Homework 3 – Numerical Methods in Financial Engineering

Hugo MOREL, David PENG and Léo PIACENTINO

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1 Exercise 1: Truncation error

Consider a function, $U(t,S) \in C^{1,2}(\Gamma) \cap C(\overline{\Gamma})$ where,

$$\Gamma = (0,T] \times (S_{\min}, S_{\max})$$

$$\overline{\Gamma} = [0,T] \times [S_{\min}, S_{\max}]$$

Suppose that,

$$\frac{\delta U}{\delta t}(t,S) - \frac{\tilde{\sigma}(t,S)^2}{2} S^2 \frac{\delta^2 U}{\delta S^2}(t,S) - r(t) S \frac{\delta U}{\delta S}(t,S) + r(t)U(t,S) \leq 0, \forall S \in (S_{\min}, S_{\max}), t \in (0,T].$$

We say that the (weak) maximum principle holds if,

$$\max_{(t,S)\in\delta\overline{\Gamma}} U(t,S) = \max_{(t,S)\in\delta\Gamma} U(t,S)$$

where $\delta\Gamma = \frac{\delta\overline{\Gamma}}{\delta\Gamma} = \{0\} \times [S_{\min}, S_{\max}] \cup [0,T] \times S_{min} \cup [0,T] \times S_{max}$

1.1 Suppose that the (weak) maximum principle holds. Prove Proposition 1 in slides 20 of lecture 3.

Let us consider a put option with payoff $\Psi(x) = (K-S)^+$. Hence, $S = S_{min}(S_{min} \leq K)$ so that the estimate is

$$K - S \leq p(S) \leq K$$

Hence we can write,

$$Ke^{-r(T-t)} - Se^{-D(T-t)} \leq u(S,t) \leq Ke^{-r(T-t)}$$

$$\begin{aligned}
\max_{t\in[0,T]} |u(S_{\min},t) - p(S_{\min})| &= \max_{t\in[0,T]} |u(S_{\min},t) - (K - S_{\min})| \\
&= \max_{t\in[0,T]} \max\left(|Ke^{-r(T-T)} - K + S_{\min}|, |Ke^{-r(T-t)} - S_{\min}e^{-D(T-t)} - K + S_{\min}|\right) \\
&= \max\left(\max_{t\in[0,T]} |Ke^{-r(T-T)} - K + S_{\min}|, \max_{t\in[0,T]} |Ke^{-r(T-t)} - S_{\min}e^{-D(T-t)} - K + S_{\min}|\right)
\end{aligned}$$

Let's now consider the first term,

$$M_1 = \max_{t\in[0,T]} \left(|Ke^{-r(T-T)} - K + S_{\min}\right)$$

The function inside the absolute value signs increasing when t increases, therefore its largest and smallest values are attained at $t = T$ and $t = 0$. Thus

$$\begin{aligned} M_1 &= \max\left(|Ke^{-rT}e^{-r0} - K + S_{\min}|, |Ke^{-rT}e^{-rT} - K + S_{\min}|\right) \\ &= \max\left(|Ke^{-rT} - K + S_{\min}|, S_{\min}\right) \end{aligned}$$

Taking the second term of the previous equation,

$$M_2 = \max_{t\in[0,T]} \left(|Ke^{-r(T-t)} - S_{\min}e^{-D(T-t)} - K + S_{\min}|\right)$$

And let,

$$f(t) = K(e^{-rT}e^{rt}) - S_{\min}e^{-DT}e^{Dt} - K + S_{\min}$$

The extreme values of this function can occur at $t = 0, t = T$ and at the critical points $t = t_c$ but since $f(T) = 0$ we have to consider only $t = 0$ and $t = t_c$. Finding the critical points of the function $f(t)$, we get

$$\begin{aligned} f'(t_c) = Kre^{-r(T-t_c)} - DS_{\min}e^{-D(T-t_c)} = 0 \\ t_c = \frac{1}{r-D}\ln(\frac{DS_{\min}}{rK}) + T \end{aligned}$$

if $r \neq 0, D \neq r$ and no critical points otherwise. After some calculus we get,

$$f(t_c) = S_{\min}(\frac{D}{r} - 1)(\frac{DS_{\min}}{rK})^{(\frac{D}{r-D})} - K + S_{\min}$$

Therefore for $0 < t_c < T$ we have,

$$M_2 = \max\left(|Ke^{-rT} - S_{\min}e^{DT} - K + S_{\min}|, |S_{\min}(\frac{D}{r} - 1)(\frac{DS_{\min}}{rK})^{(\frac{D}{r-D})} - K + S_{\min}|\right)$$

and for $t_c \notin [0, T]$ we have

$$M_2 = |Ke^{-rT} - S_{\min}e^{DT} - K + S_{\min}|$$

This leads to obtaining the following estimate

$$\max_{t\in[0,T]} |u(S_{\min}, t) - p(S_{\min}) \leq C_1, t \in [0, T]|$$

where $C_1 = \max(M_1, M_2)$

Getting the estimates for the upper boundary $S_{max}(S_{max} > K)$. Now we use the estimates

$$0 \leq p(S) \leq K$$

one can write as before,

$$0 \leq u(S,t) \leq Ke^{-r(T-t)}$$

Note that $p(S_max) = 0$, thus

$$\max_{t\in[0,T]} |u(S_{\max},t) - p(S_{\max})| = \max_{t\in[0,T]} |u(S_{\max},t)| \qquad = \max_{t\in[0,T]} |Ke^{-r(T-t)}| = K$$

Hence,

$$|u(S_{\max},t) - p(S_{\max})| \leq C_2$$

where $C_2 = K$

1.2 Prove Proposition 2 in slide 21 of lecture 3.

2 Exercice 2

2.1 Question 1 : Explicit Euler Method:

Here is the find of the Explicit Euler method:

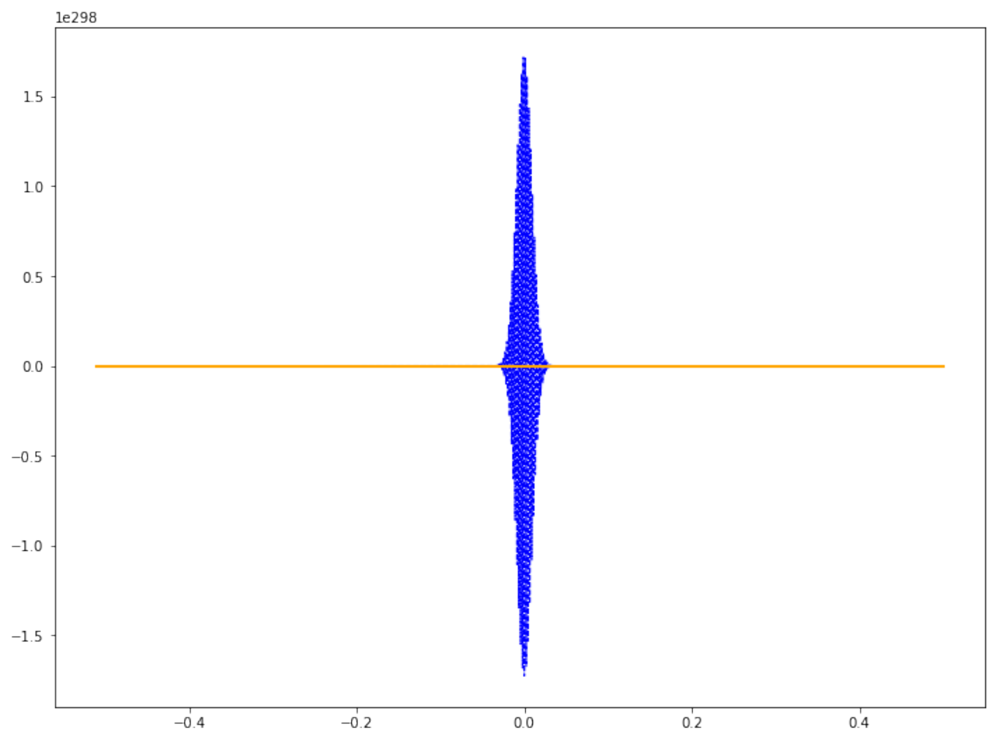

Figure 1: Explicit Euler Method

As one can see, the explicit Euler method is not stable. Indeed, the value of the approximated puts options are oscillate with a large amplitude.

2.2 Question 2 : The θ-method (Crank-Nicolson):

The θ-method requires the Implicit Euler method that we computed in the jupyter notebook, here is the Crank-Nicolson method:

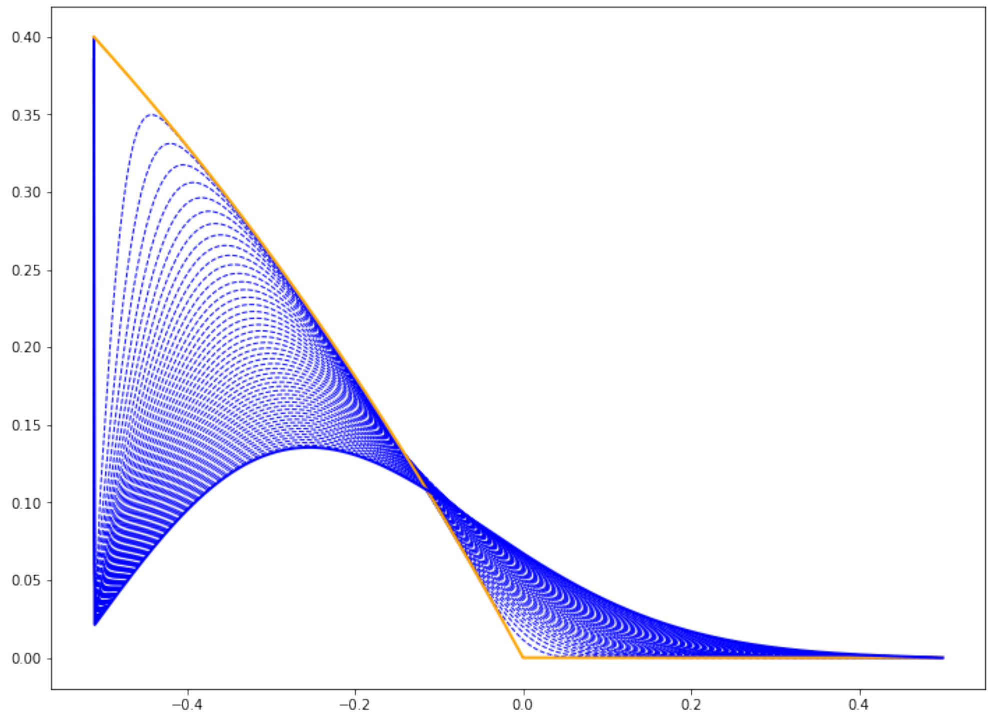

Figure 2: θ-method (Crank-Nicolson)

Here, we used Crank-Nicolson which is a mix of the Euler Implicit and Explicit schemes. We have seen that the Explicit Method is not stable. Indeed, we have not verified key assumption of consistency with C.F.L (Courant–Friedrichs–Lewy) condition. On the other hand, the Implicit Scheme does not require any assumption or condition.

One can see that the θ-method seems to converge toward put price. However, the error seems relatively high. Unfortunately, we didn't manage to find exploitable results concerning committed error.

3 Exercice 3 : Stability of the Implicit Euler method

Let $\mathscr{L} = \left\{(h, k) \in \mathbb{R}_+^* \times \mathbb{R}_+^*; \frac{k}{h^2} \leq \frac{1}{2}\right\}$ be the the condition of stability.

Let us choose the general form,

$$\frac{1}{k}U^{j+1} - \frac{1}{k}U^j + A_h U^j = F^j$$

for which we have consistency in the $\|.\|_{\infty,h}$ norms and $G^j = F^j$ it can be rewritten as

$$U^{j+1} = \mathscr{A}_{k,h} U^j + kF^j$$

where $\mathscr{A}_{k,h} = \mathbf{I} - kA_h$. Therefore

$$\|U^{j+1}\|_{\infty,h} \leq \||\mathscr{A}_{k,h}\||_{\infty,h}\|U^j\|_{\infty,h} + k\|F^j\|_{\infty,h}$$

Let us set $r = \frac{k}{h^2}$. By direct inspection knowing that the matrix is tri-diagonal we see that

$$\mathscr{A}_{k,h} = \begin{pmatrix} 1-2r & r & 0 & \dots & 0 \\ r & 1-2r & r & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \dots \\ \vdots & \ddots & \ddots & \ddots & r \\ 0 & \dots & 0 & r & 1-2r \end{pmatrix}$$

It follows that,

$$\||\mathscr{A}_{k,h}\||_{\infty,h} = |1-2r| + 2r = \begin{cases} 1, & \text{if } r \leq \frac{1}{2} \\ 4r-1, & \text{if } r > \frac{1}{2} \end{cases}$$

Therefor, if $r \leq \frac{1}{2}$, we have that

$$\begin{aligned} \|U^{j+1}\|_{\infty,h} &\leq \|U^j\|_{\infty,h} + k\|F^j\|_{\infty,h} \\ &\leq \|U^j\|_{\infty,h} + k \max_{n \leq T/k} \|F^j\|_{\infty,h} \end{aligned}$$

Iterating backwards, we obtain that for all j such that $j \leq \frac{T}{k}$,

$$\begin{aligned} \|U^j\|_{\infty,h} &\leq \|U^0\|_{\infty,h} + jk \max_{n \leq T/k} \|F^j\|_{\infty,h} \\ &\leq \|U^0\|_{\infty,h} + T \max_{n \leq T/k} \|F^j\|_{\infty,h} \end{aligned}$$

Hence, is the stability of the scheme for the ∞-norm under the condition $\mathscr{L}$