Homework 2
Problem 3

Consider 3-D gravitational acceleration : $\ddot{\vec{r}} = \frac{-\mu}{(\vec{r}^T\vec{r})^{3/2}} \cdot \vec{r}$

Take state $\vec{x} = [r \;\; \dot{r}]^T$. Find linearized dynamics $F = \frac{\partial f(\vec{x})}{\partial \vec{x}}$, $f(\vec{x}) = \dot{\vec{x}} = [\dot{\vec{r}} \;\; \ddot{\vec{r}}]^T$

$$\frac{\partial \ddot{\vec{r}}}{\partial \vec{r}} = -\mu \frac{\partial}{\partial \vec{r}}\left((\vec{r}^T\vec{r})^{-3/2} \cdot \vec{r}\right) = -\mu\left(\vec{r} \cdot \underbrace{\frac{\partial}{\partial \vec{r}}(\vec{r}^T \cdot \vec{r})^{-3/2}} + (\vec{r}^T \cdot \vec{r})^{-3/2} \cdot \underbrace{\frac{\partial \vec{r}}{\partial \vec{r}}}\right) = -\mu\left(\vec{r} \cdot -\frac{3}{2}(\vec{r}^T \cdot \vec{r})^{-5/2} \cdot \underbrace{\frac{\partial(\vec{r}^T \cdot \vec{r})}{\partial \vec{r}}} + (\vec{r}^T \cdot \vec{r})^{-3/2} \cdot \underbrace{I}\right)$$

$$\frac{\partial(\vec{r}^T \cdot \vec{r})}{\partial \vec{r}} = \left[\frac{(r_1^2 + r_2^2 + r_3^2)}{\partial r_1} \quad \frac{(r_1^2 + r_2^2 + r_3^2)}{\partial r_2} \quad \frac{(r_1^2 + r_2^2 + r_3^2)}{\partial r_3}\right] = [2r_1 \quad 2r_2 \quad 2r_3] = 2\vec{r}^T$$

$$\frac{\partial \vec{r}}{\partial \vec{r}} = \begin{bmatrix} \frac{\partial(r_1)}{\partial r_1} & \frac{\partial(r_2)}{\partial r_1} & \frac{\partial(r_3)}{\partial r_1} \\ \frac{\partial(r_1)}{\partial r_2} & \frac{\partial(r_2)}{\partial r_2} & \frac{\partial(r_3)}{\partial r_2} \\ \frac{\partial(r_1)}{\partial r_3} & \frac{\partial(r_2)}{\partial r_3} & \frac{\partial(r_3)}{\partial r_3} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

$$\frac{\partial \ddot{\vec{r}}}{\partial \vec{r}} = -\mu\left(\vec{r} \cdot -\frac{3}{2}(\vec{r}^T \cdot \vec{r})^{-5/2} \cdot 2\vec{r}^T + (\vec{r}^T \cdot \vec{r})^{-3/2} \cdot I\right) = \underbrace{-\mu\left(-3(\vec{r}^T \cdot \vec{r})^{-5/2} \cdot \vec{r} \cdot \vec{r}^T + (\vec{r}^T \cdot \vec{r})^{-3/2} \cdot I\right)}$$

$$\frac{\partial \dot{\vec{r}}}{\partial \vec{r}} = 0 \in R^{3\times3} \quad ; \quad \frac{\partial \dot{\vec{r}}}{\partial \dot{\vec{r}}} = I \in R^{3\times3}$$

$$\frac{\partial \ddot{\vec{r}}}{\partial \dot{\vec{r}}} = 0 \in R^{3\times3}$$

$$F = \begin{bmatrix} \frac{\partial \dot{\vec{r}}}{\partial \vec{r}} & \frac{\partial \dot{\vec{r}}}{\partial \dot{\vec{r}}} \\ \frac{\partial \ddot{\vec{r}}}{\partial \vec{r}} & \frac{\partial \ddot{\vec{r}}}{\partial \dot{\vec{r}}} \end{bmatrix} = \begin{bmatrix} \{0\} & \{I\} \\ \{-\mu\left(-3(\vec{r}^T\vec{r})^{-5/2} \cdot \vec{r} \cdot \vec{r}^T + (\vec{r}^T\vec{r})^{-3/2} \cdot I\right)\} & \{0\} \end{bmatrix}$$