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Context-Free Grammars

- We shall now study languages that are not regular

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Context-Free Grammars

- We shall now study languages that are not regular
- A context-free language is generated by a *context-free grammar*

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Context-Free Grammars

- We shall now study languages that are not regular
- A context-free language is generated by a *context-free grammar*
- CFG are used to describe the syntax of programming languages:

```
<definition> → (define <identifier> <expression>)
             → (define (<identifier> <params>) <expression>)
    <params> → ε
             → <identifier> <parameters>
<expression> → <number>
             → <identifier>
             → (+ <expression> <expression>)
             → (- <expression> <expression>)
             → (* <expression> <expression>)
             → (quotient <expression> <expression>)
             → (remainder <expression> <expression>)
```

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Context-Free Grammars

- A context-free grammar is an instance of (make-cfg N Σ R S)
- N is the set of capital letters in the Roman alphabet representing the nonterminal symbols (i.e., syntactic categories)
- Σ is the set of lowercase symbols in the Roman alphabet called the alphabet (or terminal symbols)
- S is the starting nonterminal symbol
- R is the set of production rules
    - Each production rule is of the form (N $\rightarrow$ (N $\cup$ Σ $\cup$ ϵ)$^+$)
    - There is a single nonterminal on the left hand side of a production rule
    - There is a symbol consisting of one or more nonterminals, terminals, or ϵ on the right hand side of a production rule

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    - Each production rule is of the form $(N \rightarrow (N \cup \Sigma \cup \epsilon)^+)$
    - There is a single nonterminal on the left hand side of a production rule
    - There is a symbol consisting of one or more nonterminals, terminals, or ϵ on the right hand side of a production rule
- A derivation consists of 1 or more derivation steps
- A derivation step is the application of a production rule: $\rightarrow_G$ or $\rightarrow$ if G is clear from the context
- Zero or more derivation steps is denoted by $\rightarrow_G^*$ or $\rightarrow^*$ if G is clear from the context

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- S is the starting nonterminal symbol
- R is the set of production rules
  - Each production rule is of the form (N → (N ∪ Σ ∪ ϵ)$^{+}$)
  - There is a single nonterminal on the left hand side of a production rule
  - There is a symbol consisting of one or more nonterminals, terminals, or ϵ on the right hand side of a production rule
- A derivation consists of 1 or more derivation steps
- A derivation step is the application of a production rule: $\rightarrow_G$ or $\rightarrow$ if G is clear from the context
- Zero or more derivation steps is denoted by $\rightarrow_G^*$ or $\rightarrow^*$ if G is clear from the context
- L(G) denotes the language generated by G: {w | w∈Σ* ∧ S $\rightarrow_G^*$ w}
- A language, L, is context-free if L = L(G) for some context-free grammar G

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Context-Free Grammars

- L = $\{a^n b^n \mid n \geq 0\}$
- Follow the steps of the DR for grammars

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Context-Free Grammars

- $L = \{a^n b^n \mid n \geq 0\}$
- Follow the steps of the DR for grammars
- Name = a2nb2n and $\Sigma = \{a\ b\}$

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Context-Free Grammars

- L = $\{a^n b^n \mid n \geq 0\}$
- Follow the steps of the DR for grammars
- Name = a2nb2n and Σ = {a b}
- S, generates all words that start with n as followed by n bs
- How can this be done?

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- S, generates all words that start with n as followed by n bs
- How can this be done?
- First a has a matching b at the end of the word
- The same is true for the remaining part of the word

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Context-Free Grammars

- $L = \{a^n b^n \mid n \geq 0\}$
- Follow the steps of the DR for grammars
- Name $=$ a2nb2n and $\Sigma = \{a\ b\}$
- S, generates all words that start with n as followed by n bs
- How can this be done?
- First a has a matching b at the end of the word
- The same is true for the remaining part of the word
- This suggests S ought to generate ϵ
- It also suggests that nonempty words in L may be generated from the outside in

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Context-Free Grammars

- L = $\{a^n b^n \mid n \geq 0\}$
- Follow the steps of the DR for grammars
- Name = a2nb2n and Σ = {a b}
- S, generates all words that start with n as followed by n bs
- How can this be done?
- First a has a matching b at the end of the word
- The same is true for the remaining part of the word
- This suggests S ought to generate ϵ
- It also suggests that nonempty words in L may be generated from the outside in
- Generate an a as the first letter in the word, generate a b for the last letter in the word, and use S to generate the rest of the word in the middle

Context-Free Grammars

- Production Rules

    `` `(S ,ARROW ,EMP) ``

    `` `(S ,ARROW aSb) ``

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Context-Free Grammars

- Production Rules

  `` `(S ,ARROW ,EMP) ``

  `` `(S ,ARROW aSb) ``

- Tests

```
;; Tests for a2nb2n
#:rejects '((b b b) (a b a))
#:accepts '(() (a b) (a a b b) (a a a b b b) (a a a a a b b b b b))
      Both methods are acceptable
(check-not-derive? a2nb2n '(b b b))
(check-not-derive? a2nb2n '(a b a))
(check-derive? a2nb2n '())
(check-derive? a2nb2n '(a b))
(check-derive? a2nb2n '(a a b b))
(check-derive? a2nb2n '(a a a b b b))
(check-derive? a2nb2n '(a a a a a b b b b b))
```

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Context-Free Grammars

- ```
  #lang fsm
  ;; L = a^nb^n Syntactic Categories
  ;;  S = words that start with n a and end with n b
  (define a2nb2n
    (make-cfg '(S)
              '(a b)
              `((S ,ARROW ,EMP)
                (S ,ARROW aSb))
              'S
              #:rejects '((b b b) (a b a))
              #:accepts '(() (a b) (a a b b) (a a a b b b)
                          (a a a a a b b b b b)))))
  ```

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Context-Free Grammars

- ```
  #lang fsm
  ;; L = a^nb^n Syntactic Categories
  ;; S = words that start with n a and end with n b
  (define a2nb2n
    (make-cfg '(S)
              '(a b)
              `((S ,ARROW ,EMP)
                (S ,ARROW aSb))
              'S
              #:rejects '((b b b) (a b a))
              #:accepts '(() (a b) (a a b b) (a a a b b b)
                          (a a a a a b b b b b))))
  ```

- Illustrate derivation tree using grammar-viz

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Context-Free Grammars

- > (grammar-test a2nb2n 10)
  ```
  '(((b a b b b a) "(b a b b b a) is not in L(G)")
    (() "The word () is too short to test.")
    ((a a b b) (S -> aSb -> aaSbb -> aabb))
    ((a a) "(a a) is not in L(G)")
    ((a b b b b b a b) "(a b b b b b a b) is not in L(G)")
    ((a a a b a a) "(a a a b a a) is not in L(G)")
    ((a a a b b a a) "(a a a b b a a) is not in L(G)")
    ((a b) (S -> aSb -> ab))
    ((a) "The word (a) is too short to test.")
    ((b b b b) "(b b b b) is not in L(G)"))
  ```

- Be mindful that trying to randomly generate words in the language is not easy

- Most of the time a word that is too short or that is not in the language is generated

- It is important for unit tests to be thorough

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Context-Free Grammars

- ```
  #lang fsm
  ;; L = a^nb^n Syntactic Categories
  ;; S = words that start with n a and end with n b
  (define a2nb2n
    (make-cfg '(S)
              '(a b)
              `((S ,ARROW ,EMP)
                (S ,ARROW aSb))
              'S
              #:rejects '((b b b) (a b a))
              #:accepts '(() (a b) (a a b b) (a a a b b b)
                          (a a a a a b b b b b))))
  ```

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Context-Free Grammars

- ```
  #lang fsm
  ;; L = a^nb^n Syntactic Categories
  ;; S = words that start with n a and end with n b
  (define a2nb2n
    (make-cfg '(S)
              '(a b)
              `((S ,ARROW ,EMP)
                (S ,ARROW aSb))
              'S
              #:rejects '((b b b) (a b a))
              #:accepts '(() (a b) (a a b b) (a a a b b b)
                             (a a a a a b b b b b))))
  ```

- ```
  ;; word --> Boolean
  ;; Purpose: Determine if given word is in L(a2nb2n)
  (define (S-INV w)
    (let* [(as (takef w (λ (s) (eq? s 'a))))
           (bs (drop w (length as)))]
      (and (equal? w (append as bs))
           (= (length as) (length bs)))))
  (check-equal? (S-INV '(a)) #f)
  (check-equal? (S-INV '(b b a a)) #f)
  (check-equal? (S-INV '(a a a b b)) #f)
  (check-equal? (S-INV '()) #t)
  (check-equal? (S-INV '(a a b b)) #t)
  (check-equal? (S-INV '(a b)) #t)
  ```

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Context-Free Grammars

- Proof invariants hold by induction on, h, the height of the derivation tree.

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Context-Free Grammars

- Proof invariants hold by induction on, h, the height of
  the derivation tree.
- Base Case: h=1

  This means the only rule used is (S ,ARROW ,EMP). Observe
  that 0 as followed by 0 bs is generated. Thus, S-INV holds.

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  This means the only rule used is (S ,ARROW ,EMP). Observe that 0 as followed by 0 bs is generated. Thus, S-INV holds.
- Inductive Step

  Assume: invariants hold for h=k
    Show: invariants hold for h=k+1

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- Inductive Step

  Assume: invariants hold for h=k
    Show: invariants hold for h=k+1
- k>=0 ==> k+1>0

  This means that (S ,ARROW aSb) is used to derive the word. By inductive hypothesis, the S in aSb generates a^rb^r. Using this rule adds an a to the front and a b to the back yielding a^r+1b^r+1. Thus, S-INV holds.

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  This means that (S ,ARROW aSb) is used to derive the word. By inductive hypothesis, the S in aSb generates a^rb^r. Using this rule adds an a to the front and a b to the back yielding a^r+1b^r+1. Thus, S-INV holds.
- L=L(a2nb2n)

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- Inductive Step

  Assume: invariants hold for h=k
    Show: invariants hold for h=k+1
- k>=0 ==> k+1>0

  This means that (S ,ARROW aSb) is used to derive the word. By inductive hypothesis, the S in aSb generates $a^r b^r$. Using this rule adds an a to the front and a b to the back yielding $a^{r+1} b^{r+1}$. Thus, S-INV holds.
- L=L(a2nb2n)

- w∈L <==> w∈L(a2nb2n)
  (⇒) Assume w∈L
  This means $w = a^n b^n$. Given that invariants always hold, w is generated using a2nb2n by first applying (S ,ARROW aSb) r times and then using (S ,ARROW ,EMP). Thus, w∈L(a2nb2n).

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Context-Free Grammars

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  This means the only rule used is (S ,ARROW ,EMP). Observe that 0 as followed by 0 bs is generated. Thus, S-INV holds.
- Inductive Step

  Assume: invariants hold for h=k
    Show: invariants hold for h=k+1
- k>=0 ==> k+1>0

  This means that (S ,ARROW aSb) is used to derive the word. By inductive hypothesis, the S in aSb generates $a^r b^r$. Using this rule adds an a to the front and a b to the back yielding $a^{r+1} b^{r+1}$. Thus, S-INV holds.
- L=L(a2nb2n)

- $w \in L \iff w \in L(a2nb2n)$
  ($\Rightarrow$) Assume $w \in L$
  This means $w = a^n b^n$. Given that invariants always hold, w is generated using a2nb2n by first applying (S ,ARROW aSb) r times and then using (S ,ARROW ,EMP). Thus, $w \in L(a2nb2n)$.
- ($\Leftarrow$) Assume $w \in L(a2nb2n)$
  Given that invariants always hold, this means $w = a^n b^n$.
  Thus, $w \in L$.

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  Assume: invariants hold for h=k
    Show: invariants hold for h=k+1
- k>=0 ==> k+1>0

  This means that (S ,ARROW aSb) is used to derive the word. By inductive hypothesis, the S in aSb generates $a^r b^r$. Using this rule adds an a to the front and a b to the back yielding $a^{r+1}b^{r+1}$. Thus, S-INV holds.
- L=L(a2nb2n)

- w∈L <==> w∈L(a2nb2n)
  (⇒) Assume w∈L
  This means $w=a^n b^n$. Given that invariants always hold, w is generated using a2nb2n by first applying (S ,ARROW aSb) r times and then using (S ,ARROW ,EMP). Thus, w∈L(a2nb2n).
- (⇐) Assume w∈L(a2nb2n)
  Given that invariants always hold, this means $w=a^n b^n$. Thus, w∈L.
- w∉L <==> w∉in L(a2nb2n)
  (⇒) Assume w∉L
  This means $w!=a^n b^n$. Given that invariants always hold, S cannot generate w. Thus, w∉L(a2nb2n).

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  This means the only rule used is (S ,ARROW ,EMP). Observe that 0 as followed by 0 bs is generated. Thus, S-INV holds.
- Inductive Step

  Assume: invariants hold for h=k
    Show: invariants hold for h=k+1
- k>=0 ==> k+1>0

  This means that (S ,ARROW aSb) is used to derive the word. By inductive hypothesis, the S in aSb generates a^rb^r. Using this rule adds an a to the front and a b to the back yielding $a^{r+1}b^{r+1}$. Thus, S-INV holds.
- L=L(a2nb2n)

- w∈L <==> w∈L(a2nb2n)
  (⇒) Assume w∈L
  This means $w=a^nb^n$. Given that invariants always hold, w is generated using a2nb2n by first applying (S ,ARROW aSb) r times and then using (S ,ARROW ,EMP). Thus, w∈L(a2nb2n).
- (⇐) Assume w∈L(a2nb2n)
  Given that invariants always hold, this means $w=a^nb^n$. Thus, w∈L.
- w∉L <==> w∉in L(a2nb2n)
  (⇒) Assume w∉L
  This means $w!=a^nb^n$. Given that invariants always hold, S cannot generate w. Thus, w∉L(a2nb2n).
- (⇐) Assume w∉L(a2nb2n)
  Given that invariants always hold, this means $w!=a^nb^n$. Thus, w∉L.

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- $L = \{w \mid w \in (a\ b)^* \wedge w \text{ has more bs than as}\}$

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Context-Free Grammars

- $L = \{w \mid w \in (a\ b)^* \land w \text{ has more bs than as}\}$
- Every word in the language must have at least one more b than as
- Suggests that at least a b must be generated
- After that, for every a generated there must be one or more bs generated

Context-Free Grammars

- `#lang fsm`

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Context-Free Grammars

- `#lang fsm`

- ```
  ;; Syntactic Categories
  ;;  S = words such that number of b > number of a
  ```

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Context-Free Grammars

- `#lang fsm`
- `;; Syntactic Categories`
  `;;  S = words such that number of b > number of a`
-     `;;  A = words such that number of b >= number of a`

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Context-Free Grammars

- `#lang fsm`
- `;; Syntactic Categories`
  `;;  S = words such that number of b > number of a`
-     `;;  A = words such that number of b >= number of a`
- `;; L = w | w in (a b)* AND  w has more b than a`
  `(define numb>numa`
  `  (make-cfg '(S A)`
  `            '(a b)`

- `            'S))`

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Context-Free Grammars

- `#lang fsm`
- ```
  ;; Syntactic Categories
  ;;  S = words such that number of b > number of a
  ```
- ```
       ;; A = words such that number of b >= number of a
  ```
- ```
  ;; L = w | w in (a b)* AND  w has more b than a
  (define numb>numa
    (make-cfg '(S A)
              '(a b)
  ```
- ```
              `((S ,ARROW b)
                (S ,ARROW AbA)
                (A ,ARROW AaAbA)
                (A ,ARROW AbAaA)
                (A ,ARROW ,EMP)
                (A ,ARROW bA))
  ```

- ```
              'S))
  ```

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Context-Free Grammars

- `#lang fsm`
- `;; Syntactic Categories`
  `;;  S = words such that number of b > number of a`
- `    ;;  A = words such that number of b >= number of a`
- `;; L = w | w in (a b)* AND  w has more b than a`
  `(define numb>numa`
  `  (make-cfg '(S A)`
  `            '(a b)`
- `` `((S ,ARROW b) ``
  `     (S ,ARROW AbA)`
  `     (A ,ARROW AaAbA)`
  `     (A ,ARROW AbAaA)`
  `     (A ,ARROW ,EMP)`
  `     (A ,ARROW bA))`
- `            #:rejects '((a b) (a b a) (a a a a a))`
  `            #:accepts '((b b b) (b b a b a a b) (a a a b b b b))`
- `            'S))`

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Context-Free Grammars

- > (grammar-test numb>numa 5)
  '(((a b b b a b b b)
     (S -> AbA -> AaAbAbA -> AaAbAaAbAbA -> aAbAaAbAbA
      -> abAaAbAbA -> abbAaAbAbA -> abbbAaAbAbA -> abbbaAbAbA
      -> abbbabAbA -> abbbabbbA -> abbbabbbA -> abbbabbbbA
      -> abbbabbbb))
    ((a a b b b)
     (S -> AbA -> AaAbAbA -> aAbAbA -> aAaAbAbAbA -> aaAbAbAbA
      -> aabAbAbA -> aabbAbA -> aabbbA -> aabbb))
    (() "The word () is too short to test.")
    ((a b b a b a a b) "(a b b a b a a b) is not in L(G)")
    ((a b b b b a b)
     (S -> AbA -> AaAbAbA -> aAbAbA -> abAbA -> abAbAaAbA
      -> abbAaAbA -> abbbAaAbA -> abbbbAaAbA -> abbbbaAbA
      -> abbbbabA -> abbbbab)))
- Be mindful about tests taking very long to run
- Interrupt execution if necessary

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Context-Free Grammars

- HOMEWORK: 1, 3, 5, 8

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Context-Free Grammars

- Are all regular languages context-free?

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Context-Free Grammars

- Are all regular languages context-free?

Theorem
All regular languages are context-free.

-

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Context-Free Grammars

- Are all regular languages context-free?

Theorem
All regular languages are context-free.

-

Proof.
Assume L is a regular language. This means that there exist a regular grammar:

$G = (make-rg\ N\ \Sigma\ P\ S)$,

such that $L = L(G)$. All the production rules in P are of the form:

1. $A \rightarrow \epsilon$
2. $A \rightarrow a$, where $a \in \Sigma$
3. $A \rightarrow aB$, where $a \in \Sigma \land B \in N$

This means $P \subset (N \rightarrow (N \cup \Sigma \cup \epsilon)^+)$. That is, $p \in P$ is a valid production rule for a context-free grammar. Given that G's N, Σ, and S may also be used in a cfg, we may conclude that L is a context-free language. □

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Context-Free Grammars

-
  ```
  (define MULT3-as (make-rg '(S B C)
                            '(a b)
                            `((S ,ARROW ,EMP)
                              (S ,ARROW aB)
                              (S ,ARROW bS)
                              (B ,ARROW aC)
                              (B ,ARROW bB)
                              (C ,ARROW aS)
                              (C ,ARROW bC))
                            'S))
  ```

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- ```
  (define MULT3-as (make-rg '(S B C)
                            '(a b)
                            `((S ,ARROW ,EMP)
                              (S ,ARROW aB)
                              (S ,ARROW bS)
                              (B ,ARROW aC)
                              (B ,ARROW bB)
                              (C ,ARROW aS)
                              (C ,ARROW bC))
                            'S))
  ```

- ```
  (define MULT3-as (make-cfg '(S B C)
                             '(a b)
                             `((S ,ARROW ,EMP)
                               (S ,ARROW aB)
                               (S ,ARROW bS)
                               (B ,ARROW aC)
                               (B ,ARROW bB)
                               (C ,ARROW aS)
                               (C ,ARROW bC))
                             'S))
  ```

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Context-Free Grammars

- Consider the language of words containing balanced parenthesis

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Context-Free Grammars

- Consider the language of words containing balanced parenthesis
- Opening parenthesis: o and closing parenthesis: c

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Context-Free Grammars

- Consider the language of words containing balanced parenthesis
- Opening parenthesis: o and closing parenthesis: c
- #lang fsm

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Context-Free Grammars

- Consider the language of words containing balanced parenthesis
- Opening parenthesis: o and closing parenthesis: c
- `#lang fsm`

- ```
  ;; Syntactic categories: S = words with balanced parenthesis
  ;; L = {w | w has balanced parenthesis}, where o = ( and c = )
  (define BP (make-cfg '(S)
                           '(o c)
  ```

- ```
                           'S))
  ```

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Context-Free Grammars

- Consider the language of words containing balanced parenthesis
- Opening parenthesis: o and closing parenthesis: c
- #lang fsm

- ```
  ;; Syntactic categories: S = words with balanced parenthesis
  ;; L = {w | w has balanced parenthesis}, where o = ( and c = )
  (define BP (make-cfg '(S)
  ```
- ```
                       '(o c)
                       `((S ,ARROW ,EMP)
                         (S ,ARROW SS)
                         (S ,ARROW oSc))
  ```
- ```
                       'S))
  ```

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Context-Free Grammars

- Consider the language of words containing balanced parenthesis
- Opening parenthesis: o and closing parenthesis: c
- #lang fsm

- ```
  ;; Syntactic categories: S = words with balanced parenthesis
  ;; L = {w | w has balanced parenthesis}, where o = ( and c = )
  (define BP (make-cfg '(S)
                       '(o c)
  ```
- ```
                       `((S ,ARROW ,EMP)
                         (S ,ARROW SS)
                         (S ,ARROW oSc))
  ```
- ```
                       'S))
  ```
- ```
  (check-equal? (grammar-derive BP (parensstr->los ""))
                "The word () is too short to test.")
  (check-equal? (grammar-derive BP (parensstr->los "))(("))
                "(c c o o) is not in L(G).")
  (check-equal? (grammar-derive BP (parensstr->los "()("))
                "(o c o) is not in L(G).")
  (check-equal? (grammar-derive BP (parensstr->los "()"))
                '(S -> oSc -> oc))
  (check-equal? (grammar-derive BP (parensstr->los "(())()"))
                '(S -> SS -> oScS -> ooSccS -> ooccS -> ooccoSc -> ooccoc))
  ```

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- Consider deriving the "(()())":

  '(S -> oSc -> oSSc -> ooScSc -> oocSc -> oocoScc -> oococc)

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Context-Free Grammars

- Consider deriving the "(()())":

    '(S -> oSc -> oSSc -> ooScSc -> oocSc -> oocoScc -> oococc)

- A derivation may be visualized as a tree:

- Such a visualization is called a parse tree

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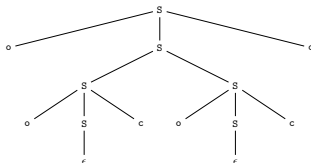
Context-Free Grammars

- Consider deriving the "(()())":

    '(S -> oSc -> oSSc -> ooScSc -> oocSc -> oocoScc -> oococc)

- A derivation may be visualized as a tree:

- Such a visualization is called a parse tree
- The leaves are terminal symbols or ϵ
- The interior nodes are rooted at a nonterminal
- The edges out of an interior node indicate what is generated from the nonterminal
- The *yield* of a parse tree is the concatenation of all the leaves from left to right

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Context-Free Grammars

- A parse tree is formally defined as follows:
    - For each a$\in\Sigma$ the following is a parse tree: a
    - For A $\rightarrow$ ϵ the following is a parse tree:

A
|
ϵ

    - If $T_0T_1\ldots T_{n-1}$ are the parse trees for $a_0\ldots a_{n-1}$, where a_i is a nonterminal or a terminal symbol, then for A $\rightarrow$ $a_0a_1\ldots a_{n-1}$ the following is a parse tree:

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Context-Free Grammars

- Parse trees represent derivations eliminating irrelevant differences like the order in which production rules are applied

- Two derivations are similar if they are captured by the same parse tree:

  ```
  S -> oSc -> oSSc -> ooScSc -> oocSc -> oocoScc -> oococc
  S -> oSc -> oSSc -> oSoScc -> oSocc -> ooScocc -> oococc
  ```

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Context-Free Grammars

- Parse trees represent derivations eliminating irrelevant differences like the order in which production rules are applied
- Two derivations are similar if they are captured by the same parse tree:

  ```
  S -> oSc -> oSSc -> ooScSc -> oocSc -> oocoScc -> oococc
  S -> oSc -> oSSc -> oSoScc -> oSocc -> ooScocc -> oococc
  ```

- Both of these derivations are captured by the previous parse tree
- Use the same production rules on the same nonterminals. The only difference is the order in which the production rules are used

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Context-Free Grammars

- Parse trees represent derivations eliminating irrelevant differences like the order in which production rules are applied

- Two derivations are similar if they are captured by the same parse tree:

  ```
  S -> oSc -> oSSc -> ooScSc -> oocSc -> oocoScc -> oocooc
  S -> oSc -> oSSc -> oSoScc -> oSocc -> ooScocc -> oocooc
  ```

- Both of these derivations are captured by the previous parse tree

- Use the same production rules on the same nonterminals. The only difference is the order in which the production rules are used

- The first is a *leftmost derivation*

- The second is a *rightmost derivation*

- Every parse tree has a unique leftmost and a unique rightmost derivation

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Context-Free Grammars

- Equivalent:
  - $A \rightarrow_G^* w$
  - $\exists$ a parse tree with root A and yield w
  - $\exists$ a leftmost derivation $A \xrightarrow{L}_G^* w$
  - $\exists$ a rightmost derivation $A \xrightarrow{R}_G^* w$

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Context-Free Grammars

- Not all derivations of a word may be similar

    S -> SS -> S -> oSSc -> ooScSc -> oocSc -> oocoScc -> oococc

- Not captured by:

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Context-Free Grammars

- Not all derivations of a word may be similar

  S -> SS -> S -> oSSc -> ooScSc -> oocSc -> oocoScc -> oococc

- Not captured by:

- Captured by:

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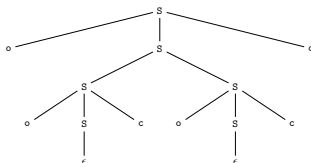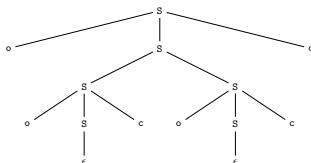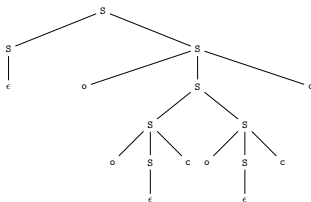
Context-Free Grammars

- Not all derivations of a word may be similar

  S -> SS -> S -> oSSc -> ooScSc -> oocSc -> oocoScc -> oococc

- Not captured by:

- Captured by:

- A cfg, G, is called *ambiguous* if for any w∈L(G) there are 2 or more parse trees

- BP is ambiguous

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Context-Free Grammars

- Ambiguity is a problem when meaning must be assigned to a word as done in the implementation of programming languages

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Context-Free Grammars

- Ambiguity is a problem when meaning must be assigned to a word as done in the implementation of programming languages

- Arithmetic expressions grammar:

```
#lang fsm
(define AE (make-cfg '(S I)
                     '(p t x y z)
                     `((S ,ARROW SpS)
                       (S ,ARROW StS)
                       (S ,ARROW I)
                       (I ,ARROW x)
                       (I ,ARROW y)
                       (I ,ARROW z))
                     'S))
;; Tests for AE
(check-equal? (grammar-derive AE '(x p x))
              '(S -> SpS -> IpS -> xpS -> xpI -> xpx))
(check-equal? (grammar-derive AE '(x t z))
              '(S -> StS -> ItS -> xtS -> xtI -> xtz))
(check-equal? (grammar-derive AE '(x t z p y))
              '(S -> SpS -> StSpS -> ItSpS -> xtSpS -> xtIpS -> xtz
                -> xtzpI -> xtzpy))
```

- p stands for +, t stands for *

- x, y, and z are variables

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Context-Free Grammars

- '(x t z p y), has two derivations that are not similar:

```
S -> SpS -> StSpS -> ItSpS -> xtSpS -> xtIpS -> xtzpS
  -> xtzpI -> xtzpy

S -> StS -> ItS -> xtS -> xtSpS -> xtIpS -> xtzpS -> xtzpI
  -> xtzpy
```

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Context-Free Grammars

- '(x t z p y), has two derivations that are not similar:

  ```
  S -> SpS -> StSpS -> ItSpS -> xtSpS -> xtIpS -> xtzpS
     -> xtzpI -> xtzpy

  S -> StS -> ItS -> xtS -> xtSpS -> xtIpS -> xtzpS -> xtzpI
     -> xtzpy
  ```

- Parse trees:

- Suggests that x and z are multiplied and then their product and y are added

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Context-Free Grammars

- '(x t z p y), has two derivations that are not similar:

```
S -> SpS -> StSpS -> ItSpS -> xtSpS -> xtIpS -> xtzpS
  -> xtzpI -> xtzpy

S -> StS -> ItS -> xtS -> xtSpS -> xtIpS -> xtzpS -> xtzpI
  -> xtzpy
```

- Parse trees:

- Suggests that x and z are multiplied and then their product and y are added

- This parse tree suggests that first z and y are added and then their sum is multiplied by x

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Context-Free Grammars

- '(x t z p y), has two derivations that are not similar:

      S -> SpS -> StSpS -> ItSpS -> xtSpS -> xtIpS -> xtzpS
        -> xtzpI -> xtzpy

      S -> StS -> ItS -> xtS -> xtSpS -> xtIpS -> xtzpS -> xtzpI
        -> xtzpy

- Parse trees:

- Suggests that x and z are multiplied and then their product and y are added

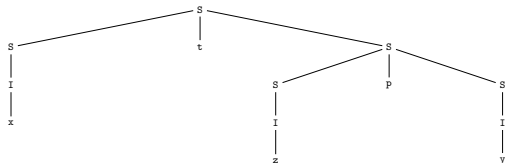- This parse tree suggests that first z and y are added and then their sum is multiplied by x
- A word may have different meanings when a grammar is ambiguous

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Context-Free Grammars

- Disambiguate `AE`: first produce the summing subexpressions, if any, and then produce the product subexpressions

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Context-Free Grammars

- Disambiguate AE: first produce the summing subexpressions, if any, and then produce the product subexpressions

- A new syntactic category, T, is introduced to generate product subexpressions

- A product subexpression may only be generated after summing subexpressions, if any, are generated:

- ```
#lang fsm
(define AE2 (make-cfg '(S I T)
                      '(p t x y z)
                      `((S ,ARROW SpS)
                        (S ,ARROW T)
                        (S ,ARROW I)
                        (T ,ARROW TtT)
                        (T ,ARROW I)
                        (I ,ARROW x)
                        (I ,ARROW y)
                        (I ,ARROW z))
                      'S))
```

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# Context-Free Grammars

- Disambiguate AE: first produce the summing subexpressions, if any, and then produce the product subexpressions

- A new syntactic category, T, is introduced to generate product subexpressions

- A product subexpression may only be generated after summing subexpressions, if any, are generated:

- ```
  #lang fsm
  (define AE2 (make-cfg '(S I T)
                        '(p t x y z)
                        `((S ,ARROW SpS)
                          (S ,ARROW T)
                          (S ,ARROW I)
                          (T ,ARROW TtT)
                          (T ,ARROW I)
                          (I ,ARROW x)
                          (I ,ARROW y)
                          (I ,ARROW z))
                        'S))
  ```

- Unfortunately, some context-free languages can only be generated by an ambiguous cfg

- Such grammars are called *inherently ambiguous*

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Pushdown
Automata

Equivalence of
pdas and cfgs

Properties of
CFLs

Deterministic
pdas

Context-Free Grammars

- HOMEWORK: 10, 12, 14
- Quiz: in class (1 week)

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Pushdown Automata

- Would like to have a machine that decides if a given word is a member of a CFL
- What should such a machine look like?

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Pushdown Automata

- Would like to have a machine that decides if a given word is a member of a CFL

- What should such a machine look like?

- Write a program to decide $a^n b^n$

- Call an auxiliary function that takes as input w's sub-word without the leading as, if any, and an accumulator with w's leading as

- 3 conditions:

  1 If the given word is empty then the value from testing if the given accumulator is empty is returned.

  2 If the first element of the given word is a then false is returned.

  3 Otherwise, return the conjunction of testing if the accumulator is not empty and checking the rest of both the given word and the given accumulator.

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Pushdown Automata

- ```
  #lang fsm
  ;; word → Boolean
  ;; Purpose: Decide if given word is in a^nb^n
  (define (is-in-a^nb^n? w)
  ```

- ```
      ;; Tests for is-in-anbn?
      (check-pred (λ (w) (not (is-in-a^nb^n? w))) '(a))
      (check-pred (λ (w) (not (is-in-a^nb^n? w))) '(b b))
      (check-pred (λ (w) (not (is-in-a^nb^n? w))) '(a b b))
      (check-pred (λ (w) (not (is-in-a^nb^n? w))) '(a b a a b b))
      (check-pred is-in-a^nb^n? '())
      (check-pred is-in-a^nb^n? '(a a b b))
  ```

```
#lang fsm
;; word → Boolean
;; Purpose: Decide if given word is in a^nb^n
(define (is-in-a^nb^n? w)
```

```
(check (dropf w (λ (s) (eq? s 'a)))    ;; everything after initial as
       (takef w (λ (s) (eq? s 'a))))) ;; the initial as
```

```
;; Tests for is-in-anbn?
(check-pred (λ (w) (not (is-in-a^nb^n? w))) '(a))
(check-pred (λ (w) (not (is-in-a^nb^n? w))) '(b b))
(check-pred (λ (w) (not (is-in-a^nb^n? w))) '(a b b))
(check-pred (λ (w) (not (is-in-a^nb^n? w))) '(a b a a b b))
(check-pred is-in-a^nb^n? '())
(check-pred is-in-a^nb^n? '(a a b b))
```

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- ```
  #lang fsm
  ;; word → Boolean
  ;; Purpose: Decide if given word is in a^nb^n
  (define (is-in-a^nb^n? w)
  ```

- ```
    ;; word (listof symbol) → Boolean
    ;; Purpose: Determine if first given word has only bs that
    ;;          match as in the second given word
    ;; Accumulator Invariant
    ;;   acc = the unmatched as at the beginning of w
    ;; Assume: w in (a b)*
    (define (check wrd acc)
      (cond [(empty? wrd) (empty? acc)]
            [(eq? (first wrd) 'a) #f]
            [else (and (not (empty? acc))
                       (check (rest wrd) (rest acc)))]))
  ```

- ```
  (check (dropf w (λ (s) (eq? s 'a)))   ;; everything after initial as
         (takef w (λ (s) (eq? s 'a))))) ;; the initial as
  ```

- ```
        ;; Tests for is-in-anbn?
        (check-pred (λ (w) (not (is-in-a^nb^n? w))) '(a))
        (check-pred (λ (w) (not (is-in-a^nb^n? w))) '(b b))
        (check-pred (λ (w) (not (is-in-a^nb^n? w))) '(a b b))
        (check-pred (λ (w) (not (is-in-a^nb^n? w))) '(a b a a b b))
        (check-pred is-in-a^nb^n? '())
        (check-pred is-in-a^nb^n? '(a a b b))
  ```

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- ```
  (check (dropf w (λ (s) (eq? s 'a)))    ;; everything after initial as
         (takef w (λ (s) (eq? s 'a)))))) ;; the initial as
  ```

- ```
      ;; Tests for is-in-anbn?
      (check-pred (λ (w) (not (is-in-a^nb^n? w))) '(a))
      (check-pred (λ (w) (not (is-in-a^nb^n? w))) '(b b))
      (check-pred (λ (w) (not (is-in-a^nb^n? w))) '(a b b))
      (check-pred (λ (w) (not (is-in-a^nb^n? w))) '(a b a a b b))
      (check-pred is-in-a^nb^n? '())
      (check-pred is-in-a^nb^n? '(a a b b))
  ```

- The acc is a stack

- The program first pushes all the as at the beginning of the given word onto the accumulator

- The auxiliary function pops an a for each recursive

- Suggests extending an ndfa with a stack to remember part of the consumed input

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Pushdown Automata

- A (nondeterministic) pushdown automaton, pda, is an instance of:

      (make-ndpda K Σ Γ S F δ)

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Pushdown Automata

- A (nondeterministic) pushdown automaton, pda, is an instance of:

  (make-ndpda K Σ Γ S F δ)

- The transition relation, δ, is a finite subset of:

  ((K $\times$ (Σ $\cup$ {EMP}) $\times$ Γ$^+ \cup \{EMP\}$) $\times$ (K $\times$ Γ$^+ \cup \{EMP\}$)).

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  instance of:

  (make-ndpda K Σ Γ S F δ)

- The transition relation, δ, is a finite subset of:

  $((K \times (\Sigma \cup \{\text{EMP}\}) \times \Gamma^+ \cup \{EMP\}) \times (K \times \Gamma^+ \cup \{EMP\})).$

- ((A a p) (B g)) means that the machine is in state A, reads a from the
  input tape, pops p off the stack, pushes g onto the stack, and moves to B

- May be nondeterministic

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- The transition relation, δ, is a finite subset of:

    $((K \times (\Sigma \cup \{EMP\}) \times \Gamma^+ \cup \{EMP\}) \times (K \times \Gamma^+ \cup \{EMP\}))$.

- ((A a p) (B g)) means that the machine is in state A, reads a from the input tape, pops p off the stack, pushes g onto the stack, and moves to B

- May be nondeterministic

- ((P EMP EMP) (Q j)) is a push operation that does not consult the input tape nor the stack

- ((P EMP j) (Q EMP)) is a pop operation

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Pushdown Automata

- A pda configuration is a member of ($K \times \Sigma^* \times \Gamma^*$)

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Pushdown Automata

- A pda configuration is a member of $(K \times \Sigma^* \times \Gamma^*)$
- A computation step moves the machine from a starting configuration to a new configuration using a single rule denoted as:

    $(P, xw, a) \vdash (Q, w, g)$

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- Zero or more steps are denoted using $\vdash^*$

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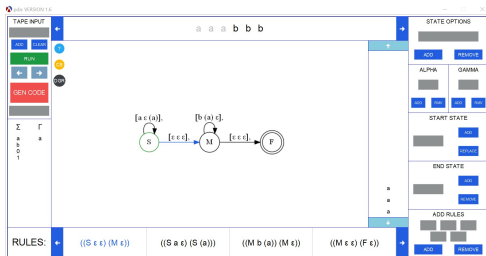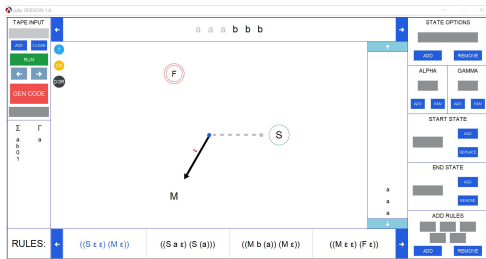
    $(P, xw, a) \vdash (Q, w, g)$
- Zero or more steps are denoted using $\vdash^*$
- A computation of length n on a word, w, is denoted by:

    $C_0 \vdash C_1 \vdash C_2 \vdash \ldots \vdash C_n$, where C_i is a pda configuration.

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- A computation of length n on a word, w, is denoted by:

  $C_0 \vdash C_1 \vdash C_2 \vdash \ldots \vdash C_n$, where C_i is a pda configuration.

- If a pda, M, starting in the starting state consumes all the input and reaches a final state with the stack empty then M accepts. Otherwise, M rejects

- A word, w, is in the language of M, L(M), if there is a computation from the starting state that consumes w and M accepts

- It does not matter that there may be computations that reject w.

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- A state invariant predicate may be associated with each state
- An invariant predicate has two inputs: the consumed input and the stack
- It must test and relate the invariant conditions for and between the consumed input and the stack.

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Pushdown Automata

- Design and implement a pda for $L = a^n b^n$
- How may the stack be used?

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Pushdown Automata

- Design and implement a pda for $L = a^n b^n$
- How may the stack be used?
- Accumulate the read as

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Pushdown Automata

- Design and implement a pda for $L = a^n b^n$
- How may the stack be used?
- Accumulate the read as
- Match bs by popping an as

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Pushdown Automata

- Design and implement a pda for $L = a^nb^n$
- How may the stack be used?
- Accumulate the read as
- Match bs by popping an as
- After all the input is read, the machine ought to move to a final state
- It accepts if the stack is empty. Otherwise, it rejects

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- Name: a^nb^n Σ = {a b}

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Pushdown Automata

- Name: a^nb^n Σ = {a b}

- ;; Tests for a^nb^n
  ```
  (check-equal? (sm-apply a^nb^n '(a)) 'reject)
  (check-equal? (sm-apply a^nb^n '(b b)) 'reject)
  (check-equal? (sm-apply a^nb^n '(a b b)) 'reject)
  (check-equal? (sm-apply a^nb^n '(a b a a b b)) 'reject)
  (check-equal? (sm-apply a^nb^n '()) 'accept)
  (check-equal? (sm-apply a^nb^n '(a a b b)) 'accept)
  ```

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- Name: a^nb^n Σ = {a b}
- ;; Tests for a^nb^n
  (check-equal? (sm-apply a^nb^n '(a)) 'reject)
  (check-equal? (sm-apply a^nb^n '(b b)) 'reject)
  (check-equal? (sm-apply a^nb^n '(a b b)) 'reject)
  (check-equal? (sm-apply a^nb^n '(a b a a b b)) 'reject)
  (check-equal? (sm-apply a^nb^n '()) 'accept)
  (check-equal? (sm-apply a^nb^n '(a a b b)) 'accept)
- States

      ;; States
      ;; S: ci = a* = stack, start state

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- Name: a^nb^n Σ = {a b}

- ```
  ;; Tests for a^nb^n
  (check-equal? (sm-apply a^nb^n '(a)) 'reject)
  (check-equal? (sm-apply a^nb^n '(b b)) 'reject)
  (check-equal? (sm-apply a^nb^n '(a b b)) 'reject)
  (check-equal? (sm-apply a^nb^n '(a b a a b b)) 'reject)
  (check-equal? (sm-apply a^nb^n '()) 'accept)
  (check-equal? (sm-apply a^nb^n '(a a b b)) 'accept)
  ```

- States

  ```
  ;; States
  ;; S:   ci = a* = stack, start state
  ```

- ```
  ;; M:   ci = (append (listof a) (listof b))
  ;;      ∧ stack = a*
  ;;      ∧ |ci as| = |stack| + |ci bs|
  ```

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Pushdown Automata

- Name: $a^n b^n$ Σ = {a b}

- ;; Tests for $a^n b^n$
  ```
  (check-equal? (sm-apply a^nb^n '(a)) 'reject)
  (check-equal? (sm-apply a^nb^n '(b b)) 'reject)
  (check-equal? (sm-apply a^nb^n '(a b b)) 'reject)
  (check-equal? (sm-apply a^nb^n '(a b a a b b)) 'reject)
  (check-equal? (sm-apply a^nb^n '()) 'accept)
  (check-equal? (sm-apply a^nb^n '(a a b b)) 'accept)
  ```

- States
  ```
  ;; States
  ;; S:   ci = a* = stack, start state
  ```

- ```
  ;; M:   ci = (append (listof a) (listof b))
  ;;      ∧ stack = a*
  ;;      ∧ |ci as| = |stack| + |ci bs|
  ```

- ```
  ;; F:    ci = (append (listof a) (listof b))
  ;;           ∧ |stack| = 0
  ;;      ∧ |ci as|=|ci bs|, final state
  ```

Pushdown Automata

- Transition Relation
- Push all a and nondeterministically move to M
        ((S ,EMP ,EMP) (M ,EMP))   ((S a ,EMP) (S (a)))

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Pushdown Automata

- Transition Relation
- Push all a and nondeterministically move to M

    ((S ,EMP ,EMP) (M ,EMP))   ((S a ,EMP) (S (a)))

- Match all b and nondeterministically move to F

    ((M b (a)) (M ,EMP))   ((M ,EMP ,EMP) (F ,EMP))

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-
```
;; L = {a^nb^n | n >= 0}
;; States
;;  S ci = (listof a) = stack, start state
;;  M ci = (append (listof a) (listof b)) AND
;;    (length ci as) = (length stack) + (length ci bs)
;;  F ci = (append (listof a) (listof b)) and all as and bs matched,
;;    final state
;; The stack is a (listof a)
(define a^nb^n (make-ndpda '(S M F)  '(a b)  '(a)  'S  '(F)
                              `(((S ,EMP ,EMP) (M ,EMP))
                                ((S a ,EMP) (S (a)))
                                ((M b (a)) (M ,EMP))
                                ((M ,EMP ,EMP) (F ,EMP)))))

;; Tests for a^nb^n
(check-equal? (sm-apply a^nb^n '(a)) 'reject)
(check-equal? (sm-apply a^nb^n '(b b)) 'reject)
(check-equal? (sm-apply a^nb^n '(a b b)) 'reject)
(check-equal? (sm-apply a^nb^n '(a b a a b b)) 'reject)
(check-equal? (sm-apply a^nb^n '()) 'accept)
(check-equal? (sm-apply a^nb^n '(a a b b)) 'accept)
```

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-
```
;; L = {a^nb^n | n >= 0}
;; States
;; S ci = (listof a) = stack, start state
;; M ci = (append (listof a) (listof b)) AND
;;    (length ci as) = (length stack) + (length ci bs)
;; F ci = (append (listof a) (listof b)) and all as and bs matched,
;;    final state
;; The stack is a (listof a)
(define a^nb^n (make-ndpda '(S M F)  '(a b)  '(a)  'S  '(F)
                           `(((S ,EMP ,EMP) (M ,EMP))
                             ((S a ,EMP) (S (a)))
                             ((M b (a)) (M ,EMP))
                             ((M ,EMP ,EMP) (F ,EMP)))))

;; Tests for a^nb^n
(check-equal? (sm-apply a^nb^n '(a)) 'reject)
(check-equal? (sm-apply a^nb^n '(b b)) 'reject)
(check-equal? (sm-apply a^nb^n '(a b b)) 'reject)
(check-equal? (sm-apply a^nb^n '(a b a a b b)) 'reject)
(check-equal? (sm-apply a^nb^n '()) 'accept)
(check-equal? (sm-apply a^nb^n '(a a b b)) 'accept)
```

-
```
> (sm-test a^nb^n 10)
'(((b b b a a a a a) reject)
  ((a b) accept)
  ((b b b a a b a b a) reject)
  ((a a a a b) reject)
  ((a a b) reject)
  ((b b b b b a a a) reject)
  ((b) reject)
  ((a b b) accept)
  ((b b b a b b b a) reject)
  ((b a a a a) reject))
```

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- ```
;; word stack → Boolean
;; Purpose: Determine if ci and stack are the same (listof a)
(define (S-INV ci stck)
 (and (= (length ci) (length stck))
      (andmap (λ (i g) (and (eq? i 'a) (eq? g 'a))) ci stck)))
;; Tests for S-INV
(check-equal? (S-INV '() '(a a)) #f) (check-equal? (S-INV '(a) '()) #f)
(check-equal? (S-INV '() '()) #t)
(check-equal? (S-INV '(a a a) '(a a a)) #t)
```

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# Pushdown Automata

- ```
  ;; word stack → Boolean
  ;; Purpose: Determine if ci and stack are the same (listof a)
  (define (S-INV ci stck)
   (and (= (length ci) (length stck))
        (andmap (λ (i g) (and (eq? i 'a) (eq? g 'a))) ci stck)))
  ;; Tests for S-INV
  (check-equal? (S-INV '() '(a a)) #f) (check-equal? (S-INV '(a) '()) #f)
  (check-equal? (S-INV '() '()) #t)
  (check-equal? (S-INV '(a a a) '(a a a)) #t)
  ```
- ```
  ;; word stack → Boolean
  ;; Purpose: Determine if ci = EMP or a+b+ AND the stack
  ;;          only contains a AND |ci as| = |stack| + |ci bs|
  (define (M-INV ci stck)
    (let* [(as (takef ci (λ (s) (eq? s 'a))))
           (bs (takef (drop ci (length as)) (λ (s) (eq? s 'b))))]
      (and (equal? (append as bs) ci)
           (andmap (λ (s) (eq? s 'a)) stck)
           (= (length as) (+ (length bs) (length stck))))))
  ;; Tests for M-INV
  (check-equal? (M-INV '(a a b) '(a a)) #f)
  (check-equal? (M-INV '(a a a b) '(a)) #f)
  (check-equal? (M-INV '() '()) #t)
  (check-equal? (M-INV '(a) '(a)) #t)
  (check-equal? (M-INV '(a a a b b) '(a)) #t)
  ```

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- ```
;; word stack → Boolean
;; Purpose: Determine if ci and stack are the same (listof a)
(define (S-INV ci stck)
 (and (= (length ci) (length stck))
      (andmap (λ (i g) (and (eq? i 'a) (eq? g 'a))) ci stck)))
;; Tests for S-INV
(check-equal? (S-INV '() '(a a)) #f) (check-equal? (S-INV '(a) '()) #f)
(check-equal? (S-INV '() '()) #t)
(check-equal? (S-INV '(a a a) '(a a a)) #t)
```

- ```
;; word stack → Boolean
;; Purpose: Determine if ci = EMP or a+b+ AND the stack
;;   only contains a AND |ci as| = |stack| + |ci bs|
(define (M-INV ci stck)
  (let* [(as (takef ci (λ (s) (eq? s 'a))))
         (bs (takef (drop ci (length as)) (λ (s) (eq? s 'b))))]
    (and (equal? (append as bs) ci)
         (andmap (λ (s) (eq? s 'a)) stck)
         (= (length as) (+ (length bs) (length stck))))))
;; Tests for M-INV
(check-equal? (M-INV '(a a b) '(a a)) #f)
(check-equal? (M-INV '(a a a b) '(a)) #f)
(check-equal? (M-INV '() '()) #t)
(check-equal? (M-INV '(a) '(a)) #t)
(check-equal? (M-INV '(a a a b b) '(a)) #t)
```

- ```
;; word stack → Boolean Purpose: Determine if ci=a^nb^n & empty stack
(define (F-INV ci stck)
  (let* [(as (takef ci (λ (s) (eq? s 'a))))
         (bs (takef (drop ci (length as)) (λ (s) (eq? s 'b))))]
    (and (empty? stck) (equal? (append as bs) ci)  (= (length as) (length bs)))))
;; Tests for F-INV
(check-equal? (F-INV '(a a b) '()) #f)
(check-equal? (F-INV '(a a b b a b) '(a a a)) #f)
(check-equal? (F-INV '() '()) #t)
(check-equal? (F-INV '(a a b b) '()) #t)
```

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- $L = a^n b^n$ ci = the consumed input $w \in (sm\text{-}sigma\ M)^*$
  $F = (sm\text{-}finals\ M)$ $P = a^n b^n$

Theorem
The state invariants hold when P is applied to w.

  -
  - The proof is by induction on, n, the number of transitions to consume w

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- Base Case
    - When P starts, S-INV holds because $ci = $ '()and the stack = '()
    - Observe that empty transitions into M and F may lead to accept and, therefore, we must establish that the invariants for these states also hold
    - After using ((S EMP EMP) (M EMP)), M-INV holds because $ci=$'()and the stack='()
    - After using ((M EMP EMP) (F EMP)), F-INV also holds because $ci=$'()and the stack='()

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- Assume INVs hold for k steps. Show they hold for the k+1 step

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Pushdown Automata

- Assume INVs hold for k steps. Show they hold for the k+1 step
- Proof invariants hold after each nonempty transition:
  `((S a EMP) (S (a)))`
    - By inductive hypothesis, `S-INV` holds
    - After consuming an a and pushing an a, P may reach S and by empty transition M
    - Note that an empty transition into F with a nonempty stack cannot lead to an accept. Therefore, we do not concern ourselves about P making such a transition
    - `S-INV` and `M-INV` hold because both the length of the consumed input and of the stack increased by 1, thus, remaining of equal length and because both continue to only contain as

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Pushdown Automata

- Assume INVs hold for k steps. Show they hold for the k+1 step
- Proof invariants hold after each nonempty transition:
  ((S a EMP) (S (a)))
    - By inductive hypothesis, S-INV holds
    - After consuming an a and pushing an a, P may reach S and by empty transition M
    - Note that an empty transition into F with a nonempty stack cannot lead to an accept.
      Therefore, we do not concern ourselves about P making such a transition
    - S-INV and M-INV hold because both the length of the consumed input and of the stack
      increased by 1, thus, remaining of equal length and because both continue to only contain as

- ((M b (a)) (M ,EMP)):
    - By inductive hypothesis, M-INV holds
    - After consuming a b and popping an a, P may reach M or nondeterministically reach F
      because it may eventually accept
    - M-INV holds because c_i continues to be as followed by bs, the stack can only contain as, and
      the number of as in c_i remains equal to the sum of the number of bs in c_i and the length of
      the stack.
    - F-INV holds because for a computation that ends with an accept the read b is the last
      symbol in the given word and popping an a makes the stack empty, c_i continues to be the
      read as followed by the read bs, and, given that the stack is empty, the number of as equals
      the number of bs in the consumed input.

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- Proving L = L(P)

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- Proving L = L(P)

Lemma
$w \in L \Leftrightarrow w \in L(P)$

Proof.

($\Rightarrow$) Assume $w \in L$. This means that $w = a^n b^n$. Given that state invariants always hold, there is a computation that has P consume all the as, then consume all the bs, and then reach F with an empty stack. Therefore, $w \in L(P)$.

($\Leftarrow$) Assume $w \in L(P)$. This means that M halts in F, the only final state, with an empty stack having consumed w. Given that the state invariants always hold we may conclude that $w = a^n b^n$. Therefore, $w \in L$. $\qquad \square$

-

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- Proving L = L(P)

Lemma
$w \in L \Leftrightarrow w \in L(P)$

Proof.

($\Rightarrow$) Assume $w \in L$. This means that $w = a^n b^n$. Given that state invariants always hold, there is a computation that has P consume all the as, then consume all the bs, and then reach F with an empty stack. Therefore, $w \in L(P)$.

($\Leftarrow$) Assume $w \in L(P)$. This means that M halts in F, the only final state, with an empty stack having consumed w. Given that the state invariants always hold we may conclude that $w = a^n b^n$. Therefore, $w \in L$. □

-

Lemma
$w \notin L \Leftrightarrow w \notin L(P)$

Proof.

($\Rightarrow$) Assume $w \notin L$

-
  - $w \neq a^n b^n$
  - 3 possibilities for the structure of w: not $a^n b^m$, w in $a^n b^m$, where $n > m$, or w in $a^n b^m$, where $n < m$
  - In the first case, w either starts with a b or has at least one a after a b: cannot transition out of S or cannot transition out of M
  - In the second case, never transition into F with an empty stack
  - In the third case, can't read all the input

($\Leftarrow$) Assume $w \notin L(P)$. This means that P cannot transition into F with an empty stack having consumed w. Given that the state invariants always hold, w must either start with a b, have an a after a b, have too many as, or have too many bs. In all cases, $w \notin L$. □

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-      ;; L = wcw^R | w in (a b)*
- What is the design idea?

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-        ;; L = wcw^R | w in (a b)*
- What is the design idea?
- Read w and push it on the stack

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-      ;; L = wcw^R | w in (a b)*
- What is the design idea?
- Read w and push it on the stack
- Read c

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-     ;; L = wcw^R | w in (a b)*
- What is the design idea?
- Read w and push it on the stack
- Read c
- Match elements of w^R with the stack elements
- If all the elements after the c match all the elements on the stack then the machine accepts.

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- ;; L = wcw^R | w in (a b)*

- (define wcw^r (make-ndpda

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- ```
  ;; L = wcw^R | w in (a b)*
  ```

- ```
  (define wcw^r (make-ndpda
  ```

- ```
  ;; Tests for wcw^r
  (check-equal? (sm-apply wcw^r '(a)) 'reject)
  (check-equal? (sm-apply wcw^r '(a c)) 'reject)
  (check-equal? (sm-apply wcw^r '(b c a)) 'reject)
  (check-equal? (sm-apply wcw^r '(a a b c b a b)) 'reject)
  (check-equal? (sm-apply wcw^r '(c)) 'accept)
  (check-equal? (sm-apply wcw^r '(a c a)) 'accept)
  (check-equal? (sm-apply wcw^r '(a b b b c b b b a)) 'accept)
  ```

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- `;; L = wcw^R | w in (a b)*`

- ```
  ;; States
  ;; S ci is empty and stack is empty
  ```

- `(define wcw^r (make-ndpda`

- ```
  ;; Tests for wcw^r
  (check-equal? (sm-apply wcw^r '(a)) 'reject)
  (check-equal? (sm-apply wcw^r '(a c)) 'reject)
  (check-equal? (sm-apply wcw^r '(b c a)) 'reject)
  (check-equal? (sm-apply wcw^r '(a a b c b a b)) 'reject)
  (check-equal? (sm-apply wcw^r '(c)) 'accept)
  (check-equal? (sm-apply wcw^r '(a c a)) 'accept)
  (check-equal? (sm-apply wcw^r '(a b b b c b b b a)) 'accept)
  ```

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- ```
;; L = wcw^R | w in (a b)*
```

- ```
;; States
;; S ci is empty and stack is empty
```

- ```
;; P ci = stack^R AND c not in ci
```

- ```
(define wcw^r (make-ndpda
```

- ```
;; Tests for wcw^r
(check-equal? (sm-apply wcw^r '(a)) 'reject)
(check-equal? (sm-apply wcw^r '(a c)) 'reject)
(check-equal? (sm-apply wcw^r '(b c a)) 'reject)
(check-equal? (sm-apply wcw^r '(a a b c b a b)) 'reject)
(check-equal? (sm-apply wcw^r '(c)) 'accept)
(check-equal? (sm-apply wcw^r '(a c a)) 'accept)
(check-equal? (sm-apply wcw^r '(a b b b c b b b a)) 'accept)
```

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```
;; L = wcw^R | w in (a b)*

;; States
;; S ci is empty and stack is empty

;; P ci = stack^R AND c not in ci

;; Q ci = (append w (list 'c) v) AND
;;    w = stack^R v^R


(define wcw^r (make-ndpda
```

```
;; Tests for wcw^r
(check-equal? (sm-apply wcw^r '(a)) 'reject)
(check-equal? (sm-apply wcw^r '(a c)) 'reject)
(check-equal? (sm-apply wcw^r '(b c a)) 'reject)
(check-equal? (sm-apply wcw^r '(a a b c b a b)) 'reject)
(check-equal? (sm-apply wcw^r '(c)) 'accept)
(check-equal? (sm-apply wcw^r '(a c a)) 'accept)
(check-equal? (sm-apply wcw^r '(a b b b c b b b a)) 'accept)
```

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```
;; L = wcw^R | w in (a b)*

;; States
;; S ci is empty and stack is empty

;; P ci = stack^R AND c not in ci

;; Q ci = (append w (list 'c) v) AND
;;     w = stack^R v^R

;; F stack = '() AND ci = (append w (list c) w^R)

(define wcw^r (make-ndpda
```

```
;; Tests for wcw^r
(check-equal? (sm-apply wcw^r '(a)) 'reject)
(check-equal? (sm-apply wcw^r '(a c)) 'reject)
(check-equal? (sm-apply wcw^r '(b c a)) 'reject)
(check-equal? (sm-apply wcw^r '(a a b c b a b)) 'reject)
(check-equal? (sm-apply wcw^r '(c)) 'accept)
(check-equal? (sm-apply wcw^r '(a c a)) 'accept)
(check-equal? (sm-apply wcw^r '(a b b b c b b b a)) 'accept)
```

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```
;; L = wcw^R | w in (a b)*
```

```
;; States
;; S ci is empty and stack is empty
```

```
;; P ci = stack^R AND c not in ci
```

```
;; Q ci = (append w (list 'c) v) AND
;;    w = stack^R v^R
```

```
;; F stack = '() AND ci = (append w (list c) w^R)
```

```
(define wcw^r (make-ndpda
```

```
                    `(((S ,EMP ,EMP) (P ,EMP))
```

```
;; Tests for wcw^r
(check-equal? (sm-apply wcw^r '(a)) 'reject)
(check-equal? (sm-apply wcw^r '(a c)) 'reject)
(check-equal? (sm-apply wcw^r '(b c a)) 'reject)
(check-equal? (sm-apply wcw^r '(a a b c b a b)) 'reject)
(check-equal? (sm-apply wcw^r '(c)) 'accept)
(check-equal? (sm-apply wcw^r '(a c a)) 'accept)
(check-equal? (sm-apply wcw^r '(a b b b c b b b a)) 'accept)
```

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- ;; L = wcw^R | w in (a b)*
- ;; States
  ;; S ci is empty and stack is empty
- ;; P ci = stack^R AND c not in ci
- ;; Q ci = (append w (list 'c) v) AND
  ;;   w = stack^R v^R
- ;; F stack = '() AND ci = (append w (list c) w^R)
- (define wcw^r (make-ndpda

- ``(((S ,EMP ,EMP) (P ,EMP))
- ((P a ,EMP) (P (a)))
  ((P b ,EMP) (P (b)))

- ;; Tests for wcw^r
  (check-equal? (sm-apply wcw^r '(a)) 'reject)
  (check-equal? (sm-apply wcw^r '(a c)) 'reject)
  (check-equal? (sm-apply wcw^r '(b c a)) 'reject)
  (check-equal? (sm-apply wcw^r '(a a b c b a b)) 'reject)
  (check-equal? (sm-apply wcw^r '(c)) 'accept)
  (check-equal? (sm-apply wcw^r '(a c a)) 'accept)
  (check-equal? (sm-apply wcw^r '(a b b b c b b b a)) 'accept)

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```
;; L = wcw^R | w in (a b)*

;; States
;; S ci is empty and stack is empty

;; P ci = stack^R AND c not in ci

;; Q ci = (append w (list 'c) v) AND
;;     w = stack^R v^R

;; F stack = '() AND ci = (append w (list c) w^R)

(define wcw^r (make-ndpda
```

```
`(((S ,EMP ,EMP) (P ,EMP))

  ((P a ,EMP) (P (a)))
  ((P b ,EMP) (P (b)))

  ((P c ,EMP) (Q ,EMP))
```

```
;; Tests for wcw^r
(check-equal? (sm-apply wcw^r '(a)) 'reject)
(check-equal? (sm-apply wcw^r '(a c)) 'reject)
(check-equal? (sm-apply wcw^r '(b c a)) 'reject)
(check-equal? (sm-apply wcw^r '(a a b c b a b)) 'reject)
(check-equal? (sm-apply wcw^r '(c)) 'accept)
(check-equal? (sm-apply wcw^r '(a c a)) 'accept)
(check-equal? (sm-apply wcw^r '(a b b b c b b b a)) 'accept)
```

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```
;; L = wcw^R | w in (a b)*

;; States
;; S ci is empty and stack is empty

;; P ci = stack^R AND c not in ci

;; Q ci = (append w (list 'c) v) AND
;;    w = stack^R v^R

;; F stack = '() AND ci = (append w (list c) w^R)

(define wcw^r (make-ndpda

                      `(((S ,EMP ,EMP) (P ,EMP))
                        ((P a ,EMP) (P (a)))
                        ((P b ,EMP) (P (b)))
                        ((P c ,EMP) (Q ,EMP))
                        ((Q a (a)) (Q ,EMP))
                        ((Q b (b)) (Q ,EMP))
                        ((Q ,EMP ,EMP) (F ,EMP)))))

;; Tests for wcw^r
(check-equal? (sm-apply wcw^r '(a)) 'reject)
(check-equal? (sm-apply wcw^r '(a c)) 'reject)
(check-equal? (sm-apply wcw^r '(b c a)) 'reject)
(check-equal? (sm-apply wcw^r '(a a b c b a b)) 'reject)
(check-equal? (sm-apply wcw^r '(c)) 'accept)
(check-equal? (sm-apply wcw^r '(a c a)) 'accept)
(check-equal? (sm-apply wcw^r '(a b b b c b b b a)) 'accept)
```

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```
;; L = wcw^R | w in (a b)*

;; States
;; S ci is empty and stack is empty

;; P ci = stack^R AND c not in ci

;; Q ci = (append w (list 'c) v) AND
;;      w = stack^R v^R

;; F stack = '() AND ci = (append w (list c) w^R)

(define wcw^r (make-ndpda
                    '(S P Q F)
                    '(a b c)
                    '(a b)
                    'S
                    '(F)
                    `(((S ,EMP ,EMP) (P ,EMP))
                      ((P a ,EMP) (P (a)))
                      ((P b ,EMP) (P (b)))
                      ((P c ,EMP) (Q ,EMP))
                      ((Q a (a)) (Q ,EMP))
                      ((Q b (b)) (Q ,EMP))
                      ((Q ,EMP ,EMP) (F ,EMP)))))

;; Tests for wcw^r
(check-equal? (sm-apply wcw^r '(a)) 'reject)
(check-equal? (sm-apply wcw^r '(a c)) 'reject)
(check-equal? (sm-apply wcw^r '(b c a)) 'reject)
(check-equal? (sm-apply wcw^r '(a a b c b a b)) 'reject)
(check-equal? (sm-apply wcw^r '(c)) 'accept)
(check-equal? (sm-apply wcw^r '(a c a)) 'accept)
(check-equal? (sm-apply wcw^r '(a b b b c b b b a)) 'accept)
```

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•
```
;; word stack → Boolean
;; Purpose: Determine in the given word and stack are empty
(define (S-INV ci s) (and (empty? ci) (empty? s)))

;; Tests for S-INV
(check-equal? (S-INV '() '(a a)) #f)
(check-equal? (S-INV '(a c a) '()) #f)
(check-equal? (S-INV '(a c a) '(b b)) #f)
(check-equal? (S-INV '() '()) #t)
```

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•
```
;; word stack → Boolean
;; Purpose: Determine if the given ci is the reverse of
;;          the given stack AND c is not in ci
(define (P-INV ci s)
  (and (equal? ci (reverse s)) (not (member 'c ci))))

;; Tests for P-INV
(check-equal? (P-INV '(a c a) '(a c a)) #f)
(check-equal? (P-INV '(a a) '(a b)) #f)
(check-equal? (P-INV '() '()) #t)
(check-equal? (P-INV '(a b) '(b a)) #t)
(check-equal? (P-INV '(a b a a) '(a a b a)) #t)
```

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-

```
;; word stack → Boolean
;; Purpose: Determine if ci=s^Rv^Rcv
(define (Q-INV ci s)
  (let* [(w (takef ci (λ (s) (not (eq? s 'c)))))
         (v (if (member 'c ci)
                (drop ci (add1 (length w)))
                '()))]
    (and (equal? ci (append w (list 'c) v))
         (equal? w (append (reverse s) (reverse v))))))

;; Tests for Q-INV
(check-equal? (Q-INV '(a a) '()) #f)
(check-equal? (Q-INV '(b b c a) '(b a)) #f)
(check-equal? (Q-INV '(c) '()) #t)
(check-equal? (Q-INV '(b a c) '(a b)) #t)
(check-equal? (Q-INV '(a b c b) '(a)) #t)
(check-equal? (Q-INV '(a b b c b) '(b a)) #t)
```

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-
```
;; word stack → Boolean
;; Purpose: Determine if ci=s^Rv^Rcv AND stack is empty
(define (F-INV ci s)
  (let* [(w (takef ci (λ (s) (not (eq? s 'c)))))]
    (and (empty? s)
         (equal? ci (append w (list 'c) (reverse w))))))

;; Tests for F-INV
(check-equal? (F-INV '() '()) #f)
(check-equal? (F-INV '(b b) '()) #f)
(check-equal? (F-INV '(b a c) '(b a)) #f)
(check-equal? (F-INV '(c) '()) #t)
(check-equal? (F-INV '(b a c a b) '()) #t)
(check-equal? (F-INV '(a b b c b b a) '()) #t)
```

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- $L = wcw^R$ $M = wcw^R$ $w \in$ (sm-sigma M)*
  s = the stack   F = (sm-finals M)   ci = consumed input

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Pushdown Automata

- $L = wcw^R$ $M = wcw\hat{}R$ $w \in (\text{sm-sigma M})^*$
  $s = $ the stack $F = (\text{sm-finals M})$ $ci = $ consumed input

Theorem
The state invariants hold when M accepts w.

-

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- Base Case

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- Base Case
- When M starts S-INV holds because $ci = $ '() and $s = $ '()

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- Base Case
- When M starts S-INV holds because ci = '() and s = '()
- A computation that leads to accept may also reach P by consuming no input
- ci = '() and s = '() when the machine moves to P
- P-INV holds because c is not a member of ci and ci = (reverse s)

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Pushdown Automata

- Inductive Step
- <u>((P a ,EMP) (P (a)))</u>
    - By inductive hypothesis, `P-INV` holds: ci does not contain c and that $ci = (reverse\ s)$
    - Reading and pushing an a, ci still does not contain c and we still have that $ci = (reverse\ s)$
    - `P-INV` holds

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Pushdown Automata

- Inductive Step
- `((P a ,EMP) (P (a)))`
    - By inductive hypothesis, `P-INV` holds: ci does not contain c and that ci = (reverse s)
    - Reading and pushing an a, ci still does not contain c and we still have that ci = (reverse s)
    - `P-INV` holds
- `((P b ,EMP) (P (b)))`:
    - By inductive hypothesis, `P-INV` holds: ci does not contain c and that ci = (reverse s)
    - By reading and pushing a b, ci still does not contain c and we still have that ci = (reverse s)
    - `P-INV` holds

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- Inductive Step

- ((P c ,EMP) (Q ,EMP)):
  - By inductive hypothesis, P-INV holds: c_i does not contain c and that c_i = (reverse s)
  - By reading c, c_i contains c
  - After the transition c_i = (reverse s)c = (reverse s)(reverse '())c(reverse '()) = (reverse s)(reverse v)cv
  - That is, v = '(). Thus, Q-INV holds
  - After using this transition rule, M may also reach F by consuming no input on a computation that leads to accept
  - Given that Q-INV holds and M only moves to F if v = s = '(), we have that c_i = c = '()c'() = wc(reverse w). That is, w = EMP. Thus, F-INV holds

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- Inductive Step

- `((P c ,EMP) (Q ,EMP))`:
  - By inductive hypothesis, `P-INV` holds: c_i does not contain c and that c_i = (reverse s)
  - By reading c, c_i contains c
  - After the transition c_i = (reverse s)c = (reverse s)(reverse `'()`)c(reverse `'()`) = (reverse s)(reverse v)cv
  - That is, v = `'()`. Thus, `Q-INV` holds
  - After using this transition rule, M may also reach F by consuming no input on a computation that leads to accept
  - Given that `Q-INV` holds and M only moves to F if v = s = `'()`, we have that c_i = c = `'()`c`'()` = wc(reverse w). That is, w = EMP. Thus, `F-INV` holds

- `((Q a (a)) (Q ,EMP))`:
  - By inductive hypothesis, `Q-INV` holds: c_i = (reverse s)(reverse v)cv
  - Reading and popping a moves the a on the top of the stack to v.
  - Thus, c_i = (reverse s)(reverse v)cv after using this transition and `Q-INV` holds.
  - If a is the last element of the input and the stack is empty then M moves to F. This means that c_i = (reverse s)(reverse v)cv = EMP(reverse v)cv = (reverse v)cv = (reverse w)cw. That is, v = w. Therefore, `F-INV` holds

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- Inductive Step

- `((P c ,EMP) (Q ,EMP))`:
  - By inductive hypothesis, `P-INV` holds: c_i does not contain c and that c_i = (reverse s)
  - By reading c, c_i contains c
  - After the transition c_i = (reverse s)c = (reverse s)(reverse `'()`)c(reverse `'()`) = (reverse s)(reverse v)cv
  - That is, v = `'()`. Thus, `Q-INV` holds
  - After using this transition rule, M may also reach F by consuming no input on a computation that leads to accept
  - Given that Q-INV holds and M only moves to F if v = s = `'()`, we have that c_i = c = `'()`c`'()` = wc(reverse w). That is, w = EMP. Thus, F-INV holds

- `((Q a (a)) (Q ,EMP))`:
  - By inductive hypothesis, Q-INV holds: c_i = (reverse s)(reverse v)cv
  - Reading and popping a moves the a on the top of the stack to v.
  - Thus, c_i = (reverse s)(reverse v)cv after using this transition and Q-INV holds.
  - If a is the last element of the input and the stack is empty then M moves to F. This means that c_i = (reverse s)(reverse v)cv = EMP(reverse v)cv = (reverse v)cv = (reverse w)cw. That is, v = w. Therefore, F-INV holds

- `((Q b (b)) (Q ,EMP))`: idem, but with b.

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- `L = L(M)`

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- `L = L(M)`

Lemma
$w \in L \Leftrightarrow w \in L(M)$

-

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- L = L(M)

Lemma
$w \in L \Leftrightarrow w \in L(M)$

-

- ($\Rightarrow$) Assume w∈L
    - $w = vcv^R$
    - Given that state invariants always hold, there is a computation that has M consume w and reach F with an empty stack
    - w∈L(M)

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- L = L(M)

Lemma
$w \in L \Leftrightarrow w \in L(M)$

-

- ($\Rightarrow$) Assume $w \in L$
    - $w = vcv^R$
    - Given that state invariants always hold, there is a computation that has M consume w and reach F with an empty stack
    - $w \in L(M)$

- ($\Leftarrow$) Assume $w \in L(M)$
    - M halts in F with an empty stack having consumed w
    - Given that the state invariants always hold: $w = vcv^R$
    - $w \in L$

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Lemma
$w \notin L \Leftrightarrow w \notin L(M)$

- •

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Lemma
$w \notin L \Leftrightarrow w \notin L(M)$

-
- ($\Rightarrow$) Assume $w \notin L$
  - $w \neq v^R cv$
  - Given that the state invariant predicates always hold, there is no computation that has M consume w and end in F with an empty stack
  - $w \notin L(M)$

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Lemma
$w{\notin}L \Leftrightarrow w{\notin}L(M)$

-

- ($\Rightarrow$) Assume $w{\notin}L$
  - $w \neq v^R cv$
  - Given that the state invariant predicates always hold, there is no computation that has M consume w and end in F with an empty stack
  - $w{\notin}L(M)$

- ($\Leftarrow$) Assume $w{\notin}L(M)$
  - M cannot transition into F with an empty stack having consumed w
  - Given that the state invariants always hold: $w \neq vcv^R$
  - $w{\notin}L$

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- HOMEWORK: 1–5, 8
- In class quiz (1 week from today)

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- For every language, L, decided by an ndfa, is there is a pda that decides L?

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- For every language, L, decided by an ndfa, is there is a pda that decides L?
- If a language, L, is regular then there is a ndfa that decides L
- Intuitively, an ndfa may be thought of as a pda that never operates on its stack
- We ought to be able to build a pda from an ndfa

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Pushdown Automata

- Design Idea
- Given an ndfa, M, a pda is constructed using:
- M's states, alphabet, starting state, and final states

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Pushdown Automata

- Design Idea
- Given an ndfa, M, a pda is constructed using:
- M's states, alphabet, starting state, and final states
- The stack alphabet for the constructed pda may be '() because nothing shall every be pushed onto the stack

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Pushdown Automata

- Design Idea

- Given an `ndfa`, `M`, a `pda` is constructed using:

- `M`'s states, alphabet, starting state, and final states

- The stack alphabet for the constructed `pda` may be `'()` because nothing shall every be pushed onto the stack

- The `pda`'s transition relation is built using `M`'s transition relation:

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- Design Idea
- Given an ndfa, M, a pda is constructed using:
- M's states, alphabet, starting state, and final states
- The stack alphabet for the constructed pda may be '() because nothing shall every be pushed onto the stack
- The pda's transition relation is built using M's transition relation:
- Each of M's rules is converted to a pda-rule

Pushdown Automata

- Design Idea
- Given an ndfa, M, a pda is constructed using:
- M's states, alphabet, starting state, and final states
- The stack alphabet for the constructed pda may be '() because nothing shall every be pushed onto the stack
- The pda's transition relation is built using M's transition relation:
- Each of M's rules is converted to a pda-rule
- For an ndfa-rule, (P a R), the pda-rule ((P a EMP) (R EMP)) is generated
- The pda moves from P to R without modifying the stack

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- ```
  ;; ndfa → pda
  ;; Purpose: Convert the given ndfa to a pda
  (define (ndfa->pda M)
  ```

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- ```
;; ndfa → pda
;; Purpose: Convert the given ndfa to a pda
(define (ndfa->pda M)
```

- ```
;; Sample pda
(define ALOM-PDA (ndfa->pda AT-LEAST-ONE-MISSING))
(define LNDFA-PDA (ndfa->pda LNDFA))

(check-equal? (sm-testequiv? ALOM-PDA AT-LEAST-ONE-MISSING) #t)
(check-equal? (sm-testequiv? LNDFA-PDA LNDFA) #t)
```

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# Pushdown Automata

- ```
;; ndfa → pda
;; Purpose: Convert the given ndfa to a pda
(define (ndfa->pda M)
```

- ```
  (let [(states (sm-states M))
        (sigma  (sm-sigma M))
        (start  (sm-start M))
        (finals (sm-finals M))
        (rules  (sm-rules M))]
    (make-ndpda states
                sigma
                '()
                start
                finals
```

- ```
;; Sample pda
(define ALOM-PDA  (ndfa->pda AT-LEAST-ONE-MISSING))
(define LNDFA-PDA (ndfa->pda LNDFA))

(check-equal? (sm-testequiv? ALOM-PDA AT-LEAST-ONE-MISSING) #t)
(check-equal? (sm-testequiv? LNDFA-PDA LNDFA) #t)
```

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- ```
  ;; ndfa → pda
  ;; Purpose: Convert the given ndfa to a pda
  (define (ndfa->pda M)
  ```

- ```
    (let [(states (sm-states M))
          (sigma  (sm-sigma M))
          (start  (sm-start M))
          (finals (sm-finals M))
          (rules  (sm-rules M))]
      (make-ndpda states
                  sigma
                  '()
                  start
                  finals
  ```

- ```
                  (map (λ (r) (list (list (first r) (second r) EMP)
                                    (list (third r) EMP)))
                       rules)))))
  ```

- ```
  ;; Sample pda
  (define ALOM-PDA  (ndfa->pda AT-LEAST-ONE-MISSING))
  (define LNDFA-PDA (ndfa->pda LNDFA))

  (check-equal? (sm-testequiv? ALOM-PDA AT-LEAST-ONE-MISSING) #t)
  (check-equal? (sm-testequiv? LNDFA-PDA LNDFA) #t)
  ```

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- HOMEWORK: 11

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Equivalence of pdas and cfgs

Theorem

The set of languages accepted by pdas is exactly the context-free languages.

- •

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Equivalence of pdas and cfgs

Theorem
The set of languages accepted by pdas is exactly the context-free languages.

-

- We must show:
  1. Given a cfg, G, show how to build a pda for L(G)
  2. Given a pda, P, show how to build a cfg for L(P)

Equivalence of `pdas` and `cfgs`

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Lemma
L is a context-free language $\Rightarrow \exists$ `pda` *that accepts L*

- Design Idea

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Equivalence of pdas and cfgs

Lemma
L is a context-free language $\Rightarrow \exists$ pda that accepts L

- Design Idea
- P = (make-ndpda K Σ S F Δ), such that L(P) = L(G)

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Lemma
L is a context-free language $\Rightarrow \exists$ pda that accepts L

- Design Idea
- P = (make-ndpda K Σ S F Δ), such that L(P) = L(G)
- Read the input from left to right: simulate the leftmost derivation of w

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Equivalence of `pdas` and `cfgs`

Lemma
L is a context-free language $\Rightarrow \exists$ `pda` that accepts L

- Design Idea
- `P = (make-ndpda K` Σ `S F` Δ`)`, such that `L(P) = L(G)`
- Read the input from left to right: simulate the leftmost derivation of `w`
- To start, `P` pushes S_G onto the stack
- For each `cfg`-rule, `A` $\rightarrow$`x`, `P` pops `A` off the stack and pushes `x` onto the stack
- For each a$\in\Sigma_G$, `P` reads a from the tape and pops a off the stack

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Lemma
L is a context-free language $\Rightarrow \exists$ pda that accepts L

- Design Idea
- P = (make-ndpda K Σ S F Δ), such that L(P) = L(G)
- Read the input from left to right: simulate the leftmost derivation of w
- To start, P pushes S_G onto the stack
- For each cfg-rule, A $\rightarrow$x, P pops A off the stack and pushes x onto the stack
- For each a$\in\Sigma_G$, P reads a from the tape and pops a off the stack
- Suggests that P only needs two states:
  - Starting state, S, in which P pushes S_G and moves to, Q
  - P remains in Q, which is the final state, for the rest of the computation

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Equivalence of pdas and cfgs

- P is constructed as follows:

    (make-ndpda '(S Q) Σ_G $V_G \cup \Sigma_G$ 'S '(Q) Δ)

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- P is constructed as follows:

    (make-ndpda '(S Q) Σ_G $V_G \cup \Sigma_G$ 'S '(Q) Δ)

- There are three types of rule in Δ:

    1. ((S EMP (list S_G)) (Q EMP))
    2. ((Q EMP (list 'A)) (Q (symbol->fsmlos x))), for $(A \rightarrow x) \in R_G$
    3. ((Q a (a)) (Q EMP)), for $a \in \Sigma_G$

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    3. ((Q a (a)) (Q EMP)), for a$\in \Sigma_G$

- We are careful to say nothing about the behavior of P if it is ever applied to a word not in L(G)

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- ;; cfg → pda   Purpose: Transform the given cfg into a pda
  (define (cfg2pda G)

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- ```
  ;; cfg → pda      Purpose: Transform the given cfg into a pda
  (define (cfg2pda G)
  ```

- ```
  ;; Tests
  (define a2nb2n-pda (cfg2pda a2nb2n))
  (define numb>numa-pda (cfg2pda numb>numa))

  (check-equal? (sm-apply a2nb2n-pda '(b b)) 'reject)
  (check-equal? (sm-apply a2nb2n-pda '(a a b b a b)) 'reject)
  (check-equal? (sm-apply a2nb2n-pda '()) 'accept)
  (check-equal? (sm-apply a2nb2n-pda '(a a a b b b)) 'accept)

  (check-equal? (sm-apply numb>numa-pda '(b b b)) 'accept)
  (check-equal? (sm-apply numb>numa-pda '(b b a)) 'accept)
  (check-equal? (sm-apply numb>numa-pda '(b b a b b)) 'accept)
  (check-equal? (sm-apply numb>numa-pda '(a b b a b)) 'accept)
  ```

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- ;; cfg → pda   Purpose: Transform the given cfg into a pda
  (define (cfg2pda G)

- (let [(nts (grammar-nts G))   (sigma (grammar-sigma G))
        (start (grammar-start G)) (rules (grammar-rules G))]
    (make-ndpda '(S Q)
                sigma
                (append nts sigma)
                'S
                (list 'Q)

- ;; Tests
  (define a2nb2n-pda (cfg2pda a2nb2n))
  (define numb>numa-pda (cfg2pda numb>numa))

  (check-equal? (sm-apply a2nb2n-pda '(b b)) 'reject)
  (check-equal? (sm-apply a2nb2n-pda '(a a b b a b)) 'reject)
  (check-equal? (sm-apply a2nb2n-pda '()) 'accept)
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    (make-ndpda '(S Q)
                sigma
                (append nts sigma)
                'S
                (list 'Q)
                (append
                 (list (list (list 'S EMP EMP) (list 'Q (list start))))
                 (map (λ (r)
                        (list (list 'Q EMP (list (first r)))
                              (list 'Q
                                    (if (eq? (third r) EMP)
                                        EMP
                                        (symbol->fsmlos (third r))))))
                      rules)
                 (map (λ (a) (list (list 'Q a (list a)) (list 'Q EMP)))
                      sigma)))))
;; Tests
(define a2nb2n-pda (cfg2pda a2nb2n))
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(check-equal? (sm-apply a2nb2n-pda '(b b)) 'reject)
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(check-equal? (sm-apply a2nb2n-pda '()) 'accept)
(check-equal? (sm-apply a2nb2n-pda '(a a a b b)) 'accept)

(check-equal? (sm-apply numb>numa-pda '(b b b)) 'accept)
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```

- Tests using words not in the corresponding language are only written for
  a2nb2n-pda
- This is because the design specifies nothing about the behavior of a pda
  when given a word that is not in the language

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- Tests using words not in the corresponding language are only written for
  a2nb2n-pda
- This is because the design specifies nothing about the behavior of a pda
  when given a word that is not in the language
- Sometimes a constructed pda is capable of rejecting and other times it is
  incapable of rejecting
- A pda produced by the constructor is guaranteed to accept if the given
  word is in its language but no such guarantee is made for rejection

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(check-equal? (sm-apply a2nb2n-pda '(a a a b b b)) 'accept)

(check-equal? (sm-apply numb>numa-pda '(b b b)) 'accept)
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```

- Tests using words not in the corresponding language are only written for a2nb2n-pda
- This is because the design specifies nothing about the behavior of a pda when given a word that is not in the language
- Sometimes a constructed pda is capable of rejecting and other times it is incapable of rejecting
- A pda produced by the constructor is guaranteed to accept if the given word is in its language but no such guarantee is made for rejection
- When a machine can guarantee accepting any word in a given language, L, but cannot guarantee rejecting any word not in L, we say that the machine *semidecides* L

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- For an intuitive understanding of why a pda may only semidecide a language, it is necessary to understand how nondeterminism is implemented

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- For an intuitive understanding of why a pda may only semidecide a language, it is necessary to understand how nondeterminism is implemented

- Breadth-first search for a computation to accept the given word

- The search space is a tree of derivations defined by the machine's rules

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- For an intuitive understanding of why a pda may only semidecide a language, it is necessary to understand how nondeterminism is implemented

- Breadth-first search for a computation to accept the given word

- The search space is a tree of derivations defined by the machine's rules

- If a computation ends at level k then breadth-first search is guaranteed to reach it. If the computation leads to accept then the machine accepts

- Otherwise, the breath-first process continues to extend the search tree to find a computation that leads to accept

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- For an intuitive understanding of why a pda may only semidecide a language, it is necessary to understand how nondeterminism is implemented
- Breadth-first search for a computation to accept the given word
- The search space is a tree of derivations defined by the machine's rules
- If a computation ends at level k then breadth-first search is guaranteed to reach it. If the computation leads to accept then the machine accepts
- Otherwise, the breath-first process continues to extend the search tree to find a computation that leads to accept
- If all paths in the search tree are finite and reject then the machine rejects
- If there is a path in the search tree that is not finite BFS may continue to search for a computation that leads to an accept forever
- Thus, the machine can accept but can never reject

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- To understand why a2nb2n-pda decides its language:

```
'(((P ε ε)   (Q (S)))
  ((Q ε (S)) (Q ε))
  ((Q ε (S)) (Q (a S b)))
  ((Q a (a)) (Q ε))
  ((Q b (b)) (Q ε)))
```

- After popping S, the machine must read input before popping another nonterminal or must only read input (when there are only bs in the stack)

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  ```

- After popping S, the machine must read input before popping another nonterminal or must only read input (when there are only bs in the stack)
- The input becomes empty and the stack is examined to decide if the word is rejected or accepted
- The machine cannot consume the whole input which means the computation ends in reject
- The machine can always say accept or reject, because all the paths in the search space are finite
- We say the machine *decides* its language.

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- The machine can always say accept or reject, because all the paths in the search space are finite

- We say the machine *decides* its language.

- In contrast, the transition relation for numb>numa is:

  ```
  '(((P ε ε) (Q (S)))
    ((Q ε (S)) (Q (b)))
    ((Q ε (S)) (Q (A b A)))
    ((Q ε (A)) (Q (A a A b A)))
    ((Q ε (A)) (Q (A b A a A)))
    ((Q ε (A)) (Q ε))
    ((Q ε (A)) (Q (b A)))
    ((Q a (a)) (Q ε))
    ((Q b (b)) (Q ε)))
  ```

- There are computations for which the stack always grows using the fourth or fifth rules without consuming any input

- This means that there are computations that are infinite

- A pda can only reject if all possible computations reject: impossible for computations that are infinite

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- To understand why a2nb2n-pda decides its language:

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- After popping S, the machine must read input before popping another nonterminal or must only read input (when there are only bs in the stack)
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```

- There are computations for which the stack always grows using the fourth or fifth rules without consuming any input
- This means that there are computations that are infinite
- A pda can only reject if all possible computations reject: impossible for computations that are infinite
- Accept if in Q with an empty stack and no more input to read, but it can never determine that all possible computations reject
- We say the machine *semidecides* its language.

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Let:

```
G = (make-cfg N Σ R S)      P = (cfg2pda G)
```

Lemma

$S \xrightarrow{L}_G \omega\alpha \Leftrightarrow (Q\ \omega\ S) \vdash^*_M (Q\ \epsilon\ \alpha)$, where $\alpha \in \{N(N \cup \Sigma)^* \cup \{\epsilon\}\} \wedge \omega \in \Sigma^*$

- A leftmost derivation of $\omega\alpha$ is either empty or starts with the leftmost
  nonterminal in $\omega\alpha$ is logically equivalent to M starting in Q, reading ω, and
  popping S to end in Q with α on the stack

- Observe that if the stack is not empty after consuming ω then the topmost
  element is the next nonterminal to substitute in the leftmost derivation

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Let:

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$S \, ^L\!\rightarrow_G \omega\alpha \Leftrightarrow (Q \, \omega \, S) \vdash^*_M (Q \, \epsilon \, \alpha)$, where $\alpha \in \{N(N \cup \Sigma)^* \cup \{\epsilon\}\} \land \omega \in \Sigma^*$

• ($\Rightarrow$) The proof is by induction on n = the length of the leftmost derivation of ω.

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- $(\Rightarrow)$ The proof is by induction on $n =$ the length of the leftmost derivation of ω.
- Base case, $n = 0$
- This means that $\omega = \epsilon$ and $\alpha = S$
- We have that $(Q \; \omega \; S) = (Q \; \epsilon \; S) \vdash^*_M (Q \; \epsilon \; S) = (Q \; \epsilon \; \alpha)$.

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- ($\Rightarrow$) The proof is by induction on n = the length of the leftmost derivation of ω.
- Base case, $n = 0$
- This means that $\omega = \epsilon$ and $\alpha = S$
- We have that $(Q\ \omega\ S) = (Q\ \epsilon\ S) \vdash^*_M (Q\ \epsilon\ S) = (Q\ \epsilon\ \alpha)$.
- For the inductive step:
- Assume: $S \xrightarrow{L}_G \omega\alpha \Leftrightarrow (Q\ \omega\ S) \vdash^*_M (Q\ \epsilon\ \alpha)$, for $n = k$.
- Show: $S \xrightarrow{L}_G \omega\alpha \Leftrightarrow (Q\ \omega\ S) \vdash^*_M (Q\ \epsilon\ \alpha)$, for $n = k + 1$.

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- Show: $S \xrightarrow{L}_G \omega\alpha \Leftrightarrow (Q\ \omega\ S) \vdash^*_M (Q\ \epsilon\ \alpha)$, for $n = k + 1$.
- The derivation of length $k+1$ looks as follows:
- $S \rightarrow u_1 \rightarrow u_2 \rightarrow \ldots \rightarrow u_n \rightarrow u_{n+1} = \omega\alpha$

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- The derivation of length k+1 looks as follows:
- $S \rightarrow u_1 \rightarrow u_2 \rightarrow \ldots \rightarrow u_n \rightarrow u_{n+1} = \omega\alpha$
- Observe that $u_n = xA\beta$, where $x \in \Sigma^* \wedge A \in N \wedge \beta \in \{N \cup \Sigma\}^*$

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- For the inductive step:
- Assume: $S \xrightarrow{L}_G \omega\alpha \Leftrightarrow (Q\ \omega\ S) \vdash^*_M (Q\ \epsilon\ \alpha)$, for $n = k$.
- Show: $S \xrightarrow{L}_G \omega\alpha \Leftrightarrow (Q\ \omega\ S) \vdash^*_M (Q\ \epsilon\ \alpha)$, for $n = k + 1$.
- The derivation of length k+1 looks as follows:
- $S \to u_1 \to u_2 \to \ldots \to u_n \to u_{n+1} = \omega\alpha$
- Observe that $u_n = xA\beta$, where $x \in \Sigma^* \wedge A \in N \wedge \beta \in \{N \cup \Sigma\}^*$
- The derivation step from u_n to u_{n+1} substitutes A using a rule: $A \to \theta$, where $\theta \in \{N \cup \Sigma\}^*$
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Let:

```
G = (make-cfg N Σ R S)     P = (cfg2pda G)
```

Lemma

$S \,^L\!\!\to_G \omega\alpha \Leftrightarrow (Q \; \omega \; S) \vdash^*_M (Q \; \epsilon \; \alpha)$, where $\alpha \in \{N(N \cup \Sigma)^* \cup \{\epsilon\}\} \wedge \omega \in \Sigma^*$

- ($\Rightarrow$) The proof is by induction on n = the length of the leftmost derivation of ω.
- Base case, n = 0
- This means that $\omega = \epsilon$ and $\alpha = S$
- We have that $(Q \; \omega \; S) = (Q \; \epsilon \; S) \vdash^*_M (Q \; \epsilon \; S) = (Q \; \epsilon \; \alpha)$.
- For the inductive step:
- Assume: $S \,^L\!\!\to_G \omega\alpha \Leftrightarrow (Q \; \omega \; S) \vdash^*_M (Q \; \epsilon \; \alpha)$, for n = k.
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- The derivation step from u_n to u_{n+1} substitutes A using a rule: $A \to \theta$, where $\theta \in \{N \cup \Sigma\}^*$
- $u_{n+1} = x\theta\beta$.
- By inductive hypothesis, we have $(Q \; x \; S) \vdash^* (Q \; \epsilon \; A\beta)$

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- By construction of P, there is a type 2 rule such that $(Q\ \epsilon\ A\beta) \vdash (Q\ \epsilon\ \theta\beta)$
- The leading terminals of θ, y, are in ω and that α starts with, B, the leftmost nonterminal in θ and includes β
- That is, $\omega = xy$ and $\alpha = B\beta$, where $y \in \Sigma^*$ and $\beta \in N$

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- The derivation step from u_n to u_{n+1} substitutes A using a rule: $A \to \theta$, where $\theta \in \{N \cup \Sigma\}^*$
- $u_{n+1} = x\theta\beta$.
- By inductive hypothesis, we have $(Q \, x \, S) \vdash^* (Q \, \epsilon \, A\beta)$
- By construction of P, there is a type 2 rule such that $(Q \, \epsilon \, A\beta) \vdash (Q \, \epsilon \, \theta\beta)$
- The leading terminals of θ, y, are in ω and that α starts with, B, the leftmost nonterminal in θ and includes β
- That is, $\omega = xy$ and $\alpha = B\beta$, where $y \in \Sigma^*$ and $\beta \in N$
- This means that $y\alpha = \theta\beta$
- By construction of P there are type 3 rules that consume y to yield the following computation: $(Q \, \omega \, S) = (Q \, xy \, S) \vdash^* (Q \, y \, \theta\beta) = (Q \, y \, y\alpha) \vdash^* (Q \, \epsilon \, \alpha)$.

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- ($\Leftarrow$) Assume $(Q\ \omega\ S) \vdash^* (Q\ \epsilon\ \alpha)$. The proof is by induction on, n, the number of type 2 transitions used by P

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- Thus, we have that $S \xrightarrow{L} \omega\alpha = \epsilon S = S$.

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- Thus, we have that $S \overset{L}{\to} \omega\alpha = \epsilon S = S$.
- For the inductive step:
- Assume: $(Q\ \omega\ S) \vdash^*_M (Q\ \epsilon\ \alpha) \Rightarrow S \overset{L}{\to}_G \omega\alpha$, for $n = k$.
- Show: $(Q\ \omega\ S) \vdash^*_M (Q\ \epsilon\ \alpha) \Rightarrow S \overset{L}{\to}_G \omega\alpha$, for $n = k + 1$.

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- Show: $(Q \ \omega \ S) \vdash^*_M (Q \ \epsilon \ \alpha) \Rightarrow S \xrightarrow{L}_G \omega\alpha$, for $n = k + 1$.
- Let $\omega = xy$ and $(A \rightarrow \theta) \in R$
- A computation using $n + 1$ type 2 rules looks as follows:
  $(Q \ \omega \ S) = (Q \ xy \ S) \vdash^* (Q \ y \ A\beta) \vdash (Q \ y \ \theta\beta) \vdash^* (Q \ \epsilon \ \alpha)$

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- ($\Leftarrow$) Assume $(Q\ \omega\ S) \vdash^* (Q\ \epsilon\ \alpha)$. The proof is by induction on, n, the number of type 2 transitions used by P
- For the base case, $n = 0$
- $(Q\ \omega\ S) \vdash^*(Q\ \epsilon\ \alpha)$ means that $\omega = \epsilon$ and $\alpha = S$.
- Thus, we have that $S\ ^L\to\ \omega\alpha = \epsilon S = S$.
- For the inductive step:
- Assume: $(Q\ \omega\ S) \vdash^*_M (Q\ \epsilon\ \alpha) \Rightarrow S\ ^L\to_G\ \omega\alpha$, for $n = k$.
- Show: $(Q\ \omega\ S) \vdash^*_M (Q\ \epsilon\ \alpha) \Rightarrow S\ ^L\to_G\ \omega\alpha$, for $n = k + 1$.
- Let $\omega = xy$ and $(A \to \theta)\in R$
- A computation using $n + 1$ type 2 rules looks as follows:
  $(Q\ \omega\ S) = (Q\ xy\ S) \vdash^* (Q\ y\ A\beta) \vdash (Q\ y\ \theta\beta) \vdash^* (Q\ \epsilon\ \alpha)$
- By inductive hypothesis, we have $S\ ^L\to^*_G\ xA\beta$
- Using the rule for A above yields $S\ ^L\to^*_G\ x\theta\beta$

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$S \xrightarrow{L}_G \omega\alpha \Leftrightarrow (Q\ \omega\ S) \vdash^*_M (Q\ \epsilon\ \alpha)$, where $\alpha \in \{N(N \cup \Sigma)^* \cup \{\epsilon\}\} \wedge \omega \in \Sigma^*$

- ($\Leftarrow$) Assume $(Q\ \omega\ S) \vdash^* (Q\ \epsilon\ \alpha)$. The proof is by induction on, n, the number of type 2 transitions used by P
- For the base case, $n = 0$
- $(Q\ \omega\ S) \vdash^*(Q\ \epsilon\ \alpha)$ means that $\omega = \epsilon$ and $\alpha = S$.
- Thus, we have that $S \xrightarrow{L} \omega\alpha = \epsilon S = S$.
- For the inductive step:
- Assume: $(Q\ \omega\ S) \vdash^*_M (Q\ \epsilon\ \alpha) \Rightarrow S \xrightarrow{L}_G \omega\alpha$, for $n = k$.
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  $(Q\ \omega\ S) = (Q\ xy\ S) \vdash^* (Q\ y\ A\beta) \vdash (Q\ y\ \theta\beta) \vdash^* (Q\ \epsilon\ \alpha)$
- By inductive hypothesis, we have $S \xrightarrow{L}^*_G xA\beta$
- Using the rule for A above yields $S \xrightarrow{L}^*_G x\theta\beta$
- Observe that $(Q\ y\ \theta\beta) \vdash^* (Q\ \epsilon\ \alpha)$ only uses type 3 rules
- This means that $y\alpha = \theta\beta$
- Thus, we have $S \xrightarrow{L}^*_G x\theta\beta = xy\alpha = \omega\alpha$

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Equivalence of pdas and cfgs

- We need to prove that if P = (make-ndpda K Σ Γ S F Δ) semidecides L then L is a context-free language
- We shall build a cfg, G, such that L(P) = L(G)

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- We need to prove that if P = (make-ndpda K Σ Γ S F Δ) semidecides L then L is a context-free language
- We shall build a `cfg`, G, such that $L(P) = L(G)$
- Useful to restrict the structure of P's transition rules to have what we shall call a *simple* pda
- A pda is simple if all transition rules have the following structure:

    $((Q\ a\ \beta)\ (P\ \theta))$, such that $Q \neq S$, $\beta \in \Gamma$, $\wedge\ |\theta| \leq 2$

- Always pops the topmost stack element and pushes zero, one, or two elements onto the stack
- No restrictions on S, because whenever a pda starts the stack is empty

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- To transform P into a simple pda, P′, the components of P′ are defined as follows:

$$
\begin{aligned}
K' &= K \cup \{X\ Y\},\ such\ that\ X,Y \notin K\\
\Sigma' &= \Sigma\\
\Gamma' &= \Gamma \cup \{Z\},\ such\ that\ Z \notin \Gamma\\
S' &= X\\
F' &= (Y)\\
\Delta' &= T(\Delta)\ \cup\ \{((X\ EMP\ EMP)(S\ (Z)))\}\ \cup\ \{\forall W \in F\ ((W\ EMP\ (Z))(Y\ EMP))\}
\end{aligned}
$$

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\end{aligned}
$$

- The new states, X and Y: start and only final states

- New stack symbol, Z, is the stack bottom symbol: whenever P would have an empty stack, P′ only has Z on the stack

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- New stack symbol, Z, is the stack bottom symbol: whenever P would have an empty stack, P$'$ only has Z on the stack

- P$'$ simulates P's computation

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- The new states, X and Y: start and only final states
- New stack symbol, Z, is the stack bottom symbol: whenever P would have an empty stack, P' only has Z on the stack
- P' simulates P's computation
- T replaces all transitions rules that violate simplicity with equivalent rules that satisfy simplicity
- T replaces rules, $((Q\ a\ \beta)\ (P\ \theta))$, that violate simplicity in three steps:

  1. *Replace $|\beta| \geq 2$ rules*
  2. *Replace $|\beta| = 0$ rules without new rules with $|\beta| \geq 2$*
  3. *Replace $|\theta| > 2$ rules without new rules with $|\beta| \neq 1$*

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- New stack symbol, Z, is the stack bottom symbol: whenever P would have an empty stack, P′ only has Z on the stack
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- T replaces all transitions rules that violate simplicity with equivalent rules that satisfy simplicity
- T replaces rules, ((Q a β) (P θ)), that violate simplicity in three steps:

    1. *Replace $|\beta| \geq 2$ rules*
    2. *Replace $|\beta| = 0$ rules without new rules with $|\beta| \geq 2$*
    3. *Replace $|\theta| > 2$ rules without new rules with $|\beta| \neq 1$*

- *Implementation in the book*

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- Convert a simple pda to a cfg
- This is done by exploiting the properties of transition rules in a simple pda
- Unfortunately, the details get messy and the algorithm is not as intuitive as the algorithm to build a pda from a cfg

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Equivalence of pdas and cfgs

- The transformation of a pda to a simple pda simplifies the construction of a cfg for L(P)
- The simplification stems from the fact that every pda-rule pops an element and pushes at most two elements

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- The simplification stems from the fact that every pda-rule pops an element and pushes at most two elements

- The middle part simulates the given pda
- The grammar that is constructed shall simulate the simple pda
- Generate all words that take the simple pda from A to F' by popping Z

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Properties of
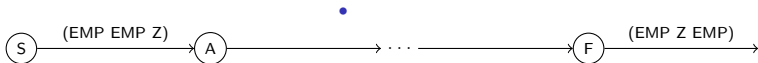CFLs

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- The transformation of a pda to a simple pda simplifies the construction of a cfg for L(P)
- The simplification stems from the fact that every pda-rule pops an element and pushes at most two elements

- The middle part simulates the given pda
- The grammar that is constructed shall simulate the simple pda
- Generate all words that take the simple pda from A to F$'$ by popping Z
- We may represent these words as a triple: (S Z F$'$)
- This means that the starting nonterminal for the constructed grammar, S, must generate (S Z F$'$):
  S $\rightarrow$ (A Z F$'$)

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Equivalence of pdas and cfgs

- Triples need to be converted to symbols
- We shall talk about all words that take the simple pda from some state Q to some state R by popping θ: (Q θ R)
- All such triples represent the nonterminals of the grammar that is constructed and are formally defined as follows:

```
;; An list nonterminal, lnt, is a (list state symbol state).

;; Interpretation:
;;   All words that take the pda from the first state to
;;   the second state by popping the symbol off the stack.
```

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- We define a cfg-rule's left-hand side as follows:

```
;; A lhs is either:
;;  1. symbol
;;  2. lnt

;; Interpretation:
;;  The left-hand side of a cfg-rule is represented as either a
;;  a symbol or an lnt.
```

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  ;; 2. lnt

  ;; Interpretation:
  ;;   The left-hand side of a cfg-rule is represented as either a
  ;;   a symbol or an lnt.
  ```

- How can the right-hand side of a cfg-rule be represented?
- Simple pda rules vary in what is pushed
- Rules that push 0 elements: ((Q a β) (R EMP))
- After using this rule the machine may transition in zero or more steps to an arbitrary state V:

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  ;;   a symbol or an lnt.
  ```
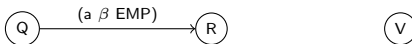
- How can the right-hand side of a cfg-rule be represented?
- Simple pda rules vary in what is pushed
- Rules that push 0 elements: ((Q a β) (R EMP))
- After using this rule the machine may transition in zero or more steps to an arbitrary state V:

- The grammar needs to generate all words that take the simple pda from Q to V by popping β and pushing nothing
- Such words are generated by (Q β V)
- To get to V, the machine reads a to reach R and then pops nothing to reach V in zero or more moves

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;; An list nonterminal, lnt, is a (list state symbol state).

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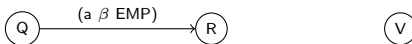- We define a cfg-rule's left-hand side as follows:

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;; A lhs is either:
;; 1. symbol
;; 2. lnt

;; Interpretation:
;;   The left-hand side of a cfg-rule is represented as either a
;;   a symbol or an lnt.
```

- How can the right-hand side of a cfg-rule be represented?
- Simple pda rules vary in what is pushed
- Rules that push 0 elements: ((Q a β) (R EMP))
- After using this rule the machine may transition in zero or more steps to an arbitrary state V:

- The grammar needs to generate all words that take the simple pda from Q to V by popping β and pushing nothing
- Such words are generated by (Q β V)
- To get to V, the machine reads a to reach R and then pops nothing to reach V in zero or more moves
- The needed rule is:

    (Q β V) $\rightarrow$ a(R EMP V)

- The right-hand side of a cfg-rule may be represented by a symbol followed by an lnt.

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- Consider the cfg-rule needed when a single element is pushed:

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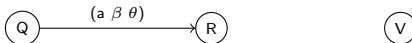
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- Consider the cfg-rule needed when a single element is pushed:

- The grammar needs to generate all words that take the simple pda from Q to V by popping β and pushing θ
- Similar to pushing nothing except that after reaching R θ must be popped
- The needed cfg-rule is:

  (Q β V) $\rightarrow$ a(R θ V)

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- Consider the pda-rules that push two elements: $((Q \ a \ \beta) \ (R \ \theta\tau))$.
- After popping the first element the machine may be in an arbitrary state W. Graphically, consider the following:

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- Consider the pda-rules that push two elements: ((Q a β) (R $\theta\tau$)).
- After popping the first element the machine may be in an arbitrary state W. Graphically, consider the following:

- The grammar must be able to generate a word that starts with a followed by a word that takes the simple pda from R to W by popping θ and ending with a word that takes it from W to V by popping τ
- The needed cfg-rule may be represented as follows:

    (Q β V) $\rightarrow$ a(R θ W)(W τ V)

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- Finally, we must consider the stopping conditions for word generation by the grammar
- The three types of production rules generated above always generate at least one lnt

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Equivalence of pdas and cfgs

- Finally, we must consider the stopping conditions for word generation by the grammar
- The three types of production rules generated above always generate at least one lnt
- Once all the terminal symbols are generated, remaining nonterminals must be eliminated
- If simple pda is in an arbitrary state Q then it may remain in Q without popping or consuming any input:

    $\forall Q \in$ (sm-states P$'$) (Q EMP Q) $\rightarrow$ EMP

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- If simple pda is in an arbitrary state Q then it may remain in Q without popping or consuming any input:

    $\forall$Q$\in$(sm-states P$'$) (Q EMP Q) $\rightarrow$ EMP

- The representation of the right-hand side of a cfg-rule has variety and may be defined as follows:

```
;; A rhs is either:
;; 1. symbol
;; 2. (list symbol lnt)
;; 3. (list symbol lnt lnt)
;; Interpretation:
;;   A cfg-rule right hand side is represented as either a
;;   symbol or a list with a symbol and either one or two lnt.
```

- We can now define the representation of a cfg-rule as follows:

```
;; A cfg-rl is a (list lhs ARROW rhs)
;; Interpretation:
;;   A cfg-rl represents a cfg-fule as a list with an lhs,
;;   an ARROW, and a rhs.
```

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- Finally, we must consider the stopping conditions for word generation by the grammar
- The three types of production rules generated above always generate at least one lnt
- Once all the terminal symbols are generated, remaining nonterminals must be eliminated
- If simple pda is in an arbitrary state Q then it may remain in Q without popping or consuming any input:

  $\forall$Q$\in$(sm-states P$'$) (Q EMP Q) $\rightarrow$ EMP

- The representation of the right-hand side of a cfg-rule has variety and may be defined as follows:
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;; A rhs is either:
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;; 2. (list symbol lnt)
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;; Interpretation:
;; A cfg-rule right hand side is represented as either a
;; symbol or a list with a symbol and either one or two lnt.
```
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;; A cfg-rl is a (list lhs ARROW rhs)
;; Interpretation:
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;; an ARROW, and a rhs.
```
- Implementation in the textbook

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Properties of CFLs

- For every context-free L there is a cfg that generates its members and a pda that accepts its members

- We e shall study closure properties of context-free languages and how to prove that a language is not context-free

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- Context-free languages are closed under union

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Properties of CFLs

- Context-free languages are closed under union
- Design Idea

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Properties of CFLs

- Context-free languages are closed under union
- Design Idea
- If L_1 and L_2 are context-free languages then there are cfgs, G_1 and G_2, such that L_1 = $L(G_1)$ and L_2 = $L(G_2)$
- Without loss of generality, we assume that (grammar-nts G_1) $\cap$ (grammar-nts G_2) = $\varnothing$

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Properties of CFLs

- Context-free languages are closed under union
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- If L_1 and L_2 are context-free languages then there are cfgs, G_1 and G_2, such that L_1 = $L(G_1)$ and L_2 = $L(G_2)$
- Without loss of generality, we assume that (grammar-nts G_1) $\cap$ (grammar-nts G_2) = $\varnothing$
- The grammar for L_1 $\cup$ L_2 nondeterministically decides to simulate a derivation using G_1 or G_2.

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Properties of CFLs

- Context-free languages are closed under union

- Design Idea

- If L_1 and L_2 are context-free languages then there are cfgs, G_1 and G_2, such that $L_1 = L(G_1)$ and $L_2 = L(G_2)$

- Without loss of generality, we assume that `(grammar-nts` G_1`) ∩ (grammar-nts` G_2`) = ∅`

- The grammar for $L_1 ∪ L_2$ nondeterministically decides to simulate a derivation using G_1 or G_2.

- A cfg for $L_1 ∪ L_2$ is built using the following components:

V	=	`(grammar-nts` G_1`) ∪ (grammar-nts` G_2`) ∪ {`A`}`
Σ	=	`(grammar-sigma` G_1`) ∪ (grammar-sigma` G_2`)`
R	=	`(grammar-rules` G_1`) ∪ (grammar-rules` G_2`) ∪`
		`{(S ARROW (grammar-start` G_1`)) (S ARROW (grammar-start` G_2`))}`
S	=	A `= (generate-symbol 'S (append (grammar-nts` G_1`) (grammar-nts` G_2`)))`

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Properties of CFLs

- ```
  ;; cfg cfg → cfg
  ;; Purpose: Construct a grammar for the union of the given grammars
  (define (cfg-union G1 G2)
  ```

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- ```
;; cfg cfg → cfg
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(define (cfg-union G1 G2)
```

- ```
(define a2nb2nUnumb>numa (cfg-union a2nb2n numb>numa))
(define numb>numaUMULT3-as (cfg-union numb>numa MULT3-as))

(check-equal? (grammar-derive a2nb2nUnumb>numa '(b a a a b b))
              "(b a a a b b) is not in L(G).")
(check-equal? (last (grammar-derive a2nb2nUnumb>numa '(a b)))
              'ab)
(check-equal? (last (grammar-derive a2nb2nUnumb>numa '(a a b b)))
              'aabb)
(check-equal? (last (grammar-derive a2nb2nUnumb>numa '(b b)))
              'bb)
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```

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# Properties of CFLs

```
;; cfg cfg → cfg
;; Purpose: Construct a grammar for the union of the given grammars
(define (cfg-union G1 G2)
  (let* [(H (grammar-rename-nts (grammar-nts G1) G2))
```

```
(define a2nb2nUnumb>numa (cfg-union a2nb2n numb>numa))
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Properties of CFLs

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-           (G-nts (grammar-nts G1))   (H-nts (grammar-nts H))
            (G-sigma (grammar-sigma G1)) (H-sigma (grammar-sigma H))
            (G-rules (grammar-rules G1)) (H-rules (grammar-rules H))
            (G-start (grammar-start G1)) (H-start (grammar-start H))

- (define a2nb2nUnumb>numa (cfg-union a2nb2n numb>numa))
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    (let* [(H (grammar-rename-nts (grammar-nts G1) G2))
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- ```
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           (G-sigma (grammar-sigma G1)) (H-sigma (grammar-sigma H))
           (G-rules (grammar-rules G1)) (H-rules (grammar-rules H))
           (G-start (grammar-start G1)) (H-start (grammar-start H))
  ```
- ```
           (A (generate-symbol 'S (append G-nts H-nts)))]
  ```

- ```
  (define a2nb2nUnumb>numa (cfg-union a2nb2n numb>numa))
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                'bbb)
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  ```

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         (G-rules (grammar-rules G1)) (H-rules (grammar-rules H))
         (G-start (grammar-start G1)) (H-start (grammar-start H))
         (A (generate-symbol 'S (append G-nts H-nts)))]
    (make-cfg (cons A (append G-nts H-nts))
              (remove-duplicates (append G-sigma H-sigma))
              (append (list (list A ARROW G-start)
                            (list A ARROW H-start))
                      G-rules
                      H-rules)
              A)))
(define a2nb2nUnumb>numa (cfg-union a2nb2n numb>numa))
(define numb>numaUMULT3-as (cfg-union numb>numa MULT3-as))

(check-equal? (grammar-derive a2nb2nUnumb>numa '(b a a a b b))
              "(b a a a b b) is not in L(G).")
(check-equal? (last (grammar-derive a2nb2nUnumb>numa '(a b)))
              'ab)
(check-equal? (last (grammar-derive a2nb2nUnumb>numa '(a a b b)))
              'aabb)
(check-equal? (last (grammar-derive a2nb2nUnumb>numa '(b b)))
              'bb)
(check-equal? (last (grammar-derive a2nb2nUnumb>numa '(b a a b b)))
              'baabb)
(check-equal? (grammar-derive numb>numaUMULT3-as '(a a a a b b))
              "(a a a a b b) is not in L(G).")
(check-equal? (last (grammar-derive numb>numaUMULT3-as '(b b b)))
              'bbb)
(check-equal? (last (grammar-derive numb>numaUMULT3-as '(a b a b a b)))
```

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Properties of CFLs

Theorem
Context-free languages are closed under union.

- Assume S $\rightarrow_G$ w

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Context-free languages are closed under union.

- Assume $S \rightarrow_G w$
- By construction of G, the derivation of w starts with (grammar-start G) and then either (grammar-start G1) is generated or (grammar-start G2) is generated

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- By construction of G, the derivation of w starts with (grammar-start G)
  and then either (grammar-start G1) is generated or (grammar-start G2)
  is generated

- If (grammar-start G1) is generated then G simulates G1

- If (grammar-start G2) is generated then G simulates G2

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- If (grammar-start G2) is generated then G simulates G2

- Recall that (grammar-nts G1)∩(grammar-nts G2) = ∅

- This means that either G1 generates w or G2 generates w

- Thus, w ∈ L(G1)∪L(G2).

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- HOMEWORK: 1

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- Context-free languages are closed under concatenation.

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Properties of CFLs

- Context-free languages are closed under concatenation.
- Design Idea
- There are cfgs, G_1 and G_2, such that $L_1 = L(G_1)$ and $L_2 = L(G_2)$

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- There are cfgs, G_1 and G_2, such that L_1 = $L(G_1)$ and L_2 = $L(G_2)$
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- We shall build a grammar, G, for L_1L_2.

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- Every word in L_1L_2 may be written as w = xy, such that $x \in L_1$ and $y \in L_2$

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- Context-free languages are closed under concatenation.
- Design Idea
- There are cfgs, G_1 and G_2, such that $L_1 = L(G_1)$ and $L_2 = L(G_2)$
- Without loss of generality, we assume that $(\text{grammar-nts } G_1) \cap (\text{grammar-nts } G_2) = \varnothing$
- We shall build a grammar, G, for L_1L_2.
- Every word in L_1L_2 may be written as w = xy, such that $x \in L_1$ and $y \in L_2$
- A cfg for L_1L_2 is built using the following components:

V	=	$(\text{grammar-nts } G_1) \cup (\text{grammar-nts } G_2) \cup \{A\}$
Σ	=	$(\text{grammar-sigma } G_1) \cup (\text{grammar-sigma } G_2)$
R	=	$(\text{grammar-rules } G_1) \cup (\text{grammar-rules } G_2) \cup$
		$\{S \text{ ARROW } (\text{grammar-start } G_1)(\text{grammar-start } G_2)\}$
S	=	A = (generate-symbol 'S (append (grammar-nts G_1) (grammar

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Properties of CFLs

- ```
  ;; cfg cfg → cfg
  ;; Purpose: Build a cfg for the concatenation of the given cfgs
  (define (cfg-concat G1 G2)
  ```

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  ;; cfg cfg → cfg
  ;; Purpose: Build a cfg for the concatenation of the given cfgs
  (define (cfg-concat G1 G2)
  ```

- ```
  ;; Tests for cfg-concat
  (define a2nb2n-numb>numa (cfg-concat a2nb2n numb>numa))
  (define MULT3-as-a2nb2n (cfg-concat MULT3-as a2nb2n))

  (check-equal? (grammar-derive a2nb2n-numb>numa '(b a))
                "(b a) is not in L(G).")
  (check-equal? (grammar-derive a2nb2n-numb>numa '(a a a))
                "(a a a) is not in L(G).")
  (check-equal? (grammar-derive a2nb2n-numb>numa '(a a b b))
  ```

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- ```
  ;; cfg cfg → cfg
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  (define (cfg-concat G1 G2)
  ```
- ```
    (let* [(G2 (grammar-rename-nts (grammar-nts G1) G2))
  ```

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  (let* [(G2 (grammar-rename-nts (grammar-nts G1) G2))
         (G1-nts  (grammar-nts G1)) (G2-nts (grammar-nts G2))
         (G1-sigma (grammar-sigma G1)) (G2-sigma (grammar-sigma G2)
         (G1-rules (grammar-rules G1)) (G2-rules (grammar-rules G2)
         (G1-start (grammar-start G1)) (G2-start (grammar-start G2)
```

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;; Tests for cfg-concat
(define a2nb2n-numb>numa (cfg-concat a2nb2n numb>numa))
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```

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         (G1-sigma (grammar-sigma G1)) (G2-sigma (grammar-sigma G2)
         (G1-rules (grammar-rules G1)) (G2-rules (grammar-rules G2)
         (G1-start (grammar-start G1)) (G2-start (grammar-start G2)
         (A (generate-symbol 'S (append G1-nts G2-nts)))]
```

```
;; Tests for cfg-concat
(define a2nb2n-numb>numa (cfg-concat a2nb2n numb>numa))
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```

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Properties of CFLs

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- (G1-nts (grammar-nts G1)) (G2-nts (grammar-nts G2))
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       (G1-rules (grammar-rules G1)) (G2-rules (grammar-rules G2))
       (G1-start (grammar-start G1)) (G2-start (grammar-start G2))
- (A (generate-symbol 'S (append G1-nts G2-nts)))]
- (make-cfg (cons A (append G1-nts G2-nts))
           (remove-duplicates (append G1-sigma G2-sigma))
           (append
             (list (list A ARROW (los->symbol (list G1-start G2-
             G1-rules
             G2-rules)
           A)))
- ;; Tests for cfg-concat
  (define a2nb2n-numb>numa (cfg-concat a2nb2n numb>numa))
  (define MULT3-as-a2nb2n (cfg-concat MULT3-as a2nb2n))

  (check-equal? (grammar-derive a2nb2n-numb>numa '(b a))
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                "(a a a) is not in L(G).")
  (check-equal? (grammar-derive a2nb2n-numb>numa '(a a b b))

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Properties of CFLs

Theorem
Context-free languages are closed under concatenation.

- HOMEWORK: 2 (proof), 3

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Properties of CFLs

- Context-free languages are closed under Kleene star

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Properties of CFLs

- Context-free languages are closed under Kleene star
- Design idea
- Given that L is a context-free language, there exists a G = (make-cfg N Σ R S) such that L = L(G)

Properties of CFLs

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- Given that L is a context-free language, there exists a G = (make-cfg N Σ R S) such that L = L(G)
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Properties of CFLs

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- Given that L is a context-free language, there exists a G = (make-cfg N Σ R S) such that L = L(G)
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- Each word in L is generated by S
- A cfg for L* must generate 0 or more Ss.

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Properties of CFLs

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- Given that L is a context-free language, there exists a G = (make-cfg N Σ R S) such that L = L(G)
- A cfg for L* must generate 0 or more words in L
- Each word in L is generated by S
- A cfg for L* must generate 0 or more Ss.
- A cfg for L* may be constructed using the following components:

$$
\begin{aligned}
V &= \text{(grammar-nts G)} \cup \{A\} \\
\Sigma &= \text{(grammar-sigma G)} \\
R &= \text{(grammar-rules G)} \cup \{\text{(S ARROW EMP) (S ARROW SS}\} \\
S &= A = \text{(generate-symbol 'S (grammar-nts G))}
\end{aligned}
$$

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Properties of CFLs

- QUIZ: 4-5 (due in a week)

Properties of CFLs

- Like regular languages, context-free languages are infinite and exhibit periodicity

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Properties of CFLs

- Like regular languages, context-free languages are infinite and exhibit periodicity
- There are languages that are not context-free
- We shall prove a pumping theorem for context-free languages

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Properties of CFLs

- To simplify our work, bound a parse tree's yield length

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Properties of CFLs

- To simplify our work, bound a parse tree's yield length

- Define κ as the longest righthand side of any rule:

```
;; L = a^nb^n
(define a2nb2n (make-cfg '(S)
                         '(a b)
                         `((S ,ARROW ,EMP)
                           (S ,ARROW aSb))
                         'S))

;; L = w | w in (a b)* AND  w has more b than a
(define numb>numa (make-cfg '(S A)
                            '(a b)
                            `((S ,ARROW b)
                              (S ,ARROW AbA)
                              (A ,ARROW AaAbA)
                              (A ,ARROW AbAaA)
                              (A ,ARROW ,EMP)
                              (A ,ARROW bA))
                            'S))
```

- For a2nb2n, κ is 3 because its longest righthand side is aSb

- For numb>numa, κ is 5.

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- To simplify our work, bound a parse tree's yield length
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  ;; L = w | w in (a b)* AND  w has more b than a
  (define numb>numa (make-cfg '(S A)
                              '(a b)
                              `((S ,ARROW b)
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                                (A ,ARROW AaAbA)
                                (A ,ARROW AbAaA)
                                (A ,ARROW ,EMP)
                                (A ,ARROW bA))
                              'S))
  ```

- For a2nb2n, κ is 3 because its longest righthand side is aSb
- For numb>numa, κ is 5.
- Given a cfg G, the yield's length for any parse tree is bounded by the height of the tree and κ

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Properties of CFLs

Lemma
T's yield's length $\leq \kappa^h$.

- The proof is by induction on h

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Lemma
T's yield's length $\leq \kappa^h$.

- The proof is by induction on h
- Base case: $h = 1$
- A parse tree of height 1 means that a single rule has been used
- The yield's length $\leq \kappa = \kappa^1 = \kappa^h$

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- A parse tree of height 1 means that a single rule has been used
- The yield's length $\leq \kappa = \kappa^1 = \kappa^h$
- Inductive Step
- Assume: T's yield's length $\leq \kappa^n$, for $h = n$
- Show: T's yield's length $\leq \kappa^{n+1}$, for $h = n + 1$

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- Inductive Step
- Assume: T's yield's length $\leq \kappa^n$, for $h = n$
- Show: T's yield's length $\leq \kappa^{n+1}$, for $h = n + 1$
- Consider the structure of a parse tree of height $n + 1$
    - Root
    - At most κ subtrees of height n

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- Assume: T's yield's length $\leq \kappa^n$, for $h = n$
- Show: T's yield's length $\leq \kappa^{n+1}$, for $h = n + 1$
- Consider the structure of a parse tree of height $n + 1$
    - Root
    - At most κ subtrees of height n
- Let T_i be any of these subtrees
- By inductive hypothesis, T_i's yield's length $\leq \kappa^n$
- If every T_i's yield's length is κ^n then T's yield's length is at most $\kappa * \kappa^n$
- Thus, T's yield's length $\leq \kappa^{n+1}$. $\square$

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T's yield's length $\leq \kappa^h$.

- The proof is by induction on h
- Base case: $h = 1$
- A parse tree of height 1 means that a single rule has been used
- The yield's length $\leq \kappa = \kappa^1 = \kappa^h$
- Inductive Step
- Assume: T's yield's length $\leq \kappa^n$, for $h = n$
- Show: T's yield's length $\leq \kappa^{n+1}$, for $h = n + 1$
- Consider the structure of a parse tree of height $n + 1$
    - Root
    - At most κ subtrees of height n
- Let T_i be any of these subtrees
- By inductive hypothesis, T_i's yield's length $\leq \kappa^n$
- If every T_i's yield's length is κ^n then T's yield's length is at most $\kappa * \kappa^n$
- Thus, T's yield's length $\leq \kappa^{n+1}$. $\square$
- Let $w \in L(G)$ such that $|w| > \kappa^h$
- w's parse tree must have a path longer than h

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Theorem

If $w \in L(G) \wedge |w| > \kappa^{|V|}$ then $w = uvxyz$ such that $(v \neq EMP \vee y \neq EMP)$, $\exists k$ $|vxy| \leq k$, and $\forall n \geq 0$ $uv^n xy^n z \in L(G)$.

- Proof

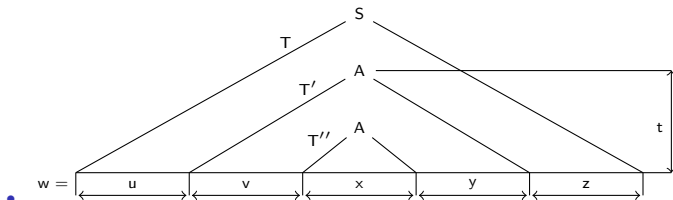Part III:
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- Proof
- Let T be the parse tree for w rooted at S with the smallest number of leaves

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- Proof
- Let T be the parse tree for w rooted at S with the smallest number of leaves
- $|w| > \kappa^{|V|} \Rightarrow$ path in T of length at least $|V| + 1$
- $|V| + 2$ nodes of which the last must be a terminal symbol
- $|V| + 1$ nonterminals
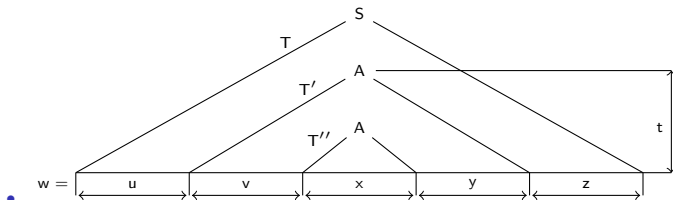- Must have a repeated nonterminal

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- Must have a repeated nonterminal

-
- Observe that the part of T' excluding T'' may be repeated an arbitrary number of times or may be excluded resulting in a word that is in L(G)
- $uv^n xy^n z \in L(G)$ for $n \geq 0$

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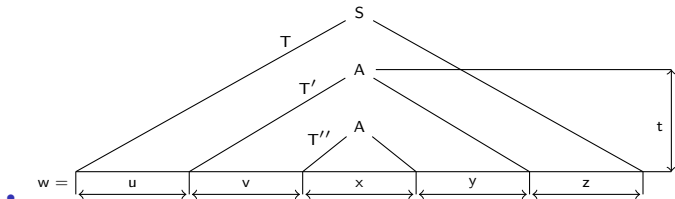- Must have a repeated nonterminal

- Observe that the part of T' excluding T'' may be repeated an arbitrary number of times or may be excluded resulting in a word that is in L(G)
- $uv^n xy^n z \in L(G)$ for $n \geq 0$
- Let t be the height of T'
- The yield of T', vxy, has a length of at most κ^t
- This is the constant k that bounds the length of the subword that contains the parts that may be repeated an arbitrary number of times: $|vxy| \leq \kappa^t = k$

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- The yield of T', vxy, has a length of at most κ^t
- This is the constant k that bounds the length of the subword that contains the parts that may be repeated an arbitrary number of times: $|vxy| \leq \kappa^t$
- Finally, observe that if $vy = EMP$ then there is a parse tree with fewer leaves that generates w:

  $S \rightarrow^* uAz \rightarrow^* uxz = w$
- Smaller parse tree omits the derivation of A from A
- Contradicts our assumption that T is the parse tree for w with the smallest number of leaves
- Therefore, $vy \neq EMP \Rightarrow 0 < |vxy| \leq k < \kappa^{|V|}$ □

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Theorem
$L = \{a^n b^n c^n \mid n \geq 0\}$ is not context-free.

- Let $w = a^k b^k c^k$

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Theorem

$L = \{a^n b^n c^n \mid n \geq 0\}$ is not context-free.

- Let $w = a^k b^k c^k$
- We must determine the possible values vxy may take
- vxy may not contains as, bs, and cs because the length of vxy must be less than or equal to k
- vxy may contain only one letter variety or may contain two letter varieties by straddling the border between as and bs or between bs and cs

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Theorem
$L = \{a^n b^n c^n \mid n \geq 0\}$ *is not context-free.*

- Let $w = a^k b^k c^k$
- We must determine the possible values vxy may take
- vxy may not contains as, bs, and cs because the length of vxy must be less than or equal to k
- vxy may contain only one letter variety or may contain two letter varieties by straddling the border between as and bs or between bs and cs
- We prove that L is not context-free as follows:

vxy	Argument
a^+	Pump up once on v and y. The resulting word has more as than bs and than cs. Therefore, it is not in L.
b^+	Pump up once on v and y. The resulting word has more bs than as and than cs. Therefore, it is not in L.
c^+	Pump up once on v and y. The resulting word has more cs than as and than bs. Therefore, it is not in L.
a^+b^+	Pump up once on v and y. The resulting word has a b before an a. Therefore, it is not in L.
b^+c^+	Pump up once on v and y. The resulting word has a c before a b. Therefore, it is not in L.

- No matter where the vxy window is placed in w, a word that is not in w may be generated
- We may conclude that L is not context-free. □

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Theorem

$L = \{a^n \mid n \text{ is a square}\}$ *is not context-free.*

- Let $w = a^{k^2}$

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Theorem

$L = \{a^n \mid n \text{ is a square}\}$ is not context-free.

- Let $w = a^{k^2}$
- $vy = a^i$, where $i > 0$

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Theorem
$L = \{a^n \mid n \text{ is a square}\}$ is not context-free.

- Let $w = a^{k^2}$
- $vy = a^i$, where $i > 0$
- If we pump up once the resulting word is a^{K^2+i}

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- Let $w = a^{k^2}$

- $vy = a^i$, where $i > 0$

- If we pump up once the resulting word is $a^{K^2 + i}$

- Observe that there are no words in L with a length greater than $|a^{K^2}|$ and less than $|a^{(k+1)^2}|$

- After a^{k^2} the next word in L has length $K^2 + 2K + 1$

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- After a^{k^2} the next word in L has length $K^2 + 2K + 1$

- In order for a^{k^2+i} to be in L, i must at a minimum be of length $2k + 1$

- Impossible because $|vxy|$ must be less than or equal to k

- i cannot be greater than or equal to $2k + 1$

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- Let $w = a^{k^2}$
- $vy = a^i$, where $i > 0$
- If we pump up once the resulting word is a^{K^2+i}
- Observe that there are no words in L with a length greater than $|a^{K^2}|$ and less than $|a^{(k+1)^2}|$
- After a^{k^2} the next word in L has length $K^2 + 2K + 1$
- In order for a^{k^2+i} to be in L, i must at a minimum be of length $2k + 1$
- Impossible because |vxy| must be less than or equal to k
- i cannot be greater than or equal to $2k + 1$
- No matter where the vxy window is placed in w, a word that is not in w may be generated
- Therefore, we may conclude that L is not context-free $\quad\square$

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Theorem
Context-free languages are not closed under intersection.

- Proof

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Theorem
Context-free languages are not closed under intersection.

- Proof
- Let $L_1 = \{a^n b^n c^m \mid m,n \geq 0\}$
- $L_2 = \{a^m b^n c^n \mid m,n \geq 0\}$
- Both are context-free

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- Both are context-free
- Consider $L = L_1 \cap L_2$

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Context-free languages are not closed under intersection.

- Proof
- Let $L_1 = \{a^n b^n c^m \mid m,n \geq 0\}$
- $L_2 = \{a^m b^n c^n \mid m,n \geq 0\}$
- Both are context-free
- Consider $L = L_1 \cap L_2$
- $L = \{a^n b^n c^n \mid n \geq 0\}$, which we know is not context-free
- Therefore, context-free languages are not closed under intersection.

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Theorem
Context-free languages are not closed under complement.

- Proof

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Theorem
Context-free languages are not closed under complement.

- Proof
- $L_1 \cap L_2 = $ (complement ((complement L_1) $\cup$ (complement L_2)))
- If context-free languages were closed under complement then they would also be closed under intersection, which we know they are not
- Therefore, context free languages are not closed under complement

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Properties of CFLs

- Although context-free languages are not closed under intersection, there exist context-free languages that result in a context-free language when intersected

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Properties of CFLs

- Although context-free languages are not closed under intersection, there exist context-free languages that result in a context-free language when intersected

- Recall that all regular languages are context-free

- The intersection of a context-free language and a regular language is a context-free language

- How can we prove this?

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Properties of CFLs

- Although context-free languages are not closed under intersection, there exist context-free languages that result in a context-free language when intersected
- Recall that all regular languages are context-free
- The intersection of a context-free language and a regular language is a context-free language
- How can we prove this?
- Develop a new constructor for a context-free language: constructive proof

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- Design Idea

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- Design Idea
- Have a pda: M_1 = (make-ndpda K_1 Σ_1 Γ S_1 F_1 R_1)
- Have a dfa: M_2 = (make-dfa K_2 Σ_2 S_2 F_2 R_2)

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- Build a pda that simultaneously simulates M_1 and M_2
- Only accept if both M_1 and M_2 accept.

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- Build a pda that simultaneously simulates M_1 and M_2
- Only accept if both M_1 and M_2 accept.
- Basic idea: Each state of the new pda be a *super state* that represents a state in M_1 and a state in M_2
- Simulate all the transitions of M_1 and track the transitions made by M_2
- If M_1 has a rule that moves from P to Q on an a then for every state Y$\in$$M_2$ the new pda has a rule that moves from super state (P Y) to super state (Q T)
- If M_1 has a rule that moves from P to Q on EMP then for every state Y$\in$$M_2$ the new pda has a rule that moves from super state (P Y) to super state (Q Y) to simulate that M_2 does not change state

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- If M_1 has a rule that moves from P to Q on EMP then for every state Y∈M_2 the new pda has a rule that moves from super state (P Y) to super state (Q Y) to simulate that M_2 does not change state
-
  ```
  ;; Push or pop elements, stacke, is either
  ;;  1. EMP
  ;;  2. (listof symbol)

  ;; A super state, ss, is a (list state state)

  ;; A pda super rule, ssrule, is:
  ;; (list (list ss symbol stacke) (list ss stacke))
  ```

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- ;; pda dfa → pda Purpose: Build intersection pda from given machines
  (define (pda-intersect-dfa a-pda a-dfa)

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Properties of CFLs

- ```
;; pda dfa → pda Purpose: Build intersection pda from given machines
(define (pda-intersect-dfa a-pda a-dfa)
```

- ```
(let* [(pda-sts (sm-states a-pda))  (dfa-sts (sm-states a-dfa))
       (pda-rls (sm-rules a-pda))   (dfa-rls (sm-rules a-dfa))
```

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       (pda-rls (sm-rules a-pda))   (dfa-rls (sm-rules a-dfa))
```
- ```
       (K (cartesian-product pda-sts dfa-sts))
       (start (list (sm-start a-pda) (sm-start a-dfa)))
       (F (cartesian-product (sm-finals a-pda) (sm-finals a-dfa)))
```

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# Properties of CFLs

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(define (pda-intersect-dfa a-pda a-dfa)
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       (pda-rls (sm-rules a-pda))   (dfa-rls (sm-rules a-dfa))
```
- ```
     (K (cartesian-product pda-sts dfa-sts))
     (start (list (sm-start a-pda) (sm-start a-dfa)))
     (F (cartesian-product (sm-finals a-pda) (sm-finals a-dfa)))
```
- ```
     (non-EMP-pda-rls (filter
                        (λ (r) (not (eq? (second (first r)) EMP))) pda-rls))
     (EMP-pda-rls (filter (λ (r) (eq? (second (first r)) EMP)) pda-rls))
     (non-EMP-rls (make-nonEMP-rls non-EMP-pda-rls dfa-sts dfa-rls))
     (EMP-rls (make-EMP-rls EMP-pda-rls dfa-sts))
```

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(let* [(pda-sts (sm-states a-pda)) (dfa-sts (sm-states a-dfa))
       (pda-rls (sm-rules a-pda))   (dfa-rls (sm-rules a-dfa))
```
- ```
       (K (cartesian-product pda-sts dfa-sts))
       (start (list (sm-start a-pda) (sm-start a-dfa)))
       (F (cartesian-product (sm-finals a-pda) (sm-finals a-dfa)))
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       (non-EMP-pda-rls (filter
                         (λ (r) (not (eq? (second (first r)) EMP))) pda-rls))
       (EMP-pda-rls (filter (λ (r) (eq? (second (first r)) EMP)) pda-rls))
       (non-EMP-rls (make-nonEMP-rls non-EMP-pda-rls dfa-sts dfa-rls))
       (EMP-rls (make-EMP-rls EMP-pda-rls dfa-sts))
```
- ```
       (ss-tbl (make-ss-table K))
       (ss->state (λ (ss) (second (assoc ss ss-tbl))))]
```

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- ```
;; pda dfa → pda Purpose: Build intersection pda from given machines
(define (pda-intersect-dfa a-pda a-dfa)
```

- ```
(let* [(pda-sts (sm-states a-pda))  (dfa-sts (sm-states a-dfa))
       (pda-rls (sm-rules a-pda))   (dfa-rls (sm-rules a-dfa))
```
- ```
(K (cartesian-product pda-sts dfa-sts))
(start (list (sm-start a-pda) (sm-start a-dfa)))
(F (cartesian-product (sm-finals a-pda) (sm-finals a-dfa)))
```
- ```
(non-EMP-pda-rls (filter
                  (λ (r) (not (eq? (second (first r)) EMP))) pda-rls))
(EMP-pda-rls (filter (λ (r) (eq? (second (first r)) EMP)) pda-rls))
(non-EMP-rls (make-nonEMP-rls non-EMP-pda-rls dfa-sts dfa-rls))
(EMP-rls (make-EMP-rls EMP-pda-rls dfa-sts))
```
- ```
(ss-tbl (make-ss-table K))
(ss->state (λ (ss) (second (assoc ss ss-tbl))))]
```
- ```
(make-ndpda (map ss->state K)  (sm-sigma a-pda)  (sm-gamma a-pda)
            (ss->state start)  (map ss->state F)
            (convert-ssrules (append non-EMP-rls EMP-rls) ss->state))))
```

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- ;; pda dfa → pda Purpose: Build intersection pda from given machines
  (define (pda-intersect-dfa a-pda a-dfa)
- ;; (listof state) (listof state) → (listof ss)
  ;; Purpose: Create a list of super states
  (define (cartesian-product pda-sts dfa-sts) ...)

  ;; (listof pda-rule) (listof state) → (listof ssrule)
  ;; Purpose: Create ssrules for given empty transition pda-rules
  (define (make-EMP-rls EMP-pda-rls dfa-sts) ...)
  ;; (listof pda-rule) (listof state) (listof dfa-rls) → (listof ssrule)
  ;; Purpose: Create ssrules for given nonempty transition pda-rules
  (define (make-nonEMP-rls non-EMP-pda-rls dfa-sts dfa-rls) ...)

  ;; (listof ss) → (listof (list ss state))
  ;; Purpose: Create table associating given super states with a new state
  (define (make-ss-table K) ...)
  ;; (listof ssrule) (ss → state) → (listof pda-rule)
  ;; Purpose: Convert ssrules to pda-rules
  (define (convert-ssrules ss-rls ss->state) ...)
- (let* [(pda-sts (sm-states a-pda)) (dfa-sts (sm-states a-dfa))
         (pda-rls (sm-rules a-pda)) (dfa-rls (sm-rules a-dfa))
- (K (cartesian-product pda-sts dfa-sts))
         (start (list (sm-start a-pda) (sm-start a-dfa)))
         (F (cartesian-product (sm-finals a-pda) (sm-finals a-dfa)))
- (non-EMP-pda-rls (filter
                            (λ (r) (not (eq? (second (first r)) EMP))) pda-rls))
         (EMP-pda-rls (filter (λ (r) (eq? (second (first r)) EMP)) pda-rls))
         (non-EMP-rls (make-nonEMP-rls non-EMP-pda-rls dfa-sts dfa-rls))
         (EMP-rls (make-EMP-rls EMP-pda-rls dfa-sts))
- (ss-tbl (make-ss-table K))
         (ss->state (λ (ss) (second (assoc ss ss-tbl))))]
- (make-ndpda (map ss->state K) (sm-sigma a-pda) (sm-gamma a-pda)
               (ss->state start) (map ss->state F)
               (convert-ssrules (append non-EMP-rls EMP-rls) ss->state))))

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-
```
;; (listof state) (listof state) → (listof ss)
;; Purpose: Create a list of super states
(define (cartesian-product pda-sts dfa-sts)
  (for*/list [(s1 pda-sts) (s2 dfa-sts)] (list s1 s2)))
```

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- ```
  ;; (listof state) (listof state) → (listof ss)
  ;; Purpose: Create a list of super states
  (define (cartesian-product pda-sts dfa-sts)
    (for*/list [(s1 pda-sts) (s2 dfa-sts)] (list s1 s2)))
  ```

- ```
  (define (make-EMP-rls EMP-pda-rls dfa-sts)
    (for*/list [(r EMP-pda-rls) (s dfa-sts)]
      (list (list (list (first (first r)) s)
                  EMP
                  (third (first r)))
            (list (list (first (second r)) s)
                  (second (second r))))))
  ```

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- ```
;; (listof state) (listof state) → (listof ss)
;; Purpose: Create a list of super states
(define (cartesian-product pda-sts dfa-sts)
  (for*/list [(s1 pda-sts) (s2 dfa-sts)] (list s1 s2)))
```

- ```
(define (make-EMP-rls EMP-pda-rls dfa-sts)
  (for*/list [(r EMP-pda-rls) (s dfa-sts)]
    (list (list (list (first (first r)) s)
                EMP
                (third (first r)))
          (list (list (first (second r)) s)
                (second (second r))))))
```

- ```
(define (make-nonEMP-rls non-EMP-pda-rls dfa-sts dfa-rls)
  (for*/list [(r non-EMP-pda-rls) (s dfa-sts)]
    (list (list
            (list (first (first r)) s)
            (second (first r))
            (third (first r)))
          (list
            (list
              (first (second r))
              (third
              (first
                (filter (λ (rl)
                          (and (eq? (first rl) s)
                               (eq? (second rl)
                                    (second (first r)))))
                        dfa-rls))))
            (second (second r))))))
```

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# Properties of CFLs

- 
```
(define (make-ss-table K)
 (for*/list [(ss K)] (list ss (generate-symbol 'T '(T)))))

(define (convert-ssrules ss-rls ss->state)
 (for*/list [(r ss-rls)]
  (list
   (list (ss->state (first (first r)))
         (second (first r))
         (third (first r)))
   (list (ss->state (first (second r)))
         (second (second r))))))
```

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# Properties of CFLs

- ```
(define (make-ss-table K)
 (for*/list [(ss K)] (list ss (generate-symbol 'T '(T)))))

(define (convert-ssrules ss-rls ss->state)
 (for*/list [(r ss-rls)]
  (list
   (list (ss->state (first (first r)))
         (second (first r))
         (third (first r)))
   (list (ss->state (first (second r)))
         (second (second r)))))))
```

- ```
;; Tests for pda-intersect-dfa
(define a^nb^nIM (pda-intersect-dfa a^nb^n M))
(define a^nb^nINO-ABBA (pda-intersect-dfa a^nb^n NO-ABAA))

(check-equal? (sm-apply a^nb^nIM '()) 'reject)
(check-equal? (sm-apply a^nb^nIM '(a b b)) 'reject)
(check-equal? (sm-apply a^nb^nIM '(a a b b)) 'reject)
(check-equal? (sm-apply a^nb^nIM '(a b)) 'accept)

(check-equal? (sm-apply a^nb^nINO-ABBA '(b b)) 'reject)
(check-equal? (sm-apply a^nb^nINO-ABBA '(a a b)) 'reject)
(check-equal? (sm-apply a^nb^nINO-ABBA '()) 'accept)
(check-equal? (sm-apply a^nb^nINO-ABBA '(a a b b)) 'accept)
```

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# Properties of CFLs

## Theorem
*The intersection of a context-free language and a regular language is context-free.*

- Proof (sketch)

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# Properties of CFLs

## Theorem
*The intersection of a context-free language and a regular language is context-free.*

- Proof (sketch)
- Let Q be a pda, let D be a dfa, and let M = (pda-intersect-dfa Q D)

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# Properties of CFLs

## Theorem
*The intersection of a context-free language and a regular language is context-free.*

- Proof (sketch)
- Let Q be a pda, let D be a dfa, and let M = (pda-intersect-dfa Q D)
- By construction, M simulates the transitions of both Q and D
- M starts its simulations in a state that represents Q's and D's starting states
- Only accepts if both Q and D accept
- This means that all accepted strings must be in L(Q) and L(D)

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# Properties of CFLs

## Theorem
*The intersection of a context-free language and a regular language is context-free.*

- Proof (sketch)
- Let Q be a pda, let D be a dfa, and let M = (pda-intersect-dfa Q D)
- By construction, M simulates the transitions of both Q and D
- M starts its simulations in a state that represents Q's and D's starting states
- Only accepts if both Q and D accept
- This means that all accepted strings must be in L(Q) and L(D)
- $L(M) = L(Q) \cap L(D)$.   □

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# Properties of CFLs

## Theorem

$L = \{\ w \in \{a\ b\}^*|\ w$ does not contain abaa and has equal number of $a$s and $b$s.$\}$
is context-free.

- The previous theorem may be used to prove a language, L, is context-free

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# Properties of CFLs

## Theorem

$L = \{ w \in \{a\ b\}^* |\ w$ does not contain abaa and has equal number of as and bs. $\}$ is context-free.

- The previous theorem may be used to prove a language, L, is context-free
- The language of all words that do not contain abaa is decided by NO-ABAA (from dfa chapter)
- L(NO-ABAA) is regular

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# Properties of CFLs

## Theorem

$L = \{ \; w \in \{a \; b\}^* | \; w \; does \; not \; contain \; abaa \; and \; has \; equal \; number \; of \; as \; and \; bs. \}$
*is context-free.*

- The previous theorem may be used to prove a language, L, is context-free

- The language of all words that do not contain abaa is decided by
  NO-ABAA (from dfa chapter)

- L(NO-ABAA) is regular

- The language, E, of all words that have an equal number of as and bs is
  generated by the following context-free grammar:

     S → EMP
       → aSb
       → bSa
       → aSbSbSa
       → bSaSaSb

- E is context-free

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# Properties of CFLs

## Theorem

$L = \{ w \in \{a\ b\}^* |\ w$ does not contain abaa and has equal number of as and bs.$\}$ is context-free.

- The previous theorem may be used to prove a language, L, is context-free
- The language of all words that do not contain abaa is decided by NO-ABAA (from dfa chapter)
- L(NO-ABAA) is regular
- The language, E, of all words that have an equal number of as and bs is generated by the following context-free grammar:

    S → EMP
      → aSb
      → bSa
      → aSbSbSa
      → bSaSaSb

- E is context-free
- Observe that $L = L(E) \cap L(\text{NO-ABAA})$
- By previous theorem, we may conclude that L is context-free.    □

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# Properties of CFLs

- HOMEWORK: 12–14

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# Deterministic pdas

- Now, we explore deterministic pdas in more detail

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# Deterministic pdas

- Now, we explore deterministic pdas in more detail
- From a given configuration, $C_i$, there is at most one transition that is applicable
- The pda never has a choice of what to do next

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# Deterministic pdas

- Now, we explore deterministic pdas in more detail
- From a given configuration, $C_i$, there is at most one transition that is applicable
- The pda never has a choice of what to do next
- Such a pda must satisfy the following constraints:
  1. There are no pair of transitions that offer a choice on which to use
  2. For all accepting states, Q, there are no transitions of this nature: ((Q EMP EMP) (P a)). That is, the machine does not have to choose between continuing with the computation or accepting.

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# Deterministic pdas

- Now, we explore deterministic pdas in more detail
- From a given configuration, $C_i$, there is at most one transition that is applicable
- The pda never has a choice of what to do next
- Such a pda must satisfy the following constraints:
  1. There are no pair of transitions that offer a choice on which to use
  2. For all accepting states, Q, there are no transitions of this nature: ((Q EMP EMP) (P a)). That is, the machine does not have to choose between continuing with the computation or accepting.
- Given a nondeterministic pda, can a deterministic pda that decides the same language be constructed?

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# Deterministic pdas

$L = wcw^R$

- The nondeterministic pda's we previously designed may be summarized as follows:

    1. Nondeterministically move from S, the starting state, to P
    2. In P, accumulate w on the stack in reversed order and move to Q upon reading a c without modifying the stack.
    3. In Q, match the elements of $w^R$ with the elements on the stack and nondeterministically move to, F, the final state.

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# Deterministic pdas

$L = wcw^R$

- The nondeterministic pda's we previously designed may be summarized as follows:

  ❶ Nondeterministically move from S, the starting state, to P
  ❷ In P, accumulate w on the stack in reversed order and move to Q upon reading a c without modifying the stack.
  ❸ In Q, match the elements of $w^R$ with the elements on the stack and nondeterministically move to, F, the final state.

- A deterministic pda for L must avoid nondeterministic transitions

- Observe that the nondeterministic pda operates in two phases: before reading c, it pushes w onto the stack and after reading c, it matches $w^R$ with the elements on the stack

- The phases are clearly delineated around reading c

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# Deterministic pdas

$L = wcw^R$

- The nondeterministic pda's we previously designed may be summarized as follows:

  1. Nondeterministically move from S, the starting state, to P
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  3. In Q, match the elements of $w^R$ with the elements on the stack and nondeterministically move to, F, the final state.

- A deterministic pda for L must avoid nondeterministic transitions

- Observe that the nondeterministic pda operates in two phases: before reading c, it pushes w onto the stack and after reading c, it matches $w^R$ with the elements on the stack

- The phases are clearly delineated around reading c

- A deterministic pda may decide L as follows:

  1. Before reading c, push read elements onto the stack
  2. After reading c, match read elements with elements onto the stack

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# Deterministic pdas

L = wcw$^R$

- Name is dwcw^r, input alphabet is {a b c}, and the stack alphabet is {a b}.

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# Deterministic pdas

$L = wcw^R$

- Name is dwcw^r, input alphabet is {a b c}, and the stack alphabet is {a b}.

- Sample tests:

```
(check-equal? (sm-apply dwcw^r '())) 'reject)
(check-equal? (sm-apply dwcw^r '(a b b a)) 'reject)
(check-equal? (sm-apply dwcw^r '(a a)) 'reject)
(check-equal? (sm-apply dwcw^r '(a b b c b b a b))
              'reject)
(check-equal? (sm-apply dwcw^r '(a c a)) 'accept)
(check-equal? (sm-apply dwcw^r '(b a c a b)) 'accept)
(check-equal? (sm-apply dwcw^r '(b b a c a b b)) 'accept)
```

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Deterministic pdas

$L = wcw^R$

- S: $ci = \{a\ b\}^*$ AND stack = ci^R, starting state

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$L = wcw^R$

- S: $c_i = \{a\ b\}^*$ AND stack = c_i^R, starting state
- F: $c_i = xycy^R$ AND stack = x^R, final state

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$L = wcw^R$

- S: $c_i = \{a\ b\}^*$ AND stack = c_i^R, starting state
- F: $c_i = xycy^R$ AND stack = x^R, final state
- ((S a ,EMP) (S (a)))   ((S b ,EMP) (S (b)))   ((S c ,EMP) (F ,EMP))

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$L = wcw^R$

- S: $c_i = \{a\ b\}^*$ AND stack = c_i^R, starting state
- F: $c_i = xycy^R$ AND stack = x^R, final state
- ((S a ,EMP) (S (a)))   ((S b ,EMP) (S (b)))   ((S c ,EMP) (F ,EMP))
- ((F a (a)) (F ,EMP))   ((F b (b)) (F ,EMP))
- There are no nondeterministic choices that the machine must make

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$L = wcw^R$

- S: $ci = \{a\ b\}^*$ AND stack $= ci^R$, starting state
- F: $ci = xycy^R$ AND stack $= x^R$, final state
- ((S a ,EMP) (S (a)))   ((S b ,EMP) (S (b)))   ((S c ,EMP) (F ,EMP))
- ((F a (a)) (F ,EMP))   ((F b (b)) (F ,EMP))
- There are no nondeterministic choices that the machine must make
- ```
;; word stack → Boolean
(define (S-INV ci s)
  (and (andmap (λ (s) (or (eq? s 'a) (eq? s 'b))) ci)
       (equal? ci (reverse s))))

(check-equal? (S-INV '(c) '()) #f)
(check-equal? (S-INV '(b a) '(b a)) #f)
(check-equal? (S-INV '() '()) #t)
(check-equal? (S-INV '(a b b) '(b b a)) #t)
```

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$L = wcw^R$

- S: $ci = \{a\ b\}^*$ AND stack = ci^R, starting state
- F: $ci = xycy^R$ AND stack = x^R, final state
- ((S a ,EMP) (S (a)))   ((S b ,EMP) (S (b)))   ((S c ,EMP) (F ,EMP))
- ((F a (a)) (F ,EMP))   ((F b (b)) (F ,EMP))
- There are no nondeterministic choices that the machine must make
-     ;; word stack → Boolean
      (define (S-INV ci s)
        (and (andmap (λ (s) (or (eq? s 'a) (eq? s 'b))) ci)
             (equal? ci (reverse s))))

      (check-equal? (S-INV '(c) '()) #f)
      (check-equal? (S-INV '(b a) '(b a)) #f)
      (check-equal? (S-INV '() '()) #t)
      (check-equal? (S-INV '(a b b) '(b b a)) #t)
-     ;; word stack → Boolean
      (define (F-INV ci s)
        (and (member 'c ci)
             (let* [(w (takef ci (λ (x) (not (eq? x 'c)))))
                    (x (take ci (length s)))
                    (y^R (drop ci (add1 (length w))))]
               (and (equal?
                      ci
                      (append x (reverse y^R) (list 'c) y^R))
                    (equal? s (reverse x)))))))

      (check-equal? (F-INV '(a b c) '()) #f)
      (check-equal? (F-INV '(a a c) '(a)) #f)
      (check-equal? (F-INV '(a b) '(b a)) #f)
      (check-equal? (F-INV '(c) '()) #t)
      (check-equal? (F-INV '(b a c) '(a b)) #t)
      (check-equal? (F-INV '(a b b c b) '(b a)) #t)

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```
L = wcw^R        M = dwcw^r           v ∈ (sm-sigma M)*
s = the stack    ci = consumed input
```

Theorem
The state invariants hold when M accepts v.

- For the base case, when M starts $c_i =$ '() and $s =$ '(). This means that c_i only contains as and bs and that c_i equals s reversed. Thus, S-INV holds.

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- Proof invariants hold after each transition that consumes input:

- <u>((S a ,EMP) (S (a)))</u>: By inductive hypothesis S-INV holds. This means that $c_i \in \{a\ b\}^*$ and that $c_i = s^R$. After using this rule, the consumed input only contains as and bs and the consumed input is equal to the stack reversed. Thus, S-INV holds.

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- <u>((S b ,EMP) (S (b)))</u>: By inductive hypothesis S-INV holds. This means that $c_i \in \{a\ b\}^*$ and that $c_i = s^R$. After using this rule, the consumed input only contains as and bs and the consumed input is equal to the stack reversed. Thus, S-INV holds.

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- `((S c ,EMP) (F ,EMP))`: By inductive hypothesis S-INV holds. This means that $c_i \in \{a\ b\}^*$ and that $c_i = s^R$. After using this rule, $c_i = xycy^R$, where $x \in \{a\ b\}^*$ and $y = $ '(), and stack$=x^R$. Thus, F-INV holds.

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- ((F a (a)) (F ,EMP)): By inductive hypothesis F-INV holds. This means that $c_i = xycy^R$, where $x, y \in \{a\ b\}^*$, and stack=x^R. Using this rule means that x's last element is an a. That is, x=x'a. After using this rule, c_i=x'aycy^{R}a=x'ayc(ay)^R and stack=x'^{R}. Thus, F-INV holds.

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L = wcw^R

Lemma
$w \in L \Leftrightarrow w \in L(M)$

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Lemma
$w \in L \Leftrightarrow w \in L(M)$

•

Proof.

($\Rightarrow$) Assume $w \in L$. This means that $w = vcv^R$. Given that state invariants always hold, there is a
computation that has M consume w and reach F with an empty stack. Therefore, $w \in L(M)$.
($\Leftarrow$) Assume $w \in L(M)$. This means that M halts in F, the only final state, with an empty stack having
consumed w. Given that the state invariants always hold we may conclude that $w = vcv^R$. Therefore, $w \in L$.
□ □

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□   □

•

Lemma
$w \notin L \Leftrightarrow w \notin L(M)$

•

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□   □
   •

Lemma
$w \notin L \Leftrightarrow w \notin L(M)$
   •

Proof.

($\Rightarrow$) Assume $w \notin L$. This means $w \neq v^R cv$. Given that the state invariant predicates always hold, there is no computation that has M consume w and end in F with an empty stack. Therefore, $w \notin L(M)$.
($\Leftarrow$) Assume $w \notin L(M)$. This means that M does not halt in F with an empty stack having consumed w. Given that the state invariants always hold, this means that $w \neq vcv^R$. Thus, $w \notin L$.   □   □
   •

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L = $\{a^m b^n c^p |\ m \neq n\ \wedge\ m,n,p > 0\}$

- Intuitively: push all the as onto the stack, match bs, either pop excess as or read excess bs, and read cs
- Challenge: deterministically determine if excess as need to be popped or excess bs need to be read

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- This is a problem because the machine cannot nondeterministically decide to keep matching as on the stack or read excess bs

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- If the machine could sense that the stack is empty then the decision becomes deterministic
- If the stack is empty read excess bs
- Otherwise, match as as bs are read

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- Otherwise, match as as bs are read
- Another potential problem is deterministically transitioning to the final state
- To prevent nondeterministic transitions of this nature, a special end of input symbol, z, not in the input alphabet of the machine, shall be used
- All input: wz

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- How can the machine sense that the stack is empty?
- Use a technique developed to construct simple pdas: the machine shall use a bottom of the stack symbol
- For our purposes, y serves as the bottom of the stack symbol.

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- All input: wz
- How can the machine sense that the stack is empty?
- Use a technique developed to construct simple pdas: the machine shall use a bottom of the stack symbol
- For our purposes, y serves as the bottom of the stack symbol.

- The machine's phases may be outlined as follows:

  1. When started, if an a is read then (a y) is pushed onto the stack.
  2. All the remaining as read are pushed onto the stack.
  3. Match a read b by popping an a
  4. Decide if there are more as or more bs:
     1. If a c is read and there is an a on the stack then there are more as than bs. The machine transitions to a state to read the remaining cs and pop the remaining as. Upon reading z, the machine pops y and moves to the final state.
     2. If a b is read and y is at the top of the stack then there are more bs than as. The machine transitions to read the bs with changing the stack. It then transitions to read the cs without altering the stack. Upon reading z, the machine pops y and moves to the final state

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- ```
  ;; State Documentation
  ;; S: ci = stack = '(), starting state
  ;; A: ci = a^+ and stack = a^+y
  ;; B: ci = a^ib^j and stack = a^(i-j)y and 0 < i <= j
  ;; C: ci = a^ib^jc^k and stack = a^(i-j-k)y and i > j and i,j,k > 0
  ;; D: ci = a^ib^j and stack = y and j > i and i,j > 0
  ;; E: ci = a^ib^jc^+ and stack = y, j != i and i,j > 0
  ;; G: ci = a^ib^jc^+z and stack = a*y and i != j and i,j > 0
  ;; F: ci = a^ib^jc^+z and stack = '() and i != j, and i,j > 0, final state
  ```

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L = {$a^m b^n c^p$| $m \neq n \wedge m,n,p>0$}

- #lang fsm

```
(define ambncp (make-ndpda '(S A B C D E F G)
                           '(a b c z)
                           '(a y)
                           'S
                           '(F)
```

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                             '(a b c z)
                             '(a y)
                             'S
                             '(F)
  ```

- ```
  (check-equal? (sm-apply ambncp '(z)) 'reject)
  (check-equal? (sm-apply ambncp '(a b b c c)) 'reject)
  (check-equal? (sm-apply ambncp '(a a b b c c z)) 'reject)
  (check-equal? (sm-apply ambncp '(a b a z)) 'reject)
  (check-equal? (sm-apply ambncp '(a b b z)) 'reject)
  (check-equal? (sm-apply ambncp '(a a a b b c c z)) 'accept)
  (check-equal? (sm-apply ambncp '(a a a b c z)) 'accept)
  (check-equal? (sm-apply ambncp '(a a a b b b c c c z)) 'accept)
  ```

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                           '(a b c z)
                           '(a y)
                           'S
                           '(F)
```
- `(((S a ,EMP) (A (a y)))

- (check-equal? (sm-apply ambncp '(z)) 'reject)
  (check-equal? (sm-apply ambncp '(a b b c c)) 'reject)
  (check-equal? (sm-apply ambncp '(a a b b c c z)) 'reject)
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  (define ambncp (make-ndpda '(S A B C D E F G)
                             '(a b c z)
                             '(a y)
                             'S
                             '(F)
  ```
-
-
  ```
                      `(((S a ,EMP) (A (a y)))
                        ((A a ,EMP) (A (a)))
                        ((A b (a)) (B ,EMP))
  ```

-
  ```
  (check-equal? (sm-apply ambncp '(z)) 'reject)
  (check-equal? (sm-apply ambncp '(a b b c c)) 'reject)
  (check-equal? (sm-apply ambncp '(a a b b c c z)) 'reject)
  (check-equal? (sm-apply ambncp '(a b a z)) 'reject)
  (check-equal? (sm-apply ambncp '(a b b z)) 'reject)
  (check-equal? (sm-apply ambncp '(a a a b b c c z)) 'accept)
  (check-equal? (sm-apply ambncp '(a a a b c z)) 'accept)
  (check-equal? (sm-apply ambncp '(a a a b b b c c c z)) 'accept)
  ```

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```
L = {aᵐbⁿcᵖ| m≠n ∧ m,n,p>0}
```

- #lang fsm

```
(define ambncp (make-ndpda '(S A B C D E F G)
                            '(a b c z)
                            '(a y)
                            'S
                            '(F)
```
- `(((S a ,EMP) (A (a y)))
- ((A a ,EMP) (A (a)))
  ((A b (a)) (B ,EMP))
- ((B b (a)) (B ,EMP))
  ((B c (a)) (C ,EMP))

- ((B b (y)) (D (y)))

- ```
(check-equal? (sm-apply ambncp '(z)) 'reject)
(check-equal? (sm-apply ambncp '(a b b c c)) 'reject)
(check-equal? (sm-apply ambncp '(a a b b c c z)) 'reject)
(check-equal? (sm-apply ambncp '(a b a z)) 'reject)
(check-equal? (sm-apply ambncp '(a b b z)) 'reject)
(check-equal? (sm-apply ambncp '(a a a b b c c z)) 'accept)
(check-equal? (sm-apply ambncp '(a a a b c z)) 'accept)
(check-equal? (sm-apply ambncp '(a a a b b b b c c c c z)) 'accept)
```

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L = {a^m b^n c^p| m≠n ∧ m,n,p>0}

- #lang fsm

  ```
  (define ambncp (make-ndpda '(S A B C D E F G)
                             '(a b c z)
                             '(a y)
                             'S
                             '(F)
  ```
- `` `(((S a ,EMP) (A (a y))) ``
- ```
  ((A a ,EMP) (A (a)))
  ((A b (a)) (B ,EMP))
  ```
- ```
  ((B b (a)) (B ,EMP))
  ((B c (a)) (C ,EMP))
  ```
- ```
  ((C c (a)) (C ,EMP))
  ((C z ,EMP) (G ,EMP))
  ((C c (y)) (E (y)))
  ```
- ```
  ((B b (y)) (D (y)))
  ```

- ```
  (check-equal? (sm-apply ambncp '(z)) 'reject)
  (check-equal? (sm-apply ambncp '(a b b c c)) 'reject)
  (check-equal? (sm-apply ambncp '(a a b b c c z)) 'reject)
  (check-equal? (sm-apply ambncp '(a b a z)) 'reject)
  (check-equal? (sm-apply ambncp '(a b b z)) 'reject)
  (check-equal? (sm-apply ambncp '(a a a b b c c z)) 'accept)
  (check-equal? (sm-apply ambncp '(a a a b c z)) 'accept)
  (check-equal? (sm-apply ambncp '(a a a b b b c c c z)) 'accept)
  ```

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```
L = {aᵐbⁿcᵖ| m≠n ∧ m,n,p>0}
```

- `#lang fsm`

```
(define ambncp (make-ndpda '(S A B C D E F G)
                            '(a b c z)
                            '(a y)
                            'S
                            '(F)
```
- `` `(((S a ,EMP) (A (a y))) ``
- `  ((A a ,EMP) (A (a)))`
  `  ((A b (a)) (B ,EMP))`
- `  ((B b (a)) (B ,EMP))`
  `  ((B c (a)) (C ,EMP))`
- `  ((C c (a)) (C ,EMP))`
  `  ((C z ,EMP) (G ,EMP))`
  `  ((C c (y)) (E (y)))`
- `  ((B b (y)) (D (y)))`
- `  ((D b (y)) (D (y)))`
  `  ((D c (y)) (E (y)))`

- ```
  (check-equal? (sm-apply ambncp '(z)) 'reject)
  (check-equal? (sm-apply ambncp '(a b b c c)) 'reject)
  (check-equal? (sm-apply ambncp '(a a b b c c z)) 'reject)
  (check-equal? (sm-apply ambncp '(a b a z)) 'reject)
  (check-equal? (sm-apply ambncp '(a b b z)) 'reject)
  (check-equal? (sm-apply ambncp '(a a a b b c c z)) 'accept)
  (check-equal? (sm-apply ambncp '(a a a b c z)) 'accept)
  (check-equal? (sm-apply ambncp '(a a a b b b b c c c c z)) 'accept)
  ```

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```
L = {aᵐbⁿcᵖ| m≠n ∧ m,n,p>0}
```

- #lang fsm

```
(define ambncp (make-ndpda '(S A B C D E F G)
                           '(a b c z)
                           '(a y)
                           'S
                           '(F)
```
-
- `` `(((S a ,EMP) (A (a y)))
- ``   ((A a ,EMP) (A (a)))``
      `((A b (a)) (B ,EMP))`
- `  ((B b (a)) (B ,EMP))`
  `  ((B c (a)) (C ,EMP))`
- `  ((C c (a)) (C ,EMP))`
  `  ((C z ,EMP) (G ,EMP))`
  `  ((C c (y)) (E (y)))`
- `  ((B b (y)) (D (y)))`
- `  ((D b (y)) (D (y)))`
  `  ((D c (y)) (E (y)))`
- `  ((E c (y)) (E (y)))`
  `  ((E z (y)) (F ,EMP))`

- ```
  (check-equal? (sm-apply ambncp '(z)) 'reject)
  (check-equal? (sm-apply ambncp '(a b b c c)) 'reject)
  (check-equal? (sm-apply ambncp '(a a b b c c z)) 'reject)
  (check-equal? (sm-apply ambncp '(a b a z)) 'reject)
  (check-equal? (sm-apply ambncp '(a b b z)) 'reject)
  (check-equal? (sm-apply ambncp '(a a a b b c c z)) 'accept)
  (check-equal? (sm-apply ambncp '(a a a b c z)) 'accept)
  (check-equal? (sm-apply ambncp '(a a a b b b b c c c c z)) 'accept)
  ```

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L = {a^{m}b^{n}c^p| m≠n ∧ m,n,p>0}

- #lang fsm

  ```
  (define ambncp (make-ndpda '(S A B C D E F G)
                             '(a b c z)
                             '(a y)
                             'S
                             '(F)
  ```
- `(((S a ,EMP) (A (a y)))
- ((A a ,EMP) (A (a)))
  ((A b (a)) (B ,EMP))
- ((B b (a)) (B ,EMP))
  ((B c (a)) (C ,EMP))
- ((C c (a)) (C ,EMP))
  ((C z ,EMP) (G ,EMP))
  ((C c (y)) (E (y)))
- ((B b (y)) (D (y)))
- ((D b (y)) (D (y)))
  ((D c (y)) (E (y)))
- ((E c (y)) (E (y)))
  ((E z (y)) (F ,EMP))
- ((G ,EMP (a)) (G ,EMP))
  ((G ,EMP (y)) (F ,EMP)))))

- ```
  (check-equal? (sm-apply ambncp '(z)) 'reject)
  (check-equal? (sm-apply ambncp '(a b b c c)) 'reject)
  (check-equal? (sm-apply ambncp '(a a b b c c z)) 'reject)
  (check-equal? (sm-apply ambncp '(a b a z)) 'reject)
  (check-equal? (sm-apply ambncp '(a b b z)) 'reject)
  (check-equal? (sm-apply ambncp '(a a a b b c c z)) 'accept)
  (check-equal? (sm-apply ambncp '(a a a b c z)) 'accept)
  (check-equal? (sm-apply ambncp '(a a a b b b b c c c c z)) 'accept)
  ```

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$L = \{a^m b^n c^p \mid m \neq n \land m,n,p > 0\}$ ci = consumed input P = ambncp

Theorem
The state invariants hold when P is applied to w.

- When P starts, S-INV holds because ci = '() and the stack = '(). This establishes the base case.

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`L = {`$a^m b^n c^p$`| m`$\neq$`n `$\land$` m,n,p>0} ci = consumed input P = ambncp`

Theorem

The state invariants hold when P is applied to w.

- When P starts, S-INV holds because `ci = '()` and the stack `= '()`. This establishes the base case.
- Proof that invariants hold after each nonempty transition:
- `((S a ,EMP) (A (a y)))`: By inductive hypothesis, S-INV holds. This means `ci = stack = '()`. Using this transition adds an a to ci and pushes y and then a onto the stack. Therefore, after using this rule, `ci = `(a^+)` and stack = `$(a^+$` y)`. Thus, A-INV holds.

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L = {$a^m b^n c^p$| m$\neq$n $\wedge$ m,n,p>0} ci = consumed input P = ambncp

Theorem

The state invariants hold when P is applied to w.

- When P starts, S-INV holds because ci = '() and the stack = '(). This establishes the base case.
- Proof that invariants hold after each nonempty transition:
- ((S a ,EMP) (A (a y))): By inductive hypothesis, S-INV holds. This means ci = stack = '().
  Using this transition adds an a to ci and pushes y and then a onto the stack. Therefore, after using
  this rule, ci = (a^+) and stack = (a^+ y). Thus, A-INV holds.
- ((A a ,EMP) (A (a))): By inductive hypothesis, A-INV holds. This means ci = (a^+) and stack =
  (a^+ y). Using this transition adds an a to ci and pushes an a onto the stack. Therefore, after using
  this rule, ci = (a^+) and stack = (a^+ y). Thus, A-INV holds.

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L = $\{a^m b^n c^p|$ m$\neq$n $\wedge$ m,n,p>0$\}$ ci = consumed input P = ambncp

Theorem

The state invariants hold when P is applied to w.

- When P starts, S-INV holds because ci = '() and the stack = '(). This establishes the base case.
- Proof that invariants hold after each nonempty transition:
- `((S a ,EMP) (A (a y)))`: By inductive hypothesis, S-INV holds. This means ci = stack = '(). Using this transition adds an a to ci and pushes y and then a onto the stack. Therefore, after using this rule, ci = (a^+) and stack = $(a^+$ y). Thus, A-INV holds.
- `((A a ,EMP) (A (a)))`: By inductive hypothesis, A-INV holds. This means ci = (a^+) and stack = $(a^+$ y). Using this transition adds an a to ci and pushes an a onto the stack. Therefore, after using this rule, ci = (a^+) and stack = $(a^+$ y). Thus, A-INV holds.
- `((A b (a)) (B ,EMP))`: By inductive hypothesis, A-INV holds. This means ci = (a^+) = (a^i) and stack = $(a^+$ y). Using this transition adds a b to ci and pops an a off the stack. Therefore, after using this rule, ci = $(a^i\ b^1)$ = $(a^i\ b^j)$, stack = $(a^{i-1}$ y) = $(a^{i-j}$ y) and $0 < j \leq i$. Thus, B-INV holds.

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$L = \{a^m b^n c^p \mid m \neq n \land m,n,p > 0\}$ ci = consumed input P = ambncp

Theorem

The state invariants hold when P is applied to w.

- When P starts, S-INV holds because ci = '() and the stack = '(). This establishes the base case.
- Proof that invariants hold after each nonempty transition:
- ((S a ,EMP) (A (a y))): By inductive hypothesis, S-INV holds. This means ci = stack = '().
  Using this transition adds an a to ci and pushes y and then a onto the stack. Therefore, after using
  this rule, ci = (a^+) and stack = $(a^+$ y). Thus, A-INV holds.
- ((A a ,EMP) (A (a))): By inductive hypothesis, A-INV holds. This means ci = (a^+) and stack =
  $(a^+$ y). Using this transition adds an a to ci and pushes an a onto the stack. Therefore, after using
  this rule, ci = (a^+) and stack = $(a^+$ y). Thus, A-INV holds.
- ((A b (a)) (B ,EMP)): By inductive hypothesis, A-INV holds. This means ci = (a^+) = (a^i) and
  stack = $(a^+$ y). Using this transition adds a b to ci and pops an a off the stack. Therefore, after
  using this rule, ci = $(a^i$ $b^1)$ = $(a^i$ $b^j)$, stack = $(a^{i-1}$ y) = $(a^{i-j}$ y) and $0 < j \leq i$. Thus, B-INV
  holds.
- ((B b (a)) (B ,EMP)): By inductive hypothesis, B-INV holds. This means ci = $(a^i$ $b^j)$, stack =
  $(a^{i-j}$ y) and $0 < j \leq i$. Given that there is an a on the stack, j must be strictly less than i. That is,
  j < i. Using this transition adds a b to ci and pops an a off the stack. Therefore, after using this
  rule, ci = $(a^i$ $b^{j+1})$, stack = $(a^{i-(j+1)}$ y), $0 < j+1$, and $j+1 \leq i$. Without loss of generality, we
  may think of j+1 as a new value for j. Thus, B-INV holds.

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$L = \{a^m b^n c^p \mid m \neq n \land m,n,p > 0\}$ ci = consumed input P = ambncp

Theorem

The state invariants hold when P is applied to w.

- When P starts, S-INV holds because ci = '() and the stack = '(). This establishes the base case.
- Proof that invariants hold after each nonempty transition:
- ((S a ,EMP) (A (a y))): By inductive hypothesis, S-INV holds. This means ci = stack = '(). Using this transition adds an a to ci and pushes y and then a onto the stack. Therefore, after using this rule, ci = (a^+) and stack = $(a^+ y)$. Thus, A-INV holds.
- ((A a ,EMP) (A (a))): By inductive hypothesis, A-INV holds. This means ci = (a^+) and stack = $(a^+ y)$. Using this transition adds an a to ci and pushes an a onto the stack. Therefore, after using this rule, ci = (a^+) and stack = $(a^+ y)$. Thus, A-INV holds.
- ((A b (a)) (B ,EMP)): By inductive hypothesis, A-INV holds. This means ci = (a^+) = (a^i) and stack = $(a^+ y)$. Using this transition adds a b to ci and pops an a off the stack. Therefore, after using this rule, ci = $(a^i b^1)$ = $(a^i b^j)$, stack = $(a^{i-1} y)$ = $(a^{i-j} y)$ and $0 < j \leq i$. Thus, B-INV holds.
- ((B b (a)) (B ,EMP)): By inductive hypothesis, B-INV holds. This means ci = $(a^i b^j)$, stack = $(a^{i-j} y)$ and $0 < j \leq i$. Given that there is an a on the stack, j must be strictly less than i. That is, $j < i$. Using this transition adds a b to ci and pops an a off the stack. Therefore, after using this rule, ci = $(a^i b^{j+1})$, stack = $(a^{i-(j+1)} y)$, $0 < j+1$, and $j+1 \leq i$. Without loss of generality, we may think of j+1 as a new value for j. Thus, B-INV holds.
- ((B c (a)) (C ,EMP)): By inductive hypothesis, B-INV holds. This means ci = $(a^i b^j)$, stack = $(a^{i-j} y)$, and $0 < j \leq i$. Given that there is an a on the stack, $j < i$. Using this transition , adds a c to ci and pops an a. Therefore, after using this rule, ci = $(a^i b^j c^1)$ = $(a^i b^j c^k)$, stack = $(a^{i-j-1} y)$ = $(a^{i-j-k} y)$, $i > j$, and $i,j,k > 0$. Thus, C-INV holds.

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L = $\{a^m b^n c^p | \ m{\neq}n \ \wedge \ m,n,p{>}0\}$ ci = consumed input P = ambncp

Theorem

The state invariants hold when P is applied to w.

- When P starts, S-INV holds because ci = '() and the stack = '(). This establishes the base case.
- Proof that invariants hold after each nonempty transition:
- ((S a ,EMP) (A (a y))): By inductive hypothesis, S-INV holds. This means ci = stack = '().
  Using this transition adds an a to ci and pushes y and then a onto the stack. Therefore, after using
  this rule, ci = (a^+) and stack = $(a^+ \ y)$. Thus, A-INV holds.
- ((A a ,EMP) (A (a))): By inductive hypothesis, A-INV holds. This means ci = (a^+) and stack =
  $(a^+ \ y)$. Using this transition adds an a to ci and pushes an a onto the stack. Therefore, after using
  this rule, ci = (a^+) and stack = $(a^+ \ y)$. Thus, A-INV holds.
- ((A b (a)) (B ,EMP)): By inductive hypothesis, A-INV holds. This means ci = (a^+) = (a^i) and
  stack = $(a^+ \ y)$. Using this transition adds a b to ci and pops an a off the stack. Therefore, after
  using this rule, ci = $(a^i \ b^1)$ = $(a^i \ b^j)$, stack = $(a^{i-1} \ y)$ = $(a^{i-j} \ y)$ and $0 < j \leq i$. Thus, B-INV
  holds.
- ((B b (a)) (B ,EMP)): By inductive hypothesis, B-INV holds. This means ci = $(a^i \ b^j)$, stack =
  $(a^{i-j} \ y)$ and $0 < j \leq i$. Given that there is an a on the stack, j must be strictly less than i. That is,
  $j < i$. Using this transition adds a b to ci and pops an a off the stack. Therefore, after using this
  rule, ci = $(a^i \ b^{j+1})$, stack = $(a^{i-(j+1)} \ y)$, $0 < j+1$, and $j+1 \leq i$. Without loss of generality, we
  may think of j+1 as a new value for j. Thus, B-INV holds.
- ((B c (a)) (C ,EMP)): By inductive hypothesis, B-INV holds. This means ci = $(a^i \ b^j)$, stack =
  $(a^{i-j} \ y)$, and $0 < j \leq i$. Given that there is an a on the stack, $j < i$. Using this transition , adds a
  c to ci and pops an a. Therefore, after using this rule, ci = $(a^i \ b^j \ c^1)$ = $(a^i \ b^j \ c^k)$, stack =
  $(a^{i-j-1} \ y)$ = $(a^{i-j-k} \ y)$, $i > j$, and i,j,k > 0. Thus, C-INV holds.
- ((B b (y)) (D (y))): By inductive hypothesis, B-INV holds. This means ci = $(a^i \ b^j)$, stack =
  $(a^{i-j} \ y)$, and $0 < j \leq i$. Observe that if the stack does not contain an a then j = i. Using this
  transition adds a b to ci without modifying the stack. Therefore, ci = $(a^i \ b^{j+1})$ and stack = (y) and
  $j+1 > i$ and i,j+1 > 0. Without loss of generality, we may think of j+1 as a new value for j.
  Therefore, D-INV holds.

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$L = \{a^m b^n c^p |\ m \neq n\ \wedge\ m,n,p > 0\}$

- <u>((C c (a)) (C ,EMP)):</u> By inductive hypothesis, C-INV holds. This means $ci = (a^i\ b^j\ c^k)$, stack $= (a^{i-j-k}\ y)$, $i > j$, and $i,j,k > 0$. Using this transition adds a c to ci and pops an a from the stack. Therefore, $ci = (a^i\ b^j\ c^{k+1})$, stack $= (a^{i-j-(k+1)}\ y)$, $i > j$, and $i,j,k+1 > 0$. Without loss of generality, we may think of $k+1$ as a new value for k. Therefore, C-INV holds.

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$L = \{a^m b^n c^p \mid m \neq n \land m,n,p > 0\}$

- $\underline{((C \ c \ (a)) \ (C \ ,EMP))}$: By inductive hypothesis, C-INV holds. This means $ci = (a^i \ b^j \ c^k)$, stack $= (a^{i-j-k} \ y)$, $i > j$, and $i,j,k > 0$. Using this transition adds a c to ci and pops an a from the stack. Therefore, $ci = (a^i \ b^j \ c^{k+1})$, stack $= (a^{i-j-(k+1)} \ y)$, $i > j$, and $i,j,k+1 > 0$. Without loss of generality, we may think of $k+1$ as a new value for k. Therefore, C-INV holds.

- $\underline{((C \ z \ ,EMP) \ (G \ ,EMP))}$: By inductive hypothesis, C-INV holds. This means $ci = (a^i \ b^j \ c^k)$, stack $= (a^{i-j-k} \ y)$, $i > j$, and $i,j,k > 0$. Observe that $i \neq j$ and that $c^k \in c^+$. Using this transition adds z to ci without changing the stack. Therefore, $ci = (a^i \ b^j \ c^+ \ z)$, stack $= (a^* \ y)$, $i \neq j$, and $i,j > 0$. Thus, G-INV holds.

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$L = \{a^m b^n c^p \mid m \neq n \land m,n,p > 0\}$

- $((C\ c\ (a))\ (C\ ,EMP))$: By inductive hypothesis, C-INV holds. This means $ci = (a^i\ b^j\ c^k)$, stack $= (a^{i-j-k}\ y)$, $i > j$, and $i,j,k > 0$. Using this transition adds a c to ci and pops an a from the stack. Therefore, $ci = (a^i\ b^j\ c^{k+1})$, stack $= (a^{i-j-(k+1)}\ y)$, $i > j$, and $i,j,k+1 > 0$. Without loss of generality, we may think of $k+1$ as a new value for k. Therefore, C-INV holds.

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- <u>((E z (y)) (F ,EMP))</u>: By inductive hypothesis, E-INV holds. This means $ci = (a^i\ b^j\ c^+)$, stack $= (y)$, $j \neq i$, and $i,j > 0$. Using this transition adds z to ci and pops y from the stack. Therefore, after using this transition, $ci = (a^i\ b^j\ c^+\ z)$, stack $=$ '(), $i \neq j$, and $i,j > 0$. Thus, F-INV holds.

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- This establishes the inductive step and concludes the proof.

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$L = \{a^m b^n c^p |\ m \neq n\ \wedge\ m,n,p > 0\}$

Lemma
$w \in L \Leftrightarrow w \in L(P)$

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$w \in L \Leftrightarrow w \in L(P)$

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Proof.

($\Rightarrow$) Assume $w \in L$. This means that $w = a^m b^n c^p$, where $m \neq n$ and $m,n,p > 0$. Given that state invariants always hold, there is a computation that has P consume w reaching F with an empty stack. Therefore, $w \in L(P)$.

($\Leftarrow$) Assume $w \in L(P)$. This means that M halts in F, the only final state, with an empty stack having consumed w. Given that the state invariants always hold we may conclude that $w = a^m b^n c^p$, where $m \neq n$ and $m,n,p > 0$. Therefore, $w \in L$. □

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Lemma
$w \notin L \Leftrightarrow w \notin L(P)$

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w∈L ⇔ w∈L(P)

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($\Rightarrow$) Assume w∈L. This means that $w = a^m b^n c^p$, where $m \neq n$ and $m,n,p > 0$. Given that state invariants always hold, there is a computation that has P consume w reaching F with an empty stack. Therefore, w∈L(P).

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Lemma
w∉L ⇔ w∉L(P)

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Proof.

($\Rightarrow$) Assume w∉L. This means that $w \neq a^m b^n c^p$, $m \neq n$, or $m,n,p \not> 0$. Given that state predicate invariants always hold, P cannot consume w and end in F with an empty stack. Therefore, w∉L(P).

($\Leftarrow$) Assume w∉L(P). This means that P cannot transition into F with an empty stack having consumed w. Given that the state predicate invariants always hold, $w \neq a^m b^n c^p$, $m \neq n$, or $m,n,p \not> 0$. Therefore, w∉L. $\qquad\square$

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- Are all CFLs deterministic CFLs?

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- Are all CFLs deterministic CFLs?
- Change the meaning of accept for, M, a deterministic pda
- A language, L, is deterministic context-free if $Lz = L(M)$
- If M accepts Lz then a nondeterministic pda, M', may be constructed that nondeterministically "senses" the end of the input and transitions to states that consume no more input

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- We shall build on two important concepts
- M is a simple pda with a bottom-of-the-stack symbol

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- We shall build on two important concepts

- M is a simple pda with a bottom-of-the-stack symbol

- Second, we define a configuration, C = (Q w s), as a *dead end* if in zero or more steps it leads to a configuration, E = (R w' s'), in which no input has been consumed (i.e., w = w') or the stack has not shrunk (i.e., $|s'| \geq |s|$)

- If a pda does not have a dead-end configuration then it will eventually read all its input

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- Second, we define a configuration, $C = (Q\ w\ s)$, as a *dead end* if in zero or more steps it leads to a configuration, $E = (R\ w'\ s')$, in which no input has been consumed (i.e., $w = w'$) or the stack has not shrunk (i.e., $|s'| \geq |s|$)

- If a pda does not have a dead-end configuration then it will eventually read all its input

- If M is simple then determining if a configuration is a dead end only depends on the current state, the next element to read, and the top stack symbol

- Let D be the set of dead-end configurations. Observe that D is finite.

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- Recall that context-free languages are not closed under complement
- To establish that not all context-free languages are deterministic, we shall prove that deterministic context-free languages are closed under complement.

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Theorem
Deterministic context-free languages are closed under complement.

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Theorem
Deterministic context-free languages are closed under complement.

-
- Assume L is a context-free language such that Lz is accepted by a deterministic simple pda M = (make-ndpda K Σ Γ S F R)
- Intuitively, we would like to build a pda for L's complement by simply switching the role of M's accepting and rejecting states
- Unfortunately, this does not work because pdas may reject without reading all the input: M is in a configuration for which no transition rules apply or M is in a dead-end configuration

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- Intuitively, we would like to build a pda for L's complement by simply switching the role of M's accepting and rejecting states
- Unfortunately, this does not work because pdas may reject without reading all the input: M is in a configuration for which no transition rules apply or M is in a dead-end configuration
- We shall convert M into, M', an equivalent deterministic pda without dead-end configurations

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- Recall that context-free languages are not closed under complement
- To establish that not all context-free languages are deterministic, we shall prove that deterministic context-free languages are closed under complement.

Theorem
Deterministic context-free languages are closed under complement.

-
- Assume L is a context-free language such that Lz is accepted by a deterministic simple pda M = (make-ndpda K Σ Γ S F R)
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- M' is constructed as follows:

  1. $\forall$ (Q a x)$\in$D, remove rules of the form ((Q a x) (H y))
     in M and add ((Q a x) (U EMP)), where U is a new
     non-accepting state.
  2. $\forall$ a$\in$Σ, add the rule ((U a EMP) (U EMP))
  3. $\forall$ b$\in$Γ, add the rules ((U z EMP) (V EMP)) and
  ((V EMP b) (V EMP)), where V is a new non-accepting state.

- M' is deterministic given that M is deterministic and the added rules do not make M' nondeterministic
- M' rejects by moving to a non-accepting state whenever M does not read the entire input
- M' has no dead-end configurations
- Thus, M' always reads the entire input

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- Thus, M' always reads the entire input
- Given that M' reads the entire input, reversing the role of accepting and non-accepting states results in a machine that accepts (Σ*-L)z = L's complement □

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- This a rather stunning result
- It allows us to prove that there are context-free languages that are not deterministic

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- This a rather stunning result
- It allows us to prove that there are context-free languages that are not deterministic
- $L = \{a^i b^j c^k \mid i \neq j \vee i \neq k\}$

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- L = $\{a^i b^j c^k \mid i \neq j \lor i \neq k\}$
- Assume L is a deterministic context-free language

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- L = $\{a^i b^j c^k \mid i \neq j \lor i \neq k\}$
- Assume L is a deterministic context-free language
- The theorem informs us that L's complement, $\bar{L}$, is also a deterministic context-free language

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- The deterministic context-free languages are a proper subset of the context-free languages

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- **In the context of pdas, nondeterminism is more powerful than determinism**

Deterministic pdas

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- The deterministic context-free languages are a proper subset of the context-free languages
- **In the context of $pdas$, nondeterminism is more powerful than determinism**
- This makes software development more difficult: parsers decide if a program is valid

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Theorem
Deterministic context-free languages are not closed under union.

- Assume deterministic context-free languages are closed under union

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Theorem

Deterministic context-free languages are not closed under union.

- Assume deterministic context-free languages are closed under union
- Consider the following deterministic context-free languages:
  - $L = \{a^m b^n c^p \mid m \neq n \wedge m,n,p > 0\}$
  - $S = \{a^m b^n c^p \mid n \neq p \wedge m,n,p > 0\}$

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- Let LUS = $L \cup S$
- Given our assumption, LUS is a deterministic context-free language

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    - $L = \{a^m b^n c^p |\ m \neq n \wedge m,n,p > 0\}$
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- Let LUS = $L \cup S$
- Given our assumption, LUS is a deterministic context-free language
- Deterministic context-free languages are closed under complement
- Thus, the complement of LSU, NOT-LSU, is also a deterministic context-free language

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- Let LUS = $L \cup S$
- Given our assumption, LUS is a deterministic context-free language
- Deterministic context-free languages are closed under complement
- Thus, the complement of LSU, NOT-LSU, is also a deterministic context-free language
- NOT-LSU is the union of the following two languages:
  - $\{a^m b^n c^p \mid m = n = p \wedge m,n,p > 0\}$
  - $\{w \mid w \in \{a\ b\ c\}^* \wedge$ w has letters out of order $\vee$ w is missing at least one element in $\{a\ b\ c\}\}$

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- Let LUS = $L \cup S$
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  - $\{w \mid w \in \{a\ b\ c\}^* \land w$ has letters out of order $\lor w$ is missing at least one element in $\{a\ b\ c\}\}$
- The intersection of a context-free language and a regular language is a context-free language
- NOT-LSU $\cap$ $a^* b^* c^* = a^n b^n c^n$, where $n > 0$
- This is not a context-free language
- Our assumption is wrong and deterministic context-free languages are not closed under union

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Theorem
Deterministic context-free languages are not closed under intersection.

- Assume deterministic context-free languages are closed under intersection

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Theorem
Deterministic context-free languages are not closed under intersection.

- Assume deterministic context-free languages are closed under intersection

- Consider the following languages:
  - $L = \{a^m b^n c^p | \ m = n \wedge m,n,p \geq 0\}$
  - $S = \{a^m b^n c^p | \ n = p \wedge m,n,p \geq 0\}$

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Theorem

Deterministic context-free languages are not closed under intersection.

- Assume deterministic context-free languages are closed under intersection

- Consider the following languages:
    - $L = \{a^m b^n c^p \mid m = n \wedge m,n,p \geq 0\}$
    - $S = \{a^m b^n c^p \mid n = p \wedge m,n,p \geq 0\}$
- The intersection of L and S is $\{a^n b^n c^n \mid n \geq 0\}$
- This language is not context-free and, therefore, deterministic context-free languages are not closed under intersection

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- HOMEWORK: 8, 10
- QUIZ: 9 (due in 1 week)