Homework 2
Problem 3

Consider 3-D gravitational acceleration : $\ddot{\vec{r}} = \frac{-\mu}{(\vec{r}^{T}\vec{r})^{3/2}} \cdot \vec{r}$

Take state $\vec{x} = [r \ \ \dot{r}]^T$. Find linearized dynamics $F = \frac{\partial f(\vec{x})}{\partial \vec{x}}$, $f(\vec{x}) = \dot{\vec{x}} = [\dot{\vec{r}} \ \ \ddot{\vec{r}}]^T$

$$\frac{\partial \ddot{\vec{r}}}{\partial \vec{r}} = -\mu \frac{\partial}{\partial \vec{r}} \left(\left((\vec{r}^{T}\vec{r})^{-3/2} \cdot \vec{r} \right) \right) = \mu \left(\frac{\partial}{\partial \vec{r}} (\vec{r}^{T}\vec{r})^{-3/2} \cdot \vec{r} + (\vec{r}^{T}\vec{r})^{-3/2} \right) = \mu \left(-\frac{3}{2} (\vec{r}^{T}\vec{r})^{-5/2} \cdot \frac{\partial}{\partial \vec{r}} (\vec{r}^{T}\vec{r}) \cdot \vec{r} + (\vec{r}^{T} \cdot \vec{r})^{-3/2} \right)$$

$\frac{\partial}{\partial \vec{r}} (\vec{r}^{T} \cdot \vec{r}) = \vec{r}^{T} (I + I^{T}) = 2 \vec{r}^{T}$, found from atmos.washington.edu/~dennis/MatrixCalculus.pdf

$$\frac{\partial \ddot{\vec{r}}}{\partial \vec{r}} = \mu \left(-\frac{3}{2} (\vec{r}^{T} \cdot \vec{r})^{-5/2} \cdot 2 \cdot \vec{r}^{T} \cdot \vec{r} + (\vec{r}^{T} \cdot \vec{r})^{-3/2} \right) = \mu \left(-3 (\vec{r}^{T}\vec{r})^{-5/2} \cdot \vec{r}^{T} \cdot \vec{r} + (\vec{r}^{T} \cdot \vec{r})^{-3/2} \right)$$

$$\frac{\partial \ddot{\vec{r}}}{\partial \vec{r}} = \mu \left(-3 (\vec{r}^{T} \cdot \vec{r})^{-3/2} + (\vec{r}^{T} \cdot \vec{r})^{-3/2} \right) = \mu \left((-3+1) (\vec{r}^{T} \cdot \vec{r})^{-3/2} \right) = \frac{-2\mu}{(\vec{r}^{T} \cdot \vec{r})^{3/2}}$$

$$\boxed{F = \begin{bmatrix} \frac{\partial \dot{\vec{r}}}{\partial \vec{r}} & \frac{\partial \dot{\vec{r}}}{\partial \dot{\vec{r}}} \\ \frac{\partial \ddot{\vec{r}}}{\partial \vec{r}} & \frac{\partial \ddot{\vec{r}}}{\partial \dot{\vec{r}}} \end{bmatrix} = \begin{bmatrix} \{0\} & \{1\} \\ \left\{ \frac{-2 \cdot \mu}{(\vec{r}^{T} \cdot \vec{r})^{3/2}} \right\} & \{0\} \end{bmatrix}}$$