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Chapter 20: Homogeneous and Homothetic Functions

Definition and Examples

<u>Definition</u>: For any scalar k, a real-valued function $f(x_1, \ldots, x_n)$ is **homogeneous of degree** k if:

$$f(tx_1, \ldots, tx_n) = t^k f(x_1, \ldots, x_n), \quad \forall x_1, \ldots, x_n; \ \forall t > 0 \quad \mathbf{(1)}$$

- Examples:

1. $x_1^2 x_2 + 3x_1 x_2^2 + x_2^3$ is homogeneous of degree three, since each term is homogeneous of degree three.
    - Replacing x_1, x_2, x_3 by tx_1, tx_2, tx_3:
    $$\begin{aligned}(tx_1)^2(tx_2) + 3(tx_1)(tx_2)^2 + (tx_2)^3 &= t^2x_1^2tx_2 + 3tx_1t^2x_2^2 + t^3x_2^3 \\ &= t^3\left(x_1^2x_2 + 3x_1x_2^2 + x_2^3\right) \quad \blacksquare\end{aligned}$$
2. $x_1^7 x_2 x_3^2 + 5x_1^6 - x_2^5 x_3^5$ is homogeneous of degree ten, since each term is homogeneous of degree ten.
    - Replacing x_1, x_2, x_3 by tx_1, tx_2, tx_3:

$$\begin{aligned}&(tx_1)^7(tx_2)(tx_3)^2 + (tx_1)^6(tx_2)^4 + (tx_2)^5(tx_3)^5 \\ &\qquad = t^{10}\left(x_1^7x_2x_3^2 + 5x_1^6 - x_2^5x_3^5\right) \quad \blacksquare\end{aligned}$$

3. $4x_1^2 - 5x_1x_2^2$ is *not* homogeneous since the first term has degree five and the second has degree three.
4. A linear function $z = \sum_i a_i x_i$ is homogeneous of degree one.
5. A quadratic form $z = \sum_{i,j} a_{ij} x_i x_j$ is homogeneous of degree two.
6. The function $f_1(x_1, x_2) = 30x_1^{1/2}x_2^{3/2} - 2x_1^3x_2^{-1}$ is homogeneous of degree two.
7. The function $f_2(x_1, x_2) = x_1^{1/2}x_2^{1/4} + x_1^2x_2^{-5/4}$ is homogeneous of degree three-quarters.
8. The function $f_3(x_1, x_2) = \dfrac{x_1^7 - 3x_1^2x_2^5}{x_1^4 + 2x_1^2x_2^2 + x_2^4}$ is homogeneous of degree three.

- However, the only homogeneous functions of one variable are the functions of the form $z = ax^k, \ \forall k \in \mathbb{R}$.
    - To prove this statement, let $z = f(x)$ be an arbitrary homogeneous function of one variable.
    - Let $a \equiv f(1)$ and let x be arbitrary.
    - Then, $f(x) = f(x \cdot 1) = x^k f(1) = ax^k$.

Homogeneous Functions in Economics

Economists often find it convenient to work with homogeneous functions as production functions.

- For example, if $q = f(x_1, \ldots, x_n)$ is production function which is homogeneous fo degree one, then:

$$f(tx_1, \ldots, tx_n) = tf(x_1, \ldots, x_n) \quad \mathbf{(2)}$$

for all input bundles $(x_1, \ldots, x_n)$ and all $t > 0$.

  - Taking $t = 2$, eq. $\mathbf{(2)}$ says that if the firm doubles all inputs, it doubles its output too.
    - For $t = 3$, if it triples each input, it triples the corresponding output.
    - Such a firm is said to exhibit **constant returns to scale**.
  - Suppose, on the other hand, the production function is homogeneous of degree $k > 1$.
    - If such a firm were to double the amount of each input, its output would rise by a factor of 2^k.
    - Since $k > 1$, its output would more than double.
    - Such a firm is said to exhibit **increasing returns to scale**.
  - Finally, a firm which has a production function that is homogeneous of degree $k < 1$, will have its output increase by a factor less than two when it doubles all its inputs.
    - Such a firm exhibits **decreasing returns to scale**.

A specific homogeneous functional form which economists frequently use as a production or utility function is the **Cobb-Douglas function**:

$$q = Ax_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}, \quad \mathbf{(3)}$$

a monomial with exponents $a_1, \ldots, a_n$ that are usually positive fractions.

- Economists interested in estimating the production function of a specific firm or industry will often try to find the Cobb-Douglas production function which best fits the firm's input-output data.
  - They can often use linear ordinary least squares techniques since by taking the logarithm of both sides of function $\mathbf{(3)}$, they can work with the log of the output as linear function of the logs of the inputs: $\log q = \log A + a_1 \log x_1 + \cdots + a_n \log x_n$.
- Notice that a Cobb Douglas production function exhibits decreasing, constant, or increasing returns to scale according to whether the sum of its exponents is less than, equal to, or greater than 1.
  - Economists have usually found in their empirical studies that this sum is very close to 1.

While production functions are often homogeneous *by assumption*, demand functions are homogeneous *by nature* (at least if we ignore the "money illusion").

- Recall that a demand function $\mathbf{x} = D(\mathbf{p}, I)$ associates to each price vector $\mathbf{p} = (p_1, \ldots, p_n)$ and income level I, an individual's most-preferred consumption bundle $\mathbf{x}$ at those prices and income.
  - It is the solution of the basic consumer maximization problem: $\mathbf{x} = D(\mathbf{p}, I)$ maximizes $U(\mathbf{x})$ subject to the constraints $x_i \geq 0, \ \forall i$ and $\mathbf{p} \cdot \mathbf{x} \leq I$ $\mathbf{(4)}$.
  - Notice that if all the prices and the consumer's income tripled, constraint $\mathbf{(4)}$ would not change.
    - We could just divide the new inequality through by 3 to return to the original inequality.
- In terms of demand function: $D(t\mathbf{p}, tI) = D(\mathbf{p}, I), \ \forall \mathbf{p}, I$ $\mathbf{(5)}$.
  - Since $t^0 = 1$, eq. $\mathbf{(5)}$ states that demand is homogeneous of degree 0 in $\mathbf{p}$ and I.
  - Since each individual demand function is homogeneous of degree zero, the sum of these individual demands, aggregate demand, is also homogeneous of degree zero.

Finally, a similar, straightforward calculation shows that for a firm in a competitive market, the (minimal) cost function is a homogeneous function of input prices and the optimal function is a homogeneous function of output price.