The Discrete Fourier Transform

Given an analog signal $y(t),\ t \in [0, L]$, sample it at the rate f_S to obtain $f_S L = N$ samples.

The DFT is a frequency detector.

Let $f_1 = 1/L = f_S/N$ be the fundamental frequency.

The DFT detects the presence of the following frequencies in the signal $f_k = k \cdot f_1,\ k = 0, 1, \ldots, (N-1)$.

The DFT converts the sequence of N samples $\left[y[0], y[1], \ldots, y[N-1]\right]$ into another sequence of N complex numbers called Fourier coefficients $\left[Y[0], Y[1], \ldots, Y[N-1]\right]$ where $Y[k]$ expresses the degrees to which the frequency $f_k = kf_1$ is present in the signal.

$$Y[k] = \sum_{n=0}^{N-1} y[n] \exp\{-i2\pi kn/N\}, \quad k = 0, 1, \ldots, (N-1)$$

The Inverse Discrete Fourier Transform (IDFT) converts every $Y[k]$ back into $y[k]$:

$$y[n] = \frac{1}{N} \sum_{k=0}^{N-1} Y[k] \exp\{i2\pi kn/N\}, \quad n = 0, 1, \ldots, (N-1)$$

The magnitude $|Y[k]|$ measures the magnitude of the frequency $f_k = kf_1$ in the signal via $a_k = \dfrac{2|Y[k]|}{N}$, where a_k is the peak amplitude of sinusoid of frequency f_k.

The argument of $Y[k]$ indicates the phase of the sinusoid of frequency f_k with respect to the cosine function.

Applications of the Inverse DFT:

As long as the original has no frequency component higher than the Nyquist frequency, the IDFT allows us to perfectly reconstruct the original.

This means that a signal is determined by its frequency content.

- JPEG, MP3 and MPEG uses DFT to compress pictures, audio and video files, respectively.
- Filter out noise or unwanted frequency.

Complex exponentials:

When $z = i\theta$ is purely imaginary, it can be shown that $e^{i\theta} = \cos(\theta) + i\sin(\theta)$.

- This is Euler's identity.
- $e^{i\theta}$ is the complex exponential.

Note that, since $\sin^2(\theta) + \cos^2(\theta) = 1,\ \forall\, \theta \in \mathbb{R}$, then $z = e^{i\theta} = \cos(\theta) + i\sin(\theta)$ lies on the unit circle on the complex plane.

Complex sinusoids:

Recall that real sinusoids with frequency f Hz have the forms $y(t) = a\sin(2\pi ft)$ and $y(t) = a\cos(2\pi ft)$.

A complex sinusoid with frequency f Hz has the form:

$$y = e^{i2\pi ft} = \cos(2\pi ft) + i\sin(2\pi ft), \quad t \in [0, \infty)$$

- Note that a complex sinusoid is a complex-valued function: for each t, the output $y(t)$ is a complex number.
- It traces out the curve that is the unit circle on the complex plane at the frequency f cycles per second (Hz).
- If $f > 0$, the curve is traced in the counterclockwise direction; if $f < 0$, it is traced in the clockwise direction.

Why does the DFT work (A geometric explanation)

Dot product measures similarity.

- Consider $u = (1, 1)$. Consider the three vectors: $v_1 = (3, 4)$, $v_2 = (5, 0)$ and $v_3 = (4, -3)$.
- The dot products:
  - $u \cdot v_1 = 1 \cdot 3 + 1 \cdot 4 = 7$;
  - $u \cdot v_2 = 1 \cdot 5 + 1 \cdot 0 = 5$;
  - $u \cdot v_3 = 1 \cdot 4 + 1 \cdot (-3) = 1$.

Basis vectors (x-y plane example):

- $\vec{u} = (1, 0), \vec{v} = (0, 1)$.
  - Building blocks for all other vectors.
  - $\vec{w} = (3, 5) = 3\vec{u} + 5\vec{v}$.
    - $\vec{w} \cdot \vec{u} = 3 \cdot 1 + 5 \cdot 0 = 3$.
    - $\vec{w} \cdot \vec{v} = 3 \cdot 0 + 5 \cdot 1 = 5$.
  - The dot products $\vec{w} \cdot \vec{u}$ e $\vec{w} \cdot \vec{v}$ are scalar projections.

Analogously, DFT = Projection to sinusoids.

- $(e^{-i2\pi k0/N}, e^{-i2\pi k1/N}, \ldots, e^{-i2\pi k(N-1)/N}),\ k \in \{0, 1, 2, \ldots, (N-1)\}$ are basis vectors for the space of all N samples.
- DFT: dot product of $\{y[n]\}$ with sinusoid of frequency f_k.

Reconstruction of a Signal with IDFT:

The signal:

$$y = 3\sin(2\pi \cdot 10t) + 2\cos(2\pi \cdot 20t + \pi/4), \quad t \in [0, 1/10]$$

Sample at $80\ \mathrm{Hz}$ for $L = 1/10$ seconds (8 samples):

$$y[n] : [1.41, 0.71, 1.59, 3.54, 1.41, -3.54, -4.41, -0.71]$$

DFT yields:

$$Y[n] : [0, -12i, 5.66 + 5.66i, 0, 0, 0, 5.66 - 5.66i, 12i]$$

Taking the magnitudes:

$$|Y[n]| : [0, 12, 8, 0, 0, 0, 8, 12]$$

Frequency in Herz: $[0, 10, 20, 30, 40, -30, -20, -10]$.

The amplitude is given by $a_k = \dfrac{2|Y[k]|}{N}$:

- Then, $a_1 = \dfrac{2 \cdot 12}{8} = 3$ (coefficient of the sine function).
  - From the list of frequencies, we know it is related to $f_k = 10\ \mathrm{Hz}$.

- The argument $\arg(Y[1]) = \arg(12i) = -\frac{\pi}{2}$.
    - The argument yields the phase of the sinusoid with respect to the <u>cosine</u> function.
    - Phasing the cosine function by $-\frac{\pi}{2}$, one obtains the <u>sine</u> function.
    - Putting it all together: $a_k \cdot \cos(2\pi \cdot f_k t + \phi) = 3\cos(2\pi \cdot 10t - \pi/2) = 3\sin(2\pi \cdot 10t)$.
- Then, $a_2 = \frac{2 \cdot 8}{8} = 2$ (coefficient of the cosine function).
    - This is the coefficient related to $f_k = 20$.
- The argument $\arg(Y[2]) = \arg(5.66 + 5.66i) = \frac{\pi}{4}$.
    - Putting it all together: $2\cos(2\pi \cdot 20t + \pi/4)$.
- The other frequency components are either the (aliased) negative equivalent frequencies ($-10, -20, -30$) or have zero magnitude associated with them $(0, 30, 40)$.