

# Part III: Context-Free Languages

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# Outline

- 1 Context-Free Grammars
- 2 Pushdown Automata
- 3 Equivalence of pdas and cfigs
- 4 Properties of CFLs
- 5 Deterministic pdas

# Context-Free Grammars

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- A context-free language is generated by a *context-free grammar*
- CFG are used to describe the syntax of programming languages:

```
<definition> → (define <identifier> <expression>)
               → (define (<identifier> <params>) <expression>)
<params>      → ε
               → <identifier> <parameters>
<expression> → <number>
               → <identifier>
               → (+ <expression> <expression>)
               → (- <expression> <expression>)
               → (* <expression> <expression>)
               → (quotient <expression> <expression>)
               → (remainder <expression> <expression>)
```

# Context-Free Grammars

- A context-free grammar is an instance of (make-cfg  $N \Sigma R S$ )
- $N$  is the set of capital letters in the Roman alphabet representing the nonterminal symbols (i.e., syntactic categories)
- $\Sigma$  is the set of lowercase symbols in the Roman alphabet called the alphabet (or terminal symbols)
- $S$  is the starting nonterminal symbol
- $R$  is the set of production rules
  - Each production rule is of the form  $(N \rightarrow (N \cup \Sigma \cup \epsilon)^+)$
  - There is a single nonterminal on the left hand side of a production rule
  - There is a symbol consisting of one or more nonterminals, terminals, or  $\epsilon$  on the right hand side of a production rule

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- A derivation consists of 1 or more derivation steps
- A derivation step is the application of a production rule:  $\rightarrow_G$  or  $\rightarrow$  if  $G$  is clear from the context
- Zero or more derivation steps is denoted by  $\rightarrow_G^*$  or  $\rightarrow^*$  if  $G$  is clear from the context

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- Zero or more derivation steps is denoted by  $\rightarrow_G^*$  or  $\rightarrow^*$  if  $G$  is clear from the context
- $L(G)$  denotes the language generated by  $G$ :  $\{w \mid w \in \Sigma^* \wedge S \rightarrow_G^* w\}$
- A language,  $L$ , is context-free if  $L = L(G)$  for some context-free grammar  $G$



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- Follow the steps of the DR for grammars

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- How can this be done?

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- It also suggests that nonempty words in  $L$  may be generated from the outside in

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- It also suggests that nonempty words in  $L$  may be generated from the outside in
- Generate an  $a$  as the first letter in the word, generate a  $b$  for the last letter in the word, and use  $S$  to generate the rest of the word in the middle

# Context-Free Grammars

- Production Rules

$\rightarrow (S, \text{ARROW}, \text{EMP})$

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- Tests

;; Tests for  $a^2nb^2n$

```
(check-equal? (grammar-derive a2nb2n '(b b b))  
              "(b b b) is not in L(G)")
```

```
(check-equal? (grammar-derive a2nb2n '(a b a))  
              "(a b a) is not in L(G)")
```

```
(check-equal? (grammar-derive a2nb2n '(a b))  
              '(S -> aSb -> ab))
```

```
(check-equal? (grammar-derive a2nb2n '(a a a b b b))  
              '(S -> aSb -> aaSbb -> aaaSbbb -> aaabbbb))
```



# Context-Free Grammars

- `#lang fsm`  
;; L =  $a^n b^n$  Syntactic Categories  
;; S = words that start with n a and end with n b  
(define a2nb2n (make-cfg '(S)  
                          '(a b)  
                          `((S ,ARROW ,EMP) (S ,ARROW aSb))  
                          'S))  
(check-equal? (grammar-derive a2nb2n '(b b b)) ;; Tests for a2nb2n  
              "(b b b) is not in L(G)")  
(check-equal? (grammar-derive a2nb2n '(a b a))  
              "(a b a) is not in L(G)")  
(check-equal? (grammar-derive a2nb2n '(a b))  
              '(S -> aSb -> ab))  
(check-equal? (grammar-derive a2nb2n '(a a a b b b))  
              '(S -> aSb -> aaSbb -> aaaSbbb -> aaabbbb))

# Context-Free Grammars

- ```
> (grammar-test a2nb2n 10)
'(((b a b b b a) "(b a b b b a) is not in L(G)")
  (() "The word () is too short to test.")
  ((a a b b) (S -> aSb -> aaSbb -> aabb))
  ((a a) "(a a) is not in L(G)")
  ((a a b b b b b a b) "(a a b b b b b a b) is not in L(G)")
  ((a a a b a a) "(a a a b a a) is not in L(G)")
  ((a a a b b a a) "(a a a b b a a) is not in L(G)")
  ((a b) (S -> aSb -> ab))
  ((a) "The word (a) is too short to test.")
  ((b b b b) "(b b b b) is not in L(G)"))
```
- Be mindful that trying to randomly generate words in the language is not easy
- Most of the time a word that is too short or that is not in the language is generated
- It is important for unit tests to be thorough

# Context-Free Grammars

- $L = \{w \mid w \in (a b)^* \wedge w \text{ has more bs than as}\}$

# Context-Free Grammars

- $L = \{w \mid w \in (a b)^* \wedge w \text{ has more } b\text{'s than } a\text{'s}\}$
- Every word in the language must have at least one more b than a
- Suggests that at least a b must be generated
- After that, for every a generated there must be one or more bs generated

# Context-Free Grammars

- `#lang fsm`

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- `#lang fsm`
- `;; Syntactic Categories`  
`;; S = words such that number of b > number of a`

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# Context-Free Grammars

- `#lang fsm`
- `;; Syntactic Categories`
- `;; S = words such that number of b > number of a`
- `;; A = words such that number of b >= number of a`
- `;; L = w | w in (a b)* AND w has more b than a`  
`(define numb>numa (make-cfg '(S A)`  
`'(a b)`
- `'S))`



# Context-Free Grammars

- `#lang fsm`
- `;; Syntactic Categories`
- `;; S = words such that number of b > number of a`
- `;; A = words such that number of b >= number of a`
- `;; L = w | w in (a b)* AND w has more b than a`
- `(define numb>numa (make-cfg '(S A)`  
    `'(a b)`  
    `'((S ,ARROW b)`  
      `(S ,ARROW AbA)`  
      `(A ,ARROW AaAbA)`  
      `(A ,ARROW AbAaA)`  
      `(A ,ARROW ,EMP)`  
      `(A ,ARROW bA))`  
    `'S))`

# Context-Free Grammars

- ```
#lang fsm
```
- ```
;; Syntactic Categories
```
- ```
;; S = words such that number of b > number of a
```
- ```
;; A = words such that number of b >= number of a
```
- ```
;; L = w | w in (a b)* AND w has more b than a
```
- ```
(define numb>numa (make-cfg '(S A)
```
- ```
  '(a b)
```
- ```
  `((S ,ARROW b)
```
- ```
    (S ,ARROW AbA)
```
- ```
    (A ,ARROW AaAbA)
```
- ```
    (A ,ARROW AbAaA)
```
- ```
    (A ,ARROW ,EMP)
```
- ```
    (A ,ARROW bA))
```
- ```
'S))
```
- ```
;; Tests for numb>numa
```
- ```
(check-equal? (grammar-derive numb>numa '(a b))
```
- ```
  "(a b) is not in L(G)")
```
- ```
(check-equal? (grammar-derive numb>numa '(a b a))
```
- ```
  "(a b a) is not in L(G)")
```
- ```
(check-equal? (grammar-derive numb>numa '(a a a a a))
```
- ```
  "(a a a a a) is not in L(G)")
```
- ```
(check-equal? (grammar-derive numb>numa '(b b b))
```
- ```
  '(S -> AbA -> bA -> bbA -> bbbA -> bbb))
```
- ```
(check-equal? (grammar-derive numb>numa '(b b a b a a b))
```
- ```
  '(S -> AbA -> AbAaAbA -> bAaAbA -> bAbAaAaAbA
```
- ```
    -> bAbAaAbAaAaAbA -> bbAaAbAaAaAbA -> bbaAbAaAaAbA
```
- ```
    -> bbabAaAaAbA -> bbabaAaAbA -> bbabaaAbA
```
- ```
    -> bbabaabA -> bbabaab))
```
- ```
(check-equal? (grammar-derive numb>numa '(a a a b b b b))
```
- ```
  '(S -> AbA -> AaAbAbA -> aAbAbA -> aAaAbAbAbA
```
- ```
    -> aaAbAbAbAbA -> aaAaAbAbAbAbA -> aaaAbAbAbAbA
```
- ```
    -> aaabAbAbAbA -> aaabbbAbAbA -> aaabbbbAbA
```
- ```
    -> aaabbbbA -> aaabbbb))
```

# Context-Free Grammars

- ```
> (grammar-test numb>numa 5)
'(((a b b b a b b b b)
  (S -> AbA -> AaAbAbA -> AaAbAaAbAbA -> aAbAaAbAbA
    -> abAaAbAbA -> abbAaAbAbA -> abbbAaAbAbA -> abbbbaAbAbA
    -> abbbbabAbA -> abbbbabba -> abbbbabbbA -> abbbbabbbbA
    -> abbbabbbb))
  ((a a b b b)
    (S -> AbA -> AaAbAbA -> aAbAbA -> aAaAbAbAbA -> aaAbAbAbA
      -> aabAbAbA -> aabbAbA -> aabbbbA -> aabbbb))
  (()) "The word () is too short to test.")
  ((a b b a b a a b) "(a b b a b a a b) is not in L(G)")
  ((a b b b b a b)
    (S -> AbA -> AaAbAbA -> aAbAbA -> abAbA -> abAbAaAbA
      -> abbAaAbA -> abbbAaAbA -> abbbbAaAbA -> abbbbaAbA
      -> abbbbabA -> abbbbab)))
```
- Be mindful about tests taking very long to run
- Interrupt execution if necessary

# Context-Free Grammars

- HOMEWORK: 1, 3, 5, 8

# Context-Free Grammars

- Are all regular languages context-free?

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## Theorem

*All regular languages are context-free.*

-

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## Theorem

*All regular languages are context-free.*

- 

## Proof.

Assume  $L$  is a regular language. This means that there exist a regular grammar:

- $G = (\text{make-rg } N \ \Sigma \ P \ S),$

such that  $L = L(G)$ . All the production rules in  $P$  are of the form:

1.  $A \rightarrow \epsilon$
2.  $A \rightarrow a$ , where  $a \in \Sigma$
3.  $A \rightarrow aB$ , where  $a \in \Sigma \wedge B \in N$

This means  $P \subseteq (N \rightarrow (N \cup \Sigma \cup \epsilon)^+)$ . That is,  $p \in P$  is a valid production rule for a context-free grammar. Given that  $G$ 's  $N$ ,  $\Sigma$ , and  $S$  may also be used in a cfg, we may conclude that  $L$  is a context-free language.  $\square$

# Context-Free Grammars

- ```
(define MULT3-as (make-rg '(S B C)
                             '(a b)
                             `((S ,ARROW ,EMP)
                               (S ,ARROW aB)
                               (S ,ARROW bS)
                               (B ,ARROW aC)
                               (B ,ARROW bB)
                               (C ,ARROW aS)
                               (C ,ARROW bC))
                             'S))
```



# Context-Free Grammars

- ```
(define MULT3-as (make-rg '(S B C)
  '(a b)
  `((S ,ARROW ,EMP)
    (S ,ARROW aB)
    (S ,ARROW bS)
    (B ,ARROW aC)
    (B ,ARROW bB)
    (C ,ARROW aS)
    (C ,ARROW bC))
  'S))
```
- ```
(define MULT3-as (make-cfg '(S B C)
  '(a b)
  `((S ,ARROW ,EMP)
    (S ,ARROW aB)
    (S ,ARROW bS)
    (B ,ARROW aC)
    (B ,ARROW bB)
    (C ,ARROW aS)
    (C ,ARROW bC))
  'S))
```

# Context-Free Grammars

- Consider the language of words containing balanced parenthesis

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- `#lang fsm`

# Context-Free Grammars

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- Opening parenthesis: o and closing parenthesis: c
- #lang fsm

- ```
;; Syntactic categories: S = words with balanced parenthesis
;; L = {w | w has balanced parenthesis}, where o = ( and c = )
(define BP (make-cfg '(S)
                     '(o c)
```

- ```
'S))
```

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- Consider the language of words containing balanced parenthesis
- Opening parenthesis: o and closing parenthesis: c
- #lang fsm

- ```
;; Syntactic categories: S = words with balanced parenthesis
;; L = {w | w has balanced parenthesis}, where o = ( and c = )
(define BP (make-cfg '(S)
  '(o c)
  `((S ,ARROW ,EMP)
    (S ,ARROW SS)
    (S ,ARROW oSc))
  'S))
```

# Context-Free Grammars

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- ;; Syntactic categories: S = words with balanced parenthesis  
;; L = {w | w has balanced parenthesis}, where o = ( and c = )  
(define BP (make-cfg ' (S)  
                          ' (o c)  
                          ' ((S ,ARROW ,EMP)  
                              (S ,ARROW SS)  
                              (S ,ARROW oSc))  
                          'S))  
• (check-equal? (grammar-derive BP (parensstr->los ""))  
                  "The word () is too short to test.")  
• (check-equal? (grammar-derive BP (parensstr->los "))((")  
                  "(c c o o) is not in L(G).")  
• (check-equal? (grammar-derive BP (parensstr->los "()((")  
                  "(o c o) is not in L(G).")  
• (check-equal? (grammar-derive BP (parensstr->los "()((")  
                  '(S -> oSc -> oc))  
• (check-equal? (grammar-derive BP (parensstr->los "((()))()  
                  '(S -> SS -> oScS -> ooScSc -> oocScS -> oocccoc))

# Context-Free Grammars

- Consider the language of words containing balanced parenthesis
- Opening parenthesis: o and closing parenthesis: c
- #lang fsm
- ;; string  $\rightarrow$  (listof (o or c)) throws error
- ;; Purpose: Converts given string of parens to a list of parens
- (define (parensstr $\rightarrow$ los s)
 (map ( $\lambda$  (p)
 (cond [(eq? p #\ ( ) 'o]
 [(eq? p #\ ) 'c]
 [else (error (format "parensstr $\rightarrow$ los: non-parens symbol in: ~s"
 s))]))
 (string $\rightarrow$ list s)))
 )
 )
- ;; Tests for parensstr $\rightarrow$ los
 (check-equal? (parensstr $\rightarrow$ los "") '())
 (check-equal? (parensstr $\rightarrow$ los "(()())") '(o o c c o c))
 )
- ;; Syntactic categories: S = words with balanced parenthesis
 ;; L = {w | w has balanced parenthesis}, where o = ( and c = )
 (define BP (make-cfg 'S)
 '(o c)
 `((S ,ARROW ,EMP)
 (S ,ARROW SS)
 (S ,ARROW oSc))
 'S))
- (check-equal? (grammar-derive BP (parensstr $\rightarrow$ los ""))
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 (check-equal? (grammar-derive BP (parensstr $\rightarrow$ los "())")
 '(S  $\rightarrow$  oSc  $\rightarrow$  oc))
 (check-equal? (grammar-derive BP (parensstr $\rightarrow$ los "(()())")
 '(S  $\rightarrow$  SS  $\rightarrow$  oScS  $\rightarrow$  ooScS  $\rightarrow$  oocS  $\rightarrow$  oocSc  $\rightarrow$  ooccc))
 )



# Context-Free Grammars

- Consider deriving the " $(()())$ ":

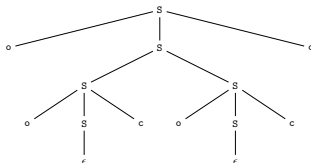
$(S \rightarrow oSc \rightarrow oSSc \rightarrow ooScSc \rightarrow oocSc \rightarrow oocoSc c \rightarrow oococc)$

# Context-Free Grammars

- Consider deriving the " $()()$ ":

$(S \rightarrow oSc \rightarrow oSSc \rightarrow ooScSc \rightarrow oocSc \rightarrow oocoScc \rightarrow oococc)$

- A derivation may be visualized as a tree:



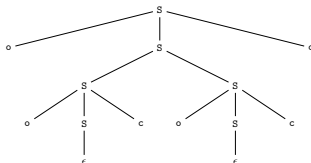
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# Context-Free Grammars

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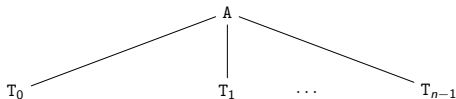
- Such a visualization is called a parse tree
- The leaves are terminal symbols or  $\epsilon$
- The interior nodes are rooted at a nonterminal
- The edges out of an interior node indicate what is generated from the nonterminal
- The *yield* of a parse tree is the concatenation of all the leaves from left to right

# Context-Free Grammars

- A parse tree is formally defined as follows:
  - For each  $a \in \Sigma$  the following is a parse tree:  $a$
  - For  $A \rightarrow \epsilon$  the following is a parse tree:



- If  $T_0 T_1 \dots T_{n-1}$  are the parse trees for  $a_0 \dots a_{n-1}$ , where  $a_i$  is a nonterminal or a terminal symbol, then for  $A \rightarrow a_0 a_1 \dots a_{n-1}$  the following is a parse tree:



# Context-Free Grammars

- Parse trees represent derivations eliminating irrelevant differences like the order in which production rules are applied
- Two derivations are similar if they are captured by the same parse tree:

$S \rightarrow oSc \rightarrow oSSc \rightarrow ooScSc \rightarrow oocSc \rightarrow oocoScc \rightarrow oococc$

$S \rightarrow oSc \rightarrow oSSc \rightarrow oSoScc \rightarrow oSocc \rightarrow ooScocc \rightarrow oococc$

# Context-Free Grammars

- Parse trees represent derivations eliminating irrelevant differences like the order in which production rules are applied
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$$\begin{array}{l} S \rightarrow oSc \rightarrow oSSc \rightarrow ooScSc \rightarrow oocSc \rightarrow oocoScc \rightarrow oococc \\ S \rightarrow oSc \rightarrow oSSc \rightarrow oSoScc \rightarrow oSocc \rightarrow ooScocc \rightarrow oococc \end{array}$$
- Both of these derivations are captured by the previous parse tree
- Use the same production rules on the same nonterminals. The only difference is the order in which the production rules are used

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- Two derivations are similar if they are captured by the same parse tree:  
$$\begin{array}{l} S \rightarrow oSc \rightarrow oSSc \rightarrow ooScSc \rightarrow oocSc \rightarrow oocoSc \rightarrow oococc \\ S \rightarrow oSc \rightarrow oSSc \rightarrow oSoSc \rightarrow oSocc \rightarrow ooScocc \rightarrow oococc \end{array}$$
- Both of these derivations are captured by the previous parse tree
- Use the same production rules on the same nonterminals. The only difference is the order in which the production rules are used
- The first is a *leftmost derivation*
- The second is a *rightmost derivation*
- Every parse tree has a unique leftmost and a unique rightmost derivation

# Context-Free Grammars

- Equivalent:

- $A \rightarrow_G^* w$
- $\exists$  a parse tree with root  $A$  and yield  $w$
- $\exists$  a leftmost derivation  $A \xrightarrow_G^L{}^* w$
- $\exists$  a rightmost derivation  $A \xrightarrow_G^R{}^* w$

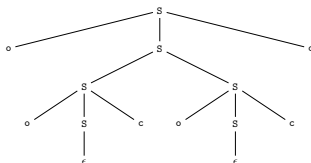


# Context-Free Grammars

- Not all derivations of a word may be similar

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- Not captured by:

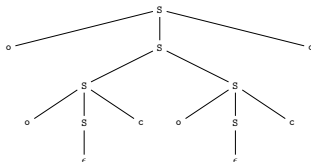


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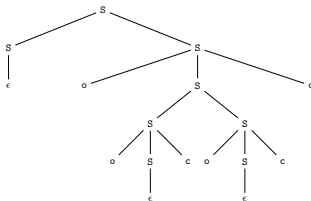
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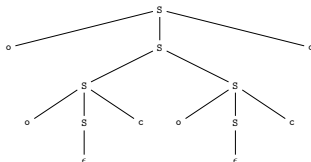


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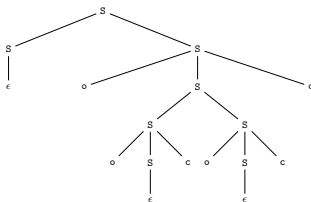
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- A cfg,  $G$ , is called *ambiguous* if for any  $w \in L(G)$  there are 2 or more parse trees
- BP is ambiguous

# Context-Free Grammars

- Ambiguity is a problem when meaning must be assigned to a word as done in the implementation of programming languages

# Context-Free Grammars

- Ambiguity is a problem when meaning must be assigned to a word as done in the implementation of programming languages
- Arithmetic expressions grammar:

```
#lang fsm
```

```
(define AE (make-cfg '(S I)
```

```
  '(p t x y z)
```

```
  `((S ,ARROW SpS)
```

```
    (S ,ARROW StS)
```

```
    (S ,ARROW I)
```

```
    (I ,ARROW x)
```

```
    (I ,ARROW y)
```

```
    (I ,ARROW z))
```

```
  'S))
```

```
;; Tests for AE
```

```
(check-equal? (grammar-derive AE '(x p x))
```

```
  '(S -> SpS -> IpS -> xpS -> xpI -> xpx))
```

```
(check-equal? (grammar-derive AE '(x t z))
```

```
  '(S -> StS -> ItS -> xtS -> xtI -> xtz))
```

```
(check-equal? (grammar-derive AE '(x t z p y))
```

```
  '(S -> SpS -> StSpS -> ItSpS -> xtSpS -> xtIpS -> xtztz  
    -> xtzpI -> xtzpy))
```

- p stands for +, t stands for \*
- x, y, and z are variables

## Context-Free Grammars

- $(x t z p y)$ , has two derivations that are not similar:  
$$\begin{aligned} S &\rightarrow SpS \rightarrow StSpS \rightarrow ItSpS \rightarrow xtSpS \rightarrow xtIpS \rightarrow xtzpS \\ &\rightarrow xtzpI \rightarrow xtzpy \end{aligned}$$
  
$$\begin{aligned} S &\rightarrow StS \rightarrow ItS \rightarrow xtS \rightarrow xtSpS \rightarrow xtIpS \rightarrow xtzpS \rightarrow xtzpI \\ &\rightarrow xtzpy \end{aligned}$$

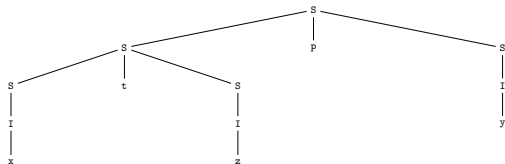
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- Suggests that x and z are multiplied and then their product and y are added

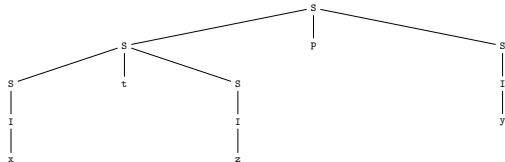
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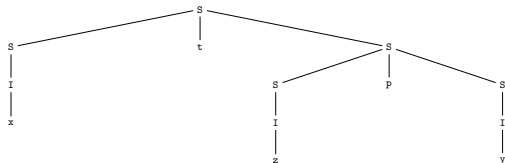
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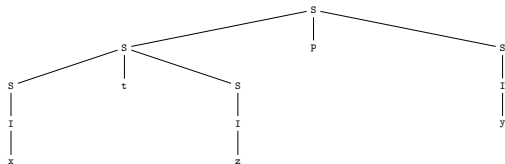
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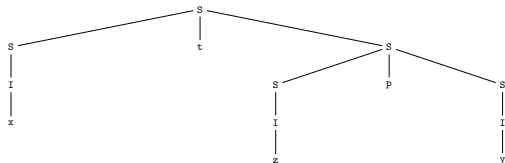
# Context-Free Grammars

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$$S \rightarrow StS \rightarrow ItS \rightarrow xtS \rightarrow xtSpS \rightarrow xtIpS \rightarrow xtzpS \rightarrow xtzpI$$
$$\rightarrow xtzpy$$

- Parse trees:



- Suggests that x and z are multiplied and then their product and y are added



- This parse tree suggests that first z and y are added and then their sum is multiplied by x
- A word may have different meanings when a grammar is ambiguous

# Context-Free Grammars

- Disambiguate AE: first produce the summing subexpressions, if any, and then produce the product subexpressions

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- Disambiguate AE: first produce the summing subexpressions, if any, and then produce the product subexpressions
- A new syntactic category, T, is introduced to generate product subexpressions
- A product subexpression may only be generated after summing subexpressions, if any, are generated:
- `#lang fsm`  

```
(define AE2 (make-cfg '(S I T)
  '(p t x y z)
  `((S ,ARROW SpS)
    (S ,ARROW T)
    (S ,ARROW I)
    (T ,ARROW TtT)
    (T ,ARROW I)
    (I ,ARROW x)
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    (I ,ARROW z))
  'S))
```

# Context-Free Grammars

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                          (I ,ARROW z))
                        'S))
```
- Unfortunately, some context-free languages can only be generated by an ambiguous cfg
- Such grammars are called *inherently ambiguous*

# Context-Free Grammars

- HOMEWORK: 10, 12, 14
- Quiz: in class (1 week)

# Pushdown Automata

- Would like to have a machine that decides if a given word is a member of a CFL
- What should such a machine look like?

# Pushdown Automata

- Would like to have a machine that decides if a given word is a member of a CFL
- What should such a machine look like?
- Write a program to decide  $a^n b^n$
- Call an auxiliary function that takes as input  $w$ 's sub-word without the leading  $a$ s, if any, and an accumulator with  $w$ 's leading  $a$ s
- 3 conditions:
  - ① If the given word is empty then the value from testing if the given accumulator is empty is returned.
  - ② If the first element of the given word is a then false is returned.
  - ③ Otherwise, return the conjunction of testing if the accumulator is not empty and checking the rest of both the given word and the given accumulator.

# Pushdown Automata

- ```
#lang fsm
;; word → Boolean
;; Purpose: Decide if given word is in  $a^n b^n$ 
(define (is-in- $a^n b^n?$  w)
```
- ```
;; Tests for is-in- $a^n b^n?$ 
(check-pred (λ (w) (not (is-in- $a^n b^n?$  w))) '(a))
(check-pred (λ (w) (not (is-in- $a^n b^n?$  w))) '(b b))
(check-pred (λ (w) (not (is-in- $a^n b^n?$  w))) '(a b b))
(check-pred (λ (w) (not (is-in- $a^n b^n?$  w))) '(a b a a b b))
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(check-pred is-in- $a^n b^n?$  '(a a b b))
```



# Pushdown Automata

- ```
#lang fsm
;; word → Boolean
;; Purpose: Decide if given word is in  $a^nb^n$ 
(define (is-in- $a^nb^n?$  w)
```
- ```
(check (dropf w (λ (s) (eq? s 'a))) ;; everything after initial as
      (takef w (λ (s) (eq? s 'a))))) ;; the initial as
```
- ```
;; Tests for is-in-anbn?
(check-pred (λ (w) (not (is-in- $a^nb^n?$  w))) '(a))
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```

# Pushdown Automata

- ```
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;; word → Boolean
;; Purpose: Decide if given word is in anbn
(define (is-in-anbn? w)
```
- ```
;; word (listof symbol) → Boolean
;; Purpose: Determine if first given word has only bs that
;;       match as in the second given word
;; Accumulator Invariant
;;   acc = the unmatched as at the beginning of w
;; Assume: w in (a b)*
(define (check wrd acc)
  (cond [(empty? wrd) (empty? acc)]
        [(eq? (first wrd) 'a) #f]
        [else (and (not (empty? acc))
                    (check (rest wrd) (rest acc)))]))
```
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```
- The acc is a stack
- The program first pushes all the as at the beginning of the given word onto the accumulator
- The auxiliary function pops an a for each recursive
- Suggests extending an ndfa with a stack to remember part of the consumed input

# Pushdown Automata

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- $((A \ a \ p) \ (B \ g))$  means that the machine is in state  $A$ , reads  $a$  from the input tape, pops  $p$  off the stack, pushes  $g$  onto the stack, and moves to  $B$
- May be nondeterministic

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- May be nondeterministic
- $((P \ EMP \ EMP) \ (Q \ j))$  is a push operation that does not consult the input tape nor the stack
- $((P \ EMP \ j) \ (Q \ EMP))$  is a pop operation

# Pushdown Automata

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- A computation of length  $n$  on a word,  $w$ , is denoted by:

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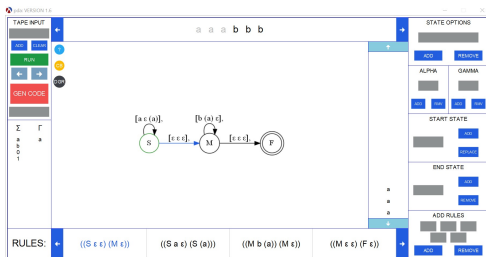
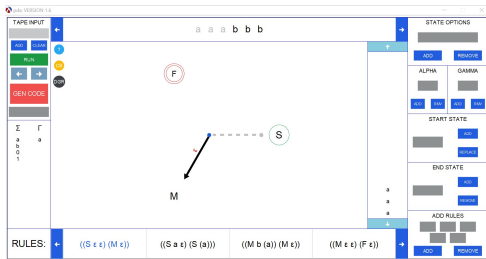
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- If a pda,  $M$ , starting in the starting state consumes all the input and reaches a final state with the stack empty then  $M$  accepts. Otherwise,  $M$  rejects
- A word,  $w$ , is in the language of  $M$ ,  $L(M)$ , if there is a computation from the starting state that consumes  $w$  and  $M$  accepts
- It does not matter that there may be computations that reject  $w$ .

# Pushdown Automata



# Pushdown Automata

- A state invariant predicate may be associated with each state
- An invariant predicate has two inputs: the consumed input and the stack
- It must test and relate the invariant conditions for and between the consumed input and the stack.

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- Design and implement a pda for  $L = a^n b^n$
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- Match  $b$ s by popping an  $a$ s

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- Design and implement a pda for  $L = a^n b^n$
- How may the stack be used?
- Accumulate the read  $a$ s
- Match  $b$ s by popping an  $a$ s
- After all the input is read, the machine ought to move to a final state
- It accepts if the stack is empty. Otherwise, it rejects

# Pushdown Automata

- Name:  $a^n b^n$   $\Sigma = \{a, b\}$

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- Name:  $a^n b^n$   $\Sigma = \{a, b\}$
- ;; Tests for  $a^n b^n$   

```
(check-equal? (sm-apply anbn '(a)) 'reject)
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(check-equal? (sm-apply anbn '(a b b)) 'reject)
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```

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```
- States

```
;; States
;; S:  ci = a* = stack, start state
```

# Pushdown Automata

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```
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```
;; States
;; S:  ci = a* = stack, start state

;; M:  ci = (append (listof a) (listof b))
;;      ^ stack = a*
;;      ^ |ci as| = |stack| + |ci bs|
```

# Pushdown Automata

- Name:  $a^n b^n \Sigma = \{a^n b^n\}$
- ;; Tests for  $a^n b^n$   
(check-equal? (sm-apply a<sup>n</sup>b<sup>n</sup> '(a)) 'reject)  
(check-equal? (sm-apply a<sup>n</sup>b<sup>n</sup> '(b b)) 'reject)  
(check-equal? (sm-apply a<sup>n</sup>b<sup>n</sup> '(a b b)) 'reject)  
(check-equal? (sm-apply a<sup>n</sup>b<sup>n</sup> '(a b a a b b)) 'reject)  
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- ;; M: ci = (append (listof a) (listof b))  
;;     ^ stack = a\*  
;;     ^ |ci as| = |stack| + |ci bs|
- ;; F: ci = (append (listof a) (listof b))  
;;     ^ |stack| = 0  
;;     ^ |ci as|=|ci bs|, final state

# Pushdown Automata

- Transition Relation
- Push all  $a$  and nondeterministically move to  $M$

$$((S, \text{EMP}, \text{EMP}) (M, \text{EMP})) \quad ((S a, \text{EMP}) (S(a)))$$



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- Transition Relation
- Push all  $a$  and nondeterministically move to  $M$

$$((S, \text{EMP}, \text{EMP}) (M, \text{EMP})) \quad ((S a, \text{EMP}) (S(a)))$$

- Match all  $b$  and nondeterministically move to  $F$

$$((M b(a)) (M, \text{EMP})) \quad ((M, \text{EMP}, \text{EMP}) (F, \text{EMP}))$$

# Pushdown Automata

- ```
;; L = {anbn | n >= 0}
;; States
;; S ci = (listof a) = stack, start state
;; M ci = (append (listof a) (listof b)) AND
;;   (length ci as) = (length stack) + (length ci bs)
;; F ci = (append (listof a) (listof b)) and all as and bs matched,
;;   final state
;; The stack is a (listof a)
(define anbn (make-ndpda '(S M F) '(a b) '(a) 'S '(F)
  `(((S ,EMP ,EMP) (M ,EMP))
    ((S a ,EMP) (S (a)))
    ((M b (a)) (M ,EMP))
    ((M ,EMP ,EMP) (F ,EMP)))))

;; Tests for anbn
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(define anbn (make-ndpda '(S M F) '(a b) '(a) 'S '(F)
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(check-equal? (sm-apply anbn '(a)) 'reject)
(check-equal? (sm-apply anbn '(b b)) 'reject)
(check-equal? (sm-apply anbn '(a b b)) 'reject)
(check-equal? (sm-apply anbn '(a b a a b b)) 'reject)
(check-equal? (sm-apply anbn '()) 'accept)
(check-equal? (sm-apply anbn '(a a b b)) 'accept)
```
- ```
> (sm-test anbn 10)
'(((b b b a a a a) reject)
  ((a b) accept)
  ((b b b a a b a b a) reject)
  ((a a a a b) reject)
  ((a a b) reject)
  ((b b b b b a a a) reject)
  ((b) reject)
  ((a a b b) accept)
  ((b b b a b b b a) reject)
  ((b a a a a) reject))
```

# Pushdown Automata

- ```
;; word stack → Boolean
;; Purpose: Determine if ci and stack are the same (listof a)
(define (S-INV ci stck)
  (and (= (length ci) (length stck))
        (andmap (λ (i g) (and (eq? i 'a) (eq? g 'a))) ci stck)))
;; Tests for S-INV
(check-equal? (S-INV '() '(a a)) #f) (check-equal? (S-INV '(a) '()) #f)
(check-equal? (S-INV '() '()) #t)
(check-equal? (S-INV '(a a a) '(a a a)) #t)
```

# Pushdown Automata

- ```
;; word stack → Boolean
;; Purpose: Determine if ci and stack are the same (listof a)
(define (S-INV ci stck)
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;; Tests for S-INV
(check-equal? (S-INV '() '(a a)) #f) (check-equal? (S-INV '(a) '()) #f)
(check-equal? (S-INV '() '()) #t)
(check-equal? (S-INV '(a a a) '(a a a)) #t)
```
- ```
;; word stack → Boolean
;; Purpose: Determine if ci = EMP or a+b+ AND the stack
;;           only contains a AND |ci as| = |stack| + |ci bs|
(define (M-INV ci stck)
  (let* [(as (takef ci (λ (s) (eq? s 'a))))
        (bs (takef (drop ci (length as)) (λ (s) (eq? s 'b))))]
    (and (equal? (append as bs) ci)
          (andmap (λ (s) (eq? s 'a)) stck)
          (= (length as) (+ (length bs) (length stck))))))
;; Tests for M-INV
(check-equal? (M-INV '(a a b) '(a a)) #f)
(check-equal? (M-INV '(a a a b) '(a)) #f)
(check-equal? (M-INV '() '()) #t)
(check-equal? (M-INV '(a) '(a)) #t)
(check-equal? (M-INV '(a a a b b) '(a)) #t)
```

# Pushdown Automata

- ;; word stack  $\rightarrow$  Boolean  
;; Purpose: Determine if ci and stack are the same (listof a)  
(define (S-INV ci stck)  
  (and (= (length ci) (length stck))  
        (andmap ( $\lambda$  (i g) (and (eq? i 'a) (eq? g 'a))) ci stck)))  
;; Tests for S-INV  
(check-equal? (S-INV '() '(a a)) #f) (check-equal? (S-INV '(a) '()) #f)  
(check-equal? (S-INV '() '()) #t)  
(check-equal? (S-INV '(a a a) '(a a a)) #t)
- ;; word stack  $\rightarrow$  Boolean  
;; Purpose: Determine if ci = EMP or a+b+ AND the stack  
;;       only contains a AND |ci as| = |stack| + |ci bs|  
(define (M-INV ci stck)  
  (let\* [(as (takef ci ( $\lambda$  (s) (eq? s 'a))))  
        (bs (takef (drop ci (length as)) ( $\lambda$  (s) (eq? s 'b))))]  
    (and (equal? (append as bs) ci)  
          (andmap ( $\lambda$  (s) (eq? s 'a)) stck)  
          (= (length as) (+ (length bs) (length stck)))))  
;; Tests for M-INV  
(check-equal? (M-INV '(a a b) '(a a)) #f)  
(check-equal? (M-INV '(a a a b) '(a)) #f)  
(check-equal? (M-INV '() '()) #t)  
(check-equal? (M-INV '(a) '(a)) #t)  
(check-equal? (M-INV '(a a a b b) '(a)) #t)
- ;; word stack  $\rightarrow$  Boolean Purpose: Determine if ci=a<sup>n</sup>b<sup>n</sup> & empty stack  
(define (F-INV ci stck)  
  (let\* [(as (takef ci ( $\lambda$  (s) (eq? s 'a))))  
        (bs (takef (drop ci (length as)) ( $\lambda$  (s) (eq? s 'b))))]  
    (and (empty? stck) (equal? (append as bs) ci) (= (length as) (length bs))))  
;; Tests for F-INV  
(check-equal? (F-INV '(a a b) '()) #f)  
(check-equal? (F-INV '(a a b b a b) '(a a a)) #f)  
(check-equal? (F-INV '() '()) #t)  
(check-equal? (F-INV '(a a b b) '()) #t)

# Pushdown Automata

- $L = a^n b^n$        $ci = \text{the consumed input}$      $w \in (\Sigma^+ M)^*$   
 $F = \{ \text{sm-finals } M \}$      $P = a^n b^n$

## Theorem

*The state invariants hold when  $P$  is applied to  $w$ .*

- 
- The proof is by induction on,  $n$ , the number of transitions to consume  $w$

# Pushdown Automata

- Base Case
  - When P starts, S-INV holds because  $ci = '()$  and the stack =  $'()$
  - Observe that empty transitions into M and F may lead to accept and, therefore, we must establish that the invariants for these states also hold
  - After using  $((S \text{ EMP EMP}) (M \text{ EMP}))$ , M-INV holds because  $ci = '()$  and the stack =  $'()$
  - After using  $((M \text{ EMP EMP}) (F \text{ EMP}))$ , F-INV also holds because  $ci = '()$  and the stack =  $'()$



# Pushdown Automata

- Assume INVs hold for  $k$  steps. Show they hold for the  $k+1$  step

# Pushdown Automata

- Assume INVs hold for  $k$  steps. Show they hold for the  $k+1$  step
- Proof invariants hold after each nonempty transition:  
 $((S \text{ a EMP}) (S (a)))$ 
  - By inductive hypothesis,  $S\text{-INV}$  holds
  - After consuming an  $a$  and pushing an  $a$ ,  $P$  may reach  $S$  and by empty transition  $M$
  - Note that an empty transition into  $F$  with a nonempty stack cannot lead to an accept. Therefore, we do not concern ourselves about  $P$  making such a transition
  - $S\text{-INV}$  and  $M\text{-INV}$  hold because both the length of the consumed input and of the stack increased by 1, thus, remaining of equal length and because both continue to only contain  $a$ s

# Pushdown Automata

- Assume INVs hold for  $k$  steps. Show they hold for the  $k+1$  step
- Proof invariants hold after each nonempty transition:  
 $((S \ a \ EMP) \ (S \ (a)))$ 
  - By inductive hypothesis,  $S$ -INV holds
  - After consuming an  $a$  and pushing an  $a$ ,  $P$  may reach  $S$  and by empty transition  $M$
  - Note that an empty transition into  $F$  with a nonempty stack cannot lead to an accept. Therefore, we do not concern ourselves about  $P$  making such a transition
  - $S$ -INV and  $M$ -INV hold because both the length of the consumed input and of the stack increased by 1, thus, remaining of equal length and because both continue to only contain  $a$ s
- $((M \ b \ (a)) \ (M \ ,EMP))$ :
  - By inductive hypothesis,  $M$ -INV holds
  - After consuming an  $a$  and popping an  $a$ ,  $P$  may reach  $M$  or nondeterministically reach  $F$  because it may eventually accept
  - $M$ -INV holds because  $c_i$  continues to be  $a$ s followed by  $b$ s, the stack can only contain  $a$ s, and the number of  $a$ s in  $c_i$  remains equal to the sum of the number of  $b$ s in  $c_i$  and the length of the stack.
  - $F$ -INV holds because for a computation that ends with an accept the read  $b$  is the last symbol in the given word and popping an  $a$  makes the stack empty,  $c_i$  continues to be the read  $a$ s followed by the read  $b$ s, and, given that the stack is empty, the number of  $a$ s equals the number of  $b$ s in the consumed input.

# Pushdown Automata

- Proving  $L = L(P)$

# Pushdown Automata

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## Lemma

$$w \in L \Leftrightarrow w \in L(P)$$

## Proof.

( $\Rightarrow$ ) Assume  $w \in L$ . This means that  $w = a^n b^n$ . Given that state invariants always hold, there is a computation that has P consume all the as, then consume all the bs, and then reach F with an empty stack. Therefore,  $w \in L(P)$ .

( $\Leftarrow$ ) Assume  $w \in L(P)$ . This means that M halts in F, the only final state, with an empty stack having consumed w. Given that the state invariants always hold we may conclude that  $w = a^n b^n$ . Therefore,  $w \in L$ . □

-

# Pushdown Automata

- Proving  $L = L(P)$

## Lemma

$$w \in L \Leftrightarrow w \in L(P)$$

## Proof.

( $\Rightarrow$ ) Assume  $w \in L$ . This means that  $w = a^n b^n$ . Given that state invariants always hold, there is a computation that has  $P$  consume all the  $a$ s, then consume all the  $b$ s, and then reach  $F$  with an empty stack. Therefore,  $w \in L(P)$ .

( $\Leftarrow$ ) Assume  $w \in L(P)$ . This means that  $M$  halts in  $F$ , the only final state, with an empty stack having consumed  $w$ . Given that the state invariants always hold we may conclude that  $w = a^n b^n$ . Therefore,  $w \in L$ . □

•

## Lemma

$$w \notin L \Leftrightarrow w \notin L(P)$$

## Proof.

( $\Rightarrow$ ) Assume  $w \notin L$

- - $w \neq a^n b^n$
  - 3 possibilities for the structure of  $w$ : not  $a^n b^m$ ,  $w$  in  $a^n b^m$ , where  $n > m$ , or  $w$  in  $a^n b^m$ , where  $n < m$
  - In the first case,  $w$  either starts with a  $b$  or has at least one  $a$  after a  $b$ : cannot transition out of  $S$  or cannot transition out of  $M$
  - In the second case, never transition into  $F$  with an empty stack
  - In the third case, can't read all the input

( $\Leftarrow$ ) Assume  $w \notin L(P)$ . This means that  $P$  cannot transition into  $F$  with an empty stack having consumed  $w$ . Given that the state invariants always hold,  $w$  must either start with a  $b$ , have an  $a$  after a  $b$ , have too many  $a$ s, or have too many  $b$ s. In all cases,  $w \notin L$ .

# Pushdown Automata

- $L = \{wcw^R \mid w \in (a\ b)^*\}$
- What is the design idea?

# Pushdown Automata

- $L = \{wcw^R \mid w \in (a\ b)^*\}$
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# Pushdown Automata

- $L = \{wcw^R \mid w \in (a b)^*\}$
- What is the design idea?
- Read  $w$  and push it on the stack
- Read  $c$

# Pushdown Automata

- $L = \{wcw^R \mid w \in (a\ b)^*\}$
- What is the design idea?
- Read  $w$  and push it on the stack
- Read  $c$
- Match elements of  $w^R$  with the stack elements
- If all the elements after the  $c$  match all the elements on the stack then the machine accepts.

# Pushdown Automata

- $L = \{wcw^R \mid w \text{ in } (a \ b)^*\}$

- `(define wcwr (make-ndpda`

# Pushdown Automata

- ```
;; L = w c w^R | w in (a b)*
```

- ```
(define w c w^r (make-ndpda
```

- ```
;; Tests for w c w^r
(check-equal? (sm-apply w c w^r '(a)) 'reject)
(check-equal? (sm-apply w c w^r '(a c)) 'reject)
(check-equal? (sm-apply w c w^r '(b c a)) 'reject)
(check-equal? (sm-apply w c w^r '(a a b c b a b)) 'reject)
(check-equal? (sm-apply w c w^r '(c)) 'accept)
(check-equal? (sm-apply w c w^r '(a c a)) 'accept)
(check-equal? (sm-apply w c w^r '(a b b b c b b b a)) 'accept)
```

# Pushdown Automata

- ```
;; L = w c w^R | w in (a b)*
```
- ```
;; States
```

```
;; S ci is empty and stack is empty
```

- ```
(define w c w^r (make-ndpda
```

- ```
;; Tests for w c w^r
```

```
(check-equal? (sm-apply w c w^r '(a)) 'reject)
```

```
(check-equal? (sm-apply w c w^r '(a c)) 'reject)
```

```
(check-equal? (sm-apply w c w^r '(b c a)) 'reject)
```

```
(check-equal? (sm-apply w c w^r '(a a b c b a b)) 'reject)
```

```
(check-equal? (sm-apply w c w^r '(c)) 'accept)
```

```
(check-equal? (sm-apply w c w^r '(a c a)) 'accept)
```

```
(check-equal? (sm-apply w c w^r '(a b b b c b b b a)) 'accept)
```

# Pushdown Automata

- ```
;; L =  $wcw^R$  |  $w$  in  $(a\ b)^*$ 
```
- ```
;; States
```
- ```
;; S ci is empty and stack is empty
```
- ```
;; P ci =  $stack^R$  AND  $c$  not in  $ci$ 
```
- ```
(define wcwr (make-ndpda
```
- ```
;; Tests for wcwr
(check-equal? (sm-apply wcwr '(a)) 'reject)
(check-equal? (sm-apply wcwr '(a c)) 'reject)
(check-equal? (sm-apply wcwr '(b c a)) 'reject)
(check-equal? (sm-apply wcwr '(a a b c b a b)) 'reject)
(check-equal? (sm-apply wcwr '(c)) 'accept)
(check-equal? (sm-apply wcwr '(a c a)) 'accept)
(check-equal? (sm-apply wcwr '(a b b b c b b b a)) 'accept)
```

# Pushdown Automata

- ```
;; L = wcw^R | w in (a b)*
;; States
;; S ci is empty and stack is empty
;; P ci = stack^R AND c not in ci
;; Q ci = (append w (list 'c) v) AND
      w = stack^R v^R

(define wcw^r (make-ndpda

;; Tests for wcw^r
(check-equal? (sm-apply wcw^r '(a)) 'reject)
(check-equal? (sm-apply wcw^r '(a c)) 'reject)
(check-equal? (sm-apply wcw^r '(b c a)) 'reject)
(check-equal? (sm-apply wcw^r '(a a b c b a b)) 'reject)
(check-equal? (sm-apply wcw^r '(c)) 'accept)
(check-equal? (sm-apply wcw^r '(a c a)) 'accept)
(check-equal? (sm-apply wcw^r '(a b b b c b b b a)) 'accept)
```





# Pushdown Automata

- ;; L =  $wcw^R$  |  $w$  in  $(a\ b)^*$
- ;; States  
;; S ci is empty and stack is empty
- ;; P ci =  $stack^R$  AND c not in ci
- ;; Q ci = (append w (list 'c) v) AND  
;; w =  $stack^R v^R$
- ;; F stack = '() AND ci = (append w (list c)  $w^R$ )
- (define wcw<sup>r</sup> (make-ndpda
- `(((S ,EMP ,EMP) (P ,EMP))
- ;; Tests for wcw<sup>r</sup>  
(check-equal? (sm-apply wcw<sup>r</sup> '(a)) 'reject)  
(check-equal? (sm-apply wcw<sup>r</sup> '(a c)) 'reject)  
(check-equal? (sm-apply wcw<sup>r</sup> '(b c a)) 'reject)  
(check-equal? (sm-apply wcw<sup>r</sup> '(a a b c b a b)) 'reject)  
(check-equal? (sm-apply wcw<sup>r</sup> '(c)) 'accept)  
(check-equal? (sm-apply wcw<sup>r</sup> '(a c a)) 'accept)  
(check-equal? (sm-apply wcw<sup>r</sup> '(a b b b c b b b a)) 'accept)

# Pushdown Automata

- ```
;; L = wcw^R | w in (a b)*
;; States
;; S ci is empty and stack is empty
;; P ci = stack^R AND c not in ci
;; Q ci = (append w (list 'c) v) AND
;;       w = stack^R v^R
;; F stack = '() AND ci = (append w (list c) w^R)
(define wcw^R (make-ndpda
```

- $((S, EMP, EMP) (P, EMP))$
- $((P a, EMP) (P (a)))$   
 $((P b, EMP) (P (b)))$

- ```
;; Tests for wcw~r
(check-equal? (sm-apply wcw~r '(a)) 'reject)
(check-equal? (sm-apply wcw~r '(a c)) 'reject)
(check-equal? (sm-apply wcw~r '(b c a)) 'reject)
(check-equal? (sm-apply wcw~r '(a a b c b a b)) 'reject)
(check-equal? (sm-apply wcw~r '(c)) 'accept)
(check-equal? (sm-apply wcw~r '(a c a)) 'accept)
(check-equal? (sm-apply wcw~r '(a b b b c b b b a)) 'accept)
```

# Pushdown Automata

- ```
;; L = w c w^R | w in (a b)*
```
- ```
;; States
```
- ```
;; S ci is empty and stack is empty
```
- ```
;; P ci = stack^R AND c not in ci
```
- ```
;; Q ci = (append w (list 'c) v) AND
```
- ```
;; w = stack^R v^R
```
- ```
;; F stack = '() AND ci = (append w (list c) w^R)
```
- ```
(define w c w^r (make-ndpda
```

- ```
`(((S ,EMP ,EMP) (P ,EMP))
```
- ```
((P a ,EMP) (P (a)))
```
- ```
((P b ,EMP) (P (b)))
```
- ```
((P c ,EMP) (Q ,EMP))
```

- ```
;; Tests for w c w^r
(check-equal? (sm-apply w c w^r '(a)) 'reject)
(check-equal? (sm-apply w c w^r '(a c)) 'reject)
(check-equal? (sm-apply w c w^r '(b c a)) 'reject)
(check-equal? (sm-apply w c w^r '(a a b c b a b)) 'reject)
(check-equal? (sm-apply w c w^r '(c)) 'accept)
(check-equal? (sm-apply w c w^r '(a c a)) 'accept)
(check-equal? (sm-apply w c w^r '(a b b b c b b b a)) 'accept)
```

# Pushdown Automata

- ```
;; L = w c w^R | w in (a b)*
```
- ```
;; States
```
- ```
;; S ci is empty and stack is empty
```
- ```
;; P ci = stack^R AND c not in ci
```
- ```
;; Q ci = (append w (list 'c) v) AND
```
- ```
;; w = stack^R v^R
```
- ```
;; F stack = '() AND ci = (append w (list c) w^R)
```
- ```
(define w c w^r (make-ndpda
```

- ```
`(((S ,EMP ,EMP) (P ,EMP))
```
- ```
((P a ,EMP) (P (a)))
```
- ```
((P b ,EMP) (P (b)))
```
- ```
((P c ,EMP) (Q ,EMP))
```
- ```
((Q a (a)) (Q ,EMP))
```
- ```
((Q b (b)) (Q ,EMP))
```
- ```
((Q ,EMP ,EMP) (F ,EMP))))
```
- ```
;; Tests for w c w^r
```

```
(check-equal? (sm-apply w c w^r '(a)) 'reject)
```

```
(check-equal? (sm-apply w c w^r '(a c)) 'reject)
```

```
(check-equal? (sm-apply w c w^r '(b c a)) 'reject)
```

```
(check-equal? (sm-apply w c w^r '(a a b c b a b)) 'reject)
```

```
(check-equal? (sm-apply w c w^r '(c)) 'accept)
```

```
(check-equal? (sm-apply w c w^r '(a c a)) 'accept)
```

```
(check-equal? (sm-apply w c w^r '(a b b b c b b b a)) 'accept)
```

# Pushdown Automata

```
• ;; L = wcw~R | w in (a b)*  
• ;; States  
  ;; S ci is empty and stack is empty  
• ;; P ci = stack~R AND c not in ci  
• ;; Q ci = (append w (list 'c) v) AND  
  ;; w = stack~R v~R  
• ;; F stack = '() AND ci = (append w (list c) w~R)  
• (define wcw~r (make-ndpda  
•   '(S P Q F)  
   '(a b c)  
   '(a b)  
   'S  
   'F)  
•   `(((S ,EMP ,EMP) (P ,EMP))  
•     ((P a ,EMP) (P (a)))  
•     ((P b ,EMP) (P (b)))  
•     ((P c ,EMP) (Q ,EMP))  
•     ((Q a (a)) (Q ,EMP))  
•     ((Q b (b)) (Q ,EMP))  
•     ((Q ,EMP ,EMP) (F ,EMP))))  
• ;; Tests for wcw~r  
  (check-equal? (sm-apply wcw~r '(a)) 'reject)  
  (check-equal? (sm-apply wcw~r '(a c)) 'reject)  
  (check-equal? (sm-apply wcw~r '(b c a)) 'reject)  
  (check-equal? (sm-apply wcw~r '(a a b c b a b)) 'reject)  
  (check-equal? (sm-apply wcw~r '(c)) 'accept)  
  (check-equal? (sm-apply wcw~r '(a c a)) 'accept)  
  (check-equal? (sm-apply wcw~r '(a b b b c b b a)) 'accept)
```

# Pushdown Automata

- ```
;; word stack → Boolean
;; Purpose: Determine in the given word and stack are empty
(define (S-INV ci s) (and (empty? ci) (empty? s)))

;; Tests for S-INV
(check-equal? (S-INV '() '(a a)) #f)
(check-equal? (S-INV '(a c a) '()) #f)
(check-equal? (S-INV '(a c a) '(b b)) #f)
(check-equal? (S-INV '() '()) #t)
```

# Pushdown Automata

- ```
;; word stack → Boolean
;; Purpose: Determine if the given ci is the reverse of
;;           the given stack AND c is not in ci
(define (P-INV ci s)
  (and (equal? ci (reverse s)) (not (member 'c ci))))

;; Tests for P-INV
(check-equal? (P-INV '(a c a) '(a c a)) #f)
(check-equal? (P-INV '(a a) '(a b)) #f)
(check-equal? (P-INV '() '()) #t)
(check-equal? (P-INV '(a b) '(b a)) #t)
(check-equal? (P-INV '(a b a a) '(a a b a)) #t)
```

# Pushdown Automata

- ```
;; word stack → Boolean
;; Purpose: Determine if ci=s~Rv~Rcv
(define (Q-INV ci s)
  (let* [(w (takef ci (λ (s) (not (eq? s 'c))))))
        (v (if (member 'c ci)
                (drop ci (add1 (length w)))
                '()))]
    (and (equal? ci (append w (list 'c) v))
         (equal? w (append (reverse s) (reverse v))))))

;; Tests for Q-INV
(check-equal? (Q-INV '(a a) '()) #f)
(check-equal? (Q-INV '(b b c a) '(b a)) #f)
(check-equal? (Q-INV '(c) '()) #t)
(check-equal? (Q-INV '(b a c) '(a b)) #t)
(check-equal? (Q-INV '(a b c b) '(a)) #t)
(check-equal? (Q-INV '(a b b c b) '(b a)) #t)
```



# Pushdown Automata

- ```
;; word stack → Boolean
;; Purpose: Determine if ci=s^Rv^Rcv AND stack is empty
(define (F-INV ci s)
  (let* [(w (takef ci (λ (s) (not (eq? s 'c)))))]
    (and (empty? s)
         (equal? ci (append w (list 'c) (reverse w)))))

;; Tests for F-INV
(check-equal? (F-INV '() '()) #f)
(check-equal? (F-INV '(b b) '()) #f)
(check-equal? (F-INV '(b a c) '(b a)) #f)
(check-equal? (F-INV '(c) '()) #t)
(check-equal? (F-INV '(b a c a b) '()) #t)
(check-equal? (F-INV '(a b b c b b a) '()) #t)
```

# Pushdown Automata

- $L = \{wcw^R \mid w \in (\Sigma^+)^*\}$   
 $s = \text{the stack}$        $F = \text{finals}$        $ci = \text{consumed input}$

# Pushdown Automata

- |                                         |                                 |                              |
|-----------------------------------------|---------------------------------|------------------------------|
| $L = \{wcw^R \mid w \in (\Sigma^+)^*\}$ | $M = \text{pushdown automaton}$ | $w \in (\Sigma^+)^*$         |
| $s = \text{the stack}$                  | $F = \text{final states}$       | $ci = \text{consumed input}$ |

## Theorem

*The state invariants hold when  $M$  accepts  $w$ .*

-

# Pushdown Automata

- Base Case

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- When  $M$  starts S-INV holds because  $ci = '()$  and  $s = '()$

# Pushdown Automata

- Base Case
- When  $M$  starts  $S$ -INV holds because  $ci = '()$  and  $s = '()$
- A computation that leads to accept may also reach  $P$  by consuming no input
- $ci = '()$  and  $s = '()$  when the machine moves to  $P$
- $P$ -INV holds because  $c$  is not a member of  $ci$  and  $ci = (\text{reverse } s)$

# Pushdown Automata

- Inductive Step
- $((P \text{ a }, EMP) (P (a)))$ 
  - By inductive hypothesis, P-INV holds:  $c_i$  does not contain  $c$  and that  $c_i = (\text{reverse } s)$
  - Reading and pushing an  $a$ ,  $c_i$  still does not contain  $c$  and we still have that  $c_i = (\text{reverse } s)$
  - P-INV holds

# Pushdown Automata

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- $((P \text{ a }, EMP) (P (a)))$ 
  - By inductive hypothesis, P-INV holds:  $c_i$  does not contain  $c$  and that  $c_i = (\text{reverse } s)$
  - Reading and pushing an  $a$ ,  $c_i$  still does not contain  $c$  and we still have that  $c_i = (\text{reverse } s)$
  - P-INV holds
- $((P \text{ b }, EMP) (P (b)))$ :
  - By inductive hypothesis, P-INV holds:  $c_i$  does not contain  $c$  and that  $c_i = (\text{reverse } s)$
  - By reading and pushing a  $b$ ,  $c_i$  still does not contain  $c$  and we still have that  $c_i = (\text{reverse } s)$
  - P-INV holds



# Pushdown Automata

- Inductive Step
- $((P, c, EMP), (Q, EMP))$ :
  - By inductive hypothesis,  $P-INV$  holds:  $c_i$  does not contain  $c$  and that  $c_i = (\text{reverse } s)$
  - By reading  $c$ ,  $c_i$  contains  $c$
  - After the transition  $c_i = (\text{reverse } s)c = (\text{reverse } s)(\text{reverse } '())c(\text{reverse } '()) = (\text{reverse } s)(\text{reverse } v)cv$
  - That is,  $v = '()$ . Thus,  $Q-INV$  holds
  - After using this transition rule,  $M$  may also reach  $F$  by consuming no input on a computation that leads to accept
  - Given that  $Q-INV$  holds and  $M$  only moves to  $F$  if  $v = s = '()$ , we have that  $c_i = c = '()c'() = wc(\text{reverse } w)$ . That is,  $w = EMP$ . Thus,  $F-INV$  holds

# Pushdown Automata

## • Inductive Step

### • $((P \text{ } c, EMP) (Q, EMP))$ :

- By inductive hypothesis, P-INV holds:  $ci$  does not contain  $c$  and that  $ci = (\text{reverse } s)$
- By reading  $c$ ,  $ci$  contains  $c$
- After the transition  $ci = (\text{reverse } s)c = (\text{reverse } s)(\text{reverse } '())c(\text{reverse } '()) = (\text{reverse } s)(\text{reverse } v)cv$
- That is,  $v = '()$ . Thus, Q-INV holds
- After using this transition rule,  $M$  may also reach  $F$  by consuming no input on a computation that leads to accept
- Given that Q-INV holds and  $M$  only moves to  $F$  if  $v = s = '()$ , we have that  $ci = c = '()c'() = wc(\text{reverse } w)$ . That is,  $w = EMP$ . Thus, F-INV holds

### • $((Q \text{ } a (a)) (Q, EMP))$ :

- By inductive hypothesis, Q-INV holds:  $ci = (\text{reverse } s)(\text{reverse } v)cv$
- Reading and popping  $a$  moves the  $a$  on the top of the stack to  $v$ .
- Thus,  $ci = (\text{reverse } s)(\text{reverse } v)cv$  after using this transition and Q-INV holds.
- If  $a$  is the last element of the input and the stack is empty then  $M$  moves to  $F$ . This means that  $ci = (\text{reverse } s)(\text{reverse } v)cv = EMP(\text{reverse } v)cv = (\text{reverse } v)cv = (\text{reverse } w)cw$ . That is,  $v = w$ . Therefore, F-INV holds

# Pushdown Automata

- Inductive Step
- $((P \text{ } c, \text{EMP}) (Q, \text{EMP}))$ :
  - By inductive hypothesis, P-INV holds:  $c_i$  does not contain  $c$  and that  $c_i = (\text{reverse } s)$
  - By reading  $c$ ,  $c_i$  contains  $c$
  - After the transition  $c_i = (\text{reverse } s)c = (\text{reverse } s)(\text{reverse } '())c(\text{reverse } '()) = (\text{reverse } s)(\text{reverse } v)cv$
  - That is,  $v = '()$ . Thus, Q-INV holds
  - After using this transition rule, M may also reach F by consuming no input on a computation that leads to accept
  - Given that Q-INV holds and M only moves to F if  $v = s = '()$ , we have that  $c_i = c = '()c'() = wc(\text{reverse } w)$ . That is,  $w = \text{EMP}$ . Thus, F-INV holds
- $((Q \text{ } a \text{ } (a)) (Q, \text{EMP}))$ :
  - By inductive hypothesis, Q-INV holds:  $c_i = (\text{reverse } s)(\text{reverse } v)cv$
  - Reading and popping  $a$  moves the  $a$  on the top of the stack to  $v$ .
  - Thus,  $c_i = (\text{reverse } s)(\text{reverse } v)cv$  after using this transition and Q-INV holds.
  - If  $a$  is the last element of the input and the stack is empty then M moves to F. This means that  $c_i = (\text{reverse } s)(\text{reverse } v)cv = \text{EMP}(\text{reverse } v)cv = (\text{reverse } v)cv = (\text{reverse } w)cw$ . That is,  $v = w$ . Therefore, F-INV holds
- $((Q \text{ } b \text{ } (b)) (Q, \text{EMP}))$ : idem, but with  $b$ .

# Pushdown Automata

- $L = L(M)$

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## Lemma

$$w \in L \Leftrightarrow w \in L(M)$$

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- ( $\Rightarrow$ ) Assume  $w \in L$ 
  - $w = vcv^R$
  - Given that state invariants always hold, there is a computation that has  $M$  consume  $w$  and reach  $F$  with an empty stack
  - $w \in L(M)$

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- $(\Rightarrow)$  Assume  $w \in L$ 
  - $w = vcv^R$
  - Given that state invariants always hold, there is a computation that has  $M$  consume  $w$  and reach  $F$  with an empty stack
  - $w \in L(M)$
- $(\Leftarrow)$  Assume  $w \in L(M)$ 
  - $M$  halts in  $F$  with an empty stack having consumed  $w$
  - Given that the state invariants always hold:  $w = vcv^R$
  - $w \in L$

# Pushdown Automata

## Lemma

$$w \notin L \Leftrightarrow w \notin L(M)$$





# Pushdown Automata

## Lemma

$$w \notin L \Leftrightarrow w \notin L(M)$$

- 
- $(\Rightarrow)$  Assume  $w \notin L$ 
  - $w \neq v^Rcv$
  - Given that the state invariant predicates always hold, there is no computation that has  $M$  consume  $w$  and end in  $F$  with an empty stack
  - $w \notin L(M)$

# Pushdown Automata

## Lemma

$$w \notin L \Leftrightarrow w \notin L(M)$$

- 
- ( $\Rightarrow$ ) Assume  $w \notin L$ 
  - $w \neq v^R cv$
  - Given that the state invariant predicates always hold, there is no computation that has  $M$  consume  $w$  and end in  $F$  with an empty stack
  - $w \notin L(M)$
- ( $\Leftarrow$ ) Assume  $w \notin L(M)$ 
  - $M$  cannot transition into  $F$  with an empty stack having consumed  $w$
  - Given that the state invariants always hold:  $w \neq vcv^R$
  - $w \notin L$

# Pushdown Automata

- HOMEWORK: 1–5, 8
- In class quiz (1 week from today)

# Pushdown Automata

- For every language,  $L$ , decided by an  $\text{ndfa}$ , is there is a  $\text{pda}$  that decides  $L$ ?

# Pushdown Automata

- For every language,  $L$ , decided by an  $\text{ndfa}$ , is there is a  $\text{pda}$  that decides  $L$ ?
- If a language,  $L$ , is regular then there is a  $\text{ndfa}$  that decides  $L$
- Intuitively, an  $\text{ndfa}$  may be thought of as a  $\text{pda}$  that never operates on its stack
- We ought to be able to build a  $\text{pda}$  from an  $\text{ndfa}$

# Pushdown Automata

- Design Idea
- Given an ndfa,  $M$ , a pda is constructed using:
- $M$ 's states, alphabet, starting state, and final states

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- Given an ndfa,  $M$ , a pda is constructed using:
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- The pda's transition relation is built using  $M$ 's transition relation:



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- Each of  $M$ 's rules is converted to a pda-rule

# Pushdown Automata

- Design Idea
- Given an ndfa,  $M$ , a pda is constructed using:
- $M$ 's states, alphabet, starting state, and final states
- The stack alphabet for the constructed pda may be  $\{ \epsilon \}$  because nothing shall ever be pushed onto the stack
- The pda's transition relation is built using  $M$ 's transition relation:
- Each of  $M$ 's rules is converted to a pda-rule
- For an ndfa-rule,  $(P \xrightarrow{a} R)$ , the pda-rule  $((P \xrightarrow{a \epsilon} EMP) (R \xrightarrow{\epsilon} EMP))$  is generated
- The pda moves from  $P$  to  $R$  without modifying the stack

# Pushdown Automata

- `;; ndfa  $\rightarrow$  pda`  
`;; Purpose: Convert the given ndfa to a pda`  
`(define (ndfa $\rightarrow$ pda M)`

# Pushdown Automata

- ```
;; ndfa → pda
;; Purpose: Convert the given ndfa to a pda
(define (ndfa->pda M)
```
- ```
;; Sample pda
(define ALOM-PDA (ndfa->pda AT-LEAST-ONE-MISSING))
(define LNDFPDA-PDA (ndfa->pda LNDFPDA))

(check-equal? (sm-testequiv? ALOM-PDA AT-LEAST-ONE-MISSING) #t)
(check-equal? (sm-testequiv? LNDFPDA-PDA LNDFPDA) #t)
```

# Pushdown Automata

- `;; ndfa → pda`  
`;; Purpose: Convert the given ndfa to a pda`  
`(define (ndfa->pda M)`
- `(let [(states (sm-states M))`  
`(sigma (sm-sigma M))`  
`(start (sm-start M))`  
`(finals (sm-finals M))`  
`(rules (sm-rules M))]`  
`(make-ndpda states`  
`sigma`  
`'()`  
`start`  
`finals`
- `;; Sample pda`  
`(define ALOM-PDA (ndfa->pda AT-LEAST-ONE-MISSING))`  
`(define LNDFPDA-PDA (ndfa->pda LNDFPDA))`  
  
`(check-equal? (sm-testequiv? ALOM-PDA AT-LEAST-ONE-MISSING) #t)`  
`(check-equal? (sm-testequiv? LNDFPDA-PDA LNDFPDA) #t)`

# Pushdown Automata

- `;; ndfa → pda`  
`;; Purpose: Convert the given ndfa to a pda`  
`(define (ndfa->pda M)`
- `(let [(states (sm-states M))`  
`(sigma (sm-sigma M))`  
`(start (sm-start M))`  
`(finals (sm-finals M))`  
`(rules (sm-rules M))]`  
`(make-ndpda states`  
`sigma`  
`'()`  
`start`  
`finals`
- `(map (λ (r) (list (list (first r) (second r) EMP)`  
`(list (third r) EMP))))`  
`rules))))`
- `;; Sample pda`  
`(define ALOM-PDA (ndfa->pda AT-LEAST-ONE-MISSING))`  
`(define LNDFPDA-PDA (ndfa->pda LNDFPDA))`  
  
`(check-equal? (sm-testequiv? ALOM-PDA AT-LEAST-ONE-MISSING) #t)`  
`(check-equal? (sm-testequiv? LNDFPDA-PDA LNDFPDA) #t)`

# Pushdown Automata

- HOMEWORK: 11

# Equivalence of pdas and cfls

## Theorem

*The set of languages accepted by pdas is exactly the context-free languages.*





# Equivalence of pdas and cfgs

## Theorem

*The set of languages accepted by pdas is exactly the context-free languages.*

- 
- We must show:
  - ① Given a cfg,  $G$ , show how to build a pda for  $L(G)$
  - ② Given a pda,  $P$ , show how to build a cfg for  $L(P)$

# Equivalence of pdas and cfls

## Lemma

*$L$  is a context-free language  $\Rightarrow \exists$  pda that accepts  $L$*

- Design Idea

# Equivalence of pdas and cfls

## Lemma

*$L$  is a context-free language  $\Rightarrow \exists$  pda that accepts  $L$*

- Design Idea
- $P = (\text{make-ndpda } K \ \Sigma \ S \ F \ \Delta)$ , such that  $L(P) = L(G)$

# Equivalence of pdas and cfgs

## Lemma

*$L$  is a context-free language  $\Rightarrow \exists$  pda that accepts  $L$*

- Design Idea
- $P = (\text{make-ndpda } K \ \Sigma \ S \ F \ \Delta)$ , such that  $L(P) = L(G)$
- Read the input from left to right: simulate the leftmost derivation of  $w$

# Equivalence of pdas and cfgs

## Lemma

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- Design Idea
- $P = (\text{make-ndpda } K \ \Sigma \ S \ F \ \Delta)$ , such that  $L(P) = L(G)$
- Read the input from left to right: simulate the leftmost derivation of  $w$
- To start,  $P$  pushes  $S_G$  onto the stack
- For each cfg-rule,  $A \rightarrow x$ ,  $P$  pops  $A$  off the stack and pushes  $x$  onto the stack
- For each  $a \in \Sigma_G$ ,  $P$  reads  $a$  from the tape and pops  $a$  off the stack

# Equivalence of pdas and cfgs

## Lemma

*$L$  is a context-free language  $\Rightarrow \exists$  pda that accepts  $L$*

- Design Idea
- $P = (\text{make-ndpda } K \ \Sigma \ S \ F \ \Delta)$ , such that  $L(P) = L(G)$
- Read the input from left to right: simulate the leftmost derivation of  $w$
- To start,  $P$  pushes  $S_G$  onto the stack
- For each cfg-rule,  $A \rightarrow x$ ,  $P$  pops  $A$  off the stack and pushes  $x$  onto the stack
- For each  $a \in \Sigma_G$ ,  $P$  reads  $a$  from the tape and pops  $a$  off the stack
- Suggests that  $P$  only needs two states:
  - Starting state,  $S$ , in which  $P$  pushes  $S_G$  and moves to,  $Q$
  - $P$  remains in  $Q$ , which is the final state, for the rest of the computation

# Equivalence of pdas and cfigs

- $P$  is constructed as follows:

$(\text{make-ndpda } \langle (S \rightarrow Q) \rangle \Sigma_G \cup V_G \cup \Sigma_G \langle S \rightarrow (Q) \rangle \Delta)$

# Equivalence of pdas and cfgs

- P is constructed as follows:

$(\text{make-ndpda } \langle (S \ Q) \ \Sigma_G \ V_G \cup \Sigma_G \ 'S \ ' (Q) \ \Delta \rangle)$

- There are three types of rule in  $\Delta$ :

1.  $((S \ \text{EMP} \ (\text{list } S_G)) \ (Q \ \text{EMP}))$
2.  $((Q \ \text{EMP} \ (\text{list } 'A)) \ (Q \ (\text{symbol} \rightarrow \text{fsmlos } x))), \text{ for } (A \rightarrow x) \in R_G$
3.  $((Q \ a \ (a)) \ (Q \ \text{EMP})), \text{ for } a \in \Sigma_G$



# Equivalence of pdas and cfgs

- P is constructed as follows:

$(\text{make-ndpda } \langle (S \ Q) \ \Sigma_G \ V_G \cup \Sigma_G \ 'S \ ' (Q) \ \Delta \rangle)$

- There are three types of rule in  $\Delta$ :

1.  $((S \ \text{EMP} \ (\text{list } S_G)) \ (Q \ \text{EMP}))$
2.  $((Q \ \text{EMP} \ (\text{list } 'A)) \ (Q \ (\text{symbol} \rightarrow \text{fsmlos } x)))$ , for  $(A \rightarrow x) \in R_G$
3.  $((Q \ a \ (a)) \ (Q \ \text{EMP}))$ , for  $a \in \Sigma_G$

- We are careful to say nothing about the behavior of P if it is ever applied to a word not in  $L(G)$

# Equivalence of pdas and cfls

- `;; cfg  $\rightarrow$  pda`      Purpose: Transform the given cfg into a pda  
  `(define (cfg2pda G)`

# Equivalence of pdas and cfgs

- `;; cfg → pda`      Purpose: Transform the given cfg into a pda  
    (define (cfg2pda G)

- `;; Tests`  
    (define a2nb2n-pda (cfg2pda a2nb2n))  
    (define numb>numa-pda (cfg2pda numb>numa))  
  
    (check-equal? (sm-apply a2nb2n-pda '(b b)) 'reject)  
    (check-equal? (sm-apply a2nb2n-pda '(a a b b a b)) 'reject)  
    (check-equal? (sm-apply a2nb2n-pda '()) 'accept)  
    (check-equal? (sm-apply a2nb2n-pda '(a a a b b b)) 'accept)  
  
    (check-equal? (sm-apply numb>numa-pda '(b b b)) 'accept)  
    (check-equal? (sm-apply numb>numa-pda '(b b a)) 'accept)  
    (check-equal? (sm-apply numb>numa-pda '(b b a b b)) 'accept)  
    (check-equal? (sm-apply numb>numa-pda '(a b b a b)) 'accept)

# Equivalence of pdas and cfgs

- `;; cfg → pda`      Purpose: Transform the given cfg into a pda  
`(define (cfg2pda G)`
- `(let [(nts (grammar-nts G))      (sigma (grammar-sigma G))`  
`(start (grammar-start G)) (rules (grammar-rules G))]`  
`(make-ndpda '(S Q)`  
`sigma`  
`(append nts sigma)`  
`'S`  
`(list 'Q)`
- `;; Tests`  
`(define a2nb2n-pda (cfg2pda a2nb2n))`  
`(define numb>numa-pda (cfg2pda numb>numa))`  
  
`(check-equal? (sm-apply a2nb2n-pda '(b b)) 'reject)`  
`(check-equal? (sm-apply a2nb2n-pda '(a a b b a b)) 'reject)`  
`(check-equal? (sm-apply a2nb2n-pda '()) 'accept)`  
`(check-equal? (sm-apply a2nb2n-pda '(a a a b b b)) 'accept)`  
  
`(check-equal? (sm-apply numb>numa-pda '(b b b)) 'accept)`  
`(check-equal? (sm-apply numb>numa-pda '(b b a)) 'accept)`  
`(check-equal? (sm-apply numb>numa-pda '(b b a b b)) 'accept)`  
`(check-equal? (sm-apply numb>numa-pda '(a b b a b)) 'accept)`

# Equivalence of pdas and cfgs

- ;; cfg  $\rightarrow$  pda      Purpose: Transform the given cfg into a pda  
(define (cfg2pda G)  
  - (let [(nts (grammar-nts G))      (sigma (grammar-sigma G))  
(start (grammar-start G)) (rules (grammar-rules G))]  
(make-ndpda '(S Q)  
sigma  
(append nts sigma)  
'S  
(list 'Q)  
(append  
(list (list (list 'S EMP EMP) (list 'Q (list start))))  
(map (lambda (r)  
(list (list 'Q EMP (list (first r)))  
(list 'Q  
(if (eq? (third r) EMP)  
EMP  
(symbol->fsmlos (third r))))))  
rules)  
(map (lambda (a) (list (list 'Q a (list a)) (list 'Q EMP)))  
sigma))))))
- ;; Tests  
(define a2nb2n-pda (cfg2pda a2nb2n))  
(define numb>numa-pda (cfg2pda numb>numa))  
  
(check-equal? (sm-apply a2nb2n-pda '(b b)) 'reject)  
(check-equal? (sm-apply a2nb2n-pda '(a a b b a b)) 'reject)  
(check-equal? (sm-apply a2nb2n-pda '()) 'accept)  
(check-equal? (sm-apply a2nb2n-pda '(a a a b b b)) 'accept)  
  
(check-equal? (sm-apply numb>numa-pda '(b b b)) 'accept)  
(check-equal? (sm-apply numb>numa-pda '(b b a)) 'accept)  
(check-equal? (sm-apply numb>numa-pda '(b b a b b)) 'accept)  
(check-equal? (sm-apply numb>numa-pda '(a b b a b)) 'accept)

## Equivalence of pdas and cfgs

;; Tests

```
(define a2nb2n-pda (cfg2pda a2nb2n))
```

```
(define numb>numa-pda (cfg2pda numb>numa))
```

```
(check-equal? (sm-apply a2nb2n-pda '(b b)) 'reject)
```

```
(check-equal? (sm-apply a2nb2n-pda '(a a b b a b)) 'reject)
```

```
(check-equal? (sm-apply a2nb2n-pda '()) 'accept)
```

```
(check-equal? (sm-apply a2nb2n-pda '(a a a b b b)) 'accept)
```

```
(check-equal? (sm-apply numb>numa-pda '(b b b)) 'accept)
```

```
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```

```
(check-equal? (sm-apply numb>numa-pda '(b b a b b)) 'accept)
```

```
(check-equal? (sm-apply numb>numa-pda '(a b b a b)) 'accept)
```

- Tests using words not in the corresponding language are only written for a2nb2n-pda
- This is because the design specifies nothing about the behavior of a pda when given a word that is not in the language

# Equivalence of pdas and cfgs

;; Tests

```
(define a2nb2n-pda (cfg2pda a2nb2n))
```

```
(define numb>numa-pda (cfg2pda numb>numa))
```

```
(check-equal? (sm-apply a2nb2n-pda '(b b)) 'reject)
```

```
(check-equal? (sm-apply a2nb2n-pda '(a a b b a b)) 'reject)
```

```
(check-equal? (sm-apply a2nb2n-pda '()) 'accept)
```

```
(check-equal? (sm-apply a2nb2n-pda '(a a a b b b)) 'accept)
```

```
(check-equal? (sm-apply numb>numa-pda '(b b b)) 'accept)
```

```
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```

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```

- Tests using words not in the corresponding language are only written for a2nb2n-pda
- This is because the design specifies nothing about the behavior of a pda when given a word that is not in the language
- Sometimes a constructed pda is capable of rejecting and other times it is incapable of rejecting
- A pda produced by the constructor is guaranteed to accept if the given word is in its language but no such guarantee is made for rejection

## Equivalence of pdas and cfgs

;; Tests

```
(define a2nb2n-pda (cfg2pda a2nb2n))
```

```
(define numb>numa-pda (cfg2pda numb>numa))
```

```
(check-equal? (sm-apply a2nb2n-pda '(b b)) 'reject)
```

```
(check-equal? (sm-apply a2nb2n-pda '(a a b b a b)) 'reject)
```

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- This is because the design specifies nothing about the behavior of a pda when given a word that is not in the language
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- A pda produced by the constructor is guaranteed to accept if the given word is in its language but no such guarantee is made for rejection
- When a machine can guarantee accepting any word in a given language, L, but cannot guarantee rejecting any word not in L, we say that the machine *semidecides* L



# Equivalence of pdas and cfigs

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- Otherwise, the breath-first process continues to extend the search tree to find a computation that leads to accept
- If all paths in the search tree are finite and reject then the machine rejects
- If there is a path in the search tree that is not finite BFS may continue to search for a computation that leads to an accept forever
- Thus, the machine can accept but can never reject

# Equivalence of pdas and cfls

- To understand why a 2nb2n-pda decides its language:

'(((P ∈ ε) (Q (S)))  
((Q ∈ (S)) (Q ∈))  
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- Accept if in Q with an empty stack and no more input to read, but it can never determine that all possible computations reject
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# Equivalence of pdas and cfgs

Let:

$$G = (\text{make-cfg } N \ \Sigma \ R \ S) \qquad P = (\text{cfg2pda } G)$$

## Lemma

$S \xrightarrow{L}_G \omega\alpha \Leftrightarrow (Q \ \omega \ S) \vdash^*_M (Q \ \epsilon \ \alpha)$ , where  $\alpha \in \{N(N \cup \Sigma)^* \cup \{\epsilon\}\} \wedge \omega \in \Sigma^*$

- A leftmost derivation of  $\omega\alpha$  is either empty or starts with the leftmost nonterminal in  $\omega\alpha$  is logically equivalent to  $M$  starting in  $Q$ , reading  $\omega$ , and popping  $S$  to end in  $Q$  with  $\alpha$  on the stack
- Observe that if the stack is not empty after consuming  $\omega$  then the topmost element is the next nonterminal to substitute in the leftmost derivation

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- By construction of  $P$  there are type 3 rules that consume  $y$  to yield the following computation:  $(Q \omega S) = (Q xy S) \vdash^* (Q y \theta\beta) = (Q y y\alpha) \vdash^* (Q \in \alpha)$ .

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- For the base case,  $n = 0$
- $(Q \ \omega \ S) \vdash^* (Q \ \epsilon \ \alpha)$  means that  $\omega = \epsilon$  and  $\alpha = S$ .
- Thus, we have that  $S \xrightarrow{L}_G \omega\alpha = \epsilon S = S$ .

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$S \xrightarrow{L}_G \omega\alpha \Leftrightarrow (Q \ \omega \ S) \vdash^*_M (Q \ \epsilon \ \alpha)$ , where  $\alpha \in \{N(N \cup \Sigma)^* \cup \{\epsilon\}\} \wedge \omega \in \Sigma^*$

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- Show:  $(Q \ \omega \ S) \vdash^*_M (Q \ \epsilon \ \alpha) \Rightarrow S \xrightarrow{L}_G \omega\alpha$ , for  $n = k + 1$ .

# Equivalence of pdas and cfgs

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- For the inductive step:
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- Let  $\omega = xy$  and  $(A \rightarrow \theta) \in R$
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- By inductive hypothesis, we have  $S \xrightarrow{L}_G^* xA\beta$
- Using the rule for  $A$  above yields  $S \xrightarrow{L}_G^* x\theta\beta$
- Observe that  $(Q y \theta\beta) \vdash^* (Q \epsilon \alpha)$  only uses type 3 rules
- This means that  $y\alpha = \theta\beta$
- Thus, we have  $S \xrightarrow{L}_G^* x\theta\beta = xy\alpha = \omega\alpha$

# Equivalence of pdas and cfigs

- We need to prove that if  $P = (\text{make-ndpda } K \Sigma \Gamma S F \Delta)$  semidecides  $L$  then  $L$  is a context-free language
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- We shall build a cfg,  $G$ , such that  $L(P) = L(G)$
- Useful to restrict the structure of  $P$ 's transition rules to have what we shall call a *simple* pda
- A pda is simple if all transition rules have the following structure:  
 $((Q \text{ a } \beta) (P \theta))$ , such that  $Q \neq S$ ,  $\beta \in \Gamma$ ,  $\wedge |\theta| \leq 2$
- Always pops the topmost stack element and pushes zero, one, or two elements onto the stack
- No restrictions on  $S$ , because whenever a pda starts the stack is empty

# Equivalence of pdas and cfgs

- To transform  $P$  into a simple pda,  $P'$ , the components of  $P'$  are defined as follows:

$$\begin{aligned}K' &= K \cup \{X Y\}, \text{ such that } X, Y \notin K \\ \Sigma' &= \Sigma \\ \Gamma' &= \Gamma \cup \{Z\}, \text{ such that } Z \notin \Gamma \\ S' &= X \\ F' &= (Y) \\ \Delta' &= T(\Delta) \cup \{((X \text{ EMP EMP})(S(Z))))\} \cup \{\forall W \in F ((W \text{ EMP } (Z))(Y \text{ EMP}))\}\end{aligned}$$

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- $T$  replaces all transitions rules that violate simplicity with equivalent rules that satisfy simplicity
- $T$  replaces rules,  $((Q \text{ a } \beta) (P \theta))$ , that violate simplicity in three steps:

- 1 *Replace  $|\beta| \geq 2$  rules*
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- Implementation in the book*



# Equivalence of pdas and cfgs

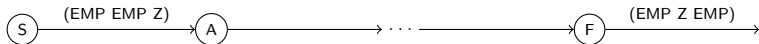
- Convert a simple pda to a cfg
- This is done by exploiting the properties of transition rules in a simple pda
- Unfortunately, the details get messy and the algorithm is not as intuitive as the algorithm to build a pda from a cfg

# Equivalence of pdas and cfgs

- The transformation of a pda to a simple pda simplifies the construction of a cfg for  $L(P)$
- The simplification stems from the fact that every pda-rule pops an element and pushes at most two elements

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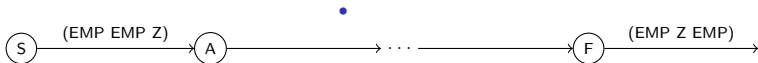
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- The grammar that is constructed shall simulate the simple pda
- Generate all words that take the simple pda from A to  $F'$  by popping Z

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- The grammar that is constructed shall simulate the simple pda
- Generate all words that take the simple pda from A to  $F'$  by popping Z
- We may represent these words as a triple:  $(S \ Z \ F')$
- This means that the starting nonterminal for the constructed grammar, S, must generate  $(S \ Z \ F')$ :

$$S \rightarrow (A \ Z \ F')$$

# Equivalence of pdas and cfigs

- Triples need to be converted to symbols
- We shall talk about all words that take the simple pda from some state  $Q$  to some state  $R$  by popping  $\theta$ :  $(Q \ \theta \ R)$
- All such triples represent the nonterminals of the grammar that is constructed and are formally defined as follows:
  - ;; An list nonterminal, lnt, is a (list state symbol state).
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- We define a cfg-rule's left-hand side as follows:

;; A lhs is either:

;; 1. symbol

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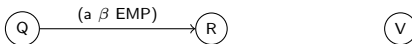
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- How can the right-hand side of a cfg-rule be represented?
- Simple pda rules vary in what is pushed
- Rules that push 0 elements:  $((Q \alpha \beta) (R \text{ EMP}))$
- After using this rule the machine may transition in zero or more steps to an arbitrary state  $V$ :



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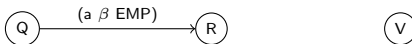
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- The grammar needs to generate all words that take the simple pda from  $Q$  to  $V$  by popping  $\beta$  and pushing nothing
- Such words are generated by  $\langle Q \beta V \rangle$
- To get to  $V$ , the machine reads  $a$  to reach  $R$  and then pops nothing to reach  $V$  in zero or more moves



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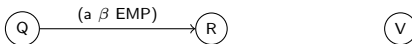
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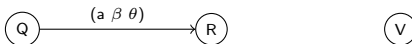
- The grammar needs to generate all words that take the simple pda from  $Q$  to  $V$  by popping  $\beta$  and pushing nothing
- Such words are generated by  $(Q \ \beta \ V)$
- To get to  $V$ , the machine reads  $a$  to reach  $R$  and then pops nothing to reach  $V$  in zero or more moves
- The needed rule is:

$(Q \ \beta \ V) \rightarrow a(R \ \text{EMP} \ V)$

- The right-hand side of a cfg-rule may be represented by a symbol followed by an lnt.

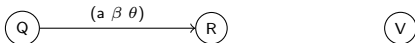
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# Equivalence of pdas and cfgs

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- The grammar needs to generate all words that take the simple pda from Q to V by popping  $\beta$  and pushing  $\theta$
- Similar to pushing nothing except that after reaching R  $\theta$  must be popped
- The needed cfg-rule is:

$$(Q \beta V) \rightarrow a(R \theta V)$$

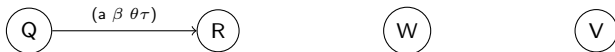
## Equivalence of pdas and cfgs

- Consider the pda-rules that push two elements:  $((Q \ a \ \beta) \ (R \ \theta\tau))$ .
- After popping the first element the machine may be in an arbitrary state  $W$ .  
Graphically, consider the following:



## Equivalence of pdas and cfgs

- Consider the pda-rules that push two elements:  $((Q \ a \ \beta) \ (R \ \theta \tau))$ .
- After popping the first element the machine may be in an arbitrary state  $W$ . Graphically, consider the following:



- The grammar must be able to generate a word that starts with  $a$  followed by a word that takes the simple pda from  $R$  to  $W$  by popping  $\theta$  and ending with a word that takes it from  $W$  to  $V$  by popping  $\tau$
- The needed cfg-rule may be represented as follows:

$$(Q \ \beta \ V) \rightarrow a(R \ \theta \ W)(W \ \tau \ V)$$

## Equivalence of pdas and cfgs

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- We can now define the representation of a cfg-rule as follows:

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- Implementation in the textbook

# Equivalence of pdas and cfls

- HOMEWORK: 7

# Properties of CFLs

- For every context-free  $L$  there is a cfg that generates its members and a pda that accepts its members
- We shall study closure properties of context-free languages and how to prove that a language is not context-free

# Properties of CFLs

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- The grammar for  $L_1 \cup L_2$  nondeterministically decides to simulate a derivation using  $G_1$  or  $G_2$ .

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- Context-free languages are closed under union
- Design Idea
- If  $L_1$  and  $L_2$  are context-free languages then there are cfls,  $G_1$  and  $G_2$ , such that  $L_1 = L(G_1)$  and  $L_2 = L(G_2)$
- Without loss of generality, we assume that  $(\text{grammar-nts } G_1) \cap (\text{grammar-nts } G_2) = \emptyset$
- The grammar for  $L_1 \cup L_2$  nondeterministically decides to simulate a derivation using  $G_1$  or  $G_2$ .
- A cfl for  $L_1 \cup L_2$  is built using the following components:


|          |     |                                                                                                                                                                            |
|----------|-----|----------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| $V$      | $=$ | $(\text{grammar-nts } G_1) \cup (\text{grammar-nts } G_2) \cup \{A\}$                                                                                                      |
| $\Sigma$ | $=$ | $(\text{grammar-sigma } G_1) \cup (\text{grammar-sigma } G_2)$                                                                                                             |
| $R$      | $=$ | $(\text{grammar-rules } G_1) \cup (\text{grammar-rules } G_2) \cup$<br>$\{(S \text{ ARROW } (\text{grammar-start } G_1)) (S \text{ ARROW } (\text{grammar-start } G_2))\}$ |
| $S$      | $=$ | $A = (\text{generate-symbol 'S } (\text{append } (\text{grammar-nts } G_1) (\text{grammar-nts } G_2)))$                                                                    |



# Properties of CFLs

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              (remove-duplicates (append G-sigma H-sigma))  
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- If (grammar-start  $G1$ ) is generated then  $G$  simulates  $G1$
- If (grammar-start  $G2$ ) is generated then  $G$  simulates  $G2$
- Recall that  $(\text{grammar-nts } G1) \cap (\text{grammar-nts } G2) = \emptyset$
- This means that either  $G1$  generates  $w$  or  $G2$  generates  $w$
- Thus,  $w \in L(G1) \cup L(G2)$ .

# Properties of CFLs

- HOMEWORK: 1

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  - $R = (\text{grammar-rules } G_1) \cup (\text{grammar-rules } G_2) \cup \{S \rightarrow (\text{grammar-start } G_1)(\text{grammar-start } G_2)\}$
  - $S = A = (\text{generate-symbol 'S' (append (grammar-nts } G_1) (\text{grammar-nts } G_2)))$



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```
(check-equal? (grammar-derive a2nb2n-numb>numa '(b a))  
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- HOMEWORK: 2 (proof), 3

# Properties of CFLs

- Context-free languages are closed under Kleene star



# Properties of CFLs

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$$S = A = (\text{generate-symbol 'S' } (\text{grammar-nts } G))$$

# Properties of CFLs

- QUIZ: 4-5 (due in a week)

# Properties of CFLs

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# Properties of CFLs

- Like regular languages, context-free languages are infinite and exhibit periodicity
- There are languages that are not context-free
- We shall prove a pumping theorem for context-free languages

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## Properties of CFLs

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- Define  $\kappa$  as the longest righthand side of any rule:

```
;; L = anbn
```

```
(define a2nb2n (make-cfg '(S)
                           '(a b)
                           `((S ,ARROW ,EMP)
                             (S ,ARROW aSb))
                           'S))
```

```
;; L = w | w in (a b)* AND w has more b than a
```

```
(define numb>numa (make-cfg '(S A)
                              '(a b)
                              `((S ,ARROW b)
                                (S ,ARROW AbA)
                                (A ,ARROW AaAbA)
                                (A ,ARROW AbAaA)
                                (A ,ARROW ,EMP)
                                (A ,ARROW bA))
                              'S))
```

- For a2nb2n,  $\kappa$  is 3 because its longest righthand side is aSb
- For numb>numa,  $\kappa$  is 5.

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(define numb>numa (make-cfg '(S A)
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                              `((S ,ARROW b)
                                (S ,ARROW AbA)
                                (A ,ARROW AaAbA)
                                (A ,ARROW AbAaA)
                                (A ,ARROW ,EMP)
                                (A ,ARROW bA))
                              'S))
```

- For a2nb2n,  $\kappa$  is 3 because its longest righthand side is aSb
- For numb>numa,  $\kappa$  is 5.
- Given a cfg G, the yield's length for any parse tree is bounded by the height of the tree and  $\kappa$

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  - At most  $\kappa$  subtrees of height  $n$
- Let  $T_i$  be any of these subtrees
- By inductive hypothesis,  $T_i$ 's yield's length  $\leq \kappa^n$
- If every  $T_i$ 's yield's length is  $\leq \kappa^n$  then  $T$ 's yield's length is at most  $\kappa * \kappa^n$
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- Thus,  $T$ 's yield's length  $\leq \kappa^{n+1}$ .  $\square$
- Let  $w \in L(G)$  such that  $|w| > \kappa^h$
- $w$ 's parse tree must have a path longer than  $h$



# Properties of CFLs

## Theorem

If  $w \in L(G) \wedge |w| > \kappa^{|V|}$  then  $w = uvxyz$  such that ( $v \neq \text{EMP} \vee y \neq \text{EMP}$ ),  $\exists k \ |vxy| \leq k$ , and  $\forall n \geq 0$   
 $uv^nxy^nz \in L(G)$ .

- Proof

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If  $w \in L(G) \wedge |w| > \kappa^{|V|}$  then  $w = uvxyz$  such that ( $v \neq \epsilon \vee y \neq \epsilon$ ),  $\exists k \mid |vxy| \leq k$ , and  $\forall n \geq 0$   $uv^nxy^n z \in L(G)$ .

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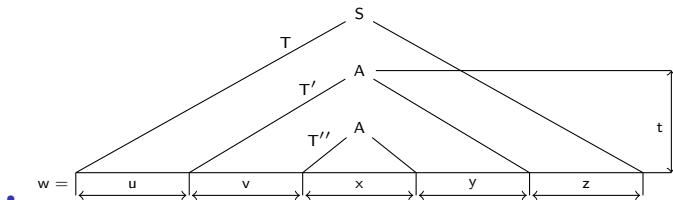
- Proof
- Let  $T$  be the parse tree for  $w$  rooted at  $S$  with the smallest number of leaves
- $|w| > \kappa^{|V|} \Rightarrow$  path in  $T$  of length at least  $|V| + 1$
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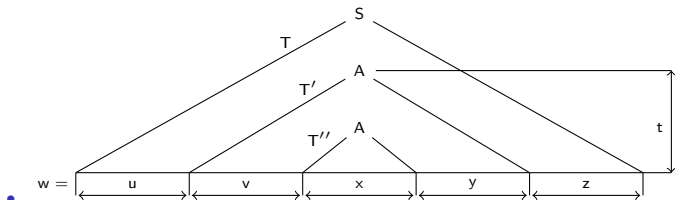
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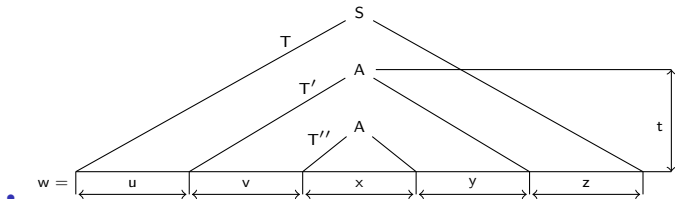
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- Let  $t$  be the height of  $T'$
- The yield of  $T'$ ,  $vxy$ , has a length of at most  $\kappa^t$
- This is the constant  $k$  that bounds the length of the subword that contains the parts that may be repeated an arbitrary number of times:  $|vxy| \leq \kappa^t = k$

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- Finally, observe that if  $vy = \text{EMP}$  then there is a parse tree with fewer leaves that generates  $w$ :  

$$S \xrightarrow{*} uAz \xrightarrow{*} uxz = w$$
- Smaller parse tree omits the derivation of  $A$  from  $A$
- Contradicts our assumption that  $T$  is the parse tree for  $w$  with the smallest number of leaves
- Therefore,  $vv \neq \text{EMP} \Rightarrow 0 < |vxy| \leq k \leq \kappa^{|V|}$

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## Theorem

$L = \{a^n b^n c^n \mid n \geq 0\}$  is not context-free.

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- Let  $w = a^k b^k c^k$
- We must determine the possible values  $vxy$  may take
- $vxy$  may not contain  $as$ ,  $bs$ , and  $cs$  because the length of  $vxy$  must be less than or equal to  $k$
- $vxy$  may contain only one letter variety or may contain two letter varieties by straddling the border between  $as$  and  $bs$  or between  $bs$  and  $cs$



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- $vxy$  may contain only one letter variety or may contain two letter varieties by straddling the border between as and bs or between bs and cs
- We prove that  $L$  is not context-free as follows:

$vxy$	Argument
$a^+$	Pump up once on $v$ and $y$ . The resulting word has more as than bs and than cs. Therefore, it is not in $L$ .
$b^+$	Pump up once on $v$ and $y$ . The resulting word has more bs than as and than cs. Therefore, it is not in $L$ .
$c^+$	Pump up once on $v$ and $y$ . The resulting word has more cs than as and than bs. Therefore, it is not in $L$ .
$a^+b^+$	Pump up once on $v$ and $y$ . The resulting word has a $b$ before an $a$ . Therefore, it is not in $L$ .
$b^+c^+$	Pump up once on $v$ and $y$ . The resulting word has a $c$ before a $b$ . Therefore, it is not in $L$ .

- No matter where the  $vxy$  window is placed in  $w$ , a word that is not in  $w$  may be generated
- We may conclude that  $L$  is not context-free.  $\square$

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- If we pump up once the resulting word is  $a^{k^2+i}$

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- $vy = a^i$ , where  $i > 0$
- If we pump up once the resulting word is  $a^{k^2+i}$
- Observe that there are no words in  $L$  with a length greater than  $|a^{k^2}|$  and less than  $|a^{(k+1)^2}|$
- After  $a^{k^2}$  the next word in  $L$  has length  $k^2 + 2k + 1$

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- In order for  $a^{k^2+i}$  to be in  $L$ ,  $i$  must at a minimum be of length  $2k + 1$
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- Therefore, we may conclude that  $L$  is not context-free  $\square$

# Properties of CFLs

- HOMEWORK: 6, 7, 9, 10



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- Consider  $L = L_1 \cap L_2$
- $L = \{a^n b^n c^n \mid n \geq 0\}$ , which we know is not context-free
- Therefore, context-free languages are not closed under intersection.

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*Context-free languages are not closed under complement.*

- Proof
- $L_1 \cap L_2 = (\text{complement } ((\text{complement } L_1) \cup (\text{complement } L_2)))$
- If context-free languages were closed under complement then they would also be closed under intersection, which we know they are not
- Therefore, context free languages are not closed under complement

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- Although context-free languages are not closed under intersection, there exist context-free languages that result in a context-free language when intersected
- Recall that all regular languages are context-free
- The intersection of a context-free language and a regular language is a context-free language
- How can we prove this?
- Develop a new constructor for a context-free language: constructive proof

- Design Idea

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- Design Idea
- Have a pda:  $M_1 = (\text{make-ndpda } K_1 \ \Sigma_1 \ \Gamma \ S_1 \ F_1 \ R_1)$
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- Basic idea: Each state of the new pda be a *super state* that represents a state in  $M_1$  and a state in  $M_2$
- Simulate all the transitions of  $M_1$  and track the transitions made by  $M_2$
- If  $M_1$  has a rule that moves from  $P$  to  $Q$  on an  $a$  then for every state  $Y \in M_2$  the new pda has a rule that moves from super state  $(P \ Y)$  to super state  $(Q \ Y)$
- If  $M_1$  has a rule that moves from  $P$  to  $Q$  on  $\epsilon$  then for every state  $Y \in M_2$  the new pda has a rule that moves from super state  $(P \ Y)$  to super state  $(Q \ Y)$  to simulate that  $M_2$  does not change state

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- If  $M_1$  has a rule that moves from P to Q on EMP then for every state  $Y \in M_2$  the new pda has a rule that moves from super state (P Y) to super state (Q Y) to simulate that  $M_2$  does not change state
- ```
;; Push or pop elements, stacke, is either
;; 1. EMP
;; 2. (listof symbol)

;; A super state, ss, is a (list state state)

;; A pda super rule, ssrule, is:
;; (list (list ss symbol stacke) (list ss stacke))
```

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- `;; pda dfa  $\rightarrow$  pda` Purpose: Build intersection pda from given machines  
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- `(ss-tbl (make-ss-table K))`  
`(ss->state (λ (ss) (second (assoc ss ss-tbl))))]`

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(start (list (sm-start a-pda) (sm-start a-dfa)))  
(F (cartesian-product (sm-finals a-pda) (sm-finals a-dfa)))  
(non-EMP-pda-rls (filter  
                  (λ (r) (not (eq? (second (first r)) EMP))) pda-rls))  
(EMP-pda-rls (filter (λ (r) (eq? (second (first r)) EMP)) pda-rls))  
(non-EMP-rls (make-nonEMP-rls non-EMP-pda-rls dfa-sts dfa-rls))  
(EMP-rls (make-EMP-rls EMP-pda-rls dfa-sts))  
(ss-tbl (make-ss-table K))  
(ss->state (λ (ss) (second (assoc ss ss-tbl))))])  
• (make-ndpda (map ss->state K) (sm-sigma a-pda) (sm-gamma a-pda)  
             (ss->state start) (map ss->state F)  
             (convert-ssrules (append non-EMP-rls EMP-rls) ss->state))))

## Properties of CFLs

- `;; pda dfa → pda` Purpose: Build intersection pda from given machines  
(define (pda-intersect-dfa a-pda a-dfa)
- `;; (listof state) (listof state) → (listof ss)`  
`;; Purpose: Create a list of super states`  
(define (cartesian-product pda-sts dfa-sts) ...)  
  
`;; (listof pda-rule) (listof state) → (listof ssrule)`  
`;; Purpose: Create ssrules for given empty transition pda-rules`  
(define (make-EMP-rls EMP-pda-rls dfa-sts) ...)  
`;; (listof pda-rule) (listof state) (listof dfa-rls) → (listof ssrule)`  
`;; Purpose: Create ssrules for given nonempty transition pda-rules`  
(define (make-nonEMP-rls non-EMP-pda-rls dfa-sts dfa-rls) ...)  
  
`;; (listof ss) → (listof (list ss state))`  
`;; Purpose: Create table associating given super states with a new state`  
(define (make-ss-table K) ...)  
`;; (listof ssrule) (ss → state) → (listof pda-rule)`  
`;; Purpose: Convert ssrules to pda-rules`  
(define (convert-ssrules ss-rls ss->state) ...)  
  
• (let\* [(pda-sts (sm-states a-pda)) (dfa-sts (sm-states a-dfa))  
 (pda-rls (sm-rules a-pda)) (dfa-rls (sm-rules a-dfa))  
 (K (cartesian-product pda-sts dfa-sts))  
 (start (list (sm-start a-pda) (sm-start a-dfa)))  
 (F (cartesian-product (sm-finals a-pda) (sm-finals a-dfa)))  
 (non-EMP-pda-rls (filter  
 (λ (r) (not (eq? (second (first r)) EMP))) pda-rls))  
 (EMP-pda-rls (filter (λ (r) (eq? (second (first r)) EMP)) pda-rls))  
 (non-EMP-rls (make-nonEMP-rls non-EMP-pda-rls dfa-sts dfa-rls))  
 (EMP-rls (make-EMP-rls EMP-pda-rls dfa-sts))  
 (ss-tbl (make-ss-table K))  
 (ss->state (λ (ss) (second (assoc ss ss-tbl))))])  
 (make-ndpda (map ss->state K) (sm-sigma a-pda) (sm-gamma a-pda)  
 (ss->state start) (map ss->state F)  
 (convert-ssrules (append non-EMP-rls EMP-rls) ss->state))))

# Properties of CFLs

- ```
;; (listof state) (listof state) → (listof ss)
;; Purpose: Create a list of super states
(define (cartesian-product pda-sts dfa-sts)
  (for*/list [(s1 pda-sts) (s2 dfa-sts)] (list s1 s2)))
```

# Properties of CFLs

- ;; (listof state) (listof state)  $\rightarrow$  (listof ss)  
;; Purpose: Create a list of super states  
(define (cartesian-product pda-sts dfa-sts)  
 (for\*/list [(s1 pda-sts) (s2 dfa-sts)] (list s1 s2)))
- (define (make-EMP-rls EMP-pda-rls dfa-sts)  
 (for\*/list [(r EMP-pda-rls) (s dfa-sts)]  
 (list (list (list (first (first r)) s)  
 EMP  
 (third (first r)))  
 (list (list (first (second r)) s)  
 (second (second r)))))))

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 (for\*/list [(r EMP-pda-rls) (s dfa-sts)]  
 (list (list (list (first (first r)) s)  
 EMP  
 (third (first r)))  
 (list (list (first (second r)) s)  
 (second (second r)))))))
- (define (make-nonEMP-rls non-EMP-pda-rls dfa-sts dfa-rls)  
 (for\*/list [(r non-EMP-pda-rls) (s dfa-sts)]  
 (list (list  
 (list (first (first r)) s)  
 (second (first r))  
 (third (first r)))  
 (list  
 (list  
 (first (second r))  
 (third  
 (first  
 (filter ( $\lambda$  (rl)  
 (and (eq? (first rl) s)  
 (eq? (second rl)  
 (second (first r))))))  
 dfa-rls))))  
 (second (second r)))))))



## Properties of CFLs

- ```
(define (make-ss-table K)
  (for*/list [(ss K)] (list ss (generate-symbol 'T '(T)))))

(define (convert-ssrules ss-rls ss->state)
  (for*/list [(r ss-rls)]
    (list
      (list (ss->state (first (first r)))
            (second (first r))
            (third (first r)))
      (list (ss->state (first (second r)))
            (second (second r))))))
```

## Properties of CFLs

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(define (make-ss-table K)
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      (list (ss->state (first (first r)))
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      (list (ss->state (first (second r)))
            (second (second r))))))
```
- ```
;; Tests for pda-intersect-dfa
(define a^nb^nIM (pda-intersect-dfa a^nb^n M))
(define a^nb^nINO-ABBA (pda-intersect-dfa a^nb^n NO-ABAA))

(check-equal? (sm-apply a^nb^nIM '()) 'reject)
(check-equal? (sm-apply a^nb^nIM '(a b b)) 'reject)
(check-equal? (sm-apply a^nb^nIM '(a a b b)) 'reject)
(check-equal? (sm-apply a^nb^nIM '(a b)) 'accept)

(check-equal? (sm-apply a^nb^nINO-ABBA '(b b)) 'reject)
(check-equal? (sm-apply a^nb^nINO-ABBA '(a a b)) 'reject)
(check-equal? (sm-apply a^nb^nINO-ABBA '()) 'accept)
(check-equal? (sm-apply a^nb^nINO-ABBA '(a a b b)) 'accept)
```

# Properties of CFLs

## Theorem

*The intersection of a context-free language and a regular language is context-free.*

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- Proof (sketch)
- Let  $Q$  be a pda, let  $D$  be a dfa, and let  $M = (\text{pda-intersect-dfa } Q \ D)$
- By construction,  $M$  simulates the transitions of both  $Q$  and  $D$
- $M$  starts its simulations in a state that represents  $Q$ 's and  $D$ 's starting states
- Only accepts if both  $Q$  and  $D$  accept
- This means that all accepted strings must be in  $L(Q)$  and  $L(D)$

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- This means that all accepted strings must be in  $L(Q)$  and  $L(D)$
- $L(M) = L(Q) \cap L(D)$ .  $\square$

# Properties of CFLs

## Theorem

$L = \{ w \in \{a b\}^* \mid w \text{ does not contain } abaa \text{ and has equal number of } a\text{'s and } b\text{'s.} \}$   
*is context-free.*

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- The language of all words that do not contain  $abaa$  is decided by NO-ABAA (from dfa chapter)
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- The language of all words that do not contain  $abaa$  is decided by NO-ABAA (from dfa chapter)
- $L(\text{NO-ABAA})$  is regular
- The language,  $E$ , of all words that have an equal number of  $a$ 's and  $b$ 's is generated by the following context-free grammar:

$$\begin{aligned} S &\rightarrow EMP \\ &\rightarrow aSb \\ &\rightarrow bSa \\ &\rightarrow aSbSbSa \\ &\rightarrow bSaSaSb \end{aligned}$$

- $E$  is context-free

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## Theorem

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$$\begin{aligned} S &\rightarrow EMP \\ &\rightarrow aSb \\ &\rightarrow bSa \\ &\rightarrow aSbSbSa \\ &\rightarrow bSaSaSb \end{aligned}$$

- $E$  is context-free
- Observe that  $L = L(E) \cap L(\text{NO-ABAA})$
- By previous theorem, we may conclude that  $L$  is context-free.  $\square$

# Properties of CFLs

- HOMEWORK: 12–14

# Deterministic pdas

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- Such a pda must satisfy the following constraints:
  - ① There are no pair of transitions that offer a choice on which to use
  - ② For all accepting states,  $Q$ , there are no transitions of this nature:  $((Q \text{ EMP EMP}) (P \text{ a}))$ . That is, the machine does not have to choose between continuing with the computation or accepting.

# Deterministic pdas

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- The pda never has a choice of what to do next
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  - ② For all accepting states,  $Q$ , there are no transitions of this nature:  $((Q \text{ EMP EMP}) (P \text{ a}))$ . That is, the machine does not have to choose between continuing with the computation or accepting.
- Given a nondeterministic pda, can a deterministic pda that decides the same language be constructed?

## Deterministic pdas

$$L = wcw^R$$

- The nondeterministic pda's we previously designed may be summarized as follows:
  - ① Nondeterministically move from  $S$ , the starting state, to  $P$
  - ② In  $P$ , accumulate  $w$  on the stack in reversed order and move to  $Q$  upon reading a  $c$  without modifying the stack.
  - ③ In  $Q$ , match the elements of  $w^R$  with the elements on the stack and nondeterministically move to,  $F$ , the final state.



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- A deterministic pda for  $L$  must avoid nondeterministic transitions
- Observe that the nondeterministic pda operates in two phases: before reading  $c$ , it pushes  $w$  onto the stack and after reading  $c$ , it matches  $w^R$  with the elements on the stack
- The phases are clearly delineated around reading  $c$

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- A deterministic pda for  $L$  must avoid nondeterministic transitions
- Observe that the nondeterministic pda operates in two phases: before reading  $c$ , it pushes  $w$  onto the stack and after reading  $c$ , it matches  $w^R$  with the elements on the stack
- The phases are clearly delineated around reading  $c$
- A deterministic pda may decide  $L$  as follows:
  - ① Before reading  $c$ , push read elements onto the stack
  - ② After reading  $c$ , match read elements with elements onto the stack

# Deterministic pdas

$$L = wcw^R$$

- Name is  $dwcw^R$ , input alphabet is  $\{a \ b \ c\}$ , and the stack alphabet is  $\{a \ b\}$ .

# Deterministic pdas

$$L = wcw^R$$

- Name is  $dwcw^R$ , input alphabet is  $\{a \ b \ c\}$ , and the stack alphabet is  $\{a \ b\}$ .
- Sample tests:

```
(check-equal? (sm-apply dwcw^r '())) 'reject)
(check-equal? (sm-apply dwcw^r '(a b b a)) 'reject)
(check-equal? (sm-apply dwcw^r '(a a)) 'reject)
(check-equal? (sm-apply dwcw^r '(a b b c b b a b))
              'reject)
(check-equal? (sm-apply dwcw^r '(a c a)) 'accept)
(check-equal? (sm-apply dwcw^r '(b a c a b)) 'accept)
(check-equal? (sm-apply dwcw^r '(b b a c a b b)) 'accept)
```

# Deterministic pdas

$$L = wcw^R$$



S:  $ci = \{a\ b\}^*$  AND  $stack = ci^R$ , starting state

# Deterministic pdas

$$L = wcw^R$$

- S:  $ci = \{a\ b\}^*$  AND  $stack = ci^R$ , starting state
- F:  $ci = xycy^R$  AND  $stack = x^R$ , final state

## Deterministic pdas

$$L = wcw^R$$

- $S: ci = \{a\ b\}^*$  AND  $stack = ci^R$ , starting state
- $F: ci = xycy^R$  AND  $stack = x^R$ , final state
- $((S\ a\ ,EMP)\ (S\ (a)))\quad ((S\ b\ ,EMP)\ (S\ (b)))\quad ((S\ c\ ,EMP)\ (F\ ,EMP))$

## Deterministic pdas

$$L = wcw^R$$

- $S: ci = \{a\ b\}^*$  AND  $stack = ci^R$ , starting state
- $F: ci = xycy^R$  AND  $stack = x^R$ , final state
- $((S\ a\ ,EMP)\ (S\ (a)))\quad ((S\ b\ ,EMP)\ (S\ (b)))\quad ((S\ c\ ,EMP)\ (F\ ,EMP))$
- $((F\ a\ (a))\ (F\ ,EMP))\quad ((F\ b\ (b))\ (F\ ,EMP))$
- There are no nondeterministic choices that the machine must make



## Deterministic pdas

$$L = w c w^R$$

- S:  $ci = \{a\ b\}^*$  AND  $stack = ci^R$ , starting state
- F:  $ci = xycy^R$  AND  $stack = x^R$ , final state
- $((S\ a\ ,EMP)\ (S\ (a)))\ ((S\ b\ ,EMP)\ (S\ (b)))\ ((S\ c\ ,EMP)\ (F\ ,EMP))$
- $((F\ a\ (a))\ (F\ ,EMP))\ ((F\ b\ (b))\ (F\ ,EMP))$
- There are no nondeterministic choices that the machine must make
- ```
;; word stack → Boolean
(define (S-INV ci s)
  (and (andmap (λ (s) (or (eq? s 'a) (eq? s 'b))) ci)
    (equal? ci (reverse s))))

(check-equal? (S-INV '(c) '()) #f)
(check-equal? (S-INV '(b a) '(b a)) #f)
(check-equal? (S-INV '() '()) #t)
(check-equal? (S-INV '(a b b) '(b b a)) #t)
```

# Deterministic pdas

$L = w c w^R$

- $S: ci = \{a\}^* \text{ AND } stack = ci^R, \text{ starting state}$
- $F: ci = x y c^R \text{ AND } stack = x^R, \text{ final state}$
- $((S\ a\ ,EMP)\ (S\ (a)))\ ((S\ b\ ,EMP)\ (S\ (b)))\ ((S\ c\ ,EMP)\ (F\ ,EMP))$
- $((F\ a\ (a))\ (F\ ,EMP))\ ((F\ b\ (b))\ (F\ ,EMP))$
- There are no nondeterministic choices that the machine must make
- ```
;; word stack → Boolean
(define (S-INV ci s)
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    (equal? ci (reverse s))))

(check-equal? (S-INV '(c) '()) #f)
(check-equal? (S-INV '(b a) '(b a)) #f)
(check-equal? (S-INV '() '()) #t)
(check-equal? (S-INV '(a b b) '(b b a)) #t)
```
- ```
;; word stack → Boolean
(define (F-INV ci s)
  (and (member 'c ci)
    (let* [(w (takef ci (λ (x) (not (eq? x 'c)))))
      (x (take ci (length s)))
      (y^R (drop ci (add1 (length w))))]
      (and (equal?
        ci
        (append x (reverse y^R) (list 'c) y^R))
        (equal? s (reverse x))))))

(check-equal? (F-INV '(a b c) '()) #f)
(check-equal? (F-INV '(a a c) '(a)) #f)
(check-equal? (F-INV '(a b) '(b a)) #f)
(check-equal? (F-INV '(c) '()) #t)
(check-equal? (F-INV '(b a c) '(a b)) #t)
(check-equal? (F-INV '(a b b c b) '(b a)) #t)
```

# Deterministic pdas

$L = w c w^R$        $M = d w c w^R$        $v \in (\text{sm-sigma } M)^*$   
 $s = \text{the stack}$        $ci = \text{consumed input}$

## Theorem

*The state invariants hold when  $M$  accepts  $v$ .*

- For the base case, when  $M$  starts  $ci = '()$  and  $s = '()$ . This means that  $ci$  only contains  $as$  and  $bs$  and that  $ci$  equals  $s$  reversed. Thus, S-INV holds.

# Deterministic pdas

$L = \{wcw^R \mid w \in \{a,b\}^*\}$        $M = \text{dwcw}^R$        $v \in (\Sigma^* M)^*$   
 $s = \text{the stack}$        $ci = \text{consumed input}$

## Theorem

*The state invariants hold when  $M$  accepts  $v$ .*

- For the base case, when  $M$  starts  $ci = \epsilon$  and  $s = \epsilon$ . This means that  $ci$  only contains  $a$ s and  $b$ s and that  $ci$  equals  $s$  reversed. Thus, S-INV holds.
- Proof invariants hold after each transition that consumes input:
- $((S \xrightarrow{a, EMP} S(a)))$ : By inductive hypothesis S-INV holds. This means that  $ci \in \{a, b\}^*$  and that  $ci = s^R$ . After using this rule, the consumed input only contains  $a$ s and  $b$ s and the consumed input is equal to the stack reversed. Thus, S-INV holds.

# Deterministic pdas

$L = w c w^R$        $M = d w c w^R$        $v \in (sm\text{-}\sigma(M))^*$   
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- Proof invariants hold after each transition that consumes input:
- $((S\ a, EMP) (S\ (a)))$ : By inductive hypothesis S-INV holds. This means that  $ci \in \{a\ b\}^*$  and that  $ci = s^R$ . After using this rule, the consumed input only contains  $as$  and  $bs$  and the consumed input is equal to the stack reversed. Thus, S-INV holds.
- $((S\ b, EMP) (S\ (b)))$ : By inductive hypothesis S-INV holds. This means that  $ci \in \{a\ b\}^*$  and that  $ci = s^R$ . After using this rule, the consumed input only contains  $as$  and  $bs$  and the consumed input is equal to the stack reversed. Thus, S-INV holds.

# Deterministic pdas

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- Proof invariants hold after each transition that consumes input:
- $\frac{((S\ a, EMP)\ (S\ (a)))}{}$ : By inductive hypothesis S-INV holds. This means that  $ci \in \{a\ b\}^*$  and that  $ci = s^R$ . After using this rule, the consumed input only contains  $as$  and  $bs$  and the consumed input is equal to the stack reversed. Thus, S-INV holds.
- $\frac{((S\ b, EMP)\ (S\ (b)))}{}$ : By inductive hypothesis S-INV holds. This means that  $ci \in \{a\ b\}^*$  and that  $ci = s^R$ . After using this rule, the consumed input only contains  $as$  and  $bs$  and the consumed input is equal to the stack reversed. Thus, S-INV holds.
- $\frac{((S\ c, EMP)\ (F, EMP))}{}$ : By inductive hypothesis S-INV holds. This means that  $ci \in \{a\ b\}^*$  and that  $ci = s^R$ . After using this rule,  $ci = xycy^R$ , where  $x \in \{a\ b\}^*$  and  $y = '()$ , and  $stack = x^R$ . Thus, F-INV holds.

# Deterministic pdas

$L = w c w^R$        $M = d w c w^R$        $v \in (sm\text{-}\sigma M)^*$   
 $s = \text{the stack}$        $ci = \text{consumed input}$

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- For the base case, when  $M$  starts  $ci = '()$  and  $s = '()$ . This means that  $ci$  only contains  $as$  and  $bs$  and that  $ci$  equals  $s$  reversed. Thus, S-INV holds.
- Proof invariants hold after each transition that consumes input:
- $((S\ a, EMP) (S\ (a)))$ : By inductive hypothesis S-INV holds. This means that  $ci \in \{a\ b\}^*$  and that  $ci = s^R$ . After using this rule, the consumed input only contains  $as$  and  $bs$  and the consumed input is equal to the stack reversed. Thus, S-INV holds.
- $((S\ b, EMP) (S\ (b)))$ : By inductive hypothesis S-INV holds. This means that  $ci \in \{a\ b\}^*$  and that  $ci = s^R$ . After using this rule, the consumed input only contains  $as$  and  $bs$  and the consumed input is equal to the stack reversed. Thus, S-INV holds.
- $((S\ c, EMP) (F\ , EMP))$ : By inductive hypothesis S-INV holds. This means that  $ci \in \{a\ b\}^*$  and that  $ci = s^R$ . After using this rule,  $ci = xycy^R$ , where  $x \in \{a\ b\}^*$  and  $y = '()$ , and  $stack = x^R$ . Thus, F-INV holds.
- $((F\ a\ (a)) (F\ , EMP))$ : By inductive hypothesis F-INV holds. This means that  $ci = xycy^R$ , where  $x, y \in \{a\ b\}^*$ , and  $stack = x^R$ . Using this rule means that  $x$ 's last element is an  $a$ . That is,  $x = x'a$ . After using this rule,  $ci = x'a ycy^R = x'a y c(a y)^R$  and  $stack = x'^R$ . Thus, F-INV holds.

# Deterministic pdas

$L = w c w^R$        $M = d w c w^R$        $v \in (\Sigma^* M)^*$   
 $s = \text{the stack}$        $ci = \text{consumed input}$

## Theorem

*The state invariants hold when  $M$  accepts  $v$ .*

- For the base case, when  $M$  starts  $ci = \epsilon$  and  $s = \epsilon$ . This means that  $ci$  only contains  $a$ s and  $b$ s and that  $ci$  equals  $s$  reversed. Thus, S-INV holds.
- Proof invariants hold after each transition that consumes input:
- $((S\ a, EMP)\ (S\ (a)))$ : By inductive hypothesis S-INV holds. This means that  $ci \in \{a\ b\}^*$  and that  $ci = s^R$ . After using this rule, the consumed input only contains  $a$ s and  $b$ s and the consumed input is equal to the stack reversed. Thus, S-INV holds.
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- $((S\ c, EMP)\ (F, EMP))$ : By inductive hypothesis S-INV holds. This means that  $ci \in \{a\ b\}^*$  and that  $ci = s^R$ . After using this rule,  $ci = x y c y^R$ , where  $x \in \{a\ b\}^*$  and  $y = \epsilon$ , and  $stack = x^R$ . Thus, F-INV holds.
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- $((F\ b\ (b))\ (F, EMP))$ : By inductive hypothesis F-INV holds. This means that  $ci = x y c y^R$ , where  $x, y \in \{a\ b\}^*$ , and  $stack = x^R$ . Using this rule means that  $x$ 's last element is a  $b$ . That is,  $x = x' b$ . After using this rule,  $ci = x' b y c y^R b = x' b y c (b y)^R$  and  $stack = x'^R$ . Thus, F-INV holds.



# Deterministic pdas

$$L = wcw^R$$

## Lemma

$$w \in L \Leftrightarrow w \in L(M)$$



# Deterministic pdas

$$L = w c w^R$$

## Lemma

$$w \in L \Leftrightarrow w \in L(M)$$

•

## Proof.

( $\Rightarrow$ ) Assume  $w \in L$ . This means that  $w = v c v^R$ . Given that state invariants always hold, there is a computation that has  $M$  consume  $w$  and reach  $F$  with an empty stack. Therefore,  $w \in L(M)$ .

( $\Leftarrow$ ) Assume  $w \in L(M)$ . This means that  $M$  halts in  $F$ , the only final state, with an empty stack having consumed  $w$ . Given that the state invariants always hold we may conclude that  $w = v c v^R$ . Therefore,  $w \in L$ .

□

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□

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•

## Lemma

$$w \notin L \Leftrightarrow w \notin L(M)$$

•

# Deterministic pdas

$$L = w\bar{c}w^R$$

## Lemma

$$w \in L \Leftrightarrow w \in L(M)$$

•

## Proof.

( $\Rightarrow$ ) Assume  $w \in L$ . This means that  $w = v\bar{c}v^R$ . Given that state invariants always hold, there is a computation that has  $M$  consume  $w$  and reach  $F$  with an empty stack. Therefore,  $w \in L(M)$ .

( $\Leftarrow$ ) Assume  $w \in L(M)$ . This means that  $M$  halts in  $F$ , the only final state, with an empty stack having consumed  $w$ . Given that the state invariants always hold we may conclude that  $w = v\bar{c}v^R$ . Therefore,  $w \in L$ . □

□

•

## Lemma

$$w \notin L \Leftrightarrow w \notin L(M)$$

•

## Proof.

( $\Rightarrow$ ) Assume  $w \notin L$ . This means  $w \neq v\bar{c}v^R$ . Given that the state invariant predicates always hold, there is no computation that has  $M$  consume  $w$  and end in  $F$  with an empty stack. Therefore,  $w \notin L(M)$ .

( $\Leftarrow$ ) Assume  $w \notin L(M)$ . This means that  $M$  does not halt in  $F$  with an empty stack having consumed  $w$ .

Given that the state invariants always hold, this means that  $w \neq v\bar{c}v^R$ . Thus,  $w \notin L$ . □

•

## Deterministic pdas

$$L = \{a^m b^n c^p \mid m \neq n \wedge m, n, p > 0\}$$

- Intuitively: push all the as onto the stack, match bs, either pop excess as or read excess bs, and read cs
- Challenge: deterministically determine if excess as need to be popped or excess bs need to be read

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- To prevent nondeterministic transitions of this nature, a special end of input symbol,  $z$ , not in the input alphabet of the machine, shall be used
- All input:  $wz$



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- For our purposes,  $y$  serves as the bottom of the stack symbol.

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- Another potential problem is deterministically transitioning to the final state
- To prevent nondeterministic transitions of this nature, a special end of input symbol, z, not in the input alphabet of the machine, shall be used
- All input: wz
- How can the machine sense that the stack is empty?
- Use a technique developed to construct simple pdas: the machine shall use a bottom of the stack symbol
- For our purposes, y serves as the bottom of the stack symbol.
- The machine's phases may be outlined as follows:
  - 1 When started, if an a is read then (a y) is pushed onto the stack.
  - 2 All the remaining as read are pushed onto the stack.
  - 3 Match a read b by popping an a
  - 4 Decide if there are more as or more bs:
    - 1 If a c is read and there is an a on the stack then there are more as than bs. The machine transitions to a state to read the remaining cs and pop the remaining as. Upon reading z, the machine pops y and moves to the final state.
    - 2 If a b is read and y is at the top of the stack then there are more bs than as. The machine transitions to read the bs with changing the stack. It then transitions to read the cs without altering the stack. Upon reading z, the machine pops y and moves to the final state

# Deterministic pdas

$$L = \{a^m b^n c^p \mid m \neq n \wedge m, n, p > 0\}$$

- ```
;; State Documentation
;; S: ci = stack = '()', starting state
;; A: ci = a^+ and stack = a^+y
;; B: ci = a^i b^j and stack = a^(i-j)y and 0 < i <= j
;; C: ci = a^i b^j c^k and stack = a^(i-j-k)y and i > j and i, j, k > 0
;; D: ci = a^i b^j and stack = y and j > i and i, j > 0
;; E: ci = a^i b^j c^+ and stack = y, j != i and i, j > 0
;; G: ci = a^i b^j c^+z and stack = a*y and i != j and i, j > 0
;; F: ci = a^i b^j c^+z and stack = '()' and i != j, and i, j > 0, final state
```

# Deterministic pdas

$$L = \{a^m b^n c^p \mid m \neq n \wedge m, n, p > 0\}$$

- #lang fsm

```
(define ambncp (make-ndpda '(S A B C D E F G)
                             '(a b c z)
                             '(a y)
                             'S
                             '(F)))
```

# Deterministic pdas

$$L = \{a^m b^n c^p \mid m \neq n \wedge m, n, p > 0\}$$

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```
(define ambncp (make-ndpda '(S A B C D E F G)
                             '(a b c z)
                             '(a y)
                             'S
                             '(F)))
```

- ```
(check-equal? (sm-apply ambncp '(z)) 'reject)
(check-equal? (sm-apply ambncp '(a b b c c)) 'reject)
(check-equal? (sm-apply ambncp '(a a b b c c z)) 'reject)
(check-equal? (sm-apply ambncp '(a b a z)) 'reject)
(check-equal? (sm-apply ambncp '(a b b z)) 'reject)
(check-equal? (sm-apply ambncp '(a a a b b c c z)) 'accept)
(check-equal? (sm-apply ambncp '(a a a b c z)) 'accept)
(check-equal? (sm-apply ambncp '(a a a b b b b c c c z)) 'accept)
```

# Deterministic pdas

$$L = \{a^m b^n c^p \mid m \neq n \wedge m, n, p > 0\}$$

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```
(define ambncp (make-ndpda '(S A B C D E F G)
                              '(a b c z)
                              '(a y)
                              'S
                              '(F)
                              `(((S a ,EMP) (A (a y)))
```

- ```
(check-equal? (sm-apply ambncp '(z)) 'reject)
(check-equal? (sm-apply ambncp '(a b b c c)) 'reject)
(check-equal? (sm-apply ambncp '(a a b b c c z)) 'reject)
(check-equal? (sm-apply ambncp '(a b a z)) 'reject)
(check-equal? (sm-apply ambncp '(a b b z)) 'reject)
(check-equal? (sm-apply ambncp '(a a a b b c c z)) 'accept)
(check-equal? (sm-apply ambncp '(a a a b c z)) 'accept)
(check-equal? (sm-apply ambncp '(a a a b b b b c c c z)) 'accept)
```

# Deterministic pdas

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(define ambncp (make-ndpda '(S A B C D E F G)
                              '(a b c z)
                              '(a y)
                              'S
                              '(F)
                              `(((S a ,EMP) (A (a y)))
                                ((A a ,EMP) (A (a)))
                                ((A b (a)) (B ,EMP))
```

- ```
(check-equal? (sm-apply ambncp '(z)) 'reject)
(check-equal? (sm-apply ambncp '(a b b c c)) 'reject)
(check-equal? (sm-apply ambncp '(a a b b c c z)) 'reject)
(check-equal? (sm-apply ambncp '(a b a z)) 'reject)
(check-equal? (sm-apply ambncp '(a b b z)) 'reject)
(check-equal? (sm-apply ambncp '(a a a b b c c z)) 'accept)
(check-equal? (sm-apply ambncp '(a a a b c z)) 'accept)
(check-equal? (sm-apply ambncp '(a a a b b b b c c c z)) 'accept)
```



# Deterministic pdas

$$L = \{a^m b^n c^p \mid m \neq n \wedge m, n, p > 0\}$$

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```
(define ambncp (make-ndpda '(S A B C D E F G)
  '(a b c z)
  '(a y)
  'S
  '(F)
  `(((S a ,EMP) (A (a y)))
    ((A a ,EMP) (A (a)))
    ((A b (a)) (B ,EMP))
    ((B b (a)) (B ,EMP))
    ((B c (a)) (C ,EMP))

    ((B b (y)) (D (y)))
```

- ```
(check-equal? (sm-apply ambncp '(z)) 'reject)
(check-equal? (sm-apply ambncp '(a b b c c)) 'reject)
(check-equal? (sm-apply ambncp '(a a b b c c z)) 'reject)
(check-equal? (sm-apply ambncp '(a b a z)) 'reject)
(check-equal? (sm-apply ambncp '(a b b z)) 'reject)
(check-equal? (sm-apply ambncp '(a a a b b c c z)) 'accept)
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# Deterministic pdas

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```
(define ambncp (make-ndpda '(S A B C D E F G)
                             '(a b c z)
                             '(a y)
                             'S
                             '(F)
                             `(((S a ,EMP) (A (a y)))
                               ((A a ,EMP) (A (a)))
                               ((A b (a)) (B ,EMP))
                               ((B b (a)) (B ,EMP))
                               ((B c (a)) (C ,EMP))
                               ((C c (a)) (C ,EMP))
                               ((C z ,EMP) (G ,EMP))
                               ((C c (y)) (E (y)))
                               ((B b (y)) (D (y)))
```

- ```
(check-equal? (sm-apply ambncp '(z)) 'reject)
(check-equal? (sm-apply ambncp '(a b b c c)) 'reject)
(check-equal? (sm-apply ambncp '(a a b b c c z)) 'reject)
(check-equal? (sm-apply ambncp '(a b a z)) 'reject)
(check-equal? (sm-apply ambncp '(a b b z)) 'reject)
(check-equal? (sm-apply ambncp '(a a a b b c c z)) 'accept)
(check-equal? (sm-apply ambncp '(a a a b c z)) 'accept)
(check-equal? (sm-apply ambncp '(a a a b b b b c c c z)) 'accept)
```

# Deterministic pdas

$$L = \{a^m b^n c^p \mid m \neq n \wedge m, n, p > 0\}$$

- #lang fsm
- ```
(define ambncp (make-ndpda '(S A B C D E F G)
                              '(a b c z)
                              '(a y)
                              'S
                              '(F)
                              `(((S a ,EMP) (A (a y)))
                                ((A a ,EMP) (A (a)))
                                ((A b (a)) (B ,EMP))
                                ((B b (a)) (B ,EMP))
                                ((B c (a)) (C ,EMP))
                                ((C c (a)) (C ,EMP))
                                ((C z ,EMP) (G ,EMP))
                                ((C c (y)) (E (y)))
                                ((B b (y)) (D (y)))
                                ((D b (y)) (D (y)))
                                ((D c (y)) (E (y)))

                                (check-equal? (sm-apply ambncp '(z)) 'reject)
                                (check-equal? (sm-apply ambncp '(a b b c c)) 'reject)
                                (check-equal? (sm-apply ambncp '(a a b b c c z)) 'reject)
                                (check-equal? (sm-apply ambncp '(a b a z)) 'reject)
                                (check-equal? (sm-apply ambncp '(a b b z)) 'reject)
                                (check-equal? (sm-apply ambncp '(a a a b b c c z)) 'accept)
                                (check-equal? (sm-apply ambncp '(a a a b c z)) 'accept)
                                (check-equal? (sm-apply ambncp '(a a a b b b b c c c c z)) 'accept))
```

# Deterministic pdas

$$L = \{a^m b^n c^p \mid m \neq n \wedge m, n, p > 0\}$$

- #lang fsm

```
(define ambncp (make-ndpda '(S A B C D E F G)
                             '(a b c z)
                             '(a y)
                             'S
                             '(F)
                             `(((S a ,EMP) (A (a y)))
                               ((A a ,EMP) (A (a)))
                               ((A b (a)) (B ,EMP))
                               ((B b (a)) (B ,EMP))
                               ((B c (a)) (C ,EMP))
                               ((C c (a)) (C ,EMP))
                               ((C z ,EMP) (G ,EMP))
                               ((C c (y)) (E (y)))
                               ((B b (y)) (D (y)))
                               ((D b (y)) (D (y)))
                               ((D c (y)) (E (y)))
                               ((E c (y)) (E (y)))
                               ((E z (y)) (F ,EMP))
```

- ```
(check-equal? (sm-apply ambncp '(z)) 'reject)
(check-equal? (sm-apply ambncp '(a b b c c)) 'reject)
(check-equal? (sm-apply ambncp '(a a b b c c z)) 'reject)
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(check-equal? (sm-apply ambncp '(a a a b b c c z)) 'accept)
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(check-equal? (sm-apply ambncp '(a a a b b b b c c c c z)) 'accept)
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                              'S
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                              `(((S a ,EMP) (A (a y)))
                                ((A a ,EMP) (A (a)))
                                ((A b (a)) (B ,EMP))
                                ((B b (a)) (B ,EMP))
                                ((B c (a)) (C ,EMP))
                                ((C c (a)) (C ,EMP))
                                ((C z ,EMP) (G ,EMP))
                                ((C c (y)) (E (y)))
                                ((B b (y)) (D (y)))
                                ((D b (y)) (D (y)))
                                ((D c (y)) (E (y)))
                                ((E c (y)) (E (y)))
                                ((E z (y)) (F ,EMP))
                                ((G ,EMP (a)) (G ,EMP))
                                ((G ,EMP (v)) (F ,EMP))))))
```

```
(check-equal? (sm-apply ambncp '(z)) 'reject)
(check-equal? (sm-apply ambncp '(a b b c c)) 'reject)
(check-equal? (sm-apply ambncp '(a a b b c c z)) 'reject)
(check-equal? (sm-apply ambncp '(a b a z)) 'reject)
(check-equal? (sm-apply ambncp '(a b b z)) 'reject)
(check-equal? (sm-apply ambncp '(a a a b b c c z)) 'accept)
(check-equal? (sm-apply ambncp '(a a a b c c)) 'accept)
(check-equal? (sm-apply ambncp '(a a a b b b b c c c z)) 'accept)
```

# Deterministic pdas

$$L = \{a^m b^n c^p \mid m \neq n \wedge m, n, p > 0\} \quad \text{ci} = \text{consumed input } P = \text{ambncp}$$

## Theorem

*The state invariants hold when P is applied to w.*

- When P starts, S-INV holds because  $ci = '()$  and the stack =  $'()$ . This establishes the base case.

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## Theorem

*The state invariants hold when P is applied to w.*

- When P starts, S-INV holds because  $ci = '()$  and the  $stack = '()$ . This establishes the base case.
- Proof that invariants hold after each nonempty transition:
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# Deterministic pdas

$$L = \{a^m b^n c^p \mid m \neq n \wedge m, n, p > 0\} \quad \text{ci} = \text{consumed input} \quad P = \text{ambncp}$$

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- $((C \ z \ ,EMP) \ (G \ ,EMP))$ : By inductive hypothesis, C-INV holds. This means  $ci = (a^i \ b^j \ c^k)$ ,  $stack = (a^{i-j-k} \ y)$ ,  $i > j$ , and  $i, j, k > 0$ . Observe that  $i \neq j$  and that  $c^k \in c^+$ . Using this transition adds  $z$  to  $ci$  without changing the stack. Therefore,  $ci = (a^i \ b^j \ c^+ \ z)$ ,  $stack = (a^* \ y)$ ,  $i \neq j$ , and  $i, j > 0$ . Thus, G-INV holds.
- $((C \ c \ (y)) \ (E \ (y)))$ : By inductive hypothesis, C-INV holds. This means  $ci = (a^i \ b^j \ c^k)$ ,  $stack = (a^{i-j-k} \ y)$ ,  $i > j$ , and  $i, j, k > 0$ . Note that  $c^k \in c^+$ . Using this transition means that a  $c$  is added to  $ci$  without changing the stack. Therefore, after using this transition,  $ci = (a^i \ b^j \ c^+)$ ,  $stack = (y)$ ,  $j \neq i$ , and  $i, j > 0$ . Thus, E-INV holds.
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- $((E \ z \ (y)) \ (F \ ,EMP))$ : By inductive hypothesis, E-INV holds. This means  $ci = (a^i \ b^j \ c^+)$ ,  $stack = (y)$ ,  $j \neq i$ , and  $i, j > 0$ . Using this transition adds  $z$  to  $ci$  and pops  $y$  from the stack. Therefore, after using this transition,  $ci = (a^i \ b^j \ c^+ \ z)$ ,  $stack = '()$ ,  $i \neq j$ , and  $i, j > 0$ . Thus, F-INV holds.

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- This establishes the inductive step and concludes the proof.

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( $\Rightarrow$ ) Assume  $w \in L$ . This means that  $w = a^m b^n c^p$ , where  $m \neq n$  and  $m, n, p > 0$ . Given that state invariants always hold, there is a computation that has  $P$  consume  $w$  reaching  $F$  with an empty stack. Therefore,  $w \in L(P)$ .

( $\Leftarrow$ ) Assume  $w \in L(P)$ . This means that  $M$  halts in  $F$ , the only final state, with an empty stack having consumed  $w$ . Given that the state invariants always hold we may conclude that  $w = a^m b^n c^p$ , where  $m \neq n$  and  $m, n, p > 0$ . Therefore,  $w \in L$ . □

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( $\Leftarrow$ ) Assume  $w \notin L(P)$ . This means that  $P$  cannot transition into  $F$  with an empty stack having consumed  $w$ . Given that the state predicate invariants always hold,  $w \neq a^m b^n c^p$ ,  $m \neq n$ , or  $m, n, p \not> 0$ . Therefore,  $w \notin L$ .  $\square$

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# Deterministic pdas

- HOMEWORK: 5–7

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- $M$  is a simple pda with a bottom-of-the-stack symbol
- Second, we define a configuration,  $C = (Q \ w \ s)$ , as a *dead end* if in zero or more steps it leads to a configuration,  $E = (R \ w' \ s')$ , in which no input has been consumed (i.e.,  $w = w'$ ) or the stack has not shrunk (i.e.,  $|s'| \geq |s|$ )
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- If a pda does not have a dead-end configuration then it will eventually read all its input
- If  $M$  is simple then determining if a configuration is a dead end only depends on the current state, the next element to read, and the top stack symbol
- Let  $D$  be the set of dead-end configurations. Observe that  $D$  is finite.

# Deterministic pdas

- Recall that context-free languages are not closed under complement
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- Assume  $L$  is a context-free language such that  $L^c$  is accepted by a deterministic simple pda  $M = (\text{make\_ndpda } K \ \Sigma \ \Gamma \ S \ F \ R)$
- Intuitively, we would like to build a pda for  $L$ 's complement by simply switching the role of  $M$ 's accepting and rejecting states
- Unfortunately, this does not work because pdas may reject without reading all the input:  $M$  is in a configuration for which no transition rules apply or  $M$  is in a dead-end configuration

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- $M'$  is constructed as follows:
  - $\forall (Q a x) \in D$ , remove rules of the form  $((Q a x) (H y))$  in  $M$  and add  $((Q a x) (U \text{EMP}))$ , where  $U$  is a new non-accepting state.
  - $\forall a \in \Sigma$ , add the rule  $((U a \text{EMP}) (U \text{EMP}))$
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- $M'$  is deterministic given that  $M$  is deterministic and the added rules do not make  $M'$  nondeterministic
- $M'$  rejects by moving to a non-accepting state whenever  $M$  does not read the entire input
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- Thus,  $M'$  always reads the entire input
- Given that  $M'$  reads the entire input, reversing the role of accepting and non-accepting states results in a machine that accepts  $(\Sigma^* - L)^c = L$ 's complement

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- **In the context of *pdas*, nondeterminism is more powerful than determinism**
- This makes software development more difficult: parsers decide if a program is valid

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- NOT-LSU is the union of the following two languages:
  - $\{a^m b^n c^p \mid m = n = p \wedge m, n, p > 0\}$
  - $\{w \mid w \in \{a b c\}^* \wedge w \text{ has letters out of order} \vee w \text{ is missing at least one element in } \{a b c\}\}$

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- Given our assumption, LUS is a deterministic context-free language
- Deterministic context-free languages are closed under complement
- Thus, the complement of LSU, NOT-LSU, is also a deterministic context-free language
- NOT-LSU is the union of the following two languages:
  - $\{a^m b^n c^p \mid m = n = p \wedge m, n, p > 0\}$
  - $\{w \mid w \in \{a b c\}^* \wedge w \text{ has letters out of order } \vee w \text{ is missing at least one element in } \{a b c\}\}$
- The intersection of a context-free language and a regular language is a context-free language

## Deterministic pdas

### Theorem

*Deterministic context-free languages are not closed under union.*

- Assume deterministic context-free languages are closed under union
- Consider the following deterministic context-free languages:
  - $L = \{a^m b^n c^p \mid m \neq n \wedge m, n, p > 0\}$
  - $S = \{a^m b^n c^p \mid n \neq p \wedge m, n, p > 0\}$
- Let  $LUS = L \cup S$
- Given our assumption,  $LUS$  is a deterministic context-free language
- Deterministic context-free languages are closed under complement
- Thus, the complement of  $LSU$ ,  $NOT-LSU$ , is also a deterministic context-free language
- $NOT-LSU$  is the union of the following two languages:
  - $\{a^m b^n c^p \mid m = n = p \wedge m, n, p > 0\}$
  - $\{w \mid w \in \{a b c\}^* \wedge w \text{ has letters out of order } \vee w \text{ is missing at least one element in } \{a b c\}\}$
- The intersection of a context-free language and a regular language is a context-free language
- $NOT-LSU \cap a^* b^* c^* = a^n b^n c^n$ , where  $n > 0$
- This is not a context-free language
- Our assumption is wrong and deterministic context-free languages are not closed under union

# Deterministic pdas

## Theorem

*Deterministic context-free languages are not closed under intersection.*

- Assume deterministic context-free languages are closed under intersection

# Deterministic pdas

## Theorem

*Deterministic context-free languages are not closed under intersection.*

- Assume deterministic context-free languages are closed under intersection
- Consider the following languages:
  - $L = \{a^m b^n c^p \mid m = n \wedge m, n, p \geq 0\}$
  - $S = \{a^m b^n c^p \mid n = p \wedge m, n, p \geq 0\}$



# Deterministic pdas

## Theorem

*Deterministic context-free languages are not closed under intersection.*

- Assume deterministic context-free languages are closed under intersection
- Consider the following languages:
  - $L = \{a^m b^n c^p \mid m = n \wedge m, n, p \geq 0\}$
  - $S = \{a^m b^n c^p \mid n = p \wedge m, n, p \geq 0\}$
- The intersection of  $L$  and  $S$  is  $\{a^n b^n c^n \mid n \geq 0\}$
- This language is not context-free and, therefore, deterministic context-free languages are not closed under intersection

# Deterministic pdas

- HOMEWORK: 8, 10
- QUIZ: 9 (due in 1 week)