Algebraic Logic

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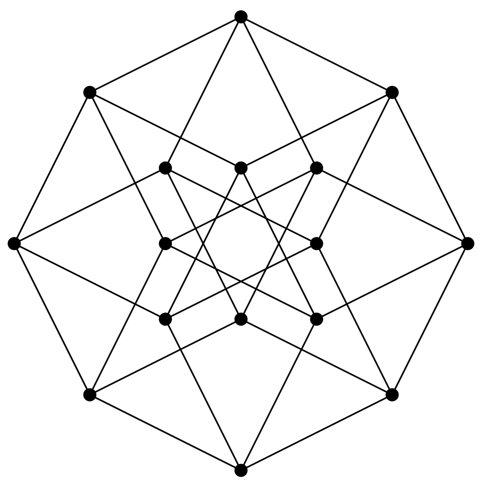


Contents

CHAPTER 1

Lattices and closure operators

1.1 Partially ordered sets

Given a binary relation R on a set X and $x, y \in X$, we write xRy when the ordered pair $\langle x, y \rangle$ belongs to the relation R. A binary relation R on a set X is said to be

(i) *reflexive* when xRx for every $x \in X$;

(ii) *transitive* when for every $x, y, z \in X$, if xRy and yRz, then xRz;

(iii) *antisymmetric* when for every $x, y \in X$, if xRy and yRx, then $x = y$.

Definition 1.1. A binary relation $\leqslant$ on a set X is said to be a *partial order* when it is reflexive, transitive, and antisymmetric. In this case, the pair $\mathbb{X} = \langle X; \leqslant \rangle$ is said to be a *poset* (a shorthand for a *partially ordered set*) and X is called the *universe* of $\mathbb{X}$.

Given a poset $\mathbb{X}$ and $x, y \in X$, we will often say that x is *below* y to indicate that $x \leqslant y$. Furthermore, we write $x < y$ when both $x \leqslant y$ and $x \neq y$, or equivalently $x \leqslant y$ and $y \nleqslant x$. Lastly, given $x, y, z \in X$, we will write $x \leqslant y, z$ as a shorthand for $x \leqslant y$ and $x \leqslant z$. A similar reading applies to expressions of the form $x, y \leqslant z$. Notice that the universe of $\mathbb{X}$ might be empty, in which case the relation $\leqslant$ is also empty.

Definition 1.2. Let $\mathbb{X}$ be a poset. A set $V \subseteq X$ is said to be

(i) an *upset* of $\mathbb{X}$ when for every $x, y \in X$,

$$\text{if } x \in V \text{ and } x \leqslant y, \text{ then } y \in V;$$

(ii) a *downset* of $\mathbb{X}$ when for every $x, y \in X$,

$$\text{if } x \in V \text{ and } y \leqslant x, \text{ then } y \in V.$$

The families of upsets and downsets of $\mathbb{X}$ will be denoted by $\mathsf{Up}(\mathbb{X})$ and $\mathsf{Down}(\mathbb{X})$.

Example 1.3 (Posets). Given a set X, the *identity relation* id_X is defined for all $x, y \in X$ as

$$\langle x, y\rangle \in \mathrm{id}_X \iff x = y.$$

The pair $\langle X; \mathrm{id}_X\rangle$ is a poset, known as the *discrete poset* with universe X.

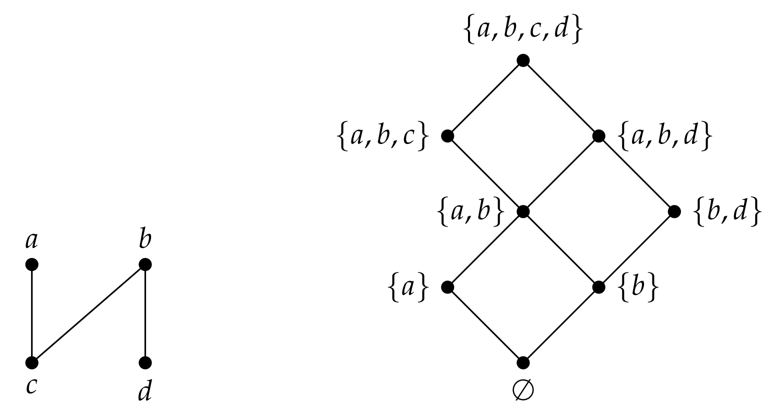
Given a poset $\mathbb{X}$, the pairs $\langle \mathsf{Up}(\mathbb{X}); \subseteq\rangle$ and $\langle \mathsf{Down}(\mathbb{X}); \subseteq\rangle$ are also posets. When $\mathbb{X} = \langle X; \mathrm{id}_X\rangle$, both $\langle \mathsf{Up}(\mathbb{X}); \subseteq\rangle$ and $\langle \mathsf{Down}(\mathbb{X}); \subseteq\rangle$ coincide with the poset $\langle \mathcal{P}(X); \subseteq\rangle$, which is known as the *powerset lattice* of X.

Further examples include the sets $\mathbb{N}$ and $\mathbb{Z}$ of natural and integer numbers, respectively, endowed with the standard order. Lastly, the set $\mathbb{N}$ endowed with the *divisibility relation*

$$| := \{\langle m, n\rangle \in \mathbb{N} \times \mathbb{N} : \text{there exists } k \in \mathbb{N} \text{ s.t. } n = m \cdot k\}$$

is also a poset, known as the *divisibility lattice*. However, the set $\mathbb{Z}$ fails to be a poset when we endow it with the analogous divisibility relation. This is because the divisibility relation is not antisymmetric on $\mathbb{Z}$, as n and $-n$ are distinct and divide each other, for every positive integer n. ⊠

An attractive feature of posets is that they can often be represented by pictures consisting of dots connected by lines, known as *Hasse diagrams*. This is always true for finite posets $\mathbb{X}$ of a manageable size, whose Hasse diagrams can be obtained as follows. First, we depict the elements of X as dots, making sure that if $x < y$, then the dot corresponding to x lies below that corresponding to y. Then we connect two dots with a line whenever they correspond to two points $x, y \in X$ such that $x < y$ and that there is no $z \in X$ such that $x < z < y$. As a result, $x < y$ for two elements $x, y \in X$ precisely when there exists an ascending path from x to y. The picture below illustrates this method by depicting the Hasse diagram of poset $\mathbb{X}$ and that of the poset $\langle \mathsf{Up}(\mathbb{X}); \subseteq\rangle$ of its upsets.



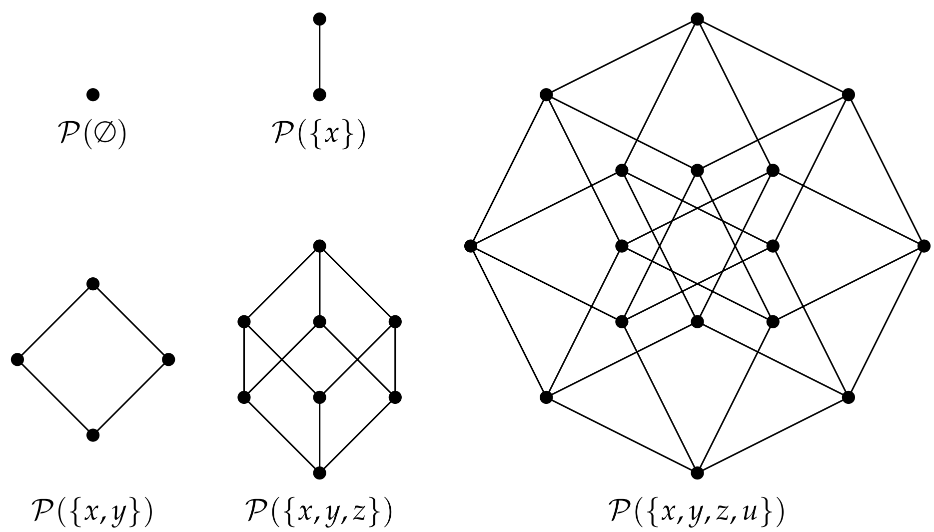
$\mathbb{X}$ $\qquad$ $\mathsf{Up}(\mathbb{X})$

The left hand side of this picture indicates that $\mathbb{X}$ is the poset with universe $\{a, b, c, d\}$ and order relation

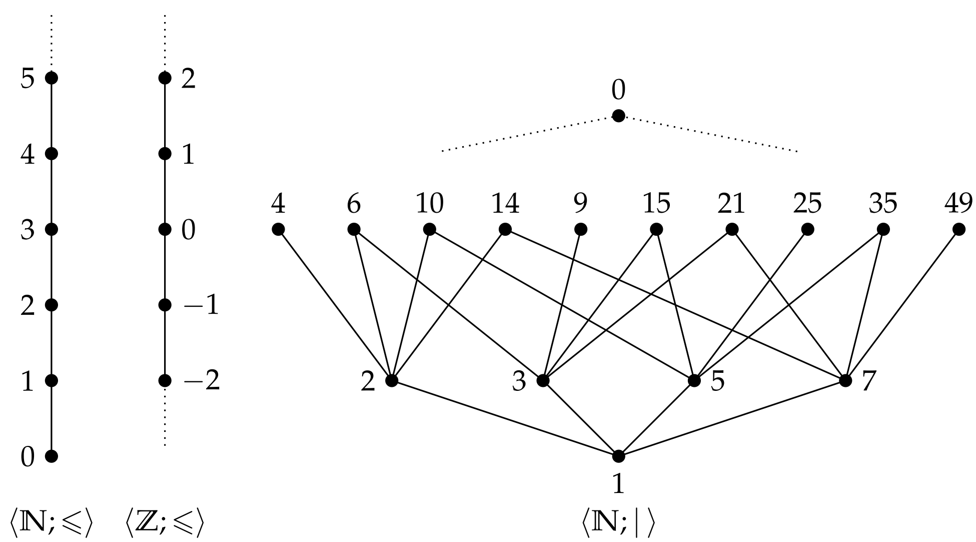
$$\leqslant \, = \mathrm{id}_X \cup \{\langle c, a\rangle, \langle c, b\rangle, \langle d, b\rangle\}.$$

The structure of $\mathsf{Up}(\mathbb{X})$ can be inferred in a similar way from the the right hand side of the picture.

Finite posets can be faithfully reconstructed from their Hasse diagrams. In other words, to define a finite poset it suffices to draw its Hasse diagram. As an example, the Hasse diagrams of the powerset lattices $\langle \mathcal{P}(X); \subseteq \rangle$ with X of cardinality $\leqslant 4$ are shown below.



Hasse diagrams can also be employed to describe infinite posets, provided that their structure is regular enough to be indicated by dotted lines. For instance, the posets $\langle \mathbb{N}; \leqslant \rangle$ and $\langle \mathbb{Z}; \leqslant \rangle$ are depicted below, together with a portion of the divisibility lattice $\langle \mathbb{N}; | \rangle$.



In order to relate distinct posets, it is convenient to introduce maps that preserve their structure. In this context, we will need to distinguish between the order relations of different posets. To this end, we use the symbol $\leqslant^{\mathbb{X}}$ to denote the order relation of a poset $\mathbb{X}$.

Definition 1.4. Let $\mathbb{X}$ and $\mathbb{Y}$ be posets. A map $f\colon \mathbb{X} \to \mathbb{Y}$ is said to be

(i) *order preserving* when for every $x, y \in X$,

$$\text{if } x \leqslant^{\mathbb{X}} y, \text{ then } f(x) \leqslant^{\mathbb{Y}} f(y);$$

(ii) *order reflecting* when for every $x, y \in X$,

$$\text{if } f(x) \leqslant^{\mathbb{Y}} f(y), \text{ then } x \leqslant^{\mathbb{X}} y;$$

(iii) an *order embedding* when it is both order preserving and order reflecting;

(iv) an *order isomorphism* when it is a surjective order embedding.

For instance, the inclusion map $i\colon \mathbb{N} \to \mathbb{Z}$ is an order embedding of $\langle \mathbb{N}; \leqslant \rangle$ into $\langle \mathbb{Z}; \leqslant \rangle$, while the function $f\colon \mathbb{Z} \to \mathbb{N}$ defined by the rule

$$f(n) := \text{the greatest element between } 0 \text{ and } n$$

is an order preserving map from $\langle \mathbb{Z}; \leqslant \rangle$ to $\langle \mathbb{N}; \leqslant \rangle$ that is not order reflecting.

Notice that order embeddings are necessarily injective. To prove this, consider an order embedding $f\colon \mathbb{X} \to \mathbb{Y}$ and two distinct elements $x, y \in X$. Since $\leqslant^{\mathbb{X}}$ is antisymmetric and $x \neq y$, either $x \nleqslant^{\mathbb{X}} y$ or $y \nleqslant^{\mathbb{X}} x$. But f is order reflecting, so either $f(x) \nleqslant^{\mathbb{Y}} f(y)$ or $f(y) \nleqslant^{\mathbb{Y}} f(x)$. In either case, $f(x) \neq f(y)$ by the reflexivity of the relation $\leqslant^{\mathbb{Y}}$, thus f is indeed injective.

As a consequence, an order isomorphism $f\colon \mathbb{X} \to \mathbb{Y}$ is precisely a bijection such that for every $x, y \in X$,

$$x \leqslant^{\mathbb{X}} y \Longleftrightarrow f(x) \leqslant^{\mathbb{Y}} f(y).$$

It follows that $f\colon \mathbb{X} \to \mathbb{Y}$ is an order isomorphism iff so is $f^{-1}\colon \mathbb{Y} \to \mathbb{X}$. We will write $\mathbb{X} \cong \mathbb{Y}$ to indicate that there exists an order isomorphism between $\mathbb{X}$ and $\mathbb{Y}$. For instance, let $2\mathbb{Z}$ be the set of even integers. The map $f\colon \mathbb{Z} \to 2\mathbb{Z}$ defined by the rule $f(n) := 2n$ is an order isomorphism from $\langle \mathbb{Z}; \leqslant \rangle$ to the poset consisting of $2\mathbb{Z}$ ordered under the standard order.

When there exists an order embedding $f\colon \mathbb{X} \to \mathbb{Y}$, the poset $\mathbb{Y}$ contains a copy of $\mathbb{X}$, consisting of the elements of the image $f[X]$ endowed with the restriction of the order relation $\leqslant^{\mathbb{Y}}$ to $f[X]$, as we proceed to explain.

Definition 1.5. A poset $\mathbb{X}$ is said to be a *subposet* of a poset $\mathbb{Y}$ when $X \subseteq Y$ and $\leqslant^{\mathbb{X}}$ is the restriction of $\leqslant^{\mathbb{Y}}$ to X.

In this case, if no confusion is likely to arise, we will use the same notation for the order relation of $\mathbb{X}$ and $\mathbb{Y}$.

Proposition 1.6. *A poset $\mathbb{X}$ is isomorphic to a subposet of $\mathbb{Y}$ iff there exists an order embedding $f\colon \mathbb{X} \to \mathbb{Y}$. In this case, $\mathbb{X}$ is isomorphic to $\langle f[X]; \leqslant \rangle$, where $\leqslant$ is the restriction of $\leqslant^{\mathbb{Y}}$ to $f[X]$.*

Proof. First, suppose that $\mathbb{X}$ is isomorphic to a subposet $\mathbb{Z}$ of $\mathbb{Y}$. Then there exists an order isomorphism $f\colon \mathbb{X} \to \mathbb{Z}$. Since $\mathbb{Z}$ is a subposet of $\mathbb{Y}$, the map $f\colon \mathbb{X} \to \mathbb{Y}$ is an order embedding. To prove the converse, consider an order embedding $f\colon \mathbb{X} \to \mathbb{Y}$ and let $\langle f[X]; \leqslant\rangle$ be the poset defined in the statement. The map $f\colon \mathbb{X} \to \langle f[X]; \leqslant\rangle$ is clearly surjective. Furthermore, it is an order embedding, since so is $f\colon \mathbb{X} \to \mathbb{Y}$. Thus, we conclude that $f\colon \mathbb{X} \to \langle f[X]; \leqslant\rangle$ is an order isomorphism. ⊠

The following notions are instrumental to comparing the elements of a given poset.

Definition 1.7. Two elements x and y of a poset $\mathbb{X}$ are said to be *comparable* when either $x \leqslant y$ or $y \leqslant x$. They are said to be *incomparable* otherwise. Accordingly, we say that $\mathbb{X}$ is

(i) a *chain* when every two elements of X are comparable;

(ii) an *antichain* when every two distinct elements of X are incomparable, that is, if $\mathbb{X}$ is the discrete poset with universe X.

By extension, a subset $Y \subseteq X$ is said to be a *chain* (resp. an *antichain*) in $\mathbb{X}$ when the subposet of $\mathbb{X}$ with universe Y is a chain (resp. an antichain). Chains are sometimes also called *linearly ordered* posets, while antichains are sometimes called *discretely ordered* posets.

For instance, the number systems $\langle \mathbb{N}; \leqslant\rangle$ and $\langle \mathbb{Z}; \leqslant\rangle$ are both chains. Most posets, however, are neither chains nor antichains. For example, the powerset lattice $\langle \mathcal{P}(X); \subseteq\rangle$ is neither a chain nor an antichain for every set X with at least two elements. To see this, consider two distinct elements $x, y \in X$. Then the sets $\{x\}$ and $\{y\}$ are incomparable elements of $\langle \mathcal{P}(X); \subseteq\rangle$, so this poset is not a chain. Similarly, $\emptyset$ and $\{x\}$ are two distinct, but comparable elements. Consequently, $\langle \mathcal{P}(X); \subseteq\rangle$ is not an antichain.

The following special elements play a fundamental role in order theory:

Definition 1.8. An element x of a poset $\mathbb{X}$ is said to be

(i) a *greatest element* of $\mathbb{X}$ when $y \leqslant x$ for every $y \in X$;

(ii) a *least* element of $\mathbb{X}$ when $x \leqslant y$ for every $y \in X$;

(iii) a *maximal element* of $\mathbb{X}$ when there is no $y \in X$ such that $x < y$;

(iv) a *minimal element* of $\mathbb{X}$ when there is no $y \in X$ such that $y < x$;

(v) an *upper bound of a set* $Y \subseteq X$ in $\mathbb{X}$ when $y \leqslant x$ for every $y \in Y$;

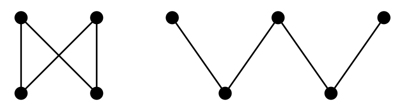
(vi) a *lower bound of a set* $Y \subseteq X$ in $\mathbb{X}$ when $x \leqslant y$ for every $y \in Y$.

When $\mathbb{X}$ has both a greatest and least element, it is said to be *bounded*.

Every poset has at most one greatest (resp. least) element. To prove this, let x and y be greatest elements of a poset $\mathbb{X}$. Since x is a greatest element of $\mathbb{X}$, we have $y \leqslant x$. Similarly, the assumption that y is a greatest element yields $x \leqslant y$. By antisymmetry we conclude that $x = y$. The case of least elements is analogous. Accordingly, when they exist, we will denote the greatest and least elements of a poset by 1 and 0.

While a greatest element is always maximal (resp. a least element is always minimal), the converse need not hold in general. For instance, every element of the poset $\langle X; \mathrm{id}_X \rangle$ is both maximal and minimal, but none of them is a greatest or least element of $\mathbb{X}$, provided that X has at least two elements.

In general, maximal and minimal elements need not exist. A counterexample is the poset $\langle \mathbb{Z}; \leqslant \rangle$, which lacks both maximal and minimal element. Furthermore, the existence of maximal (resp. minimal) elements does not imply the existence of a greatest (resp. least) element. For instance, the following poset has five maximal and four minimal elements, but lacks both a greatest and a least element.



As we mentioned, infinite posets, such as $\langle \mathbb{Z}; \leqslant \rangle$, may lack maximal and minimal elements. However, this cannot happen in the finite case.

Proposition 1.9. *Every nonempty finite poset has both a maximal and a minimal element.*

Proof. Suppose, with a view to contradiction, that a finite nonempty poset $\mathbb{X}$ lacks either maximal or minimal elements. We deal with the case when $\mathbb{X}$ has no maximal elements, as the proof of the other case is symmetric. As $\mathbb{X}$ is nonempty, it has an element $x_1 \in X$. Since x_1 is not maximal, there exists some $x_2 \in X$ such that $x_1 < x_2$. Similarly, since x_2 is not maximal, there exists some $x_3 \in X$ such that $x_2 < x_3$. Iterating this construction, we produce an infinite sequence

$$x_1 < x_2 < x_3 < \cdots < x_n < \cdots$$

of distinct elements of $\mathbb{X}$, contradicting the assumption that $\mathbb{X}$ is finite. ⊠

Our proofs will often rely on the existence of a maximal element of some poset. Since a poset may lack maximal elements, the following principle plays a fundamental role in such proofs. Notably, it is equivalent (over Zermelo–Fraenkel set theory) to the Axiom of Choice.

Zorn's Lemma. *Let $\mathbb{X}$ be a poset. If every chain in a poset $\mathbb{X}$ has an upper bound in $\mathbb{X}$, then $\mathbb{X}$ has a maximal element.*

An elementary observation is that if $\mathbb{X} = \langle X; \leqslant \rangle$ is a poset, then so is $\mathbb{X}^\partial := \langle X; \geqslant \rangle$, where $x \geqslant y$ iff $y \leqslant x$. The poset $\mathbb{X}^\partial$ is called the *(order) dual* of $\mathbb{X}$.

Similarly, each statement Φ about posets has an order dual statement Φ^∂ obtained by replacing every occurrence of $\leqslant$ (resp. $\geqslant$) in Φ by $\geqslant$ (resp. $\leqslant$). Then Φ^∂ holds in a poset $\mathbb{X}$ iff Φ holds in $\mathbb{X}^\partial$. This definition could be made more precise by given a formal account of what counts as a statement about posets. However, doing so would take us to far afield from the subject at hand, therefore we shall content ourselves with relying on the reader's informal understanding of what counts as a statement about posets. The purpose of introducing Φ^∂ is that it allows us to formulate the following extremely useful principle, which will often allow us to cut our work down to half.

Duality Principle. *If a statement Φ is true in all posets, then Φ^∂ is also true in all posets.*

Proof. If Φ holds in each poset $\mathbb{X}$, then in particular it holds in each poset of the form $\mathbb{X}^\partial$. But Φ holds in $\mathbb{X}^\partial$ if and only if Φ^∂ holds in $\mathbb{X}$, so Φ^∂ indeed holds in every poset. $\boxtimes$

For example, applying the Duality Principle to Zorn's Lemma proves that if every chain in a poset $\mathbb{X}$ has a lower bound in $\mathbb{X}$, then $\mathbb{X}$ has a minimal element.

1.2 Lattices as partially ordered sets

Posets in which certain optimal upper and lower bounds exist will be called *lattices*. The next definition explains what "optimal" means in this context.

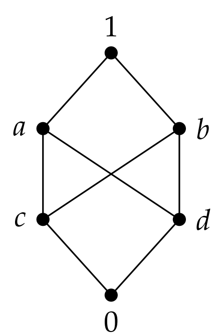
Definition 1.10. Let $\mathbb{X}$ be a poset and $Y \subseteq X$. An element $x \in X$ is said to be

(i) a *meet of Y in $\mathbb{X}$* when x is the greatest lower bound of Y in $\mathbb{X}$;

(ii) a *join of Y in $\mathbb{X}$* when x is the least upper bound of Y in $\mathbb{X}$.

When the poset $\mathbb{X}$ is clear from the context, we simply say that x is a meet or a join of Y. The terms *infimum* and *supremum* are sometimes used as synonyms for *meet* and *join*.

Not every subset of a poset has a meet or a join. For instance, in the poset of natural numbers $\langle \mathbb{N}; \leqslant \rangle$, the join of $\mathbb{N}$ does not exist, because $\mathbb{N}$ has no upper bound in $\langle \mathbb{N}; \leqslant \rangle$. However, even if a set has an upper bound, the least such may not exist. For instance, while every subset of the poset $\mathbb{X}$ depicted below has an upper bound (namely, the greatest element 1), the least upper bound of $Y := \{c, d\}$ in $\mathbb{X}$ does not exist, because the set $\{1, a, b\}$ of upper bounds of Y lacks a least element.



While meets and joins need not exist in general, they are necessarily unique whenever they do exist.

Proposition 1.11. *Let $\mathbb{X}$ be a poset and $Y \subseteq X$. When the meet (resp. join) of Y exists, it is unique.*

Proof. Suppose that $x, y \in X$ are both meets of Y. As x is a meet of Y, it is a lower bound of Y. Moreover, being a meet of Y, the element y is the greatest lower bound of Y. As a consequence, $x \leqslant y$. A similar argument shows that $y \leqslant x$. As the relation $\leqslant$ is antisymmetric, we conclude that $x = y$. The case of joins is handled similarly. $\boxtimes$

Accordingly, given a poset $\mathbb{X}$ and a set $Y \subseteq X$,

(i) if the meet of Y in $\mathbb{X}$ exists, we denote it by $\bigwedge Y$;

(ii) if the join of Y in $\mathbb{X}$ exists, we denote it by $\bigvee Y$.

When it is convenient to stress that these meets and joins are considered in the poset $\mathbb{X}$, we will add the appropriate superscript and write

$$\bigwedge^{\mathbb{X}} Y \text{ and } \bigvee^{\mathbb{X}} Y.$$

When $Y = \{x, y\}$, we will write $x \wedge y$ and $x \vee y$ instead of $\bigwedge Y$ and $\bigvee Y$. Lastly, given a family $\{y_i : i \in I\}$ of elements of X, we will use $\bigwedge_{i \in I} y_i$ and $\bigvee i \in I y_i$ as a shorthand for, respectively, $\bigwedge\{y_i : i \in I\}$ and $\bigvee\{y_i : i \in I\}$.

It will often be convenient to paraphrase the definition of meets and joins in a poset $\mathbb{X}$ as follows. The meet of $Y \subseteq X$, when it exists, is the unique element $\bigwedge Y \in X$ such that

$$x \leqslant \bigwedge Y \iff x \leqslant y \text{ for each } y \in Y.$$

Similarly, the join of $Y \subseteq X$, if it exists, is the unique element $\bigvee Y \in X$ such that

$$\bigvee Y \leqslant x \iff y \leqslant x \text{ for each } y \in Y.$$

Definition 1.12. A nonempty poset $\mathbb{X}$ is said to be

(i) a *meet semilattice* when the meet of $\{x, y\}$ exists for every pair of elements $x, y \in X$;

(ii) a *join semilattice* when the join of $\{x, y\}$ exists for every pair of elements $x, y \in X$;

(iii) a *lattice* when it is both a meet semilattice and a join semilattice.

Notice that if the meet (resp. join) of a set $Y \subseteq X$ exists in a poset $\mathbb{X}$ and equals some $y \in X$, then the join (resp. meet) of the set Y exists in the order dual poset $\mathbb{X}^\partial$ and also equals y. In particular, a poset $\mathbb{X}$ is a meet (resp. join) semilattice iff its order dual $\mathbb{X}^\partial$ is a join (resp. meet) semilattice. As a consequence, we obtain the following variant of the Duality Principle for lattices, which is a special case of the Duality Principle for posets.

Duality Principle. *If a statement Φ is true in all lattices, then the statement obtained by replacing $\bigwedge$ with $\bigvee$, $\bigvee$ with $\bigwedge$, $\leqslant$ with $\geqslant$, and $\geqslant$ with $\leqslant$ in Φ is also true in all lattices.*

In a lattice $\mathbb{X}$, the meet and the join of every finite nonempty set $\{x_1, \ldots, x_n\} \subseteq X$ exist and coincide with

$$x_1 \wedge (x_2 \wedge \ldots (x_{n-1} \wedge x_n) \ldots) \text{ and } x_1 \vee (x_2 \vee \ldots (x_{n-1} \vee x_n) \ldots).$$

On the other hand, meet and joins of arbitrary subsets of X need not exists in $\mathbb{X}$. This makes the following concept interesting:

Definition 1.13. A lattice $\mathbb{X}$ is said to be *complete* when $\bigwedge Y$ and $\bigvee Y$ exist for every $Y \subseteq X$.

Every complete lattice $\mathbb{X}$ is bounded, because $\bigvee X$ and $\bigwedge X$ are, respectively, the greatest and the least element of $\mathbb{X}$.

Example 1.14 (Upsets & downsets)**.** Given a poset $\mathbb{X}$, the pairs $\langle \mathsf{Up}(\mathbb{X}); \subseteq \rangle$ and $\langle \mathsf{Down}(\mathbb{X}); \subseteq \rangle$ are complete lattices in which meets and joins are intersections and unions. To prove that this holds for $\langle \mathsf{Up}(\mathbb{X}); \subseteq \rangle$, it suffices to show that

$$\bigwedge^{\mathsf{Up}(\mathbb{X})} U = \bigcap_{i \in I} U_i \quad \text{and} \quad \bigvee_{i \in I}^{\mathsf{Up}(\mathbb{X})} U = \bigcup_{i \in I} U_i$$

for every family $U = \{U_i : i \in I\} \subseteq \mathsf{Up}(\mathbb{X})$. As an example, we detail the proof of the first equality. We need to show that $\bigcap_{i \in I} U_i$ is the greatest lower bound of U in $\langle \mathsf{Up}(\mathbb{X}); \subseteq \rangle$. Observe first that $\bigcap_{i \in I} U_i$ is still an upset of $\mathbb{X}$. Then notice that $\bigcap_{i \in I} U_i \subseteq U_j$ for all $j \in I$, so $\bigcap_{i \in I} U_i$ is a lower bound of U. To prove that it is the greatest one, consider a lower bound V of U. Then $V \subseteq U_i$ for all $i \in I$. Consequently, $V \subseteq \bigcap_{i \in I} U_i$ as desired. The proof that $\langle \mathsf{Down}(\mathbb{X}); \subseteq \rangle$ is also a complete lattice is analogous.

Now, recall that, when $\mathbb{X} = \langle X; \mathrm{id}_X \rangle$, the lattices $\langle \mathsf{Up}(\mathbb{X}); \subseteq \rangle$ and $\langle \mathsf{Down}(\mathbb{X}); \subseteq \rangle$ coincide with the powerset lattice $\langle \mathcal{P}(X); \subseteq \rangle$. The above discussion implies that $\langle \mathcal{P}(X); \subseteq \rangle$ is also a complete lattice in which meets and joins are intesections and unions. $\boxtimes$

Example 1.15 (Chains are lattices)**.** Every nonempty chain $\mathbb{X}$ is a lattice. To prove this, observe that every two elements x and y of $\mathbb{X}$ are comparable because $\mathbb{X}$ is a chain. Accordingly, we will prove that

$$\begin{aligned} x \wedge y &= \text{the least element among } x \text{ and } y; \\ x \vee y &= \text{the greatest element among } x \text{ and } y. \end{aligned}$$

To prove the first equality above, let z be the least element among x and y. Clearly, $z \leqslant x, y$, whence z is a lower bound of $\{x, y\}$. Then let $u \in X$ be another lower bound of $\{x, y\}$. As $z \in \{x, y\}$, this implies $u \leqslant z$. Hence, we conclude that z is the greatest lower bound of $\{x, y\}$, that is, $z = x \wedge y$. The case of joins is handled similarly. $\boxtimes$

Example 1.16 (Chains need not be complete)**.** While every nonempty chain is a lattice, it need not be a complete lattice. One reason is that every complete lattice is bounded, but a chain may lack a greatest or a least element, as in the case of $\langle \mathbb{N}; \leqslant \rangle$ and $\langle \mathbb{Z}; \leqslant \rangle$. In a chain without a greatest element, some subsets do not have a least upper bound simply because they have no upper bound at all. However, it may also happen that a subset which has an upper bound fails to have a least upper bound.

For instance, consider the chain of rational numbers $\langle \mathbb{Q}; \leqslant \rangle$ with the standard order and recall that $\mathbb{Q}$ is *dense* in $\mathbb{R}$, in the sense that for every $x, y \in \mathbb{R}$ with $x < y$ there exists $z \in \mathbb{Q}$ such that $x < z < y$. Then take

$$X := \{x \in \mathbb{Q} : x^2 < 2\} = \{x \in \mathbb{Q} : -\sqrt{2} < x < +\sqrt{2}\}.$$

We claim that the join $\bigvee X$ does not exist in $\langle \mathbb{Q}; \leqslant \rangle$. Indeed, suppose for the sake of contradiction that this join is some $y \in \mathbb{Q}$. It cannot be that $\sqrt{2} < y$: in that case the density of $\mathbb{Q}$ in $\mathbb{R}$ guarantees the existence of some $z \in \mathbb{Q}$ with $\sqrt{2} < z < y$, but then z is also an upper bound of X, so y is not the least upper bound of X. It also cannot be that $y < \sqrt{2}$: in that case the density of $\mathbb{Q}$ in $\mathbb{R}$ guarantees the existence of some positive $z \in \mathbb{Q}$ with $y < z < \sqrt{2}$, but then $z \in X$, so y is an an upper bound of X. We have therefore reached a contradiction, since for each rational y either $\sqrt{2} < y$ or $y < \sqrt{2}$. A similar reasoning shows that the meet $\bigwedge X$ also does not exist in $\langle \mathbb{Q}; \leqslant \rangle$. ⊠

The task of proving that a given poset is a complete lattice is simplified by following observation:

Proposition 1.17. *Let $\mathbb{X}$ be a poset. If $\bigwedge Y$ (resp. $\bigvee Y$) exists for all $Y \subseteq X$, then $\mathbb{X}$ is a complete lattice.*

Proof. Suppose that $\bigwedge Y$ exists for all $Y \subseteq X$. It suffices to show that the join of every subset $Y \subseteq X$ exists as well. To this end, consider $Y \subseteq X$ and let $U(Y)$ be the set of upper bounds of Y in $\mathbb{X}$. By assumption, the element $x := \bigwedge U(Y)$ exists in $\mathbb{X}$. We will prove that x is the join of Y in $\mathbb{X}$. First, the definition of x guarantees that x is a lower bound of $U(Y)$. Therefore, it only remains to prove that $x \in U(Y)$. Accordingly, consider an element $y \in Y$. Clearly, y is a lower bound of $U(Y)$. Together with the assumption that x is the greatest lower bound of $U(Y)$, this implies $y \leqslant x$. Since $y \in U(Y)$ and $x \leqslant y$, we obtain $x \in U(Y)$. Hence, we conclude that $x = \bigvee Y$ and, therefore, that $\mathbb{X}$ is a complete lattice. By the Duality Principle, the variant of the statement involving joins holds too. ⊠

In order to apply the above test to a poset $\mathbb{X}$, it is sometimes useful to distinguish the case where $Y = \emptyset$ from the one in which Y is nonempty. Because of this, the following description of the meet and the join of the empty set is of special interest.

Proposition 1.18. *The following conditions hold for a poset $\mathbb{X}$:*

(i) *$\mathbb{X}$ has a greatest element 1 if and only if $\bigwedge \emptyset$ exists in $\mathbb{X}$. In this case, $\bigwedge \emptyset = 1$;*

(ii) *$\mathbb{X}$ has a least element 0 if and only if $\bigvee \emptyset$ exists in $\mathbb{X}$. In this case, $\bigvee \emptyset = 0$.*

Proof. We detail only the proof of condition (i), as condition (ii) will follow from the Duality Principle. First, suppose that $\bigwedge \emptyset$ exists in $\mathbb{X}$. Since every element of $\mathbb{X}$ is vacuously a lower bound of $\emptyset$ in $\mathbb{X}$ and $\bigwedge \emptyset$ is the greatest such lower bound, we obtain that $x \leqslant \bigwedge \emptyset$ for all $x \in X$. Hence, $\bigwedge \emptyset$ is the greatest element of $\mathbb{X}$. Conversely, suppose that $\mathbb{X}$ has a greatest element 1. Then 1 is vacuously a lower bound of $\emptyset$ in $\mathbb{X}$. Furthermore, being the greatest element of $\mathbb{X}$, it is obviously the the greatest such lower bound. Thus, $1 = \bigwedge \emptyset$ as desired. ⊠

Corollary 1.19. *A poset $\mathbb{X}$ is a complete lattice iff it has a greatest (resp. least) element and the meet (resp. join) of every nonempty $Y \subseteq X$ exists in $\mathbb{X}$.*

Proof. Immediate from Propositions 1.17 and 1.18. ⊠

Corollary 1.19 is instrumental in proving that concrete posets are complete lattices, as we proceed to illustrate.

Example 1.20 (Divisibility lattice)**.** We will prove that the divisibility lattice $\langle \mathbb{N}; | \rangle$ is complete. Since $\langle \mathbb{N}; | \rangle$ has a greatest element, namely 0, it suffices to show that every nonempty $X \subseteq \mathbb{N}$ has a meet in $\langle \mathbb{N}; | \rangle$. To this end, observe that there is a greatest common divisor n of the elements of X: if $X = \{0\}$, then $n = 0$, otherwise n is the (finite) product of all powers of primes p^k such that p^k divides all elements of X but p^{k+1} does not divide some element of X. The definition of the order of $\langle \mathbb{N}; | \rangle$ guarantees that n is the greatest lower bound (that is, the meet) of X. ⊠

Another family of complete lattices arises in relation to the following concept:

Definition 1.21. A *topology* on a set X is a family $\tau \subseteq \mathcal{P}(X)$ such that

(i) $\emptyset, X \in \tau$;

(ii) If $Y, Z \in \tau$, then $Y \cap Z \in \tau$;

(iii) If $\{Y_i : i \in I\} \subseteq \tau$, then $\bigcup_{i \in I} Y_i \in \tau$.

In this case, the pair $\langle X; \tau \rangle$ is said to be a *topological space*. Furthermore, the elements of τ are called *open* and their complements relative to X *closed*. It follows that the collection of closed sets of a topological space $\langle X; \tau \rangle$ contains $\emptyset$ and X and is closed under binary unions and arbitrary intersections.

Example 1.22 (Order topology)**.** Consider a chain $\mathbb{X}$. By analogy with the open intervals of the real line, we may define the *open intervals* of $\mathbb{X}$ to be sets of the following three kinds:

$$\begin{aligned} (x, y) &:= \{z \in X : x < z < y\}, \\ (-\infty, x) &:= \{z \in X : z < x\}, \\ (x, +\infty) &:= \{z \in X : x < z\}. \end{aligned}$$

The collection of arbitrary unions of opens intervals of these three forms a topology called the *order topology* on $\mathbb{X}$. ⊠

The order topology on the chain $\langle \mathbb{R}; \leqslant \rangle$ of the real numbers is often called the *standard topology* on $\mathbb{R}$. Every open set of the standard topology of $\mathbb{R}$ is in fact a union of sets of the form (x, y), since subsets of $\mathbb{R}$ of the form $(z, +\infty)$ and $(-\infty, z)$ can be obtained as unions of subsets of $\mathbb{R}$ of the form (x, y) and so can the whole set $\mathbb{R}$.

Example 1.23 (Lattices of closed sets)**.** Let $\langle X; \tau \rangle$ be a topological space and $\mathsf{Cl}(X; \tau)$ the family of its closed sets. We will prove that the pair $\langle \mathsf{Cl}(X; \tau); \subseteq \rangle$ is a complete lattice. In view of Proposition 1.17, it suffices to show that arbitrary meets exist in $\langle \mathsf{Cl}(X; \tau); \subseteq \rangle$. But this follows immediately from the fact that the collection $\mathsf{Cl}(X; \tau)$ is closed under arbitrary intersections. Consequently, $\langle \mathsf{Cl}(X; \tau); \subseteq \rangle$ is a complete lattice in which arbitrary meets coincide with intersections. However, arbitrary joins in $\langle \mathsf{Cl}(X; \tau); \subseteq \rangle$ need not coincide with unions. ⊠

As we mentioned, lattices may fail to be complete, easy counterexamples being number systems. However, this cannot happen in the finite case.

Proposition 1.24. *Every finite lattice is complete.*

Proof. In view of Proposition 1.17, it suffices to show that $\bigwedge Y$ exists in $\mathbb{X}$ for all $Y \subseteq X$. There are two cases: either $Y = \emptyset$ or X is nonempty. Suppose first that $Y = \emptyset$. Then take an enumeration $X = \{x_1, \dots, x_n\}$ and observe that

$$1 \coloneqq x_1 \vee \cdots \vee x_n$$

is the greatest element of $\mathbb{X}$. By condition (i) of Proposition 1.18, we conclude that $\bigwedge Y$ exists and coincides with 1. Then we consider the case where Y is nonempty and take an enumeration $Y = \{y_1, \dots, y_m\}$. Clearly, $y_1 \wedge \cdots \wedge y_m$ is the meet of Y in $\mathbb{X}$. ⊠

1.3 Lattices as algebraic structures

Lattices and semilattices can also be viewed as algebraic structures, as we proceed to explain. To this end, we recall some fundamentals of general algebraic systems.

Definition 1.25.

(i) An *operation* of arity n on a set A is a function $f \colon A^n \to A$.

(ii) A *type* is a map $\rho \colon \mathcal{F} \to \mathbb{N}$, where $\mathcal{F}$ is a set of function symbols. In this case, $\rho(f)$ is said to be the *arity* of the function symbol f, for every $f \in \mathcal{F}$. Function symbols of arity zero are called *constants*.

(iii) An *algebra* of type ρ is a pair $\boldsymbol{A} = \langle A; F \rangle$ where A is a nonempty set and $F = \{f^{\boldsymbol{A}} : f \in \mathcal{F}\}$ is a set of operations on A whose arity is determined by ρ, in the sense that each $f^{\boldsymbol{A}}$ has arity $\rho(f)$. The set A is called the *universe* of $\boldsymbol{A}$.

When $\mathcal{F} = \{f_1, \dots, f_n\}$, we shall write $\langle A; f_1^{\boldsymbol{A}}, \dots, f_n^{\boldsymbol{A}} \rangle$ instead of $\langle A; F \rangle$. In this case, we often drop the superscripts, and write simply $\langle A; f_1, \dots, f_n \rangle$.

Example 1.26 (Groups)**.** Classical examples of algebras are groups and rings. For instance, the type of groups ρ_G consists of a binary symbol $+$, a unary symbol $-$, and a constant symbol 0. A group is then an algebra $\langle A; +, -, 0 \rangle$ of type ρ_G in which $+$ is associative, 0 is a neutral element for $+$, and $-$ produces additive inverses. More explicitly, a group is an algebra $\boldsymbol{A}$ of type ρ_G such that for every $a, b, c \in A$,

$$a + (b + c) = (a + b) + c \qquad a + 0 = a \qquad 0 + a = a \qquad a + -a = 0 \qquad -a + a = 0.$$ ⊠

Many other important classes of algebras are also defined by some system of equations. It will therefore be convenient to make the notion of an equation precise.

Definition 1.27. Given a type $\rho \colon \mathcal{F} \to \mathbb{N}$ and a set of variables X disjoint from $\mathcal{F}$, the set of *terms of type ρ over X* is the least set $T_\rho(X)$ such that

(i) $X \subseteq T_\rho(X)$;

(ii) if $c \in \mathcal{F}$ is a constant, then $c \in T_\rho(X)$;

(iii) if $\varphi_1, \dots, \varphi_n \in T_\rho(X)$ and $f \in \mathcal{F}$ is n-ary, then $f(\varphi_1 \dots \varphi_n) \in T_\rho(X)$.

When f is a binary operation, such as $+$, we often write $\varphi_1 + \varphi_2$ instead of $f(\varphi_1, \varphi_2)$.

Definition 1.28. Given a term $\varphi \in T_\rho(X)$, we write $\varphi(x_1, \dots, x_n)$ to indicate that the variables occurring in φ are among $x_1, \dots, x_n$. Furthermore, given an algebra $\boldsymbol{A}$ of type ρ and elements $a_1, \dots, a_n \in A$, we define an element

$$\varphi^{\boldsymbol{A}}(a_1, \dots, a_n)$$

of A by recursion on the construction of φ as follows:

(i) if φ is a variable x_i, then $\varphi^{\boldsymbol{A}}(a_1, \dots, a_n) := a_i$;

(ii) if φ is a constant c, then $c^{\boldsymbol{A}}$ is the interpretation of c in $\boldsymbol{A}$;

(iii) if $\varphi = f(\psi_1, \dots, \psi_m)$, then

$$\varphi^{\boldsymbol{A}}(a_1, \dots, a_n) := f^{\boldsymbol{A}}(\psi_1^{\boldsymbol{A}}(a_1, \dots, a_n), \dots, \psi_m^{\boldsymbol{A}}(a_1, \dots, a_n)).$$

Definition 1.29. An *equation of type ρ over X* is an expression of the form $\varphi \approx \psi$, where $\varphi, \psi \in T_\rho(X)$. An equation $\varphi \approx \psi$ is *valid* in an algebra $\boldsymbol{A}$ of type ρ when

$$\varphi^{\boldsymbol{A}}(a_1, \dots, a_n) = \psi^{\boldsymbol{A}}(a_1, \dots, a_n) \text{ for every } a_1, \dots, a_n \in A,$$

in which case we say that $\boldsymbol{A}$ *validates* $\varphi \approx \psi$. Alternatively, we say that $\boldsymbol{A}$ *satisfies* $\varphi \approx \psi$, or that the equation $\varphi \approx \psi$ *holds* in $\boldsymbol{A}$.

We can now rephrase the definition of groups as follows: groups are precisely the algebras of type ρ_G that validate the equations

$$x + (y + z) \approx (x + y) + z \qquad x + 0 \approx x \qquad 0 + x \approx x \qquad x + -x \approx 0 \qquad -x + x \approx 0.$$

Lattices and semilattices admit a similar definition as algebras which satisfy certain equations. The type of meet semilattices ρ_M consists of a single binary symbol $\wedge$, while the type ρ_J of join semilattices consists of a single binary symbol $\vee$. The type of lattices ρ_L then consists of two binary symbols $\wedge$ and $\vee$.

Definition 1.30.

(i) A *meet semilattice* is an algebra $\boldsymbol{A} = \langle A; \wedge \rangle$ of type ρ_M satisfying the equations:

$$x \wedge x \approx x \qquad \textit{(idempotent law)}$$
$$x \wedge y \approx y \wedge x \qquad \textit{(commutative law)}$$
$$x \wedge (y \wedge z) \approx (x \wedge y) \wedge z. \qquad \textit{(associative law)}$$

(ii) A *join semilattice* is an algebra $\boldsymbol{A} = \langle A; \vee \rangle$ of type ρ_J satisfying the equations:

$$x \vee x \approx x \qquad \textit{(idempotent law)}$$
$$x \vee y \approx y \vee x \qquad \textit{(commutative law)}$$
$$x \vee (y \vee z) \approx (x \vee y) \vee z. \qquad \textit{(associative law)}$$

(iii) A *lattice* is an algebra $\boldsymbol{A} = \langle A; \vee, \wedge \rangle$ of type ρ_L such that $\langle A; \wedge \rangle$ is a meet semilattice (that is, it satisfies the above equations for $\wedge$), $\langle A; \vee \rangle$ is a join semilattice (that is, it satisfies the above equations for $\vee$), and moreover $\boldsymbol{A}$ satisfies the equations:

$$x \wedge (y \vee x) \approx x \quad \text{and} \quad x \vee (y \wedge x) \approx x. \qquad \textit{(absorption laws)}$$

In order to prove that this *algebraic definition* of meet and join semilattices (Definition 1.30) is equivalent to their *order theoretic definition* (Definition 1.12), we now explain how to translate between the two notions. While from a purely formal perspective, meet and join semilattices in the sense of Definition 1.30 are essentially the same objects (differing only in the symbol which represents their unique binary operation), we shall give these two classes of algebras a different order theoretic interpretation.

We first describe the translation from the algebraic definition to the order theoretic definition. Consider a meet semilattice $\boldsymbol{A} = \langle A; \wedge \rangle$ in the sense of Definition 1.30. The operation $\wedge$ determines a binary relation $\leqslant_\wedge$ on A as follows:

$$a \leqslant_\wedge b \iff a \wedge b = a.$$

We shall prove that $\boldsymbol{A}^p := \langle A; \leqslant_\wedge \rangle$ is a poset which is a meet semilattice in the sense of Definition 1.12. Similarly, consider a join semilattice $\boldsymbol{A} = \langle A; \vee \rangle$ in the sense of Definition 1.30. The operation $\vee$ determines a binary relation $\leqslant_\vee$ on A as follows:

$$a \leqslant_\vee b \iff a \vee b = b.$$

We shall prove that $\boldsymbol{A}^p := \langle A; \leqslant_\vee \rangle$ is a poset which is a join semilattice in the sense of Definition 1.12.

In the converse direction, consider poset $\mathbb{X} = \langle X; \leqslant \rangle$ which is a meet semilattice in the sense of Definition 1.12. Taking $\wedge \colon X \times X \to X$ to be the binary operation such that $x \wedge y$ the the meet of the set $\{x, y\}$ in $\mathbb{X}$, we shall prove that the algebra $\mathbb{X}^a := \langle X; \wedge \rangle$ is a meet semilattice in the sense of Definition 1.30. Similarly, consider a poset $\mathbb{X} = \langle X; \leqslant \rangle$ which is a join semilattice in the sense of Definition 1.12. Taking $\vee \colon X \times X \to X$ to be the binary operation such that $x \vee y$ is the join of the set $\{x, y\}$ in $\mathbb{X}$, we shall prove that the algebra $\mathbb{X}^a := \langle X; \vee \rangle$ is a join semilattice in the sense of Definition 1.30.

Proposition 1.31. *Let $\mathbb{X}$ and $\boldsymbol{A}$ be meet (resp. join) semilattices in the sense of Definitions 1.12 and 1.30, respectively. The following conditions hold:*

(i) *$\mathbb{X}^a$ is a meet (resp. join) semilattice in the sense of Definition 1.30;*

(ii) *$\boldsymbol{A}^p$ is a meet (resp. join) semilattice in the sense of Definition 1.12;*

(iii) *$\mathbb{X} = \mathbb{X}^{ap}$ and $\boldsymbol{A} = \boldsymbol{A}^{pa}$.*

Proof. We only prove the claims for meet semilattices. The claims for join semilattices are proved by replacing $\wedge$ by $\vee$, $\leqslant$ by $\geqslant$, and $\leqslant_\wedge$ by $\geqslant_\vee$ throughout the proof. (Notice that one cannot simply apply the Duality Principle here, because this principle only applies to the relational definition of meet and join semilattices.)

(i): Consider $x, y, z \in X$. We begin by proving that $x = x \wedge x$. To this end, recall that $x \wedge x$ is, by definition, a lower bound of $\{x\}$. Consequently, $x \wedge x \leqslant x$. Since $\leqslant$ is reflexive, we have $x \leqslant x$, so x is also a lower bound of $\{x\}$. As $x \wedge x$ is, by definition, the greatest lower bound of $\{x\}$, this yields $x \leqslant x \wedge x$. By applying the antisymmetry of $\leqslant$ to the inequalities $x \wedge x \leqslant x$ and $x \leqslant x \wedge x$, we conclude that $x = x \wedge x$, as desired. Hence, $\mathbb{X}^a$ validates the idempotent law.

The fact that $\mathbb{X}^a$ validates also the commutative law is an immediate consequence of the equality $\{x, y\} = \{y, x\}$.

To prove the associative law, observe that $x \wedge (y \wedge z) \leqslant x, y \wedge z$, as $x \wedge (y \wedge z)$ is a lower bound of $\{x, y \wedge z\}$. Moreover, $y \wedge z \leqslant y, z$, as $y \wedge z$ is a lower bound of $\{y, z\}$. By the transitivity of $\leqslant$, we obtain $x \wedge (y \wedge z) \leqslant x, y, z$. In particular, $x \wedge (y \wedge z)$ is a lower bound of $\{x, y\}$. Since $x \wedge y$ is the greatest such lower bound, this yields $x \wedge (y \wedge z) \leqslant x \wedge y$. Thus, we obtain $x \wedge (y \wedge z) \leqslant x \wedge y, z$, that is, $x \wedge (y \wedge z)$ is a lower bound of $\{x \wedge y, z\}$. Since $(x \wedge y) \wedge z$ is the greatest such lower bound, we conclude that

$$x \wedge (y \wedge z) \leqslant (x \wedge y) \wedge z.$$

Similarly, we obtain $(x \wedge y) \wedge z \leqslant x \wedge (y \wedge z)$ and, therefore, $x \wedge (y \wedge z) = (x \wedge y) \wedge z$, by the antisymmetry of $\leqslant$. Hence, $\mathbb{X}^a$ validates the associative law in $\mathbb{X}^a$.

(ii): We begin by proving that $\boldsymbol{A}^p = \langle A; \leqslant_\wedge \rangle$ is a poset. To this end, consider $a, b, c \in A$. By the idempotent law, $a \wedge a = a$. Consequently, the definition of $\leqslant_\wedge$ guarantees that $a \leqslant_\wedge a$ and, therefore, that $\leqslant_\wedge$ is reflexive. To prove that $\leqslant_\wedge$ is antisymmetric, suppose that $a \leqslant_\wedge b$ and $b \leqslant_\wedge a$. By the definition of $\leqslant_\wedge$, we have $a \wedge b = a$ and $b \wedge a = b$. Therefore, we can apply the commutative law, obtaining

$$a = a \wedge b = b \wedge a = b.$$

To prove that $\leqslant_\wedge$ is transitive, suppose that $a \leqslant_\wedge b$ and $b \leqslant_\wedge c$, that is,

$$a \wedge b = a \quad \text{and} \quad b \wedge c = b.$$

Together with the associative law, this yields

$$a \wedge c = (a \wedge b) \wedge c = a \wedge (b \wedge c) = a \wedge b = a.$$

By the definition of $\leqslant_\wedge$, this amounts to $a \leqslant_\wedge c$. Hence, we conclude that $\boldsymbol{A}^p$ is a poset.

To prove that $\boldsymbol{A}^p$ is a meet semilattice in the sense of Definition 1.12, it only remains to show that $a \wedge b$ is the meet of $\{a, b\}$ in $\boldsymbol{A}^p$, for all $a, b \in A$. To this end, observe that by applying in succession the commutative, associative, and idempotent law, we obtain

$$(a \wedge b) \wedge a = a \wedge (a \wedge b) = (a \wedge a) \wedge b = a \wedge b.$$

By the definition of $\leqslant_\wedge$, this amounts to $a \wedge b \leqslant_\wedge a$. A similar argument yields $a \wedge b \leqslant_\wedge b$. Thus, $a \wedge b$ is a lower bound of $\{a, b\}$ in $\boldsymbol{A}^p$. To prove that it is the greatest one, suppose that c is also a lower bound of $\{a, b\}$ in $\boldsymbol{A}^p$, that is, $c \leqslant_\wedge a, b$. By definition of $\leqslant_\wedge$, this means that $c \wedge a = c$ and $c \wedge b = c$. Together with the associative law, this implies that

$$c \wedge (a \wedge b) = (c \wedge a) \wedge b = c \wedge b = c,$$

that is, $c \leqslant_\wedge a \wedge b$. Thus, we conclude that $a \wedge b$ is the meet of $\{a, b\}$ in $\boldsymbol{A}^p$.

(iii): To prove that $\mathbb{X} = \mathbb{X}^{ap}$, consider $x, y \in X$. We begin by showing that

$$x \leqslant^{\mathbb{X}} y \iff x \text{ is the meet of } \{x, y\} \text{ in } \mathbb{X}. \tag{1.1}$$

Suppose first that $x \leqslant^{\mathbb{X}} y$. As $\leqslant^{\mathbb{X}}$ is reflexive, this implies that x is a lower bound of $\{x, y\}$ in $\mathbb{X}$. To prove that it is the greatest one, consider another such lower bound z. Then $z \leqslant x$, as desired. Thus, x is the meet of $\{x, y\}$ in $\mathbb{X}$. Conversely, if x is the meet of $\{x, y\}$ in $\mathbb{X}$, then clearly $x \leqslant^{\mathbb{X}} y$.

In view of condition (1.1), we obtain

$$\begin{aligned} x \leqslant^{\mathbb{X}} y &\iff x \text{ is the meet of } \{x, y\} \text{ in } \mathbb{X} \\ &\iff x = x \wedge^{\mathbb{X}^a} y \\ &\iff x \leqslant^{\mathbb{X}^{ap}} y, \end{aligned}$$

where the last two equivalences follow from the definitions of $\mathbb{X}^a$ and $\mathbb{X}^{ap}$, respectively. Hence, we conclude that $\mathbb{X} = \mathbb{X}^{ap}$, as desired.

To prove that $\boldsymbol{A} = \boldsymbol{A}^{pa}$, consider $a, b \in A$. Since $a \wedge^{\boldsymbol{A}^{pa}} b$ is the meet of $\{a, b\}$ in $\boldsymbol{A}^p$, we have $a \wedge^{\boldsymbol{A}^{pa}} b \leqslant_\wedge^{\boldsymbol{A}^p} a, b$. By the definition of the relation $\leqslant_\wedge^{\boldsymbol{A}^p}$, this amounts to

$$(a \wedge^{\boldsymbol{A}^{pa}} b) \wedge^{\boldsymbol{A}} a = a \wedge^{\boldsymbol{A}^{pa}} b \;\text{ and }\; (a \wedge^{\boldsymbol{A}^{pa}} b) \wedge^{\boldsymbol{A}} b = a \wedge^{\boldsymbol{A}^{pa}} b.$$

Together with the fact that the associative law are valid in $\boldsymbol{A}$, this yields

$$(a \wedge^{\boldsymbol{A}^{pa}} b) \wedge^{\boldsymbol{A}} (a \wedge^{\boldsymbol{A}} b) = ((a \wedge^{\boldsymbol{A}^{pa}} b) \wedge^{\boldsymbol{A}} b) \wedge^{\boldsymbol{A}} b = (a \wedge^{\boldsymbol{A}^{pa}} b) \wedge^{\boldsymbol{A}} b = a \wedge^{\boldsymbol{A}^{pa}} b,$$

which, by the definition of $\boldsymbol{A}^p$, amounts to $a \wedge^{\boldsymbol{A}^{pa}} b \leqslant_\wedge^{\boldsymbol{A}^p} a \wedge^{\boldsymbol{A}} b$.

Lastly, applying in succession the commutative, associative, and idempotent law in $\boldsymbol{A}$, we obtain

$$(a \wedge^{\boldsymbol{A}} b) \wedge^{\boldsymbol{A}} a = a \wedge^{\boldsymbol{A}} (a \wedge^{\boldsymbol{A}} b) = (a \wedge^{\boldsymbol{A}} a) \wedge^{\boldsymbol{A}} b = a \wedge^{\boldsymbol{A}} b.$$

By the definition of $\boldsymbol{A}^p$, this amounts to $a \wedge^{\boldsymbol{A}} b \leqslant_\wedge^{\boldsymbol{A}^p} a$. Similarly, we obtain $a \wedge^{\boldsymbol{A}} b \leqslant_\wedge^{\boldsymbol{A}^p} b$, so $a \wedge^{\boldsymbol{A}} b$ is a lower bound of $\{a, b\}$ in $\boldsymbol{A}^p$. Since $a \wedge^{\boldsymbol{A}^{pa}} b$ is the greatest such lower bound, we obtain that $a \wedge^{\boldsymbol{A}} b \leqslant_\wedge^{\boldsymbol{A}^p} a \wedge^{\boldsymbol{A}^{pa}} b$. As we already proved that $a \wedge^{\boldsymbol{A}^{pa}} b \leqslant_\wedge^{\boldsymbol{A}^p} a \wedge^{\boldsymbol{A}} b$, we can apply the antisymmetry of $\leqslant_\wedge^{\boldsymbol{A}^p}$, obtaining $a \wedge^{\boldsymbol{A}} b = a \wedge^{\boldsymbol{A}^{pa}} b$. Thus $\boldsymbol{A} = \boldsymbol{A}^{pa}$. ⊠

It remains to provide a translation between the two definitions of lattices. To this end, consider a lattice $\boldsymbol{A} = \langle A; \wedge, \vee \rangle$ in the sense of Definition 1.30. Then

$$a \leqslant_\wedge b \iff a \wedge b = a \iff a \vee b = b \iff a \leqslant_\vee b.$$

This is a consequence of absorption: if $a \wedge b = a$, then $a \vee b = (a \wedge b) \vee b = (b \wedge a) \vee b$, and conversely if $a \vee b = b$, then $a \wedge b = a \wedge (a \vee b) = a \wedge (b \vee a) = a$. Because the two partial orders $\leqslant_\wedge$ and $\leqslant_\vee$ agree, we can simply talk about the partial order $\leqslant$ and define the poset $\boldsymbol{A}^p$ as $\boldsymbol{A}^p \coloneqq \langle A; \leqslant \rangle$.

In the converse direction, consider a lattice $\mathbb{X} = \langle X; \leqslant \rangle$ which is a lattice in the sense of Definition 1.12. Taking again $\wedge$ and $\vee$ to be, respectively, the operations of taking the meet and the join of two elements in $\mathbb{X}$, we define $\mathbb{X}^a \coloneqq \langle X; \wedge, \vee \rangle$.

Proposition 1.32. *Let $\mathbb{X}$ and $\boldsymbol{A}$ be lattices in the sense of Definitions 1.12 and 1.30, respectively. The following conditions hold:*

(i) *$\mathbb{X}^a$ is a lattice in the sense of Definition 1.30;*

(ii) *$\boldsymbol{A}^p$ is a lattice in the sense of Definition 1.12;*

(iii) *$\mathbb{X} = \mathbb{X}^{ap}$ and $\boldsymbol{A} = \boldsymbol{A}^{pa}$.*

Proof. (i): Because $\mathbb{X}$ is both a meet and a join semilattice in the sense of Definition 1.12, by Proposition 1.31 the algebra $\mathbb{X}^a$ satisfies idempotence, commutativity, and associativity. It remains to prove that it satisfies absorption. To this end, observe that $x \leqslant x, y \vee x$, because $\leqslant$ is reflexive and $y \vee x$ an upper bound of $\{y, x\}$. Then x is a lower bound for $\{x, y \vee x\}$. Since $x \wedge (y \vee x)$ is the greatest such lower bound, this implies $x \leqslant x \wedge (y \vee x)$. Furthermore, we have $x \wedge (y \vee x) \leqslant x$, because $x \wedge (y \vee x)$ is a lower bound of $\{x, y \vee x\}$. By the antisymmetry of $\leqslant$, we conclude that $x \wedge (y \vee x) = x$. A similar argument shows that $x \vee (y \wedge x) = x$, thus establishing the validity of the absorption laws in $\mathbb{X}^a$. Hence, we conclude that $\mathbb{X}^a$ is a lattice in the sense of Definition 1.30.

(ii): Because $\langle A; \wedge \rangle$ and $\langle A; \vee \rangle$ are, respectively, a meet and a join semilattice in the sense of Definition 1.30, by Proposition 1.31 the poset $\boldsymbol{A}^p$ is both a meet and a join semilattice in the sense of Definition 1.12. In other words, it is a lattice in the sense of that definition.

(iii): The equality $\mathbb{X} = \mathbb{X}^{ap}$ follows from the corresponding equality for either meet or join semilattices. This is because each lattice is in particular a meet and a join semilattice, and moreover if $\langle A; \vee, \wedge \rangle$ is a lattice, then the posets $\langle A; \wedge \rangle^p$ and $\langle A; \vee \rangle^p$ coincide. Likewise, the equality $\boldsymbol{A} = \boldsymbol{A}^{pa}$ follows from the corresponding equalities for meet semilattices and for join semilattices. ⊠

In view of Propositions 1.31 and 1.32, from now on we shall treat lattices and semilattices both as posets and algebras without further notice.

1.4 Closure operators

The following concept will play a fundamental role in this book.

Definition 1.33. A map $C \colon \mathcal{P}(X) \to \mathcal{P}(X)$ is said to be a *closure operator* on a set X when it satisfies the following conditions for every $Y, Z \subseteq X$:

(i) *Extensivity*: $Y \subseteq C(Y)$;

(ii) *Idempotence*: $C(C(Y)) = C(Y)$;

(iii) *Monotonicity*: if $Y \subseteq Z$, then $C(Y) \subseteq C(Z)$.

We say that a set $Y \subseteq X$ is *closed* when $Y = C(Y)$.

Notice that the inclusion $C(Y) \subseteq C(C(Y))$ for all $Y \subseteq X$ follows from extensivity and monotonicity. To prove that a map $C\colon \mathcal{P}(X) \to \mathcal{P}(X)$ is a closure operator, it therefore suffices to prove extensivity, monotonicity, and the inclusion $C(C(Y)) \subseteq C(Y)$ for all $Y \subseteq X$. When no confusion shall occur, given a closure operator C on X and $x_1, \dots, x_n \in X$, we shall write $C(x_1, \dots, x_n)$ as a shorthand for $C(\{x_1, \dots, x_n\})$.

Example 1.34 (Topological closure)**.** Given a topological space $\langle X; \tau \rangle$, let

$$\overline{(-)}\colon \mathcal{P}(X) \to \mathcal{P}(X)$$

be the map defined for every $Y \subseteq X$ as

$$\overline{Y} := \bigcap \{Z \subseteq X : Z \in \mathsf{Cl}(X; \tau) \text{ and } Y \subseteq Z\}.$$

Notice that $\overline{Y}$ is closed, as arbitrary intersections of closed are closed. Therefore, $\overline{Y}$ is the least closed set extending Y. Because of this, $\overline{Y}$ is called the *topological closure* of Y. It is easy to see that $\overline{(-)}$ is a closure operator on X, whose closed sets are precisely the closed sets of the topological space $\langle X; \tau \rangle$. The closure operators of this form are called *topological*. ⊠

Example 1.35 (Upward & downward closure)**.** Given a poset $\mathbb{X}$, let

$$\uparrow^{\mathbb{X}}\colon \mathcal{P}(X) \to \mathcal{P}(X) \text{ and } \downarrow^{\mathbb{X}}\colon \mathcal{P}(X) \to \mathcal{P}(X)$$

be the maps defined for every $Y \subseteq X$ as

$$\uparrow^{\mathbb{X}} Y := \{x \in X : \text{there exists } y \in Y \text{ such that } y \leqslant x\};$$
$$\downarrow^{\mathbb{X}} Y := \{x \in X : \text{there exists } y \in Y \text{ such that } x \leqslant y\}.$$

It is easy to see that $\uparrow^{\mathbb{X}}$ and $\downarrow^{\mathbb{X}}$ are closure operators on X. The closed sets of $\uparrow^{\mathbb{X}}$ and $\downarrow^{\mathbb{X}}$ consist, respectively, of the upsets of $\mathbb{X}$ and the downsets of $\mathbb{X}$. Furthermore, $\uparrow^{\mathbb{X}} Y$ and $\downarrow^{\mathbb{X}} Y$ are, respectively, the least upset and the least downset of $\mathbb{X}$ containing Y.

When the poset $\mathbb{X}$ is clear from the context, we will drop the superscripts in $\uparrow^{\mathbb{X}}$ and $\downarrow^{\mathbb{X}}$ and write simply $\uparrow$ and $\downarrow$. Similarly, given $x \in X$, we will write $\uparrow^{\mathbb{X}} x$ and $\downarrow^{\mathbb{X}} x$ as a shorthand for $\uparrow^{\mathbb{X}}\{x\}$ and $\downarrow^{\mathbb{X}}\{x\}$. Lastly, an upset (resp. a downset) of $\mathbb{X}$ will be called *principal* when it is of the form $\uparrow x$ (resp. $\downarrow x$) for some $x \in X$. ⊠

The upward and downward closure operators on a poset are in fact special cases of topological closure operators.

Example 1.36 (Upset & downset topology)**.** The collection of all upsets of a poset $\mathbb{X}$ forms a topology called the *upset topology* on $\mathbb{X}$. The upset topology is sometimes also called the *Alexandroff topology*. The closed sets of the upset topology are the downsets of $\mathbb{X}$, since these are precisely the complements of the upsets of $\mathbb{X}$. The topological closure operator associated with the upset topology is therefore $\downarrow$. A analogous result holds if we replace the role of upsets and downsets in this example. ⊠

Closure operators admit the following alternative presentations:

Definition 1.37. A *closure system* on a set X is a family $\mathcal{S} \subseteq \mathcal{P}(X)$ closed under arbitrary intersections, that is, such that for every $\{Y_i : i \in I\} \subseteq \mathcal{P}(X)$,

$$\text{if } Y_i \in \mathcal{S} \text{ for every } i \in I, \text{ then } \bigcap_{i \in I} Y_i \in \mathcal{S}.$$

The intersection of the empty family of subsets of X is understood here as $\bigcap \emptyset := X$. Consequently, the set X belongs to every closure system on X.

Definition 1.38. A *consequence relation* on a set X is a relation $\vdash\, \subseteq \mathcal{P}(X) \times X$ such that

(i) *Reflexivity*: if $Y \subseteq X$ and $y \in Y$, then $Y \vdash y$;

(ii) *Transitivity*: for all $Y, Z \subseteq X$ and $x \in X$, if $Y \vdash z$ for all $z \in Z$ and $Z \vdash x$, then $Y \vdash x$.

In this case, for every $x \in X$ and $Y, Z \subseteq X$ such that $Y \subseteq Z$ the following holds:

$$\text{if } Y \vdash x, \text{ then } Z \vdash x. \qquad (\textit{monotonicity})$$

This is because from $Y \subseteq Z$ and reflexivity it follows $Z \vdash y$ for each $y \in Y$, so by transitivity $Y \vdash x$ implies $Z \vdash x$.

The next result explains how to translate between closure operators, closure systems, and consequence relations.

Theorem 1.39. *Given a set X, the following conditions hold:*

(i) *If C is a closure operator on X, the family of its closed sets $\mathcal{S}_C$ is a closure system on X. Conversely, if $\mathcal{S}$ is a closure system on X, the map $C_{\mathcal{S}} \colon \mathcal{P}(X) \to \mathcal{P}(X)$ defined as*

$$C_{\mathcal{S}}(Y) := \bigcap \{Z \in \mathcal{S} : Y \subseteq Z\}$$

is a closure operator on X. These transformations are inverse to each other;

(ii) *If C is a closure operator on X, the relation*

$$\vdash_C := \{\langle Y, x\rangle \in \mathcal{P}(X) \times X : x \in C(Y)\}$$

is a consequence relation on X. Conversely, if $\vdash$ is a consequence relation on X, the map $C_{\vdash} \colon \mathcal{P}(X) \to \mathcal{P}(X)$ defined as

$$C_{\vdash}(Y) := \{x \in X : Y \vdash x\}$$

is a closure operator on X. These transformations are inverse to each other.

Proof. (i): Let $\{Y_i : i \in I\}$ be a family of closed sets of the closure operator C. We will prove that $Y := \bigcap_{i \in I} Y_i$ is also closed. By monotonicity $C(Y) \subseteq C(Y_i)$ for each $i \in I$ and thus $C(Y) \subseteq \bigcap_{i \in I} C(Y_i) = \bigcap_{i \in I} Y_i$, where the last equality holds because each Y_i is closed. Therefore, by the definition of Y we obtain $C(Y) \subseteq Y$. Since $Y \subseteq C(Y)$ by extensivity, we conclude that $Y = C(Y)$. The closed sets of C thus form a closure system. Conversely, if $\mathcal{S}$

is a closure system, then the map $C_{\mathcal{S}}$ associated with it is extensive and order preserving by definition. Moreover, if $Y \subseteq X$ and $Z \in \mathcal{S}$, then $Y \subseteq Z$ iff $C_{\mathcal{S}}(Y) \subseteq Z$. It follows that

$$C_{\mathcal{S}}(C_{\mathcal{S}}(Y)) = \bigcap\{Z \in \mathcal{S} : C_{\mathcal{S}}(Y) \subseteq Z\} = \bigcap\{Z \in \mathcal{S} : Y \subseteq Z\} = C_{\mathcal{S}}(Y).$$

To prove that these transformations are inverse to each other, let C' be the closure operator obtained from a closure operator C on X by composing the two constructions. That is,

$$C'(Y) = \bigcap\{Z \subseteq X : Y \subseteq Z \text{ and } Z = C(Z)\}.$$

Then $C'(Y) \subseteq C(Y)$ because $Y \subseteq Z = C(Z)$ holds in particular for $Z := C(Y)$, and $C(Y) \subseteq C'(Y)$ because $Y \subseteq Z = C(Z)$ implies $C(Y) \subseteq C(Z) = Z$. Thus $C' = C$.

In the opposite direction, let $\mathcal{S}'$ be the closure system obtained from a closure system $\mathcal{S}$ on X by composing the two constructions. That is,

$$Y \in \mathcal{S}' \iff Y = \bigcap\{Z \in \mathcal{S} : Y \subseteq Z\}.$$

Then $Y \in \mathcal{S}$ implies $Y \in \mathcal{S}'$, since $Y \subseteq \{Z \in \mathcal{S} : Y \subseteq Z\}$ holds for each $Y \subseteq X$ and $\{Z \in \mathcal{S} : Y \subseteq Z\} \subseteq Y$ holds for each $Y \in \mathcal{S}$ because we can take $Z := Y$. Conversely, if $Y \in \mathcal{S}'$, then Y is the intersection of a family of sets in $\mathcal{S}$, since $Y = \bigcap\{Z \in \mathcal{S} : Y \subseteq Z\}$. But $\mathcal{S}$ is closed under arbitrary intersections, so $Y \in \mathcal{S}$. Thus $\mathcal{S}' = \mathcal{S}$.

(ii): If C is a closure operator on X and $y \in Y \subseteq X$, then $y \in C(Y)$ by extensivity, so $Y \vdash_C y$ holds in the associated consequence relation. Moreover, if $Y \vdash_C z$ for each $z \in Z \subseteq X$ and $Z \vdash_C x$, then $Z \subseteq C(Y)$ and $x \in C(Z)$, so $x \in C(Z) \subseteq C(C(Y)) = C(Y)$ by monotonicity and idempotence. Consequently, $Y \vdash_C x$. Conversely, if $\vdash$ is a consequence relation on X, then the asssociated map $C_\vdash$ is extensive, since by reflexivity $y \in Y \subseteq X$ implies $Y \vdash y$ and thus $y \in C_\vdash(Y)$. Moreover, $C_\vdash(Y) \vdash x$ implies $Y \vdash x$ for every $Y \subseteq X$ and $x \in X$, since by transitivity $Y \vdash z$ for each $z \in C_\vdash(Y)$ and $C_\vdash(Y) \vdash x$ imply that $Y \vdash x$. Consequently,

$$C_\vdash(C_\vdash(Y)) = \{x \in X : C_\vdash(Y) \vdash x\} \subseteq \{x \in X : Y \vdash x\} = C_\vdash(Y).$$

Finally, $Y \subseteq Z$ for $Y, Z \subseteq X$ implies $C_\vdash(Y) \subseteq C_\vdash(Z)$ by the monotonicity of $\vdash$.

One can immediately see from their definitions that the two transformations are inverse to each other. Let C' be the closure operator obtained from a closure operator C on X by composing the two constructions. Then $C'(Y) = \{x \in X : Y \vdash_C x\} = \{x \in X : x \in C(Y)\} = C(Y)$ for every $Y \subseteq X$. In the opposite direction, let $\vdash'$ be the consequence relation obtained from the consequence relation $\vdash$ on X by composing the two constructions. Then $Y \vdash' x$ iff $x \in C_\vdash(Y)$ iff $Y \vdash x$. ⊠

In view of the above result, given a closure operator C on a set X, we refer to

$$\mathcal{S}_C := \{Y \subseteq X : Y = C(Y)\} \quad \text{and} \quad \vdash_C := \{\langle Y, x\rangle \in \mathcal{P}(X) \times X : x \in C(Y)\}$$

as to the closure system and the consequence relation *associated with* C. We will drop the subscript $(-)_C$ whenever there is no danger of confusion.

For instance, the upsets and the downsets of a poset $\mathbb{X}$ are precisely the closed sets of the closure operators $\uparrow$ and $\downarrow$. It follows that $\mathsf{Up}(\mathbb{X})$ and $\mathsf{Down}(\mathbb{X})$ are the closure systems associated with $\uparrow$ and $\downarrow$. Similarly, the closed sets of a topological space $\langle X; \tau \rangle$ are precisely the closed sets of the closure operator $\overline{(-)}$. Consequently, the family $\mathsf{Cl}(X; \tau)$ of closed sets of $\langle X; \tau \rangle$ is the closure system associated with $\overline{(-)}$.

Every closure system can be viewed as a poset ordered under the inclusion relation. This poset turns out to be a complete lattice.

Proposition 1.40. *Let $\mathcal{S}$ be the closure system associated with a closure operator C on a set X. Then $\langle \mathcal{S}; \subseteq \rangle$ is a complete lattice in which meets are intersections and joins are closures of unions. That is, for every $\{Y_i : i \in I\} \subseteq \mathcal{S}$*

$$\bigvee_{i \in I} Y_i = C(\bigcup_{i \in I} Y_i).$$

Proof. Consider a family $\{Y_i : i \in I\}$ of sets in $\mathcal{S}$. The intersection $Y := \bigcap_{i \in I} Y_i$ is the greatest lower bound of the family $\{Y_i : i \in I\}$ in $\langle \mathcal{P}(X); \subseteq \rangle$. Because $Y \in \mathcal{S}$, it is also the greatest lower bound of this family in $\langle \mathcal{S}; \subseteq \rangle$. It follows that the meet of Y in $\langle \mathcal{S}; \subseteq \rangle$ exists and coincides with the intersection $\bigcap Y$. To prove that the join $\bigvee_{i \in I} Y_i$ exists and equals $C(\bigcup_{i \in I} Y_i)$, it suffices to prove that each $Z \in \mathcal{S}$ satisfies the following equivalence:

$$C(\bigcup_{i \in I} Y_i) \subseteq Z \iff Y_i \subseteq Z \text{ for each } i \in I.$$

The left to right implication holds because $Y_j \subseteq \bigcup_{i \in I} Y_i \subseteq C(\bigcup_{i \in I} Y_i)$ for each $j \in I$. The right to left implication holds because if $Y_i \subseteq Z \in \mathcal{S}$ for each $i \in I$, then $\bigcup_{i \in I} Y_i \subseteq Z$, so $C(\bigcup_{i \in I} Y_i) \subseteq C(Z) = Z$. ⊠

In view of Proposition 1.40, we will often treat closure systems $\mathcal{S}$ as complete lattices and write $\mathcal{S}$ as a shorthand for $\langle \mathcal{S}; \subseteq \rangle$. Notably, not only is every closure system a complete lattice, but (up to isomorphism) every complete lattice arises in this way.

Theorem 1.41 (Representation theorem)**.** *Every complete lattice is isomorphic to a closure system.*

Proof. Consider a complete lattice $\mathbb{X}$. We will show that

$$\mathcal{S} := \{\downarrow x : x \in X\}$$

is a closure system on X such that $\mathbb{X} \cong \mathcal{S}$.

We begin by proving that $\mathcal{S}$ is a closure system on X. Notice that $\mathcal{S} \subseteq \mathcal{P}(X)$. Now consider a family $\{\downarrow x_i : i \in I\} \subseteq \mathcal{S}$. We need to prove that its intersection belongs to $\mathcal{S}$. To this end, observe that for every $y \in X$,

$$y \in \bigcap_{i \in I} \downarrow x_i \iff y \leqslant x_i \text{ for every } i \in I \iff y \leqslant \bigwedge_{i \in I}^{\mathbb{X}} x_i.$$

These equivalences hold even if I is empty. As a consequence,

$$\bigcap_{i \in I} \downarrow x_i = \downarrow\Big(\bigwedge_{i \in I}^{\mathbb{X}} x_i\Big) \in \mathcal{S}.$$

Hence, $\mathcal{S}$ is a closure system on X as desired.

It only remains to show that $\mathbb{X}$ is isomorphic to $\mathcal{S}$. To this end, consider the map $f\colon X \to \mathcal{S}$ defined by the rule

$$f(x) := {\downarrow}x \text{ for every } x \in X.$$

Clearly, f is a well-defined surjection. Moreover, for every $x, y \in X$,

$$x \leqslant y \iff {\downarrow}x \subseteq {\downarrow}y \iff f(x) \subseteq f(y).$$

Hence, f is an order embedding from $\mathbb{X}$ into $\mathcal{S}$. Since f is surjective, we conclude that it is an order isomorphism. ⊠

1.5 Finitarity and compactness

Among all closure operators, the following are of special interest:

Definition 1.42. Given a closure operator C on a set X,

(i) a closed set Y of C is said to be *finitely generated* when there exists a finite $Z \subseteq X$ such that $Y = C(Z)$;

(ii) C is said to be *finitary* when for every $Y \cup \{x\} \subseteq X$ such that $x \in C(Y)$ there exists a finite $Z \subseteq Y$ such that $x \in C(Z)$.

Consequently, a closure operator is finitary precisely when each of its closed sets is the union of all the finitely generated closed sets contained into it.

Example 1.43 (Upward & downward closure)**.** Given a poset $\mathbb{X}$, the closure operators

$${\uparrow}\colon \mathcal{P}(X) \to \mathcal{P}(X) \text{ and } {\downarrow}\colon \mathcal{P}(X) \to \mathcal{P}(X)$$

are finitary. This is because for every $Y \cup \{x\} \subseteq X$,

$$\begin{aligned} x \in {\uparrow}Y &\iff \text{there exists } y \in Y \text{ such that } x \in {\uparrow}y;\\ x \in {\downarrow}Y &\iff \text{there exists } y \in Y \text{ such that } x \in {\downarrow}y. \end{aligned}$$ ⊠

Example 1.44 (Topological closure)**.** On the other hand, the closure operator of topological closure often fails to be finitary. For instance, this is the case for the standard topology on $\mathbb{R}$. To prove this, recall that the open sets of this topology are unions of sets of the form (y, z) for $y, z \in \mathbb{R}$. Consequently, every open set containing an element $x \in \mathbb{R}$ must also contain some element $> x$. We will prove that the closure operator $\overline{(-)}\colon \mathcal{P}(\mathbb{R}) \to \mathcal{P}(\mathbb{R})$ is not finitary by showing that $x \in \overline{(x, +\infty)}$, but $x \notin \overline{X}$ for every finite $X \subseteq (x, +\infty)$.

First observe that the set

$$Y := \mathbb{R} - \overline{(x, +\infty)}$$

is open because $\overline{(x, +\infty)}$ is closed. Moreover, Y does not contain any element $> x$ as $(x, +\infty) \subseteq \overline{(x, +\infty)}$ and $Y \cap \overline{(x, +\infty)} = \emptyset$. Consequently, $x \notin Y$. Hence, we conclude that

$$x \in Y^c = \overline{(x, +\infty)}.$$

Then consider a finite set $X = \{x_1, \dots, x_n\} \subseteq (x, +\infty)$ with $n \in \mathbb{N}$, with the understanding that $X = \emptyset$ if $n = 0$. Each singleton set $\{y\}$ with $y \in \mathbb{R}$ is closed in the standard topology in $\mathbb{R}$, since its complement $\mathbb{R} - \{y\} = (-\infty, y) \cup (y, +\infty)$ is open. Because the family of closed sets of any topology contains the empty set and is closed under binary unions, $X = \{x_1\} \cup \cdots \cup \{x_n\}$ is closed. That is, $\overline{X} = X$, so $x \notin \overline{X}$. ⊠

In view of Theorem 1.39, it makes sense to characterize the finitary closure operators in terms of the structure of the closure systems and consequence relations associated with them. The following concepts are instrumental to this purpose.

Definition 1.45. Given a poset $\mathbb{X}$, a nonempty set $Y \subseteq X$ is said to be

(i) *upward directed* in $\mathbb{X}$ when for every $x, y \in Y$ there exists $z \in Y$ such that $x, y \leqslant z$;

(ii) *downward directed* in $\mathbb{X}$ when for every $x, y \in Y$ there exists $z \in Y$ such that $z \leqslant x, y$.

A closure system $\mathcal{S}$ is said to be *inductive* when $\bigcup Y \in \mathcal{S}$ for every family $Y \subseteq \mathcal{S}$ that is upward directed in $\langle \mathcal{S}; \subseteq \rangle$.

Definition 1.46. A consequence relation $\vdash$ on a set X is said to be *finitary* when for every $Y \cup \{x\} \subseteq X$,

$$\text{if } Y \vdash x \text{, there exists a finite } Z \subseteq Y \text{ such that } Z \vdash x.$$

Finitary closure operators, inductive closure systems, and finitary consequence relations are related as follows:

Theorem 1.47. *The following conditions are equivalent for a closure operator C on a set X:*

(i) *C is finitary;*

(ii) *The closure system associated with C is inductive;*

(iii) *The consequence relation associated with C is finitary.*

Proof. Recall that the closure system and the consequence relation associated with C are, respectively,

$$\mathcal{S}_C := \{Y \subseteq X : Y = C(Y)\} \text{ and } \vdash_C := \{\langle Y, x \rangle \in \mathcal{P}(X) \times X : x \in C(Y)\}.$$

(i)⇒(ii): Let $\{Y_i : i \in I\} \subseteq \mathcal{S}_C$ be an upward directed family. We need to show that $\bigcup_{i \in I} Y_i$ is a closed set of C, that is, $C(\bigcup_{i \in I} Y_i) \subseteq \bigcup_{i \in I} Y_i$. To this end, consider an element $x \in C(\bigcup_{i \in I} Y_i)$. Since the closure operator C is finitary, there is a finite $Z \subseteq \bigcup_{i \in I} Y_i$ such that $x \in C(Z)$. Furthermore, there exists a finite subfamily $\{Y_1, \dots, Y_n\} \subseteq \{Y_i : i \in I\}$ such that $Z \subseteq Y_1 \cup \cdots \cup Y_n$, as $Z \subseteq \bigcup_{i \in I} Y_i$ and Z is finite. Lastly, the assumption that the family $\{Y_i : i \in I\}$ is upward directed in $\langle \mathcal{S}_C; \subseteq \rangle$ guarantees that existence of some $j \in I$ such that $Y_1, \dots, Y_n \subseteq Y_j$. Consequently, $Z \subseteq Y_1 \cup \cdots \cup Y_n \subseteq Y_j$. Together with the assumption that Y_j belongs to $\mathcal{S}_C$ and, therefore, that Y_j is a closed set of C, this yields $C(Z) \subseteq C(Y_j) = Y_j$. Since $x \in C(Z)$, we conclude that $x \in Y_j \subseteq \bigcup_{i \in I} Y_i$.

(ii)⇒(iii): Let $Y \cup \{x\} \subseteq X$ be such that $Y \vdash_C x$. By the definition of $\vdash_C$ this means that $x \in C(Y)$. Then observe that

$$U := \{C(Z) : Z \subseteq Y \text{ is finite}\}$$

is a family of elements of $\mathcal{S}_C$ because each $C(Z)$ is a closed set. We will prove that U is upward directed. First, observe that U is nonempty because $C(\emptyset) \in U$. Then consider two finite subsets Z_1 and Z_2 of Y. We need to find a finite subset Z of Y such that $C(Z)$ contains $C(Z_1)$ and $C(Z_2)$. Taking $Z := Z_1 \cup Z_2$, we are done. Therefore, U is an upward directed family of elements of $\mathcal{S}_C$. Since the closure system $\mathcal{S}_C$ is inductive, we obtain $\bigcup U \in \mathcal{S}$. As $\mathcal{S}_C$ is the family of closed sets of C, this amounts to $\bigcup U = C(\bigcup U)$.

Now, observe that $Y \subseteq \bigcup U$ because $y \in C(y)$ and $C(y) \in U$ for every $y \in Y$. Consequently, $C(Y) \subseteq C(\bigcup U) = \bigcup U$. Together with the assumption that $x \in C(Y)$, this yields $x \in \bigcup U$. Therefore, there exists a finite $Z \subseteq Y$ such that $x \in C(Z)$. By the definition of $\vdash_C$ this amounts to $Z \vdash_C x$ as desired.

(iii)⇒(i): Let $Y \cup \{x\} \subseteq X$ be such that $x \in C(Y)$. By the definition of $\vdash_C$ we have $Y \vdash_C x$. Since $\vdash_C$ is finitary, there exists a finite $Z \subseteq Y$ such that $Z \vdash_C x$. Again by the definition of $\vdash_C$ this amounts to $x \in C(Z)$. ⊠

The finitely generated closed sets of a finitary closure operator can be described in purely lattice theoretic therms, as we proceed to explain.

Definition 1.48. An element x of a complete lattice $\mathbb{X}$ is said to be *compact* when for every $Y \subseteq X$,

$$\text{if } x \leqslant \bigvee Y, \text{ there exists a finite } Z \subseteq Y \text{ such that } x \leqslant \bigvee Z.$$

Notably, compact elements are closed under binary joins:

Proposition 1.49 (Compact elements)**.** *The set* $\mathsf{Comp}(\mathbb{X})$ *of compact elements of a complete lattice* $\mathbb{X}$ *is closed under binary joins in* $\mathbb{X}$ *and contains the least element of* $\mathbb{X}$*.*

Proof. Consider $x, y \in \mathsf{Comp}(\mathbb{X})$ and a set $Z \subseteq X$ such that

$$x \vee y \leqslant \bigvee Z.$$

Clearly, $x, y \leqslant \bigvee Z$. As x and y are compact, there exist finite subsets $Z_1, Z_2 \subseteq Z$ such that $x \leqslant \bigvee Z_1$ and $y \leqslant \bigvee Z_2$. Then

$$x \vee y \leqslant \bigvee (Z_1 \cup Z_2).$$

Since the union $Z_1 \cup Z_2$ is also finite, we conclude that $x \vee y \in \mathsf{Comp}(\mathbb{X})$. The least element 0 of $\mathbb{X}$ is compact because $0 \leqslant \bigvee Y$ indeed implies $0 \leqslant \bigvee \emptyset$ for the finite set $\emptyset \subseteq Y$. ⊠

The set $\mathsf{Comp}(\mathbb{X})$ of compact elements of a complete lattice $\mathbb{X}$ therefore becomes a join semilattice $\langle \mathsf{Comp}(\mathbb{X}); \vee \rangle$ in the sense of Definition 1.30 when we endow it with the restriction of the binary join operation of $\mathbb{X}$. Moreover, $\langle \mathsf{Comp}(\mathbb{X}); \vee \rangle$ is a join subsemilattice of $\langle \mathbb{X}; \vee \rangle$ with the same least element as $\mathbb{X}$.

Recall from Proposition 1.40 that every closure system can be viewed as a complete lattice. If the closure system is finitary, then its finitely generated closed sets and compact elements are related as follows:

Proposition 1.50. *Let C be a finitary closure operator on a set X and $\mathcal{S}$ the associated closure system. A closed set $Y \subseteq X$ is finitely generated iff it is a compact element of $\langle \mathcal{S}; \subseteq \rangle$.*

Proof. Recall from Proposition 1.40 that the join of a family $\{Z_i :\in I\} \subseteq \mathcal{S}$ in $\langle \mathcal{S}; \subseteq \rangle$ can be described as follows:

$$\bigvee_{i \in I} Z_i = C(\bigcup_{i \in I} Z_i).$$

This fact will be used repeatedly in the proof.

Let Y be a finitely generated closed set of C. Then there exists a finite $Z \subseteq Y$ such that $Y = C(Z)$. Since Y is closed, it belongs to $\mathcal{S}$. Therefore, it suffices to prove that Y is compact in $\langle \mathcal{S}; \subseteq \rangle$. Accordingly, let $\{Z_i : i \in I\} \subseteq \mathcal{S}$ be such that

$$Y \subseteq \bigvee_{i \in I} Z_i = C(\bigcup_{i \in I} Z_i).$$

As $Z \subseteq C(Z) = Y$, this implies $Z \subseteq C(\bigcup_{i \in I} Z_i)$. Since C is finitary, for every $z \in Z$ there exists a finite $J_z \subseteq I$ such that

$$z \in C(\bigcup_{j \in J_z} Z_j).$$

Notice that the union $J := \bigcup\{J_z : z \in Z\}$ is also finite because Z is finite. Moreover, for every $z \in Z$,

$$z \in C(\bigcup_{j \in J_z} Z_j) \subseteq C(\bigcup_{j \in J} Z_j)$$

and, therefore, $Z \subseteq C(\bigcup_{j \in J} Z_j)$. Since $Y = C(Z)$, this yields

$$Y = C(Z) \subseteq C(C(\bigcup_{j \in J} Z_j)) = C(\bigcup_{j \in J} Z_j) = \bigvee_{j \in J} Z_j.$$

As J is finite, we conclude that Y is compact.

Conversely, consider a compact element Y of $\langle \mathcal{S}; \subseteq \rangle$. Since $\mathcal{S}$ is the family of closed sets of C, the set Y is closed. To prove that it is finitely generated, observe that

$$Y \subseteq \bigcup_{y \in Y} C(y) \subseteq C(\bigcup_{y \in Y} C(y)) = \bigvee_{y \in Y} C(y).$$

As Y is compact, there exists a finite $Z \subseteq Y$ such that

$$Y \subseteq \bigvee_{z \in Z} C(z) = C(\bigcup_{z \in Z} C(z)). \tag{1.2}$$

Since $C(z) \subseteq C(Z)$ for every $z \in Z$, we have $\bigcup_{z \in Z} C(z) \subseteq C(Z)$ and, therefore,

$$C(\bigcup_{z \in Z} C(z)) \subseteq C(C(Z)) = C(Z).$$

Together with condition (1.2), this yields $Y \subseteq C(Z)$. Moreover, $C(Z) \subseteq C(Y) = Y$ because Y is closed and $Z \subseteq Y$. Consequently, $Y = C(Z)$. As Z is finite, we conclude that Y is finitely generated. ⊠

For instance, let $\mathbb{X}$ be a poset. Recall that the closure operators $\uparrow$ and $\downarrow$ are finitary. By Theorem 1.47 the closure systems $\mathsf{Up}(\mathbb{X})$ and $\mathsf{Down}(\mathbb{X})$ associated with these closure operators are inductive. Furthermore, the compact elements of $\langle \mathsf{Up}(\mathbb{X}); \subseteq \rangle$ are the subsets of X that are either empty or of the form $\uparrow\{x_1, \dots, x_n\}$ for some $x_1, \dots, x_n \in X$ by Proposition 1.50. Similarly, the compact elements of $\langle \mathsf{Down}(\mathbb{X}); \subseteq \rangle$ are the subsets of X that are either empty of the form $\downarrow\{x_1, \dots, x_n\}$.

1.6 Algebraic lattices

Recall from Theorem 1.41 that every complete lattice can be represented as a closure system. Therefore, it is natural to wonder which are the complete lattices isomorphic to the *inductive* closure systems. Before we describe these lattices, let us first introduce a useful auxiliary notion.

Definition 1.51. A subset D of a complete lattice $\mathbb{X}$ is said to be *join dense* (resp. *meet dense*) when each element of $\mathbb{X}$ is a join (resp. a meet) of some subset of D. That is, for each $x \in \mathbb{X}$ there exists some $Y \subseteq D$ such that $x = \bigvee Y$ (resp. $x = \bigwedge Y$).

The next result reviews various equivalent formulations of the notion of join density. As expected, the dual statements characterize meet density.

Proposition 1.52. *The following conditions are equivalent for a subset D of a complete lattice $\mathbb{X}$:*

(i) *D is join dense in $\mathbb{X}$;*

(ii) *Each $x \in X$ is the join of all elements of D below x. That is, $x = \bigvee(\downarrow x \cap D)$;*

(iii) *For every $x, y \in X$,*

$$x \nleqslant y \iff \text{there exists } d \in D \text{ such that } d \leqslant x \text{ and } d \nleqslant y.$$

Proof. (i)$\Rightarrow$(ii): Consider $x \in X$. Since D is join dense in $\mathbb{X}$, there exists $Y \subseteq D$ such that $x = \bigvee Y$. Then $y \leqslant x$ for each $y \in Y$, so $Y \subseteq \downarrow x \cap D$. Each upper bound of the set $\downarrow x \cap D$ is therefore an upper bound of Y. Since x is an upper bound of $\downarrow x \cap D$ and the least upper bound of Y, it must in fact be the least upper bound of $\downarrow x \cap D$.

(ii)$\Rightarrow$(iii): The right to left implication of condition (iii) always holds by transitivity. To prove the left to tight implication, consider some $x, y \in X$ such that $x \nleqslant y$. Because x is the join of $\downarrow x \cap D$ by condition (ii), we have $\bigvee(\downarrow x \cap D) \nleqslant y$. Thus, there exists $d \in D$ such that $d \leqslant x$ and $d \nleqslant y$.

(iii)$\Rightarrow$(i): Consider some $x \in X$ and let $Y := \downarrow x \cap D$. We need to prove that $x = \bigvee Y$. Clearly x is an upper bound of Y, since x is an upper bound of $\downarrow x$. We show that x is the least upper bound. To this end, consider an upper bound y of Y. By the definition of Y, we see that $d \leqslant x$ implies $d \leqslant y$ for each $d \in D$. Thus $x \leqslant y$ by condition (iii). ⊠

Having identified compact elements as the abstract counterpart of finitely generated sets in inductive closure systems (Proposition 1.50), we can now further identify the abstract counterpart of inductive closure systems as the so-called algebraic lattices.

Definition 1.53. A complete lattice $\mathbb{X}$ is said to be *algebraic* when the set $\mathsf{Comp}(\mathbb{X})$ of compact elements of $\mathbb{X}$ is join dense in $\mathbb{X}$.

In order to connect algebraic lattices and inductive closure systems, a last ingredient is needed.

Definition 1.54. Let $\boldsymbol{A}$ be a join semilattice with least element 0. An *ideal* of $\boldsymbol{A}$ is a downset containing 0 such that for every $a, b \in A$,

$$\text{if } a, b \in I, \text{ then } a \vee b \in I.$$

The set of ideals of $\boldsymbol{A}$ will be denoted by $\mathsf{Id}(\boldsymbol{A})$.

Equivalently, the ideals of $\boldsymbol{A}$ are the upward directed downsets of $\boldsymbol{A}$.

Proposition 1.55. *If $\boldsymbol{A}$ is a join semilattice with a least element, then $\mathsf{Id}(\boldsymbol{A})$ is an inductive closure system on $\boldsymbol{A}$.*

Proof. We begin by proving that $\mathsf{Id}(\boldsymbol{A})$ is a closure system. Consider a family $\{I_j : j \in J\}$ of ideals of $\boldsymbol{A}$. We will prove that the intersection I of the various I_j is an ideal of $\boldsymbol{A}$. Since each I_j contains the least element 0 of $\boldsymbol{A}$, their intersection I contains 0 too. Furthermore, as each I_j is a downset and intersections of downsets are still downsets, the set I is also a downset. Lastly, consider $a, b \in I$. Then a and b belong to each I_j and so does $a \vee b$ because I_j is closed under binary joins. Thus, $a \vee b \in I$ as desired. Hence, we conclude that I is an ideal. This establishes that $\mathsf{Id}(\boldsymbol{A})$ is a closure system.

To prove that the closure system $\mathsf{Id}(\boldsymbol{A})$ is inductive, consider an upward directed family $\{I_j : j \in J\} \subseteq \mathsf{Id}(\boldsymbol{A})$ and let I be its union. Since upward directed families are nonempty by definition, there exists some $j \in J$. As $0 \in I_j$ and $I_j \subseteq I$, we obtain $0 \in I$. To prove that I is a downset, consider $a, b \in A$ such that $a \in I$ and $b \leqslant a$. Since $a \in I$, there exists $j \in J$ such that $a \in I_j$. As I_j is a downset and $b \leqslant a$, this implies $b \in I_j \subseteq I$. It only remains to prove that I is closed under binary joins. To this end, consider $a, b \in I$. Then there are $j_1, j_2 \in J$ such that $a \in I_{j_1}$ and $b \in I_{j_2}$. Since the family $\{I_j : j \in J\}$ is upward directed, there exists $j \in J$ such that $I_{j_1}, I_{j_2} \subseteq I_j$. This implies $a, b \in I_j$ and, therefore, $a \vee b \in I_j$ because I_j is closed under binary joins. As $I_j \subseteq I$, we conclude that $a \vee b \in I$ as desired. ⊠

Let $\mathbb{X}$ be a complete lattice. Recall from Proposition 1.49 that $\langle \mathsf{Comp}(\mathbb{X}); \vee \rangle$ is a join semilattice whose least element is the least element of $\mathbb{X}$. Therefore, $\mathsf{Id}(\langle \mathsf{Comp}(\mathbb{X}); \vee \rangle)$ is an inductive closure system by Proposition 1.55. Bearing this in mind, algebraic lattices and inductive closure systems are related as follows:

Theorem 1.56 (Representation theorem)**.** *Every inductive closure system is an algebraic lattice. Conversely, if $\mathbb{X}$ is an algebraic lattice, then $\mathsf{Id}(\langle \mathsf{Comp}(\mathbb{X}); \vee \rangle)$ is an inductive closure system isomorphic to $\mathbb{X}$.*

Proof. Let $\mathcal{S}$ be an inductive closure system on a set X. By Proposition 1.40 the pair $\langle \mathcal{S}; \subseteq \rangle$ is a complete lattice. Therefore, to prove that $\langle \mathcal{S}; \subseteq \rangle$ is an algebraic lattice, it suffices to show that each $Y \in \mathcal{S}$ is a join of compact elements.

To this end, recall from condition (i) of Theorem 1.39 that $\mathcal{S}$ is the closure system associated with some closure operator C on X. Since $\mathcal{S}$ is inductive, C is finitary by Theorem 1.47. Therefore, we can apply Proposition 1.50 obtaining that the compact elements of $\mathcal{S}$ are precisely the finitely generated closed sets of C.

Then consider an element $Y \in \mathcal{S}$. Since $\mathcal{S}$ is the family of closed sets of C, we have $Y = C(Y)$. In turn, this yields

$$Y = C(\bigcup_{y \in Y} C(y)) = \bigvee_{y \in Y} C(y).$$

Since each $C(y)$ is finitely generated, it is also compact in $\langle \mathcal{S}; \subseteq \rangle$. Consequently, Y is a join of compact elements as desired.

To prove the converse, consider an algebraic lattice $\mathbb{X}$. Then $\mathsf{Id}(\langle \mathsf{Comp}(\mathbb{X}); \vee \rangle)$ is an inductive closure system. We will prove that the map $f \colon \mathbb{X} \to \mathsf{Id}(\langle \mathsf{Comp}(\mathbb{X}); \vee \rangle)$ defined by the rule

$$f(x) := \{y \in \mathsf{Comp}(\mathbb{X}) : y \leqslant x\}$$

is an order isomorphism. We begin by showing that $f(x)$ is an ideal of $\langle \mathsf{Comp}(\mathbb{X}); \vee \rangle$ for every $x \in X$. Clearly, $f(x)$ is a downset of $\mathsf{Comp}(\mathbb{X})$ containing the least element of $\mathsf{Comp}(\mathbb{X})$. To prove that $f(x)$ is closed under binary joins, consider $y, z \in f(x)$. The definition of $f(x)$ guarantees that $y, z \leqslant x$ and $y, z \in \mathsf{Comp}(\mathbb{X})$. From $y, z \leqslant x$ it follows $y \vee z \leqslant x$. Furthermore, since $y, z \in \mathsf{Comp}(\mathbb{X})$ and $\mathsf{Comp}(\mathbb{X})$ is closed under binary joins, we have $y \vee z \in \mathsf{Comp}(\mathbb{X})$. Hence, $y \vee z \in f(x)$ and $f(x)$ is an ideal of $\langle \mathsf{Comp}(\mathbb{X}); \vee \rangle$ as desired. It follows that the map $f \colon \mathbb{X} \to \mathsf{Id}(\langle \mathsf{Comp}(\mathbb{X}); \vee \rangle)$ is well defined.

To prove that f is an order embedding, consider $x, y \in X$. Clearly, if $x \leqslant y$, then $f(x) \subseteq f(y)$. To prove that $f(x) \subseteq f(y)$ implies $x \leqslant y$, we reason by contraposition. Suppose that $x \nleqslant y$. Then recall that $\mathsf{Comp}(\mathbb{X})$ is join dense in $\mathbb{X}$ because the lattice $\mathbb{X}$ is algebraic. Therefore, we can apply condition (iii) of Proposition 1.52 to the assumption that $x \nleqslant y$ obtaining that there exists $z \in \mathsf{Comp}(\mathbb{X})$ such that $z \leqslant x$ and $z \nleqslant y$. This implies that $z \in f(x) - f(y)$ and, therefore, $f(x) \nsubseteq f(y)$. Hence, we conclude that f is an order embedding.

It only remains to prove that f is surjective. Consider an ideal I of $\langle \mathsf{Comp}(\mathbb{X}); \vee \rangle$. We will show that $f(x) = I$ for $x := \bigvee I$. Clearly,

$$I \subseteq \{y \in \mathsf{Comp}(\mathbb{X}) : y \leqslant \bigvee I\} = \{y \in \mathsf{Comp}(\mathbb{X}) : y \leqslant x\} = f(x).$$

To prove that $f(x) \subseteq I$, consider $y \in f(x)$. Then $y \in \mathsf{Comp}(\mathbb{X})$ and $y \leqslant x = \bigvee I$. Since y is compact, there are $z_1, \dots, z_n \in I$ such that $y \leqslant z_1 \vee \dots \vee z_n$. Consequently,

$$y \vee z_1 \vee \dots \vee z_n = z_1 \vee \dots \vee z_n \in I.$$

As I is an ideal, this yields $y \in I$. Thus, we conclude that $f(x) = I$. ⊠

Recall that the closure systems $\mathsf{Up}(\mathbb{X})$ and $\mathsf{Down}(\mathbb{X})$ are inductive for every poset $\mathbb{X}$. From Theorem 1.56 it follows that they are also algebraic lattices. Powerset lattices $\langle \mathcal{P}(X); \subseteq \rangle$ are algebraic too because they coincide with the closure systems of the form

$\mathsf{Up}(\mathbb{X})$, where $\mathbb{X}$ is the discrete poset with universe X. Lastly, every finite lattice is algebraic because finite lattices are complete and all their elements are compact.

At this stage it is important to notice that, while algebraic lattices are precisely the posets isomorphic to inductive closure systems, there are closure systems that fail to be inductive but that nonetheless are algebraic lattices.

Example 1.57. Consider the set $X := \mathbb{N} \cup \{\infty\}$ and let $C \colon \mathcal{P}(X) \to \mathcal{P}(X)$ be the map defined for every $Y \subseteq X$ as

$$C(Y) := \begin{cases} Y & \text{if } Y \subsetneq \mathbb{N}; \\ X & \text{otherwise.} \end{cases}$$

It is easy to see that C is a closure operator on X. Furthermore, C is not finitary because $\infty \in C(\mathbb{N})$, but $\infty \notin Y = C(Y)$ for every finite $Y \subseteq \mathbb{N}$. Consequently, the closure system $\mathcal{S}_C$ associated with C is not inductive. However, that restriction map $(-) \cap \mathbb{N} \colon \mathcal{S}_C \to \mathcal{P}(\mathbb{N})$ is an order isomorphism from $\mathcal{S}_C$ to the powerset lattice $\langle \mathcal{P}(\mathbb{N}); \subseteq \rangle$. Since the latter is an algebraic lattice, so is the closure system $\mathcal{S}_C$. ⊠

We close this section with a useful result on the structure of algebraic lattices.

Definition 1.58. An element x of a lattice $\mathbb{X}$ is said to be

(i) *meet irreducible* in $\mathbb{X}$ when it is not the greatest element of $\mathbb{X}$ and for every $y, z \in X$,

$$\text{if } x = y \wedge z \text{, then } x = y \text{ or } x = z;$$

(ii) *join irreducible* in $\mathbb{X}$ when it is not the least element of $\mathbb{X}$ and for every $y, z \in X$,

$$\text{if } x = y \vee z \text{, then } x = y \text{ or } x = z;$$

(iii) *completely meet irreducible* in $\mathbb{X}$ when for every $Y \subseteq X$,

$$\text{if } x = \bigwedge Y \text{, then } x \in Y;$$

(iv) *completely join irreducible* in $\mathbb{X}$ when for every $Y \subseteq X$,

$$\text{if } x = \bigvee Y \text{, then } x \in Y.$$

The sets of meet irreducible and completely meet irreducible elements of a poset $\mathbb{X}$ will be denoted by $\mathsf{M}(\mathbb{X})$ and $\mathsf{M}_\infty(\mathbb{X})$. Similarly, we will denote the sets of join irreducible and completely join irreducible elements of $\mathbb{X}$ by $\mathsf{J}(\mathbb{X})$ and $\mathsf{J}_\infty(\mathbb{X})$.

Observe that the (completely) meet irreducible elements of $\mathbb{X}$ are precisely the (completely) join irreducible elements of $\mathbb{X}^\partial$, and vice versa.

The meet irreducibility (resp. join irreducibility) of x can equivalently be defined as follows: for every *finite* $Y \subseteq X$, if $x = \bigwedge Y$ (resp. if $x = \bigvee Y$), then $x \in Y$. In other words, restricting to finite sets $Y \subseteq X$ in the definition of completely meet (resp. join) irreducible elements yields precisely the meet (resp. join) irreducible elements. Indeed, if x satisfies this alternative definition of meet irreducibility, then x cannot be the greatest element of $\mathbb{X}$, since otherwise $x = \bigwedge \varnothing$ but $x \notin \varnothing$. Moreover, for such an element x the equality $x = y \wedge z$ implies that $x = y$ or $x = z$, since $x = \bigwedge\{y, z\}$ implies $x \in \{y, z\}$. Proving that conversely each meet irreducible $x \in X$ satisfies the above condition is a simple induction on the cardinality of Y. The equivalence of the two definitions of join irreducibility then follows from the Duality Principle.

The inclusion $\mathsf{M}_\infty(\mathbb{X}) \subseteq \mathsf{M}(\mathbb{X})$ and $\mathsf{J}_\infty(\mathbb{X}) \subseteq \mathsf{J}(\mathbb{X})$ are now immediate. On the other hand, meet (resp. join) irreducible elements need not be completely meet (resp. join) irreducible. For instance, every element of a chain $\mathbb{X}$ other than the greatest element is meet irreducible. To prove this, suppose that $x \in \mathbb{X}$ is not the greatest element of $\mathbb{X}$ and that $x = y \wedge z$ for some $y, z \in X$. Since $\mathbb{X}$ is a chain, $y \wedge z$ is the least element between y and z. Consequently, $x = y$ or $x = z$, so x is indeed meet irreducible. Similarly, every element of $\mathbb{X}$ other than the minimum is join irreducible. On the other hand, in the chain of real numbers $\langle \mathbb{R}; \leqslant \rangle$ no element is completely meet irreducible or completely join irreducible. This is because every real number x is the meet of the set of all elements $> x$ and the join of the set of all elements $< x$.

Example 1.59. A lattice may lack meet and join irreducible elements altogether. For instance, the direct product $\langle \mathbb{Z}; \leqslant \rangle \times \langle \mathbb{Z}; \leqslant \rangle$ is the lattice with universe $\mathbb{Z} \times \mathbb{Z}$ and order relation $\sqsubseteq$ defined as

$$\langle m_1, m_2 \rangle \sqsubseteq \langle n_1, n_2 \rangle \iff m_1 \leqslant n_1 \text{ and } m_2 \leqslant n_2.$$

No element of $\langle \mathbb{Z}; \leqslant \rangle \times \langle \mathbb{Z}; \leqslant \rangle$ is meet irreducible or join irreducible because

$$\langle m, n \rangle = \langle m, n+1 \rangle \wedge \langle m+1, n \rangle \quad \text{and} \quad \langle m, n \rangle = \langle m, n-1 \rangle \vee \langle m-1, n \rangle$$

for every $m, n \in \mathbb{Z}$. ⊠

This makes the following property of algebraic lattices appealing.

Theorem 1.60. *In $\mathbb{X}$ an algebraic lattice, the set of completely meet irreducible elements $\mathsf{M}_\infty(\mathbb{X})$ is meet dense.*

Proof. Consider an element $x \in X$ and let

$$Y := \{y \in \mathsf{M}_\infty(\mathbb{X}) : x \leqslant y\}.$$

We will prove that $x = \bigwedge Y$. The definition of Y guarantees that $x \leqslant \bigwedge Y$. Then we turn prove that $\bigwedge Y \leqslant x$. Since $\mathbb{X}$ is algebraic, $\bigwedge Y$ is the join of a set of compact elements. Consequently, it suffices to show that every compact element below $\bigwedge Y$ is also below x.

Suppose, on the contrary, that there exists a compact element $y \leqslant \bigwedge Y$ such that $y \nleqslant x$. Then consider the set

$$Z := \{z \in X : x \leqslant z \text{ and } y \nleqslant z\}.$$

We will use Zorn's Lemma to establish the existence of a maximal element in the subposet $\mathbb{Z}$ of $\mathbb{X}$ with universe Z. To this end, we need to prove that every chain C in $\mathbb{Z}$ has an upper bound in $\mathbb{Z}$. We will do this by showing that $\bigvee C \in Z$. Accordingly, consider a chain C in $\mathbb{Z}$. We may assume without loss of generality that $x \in C$ because x is the least element of $\mathbb{Z}$. As a consequence, we have $x \leqslant \bigvee C$. To prove that $\bigvee C \in Z$, it only remains to show that $y \nleqslant \bigvee C$. Suppose the contrary, with a view to contradiction. Since y is compact and $y \leqslant \bigvee C$, there exists a finite subset $D \subseteq C$ such that $y \leqslant \bigvee D$. We may assume that $x \in D$ because $x \in C$. Then consider an enumeration $D = \{z_1, \ldots, z_n\}$. Notice that $\{z_1, \ldots, z_n\}$ is a finite chain in $\mathbb{Z}$ because C is a chain in $\mathbb{Z}$ and $D \subseteq C$. Therefore, $\{z_1, \ldots, z_n\}$ has a greatest element z_k. As a consequence,

$$y \leqslant \bigvee D = z_1 \vee \cdots \vee z_n = z_k.$$

On the other hand, the assumption that $z_k \in Z$ implies $y \nleqslant z_k$, a contraction. Hence, we conclude that $y \nleqslant \bigvee C$. This implies $\bigvee C \in Z$ as desired. Hence, we can apply Zorn's Lemma obtaining that $\mathbb{Z}$ has a maximal element z.

We will prove that $z \in \mathsf{M}_\infty(\mathbb{X})$. Consider therefore some $U \subseteq X$ such that $z = \bigwedge U$. Since $z \in Z$ we have $y \nleqslant z$. Together with $z = \bigwedge U$, this implies that there exists $u \in U$ such that $y \nleqslant u$. Moreover, from $z = \bigwedge U$ and $u \in U$ it follows that $z \leqslant u$. As $z \in Z$, this yields $x \leqslant z \leqslant u$. Therefore, $x \leqslant u$ and $y \nleqslant u$, that is, $u \in Z$. Since $z \leqslant u$ and z is maximal in $\mathbb{Z}$, we conclude that $z = u$, so $z \in U$.

Lastly, from $z \in Z$ it follows that $x \leqslant z$. Thus, $z \in Y$. Recall that $y \leqslant \bigwedge Y$ by assumption. As $z \in Y$, we obtain $y \leqslant z$, a contradiction with $z \in Z$. Hence, we conclude that $\bigwedge Y \leqslant x$ and therefore $x = \bigwedge Y$. ⊠

Example 1.61 (Completely meet irreducible upsets)**.** Recall that $\langle \mathsf{Up}(\mathbb{X}); \subseteq \rangle$ is an algebraic lattice for every poset $\mathbb{X}$. Bearing this in mind, we will exemplify Theorem 1.60 by characterizing the completely meet irreducible elements of $\langle \mathsf{Up}(\mathbb{X}); \subseteq \rangle$ and explaining why every element of $\mathsf{Up}(\mathbb{X})$ is a meet of completely meet irreducible ones. More precisely, we will prove that

$$\mathsf{M}_\infty(\mathsf{Up}(\mathbb{X})) = \{X - {\downarrow} x : x \in X\} \tag{1.3}$$

and that every upset of $\mathbb{X}$ is an intersection of elements of $\mathsf{M}_\infty(\mathsf{Up}(\mathbb{X}))$.

Observe that the map $i \colon Y \mapsto X - Y$ is an isomorphism between the poset $\langle \mathsf{Up}(\mathbb{X}); \subseteq \rangle$ and the poset $\langle \mathsf{Down}(\mathbb{X}); \subseteq \rangle^\partial = \langle \mathsf{Down}(\mathbb{X}); \supseteq \rangle$: if Y is an upset of $\mathbb{X}$, then $X - Y$ is a downset of $\mathbb{X}$, and moreover $Y \subseteq Z$ for upsets Y, Z of $\mathbb{X}$ iff $X - Y \supseteq X - Z$. An upset Y of $\mathbb{X}$ is therefore completely meet irreducible in $\mathsf{Up}(\mathbb{X})$ iff $i(Y)$ is completely meet irreducible in $\langle \mathsf{Down}(\mathbb{X}), \subseteq \rangle^\partial$, or in other words iff $i(Y)$ is completely join irreducible in $\langle \mathsf{Down}(\mathbb{X}), \subseteq \rangle$. The meet irreducibles of $\mathsf{Up}(\mathbb{X})$ are therefore precisely the complements of the join irreducibles of $\mathsf{Down}(\mathbb{X})$.

We now show that the join irreducibles of $\mathsf{Down}(\mathbb{X})$ are precisely the principal downsets. We begin by proving that each principal downset ${\downarrow} x$ of $\mathbb{X}$ is indeed completely join irreducible. For suppose that ${\downarrow} x = \bigcup_{i \in i} Y_i$ for some family $\{Y_i : i \in I\}$ of downsets of $\mathbb{X}$. Then $x \in Y_j$ for some $j \in I$. Since Y_j is a downset, from $x \in Y_j$ it follows ${\downarrow} x \subseteq Y_j$. On the other hand, $Y_j \subseteq \bigcup_{i \in i} Y_i = {\downarrow} x$, so indeed $Y_j = {\downarrow} x$. Conversely, each downset Y of $\mathbb{X}$ is a union, and therefore a join, of principal downsets, namely $Y = \bigcup_{y \in Y} {\downarrow} y$. If Y is completely join irreducible, then $Y = {\downarrow} y$ for some $y \in Y$. In other words, each completely join irreducible downset of $\mathbb{X}$ must be principal.

Putting the last two paragraphs together, the meet irreducible elements of $\langle \mathsf{Up}(\mathbb{X}); \subseteq \rangle$ are precisely the complements of principal downsets of $\mathbb{X}$. In other words, these are the sets of the form $X - {\downarrow}x$ for $x \in X$. $\boxtimes$