.10 it follows that

$$\boldsymbol{A} \in \mathsf{M}_{\mathrm{RFSI}} \iff \mathrm{id}_A \in \mathsf{Irr}_\mathsf{M}(\boldsymbol{A});$$
$$\boldsymbol{A}{\restriction}_{\mathscr{L}_\mathsf{K}} \in \mathsf{K}_{\mathrm{RFSI}} \iff \mathrm{id}_A \in \mathsf{Irr}_\mathsf{K}(\boldsymbol{A}{\restriction}_{\mathscr{L}_\mathsf{K}}).$$

Observe that $\mathsf{Con}_\mathsf{M}(\boldsymbol{A}) = \mathsf{Con}_\mathsf{K}(\boldsymbol{A}{\restriction}_{\mathscr{L}_\mathsf{K}})$ because M is a congruence preserving pp expansion of K by assumption. Therefore, $\mathsf{Irr}_\mathsf{M}(\boldsymbol{A}) = \mathsf{Irr}_\mathsf{K}(\boldsymbol{A}{\restriction}_{\mathscr{L}_\mathsf{K}})$. Together with the above display, this establishes (58), as desired. The proof that $\mathsf{M}_{\mathrm{RSI}} = \{\boldsymbol{A} \in \mathsf{M} : \boldsymbol{A}{\restriction}_{\mathscr{L}_\mathsf{K}} \in \mathsf{K}_{\mathrm{RSI}}\}$ is analogous and, therefore, omitted.

It only remains to prove the last part of the statement. Suppose that K is a variety. From Theorem 10.3(ii) it follows that M is a quasivariety. Therefore, it remains to show that M is closed under homomorphic images. By Proposition 2.5 it suffices to prove that $\mathsf{Con}(\boldsymbol{A}) = \mathsf{Con}_\mathsf{M}(\boldsymbol{A})$ for every $\boldsymbol{A} \in \mathsf{M}$. To this end, observe that for every $\boldsymbol{A} \in \mathsf{M}$,

$$\mathsf{Con}(\boldsymbol{A}) \subseteq \mathsf{Con}(\boldsymbol{A}{\restriction}_{\mathscr{L}_\mathsf{K}}) = \mathsf{Con}_\mathsf{K}(\boldsymbol{A}{\restriction}_{\mathscr{L}_\mathsf{K}}) = \mathsf{Con}_\mathsf{M}(\boldsymbol{A}),$$

where the first step holds by Remark 12.2, the second follows from $\boldsymbol{A}{\restriction}_{\mathscr{L}_\mathsf{K}} \in \mathsf{K}$ (see Proposition 10.2) and the assumption that K is a variety, and the third from $\boldsymbol{A} \in \mathsf{M}$ and the assumption that M is a congruence preserving pp expansion of K. Thus, $\mathsf{Con}(\boldsymbol{A}) = \mathsf{Con}_\mathsf{M}(\boldsymbol{A})$, as desired. $\boxtimes$

Our next goal is to prove Theorem 12.10. To this end, it is convenient to establish the following technical observation first.

Proposition 12.15. *Let K be a relatively congruence distributive quasivariety for which $\mathsf{K}_{\mathrm{RFSI}}$ is closed under nontrivial subalgebras. Moreover, consider $\boldsymbol{A} \in \mathsf{K}$ and $\boldsymbol{B} \leqslant \boldsymbol{A} \times \boldsymbol{A}$ with projection maps $p_1, p_2 \colon \boldsymbol{B} \to \boldsymbol{A}$ such that for every $\langle a, b \rangle \in B$ we have $\langle a, a \rangle, \langle b, b \rangle \in B$. Assume that K has the strong epimorphism surjectivity property. Then $B = \mathsf{Cg}_\mathsf{K}^{\boldsymbol{A}}(B) \cap (p_1[B] \times p_2[B])$.*

Proof. As the inclusion $B \subseteq \mathsf{Cg}_\mathsf{K}^{\boldsymbol{A}}(B) \cap (p_1[B] \times p_2[B])$ always holds, we detail the proof of the reverse inclusion. Consider $\langle a, b \rangle \in \mathsf{Cg}_\mathsf{K}^{\boldsymbol{A}}(B) \cap (p_1[B] \times p_2[B])$. We need to prove that $\langle a, b \rangle \in B$. Suppose the contrary, with a view to contradiction.

From Proposition 2.8 it follows that $\mathsf{Cg}_\mathsf{K}^{\boldsymbol{A}}(B)$ is the universe of a member $\mathsf{Cg}_\mathsf{K}^{\boldsymbol{A}}(B)^*$ of K such that $\mathsf{Cg}_\mathsf{K}^{\boldsymbol{A}}(B)^* \leqslant \boldsymbol{A} \times \boldsymbol{A}$ is a subdirect product. Together with $B \subseteq \mathsf{Cg}_\mathsf{K}^{\boldsymbol{A}}(B)$ and the assumption that $\boldsymbol{B} \leqslant \boldsymbol{A} \times \boldsymbol{A}$, this yields $\boldsymbol{B} \leqslant \mathsf{Cg}_\mathsf{K}^{\boldsymbol{A}}(B)^*$. As $\boldsymbol{B} \leqslant \mathsf{Cg}_\mathsf{K}^{\boldsymbol{A}}(B)^* \in \mathsf{K}$ and $\langle a, b \rangle \in \mathsf{Cg}_\mathsf{K}^{\boldsymbol{A}}(B) - B$, we can apply the assumption that K has the strong epimorphism surjectivity property, obtaining a pair of homomorphisms $g, h \colon \mathsf{Cg}_\mathsf{K}^{\boldsymbol{A}}(B)^* \to \boldsymbol{C}$ with $\boldsymbol{C} \in \mathsf{K}$ such that

$$g{\restriction}_B = h{\restriction}_B \quad \text{and} \quad g(\langle a, b \rangle) \neq h(\langle a, b \rangle). \tag{59}$$

In view of Remark 6.2, we may further assume that $\boldsymbol{C} \in \mathsf{K}_{\mathrm{RFSI}}$.

Since $\langle a, b \rangle \in p_1[B] \times p_2[B]$ by assumption, there exist $c, d \in A$ such that $\langle a, c \rangle, \langle d, b \rangle \in B$. By the assumptions this yields $\langle a, a \rangle, \langle b, b \rangle \in B$. Consequently, the left hand side of (59) guarantees that $g(\langle a, a \rangle) = h(\langle a, a \rangle)$ and $g(\langle b, b \rangle) = h(\langle b, b \rangle)$. By the right hand side of (59) this implies that

$$\text{either } g(\langle a, b \rangle) \neq g(\langle a, a \rangle) \text{ or } h(\langle a, b \rangle) \neq h(\langle a, a \rangle) \tag{60}$$

and

$$\text{either } g(\langle a, b \rangle) \neq g(\langle b, b \rangle) \text{ or } h(\langle a, b \rangle) \neq h(\langle b, b \rangle). \tag{61}$$

We rely on the next observation.

Claim 12.16. *For every $\phi \in \{\mathsf{Ker}(g), \mathsf{Ker}(h)\}$ there exists $\eta \in \mathsf{Con}_{\mathsf{K}}(\boldsymbol{A})$ such that*

$$\phi \in \{(\eta \times A^2)\!\restriction_{\mathsf{Cg}_{\mathsf{K}}^{\boldsymbol{A}}(B)}, (A^2 \times \eta)\!\restriction_{\mathsf{Cg}_{\mathsf{K}}^{\boldsymbol{A}}(B)}\}.$$

Proof of the Claim. By symmetry it suffices to show that the statement holds for the case where $\phi = \mathsf{Ker}(g)$. Since $g \colon \mathsf{Cg}_{\mathsf{K}}^{\boldsymbol{A}}(B)^* \to \boldsymbol{C}$ is a homomorphism with $\boldsymbol{C} \in \mathsf{K}$, we have $\mathsf{Ker}(g) \in \mathsf{Con}_{\mathsf{K}}(\mathsf{Cg}_{\mathsf{K}}^{\boldsymbol{A}}(B)^*)$. Moreover, recall that $\boldsymbol{C} \in \mathsf{K}_{\mathrm{RFSI}}$ and that $\mathsf{K}_{\mathrm{RFSI}}$ is closed under nontrivial subalgebras by assumption. Therefore, from $\mathsf{Cg}_{\mathsf{K}}^{\boldsymbol{A}}(B)^*/\mathsf{Ker}(g) \in \mathbb{IS}(\boldsymbol{C})$ it follows that $\mathsf{Cg}_{\mathsf{K}}^{\boldsymbol{A}}(B)^*/\mathsf{Ker}(g)$ is either trivial or belongs to $\mathsf{K}_{\mathrm{RFSI}}$. Lastly, recall that $\mathsf{Cg}_{\mathsf{K}}^{\boldsymbol{A}}(B)^* \leqslant \boldsymbol{A} \times \boldsymbol{A}$ is a subdirect product with $\boldsymbol{A} \in \mathsf{K}$. As K is a relatively congruence distributive quasivariety by assumption, we can apply Corollary 2.18, obtaining the desired conclusion. $\boxtimes$

By symmetry and Claim 12.16 we may assume that there exist $\eta_1, \eta_2 \in \mathsf{Con}_{\mathsf{K}}(\boldsymbol{A})$ such that $\mathsf{Ker}(g) = (\eta_1 \times A^2)\!\restriction_{\mathsf{Cg}_{\mathsf{K}}^{\boldsymbol{A}}(B)}$ and $\mathsf{Ker}(h) \in \{(\eta_2 \times A^2)\!\restriction_{\mathsf{Cg}_{\mathsf{K}}^{\boldsymbol{A}}(B)}, (A^2 \times \eta_2)\!\restriction_{\mathsf{Cg}_{\mathsf{K}}^{\boldsymbol{A}}(B)}\}$. We have two cases depending on whether $\mathsf{Ker}(h)$ is $(\eta_2 \times A^2)\!\restriction_{\mathsf{Cg}_{\mathsf{K}}^{\boldsymbol{A}}(B)}$ or $(A^2 \times \eta_2)\!\restriction_{\mathsf{Cg}_{\mathsf{K}}^{\boldsymbol{A}}(B)}$.

First, suppose that $\mathsf{Ker}(h) = (\eta_2 \times A^2)\!\restriction_{\mathsf{Cg}_{\mathsf{K}}^{\boldsymbol{A}}(B)}$. Then

$$\mathsf{Ker}(g) = (\eta_1 \times A^2)\!\restriction_{\mathsf{Cg}_{\mathsf{K}}^{\boldsymbol{A}}(B)} \text{ and } \mathsf{Ker}(h) = (\eta_2 \times A^2)\!\restriction_{\mathsf{Cg}_{\mathsf{K}}^{\boldsymbol{A}}(B)}. \tag{62}$$

Recall that $\langle a, b \rangle \in \mathsf{Cg}_{\mathsf{K}}^{\boldsymbol{A}}(B)$ and, therefore, $a, b \in A$. Together with $\eta_1, \eta_2 \in \mathsf{Con}(\boldsymbol{A})$, this implies $\langle a, a \rangle \in \eta_1 \cap \eta_2$ and $\langle a, b \rangle \in A^2$. Thus, $\langle \langle a, a \rangle, \langle a, b \rangle \rangle \in (\eta_1 \times A^2) \cap (\eta_2 \times A^2)$. Since $\langle a, a \rangle, \langle a, b \rangle \in \mathsf{Cg}_{\mathsf{K}}^{\boldsymbol{A}}(B)$, we obtain

$$\langle \langle a, a \rangle, \langle a, b \rangle \rangle \in ((\eta_1 \times A^2) \cap (\eta_2 \times A^2))\!\restriction_{\mathsf{Cg}_{\mathsf{K}}^{\boldsymbol{A}}(B)} = (\eta_1 \times A^2)_{\mathsf{Cg}_{\mathsf{K}}^{\boldsymbol{A}}(B)} \cap (\eta_2 \times A^2)_{\mathsf{Cg}_{\mathsf{K}}^{\boldsymbol{A}}(B)}.$$

In view of (62), this amounts to $\langle \langle a, a \rangle, \langle a, b \rangle \rangle \in \mathsf{Ker}(g) \cap \mathsf{Ker}(h)$, that is,

$$g(\langle a, a \rangle) = g(\langle a, b \rangle) \text{ and } h(\langle a, a \rangle) = h(\langle a, b \rangle),$$

a contradiction with (60).

It only remains to consider the case where $\mathsf{Ker}(h) = (A^2 \times \eta_2)\!\restriction_{\mathsf{Cg}_{\mathsf{K}}^{\boldsymbol{A}}(B)}$. In this case, we have

$$\mathsf{Ker}(g) = (\eta_1 \times A^2)\!\restriction_{\mathsf{Cg}_{\mathsf{K}}^{\boldsymbol{A}}(B)} \text{ and } \mathsf{Ker}(h) = (A^2 \times \eta_2)\!\restriction_{\mathsf{Cg}_{\mathsf{K}}^{\boldsymbol{A}}(B)}. \tag{63}$$

We rely on the following observation.

Claim 12.17. *We have $\mathsf{Cg}_{\mathsf{K}}^{\boldsymbol{A}}(B) \subseteq \eta_1$.*

Proof of the Claim. As $\eta_1 \in \mathsf{Con}_\mathsf{K}(\boldsymbol{A})$, it will be enough to show that $B \subseteq \eta_1$. To this end, consider $\langle p, q \rangle \in B$. By assumption we also have $\langle q, q \rangle \in B$. Since $\boldsymbol{B} \leqslant \boldsymbol{A} \times \boldsymbol{A}$ by assumption, we have $p, q \in A$. Together with $\eta_2 \in \mathsf{Con}(\boldsymbol{A})$, this yields $\langle q, q \rangle \in \eta_2$ and, therefore, $\langle \langle p, q \rangle, \langle q, q \rangle \rangle \in (A^2 \times \eta_2)$. As $\langle p, q \rangle \in B \subseteq \mathsf{Cg}^{\boldsymbol{A}}_\mathsf{K}(B)$ and $\langle q, q \rangle \in \mathsf{Cg}^{\boldsymbol{A}}_\mathsf{K}(B)$ (the latter because $\mathsf{Cg}^{\boldsymbol{A}}_\mathsf{K}(B)$ is a congruence of $\boldsymbol{A}$), we obtain

$$\langle \langle p, q \rangle, \langle q, q \rangle \rangle \in (A^2 \times \eta_2){\restriction}_{\mathsf{Cg}^{\boldsymbol{A}}_\mathsf{K}(B)} = \mathsf{Ker}(h),$$

where the last equality holds by the right hand side of (63). The left hand side of (59) implies $\mathsf{Ker}(g){\restriction}_B = \mathsf{Ker}(h){\restriction}_B$. Together with the above display and $\langle p, q \rangle, \langle q, q \rangle \in B$, this implies $\langle \langle p, q \rangle, \langle q, q \rangle \rangle \in \mathsf{Ker}(g)$. Since $\mathsf{Ker}(g) = (\eta_1 \times A^2){\restriction}_{\mathsf{Cg}^{\boldsymbol{A}}_\mathsf{K}(B)}$ by the left hand side of (63), we conclude that $\langle p, q \rangle \in \eta_1$. ⊠

Now, recall that $\langle a, b \rangle \in \mathsf{Cg}^{\boldsymbol{A}}_\mathsf{K}(B)$. By Claim 12.17 we obtain $\langle a, b \rangle \in \eta_1$. Since $b \in A$, this implies $\langle \langle a, b \rangle, \langle b, b \rangle \rangle \in (\eta_1 \times A^2)$. On the other hand, from $a, b \in A$ and $\eta_2 \in \mathsf{Con}(\boldsymbol{A})$ it follows that $\langle \langle a, b \rangle, \langle b, b \rangle \rangle \in (A^2 \times \eta_2)$. As $\langle a, b \rangle, \langle b, b \rangle \in \mathsf{Cg}^{\boldsymbol{A}}_\mathsf{K}(B)$, we obtain

$$\langle \langle a, b \rangle, \langle b, b \rangle \rangle \in (\eta_1 \times A^2){\restriction}_{\mathsf{Cg}^{\boldsymbol{A}}_\mathsf{K}(B)} \quad \text{and} \quad \langle \langle a, b \rangle, \langle b, b \rangle \rangle \in (A^2 \times \eta_2){\restriction}_{\mathsf{Cg}^{\boldsymbol{A}}_\mathsf{K}(B)}.$$

By (63) this amounts to $\langle \langle a, b \rangle, \langle b, b \rangle \rangle \in \mathsf{Ker}(g) \cap \mathsf{Ker}(h)$, that is,

$$g(\langle a, b \rangle) = g(\langle b, b \rangle) \quad \text{and} \quad h(\langle a, b \rangle) = h(\langle b, b \rangle),$$

a contradiction with (61). ⊠

We will also make use of the next observation from [29, Thm. 6.1].

Theorem 12.18. *Let* K *be a relatively congruence distributive quasivariety for which* $\mathsf{K}_{\mathrm{RFSI}}$ *is closed under nontrivial subalgebras. If* K *has the weak epimorphism surjectivity property, then the variety* $\mathbb{V}(\mathsf{K})$ *is arithmetical.*

As a last step before proving Theorem 12.10, we establish the following result on the strong epimorphism surjectivity property.

Theorem 12.19. *Let* K *be a relatively congruence distributive quasivariety for which* $\mathsf{K}_{\mathrm{RFSI}}$ *is closed under nontrivial subalgebras. If* K *has the strong epimorphism surjectivity property, then* K *is an arithmetical variety with the congruence extension property.*

Proof. Suppose that K has the strong epimorphism surjectivity property.

We begin by showing that K is a variety. As K is a quasivariety, by Proposition 2.5 it suffices to show that $\mathsf{Con}(\boldsymbol{A}) = \mathsf{Con}_\mathsf{K}(\boldsymbol{A})$ for every $\boldsymbol{A} \in \mathsf{K}$. To this end, consider $\boldsymbol{A} \in \mathsf{K}$ and $\theta \in \mathsf{Con}(\boldsymbol{A})$. We will show that $\theta = \mathsf{Cg}^{\boldsymbol{A}}_\mathsf{K}(\theta)$ which, in turns, implies $\theta \in \mathsf{Con}_\mathsf{K}(\boldsymbol{A})$, as desired. In view of Proposition 2.8, the congruence θ is the universe of a subalgebra $\boldsymbol{B} \leqslant \boldsymbol{A} \times \boldsymbol{A}$. Furthermore, for every $\langle a, b \rangle \in B = \theta$ we have $\langle a, a \rangle, \langle b, b \rangle \in \theta = B$ because θ is a congruence of $\boldsymbol{A}$. Therefore, we can apply Proposition 12.15, obtaining

$$\theta = B = \mathsf{Cg}^{\boldsymbol{A}}_\mathsf{K}(B) \cap (p_1[B] \times p_2[B]) = \mathsf{Cg}^{\boldsymbol{A}}_\mathsf{K}(\theta) \cap (p_1[\theta] \times p_2[\theta]).$$

Since θ is a reflexive relation on $\boldsymbol{A}$, we have $p_1[\theta] \times p_2[\theta] = A \times A$. Therefore, the above display yields $\theta = \mathsf{Cg}^{\boldsymbol{A}}_\mathsf{K}(\theta) \cap (A \times A) = \mathsf{Cg}^{\boldsymbol{A}}_\mathsf{K}(\theta)$. It follows that $\theta \in \mathsf{Con}_\mathsf{K}(\boldsymbol{A})$. Hence, we conclude that K is a variety.

Next we prove that the variety K has the congruence extension property. It will be enough to show that $\theta = \mathsf{Cg}^{\boldsymbol{A}}(\theta){\restriction}_C$ for all $\boldsymbol{C} \leqslant \boldsymbol{A} \in \mathsf{K}$ and $\theta \in \mathsf{Con}(\boldsymbol{C})$. Accordingly, consider $\boldsymbol{C} \leqslant \boldsymbol{A} \in \mathsf{K}$ and $\theta \in \mathsf{Con}(\boldsymbol{C})$. In view of Proposition 2.8, the congruence θ is the universe of a subalgebra $\boldsymbol{B} \leqslant \boldsymbol{C} \times \boldsymbol{C}$. Furthermore, for every $\langle a, b \rangle \in B = \theta$ we have $\langle a, a \rangle, \langle b, b \rangle \in \theta = B$ because θ is a congruence of $\boldsymbol{C}$. As $\boldsymbol{C} \leqslant \boldsymbol{A}$, we have $\boldsymbol{C} \times \boldsymbol{C} \leqslant \boldsymbol{A} \times \boldsymbol{A}$, whence $\boldsymbol{B} \leqslant \boldsymbol{A} \times \boldsymbol{A}$. Therefore, we can apply Proposition 12.15, obtaining

$$\theta = B = \mathsf{Cg}_{\mathsf{K}}^{\boldsymbol{A}}(B) \cap (p_1[B] \times p_2[B]) = \mathsf{Cg}_{\mathsf{K}}^{\boldsymbol{A}}(\theta) \cap (p_1[\theta] \times p_2[\theta]).$$

Since θ is a reflexive relation on $\boldsymbol{C}$, we have $p_1[\theta] \times p_2[\theta] = C \times C$. Therefore, the above display yields $\theta = \mathsf{Cg}_{\mathsf{K}}^{\boldsymbol{A}}(\theta) \cap (C \times C) = \mathsf{Cg}^{\boldsymbol{A}}(\theta){\restriction}_C$. Thus, we conclude that K has the congruence extension property.

It only remains to prove that the variety K is arithmetical. As K has the strong epimorphism surjectivity property by assumption, it has the weak epimorphism surjectivity property as well. Moreover, $\mathsf{K} = \mathbb{V}(\mathsf{K})$ because K is a variety. Therefore, from Theorem 12.18 it follows that K is arithmetical. ⊠

We are now ready to prove Theorem 12.10.

Proof. Let M be a congruence preserving Beth companion of K. Since M is a pp expansion of K and K is a quasivariety by assumption, we obtain that M is also a quasivariety (see Theorem 10.3(ii)). In addition, as M is a Beth companion of K and K is a quasivariety, M has the strong epimorphism surjectivity property by Theorem 11.6. From the assumption that K is relatively congruence distributive and Theorem 12.8 it follows that M is relatively congruence distributive as well. Lastly, we will prove that M_{RFSI} is closed under nontrivial subalgebras. Consider $\boldsymbol{A} \leqslant \boldsymbol{B} \in \mathsf{M}_{\text{RFSI}}$ with $\boldsymbol{A}$ nontrivial. By Theorem 12.9 we have $\boldsymbol{B}{\restriction}_{\mathscr{L}_{\mathsf{K}}} \in \mathsf{K}_{\text{RFSI}}$. Furthermore, $\boldsymbol{A} \leqslant \boldsymbol{B}$ implies $\boldsymbol{A}{\restriction}_{\mathscr{L}_{\mathsf{K}}} \leqslant \boldsymbol{B}{\restriction}_{\mathscr{L}_{\mathsf{K}}}$. As $\boldsymbol{A}$ is nontrivial, so is $\boldsymbol{A}{\restriction}_{\mathscr{L}_{\mathsf{K}}}$. Therefore, $\boldsymbol{A}{\restriction}_{\mathscr{L}_{\mathsf{K}}} \leqslant \boldsymbol{B}{\restriction}_{\mathscr{L}_{\mathsf{K}}} \in \mathsf{K}_{\text{RFSI}}$ and the assumption that K_{RFSI} is closed under nontrivial subalgebras guarantee that $\boldsymbol{A}{\restriction}_{\mathscr{L}_{\mathsf{K}}} \in \mathsf{K}_{\text{RFSI}}$. Together with $\boldsymbol{A} \in \mathsf{M}$, this allows us to apply Theorem 12.9, obtaining $\boldsymbol{A} \in \mathsf{M}_{\text{RFSI}}$. Hence, we conclude that M_{RFSI} is closed under nontrivial subalgebras.

Therefore, M is a quasivariety with the strong epimorphism surjectivity property that, moreover, is relatively congruence distributive and such that M_{RFSI} is closed under nontrivial subalgebras. Thus, from Theorem 12.19 it follows that M is an arithmetical variety with the congruence extension property. Since M is a variety, we obtain $\mathsf{M}_{\text{FSI}} = \mathsf{M}_{\text{RFSI}}$. As M_{RFSI} is closed under nontrivial subalgebras, we conclude that so is M_{FSI}. ⊠

Our last goal is to prove Corollary 12.11. To this end, we rely on the following consequence of Theorem 12.19 which, in the context of varieties, originates in [10, Cor. 3(i)] (see also [29, Example 6.5]).

Corollary 12.20. *Every relatively filtral quasivariety with the strong epimorphism surjectivity property is a discriminator variety.*

Proof. Let K be a relatively filtral quasivariety with the strong epimorphism surjectivity property. As K is relatively filtral, it is relatively congruence distributive and K_{RFSI} is closed under nontrivial subalgebras (see, e.g., [24, Cor. 6.5(i, iv)]). Therefore, from Theorem 12.19

it follows that K is an arithmetical variety. Since K is a relatively filtral quasivariety, this yields that K is a congruence permutable filtral variety. As congruence permutable filtral varieties coincide with discriminator varieties (see, e.g., [18, 52]), we conclude that K is a discriminator variety. ⊠

Furthermore, we recall that a join semilattice $\langle A; \vee \rangle$ is said to be *dually Brouwerian* when for all $a, b \in A$ there exists the smallest $c \in A$ such that $a \leqslant b \vee c$. Moreover, an element a of a complete lattice $\mathbf{A}$ is *compact* when for every $X \subseteq A$ such that $a \leqslant \bigvee X$ there exists a finite $Y \subseteq X$ such that $a \leqslant \bigvee Y$. With every quasivariety K and algebra $\mathbf{A}$ we associate a join semilattice $\mathsf{Comp}_\mathsf{K}(\mathbf{A})$ whose universe is the set of compact elements of $\mathsf{Con}_\mathsf{K}(\mathbf{A})$ and whose join operation $+$ is defined by the rule $\theta + \phi = \mathsf{Cg}_\mathsf{K}^{\mathbf{A}}(\theta \cup \phi)$. Lastly, a member $\mathbf{A}$ of a quasivariety K is said to be *simple relative to* K when $\mathsf{Con}_\mathsf{K}(\mathbf{A})$ has exactly two elements, and we say that K is *relatively semisimple* when every member of $\mathsf{K}_{\mathrm{RSI}}$ is simple relative to K. When K is a variety, we simply say that $\mathbf{A}$ is *simple* and K *semisimple*. We will rely on the fact that a quasivariety K is relatively filtral if and only if it is relatively semisimple and $\mathsf{Comp}_\mathsf{K}(\mathbf{A})$ is dually Brouwerian for every $\mathbf{A} \in \mathsf{K}$ (see [24, Thm. 6.3] and [78, Thms. 5 & 8]).[9] Bearing this in mind, we will now prove Corollary 12.11.

Proof. Let M be a Beth companion of a relatively filtral quasivariety K. Since every relatively filtral quasivariety has the relative congruence extension property (see, e.g., [24, Cor. 6.5(i)]), so does K. As M is a pp expansion of K, we conclude that M is congruence preserving by Theorem 12.4(ii).

We will show that M is also a relatively filtral quasivariety. The fact that M is a quasivariety is a consequence of Theorem 10.3(ii) and the assumption that M is a pp expansion of the quasivariety K. Therefore, it only remains to prove that M is relatively filtral, i.e., that it is relatively semisimple and $\mathsf{Comp}_\mathsf{M}(\mathbf{A})$ is dually Brouwerian for every $\mathbf{A} \in \mathsf{M}$. To show that M is relatively semisimple, consider $\mathbf{A} \in \mathsf{M}_{\mathrm{RSI}}$. Since M is a congruence preserving pp expansion of K, we can apply Theorem 12.9, obtaining $\mathbf{A}{\restriction}_{\mathscr{L}_\mathsf{K}} \in \mathsf{K}_{\mathrm{RSI}}$. As the quasivariety K is relatively semisimple (because it is relatively filtral by assumption), we obtain that $\mathsf{Con}_\mathsf{K}(\mathbf{A}{\restriction}_{\mathscr{L}_\mathsf{K}})$ has exactly two elements. Moreover, $\mathsf{Con}_\mathsf{M}(\mathbf{A}) = \mathsf{Con}_\mathsf{K}(\mathbf{A}{\restriction}_{\mathscr{L}_\mathsf{K}})$ because $\mathbf{A} \in \mathsf{M}$ and M is a congruence preserving pp expansion of K. Therefore, $\mathsf{Con}_\mathsf{M}(\mathbf{A})$ has exactly two elements, whence $\mathbf{A}$ is simple relative to M. Hence, we conclude that M is relatively semisimple, as desired. Next we prove that $\mathsf{Comp}_\mathsf{M}(\mathbf{A})$ is dually Brouwerian for every $\mathbf{A} \in \mathsf{M}$. Consider $\mathbf{A} \in \mathsf{M}$ and observe that $\mathbf{A}{\restriction}_{\mathscr{L}_\mathsf{K}} \in \mathsf{K}$ by Proposition 10.2. Since M is congruence preserving, we have $\mathsf{Con}_\mathsf{M}(\mathbf{A}) = \mathsf{Con}_\mathsf{K}(\mathbf{A}{\restriction}_{\mathscr{L}_\mathsf{K}})$, whence $\mathsf{Comp}_\mathsf{M}(\mathbf{A}) = \mathsf{Comp}_\mathsf{K}(\mathbf{A}{\restriction}_{\mathscr{L}_\mathsf{K}})$. As $\mathsf{Comp}_\mathsf{K}(\mathbf{A}{\restriction}_{\mathscr{L}_\mathsf{K}})$ is dually Brouwerian (because K is relatively filtral and $\mathbf{A}{\restriction}_{\mathscr{L}_\mathsf{K}} \in \mathsf{K}$), we conclude that so is $\mathsf{Comp}_\mathsf{M}(\mathbf{A})$. Thus, we conclude that M is a relatively filtral quasivariety.

Lastly, observe that M has the strong epimorphism surjectivity property because it is a Beth companion of the quasivariety K (see Theorem 11.6). Thus, M is a relatively filtral quasivariety with the strong epimorphism surjectivity property. By Corollary 12.20 we conclude that M is a discriminator variety. ⊠

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[9]This description of filtrality originated in the context of varieties (see [51, 52]) and was later extended to quasivarieties in [24].

13. Absolutely closed and primal algebras

The aim of this section is to provide additional criteria to establish whether a pp expansion of a quasivariety is a Beth companion. The two main results are Theorems 13.3 and 13.25 involving absolutely closed algebras and primal algebras, respectively. We will employ these results to determine the Beth companions of torsion-free Abelian groups (Example 13.6), Abelian ℓ-groups (Example 13.13), MV-algebras (Example 13.17), and varieties of MV-algebras generated by a finite chain (Example 13.26).

We begin by recalling the definition of an absolutely closed algebra (see [71, p. 236]).

Definition 13.1. Let K be a class of algebras. A member $\boldsymbol{A}$ of K is called *absolutely closed* in K when $\mathsf{d}_\mathsf{K}(\boldsymbol{A}, \boldsymbol{B}) = A$ for every $\boldsymbol{B} \in \mathsf{K}$ such that $\boldsymbol{A} \leqslant \boldsymbol{B}$. The class of absolutely closed members of K will be denoted by K_{AC}.

On the one hand, absolutely closed algebras are related to the reducts of the members of Beth companions, as the following result states.

Theorem 13.2. *Let K be a quasivariety with a Beth companion $\mathsf{M} = \mathbb{S}(\mathsf{K}[\mathscr{L}_\mathcal{F}])$. Then*

$$\mathsf{K}[\mathscr{L}_\mathcal{F}]{\restriction}_{\mathscr{L}_\mathsf{K}} \subseteq \mathsf{K}_{\mathrm{AC}} \subseteq \mathsf{M}{\restriction}_{\mathscr{L}_\mathsf{K}}.$$

Moreover, if M is an equational Beth companion of K, then $\mathsf{M}{\restriction}_{\mathscr{L}_\mathsf{K}} = \mathsf{K}_{\mathrm{AC}}$.

On the other hand, the next result shows that absolutely closed algebras provide a sufficient condition for a pp expansion of a quasivariety to be a Beth companion.

Theorem 13.3. *Let M be a pp expansion of a quasivariety K such that $\mathsf{M}{\restriction}_{\mathscr{L}_\mathsf{K}} \subseteq \mathsf{K}_{\mathrm{AC}}$. Then M is a Beth companion of K.*

We postpone the proofs of Theorems 13.2 and 13.3 and proceed to describe some of their applications. To this end, it is convenient to introduce injective algebras and absolute retracts (see, e.g., [76, pp. 80–81]) and relate them to absolutely closed algebras.

Definition 13.4. A member $\boldsymbol{A}$ of a quasivariety K is said to be:

(i) *injective in* K when for all $\boldsymbol{B}, \boldsymbol{C} \in \mathsf{K}$ such that $\boldsymbol{B} \leqslant \boldsymbol{C}$ and homomorphism $f \colon \boldsymbol{B} \to \boldsymbol{A}$ there exists a homomorphism $g \colon \boldsymbol{C} \to \boldsymbol{A}$ such that $g{\restriction}_B = f$;

$$
\begin{array}{ccc}
\boldsymbol{B} & \lhook\joinrel\longrightarrow & \boldsymbol{C} \\
{\scriptstyle f}\downarrow & {\scriptstyle g}\nearrow & \\
\boldsymbol{A} & &
\end{array}
$$

(ii) an *absolute retract in* K when for every $\boldsymbol{B} \in \mathsf{K}$ such that $\boldsymbol{A} \leqslant \boldsymbol{B}$ there exists a homomorphism $g \colon \boldsymbol{B} \to \boldsymbol{A}$ such that $g{\restriction}_A$ is the identity map on A.

$$
\begin{array}{ccc}
\boldsymbol{A} & \lhook\joinrel\longrightarrow & \boldsymbol{B} \\
{\scriptstyle id}\downarrow & {\scriptstyle g}\nearrow & \\
\boldsymbol{A} & &
\end{array}
$$

Proposition 13.5. *The following conditions hold for a class of algebras K and $\boldsymbol{A} \in \mathsf{K}$:*

(i) *if $\boldsymbol{A}$ is injective in K, then it is an absolute retract in K;*

(ii) *if $\boldsymbol{A}$ is an absolute retract in K, then it is absolutely closed in K.*

Proof. (i): See, e.g., [76, Prop. 1.1].

(ii): Suppose that $\boldsymbol{A}$ is an absolute retract in K and consider $\boldsymbol{B} \in \mathsf{K}$ such that $\boldsymbol{A} \leqslant \boldsymbol{B}$. Since $\boldsymbol{A}$ is an absolute retract in K, there exists a homomorphism $g \colon \boldsymbol{B} \to \boldsymbol{A}$ such that $g{\restriction}_A$ is the identity on A. Let $h = i \circ g$, where i is the inclusion map of $\boldsymbol{A}$ into $\boldsymbol{B}$. Then $h \colon \boldsymbol{B} \to \boldsymbol{B}$ is a homomorphism such that $h(a) = g(a) = a$ for every $a \in A$. Consider the identity map $id \colon \boldsymbol{B} \to \boldsymbol{B}$. As $h(a) = a = id(a)$ for every $a \in A$, we have $h{\restriction}_A = id{\restriction}_A$. Then $h(b) = id(b) = b$ for every $b \in \mathsf{d}_\mathsf{K}(\boldsymbol{A}, \boldsymbol{B})$. Since the image of h is A, it follows that $\mathsf{d}_\mathsf{K}(\boldsymbol{A}, \boldsymbol{B}) = A$. Therefore, $\boldsymbol{A}$ is absolutely closed in K. $\boxtimes$

We are now ready to illustrate how Theorem 13.3 can be applied to describe the Beth companions of concrete classes of algebras.

Example 13.6 (Torsion-free Abelian groups)**.** An Abelian group[10] $\boldsymbol{A} = \langle A; +, -, 0 \rangle$ is said to be *torsion-free* when 0 is its only element of finite order. Torsion-free Abelian groups form a quasivariety TFAG axiomatized relative to Abelian groups by the quasiequations $nx \approx 0 \to x \approx 0$ for every $n \in \mathbb{Z}^+$. Our aim is to describe the Beth companion of TFAG.

To this end, let $\boldsymbol{A} \in \mathsf{TFAG}$, $a \in A$, and $n \in \mathbb{Z}^+$. We say that an element $b \in A$ is the result of *dividing a by n* if $nb = a$. An Abelian group $\boldsymbol{A}$ is called *divisible* when for all $a \in A$ and $n \in \mathbb{Z}^+$ there exists some $b \in A$ such that $nb = a$. In view of the next result, "dividing by n" is an extendable implicit operation of TFAG.

Proposition 13.7. *For each $n \in \mathbb{Z}^+$ there exists a unary $f_n \in \mathsf{ext}_{eq}(\mathsf{TFAG})$ such that for all $\boldsymbol{A} \in \mathsf{TFAG}$ and $a \in \mathsf{dom}(f_n^{\boldsymbol{A}})$,*

$$\mathsf{dom}(f_n^{\boldsymbol{A}}) = \{c \in A : nb = c \text{ for some } b \in A\};$$
$$f_n^{\boldsymbol{A}}(a) = \text{the result of dividing } a \text{ by } n.$$

Proof. Let $\varphi_n(x, y) = x \approx ny$. Moreover, let $\mathbb{Z}$ and $\mathbb{Q}$ be the additive groups of the integers and the rationals, respectively. The fundamental theorem of finitely generated Abelian groups (see, e.g., [45, Thm. 5.2.3]) implies that every finitely generated torsion-free Abelian group is isomorphic to $\mathbb{Z}^m$ for some $m \in \mathbb{N}$. Proposition 2.16 implies that TFAG is generated as a quasivariety by its finitely generated members. Since $\mathbb{Z} \leqslant \mathbb{Q}$, we obtain that TFAG is also generated as a quasivariety by $\mathbb{Q}$. Observe that for all $\frac{a}{b}, \frac{c}{d} \in \mathbb{Q}$ and $n \in \mathbb{Z}^+$ we have that $\mathbb{Q} \vDash \varphi_n(\frac{a}{b}, \frac{c}{d})$ implies $\frac{c}{d} = \frac{a}{nb}$. Therefore, each φ_n is functional in $\mathbb{Q}$. As TFAG is the quasivariety generated by $\mathbb{Q}$, Corollary 3.11 yields that φ_n defines a member f_n of $\mathsf{imp}_{eq}(\mathsf{TFAG})$. From the definition of φ_n it follows that the two displays in the statement hold. Moreover, $f_n^{\mathbb{Q}}$ is total because every rational can be divided by n in $\mathbb{Q}$. As TFAG is generated as a quasivariety by $\mathbb{Q}$, Proposition 8.11(ii) guarantees that f_n is extendable. $\boxtimes$

Corollary 13.8. TFAG *lacks the strong epimorphism surjectivity property.*

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[10]In this and the next example, we temporarily switch to the additive notation for Abelian groups, as it will be more convenient.

Proof. Let f_2 be the member of $\mathsf{ext}_{eq}(\mathsf{TFAG})$ given by Proposition 13.7. Since $\mathbb{Z} \leqslant \mathbb{Q} \in \mathsf{TFAG}$ and $f_2^{\mathbb{Q}}(1) = \frac{1}{2} \notin \mathbb{Z}$, from Theorem 4.10 it follows that $\frac{1}{2} \in \mathsf{d}_{\mathsf{TFAG}}(\mathbb{Z}, \mathbb{Q}) - \mathbb{Z}$. Therefore, TFAG lacks the strong epimorphism surjectivity property by Proposition 6.6. $\boxtimes$

Let $\mathcal{F} = \{f_n : n \in \mathbb{Z}^+\}$ be the set of implicit operations given by Proposition 13.7. By the same proposition we have $\mathcal{F} \subseteq \mathsf{ext}_{eq}(\mathsf{TFAG})$. Denote by $\mathscr{L}$ the language of groups and let $\mathscr{L}_{\mathcal{F}} = \mathscr{L} \cup \{d_n : n \in \mathbb{Z}^+\}$ be an $\mathcal{F}$-expansion of $\mathscr{L}$ in which the role of g_{f_n} is played by d_n. Then $\mathsf{DAG} = \mathbb{S}(\mathsf{TFAG}[\mathscr{L}_{\mathcal{F}}])$ is an equational pp expansion of TFAG. We will prove that it is an equational Beth companion of TFAG. To this end, we rely on the following observation.

Proposition 13.9. $\mathsf{DAG}\!\restriction_{\mathscr{L}}$ *is the class of divisible torsion-free Abelian groups. Moreover, every member of* $\mathsf{DAG}\!\restriction_{\mathscr{L}}$ *is injective in* TFAG.

Proof. Let $\boldsymbol{A} \in \mathsf{TFAG}$. As $\mathcal{F} \subseteq \mathsf{ext}_{eq}(\mathsf{TFAG})$, from Theorem 10.4 it follows that $\mathsf{DAG} = \mathsf{TFAG}[\mathscr{L}_{\mathcal{F}}]$. Then $\boldsymbol{A} \in \mathsf{DAG}\!\restriction_{\mathscr{L}}$ if and only if $f_n^{\boldsymbol{A}}$ is total for every $n \in \mathbb{Z}^+$. Therefore, $\boldsymbol{A} \in \mathsf{DAG}\!\restriction_{\mathscr{L}}$ if and only if $\boldsymbol{A}$ is divisible. Since $\mathsf{DAG}\!\restriction_{\mathscr{L}} \subseteq \mathsf{TFAG}$, we conclude that the members of $\mathsf{DAG}\!\restriction_{\mathscr{L}}$ are exactly the divisible torsion-free Abelian groups. It only remains to prove that every member of $\mathsf{DAG}\!\restriction_{\mathscr{L}}$ is injective in TFAG. It is well known that the injective members of the variety AG of Abelian groups are exactly the divisible Abelian groups (see, e.g., [106, Cor. 2.3.2]). Since $\mathsf{TFAG} \subseteq \mathsf{AG}$, from the definition of an injective algebra it follows that every divisible member of AG that is torsion-free is also injective in TFAG. As all the members of $\mathsf{DAG}\!\restriction_{\mathscr{L}}$ are divisible, we conclude they are injective in TFAG. $\boxtimes$

We are now ready to establish the desired description of the Beth companion of TFAG.

Theorem 13.10. DAG *is a variety and an equational Beth companion of* TFAG.

Proof. Let Σ be a set of equations axiomatizing the variety of Abelian groups. Since $\varphi_n = x \approx ny$ is the equation defining f_n (see the proof of Proposition 13.7), from Theorem 10.4 it follows that DAG is axiomatized by the set of quasiequations

$$\Gamma = \Sigma \cup \{nx \approx 0 \to x \approx 0 : n \in \mathbb{Z}^+\} \cup \{x \approx nd_n(x) : n \in \mathbb{Z}^+\}.$$

We will show that DAG is also axiomatized by the set of equations

$$\Gamma' = \Sigma \cup \{x \approx d_n(nx) : n \in \mathbb{Z}^+\} \cup \{x \approx nd_n(x) : n \in \mathbb{Z}^+\}.$$

It will be enough to prove that Γ and Γ' have the same models. First, let $\boldsymbol{A}$ be a model of Γ. Then $\boldsymbol{A}\!\restriction_{\mathscr{L}}$ is a torsion-free Abelian group. Consider $a \in A$ and $n \in \mathbb{Z}^+$. Since $\boldsymbol{A} \vDash x \approx nd_n(x)$, we have $na = nd_n^{\boldsymbol{A}}(na)$. Then $0 = n(d_n^{\boldsymbol{A}}(na) - a)$, which implies $a = d_n^{\boldsymbol{A}}(na)$ because $\boldsymbol{A}\!\restriction_{\mathscr{L}}$ is torsion-free. So, $\boldsymbol{A} \vDash \Gamma'$. Conversely, suppose that $\boldsymbol{A}$ is a model of Γ'. Let $a \in A$ and $n \in \mathbb{Z}^+$. If $na = 0$, then

$$a = d_n^{\boldsymbol{A}}(na) = d_n^{\boldsymbol{A}}(0) = d_n^{\boldsymbol{A}}(n0) = 0,$$

where the first and last equalities hold because $\boldsymbol{A} \vDash x \approx d_n(nx)$, the second because $na = 0$ by assumption, and the third because $0 = n0$. Therefore, $\boldsymbol{A} \vDash nx \approx 0 \to x \approx 0$. Hence, $\boldsymbol{A}$ is a model of Γ. We conclude that the set of equations Γ' axiomatizes DAG, which is then a variety.

Lastly, by Proposition 13.9 every member of $\mathsf{DAG}{\restriction}_{\mathscr{L}}$ is injective in TFAG. Then Proposition 13.5 implies $\mathsf{DAG}{\restriction}_{\mathscr{L}} \subseteq \mathsf{TFAG}_{\text{AC}}$. Since DAG is a pp expansion of TFAG, from Theorem 13.3 it follows that DAG is a Beth companion of TFAG which, moreover, is equational because DAG is an equational pp expansion of TFAG. ⊠

For the next pair of examples, it is convenient to introduce the following class of algebras (see [71, p. 236]).

Definition 13.11. A member $\boldsymbol{A}$ of a class of algebras K is called *saturated* in K when there exists no $\boldsymbol{B} \in \mathsf{K}$ such that $\boldsymbol{A}$ is a proper K-epic subalgebra of $\boldsymbol{B}$.

In quasivarieties, saturated algebras are also called *epicomplete* (see, e.g., [8, p. 176]). Saturated and absolutely closed algebras are related as follows.

Proposition 13.12. *The following conditions hold for a class of algebras K:*

(i) *every algebra that is absolutely closed in K is saturated in K;*
(ii) *if K is a quasivariety with the amalgamation property, then every algebra that is saturated in K is absolutely closed in K.*

Proof. (i): Suppose that $\boldsymbol{A}$ is absolutely closed in K. Consider $\boldsymbol{B} \in \mathsf{K}$ such that $\boldsymbol{A}$ is a proper subalgebra of $\boldsymbol{B}$. Assume, with a view to contradiction, that $\boldsymbol{A}$ is a K-epic subalgebra of $\boldsymbol{B}$. Then $f = g$ for every $\boldsymbol{C} \in \mathsf{K}$ and pair of homomorphisms $f, g \colon \boldsymbol{B} \to \boldsymbol{C}$ such that $f{\restriction}_A = g{\restriction}_A$. Therefore, $\mathsf{d}_{\mathsf{K}}(\boldsymbol{A}, \boldsymbol{B}) = B$. Since $\boldsymbol{A}$ is absolutely closed in K, we have that $\mathsf{d}_{\mathsf{K}}(\boldsymbol{A}, \boldsymbol{B}) = A$. We conclude that $\boldsymbol{A} = \boldsymbol{B}$, which contradicts the assumption that $\boldsymbol{A}$ is a proper subalgebra of $\boldsymbol{B}$.

(ii): Assume that K has the amalgamation property and that $\boldsymbol{A}$ is saturated in K. Let $\boldsymbol{B} \in \mathsf{K}$ be such that $\boldsymbol{A} \leqslant \boldsymbol{B}$. To prove that $\boldsymbol{A}$ is absolutely closed in K, we need to show that $\mathsf{d}_{\mathsf{K}}(\boldsymbol{A}, \boldsymbol{B}) = A$. Consider the subalgebra $\boldsymbol{D}$ of $\boldsymbol{B}$ with universe $\mathsf{d}_{\mathsf{K}}(\boldsymbol{A}, \boldsymbol{B})$. Then $\boldsymbol{A} \leqslant \boldsymbol{D} \leqslant \boldsymbol{B}$. Since K has the amalgamation property and is a quasivariety by assumption, Proposition 4.11 implies $\mathsf{d}_{\mathsf{K}}(\boldsymbol{A}, \boldsymbol{D}) = \mathsf{d}_{\mathsf{K}}(\boldsymbol{A}, \boldsymbol{B}) \cap D$. As $\mathsf{d}_{\mathsf{K}}(\boldsymbol{A}, \boldsymbol{B}) = D$, we obtain $\mathsf{d}_{\mathsf{K}}(\boldsymbol{A}, \boldsymbol{D}) = D$. Furthermore, $\boldsymbol{D} \in \mathsf{K}$ because $\boldsymbol{D} \leqslant \boldsymbol{B} \in \mathsf{K}$ and K is a quasivariety. It follows that $\boldsymbol{A}$ is a K-epic subalgebra of $\boldsymbol{D}$. Then the assumption that $\boldsymbol{A}$ is saturated in K and $\boldsymbol{D} \in \mathsf{K}$ let us conclude that $\boldsymbol{A} = \boldsymbol{D}$. Therefore, $\mathsf{d}_{\mathsf{K}}(\boldsymbol{A}, \boldsymbol{B}) = A$, as desired. ⊠

Example 13.13 (ℓ-**groups**)**.** An Abelian ℓ-group is an algebra $\langle A; +, -, \wedge, \vee, 0 \rangle$ such that $\langle A; +, -, 0 \rangle$ is an Abelian group, $\langle A; \wedge, \vee \rangle$ is a lattice, and

$$a \leqslant b \text{ implies } a + c \leqslant b + c$$

for all $a, b, c \in A$, where $\leqslant$ denotes the partial order on A induced by its lattice structure (see, e.g., [80]). The class $\ell\mathsf{AG}$ of Abelian ℓ-groups forms a variety (see, e.g., [16, Cor. 1 of Thm. XIII.2]). Our aim is to describe the Beth companion of $\ell\mathsf{AG}$.

To this end, given $\boldsymbol{A} \in \ell\mathsf{AG}$, $a, b \in A$, and $n \in \mathbb{Z}^+$, we say that b is the result of *dividing a by n* if $nb = a$. We say that an Abelian ℓ-group is *divisible* when so is its group reduct.

Proposition 13.14. *For each $n \in \mathbb{Z}^+$ there exists a unary $f_n \in \mathsf{ext}_{eq}(\ell\mathsf{AG})$ such that for all $\boldsymbol{A} \in \ell\mathsf{AG}$ and $a \in \mathsf{dom}(f_n^{\boldsymbol{A}})$,*

$$\mathsf{dom}(f_n^{\boldsymbol{A}}) = \{c \in A : nb = c \text{ for some } b \in A\};$$
$$f_n^{\boldsymbol{A}}(a) = \text{the result of dividing } a \text{ by } n.$$

Proof. Let $\varphi_n(x, y) = x \approx ny$. The proof of Proposition 13.7 shows that φ_n defines a unary implicit operation of TFAG. As the class of group reducts of $\ell\mathsf{AG}$ is TFAG (see [16, Cor. of Thm. XIII.11]), the equation φ_n defines also a unary $f_n \in \mathsf{imp}_{eq}(\ell\mathsf{AG})$. Clearly, the two displays in the statement hold for f_n. Therefore, it only remains to show that f_n is extendable. Recall from [15, Lem. 1, p. 317] that the members of $\ell\mathsf{AG}_{\mathrm{RFSI}}$ are nontrivial and linearly ordered. Moreover, the class of nontrivial linearly ordered members of $\ell\mathsf{AG}_{\mathrm{RFSI}}$ is $\mathbb{U}(\mathbb{Q})$, where $\mathbb{Q}$ denotes the additive group of the rationals equipped with the lattice structure induced by the standard order of $\mathbb{Q}$ by [60]. Therefore, the Subdirect Decomposition Theorem 2.9 yields that $\ell\mathsf{AG}$ is generated as a quasivariety by $\mathbb{Q}$. Arguing as in the proof of Proposition 13.7, we conclude that each f_n is extendable.   ⊠

The next observation can be traced back at least to [97].

Corollary 13.15. *$\ell\mathsf{AG}$ lacks the strong epimorphism surjectivity property.*

Proof. Analogous to the proof of Corollary 13.8 with the sole difference that $\mathbb{Q}$ is the algebra employed in the proof of Proposition 13.14 and $\mathbb{Z}$ the subalgebra of $\mathbb{Q}$ whose universe is the set of integers.   ⊠

Set $\mathcal{F} = \{f_n : n \in \mathbb{Z}^+\}$ be the set of implicit operations given by Proposition 13.14. By the same proposition we have $\mathcal{F} \subseteq \mathsf{ext}_{pp}(\mathsf{TFAG})$. Let $\mathscr{L}_{\mathcal{F}}$ be an $\mathcal{F}$-expansion of $\mathscr{L}_{\ell\mathsf{AG}}$. Then $\ell\mathsf{DAG} = \mathbb{S}(\ell\mathsf{AG}[\mathscr{L}_{\mathcal{F}}])$ is an equational pp expansion of $\ell\mathsf{AG}$.

Theorem 13.16. *$\ell\mathsf{DAG}$ is a variety and an equational Beth companion of $\ell\mathsf{AG}$.*

Proof. Since $\mathcal{F} \subseteq \mathsf{ext}_{eq}(\ell\mathsf{AG})$ and $\ell\mathsf{AG}$ is a variety, Theorem 10.3(iii) implies that $\ell\mathsf{DAG}$ is also a variety. Moreover, from Theorem 10.4 it follows that $\ell\mathsf{DAG} = \ell\mathsf{AG}[\mathscr{L}_{\mathcal{F}}]$. Therefore, $\ell\mathsf{DAG}\restriction_{\mathscr{L}_{\ell\mathsf{AG}}}$ is the class of divisible Abelian ℓ-groups. Then [2, Thm. 2.1] yields that $\ell\mathsf{DAG}\restriction_{\mathscr{L}_{\ell\mathsf{AG}}}$ is the class of members of $\ell\mathsf{AG}$ that are saturated in $\ell\mathsf{AG}$. Since $\ell\mathsf{Ab}$ has the amalgamation property (see [98, Thm. 2.3]), we can apply Proposition 13.12(ii), obtaining $\ell\mathsf{DAG}\restriction_{\mathscr{L}_{\ell\mathsf{AG}}} \subseteq \ell\mathsf{AG}_{\mathrm{AC}}$. Thus, Theorem 13.3 implies that $\ell\mathsf{DAG}$ is a Beth companion of $\ell\mathsf{AG}$ which, moreover, is equational because $\ell\mathsf{DAG}$ is an equational pp expansion of $\ell\mathsf{AG}$.   ⊠

Example 13.17 (MV-algebras)**.** An *MV-algebra* is an algebra $\boldsymbol{A} = \langle A; \oplus, \neg, 0 \rangle$ comprising a commutative monoid $\langle A; \oplus, 0 \rangle$ and satisfying the equations

$$\neg\neg x \approx x, \qquad x \oplus \neg 0 \approx \neg 0, \qquad \neg(\neg x \oplus y) \oplus y \approx \neg(\neg y \oplus x) \oplus x.$$

From a logical standpoint, the interest of MV-algebras derives from the fact that they algebraize the infinite-valued Łukasiewicz logic (see, e.g., [36]).

The variety MV of MV-algebras is generated by the algebra $[0, 1] = \langle [0, 1], \oplus, \neg, 0 \rangle$ with universe the real unit interval $[0, 1] = \{a \in \mathbb{R} : 0 \leqslant a \leqslant 1\}$ and equipped with the operations

$$a \oplus b = \min\{a + b, 1\} \quad \text{and} \quad \neg a = 1 - a$$

for all $a, b \in [0, 1]$, where $+$ and $-$ denote the standard addition and subtraction in $\mathbb{R}$ (see, e.g., [36, Prop. 8.1.1]). Our aim is to describe the Beth companion of MV.

To this end, we will employ the following abbreviations

$$1 = \neg 0 \qquad \text{and} \qquad x \odot y = \neg(\neg x \oplus \neg y),$$

and for every $n \in \mathbb{N}$ we recursively define $n.x$ by setting

$$0.x = 0 \qquad \text{and} \qquad (n+1).x = (n.x) \oplus x.$$

Let $A \in \mathsf{MV}$ and $a \in A$. For every $n \in \mathbb{Z}^+$ we say that $b \in A$ is the result of *dividing a by n* when $n.b = a$ and $b \odot ((n-1).b) = 0$. An MV-algebra $\boldsymbol{A}$ is called *divisible* when for all $a \in A$ and $n \in \mathbb{Z}^+$ there exists $b \in A$ obtained as the result of dividing a by n.

Proposition 13.18. *For each $n \in \mathbb{Z}^+$ there exists a unary $f_n \in \mathsf{ext}_{eq}(\mathsf{MV})$ such that for all $\boldsymbol{A} \in \mathsf{MV}$ and $a \in \mathsf{dom}(f_n^{\boldsymbol{A}})$,*

$$\mathsf{dom}(f_n^{\boldsymbol{A}}) = \{c \in A : b \text{ is the result of diving } c \text{ by } n \text{ for some } b \in A\};$$

$$f_n^{\boldsymbol{A}}(a) = \text{the result of dividing } a \text{ by } n.$$

Proof. Consider the conjunction of equations

$$\varphi_n(x, y) = (n.y \approx x) \sqcap (y \odot ((n-1).y) \approx 0).$$

We will rely on the following observation.

Claim 13.19. *For all $a, b \in [0, 1]$ the following conditions hold in $[0, 1]$:*

(i) $n.b = a$ *if and only if either* $\left(a < 1 \text{ and } b = \frac{a}{n}\right)$ *or* $\left(a = 1 \text{ and } b \geqslant \frac{1}{n}\right)$;

(ii) $b \odot ((n-1).b) = 0$ *if and only if* $b \leqslant \frac{1}{n}$;

(iii) $[0, 1] \vDash \varphi_n(a, b)$ *if and only if* $b = \frac{a}{n}$.

Proof of the Claim. The definitions of $\oplus$ and $\neg$ on $[0, 1]$ yield $n.c = \min\{nc, 1\}$ and $c \odot d = \max\{c + d - 1, 0\}$ for all $c, d \in I$ and $n \in \mathbb{N}$.

(i): Suppose that $n.b = a$. Then $\min\{nb, 1\} = a$. If $a < 1$, then $nb = a$. If $a = 1$, then $\min\{nb, 1\} = 1$. So, $nb \geqslant 1$, which yields $b \geqslant \frac{1}{n}$. To prove the reverse implication, we have to consider two cases. First, assume that $a < 1$ and $b = \frac{a}{n}$. Then $n.b = \min\{nb, 1\} = \min\{a, 1\} = a$. Next we consider the case where $a = 1$ and $b \geqslant \frac{1}{n}$. We have $nb \geqslant 1$, and hence $n.b = \min\{nb, 1\} = 1 = a$.

(ii): We have that $b \odot ((n-1).b) = 0$ if and only if $\max\{b + ((n-1).b) - 1, 0\} = 0$, which is equivalent to $b + ((n-1).b - 1) \leqslant 0$. Moreover,

$$b + ((n-1).b) - 1 = b + \min\{(n-1)b, 1\} - 1 = \min\{nb - 1, b\}.$$

Therefore, $b \odot ((n-1).b) = 0$ if and only if $\min\{nb - 1, b\} \leqslant 0$. As $b \geqslant 0$, the latter condition is equivalent to $nb - 1 \leqslant 0$, and hence to $b \leqslant \frac{1}{n}$.

(iii): Together with (i) and (ii), the definition of φ_n yields that $[0, 1] \vDash \varphi_n(a, b)$ if and only if either $\left(a < 1 \text{ and } b = \frac{a}{n} \text{ and } b \leqslant \frac{1}{n}\right)$ or $\left(a = 1 \text{ and } b = \frac{1}{n}\right)$. Since $a \in [0, 1]$, we have $\frac{a}{n} \leqslant \frac{1}{n}$. Therefore, if $b = \frac{a}{n}$, we have $b \leqslant \frac{1}{n}$. We conclude that $[0, 1] \vDash \varphi_n(a, b)$ if and only if $b = \frac{a}{n}$. $\boxtimes$

From Claim 13.19(iii) it follows that φ_n is functional and total in $[0,1]$. Since $\mathsf{MV} = \mathbb{Q}([0,1])$ (see, e.g., [56, p. 84]), Corollary 3.11 and Proposition 8.11(ii) imply that φ_n defines a unary $f_n \in \mathsf{ext}_{eq}(\mathsf{MV})$. Lastly, as φ_n defines f_n, the two displays in the statement hold. $\boxtimes$

As a consequence, we obtain the following observation from [17, 103].

Corollary 13.20. MV *lacks the strong epimorphism surjectivity property.*

Proof. Let f_2 be the member of $\mathsf{ext}_{eq}(\mathsf{MV})$ given by Proposition 13.18. Moreover, let $\boldsymbol{A}$ be the subalgebra of $[0,1]$ with universe $\{0,1\}$. Since $\boldsymbol{A} \leqslant [0,1] \in \mathsf{MV}$ and $f_2^{[0,1]}(1) = \frac{1}{2} \notin A$, from Theorem 4.10 it follows that $\frac{1}{2} \in \mathsf{d}_{\mathsf{MV}}(\boldsymbol{A}, [0,1]) - A$. Therefore, MV lacks the strong epimorphism surjectivity property by Proposition 6.6. $\boxtimes$

A *DMV-algebra* (see [54] and [53, Def. 5.1.1]) is an algebra $\boldsymbol{A} = \langle A; \oplus, \neg, \{d_n\}_{n \in \mathbb{Z}^+}, 0 \rangle$ comprising an MV-algebra $\langle A; \oplus, \neg, 0 \rangle$ and a sequence of unary operations $\{d_n\}_{n \in \mathbb{Z}^+}$ satisfying the equations

$$n.d_n(x) \approx x \quad \text{and} \quad d_n(x) \odot ((n-1).d_n(x)) \approx 0.$$

Let DMV be the variety of DMV-algebras.

Theorem 13.21. DMV *is an equational Beth companion of* MV.

Proof. Let $\mathcal{F} = \{f_n : n \in \mathbb{Z}^+\} \subseteq \mathsf{ext}_{eq}(\mathsf{MV})$ be the family of operations given by Proposition 13.18. Moreover, let d_n be a unary function symbol for each $n \in \mathbb{Z}^+$. Then the language $\mathscr{L}_{\mathcal{F}} = \mathscr{L}_{\mathsf{MV}} \cup \{d_n : n \in \mathbb{Z}^+\}$ is an $\mathcal{F}$-expansion of $\mathscr{L}_{\mathsf{MV}}$ in which the role of g_{f_n} is played by d_n. From Theorem 10.4 and the fact that each f_n is defined by the conjunction of equations φ_n in the proof of Proposition 13.18 it follows that $\mathbb{S}(\mathsf{MV}[\mathscr{L}_{\mathcal{F}}]) = \mathsf{MV}[\mathscr{L}_{\mathcal{F}}]$ is an equational pp expansion of MV axiomatized by the axioms of MV-algebras plus the equations $n.d_n(x) \approx x$ and $d_n(x) \odot ((n-1).d_n(x)) \approx 0$ for $n \in \mathbb{Z}^+$. Clearly, every member of $\mathsf{MV}[\mathscr{L}_{\mathcal{F}}]$ is a DMV-algebra. On the other hand, every DMV-algebra can be obtained by adding the implicit operations f_n to its MV-algebra reduct, which belongs to $\mathsf{MV}[\mathscr{L}_{\mathcal{F}}]\!\restriction_{\mathscr{L}_{\mathsf{MV}}}$. Therefore, $\mathsf{MV}[\mathscr{L}_{\mathcal{F}}]$ coincides with the variety DMV of DMV-algebras.

In view of Theorem 13.3, to show that DMV is a Beth companion of MV, it suffices to prove that $\mathsf{DMV}\!\restriction_{\mathsf{MV}} \subseteq \mathsf{MV}_{\mathrm{AC}}$. The definition of DMV yields that the members of $\mathsf{DMV}\!\restriction_{\mathsf{MV}}$ are divisible MV-algebras. Every divisible MV-algebra is saturated in MV (see [47, Thm. 3.18(ii)]) and MV has the amalgamation property (see [95, p. 91]). Therefore, from Proposition 13.12(ii) it follows that $\mathsf{DMV}\!\restriction_{\mathsf{MV}} \subseteq \mathsf{MV}_{\mathrm{AC}}$. Then Theorem 13.3 yields that DMV is a Beth companion of MV which, moreover, is equational because DMV is an equational pp expansion of MV. $\boxtimes$

The next concept originates in [49, 50].

Definition 13.22. A finite algebra $\boldsymbol{A}$ is said to be *primal* when for every function $f: A^n \to A$ of positive arity there exists a term $t(x_1, \ldots, x_n)$ of $\mathscr{L}_{\boldsymbol{A}}$ such that for all $a_1, \ldots, a_n \in A$,

$$f(a_1, \ldots, a_n) = t^{\boldsymbol{A}}(a_1, \ldots, a_n).$$

Examples of primal algebras include the two-element Boolean algebra and the rings of the form $\mathbb{Z}_p$ with p prime (see, e.g., [49]). Primal algebras admit the following elegant characterization (see, e.g., [21, Cor. IV.10.8]), where *rigid* means "lacking nonidentity automorphisms".

Theorem 13.23. *A finite algebra $\boldsymbol{A}$ is primal if and only if $\mathbb{V}(\boldsymbol{A})$ is arithmetical and $\boldsymbol{A}$ is simple, rigid, and lacks proper subalgebras.*

The structure theory of varieties generated by a primal algebra is very rich, as a consequence of the fact that these are precisely the varieties categorically equivalent to the variety of Boolean algebras [70] (see also [41]). In particular, since the variety of Boolean algebras has the surjective epimorphism property (see Theorem 7.5) and this property is preserved by categorical equivalences between varieties by Remark 6.4, we deduce the following.

Proposition 13.24. *Varieties generated by a primal algebra have the strong epimorphism surjectivity property.*

In the next example, we will employ the following observation.

Theorem 13.25. *Let $\boldsymbol{A}$ be an $\mathscr{L}$-algebra, $\mathcal{F} \subseteq \mathsf{imp}_{pp}(\boldsymbol{A})$, and $\mathscr{L}_{\mathcal{F}}$ an $\mathcal{F}$-expansion of $\mathscr{L}$ such that $\boldsymbol{A}[\mathscr{L}_{\mathcal{F}}]$ is defined. If $\boldsymbol{A}[\mathscr{L}_{\mathcal{F}}]$ is primal, then $\mathbb{V}(\boldsymbol{A}[\mathscr{L}_{\mathcal{F}}])$ is a Beth companion of $\mathbb{Q}(\boldsymbol{A})$. If, moreover, $\mathcal{F} \subseteq \mathsf{imp}_{eq}(\boldsymbol{A})$, then the Beth companion $\mathbb{V}(\boldsymbol{A}[\mathscr{L}_{\mathcal{F}}])$ is equational.*

Proof. Suppose that $\boldsymbol{A}[\mathscr{L}_{\mathcal{F}}]$ is primal. Then $\mathbb{V}(\boldsymbol{A}[\mathscr{L}_{\mathcal{F}}])$ has the strong epimorphism surjectivity property by Proposition 13.24. Furthermore, recall from [21, Thm. IV.9.4] that $\mathbb{Q}(\boldsymbol{A}[\mathscr{L}_{\mathcal{F}}]) = \mathbb{V}(\boldsymbol{A}[\mathscr{L}_{\mathcal{F}}])$ because $\boldsymbol{A}[\mathscr{L}_{\mathcal{F}}]$ is primal. Together with Theorem 11.6, this yields that, to conclude that $\mathbb{V}(\boldsymbol{A}[\mathscr{L}_{\mathcal{F}}])$ is a Beth companion of $\mathbb{Q}(\boldsymbol{A})$, it only remains to show that $\mathbb{Q}(\boldsymbol{A}[\mathscr{L}_{\mathcal{F}}])$ is a pp expansion of $\mathbb{Q}(\boldsymbol{A})$.

Since $\mathcal{F} \subseteq \mathsf{imp}_{pp}(\boldsymbol{A})$, for every $f \in \mathcal{F}$ there exists a pp formula φ_f functional in $\boldsymbol{A}$ that defines f. By Corollary 3.11 each φ_f defines some $f^* \in \mathsf{imp}_{pp}(\mathbb{Q}(\boldsymbol{A}))$. Let $\mathcal{F}^* = \{f^* : f \in \mathcal{F}\}$. As $\boldsymbol{A}[\mathscr{L}_{\mathcal{F}}]$ is defined, $f^{\boldsymbol{A}}$ is total for every $f \in \mathcal{F}$. Consequently, $f^{\boldsymbol{A}} = (f^*)^{\boldsymbol{A}}$ because f and f^* are both defined by φ_f. Therefore, $(f^*)^{\boldsymbol{A}}$ is total for every $f^* \in \mathcal{F}^*$. Then Proposition 8.11(ii) yields $\mathcal{F}^* \subseteq \mathsf{ext}_{pp}(\mathbb{Q}(\boldsymbol{A}))$. We can regard $\mathscr{L}_{\mathcal{F}}$ as an $\mathcal{F}^*$-expansion of $\mathscr{L}$ by setting $g_{f^*} = g_f$ for each $f \in \mathcal{F}$. Since $f^{\boldsymbol{A}} = (f^*)^{\boldsymbol{A}}$ for every $f \in \mathcal{F}$, the definition of $\boldsymbol{A}[\mathscr{L}_{\mathcal{F}}]$ is independent on whether $\mathscr{L}_{\mathcal{F}}$ is thought of as an $\mathcal{F}$-expansion or as an $\mathcal{F}^*$-expansion. Thus, Theorem 10.5 implies that $\mathbb{Q}(\boldsymbol{A}[\mathscr{L}_{\mathcal{F}}])$ is a pp expansion of $\mathbb{Q}(\boldsymbol{A})$ induced by $\mathcal{F}^*$ and $\mathscr{L}_{\mathcal{F}}$.

The last part of the statement follows immediately from the construction described above.   ⊠

Example 13.26 (Varieties of MV-algebras generated by a finite chain)**.** For $n \in \mathbb{Z}^+$, we denote by $Ł_n$ the subalgebra of the real unit interval $[0,1]$ (cf. Example 13.17) with universe $\{\frac{m}{n} : m \in \mathbb{N},\ m \leqslant n\}$. Notice that $Ł_n$ is a finite MV-algebra of $n+1$ elements. We consider the variety

$$\mathsf{MV}_n = \mathbb{V}(Ł_n) = \mathbb{Q}(Ł_n),$$

where the second equality in the above display holds by [57, Lem. 1.6]. One of the reasons the varieties MV_n are of interest is that they are precisely the proper nontrivial subvarieties of MV with the amalgamation property (see [43, Thm. 13]) or, equivalently, the subvarieties of MV generated by a finite subdirectly irreducible algebra (see, e.g., [34, Lem. 3] and [36, Prop. 3.6.5]).

For each $n \in \mathbb{Z}^+$ consider the conjunction of equations

$$\psi_n(x, y) = (n.y \approx 1) \sqcap (y \odot ((n-1).y) \approx 0).$$

As $\mathrm{L}_n \leqslant [0, 1]$, Claim 13.19(i, ii) implies that for all $a, b \in \mathrm{L}_n$,

$$\mathrm{L}_n \vDash \psi_n(a, b) \iff b = \frac{1}{n}.$$

Therefore, ψ_n defines a total unary $c_n \in \mathsf{imp}_{eq}(\mathrm{L}_n)$ such that $c_n^{\mathrm{L}_n}$ is the constant map with value $\frac{1}{n}$. Let $\mathscr{L}_n$ be a c_n-expansion of $\mathscr{L}_{\mathsf{MV}_n}$. Since $c_n^{\mathrm{L}_n}$ is total, the algebra $\mathrm{L}_n[\mathscr{L}_n]$ is defined and can thought of as the result of adding a constant for the element $\frac{1}{n}$ to L_n. Then let $\mathsf{DMV}_n = \mathbb{V}(\mathrm{L}_n[\mathscr{L}_n])$.

Theorem 13.27. DMV_n *is a an equational Beth companion of* MV_n.

Proof. Recall that $\mathsf{MV}_n = \mathbb{Q}(\mathrm{L}_n)$. Furthermore, $c_n \in \mathsf{imp}_{eq}(\mathrm{L}_n)$ and $\mathrm{L}_n[\mathscr{L}_n]$ is defined. Therefore, in view of Theorem 13.25, it suffices to show that $\mathrm{L}_n[\mathscr{L}_n]$ is a primal algebra. To this end, we will employ Theorem 13.23.

First, observe that $\mathrm{L}_n[\mathscr{L}_n]$ is finite because so is L_n. Moreover, recall from [36, Cor. 3.5.4] that L_n is simple. As $\mathrm{L}_n[\mathscr{L}_n]$ is obtained by adding a constant operation to L_n, we have $\mathsf{Con}(\mathrm{L}_n) = \mathsf{Con}(\mathrm{L}_n[\mathscr{L}_n])$. Hence, $\mathrm{L}_n[\mathscr{L}_n]$ is simple too. Similarly, since $\mathrm{L}_n[\mathscr{L}_n]$ is obtained by adding to L_n a constant operation with value $\frac{1}{n}$ and in L_n we have $\frac{m}{n} = m.\frac{1}{n}$ for every $0 \leqslant m \leqslant n$, the algebra $\mathrm{L}_n[\mathscr{L}_n]$ is rigid and lacks proper subalgebras. Lastly, recall that $\mathsf{MV}_n = \mathbb{V}(\mathrm{L}_n)$ is arithmetical (see, e.g., [55, Prop. 7.6]). Therefore, $\mathbb{V}(\boldsymbol{A})$ is arithmetical for every expansion $\boldsymbol{A}$ of L_n by [21, Thm. II.12.5]. In particular, $\mathbb{V}(\mathrm{L}_n[\mathscr{L}_n])$ is arithmetical, as desired. $\boxtimes$

We now turn our attention to proving Theorems 13.2 and 13.3. We begin by establishing the following pair of results.

Proposition 13.28. *Let* $\mathsf{M} = \mathbb{S}(\mathsf{K}[\mathscr{L}_\mathcal{F}])$ *be a pp expansion of a universal class* K. *Then the following conditions hold:*

  (i) $\mathsf{d}_\mathsf{K}(\boldsymbol{A}{\restriction}_{\mathscr{L}_\mathsf{K}}, \boldsymbol{B}{\restriction}_{\mathscr{L}_\mathsf{K}}) = \mathsf{d}_\mathsf{M}(\boldsymbol{A}, \boldsymbol{B})$ *for all* $\boldsymbol{A} \leqslant \boldsymbol{B} \in \mathsf{K}[\mathscr{L}_\mathcal{F}]$;
  (ii) M *is a Beth companion of* K *if and only if* $\mathsf{d}_\mathsf{K}(\boldsymbol{A}{\restriction}_{\mathscr{L}_\mathsf{K}}, \boldsymbol{B}{\restriction}_{\mathscr{L}_\mathsf{K}}) = A$ *for all* $\boldsymbol{A} \leqslant \boldsymbol{B} \in \mathsf{M}$.

Proof. (i): Let $\boldsymbol{A} \leqslant \boldsymbol{B} \in \mathsf{K}[\mathscr{L}_\mathcal{F}]$. To establish the inclusion from left to right, consider $b \in B - \mathsf{d}_\mathsf{M}(\boldsymbol{A}, \boldsymbol{B})$. Then there exists $\boldsymbol{C} \in \mathsf{M}$ and a pair of homomorphisms $g, h \colon \boldsymbol{B} \to \boldsymbol{C}$ such that $g{\restriction}_A = h{\restriction}_A$ and $g(b) \neq h(b)$. From Proposition 10.2 it follows that $\boldsymbol{C}{\restriction}_{\mathscr{L}_\mathsf{K}} \in \mathsf{K}$. Together with the fact that $g, h \colon \boldsymbol{B}{\restriction}_{\mathscr{L}_\mathsf{K}} \to \boldsymbol{C}{\restriction}_{\mathscr{L}_\mathsf{K}}$ are homomorphisms such that $g{\restriction}_A = h{\restriction}_A$ and $g(b) \neq h(b)$, this yields $b \in B - \mathsf{d}_\mathsf{K}(\boldsymbol{A}{\restriction}_{\mathscr{L}_\mathsf{K}}, \boldsymbol{B}{\restriction}_{\mathscr{L}_\mathsf{K}})$. Hence, $\mathsf{d}_\mathsf{K}(\boldsymbol{A}{\restriction}_{\mathscr{L}_\mathsf{K}}, \boldsymbol{B}{\restriction}_{\mathscr{L}_\mathsf{K}}) \subseteq \mathsf{d}_\mathsf{M}(\boldsymbol{A}, \boldsymbol{B})$. To prove the reverse inclusion, consider $b \in B - \mathsf{d}_\mathsf{K}(\boldsymbol{A}{\restriction}_{\mathscr{L}_\mathsf{K}}, \boldsymbol{B}{\restriction}_{\mathscr{L}_\mathsf{K}})$. Then there exist $\boldsymbol{C} \in \mathsf{K}$ and a pair of homomorphisms $g, h \colon \boldsymbol{B}{\restriction}_{\mathscr{L}_\mathsf{K}} \to \boldsymbol{C}$ such that $g{\restriction}_A = h{\restriction}_A$ and $g(b) \neq h(b)$. Since $\mathsf{M} = \mathbb{S}(\mathsf{K}[\mathscr{L}_\mathcal{F}])$ is a pp expansion of the universal class K by assumption, Proposition 9.6 guarantees that K is the class of $\mathscr{L}_\mathsf{K}$-subreducts of $\mathsf{K}[\mathscr{L}_\mathcal{F}]$. So, there exists $\boldsymbol{D} \in \mathsf{K}[\mathscr{L}_\mathcal{F}]$ such that $\boldsymbol{C} \leqslant \boldsymbol{D}{\restriction}_{\mathscr{L}_\mathsf{K}}$. As $\boldsymbol{B}, \boldsymbol{D} \in \mathsf{K}[\mathscr{L}_\mathcal{F}]$, the homomorphisms $g, h \colon \boldsymbol{B}{\restriction}_{\mathscr{L}_\mathsf{K}} \to \boldsymbol{D}{\restriction}_{\mathscr{L}_\mathsf{K}}$ can by viewed as homomorphisms $g, h \colon \boldsymbol{B} \to \boldsymbol{D}$ by Proposition 9.5. Therefore, from $\boldsymbol{D} \in \mathsf{M}$,

$g{\restriction}_A = h{\restriction}_A$, and $g(b) \neq h(b)$ it follows that $b \in B - \mathsf{d_M}(\boldsymbol{A}, \boldsymbol{B})$. Hence, conclude that $\mathsf{d_M}(\boldsymbol{A}, \boldsymbol{B}) \subseteq \mathsf{d_K}(\boldsymbol{A}{\restriction}_{\mathscr{L}_\mathsf{K}}, \boldsymbol{B}{\restriction}_{\mathscr{L}_\mathsf{K}})$.

(ii): Assume that M is a Beth companion of K and consider $\boldsymbol{A} \leqslant \boldsymbol{B} \in \mathsf{M}$. We will show that $\mathsf{d_K}(\boldsymbol{A}{\restriction}_{\mathscr{L}_\mathsf{K}}, \boldsymbol{B}{\restriction}_{\mathscr{L}_\mathsf{K}}) = A$. Since $\mathsf{M} = \mathbb{S}(\mathsf{K}[\mathscr{L}_\mathcal{F}])$, there exists $\boldsymbol{C} \in \mathsf{K}[\mathscr{L}_\mathcal{F}]$ such that $\boldsymbol{B} \leqslant \boldsymbol{C}$. As $\boldsymbol{C} \in \mathsf{K}[\mathscr{L}_\mathcal{F}]$, from (i) it follows that $\mathsf{d_K}(\boldsymbol{A}{\restriction}_{\mathscr{L}_\mathsf{K}}, \boldsymbol{C}{\restriction}_{\mathscr{L}_\mathsf{K}}) = A$. Moreover, Corollary 4.6(i) yields $\mathsf{d_K}(\boldsymbol{A}{\restriction}_{\mathscr{L}_\mathsf{K}}, \boldsymbol{B}{\restriction}_{\mathscr{L}_\mathsf{K}}) \subseteq \mathsf{d_K}(\boldsymbol{A}{\restriction}_{\mathscr{L}_\mathsf{K}}, \boldsymbol{C}{\restriction}_{\mathscr{L}_\mathsf{K}})$. Therefore, $\mathsf{d_K}(\boldsymbol{A}{\restriction}_{\mathscr{L}_\mathsf{K}}, \boldsymbol{B}{\restriction}_{\mathscr{L}_\mathsf{K}}) = A$.

To prove the converse implication assume that $\mathsf{d_K}(\boldsymbol{A}{\restriction}_{\mathscr{L}_\mathsf{K}}, \boldsymbol{B}{\restriction}_{\mathscr{L}_\mathsf{K}}) = A$ for all $\boldsymbol{A} \leqslant \boldsymbol{B} \in \mathsf{M}$. By Proposition 6.6 and Theorem 11.6, to show that M is a Beth companion of K, it suffices to establish that $\mathsf{d_M}(\boldsymbol{C}, \boldsymbol{D}) = C$ for all $\boldsymbol{C} \leqslant \boldsymbol{D} \in \mathsf{M}$. Consider $\boldsymbol{C} \leqslant \boldsymbol{D} \in \mathsf{M}$. As $\mathsf{M} = \mathbb{S}(\mathsf{K}[\mathscr{L}_\mathcal{F}])$, there exists $\boldsymbol{E} \in \mathsf{K}[\mathscr{L}_\mathcal{F}]$ such that $\boldsymbol{D} \leqslant \boldsymbol{E}$. Then

$$\mathsf{d_M}(\boldsymbol{C}, \boldsymbol{D}) \subseteq \mathsf{d_M}(\boldsymbol{C}, \boldsymbol{E}) = \mathsf{d_K}(\boldsymbol{C}{\restriction}_{\mathscr{L}_\mathsf{K}}, \boldsymbol{E}{\restriction}_{\mathscr{L}_\mathsf{K}}),$$

where the first equality holds by Corollary 4.6(i) and the second follows from (i) because $\boldsymbol{E} \in \mathsf{K}[\mathscr{L}_\mathcal{F}]$. Our assumption implies that $\mathsf{d_K}(\boldsymbol{C}{\restriction}_{\mathscr{L}_\mathsf{K}}, \boldsymbol{E}{\restriction}_{\mathscr{L}_\mathsf{K}}) = C$. Thus, $\mathsf{d_M}(\boldsymbol{C}, \boldsymbol{D}) = C$. ⊠

Proposition 13.29. *Let* M_1 *and* M_2 *be a pair of Beth companions of a quasivariety* K. *Then* $\mathsf{M}_1{\restriction}_{\mathscr{L}_\mathsf{K}} = \mathsf{M}_2{\restriction}_{\mathscr{L}_\mathsf{K}}$.

Proof. Since M_1 and M_2 are Beth companions of K, by Theorem 11.7 there exists a pair of maps $\tau \colon \mathscr{L}_{\mathsf{M}_1} \to T_2$ and $\rho \colon \mathscr{L}_{\mathsf{M}_2} \to T_1$ witnessing that M_1 and M_2 are faithfully term equivalent relative to K, where T_i is the set of terms of M_i in a countably infinite set of variables for $i = 1, 2$. By symmetry it suffices to show $\mathsf{M}_1{\restriction}_{\mathscr{L}_\mathsf{K}} \subseteq \mathsf{M}_2{\restriction}_{\mathscr{L}_\mathsf{K}}$. To this end, consider $\boldsymbol{A} \in \mathsf{M}_1$. The definition of a faithful term equivalence yields $\rho(\boldsymbol{A}) \in \mathsf{M}_2$ and $\rho(\boldsymbol{A}){\restriction}_{\mathscr{L}_\mathsf{K}} = \boldsymbol{A}{\restriction}_{\mathscr{L}_\mathsf{K}}$ (see Remark 11.12(iii)). Therefore, $\boldsymbol{A}{\restriction}_{\mathscr{L}_\mathsf{K}} = \rho(\boldsymbol{A}){\restriction}_{\mathscr{L}_\mathsf{K}} \in \mathsf{M}_2{\restriction}_{\mathscr{L}_\mathsf{K}}$, as desired. ⊠

We are now ready to prove Theorem 13.2.

Proof. We first prove the inclusion $\mathsf{K}[\mathscr{L}_\mathcal{F}]{\restriction}_{\mathscr{L}_\mathsf{K}} \subseteq \mathsf{K}_{\text{AC}}$. Consider $\boldsymbol{A} \in \mathsf{K}[\mathscr{L}_\mathcal{F}]$. To show that $\boldsymbol{A}{\restriction}_{\mathscr{L}_\mathsf{K}}$ is absolutely closed in K, let $\boldsymbol{B} \in \mathsf{K}$ with $\boldsymbol{A}{\restriction}_{\mathscr{L}_\mathsf{K}} \leqslant \boldsymbol{B}$. We need to prove that $\mathsf{d_K}(\boldsymbol{A}{\restriction}_{\mathscr{L}_\mathsf{K}}, \boldsymbol{B}) = A$. Since M is a pp expansion of K, Proposition 10.2 guarantees the existence of $\boldsymbol{C} \in \mathsf{M}$ such that $\boldsymbol{B} \leqslant \boldsymbol{C}{\restriction}_{\mathscr{L}_\mathsf{K}}$. Then Corollary 4.6(i) yields $\mathsf{d_K}(\boldsymbol{A}{\restriction}_{\mathscr{L}_\mathsf{K}}, \boldsymbol{B}) \subseteq \mathsf{d_K}(\boldsymbol{A}{\restriction}_{\mathscr{L}_\mathsf{K}}, \boldsymbol{C}{\restriction}_{\mathscr{L}_\mathsf{K}})$. As M is a Beth companion of K, from Proposition 13.28(ii) it follows that $\mathsf{d_K}(\boldsymbol{A}{\restriction}_{\mathscr{L}_\mathsf{K}}, \boldsymbol{C}{\restriction}_{\mathscr{L}_\mathsf{K}}) = A$. Therefore, $\mathsf{d_K}(\boldsymbol{A}{\restriction}_{\mathscr{L}_\mathsf{K}}, \boldsymbol{B}) = A$, as desired

To prove the inclusion $\mathsf{K}_{\text{AC}} \subseteq \mathsf{M}{\restriction}_{\mathscr{L}_\mathsf{K}}$, consider $\boldsymbol{A} \in \mathsf{K}_{\text{AC}}$. Since $\mathcal{F} \subseteq \mathsf{ext}(\mathsf{K})$, Proposition 9.6 implies that there exists $\boldsymbol{B} \in \mathsf{K}[\mathscr{L}_\mathcal{F}]$ such that $\boldsymbol{A} \leqslant \boldsymbol{B}{\restriction}_{\mathscr{L}_\mathsf{K}}$. We show that A is the universe of a subalgebra of $\boldsymbol{B}$. From $\boldsymbol{A} \leqslant \boldsymbol{B}{\restriction}_{\mathscr{L}_\mathsf{K}}$ it follows that A is closed under the operations of the language of K. Recall that every operation symbol of $\mathscr{L}_\mathcal{F} - \mathscr{L}_\mathsf{K}$ is of the form g_f for some $f \in \mathcal{F}$. Consider an n-ary $f \in \mathcal{F}$. We will show that A is closed under $g_f^{\boldsymbol{B}}$. Since $\boldsymbol{B} \in \mathsf{K}[\mathscr{L}_\mathcal{F}]$, we have $g_f^{\boldsymbol{B}} = f^{\boldsymbol{B}{\restriction}_{\mathscr{L}_\mathsf{K}}}$. Let $a_1, \ldots, a_n \in A$. Since $f^{\boldsymbol{B}{\restriction}_{\mathscr{L}_\mathsf{K}}}$ is total, Theorem 4.10 yields $f^{\boldsymbol{B}{\restriction}_{\mathscr{L}_\mathsf{K}}}(a_1, \ldots, a_n) \in \mathsf{d_K}(\boldsymbol{A}, \boldsymbol{B}{\restriction}_{\mathscr{L}_\mathsf{K}})$. As $\boldsymbol{A}$ is absolutely closed in K by assumption, we have $\mathsf{d_K}(\boldsymbol{A}, \boldsymbol{B}{\restriction}_{\mathscr{L}_\mathsf{K}}) = A$, and hence $f^{\boldsymbol{B}{\restriction}_{\mathscr{L}_\mathsf{K}}}(a_1, \ldots, a_n) \in A$. We have shown that A is the universe of a subalgebra of $\boldsymbol{B}$. Therefore, we can expand $\boldsymbol{A}$ to an $\mathscr{L}_\mathsf{M}$-algebra $\boldsymbol{A}^*$ by setting $g_f^{\boldsymbol{A}^*} = g_f^{\boldsymbol{B}}{\restriction}_A$ for every $g_f \in \mathscr{L}_\mathcal{F} - \mathscr{L}_\mathsf{K}$. The definition of $\boldsymbol{A}^*$ guarantees that $\boldsymbol{A}^* \leqslant \boldsymbol{B}$. Since $\boldsymbol{B} \in \mathsf{M}$ and M is a universal class by Theorem 10.3(i), we obtain $\boldsymbol{A}^* \in \mathbb{S}(\mathsf{M}) \subseteq \mathsf{M}$. Thus, we conclude that $\boldsymbol{A} = \boldsymbol{A}^*{\restriction}_{\mathscr{L}_\mathsf{K}} \in \mathsf{M}{\restriction}_{\mathscr{L}_\mathsf{K}}$.

It remains to show that, when M is an equational Beth companion of K, we have $\mathsf{K}_{\mathrm{AC}} = \mathsf{M}{\upharpoonright}_{\mathscr{L}_{\mathsf{K}}}$. Suppose that M is an equational Beth companion of K. Then M is faithfully term equivalent to a Beth companion M^* of K induced by a family of operations defined by conjunctions of equations. Moreover, Theorem 10.4 yields that M^* is of the form $\mathsf{K}[\mathscr{L}_{\mathcal{F}^*}]$ for some $\mathcal{F}^* \subseteq \mathsf{ext}_{eq}(\mathsf{K})$. Then $\mathsf{K}[\mathscr{L}_{\mathcal{F}^*}]{\upharpoonright}_{\mathscr{L}_{\mathsf{K}}} = \mathsf{M}^*{\upharpoonright}_{\mathscr{L}_{\mathsf{K}}}$. Since $\mathsf{K}[\mathscr{L}_{\mathcal{F}^*}]{\upharpoonright}_{\mathscr{L}_{\mathsf{K}}} \subseteq \mathsf{K}_{\mathrm{AC}} \subseteq \mathsf{M}^*{\upharpoonright}_{\mathscr{L}_{\mathsf{K}}}$ by the first two paragraphs of this proof, we have $\mathsf{K}_{\mathrm{AC}} = \mathsf{M}^*{\upharpoonright}_{\mathscr{L}_{\mathsf{K}}}$. As M and M^* are Beth companions of K, Proposition 13.29 yields $\mathsf{M}{\upharpoonright}_{\mathscr{L}_{\mathsf{K}}} = \mathsf{M}^*{\upharpoonright}_{\mathscr{L}_{\mathsf{K}}}$. Hence, we conclude that $\mathsf{K}_{\mathrm{AC}} = \mathsf{M}{\upharpoonright}_{\mathscr{L}_{\mathsf{K}}}$.   ⊠

Lastly, we prove Theorem 13.3.

Proof. In view of Proposition 13.28(ii), to prove that M is a Beth companion of K, it suffices to show that $\mathsf{d}_{\mathsf{K}}(\boldsymbol{A}{\upharpoonright}_{\mathscr{L}_{\mathsf{K}}}, \boldsymbol{B}{\upharpoonright}_{\mathscr{L}_{\mathsf{K}}}) = A$ for all $\boldsymbol{A} \leqslant \boldsymbol{B} \in \mathsf{M}$. To this end, consider $\boldsymbol{A} \leqslant \boldsymbol{B} \in \mathsf{M}$. Since M is a universal class by Theorem 10.3(i), we obtain $\boldsymbol{A} \in \mathsf{M}$. Therefore, the assumption that $\mathsf{M}{\upharpoonright}_{\mathscr{L}_{\mathsf{K}}} \subseteq \mathsf{K}_{\mathrm{AC}}$ implies $\boldsymbol{A}{\upharpoonright}_{\mathscr{L}_{\mathsf{K}}} \in \mathsf{K}_{\mathrm{AC}}$. Hence, $\mathsf{d}_{\mathsf{K}}(\boldsymbol{A}{\upharpoonright}_{\mathscr{L}_{\mathsf{K}}}, \boldsymbol{B}{\upharpoonright}_{\mathscr{L}_{\mathsf{K}}}) = A$.   ⊠

14. Classes without a Beth companion

We close this work by providing some examples of classes of algebras lacking a Beth companion. The two main results of the section concern the varieties of monoids and of commutative monoids (Theorem 14.1), and certain quasivarieties of Heyting algebras (Theorem 14.11). We begin with the result on monoids.

Theorem 14.1. *The varieties of monoids and of commutative monoids lack a Beth companion.*

The theorem above is the starting point of a complete description of the varieties of commutative monoids admitting a Beth companion. As the methods utilized in its proof go beyond the theory of implicit operations developed here, we will provide such a description in the separate work [31].

In order to prove Theorem 14.1, we first need to introduce the notion of a dominion base and establish some technical results about dominion bases and implicit operations.

Definition 14.2. Let K be a class of algebras and $\Delta \subseteq \mathsf{imp}_{pp}(\mathsf{K})$. We say that Δ is a *dominion base* for K when for all $\boldsymbol{A} \leqslant \boldsymbol{B} \in \mathsf{K}$ and $b \in \mathsf{d}_{\mathsf{K}}(\boldsymbol{A}, \boldsymbol{B})$ there exist $f \in \Delta$ and $\langle a_1, \ldots, a_n \rangle \in \mathsf{dom}(f^{\boldsymbol{B}}) \cap A^n$ such that $f^{\boldsymbol{B}}(a_1, \ldots, a_n) = b$.

Theorem 4.10 states that $\mathsf{imp}_{pp}(\mathsf{K})$ is a dominion base for every elementary class K, and Isbell's Zigzag Theorem 4.9 states that Isbell's operations (see Example 3.14) form a dominion base for the varieties of monoids and of commutative monoids. The following result illustrates how having a concrete and transparent dominion base simplifies the task of finding interpolants for implicit operations.

Theorem 14.3. *Let K be a quasivariety with dominion base Δ and $f \in \mathsf{imp}_{pp}(\mathsf{K})$ of arity n. Then there exist $g \in \Delta$ and n-ary terms $t_1, \ldots, t_m$ of K such that the composition $g(t_1^{\mathsf{K}}, \ldots, t_m^{\mathsf{K}})$ interpolates f in K.*

Proof. Let φ be a pp formula defining f. Then

$$\varphi(x_1, \ldots, x_n, y) = \exists z_1, \ldots, z_k \psi(z_1, \ldots, z_k, x_1, \ldots, x_n, y),$$

where ψ is a conjunction of equations. Let $X = \{z_1, \ldots, z_k, x_1, \ldots, x_n, y\}$. Since K is a quasivariety, the free algebra $\boldsymbol{F}_{\mathsf{K}}(X)$ belongs to K (see Theorem 2.19). We denote by θ the K-congruence of $\boldsymbol{F}_{\mathsf{K}}(X)$ generated by the pairs $\langle s_1, s_2 \rangle$, where $s_1 \approx s_2$ is an equation in ψ. Consider $\boldsymbol{B} = \boldsymbol{F}_{\mathsf{K}}(X)/\theta$ and $\boldsymbol{A} = \mathsf{Sg}^{\boldsymbol{B}}(x_1/\theta, \ldots, x_n/\theta)$. The definition of θ implies that

$$\boldsymbol{B} \vDash \psi(z_1/\theta, \ldots, z_k/\theta, x_1/\theta, \ldots, x_n/\theta, y/\theta).$$

Therefore, $\langle x_1/\theta, \ldots, x_n/\theta \rangle \in \mathsf{dom}(f^{\boldsymbol{B}})$ and $f^{\boldsymbol{B}}(x_1/\theta, \ldots, x_n/\theta) = y/\theta$ because f is defined by φ. As $f \in \mathsf{imp}_{pp}(\mathsf{K})$ by assumption and $x_1/\theta, \ldots, x_n/\theta \in A$ by the definition of $\boldsymbol{A}$, it follows from Theorem 4.10 that $y/\theta \in \mathsf{d}_{\mathsf{K}}(\boldsymbol{A}, \boldsymbol{B})$. Since Δ is a dominion base for K and $\boldsymbol{A}$ is generated by $x_1/\theta, \ldots, x_n/\theta$, there exist an m-ary $g \in \Delta$ and terms $t_1(x_1, \ldots, x_n), \ldots, t_m(x_1, \ldots, x_n)$ of K such that

$$\langle t_1^{\boldsymbol{B}}(x_1/\theta, \ldots, x_n/\theta), \ldots, t_m^{\boldsymbol{B}}(x_1/\theta, \ldots, x_n/\theta) \rangle \in \mathsf{dom}(g^{\boldsymbol{B}}) \tag{64}$$

and

$$g^{\boldsymbol{B}}(t_1^{\boldsymbol{B}}(x_1/\theta, \ldots, x_n/\theta), \ldots, t_m^{\boldsymbol{B}}(x_1/\theta, \ldots, x_n/\theta)) = y/\theta. \tag{65}$$

We will prove that $g(t_1^{\mathsf{K}}, \ldots, t_m^{\mathsf{K}})$ interpolates f in K. To this end, consider $\boldsymbol{C} \in \mathsf{K}$ and $c_1, \ldots, c_n, d \in C$ such that $\langle c_1, \ldots, c_n \rangle \in \mathsf{dom}(f^{\boldsymbol{C}})$ and $f^{\boldsymbol{C}}(c_1, \ldots, c_n) = d$. We need to show that

$$\langle t_1^{\boldsymbol{C}}(c_1, \ldots, c_n), \ldots, t_m^{\boldsymbol{C}}(c_1, \ldots, c_n) \rangle \in \mathsf{dom}(g^{\boldsymbol{C}}) \quad \text{and} \quad g^{\boldsymbol{C}}(t_1^{\boldsymbol{C}}(c_1, \ldots, c_n), \ldots, t_m^{\boldsymbol{C}}(c_1, \ldots, c_n)) = d.$$

Since f is defined by φ, from $f^{\boldsymbol{C}}(c_1, \ldots, c_n) = d$ it follows that $\boldsymbol{C} \vDash \psi(e_1, \ldots, e_k, c_1, \ldots, c_n, d)$ for some $e_1, \ldots, e_k \in C$. Therefore,

$$\boldsymbol{C} \vDash s_1(e_1, \ldots, e_k, c_1, \ldots, c_n, d) \approx s_2(e_1, \ldots, e_k, c_1, \ldots, c_n, d) \tag{66}$$

for every equation $s_1 \approx s_2$ in ψ. As $X = \{z_1, \ldots, z_k, x_1, \ldots, x_n, y\}$ is a set of free generators for $\boldsymbol{F}_{\mathsf{K}}(X)$, there exists a homomorphism $h \colon \boldsymbol{F}_{\mathsf{K}}(X) \to \boldsymbol{C}$ such that $h(z_i) = e_i$ and $h(x_i) = c_i$ for each i, and $h(y) = d$. The definition of θ and (66) yield $\theta \subseteq \mathsf{Ker}(h)$. Since $\boldsymbol{B} = \boldsymbol{F}_{\mathsf{K}}(X)/\theta$, Proposition 2.6 implies that the homomorphism h induces a homomorphism $k \colon \boldsymbol{B} \to \boldsymbol{C}$ such that $k(z_i/\theta) = e_i$ and $k(x_i/\theta) = c_i$ for each i, and $k(y/\theta) = d$. Since k is a homomorphism, we obtain

$$t_i^{\boldsymbol{C}}(c_1, \ldots, c_n) = t_i^{\boldsymbol{B}}(k(x_1/\theta), \ldots, k(x_n/\theta)) = k(t_i^{\boldsymbol{B}}(x_1/\theta, \ldots, x_n/\theta))$$

for each $i \leqslant m$. So, (64) implies $\langle t_1^{\boldsymbol{C}}(c_1, \ldots, c_n), \ldots, t_m^{\boldsymbol{C}}(c_1, \ldots, c_n) \rangle \in \mathsf{dom}(g^{\boldsymbol{C}})$ because $g \in \mathsf{imp}_{pp}(\mathsf{K})$. We also have

$$
\begin{aligned}
g^{\boldsymbol{C}}(t_1^{\boldsymbol{C}}(c_1, \ldots, c_n), &\ldots, t_m^{\boldsymbol{C}}(c_1, \ldots, c_n)) \\
&= g^{\boldsymbol{C}}(t_1^{\boldsymbol{C}}(k(x_1/\theta), \ldots, k(x_n/\theta)), \ldots, t_m^{\boldsymbol{B}}(k(x_1/\theta), \ldots, k(x_n/\theta))) \\
&= k(g^{\boldsymbol{B}}(t_1^{\boldsymbol{B}}(x_1/\theta, \ldots, x_n/\theta), \ldots, t_m^{\boldsymbol{B}}(x_1/\theta, \ldots, x_n/\theta))) \\
&= k(y/\theta) \\
&= d,
\end{aligned}
$$

where the first equality holds because $k(x_i/\theta) = c_i$ for each i, the second follows from the assumptions that k is a homomorphism and $g \in \mathsf{imp}_{pp}(\mathsf{K})$, the third from (65), and the

last from $k(y/\theta) = d$. Hence, we conclude that $g^{\boldsymbol{C}}(t_1^{\boldsymbol{C}}(c_1, \ldots, c_n), \ldots, t_m^{\boldsymbol{C}}(c_1, \ldots, c_n)) = d$, as desired. ⊠

We now apply Theorem 14.3 and the fact that Isbell's operations form a dominion base to deduce a useful property of implicit operations of the variety of (commutative) monoids.

Proposition 14.4. *Let* K *be either the variety of monoids or the variety of commutative monoids. For every unary* $f \in \mathsf{imp}_{pp}(\mathsf{K})$ *there exist* $l, r \in \mathbb{N}$ *such that* $a^l f^{\boldsymbol{A}}(a) = a^r$ *for all* $\boldsymbol{A} \in \mathsf{K}$ *and* $a \in \mathsf{dom}(f^{\boldsymbol{A}})$.

Proof. Consider a unary $f \in \mathsf{imp}_{pp}(\mathsf{K})$. For each $n \in \mathbb{N}$ we denote by g_n the implicit operation of K defined by the n-th Isbell's formula (see Example 3.14). As Isbell's operations form a dominion base for K, Theorem 14.3 yields $n \in \mathbb{N}$ and unary terms $t_1, \ldots, t_{2n+1}$ such that $g_n(t_1^{\mathsf{K}}, \ldots, t_{2n+1}^{\mathsf{K}})$ interpolates f in K. As unary terms of K are equivalent to powers of a variable, there exist $e_1, \ldots, e_{n+1} \in \mathbb{N}$ such that $\mathsf{K} \vDash t_i(x) \approx x^{e_i}$ for each $i \leqslant 2n + 1$.

Recall that $g_0^{\boldsymbol{A}}$ is the identity function for every $\boldsymbol{A} \in \mathsf{K}$. Thus, when $n = 0$ we have $f^{\boldsymbol{A}}(a) = t_1^{\boldsymbol{A}}(a) = a^{e_1}$ for all $\boldsymbol{A} \in \mathsf{K}$ and $a \in \mathsf{dom}(f^{\boldsymbol{A}})$, and hence we can take $l = 0$ and $r = e_1$. So, in the rest of the proof we will assume that $n > 0$. Let $l = \sum_{i=1}^{n} e_{2i}$ and $r = \sum_{i=0}^{n} e_{2i+1}$. Consider $\boldsymbol{A} \in \mathsf{K}$ and $a \in \mathsf{dom}(f^{\boldsymbol{A}})$. We show that $a^l f^{\boldsymbol{A}}(a) = a^r$. To this end, let $a_i = a^{e_i}$ for every $i \leqslant 2n + 1$. Then $f^{\boldsymbol{A}}(a) = g_n^{\boldsymbol{A}}(a_1, \ldots, a_{2n+1})$ because $g_n(t_1^{\mathsf{K}}, \ldots, t_{2n+1}^{\mathsf{K}})$ interpolates f in K and $\mathsf{K} \vDash t_i(x) \approx x^{e_i}$ for each $i \leqslant 2n + 1$. Notice that $a_1, \ldots, a_{2n+1}$ pairwise commute because they are powers of a in $\boldsymbol{A}$.

We will rely on the next fact, which was established under the commutativity assumption in [71, proof of 2.7].

Claim 14.5. *We have* $(\prod_{i=1}^{n} a_{2i}) f^{\boldsymbol{A}}(a) = \prod_{i=0}^{n} a_{2i+1}$.

Proof of the Claim. As $f^{\boldsymbol{A}}(a) = g_n^{\boldsymbol{A}}(a_1, \ldots, a_{2n+1})$, the definition of g_n implies that there exist $c_1, \ldots, c_n \in A$ satisfying

  (i) $f^{\boldsymbol{A}}(a) = a_1 c_1$;
  (ii) $a_{2i} c_i = a_{2i+1} c_{i+1}$ for every $1 \leqslant i \leqslant n - 1$;
  (iii) $a_{2n} c_n = a_{2n+1}$.

Let $c_{n+1} = 1^{\boldsymbol{A}}$. To conclude the proof of the claim, it suffices to show that for every positive $m \leqslant n$,

$$\left(\prod_{i=1}^{m} a_{2i} \right) f^{\boldsymbol{A}}(a) = \left(\prod_{i=0}^{m} a_{2i+1} \right) c_{m+1}. \tag{67}$$

This is because we set $c_{n+1} = 1^{\boldsymbol{A}}$ and, therefore, for $m = n$ we obtain $(\prod_{i=1}^{n} a_{2i}) f^{\boldsymbol{A}}(a) = (\prod_{i=0}^{n} a_{2i+1}) c_{n+1} = \prod_{i=0}^{n} a_{2i+1}$, as desired.

The proof of the above display proceeds by induction on $m \leqslant n$. First, we have

$$a_2 f^{\boldsymbol{A}}(a) = a_2 a_1 c_1 = a_1 a_2 c_1 = a_1 a_3 c_2,$$

where the first equality holds by (i), the second because a_1 and a_2 commute, and the third follows from (ii). Therefore, (67) holds for $m = 1$. Suppose now that $1 < m \leqslant n$. We

show that (67) holds for m under the assumption that it holds for $m-1$, which means that $(\prod_{i=1}^{m-1} a_{2i}) f^{\boldsymbol{A}}(a) = (\prod_{i=0}^{m-1} a_{2i+1}) c_m$. We have

$$\left(\prod_{i=1}^{m} a_{2i} \right) f^{\boldsymbol{A}}(a) = a_{2m} \left(\prod_{i=1}^{m-1} a_{2i} \right) f^{\boldsymbol{A}}(a) = a_{2m} \left(\prod_{i=0}^{m-1} a_{2i+1} \right) c_m = \left(\prod_{i=0}^{m-1} a_{2i+1} \right) a_{2m} c_m$$

$$= \left(\prod_{i=0}^{m-1} a_{2i+1} \right) a_{2m+1} c_{m+1} = \left(\prod_{i=0}^{m} a_{2i+1} \right) c_{m+1},$$

where the first and third equalities hold because a_{2m} commutes with every a_i, the second follows from the induction hypothesis, the fourth is a consequence of (ii) when $i \leqslant n-1$ and of (iii) and $c_{n+1} = 1^{\boldsymbol{A}}$ when $m = n$, and the last is straightforward. This shows that (67) holds for every $m \leqslant n$. $\boxtimes$

Since $l = \sum_{i=1}^{n} e_{2i}$ and $r = \sum_{i=0}^{n} e_{2i+1}$, we have

$$\prod_{i=1}^{n} a_{2i} = \prod_{i=1}^{n} a^{e_{2i}} = a^l \quad \text{and} \quad \prod_{i=0}^{n} a_{2i+1} = \prod_{i=0}^{n} a^{e_{2i+1}} = a^r.$$

Therefore, Claim 14.5 yields

$$a^l f^{\boldsymbol{A}}(a) = \left(\prod_{i=1}^{n} a_{2i} \right) f^{\boldsymbol{A}}(a) = \prod_{i=0}^{n} a_{2i+1} = a^r. \qquad \boxtimes$$

We will also need the following technical result.

Proposition 14.6. *Let* K *be a quasivariety with a Beth companion of the form* $\mathbb{S}(\mathsf{K}[\mathscr{L}_{\mathcal{F}}])$. *Then for every* $f \in \mathsf{imp}_{pp}(\mathsf{K})$ *there exists* $g \in \mathsf{ext}_{pp}(\mathsf{K})$ *that interpolates* f *in* $\mathsf{K}[\mathscr{L}_{\mathcal{F}}] {\restriction}_{\mathscr{L}_{\mathsf{K}}}$.

Proof. Since $f \in \mathsf{imp}_{pp}(\mathsf{K})$, by Theorem 11.6 there exists a term t of $\mathscr{L}_{\mathcal{F}}$ that interpolates f in $\mathbb{S}(\mathsf{K}[\mathscr{L}_{\mathcal{F}}])$. As f has positive arity and t interpolates f, it follows that t is not a constant. So, Proposition 10.22(ii) yields $g \in \mathsf{ext}_{pp}(\mathsf{K})$ such that

$$t^{\boldsymbol{B}} = g^{\boldsymbol{B} {\restriction}_{\mathscr{L}_{\mathsf{K}}}} \text{ for every } \boldsymbol{B} \in \mathsf{K}[\mathscr{L}_{\mathcal{F}}].$$

We will show that g interpolates f in $\mathsf{K}[\mathscr{L}_{\mathcal{F}}] {\restriction}_{\mathscr{L}_{\mathsf{K}}}$. Let $\boldsymbol{A} \in \mathsf{K}[\mathscr{L}_{\mathcal{F}}] {\restriction}_{\mathscr{L}_{\mathsf{K}}}$ and $\langle a_1, \dots, a_n \rangle \in \mathsf{dom}(f^{\boldsymbol{A}})$. As $\boldsymbol{A}[\mathscr{L}_{\mathcal{F}}] \in \mathsf{K}[\mathscr{L}_{\mathcal{F}}]$ and $\boldsymbol{A} = \boldsymbol{A}[\mathscr{L}_{\mathcal{F}}] {\restriction}_{\mathscr{L}_{\mathsf{K}}}$, the above display implies that $t^{\boldsymbol{A}[\mathscr{L}_{\mathcal{F}}]} = g^{\boldsymbol{A}}$. It follows that $g^{\boldsymbol{A}}$ is total, and hence $\langle a_1, \dots, a_n \rangle \in \mathsf{dom}(g^{\boldsymbol{A}})$. Since t interpolates f in $\mathbb{S}(\mathsf{K}[\mathscr{L}_{\mathcal{F}}])$, we obtain

$$f^{\boldsymbol{A}}(a_1, \dots, a_n) = f^{\boldsymbol{A}[\mathscr{L}_{\mathcal{F}}] {\restriction}_{\mathscr{L}_{\mathsf{K}}}}(a_1, \dots, a_n) = t^{\boldsymbol{A}[\mathscr{L}_{\mathcal{F}}]}(a_1, \dots, a_n) = g^{\boldsymbol{A}}(a_1, \dots, a_n).$$

We conclude that g interpolates f in $\mathsf{K}[\mathscr{L}_{\mathcal{F}}] {\restriction}_{\mathscr{L}_{\mathsf{K}}}$. $\boxtimes$

We are now ready to prove Theorem 14.1.

Proof. Let K be either the variety of monoids or the variety of commutative monoids and assume, with a view to contradiction, that K has a Beth companion. Corollary 11.5 implies that K has also a Beth companion of the form $\mathbb{S}(\mathsf{K}[\mathscr{L}_{\mathcal{F}}])$. Let f be the operation of taking inverses in monoids. Then $f \in \mathsf{imp}_{pp}(\mathsf{K})$ by Theorem 3.7. Proposition 14.6 yields a unary $g \in \mathsf{ext}_{pp}(\mathsf{K})$ that interpolates f in $\mathsf{K}[\mathscr{L}_{\mathcal{F}}] {\restriction}_{\mathscr{L}_{\mathsf{K}}}$. By Proposition 14.4 there exist $l, r \in \mathbb{N}$ such

that $a^l g^{\boldsymbol{A}}(a) = a^r$ for all $\boldsymbol{A} \in \mathsf{K}$ and $a \in \mathsf{dom}(g^{\boldsymbol{A}})$. Since K is a variety and $\mathbb{S}(\mathsf{K}[\mathscr{L}_{\mathcal{F}}])$ a pp expansion of K, Proposition 10.2 implies that $\mathsf{K} = \mathbb{S}((\mathbb{S}(\mathsf{K}[\mathscr{L}_{\mathcal{F}}]))\restriction_{\mathscr{L}_{\mathsf{K}}})$. Hence, $\mathsf{K} \subseteq \mathbb{S}(\mathsf{K}[\mathscr{L}_{\mathcal{F}}]\restriction_{\mathscr{L}_{\mathsf{K}}})$. Therefore, there exists an extension $\boldsymbol{B}$ of the multiplicative monoid $\mathbb{Q}$ of the rationals such that $\boldsymbol{B} \in \mathsf{K}[\mathscr{L}_{\mathcal{F}}]\restriction_{\mathscr{L}_{\mathsf{K}}}$. Then $g^{\boldsymbol{B}}$ is total because $g \in \mathsf{ext}_{pp}(\mathsf{K}) = \mathcal{F}$. Since g interpolates f in $\mathsf{K}[\mathscr{L}_{\mathcal{F}}]\restriction_{\mathscr{L}_{\mathsf{K}}}$, we obtain $g^{\boldsymbol{B}}(2) = f^{\boldsymbol{B}}(2) = f^{\mathbb{Q}}(2) = 2^{-1}$. Therefore, $2^{l-1} = 2^l g^{\boldsymbol{B}}(2) = 2^r$. So, $2^{l-1} = 2^r$ holds in $\mathbb{Q}$. In turn, this implies $l = r + 1$. Consequently,

$$a^{r+1} g^{\boldsymbol{A}}(a) = a^r \text{ for all } \boldsymbol{A} \in \mathsf{K} \text{ and } a \in \mathsf{dom}(g^{\boldsymbol{A}}). \tag{68}$$

Claim 14.7. *There exist $\boldsymbol{C} \in \mathsf{K}$ and $c \in C$ such that $c^r \neq c^{r+1}$ and $c^{r+1} = c^{r+2}$.*

Proof of the Claim. Let C be the set of symbols $\{c^i : 0 \leqslant i \leqslant r + 1\}$ and define the following binary operation on C

$$c^i \cdot c^j = \begin{cases} c^{i+j} & \text{if } i + j < r + 1; \\ c^{r+1} & \text{otherwise.} \end{cases}$$

It is straightforward to verify that this defines a commutative monoid $\boldsymbol{C}$ with neutral element c^0. So, $\boldsymbol{C} \in \mathsf{K}$. Let $c = c^1$. Then c^i is the i-th power of c^1 for every $i \leqslant r + 1$. From the definition of $\boldsymbol{C}$ it then follows immediately that $c^r \neq c^{r+1}$ and $c^{r+1} = c^{r+2}$. ⊠

Let $\boldsymbol{C}$ and $c \in C$ be as in Claim 14.7. Since $g \in \mathsf{ext}_{pp}(\mathsf{K})$, there exists $\boldsymbol{D} \in \mathsf{K}$ such that $\boldsymbol{C} \leqslant \boldsymbol{D}$ and $g^{\boldsymbol{D}}$ is total. From $c \in \mathsf{dom}(g^{\boldsymbol{D}})$ and (68) it follows $c^{r+1} g^{\boldsymbol{D}}(c) = c^r$. Thus, using $c^{r+1} = c^{r+2}$, we deduce

$$c^r = c^{r+1} g^{\boldsymbol{D}}(c) = c^{r+2} g^{\boldsymbol{D}}(c) = c c^{r+1} g^{\boldsymbol{D}}(c) = c c^r = c^{r+1},$$

a contradiction with $c^r \neq c^{r+1}$ in $\boldsymbol{C}$. ⊠

Remark 14.8. The proof of Theorem 14.1 can easily be adapted to show that both the variety of semigroups and the variety of commutative semigroups lack a Beth companion. To see this, recall that Isbell's formulas (see Example 3.14) also form a dominion base for these varieties (see [71, Thm. 2.3] for semigroups and [69, Thm. 1.1] for commutative semigroups). The changes required for adapting the proof of Theorem 14.1 are limited to the following. First, the role of the implicit operation f defined by $\varphi = (x \cdot y \approx 1) \sqcap (y \cdot x \approx 1)$ should be taken over by the implicit operation g defined by

$$\psi = (x^2 \cdot y \approx x) \sqcap (y^2 \cdot x \approx y) \sqcap (x \cdot y \approx y \cdot x),$$

which also defines inverses when they exist. In particular, g coincides with f on $\mathbb{Q}$. Moreover, the monoids $\mathbb{Q}$ and $\boldsymbol{C}$ appearing in the proof of Theorem 14.1 should be replaced by their semigroup reducts. Lastly, the proof of Claim 14.5 uses the fact that $\boldsymbol{A}$ has a neutral element, which need not be the case if $\boldsymbol{A}$ is an arbitrary semigroup. This problem can be overcome easily by adding a neutral element to $\boldsymbol{A}$ in the proof of that claim.

On the other hand, the implicit operation g becomes extendable when restricted to the quasivariety CCS of cancellative commutative semigroups. In fact, an argument similar to the one detailed in the proof of Theorem 11.9(i) shows that the pp expansion of CCS induced by g is the Beth companion of CCS and, moreover, is term equivalent to the variety of Abelian groups (inversion is given by g and the neutral element is rendered as the unary operation $x \cdot g(x)$). ⊠

The second main result of the section gives a sufficient condition for a quasivariety of Heyting algebras to lack a Beth companion. In order to state it, we first need to recall some definitions. The first is the notion of a maximal relatively subdirectly irreducible algebra in a quasivariety (see, e.g., [76, p. 81]).

Definition 14.9. Let K be a quasivariety. We say that $\boldsymbol{A} \in \mathsf{K}_{\mathrm{RSI}}$ is *maximal* when $\boldsymbol{A}$ cannot be properly embedded into any $\boldsymbol{B} \in \mathsf{K}_{\mathrm{RSI}}$. We denote by $\mathcal{M}(\mathsf{K}_{\mathrm{RSI}})$ the class of maximal members of $\mathsf{K}_{\mathrm{RSI}}$.

We will also make use of the ordered sum, an operation that has been extensively used to study implicit definability and surjectivity of epimorphisms in varieties of Heyting algebras (see, e.g., [90, p. 87] and [94, p. 9]). Intuitively, the ordered sum of two Heyting algebras $\boldsymbol{A}$ and $\boldsymbol{B}$ is the result of pasting $\boldsymbol{A}$ below $\boldsymbol{B}$ and gluing the top element $1^{\boldsymbol{A}}$ of $\boldsymbol{A}$ to the bottom element $0^{\boldsymbol{B}}$ of $\boldsymbol{B}$.

Definition 14.10. Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be a pair of Heyting algebras. Their *ordered sum* (also known as *vertical sum* or just *sum*) $\boldsymbol{A} + \boldsymbol{B}$ is the unique Heyting algebra whose universe is the disjoint union of $A - \{1^{\boldsymbol{A}}\}$ and B and whose lattice order is given by

$$c \leqslant d \iff \text{ either } (c, d \in A - \{1^{\boldsymbol{A}}\} \text{ and } c \leqslant^{\boldsymbol{A}} d)$$
$$\text{or } (c, d \in B \text{ and } c \leqslant^{\boldsymbol{B}} d)$$
$$\text{or } (c \in A - \{1^{\boldsymbol{A}}\} \text{ and } d \in B),$$

where $\leqslant^{\boldsymbol{A}}$ and $\leqslant^{\boldsymbol{B}}$ denote the lattice orders of $\boldsymbol{A}$ and $\boldsymbol{B}$, respectively.

In the following, for each $n \in \mathbb{Z}^+$ we will denote by $\boldsymbol{C}_n$ the n-element linearly ordered Heyting algebra. We are now ready to state the second main result of the section.

Theorem 14.11. *Let K be a relatively congruence distributive quasivariety of Heyting algebras. If there exists a Heyting algebra $\boldsymbol{A}$ such that $\boldsymbol{A} + \boldsymbol{C}_5 \in \mathcal{M}(\mathsf{K}_{\mathrm{RSI}})$, then K lacks a Beth companion.*

To give a better understanding of the applicability of Theorem 14.11, we rely on the next characterization of relatively congruence distributive quasivarieties of Heyting algebras, where HA stands for the variety of Heyting algebras.

Theorem 14.12. *A quasivariety K of Heyting algebras is relatively congruence distributive if and only if $\mathsf{K} = \mathbb{Q}(\mathsf{M})$ for some universal class M such that $\mathsf{M} \subseteq \mathsf{HA}_{\mathrm{FSI}}$.*

Proof. Let K be a quasivariety of Heyting algebras. Since the variety HA is congruence distributive by Theorem 7.2, from [40, Cor. 1.4] it follows that K is relatively congruence distributive if and only if $\mathsf{K}_{\mathrm{RFSI}} \subseteq \mathsf{HA}_{\mathrm{FSI}}$. Therefore, it only remains to prove that $\mathsf{K}_{\mathrm{RFSI}} \subseteq \mathsf{HA}_{\mathrm{FSI}}$ if and only if there exists a universal class M such that $\mathsf{K} = \mathbb{Q}(\mathsf{M})$ and $\mathsf{M} \subseteq \mathsf{HA}_{\mathrm{FSI}}$.

We first establish the implication from left to right. To this end, assume that $\mathsf{K}_{\mathrm{RFSI}} \subseteq \mathsf{HA}_{\mathrm{FSI}}$. Let $\mathsf{M} = \mathbb{U}(\mathsf{K}_{\mathrm{RFSI}})$. The Subdirect Decomposition Theorem 2.9 yields $\mathsf{K} = \mathbb{Q}(\mathsf{K}_{\mathrm{RFSI}})$. It follows that $\mathsf{K} = \mathbb{Q}(\mathsf{M})$ because $\mathsf{K}_{\mathrm{RFSI}} \subseteq \mathsf{M} \subseteq \mathsf{K}$. It is well known (see, e.g., [85, Prop. A.4.3]) that a Heyting algebra is finitely subdirectly irreducible if and only if its greatest element is join

irreducible, a property that can be expressed with a universal sentence. Therefore, $\mathsf{HA}_{\mathrm{FSI}}$ is a universal class by Theorem 2.1(iii). Then

$$\mathsf{M} = \mathbb{U}(\mathsf{K}_{\mathrm{RFSI}}) \subseteq \mathbb{U}(\mathsf{HA}_{\mathrm{FSI}}) = \mathsf{HA}_{\mathrm{FSI}}.$$

Thus, M has the desired properties. For the converse implication, assume that there exists a universal class M such that $\mathsf{K} = \mathbb{Q}(\mathsf{M})$ and $\mathsf{M} \subseteq \mathsf{HA}_{\mathrm{FSI}}$. Theorem 2.13 implies that $\mathsf{K}_{\mathrm{RFSI}} = \mathbb{Q}(\mathsf{M})_{\mathrm{RFSI}} \subseteq \mathbb{ISP}_{\mathrm{u}}(\mathsf{M})$. Then $\mathsf{K}_{\mathrm{RFSI}} \subseteq \mathbb{U}(\mathsf{M})$ by Theorem 2.2. As M is a universal class contained in $\mathsf{HA}_{\mathrm{FSI}}$, we obtain

$$\mathsf{K}_{\mathrm{RFSI}} \subseteq \mathbb{U}(\mathsf{M}) = \mathsf{M} \subseteq \mathsf{HA}_{\mathrm{FSI}},$$

as desired.   ⊠

Before presenting its proof, we first establish a series of consequences of Theorem 14.11. As every variety of Heyting algebras is congruence distributive (see Theorem 7.2), the following is an immediate corollary of Theorem 14.11.

Corollary 14.13. *Let K be a variety of Heyting algebras. If there exists a Heyting algebra $\boldsymbol{A}$ with $\boldsymbol{A} + \boldsymbol{C}_5 \in \mathcal{M}(\mathsf{K}_{\mathrm{SI}})$, then K lacks a Beth companion.*

We will use the following consequence of Theorem 14.11 to show that infinitely many varieties of Heyting algebras lack a Beth companion.

Corollary 14.14. *Let K be a finite set of finite Heyting algebras such that $\mathbb{Q}(\mathsf{K})$ is relatively congruence distributive. Assume that there exists a Heyting algebra $\boldsymbol{A}$ such that $\boldsymbol{A} + \boldsymbol{C}_5 \in \mathsf{K}$ and $\boldsymbol{A} + \boldsymbol{C}_5$ cannot be properly embedded into any member of K. Then $\mathbb{Q}(\mathsf{K})$ lacks a Beth companion.*

Proof. By Theorem 14.11 it is sufficient to show that $\boldsymbol{A} + \boldsymbol{C}_5 \in \mathcal{M}(\mathbb{Q}(\mathsf{K})_{\mathrm{RSI}})$. Since K is a finite set of finite Heyting algebras, $\mathbb{P}_{\mathrm{u}}(\mathsf{K}) \subseteq \mathbb{I}(\mathsf{K})$ (see Proposition 2.14). Consequently, from Theorem 2.13 it follows that $\mathbb{Q}(\mathsf{K})_{\mathrm{RSI}} \subseteq \mathbb{IS}(\mathsf{K})$. Therefore, if $\boldsymbol{A} + \boldsymbol{C}_5$ embeds properly into a member of $\mathbb{Q}(\mathsf{K})_{\mathrm{RSI}}$, then it also embeds properly into a member of K, but this contradicts our hypothesis. Thus, $\boldsymbol{A} + \boldsymbol{C}_5 \in \mathcal{M}(\mathbb{Q}(\mathsf{K})_{\mathrm{RSI}})$.   ⊠

We also obtain an analogue of Corollary 14.14 for finitely generated varieties of Heyting algebras.

Corollary 14.15. *Let K be a finite set of finite Heyting algebras. Assume that there exists a Heyting algebra $\boldsymbol{A}$ such that $\boldsymbol{A} + \boldsymbol{C}_5 \in \mathsf{K}$ and one of the following conditions holds:*

(i) *$\boldsymbol{A} + \boldsymbol{C}_5 \in \mathbb{HS}(\boldsymbol{B})$ implies $\boldsymbol{A} + \boldsymbol{C}_5 \cong \boldsymbol{B}$ for each $\boldsymbol{B} \in \mathsf{K}$;*
(ii) *all members of K have size at most $|\boldsymbol{A} + \boldsymbol{C}_5|$.*

Then $\mathbb{V}(\mathsf{K})$ lacks a Beth companion.

Proof. Suppose that (i) holds. By Corollary 14.13 it suffices to show that $\boldsymbol{A} + \boldsymbol{C}_5 \in \mathcal{M}(\mathbb{V}(\mathsf{K})_{\mathrm{SI}})$. Suppose, with a view to contradiction, that there exists $\boldsymbol{D} \in \mathbb{V}(\mathsf{K})_{\mathrm{SI}}$ into which $\boldsymbol{A} + \boldsymbol{C}_5$ properly embeds. Since K is a finite set of finite Heyting algebras, $\mathbb{P}_{\mathrm{u}}(\mathsf{K}) \subseteq \mathbb{I}(\mathsf{K})$ (see Proposition 2.14). Since $\mathbb{V}(\mathsf{K})$ is congruence distributive by Theorem 7.2, from Jónsson's Theorem 2.12 it follows that $\mathbb{V}(\mathsf{K})_{\mathrm{SI}} \subseteq \mathbb{HS}(\mathsf{K})$. Then $\boldsymbol{D} \in \mathbb{HS}(\mathsf{K})$, and so there exists $\boldsymbol{B} \in \mathsf{K}$ such that

$D \in \mathbb{HS}(B)$. Since $A + C_5$ embeds into D, we have $A + C_5 \in \mathbb{ISHS}(B)$. For every class M we have $\mathbb{SH}(M) \subseteq \mathbb{HS}(M)$ (see, e.g., [21, Lem. II.9.2]) and $\mathbb{IH}(M) = \mathbb{H}(M)$. Consequently, $\mathbb{ISHS}(B) = \mathbb{HS}(B)$, and hence $A + C_5 \in \mathbb{HS}(B)$. Then (i) implies $A + C_5 \cong B$. Recall that $A + C_5, B \in K$ and all the members of K are finite. Therefore, since $A + C_5$ properly embeds into D and $D \in \mathbb{HS}(B)$, we have $|A + C_5| < |D| \leqslant |B|$. Thus, we reached a contradiction because $A + C_5 \cong B$.

To conclude the proof, it is then sufficient to show that (ii) implies (i). Let $B \in K$ be such that $A + C_5 \in \mathbb{HS}(B)$. Then $|A + C_5| \leqslant |B|$. So, (ii) yields $|A + C_5| = |B|$. Since $A + C_5 \in \mathbb{HS}(B)$ and B is finite (the latter because $B \in K$ and K is a class of finite algebras by assumption), we obtain that $A + C_5 \cong B$. Thus, (i) holds. ⊠

Example 14.16 (Gödel algebras). A Heyting algebra is called a *Gödel algebra* when it satisfies the prelinearity axiom $(x \to y) \vee (y \to x) \approx 1$. From a logical standpoint, the interest of Gödel algebras is that they algebraize the Gödel-Dummett logic (see, e.g., [33, 61]).

The variety GA of Gödel algebras is generated by the class of all finite linearly ordered Heyting algebras or by any infinite linearly ordered Heyting algebra (see [67, Thm. 1.5]). Every proper subvariety of GA is of the form $\mathbb{V}(C_n)$ for $n \geqslant 1$ and $\mathbb{V}(C_n) \subseteq \mathbb{V}(C_m)$ if and only if $n \leqslant m$ (see [46, 62]). The next result governs the existence of a Beth companion for varieties of Gödel algebras.

Theorem 14.17. *A variety* V *of Gödel algebras lacks a Beth companion if and only if* $V = \mathbb{V}(C_n)$ *for* $n \geqslant 5$. *All the remaining varieties of Gödel algebras are their own Beth companion.*

Proof. From [90, Thm. 8.1] it follows that the varieties of Gödel algebras with the strong epimorphism surjectivity property are exactly GA and $\mathbb{V}(C_n)$ for $n \leqslant 4$. Hence, these varieties are their own Beth companions by Theorem 11.9(vi). Let $n \geqslant 5$. As $C_n \cong C_{n-4} + C_5$, from Corollary 14.15 it follows that $\mathbb{V}(C_n)$ lacks a Beth companion. ⊠

In order to prove Theorem 14.11, we first establish a series of useful results.

Proposition 14.18. *Let* K *be a quasivariety and* $f \in \mathsf{ext}_{pp}(K)$. *Then* f^A *is total for every* $A \in \mathbb{HP}(\mathcal{M}(K_{\mathrm{RSI}})) \cap K$.

Proof. First, assume that $A \in \mathcal{M}(K_{\mathrm{RSI}})$. Since $f \in \mathsf{ext}_{pp}(K)$ and $A \in K_{\mathrm{RSI}}$, by Theorem 8.4 there exists $B \in K_{\mathrm{RSI}}$ such that $A \leqslant B$ and f^B is total. The maximality of A implies that $A = B$. This shows that f^A is total for every $A \in \mathcal{M}(K_{\mathrm{RSI}})$. Then let $M = \{A \in K : f^A \text{ is total}\}$. We have $\mathcal{M}(K_{\mathrm{RSI}}) \subseteq M$. Our goal is to show that $\mathbb{HP}(\mathcal{M}(K_{\mathrm{RSI}})) \cap K \subseteq M$. Since $M \subseteq K$ and K is a quasivariety, we have $\mathbb{P}(M) \cap K = \mathbb{P}(M)$. Therefore, applying Proposition 9.7 twice with $\mathbb{O} = \mathbb{P}$ and $\mathbb{O} = \mathbb{H}$, we obtain

$$\mathbb{HP}(\mathcal{M}(K_{\mathrm{RSI}})) \cap K \subseteq \mathbb{HP}(M) \cap K \subseteq \mathbb{H}(\mathbb{P}(M) \cap K) \cap K \subseteq \mathbb{H}(M) \cap K \subseteq M. \qquad ⊠$$

Proposition 14.19. *Let* K *be a quasivariety with a Beth companion. Moreover, let* $A, B \in \mathbb{HP}(\mathcal{M}(K_{\mathrm{RSI}})) \cap K$ *be such that* $A \leqslant B$. *Then* $\mathsf{d}_K(A, B) = A$.

Proof. We may assume that K has a Beth companion of the form $\mathbb{S}(\mathsf{K}[\mathscr{L}_{\mathcal{F}}])$. Proposition 14.18 implies that $f^{\boldsymbol{A}}$ and $f^{\boldsymbol{B}}$ are total for every $f \in \mathcal{F}$. Therefore, $\boldsymbol{A}[\mathscr{L}_{\mathcal{F}}]$ and $\boldsymbol{B}[\mathscr{L}_{\mathcal{F}}]$ are defined and are members of $\mathsf{K}[\mathscr{L}_{\mathcal{F}}]$. It then follows from Proposition 13.28(ii) that

$$\mathsf{d}_{\mathsf{K}}(\boldsymbol{A}, \boldsymbol{B}) = \mathsf{d}_{\mathsf{K}}(\boldsymbol{A}[\mathscr{L}_{\mathcal{F}}]{\upharpoonright}_{\mathscr{L}_{\mathsf{K}}}, \boldsymbol{B}[\mathscr{L}_{\mathcal{F}}]{\upharpoonright}_{\mathscr{L}_{\mathsf{K}}}) = A. \qquad \boxtimes$$

Proposition 14.20. *Let K be a quasivariety. Then $\mathsf{d}_{\mathsf{K}}(\boldsymbol{A}, \boldsymbol{B}) = \mathsf{d}_{\mathsf{K}_{\mathrm{RSI}}}(\boldsymbol{A}, \boldsymbol{B})$ for all $\mathscr{L}_{\mathsf{K}}$-algebras $\boldsymbol{A}, \boldsymbol{B}$ such that $\boldsymbol{A} \leqslant \boldsymbol{B}$.*

Proof. From the definition of a dominion it follows immediately that $\mathsf{d}_{\mathsf{K}}(\boldsymbol{A}, \boldsymbol{B}) \subseteq \mathsf{d}_{\mathsf{K}_{\mathrm{RSI}}}(\boldsymbol{A}, \boldsymbol{B})$. To prove the other inclusion, let $b \in B$ and assume that $b \notin \mathsf{d}_{\mathsf{K}}(\boldsymbol{A}, \boldsymbol{B})$. Then there exist $\boldsymbol{C} \in \mathsf{K}$ and homomorphisms $g, h \colon \boldsymbol{B} \to \boldsymbol{C}$ such that $g{\upharpoonright}_A = h{\upharpoonright}_A$ and $g(b) \neq h(b)$. The Subdirect Decomposition Theorem 2.9 implies that there exists $\{\boldsymbol{C}_i : i \in I\} \subseteq \mathsf{K}_{\mathrm{RSI}}$ such that $\boldsymbol{C} \leqslant \prod_{i \in I} \boldsymbol{C}_i$ is a subdirect product. For each $i \in I$ let $p_i \colon \boldsymbol{C} \to \boldsymbol{C}_i$ be the restriction of the canonical projection. Since $g(b) \neq h(b)$, there exists $i \in I$ such that $(p_i \circ g)(b) \neq (p_i \circ h)(b)$. As $(p_i \circ g){\upharpoonright}_A = p_i \circ (g{\upharpoonright}_A) = p_i \circ (h{\upharpoonright}_A) = (p_i \circ h){\upharpoonright}_A$, the homomorphisms $p_i \circ g, p_i \circ h \colon \boldsymbol{B} \to \boldsymbol{C}_i$ witness that $b \notin \mathsf{d}_{\mathsf{K}_{\mathrm{RSI}}}(\boldsymbol{A}, \boldsymbol{B})$. Thus, $\mathsf{d}_{\mathsf{K}_{\mathrm{RSI}}}(\boldsymbol{A}, \boldsymbol{B}) \subseteq \mathsf{d}_{\mathsf{K}}(\boldsymbol{A}, \boldsymbol{B})$. $\qquad \boxtimes$

Proposition 14.21. *Let K be a relatively congruence distributive quasivariety of Heyting algebras. Let also $\boldsymbol{A}_1, \boldsymbol{A}_2, \boldsymbol{B}$ be Heyting algebras with $\boldsymbol{B} \in \mathsf{K}_{\mathrm{RFSI}}$ and $h \colon \boldsymbol{A}_1 \times \boldsymbol{A}_2 \to \boldsymbol{B}$ a homomorphism. Then there exist $i \in \{1, 2\}$ and a homomorphism $g \colon \boldsymbol{A}_i \to \boldsymbol{B}$ such that $h = g \circ \pi_i$, where $\pi_i \colon \boldsymbol{A}_1 \times \boldsymbol{A}_2 \to \boldsymbol{A}_i$ is the canonical projection map.*

Proof. Since K is relatively congruence distributive and $\boldsymbol{B} \in \mathsf{K}_{\mathrm{RFSI}}$, from [40, Cor. 1.4] it follows that $\boldsymbol{B}$ is a finitely subdirectly irreducible Heyting algebra. Therefore, the greatest element $1^{\boldsymbol{B}}$ of $\boldsymbol{B}$ is join irreducible (see, e.g., [85, Prop. A.4.3]). Let $0^{\boldsymbol{A}_i}$ and $1^{\boldsymbol{A}_i}$ be the least and greatest elements of $\boldsymbol{A}_i$ for $i = 1, 2$. Then

$$1^{\boldsymbol{B}} = h(\langle 1^{\boldsymbol{A}_1}, 1^{\boldsymbol{A}_2}\rangle) = h(\langle 0^{\boldsymbol{A}_1}, 1^{\boldsymbol{A}_2}\rangle \vee \langle 1^{\boldsymbol{A}_1}, 0^{\boldsymbol{A}_2}\rangle) = h(\langle 0^{\boldsymbol{A}_1}, 1^{\boldsymbol{A}_2}\rangle) \vee h(\langle 1^{\boldsymbol{A}_1}, 0^{\boldsymbol{A}_2}\rangle).$$

Since $1^{\boldsymbol{B}}$ is join irreducible, we have

$$h(\langle 0^{\boldsymbol{A}_1}, 1^{\boldsymbol{A}_2}\rangle) = 1^{\boldsymbol{B}} \quad \text{or} \quad h(\langle 1^{\boldsymbol{A}_1}, 0^{\boldsymbol{A}_2}\rangle) = 1^{\boldsymbol{B}}.$$

By symmetry we may assume that $h(\langle 0^{\boldsymbol{A}_1}, 1^{\boldsymbol{A}_2}\rangle) = 1^{\boldsymbol{B}}$. We will prove that

$$h(\langle a, c\rangle) = h(\langle b, c\rangle) \text{ for all } a, b \in A_1 \text{ and } c \in A_2. \tag{69}$$

To this end, observe that

$$h(\langle a, c\rangle) \to h(\langle b, c\rangle) = h(\langle a \to b, c \to c\rangle) = h(\langle a \to b, 1^{\boldsymbol{A}_2}\rangle) \geqslant h(\langle 0^{\boldsymbol{A}_1}, 1^{\boldsymbol{A}_2}\rangle) = 1^{\boldsymbol{B}},$$

and, therefore, $h(\langle a, c\rangle) \leqslant h(\langle b, c\rangle)$. An analogous argument shows that $h(\langle b, c\rangle) \leqslant h(\langle a, c\rangle)$, whence $h(\langle a, c\rangle) = h(\langle b, c\rangle)$, as desired.

Lastly, from (69) it follows that $\ker(\pi_2) \subseteq \ker(h)$. As a straightforward consequence of Proposition 2.6 (see, e.g., [11, Ex. 1.26.8]), we obtain a homomorphism $g \colon \boldsymbol{A}_2 \to \boldsymbol{B}$ such that $h = g \circ \pi_2$. $\qquad \boxtimes$

We are now ready to prove Theorem 14.11.

Proof. To simplify the notation, we let $\boldsymbol{B} = \boldsymbol{A} + \boldsymbol{C}_5$.

Claim 14.22. *There exists $D \in \mathbb{H}(B)$ such that $D \leqslant B \times B$ and $\mathsf{d}_\mathsf{K}(D, B \times B) \neq D$.*

Proof of the Claim. Let the elements of C_5 be $0^{C_5} = c_1 < c_2 < c_3 < c_4 < c_5 = 1^{C_5}$. Recall that $B = (A - \{1^A\}) \cup C_5$. We define

$$D = \{\langle a, a \rangle : a \in A - \{1^A\}\} \cup \{\langle c_1, c_1 \rangle, \langle c_2, c_3 \rangle, \langle c_4, c_4 \rangle, \langle c_5, c_5 \rangle\} \subseteq B \times B.$$

As $B = A + C_5$ and C_5 is linearly ordered, the implication $\rightarrow^B$ of B can be described in terms of the implication $\rightarrow^A$ of A as follows. For all $a, b \in B$ we have

$$a \rightarrow^B b = \begin{cases} c_5 & \text{if } a \leqslant b; \\ a \rightarrow^A b & \text{if } a \not\leqslant b \text{ and } a, b \in A - \{1^A\}; \\ b & \text{otherwise.} \end{cases}$$

It is then immediate to verify that $D \leqslant B \times B$. Moreover, a straightforward verification yields that the map $k \colon B \to B \times B$ defined as follows is a homomorphism whose image is D: for every $b \in B$,

$$k(b) = \begin{cases} \langle b, b \rangle & \text{if } b \in \{c_1, c_5\} \cup (A - \{1^A\}); \\ \langle c_2, c_3 \rangle & \text{if } b = c_2; \\ \langle c_4, c_4 \rangle & \text{if } b = c_3; \\ \langle c_5, c_5 \rangle & \text{if } b = c_4. \end{cases}$$

Thus, $D \in \mathbb{H}(B)$.

Therefore, it only remains to show that $\mathsf{d}_\mathsf{K}(D, B \times B) \neq D$. By Proposition 14.20 we have

$$\mathsf{d}_\mathsf{K}(D, B \times B) = \mathsf{d}_{\mathsf{K}_{\mathrm{RSI}}}(D, B \times B).$$

Since $\langle c_5, c_4 \rangle \in (B \times B) - D$, it suffices to show $\langle c_5, c_4 \rangle \in \mathsf{d}_{\mathsf{K}_{\mathrm{RSI}}}(D, B \times B)$. Let $g, h \colon B \times B \to E$ be a pair of homomorphisms such that $E \in \mathsf{K}_{\mathrm{RSI}}$ and $g{\restriction}_D = h{\restriction}_D$. We need to prove that $g(\langle c_5, c_4 \rangle) = h(\langle c_5, c_4 \rangle)$. Since $E \in \mathsf{K}_{\mathrm{RSI}} \subseteq \mathsf{K}_{\mathrm{RFSI}}$, Proposition 14.21 yields that both g and h factor through a projection. We have two cases: either g and h factor through the same projection or not.

First, suppose that g and h factor through π_1. Then there exists a pair of homomorphisms $g', h' \colon B \to E$ such that $g = g' \circ \pi_1$ and $h = h' \circ \pi_1$. Therefore,

$$g(\langle c_5, c_4 \rangle) = g'(c_5) = 1^E = h'(c_5) = h(\langle c_5, c_4 \rangle),$$

where the second and third equalities hold because c_5 is the greatest element of B, and the others because $g = g' \circ \pi_1$ and $h = h' \circ \pi_1$.

Next we consider the case where both g and h factor through π_2. Then there exists a pair of homomorphisms $g', h' \colon B \to E$ such that $g = g' \circ \pi_2$ and $h = h' \circ \pi_2$. Therefore,

$$g(\langle c_5, c_4 \rangle) = g'(c_4) = g(\langle c_4, c_4 \rangle) = h(\langle c_4, c_4 \rangle) = h'(c_4) = h(\langle c_5, c_4 \rangle),$$

where the middle equality holds because $\langle c_4, c_4 \rangle \in D$ and $g{\restriction}_D = h{\restriction}_D$, and the others because $g = g' \circ \pi_2$ and $h = h' \circ \pi_2$.

Lastly, suppose that g and h factor through different projections. Without loss of generality, we may assume that g factors through π_1 and h factors through π_2. Then there exists a pair

of homomorphisms $g', h' \colon \boldsymbol{B} \to \boldsymbol{E}$ such that $g = g' \circ \pi_1$ and $h = h' \circ \pi_2$. Since $g{\restriction}_D = h{\restriction}_D$ and $\langle c_2, c_3 \rangle, \langle c_4, c_4 \rangle \in D$, we obtain

$$g'(c_2) = g(\langle c_2, c_3 \rangle) = h(\langle c_2, c_3 \rangle) = h'(c_3); \tag{70}$$

$$g'(c_4) = g(\langle c_4, c_4 \rangle) = h(\langle c_4, c_4 \rangle) = h'(c_4). \tag{71}$$

Assume, with a view to contradiction, that both g' and h' are injective. As $\boldsymbol{B} \in \mathcal{M}(\mathsf{K}_{\mathrm{RSI}})$ by assumption and $\boldsymbol{E} \in \mathsf{K}_{\mathrm{RSI}}$, it follows that g' and h' are isomorphisms. Since c_3 is the only element a of $\boldsymbol{B}$ such that $\{ b \in B : a \leqslant b \}$ has size 3, every automorphism of $\boldsymbol{B}$ must fix c_3. It follows that $g'(c_3) = h'(c_3)$ because $(h')^{-1} \circ g'$ is an automorphism of $\boldsymbol{B}$. Then (70) yields $g'(c_2) = h'(c_3) = g'(c_3)$, which is impossible because g' is injective. Therefore, either g' or h' is not injective. Suppose first that g' is not injective. Then there exist $a, b \in B$ such that $a \not\leqslant b$ and $g'(a) = g'(b)$. As $a \not\leqslant b$ and g' is a homomorphism, we obtain $a \to^{\boldsymbol{B}} b \neq 1^{\boldsymbol{B}}$ and $g'(a \to^{\boldsymbol{B}} b) = g'(a) \to^{\boldsymbol{E}} g'(b) = 1^{\boldsymbol{E}}$. Note that every Heyting algebra homomorphism is order preserving because it is a lattice homomorphism. Therefore, since c_4 is the second greatest element of $\boldsymbol{B}$, we have $a \to^{\boldsymbol{B}} b \leqslant c_4$ and, consequently, $1^{\boldsymbol{E}} = g'(a \to^{\boldsymbol{B}} b) \leqslant g'(c_4)$. So, $g'(c_4) = 1^{\boldsymbol{E}}$. Then we have

$$g(\langle c_5, c_4 \rangle) = g'(c_5) = 1^{\boldsymbol{E}} = g'(c_4) = h'(c_4) = h(\langle c_5, c_4 \rangle),$$

where first and last equalities hold because $g = g' \circ \pi_1$ and $h = h' \circ \pi_2$, the second because c_5 is the greatest element of $\boldsymbol{B}$, the third because $g'(c_4) = 1^{\boldsymbol{E}}$ as we just observed, and the fourth follows from (71). Next, suppose that h' is not injective. An argument similar to the one above shows that $h'(c_4) = 1^{\boldsymbol{E}}$. Then

$$h(\langle c_5, c_4 \rangle) = h'(c_4) = 1^{\boldsymbol{E}} = g'(c_5) = g(\langle c_5, c_4 \rangle),$$

where first and last equalities hold because $g = g' \circ \pi_1$ and $h = h' \circ \pi_2$, the second because $h'(c_4) = 1^{\boldsymbol{E}}$ as we just observed, and the third because c_5 is the greatest element of $\boldsymbol{B}$. We conclude that $g(\langle c_5, c_4 \rangle) = h(\langle c_5, c_4 \rangle)$ in all possible cases. Thus, $\langle c_5, c_4 \rangle \in \mathsf{d}_{\mathsf{K}_{\mathrm{RSI}}}(\boldsymbol{D}, \boldsymbol{B} \times \boldsymbol{B})$, as desired.

$\boxtimes$

Let $\boldsymbol{D}$ be as in Claim 14.22. As $\boldsymbol{B} \in \mathsf{K}$, $\boldsymbol{D} \leqslant \boldsymbol{B} \times \boldsymbol{B}$, and K is a quasivariety, we obtain $\boldsymbol{D}, \boldsymbol{B} \times \boldsymbol{B} \in \mathsf{K}$. Since $\boldsymbol{B} \in \mathcal{M}(\mathsf{K}_{\mathrm{RSI}})$ and $\boldsymbol{D} \in \mathbb{H}(\boldsymbol{B})$, it follows that $\boldsymbol{D}, \boldsymbol{B} \times \boldsymbol{B} \in \mathbb{H}\mathbb{P}(\mathcal{M}(\mathsf{K}_{\mathrm{RSI}})) \cap \mathsf{K}$. If K had a Beth companion, then Proposition 14.19 would imply that $\mathsf{d}_{\mathsf{K}}(\boldsymbol{D}, \boldsymbol{B} \times \boldsymbol{B}) = D$, contradicting Claim 14.22. Thus, K lacks a Beth companion. $\boxtimes$

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