Homework 2 – Numerical Methods in Financial Engineering

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1 Exercise 1: Warm-up

Generate a sample of independent random variables having the density $f : x \to 2xe^{-x^2}\mathbf{1}_{x>0}$. Explain. Plot the histogram of the empirical distribution of the generated sample against the true density.

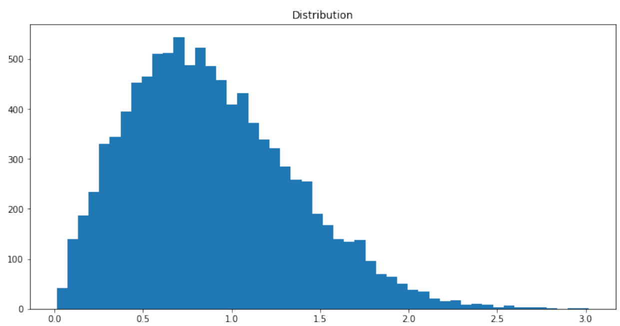

Figure 1: Histogram of the empirical distribution

One can observe that this distribution is positively skewed.

2 Exercise 2: Antithetic Variates

2.1 Prove the Lemma of Slide 40 of Lecture 2

For this exercise, we shall define different conditions and hypothesis:

- Lets X and Y two iid random variables
- Lets assume two functions f and g with the same monotony and L^2

By definition,

$$Cov(f(Y), g(Y)) = \mathbb{E}[f(Y)g(Y)] - \mathbb{E}[f(Y)]\mathbb{E}[g(Y)]$$

Since f and g are increasing (by our assumption), then

$$(f(X) - f(Y))(g(X) - g(Y)) \geq 0, \forall X, Y \in \mathbb{R}$$

By the linearity of the expectation, we have,

$$\begin{aligned} & \mathbb{E}\left[(f(X) - f(Y))(g(X) - g(Y))\right] \geq 0 \\ \iff & \mathbb{E}[f(X)g(X)] - \mathbb{E}[f(X)g(Y)] - \mathbb{E}[f(Y)g(X)] + \mathbb{E}[f(Y)g(Y)] \geq 0 \end{aligned}$$

By independence of the expectation, we have,

$$\begin{aligned}\mathbb{E}[f(X)g(X)] - \mathbb{E}[f(X)]\mathbb{E}[g(Y)] - \mathbb{E}[f(Y)]\mathbb{E}[g(X)] + \mathbb{E}[f(Y)g(Y)] \geq 0 \\ \iff 2\mathbb{E}[f(X)g(X)] - 2\mathbb{E}[f(Y)]\mathbb{E}[g(X)] \geq 0 \\ \iff 2Cov(f(X), h(X)) \geq 0 \\ \iff Cov(f(X), h(X)) \geq 0\end{aligned}$$

Similarly, if we assume that f is non-decreasing and h is non-increasing we get

$$Cov(f(X), g(X)) < 0$$

Hence,

$$Cov(g(Y), g(T(Y))) < 0$$

2.2 Show how it can be applied in the context of Black-Scholes to justify the exposition of slide 39 of Lecture 2 in the case of European call and put options

One can assume that $g(S_t)$ is monotonic, let's remind that

$$\frac{Var\left[\frac{g(S_T)+g(S_T^-)}{2}\right]}{N/2} < \frac{Var(g(S_t)}{N}$$

And,

$$S_t = S_0 e^{(R-\frac{\sigma^2}{2})t+\sigma W_t} \overset{(d)}{=} S_0 e^{(R-\frac{\sigma^2}{2})t-\sigma W_t} =: S_t^-$$

Hence, one can use Antithetic Variates to induce less calculus since $S_t \overset{(d)}{=} S_t^-$. So, we will use two times less calculus. It follow that for B&S application, we have to replace the function g by an European Call or Put.

3 Exercise 3: Antithetic Variates

Consider the Black-Scholes model,

$$dS_t = rS_t dt + \sigma S_t dW_t$$

with $S_0 > 0$

where W is a standard Brownian motion and $r, \sigma \geq 0$. For $K \geq 0$ and $T > 0$ we denote by $C_0(T, K)$ (resp.) $P_0(T, K)$) the Black-Scholes price of a European call (resp. put) option of strike K and maturity T on the underlying asset S and by $C_T(K)$ and $P_T(K)$ the corresponding payoffs, i.e.

$$C_T(K) = (S_T - K), P_T(K) = (K - S_T)$$

3.1 Show that the solution is given by

$$S_t = S_0 exp\left(\left(r - \frac{\sigma^2}{2}\right)t + \sigma W_t\right), t \geq 0$$

Set $f(S_t, t) = ln(S_t)$, by Ito's Formula:

$$\begin{aligned} df(S_t,t) &= 0d_t + \frac{1}{S_t}dS_t + \frac{1}{2}(-\frac{1}{S_t^2}d\langle S\rangle_t \\ &= \frac{1}{S_t}S_t(rdt + \sigma dW_t) - \frac{1}{2}\frac{S_t^2}{S_t^2}\sigma^2 dt \\ &= rdt + \sigma dW_t - \frac{1}{2}\sigma^2 dt \end{aligned}$$

Hence,

$$\begin{aligned} f(S_t,t) &= f(S_0,0) + \int_0^t \left(r - \frac{\sigma^2}{2}\right)ds + \int_0^t \sigma dW_s \\ \iff e^{f(S_t,t)} &= S_t = S_0 exp\left(\left(r - \frac{\sigma^2}{2}\right)t + \sigma W_t\right) \end{aligned}$$

3.2 In this question we fix $(S_0, T) = (100, 1)$ and

$$(r, \sigma, K) = (0.03, 0.3, 110)$$

3.2.1 Recall and implement the explicit formulas of C_0 and P_0. Provide the explicit values of C_0 and P_0

Recalling of formulas and implementation are present in the Jupyter Notebook. Here we have,

$$C_0 = (100 - 110)^+ = 0$$

$$P_0 = (110 - 100)^+ = 10$$

3.2.2 Estimate the values of the put and the call with a 95% confidence interval using

- i. standard Monte Carlo simulation with N sample paths,
  we have:

$$\frac{1}{N}\sum_{n=1}^{N}(S_T - K)^+ = \mathbb{E}\left[(S_T - K)^+\right] a.s$$

From the Central Limit Theorem, $(Y_n)_{n\in\mathbb{N}}$ be a sequence of $\mathbb{R}$-valued iid random variables such that $\mathbb{E}\left[|Y_1|^2\right] < \infty$, Assume $Var[Y_1] > 0$,

$$\hat{m}_N := \frac{1}{N}\sum_{n=1}^{N} Y_n$$

and

$$\hat{\sigma}_N := \sqrt{\frac{1}{N}\sum_{n=1}^{N}(Y_n - \hat{m}_N)^2}$$

Then,

$$\sqrt{N}\left(\frac{\hat{m}_N - \mathbb{E}\left[Y_1\right]}{\hat{\sigma}_N}\right)\mathbf{1}_{\hat{\sigma}_N>0} \to \mathcal{N}(0,1)$$

The asymptotic confidence intervals with $\alpha = 0.05$ gives a level of confidence of 95%, take $\alpha_C = 1 - 0.95 = 0.05$ and from tables, $C = 1.96$ when N is sufficiently large.

We get,

$$\mathbb{P}\left(\sqrt{N}|\frac{\hat{m}_N - \mathbb{E}\left[Y_1\right]}{\hat{\sigma}_N}|\leq C\right) \to 1 - \alpha_C$$
$$\iff \mathbb{E}[Y_1] \in \left[\hat{m}_N - \frac{1.96\hat{\sigma}_N}{\sqrt{N}}, \hat{m}_N + \frac{1.96\hat{\sigma}_N}{\sqrt{N}}\right]$$

with a probability of $1 - \alpha = 0.95$

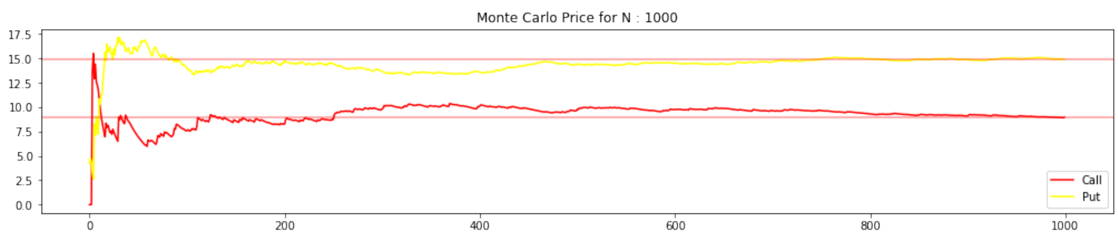

Figure 2: Monte-Carlo simulation for $N = 10^4$

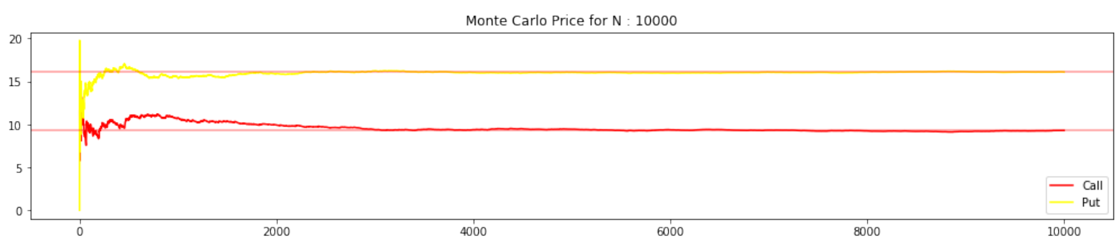

Figure 3: Monte-Carlo simulation for $N = 10^5$

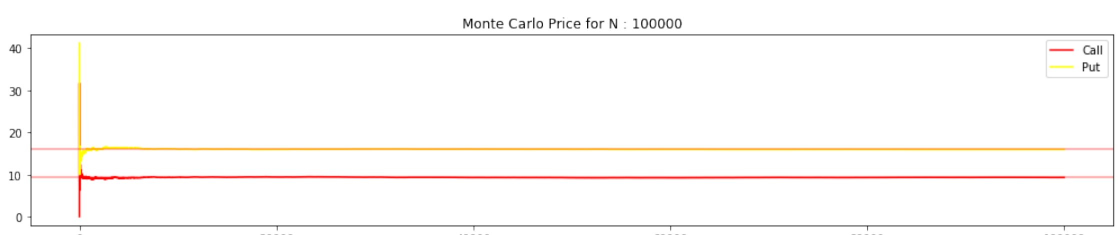

Figure 4: Monte-Carlo simulation for $N = 10^6$

- ii. Monte Carlo simulation with antithetic variates with $N/2$ sample paths, for $N = 10^4, 10^5, 10^6$

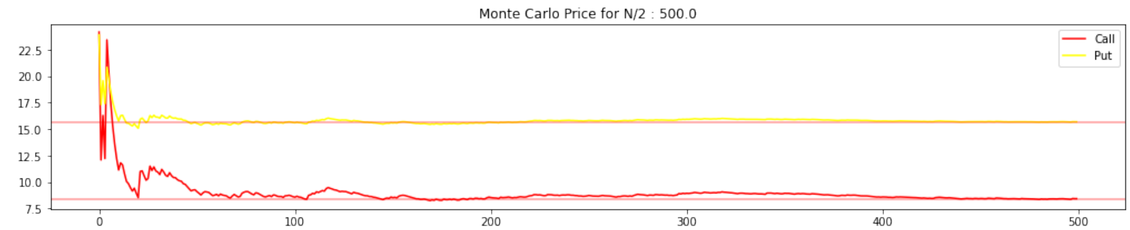

Figure 5: Monte-Carlo simulation with antithetic variates for $N = 500$

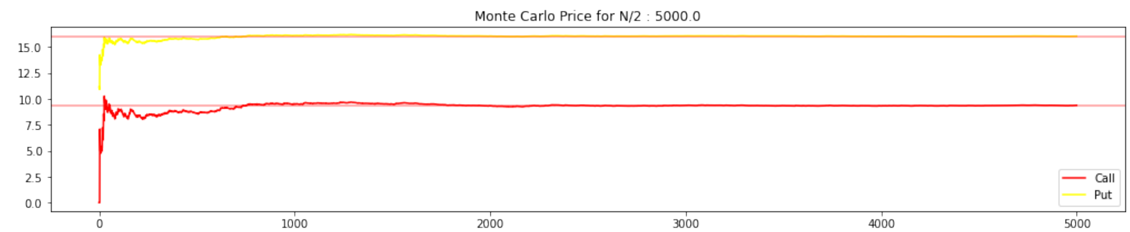

Figure 6: Monte-Carlo simulation with antithetic variates for $N = 5000$

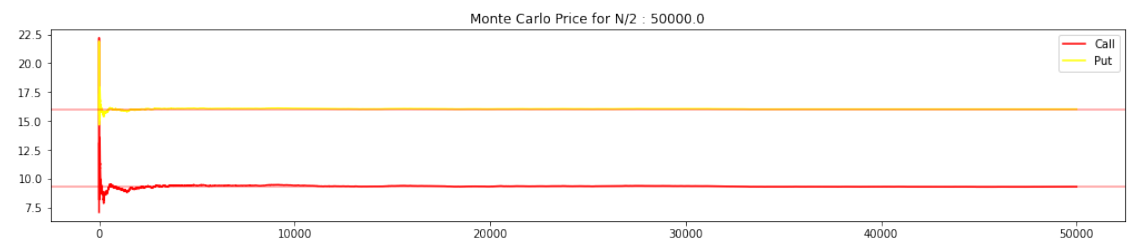

Figure 7: Monte-Carlo simulation with antithetic variates for $N = 50000$

3.2.3 Comment, can the observed result be justified theoretically?

One can observe that the results confirm the theory. Indeed, the convergence is two times faster with the antithetic variates. As an example, one can compare figures by figures in each cases. For instance, it takes about 800 iterations to converge toward the call price in the classic case when it takes only 400 iterations with antithetic variates. Moreover, we can affirm that the observed result can be justified theoretically since we demonstrated it during the exercice 2.

3.3 Applications and demonstrations

3.3.1 Fix $T = 1$, $r = 0$ and $S_0 = 100$. For each value of $\sigma \in (0.1, 0.25, 0.3, 0.5, 0.75)$, estimate by Monte-Carlo with 2×10^5 sample paths the prices of call and put options $C(T, K)$ and $P(T, K)$ for a range of strikes between $K \in (10, 200)$ with 95% confidence intervals

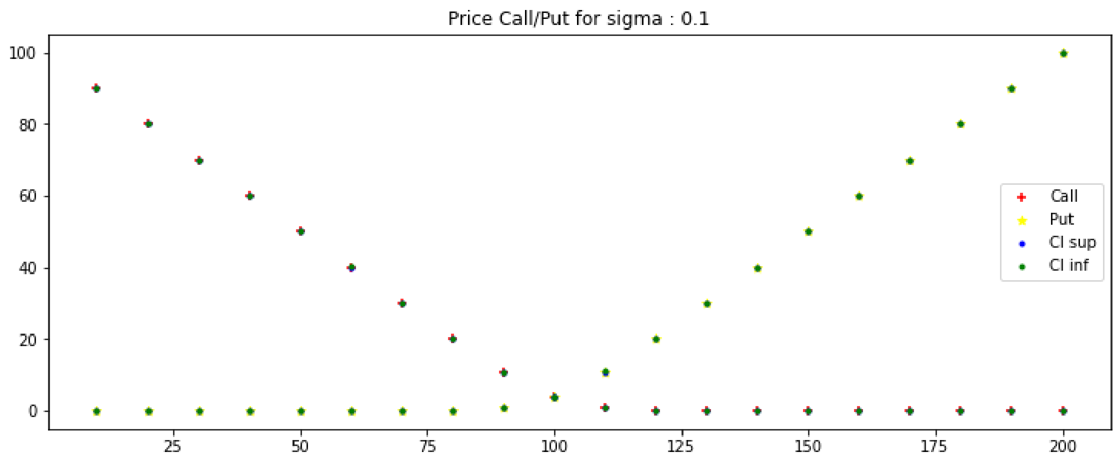

Figure 8: Monte-Carlo simulation of Call and Put prices for $\sigma = 0.1$

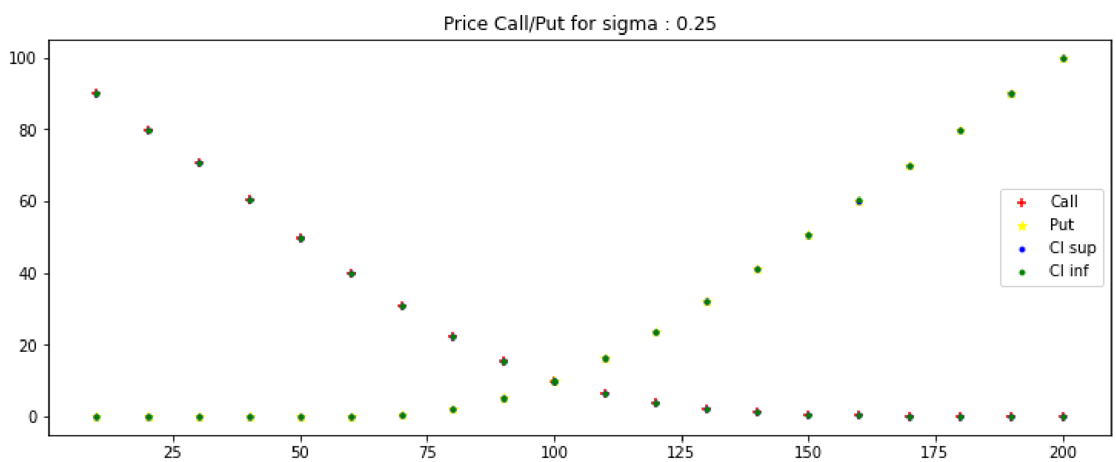

Figure 9: Monte-Carlo simulation of Call and Put prices for $\sigma = 0.25$

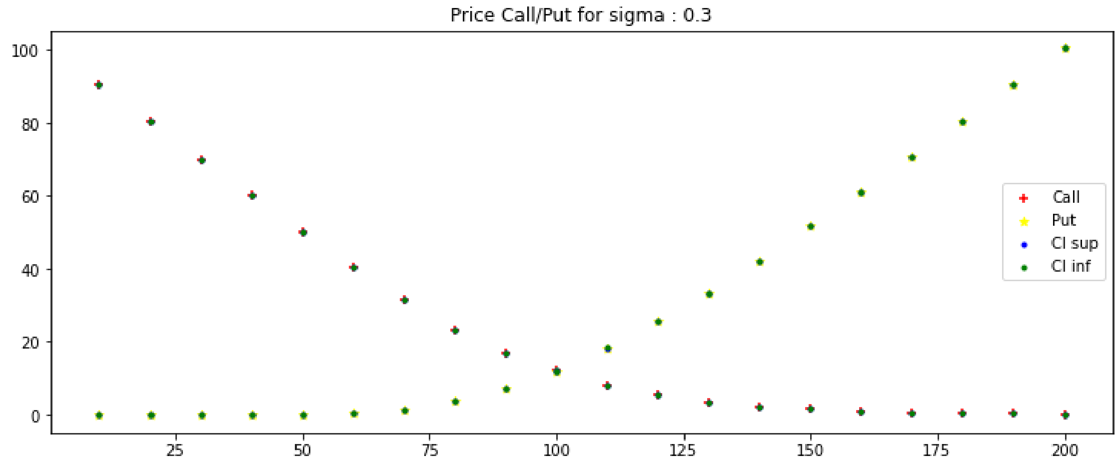

Figure 10: Monte-Carlo simulation of Call and Put prices for $\sigma = 0.3$

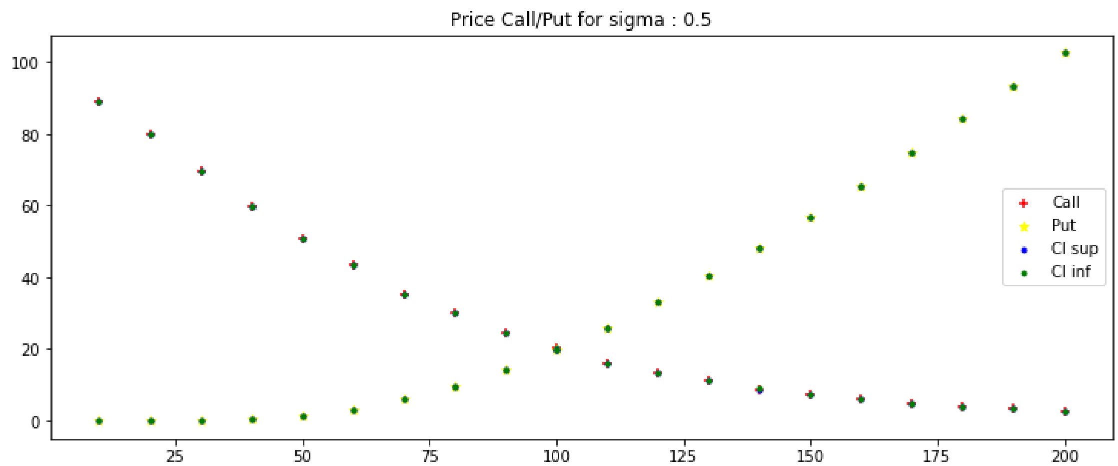

Figure 11: Monte-Carlo simulation of Call and Put prices for $\sigma = 0.5$

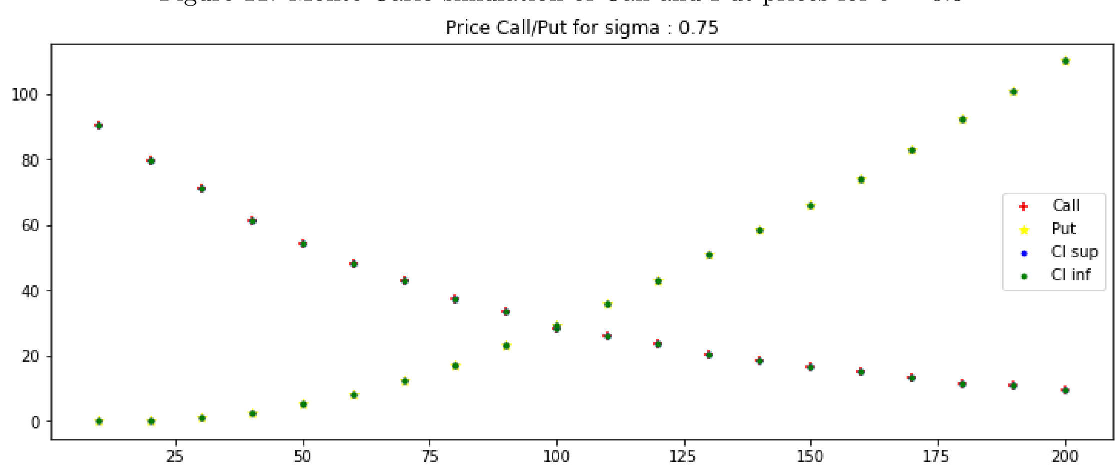

Figure 12: Monte-Carlo simulation of Call and Put prices for $\sigma = 0.75$

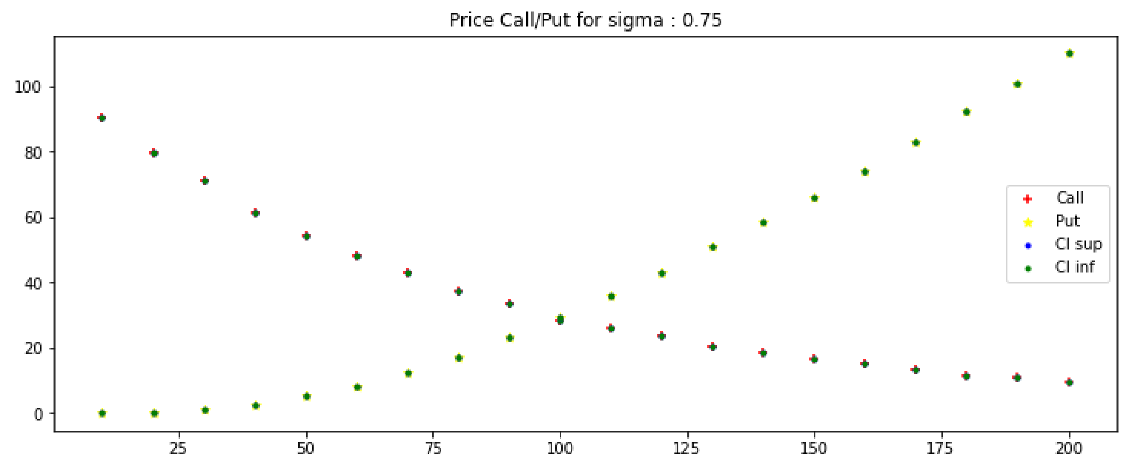

Figure 13: Monte-Carlo simulation of Call and Put prices for $\sigma = 0.75$

3.3.2 For each σ as in the previous question, plot the ratio of the empirical variance of the Monte-Carlo estimator for the call option against that of the put option as a function of the strike. (Plot the curves on the same graph with different colors for σ. Comment.

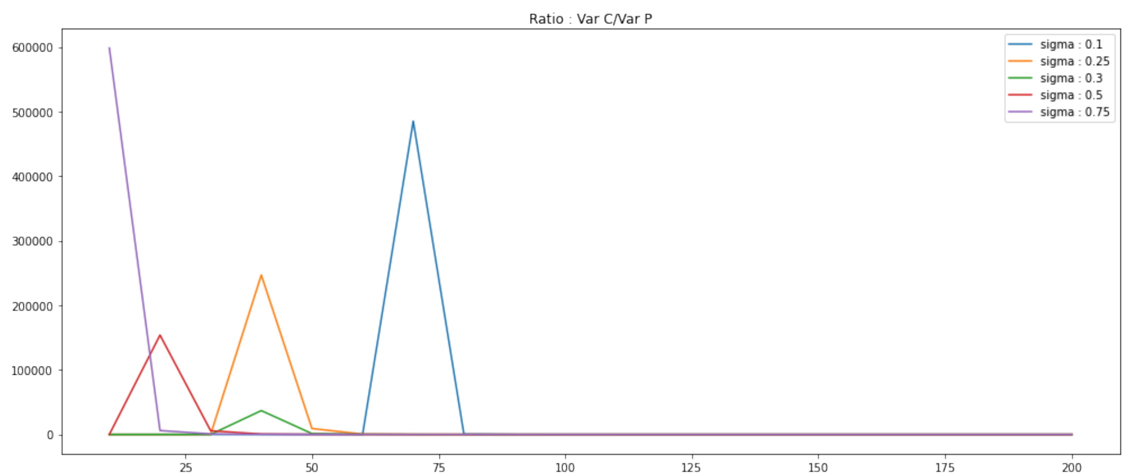

Figure 14: Ratio of the empirical variance of the Monte-Carlo estimator for each previous σ

In the very beginning, the ratio is explosive because Put's variance is close to 0 since the Put is Deep-ITM (Deep-In-The-Money). Indeed, in degenerates cases (so Deep OTM or ITM), there is no randomness for the option pay-off.

NB: We had to manage some exceptions in order not to divide by 0. Please find more precision on the Jupyter Notebook.

3.3.3 Let $F_T(K) = Var(C_T(K)) - Var(P_T(K))$. Show that

$$F_T(K) \;=\; S_0^2 e^{2rT}\left(^{\sigma^2 T} - 1\right) \;-\; 2S_0^2 e^{2rT}\left(e^{\sigma^2 T}\mathcal{N}(-d_3) - \mathcal{N}(-d_1)\right) \;+\; 2KS_0e^{rT}\left(\mathcal{N}(-d_1) - \mathcal{N}(-d_2)\right) \quad (1)$$

where we defined $\mathcal{N}(x) = \int_{\infty}^{x} e^{z^2/2}dz/\sqrt{2\pi}$ and

$$d_1 = \frac{ln(\frac{S_0 e^{rT}}{K}) + \frac{\sigma^2}{2}T}{\sigma\sqrt{T}}$$

$$d_2 = d_1 - \sigma\sqrt{T}$$
$$d_3 = d_1 + \sigma\sqrt{T}$$

3.3.4 For $T = 1$ and $r = 0$. Plot the function $K \to F_T(K)$. Comment.

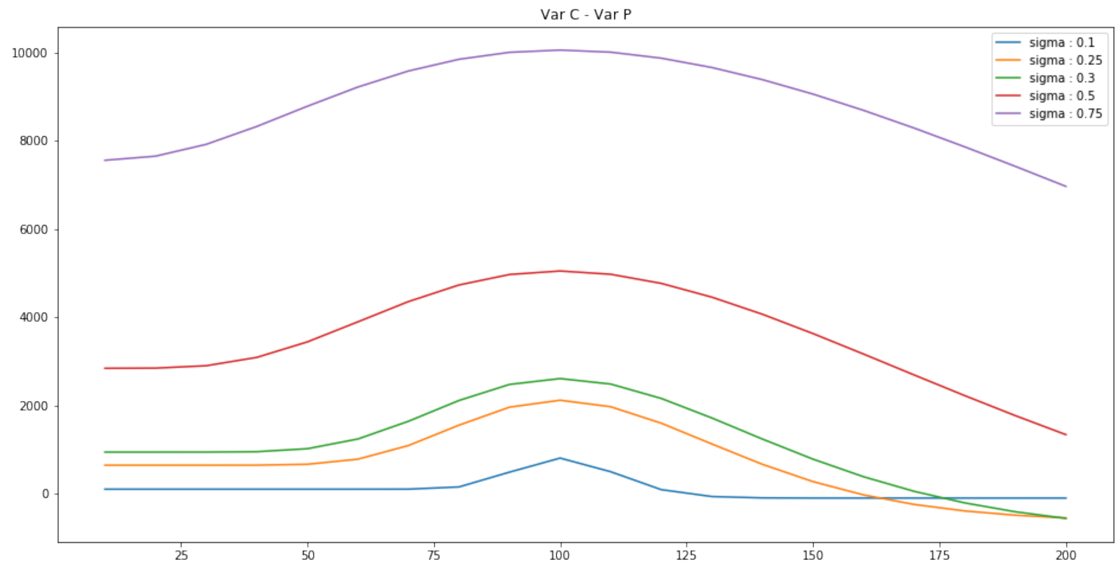

Figure 15: Plot of $F_T(K) = Var(C_T(K)) - Var(P_T(K))$

Reminding that $F_Y(K) = Var(Call) - Var(Put)$ and one can observe that:

- The difference is the largest at the money. This one is concave and maximum at this point.
- Generally, the $Var(C_T(K)) > Var(P_T(K)$
- In extreme cases, $K << S_0$ or $K >> S_0$, the difference in variance between the two prices is reducing.
- This difference is increasing in volatility. The largest is the volatility, the greatest is the difference.
- Moreover, in very degenerating cases when S_0 is large, this implies negative difference for volatility sufficiently high. Hence, when $S_0 >> K$ we have $Var(P_T(K)) > Var(C_T(K))$

This is why one can see the increasing and decreasing shape of the curves around the strike.

3.3.5 Fix $T > 0$. We now move to the mathematical study of the function F_t

- i. What are the limits $\lim_{K \to 0} F_T(K)$ and $\lim_{K \to \infty} F_T(K)$?

  Lets first tackle $\lim_{K \to 0} F_T(K)$, given that

$$\lim_{K \to 0} d_1 = \lim_{K \to 0} d_2 = \lim_{K \to 0} d_3 = +\infty$$

One can write

$$\lim_{d_1 \to +\infty} \mathcal{N}(-d_1) = \lim_{d_1 \to +\infty} \int_{\infty}^{d_1} e^{z^2/2} dz/\sqrt{2\pi} = 0$$

Hence, it follows that $\lim_{d_3 \to +\infty} \mathcal{N}(-d_3) = 0$ and

$$\lim_{K \to 0} -2_0^2 e^{2rT} \left(e^{\sigma^2 T} \mathcal{N}(-d_3) - \mathcal{N}(-d_1) \right)$$

So that,

$$\lim_{K \to 0} F_T(K) = S_0^2 e^{2rT} \left({}^{\sigma^2 T} - 1 \right)$$

Let's continue with $\lim_{K \to \infty} F_T(K)$,

$$\lim_{K \to +\infty} d_1 = \lim_{K \to +\infty} d_2 = \lim_{K \to +\infty} d_3 = -\infty$$

Hence as before,

$$\lim_{d_1 \to -\infty} \mathcal{N}(-d_1) = \lim_{d_2 \to -\infty} \mathcal{N}(-d_2) = \lim_{d_3 \to -\infty} \mathcal{N}(-d_3) = 1$$

This is why,

$$\lim_{K \to \infty} F_T(K) = 0$$

- ii. Show that

$$F_T'(K) = 2S_0 e^{rT} \left(\mathcal{N}(-d_1) - \mathcal{N}(-d_2) \right) < 0$$

One can easily see that,

$$F_T'(K) = 2S_0 e^{rT} \left(\mathcal{N}(-d_1) - \mathcal{N}(-d_2) \right)$$

Moreover,

$$2S_0 e^{rT} > 0$$

And,

$$\begin{aligned} d_1 &> d_2 \\ \iff -d1_< &- d_2 \\ \iff \mathcal{N}(-d_1) &< \mathcal{N}(-d_2) \\ \iff \mathcal{N}(-d_1) - \mathcal{N}(-d_2) &< 0 \end{aligned}$$

Hence,

$$F_T'(K) = 2S_0 e^{rT} \left(\mathcal{N}(-d_1) - \mathcal{N}(-d_2) \right) < 0$$

- iii. Show that $F''(S_0 e^{rT}) = 0$ and $F(S_0 e^{rT}) > 0$.
Here, we first compute the values of d_1, d_2 and d_3 we get

$$ln(\frac{S_0 e^{rT}}{S_0 e^{rT}}) = ln(1) = 0$$

Then,

$$\begin{aligned} d_1 &= \frac{\frac{\sigma^2}{2} T}{\sigma \sqrt{T}} = \frac{\sigma}{2} \sqrt{T} \\ d_2 &= -\frac{\sigma}{2} \sqrt{T} \\ d_3 &= \frac{3}{2} \sigma \sqrt{T} \end{aligned}$$

3.3.6 Comment on the usefulness of question 3). A detailed answer is expected based on the empirical/theoretical results together with what you have seen during the lectures.

In this section, we have seen that the Monte-Carlo methods is more than useful to compute a complex payoff option price $F_T(K)$. This complex product payoff is the difference between the variance of a call and a put.