# The Algebra of Logic

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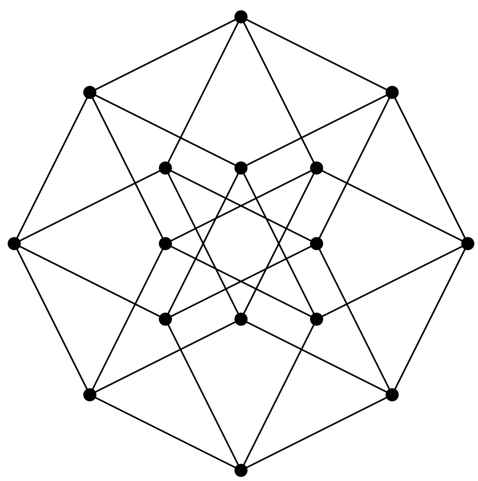


# Contents

CHAPTER 1

# Lattices and closure operators

## 1.1 Partially ordered sets

Given a binary relation $R$ on a set $X$ and $x, y \in X$, we write $xRy$ when the ordered pair $\langle x, y \rangle$ belongs to the relation $R$. A binary relation $R$ on a set $X$ is said to be

(i) *reflexive* when $xRx$ for every $x \in X$;

(ii) *transitive* when for every $x, y, z \in X$, if $xRy$ and $yRz$, then $xRz$;

(iii) *antisymmetric* when for every $x, y \in X$, if $xRy$ and $yRx$, then $x = y$.

**Definition 1.1.** A binary relation $\leqslant$ on a set $X$ is said to be a *partial order* when it is reflexive, transitive, and antisymmetric. In this case, the pair $\mathbb{X} = \langle X; \leqslant \rangle$ is said to be a *poset* (a shorthand for a *partially ordered set*) and $X$ is called the *universe* of $\mathbb{X}$.

Given a poset $\mathbb{X}$ and $x, y \in X$, we will often say that $x$ is *below* $y$ to indicate that $x \leqslant y$. Furthermore, we write $x < y$ when both $x \leqslant y$ and $x \neq y$, or equivalently $x \leqslant y$ and $y \nleqslant x$. Lastly, given $x, y, z \in X$, we will write $x \leqslant y, z$ as a shorthand for $x \leqslant y$ and $x \leqslant z$. A similar reading applies to expressions of the form $x, y \leqslant z$. Notice that the universe of $\mathbb{X}$ might be empty, in which case the relation $\leqslant$ is also empty.

**Definition 1.2.** Let $\mathbb{X}$ be a poset. A set $V \subseteq X$ is said to be

(i) an *upset* of $\mathbb{X}$ when for every $x, y \in X$,

$$\text{if } x \in V \text{ and } x \leqslant y, \text{ then } y \in V;$$

(ii) a *downset* of $\mathbb{X}$ when for every $x, y \in X$,

$$\text{if } x \in V \text{ and } y \leqslant x, \text{ then } y \in V.$$

The families of upsets and downsets of $\mathbb{X}$ will be denoted by $\mathsf{Up}(\mathbb{X})$ and $\mathsf{Down}(\mathbb{X})$.

**Example 1.3** (Posets)**.** Given a set $X$, the *identity relation* $\mathrm{id}_X$ is defined for all $x, y \in X$ as

$$\langle x, y\rangle \in \mathrm{id}_X \iff x = y.$$

The pair $\langle X; \mathrm{id}_X\rangle$ is a poset, known as the *discrete poset* with universe $X$.

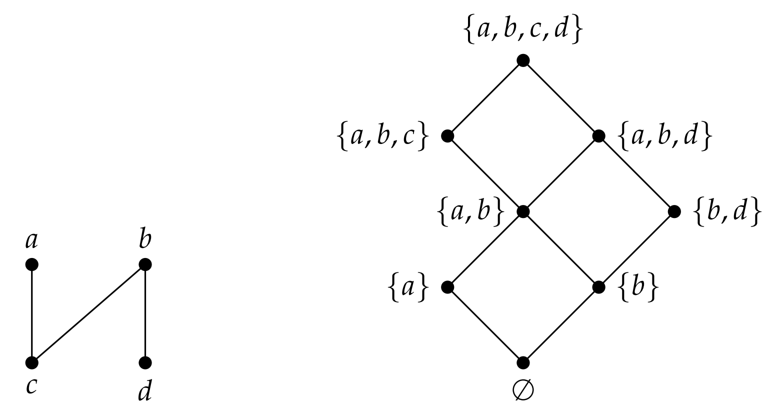
Given a poset $\mathbb{X}$, the pairs $\langle \mathsf{Up}(\mathbb{X}); \subseteq\rangle$ and $\langle \mathsf{Down}(\mathbb{X}); \subseteq\rangle$ are also posets. When $\mathbb{X} = \langle X; \mathrm{id}_X\rangle$, both $\langle \mathsf{Up}(\mathbb{X}); \subseteq\rangle$ and $\langle \mathsf{Down}(\mathbb{X}); \subseteq\rangle$ coincide with the poset $\langle \mathcal{P}(X); \subseteq\rangle$, which is known as the *powerset lattice* of $X$.

Further examples include the sets $\mathbb{N}$ and $\mathbb{Z}$ of natural and integer numbers, respectively, endowed with the standard order. Lastly, the set $\mathbb{N}$ endowed with the *divisibility relation*

$$\mid \; := \{\langle m, n\rangle \in \mathbb{N} \times \mathbb{N} : \text{there exists } k \in \mathbb{N} \text{ s.t. } n = m \cdot k\}$$

is also a poset, known as the *divisibility lattice*. However, the set $\mathbb{Z}$ fails to be a poset when we endow it with the analogous divisibility relation. This is because the divisibility relation is not antisymmetric on $\mathbb{Z}$, as $n$ and $-n$ are distinct and divide each other, for every positive integer $n$. ⊠

An attractive feature of posets is that they can often be represented by pictures consisting of dots connected by lines, known as *Hasse diagrams*. This is always true for finite posets $\mathbb{X}$ of a manageable size, whose Hasse diagrams can be obtained as follows. First, we depict the elements of $X$ as dots, making sure that if $x < y$, then the dot corresponding to $x$ lies below that corresponding to $y$. Then we connect two dots with a line whenever they correspond to two points $x, y \in X$ such that $x < y$ and that there is no $z \in X$ such that $x < z < y$. As a result, $x < y$ for two elements $x, y \in X$ precisely when there exists an ascending path from $x$ to $y$. The picture below illustrates this method by depicting the Hasse diagram of poset $\mathbb{X}$ and that of the poset $\langle \mathsf{Up}(\mathbb{X}); \subseteq\rangle$ of its upsets.



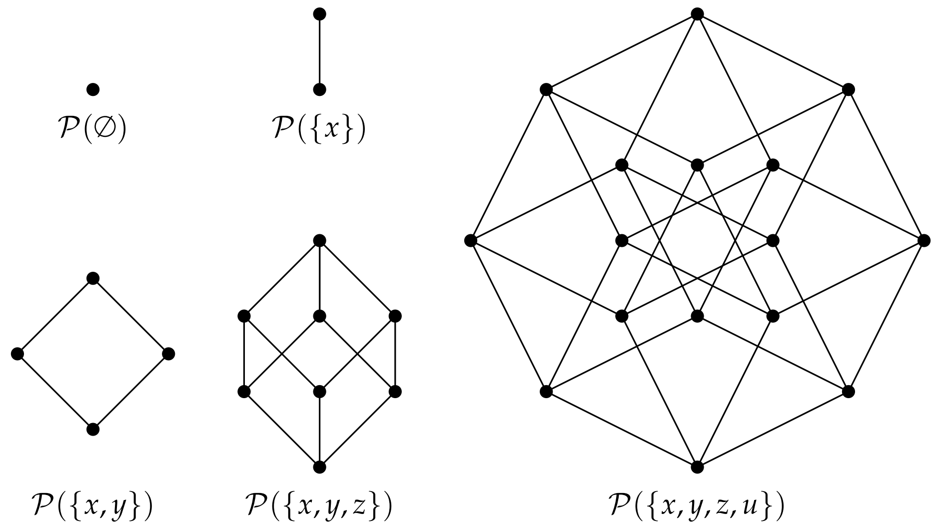

$\mathbb{X}$ $\mathsf{Up}(\mathbb{X})$

The left hand side of this picture indicates that $\mathbb{X}$ is the poset with universe $\{a, b, c, d\}$ and order relation

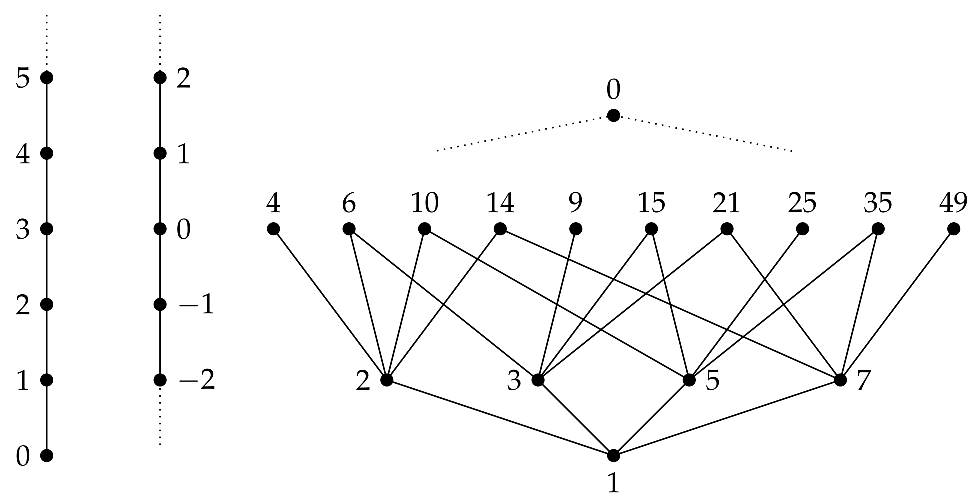
$$\leqslant \; = \mathrm{id}_X \cup \{\langle c, a\rangle, \langle c, b\rangle, \langle d, b\rangle\}.$$

The structure of $\mathsf{Up}(\mathbb{X})$ can be inferred in a similar way from the the right hand side of the picture.

Finite posets can be faithfully reconstructed from their Hasse diagrams. In other words, to define a finite poset it suffices to draw its Hasse diagram. As an example, the Hasse diagrams of the powerset lattices $\langle \mathcal{P}(X); \subseteq \rangle$ with $X$ of cardinality $\leqslant 4$ are shown below.




Hasse diagrams can also be employed to describe infinite posets, provided that their structure is regular enough to be indicated by dotted lines. For instance, the posets $\langle \mathbb{N}; \leqslant \rangle$ and $\langle \mathbb{Z}; \leqslant \rangle$ are depicted below, together with a portion of the divisibility lattice $\langle \mathbb{N}; \mid \rangle$.




$\langle \mathbb{N}; \leqslant \rangle$ $\langle \mathbb{Z}; \leqslant \rangle$ $\langle \mathbb{N}; \mid \rangle$

In order to relate distinct posets, it is convenient to introduce maps that preserve their structure. In this context, we will need to distinguish between the order relations of different posets. To this end, we use the symbol $\leqslant^{\mathbb{X}}$ to denote the order relation of a poset $\mathbb{X}$.

**Definition 1.4.** Let $\mathbb{X}$ and $\mathbb{Y}$ be posets. A map $f\colon \mathbb{X} \to \mathbb{Y}$ is said to be

(i) *order preserving* when for every $x, y \in X$,

$$\text{if } x \leqslant^{\mathbb{X}} y, \text{ then } f(x) \leqslant^{\mathbb{Y}} f(y);$$

(ii) *order reflecting* when for every $x, y \in X$,

$$\text{if } f(x) \leqslant^{\mathbb{Y}} f(y), \text{ then } x \leqslant^{\mathbb{X}} y;$$

(iii) an *order embedding* when it is both order preserving and order reflecting;

(iv) an *order isomorphism* when it is a surjective order embedding.

For instance, the inclusion map $i\colon \mathbb{N} \to \mathbb{Z}$ is an order embedding of $\langle \mathbb{N}; \leqslant \rangle$ into $\langle \mathbb{Z}; \leqslant \rangle$, while the function $f\colon \mathbb{Z} \to \mathbb{N}$ defined by the rule

$$f(n) \coloneqq \text{the greatest element between 0 and } n$$

is an order preserving map from $\langle \mathbb{Z}; \leqslant \rangle$ to $\langle \mathbb{N}; \leqslant \rangle$ that is not order reflecting.

Notice that order embeddings are necessarily injective. To prove this, consider an order embedding $f\colon \mathbb{X} \to \mathbb{Y}$ and two distinct elements $x, y \in X$. Since $\leqslant^{\mathbb{X}}$ is antisymmetric and $x \neq y$, either $x \nleqslant^{\mathbb{X}} y$ or $y \nleqslant^{\mathbb{X}} x$. But $f$ is order reflecting, so either $f(x) \nleqslant^{\mathbb{Y}} f(y)$ or $f(y) \nleqslant^{\mathbb{Y}} f(x)$. In either case, $f(x) \neq f(y)$ by the reflexivity of the relation $\leqslant^{\mathbb{Y}}$, thus $f$ is indeed injective.

As a consequence, an order isomorphism $f\colon \mathbb{X} \to \mathbb{Y}$ is precisely a bijection such that for every $x, y \in X$,

$$x \leqslant^{\mathbb{X}} y \Longleftrightarrow f(x) \leqslant^{\mathbb{Y}} f(y).$$

It follows that $f\colon \mathbb{X} \to \mathbb{Y}$ is an order isomorphism iff so is $f^{-1}\colon \mathbb{Y} \to \mathbb{X}$. We will write $\mathbb{X} \cong \mathbb{Y}$ to indicate that there exists an order isomorphism between $\mathbb{X}$ and $\mathbb{Y}$. For instance, let $2\mathbb{Z}$ be the set of even integers. The map $f\colon \mathbb{Z} \to 2\mathbb{Z}$ defined by the rule $f(n) \coloneqq 2n$ is an order isomorphism from $\langle \mathbb{Z}; \leqslant \rangle$ to the poset consisting of $2\mathbb{Z}$ ordered under the standard order.

When there exists an order embedding $f\colon \mathbb{X} \to \mathbb{Y}$, the poset $\mathbb{Y}$ contains a copy of $\mathbb{X}$, consisting of the elements of the image $f[X]$ endowed with the restriction of the order relation $\leqslant^{\mathbb{Y}}$ to $f[X]$, as we proceed to explain.

**Definition 1.5.** A poset $\mathbb{X}$ is said to be a *subposet* of a poset $\mathbb{Y}$ when $X \subseteq Y$ and $\leqslant^{\mathbb{X}}$ is the restriction of $\leqslant^{\mathbb{Y}}$ to $X$.

In this case, if no confusion is likely to arise, we will use the same notation for the order relation of $\mathbb{X}$ and $\mathbb{Y}$.

**Proposition 1.6.** *A poset $\mathbb{X}$ is isomorphic to a subposet of $\mathbb{Y}$ iff there exists an order embedding $f\colon \mathbb{X} \to \mathbb{Y}$. In this case, $\mathbb{X}$ is isomorphic to $\langle f[X]; \leqslant \rangle$, where $\leqslant$ is the restriction of $\leqslant^{\mathbb{Y}}$ to $f[X]$.*

*Proof.* First, suppose that $\mathbb{X}$ is isomorphic to a subposet $\mathbb{Z}$ of $\mathbb{Y}$. Then there exists an order isomorphism $f \colon \mathbb{X} \to \mathbb{Z}$. Since $\mathbb{Z}$ is a subposet of $\mathbb{Y}$, the map $f \colon \mathbb{X} \to \mathbb{Y}$ is an order embedding. To prove the converse, consider an order embedding $f \colon \mathbb{X} \to \mathbb{Y}$ and let $\langle f[X]; \leqslant \rangle$ be the poset defined in the statement. The map $f \colon \mathbb{X} \to \langle f[X]; \leqslant \rangle$ is clearly surjective. Furthermore, it is an order embedding, since so is $f \colon \mathbb{X} \to \mathbb{Y}$. Thus, we conclude that $f \colon \mathbb{X} \to \langle f[X]; \leqslant \rangle$ is an order isomorphism. ⊠

The following notions are instrumental to comparing the elements of a given poset.

**Definition 1.7.** Two elements $x$ and $y$ of a poset $\mathbb{X}$ are said to be *comparable* when either $x \leqslant y$ or $y \leqslant x$. They are said to be *incomparable* otherwise. Accordingly, we say that $\mathbb{X}$ is

(i) a *chain* when every two elements of $X$ are comparable;

(ii) an *antichain* when every two distinct elements of $X$ are incomparable, that is, if $\mathbb{X}$ is the discrete poset with universe $X$.

By extension, a subset $Y \subseteq X$ is said to be a *chain* (resp. an *antichain*) in $\mathbb{X}$ when the subposet of $\mathbb{X}$ with universe $Y$ is a chain (resp. an antichain). Chains are sometimes also called *linearly ordered* posets, while antichains are sometimes called *discretely ordered* posets.

For instance, the number systems $\langle \mathbb{N}; \leqslant \rangle$ and $\langle \mathbb{Z}; \leqslant \rangle$ are both chains. Most posets, however, are neither chains nor antichains. For example, the powerset lattice $\langle \mathcal{P}(X); \subseteq \rangle$ is neither a chain nor an antichain for every set $X$ with at least two elements. To see this, consider two distinct elements $x, y \in X$. Then the sets $\{x\}$ and $\{y\}$ are incomparable elements of $\langle \mathcal{P}(X); \subseteq \rangle$, so this poset is not a chain. Similarly, $\emptyset$ and $\{x\}$ are two distinct, but comparable elements. Consequently, $\langle \mathcal{P}(X); \subseteq \rangle$ is not an antichain.

The following special elements play a fundamental role in order theory:

**Definition 1.8.** An element $x$ of a poset $\mathbb{X}$ is said to be

(i) a *greatest element* of $\mathbb{X}$ when $y \leqslant x$ for every $y \in X$;

(ii) a *least* element of $\mathbb{X}$ when $x \leqslant y$ for every $y \in X$;

(iii) a *maximal element* of $\mathbb{X}$ when there is no $y \in X$ such that $x < y$;

(iv) a *minimal element* of $\mathbb{X}$ when there is no $y \in X$ such that $y < x$;

(v) an *upper bound of a set* $Y \subseteq X$ in $\mathbb{X}$ when $y \leqslant x$ for every $y \in Y$;

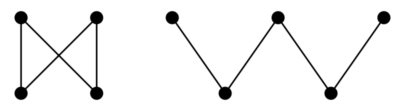
(vi) a *lower bound of a set* $Y \subseteq X$ in $\mathbb{X}$ when $x \leqslant y$ for every $y \in Y$.

When $\mathbb{X}$ has both a greatest and least element, it is said to be *bounded*.

Every poset has at most one greatest (resp. least) element. To prove this, let $x$ and $y$ be greatest elements of a poset $\mathbb{X}$. Since $x$ is a greatest element of $\mathbb{X}$, we have $y \leqslant x$. Similarly, the assumption that $y$ is a greatest element yields $x \leqslant y$. By antisymmetry we conclude that $x = y$. The case of least elements is analogous. Accordingly, when they exist, we will denote the greatest and least elements of a poset by 1 and 0.

While a greatest element is always maximal (resp. a least element is always minimal), the converse need not hold in general. For instance, every element of the poset $\langle X; \mathrm{id}_X \rangle$ is both maximal and minimal, but none of them is a greatest or least element of $\mathbb{X}$, provided that $X$ has at least two elements.

In general, maximal and minimal elements need not exist. A counterexample is the poset $\langle \mathbb{Z}; \leqslant \rangle$, which lacks both maximal and minimal element. Furthermore, the existence of maximal (resp. minimal) elements does not imply the existence of a greatest (resp. least) element. For instance, the following poset has five maximal and four minimal elements, but lacks both a greatest and a least element.



As we mentioned, infinite posets, such as $\langle \mathbb{Z}; \leqslant \rangle$, may lack maximal and minimal elements. However, this cannot happen in the finite case.

**Proposition 1.9.** *Every nonempty finite poset has both a maximal and a minimal element.*

*Proof.* Suppose, with a view to contradiction, that a finite nonempty poset $\mathbb{X}$ lacks either maximal or minimal elements. We deal with the case when $\mathbb{X}$ has no maximal elements, as the proof of the other case is symmetric. As $\mathbb{X}$ is nonempty, it has an element $x_1 \in X$. Since $x_1$ is not maximal, there exists some $x_2 \in X$ such that $x_1 < x_2$. Similarly, since $x_2$ is not maximal, there exists some $x_3 \in X$ such that $x_2 < x_3$. Iterating this construction, we produce an infinite sequence

$$x_1 < x_2 < x_3 < \cdots < x_n < \cdots$$

of distinct elements of $\mathbb{X}$, contradicting the assumption that $\mathbb{X}$ is finite. $\boxtimes$

Our proofs will often rely on the existence of a maximal element of some poset. Since a poset may lack maximal elements, the following principle plays a fundamental role in such proofs. Notably, it is equivalent (over Zermelo–Fraenkel set theory) to the Axiom of Choice.

**Zorn's Lemma.** *Let $\mathbb{X}$ be a poset. If every chain in a poset $\mathbb{X}$ has an upper bound in $\mathbb{X}$, then $\mathbb{X}$ has a maximal element.*

An elementary observation is that if $\mathbb{X} = \langle X; \leqslant \rangle$ is a poset, then so is $\mathbb{X}^\partial := \langle X; \geqslant \rangle$, where $x \geqslant y$ iff $y \leqslant x$. The poset $\mathbb{X}^\partial$ is called the *(order) dual* of $\mathbb{X}$.

Similarly, each statement $\Phi$ about posets has an order dual statement $\Phi^\partial$ obtained by replacing every occurrence of $\leqslant$ (resp. $\geqslant$) in $\Phi$ by $\geqslant$ (resp. $\leqslant$). Then $\Phi^\partial$ holds in a poset $\mathbb{X}$ iff $\Phi$ holds in $\mathbb{X}^\partial$. This definition could be made more precise by given a formal account of what counts as a statement about posets. However, doing so would take us to far afield from the subject at hand, therefore we shall content ourselves with relying on the reader's informal understanding of what counts as a statement about posets. The purpose of introducing $\Phi^\partial$ is that it allows us to formulate the following extremely useful principle, which will often allow us to cut our work down to half.

**Duality Principle.** *If a statement $\Phi$ is true in all posets, then $\Phi^\partial$ is also true in all posets.*

*Proof.* If $\Phi$ holds in each poset $\mathbb{X}$, then in particular it holds in each poset of the form $\mathbb{X}^\partial$. But $\Phi$ holds in $\mathbb{X}^\partial$ if and only if $\Phi^\partial$ holds in $\mathbb{X}$, so $\Phi^\partial$ indeed holds in every poset. ⊠

For example, applying the Duality Principle to Zorn's Lemma proves that if every chain in a poset $\mathbb{X}$ has a lower bound in $\mathbb{X}$, then $\mathbb{X}$ has a minimal element.

## 1.2 Lattices as partially ordered sets

Posets in which certain optimal upper and lower bounds exist will be called *lattices*. The next definition explains what "optimal" means in this context.

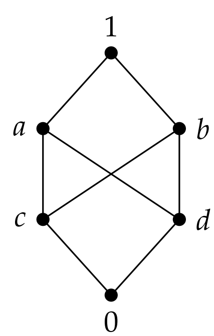
**Definition 1.10.** Let $\mathbb{X}$ be a poset and $Y \subseteq X$. An element $x \in X$ is said to be

(i) a *meet of $Y$ in $\mathbb{X}$* when $x$ is the greatest lower bound of $Y$ in $\mathbb{X}$;

(ii) a *join of $Y$ in $\mathbb{X}$* when $x$ is the least upper bound of $Y$ in $\mathbb{X}$.

When the poset $\mathbb{X}$ is clear from the context, we simply say that $x$ is a meet or a join of $Y$. The terms *infimum* and *supremum* are sometimes used as synonyms for *meet* and *join*.

Not every subset of a poset has a meet or a join. For instance, in the poset of natural numbers $\langle \mathbb{N}; \leqslant \rangle$, the join of $\mathbb{N}$ does not exist, because $\mathbb{N}$ has no upper bound in $\langle \mathbb{N}; \leqslant \rangle$. However, even if a set has an upper bound, the least such may not exist. For instance, while every subset of the poset $\mathbb{X}$ depicted below has an upper bound (namely, the greatest element 1), the least upper bound of $Y := \{c, d\}$ in $\mathbb{X}$ does not exist, because the set $\{1, a, b\}$ of upper bounds of $Y$ lacks a least element.




While meets and joins need not exist in general, they are necessarily unique whenever they do exist.

**Proposition 1.11.** *Let $\mathbb{X}$ be a poset and $Y \subseteq X$. When the meet (resp. join) of $Y$ exists, it is unique.*

*Proof.* Suppose that $x, y \in X$ are both meets of $Y$. As $x$ is a meet of $Y$, it is a lower bound of $Y$. Moreover, being a meet of $Y$, the element $y$ is the greatest lower bound of $Y$. As a consequence, $x \leqslant y$. A similar argument shows that $y \leqslant x$. As the relation $\leqslant$ is antisymmetric, we conclude that $x = y$. The case of joins is handled similarly. ⊠

Accordingly, given a poset $\mathbb{X}$ and a set $Y \subseteq X$,

(i) if the meet of $Y$ in $\mathbb{X}$ exists, we denote it by $\bigwedge Y$;

(ii) if the join of $Y$ in $\mathbb{X}$ exists, we denote it by $\bigvee Y$.

When it is convenient to stress that these meets and joins are considered in the poset $\mathbb{X}$, we will add the appropriate superscript and write

$$\bigwedge^{\mathbb{X}} Y \text{ and } \bigvee^{\mathbb{X}} Y.$$

When $Y = \{x, y\}$, we will write $x \wedge y$ and $x \vee y$ instead of $\bigwedge Y$ and $\bigvee Y$. Lastly, given a family $\{y_i : i \in I\}$ of elements of $X$, we will use $\bigwedge_{i \in I} y_i$ and $\bigvee i \in I y_i$ as a shorthand for, respectively, $\bigwedge\{y_i : i \in I\}$ and $\bigvee\{y_i : i \in I\}$.

It will often be convenient to paraphrase the definition of meets and joins in a poset $\mathbb{X}$ as follows. The meet of $Y \subseteq X$, when it exists, is the unique element $\bigwedge Y \in X$ such that

$$x \leqslant \bigwedge Y \iff x \leqslant y \text{ for each } y \in Y.$$

Similarly, the join of $Y \subseteq X$, if it exists, is the unique element $\bigvee Y \in X$ such that

$$\bigvee Y \leqslant x \iff y \leqslant x \text{ for each } y \in Y.$$

**Definition 1.12.** A nonempty poset $\mathbb{X}$ is said to be

(i) a *meet semilattice* when the meet of $\{x, y\}$ exists for every pair of elements $x, y \in X$;

(ii) a *join semilattice* when the join of $\{x, y\}$ exists for every pair of elements $x, y \in X$;

(iii) a *lattice* when it is both a meet semilattice and a join semilattice.

Notice that if the meet (resp. join) of a set $Y \subseteq X$ exists in a poset $\mathbb{X}$ and equals some $y \in X$, then the join (resp. meet) of the set $Y$ exists in the order dual poset $\mathbb{X}^\partial$ and also equals $y$. In particular, a poset $\mathbb{X}$ is a meet (resp. join) semilattice iff its order dual $\mathbb{X}^\partial$ is a join (resp. meet) semilattice. As a consequence, we obtain the following variant of the Duality Principle for lattices, which is a special case of the Duality Principle for posets.

**Duality Principle.** *If a statement $\Phi$ is true in all lattices, then the statement obtained by replacing $\bigwedge$ with $\bigvee$, $\bigvee$ with $\bigwedge$, $\leqslant$ with $\geqslant$, and $\geqslant$ with $\leqslant$ in $\Phi$ is also true in all lattices.*

In a lattice $\mathbb{X}$, the meet and the join of every finite nonempty set $\{x_1, \dots, x_n\} \subseteq X$ exist and coincide with

$$x_1 \wedge (x_2 \wedge \dots (x_{n-1} \wedge x_n) \dots) \text{ and } x_1 \vee (x_2 \vee \dots (x_{n-1} \vee x_n) \dots).$$

On the other hand, meet and joins of arbitrary subsets of $X$ need not exists in $\mathbb{X}$. This makes the following concept interesting:

**Definition 1.13.** A lattice $\mathbb{X}$ is said to be *complete* when $\bigwedge Y$ and $\bigvee Y$ exist for every $Y \subseteq X$.

Every complete lattice $\mathbb{X}$ is bounded, because $\bigvee X$ and $\bigwedge X$ are, respectively, the greatest and the least element of $\mathbb{X}$.

**Example 1.14** (Upsets & downsets)**.** Given a poset $\mathbb{X}$, the pairs $\langle \mathsf{Up}(\mathbb{X}); \subseteq \rangle$ and $\langle \mathsf{Down}(\mathbb{X}); \subseteq \rangle$ are complete lattices in which meets and joins are intersections and unions. To prove that this holds for $\langle \mathsf{Up}(\mathbb{X}); \subseteq \rangle$, it suffices to show that

$$\bigwedge^{\mathsf{Up}(\mathbb{X})} U = \bigcap_{i \in I} U_i \quad \text{and} \quad \bigvee_{i \in I}^{\mathsf{Up}(\mathbb{X})} U = \bigcup_{i \in I} U_i$$

for every family $U = \{U_i : i \in I\} \subseteq \mathsf{Up}(\mathbb{X})$. As an example, we detail the proof of the first equality. We need to show that $\bigcap_{i \in I} U_i$ is the greatest lower bound of $U$ in $\langle \mathsf{Up}(\mathbb{X}); \subseteq \rangle$. Observe first that $\bigcap_{i \in I} U_i$ is still an upset of $\mathbb{X}$. Then notice that $\bigcap_{i \in I} U_i \subseteq U_j$ for all $j \in I$, so $\bigcap_{i \in I} U_i$ is a lower bound of $U$. To prove that it is the greatest one, consider a lower bound $V$ of $U$. Then $V \subseteq U_i$ for all $i \in I$. Consequently, $V \subseteq \bigcap_{i \in I} U_i$ as desired. The proof that $\langle \mathsf{Down}(\mathbb{X}); \subseteq \rangle$ is also a complete lattice is analogous.

Now, recall that, when $\mathbb{X} = \langle X; \mathrm{id}_X \rangle$, the lattices $\langle \mathsf{Up}(\mathbb{X}); \subseteq \rangle$ and $\langle \mathsf{Down}(\mathbb{X}); \subseteq \rangle$ coincide with the powerset lattice $\langle \mathcal{P}(X); \subseteq \rangle$. The above discussion implies that $\langle \mathcal{P}(X); \subseteq \rangle$ is also a complete lattice in which meets and joins are intesections and unions. $\boxtimes$

**Example 1.15** (Chains are lattices)**.** Every nonempty chain $\mathbb{X}$ is a lattice. To prove this, observe that every two elements $x$ and $y$ of $\mathbb{X}$ are comparable because $\mathbb{X}$ is a chain. Accordingly, we will prove that

$$\begin{aligned} x \wedge y &= \text{the least element among } x \text{ and } y; \\ x \vee y &= \text{the greatest element among } x \text{ and } y. \end{aligned}$$

To prove the first equality above, let $z$ be the least element among $x$ and $y$. Clearly, $z \leqslant x, y$, whence $z$ is a lower bound of $\{x, y\}$. Then let $u \in X$ be another lower bound of $\{x, y\}$. As $z \in \{x, y\}$, this implies $u \leqslant z$. Hence, we conclude that $z$ is the greatest lower bound of $\{x, y\}$, that is, $z = x \wedge y$. The case of joins is handled similarly. $\boxtimes$

**Example 1.16** (Chains need not be complete)**.** While every nonempty chain is a lattice, it need not be a complete lattice. One reason is that every complete lattice is bounded, but a chain may lack a greatest or a least element, as in the case of $\langle \mathbb{N}; \leqslant \rangle$ and $\langle \mathbb{Z}; \leqslant \rangle$. In a chain without a greatest element, some subsets do not have a least upper bound simply because they have no upper bound at all. However, it may also happen that a subset which has an upper bound fails to have a least upper bound.

For instance, consider the chain of rational numbers $\langle \mathbb{Q}; \leqslant \rangle$ with the standard order and recall that $\mathbb{Q}$ is *dense* in $\mathbb{R}$, in the sense that for every $x, y \in \mathbb{R}$ with $x < y$ there exists $z \in \mathbb{Q}$ such that $x < z < y$. Then take

$$X := \{x \in \mathbb{Q} : x^2 < 2\} = \{x \in \mathbb{Q} : -\sqrt{2} < x < +\sqrt{2}\}.$$

We claim that the join $\bigvee X$ does not exist in $\langle \mathbb{Q}; \leqslant \rangle$. Indeed, suppose for the sake of contradiction that this join is some $y \in \mathbb{Q}$. It cannot be that $\sqrt{2} < y$: in that case the density of $\mathbb{Q}$ in $\mathbb{R}$ guarantees the existence of some $z \in \mathbb{Q}$ with $\sqrt{2} < z < y$, but then $z$ is also an upper bound of $X$, so $y$ is not the least upper bound of $X$. It also cannot be that $y < \sqrt{2}$: in that case the density of $\mathbb{Q}$ in $\mathbb{R}$ guarantees the existence of some positive $z \in \mathbb{Q}$ with $y < z < \sqrt{2}$, but then $z \in X$, so $y$ is an an upper bound of $X$. We have therefore reached a contradiction, since for each rational $y$ either $\sqrt{2} < y$ or $y < \sqrt{2}$. A similar reasoning shows that the meet $\bigwedge X$ also does not exist in $\langle \mathbb{Q}; \leqslant \rangle$. ⊠

The task of proving that a given poset is a complete lattice is simplified by following observation:

**Proposition 1.17.** *Let $\mathbb{X}$ be a poset. If $\bigwedge Y$ (resp. $\bigvee Y$) exists for all $Y \subseteq X$, then $\mathbb{X}$ is a complete lattice.*

*Proof.* Suppose that $\bigwedge Y$ exists for all $Y \subseteq X$. It suffices to show that the join of every subset $Y \subseteq X$ exists as well. To this end, consider $Y \subseteq X$ and let $U(Y)$ be the set of upper bounds of $Y$ in $\mathbb{X}$. By assumption, the element $x := \bigwedge U(Y)$ exists in $\mathbb{X}$. We will prove that $x$ is the join of $Y$ in $\mathbb{X}$. First, the definition of $x$ guarantees that $x$ is a lower bound of $U(Y)$. Therefore, it only remains to prove that $x \in U(Y)$. Accordingly, consider an element $y \in Y$. Clearly, $y$ is a lower bound of $U(Y)$. Together with the assumption that $x$ is the greatest lower bound of $U(Y)$, this implies $y \leqslant x$. Since $y \in U(Y)$ and $x \leqslant y$, we obtain $x \in U(Y)$. Hence, we conclude that $x = \bigvee Y$ and, therefore, that $\mathbb{X}$ is a complete lattice. By the Duality Principle, the variant of the statement involving joins holds too. ⊠

In order to apply the above test to a poset $\mathbb{X}$, it is sometimes useful to distinguish the case where $Y = \emptyset$ from the one in which $Y$ is nonempty. Because of this, the following description of the meet and the join of the empty set is of special interest.

**Proposition 1.18.** *The following conditions hold for a poset $\mathbb{X}$:*

(i) *$\mathbb{X}$ has a greatest element $1$ if and only if $\bigwedge \emptyset$ exists in $\mathbb{X}$. In this case, $\bigwedge \emptyset = 1$;*

(ii) *$\mathbb{X}$ has a least element $0$ if and only if $\bigvee \emptyset$ exists in $\mathbb{X}$. In this case, $\bigvee \emptyset = 0$.*

*Proof.* We detail only the proof of condition (i), as condition (ii) will follow from the Duality Principle. First, suppose that $\bigwedge \emptyset$ exists in $\mathbb{X}$. Since every element of $\mathbb{X}$ is vacuously a lower bound of $\emptyset$ in $\mathbb{X}$ and $\bigwedge \emptyset$ is the greatest such lower bound, we obtain that $x \leqslant \bigwedge \emptyset$ for all $x \in X$. Hence, $\bigwedge \emptyset$ is the greatest element of $\mathbb{X}$. Conversely, suppose that $\mathbb{X}$ has a greatest element $1$. Then $1$ is vacuously a lower bound of $\emptyset$ in $\mathbb{X}$. Furthermore, being the greatest element of $\mathbb{X}$, it is obviously the the greatest such lower bound. Thus, $1 = \bigwedge \emptyset$ as desired. ⊠

**Corollary 1.19.** *A poset $\mathbb{X}$ is a complete lattice iff it has a greatest (resp. least) element and the meet (resp. join) of every nonempty $Y \subseteq X$ exists in $\mathbb{X}$.*

*Proof.* Immediate from Propositions 1.17 and 1.18. ⊠

Corollary 1.19 is instrumental in proving that concrete posets are complete lattices, as we proceed to illustrate.

**Example 1.20** (Divisibility lattice)**.** We will prove that the divisibility lattice $\langle \mathbb{N}; \mid \rangle$ is complete. Since $\langle \mathbb{N}; \mid \rangle$ has a greatest element, namely 0, it suffices to show that every nonempty $X \subseteq \mathbb{N}$ has a meet in $\langle \mathbb{N}; \mid \rangle$. To this end, observe that there is a greatest common divisor $n$ of the elements of $X$: if $X = \{0\}$, then $n = 0$, otherwise $n$ is the (finite) product of all powers of primes $p^k$ such that $p^k$ divides all elements of $X$ but $p^{k+1}$ does not divide some element of $X$. The definition of the order of $\langle \mathbb{N}; \mid \rangle$ guarantees that $n$ is the greatest lower bound (that is, the meet) of $X$. ⊠

Another family of complete lattices arises in relation to the following concept:

**Definition 1.21.** A *topology* on a set $X$ is a family $\tau \subseteq \mathcal{P}(X)$ such that

(i) $\emptyset, X \in \tau$;

(ii) If $Y, Z \in \tau$, then $Y \cap Z \in \tau$;

(iii) If $\{Y_i : i \in I\} \subseteq \tau$, then $\bigcup_{i \in I} Y_i \in \tau$.

In this case, the pair $\langle X; \tau \rangle$ is said to be a *topological space*. Furthermore, the elements of $\tau$ are called *open* and their complements relative to $X$ *closed*. It follows that the collection of closed sets of a topological space $\langle X; \tau \rangle$ contains $\emptyset$ and $X$ and is closed under binary unions and arbitrary intersections.

**Example 1.22** (Order topology)**.** Consider a chain $\mathbb{X}$. By analogy with the open intervals of the real line, we may define the *open intervals* of $\mathbb{X}$ to be sets of the following three kinds:

$$\begin{aligned} (x, y) &:= \{z \in X : x < z < y\}, \\ (-\infty, x) &:= \{z \in X : z < x\}, \\ (x, +\infty) &:= \{z \in X : x < z\}. \end{aligned}$$

The collection of arbitrary unions of opens intervals of these three forms a topology called the *order topology* on $\mathbb{X}$. ⊠

The order topology on the chain $\langle \mathbb{R}; \leqslant \rangle$ of the real numbers is often called the *standard topology* on $\mathbb{R}$. Every open set of the standard topology of $\mathbb{R}$ is in fact a union of sets of the form $(x, y)$, since subsets of $\mathbb{R}$ of the form $(z, +\infty)$ and $(-\infty, z)$ can be obtained as unions of subsets of $\mathbb{R}$ of the form $(x, y)$ and so can the whole set $\mathbb{R}$.

**Example 1.23** (Lattices of closed sets)**.** Let $\langle X; \tau \rangle$ be a topological space and $\mathsf{Cl}(X; \tau)$ the family of its closed sets. We will prove that the pair $\langle \mathsf{Cl}(X; \tau); \subseteq \rangle$ is a complete lattice. In view of Proposition 1.17, it suffices to show that arbitrary meets exist in $\langle \mathsf{Cl}(X; \tau); \subseteq \rangle$. But this follows immediately from the fact that the collection $\mathsf{Cl}(X; \tau)$ is closed under arbitrary intersections. Consequently, $\langle \mathsf{Cl}(X; \tau); \subseteq \rangle$ is a complete lattice in which arbitrary meets coincide with intersections. However, arbitrary joins in $\langle \mathsf{Cl}(X; \tau); \subseteq \rangle$ need not coincide with unions. ⊠

As we mentioned, lattices may fail to be complete, easy counterexamples being number systems. However, this cannot happen in the finite case.

**Proposition 1.24.** *Every finite lattice is complete.*

*Proof.* In view of Proposition 1.17, it suffices to show that $\bigwedge Y$ exists in $\mathbb{X}$ for all $Y \subseteq X$. There are two cases: either $Y = \emptyset$ or $X$ is nonempty. Suppose first that $Y = \emptyset$. Then take an enumeration $X = \{x_1, \ldots, x_n\}$ and observe that

$$1 \coloneqq x_1 \vee \cdots \vee x_n$$

is the greatest element of $\mathbb{X}$. By condition (i) of Proposition 1.18, we conclude that $\bigwedge Y$ exists and coincides with 1. Then we consider the case where $Y$ is nonempty and take an enumeration $Y = \{y_1, \ldots, y_m\}$. Clearly, $y_1 \wedge \cdots \wedge y_m$ is the meet of $Y$ in $\mathbb{X}$. ⊠

## 1.3 Lattices as algebraic structures

Lattices and semilattices can also be viewed as algebraic structures, as we proceed to explain. To this end, we recall some fundamentals of general algebraic systems.

**Definition 1.25.**

(i) An *operation* of arity $n$ on a set $A$ is a function $f\colon A^n \to A$.

(ii) A *type* is a map $\rho\colon \mathcal{F} \to \mathbb{N}$, where $\mathcal{F}$ is a set of function symbols. In this case, $\rho(f)$ is said to be the *arity* of the function symbol $f$, for every $f \in \mathcal{F}$. Function symbols of arity zero are called *constants*.

(iii) An *algebra* of type $\rho$ is a pair $\boldsymbol{A} = \langle A; F \rangle$ where $A$ is a nonempty set and $F = \{f^{\boldsymbol{A}} : f \in \mathcal{F}\}$ is a set of operations on $A$ whose arity is determined by $\rho$, in the sense that each $f^{\boldsymbol{A}}$ has arity $\rho(f)$. The set $A$ is called the *universe* of $\boldsymbol{A}$.

When $\mathcal{F} = \{f_1, \ldots, f_n\}$, we shall write $\langle A; f_1^{\boldsymbol{A}}, \ldots, f_n^{\boldsymbol{A}} \rangle$ instead of $\langle A; F \rangle$. In this case, we often drop the superscripts, and write simply $\langle A; f_1, \ldots, f_n \rangle$.

**Example 1.26** (Groups)**.** Classical examples of algebras are groups and rings. For instance, the type of groups $\rho_G$ consists of a binary symbol $+$, a unary symbol $-$, and a constant symbol 0. A group is then an algebra $\langle A; +, -, 0 \rangle$ of type $\rho_G$ in which $+$ is associative, 0 is a neutral element for $+$, and $-$ produces additive inverses. More explicitly, a group is an algebra $\boldsymbol{A}$ of type $\rho_G$ such that for every $a, b, c \in A$,

$$a + (b + c) = (a + b) + c \qquad a + 0 = a \qquad 0 + a = a \qquad a + -a = 0 \qquad -a + a = 0.$$ ⊠

Many other important classes of algebras are also defined by some system of equations. It will therefore be convenient to make the notion of an equation precise.

**Definition 1.27.** Given a type $\rho\colon \mathcal{F} \to \mathbb{N}$ and a set of variables $X$ disjoint from $\mathcal{F}$, the set of *terms of type $\rho$ over $X$* is the least set $T_\rho(X)$ such that

(i) $X \subseteq T_\rho(X)$;

(ii) if $c \in \mathcal{F}$ is a constant, then $c \in T_\rho(X)$;

(iii) if $\varphi_1, \ldots, \varphi_n \in T_\rho(X)$ and $f \in \mathcal{F}$ is $n$-ary, then $f(\varphi_1 \ldots \varphi_n) \in T_\rho(X)$.

When $f$ is a binary operation, such as $+$, we often write $\varphi_1 + \varphi_2$ instead of $f(\varphi_1, \varphi_2)$.

**Definition 1.28.** Given a term $\varphi \in T_\rho(X)$, we write $\varphi(x_1, \ldots, x_n)$ to indicate that the variables occurring in $\varphi$ are among $x_1, \ldots, x_n$. Furthermore, given an algebra $\boldsymbol{A}$ of type $\rho$ and elements $a_1, \ldots, a_n \in A$, we define an element

$$\varphi^{\boldsymbol{A}}(a_1, \ldots, a_n)$$

of $A$ by recursion on the construction of $\varphi$ as follows:

(i) if $\varphi$ is a variable $x_i$, then $\varphi^{\boldsymbol{A}}(a_1, \ldots, a_n) := a_i$;

(ii) if $\varphi$ is a constant $c$, then $c^{\boldsymbol{A}}$ is the interpretation of $c$ in $\boldsymbol{A}$;

(iii) if $\varphi = f(\psi_1, \ldots, \psi_m)$, then

$$\varphi^{\boldsymbol{A}}(a_1, \ldots, a_n) := f^{\boldsymbol{A}}(\psi_1^{\boldsymbol{A}}(a_1, \ldots, a_n), \ldots, \psi_m^{\boldsymbol{A}}(a_1, \ldots, a_n)).$$

**Definition 1.29.** An *equation of type $\rho$ over $X$* is an expression of the form $\varphi \approx \psi$, where $\varphi, \psi \in T_\rho(X)$. An equation $\varphi \approx \psi$ is *valid* in an algebra $\boldsymbol{A}$ of type $\rho$ when

$$\varphi^{\boldsymbol{A}}(a_1, \ldots, a_n) = \psi^{\boldsymbol{A}}(a_1, \ldots, a_n) \text{ for every } a_1, \ldots, a_n \in A,$$

in which case we say that $\boldsymbol{A}$ *validates* $\varphi \approx \psi$. Alternatively, we say that $\boldsymbol{A}$ *satisfies* $\varphi \approx \psi$, or that the equation $\varphi \approx \psi$ *holds* in $\boldsymbol{A}$.

We can now rephrase the definition of groups as follows: groups are precisely the algebras of type $\rho_G$ that validate the equations

$$x + (y + z) \approx (x + y) + z \qquad x + 0 \approx x \qquad 0 + x \approx x \qquad x + -x \approx 0 \qquad -x + x \approx 0.$$

Lattices and semilattices admit a similar definition as algebras which satisfy certain equations. The type of meet semilattices $\rho_M$ consists of a single binary symbol $\wedge$, while the type $\rho_J$ of join semilattices consists of a single binary symbol $\vee$. The type of lattices $\rho_L$ then consists of two binary symbols $\wedge$ and $\vee$.

**Definition 1.30.**

(i) A *meet semilattice* is an algebra $\boldsymbol{A} = \langle A; \wedge \rangle$ of type $\rho_M$ satisfying the equations:

$$x \wedge x \approx x \qquad \textit{(idempotent law)}$$
$$x \wedge y \approx y \wedge x \qquad \textit{(commutative law)}$$
$$x \wedge (y \wedge z) \approx (x \wedge y) \wedge z. \qquad \textit{(associative law)}$$

(ii) A *join semilattice* is an algebra $\boldsymbol{A} = \langle A; \vee \rangle$ of type $\rho_J$ satisfying the equations:

$$x \vee x \approx x \qquad \textit{(idempotent law)}$$
$$x \vee y \approx y \vee x \qquad \textit{(commutative law)}$$
$$x \vee (y \vee z) \approx (x \vee y) \vee z. \qquad \textit{(associative law)}$$

(iii) A *lattice* is an algebra $\boldsymbol{A} = \langle A; \vee, \wedge \rangle$ of type $\rho_L$ such that $\langle A; \wedge \rangle$ is a meet semilattice (that is, it satisfies the above equations for $\wedge$), $\langle A; \vee \rangle$ is a join semilattice (that is, it satisfies the above equations for $\vee$), and moreover $\boldsymbol{A}$ satisfies the equations:

$$x \wedge (y \vee x) \approx x \quad \text{and} \quad x \vee (y \wedge x) \approx x. \qquad \textit{(absorption laws)}$$

In order to prove that this *algebraic definition* of meet and join semilattices (Definition 1.30) is equivalent to their *order theoretic definition* (Definition 1.12), we now explain how to translate between the two notions. While from a purely formal perspective, meet and join semilattices in the sense of Definition 1.30 are essentially the same objects (differing only in the symbol which represents their unique binary operation), we shall give these two classes of algebras a different order theoretic interpretation.

We first describe the translation from the algebraic definition to the order theoretic definition. Consider a meet semilattice $\boldsymbol{A} = \langle A; \wedge \rangle$ in the sense of Definition 1.30. The operation $\wedge$ determines a binary relation $\leqslant_\wedge$ on $A$ as follows:

$$a \leqslant_\wedge b \iff a \wedge b = a.$$

We shall prove that $\boldsymbol{A}^p := \langle A; \leqslant_\wedge \rangle$ is a poset which is a meet semilattice in the sense of Definition 1.12. Similarly, consider a join semilattice $\boldsymbol{A} = \langle A; \vee \rangle$ in the sense of Definition 1.30. The operation $\vee$ determines a binary relation $\leqslant_\vee$ on $A$ as follows:

$$a \leqslant_\vee b \iff a \vee b = b.$$

We shall prove that $\boldsymbol{A}^p := \langle A; \leqslant_\vee \rangle$ is a poset which is a join semilattice in the sense of Definition 1.12.

In the converse direction, consider poset $\mathbb{X} = \langle X; \leqslant \rangle$ which is a meet semilattice in the sense of Definition 1.12. Taking $\wedge \colon X \times X \to X$ to be the binary operation such that $x \wedge y$ the the meet of the set $\{x, y\}$ in $\mathbb{X}$, we shall prove that the algebra $\mathbb{X}^a := \langle X; \wedge \rangle$ is a meet semilattice in the sense of Definition 1.30. Similarly, consider a poset $\mathbb{X} = \langle X; \leqslant \rangle$ which is a join semilattice in the sense of Definition 1.12. Taking $\vee \colon X \times X \to X$ to be the binary operation such that $x \vee y$ is the join of the set $\{x, y\}$ in $\mathbb{X}$, we shall prove that the algebra $\mathbb{X}^a := \langle X; \vee \rangle$ is a join semilattice in the sense of Definition 1.30.

**Proposition 1.31.** *Let $\mathbb{X}$ and $\boldsymbol{A}$ be meet (resp. join) semilattices in the sense of Definitions 1.12 and 1.30, respectively. The following conditions hold:*

(i) *$\mathbb{X}^a$ is a meet (resp. join) semilattice in the sense of Definition 1.30;*

(ii) *$\boldsymbol{A}^p$ is a meet (resp. join) semilattice in the sense of Definition 1.12;*

(iii) *$\mathbb{X} = \mathbb{X}^{ap}$ and $\boldsymbol{A} = \boldsymbol{A}^{pa}$.*

*Proof.* We only prove the claims for meet semilattices. The claims for join semilattices are proved by replacing $\wedge$ by $\vee$, $\leqslant$ by $\geqslant$, and $\leqslant_\wedge$ by $\geqslant_\vee$ throughout the proof. (Notice that one cannot simply apply the Duality Principle here, because this principle only applies to the relational definition of meet and join semilattices.)

(i): Consider $x, y, z \in X$. We begin by proving that $x = x \wedge x$. To this end, recall that $x \wedge x$ is, by definition, a lower bound of $\{x\}$. Consequently, $x \wedge x \leqslant x$. Since $\leqslant$ is reflexive, we have $x \leqslant x$, so $x$ is also a lower bound of $\{x\}$. As $x \wedge x$ is, by definition, the greatest lower bound of $\{x\}$, this yields $x \leqslant x \wedge x$. By applying the antisymmetry of $\leqslant$ to the inequalities $x \wedge x \leqslant x$ and $x \leqslant x \wedge x$, we conclude that $x = x \wedge x$, as desired. Hence, $\mathbb{X}^a$ validates the idempotent law.

The fact that $\mathbb{X}^a$ validates also the commutative law is an immediate consequence of the equality $\{x, y\} = \{y, x\}$.

To prove the associative law, observe that $x \wedge (y \wedge z) \leqslant x, y \wedge z$, as $x \wedge (y \wedge z)$ is a lower bound of $\{x, y \wedge z\}$. Moreover, $y \wedge z \leqslant y, z$, as $y \wedge z$ is a lower bound of $\{y, z\}$. By the transitivity of $\leqslant$, we obtain $x \wedge (y \wedge z) \leqslant x, y, z$. In particular, $x \wedge (y \wedge z)$ is a lower bound of $\{x, y\}$. Since $x \wedge y$ is the greatest such lower bound, this yields $x \wedge (y \wedge z) \leqslant x \wedge y$. Thus, we obtain $x \wedge (y \wedge z) \leqslant x \wedge y, z$, that is, $x \wedge (y \wedge z)$ is a lower bound of $\{x \wedge y, z\}$. Since $(x \wedge y) \wedge z$ is the greatest such lower bound, we conclude that

$$x \wedge (y \wedge z) \leqslant (x \wedge y) \wedge z.$$

Similarly, we obtain $(x \wedge y) \wedge z \leqslant x \wedge (y \wedge z)$ and, therefore, $x \wedge (y \wedge z) = (x \wedge y) \wedge z$, by the antisymmetry of $\leqslant$. Hence, $\mathbb{X}^a$ validates the associative law in $\mathbb{X}^a$.

(ii): We begin by proving that $\boldsymbol{A}^p = \langle A; \leqslant_\wedge \rangle$ is a poset. To this end, consider $a, b, c \in A$. By the idempotent law, $a \wedge a = a$. Consequently, the definition of $\leqslant_\wedge$ guarantees that $a \leqslant_\wedge a$ and, therefore, that $\leqslant_\wedge$ is reflexive. To prove that $\leqslant_\wedge$ is antisymmetric, suppose that $a \leqslant_\wedge b$ and $b \leqslant_\wedge a$. By the definition of $\leqslant_\wedge$, we have $a \wedge b = a$ and $b \wedge a = b$. Therefore, we can apply the commutative law, obtaining

$$a = a \wedge b = b \wedge a = b.$$

To prove that $\leqslant_\wedge$ is transitive, suppose that $a \leqslant_\wedge b$ and $b \leqslant_\wedge c$, that is,

$$a \wedge b = a \quad \text{and} \quad b \wedge c = b.$$

Together with the associative law, this yields

$$a \wedge c = (a \wedge b) \wedge c = a \wedge (b \wedge c) = a \wedge b = a.$$

By the definition of $\leqslant_\wedge$, this amounts to $a \leqslant_\wedge c$. Hence, we conclude that $\boldsymbol{A}^p$ is a poset.

To prove that $\boldsymbol{A}^p$ is a meet semilattice in the sense of Definition 1.12, it only remains to show that $a \wedge b$ is the meet of $\{a, b\}$ in $\boldsymbol{A}^p$, for all $a, b \in A$. To this end, observe that by applying in succession the commutative, associative, and idempotent law, we obtain

$$(a \wedge b) \wedge a = a \wedge (a \wedge b) = (a \wedge a) \wedge b = a \wedge b.$$

By the definition of $\leqslant_\wedge$, this amounts to $a \wedge b \leqslant_\wedge a$. A similar argument yields $a \wedge b \leqslant_\wedge b$. Thus, $a \wedge b$ is a lower bound of $\{a, b\}$ in $\boldsymbol{A}^p$. To prove that it is the greatest one, suppose that $c$ is also a lower bound of $\{a, b\}$ in $\boldsymbol{A}^p$, that is, $c \leqslant_\wedge a, b$. By definition of $\leqslant_\wedge$, this means that $c \wedge a = c$ and $c \wedge b = c$. Together with the associative law, this implies that

$$c \wedge (a \wedge b) = (c \wedge a) \wedge b = c \wedge b = c,$$

that is, $c \leqslant_\wedge a \wedge b$. Thus, we conclude that $a \wedge b$ is the meet of $\{a, b\}$ in $\boldsymbol{A}^p$.

(iii): To prove that $\mathbb{X} = \mathbb{X}^{ap}$, consider $x, y \in X$. We begin by showing that

$$x \leqslant^{\mathbb{X}} y \iff x \text{ is the meet of } \{x, y\} \text{ in } \mathbb{X}. \tag{1.1}$$

Suppose first that $x \leqslant^{\mathbb{X}} y$. As $\leqslant^{\mathbb{X}}$ is reflexive, this implies that $x$ is a lower bound of $\{x, y\}$ in $\mathbb{X}$. To prove that it is the greatest one, consider another such lower bound $z$. Then $z \leqslant x$, as desired. Thus, $x$ is the meet of $\{x, y\}$ in $\mathbb{X}$. Conversely, if $x$ is the meet of $\{x, y\}$ in $\mathbb{X}$, then clearly $x \leqslant^{\mathbb{X}} y$.

In view of condition (1.1), we obtain

$$\begin{aligned} x \leqslant^{\mathbb{X}} y &\iff x \text{ is the meet of } \{x, y\} \text{ in } \mathbb{X} \\ &\iff x = x \wedge^{\mathbb{X}^a} y \\ &\iff x \leqslant^{\mathbb{X}^{ap}} y, \end{aligned}$$

where the last two equivalences follow from the definitions of $\mathbb{X}^a$ and $\mathbb{X}^{ap}$, respectively. Hence, we conclude that $\mathbb{X} = \mathbb{X}^{ap}$, as desired.

To prove that $\boldsymbol{A} = \boldsymbol{A}^{pa}$, consider $a, b \in A$. Since $a \wedge^{\boldsymbol{A}^{pa}} b$ is the meet of $\{a, b\}$ in $\boldsymbol{A}^p$, we have $a \wedge^{\boldsymbol{A}^{pa}} b \leqslant_\wedge^{\boldsymbol{A}^p} a, b$. By the definition of the relation $\leqslant_\wedge^{\boldsymbol{A}^p}$, this amounts to

$$(a \wedge^{\boldsymbol{A}^{pa}} b) \wedge^{\boldsymbol{A}} a = a \wedge^{\boldsymbol{A}^{pa}} b \;\text{ and }\; (a \wedge^{\boldsymbol{A}^{pa}} b) \wedge^{\boldsymbol{A}} b = a \wedge^{\boldsymbol{A}^{pa}} b.$$

Together with the fact that the associative law are valid in $\boldsymbol{A}$, this yields

$$(a \wedge^{\boldsymbol{A}^{pa}} b) \wedge^{\boldsymbol{A}} (a \wedge^{\boldsymbol{A}} b) = ((a \wedge^{\boldsymbol{A}^{pa}} b) \wedge^{\boldsymbol{A}} b) \wedge^{\boldsymbol{A}} b = (a \wedge^{\boldsymbol{A}^{pa}} b) \wedge^{\boldsymbol{A}} b = a \wedge^{\boldsymbol{A}^{pa}} b,$$

which, by the definition of $\boldsymbol{A}^p$, amounts to $a \wedge^{\boldsymbol{A}^{pa}} b \leqslant_\wedge^{\boldsymbol{A}^p} a \wedge^{\boldsymbol{A}} b$.

Lastly, applying in succession the commutative, associative, and idempotent law in $\boldsymbol{A}$, we obtain

$$(a \wedge^{\boldsymbol{A}} b) \wedge^{\boldsymbol{A}} a = a \wedge^{\boldsymbol{A}} (a \wedge^{\boldsymbol{A}} b) = (a \wedge^{\boldsymbol{A}} a) \wedge^{\boldsymbol{A}} b = a \wedge^{\boldsymbol{A}} b.$$

By the definition of $\boldsymbol{A}^p$, this amounts to $a \wedge^{\boldsymbol{A}} b \leqslant_\wedge^{\boldsymbol{A}^p} a$. Similarly, we obtain $a \wedge^{\boldsymbol{A}} b \leqslant_\wedge^{\boldsymbol{A}^p} b$, so $a \wedge^{\boldsymbol{A}} b$ is a lower bound of $\{a, b\}$ in $\boldsymbol{A}^p$. Since $a \wedge^{\boldsymbol{A}^{pa}} b$ is the greatest such lower bound, we obtain that $a \wedge^{\boldsymbol{A}} b \leqslant_\wedge^{\boldsymbol{A}^p} a \wedge^{\boldsymbol{A}^{pa}} b$. As we already proved that $a \wedge^{\boldsymbol{A}^{pa}} b \leqslant_\wedge^{\boldsymbol{A}^p} a \wedge^{\boldsymbol{A}} b$, we can apply the antisymmetry of $\leqslant_\wedge^{\boldsymbol{A}^p}$, obtaining $a \wedge^{\boldsymbol{A}} b = a \wedge^{\boldsymbol{A}^{pa}} b$. Thus $\boldsymbol{A} = \boldsymbol{A}^{pa}$. ⊠

It remains to provide a translation between the two definitions of lattices. To this end, consider a lattice $\boldsymbol{A} = \langle A; \wedge, \vee \rangle$ in the sense of Definition 1.30. Then

$$a \leqslant_\wedge b \iff a \wedge b = a \iff a \vee b = b \iff a \leqslant_\vee b.$$

This is a consequence of absorption: if $a \wedge b = a$, then $a \vee b = (a \wedge b) \vee b = (b \wedge a) \vee b$, and conversely if $a \vee b = b$, then $a \wedge b = a \wedge (a \vee b) = a \wedge (b \vee a) = a$. Because the two partial orders $\leqslant_\wedge$ and $\leqslant_\vee$ agree, we can simply talk about the partial order $\leqslant$ and define the poset $\boldsymbol{A}^p$ as $\boldsymbol{A}^p \coloneqq \langle A; \leqslant \rangle$.

In the converse direction, consider a lattice $\mathbb{X} = \langle X; \leqslant \rangle$ which is a lattice in the sense of Definition 1.12. Taking again $\wedge$ and $\vee$ to be, respectively, the operations of taking the meet and the join of two elements in $\mathbb{X}$, we define $\mathbb{X}^a \coloneqq \langle X; \wedge, \vee \rangle$.

**Proposition 1.32.** *Let $\mathbb{X}$ and $\boldsymbol{A}$ be lattices in the sense of Definitions 1.12 and 1.30, respectively. The following conditions hold:*

(i) *$\mathbb{X}^a$ is a lattice in the sense of Definition 1.30;*

(ii) *$\boldsymbol{A}^p$ is a lattice in the sense of Definition 1.12;*

(iii) *$\mathbb{X} = \mathbb{X}^{ap}$ and $\boldsymbol{A} = \boldsymbol{A}^{pa}$.*

*Proof.* (i): Because $\mathbb{X}$ is both a meet and a join semilattice in the sense of Definition 1.12, by Proposition 1.31 the algebra $\mathbb{X}^a$ satisfies idempotence, commutativity, and associativity. It remains to prove that it satisfies absorption. To this end, observe that $x \leqslant x, y \vee x$, because $\leqslant$ is reflexive and $y \vee x$ an upper bound of $\{y, x\}$. Then $x$ is a lower bound for $\{x, y \vee x\}$. Since $x \wedge (y \vee x)$ is the greatest such lower bound, this implies $x \leqslant x \wedge (y \vee x)$. Furthermore, we have $x \wedge (y \vee x) \leqslant x$, because $x \wedge (y \vee x)$ is a lower bound of $\{x, y \vee x\}$. By the antisymmetry of $\leqslant$, we conclude that $x \wedge (y \vee x) = x$. A similar argument shows that $x \vee (y \wedge x) = x$, thus establishing the validity of the absorption laws in $\mathbb{X}^a$. Hence, we conclude that $\mathbb{X}^a$ is a lattice in the sense of Definition 1.30.

(ii): Because $\langle A; \wedge \rangle$ and $\langle A; \vee \rangle$ are, respectively, a meet and a join semilattice in the sense of Definition 1.30, by Proposition 1.31 the poset $\boldsymbol{A}^p$ is both a meet and a join semilattice in the sense of Definition 1.12. In other words, it is a lattice in the sense of that definition.

(iii): The equality $\mathbb{X} = \mathbb{X}^{ap}$ follows from the corresponding equality for either meet or join semilattices. This is because each lattice is in particular a meet and a join semilattice, and moreover if $\langle A; \vee, \wedge \rangle$ is a lattice, then the posets $\langle A; \wedge \rangle^p$ and $\langle A; \vee \rangle^p$ coincide. Likewise, the equality $\boldsymbol{A} = \boldsymbol{A}^{pa}$ follows from the corresponding equalities for meet semilattices and for join semilattices. ⊠

In view of Propositions 1.31 and 1.32, from now on we shall treat lattices and semilattices both as posets and algebras without further notice.

## 1.4 Closure operators

The following concept will play a fundamental role in this book.

**Definition 1.33.** A map $C \colon \mathcal{P}(X) \to \mathcal{P}(X)$ is said to be a *closure operator* on a set $X$ when it satisfies the following conditions for every $Y, Z \subseteq X$:

(i) *Extensivity*: $Y \subseteq C(Y)$;

(ii) *Idempotence*: $C(C(Y)) = C(Y)$;

(iii) *Monotonicity*: if $Y \subseteq Z$, then $C(Y) \subseteq C(Z)$.

We say that a set $Y \subseteq X$ is *closed* when $Y = C(Y)$.

Notice that the inclusion $C(Y) \subseteq C(C(Y))$ for all $Y \subseteq X$ follows from extensivity and monotonicity. To prove that a map $C\colon \mathcal{P}(X) \to \mathcal{P}(X)$ is a closure operator, it therefore suffices to prove extensivity, monotonicity, and the inclusion $C(C(Y)) \subseteq C(Y)$ for all $Y \subseteq X$. When no confusion shall occur, given a closure operator $C$ on $X$ and $x_1, \ldots, x_n \in X$, we shall write $C(x_1, \ldots, x_n)$ as a shorthand for $C(\{x_1, \ldots, x_n\})$.

**Example 1.34** (Topological closure)**.** Given a topological space $\langle X; \tau \rangle$, let

$$\overline{(-)}\colon \mathcal{P}(X) \to \mathcal{P}(X)$$

be the map defined for every $Y \subseteq X$ as

$$\overline{Y} := \bigcap \{Z \subseteq X : Z \in \mathsf{Cl}(X; \tau) \text{ and } Y \subseteq Z\}.$$

Notice that $\overline{Y}$ is closed, as arbitrary intersections of closed are closed. Therefore, $\overline{Y}$ is the least closed set extending $Y$. Because of this, $\overline{Y}$ is called the *topological closure* of $Y$. It is easy to see that $\overline{(-)}$ is a closure operator on $X$, whose closed sets are precisely the closed sets of the topological space $\langle X; \tau \rangle$. The closure operators of this form are called *topological*. ⊠

**Example 1.35** (Upward & downward closure)**.** Given a poset $\mathbb{X}$, let

$$\uparrow^{\mathbb{X}}\colon \mathcal{P}(X) \to \mathcal{P}(X) \ \text{ and } \ \downarrow^{\mathbb{X}}\colon \mathcal{P}(X) \to \mathcal{P}(X)$$

be the maps defined for every $Y \subseteq X$ as

$$\uparrow^{\mathbb{X}} Y := \{x \in X : \text{there exists } y \in Y \text{ such that } y \leqslant x\};$$
$$\downarrow^{\mathbb{X}} Y := \{x \in X : \text{there exists } y \in Y \text{ such that } x \leqslant y\}.$$

It is easy to see that $\uparrow^{\mathbb{X}}$ and $\downarrow^{\mathbb{X}}$ are closure operators on $X$. The closed sets of $\uparrow^{\mathbb{X}}$ and $\downarrow^{\mathbb{X}}$ consist, respectively, of the upsets of $\mathbb{X}$ and the downsets of $\mathbb{X}$. Furthermore, $\uparrow^{\mathbb{X}} Y$ and $\downarrow^{\mathbb{X}} Y$ are, respectively, the least upset and the least downset of $\mathbb{X}$ containing $Y$.

When the poset $\mathbb{X}$ is clear from the context, we will drop the superscripts in $\uparrow^{\mathbb{X}}$ and $\downarrow^{\mathbb{X}}$ and write simply $\uparrow$ and $\downarrow$. Similarly, given $x \in X$, we will write $\uparrow^{\mathbb{X}} x$ and $\downarrow^{\mathbb{X}} x$ as a shorthand for $\uparrow^{\mathbb{X}}\{x\}$ and $\downarrow^{\mathbb{X}}\{x\}$. Lastly, an upset (resp. a downset) of $\mathbb{X}$ will be called *principal* when it is of the form $\uparrow x$ (resp. $\downarrow x$) for some $x \in X$. ⊠

The upward and downward closure operators on a poset are in fact special cases of topological closure operators.

**Example 1.36** (Upset & downset topology)**.** The collection of all upsets of a poset $\mathbb{X}$ forms a topology called the *upset topology* on $\mathbb{X}$. The upset topology is sometimes also called the *Alexandroff topology*. The closed sets of the upset topology are the downsets of $\mathbb{X}$, since these are precisely the complements of the upsets of $\mathbb{X}$. The topological closure operator associated with the upset topology is therefore $\downarrow$. A analogous result holds if we replace the role of upsets and downsets in this example. ⊠

Closure operators admit the following alternative presentations:

**Definition 1.37.** A *closure system* on a set $X$ is a family $\mathcal{S} \subseteq \mathcal{P}(X)$ closed under arbitrary intersections, that is, such that for every $\{Y_i : i \in I\} \subseteq \mathcal{P}(X)$,

$$\text{if } Y_i \in \mathcal{S} \text{ for every } i \in I, \text{ then } \bigcap_{i \in I} Y_i \in \mathcal{S}.$$

The intersection of the empty family of subsets of $X$ is understood here as $\bigcap \emptyset := X$. Consequently, the set $X$ belongs to every closure system on $X$.

**Definition 1.38.** A *consequence relation* on a set $X$ is a relation $\vdash \subseteq \mathcal{P}(X) \times X$ such that

(i) *Reflexivity*: if $Y \subseteq X$ and $y \in Y$, then $Y \vdash y$;

(ii) *Transitivity*: for all $Y, Z \subseteq X$ and $x \in X$, if $Y \vdash z$ for all $z \in Z$ and $Z \vdash x$, then $Y \vdash x$.

In this case, for every $x \in X$ and $Y, Z \subseteq X$ such that $Y \subseteq Z$ the following holds:

$$\text{if } Y \vdash x, \text{ then } Z \vdash x. \qquad (\textit{monotonicity})$$

This is because from $Y \subseteq Z$ and reflexivity it follows $Z \vdash y$ for each $y \in Y$, so by transitivity $Y \vdash x$ implies $Z \vdash x$.

The next result explains how to translate between closure operators, closure systems, and consequence relations.

**Theorem 1.39.** *Given a set $X$, the following conditions hold:*

(i) *If $C$ is a closure operator on $X$, the family of its closed sets $\mathcal{S}_C$ is a closure system on $X$. Conversely, if $\mathcal{S}$ is a closure system on $X$, the map $C_{\mathcal{S}} \colon \mathcal{P}(X) \to \mathcal{P}(X)$ defined as*

$$C_{\mathcal{S}}(Y) := \bigcap \{Z \in \mathcal{S} : Y \subseteq Z\}$$

*is a closure operator on $X$. These transformations are inverse to each other;*

(ii) *If $C$ is a closure operator on $X$, the relation*

$$\vdash_C := \{\langle Y, x \rangle \in \mathcal{P}(X) \times X : x \in C(Y)\}$$

*is a consequence relation on $X$. Conversely, if $\vdash$ is a consequence relation on $X$, the map $C_{\vdash} \colon \mathcal{P}(X) \to \mathcal{P}(X)$ defined as*

$$C_{\vdash}(Y) := \{x \in X : Y \vdash x\}$$

*is a closure operator on $X$. These transformations are inverse to each other.*

*Proof.* (i): Let $\{Y_i : i \in I\}$ be a family of closed sets of the closure operator $C$. We will prove that $Y := \bigcap_{i \in I} Y_i$ is also closed. By monotonicity $C(Y) \subseteq C(Y_i)$ for each $i \in I$ and thus $C(Y) \subseteq \bigcap_{i \in I} C(Y_i) = \bigcap_{i \in I} Y_i$, where the last equality holds because each $Y_i$ is closed. Therefore, by the definition of $Y$ we obtain $C(Y) \subseteq Y$. Since $Y \subseteq C(Y)$ by extensivity, we conclude that $Y = C(Y)$. The closed sets of $C$ thus form a closure system. Conversely, if $\mathcal{S}$

is a closure system, then the map $C_{\mathcal{S}}$ associated with it is extensive and order preserving by definition. Moreover, if $Y \subseteq X$ and $Z \in \mathcal{S}$, then $Y \subseteq Z$ iff $C_{\mathcal{S}}(Y) \subseteq Z$. It follows that

$$C_{\mathcal{S}}(C_{\mathcal{S}}(Y)) = \bigcap\{Z \in \mathcal{S} : C_{\mathcal{S}}(Y) \subseteq Z\} = \bigcap\{Z \in \mathcal{S} : Y \subseteq Z\} = C_{\mathcal{S}}(Y).$$

To prove that these transformations are inverse to each other, let $C'$ be the closure operator obtained from a closure operator $C$ on $X$ by composing the two constructions. That is,

$$C'(Y) = \bigcap\{Z \subseteq X : Y \subseteq Z \text{ and } Z = C(Z)\}.$$

Then $C'(Y) \subseteq C(Y)$ because $Y \subseteq Z = C(Z)$ holds in particular for $Z := C(Y)$, and $C(Y) \subseteq C'(Y)$ because $Y \subseteq Z = C(Z)$ implies $C(Y) \subseteq C(Z) = Z$. Thus $C' = C$.

In the opposite direction, let $\mathcal{S}'$ be the closure system obtained from a closure system $\mathcal{S}$ on $X$ by composing the two constructions. That is,

$$Y \in \mathcal{S}' \iff Y = \bigcap\{Z \in \mathcal{S} : Y \subseteq Z\}.$$

Then $Y \in \mathcal{S}$ implies $Y \in \mathcal{S}'$, since $Y \subseteq \{Z \in \mathcal{S} : Y \subseteq Z\}$ holds for each $Y \subseteq X$ and $\{Z \in \mathcal{S} : Y \subseteq Z\} \subseteq Y$ holds for each $Y \in \mathcal{S}$ because we can take $Z := Y$. Conversely, if $Y \in \mathcal{S}'$, then $Y$ is the intersection of a family of sets in $\mathcal{S}$, since $Y = \bigcap\{Z \in \mathcal{S} : Y \subseteq Z\}$. But $\mathcal{S}$ is closed under arbitrary intersections, so $Y \in \mathcal{S}$. Thus $\mathcal{S}' = \mathcal{S}$.

(ii): If $C$ is a closure operator on $X$ and $y \in Y \subseteq X$, then $y \in C(Y)$ by extensivity, so $Y \vdash_C y$ holds in the associated consequence relation. Moreover, if $Y \vdash_C z$ for each $z \in Z \subseteq X$ and $Z \vdash_C x$, then $Z \subseteq C(Y)$ and $x \in C(Z)$, so $x \in C(Z) \subseteq C(C(Y)) = C(Y)$ by monotonicity and idempotence. Consequently, $Y \vdash_C x$. Conversely, if $\vdash$ is a consequence relation on $X$, then the asssociated map $C_\vdash$ is extensive, since by reflexivity $y \in Y \subseteq X$ implies $Y \vdash y$ and thus $y \in C_\vdash(Y)$. Moreover, $C_\vdash(Y) \vdash x$ implies $Y \vdash x$ for every $Y \subseteq X$ and $x \in X$, since by transitivity $Y \vdash z$ for each $z \in C_\vdash(Y)$ and $C_\vdash(Y) \vdash x$ imply that $Y \vdash x$. Consequently,

$$C_\vdash(C_\vdash(Y)) = \{x \in X : C_\vdash(Y) \vdash x\} \subseteq \{x \in X : Y \vdash x\} = C_\vdash(Y).$$

Finally, $Y \subseteq Z$ for $Y, Z \subseteq X$ implies $C_\vdash(Y) \subseteq C_\vdash(Z)$ by the monotonicity of $\vdash$.

One can immediately see from their definitions that the two transformations are inverse to each other. Let $C'$ be the closure operator obtained from a closure operator $C$ on $X$ by composing the two constructions. Then $C'(Y) = \{x \in X : Y \vdash_C x\} = \{x \in X : x \in C(Y)\} = C(Y)$ for every $Y \subseteq X$. In the opposite direction, let $\vdash'$ be the consequence relation obtained from the consequence relation $\vdash$ on $X$ by composing the two constructions. Then $Y \vdash' x$ iff $x \in C_\vdash(Y)$ iff $Y \vdash x$. ⊠

In view of the above result, given a closure operator $C$ on a set $X$, we refer to

$$\mathcal{S}_C := \{Y \subseteq X : Y = C(Y)\} \text{ and } \vdash_C := \{\langle Y, x\rangle \in \mathcal{P}(X) \times X : x \in C(Y)\}$$

as to the closure system and the consequence relation *associated with* $C$. We will drop the subscript $(-)_C$ whenever there is no danger of confusion.

For instance, the upsets and the downsets of a poset $\mathbb{X}$ are precisely the closed sets of the closure operators $\uparrow$ and $\downarrow$. It follows that $\mathsf{Up}(\mathbb{X})$ and $\mathsf{Down}(\mathbb{X})$ are the closure systems associated with $\uparrow$ and $\downarrow$. Similarly, the closed sets of a topological space $\langle X; \tau\rangle$ are precisely the closed sets of the closure operator $\overline{(-)}$. Consequently, the family $\mathsf{Cl}(X; \tau)$ of closed sets of $\langle X; \tau\rangle$ is the closure system associated with $\overline{(-)}$.

Every closure system can be viewed as a poset ordered under the inclusion relation. This poset turns out to be a complete lattice.

**Proposition 1.40.** *Let $\mathcal{S}$ be the closure system associated with a closure operator $C$ on a set $X$. Then $\langle \mathcal{S}; \subseteq\rangle$ is a complete lattice in which meets are intersections and joins are closures of unions. That is, for every $\{Y_i : i \in I\} \subseteq \mathcal{S}$*

$$\bigvee_{i\in I} Y_i = C(\bigcup_{i\in I} Y_i).$$

*Proof.* Consider a family $\{Y_i : i \in I\}$ of sets in $\mathcal{S}$. The intersection $Y := \bigcap_{i\in I} Y_i$ is the greatest lower bound of the family $\{Y_i : i \in I\}$ in $\langle \mathcal{P}(X); \subseteq\rangle$. Because $Y \in \mathcal{S}$, it is also the greatest lower bound of this family in $\langle \mathcal{S}; \subseteq\rangle$. It follows that the meet of $Y$ in $\langle \mathcal{S}; \subseteq\rangle$ exists and coincides with the intersection $\bigcap Y$. To prove that the join $\bigvee_{i\in I} Y_i$ exists and equals $C(\bigcup_{i\in I} Y_i)$, it suffices to prove that each $Z \in \mathcal{S}$ satisfies the following equivalence:

$$C(\bigcup_{i\in I} Y_i) \subseteq Z \iff Y_i \subseteq Z \text{ for each } i \in I.$$

The left to right implication holds because $Y_j \subseteq \bigcup_{i\in I} Y_i \subseteq C(\bigcup_{i\in I} Y_i)$ for each $j \in I$. The right to left implication holds because if $Y_i \subseteq Z \in \mathcal{S}$ for each $i \in I$, then $\bigcup_{i\in I} Y_i \subseteq Z$, so $C(\bigcup_{i\in I} Y_i) \subseteq C(Z) = Z$. ⊠

In view of Proposition 1.40, we will often treat closure systems $\mathcal{S}$ as complete lattices and write $\mathcal{S}$ as a shorthand for $\langle \mathcal{S}; \subseteq\rangle$. Notably, not only is every closure system a complete lattice, but (up to isomorphism) every complete lattice arises in this way.

**Theorem 1.41** (Representation theorem)**.** *Every complete lattice is isomorphic to a closure system.*

*Proof.* Consider a complete lattice $\mathbb{X}$. We will show that

$$\mathcal{S} := \{{\downarrow}x : x \in X\}$$

is a closure system on $X$ such that $\mathbb{X} \cong \mathcal{S}$.

We begin by proving that $\mathcal{S}$ is a closure system on $X$. Notice that $\mathcal{S} \subseteq \mathcal{P}(X)$. Now consider a family $\{{\downarrow}x_i : i \in I\} \subseteq \mathcal{S}$. We need to prove that its intersection belongs to $\mathcal{S}$. To this end, observe that for every $y \in X$,

$$y \in \bigcap_{i\in I} {\downarrow}x_i \iff y \leqslant x_i \text{ for every } i \in I \iff y \leqslant \bigwedge_{i\in I}^{\mathbb{X}} x_i.$$

These equivalences hold even if $I$ is empty. As a consequence,

$$\bigcap_{i\in I} {\downarrow}x_i = {\downarrow}\Big(\bigwedge_{i\in I}^{\mathbb{X}} x_i\Big) \in \mathcal{S}.$$

Hence, $\mathcal{S}$ is a closure system on $X$ as desired.

It only remains to show that $\mathbb{X}$ is isomorphic to $\mathcal{S}$. To this end, consider the map $f\colon X \to \mathcal{S}$ defined by the rule

$$f(x) := {\downarrow}x \text{ for every } x \in X.$$

Clearly, $f$ is a well-defined surjection. Moreover, for every $x, y \in X$,

$$x \leqslant y \iff {\downarrow}x \subseteq {\downarrow}y \iff f(x) \subseteq f(y).$$

Hence, $f$ is an order embedding from $\mathbb{X}$ into $\mathcal{S}$. Since $f$ is surjective, we conclude that it is an order isomorphism. ⊠

## 1.5 Finitarity and compactness

Among all closure operators, the following are of special interest:

**Definition 1.42.** Given a closure operator $C$ on a set $X$,

(i) a closed set $Y$ of $C$ is said to be *finitely generated* when there exists a finite $Z \subseteq X$ such that $Y = C(Z)$;

(ii) $C$ is said to be *finitary* when for every $Y \cup \{x\} \subseteq X$ such that $x \in C(Y)$ there exists a finite $Z \subseteq Y$ such that $x \in C(Z)$.

Consequently, a closure operator is finitary precisely when each of its closed sets is the union of all the finitely generated closed sets contained into it.

**Example 1.43** (Upward & downward closure)**.** Given a poset $\mathbb{X}$, the closure operators

$${\uparrow}\colon \mathcal{P}(X) \to \mathcal{P}(X) \quad\text{and}\quad {\downarrow}\colon \mathcal{P}(X) \to \mathcal{P}(X)$$

are finitary. This is because for every $Y \cup \{x\} \subseteq X$,

$$\begin{aligned} x \in {\uparrow}Y &\iff \text{there exists } y \in Y \text{ such that } x \in {\uparrow}y;\\ x \in {\downarrow}Y &\iff \text{there exists } y \in Y \text{ such that } x \in {\downarrow}y. \end{aligned}$$ ⊠

**Example 1.44** (Topological closure)**.** On the other hand, the closure operator of topological closure often fails to be finitary. For instance, this is the case for the standard topology on $\mathbb{R}$. To prove this, recall that the open sets of this topology are unions of sets of the form $(y, z)$ for $y, z \in \mathbb{R}$. Consequently, every open set containing an element $x \in \mathbb{R}$ must also contain some element $> x$. We will prove that the closure operator $\overline{(-)}\colon \mathcal{P}(\mathbb{R}) \to \mathcal{P}(\mathbb{R})$ is not finitary by showing that $x \in \overline{(x, +\infty)}$, but $x \notin \overline{X}$ for every finite $X \subseteq (x, +\infty)$.

First observe that the set

$$Y := \mathbb{R} - \overline{(x, +\infty)}$$

is open because $\overline{(x, +\infty)}$ is closed. Moreover, $Y$ does not contain any element $> x$ as $(x, +\infty) \subseteq \overline{(x, +\infty)}$ and $Y \cap \overline{(x, +\infty)} = \emptyset$. Consequently, $x \notin Y$. Hence, we conclude that

$$x \in Y^c = \overline{(x, +\infty)}.$$

Then consider a finite set $X = \{x_1, \ldots, x_n\} \subseteq (x, +\infty)$ with $n \in \mathbb{N}$, with the understanding that $X = \emptyset$ if $n = 0$. Each singleton set $\{y\}$ with $y \in \mathbb{R}$ is closed in the standard topology in $\mathbb{R}$, since its complement $\mathbb{R} - \{y\} = (-\infty, y) \cup (y, +\infty)$ is open. Because the family of closed sets of any topology contains the empty set and is closed under binary unions, $X = \{x_1\} \cup \cdots \cup \{x_n\}$ is closed. That is, $\overline{X} = X$, so $x \notin \overline{X}$. ⊠

In view of Theorem 1.39, it makes sense to characterize the finitary closure operators in terms of the structure of the closure systems and consequence relations associated with them. The following concepts are instrumental to this purpose.

**Definition 1.45.** Given a poset $\mathbb{X}$, a nonempty set $Y \subseteq X$ is said to be

(i) *upward directed* in $\mathbb{X}$ when for every $x, y \in Y$ there exists $z \in Y$ such that $x, y \leqslant z$;

(ii) *downward directed* in $\mathbb{X}$ when for every $x, y \in Y$ there exists $z \in Y$ such that $z \leqslant x, y$.

A closure system $\mathcal{S}$ is said to be *inductive* when $\bigcup Y \in \mathcal{S}$ for every family $Y \subseteq \mathcal{S}$ that is upward directed in $\langle \mathcal{S}; \subseteq \rangle$.

**Definition 1.46.** A consequence relation $\vdash$ on a set $X$ is said to be *finitary* when for every $Y \cup \{x\} \subseteq X$,

$$\text{if } Y \vdash x, \text{ there exists a finite } Z \subseteq Y \text{ such that } Z \vdash x.$$

Finitary closure operators, inductive closure systems, and finitary consequence relations are related as follows:

**Theorem 1.47.** *The following conditions are equivalent for a closure operator $C$ on a set $X$:*

(i) *$C$ is finitary;*

(ii) *The closure system associated with $C$ is inductive;*

(iii) *The consequence relation associated with $C$ is finitary.*

*Proof.* Recall that the closure system and the consequence relation associated with $C$ are, respectively,

$$\mathcal{S}_C \coloneqq \{Y \subseteq X : Y = C(Y)\} \text{ and } \vdash_C \coloneqq \{\langle Y, x \rangle \in \mathcal{P}(X) \times X : x \in C(Y)\}.$$

(i)⇒(ii): Let $\{Y_i : i \in I\} \subseteq \mathcal{S}_C$ be an upward directed family. We need to show that $\bigcup_{i \in I} Y_i$ is a closed set of $C$, that is, $C(\bigcup_{i \in I} Y_i) \subseteq \bigcup_{i \in I} Y_i$. To this end, consider an element $x \in C(\bigcup_{i \in I} Y_i)$. Since the closure operator $C$ is finitary, there is a finite $Z \subseteq \bigcup_{i \in I} Y_i$ such that $x \in C(Z)$. Furthermore, there exists a finite subfamily $\{Y_1, \ldots, Y_n\} \subseteq \{Y_i : i \in I\}$ such that $Z \subseteq Y_1 \cup \cdots \cup Y_n$, as $Z \subseteq \bigcup_{i \in I} Y_i$ and $Z$ is finite. Lastly, the assumption that the family $\{Y_i : i \in I\}$ is upward directed in $\langle \mathcal{S}_C; \subseteq \rangle$ guarantees that existence of some $j \in I$ such that $Y_1, \ldots, Y_n \subseteq Y_j$. Consequently, $Z \subseteq Y_1 \cup \cdots \cup Y_n \subseteq Y_j$. Together with the assumption that $Y_j$ belongs to $\mathcal{S}_C$ and, therefore, that $Y_j$ is a closed set of $C$, this yields $C(Z) \subseteq C(Y_j) = Y_j$. Since $x \in C(Z)$, we conclude that $x \in Y_j \subseteq \bigcup_{i \in I} Y_i$.

(ii)⇒(iii): Let $Y \cup \{x\} \subseteq X$ be such that $Y \vdash_C x$. By the definition of $\vdash_C$ this means that $x \in C(Y)$. Then observe that

$$U \coloneqq \{C(Z) : Z \subseteq Y \text{ is finite}\}$$

is a family of elements of $\mathcal{S}_C$ because each $C(Z)$ is a closed set. We will prove that $U$ is upward directed. First, observe that $U$ is nonempty because $C(\emptyset) \in U$. Then consider two finite subsets $Z_1$ and $Z_2$ of $Y$. We need to find a finite subset $Z$ of $Y$ such that $C(Z)$ contains $C(Z_1)$ and $C(Z_2)$. Taking $Z \coloneqq Z_1 \cup Z_2$, we are done. Therefore, $U$ is an upward directed family of elements of $\mathcal{S}_C$. Since the closure system $\mathcal{S}_C$ is inductive, we obtain $\bigcup U \in \mathcal{S}$. As $\mathcal{S}_C$ is the family of closed sets of $C$, this amounts to $\bigcup U = C(\bigcup U)$.

Now, observe that $Y \subseteq \bigcup U$ because $y \in C(y)$ and $C(y) \in U$ for every $y \in Y$. Consequently, $C(Y) \subseteq C(\bigcup U) = \bigcup U$. Together with the assumption that $x \in C(Y)$, this yields $x \in \bigcup U$. Therefore, there exists a finite $Z \subseteq Y$ such that $x \in C(Z)$. By the definition of $\vdash_C$ this amounts to $Z \vdash_C x$ as desired.

(iii)⇒(i): Let $Y \cup \{x\} \subseteq X$ be such that $x \in C(Y)$. By the definition of $\vdash_C$ we have $Y \vdash_C x$. Since $\vdash_C$ is finitary, there exists a finite $Z \subseteq Y$ such that $Z \vdash_C x$. Again by the definition of $\vdash_C$ this amounts to $x \in C(Z)$. ⊠

The finitely generated closed sets of a finitary closure operator can be described in purely lattice theoretic therms, as we proceed to explain.

**Definition 1.48.** An element $x$ of a complete lattice $\mathbb{X}$ is said to be *compact* when for every $Y \subseteq X$,

$$\text{if } x \leqslant \bigvee Y, \text{ there exists a finite } Z \subseteq Y \text{ such that } x \leqslant \bigvee Z.$$

Notably, compact elements are closed under binary joins:

**Proposition 1.49** (Compact elements)**.** *The set* $\mathsf{Comp}(\mathbb{X})$ *of compact elements of a complete lattice* $\mathbb{X}$ *is closed under binary joins in* $\mathbb{X}$ *and contains the least element of* $\mathbb{X}$.

*Proof.* Consider $x, y \in \mathsf{Comp}(\mathbb{X})$ and a set $Z \subseteq X$ such that

$$x \vee y \leqslant \bigvee Z.$$

Clearly, $x, y \leqslant \bigvee Z$. As $x$ and $y$ are compact, there exist finite subsets $Z_1, Z_2 \subseteq Z$ such that $x \leqslant \bigvee Z_1$ and $y \leqslant \bigvee Z_2$. Then

$$x \vee y \leqslant \bigvee (Z_1 \cup Z_2).$$

Since the union $Z_1 \cup Z_2$ is also finite, we conclude that $x \vee y \in \mathsf{Comp}(\mathbb{X})$. The least element $0$ of $\mathbb{X}$ is compact because $0 \leqslant \bigvee Y$ indeed implies $0 \leqslant \bigvee \emptyset$ for the finite set $\emptyset \subseteq Y$. ⊠

The set $\mathsf{Comp}(\mathbb{X})$ of compact elements of a complete lattice $\mathbb{X}$ therefore becomes a join semilattice $\langle \mathsf{Comp}(\mathbb{X}); \vee \rangle$ in the sense of Definition 1.30 when we endow it with the restriction of the binary join operation of $\mathbb{X}$. Moreover, $\langle \mathsf{Comp}(\mathbb{X}); \vee \rangle$ is a join subsemilattice of $\langle \mathbb{X}; \vee \rangle$ with the same least element as $\mathbb{X}$.

Recall from Proposition 1.40 that every closure system can be viewed as a complete lattice. If the closure system is finitary, then its finitely generated closed sets and compact elements are related as follows:

**Proposition 1.50.** *Let $C$ be a finitary closure operator on a set $X$ and $\mathcal{S}$ the associated closure system. A closed set $Y \subseteq X$ is finitely generated iff it is a compact element of $\langle \mathcal{S}; \subseteq \rangle$.*

*Proof.* Recall from Proposition 1.40 that the join of a family $\{Z_i :\in I\} \subseteq \mathcal{S}$ in $\langle \mathcal{S}; \subseteq \rangle$ can be described as follows:

$$\bigvee_{i \in I} Z_i = C(\bigcup_{i \in I} Z_i).$$

This fact will be used repeatedly in the proof.

Let $Y$ be a finitely generated closed set of $C$. Then there exists a finite $Z \subseteq Y$ such that $Y = C(Z)$. Since $Y$ is closed, it belongs to $\mathcal{S}$. Therefore, it suffices to prove that $Y$ is compact in $\langle \mathcal{S}; \subseteq \rangle$. Accordingly, let $\{Z_i : i \in I\} \subseteq \mathcal{S}$ be such that

$$Y \subseteq \bigvee_{i \in I} Z_i = C(\bigcup_{i \in I} Z_i).$$

As $Z \subseteq C(Z) = Y$, this implies $Z \subseteq C(\bigcup_{i \in I} Z_i)$. Since $C$ is finitary, for every $z \in Z$ there exists a finite $J_z \subseteq I$ such that

$$z \in C(\bigcup_{j \in J_z} Z_j).$$

Notice that the union $J := \bigcup\{J_z : z \in Z\}$ is also finite because $Z$ is finite. Moreover, for every $z \in Z$,

$$z \in C(\bigcup_{j \in J_z} Z_j) \subseteq C(\bigcup_{j \in J} Z_j)$$

and, therefore, $Z \subseteq C(\bigcup_{j \in J} Z_j)$. Since $Y = C(Z)$, this yields

$$Y = C(Z) \subseteq C(C(\bigcup_{j \in J} Z_j)) = C(\bigcup_{j \in J} Z_j) = \bigvee_{j \in J} Z_j.$$

As $J$ is finite, we conclude that $Y$ is compact.

Conversely, consider a compact element $Y$ of $\langle \mathcal{S}; \subseteq \rangle$. Since $\mathcal{S}$ is the family of closed sets of $C$, the set $Y$ is closed. To prove that it is finitely generated, observe that

$$Y \subseteq \bigcup_{y \in Y} C(y) \subseteq C(\bigcup_{y \in Y} C(y)) = \bigvee_{y \in Y} C(y).$$

As $Y$ is compact, there exists a finite $Z \subseteq Y$ such that

$$Y \subseteq \bigvee_{z \in Z} C(z) = C(\bigcup_{z \in Z} C(z)). \tag{1.2}$$

Since $C(z) \subseteq C(Z)$ for every $z \in Z$, we have $\bigcup_{z \in Z} C(z) \subseteq C(Z)$ and, therefore,

$$C(\bigcup_{z \in Z} C(z)) \subseteq C(C(Z)) = C(Z).$$

Together with condition (1.2), this yields $Y \subseteq C(Z)$. Moreover, $C(Z) \subseteq C(Y) = Y$ because $Y$ is closed and $Z \subseteq Y$. Consequently, $Y = C(Z)$. As $Z$ is finite, we conclude that $Y$ is finitely generated. ⊠

For instance, let $\mathbb{X}$ be a poset. Recall that the closure operators $\uparrow$ and $\downarrow$ are finitary. By Theorem 1.47 the closure systems $\mathsf{Up}(\mathbb{X})$ and $\mathsf{Down}(\mathbb{X})$ associated with these closure operators are inductive. Furthermore, the compact elements of $\langle \mathsf{Up}(\mathbb{X}); \subseteq \rangle$ are the subsets of $X$ that are either empty or of the form $\uparrow\{x_1, \dots, x_n\}$ for some $x_1, \dots, x_n \in X$ by Proposition 1.50. Similarly, the compact elements of $\langle \mathsf{Down}(\mathbb{X}); \subseteq \rangle$ are the subsets of $X$ that are either empty of the form $\downarrow\{x_1, \dots, x_n\}$.

## 1.6 Algebraic lattices

Recall from Theorem 1.41 that every complete lattice can be represented as a closure system. Therefore, it is natural to wonder which are the complete lattices isomorphic to the *inductive* closure systems. Before we describe these lattices, let us first introduce a useful auxiliary notion.

**Definition 1.51.** A subset $D$ of a complete lattice $\mathbb{X}$ is said to be *join dense* (resp. *meet dense*) when each element of $\mathbb{X}$ is a join (resp. a meet) of some subset of $D$. That is, for each $x \in \mathbb{X}$ there exists some $Y \subseteq D$ such that $x = \bigvee Y$ (resp. $x = \bigwedge Y$).

The next result reviews various equivalent formulations of the notion of join density. As expected, the dual statements characterize meet density.

**Proposition 1.52.** *The following conditions are equivalent for a subset $D$ of a complete lattice $\mathbb{X}$:*

(i) *$D$ is join dense in $\mathbb{X}$;*

(ii) *Each $x \in X$ is the join of all elements of $D$ below $x$. That is, $x = \bigvee(\downarrow x \cap D)$;*

(iii) *For every $x, y \in X$,*

$$x \nleqslant y \iff \text{there exists } d \in D \text{ such that } d \leqslant x \text{ and } d \nleqslant y.$$

*Proof.* (i)$\Rightarrow$(ii): Consider $x \in X$. Since $D$ is join dense in $\mathbb{X}$, there exists $Y \subseteq D$ such that $x = \bigvee Y$. Then $y \leqslant x$ for each $y \in Y$, so $Y \subseteq \downarrow x \cap D$. Each upper bound of the set $\downarrow x \cap D$ is therefore an upper bound of $Y$. Since $x$ is an upper bound of $\downarrow x \cap D$ and the least upper bound of $Y$, it must in fact be the least upper bound of $\downarrow x \cap D$.

(ii)$\Rightarrow$(iii): The right to left implication of condition (iii) always holds by transitivity. To prove the left to tight implication, consider some $x, y \in X$ such that $x \nleqslant y$. Because $x$ is the join of $\downarrow x \cap D$ by condition (ii), we have $\bigvee(\downarrow x \cap D) \nleqslant y$. Thus, there exists $d \in D$ such that $d \leqslant x$ and $d \nleqslant y$.

(iii)$\Rightarrow$(i): Consider some $x \in X$ and let $Y := \downarrow x \cap D$. We need to prove that $x = \bigvee Y$. Clearly $x$ is an upper bound of $Y$, since $x$ is an upper bound of $\downarrow x$. We show that $x$ is the least upper bound. To this end, consider an upper bound $y$ of $Y$. By the definition of $Y$, we see that $d \leqslant x$ implies $d \leqslant y$ for each $d \in D$. Thus $x \leqslant y$ by condition (iii). ⊠

Having identified compact elements as the abstract counterpart of finitely generated sets in inductive closure systems (Proposition 1.50), we can now further identify the abstract counterpart of inductive closure systems as the so-called algebraic lattices.

**Definition 1.53.** A complete lattice $\mathbb{X}$ is said to be *algebraic* when the set $\mathsf{Comp}(\mathbb{X})$ of compact elements of $\mathbb{X}$ is join dense in $\mathbb{X}$.

In order to connect algebraic lattices and inductive closure systems, a last ingredient is needed.

**Definition 1.54.** Let $\boldsymbol{A}$ be a join semilattice with least element 0. An *ideal* of $\boldsymbol{A}$ is a downset containing 0 such that for every $a, b \in A$,

$$\text{if } a, b \in I, \text{ then } a \vee b \in I.$$

The set of ideals of $\boldsymbol{A}$ will be denoted by $\mathsf{Id}(\boldsymbol{A})$.

Equivalently, the ideals of $\boldsymbol{A}$ are the upward directed downsets of $\boldsymbol{A}$.

**Proposition 1.55.** *If $\boldsymbol{A}$ is a join semilattice with a least element, then $\mathsf{Id}(\boldsymbol{A})$ is an inductive closure system on $\boldsymbol{A}$.*

*Proof.* We begin by proving that $\mathsf{Id}(\boldsymbol{A})$ is a closure system. Consider a family $\{I_j : j \in J\}$ of ideals of $\boldsymbol{A}$. We will prove that the intersection $I$ of the various $I_j$ is an ideal of $\boldsymbol{A}$. Since each $I_j$ contains the least element 0 of $\boldsymbol{A}$, their intersection $I$ contains 0 too. Furthermore, as each $I_j$ is a downset and intersections of downsets are still downsets, the set $I$ is also a downset. Lastly, consider $a, b \in I$. Then $a$ and $b$ belong to each $I_j$ and so does $a \vee b$ because $I_j$ is closed under binary joins. Thus, $a \vee b \in I$ as desired. Hence, we conclude that $I$ is an ideal. This establishes that $\mathsf{Id}(\boldsymbol{A})$ is a closure system.

To prove that the closure system $\mathsf{Id}(\boldsymbol{A})$ is inductive, consider an upward directed family $\{I_j : j \in J\} \subseteq \mathsf{Id}(\boldsymbol{A})$ and let $I$ be its union. Since upward directed families are nonempty by definition, there exists some $j \in J$. As $0 \in I_j$ and $I_j \subseteq I$, we obtain $0 \in I$. To prove that $I$ is a downset, consider $a, b \in A$ such that $a \in I$ and $b \leqslant a$. Since $a \in I$, there exists $j \in J$ such that $a \in I_j$. As $I_j$ is a downset and $b \leqslant a$, this implies $b \in I_j \subseteq I$. It only remains to prove that $I$ is closed under binary joins. To this end, consider $a, b \in I$. Then there are $j_1, j_2 \in J$ such that $a \in I_{j_1}$ and $b \in I_{j_2}$. Since the family $\{I_j : j \in J\}$ is upward directed, there exists $j \in J$ such that $I_{j_1}, I_{j_2} \subseteq I_j$. This implies $a, b \in I_j$ and, therefore, $a \vee b \in I_j$ because $I_j$ is closed under binary joins. As $I_j \subseteq I$, we conclude that $a \vee b \in I$ as desired. ⊠

Let $\mathbb{X}$ be a complete lattice. Recall from Proposition 1.49 that $\langle \mathsf{Comp}(\mathbb{X}); \vee \rangle$ is a join semilattice whose least element is the least element of $\mathbb{X}$. Therefore, $\mathsf{Id}(\langle \mathsf{Comp}(\mathbb{X}); \vee \rangle)$ is an inductive closure system by Proposition 1.55. Bearing this in mind, algebraic lattices and inductive closure systems are related as follows:

**Theorem 1.56** (Representation theorem)**.** *Every inductive closure system is an algebraic lattice. Conversely, if $\mathbb{X}$ is an algebraic lattice, then $\mathsf{Id}(\langle \mathsf{Comp}(\mathbb{X}); \vee \rangle)$ is an inductive closure system isomorphic to $\mathbb{X}$.*

*Proof.* Let $\mathcal{S}$ be an inductive closure system on a set $X$. By Proposition 1.40 the pair $\langle \mathcal{S}; \subseteq \rangle$ is a complete lattice. Therefore, to prove that $\langle \mathcal{S}; \subseteq \rangle$ is an algebraic lattice, it suffices to show that each $Y \in \mathcal{S}$ is a join of compact elements.

To this end, recall from condition (i) of Theorem 1.39 that $\mathcal{S}$ is the closure system associated with some closure operator $C$ on $X$. Since $\mathcal{S}$ is inductive, $C$ is finitary by Theorem 1.47. Therefore, we can apply Proposition 1.50 obtaining that the compact elements of $\mathcal{S}$ are precisely the finitely generated closed sets of $C$.

Then consider an element $Y \in \mathcal{S}$. Since $\mathcal{S}$ is the family of closed sets of $C$, we have $Y = C(Y)$. In turn, this yields

$$Y = C(\bigcup_{y \in Y} C(y)) = \bigvee_{y \in Y} C(y).$$

Since each $C(y)$ is finitely generated, it is also compact in $\langle \mathcal{S}; \subseteq \rangle$. Consequently, $Y$ is a join of compact elements as desired.

To prove the converse, consider an algebraic lattice $\mathbb{X}$. Then $\mathsf{Id}(\langle \mathsf{Comp}(\mathbb{X}); \vee \rangle)$ is an inductive closure system. We will prove that the map $f \colon \mathbb{X} \to \mathsf{Id}(\langle \mathsf{Comp}(\mathbb{X}); \vee \rangle)$ defined by the rule

$$f(x) := \{y \in \mathsf{Comp}(\mathbb{X}) : y \leqslant x\}$$

is an order isomorphism. We begin by showing that $f(x)$ is an ideal of $\langle \mathsf{Comp}(\mathbb{X}); \vee \rangle$ for every $x \in X$. Clearly, $f(x)$ is a downset of $\mathsf{Comp}(\mathbb{X})$ containing the least element of $\mathsf{Comp}(\mathbb{X})$. To prove that $f(x)$ is closed under binary joins, consider $y, z \in f(x)$. The definition of $f(x)$ guarantees that $y, z \leqslant x$ and $y, z \in \mathsf{Comp}(\mathbb{X})$. From $y, z \leqslant x$ it follows $y \vee z \leqslant x$. Furthermore, since $y, z \in \mathsf{Comp}(\mathbb{X})$ and $\mathsf{Comp}(\mathbb{X})$ is closed under binary joins, we have $y \vee z \in \mathsf{Comp}(\mathbb{X})$. Hence, $y \vee z \in f(x)$ and $f(x)$ is an ideal of $\langle \mathsf{Comp}(\mathbb{X}); \vee \rangle$ as desired. It follows that the map $f \colon \mathbb{X} \to \mathsf{Id}(\langle \mathsf{Comp}(\mathbb{X}); \vee \rangle)$ is well defined.

To prove that $f$ is an order embedding, consider $x, y \in X$. Clearly, if $x \leqslant y$, then $f(x) \subseteq f(y)$. To prove that $f(x) \subseteq f(y)$ implies $x \leqslant y$, we reason by contraposition. Suppose that $x \nleqslant y$. Then recall that $\mathsf{Comp}(\mathbb{X})$ is join dense in $\mathbb{X}$ because the lattice $\mathbb{X}$ is algebraic. Therefore, we can apply condition (iii) of Proposition 1.52 to the assumption that $x \nleqslant y$ obtaining that there exists $z \in \mathsf{Comp}(\mathbb{X})$ such that $z \leqslant x$ and $z \nleqslant y$. This implies that $z \in f(x) - f(y)$ and, therefore, $f(x) \nsubseteq f(y)$. Hence, we conclude that $f$ is an order embedding.

It only remains to prove that $f$ is surjective. Consider an ideal $I$ of $\langle \mathsf{Comp}(\mathbb{X}); \vee \rangle$. We will show that $f(x) = I$ for $x := \bigvee I$. Clearly,

$$I \subseteq \{y \in \mathsf{Comp}(\mathbb{X}) : y \leqslant \bigvee I\} = \{y \in \mathsf{Comp}(\mathbb{X}) : y \leqslant x\} = f(x).$$

To prove that $f(x) \subseteq I$, consider $y \in f(x)$. Then $y \in \mathsf{Comp}(\mathbb{X})$ and $y \leqslant x = \bigvee I$. Since $y$ is compact, there are $z_1, \dots, z_n \in I$ such that $y \leqslant z_1 \vee \dots \vee z_n$. Consequently,

$$y \vee z_1 \vee \dots \vee z_n = z_1 \vee \dots \vee z_n \in I.$$

As $I$ is an ideal, this yields $y \in I$. Thus, we conclude that $f(x) = I$. ⊠

Recall that the closure systems $\mathsf{Up}(\mathbb{X})$ and $\mathsf{Down}(\mathbb{X})$ are inductive for every poset $\mathbb{X}$. From Theorem 1.56 it follows that they are also algebraic lattices. Powerset lattices $\langle \mathcal{P}(X); \subseteq \rangle$ are algebraic too because they coincide with the closure systems of the form

$\mathsf{Up}(\mathbb{X})$, where $\mathbb{X}$ is the discrete poset with universe $X$. Lastly, every finite lattice is algebraic because finite lattices are complete and all their elements are compact.

At this stage it is important to notice that, while algebraic lattices are precisely the posets isomorphic to inductive closure systems, there are closure systems that fail to be inductive but that nonetheless are algebraic lattices.

**Example 1.57.** Consider the set $X := \mathbb{N} \cup \{\infty\}$ and let $C \colon \mathcal{P}(X) \to \mathcal{P}(X)$ be the map defined for every $Y \subseteq X$ as

$$C(Y) := \begin{cases} Y & \text{if } Y \subsetneq \mathbb{N}; \\ X & \text{otherwise.} \end{cases}$$

It is easy to see that $C$ is a closure operator on $X$. Furthermore, $C$ is not finitary because $\infty \in C(\mathbb{N})$, but $\infty \notin Y = C(Y)$ for every finite $Y \subseteq \mathbb{N}$. Consequently, the closure system $\mathcal{S}_C$ associated with $C$ is not inductive. However, that restriction map $(-) \cap \mathbb{N} \colon \mathcal{S}_C \to \mathcal{P}(\mathbb{N})$ is an order isomorphism from $\mathcal{S}_C$ to the powerset lattice $\langle \mathcal{P}(\mathbb{N}); \subseteq \rangle$. Since the latter is an algebraic lattice, so is the closure system $\mathcal{S}_C$. ⊠

We close this section with a useful result on the structure of algebraic lattices.

**Definition 1.58.** An element $x$ of a lattice $\mathbb{X}$ is said to be

(i) *meet irreducible* in $\mathbb{X}$ when it is not the greatest element of $\mathbb{X}$ and for every $y, z \in X$,

$$\text{if } x = y \wedge z, \text{ then } x = y \text{ or } x = z;$$

(ii) *join irreducible* in $\mathbb{X}$ when it is not the least element of $\mathbb{X}$ and for every $y, z \in X$,

$$\text{if } x = y \vee z, \text{ then } x = y \text{ or } x = z;$$

(iii) *completely meet irreducible* in $\mathbb{X}$ when for every $Y \subseteq X$,

$$\text{if } x = \bigwedge Y, \text{ then } x \in Y;$$

(iv) *completely join irreducible* in $\mathbb{X}$ when for every $Y \subseteq X$,

$$\text{if } x = \bigvee Y, \text{ then } x \in Y.$$

The sets of meet irreducible and completely meet irreducible elements of a poset $\mathbb{X}$ will be denoted by $\mathsf{M}(\mathbb{X})$ and $\mathsf{M}_\infty(\mathbb{X})$. Similarly, we will denote the sets of join irreducible and completely join irreducible elements of $\mathbb{X}$ by $\mathsf{J}(\mathbb{X})$ and $\mathsf{J}_\infty(\mathbb{X})$.

Observe that the (completely) meet irreducible elements of $\mathbb{X}$ are precisely the (completely) join irreducible elements of $\mathbb{X}^\partial$, and vice versa.

The meet irreducibility (resp. join irreducibility) of $x$ can equivalently be defined as follows: for every *finite* $Y \subseteq X$, if $x = \bigwedge Y$ (resp. if $x = \bigvee Y$), then $x \in Y$. In other words, restricting to finite sets $Y \subseteq X$ in the definition of completely meet (resp. join) irreducible elements yields precisely the meet (resp. join) irreducible elements. Indeed, if $x$ satisfies this alternative definition of meet irreducibility, then $x$ cannot be the greatest element of $\mathbb{X}$, since otherwise $x = \bigwedge \emptyset$ but $x \notin \emptyset$. Moreover, for such an element $x$ the equality $x = y \wedge z$ implies that $x = y$ or $x = z$, since $x = \bigwedge\{y, z\}$ implies $x \in \{y, z\}$. Proving that conversely each meet irreducible $x \in X$ satisfies the above condition is a simple induction on the cardinality of $Y$. The equivalence of the two definitions of join irreducibility then follows from the Duality Principle.

The inclusion $\mathsf{M}_\infty(\mathbb{X}) \subseteq \mathsf{M}(\mathbb{X})$ and $\mathsf{J}_\infty(\mathbb{X}) \subseteq \mathsf{J}(\mathbb{X})$ are now immediate. On the other hand, meet (resp. join) irreducible elements need not be completely meet (resp. join) irreducible. For instance, every element of a chain $\mathbb{X}$ other than the greatest element is meet irreducible. To prove this, suppose that $x \in \mathbb{X}$ is not the greatest element of $\mathbb{X}$ and that $x = y \wedge z$ for some $y, z \in X$. Since $\mathbb{X}$ is a chain, $y \wedge z$ is the least element between $y$ and $z$. Consequently, $x = y$ or $x = z$, so $x$ is indeed meet irreducible. Similarly, every element of $\mathbb{X}$ other than the minimum is join irreducible. On the other hand, in the chain of real numbers $\langle \mathbb{R}; \leqslant \rangle$ no element is completely meet irreducible or completely join irreducible. This is because every real number $x$ is the meet of the set of all elements $> x$ and the join of the set of all elements $< x$.

**Example 1.59.** A lattice may lack meet and join irreducible elements altogether. For instance, the direct product $\langle \mathbb{Z}; \leqslant \rangle \times \langle \mathbb{Z}; \leqslant \rangle$ is the lattice with universe $\mathbb{Z} \times \mathbb{Z}$ and order relation $\sqsubseteq$ defined as

$$\langle m_1, m_2 \rangle \sqsubseteq \langle n_1, n_2 \rangle \iff m_1 \leqslant n_1 \text{ and } m_2 \leqslant n_2.$$

No element of $\langle \mathbb{Z}; \leqslant \rangle \times \langle \mathbb{Z}; \leqslant \rangle$ is meet irreducible or join irreducible because

$$\langle m, n \rangle = \langle m, n+1 \rangle \wedge \langle m+1, n \rangle \quad \text{and} \quad \langle m, n \rangle = \langle m, n-1 \rangle \vee \langle m-1, n \rangle$$

for every $m, n \in \mathbb{Z}$. ⊠

This makes the following property of algebraic lattices appealing.

**Theorem 1.60.** *In $\mathbb{X}$ an algebraic lattice, the set of completely meet irreducible elements $\mathsf{M}_\infty(\mathbb{X})$ is meet dense.*

*Proof.* Consider an element $x \in X$ and let

$$Y := \{y \in \mathsf{M}_\infty(\mathbb{X}) : x \leqslant y\}.$$

We will prove that $x = \bigwedge Y$. The definition of $Y$ guarantees that $x \leqslant \bigwedge Y$. Then we turn prove that $\bigwedge Y \leqslant x$. Since $\mathbb{X}$ is algebraic, $\bigwedge Y$ is the join of a set of compact elements. Consequently, it suffices to show that every compact element below $\bigwedge Y$ is also below $x$.

Suppose, on the contrary, that there exists a compact element $y \leqslant \bigwedge Y$ such that $y \nleqslant x$. Then consider the set

$$Z := \{z \in X : x \leqslant z \text{ and } y \nleqslant z\}.$$

We will use Zorn's Lemma to establish the existence of a maximal element in the subposet $\mathbb{Z}$ of $\mathbb{X}$ with universe $Z$. To this end, we need to prove that every chain $C$ in $\mathbb{Z}$ has an upper bound in $\mathbb{Z}$. We will do this by showing that $\bigvee C \in Z$. Accordingly, consider a chain $C$ in $\mathbb{Z}$. We may assume without loss of generality that $x \in C$ because $x$ is the least element of $\mathbb{Z}$. As a consequence, we have $x \leqslant \bigvee C$. To prove that $\bigvee C \in Z$, it only remains to show that $y \nleqslant \bigvee C$. Suppose the contrary, with a view to contradiction. Since $y$ is compact and $y \leqslant \bigvee C$, there exists a finite subset $D \subseteq C$ such that $y \leqslant \bigvee D$. We may assume that $x \in D$ because $x \in C$. Then consider an enumeration $D = \{z_1, \dots, z_n\}$. Notice that $\{z_1, \dots, z_n\}$ is a finite chain in $\mathbb{Z}$ because $C$ is a chain in $\mathbb{Z}$ and $D \subseteq C$. Therefore, $\{z_1, \dots, z_n\}$ has a greatest element $z_k$. As a consequence,

$$y \leqslant \bigvee D = z_1 \vee \cdots \vee z_n = z_k.$$

On the other hand, the assumption that $z_k \in Z$ implies $y \nleqslant z_k$, a contraction. Hence, we conclude that $y \nleqslant \bigvee C$. This implies $\bigvee C \in Z$ as desired. Hence, we can apply Zorn's Lemma obtaining that $\mathbb{Z}$ has a maximal element $z$.

We will prove that $z \in \mathsf{M}_\infty(\mathbb{X})$. Consider therefore some $U \subseteq X$ such that $z = \bigwedge U$. Since $z \in Z$ we have $y \nleqslant z$. Together with $z = \bigwedge U$, this implies that there exists $u \in U$ such that $y \nleqslant u$. Moreover, from $z = \bigwedge U$ and $u \in U$ it follows that $z \leqslant u$. As $z \in Z$, this yields $x \leqslant z \leqslant u$. Therefore, $x \leqslant u$ and $y \nleqslant u$, that is, $u \in Z$. Since $z \leqslant u$ and $z$ is maximal in $\mathbb{Z}$, we conclude that $z = u$, so $z \in U$.

Lastly, from $z \in Z$ it follows that $x \leqslant z$. Thus, $z \in Y$. Recall that $y \leqslant \bigwedge Y$ by assumption. As $z \in Y$, we obtain $y \leqslant z$, a contradiction with $z \in Z$. Hence, we conclude that $\bigwedge Y \leqslant x$ and therefore $x = \bigwedge Y$. ⊠

**Example 1.61** (Completely meet irreducible upsets)**.** Recall that $\langle \mathsf{Up}(\mathbb{X}); \subseteq \rangle$ is an algebraic lattice for every poset $\mathbb{X}$. Bearing this in mind, we will exemplify Theorem 1.60 by characterizing the completely meet irreducible elements of $\langle \mathsf{Up}(\mathbb{X}); \subseteq \rangle$ and explaining why every element of $\mathsf{Up}(\mathbb{X})$ is a meet of completely meet irreducible ones. More precisely, we will prove that

$$\mathsf{M}_\infty(\mathsf{Up}(\mathbb{X})) = \{X - {\downarrow} x : x \in X\} \tag{1.3}$$

and that every upset of $\mathbb{X}$ is an intersection of elements of $\mathsf{M}_\infty(\mathsf{Up}(\mathbb{X}))$.

Observe that the map $i \colon Y \mapsto X - Y$ is an isomorphism between the poset $\langle \mathsf{Up}(\mathbb{X}); \subseteq \rangle$ and the poset $\langle \mathsf{Down}(\mathbb{X}); \subseteq \rangle^\partial = \langle \mathsf{Down}(\mathbb{X}); \supseteq \rangle$: if $Y$ is an upset of $\mathbb{X}$, then $X - Y$ is a downset of $\mathbb{X}$, and moreover $Y \subseteq Z$ for upsets $Y, Z$ of $\mathbb{X}$ iff $X - Y \supseteq X - Z$. An upset $Y$ of $\mathbb{X}$ is therefore completely meet irreducible in $\mathsf{Up}(\mathbb{X})$ iff $i(Y)$ is completely meet irreducible in $\langle \mathsf{Down}(\mathbb{X}), \subseteq \rangle^\partial$, or in other words iff $i(Y)$ is completely join irreducible in $\langle \mathsf{Down}(\mathbb{X}), \subseteq \rangle$. The completely meet irreducibles of $\mathsf{Up}(\mathbb{X})$ are therefore precisely the complements of the completely join irreducibles of $\mathsf{Down}(\mathbb{X})$.

We now show that the completely join irreducibles of $\mathsf{Down}(\mathbb{X})$ are precisely the principal downsets. We begin by proving that each principal downset ${\downarrow} x$ of $\mathbb{X}$ is indeed completely join irreducible. For suppose that ${\downarrow} x = \bigcup_{i \in i} Y_i$ for some family $\{Y_i : i \in I\}$ of downsets of $\mathbb{X}$. Then $x \in Y_j$ for some $j \in I$. Since $Y_j$ is a downset, from $x \in Y_j$ it follows ${\downarrow} x \subseteq Y_j$. On the other hand, $Y_j \subseteq \bigcup_{i \in i} Y_i = {\downarrow} x$, so indeed $Y_j = {\downarrow} x$. Conversely, each downset $Y$ of $\mathbb{X}$ is a union, and therefore a join, of principal downsets, namely $Y = \bigcup_{y \in Y} {\downarrow} y$. If $Y$ is completely join irreducible, then $Y = {\downarrow} y$ for some $y \in Y$. In other words, each completely join irreducible downset of $\mathbb{X}$ must be principal.

Putting the last two paragraphs together, the completely meet irreducible elements of $\langle \mathsf{Up}(\mathbb{X}); \subseteq \rangle$ are precisely the complements of principal downsets of $\mathbb{X}$. In other words, these are the sets of the form $X - {\downarrow}x$ for $x \in X$. $\boxtimes$

CHAPTER 2

# Representation theorems

## 2.1 Distributive lattices

The following class of lattices plays a fundamental role in algebraic logic:

**Definition 2.1.** A lattice is said to be *distributive* when it validates the *distributive laws*

$$x \wedge (y \vee z) \approx (x \wedge y) \vee (x \wedge z) \quad \text{and} \quad x \vee (y \wedge z) \approx (x \vee y) \wedge (x \vee z).$$

Notably, one half of any of the distributive laws suffices.

**Proposition 2.2.** *The following conditions are equivalent for a lattice* $\boldsymbol{A}$*:*

(i) $\boldsymbol{A}$ *is distributive;*

(ii) $\boldsymbol{A}$ *validates* $x \wedge (y \vee z) \leqslant (x \wedge y) \vee (x \wedge z)$*;*

(iii) $\boldsymbol{A}$ *validates* $(x \vee y) \wedge (x \vee z) \leqslant x \vee (y \wedge z)$*.*

*Proof.* The implications (i)⇒(ii) and (i)⇒(iii) are straightforward.

(ii)⇒(i): We will prove that every lattice validates the inequalities

$$(x \wedge y) \vee (x \wedge z) \leqslant x \wedge (y \vee z) \quad \text{and} \quad x \vee (y \wedge z) \leqslant (x \vee y) \wedge (x \vee z). \tag{2.1}$$

To this end, consider a lattice $\boldsymbol{B}$ and $a, b, c \in B$. Since $a \wedge b \leqslant a$ and $a \wedge b \leqslant b \leqslant b \vee c$, we obtain

$$a \wedge b \leqslant a \wedge (b \vee c). \tag{2.2}$$

Similarly, from $a \wedge c \leqslant a$ and $a \wedge c \leqslant (b \vee c)$ it follows

$$a \wedge c \leqslant a \wedge (b \vee c). \tag{2.3}$$

By conditions (2.2) and (2.3) we obtain

$$(a \wedge b) \vee (a \wedge c) \leqslant a \wedge (b \vee c).$$

Hence, the first inequality in condition (2.1) holds in every lattice. By the Duality Principle this is true also for the second one.

Now, suppose that condition (ii) holds. Since $\boldsymbol{A}$ validates also the inequalities in (2.1), to prove that it is distributive, it only remains to show that $\boldsymbol{A}$ validates

$$(x \vee y) \wedge (x \vee z) \leqslant x \vee (y \wedge z).$$

Consider $a, b, c \in A$. Using the assignment

$$x \longmapsto a \vee b \qquad y \longmapsto a \qquad z \longmapsto c,$$

condition (ii) implies

$$(a \vee b) \wedge (a \vee c) \leqslant ((a \vee b) \wedge a) \vee ((a \vee b) \wedge c).$$

By the absorption law $a = (a \vee b) \wedge a$, so

$$(a \vee b) \wedge (a \vee c) \leqslant a \vee ((a \vee b) \wedge c) = a \vee (c \wedge (a \vee b)). \tag{2.4}$$

Now, applying condition (ii) to the case where

$$x \longmapsto c \qquad y \longmapsto a \qquad z \longmapsto b,$$

we obtain

$$c \wedge (a \vee b) \leqslant (c \wedge a) \vee (c \wedge b) = (c \wedge a) \vee (b \wedge c).$$

Since $c \wedge a \leqslant a$, this implies

$$c \wedge (a \vee b) \leqslant a \vee (b \wedge c)$$

and, therefore,

$$a \vee (c \wedge (a \vee b)) \leqslant a \vee a \vee (b \wedge c) = a \vee (b \wedge c).$$

Together with condition (2.1), this yields

$$(a \vee b) \wedge (a \vee c) \leqslant a \vee (b \wedge c).$$

(iii)⇒(i): This is obtained by applying the Duality Principle to the implication (ii)⇒(i) established above. ⊠

For the sake of simplicity, we will focus on *bounded* lattices and treat them as algebras $\boldsymbol{A} = \langle A; \wedge, \vee, 0, 1 \rangle$ where 0 and 1 are constants for the least and greatest elements. In particular, this convention applies to every finite lattice because finite lattices are always bounded.

**Example 2.3** (Upsets & downsets)**.** We will prove that the structures

$$\mathsf{Up}(\mathbb{X}) := \langle \mathsf{Up}(\mathbb{X}); \cap, \cup, \emptyset, X \rangle \quad \text{and} \quad \mathsf{Down}(\mathbb{X}) := \langle \mathsf{Down}(\mathbb{X}); \cap, \cup, \emptyset, X \rangle$$

are bounded distributive lattices (that, moreover, are complete) for every poset $\mathbb{X}$. By the Duality Principle it suffices to show that $\mathsf{Up}(\mathbb{X})$ is a bounded distributive lattice.

First recall that, when ordered under the inclusion relation, $\mathsf{Up}(\mathbb{X})$ is a complete lattice whose meet and join operations are intersections and unions and whose least and greatest elements are $\emptyset$ and $X$. Therefore, $\mathsf{Up}(\mathbb{X})$ is a well-defined bounded lattice. To prove that it is distributive, consider $U, V, Z \in \mathsf{Up}(\mathbb{X})$ and $x \in X$. We have

$$\begin{aligned} x \in U \cap (V \cup Z) &\iff x \in U \text{ and either } x \in V \text{ or } x \in Z \\ &\iff \text{either } x \in U \cap V \text{ or } x \in U \cap Z \\ &\iff x \in (U \cap V) \cup (U \cap Z). \end{aligned}$$

Therefore,

$$U \cap (V \cup Z) = (U \cap V) \cup (U \cap Z).$$

Consequently, $\mathsf{Up}(\mathbb{X})$ validates the first distributive law. From Proposition 2.2 it follows that the lattice $\mathsf{Up}(\mathbb{X})$ is distributive. ⊠

Our aim is to show that $\mathsf{Up}(\mathbb{X})$ is indeed the canonical example of a bounded distributive lattice, in the sense that every bounded distributive lattice sits inside one of this form. To this end, we will need the following concepts:

**Definition 2.4.** Let $\boldsymbol{A}$ be a bounded lattice.

(i) A subset $F \subseteq A$ is said to be a *filter* when it is an upset containing 1 and for every $a, b \in A$,
$$\text{if } a, b \in F, \text{ then } a \wedge b \in F;$$

(ii) A subset $I \subseteq A$ is said to be an *ideal* when it is a downset containing 0 and for every $a, b \in A$,
$$\text{if } a, b \in I, \text{ then } a \vee b \in I.$$

We will denote the sets of filters and ideals of $\boldsymbol{A}$ by $\mathsf{Fi}(\boldsymbol{A})$ and $\mathsf{Id}(\boldsymbol{A})$.

Notice that the filters and the ideals of $\boldsymbol{A}$ coincide with the ideals of the join semilattices $\langle A; \wedge \rangle^{\partial}$ and $\langle A; \vee \rangle$, respectively.

**Proposition 2.5** (Filter and ideal generation)**.** *If $\boldsymbol{A}$ is a bounded lattice, then $\mathsf{Fi}(\boldsymbol{A})$ and $\mathsf{Id}(\boldsymbol{A})$ are the inductive closure systems associated with the finitary closure operators*

$$\mathrm{Fg}^{\boldsymbol{A}} \colon \mathcal{P}(A) \to \mathcal{P}(A) \;\text{ and }\; \mathrm{Ig}^{\boldsymbol{A}} \colon \mathcal{P}(A) \to \mathcal{P}(A)$$

*defined by the rules*

$$\begin{aligned} \mathrm{Fg}^{\boldsymbol{A}}(X) &:= \{a \in A : \text{either } a = 1 \text{ or there are } b_1, \dots, b_n \in X \text{ such that } b_1 \wedge \dots \wedge b_n \leqslant a\}; \\ \mathrm{Ig}^{\boldsymbol{A}}(X) &:= \{a \in A : \text{either } a = 0 \text{ or there are } b_1, \dots, b_n \in X \text{ such that } a \leqslant b_1 \vee \dots \vee b_n\}. \end{aligned}$$

*Proof.* Recall that the filters of $\boldsymbol{A}$ coincide with the ideals of the join semilattice $\langle A; \wedge \rangle^{\partial}$. Therefore, we can apply Proposition 1.55 obtaining that $\mathsf{Fi}(\boldsymbol{A}) = \mathsf{Id}(\langle A; \wedge \rangle^{\partial})$ is an inductive closure system.

From Theorem 1.47 it follows that $\mathsf{Fi}(\boldsymbol{A})$ is the inductive closure system associated with the finitary closure operator $C \colon \mathcal{P}(A) \to \mathcal{P}(A)$ defined for every $X \subseteq A$ as

$$C(X) := \bigcap \{F \in \mathsf{Fi}(\boldsymbol{A}) : X \subseteq F\}.$$

We will prove that $C(X) = \mathrm{Fg}^{\boldsymbol{A}}(X)$ for every $X \subseteq A$. Since the definition of the map $\mathrm{Fg}^{\boldsymbol{A}}(-)$ guarantees that $\mathrm{Fg}^{\boldsymbol{A}}(X)$ is a filter of $\boldsymbol{A}$ containing $X$, we obtain

$$\mathrm{Fg}^{\boldsymbol{A}}(X) \in \{F \in \mathsf{Fi}(\boldsymbol{A}) : X \subseteq F\}.$$

Together with the definition of $C(X)$, this yields $C(X) \subseteq \mathrm{Fg}^{\boldsymbol{A}}(X)$. To prove the reverse inclusion, consider $a \in \mathrm{Fg}^{\boldsymbol{A}}(X)$. We have two cases: either $a = 1$ or $a \neq 1$. If $a = 1$, then $a \in F$ for every $F \in \mathsf{Fi}(\boldsymbol{A})$. By the definition of $C(X)$ we conclude that $a \in C(X)$. Then suppose that $a \neq 1$. In this case, there are $b_1, \dots, b_n \in X$ such that $b_1 \wedge \dots \wedge b_n \leqslant a$. Let $F \in \mathsf{Fi}(\boldsymbol{A})$ be such that $X \subseteq F$. Since $F$ is closed under binary meets and $b_1, \dots, b_n \in X \subseteq F$, we have $b_1 \wedge \dots \wedge b_n \in F$. As $F$ is an upset and $b_1 \wedge \dots \wedge b_n \leqslant a$, we obtain that $a \in F$. By the definition of $C$ this implies $a \in C(X)$. Hence, we conclude that $C(X) = \mathrm{Fg}^{\boldsymbol{A}}(X)$. As a consequence, $\mathsf{Fi}(\boldsymbol{A})$ is the closure system associated with the finitary closure operator $\mathrm{Fg}^{\boldsymbol{A}}(-)$.

By the Duality Principle we obtain the same analogous result connecting $\mathrm{Ig}^{\boldsymbol{A}}(-)$ and $\mathsf{Id}(\boldsymbol{A})$. ⊠

The above description of filter and ideal generation can often be simplified. For let $F$ and $I$ be a filter and an ideal of a bounded lattice $\boldsymbol{A}$. Then for every $a \in A$,

$$\mathrm{Fg}^{\boldsymbol{A}}(F \cup \{a\}) = \{c \in A : \text{there is } b \in F \text{ such that } a \wedge b \leqslant c\}$$
$$\mathrm{Ig}^{\boldsymbol{A}}(I \cup \{a\}) = \{c \in A : \text{there is } b \in F \text{ such that } c \leqslant a \vee b\}.$$

As an exemplification, we will motivate the first of the above equalities. The inclusion from right to left follows immediately from the description of filter generation given in Proposition 2.5. To prove the other inclusion, consider $c \in \mathrm{Fg}^{\boldsymbol{A}}(F \cup \{a\})$. In view of Proposition 2.5, either $c = 1$ or there are $b_1, \dots, b_n \in F$ such that $a \wedge b_1 \wedge \dots \wedge b_n \leqslant c$. Since $F$ is a filter, it contains the greatest element 1. Therefore, we may assume that the case where $a \wedge b_1 \wedge \dots \wedge b_n \leqslant c$ holds and that $b_1 = 1$. Furthermore, $b_1 \wedge \dots \wedge b_n \in F$ because $b_1, \dots, b_n \in F$ and $F$ is a filter. Consequently, letting $b := b_1 \wedge \dots \wedge b_n$, we conclude that $b \in F$ and $a \wedge b \leqslant c$ as desired.

If $\boldsymbol{A}$ is a bounded lattice and $a \in A$, then $\mathrm{Fg}^{\boldsymbol{A}}(a) = {\uparrow}a$ and $\mathrm{Ig}^{\boldsymbol{A}}(a) = {\downarrow}a$. Accordingly, ${\uparrow}a$ and ${\downarrow}a$ will be called the *principal filter* and the *principal ideal* generated by $a$. In addition, a filter (resp. an ideal) of $\boldsymbol{A}$ will be called *principal* when it is of the form ${\uparrow}a$ (resp. ${\downarrow}a$) for some $a \in A$. In finite lattices, every filter and ideal is principal as we proceed to explain.

**Proposition 2.6.** *Let $\boldsymbol{A}$ be a finite lattice. Then*

$$\mathsf{Fi}(\boldsymbol{A}) = \{{\uparrow}a : a \in A\} \ \textit{ and } \ \mathsf{Id}(\boldsymbol{A}) = \{{\downarrow}a : a \in A\}.$$

*Proof.* By the Duality Principle it suffices to prove the first equality in the statement. Furthermore, since every principal upset of $\boldsymbol{A}$ is a filter, it will be enough to prove the inclusion $\mathsf{Fi}(\boldsymbol{A}) \subseteq \{\uparrow a : a \in A\}$. Accordingly, consider $F \in \mathsf{Fi}(\boldsymbol{A})$. Since $\boldsymbol{A}$ is finite and $F$ nonempty, we can take an enumeration $F = \{a_1, \dots, a_n\}$. As $F$ is closed under binary meets, the element $a := a_1 \wedge \cdots \wedge a_n$ belongs to $F$. Moreover, the definition of $a$ guarantees that $F = \{a_1, \dots, a_n\} \subseteq \uparrow a$. As $a \in F$ and $F$ is an upset, we also have $\uparrow a \subseteq F$, so $F = \uparrow a$ as desired. ⊠

In infinite bounded lattices, however, filters and ideals need not be principal.

**Example 2.7** (Filters & ideals in chains)**.** Recall that every bounded chain is a bounded lattice. We will show that the filters and the ideals of a bounded chain $\boldsymbol{A}$ can be described as follows:

$$\mathsf{Fi}(\boldsymbol{A}) = \{U \in \mathsf{Up}(\boldsymbol{A}) : U \neq \varnothing\} \text{ and } \mathsf{Id}(\boldsymbol{A}) = \{V \in \mathsf{Down}(\boldsymbol{A}) : V \neq \varnothing\}. \tag{2.5}$$

This provides an easy method for constructing filters and ideals that are not principal. For instance, let $\langle \mathbb{Z}_\infty; \sqsubseteq \rangle$ be the poset where $\mathbb{Z}_\infty := \mathbb{Z} \cup \{-\infty, +\infty\}$ and $\sqsubseteq$ is the relation defined for every $x, y \in \mathbb{Z}_\infty$ as

$$x \sqsubseteq y \iff \text{either } x = -\infty \text{ or } y = +\infty \text{ or } x, y \in \mathbb{Z} \text{ and } x \leqslant y.$$

Clearly, $\langle \mathbb{Z}_\infty; \sqsubseteq \rangle$ is a bounded chain with least element $-\infty$ and greatest element $+\infty$. Therefore, we can apply the description of filters and ideals in bounded chains in condition (2.5) obtaining that $F := \mathbb{Z}_\infty - \{-\infty\}$ is a filter and $I := \mathbb{Z}_\infty - \{+\infty\}$ an ideal of $\langle \mathbb{Z}_\infty; \sqsubseteq \rangle$. However, $F$ and $I$ are not principal because $F$ lacks a least element and $I$ a greatest one.

We now turn to prove condition (2.5). By the Duality Principle it suffices to show that $\mathsf{Fi}(\boldsymbol{A}) = \{U \in \mathsf{Up}(\boldsymbol{A}) : U \neq \varnothing\}$. The inclusion from left to right holds because every filter is a nonempty upset. To prove the reverse inclusion, let $U$ be a nonempty upset of $\boldsymbol{A}$. Then $U$ contains the greatest element 1 of $\boldsymbol{A}$. It only remains to show that $U$ is closed under binary meets. Accordingly, consider $a, b \in U$. Since $\boldsymbol{A}$ is a chain either $a \leqslant b$ or $b \leqslant a$. In the first case $a = a \wedge b$ and in the second $b = a \wedge b$, therefore, $a \wedge b \in \{a, b\} \subseteq U$. ⊠

A filter or an ideal of a bounded lattice $\boldsymbol{A}$ is said to be *proper* when it differs from the total set $A$. Equivalently, a filter is proper when it does not contain the least element 0. Similarly, an ideal is proper when it does not contain the greatest element 1.

**Definition 2.8.** In a bounded lattice $\boldsymbol{A}$,

(i) a filter $F$ is said to be *prime* when it is proper and for every $a, b \in A$,

$$\text{if } a \vee b \in F, \text{ then either } a \in F \text{ or } b \in F;$$

(ii) an ideal $I$ is said to be *prime* when it is proper and for every $a, b \in A$,

$$\text{if } a \wedge b \in I, \text{ then either } a \in I \text{ or } b \in I.$$

The sets of prime filters and prime ideals of $\boldsymbol{A}$ will be denoted by $\mathsf{Pf}(\boldsymbol{A})$ and $\mathsf{Pi}(\boldsymbol{A})$.

Prime filters and ideals are dual concepts in the following sense:

**Proposition 2.9.** *For every bounded lattice $\boldsymbol{A}$, the map $f\colon \mathsf{Pf}(\boldsymbol{A}) \to \mathsf{Pi}(\boldsymbol{A})$ defined by the rule $f(F) \coloneqq F^c$ is an order isomorphism from $\langle \mathsf{Pf}(\boldsymbol{A}); \subseteq\rangle$ to $\langle \mathsf{Pi}(\boldsymbol{A}); \supseteq\rangle$.*

*Proof.* We begin by showing that

$$\mathsf{Pf}(\boldsymbol{A}) = \{F \subseteq A : F^c \in \mathsf{Pi}(\boldsymbol{A})\} \text{ and } \mathsf{Pi}(\boldsymbol{A}) = \{I \subseteq A : I^c \in \mathsf{Pf}(\boldsymbol{A})\}. \tag{2.6}$$

By the Duality Principle, not only it suffices to prove the first of the above equalities, but indeed it suffices to prove the inclusion from left to right in the first equality, that is,

$$\mathsf{Pf}(\boldsymbol{A}) \subseteq \{F \subseteq A : F^c \in \mathsf{Pi}(\boldsymbol{A})\}.$$

Accordingly, consider $F \in \mathsf{Pf}(\boldsymbol{A})$. Since $F$ is proper, it does not contain the least element 0. Then $0 \in F^c$. Furthermore, as $F$ is an upset, its complement $F^c$ is a downset. Lastly, consider $a, b \in F^c$. Since $F$ is prime, we have $a \vee b \notin F$, that is, $a \vee b \in F^c$. Hence, we conclude that $F^c$ is an ideal. To prove that it is prime, observe that $F^c$ is proper because $F$ is nonempty. Then consider $a, b \in A$ such that $a \wedge b \in F^c$. Since $F$ is closed under binary meets and $a \wedge b \notin F$, we obtain that either $a \in F^c$ or $b \in F^c$. Thus, $F^c$ is a prime ideal of $\boldsymbol{A}$ as desired. Hence, condition (2.6) holds.

As a consequence, the map $f\colon \mathsf{Pf}(\boldsymbol{A}) \to \mathsf{Pi}(\boldsymbol{A})$ in the statement is a well-defined bijection. Furthermore, for every $F, G \in \mathsf{Pf}(\boldsymbol{A})$ we have

$$F \subseteq G \iff F^c \supseteq G^c \iff f(F) \supseteq f(G).$$

Therefore, $f$ is an order isomorphism from $\langle \mathsf{Pf}(\boldsymbol{A}); \subseteq\rangle$ to $\langle \mathsf{Id}(\boldsymbol{A}); \supseteq\rangle$. ⊠

In finite distributive lattices, prime filters and ideals admit a very transparent description as we proceed to explain. To this end, recall that the sets meet irreducible and join irreducible elements of a lattice $\boldsymbol{A}$ are denoted by $\mathsf{M}(\boldsymbol{A})$ and $\mathsf{J}(\boldsymbol{A})$.

**Proposition 2.10.** *Let $\boldsymbol{A}$ be finite distributive lattice. Then*

$$\mathsf{Pf}(\boldsymbol{A}) = \{{\uparrow}a : a \in \mathsf{J}(\boldsymbol{A})\} \text{ and } \mathsf{Pi}(\boldsymbol{A}) = \{{\downarrow}a : a \in \mathsf{M}(\boldsymbol{A})\}.$$

*Proof.* By the Duality Principle it suffices to prove the first equality in the statement. Let $F \in \mathsf{Pf}(\boldsymbol{A})$. Since $\boldsymbol{A}$ is finite, we can apply Proposition 2.6 obtaining that $F = {\uparrow}a$ for some $a \in A$. We will prove that $a$ is join irreducible. Since $F$ is proper and $F = {\uparrow}a$, we know that $a$ is not the least element 0. Then consider $b, c \in A$ such that $b \vee c = a$. From $b \vee c = a \in {\uparrow}a = F$ and the assumption that the filter $F$ is prime it follows that either $b \in F$ or $c \in F$. By symmetry we may assume that $b \in F$. As $F = {\uparrow}a$, this implies $a \leqslant b$. Together with $b \leqslant b \vee c = a$, this implies $a = b$. Hence, we conclude that $a$ is join irreducible.

To prove the inclusion from right to left, consider a join irreducible element $a$ of $\boldsymbol{A}$. Clearly, ${\uparrow}a$ is a filter of $\boldsymbol{A}$. First observe that $a \neq 0$ because $a$ is join irreducible. Therefore, $0 \notin {\uparrow}a$. Thus, ${\uparrow}a$ is a proper filter of $\boldsymbol{A}$. Then consider $b, c \in A$ such that $b \vee c \in {\uparrow}a$. We have $a \leqslant b \vee c$, that is, $a = a \wedge (b \vee c)$. By the distributive laws this amounts to $a = (a \wedge b) \vee (a \wedge c)$. Since $a$ is join irreducible, either $a = a \wedge b$ or $a = a \wedge c$, that is, either $a \leqslant b$ or $a \leqslant c$. As a consequence, either $b \in {\uparrow}a$ or $c \in {\uparrow}a$. Hence, we conclude that ${\uparrow}a$ is a prime filter of $\boldsymbol{A}$. ⊠

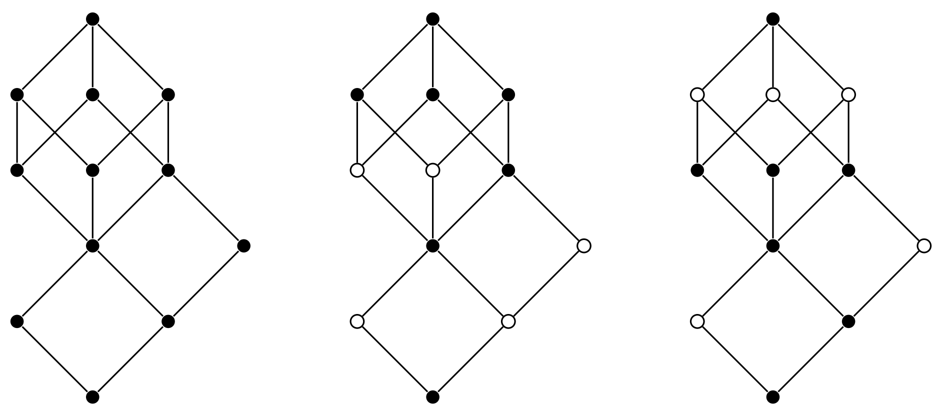
**Corollary 2.11.** *If $\boldsymbol{A} = \langle A; \leqslant \rangle$ is a finite distributive lattice, then*

$$\langle \mathsf{Pf}(\boldsymbol{A}); \subseteq \rangle \cong \langle \mathsf{J}(\boldsymbol{A}); \geqslant \rangle \ \textit{ and } \ \langle \mathsf{Pi}(\boldsymbol{A}); \subseteq \rangle \cong \langle \mathsf{M}(\boldsymbol{A}); \leqslant \rangle.$$

*Proof.* Consider the map $\uparrow(-)\colon \mathsf{J}(\boldsymbol{A}) \to \mathsf{Pf}(\boldsymbol{A})$ sends each element $a \in \mathsf{J}(\boldsymbol{A})$ to $\uparrow a$. From Proposition 2.10 it follows that $\uparrow(-)$ is a well-defined surjection. To prove that it is injective, consider $a, b \in \mathsf{J}(\boldsymbol{A})$ such that $\uparrow a = \uparrow b$. Then $a \leqslant b$ and $b \leqslant a$ which implies $a = b$. Thus, $\uparrow(-)$ is a bijection. Since for every $a, b \in \mathsf{J}(\boldsymbol{A})$ we have that $a \geqslant b$ iff $\uparrow a \subseteq \uparrow b$, we conclude that $\uparrow(-)\colon \mathsf{J}(\boldsymbol{A}) \to \mathsf{Pf}(\boldsymbol{A})$ is an order isomorphism from $\langle \mathsf{J}(\boldsymbol{A}); \geqslant \rangle$ to $\langle \mathsf{Pf}(\boldsymbol{A}); \subseteq \rangle$. The proof that $\langle \mathsf{Pi}(\boldsymbol{A}); \subseteq \rangle \cong \langle \mathsf{M}(\boldsymbol{A}); \leqslant \rangle$ is analogous. ⊠

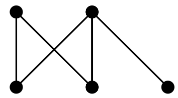
Proposition 2.10 and Corollary 2.11 make it easy to visualize the prime filters and ideals of a finite distributive lattice $\boldsymbol{A}$ as well as the posets $\langle \mathsf{Pf}(\boldsymbol{A}); \subseteq \rangle$ and $\langle \mathsf{Id}(\boldsymbol{A}); \subseteq \rangle$.

**Example 2.12** (Prime filters & prime ideals: the finite case)**.** The picture below showcases the same finite distributive lattice $\boldsymbol{A}$ three times. In the second and the third copy of $\boldsymbol{A}$, respectively, the join irreducible and the meet irreducible elements have been painted in white. The easiest way to recognize the join irreducible elements of finite lattices is to apply the following rule: an element $x$ is join irreducible iff it is not the least element and it does not posses two *immediate predecessors*, i.e., two elements $y, z < x$ such that there is no $u$ such that either $y < u < x$ or $z < u < x$. Similarly, an element is meet irreducible iff it is neither the greatest element nor it possesses two immediate successors. This allows us to determine relatively easily whether an element of a finite lattice presented by an Hasse diagram is join or meet irreducible.

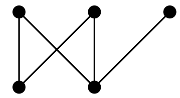


$\boldsymbol{A}$ $\qquad$ $\mathsf{J}(\boldsymbol{A})$ $\qquad$ $\mathsf{M}(\boldsymbol{A})$

In view of Proposition 2.10, the picture above allows us to recognize the prime filters and the prime ideals of $\boldsymbol{A}$: the prime filters are the principal filters generated by the join irreducible elements and the prime ideals are the principal ideals generated by the meet irreducible elements. Furthermore, applying Corollary 2.11 to the description of $\mathsf{J}(\boldsymbol{A})$ and $\mathsf{M}(\boldsymbol{A})$ in the picture above, we obtain the following description of the posets $\langle \mathsf{Pf}(\boldsymbol{A}); \subseteq \rangle$ and $\langle \mathsf{Pi}(\boldsymbol{A}); \subseteq \rangle$:



$\langle \mathsf{Pf}(\boldsymbol{A}); \subseteq \rangle$



$\langle \mathsf{Id}(\boldsymbol{A}); \subseteq \rangle$

The prime filters and ideals of an arbitrary finite distributive lattice can be identified in a similar manner. ⊠

Let $\boldsymbol{A}$ be a bounded lattice. By Proposition 2.5 both $\mathsf{Fi}(\boldsymbol{A})$ and $\mathsf{Id}(\boldsymbol{A})$ are closure systems and, therefore, complete lattices in which meets are intersections. As a consequence, the meet irreducible elements of $\mathsf{Fi}(\boldsymbol{A})$ are the proper filters of $\boldsymbol{A}$ that cannot be obtained as the intersection of two strictly larger filters. The meet irreducible elements of $\mathsf{Id}(\boldsymbol{A})$ admits an analogous description. Under the assumption that $\boldsymbol{A}$ is distributive, these are precisely the prime filters and ideals of $\boldsymbol{A}$, as we proceed to explain:

**Proposition 2.13.** *The following condition hold for a bounded distributive lattice $\boldsymbol{A}$:*

(i) *The prime filters of $\boldsymbol{A}$ are the meet irreducible elements of $\mathsf{Fi}(\boldsymbol{A})$;*

(ii) *The prime ideals of $\boldsymbol{A}$ are the meet irreducible elements of $\mathsf{Id}(\boldsymbol{A})$.*

*Proof.* As usual, it suffices to prove the condition (i). Suppose, with a view to contradiction, that there exists a prime filter $F$ of $\boldsymbol{A}$ that is not meet irreducible in $\mathsf{Fi}(\boldsymbol{A})$. As $F$ is prime, it is proper and, therefore, it is not the greatest element of $\mathsf{Fi}(\boldsymbol{A})$. Since $F$ is not meet irreducible, this means that there are two filters $G$ and $H$ different from $F$ such that $F = G \cap H$. Thus, there are $a \in G - F$ and $b \in H - F$. As $G$ and $H$ are upsets and $a, b \leqslant a \vee b$, this yields $a \vee b \in G \cap H = F$. From the assumption that $F$ is prime it follows that either $a \in F$ or $b \in F$, a contradiction. Thus, every prime filter of $\boldsymbol{A}$ is meet irreducible in $\mathsf{Fi}(\boldsymbol{A})$.

Conversely, suppose that $F$ is meet irreducible in $\mathsf{Fi}(\boldsymbol{A})$. Then $F$ is not the greatest element of $\mathsf{Fi}(\boldsymbol{A})$, that is, $F \neq A$. Therefore, $F$ is proper. To prove that it is prime, consider $a, b \in A$ such that $a \vee b \in F$. We will show that

$$F = \mathrm{Fg}^{\boldsymbol{A}}(F \cup \{a\}) \cap \mathrm{Fg}^{\boldsymbol{A}}(F \cup \{b\}). \tag{2.7}$$

The inclusion from left to right is obvious. To prove the other one, consider $c \in \mathrm{Fg}^{\boldsymbol{A}}(F \cup \{a\}) \cap \mathrm{Fg}^{\boldsymbol{A}}(F \cup \{b\})$. By the description of filter generation there are $d \in F$ and $e \in F$ such that $a \wedge d \leqslant c$ and $b \wedge e \leqslant c$. Consider the element $g := d \wedge e$. Since $F$ is closed under binary meets, we have $g \in F$. Furthermore, $a \wedge g, b \wedge g \leqslant c$. Consequently, $(a \wedge g) \vee (b \wedge g) \leqslant c$. Applying the distributive laws, we obtain

$$(a \vee b) \wedge g = (a \wedge g) \vee (b \wedge g) \leqslant c.$$

Since $g, a \vee b \in F$ and $F$ is closed under binary meets, we have $(a \vee b) \wedge g \in F$. Therefore, the above inequality implies that $c \in F$ because $F$ is an upset. This establishes condition (2.7).

Since $F$ is meet irreducible in $\mathsf{Fi}(\boldsymbol{A})$, we conclude that either $F = \mathrm{Fg}^{\boldsymbol{A}}(F \cup \{a\})$ or $F = \mathrm{Fg}^{\boldsymbol{A}}(F \cup \{b\})$. Consequently, either $a \in F$ or $b \in F$. ⊠

The importance of prime filters and ideals derives from the following result.

**Prime Filter Theorem.** *In a bounded distributive lattice, every filter is an intersection of prime filters.*

*Proof.* Let $\boldsymbol{A}$ be a bounded distributive lattice and recall that $\mathsf{Fi}(\boldsymbol{A})$ is a complete lattice in which meets are intersections. Since $\mathsf{Fi}(\boldsymbol{A})$ is an inductive closure system by Proposition 2.5, it is also an algebraic lattice by Theorem 1.56. Recall that in an algebraic lattice every element is a meet of (completely) meet irreducible ones. Since meets in $\mathsf{Fi}(\boldsymbol{A})$ are intersection, this means that every filter of $\boldsymbol{A}$ (that is, every element of $\mathsf{Fi}(\boldsymbol{A})$) is an intersection of meet irreducible elements of $\mathsf{Fi}(\boldsymbol{A})$. But the latter are precisely the prime filters of $\boldsymbol{A}$ by Proposition 2.13. Hence, we conclude that every filter of $\boldsymbol{A}$ is an intersection of prime filters. ⊠

By applying the Duality Principle to the Prime Filter Theorem we obtain that also every ideal of a bounded distributive lattice is an intersection of prime ideals.

**Corollary 2.14.** *Let $\boldsymbol{A}$ be a bounded distributive lattice and $a, b \in A$. If $a \nleqslant b$, then there exists a prime filter $F$ such that $a \in F$ and $b \notin F$.*

*Proof.* Since $a \nleqslant b$, the element $b$ does not belong to the filter $\uparrow a$. Therefore, we can apply the Prime Filter Theorem obtaining a prime filter $F$ that extends $\uparrow a$ and does not contain $b$. ⊠

Two algebras are called *similar* when they have the same type. The following concept allows us to compare similar algebras:

**Definition 2.15.** Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be similar algebras. A map $h \colon \boldsymbol{A} \to \boldsymbol{B}$ is said to be

(i) a *homomorphism* when it *preserves the basic operations* in the sense that $h(c^{\boldsymbol{A}}) = c^{\boldsymbol{B}}$ for every constant $c$ and

$$h(f^{\boldsymbol{A}}(a_1, \dots, a_n)) = f^{\boldsymbol{B}}(h(a_1), \dots, h(a_n)),$$

for every operation $f$ of positive arity $n$ and every $a_1, \dots, a_n \in A$;

(ii) an *embedding* when it is an injective homomorphism;

(iii) an *isomorphism* when it is a surjective embedding.

When $h \colon \boldsymbol{A} \to \boldsymbol{B}$ is an isomorphism, the inverse map $h^{-1} \colon \boldsymbol{B} \to \boldsymbol{A}$ is also an isomorphism. Accordingly, we write $\boldsymbol{A} \cong \boldsymbol{B}$ to indicate that there is an isomorphism between $\boldsymbol{A}$ and $\boldsymbol{B}$. Similarly, when there exists an embedding $h \colon \boldsymbol{A} \to \boldsymbol{B}$, we say that $\boldsymbol{A}$ *embeds* into $\boldsymbol{B}$.

Notably, homomorphisms $h \colon \boldsymbol{A} \to \boldsymbol{B}$ *preserve arbitrary terms* in the sense that for every term $\varphi(x_1, \dots, x_n)$ of the common type and $a_1, \dots, a_n \in A$, we have

$$h(\varphi^{\boldsymbol{A}}(a_1, \dots, a_n)) = \varphi^{\boldsymbol{B}}(h(a_1), \dots, h(a_n)).$$

As we mentioned, we view bounded lattices as algebras whose type comprises the operations of binary meets and joins and two constants, one for the least and one for the greatest element. Accordingly, a map $h\colon \boldsymbol{A} \to \boldsymbol{B}$ between two bounded lattices $\boldsymbol{A}$ and $\boldsymbol{B}$ is a homomorphism when for every $a, b \in A$,

$$h(a \wedge^{\boldsymbol{A}} b) = h(a) \wedge^{\boldsymbol{B}} h(b) \quad h(a \vee^{\boldsymbol{A}} b) = h(a) \vee^{\boldsymbol{B}} h(b) \quad h(0^{\boldsymbol{A}}) = 0^{\boldsymbol{B}} \quad h(1^{\boldsymbol{A}}) = 1^{\boldsymbol{B}}.$$

Every homomorphism $h\colon \boldsymbol{A} \to \boldsymbol{B}$ between bounded lattices is order preserving. To prove this, consider $a, b \in A$ such that $a \leqslant^{\boldsymbol{A}} b$. Then $a = a \wedge^{\boldsymbol{A}} b$. Since $h$ is a homomorphism, this yields

$$h(a) = h(a \wedge^{\boldsymbol{A}} b) = h(a) \wedge^{\boldsymbol{B}} h(b),$$

which amounts to $h(a) \leqslant^{\boldsymbol{B}} h(b)$. Similarly, every embedding $h\colon \boldsymbol{A} \to \boldsymbol{B}$ is also an order embedding. For let $a, b \in A$ be such that $h(a) \leqslant^{\boldsymbol{B}} h(b)$. Then $h(a) = h(a) \wedge^{\boldsymbol{B}} h(b)$. Since $h$ is a homomorphism, this yields

$$h(a) = h(a) \wedge^{\boldsymbol{B}} h(b) = h(a \wedge^{\boldsymbol{A}} b).$$

As $h$ is injective, we conclude that $a = a \wedge^{\boldsymbol{A}} b$, that is, $a \leqslant^{\boldsymbol{A}} b$. Lastly, while order preserving maps (resp. order embeddings) $h\colon \boldsymbol{A} \to \boldsymbol{B}$ need not be homomorphisms (resp. embeddings), it is easy to see that a map $h\colon \boldsymbol{A} \to \boldsymbol{B}$ is an isomorphism iff it is an order isomorphism.

We are now ready to prove the representation theorem for bounded distributive lattices. To this end, recall that for every poset $\mathbb{X}$ the algebra $\langle \mathsf{Up}(\mathbb{X}); \cap, \cup, \emptyset, X \rangle$ is a bounded distributive lattice.

**Birkhoff's Representation Theorem.** *Let $\boldsymbol{A}$ be a bounded distributive lattice and $\mathbb{X} := \langle \mathsf{Pf}(\boldsymbol{A}); \subseteq \rangle$. Then the map $\varepsilon\colon \boldsymbol{A} \to \mathsf{Up}(\mathbb{X})$ defined by the rule*

$$\varepsilon(a) := \{F \in \mathsf{Pf}(\boldsymbol{A}) : a \in F\}$$

*is an embedding of $\boldsymbol{A}$ into $\langle \mathsf{Up}(\mathbb{X}); \cap, \cup, \emptyset, X \rangle$.*

*Proof.* We begin by showing that the map $\varepsilon$ is well defined. To this end, consider $a \in A$. We need to prove that $\varepsilon(a)$ is an upset of $\mathbb{X}$. Accordingly, let $F \in \varepsilon(a)$ and $G \in \mathsf{Pf}(\boldsymbol{A})$ be such that $F \subseteq G$. From $F \in \varepsilon(a)$ and the definition of $\varepsilon$ it follows that $a \in F$. As $F \subseteq G$, this yields $a \in G$. Therefore, the definition of $\varepsilon$ guarantees that $G \in \varepsilon(a)$ as desired.

To prove that $\varepsilon\colon \boldsymbol{A} \to \mathsf{Up}(\mathbb{X})$ is a homomorphism, consider $a, b \in A$. We have

$$\varepsilon(a \wedge b) = \{F \in \mathsf{Pf}(\boldsymbol{A}) : a \wedge b \in F\} = \{F \in \mathsf{Pf}(\boldsymbol{A}) : a, b \in F\} = \varepsilon(a) \cap \varepsilon(b),$$

where the first and the third equalities hold by the definition of $\varepsilon$ and the second because filters are closed under binary meets. Similarly,

$$\varepsilon(a \vee b) = \{F \in \mathsf{Pf}(\boldsymbol{A}) : a \vee b \in F\} = \{F \in \mathsf{Pf}(\boldsymbol{A}) : a \in F \text{ or } b \in F\} = \varepsilon(a) \cup \varepsilon(b),$$

where the first and the third equalities hold by the definition of $\varepsilon$. To prove the second, consider $F \in \mathsf{Pf}(\boldsymbol{A})$. First suppose that $a \vee b \in F$. Since $F$ is prime, either $a \in F$ or

$b \in F$. Conversely, assume that $F$ contains either $a$ or $b$. As $F$ is an upset, this implies that $a \vee b \in F$ as desired.

Lastly, since prime filters are proper and every proper filter does not contain the least element 0, we have

$$\varepsilon(0) = \{F \in \mathsf{Pf}(\boldsymbol{A}) : 0 \in F\} = \emptyset.$$

Similarly, as every filter contains the greatest element 1 of $\boldsymbol{A}$, we have

$$\varepsilon(1) = \{F \in \mathsf{Pf}(\boldsymbol{A}) : 1 \in F\} = \mathsf{Pf}(\boldsymbol{A}) = X.$$

Thus, $\varepsilon\colon \boldsymbol{A} \to \mathsf{Up}(\mathbb{X})$ is a homomorphism.

To prove that it is injective, consider two distinct $a, b \in A$. We may assume without loss of generality that $a \nleqslant b$. From Corollary 2.14 it follows that there exists $F \in \mathsf{Pf}(\boldsymbol{A})$ such that $a \in F$ and $b \notin F$. By the definition of $\varepsilon$ this amounts to $F \in \varepsilon(a)$ and $F \notin \varepsilon(b)$. Hence, we conclude that $\varepsilon(a) \neq \varepsilon(b)$. ⊠

The embedding $\varepsilon\colon \boldsymbol{A} \to \mathsf{Up}(\mathbb{X})$ in Birkhoff's Representation Theorem need not be an isomorphism, however. This is because the bounded distributive lattice $\boldsymbol{A}$ need not be complete, while $\mathsf{Up}(\mathbb{X})$ is always complete. However, when $\boldsymbol{A}$ is finite the map $\varepsilon$ turns out to be an isomorphism, as we proceed to explain.

**Corollary 2.16.** *Let $\boldsymbol{A}$ be a finite distributive lattice and $\mathbb{X} := \langle \mathsf{Pf}(\boldsymbol{A}); \subseteq \rangle$. Then the map $\varepsilon\colon \boldsymbol{A} \to \mathsf{Up}(\mathbb{X})$ is an isomorphism from $\boldsymbol{A} = \langle A; \wedge, \vee, 0, 1\rangle$ to $\langle \mathsf{Up}(\mathbb{X}); \cap, \cup, \emptyset, X\rangle$.*

*Proof.* In view of Birkhoff's Representation Theorem, it suffices to show that $\varepsilon$ is surjective. To this end, consider $U \in \mathsf{Up}(\mathbb{X})$. If $U = \emptyset$, then $\varepsilon(0) = U$ because $\varepsilon$ is a homomorphism from $\boldsymbol{A}$ to $\mathsf{Up}(\mathbb{X})$. Then we consider the case where $U$ is nonempty. As $\boldsymbol{A}$ is finite, we can take an enumeration $U = \{F_1, \dots, F_n\}$. Moreover, since $\boldsymbol{A}$ is finite, each filter $F_i$ is principal by Proposition 2.6. Therefore, there are $a_1, \dots, a_n \in A$ such that $F_i = {\uparrow}a_i$ for every $i \leqslant n$. We will prove that $U = \varepsilon(b)$, where

$$b := a_1 \vee \cdots \vee a_n.$$

First consider $F_i \in U$. Since $F_i = {\uparrow}a_i$ and $a_i \leqslant a_1 \vee \cdots \vee a_n = b$, we obtain $b \in F_i$. By the definition of $\varepsilon$, this yields $F_i \in \varepsilon(b)$. To prove the other inclusion, consider $F \in \varepsilon(b)$. By the definition of $\varepsilon$ we have $a_1 \vee \cdots \vee a_n = b \in F$. Since $F$ is a prime filter, there exists $i \leqslant n$ such that $a_i \in F$. As $F$ is an upset of $\boldsymbol{A}$, this yields $F_i = {\uparrow}a_i \subseteq F$. Since $U$ is an upset of $\mathbb{X} = \langle \mathsf{Pf}(\boldsymbol{A}); \subseteq\rangle$ containing $F_i$ and $F \in \mathsf{Pf}(\boldsymbol{A})$, we conclude that $F \in U$. ⊠

**Corollary 2.17.** *Up to isomorphism, the finite distributive lattices are precisely the algebras of the form $\langle \mathsf{Up}(\mathbb{X}); \cap, \cup, \emptyset, X\rangle$ where $\mathbb{X}$ is a finite poset.*

## 2.2 Boolean and Heyting algebras

The following kind of lattices are the algebraic models of one of the most prominent nonclassical logics, namely, *intuitionistic logic*.

**Definition 2.18.** A *Heyting algebra* is an algebra $\boldsymbol{A} = \langle A; \wedge, \vee, \to, 0, 1\rangle$ that comprises a bounded lattice $\langle A; \wedge, \vee, 0, 1\rangle$ and a binary operation $\to$ satisfying the *residuation law*: for every $a, b, c \in A$,

$$a \wedge b \leqslant c \Longleftrightarrow a \leqslant b \to c.$$

The operation $\to$ is often called *implication*.

Examples of Heyting algebras abound. For instance, every topological space gives rise to a Heyting algebra, as we proceed to explain.

**Example 2.19** (Heyting algebras of open sets)**.** Let $\langle X; \tau\rangle$ be a topological space. As the family of open sets of $\langle X; \tau\rangle$ contains the empty set and is closed under arbitrary unions, the greatest open set contained in a subset $U \subseteq X$, known as the *interior* of $U$, exists and can be described as follows:

$$\mathsf{int}(U) \coloneqq \bigcup\{V \in \tau : V \subseteq U\}.$$

Then the family of open sets $\tau$ can be given the structure of an algebra

$$\mathsf{Op}(X; \tau) \coloneqq \langle \tau; \cap, \cup, \to, \emptyset, X\rangle$$

in which $\to$ is the binary operation defined for every $U, V \in \tau$ as

$$U \to V \coloneqq \mathsf{int}(U^c \cup V).$$

We will prove that $\mathsf{Op}(X; \tau)$ is a Heyting algebra.

First observe that the algebra $\mathsf{Op}(X; \tau)$ is well defined because $\tau$ is closed under binary intersections and unions, it contains $\emptyset$ and $X$, and the interior $\mathsf{int}(U^c \cup V)$ is always open. Since $\langle \tau; \cap, \cup, \emptyset, X\rangle$ is a obviously a bounded lattice, it only remains to prove that the residuation law holds, which in this case takes the following form:

$$U \cap V \subseteq W \Longleftrightarrow U \subseteq \mathsf{int}(V^c \cup W),$$

for every $U, V, W \in \tau$. To prove the implication from left to right, suppose that $U \cap V \subseteq W$. Then $U \subseteq V^c \cup W$. As $\mathsf{int}(V^c \cup W)$ is the greatest open set contained in $V^c \cup W$ and $U$ is an open set, this implies $U \subseteq \mathsf{int}(V^c \cup W)$. Then we turn to prove the other implication. Suppose that $U \subseteq \mathsf{int}(V^c \cup W)$ and consider $x \in U \cap V$. From

$$x \in U \subseteq \mathsf{int}(V^c \cup W) \subseteq V^c \cup W \;\text{ and }\; x \in V$$

it follows $x \in W$. Hence, we conclude that $U \cap V \subseteq W$. ⊠

While the definition of a Heyting algebra does not make this explicit, the residuation law implies the distributive laws.

**Proposition 2.20.** *Every Heyting algebra is a distributive lattice.*

*Proof.* By Proposition 2.2 it suffices to show that the inequality

$$x \wedge (y \vee z) \leqslant (x \wedge y) \vee (x \wedge z)$$

is valid in every Heyting algebra $\boldsymbol{A}$. Accordingly, consider $a, b, c \in A$ and let

$$d := (a \wedge b) \vee (a \wedge c).$$

From $(a \wedge b) \vee (a \wedge c) \leqslant d$ it follows $a \wedge b, a \wedge c \leqslant d$. By the residuation law this yields $b, c \leqslant a \to d$ and, therefore, $b \vee c \leqslant a \to d$. With another application of the residuation law we obtain $a \wedge (b \vee c) \leqslant d$. By the definition of $d$ this amounts to $a \wedge (b \vee c) \leqslant (a \wedge b) \vee (a \wedge c)$. ⊠

Given a poset $\mathbb{X}$ and a subset $Y \subseteq X$, we write $\max Y$ and $\min Y$ for the greatest and the least element of $Y$ (when they exist, of course). The next result provides a purely lattice theoretic description of the implication of a Heyting algebra.

**Proposition 2.21.** *If $\boldsymbol{A}$ is a Heyting algebra, then for every $b, c \in A$,*

$$b \to c = \max\{a \in A : a \wedge b \leqslant c\}.$$

*Furthermore, if $\boldsymbol{A}$ is a bounded lattice in which $\max\{a \in A : a \wedge b \leqslant c\}$ exists for every $b, c \in A$, then the expansion of $\boldsymbol{A}$ with the binary operation $\to$ defined by the rule*

$$b \to c := \max\{a \in A : a \wedge b \leqslant c\}$$

*is a Heyting algebra.*

*Proof.* First let $\boldsymbol{A}$ be a Heyting algebra and $b, c \in A$. From the residuation law and $b \to c \leqslant b \to c$ it follows $(b \to c) \wedge b \leqslant c$. Consequently,

$$b \to c \in \{a \in A : a \wedge b \leqslant c\}.$$

To prove that $b \to c$ is the greatest element of $\{a \in A : a \wedge b \leqslant c\}$, consider some $a \in A$ such that $a \wedge b \leqslant c$. By the residuation law we have $a \leqslant b \to c$ as desired.

Then consider a bounded lattice $\boldsymbol{A}$ in which $\max\{a \in A : a \wedge b \leqslant c\}$ exists for every $b, c \in A$. We will prove that for every $a, b, c \in A$,

$$a \wedge b \leqslant c \Longleftrightarrow a \in \{d \in A : d \wedge b \leqslant c\} \Longleftrightarrow a \leqslant \max\{d \in A : d \wedge b \leqslant c\}.$$

The first of the equivalences above is straightforward and so is the implication from left to right of the second. The implication from right to left in the second equivalence above holds too because $\{d \in A : d \wedge b \leqslant c\}$ is a downset. Therefore, letting

$$b \to c := \max\{d \in A : d \wedge b \leqslant c\},$$

the above series of equivalences amounts to the validity of the residuation law $a \wedge b \leqslant c$ iff $a \leqslant b \to c$. ⊠

From the first half of Proposition 2.21 it follows that Heyting algebras are uniquely determined by their lattice structure. More precisely, given a Heyting algebra $\boldsymbol{A} = \langle A; \wedge, \vee, \to, 0, 1\rangle$, we will refer to $\langle A; \wedge, \vee\rangle$ as to the *lattice reduct* of $\boldsymbol{A}$. We then have the following:

**Corollary 2.22.** *Two Heyting algebras are equal iff they have the same lattice reduct.*

The second half of Proposition 2.21 is instrumental in constructing examples of Heyting algebras. As an exemplification, we remark that each distributive algebraic lattice gives rise to a Heyting algebra.

**Theorem 2.23.** *Every distributive algebraic lattice can be expanded with a binary operation $\to$ that turns it into a Heyting algebra.*

*Proof.* Let $\boldsymbol{A}$ be an algebraic lattice. In view of the second half of Proposition 2.21, it suffices to prove that

$$\max\{a \in A : a \wedge b \leqslant c\}$$

exists for every $b, c \in A$. Accordingly, consider $b, c \in A$. Since $\boldsymbol{A}$ is a complete lattice, the join

$$d := \bigvee \{a \in A : a \wedge b \leqslant c\}$$

exists. As $d$ is an upper bound of $\{a \in A : a \wedge b \leqslant c\}$ by definition, to prove that $d = \max\{a \in A : a \wedge b \leqslant c\}$ it is enough to show that $d$ belongs to $\{a \in A : a \wedge b \leqslant c\}$, that is, $d \wedge b \leqslant c$.

As $\boldsymbol{A}$ is an algebraic lattice, every element is the join of the compact elements below it. Consequently, $d \wedge b \leqslant c$ iff all the compact elements below $d \wedge b$ are also below $c$. To prove that this is the case, let $e$ be a compact element such that $e \leqslant d \wedge b$. By the definition of $d$ we obtain

$$e \leqslant b \wedge d = b \wedge \bigvee \{a \in A : a \wedge b \leqslant c\} \leqslant \bigvee \{a \in A : a \wedge b \leqslant c\}.$$

Since $e$ is compact, there exists a finite $X \subseteq \{a \in A : a \wedge b \leqslant c\}$ such that $e \leqslant \bigvee X$. As $\{a \in A : a \wedge b \leqslant c\}$ contains the least element 0, we may assume that $0 \in X$ and, therefore, that of $X$ is nonempty. Then take an enumeration $X = \{a_1, \dots, a_n\}$. We have

$$e \leqslant \bigvee X = a_1 \vee \cdots \vee a_n.$$

Together with the assumption that $e \leqslant b \wedge d \leqslant b$ and that $\boldsymbol{A}$ is distributive, this yields

$$e \leqslant b \wedge (a_1 \vee \cdots \vee a_n) = (b \wedge a_1) \vee \cdots \vee (b \wedge a_n).$$

As each $a_i$ belongs to $\{a \in A : a \wedge b \leqslant c\}$, we have $a_i \wedge b \leqslant c$ and, therefore, $(b \wedge a_1) \vee \cdots \vee (b \wedge a_n) \leqslant c$. By the inequalities above, this implies $e \leqslant c$. ⊠

As every finite lattice is algebraic, from Theorem 2.23 we obtain:

**Corollary 2.24.** *Every finite distributive lattice can be expanded with a binary operation $\to$ that turns it into a Heyting algebra.*

Furthermore, the expansions mentioned in Theorem 2.23 and in the above result are unique, because Heyting algebras are uniquely determined by their lattice reducts (Corollary 2.22).

Recall that the lattices of the form $\mathsf{Up}(\mathbb{X})$ are algebraic and distributive. Therefore, they can be endowed with the structure of a Heyting algebra by Theorem 2.23. The next example provides a concrete description of the implication in the Heyting algebras of this kind.

**Example 2.25** (Heyting algebras of upsets)**.** We will prove that for every poset $\mathbb{X}$ the algebra

$$\mathsf{Up}(\mathbb{X}) := \langle \mathsf{Up}(\mathbb{X}); \cap, \cup, \to, \emptyset, X \rangle$$

where $\to$ is the operation defined for every $U, V \in \mathsf{Up}(\mathbb{X})$ as

$$U \to V := X - {\downarrow}(U \cap V^c)$$

is a Heyting algebra.

Observe that $\mathsf{Up}(\mathbb{X})$ is well defined, because $X - {\downarrow}(U \cap V^c)$ is always an upset. As $\mathsf{Up}(\mathbb{X})$ is a bounded lattice, it only remains to prove that $\mathsf{Up}(\mathbb{X})$ satisfies the residuation law, that is,

$$U \cap V \subseteq W \iff U \subseteq X - {\downarrow}(V \cap W^c),$$

for every $U, V, W \in \mathsf{Up}(\mathbb{X})$. To prove the left to right implication, we reason by contraposition. Suppose that $U \nsubseteq X - {\downarrow}(V \cap W^c)$. Then there exists $x \in U$ such that $x \notin X - {\downarrow}(V \cap W^c)$. Then $x \in {\downarrow}(V \cap W^c)$ and there exists $y \in V \cap W^c$ such that $x \leqslant y$. Since $U$ is an upset, from $x \leqslant y$ and $x \in U$ it follows that $y \in U$. Consequently, $y \in U \cap V$ and $y \notin W$. Hence, we conclude that $U \cap V \nsubseteq W$ as desired. We will prove the implication from right to left by contraposition too. Accordingly, suppose that $U \cap V \nsubseteq W$. Then there is some $x \in U \cap V$ such that $x \notin W$. As a consequence, $x \in V \cap W^c \subseteq {\downarrow}(V \cap W^c)$. Therefore, $x \notin X - {\downarrow}(V \cap W^c)$. Since $x \in U$, we conclude that $U \nsubseteq X - {\downarrow}(U \cap V^c)$. ⊠

The next result shows that $\mathsf{Up}(\mathbb{X})$ is the canonical example of a Heyting algebra.

**Theorem 2.26** (Representation theorem)**.** *Let $\boldsymbol{A}$ be a Heyting algebra and $\mathbb{X} := \langle \mathsf{Pf}(\boldsymbol{A}); \subseteq \rangle$. Then the map $\varepsilon \colon \boldsymbol{A} \to \mathsf{Up}(\mathbb{X})$ defined by the rule*

$$\varepsilon(a) := \{F \in \mathsf{Pf}(\boldsymbol{A}) : a \in F\}$$

*is an embedding of $\boldsymbol{A}$ into $\langle \mathsf{Up}(\mathbb{X}); \cap, \cup, \to, \emptyset, X \rangle$.*

*Proof.* Recall from Proposition 2.20 that $\boldsymbol{A}$ is a distributive lattice. Therefore, we can apply Birkhoff's Representation Theorem obtaining that the map $\varepsilon$ is well-defined embedding of bounded lattices. It only remains to prove that it preserve the implication. Since in $\mathsf{Up}(\mathbb{X})$ the implication is defined by the rule $U \to V := X - {\downarrow}(U \cap V^c)$, this amounts to showing that for every $a, b \in A$,

$$\varepsilon(a \to b) = X - {\downarrow}(\varepsilon(a) \cap \varepsilon(b)^c).$$

To prove the inclusion from left to right, let $F \in \varepsilon(a \to b)$. By the definition of $\varepsilon$ we have $a \to b \in F$. Then consider $G \in \varepsilon(a) \cap \varepsilon(b)^c$. We need to show that $F \nsubseteq G$. Suppose the contrary, with a view to contradiction. Then $a \to b \in F \subseteq G$. As $G$ is a filter containing $a$, this implies $a \wedge (a \to b) \in G$. As $G$ is an upset and the residuation law guarantees that $a \wedge (a \to b) \leqslant b$, we conclude that $b \in G$, a contradiction. This shows that $F \nsubseteq G$ for every $G \in \varepsilon(a) \cap \varepsilon(b)^c$ and, therefore, that $F \in X - {\downarrow}(\varepsilon(a) \cap \varepsilon(b)^c)$ as desired.

The inclusion from right to left will be proved by contraposition. To this end, consider $F \in X - \varepsilon(a \to b)$. Then $F$ is a prime filter of $\boldsymbol{A}$ such that $a \to b \notin F$. We will prove that the filter

$$G \coloneqq \mathrm{Fg}^{\boldsymbol{A}}(F \cup \{a\})$$

does not contain $b$. Suppose, on the contrary, that $b \in G$. By the description pf filter generation there is $c \in F$ such that $c \wedge a \leqslant b$. By applying the residuation law we obtain $c \leqslant a \to b$. Since $c \in F$ and $F$ is a filter, this implies $a \to b \in F$, a contradiction. Therefore, we conclude that $b \notin G$.

By the Prime Filter Theorem the filter $G$ is an intersection of prime filters. Together with $b \notin G$, this guarantees the existence of a prime filter $H$ containing $G$ such that $b \notin H$. As the definition of $G$ ensures that $a \in G$, we obtain $a \in H$ as well. Therefore, $H$ is a prime filter such that $a \in H$ and $b \notin H$. By the definition of $\varepsilon$ we obtain $H \in \varepsilon(a) \cap \varepsilon(b)^c$. As $G$ extends $F$ by the definition, we also have $F \subseteq H$. Therefore, $H \in {\downarrow}(\varepsilon(a) \cap \varepsilon(b)^c)$ as desired. ⊠

When the Heyting algebra $\boldsymbol{A}$ in the above theorem is finite, the embedding $\varepsilon\colon \boldsymbol{A} \to \mathsf{Up}(\mathbb{X})$ is a surjective and, therefore, an isomorphism by Corollary 2.16. As a consequence, we obtain the following:

**Corollary 2.27.** *Up to isomorphism, the finite Heyting algebras are precisely the algebras of the form $\langle \mathsf{Up}(\mathbb{X}); \cap, \cup, \to, \varnothing, X\rangle$ where $\mathbb{X}$ is a finite poset and $\to$ is the binary operation defined by the rule*

$$U \to V \coloneqq X - {\downarrow}(U \cap V^c).$$

Given an element $a$ of a Heyting algebra $\boldsymbol{A}$, we often write $\neg a$ as a shorthand for $a \to 0$. The element $\neg a$ will be called the *negation* of $a$. From the first half of Proposition 2.21 it follows that $\neg a$ is the greatest element of $\boldsymbol{A}$ whose meet with $a$ is 0, that is,

$$\neg a = \max\{b \in A : a \wedge b = 0\}. \tag{2.8}$$

By extension, given a term $\varphi$ in the language of Heyting algebras, we shall also write $\neg\varphi$ as a shorthand for $\varphi \to 0$.

**Definition 2.28.** A Heyting algebra is said to be a *Boolean algebra* when it validates the *excluded middle law* $x \vee \neg x \approx 1$.

While Heyting algebras are the algebraic models of intuitionistic logic, Boolean algebras are the algebraic models of *classical logic*. Boolean algebras can be characterized in terms of many logically meaningful equations. Among them, the *double negation elimination law* $x \approx \neg\neg x$ plays a fundamental role. Observe that every Heyting algebra $\boldsymbol{A}$ validates one half of it, in the sense that $a \leqslant \neg\neg a$ for every $a \in A$. This is because the residuation law yields that

$$a \leqslant \neg\neg a \iff a \wedge \neg a \leqslant 0$$

and the right hand side of the above equivalence holds by condition (2.8).

**Proposition 2.29.** *A Heyting algebra is a Boolean algebra iff it validates the double negation elimination law.*

*Proof.* Let $\boldsymbol{A}$ be a Heyting algebra. Suppose first that $\boldsymbol{A}$ is a Boolean algebra. Since one half of the double negation elimination law holds in every Heyting algebra, it suffices to show that $\neg\neg a \leqslant a$ for every $a \in A$. To this end, observe that for every $a \in A$,

$$\neg\neg a = 1 \wedge \neg\neg a = (a \vee \neg a) \wedge \neg\neg a = (a \wedge \neg\neg a) \vee (\neg a \wedge \neg\neg a) = a \vee 0 = a.$$

The above equalities are justified as follows: the first holds because 1 is the greatest element of $\boldsymbol{A}$, the second because $\boldsymbol{A}$ is a Boolean algebra and, therefore, validates the excluded middle law, the third is obtained by an application of the distributive laws, the fourth holds because $a \leqslant \neg\neg a$ and by condition (2.8) we have $\neg a \wedge \neg\neg a = 0$, and the last one because 0 is the least element of $\boldsymbol{A}$.

Conversely, suppose that the Heyting algebra $\boldsymbol{A}$ validates the double negation elimination law. We need to prove that it validates the excluded middle law as well. Consider an element $a \in A$. We begin by showing that $\neg(a \vee \neg a) = 0$. Since $\neg(a \vee \neg a)$ is the greatest element whose meet with $a \vee \neg a$ is 0 by condition (2.8), it suffices to prove that for every $b \in A$,

$$\text{if } b \wedge (a \vee \neg a) = 0, \text{ then } b = 0.$$

Accordingly, suppose that $b \wedge (a \vee \neg a) = 0$. By the distributive laws we obtain $(b \wedge a) \vee (b \wedge \neg a) = 0$. As 0 is the least element of $\boldsymbol{A}$, this amounts to $b \wedge a, b \wedge \neg a \leqslant 0$. Therefore, we can apply the residuation law obtaining $b \leqslant \neg a \wedge \neg\neg a$. Since $a \wedge \neg\neg a = 0$ by condition (2.8), we conclude that $b = 0$ as desired. This shows that $\neg(a \vee \neg a) = 0$.

Clearly, 1 is the greatest element whose meet with 0 is 0 and, therefore, $\neg 0 = 1$. Consequently, $\neg\neg(a \vee \neg a) = \neg 0 = 1$. By the double negation elimination law we conclude that $a \vee \neg a = 1$. ⊠

The primary example of a Boolean algebra is the following:

**Example 2.30** (Powerset Boolean algebras)**.** The *powerset Boolean algebra* associated with a set $X$ is the algebra

$$\mathcal{P}(X) := \langle \mathcal{P}(X); \cap, \cup, \to, \emptyset, X \rangle$$

where $\to$ is the binary operation defined by the rule

$$U \to V := U^c \cup V.$$

We will prove that $\mathcal{P}(X)$ is indeed a Boolean algebra and that it coincides with the Heyting algebra $\mathsf{Up}(\mathbb{X})$ where $\mathbb{X}$ is the discrete poset with universe $X$.

First observe that the upsets of the discrete poset $\mathbb{X}$ are precisely the subsets of $X$. Moreover, for every $U, V \in \mathcal{P}(X)$ we have

$$X - {\downarrow}(U \cap V^c) = X - (U \cap V^c) = U^c \cup V,$$

where the second equality follows from the fact that the order of $\mathbb{X}$ is the identity relation and, therefore, ${\downarrow}(U \cap V^c) = U \cap V^c$. Together with the definition of the implication in $\mathsf{Up}(\mathbb{X})$ and $\mathcal{P}(X)$, this implies that the the powerset Boolean algebra $\mathcal{P}(X)$ coincides with the Heyting algebra $\mathsf{Up}(\mathbb{X})$.

Therefore, to prove that $\mathcal{P}(X)$ is a Boolean algebra, it only remains to show that it validates the excluded middle law. To this end, it is convenient to observe that for every $U \in \mathcal{P}(X)$,

$$\neg U = U \to \emptyset = U^c \cup \emptyset = U^c.$$

As a consequence,

$$U \cup \neg U = U \cup U^c = X$$

for every $U \in \mathcal{P}(X)$, which means precisely that $\mathcal{P}(X)$ validates the excluded middle law. ⊠

The representation theory of Boolean algebras is based on the following concept:

**Definition 2.31.** A proper filter $F$ of a Heyting algebra $\boldsymbol{A}$ is said to be an *ultrafilter* when either $a \in F$ or $\neg a \in F$, for every $a \in A$.

We say that a subset $F \subseteq A$ is a *maximal proper filter* of a Heyting algebra $\boldsymbol{A}$ when $F$ is a proper filter of $\boldsymbol{A}$ and for every other proper filter $G$ of $\boldsymbol{A}$, if $F \subseteq G$, then $F = G$.

**Proposition 2.32.** *The ultrafilters of a Heyting algebra are precisely its maximal proper filters.*

*Proof.* Let $\boldsymbol{A}$ be a Heyting algebra. First consider an ultrafilter $F$ of $\boldsymbol{A}$ and let $G$ be a proper filter of $\boldsymbol{A}$ such that $F \subseteq G$. We have to prove that $F = G$. Suppose, on the contrary, that there exists some $a \in G - F$. Since $F$ is an ultrafilter, from $a \notin F$ it follows that $\neg a \in F$. Consequently, $\neg a \in F \subseteq G$ and, therefore, $G$ contains both $a$ and $\neg a$. As $G$ is a filter, this implies $a \wedge \neg a \in G$. Since $\neg a$ is the greatest element of $\boldsymbol{A}$ whose meet with $a$ is 0 by condition (2.8), this yields $0 = a \wedge \neg a \in G$. It follows that $G$ is not proper, a contradiction.

To prove the converse, consider a maximal proper filter $F$ of $\boldsymbol{A}$ and consider an element $a \in A$. To prove that $F$ is an ultrafilter, we need to show that $a \in F$ or $\neg a \in F$. If $\neg a \in F$, we are done. Then we consider the case where $\neg a \notin F$. We will prove that the filter $G := \mathrm{Fg}^{\boldsymbol{A}}(F \cup \{a\})$ is proper. Suppose, on the contrary, that $G = A$. Then $0 \in \mathrm{Fg}^{\boldsymbol{A}}(F \cup \{a\})$. By the description of filter generation there exists $b \in F$ such that $a \wedge b \leqslant 0$. By applying the residuation law we obtain $b \leqslant a \to 0 = \neg a$. Since $F$ is a filter containing $b$, this implies $\neg a \in F$, a contradiction. Hence, we conclude that $G$ is proper. As $F \subseteq G$ by the definition of $G$, the assumption that $F$ is a maximal filter implies $F = G$. Since the definition of $G$ guarantees that $a \in G$, we conclude that $a \in F$. ⊠

From Proposition 2.32 it follows that the ultrafilters of a Heyting algebra $\boldsymbol{A}$ are meet irreducible in $\mathsf{Fi}(\boldsymbol{A})$. Since the meet irreducible elements of $\mathsf{Fi}(\boldsymbol{A})$ are precisely the prime filters of $\boldsymbol{A}$ by Proposition 2.13, we obtain the following:

**Corollary 2.33.** *Every ultrafilter is prime: if $\boldsymbol{A}$ is a Heyting algebra, then* $\mathsf{Ultr}(\boldsymbol{A}) \subseteq \mathsf{Pf}(\boldsymbol{A})$.

The reverse inclusion $\mathsf{Ultr}(\boldsymbol{A}) \supseteq \mathsf{Pf}(\boldsymbol{A})$ holds precisely when $\boldsymbol{A}$ is a Boolean algebra:

**Proposition 2.34.** *The following conditions are equivalent for a Heyting algebra $\boldsymbol{A}$:*

(i) *$\boldsymbol{A}$ is a Boolean algebra;*

(ii) $\mathsf{Pf}(\boldsymbol{A}) = \mathsf{Ultr}(\boldsymbol{A})$*;*

(iii) *The poset $\langle \mathsf{Pf}(\boldsymbol{A}); \subseteq \rangle$ is discrete.*

*Proof.* (i)⇒(ii): By Corollary 2.33 it suffices to prove that $\mathsf{Pf}(\boldsymbol{A}) \subseteq \mathsf{Ultr}(\boldsymbol{A})$. Let $F$ be a prime filter of $\boldsymbol{A}$. As $F$ is prime, it is also proper. Then consider an element $a \in A$. Since $\boldsymbol{A}$ validates the excluded middle law, we have $a \vee \neg a = 1 \in F$. Because $F$ is prime, this implies that either $a \in F$ or $\neg a \in F$ as desired.

(ii)⇒(iii): Since $\mathsf{Pf}(\boldsymbol{A}) = \mathsf{Ultr}(\boldsymbol{A})$, it suffices to prove that the poset $\langle \mathsf{Ultr}(\boldsymbol{A}); \subseteq \rangle$ is discrete. To this end, consider $F, G \in \mathsf{Ultr}(\boldsymbol{A})$ such that $F \subseteq G$. Since $F$ is a maximal proper filter by Proposition 2.32 and $G$ is a proper filter by assumption, we obtain $F = G$. Thus, the poset $\langle \mathsf{Ultr}(\boldsymbol{A}); \subseteq \rangle$ is discrete.

(iii)⇒(i): We need to prove that $\boldsymbol{A}$ validates the excluded middle law. Suppose the contrary, with a view to contradiction. Then there exists some $a \in A$ such that $a \vee \neg a < 1$. Since 1 is the greatest element of $\boldsymbol{A}$, this yields $1 \nleq a \vee \neg a$. Then we can apply Corollary 2.14 obtaining a prime filter $F$ such that $a \vee \neg a \notin F$. As $F$ is a prime filter implies that $a, \neg a \notin F$. Therefore, $F$ is not an ultrafilter of $\boldsymbol{A}$. Since $F$ is proper, we can apply Proposition 2.32 obtaining that there exists a prime filter $G$ such that $F \subsetneq G$. But this contradicts the assumption that the poset $\langle \mathsf{Pf}(\boldsymbol{A}); \subseteq \rangle$ is discrete. ⊠

Recall that every Boolean algebra $\boldsymbol{A}$ is a bounded distributive lattice by Proposition 2.20. Therefore, every filter of $\boldsymbol{A}$ is an intersection of prime filters by the Prime Filter Theorem. As the prime filters of $\boldsymbol{A}$ are precisely the ultrafilters of $\boldsymbol{A}$ by Proposition 2.34, we obtain the following:

**Ultrafilter Theorem.** *In a Boolean algebra, every filter is an intersection of ultrafilters.*

We are now ready to show that powerset Boolean algebras are indeed canonical examples of Boolean algebras.

**Theorem 2.35** (Representation theorem)**.** *Let $\boldsymbol{A}$ be a Boolean algebra and $X := \mathsf{Ultr}(\boldsymbol{A})$. Then the map $\varepsilon \colon \boldsymbol{A} \to \mathcal{P}(X)$ defined by the rule*

$$\varepsilon(a) := \{F \in \mathsf{Ultr}(\boldsymbol{A}) : a \in F\}$$

*is an embedding of $\boldsymbol{A}$ into the powerset Boolean algebra $\langle \mathcal{P}(X); \cap, \cup, \to, \emptyset, X \rangle$.*

*Proof.* Let $\mathbb{X} := \langle \mathsf{Pf}(\boldsymbol{A}); \subseteq \rangle$ and $\mathsf{Up}(\mathbb{X})$ the Heyting algebra of upsets of $\mathbb{X}$. Since $\boldsymbol{A}$ is a Heyting algebra, from Theorem 2.26 it follows that the map $\varepsilon^* \colon \boldsymbol{A} \to \mathsf{Up}(\mathbb{X})$ defined by the rule $\varepsilon^*(a) := \{F \in \mathsf{Pf}(\boldsymbol{A}) : a \in F\}$ is an embedding. As $\boldsymbol{A}$ is a Boolean algebra, the poset $\mathbb{X}$ is discrete and $\mathsf{Pf}(\boldsymbol{A}) = \mathsf{Ultr}(\boldsymbol{A})$ by Proposition 2.34. As $\mathsf{Pf}(\boldsymbol{A}) = \mathsf{Ultr}(\boldsymbol{A})$, the map $\varepsilon^*$ coincides with the map $\varepsilon$ in the statement. Furthermore, since the poset $\mathbb{X}$ is discrete, the Heyting algebra $\mathsf{Up}(\mathbb{X})$ coincides with the powerset Boolean algebra $\mathcal{P}(X)$. Consequently, the fact that $\varepsilon^* \colon \boldsymbol{A} \to \mathsf{Up}(\mathbb{X})$ is an embedding amounts to the fact that the map $\varepsilon$ in the statement is an embedding of $\boldsymbol{A}$ into $\mathcal{P}(X)$ as desired. ⊠

Since the map $\varepsilon$ in the above theorem coincides with that in Corollary 2.16, when a Boolean algebra $\boldsymbol{A}$ is finite, the embedding $\varepsilon\colon \boldsymbol{A} \to \mathcal{P}(X)$ is a surjective and, therefore, an isomorphism.

**Corollary 2.36.** *Up to isomorphism, the finite Boolean algebras are precisely the powerset Boolean algebras $\langle \mathcal{P}(X); \cap, \cup, \to, \varnothing, X\rangle$ where $X$ is a finite set.*

## 2.3 Modal algebras

The algebraic models of *modal logic* are the following:

**Definition 2.37.** A *modal algebra* is an algebra $\boldsymbol{A} = \langle A; \wedge, \vee, \to, \Box, 0, 1\rangle$ that comprises a Boolean algebra $\langle A; \wedge, \vee, \to, 0, 1\rangle$ and a unary operation $\Box$ such that for every $a, b \in A$,

$$\Box(a \wedge b) = \Box a \wedge \Box b \quad \text{and} \quad \Box 1 = 1.$$

In this case, for every $a \in A$ we will write $\Diamond a$ as a shorthand for $\neg\Box\neg a$.

The unary operations $\Box$ and $\Diamond$ are often called *box* and *diamond*.

Since Boolean algebras (and, therefore, modal algebras) validate the double negation elimination law, for every $a \in A$ we have

$$\Box a = \neg\neg\Box\neg\neg a = \neg\Diamond\neg a.$$

Therefore, box and diamond are interdefinable as

$$\Diamond a = \neg\Box\neg a \quad \text{and} \quad \Box a = \neg\Diamond\neg a.$$

In fact, modal algebras can be equivalently defined as algebra $\boldsymbol{A} = \langle A; \wedge, \vee, \to, \Diamond, 0, 1\rangle$ where $\langle A; \wedge, \vee, \to, 0, 1\rangle$ is a Boolean algebra and $\Diamond$ a unary operation such that for every $a, b \in A$,

$$\Diamond(a \vee b) = \Diamond a \vee \Diamond b \quad \text{and} \quad \Diamond 0 = 0.$$

Much as in the case of Heyting algebras, every topological space gives rise to a modal algebra, as we proceed to explain.

**Example 2.38** (Modal algebras and topological spaces)**.** With every topological space $\langle X; \tau\rangle$ we associate an algebra

$$\mathsf{M}(X; \tau) \coloneqq \langle \mathcal{P}(X); \cap, \cup, \to, \Box, \varnothing, X\rangle$$

where $\langle \mathcal{P}(X); \cap, \cup, \to, \varnothing, X\rangle$ is a powerset Boolean algebra and $\Box$ is the operation of taking the interior of a set, that is, for every $U \subseteq X$,

$$\Box U \coloneqq \mathsf{int}(U).$$

We will prove that $\mathsf{M}(X; \tau)$ is a modal algebra in which $\Diamond$ is the operation of taking the closure of a set, that is, $\Diamond U = \overline{U}$.

We begin by proving that $\mathsf{M}(X;\tau)$ is a modal algebra. As $\langle \mathcal{P}(X); \cap, \cup, \to, \emptyset, X\rangle$ is a (powerset) Boolean algebra, it suffices to show that for every $U, V \subseteq X$,

$$\mathsf{int}(U \cap V) = \mathsf{int}(U) \cap \mathsf{int}(V) \;\text{ and }\; \mathsf{int}(X) = X.$$

Clearly, $X$ is the greatest open set contained in $X$. Consequently, $\mathsf{int}(X) = X$. Then we turn to prove the other equation above. From $U \cap V \subseteq U, V$ it follows that the interior of $U \cap V$ is contained in those of $U$ and $V$, that is, $\mathsf{int}(U \cap V) \subseteq \mathsf{int}(U) \cap \mathsf{int}(V)$. To prove the reverse inclusion, observe that both $\mathsf{int}(U)$ and $\mathsf{int}(V)$ are open sets and, therefore, so is their intersection $\mathsf{int}(U) \cap \mathsf{int}(V)$. Moreover, from $\mathsf{int}(U) \subseteq U$ and $\mathsf{int}(V) \subseteq V$ it follows $\mathsf{int}(U) \cap \mathsf{int}(V) \subseteq U \cap V$. Hence, $\mathsf{int}(U) \cap \mathsf{int}(V)$ is an open set contained in $U \cap V$. By the definition of the interior of a set this impies $\mathsf{int}(U) \cap \mathsf{int}(V) \subseteq \mathsf{int}(U \cap V)$. This establishes that $\mathsf{M}(X;\tau)$ is a modal algebra.

It only remains to prove that $\Diamond U = \overline{U}$ for every $U \subseteq X$. First observe that

$$\Diamond U = \neg \Box \neg U = (\mathsf{int}(U^c))^c. \tag{2.9}$$

As a consequence, $\Diamond U$ is the complement of the open set $\mathsf{int}(U^c)$ and, therefore, a closed set. We will prove that for every closed set $V$,

$$U \subseteq V \iff \Diamond U \subseteq V. \tag{2.10}$$

This is proved through the series of equivalences

$$U \subseteq V \iff V^c \subseteq U^c \iff V^c \subseteq \mathsf{int}(U^c) \iff \mathsf{int}(U^c)^c \subseteq V \iff \Diamond U \subseteq V,$$

the first and the third of which are straightforward, the second holds because $V^c$ is open (as $V^c$ is closed), and the fourth by condition (2.9).

In order to prove that $\Diamond U = \overline{U}$, it suffices to show that $U \subseteq \Diamond U \subseteq \overline{U}$ because $\Diamond U$ is closed and $\overline{U}$ is the least closed set containing $U$. Since $\Diamond U$ is closed, we can apply the right to left implication in condition (2.10) to the case where $V := \Diamond U$ obtaining $U \subseteq \Diamond U$. On the other hand, since $\overline{U}$ is a closed set containing $U$, we can apply the left to right implication in condition (2.10) to the case where $V := \overline{U}$ obtaining $\Diamond U \subseteq \overline{U}$. Hence, we conclude that $\Diamond U = \overline{U}$ as desired. ⊠

The structure theory of modal algebras is based on the following concept.

**Definition 2.39.** A *Kripke frame* is a pair $\mathbb{X} = \langle X; R\rangle$ where $X$ is a set and $R$ a binary relation on it. The set $X$ is called the *universe* of $\mathbb{X}$.

Every Kripke frame gives rise to a modal algebra, as we proceed to explain.

**Example 2.40** (Complex algebras)**.** With every Kripke frame $\mathbb{X}$ we associate an algebra

$$\mathcal{P}(\mathbb{X}) := \langle \mathcal{P}(X); \cap, \cup, \to, \Box, \emptyset, X\rangle$$

where $\langle \mathcal{P}(X); \cap, \cup, \to, \emptyset, X\rangle$ is a powerset Boolean algebra and for every $U \subseteq X$,

$$\Box U := \{x \in X : \text{if } xRy, \text{ then } y \in U\}.$$

The structure $\mathcal{P}(\mathbb{X})$ is called the *complex algebra* of $\mathbb{X}$. We will prove that it is a modal algebra in which
$$\Diamond U = \{x \in X : xRy \text{ for some } y \in U\}.$$

We begin by proving that $\mathcal{P}(\mathbb{X})$ is a modal algebra. As $\langle \mathcal{P}(X); \cap, \cup, \to, \emptyset, X\rangle$ is a (powerset) Boolean algebra, it suffices to show that for every $U, V \subseteq X$,
$$\Box(U \cap V) = \Box U \cap \Box V \text{ and } \Box X = X.$$

Applying in succession the definition of $\Box$ in $\mathcal{P}(\mathbb{X})$ and the fact that $X$ is the total set, we obtain
$$\Box X = \{x \in X : \text{if } xRy, \text{ then } y \in X\} = X.$$

On the other hand, we have
$$\begin{aligned}\Box(U \cap V) &= \{x \in X : \text{if } xRy, \text{ then } y \in U \cap V\}\\ &= \{x \in X : \text{if } xRy, \text{ then } y \in U\} \cap \{x \in X : \text{if } xRy, \text{ then } y \in V\}\\ &= \Box U \cap \Box V.\end{aligned}$$

The first and the third equalities above follow from the definition of $\Box$ in $\mathcal{P}(\mathbb{X})$ and the second is straightforward. Hence, we conclude that $\mathcal{P}(\mathbb{X})$ is indeed a modal algebra. Lastly, the definition of $\Box$ in $\mathcal{P}(\mathbb{X})$ guarantees that
$$\Diamond U = \neg\Box\neg U = \{x \in X : \text{if } xRy, \text{ then } y \in U^c\}^c = \{x \in X : xRy \text{ for some } x \in U\}. \quad \boxtimes$$

Not only does every Kripke frame $\mathbb{X}$ determine a modal algebra (namely, the complex algebra $\mathcal{P}(\mathbb{X})$), but the converse is also true. More precisely, every modal determines a Kripke frames as follows:

**Definition 2.41.** With every modal algebra $\boldsymbol{A}$ we associate the Kripke frame $\mathsf{K}(\boldsymbol{A})$ with universe the set $\mathsf{Ultr}(\boldsymbol{A})$ of ultrafilters of $\boldsymbol{A}$ and with relation
$$R_{\boldsymbol{A}} := \{\langle F, G\rangle \in \mathsf{Ultr}(\boldsymbol{A}) : \Box a \in F \text{ implies } a \in G, \text{ for every } a \in A\}.$$

We will refer to $\mathsf{K}(\boldsymbol{A})$ as to the Kripke frame *associated with* $\boldsymbol{A}$.

This concept allows us to prove that complex algebras are indeed canonical examples of modal algebras.

**Theorem 2.42** (Representation theorem)**.** *Let $\boldsymbol{A}$ be a modal algebra and $\mathbb{X} := \mathsf{K}(\boldsymbol{A})$. Then the map $\varepsilon \colon \boldsymbol{A} \to \mathcal{P}(\mathbb{X})$ defined by the rule*
$$\varepsilon(a) := \{F \in \mathsf{Ultr}(\boldsymbol{A}) : a \in F\}$$
*is an embedding of $\boldsymbol{A}$ into the complex algebra $\langle \mathcal{P}(X); \cap, \cup, \to, \Box, \emptyset, X\rangle$.*

*Proof.* Since modal algebras are Boolean algebras, we can apply Theorem 2.35 obtaining that $\varepsilon\colon \boldsymbol{A} \to \mathcal{P}(\mathbb{X})$ is an embedding of Boolean algebras. Therefore, it only remains to prove that $\varepsilon$ preserves the operation $\Box$, i.e., that for every $a \in A$,

$$\varepsilon(\Box^{\boldsymbol{A}} a) = \Box^{\mathcal{P}(\mathbb{X})} \varepsilon(a).$$

By the definition of $\varepsilon$ and of the operation $\Box$ in the complex algebra $\mathcal{P}(\mathbb{X})$, this amounts to the demand that for every $F \in \mathsf{Ultr}(\boldsymbol{A})$,

$$\Box^{\boldsymbol{A}} a \in F \iff \text{for every } G \in \mathsf{Ultr}(\boldsymbol{A}), FR_{\boldsymbol{A}}G \text{ implies } a \in G.$$

To prove the implication from left to right holds, suppose that $\Box^{\boldsymbol{A}} a \in F$ and consider $G \in \mathsf{Ultr}(\boldsymbol{A})$ such that $FR_{\boldsymbol{A}}G$. Since $\Box^{\boldsymbol{A}} a \in F$ and $FR_{\boldsymbol{A}}G$, the definition of $R_{\boldsymbol{A}}$ guarantees that $a \in G$.

To prove the implication from right to left, we reason by contraposition. Suppose that $\Box^{\boldsymbol{A}} a \notin F$. We will prove that the set

$$G := \{b \in A : \Box^{\boldsymbol{A}} b \in F\}$$

is a filter of $\boldsymbol{A}$. As $\boldsymbol{A}$ is a modal algebra and $F$ a filter, we have $\Box^{\boldsymbol{A}} 1 = 1 \in F$. Therefore, $G$ contains 1. To prove that $G$ is an upset, consider $b, c \in A$ such that $b \in G$ and $b \leqslant c$. From $b \leqslant c$ it follows $b = b \wedge c$. As $\boldsymbol{A}$ is a modal algebra, this implies

$$\Box^{\boldsymbol{A}} b = \Box^{\boldsymbol{A}}(b \wedge c) = \Box^{\boldsymbol{A}} b \wedge \Box^{\boldsymbol{A}} c,$$

that is, $\Box^{\boldsymbol{A}} b \leqslant \Box^{\boldsymbol{A}} c$. Moreover, from the assumption that $b \in G$ and the definition of $G$ it follows $\Box^{\boldsymbol{A}} b \in F$. Since $F$ is an upset and $\Box^{\boldsymbol{A}} b \leqslant \Box^{\boldsymbol{A}} c$, this yields $\Box^{\boldsymbol{A}} c \in F$ and, therefore, $c \in G$. It only remains to show that $G$ is closed under binary meets. Let $b, c \in G$. By the definition of $G$, we have $\Box^{\boldsymbol{A}} b, \Box^{\boldsymbol{A}} c \in F$. Since $\boldsymbol{A}$ is a modal algebra and $F$ is closed under binary meets,

$$\Box^{\boldsymbol{A}}(b \wedge c) = \Box^{\boldsymbol{A}} b \wedge \Box^{\boldsymbol{A}} c \in F.$$

By the definition of $G$ this implies $b \wedge c \in G$ as desired. This shows that $G$ is a filter of $\boldsymbol{A}$.

Recall that we assumed that $\Box^{\boldsymbol{A}} a \notin F$. By the definition of $G$ this amounts to $a \notin G$. Since $G$ is a filter of $\boldsymbol{A}$ and $\boldsymbol{A}$ is a Boolean algebra, we can apply the Ultrafilter Theorem obtaining that $G$ is an intersection of ultrafilters. Since $a \notin G$, this means that there exists an ultrafilter $G^+$ such that $G \subseteq G^+$ and $a \notin G^+$. From $G \subseteq G^+$ and the definition of $G$ it follows that $\Box^{\boldsymbol{A}} b \in F$ implies $b \in G^+$, for every $b \in A$. By the definition of $R_{\boldsymbol{A}}$ this amounts to $FR_{\boldsymbol{A}}G^+$. Hence, we obtained an ultrafilter $G^+$ such that $FR_{\boldsymbol{A}}G^+$ and $a \notin G^+$ as desired. ⊠

At this stage, it should not come as a surprise that in the finite case the map $\varepsilon$ in the above theorem is surjective and, therefore, an isomorphism.

**Corollary 2.43.** *Up to isomorphism, the finite modal algebras are precisely the complex algebras* $\langle \mathcal{P}(\mathbb{X}); \cap, \cup, \to, \Box, \emptyset, X \rangle$ *where* $\mathbb{X}$ *is a finite Kripke frame.*

CHAPTER 3

# Intuitionistic and modal logics

## 3.1 Propositional logics

In the context of logic, the term algebra $\boldsymbol{T}_\rho(Var)$ is often called the *algebra of formulas* (of type $\rho$) and its elements are referred to as *formulas*. An *endomorphism* of an algebra $\boldsymbol{A}$ is a homomorphism whose domain and codomain is $\boldsymbol{A}$. Endomorphisms of the algebra of formulas play a fundamental role in logic.

**Definition 3.1.** A *substitution* of type $\rho$ is an endomorphism $\sigma$ of $\boldsymbol{T}_\rho(Var)$.

When the type $\rho$ is clear from the context, we will simply say that $\sigma$ is a substitution.

In view of the fact that $\boldsymbol{T}_\rho(Var)$ is freely generated by $Var$, every function $\sigma\colon Var \to T_\rho(Var)$ can be uniquely extended to a substitution $\sigma^+$ of type $\rho$, namely the function defined by the rule

$$\varphi(x_1, \dots, x_n) \longmapsto \varphi(\sigma(x_1), \dots, \sigma(x_n)).$$

Because of this, substitutions can be presented by exhibiting functions $\sigma\colon Var \to T_\rho(Var)$.

**Definition 3.2.** A *logic* of type $\rho$ is a consequence relation $\vdash$ on the set of formulas $T_\rho(Var)$ that, moreover, is *substitution invariant* in the sense that for every substitution $\sigma$ of type $\rho$ and every set of formulas $\Gamma \cup \{\varphi\} \subseteq T_\rho(Var)$,

$$\text{if } \Gamma \vdash \varphi, \text{ then } \sigma[\Gamma] \vdash \sigma(\varphi).$$

*Remark* 3.3. As mentioned above, $\Gamma \vdash \varphi$ should be read as "$\Gamma$ proves $\varphi$" or "$\varphi$ follows from $\Gamma$". The requirement that $\vdash$ is substitution invariant, instead, is intended to capture the idea that logical inferences are true only in virtue of their form (as opposed to their content). ⊠

**Example 3.4** (Hilbert calculi)**.** We work within a fixed, but arbitrary, type $\rho$. A *rule* is an expression of the form $\Gamma \rhd \varphi$, where $\Gamma \cup \{\varphi\} \subseteq T_\rho(Var)$. In this case, $\Gamma$ is said to be the set of *premises* of the rule and $\varphi$ the *conclusion*. When $\Gamma = \emptyset$, the rule $\Gamma \rhd \varphi$ is sometimes called an *axiom*. A rule $\Gamma \rhd \varphi$ is said to be *valid* in a logic $\vdash$, when $\Gamma \vdash \varphi$. A *Hilbert calculus* is a set of rules.

Every Hilbert calculus $\mathsf{H}$ induces a logic, as we proceed to explain. Consider a set of formulas $\Gamma \cup \{\varphi\} \subseteq T_\rho(Var)$. A *proof of $\varphi$ from $\Gamma$ in* $\mathsf{H}$ is a well-ordered sequence $\langle \psi_\alpha : \alpha \leqslant \gamma \rangle$ of formulas $\psi_\alpha \in T_\rho(Var)$ whose last element $\psi_\gamma$ is $\varphi$ and such that, for every $\alpha \leqslant \gamma$, either $\psi_\alpha \in \Gamma$ or there exist a substitution $\sigma$ and a rule $\Delta \rhd \delta$ in $\mathsf{H}$ such that the formulas in $\sigma[\Delta]$ occur in the initial segment $\langle \psi_\beta : \beta < \alpha \rangle$ and $\psi_\alpha = \sigma(\delta)$.

The logic $\vdash_{\mathsf{H}}$ induced by $\mathsf{H}$ is defined, for every $\Gamma \cup \{\varphi\} \subseteq T_\rho(Var)$, as

$$\Gamma \vdash_{\mathsf{H}} \varphi \iff \text{there exists a proof of } \varphi \text{ from } \Gamma \text{ in } \mathsf{H}.$$

As expected, $\vdash_{\mathsf{H}}$ is a logic in the sense of Definition 3.2. Furthermore, it is the least logic $\vdash$ such that $\Gamma \vdash \varphi$, for every rule $\Gamma \rhd \varphi$ in $\mathsf{H}$.

A logic $\vdash$ is said to be *axiomatized* by a Hilbert calculus $\mathsf{H}$ when it coincides with $\vdash_{\mathsf{H}}$. Notice that every logic $\vdash$ is vacuously axiomatized by the Hilbert calculus

$$\{\Gamma \rhd \varphi : \Gamma \vdash \varphi\}.$$

Because of this, axiomatizations in terms of Hilbert calculi $\mathsf{H}$ acquire special interest when $\mathsf{H}$ is finite or, at least, recursive. ⊠

When no confusion shall arise, given a sequence $\vec{a}$ and a set $A$, we write $\vec{a} \in A$ to indicate that the elements of the sequence $\vec{a}$ belong to $A$. The following concept is instrumental to exhibit further examples of logics.

**Definition 3.5.** Let $\mathsf{K}$ be a class of similar algebras We define a binary relation $\vDash_{\mathsf{K}} \subseteq \mathcal{P}(E_\rho(Var)) \times E_\rho(Var)$ as follows:

$$\begin{aligned} \Theta \vDash_{\mathsf{K}} \varepsilon \approx \delta \iff & \text{for every } \boldsymbol{A} \in \mathsf{K} \text{ and every } \vec{a} \in A, \\ & \quad \text{if } \varphi^{\boldsymbol{A}}(\vec{a}) = \psi^{\boldsymbol{A}}(\vec{a}) \text{ for all } \varphi \approx \psi \in \Theta, \text{ then } \varepsilon^{\boldsymbol{A}}(\vec{a}) = \delta^{\boldsymbol{A}}(\vec{a}). \end{aligned}$$

The relation $\vDash_{\mathsf{K}}$ is known as the *equational consequence relative to* $\mathsf{K}$.

**Example 3.6** (Equationally defined logics)**.** We work within a fixed, but arbitrary, type $\rho$. Given a set of equations $\tau(x)$ in a single variable $x$ and a set of formulas $\Gamma \cup \{\varphi\} \subseteq T(Var)$, we abbreviate

$$\{\varepsilon(\varphi) \approx \delta(\varphi) \colon \varepsilon \approx \delta \in \tau\} \text{ as } \tau(\varphi), \text{ and } \bigcup_{\gamma \in \Gamma} \tau(\gamma) \text{ as } \tau[\Gamma].$$

Given a class of algebras $\mathsf{K}$ and a set of equations $\tau(x)$, we define a logic $\vdash_{\mathsf{K},\tau}$ as follows: for every $\Gamma \cup \{\varphi\} \subseteq T(Var)$,

$$\Gamma \vdash_{\mathsf{K},\tau} \varphi \iff \tau[\Gamma] \vDash_{\mathsf{K}} \tau(\varphi).$$

It is easy to prove that $\vdash_{\mathsf{K},\tau}$ is indeed a logic in the sense of Definition 3.2. Notice that, in this case, $\vdash$ is related to $\mathsf{K}$ by a *completeness theorem* witnessed by the set of equations $\tau(x)$ that allows to translate formulas into equations and, therefore, to interpret $\vdash_{\mathsf{K},\tau}$ into $\vDash_{\mathsf{K}}$.

For instance, the completeness theorem of classical propositional logic CPC with respect to the class of Boolean algebras BA states precisely that CPC coincides with $\vdash_{\mathsf{BA},\tau}$ where $\tau = \{x \approx 1\}$. Similarly, the completeness theorem of intuitionistic propositional logic IPC with respect to the class of Heyting algebras HA states precisely that IPC coincides with $\vdash_{\mathsf{HA},\tau}$ where $\tau = \{x \approx 1\}$. Because of this, CPC and IPC can be defined as follows: for every set of formulas $\Gamma \cup \{\varphi\}$ of the appropriate type,

$$\Gamma \vdash_{\mathsf{CPC}} \varphi \iff \tau[\Gamma] \vDash_{\mathsf{BA}} \tau(\varphi)$$
$$\Gamma \vdash_{\mathsf{IPC}} \varphi \iff \tau[\Gamma] \vDash_{\mathsf{HA}} \tau(\varphi),$$

where $\tau = \{x \approx 1\}$. ⊠

## 3.2 Intuitionistic logic

Motivated by the philosophy of constructivism in mathematics, intuitionistic logic identifies the principles of constructive reasoning. In this section we review its algebra-based and Kripke semantics. To this end, let $\rho_I$ be the type of Heyting algebras.

**Definition 3.7.** Intuitionistic propositional logic IPC is the logic of type $\rho_I$ axiomatized by the following Hilbert calculus, denoted by H:

$$\begin{aligned}
&\emptyset \rhd 1\\
&\emptyset \rhd x \to (y \to x)\\
&\emptyset \rhd x \to (y \to (x \wedge y))\\
&\emptyset \rhd (x \wedge y) \to x\\
&\emptyset \rhd (x \wedge y) \to y\\
&\emptyset \rhd x \to (x \vee y)\\
&\emptyset \rhd y \to (x \vee y)\\
&\emptyset \rhd (x \vee y) \to ((x \to z) \to ((y \to z) \to z))\\
&\emptyset \rhd (x \to y) \to ((x \to (y \to z)) \to (x \to z))\\
&\emptyset \rhd 0 \to x\\
x, x \to y &\rhd y.
\end{aligned}$$

*Remark* 3.8. Since the set of premises of every rule in H is finite, for every $\Gamma \cup \{\varphi\} \subseteq T(Var)$, we have

$$\Gamma \vdash_{\mathsf{IPC}} \varphi \iff \text{there exists a } \textit{finite} \text{ proof of } \varphi \text{ from } \Gamma \text{ in H}.$$ ⊠

**Definition 3.9.** We denote by HA the variety of Heyting algebras.

We recall that IPC and HA are related as follows:

**Theorem 3.10.** *Let $\tau = \{x \approx 1\}$. For every $\Gamma \cup \{\varphi\} \subseteq T(Var)$, we have*

$$\Gamma \vdash_{\mathsf{IPC}} \varphi \iff \tau[\Gamma] \vDash_{\mathsf{HA}} \tau(\varphi).$$

*Consequently, the logics* IPC *and* $\vdash_{\mathsf{HA},\tau}$ *coincide.*

Intuitionistic propositional logic admits also a Kripke semantics, as we proceeed to explain. A *valuation* in a poset $\mathbb{X}$ is a function $v\colon Var \to \mathsf{Up}(\mathbb{X})$. Given a valuation $v$ on a poset $\mathbb{X}$, we define a notion of *validity* of a formula $\varphi \in T(Var)$ at a point $w \in X$ under $v$, in symbols $w, v \Vdash \varphi$, by recursion on the construction of $\varphi$ as follows. For variables $x \in Var$ we set

$$w, v \Vdash x \Longleftrightarrow w \in v(x), \text{ for } x \in Var$$

and for constant symbols

$$w, v \Vdash 1 \text{ and } w, v \nVdash 0.$$

For complex formulas we set

$$\begin{aligned} w, v \Vdash \alpha \wedge \beta &\Longleftrightarrow w, v \Vdash \alpha \text{ and } w, v \Vdash \beta \\ w, v \Vdash \alpha \vee \beta &\Longleftrightarrow w, v \Vdash \alpha \text{ or } w, v \Vdash \beta \\ w, v \Vdash \alpha \to \beta &\Longleftrightarrow \text{ for every } u \in X \text{ such that } w \leqslant u, \text{ if } u, v \Vdash \alpha, \text{ then } u, v \Vdash \beta. \end{aligned}$$

Given a set of formulas $\Gamma$, we write $w, v \Vdash \Gamma$ to indicate that $w, v \Vdash \gamma$, for every $\gamma \in \Gamma$.

We define two logics $\vdash^{\ell}_{\mathsf{Pos}}$ and $\vdash^{g}_{\mathsf{Pos}}$ associated with the class of all posets as follows: for every $\Gamma \cup \{\varphi\} \subseteq T(Var)$,

$$\begin{aligned} \Gamma \vdash^{\ell}_{\mathsf{Pos}} \varphi \Longleftrightarrow\ & \text{for every poset } \mathbb{X}, \text{ every valuation } v \text{ in } \mathbb{X}, \text{ and every } w \in X, \\ & \text{if } w, v \Vdash \Gamma, \text{ then } w, v \Vdash \varphi; \text{ and} \\ \Gamma \vdash^{g}_{\mathsf{Pos}} \varphi \Longleftrightarrow\ & \text{for every poset } \mathbb{X} \text{ and every valuation } v \text{ in } \mathbb{X}, \\ & \text{if } w, v \Vdash \Gamma \text{ for every } w \in X, \text{ then } w, v \Vdash \varphi \text{ for every } w \in X. \end{aligned}$$

The logics $\vdash^{\ell}_{\mathsf{Pos}}$ and $\vdash^{g}_{\mathsf{Pos}}$ are known, respectively, as the *local* and the *global consequence relations* associated with the class of posets.

**Theorem 3.11.** *For every set of formulas $\Gamma \cup \{\varphi\} \subseteq T(Var)$,*

$$\Gamma \vdash_{\mathsf{IPC}} \varphi \Longleftrightarrow \Gamma \vdash^{\ell}_{\mathsf{Pos}} \varphi \Longleftrightarrow \Gamma \vdash^{g}_{\mathsf{Pos}} \varphi.$$

*Proof.* As usual, the nontrivial part of the proof consists in showing that if $\Gamma \nvdash_{\mathsf{IPC}} \varphi$, then $\Gamma \nvdash^{\ell}_{\mathsf{Pos}} \varphi$ and $\Gamma \nvdash^{g}_{\mathsf{Pos}} \varphi$. From the definition of $\vdash^{\ell}_{\mathsf{Pos}}$ and $\vdash^{g}_{\mathsf{Pos}}$ it follows immediately that $\Gamma \nvdash^{g}_{\mathsf{Pos}} \varphi$ implies $\Gamma \nvdash^{\ell}_{\mathsf{Pos}} \varphi$. Therefore, it suffices to show that $\Gamma \nvdash_{\mathsf{IPC}} \varphi$ implies $\Gamma \nvdash^{g}_{\mathsf{Pos}} \varphi$.

Accordingly, suppose that $\Gamma \nvdash_{\mathsf{IPC}} \varphi$. By Theorem 3.10 there exist a Heyting algebra $\boldsymbol{A}$ and a homomorphism $f\colon \boldsymbol{T}(Var) \to \boldsymbol{A}$ such that $f[\Gamma] \subseteq \{1\}$ and $f(\varphi) \neq 1$. Let $\mathbb{X} := \langle \mathsf{Pf}(\boldsymbol{A}); \subseteq \rangle$ and recall from the representation theorem for Heyting algebras that the map $\varepsilon\colon \boldsymbol{A} \to \mathsf{Up}(\mathbb{X})$ defined by the rule

$$\varepsilon(a) := \{F \in \mathbb{X} : a \in F\}$$

is an embedding between Heyting algebras. Therefore, the composition $\varepsilon \circ f\colon \boldsymbol{T}(Var) \to \mathsf{Up}(\mathbb{X})$ is a homomorphism between Heyting algebras.

Since $f[\Gamma] \subseteq \{1\}$ by assumption, for each $\gamma \in \Gamma$ we have

$$\varepsilon(f(\gamma)) = \varepsilon(1) = \mathbb{X}. \tag{3.1}$$

On the other hand, since $f(\varphi) \neq 1$ and $\varepsilon\colon \boldsymbol{A} \to \mathsf{Up}(\mathbb{X})$ is an embedding, we have

$$\varepsilon(f(\varphi)) \neq \mathbb{X}. \tag{3.2}$$

Now, consider the poset $\mathbb{X}$ and the valuation $v\colon Var \to \mathsf{Up}(\mathbb{X})$ defined by the rule

$$v(x) \coloneqq \varepsilon(f(x)).$$

We claim that, for every formula $\psi \in T(Var)$,

$$\varepsilon(f(\psi)) = \{F \in \mathbb{X} : F, v \Vdash \psi\}.$$

The proof proceeds by induction on the construction of $\psi$. For variables $x \in Var$ and the constant symbols 0 and 1 we have

$$\begin{aligned}
\varepsilon(f(x)) &= v(x) = \{F \in \mathbb{X} : F, v \Vdash x\} \\
\varepsilon(f(0)) &= \varepsilon(0^{\boldsymbol{A}}) = \emptyset = \{F \in \mathbb{X} : F, v \Vdash 0\} \\
\varepsilon(f(1)) &= \varepsilon(1^{\boldsymbol{A}}) = \mathbb{X} = \{F \in \mathbb{X} : F, v \Vdash 1\}.
\end{aligned}$$

Then we turn to the induction step. We have three cases:

(i) $\psi = \psi_1 \wedge \psi_2$;

(ii) $\psi = \psi_1 \vee \psi_2$;

(iii) $\psi = \psi_1 \to \psi_2$.

(i): In this case, we have

$$\begin{aligned}
\varepsilon(f(\psi_1 \wedge \psi_2)) &= \varepsilon(f(\psi_1)) \cap \varepsilon(f(\psi_2)) \\
&= \{F \in \mathbb{X} : F, v \Vdash \psi_1\} \cap \{F \in \mathbb{X} : F, v \Vdash \psi_2\} \\
&= \{F \in \mathbb{X} : F, v \Vdash \psi_1 \text{ and } F, v \Vdash \psi_2\} \\
&= \{F \in \mathbb{X} : F, v \Vdash \psi_1 \wedge \psi_2\}.
\end{aligned}$$

The first equality above follows from the fact that $\varepsilon \circ f\colon \boldsymbol{T}(Var) \to \mathsf{Up}(\mathbb{X})$ is a homomorphism, the second from the inductive hypothesis, the third is straightforward, while the last one is a consequence of the definition of $\Vdash$.

(ii): Analogous to case (i).

(iii): In this case, we have

$$\begin{aligned}
&\varepsilon(f(\psi_1 \to \psi_2)) \\
&= X \smallsetminus {\downarrow}(\varepsilon(f(\psi_1)) \smallsetminus \varepsilon(f(\psi_2))) \\
&= \{F \in \mathbb{X} : \text{for all } G \in \mathbb{X} \text{ such that } F \subseteq G, \text{ if } G \in \varepsilon(f(\psi_1)), \text{then } G \in \varepsilon(f(\psi_2))\} \\
&= \{F \in \mathbb{X} : \text{for all } G \in \mathbb{X} \text{ such that } F \subseteq G, \text{ if } G, v \Vdash \psi_1, \text{then } G, v \Vdash \psi_2\} \\
&= \{F \in \mathbb{X} : F, v \Vdash \psi_1 \to \psi_2\}.
\end{aligned}$$

Again, the first equality above follows from the fact that $\varepsilon \circ f \colon \boldsymbol{T}(Var) \to \mathsf{Up}(\mathbb{X})$ is a homomorphism, the second is straightforward, the third from the inductive hypothesis, and the last one is a consequence of the definition of $\Vdash$. This concludes the proof of the claim.

By applying the claim to conditions (3.1) and (3.2), we obtain that

$$F, v \Vdash \Gamma, \text{ for all } F \in \mathbb{X}, \text{ but there exists } F \in \mathbb{X} \text{ such that } F, v \nVdash \varphi.$$

Hence, we conclude that $\Gamma \nvdash^{g}_{\mathsf{Pos}} \varphi$, as desired. ⊠

## 3.3 Modal logic

Let $\rho_M$ be the type of modal algebras.

**Definition 3.12.** Let $\mathsf{K}$ be the least subset $\Sigma$ of $T_{\rho_M}(Var)$ such that:

(i) $\Sigma$ *contains the tautologies of* $\mathsf{CPC}$;

(ii) $\Sigma$ *contains* $\Box(x \to y) \to (\Box x \to \Box y)$;

(iii) $\Sigma$ is *closed under substitutions*: $\sigma(\varphi) \in \Sigma$, for every substitution $\sigma$ and $\varphi \in \Sigma$;

(iv) $\Sigma$ is *closed under modus ponens*: $\psi \in \Sigma$, for every $\varphi, \varphi \to \psi \in \Sigma$;

(v) $\Sigma$ is *closed under necessitation*: $\Box\varphi \in \Sigma$, for every $\varphi \in \Sigma$.

The set $\mathsf{K}$ is sometimes called the *least normal modal logic*.

Notice that, strictly speaking, $\mathsf{K}$ is not a logic, because it is a set of formulas, as opposed to a consequence relation. However, it is possible to associate two logics with $\mathsf{K}$, as we proceed to explain.

**Definition 3.13.** Let $\mathsf{K}_\ell$ and $\mathsf{K}_g$ be the logics of type $\rho_M$ defined as follows:

(i) The *local consequence* $\mathsf{K}_\ell$ of $\mathsf{K}$ is the logic axiomatized by the Hilbert calculus

$$\emptyset \rhd \varphi, \text{ for all } \varphi \in \mathsf{K} \qquad x, x \to y \rhd y.$$

(ii) The *global consequence* $\mathsf{K}_g$ of $\mathsf{K}$ is the logic axiomatized by the Hilbert calculus

$$\emptyset \rhd \varphi, \text{ for all } \varphi \in \mathsf{K} \qquad x, x \to y \rhd y \qquad x \rhd \Box x.$$

The rule $x \rhd \Box x$ is sometimes called the *necessitation rule*.

We recall that $\mathsf{MA}$ and $\mathsf{K}_g$ are related as follows:

**Theorem 3.14.** *Let $\tau = \{x \approx 1\}$. For every $\Gamma \cup \{\varphi\} \subseteq T(Var)$, we have*

$$\Gamma \vdash_{\mathsf{K}_g} \varphi \iff \tau[\Gamma] \vDash_{\mathsf{MA}} \tau(\varphi).$$

*Consequently, the logics $\mathsf{K}_g$ and $\vdash_{\mathsf{MA},\tau}$ coincide.*

Similarly, MA and $\mathsf{K}_\ell$ are related as follows:

**Theorem 3.15.** *For every $\Gamma \cup \{\varphi\} \subseteq T(Var)$, we have*

$$\Gamma \vdash_{\mathsf{K}_\ell} \varphi \iff \text{for every } \boldsymbol{A} \in \mathsf{MA}, a \in A, \text{ and homomorphism } f\colon \boldsymbol{T}(Var) \to \boldsymbol{A},$$
$$\text{if } a \leqslant f(\gamma) \text{ for each } \gamma \in \Gamma, \text{ then } a \leqslant f(\varphi).$$

As a consequence, we obtain the following:

**Corollary 3.16.** *The logics $\mathsf{K}_\ell$ and $\mathsf{K}_g$ are different.*

*Proof.* Consider the four-element Boolean algebra $\boldsymbol{A}$ with universe $\{a, c, 0, 1\}$. We endow it with a unary operation $\Box$ defined as follows:

$$\Box(b) := \begin{cases} 1 & \text{if } b = 1 \\ 0 & \text{otherwise.} \end{cases}$$

The resulting structure $\boldsymbol{B}$ is a modal algebra. Consider any homomorphism $f\colon \boldsymbol{T}(Var) \to \boldsymbol{B}$ such that $f(x) := a$. We have

$$f(\Box x) = \Box^{\boldsymbol{B}} f(x) = \Box^{\boldsymbol{B}} a = 0 < a = f(x).$$

By Theorem 3.15 this implies $x \nvdash_{\mathsf{K}_\ell} \Box x$. Furthermore, since the Hilbert calculus that axiomatizes $\mathsf{K}_g$ contains the necessitation rule, we obtain $x \vdash_{\mathsf{K}_g} \Box x$. Hence, we conclude that the logics $\mathsf{K}_\ell$ and $\mathsf{K}_g$ are indeed different. ⊠

Both the local and global consequences of K admit a Kripke semantics, as we proceed to explain. A *valuation* in a Kripke frame $\mathbb{X}$ is a function $v\colon Var \to \mathcal{P}(X)$. Given a valuation $v$ in a Kripke frame $\mathbb{X}$, we define a notion of *validity* of a formula $\varphi \in T(Var)$ at a point $w \in X$ under $v$, in symbols $w, v \Vdash \varphi$, by recursion on the construction of $\varphi$ as follows. For variables $x \in Var$ we set

$$w, v \Vdash x \iff w \in v(x), \text{ for } x \in Var$$

and for constant symbols

$$w, v \Vdash 1 \text{ and } w, v \nVdash 0.$$

For complex formulas we set

$$\begin{aligned} w, v \Vdash \alpha \wedge \beta &\iff w, v \Vdash \alpha \text{ and } w, v \Vdash \beta \\ w, v \Vdash \alpha \vee \beta &\iff w, v \Vdash \alpha \text{ or } w, v \Vdash \beta \\ w, v \Vdash \neg\alpha &\iff w, v \nVdash \alpha \\ w, v \Vdash \Box\alpha &\iff u, v \Vdash \alpha, \text{ for every } u \in X \text{ such that } \langle w, u\rangle \in R. \end{aligned}$$

Given a set of formulas $\Gamma$, we write $w, v \Vdash \Gamma$ to indicate that $w, v \Vdash \gamma$, for every $\gamma \in \Gamma$.

We define two logics $\vdash^{\ell}_{\mathsf{Frm}}$ and $\vdash^{g}_{\mathsf{Frm}}$ associated with the class of all Kripke frames as follows: for every $\Gamma \cup \{\varphi\} \subseteq T(Var)$,

$$\begin{aligned}
\Gamma \vdash^{\ell}_{\mathsf{Frm}} \varphi \iff &\text{ for every Kripke frame } \mathbb{X}, \text{ every valuation } v \text{ in } \mathbb{X}, \text{ and every } w \in X,\\
&\text{ if } w, v \Vdash \Gamma, \text{ then } w, v \Vdash \varphi; \text{ and}\\
\Gamma \vdash^{g}_{\mathsf{Frm}} \varphi \iff &\text{ for every Kripke frame } \mathbb{X} \text{ and every valuation } v \text{ in } \mathbb{X},\\
&\text{ if } w, v \Vdash \Gamma \text{ for every } w \in X, \text{ then } w, v \Vdash \varphi \text{ for every } w \in X.
\end{aligned}$$

The logics $\vdash^{\ell}_{\mathsf{Frm}}$ and $\vdash^{g}_{\mathsf{Frm}}$ are known, respectively, as the *local* and the *global consequence relations* associated with the class of Kripke frames.

**Theorem 3.17.** *For every set of formulas $\Gamma \cup \{\varphi\} \subseteq T(Var)$,*

$$\Gamma \vdash_{\mathsf{K}_{\ell}} \varphi \iff \Gamma \vdash^{\ell}_{\mathsf{Frm}} \varphi \;\; \textit{and} \;\; \Gamma \vdash_{\mathsf{K}_{g}} \varphi \iff \Gamma \vdash^{g}_{\mathsf{Frm}} \varphi.$$

*Proof.* Consider $\Gamma \cup \{\varphi\} \subseteq T(Var)$. We begin by proving that

$$\Gamma \vdash_{\mathsf{K}_{\ell}} \varphi \iff \Gamma \vdash^{\ell}_{\mathsf{Frm}} \varphi. \tag{3.3}$$

The implication from left to right follows from a standard soundness argument. To prove the other implication, we reason by contraposition. Suppose that $\Gamma \nvdash_{\mathsf{K}_{\ell}} \varphi$. In view of Theorem 3.15, there exist a modal algebra $\boldsymbol{A}$, a homomorphism $f\colon \boldsymbol{T}(Var) \to \boldsymbol{A}$, and an element $a \in A$ such that

$$a \leqslant f(\gamma) \text{ for each } \gamma \in \Gamma \text{ and } a \nleqslant f(\varphi).$$

Then the filter $\uparrow a$ does not contain $f(\varphi)$. By the Ultrafilter Theorem there exists an ultrafilter $F$ such that $a \in F$ and $f(\varphi) \notin F$. Then let $\mathbb{X}$ be the Kripke frame $\mathsf{K}(\boldsymbol{A})$ and consider the embedding $\varepsilon\colon \boldsymbol{A} \to \mathcal{P}(\mathbb{X})$ given by the representation theorem for modal algebras. We have

$$F \in \varepsilon(f(\gamma)) \text{ for each } \gamma \in \Gamma \text{ and } F \notin \varepsilon(f(\varphi)). \tag{3.4}$$

Now, consider the valuation $v\colon Var \to \mathcal{P}(\mathbb{X})$ is the Kripke frame $\mathbb{X}$ defined by the rule

$$v(x) := \{G \in \mathbb{X} : \varepsilon(f(x)) \in G\}.$$

We claim that for every formula $\psi \in T(Var)$,

$$\varepsilon(f(\psi)) = \{G \in \mathbb{X} : G, v \Vdash \psi\}.$$

We reason by induction on the construction of $\psi$. The base case is handled as in the case of $\mathsf{IPC}$ and the same applies to the inductive cases for the connectives $\wedge$ and $\vee$. Therefore, it only remains to consider the following cases:

(i) $\psi = \neg\alpha$;

(ii) $\psi = \Box\alpha$.

(i): We have

$$\begin{aligned}\varepsilon(f(\neg\alpha)) &= \neg^{\mathcal{P}(\mathbb{X})}\varepsilon(f(\alpha))\\ &= \mathbb{X} - \varepsilon(f(\alpha))\\ &= \{G \in \mathbb{X} : G, v \nVdash \alpha\}\\ &= \{G \in \mathbb{X} : G, v \Vdash \neg\alpha\}.\end{aligned}$$

The above equalities can be justified as follows. The first follows holds because $\varepsilon \circ f \colon \boldsymbol{T}(Var) \to \mathcal{P}(\mathbb{X})$ is a homomorphism, the second follows from the definition of $\neg$ in $\mathcal{P}(\mathbb{X})$, the third from the inductive hypothesis, and the last one from the definition of $\Vdash$.

(ii): In this case, we have

$$\begin{aligned}\varepsilon(f(\Box\alpha)) &= \Box^{\mathcal{P}(\mathbb{X})}\varepsilon(f(\alpha))\\ &= \{G \in \mathbb{X} : \text{for all } H \in \mathbb{X}, \text{ if } \langle G, H\rangle \in R_{\boldsymbol{A}}, \text{then } H \in \varepsilon(f(\alpha))\}\\ &= \{G \in \mathbb{X} : \text{for all } H \in \mathbb{X}, \text{ if } \langle G, H\rangle \in R_{\boldsymbol{A}}, \text{ then } H, v \Vdash \alpha\}\\ &= \{G \in \mathbb{X} : G, v \Vdash \Box\alpha\}.\end{aligned}$$

The above equalities can be justified as follows. The first follows holds because $\varepsilon \circ f \colon \boldsymbol{T}(Var) \to \mathcal{P}(\mathbb{X})$ is a homomorphism, the second follows from the definition of $\Box$ in $\mathcal{P}(\mathbb{X})$, the third from the inductive hypothesis, and the last one from the definition of $\Vdash$. This concludes the proof of the claim.

From the claim and (3.4) it follows that

$$F, v \Vdash \Gamma \text{ and } F, v \nVdash \varphi.$$

Hence, we conclude that $\Gamma \nvdash^{\ell}_{\mathsf{Frm}} \varphi$, as desired. This establishes (3.3).

The proof of the equivalence

$$\Gamma \vdash_{\mathsf{K}_g} \varphi \Longleftrightarrow \Gamma \vdash^{g}_{\mathsf{Frm}} \varphi \tag{3.5}$$

is analogous. Accordingly, we shall sketch only the proof of the implication from right to left. As usual, we reason by contraposition. Suppose that $\Gamma \nvdash_{\mathsf{K}_g} \varphi$. By Theorem 3.14 there exist a modal algebra $\boldsymbol{A}$ and a homomorphism $f \colon \boldsymbol{T}(Var) \to \boldsymbol{A}$ such that

$$f(\gamma) = 1 \text{ for each } \gamma \in \Gamma \text{ and } f(\varphi) \neq 1.$$

By the Ultrafilter Theorem, this implies that

$$\varepsilon(f(\gamma)) = \mathbb{X} \text{ for each } \gamma \in \Gamma \text{ and } \varepsilon(f(\varphi)) \neq \mathbb{X}.$$

Then we consider the valuation $v$ in $\mathbb{X}$ defined in the previous case. From the claim and the above display it follows that

$$F, v \Vdash \Gamma \text{ for each } \gamma \in \Gamma \text{ and } F \in X, \text{ and there exists } F \in \mathbb{X} \text{ such that } F, v \nVdash \varphi.$$

Hence, we conclude that $\Gamma \nvdash^{g}_{\mathsf{Frm}} \varphi$. $\boxtimes$

CHAPTER 4

# Universal algebra

## 4.1 Ultraproducts

In order to understand the relation between logic and algebra, we need to take a detour in universal algebra and the theory of quasi-varieties. We begin by reviewing a product-like construction known as *ultraproduct*. First, recall that ultrafilters on powerset Boolean algebras $\mathcal{P}(X)$ are also called *ultrafilters on* $X$. Then let $\{\boldsymbol{A}_i : i \in I\}$ be a family of similar algebras. The *equalizer* $[\![\vec{a} = \vec{c}]\!]$ of a pair of elements $\vec{a}, \vec{c} \in \prod_{i \in I} A_i$ is the set of indexes on which the sequences $\vec{a}$ and $\vec{c}$ agree, that is,

$$[\![\vec{a} = \vec{c}]\!] := \{i \in I : \vec{a}(i) = \vec{c}(i)\}.$$

Moreover, given an ultrafilter $U$ on the index set $I$, let $\theta_U$ be the binary relation on the Cartesian product $\prod_{i \in I} A_i$ defined as

$$\theta_U := \{\langle \vec{a}, \vec{c} \rangle : [\![\vec{a} = \vec{c}]\!] \in U\}.$$

**Proposition 4.1.** *If $\{\boldsymbol{A}_i : i \in I\}$ is a family of similar algebras and $U$ an ultrafilter on $I$, then $\theta_U$ is a congruence of $\prod_{i \in I} \boldsymbol{A}_i$.*

*Proof.* We begin by proving that $\theta_U$ is an equivalence relation on $\prod_{i \in I} A_i$. To this end, consider $\vec{a}, \vec{b}, \vec{c} \in \prod_{i \in I} A_i$. We have

$$[\![\vec{a} = \vec{a}]\!] = \{i \in I : \vec{a}(i) = \vec{a}(i)\} = I.$$

Observe that $I \in U$, since $U$ is a nonempty upset of $\mathcal{P}(I)$. Together with the above display, this yields $[\![\vec{a} = \vec{a}]\!] \in U$ and, therefore, $\langle \vec{a}, \vec{a} \rangle \in \theta_U$. It follows that $\theta_U$ is reflexive. To prove that it is symmetric, suppose that $\langle \vec{a}, \vec{c} \rangle \in \theta_U$. Then $[\![\vec{a} = \vec{c}]\!] \in U$. Since $[\![\vec{a} = \vec{c}]\!] = [\![\vec{c} = \vec{a}]\!]$, this implies $[\![\vec{c} = \vec{a}]\!] \in U$ and, therefore, $\langle \vec{c}, \vec{a} \rangle \in \theta_U$. Lastly, to prove that $\theta_U$ is transitive, suppose that $\langle \vec{a}, \vec{b} \rangle, \langle \vec{b}, \vec{c} \rangle \in \theta_U$, that is, $[\![\vec{a} = \vec{b}]\!], [\![\vec{b} = \vec{c}]\!] \in U$. Since $U$ is closed under binary meets,

$$[\![\vec{a} = \vec{b}]\!] \cap [\![\vec{b} = \vec{c}]\!] \in U$$

Clearly, $[\![\vec{a} = \vec{b}]\!] \cap [\![\vec{b} = \vec{c}]\!] \subseteq [\![\vec{a} = \vec{c}]\!]$. Since $U$ is an upset of $\mathcal{P}(I)$, we obtain that $[\![\vec{a} = \vec{c}]\!] \in U$, whence $\langle \vec{a}, \vec{c} \rangle \in \theta_U$. We conclude that $\theta_U$ is an equivalence relation.

To prove that $\theta_U$ is a congruence, it only remains to show that it preserves the basic operations. Accordingly, let $f$ be a basic $n$-ary operation and $\vec{a}_1, \dots, \vec{a}_n, \vec{c}_1, \dots, \vec{c}_n \in \prod_{i \in I} A_i$ such that

$$\langle \vec{a}_1, \vec{c}_1 \rangle, \dots, \langle \vec{a}_n, \vec{c}_n \rangle \in \theta_U.$$

By definition of $\theta_U$, this amounts to $[\![\vec{a}_1 = \vec{c}_1]\!], \dots, [\![\vec{a}_n = \vec{c}_n]\!] \in U$. Since $U$ is a filter, it is closed under finite meets, whence

$$[\![\vec{a}_1 = \vec{c}_1]\!] \cap \dots \cap [\![\vec{a}_n = \vec{c}_n]\!] \in U. \tag{4.1}$$

We will show that

$$[\![\vec{a}_1 = \vec{c}_1]\!] \cap \dots \cap [\![\vec{a}_n = \vec{c}_n]\!] \subseteq [\![f^{\prod_{i \in I} \boldsymbol{A}_i}(\vec{a}_1, \dots, \vec{a}_n) = f^{\prod_{i \in I} \boldsymbol{A}_i}(\vec{c}_1, \dots, \vec{c}_n)]\!]. \tag{4.2}$$

To this end, consider $j \in [\![\vec{a}_1 = \vec{c}_1]\!] \cap \dots \cap [\![\vec{a}_n = \vec{c}_n]\!]$. We have

$$\vec{a}_1(j) = \vec{c}_1(j), \dots, \vec{a}_n(j) = \vec{c}_n(j).$$

Consequently,

$$\begin{aligned} f^{\prod_{i \in I} \boldsymbol{A}_i}(\vec{a}_1, \dots, \vec{a})(j) &= f^{\boldsymbol{A}_j}(\vec{a}_1(j), \dots, \vec{a}_n(j)) \\ &= f^{\boldsymbol{A}_j}(\vec{c}_1(j), \dots, \vec{c}_n(j)) \\ &= f^{\prod_{i \in I} \boldsymbol{A}_i}(\vec{c}_1, \dots, \vec{c})(j), \end{aligned}$$

that is, $j \in [\![f^{\prod_{i \in I} \boldsymbol{A}_i}(\vec{a}_1, \dots, \vec{a}_n) = f^{\prod_{i \in I} \boldsymbol{A}_i}(\vec{c}_1, \dots, \vec{c}_n)]\!]$. This establishes (4.2). Since $U$ is an upset of $\mathcal{P}(I)$, from (4.1) and (4.2) it follows

$$[\![f^{\prod_{i \in I} \boldsymbol{A}_i}(\vec{a}_1, \dots, \vec{a}_n) = f^{\prod_{i \in I} \boldsymbol{A}_i}(\vec{c}_1, \dots, \vec{c}_n)]\!] \in U.$$

Hence, we conclude that $\langle f^{\prod_{i \in I} \boldsymbol{A}_i}(\vec{a}_1, \dots, \vec{a}_n), f^{\prod_{i \in I} \boldsymbol{A}_i}(\vec{c}_1, \dots, \vec{c}_n) \rangle \in \theta_U$, as desired. ⊠

In view of the above result, we can make the following definition.

**Definition 4.2.** An *ultraproduct* of a family of similar algebras $\{\boldsymbol{A}_i : i \in I\}$ is an algebra of the form $\prod_{i \in I} \boldsymbol{A}_i / \theta_U$, for some ultrafilter $U$ on $I$.

Given a class of similar algebras $\mathsf{K}$, we set

$$\begin{aligned} \mathbb{I}(\mathsf{K}) &:= \{\boldsymbol{A} : \boldsymbol{A} \text{ is isomorphic to some member of } \mathsf{K}\} \\ \mathbb{S}(\mathsf{K}) &:= \{\boldsymbol{A} : \boldsymbol{A} \text{ is a subalgebra of some member of } \mathsf{K}\} \\ \mathbb{P}_{\mathrm{U}}(\mathsf{K}) &:= \{\boldsymbol{A} : \boldsymbol{A} \text{ is an ultraproduct of a family } \{\boldsymbol{B}_i : i \in I\} \subseteq \mathsf{K}\}. \end{aligned}$$

The importance of ultraproducts is largely due to the following result.

**Łoś' Theorem 4.3.** *Let $\{\boldsymbol{A}_i : i \in I\}$ be a family of similar algebras, $U$ an ultrafilter on $I$ and $\phi(x_1, \dots, x_n)$ a first order formula. For every $\vec{a}_1, \dots, \vec{a}_n \in \prod_{i \in I} A_i$,*

$$\prod_{i \in I} \boldsymbol{A}_i / \theta_U \vDash \phi(\vec{a}_1 / \theta_U, \dots, \vec{a}_n / \theta_U) \iff \{i \in I : \boldsymbol{A}_i \vDash \phi(\vec{a}_1(i), \dots, \vec{a}_n(i))\} \in U.$$

**Corollary 4.4.** *Let $\{\boldsymbol{A}_i : i \in I\}$ be a family of similar algebras, $U$ an ultrafilter on $I$ and $\phi$ a sentence. If $\phi$ is valid in all the $\boldsymbol{A}_i$, then it is valid in $\prod_{i \in I} \boldsymbol{A}_i/\theta_U$.*

In view Łos' Theorem, ultraproducts are instrumental to construct nonstandard models of first order theories. For instance, let $\mathbb{N} = \langle \mathbb{N}; s, +, \cdot, 0\rangle$ be the standard model of Peano Arithmetic. If $U$ is an ultrafilter on $\mathbb{N}$, the ultraproduct $\prod_{n \in \mathbb{N}} \mathbb{N}_n/U$ is *elementarily equivalent* to $\mathbb{N}$, that is, it satisfies the same sentences as $\mathbb{N}$. On the other hand, it is not hard to see that if $U$ is free (i.e., nonprincipal), $\prod_{n \in \mathbb{N}} \mathbb{N}_n/U$ is uncountable and, therefore, contains many "infinite" (or nonstandard) natural numbers.

For the present purpose, however, we will not need the full strength of Łoś' Theorem and, therefore, we shall omit its proof. Instead, we shall focus on a particular embedding theorem for ultraproducts that depends on the following notion.

**Definition 4.5.** A *local subgraph* of an algebra $\boldsymbol{A} = \langle A; \{g_i : i \in I\}\rangle$ is a pair $\mathbb{X} = \langle X; \{g_i{\restriction}_X : i \in J\}\rangle$ where $X$ is finite subset of $A$, $J$ is a finite subset of $I$, and for each $i \in J$ we have that $g_i{\restriction}_X$ is the partial function obtained by restricting $g_i$ to $X$.

In this case, $\mathbb{X}$ is a finite *partial* algebra of finite type (even when the type of $\boldsymbol{A}$ is infinite).

Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be similar algebras and $\mathbb{X}$ a local subgraph of $\boldsymbol{A}$. A map $f \colon X \to B$ is said to be an *embedding* of $\mathbb{X}$ into $\boldsymbol{B}$ if it is injective and, for every basic $n$-ary operation $g$ of the type of $\mathbb{X}$ and $a_1, \dots, a_n \in X$ such that $g^{\boldsymbol{A}}(a_1, \dots, a_n) \in X$,

$$f(g^{\boldsymbol{A}}(a_1, \dots, a_n)) = g^{\boldsymbol{B}}(f(a_1), \dots, f(a_n)).$$

**Theorem 4.6.** *Let $\mathsf{K} \cup \{\boldsymbol{A}\}$ be a class of similar algebras. If every local subgraph of $\boldsymbol{A}$ can be embedded into some member of $\mathsf{K}$, then $\boldsymbol{A} \in \mathbb{ISP}_{\mathrm{U}}(\mathsf{K})$.*

*Proof.* Let $I$ be the set of local subgraphs of $\boldsymbol{A}$. By assumption, for every $\mathbb{X} \in I$ there are an algebra $\boldsymbol{B}_{\mathbb{X}} \in \mathsf{K}$ and an embedding $h_{\mathbb{X}} \colon \mathbb{X} \to \boldsymbol{B}_{\mathbb{X}}$. We define a partial order $\sqsubseteq$ on $I$ as follows:

$$\mathbb{X} \sqsubseteq \mathbb{Y} \iff X \subseteq Y \text{ and the type of } \mathbb{Y} \text{ extends that of } \mathbb{X}.$$

Then, for every $\mathbb{X} \in I$, define

$$J_{\mathbb{X}} := \{\mathbb{Y} \in I \colon \mathbb{X} \sqsubseteq \mathbb{Y}\}.$$

Moreover, let $\mathcal{F}$ be the filter of $\mathcal{P}(I)$ generated by $\{J_{\mathbb{X}} \colon \mathbb{X} \in I\}$. Recall that

$$\mathcal{F} = \{Y \subseteq I : J_{\mathbb{X}_1} \cap \dots \cap J_{\mathbb{X}_n} \subseteq Y, \text{ for some } \mathbb{X}_1, \dots, \mathbb{X}_n \in I\}.$$

We will prove that $\mathcal{F}$ is proper. To this end, consider $\mathbb{X}_1, \dots, \mathbb{X}_n \in I$. Then let $\mathbb{Y}$ be the local subgraph of $\boldsymbol{A}$ with universe $Y := X_1 \cup \dots \cup X_n$ and whose type in the union of the types of the various $\mathbb{X}_i$. Then

$$\mathbb{X}_i \sqsubseteq \mathbb{Y}, \text{ for every } i \leqslant n,$$

that is, $\mathbb{Y} \in J_{\mathbb{X}_1} \cap \dots \cap J_{\mathbb{X}_n}$. It follows that $\emptyset \notin \mathcal{F}$ and, therefore, that $\mathcal{F}$ is proper. As $\mathcal{F}$ is a proper filter, by the Ultrafilter Theorem it can be extended to an ultrafilter $U$ on $I$.

Now, consider a map

$$f\colon A \to \prod_{\mathbb{X}\in I} B_{\mathbb{X}}$$

such that $f(a)(\mathbb{X}) = h_{\mathbb{X}}(a)$, for every $a \in A$ and $\mathbb{X} \in I$ such that $a \in \mathbb{X}$. Moreover, let

$$f^*\colon A \to \prod_{\mathbb{X}\in I} B_{\mathbb{X}}/\theta_U$$

be the map defined by the rule

$$f^*(a) \coloneqq f(a)/\theta_U.$$

We will show $f^*$ is an embedding of $\boldsymbol{A}$ into $\prod_{\mathbb{X}\in I} \boldsymbol{B}_{\mathbb{X}}/\theta_U$.

In order to prove that $f^*$ is injective, consider a pair of distinct elements $a, c \in A$. Consider a local subgraph $\mathbb{Y}$ of $\boldsymbol{A}$ containing $a$ and $c$. We will show that

$$J_{\mathbb{Y}} \subseteq \{\mathbb{X} \in I : f(a)(\mathbb{X}) \neq f(c)(\mathbb{X})\} \tag{4.3}$$

Consider $\mathbb{X} \in J_{\mathbb{Y}}$. Then $\mathbb{Y} \sqsubseteq \mathbb{X}$ and, therefore, $a, c \in Y \subseteq X$. Since $a, c \in X$, we have

$$f(a)(\mathbb{X}) = h_{\mathbb{X}}(a) \;\text{ and }\; f(c)(\mathbb{X}) = h_{\mathbb{X}}(c).$$

Furthermore, $h_{\mathbb{X}}(a) \neq h_{\mathbb{X}}(c)$, because $h_{\mathbb{X}}$ is injective and $a \neq c$. This yields $f(a)(\mathbb{X}) \neq f(c)(\mathbb{X})$, establishing (4.3).

Recall that the definition of $U$ guarantees that $J_{\mathbb{Y}} \in \mathcal{F} \subseteq U$. Therefore, since $U$ is an upset of $\mathcal{P}(I)$, we can apply (4.3) obtaining

$$I \smallsetminus [\![f(a) = f(c)]\!] = \{\mathbb{X} \in I : f(a)(\mathbb{X}) \neq f(c)(\mathbb{X})\} \in U.$$

Since $U$ is a proper filter, this implies

$$[\![f(a) = f(c)]\!] \notin U$$

and, therefore,

$$f^*(a) = f(a)/\theta_U \neq f(c)/\theta_U = f^*(c).$$

Hence, we conclude that $f^*$ is injective.

To prove that it is a homomorphism, consider a basic $n$-ary operation $g$ and $a_1, \dots, a_n \in A$. Then consider a local subgraph $\mathbb{Y}$ of $\boldsymbol{A}$ whose universe contains $a_1, \dots, a_n, g^{\boldsymbol{A}}(a_1, \dots, a_n)$ and whose type contains $g$. We will prove that

$$J_{\mathbb{Y}} \subseteq [\![f(g^{\boldsymbol{A}}(a_1, \dots, a_n)) = g^{\prod_{\mathbb{X}\in I} \boldsymbol{B}_{\mathbb{X}}}(f(a_1), \dots, f(a_n))]\!]. \tag{4.4}$$

Consider $\mathbb{V} \in J_{\mathbb{Y}}$. Since $\mathbb{Y} \sqsubseteq \mathbb{V}$, the type of $\mathbb{V}$ contains $g$ and $a_1, \dots, a_n, g^{\boldsymbol{A}}(a_1, \dots, a_n) \in V$. Since $a_1, \dots, a_n, g^{\boldsymbol{A}}(a_1, \dots, a_n) \in V$, we have

$$\begin{aligned} f(a_1)(\mathbb{V}) &= h_{\mathbb{V}}(a_1) \\ &\vdots \\ f(a_n)(\mathbb{V}) &= h_{\mathbb{V}}(a_n) \\ f(g^{\boldsymbol{A}}(a_1, \dots, a_n))(\mathbb{V}) &= h_{\mathbb{V}}(g^{\boldsymbol{A}}(a_1, \dots, a_n)). \end{aligned}$$

Furthermore, as the type of $\mathbb{V}$ contains $g$,

$$h_{\mathbb{V}}(g^{\boldsymbol{A}}(a_1, \dots, a_n)) = g^{\boldsymbol{B}_{\mathbb{V}}}(h_{\mathbb{V}}(a_1), \dots, h_{\mathbb{V}}(a_n)).$$

From the above displays it follows

$$f(g^{\boldsymbol{A}}(a_1, \dots, a_n))(\mathbb{V}) = g^{\boldsymbol{B}_{\mathbb{V}}}(f(a_1)(\mathbb{V}), \dots, f(a_n)(\mathbb{V})) = g^{\prod_{\mathbb{X} \in I} \boldsymbol{B}_{\mathbb{X}}}(f(a_1), \dots, f(a_n))(\mathbb{V}),$$

that is, $\mathbb{V} \in [\![ f(g^{\boldsymbol{A}}(a_1, \dots, a_n)) = g^{\prod_{\mathbb{X} \in I} \boldsymbol{B}_{\mathbb{X}}}(f(a_1), \dots, f(a_n)) ]\!]$. This establishes (4.4). Lastly, as $J_{\mathbb{Y}} \in U$ and $U$ is an upset of $\mathcal{P}(I)$, condition (4.4) implies

$$[\![ f(g^{\boldsymbol{A}}(a_1, \dots, a_n)) = g^{\prod_{\mathbb{X} \in I} \boldsymbol{B}_{\mathbb{X}}}(f(a_1), \dots, f(a_n)) ]\!] \in U,$$

and, therefore,

$$\begin{aligned} f^*(g^{\boldsymbol{A}}(a_1, \dots, a_n)) &= f(g^{\boldsymbol{A}}(a_1, \dots, a_n))/\theta_U \\ &= g^{\prod_{\mathbb{X} \in I} \boldsymbol{B}_{\mathbb{X}}}(f(a_1), \dots, f(a_n))/\theta_U \\ &= g^{\prod_{\mathbb{X} \in I} \boldsymbol{B}_{\mathbb{X}}/\theta_U}(f(a_1)/\theta_U, \dots, f(a_n)/\theta_U) \\ &= g^{\prod_{\mathbb{X} \in I} \boldsymbol{B}_{\mathbb{X}}/\theta_U}(f^*(a_1), \dots, f^*(a_n)). \end{aligned}$$

Hence, we conclude that $f^*$ is a homomorphism and, therefore, an embedding of $\boldsymbol{A}$ into $\prod_{\mathbb{Y} \in I} \boldsymbol{B}_{\mathbb{Y}}/\theta_U$. As a consequence,

$$\boldsymbol{A} \in \mathbb{ISP}_{\mathrm{U}}(\{\boldsymbol{B}_{\mathbb{X}} : \mathbb{X} \in I\}) \subseteq \mathbb{ISP}_{\mathrm{U}}(\mathsf{K}). \qquad \boxtimes$$

**Corollary 4.7.** *Every algebra embeds into an ultraproduct of its finitely generated subalgebras.*