Sampling:

Given a signal $y(t),\ t \in [0, L]$, where t is continuous time measured in seconds and L is the length of the signal.

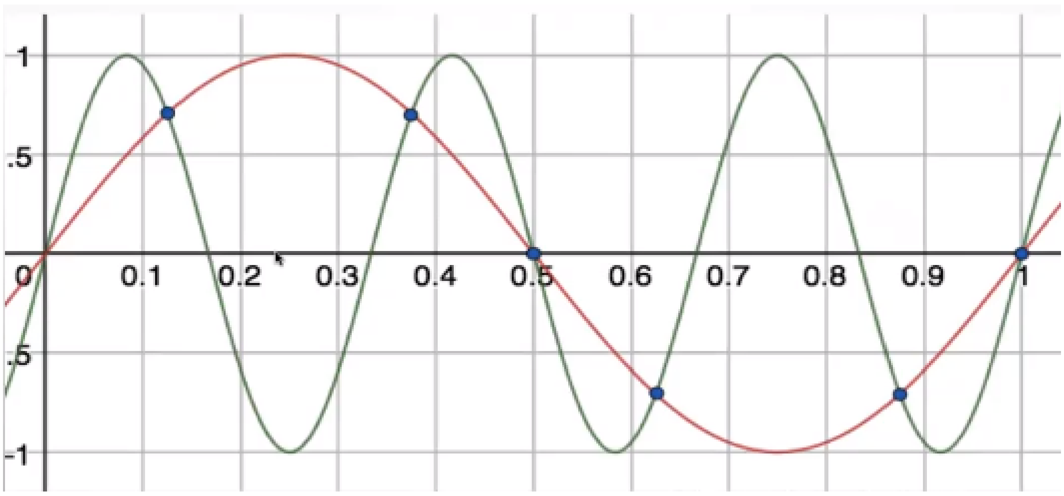
- Sample rate: $f_S = 44100$ Hz.
- Number of samples: $N = \lfloor L \cdot f_S \rfloor$ (floor function, rounds down).
    - Python: `N = int(L * fs)`.
- Time samples: $ts = \{t_0, t_1, \ldots, t_{N-1}\}$ where $\Delta t = t_i - t_{i-1} = \dfrac{1}{f_S}$ is the sampling interval.
    - Python: `ts = np.linspace(0, L, N, endpoint= False)`.
- Sampled values: $ys = \{y(t_0), y(t_1), \ldots, y(t_{N-1})\}$, or more succinctly $ys = \{y[0], y[1], \ldots, y[N-1]\}$.

Aliasing:

Given a continuous signal. Our discrete, sampled data, is approximate data.

When we only evaluate the signal at discrete times, we lose information on what happened between the samples.

This leads to a frequency ambiguity known as aliasing.



Let f_O be the observed frequency; f_S, the sampling rate and f, the actual frequency of the signal.

- If $-\dfrac{f_S}{2} < f < \dfrac{f_S}{2}$, then f is not aliased: $f = f_O$.
- Otherwise, that is $f \notin \left[-\dfrac{f_S}{2}, \dfrac{f_S}{2}\right]$, f is aliased back into it:
  $f_O \in \left[-\dfrac{f_S}{2}, \dfrac{f_S}{2}\right] : f_0 = f + k \cdot f_S,\ k \in \mathbb{Z}$.

The Sampling Theorem:

A signal $y(t),\ t \in [0, L]$ can be perfectly reconstructed from its samples taken at the sampling rate f_S, provided that the signal only contains frequencies less than $\dfrac{f_S}{2}$.

The frequency $\dfrac{f_S}{2}$ is called the Nyquist frequency.

Reconstruction of a signal:

Consider a 1-second signal given by:

$$y(t) = \sin(2\pi \cdot 1t) + \sin(2\pi \cdot 3t) + \sin(2\pi \cdot 4t),\ t \in [0, 1]$$

with frequencies 1 Hz, 3 Hz and 4 Hz.

Since the highest frequency component is 4 Hz, the sampling theorem says that this signal can be reconstructed without information loss if we sample, for example at the sampling rate of 10 Hz.

Spectrum of a real signal is symmetric:

The spectrum of every real signal $y(t),\ t \in [0, L]$ is symmetric around 0 Hz. That is, if a frequency f Hz is present in the signal, so is the frequency $-f$ Hz.

- The signal $y(t) = \sin(2\pi \cdot 1t) + \sin(2\pi \cdot 3t) + \sin(2\pi \cdot 4t),\ t \in [0, 1]$ contains the frequencies $\{-4 \text{ Hz}, -3\text{Hz}, -1\text{Hz}, 1 \text{ Hz}, 3 \text{ Hz}, 4 \text{ Hz}\}$.

The Discrete Fourier Transform:

If we sample a signal at a sampling rate f_S, the DFT can only detect frequencies between $-f_S/2$ and $f_S/2$.

Because a real signal has a symmetric spectrum. The DFT can detect frequencies between 0 and $f_S/2$.

- Python: `np.fft.rfft(samples)`. More here.

Distortion of Sound Recording:

Suppose an Analog to Digital Converter (ADC) is recording a violin at the sampling rate $f_S = 10 \text{ kHz } = 10,000 \text{ Hz}$. A tone with fundamental frequency of 750 Hz is played.

The harmonics are 750, 1500, 2250, 3000, 3750, 4500, 5250, 6000, 6750, 7500, 8250, 9000, 9750.

Nyquist frequency: $\dfrac{f_S}{2} = \dfrac{10000}{2} = 5000 \text{ Hz}.$

Harmonics above the Nyquist frequency are aliased:

- $5250 \to -4750 \text{ Hz}$ ($k = -1$).
- $6000 \to -4000 \text{ Hz}$ ($k = -1$).
- etc.

The ear cannot distinguish between negative and positive frequencies.

- Ignoring the sign and rearranging the harmonics: 750, 1500, 2250, 3000, 3750, 4500 and 250 (9750), 1000 (9000), 1750 (8250), 2500 (7500), 3250 (6750), 4000 (6000), 4750 (5250).
- These are no longer integer multiples of the fundamental frequency of 750 Hz. Thus the tone we hear in the recording is distorted.

Harmonics:

Consider the interval $[0, L]$ measured in seconds.

- Define the frequency $f_1 = 1/L$.
- This frequency is called the fundamental frequency for the interval $[0, L]$. Sinusoids, both real and complex, with frequency f_1 complete the cycle in the interval $[0, L]$.
  - Note that, since $N = f_S/L$, we also have $f_1 = 1/L = f_S/N$.

Consider the following set of N integer multiples of the fundamental frequency f_1.

$$\{f_0 = 0, f_1 = 1/L, f_2 = 2f_1, \ldots, \boldsymbol{f_S/2}, \ldots, f_{N-1} = (N-1)f_1\}$$

- In red are the aliased frequencies.

These are known as the harmonics. The DFT will try to detect the presence of these frequencies in the audio signal given by $y(t),\ t \in [0, L]$.

Why 44100 Hz?

Range of human hearing is between $20\ \mathrm{Hz}$ and $20000\ \mathrm{Hz}$.

To guarantee that every frequency in this range is properly recorded, a sampling rate of at least $40000\ \mathrm{Hz}$ is necessary.

- CD quality standard sampling rate is $44100\ \mathrm{Hz}$.
- Before sampling is taken, a low-pass filter is applied to remove all frequencies outside of the Nyquist frequency range. This prevents aliasing and corruption of the recording.
    - These low-pass filters are also called anti-aliasing filters.