VOLATILITY IS ROUGH

REPORT OF A STUDY ON ROUGH VOLATILITY: How to best estimate and forecast volatility for “high-frequency” data?

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«Estimating volatility from recent high frequency data, we revisit the question of the smoothness of the volatility process. Our main result is that log-volatility behaves essentially as a fractional Brownian motion with Hurst exponent H of order 0.1, at any reasonable time scale. » –Volatility is Rough (2014)

«The success of this model in reconciling microstructure with implied volatility surfaces provides us with an unified framework for pricing and risk management. » – CMAP's Website: https://quantreg.com/research/volatility/rough-volatility/, École Polytechnique

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1 Introduction

The purpose of this paper is to model volatility. Several previous papers have been written with this goal in mind but not all of them model it perfectly.

We can cite the paper by Dupire which models the volatility surface from the market prices of European vanilla options. It turns out that the modeling is inhomogeneous but also that the dynamics is very unrealistic which generates volatility surfaces is completely different from what can be observed.

Another type of volatility model is the stochastic volatility model with a continuous Brownian semi martingale like the Hull & White model or the Heston model. The dynamics are much more realistic than the local volatility model, but they do not correspond to observed European option prices.

What is being done a lot now is to use the local stochastic volatility (LSV) model which matches the market and generates similar dynamics.

The element that will allow us to understand the stochastic volatility is the fractional Brownian motion. Indeed, volatility is a long memory process and therefore in the literature we have several papers, including one that we will detail right after. This foundation paper is about uses fBM (fractional Brownian Motion) which allows to model a long memory by choosing a Hurst parameter $H > ½$.

The long memory process of the volatility process always has been largely accepted as a stylized fact. It first appeared that long memory referred to the slow decay of the autocorrelation function (anything slower than an exponential). However, the article quotes Rama Conte "the econometric debate on the short range or long-range nature of dependence in volatility still goes on (and may probably never be resolved)". But for some time, it has appeared more as the fact that the autocorrelation function is not integrable and even more that it decays like a power law with an exponent less than 1. To be more specific, a weakly stationary process have a long memory if its ACF has a slow decay.

In this paper it is shown that the autocorrelation function of volatility does not have this form at the usual time scales, so they try to provide explicit expressions that allows to analyze the dependence structure of the volatility process.

In the field, the volatility surface is modelized by SVI which is homogenous, and the homogenous models are Hull & White model or Heston model. But they do not correspond to the surface of the volatility. On the other side from the paper of Fukusawa which will be described later, authors found out that the surface of volatility can be modelized by the fractional Brownian Motion. They will use in this model a volatility with fractional Brownian motion with a Hurst parameter less than ½ which can be consistent with the empirically observed specific property of volatility time series but also with the shape of the volatility surface. They focus on modeling the volatility time series.

Report's objective is to understand the paper first but also to critic it as much as possible in a constructive way. To do so, we have studied the environment around the paper thanks to a short but explicit literature review. After that, we focused on the main theory, ideas and equation which are developed in the paper. Then, main results are discussed, interpreted, and implemented over another time-period. Finally, we brought a critic of the paper. Beside of this report we implement all the key results in Python.

2 Literature review

First, one can denote that this paper is fundamental in the rest of the literature linked to rough volatility. Indeed, the authors paved the way for numerous other related articles. Indeed, one can see how many papers were published from 2014 to nowadays (2021) on a website reporting the progress of related research. The explosion of articles written on this subject certainly comes from the fact that front-offices and financial markets are more and more concerned by risks. Hence, the main risk factor is nothing else than volatility. A better understanding of it would allow a better risk management and then a more efficient business. Reducing costs while developing activity leads to a higher net income for a bank. In this part, we will focus on six selected articles which seems key in our sense. The first paper review is the only paper released before 2014, all the other are released after 2014 and thus have been influenced by "Volatility is Rough". Reviews are tackled in a chronological order.

The first paper to detail should be Comte and Renault's papers: "Long Memory in Continuous-Time Stochastic Volatility Models" (1998). Indeed, the authors are consistently referring to this paper which introduced the notion of Fractional Stochastic Volatility (FSV). Their finding of interest in our case is volatility's long-memory features and properties and they effectively notice the fractality property of volatility (see 2.2 part of this paper).

In September 2016, M. Rosenbaum and O. El Euch worked on the "The Characteristic Function of Rough Heston Models". The well-known characteristic function of classical Heston Model is expressed as a solution of a Riccati equation. Hence, they find that when volatility is rough in our paper sense, a Rough Heston Models is expressed as a solution of a fractional Riccati equation.

In "The Microstructural Foundations of Leverage Effect and Rough Volatility" released in September 2016, M. Rosenbaum, M. Fukasawa and O. El Euch answered the problematic stating that microscopic interactions are impacting volatility and more precisely the leverage effect. To do so, they used fractional Brownian motion, so the RFSV model. Hence, they proved that high-frequency data generate leverage effect and a typical rough volatility.

Again, M. Rosenbaum and O. El Euch worked together on "Perfect Hedging in Rough Heston Models" in 2017, a paper that follow their two previous work in 2016. They proved that one can find an explicit hedging strategy in their previous framework of rough Heston Models.

E. Abi Jaber and O. El Euch released "Multi-factor approximation of rough volatility models" in April 2018. They achieved to use numerical methods to solve fractional Ricatti equation which are key in rough volatility model. This is a real advancement for financial market professional and applications.

In May 2018, a paper named "No-Arbitrage Implies Power-Law Market Impact and Rough Volatility" was released by P. Jusselin and M. Rosenbaum. The study the use of rough volatility for prices diffusion. Indeed, they show how a microscopic element in fractional BM impact the macroscopic price.

3 Theory

3.1 Fractional Brownian Motion to increments of log-volatility

As presented in the introduction, the volatility function takes different forms. It can have a constant form, as a deterministic function of time noted $\sigma(t)$, in the "Black & Scholes model". Or it can be a local volatility, with a deterministic function of the underlying price and time noted $\sigma(Y_t, t)$, estimated to match the observations prices in the "Dupire model". And finally, it can be a stochastic volatility modelled as a continuous Brownian semi-martingale, in the "Hull and White model". As a remark, the stochastic volatility is much more realistic than local volatility dynamic but do not perfectly fit with the observed European option prices.

Therefore, fractional stochastic volatility (FSV) has been introduced with long memory process. A fundamental paper written by Comte and Renault and detailed in the Literature Review, introduced log-volatility model using fractional Brownian Motion and ensure long memory with Hurst parameter $H > \frac{1}{2}$.

Fractional stochastic volatility is based on the Fractional Brownian Motion (fBM) introduced in 1940 by Kolmogorov, the main idea is to get Gaussian spiral process in the Hilbert space. The fBM $\{W_t^H \,; t \in R\}$ is the unique Gaussian process with mean zero and autocovariance function:

$$E[W_t^H W_s^H] = \frac{1}{2}\,\{|t|^{2H} + |s|^{2H} - |t-s|^{2H}\}$$

where $H \in (0,1)$ is called the Hurst index or parameter

In practical, different cases of H:

- If H = $\frac{1}{2}$, fBM is just Brownian Motion
- If $H > \frac{1}{2}$, increments are positively correlated, which induce a trending process
- If $H < \frac{1}{2}$, increments are negatively correlated, which induce a reverting process

Moreover, the FBM holds three principal properties:

- The process is self-similar because the covariance function is homogeneous of order 2H such that; $W_{at}^H \sim |a|^H W_t^H$
- The process has stationary increments such that; $W_t^H - W_s^H \sim W_{t-s}^H$
- For $H > \frac{1}{2}$, the process exhibits a long-range dependence:

$$\sum_{n=1}^{\infty} E[W_1^H (W_{n+1}^H - W_n^H)] = \infty$$

The authors adopted the fractional Brownian motion and stochastic volatility model from Comte and Renault, and called their model rough FSV (RFSV), furthermore in contrast to the FSV model they used a Hurst parameter $H < \frac{1}{2}$. However, the RFSV doesn't hold long memory property, but leads to conclude to the presence of long memory in data generated from it.

In page 7 of the article, authors deal with the estimation of the smoothness of the volatility. To do so, they pose the average of the incremental log-volatility under the norm $q \geq 0$:

$$m(q, \Delta) = \frac{1}{N}\sum_{k=1}^{N}\left|log(\sigma_{k\Delta}) - log\left(\sigma_{(k-1)\Delta}\right)\right|^q$$

where, Δ is the time increment and $k \in \left\{0, \left[\frac{T}{\Delta}\right]\right\}$.

Hence, thanks to this equation authors can state that this mean is the best empirical estimation of the expectation of the log volatility increment:

$$\mathbb{E}[|\log(\sigma_\Delta) - \log(\sigma_0)|^q]$$

The idea behind the previous average is to show a "simple" way of estimating expectations. Intuitively, we can expect the authors to look out the $\mathcal{L}^2$ norm in their application. Since it is the most natural and explanatory choice. Indeed, the square distance is generally used as a good measure of committed errors.

This equation,

$$\mathbb{E}[|\log(\sigma_\Delta) - \log(\sigma_0)|^q] = \mathrm{Kq}\Delta^{\zeta q}$$

where $\zeta q > 0$

is the slope of the line associated to q, is an implication, under stationarity assumptions, that the log-volatility increments enjoy the following scaling property in expectation. This is provided from the DAX and the Bund futures which are the most liquid asset, and this is deduce from the model of uncertainty zones to estimate volatility from Dayri and Rosenbaum.
In Figure 2.1 of the paper, both DAX and Bund, for a given q, all points essentially lie on a straight line. Plus, what is interested is that ζq is equivalent to the Hurst parameters in the paper's Figure 2.2 which mean that the smoothness parameter sq does not seem to depend on q.

3.2 Understanding RFSV

The scaling property presented in the previous part leads the author to set up the following model:

$$log(\sigma_{t+\Delta}) - log(\sigma_t) = \nu(W_{t+\Delta}^H - W_t^H)$$

So, $log(\sigma_t) = \nu W_t^H$

$$\sigma_t = e^{\nu W_t^H}$$

Hence more generally they state that, $\sigma_t = \sigma e^{\nu W_t^H}$. Where σ, ν are constant and W_t^H is a fractional Brownian Motion.
This means that the increments of log-volatility are driven by a linear model and its process is exponentially linear. But one can see that this model is not weakly stationary since expectation is not constant over time and the variance is not finite. Stationarity is a key property for a process to be used and applied in real market conditions. This is the reason why authors chosen to set up an Ornstein-Uhlenbeck model. The other reasoning is to capture the mean-reverting stylized fact of volatility. Hence, considering the fractality property of W_t^H, a fractal Ornstein-Uhlenbeck process (fOU) is considered.

By reading Cheridito, Kawaguchi and Maejima papers called "Factional Ornstein-Uhlenbeck Processus" written in 2003, one can better understand this type of process. They showed that this process comes from Langevin equation, as a sketch-of-the-proof:

Let $\lambda, \sigma > 0$ and $\xi \in L^0(\Omega)$,

$$X_t = \xi - \lambda \int_0^t X_s ds + N_t, t \geq 0$$

and this is then extendable for W_t^H.

Hence, from the description of the movement of a particle in a liquid:

$$\frac{dv(t)}{dt} = -\frac{f}{m} v(t) + \frac{F(t)}{m}$$

Where $m > 0$ is the mass, $f > 0$ a friction coefficient. It follows that for $H \in (0,1]$ and $a \in [-\infty, \infty)$, the path-wise Riemann-Stieltjes integral stated but not presented by the author, is given by:

$$\int_a^t e^{\lambda u} dB_u^H \, , t \geq a$$

Thus, this equation is used to derive:

$$X_t = \xi - \lambda \int_0^t X_s ds + \sigma W_t^H, t \geq 0$$

Where it can be written as follow without condition on t:

$$X_t = \nu \int_{-\infty}^t e^{-\alpha(t-s)} d\, W_t^H + m$$

For any constant m, hence applying Itô's formula,

$$dX_t = -\alpha(X_t - m)dt + \nu dW_t^H$$

Hence one can observe the mean reversion drift which fit with rough volatility empirical property.

Always in "Factional Ornstein-Uhlenbeck Processus" (2003), the authors studied the autocovariance of the log-volatility to find the following expression:

$$Cov[\log(\sigma_t), \log(\sigma_{t+\Delta}) = \frac{H(2H - 1)v^2}{2\alpha^{2H}} \left\{ e^{-\alpha\Delta}\Gamma(2H-1) + e^{-\alpha\Delta}\int_0^{\alpha\Delta} \frac{e^u}{u^{2-2H}} du + e^{\alpha\Delta}\int_{\alpha\Delta}^{\infty} \frac{e^u}{u^{2-2H}} du \right\}$$

$$Var[\log(\sigma_t)] = \frac{H(2H-1)v^2}{\alpha^{2H}}\Gamma(2H-1)$$

where Γ denotes the Gamma function.

It described the formula for the autocovariance function of the log volatility in the RFSV. We can see that the function is decaying with $\Delta \to +\infty$, in particular, $(Y^t_H)_{t\in R}$ is ergodic, and for $H \in (½, 1]$, it means long-range dependence.

The main idea of the proof is to start with the expectation of the two log which are developed started with the time 0. Then we must change variables twice.
First with w = αu, x = αv

$$\text{Cov}[\log(\sigma\_t), \log(\sigma_{t+\Delta})] = e^{-\alpha\Delta}E\left[v\int_{-\infty}^{0} e^{\alpha u} dW_t^H + e^{\alpha\Delta}\int_{-\infty}^{\frac{1}{\alpha}} e^{\alpha v} dW_v^H\right] + v^2H(2H-1)e^{-\alpha\Delta}\int_{-\infty}^{0} e^{\alpha u}\left(\int_{\frac{1}{\alpha}}^{s} e^{\alpha v}(v-u)^{2H-2}dv\right)du$$

Then with $y = x - w,\ z = x + w$, so we get

$$= \frac{H(2H-1)v^2}{2\alpha^{2H}} e^{-\alpha\Delta}\left\{\int_0^{\alpha\Delta} y^{2H-2}\left(\int_{2-y}^{y} e^z dz\right)dy + \int_{\alpha\Delta}^{\infty} y^{2H-2}\left(\int_{2-y}^{2\alpha\Delta - y} e^z dz\right)dy\right\} + O(e^{-\alpha\Delta})$$

With this log volatility equation, we can get our attention to the volatility, and compute $\mathbb{E}[\sigma_{t+\Delta}\sigma_t]$ to fall back on the autocovariance function.

$$\mathbb{E}[\sigma_{t+\Delta}\sigma_t] = \mathbb{E}\left[e^{X_t^\alpha + X_{t+\Delta}^\alpha}\right]$$

$$= \mathbb{E}\left[e^{X_t^\alpha}\right] * \mathbb{E}\left[e^{X_{t+\Delta}^\alpha}\right] * e^{Cov[X_t^\alpha, X_{t+\Delta}^\alpha]}$$

X^α is a Gaussian process, thus applying the characteristic function:

$$= e^{\mathbb{E}[X_t^\alpha] + \frac{Var[X_t^\alpha]}{2}} * e^{\mathbb{E}[X_{t+\Delta}^\alpha] + \frac{Var[X_{t+\Delta}^\alpha]}{2}} * e^{Cov[X_t^\alpha, X_{t+\Delta}^\alpha]}$$

$$= e^{\mathbb{E}[X_t^\alpha] + \frac{Var[X_t^\alpha]}{2} + \mathbb{E}[X_{t+\Delta}^\alpha] + \frac{Var[X_{t+\Delta}^\alpha]}{2} + Cov[X_t^\alpha, X_{t+\Delta}^\alpha]}$$

We saw that SDE's solution of a fractional Ornstein-Uhlenbeck process can be written as below. Furthermore, if $\alpha \to 0$ the process X_t is independent of the dt part and $E[X_t^\alpha] = E[X_{t+\Delta}^\alpha]$

$$X_t = \nu \int_{-\infty}^{t} e^{-\alpha(t-s)} d\, W_t^H + m$$

Finally, applying the Corollary 3.2 with $\alpha \to 0$, we get:

$$\mathbb{E}[\sigma_{t+\Delta}\sigma_t] = e^{2\mathbb{E}[X_t^\alpha] + 2Var[X_t^\alpha] - \frac{\nu^2\Delta^{2H}}{2}}$$

The authors displayed the logarithm of the empirical counterpart $\log(\mathbb{E}[\sigma_{t+\Delta}\sigma_t])$ against Δ^{2H} for the S&P, with a Hurst parameter $H = 0.14$ and get linearity.

3.3 On the forecasting equation

Here we will detail the equation of the expectation of a fractional Brownian Motion which is "key", to cite the authors, for their prediction method. We wanted to go deeper on this since the expression is only displayed but not explained. Hence, we will try to provide a sketch-of-the-proof and summarize the idea behind this expression:

$$\mathbb{E}[W_{t+\Delta}^H | \mathcal{F}_t] = \frac{cos(H\pi\}}{\pi} \Delta^{H+1/2} \int_{-\infty}^{t} \frac{W_s^H}{(t-s+\Delta)(t-s)^{H+1/2}} ds$$

To do so, we investigated the paper of Carl J. Nuzman and H. Vincent Poor "Linear Estimation of Self-Similar Processes via Lamperti's Transformation" (2000) and studied their Theorem 4.2. This Theorem state that (using our papers notation):

For W_s^H a fBM with parameter $H \in (0,1)$, $\Delta > 0$ and $T_0 \in \mathbb{R}$, the prediction $\widehat{W}(T_0 + \Delta) = \mathbb{E}[W(T_0 + \Delta) | W(y), t < T_0]$ is a mean-square integral:

$$\widehat{W}(T_0 + \Delta) = W(T_0) + \frac{cos(H\pi\}}{\pi} \Delta^{H+1/2} \int_0^\infty \frac{W(T_0 - t) - W(T_0)}{(t+\Delta)(t)^{H+1/2}} dt$$

And in particular for $H < 1/2$,

$$\widehat{W}(T_0 + \Delta) = \frac{cos(H\pi\}}{\pi} \Delta^{H+1/2} \int_0^\infty \frac{W(T_0 - t)}{(t+\Delta)(t)^{H+1/2}} dt$$

So, without loss of generality, one can rewrites the given expression in the paper.

This Theorem is demonstrated examining the increment of the fractional Brownian Motion:

$$Z(t) = W(T_0 - t) - W(T_0)$$

Where they previously worked on it. Hence, by using an inverse transform, they find a prediction kernel expression:

$$k_\infty(a, t) = \frac{cos(H\pi)}{\pi} \frac{\Delta^{H+1/2} t^{-H-1/2}}{\Delta + t}$$

And using the identity,

$$\int_0^\infty \frac{ds}{s^\alpha(s + \Delta)} = a^\alpha B(1 - \alpha, \alpha) = \frac{a^{-\alpha}\pi}{sin\pi\alpha}$$

One is able to find the mean-square integral form, first presented before. In particular, if $\alpha = H - ½$ we retrieve the desired form.

4 Key results and applications

4.1 Papers results

4.1.1 Smoothness of the volatility

The regression of $log(m(q, \Delta))$ by $log(\Delta)$, means how evolves the mean of log volatility under the norm-q in function of the lag. The parameter q parameter is used afterward for implementation and to determine the Hurst parameter of roughness. The results shows that the autocovariance is converging after few iterations and one also can denote that this pattern is similar through each q and indexes.

4.1.2 Distribution of the increment of the log-volatility

Although the Gaussian distribution of increments of log-volatility is a stylized fact, authors displayed the increments' histograms of the log-volatility (increments are computed as: $log(\sigma_{t+\Delta}) - log(\sigma_t)$) with the normal density in order to compare. The underlying assets have the same behavior properties, authors display the histograms of the underlying S&P index, with delta's lags equal to 1, 5, 25, 125 days.

We notice that for any delta's lag the empirical distributions of log-volatility are very close to the Gaussian density. Observations are consistent with the "The distribution of realized stock return volatility " from T. G. Andersen, T. Bollerslev, F. X. Diebold, and H. Ebens who showed that the Gaussian distribution of increments of log-volatility is a stylized fact.

4.1.3 RFSV vs FSV

One of the interesting applications is the comparison between the rough fractional stochastic volatility (RFSV) and fractional stochastic volatility (FSV). It can demonstrate the incompatibility of the classical long memory for the FSV model. They use the quantity $m(2, \Delta)$. Just recall that $m(2, \Delta) = E[(log\, \sigma t + \Delta - log\, \sigma t)2]$ and $log(m(2, \Delta)) \approx \zeta 2\, log\, \Delta + k$.

The FSV model of Comte and Renault are concave and as they said they took $H = 0.53$ (which corresponds to the FSV model parameter). Also, $\alpha = 0{,}5$ to have visible decay of volatility skew. So, the slope for the small lags is determined by the value H and is flattening until to become stationary. Hence, they use the RFSV model and it is crystal clear that to best fit the data, one must take $\alpha \ll 1/T$. Also, the parameter H must be set by the initial slope of the regression line which is $2\zeta = 2 \times 0{,}14$. Typically on the RFSV model, $\alpha = 0$ and $H = 0{,}14$.

4.1.4 Simulation of Rough Fractional Stochastic Volatility model

Here, we will briefly discuss the presented results in paper concerning simulations of rough volatility. Here again, the authors are plotting a regression of $log\big(m(q, \Delta)\big)$as function of $log(\Delta)$ in Figure 3.6. This is a comparison of the roughness obtained with two different volatility proxies: Realized Variance (RV) and Uncertainty Zone (UZ) estimator of Variance. Both are on different times scales in order to understand the fractality of the volatility process. It appears that using the shortest time-scaled proxy (UZ), one can estimate a very similar Hurst parameters ($H = 0.16$) than its true empirical value ($H = 0.13$). Results are still good for the RV proxy but suffers from a little bias explained by the integral operator.

After this, one can observe one of the main results of the paper. Figure 3.6 displays two volatility processes. The first one is the true Volatility of the S&P500 over the period and the second one comes from the RFSV model. One can observe how similar in their comportments those processes are. This is objectively striking and shows the usefulness of the model. One very interesting on the fractality of the model is the property similarity between different time-scaled data. Indeed, author are stating that the volatility over a trading has very similar pattern than the one observed for a 10-year period.

4.1.5 On the long memory feature of volatility

This part is kind of interesting in its structure. After having demonstrated that the long-memory effect (LM effect) is false and doesn't apply for RFSV model, the authors used different techniques from other papers to test for long-memory effect. Those tests were realized to show that volatility has indeed a LM effect feature. However, they showed that one can find results demonstrating long-memory effect in their model, which is not possible. In another term, the authors demonstrated how poor effectuated tests in other papers are to prove for LM effect.

4.1.6 Comparing forecasting techniques

As we see above, authors modelized the volatility. Then they can keep going and try to forecast the log-volatility and the variance with their model and compare it with AR or HAR models. After reminding formulas, estimation of AR's coefficients with R are made with a rolling time window of 500 days. AR models are estimated with order $p = 5$ and $p = 10$. They display a P- ratio which is the ratio between the mean squared error and the variance of the log-variance, which mean that the more the ratio P is high, the less the model is good. Results are interesting for the RSFV model. It consistently outperforms AR or HAR models. Plus, it only needs the Hurst parameter H to compute log-volatility, while the AR and HAR needs coefficients depending on the forecasted time horizon which must be estimated. As one can expect, the same results appeared for the forecasted variance. Here again, the P-ratio again is always better for the RFSV model than for the AR or the HAR models. To conclude, RFSV model is very efficient for forecasting whatever the index used, or the time delta used.

4.2 A numerical application to look for fractality and consistence over time

In this part, we will compare presented key results of the paper which are for the period 2010-2014 (when the paper released). Our goal is to investigate if one can find similar conclusions over a totally different time-period, from 2014 to early 2021. In a second sub-part, we will zoom over a much shorter time-period to study a market globally impacted by COVID-19.

4.2.1 From 2014 to 2021, an investigation on fractality and consistency

This part deals with a dataset starting from the next day the paper released, hence the 15th of October 2014; and finishing to the most recent available information date at the writing time, so the 13th of April 2021. Globally this part will follow the sub-outline of part 4.1 to compare results. Also, we decided to focus on the three indexes presented in the paper: DAX, S&P500 and NASDAQ-100. We are not doing our application for the bund because of a lack of data. To better understand the framework and the environment of the numerical application, here are presented the prices and the realized variance of our indices.

Figure 1 - Prices of indices: S&P500, DAX and NASDAQ

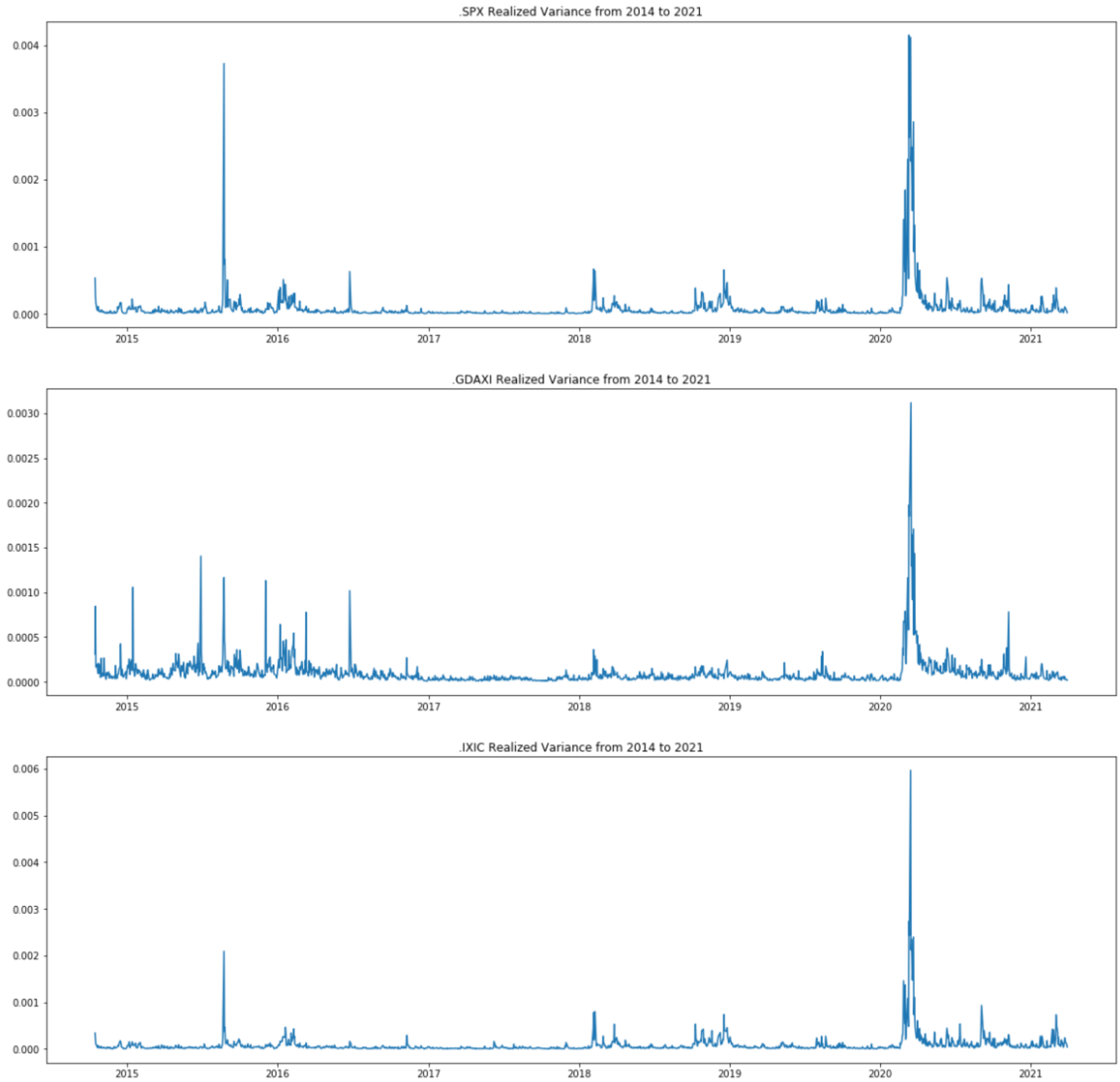

Figure 2 - Realized variance of indices: S&P500, DAX and NASDAQ (2014 to 2021)

4.2.1.1 <u>Smoothness of the volatility</u>

Firstly, we wanted to compare smoothness of the volatility papers' result and our applied result for our specific time-period.

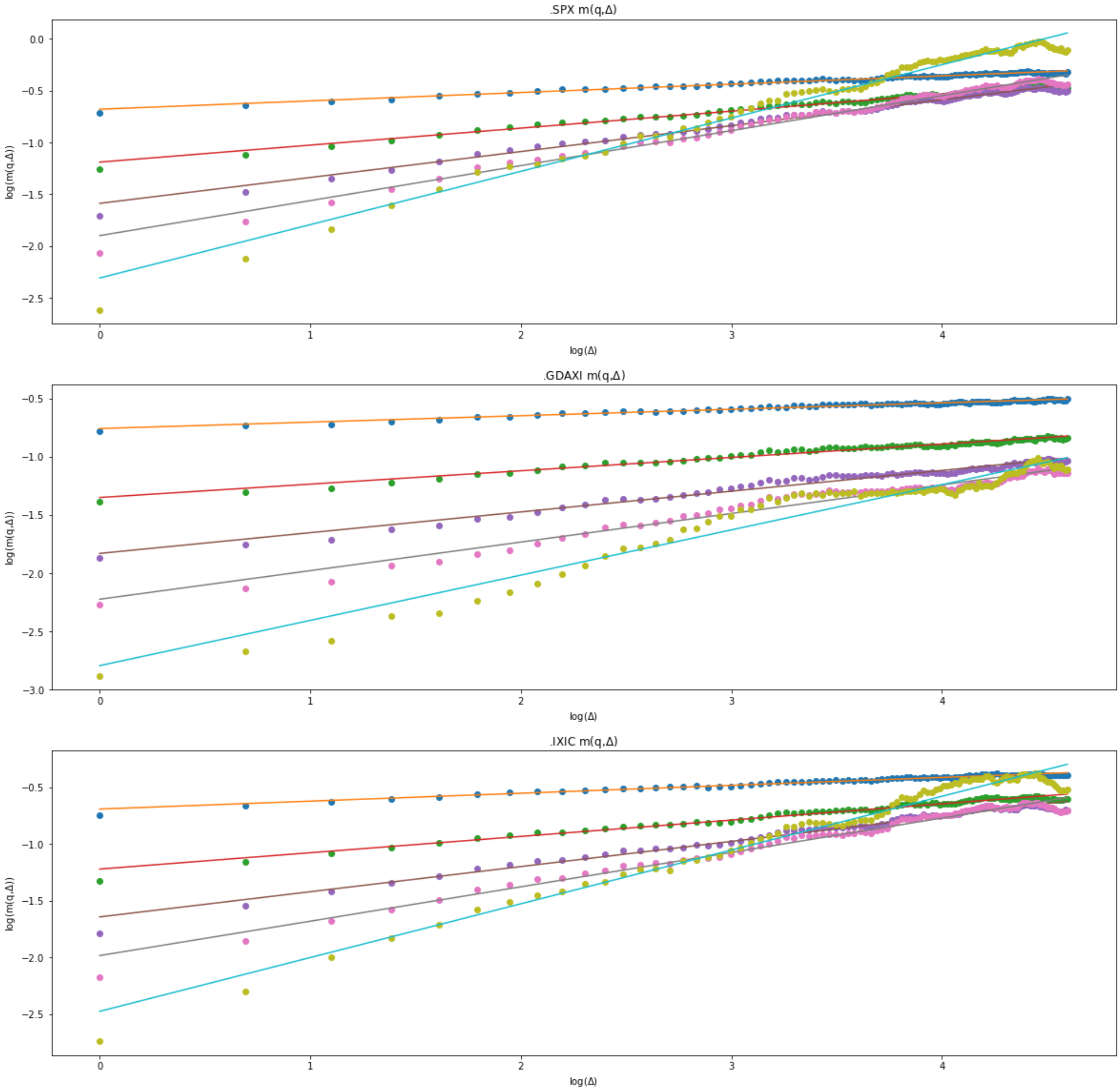

Figure 3 - Regression of the log m(q, Δ) as a function of log Δ for S&P500, DAX and NASDAQ

As one can expect the regression's of $log(m(q, \Delta))$ by $log(\Delta)$ results are very similar compared to the original paper. Indeed, the scale is similar for the three indexes, moreover all the convergence is respected independently to the q parameter after few iterations. Hence, this a key finding for consistency overtime for rough volatility. As a reminder, the period looked at here is about two time larger than the one presented on paper. This could traduce fractality pattern: results lead to the same conclusions while looking at way larger period of time.

Then, we decide to focus the estimation of H which is done by doing the regression of ζq by q. Hence, we were able to find the H parameters using the scaling relationship $\zeta q = qH$. Our H's estimations for each indexes are the following:

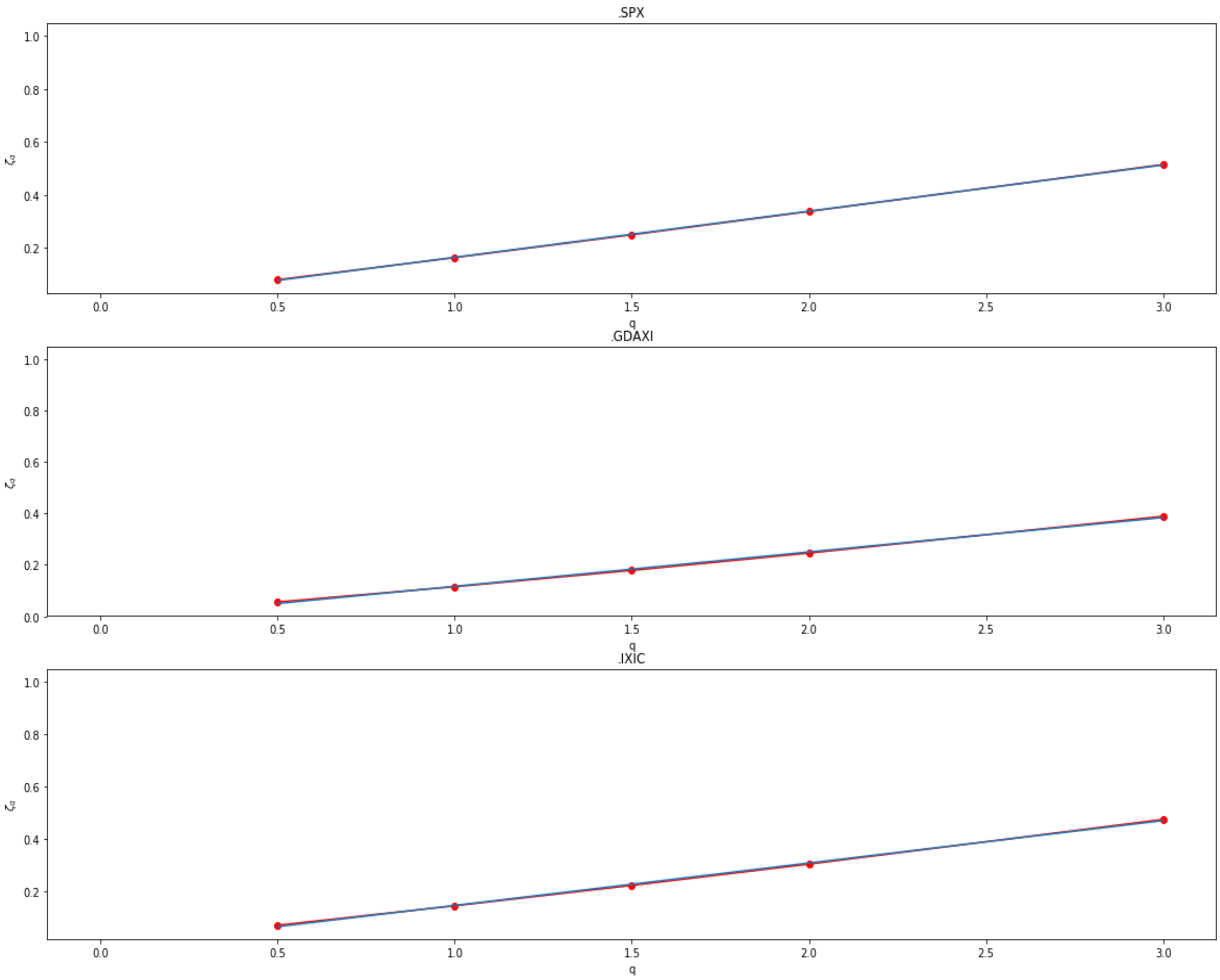

Figure 4 - ζq (blue) and empirical estimation times q (red), S&P500, DAX and NASDAQ

One can see on Figure 4 for S&P500, DAX or NASDAQ that the smoothness parameter is seems also to be independent of q. As in the paper, we find that there is a small concavity for ζq which stay marginal.

4.2.1.2 Distribution of the increment of the log-volatility

After studying smoothness, we worked on the distribution of the increment of the log-volatility computed as $log(\sigma t + \Delta) - log(\sigma t)$. We wanted to check if the distribution is still close to the gaussian distribution for the DAX, NASDAQ and S&P500 with our period of time. As a confirmation of this well-known stylized fact, all indexes are close to a gaussian distribution. With the fitted normal density superimposed in red, one is able to see that, for any Δ, delta's lags equal to (1, 5, 25, 125) days, the empirical distributions of log-volatility increments are verified as being close to Gaussian. So, we notice that for any delta's lag the empirical distributions of log-volatility are very close to the Gaussian density, with the same observations on the 1-day lag, the fit of the normal density by Δ^{H} generates (blue dashed) curves that are very close to the red fits of the normal density, consistent with the observed scaling.

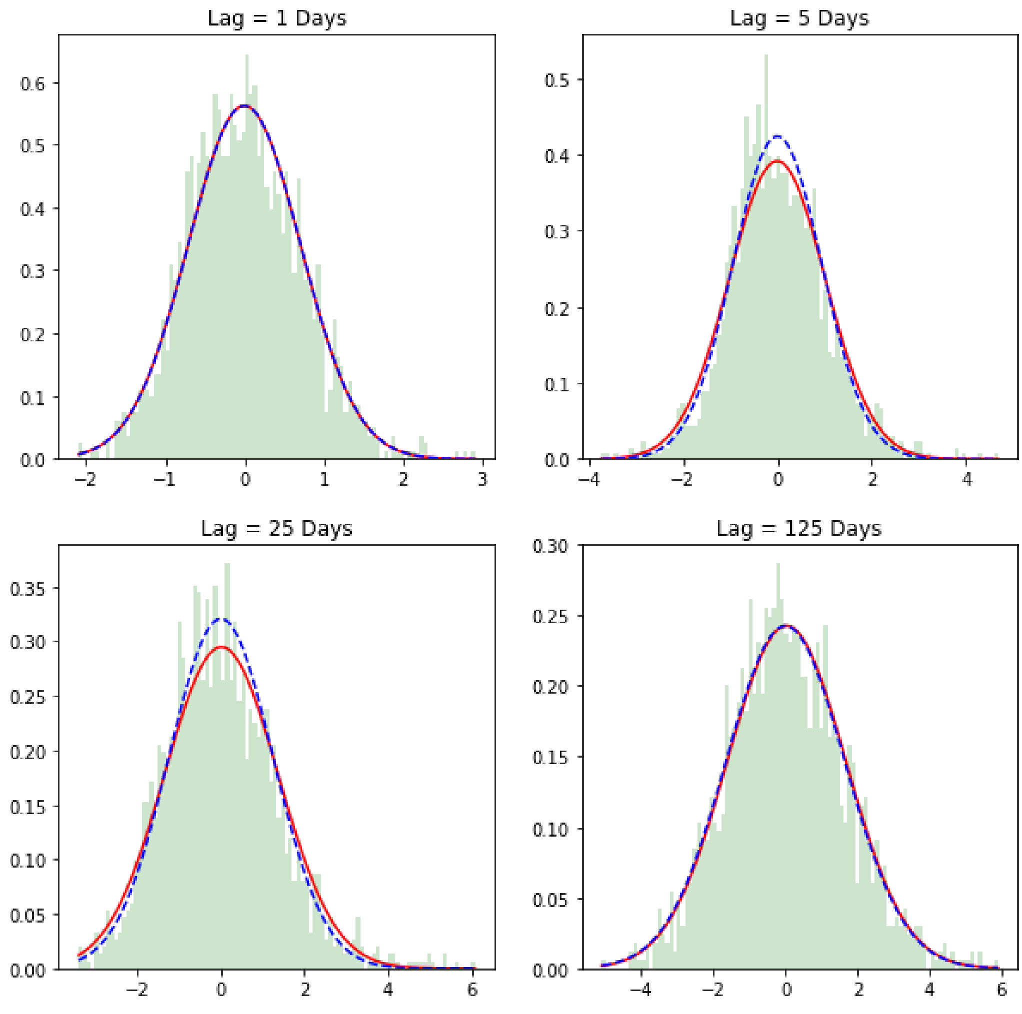

Figure 5 - Histograms for various lags Δ of the (overlapping) increments log σt+Δ − log σt of the S&P log-volatility; normal fits in red; normal fit for Δ = 1 day rescaled by ΔH in blue

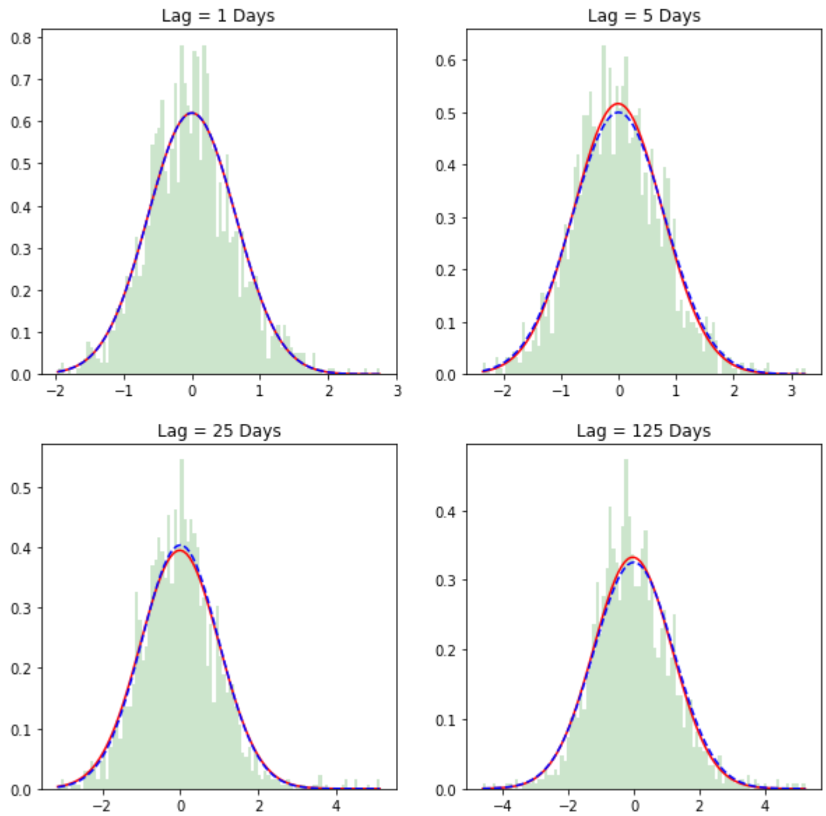

Figure 6 - Histograms for various lags Δ of the (overlapping) increments log σt+Δ − log σt of the DAX log-volatility; normal fits in red; normal fit for Δ = 1 day rescaled by ΔH in blue

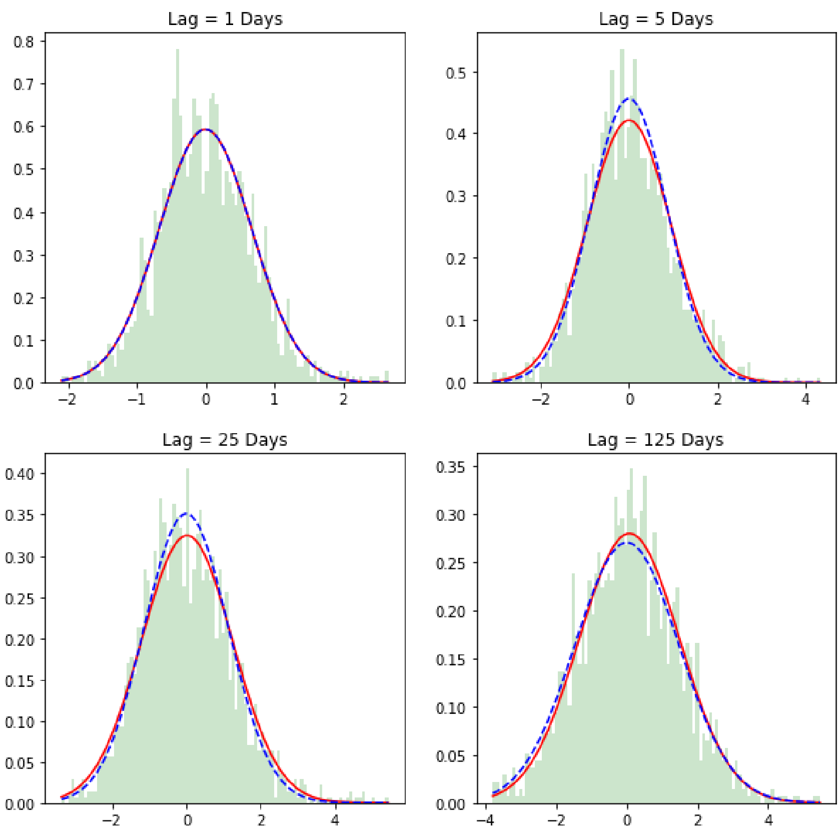

Figure 7 - Histograms for various lags Δ of the (overlapping) increments log σt+Δ − log σt of the NASDAQ log-volatility; normal fits in red; normal fit for Δ = 1 day rescaled by ΔH in blue

4.2.1.3 Studying the auto-covariance

The aim of this sub-part is to find out if the autocovariance is consistent overtime and with different data. The decay of the autocovariance is linear as showed below. This implies the fact that volatility has a no long-memory property which the same finding as the paper. Hence, there is a real form of consistency in J. Gatheral and al. paper.

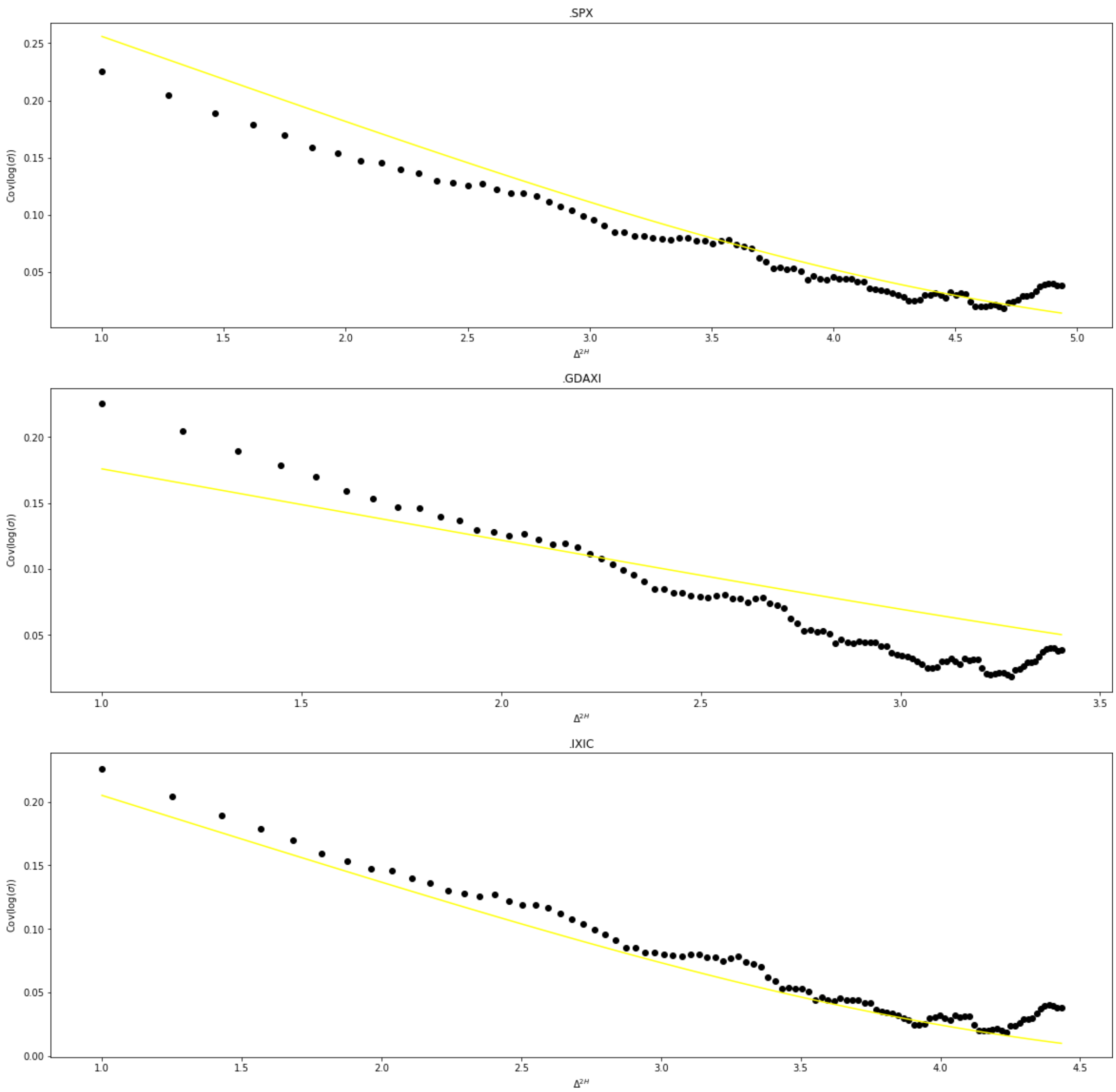

Figure 8 - Autocovariance of the log-volatility as a function of Δ2H for H = 0.14 for S&P500, DAX, NASDAQ

4.2.1.4 RFSV vs FSV

To be consistent with what we have done, we compare our RFSV model to the FSV model. So as said before, it allows us to demonstrate the incompatibility of the classical long memory for the FSV model. We use here the same parameter for FSV model to be consistent in our comparisons. The only difference resides on the alpha for the RSFV which must tend to 0 so we take 0,0001. One can see that FSV is concave as authors have, and it flattened. Plus, the scale seems to be on the same order (less than the unity). For our time-period and for the three indexes, they all present similar results: concavity and becomes flatter. Hence, RFSV again outperforms FSV in modelling volatility. This is another proof for consistency and fractality.

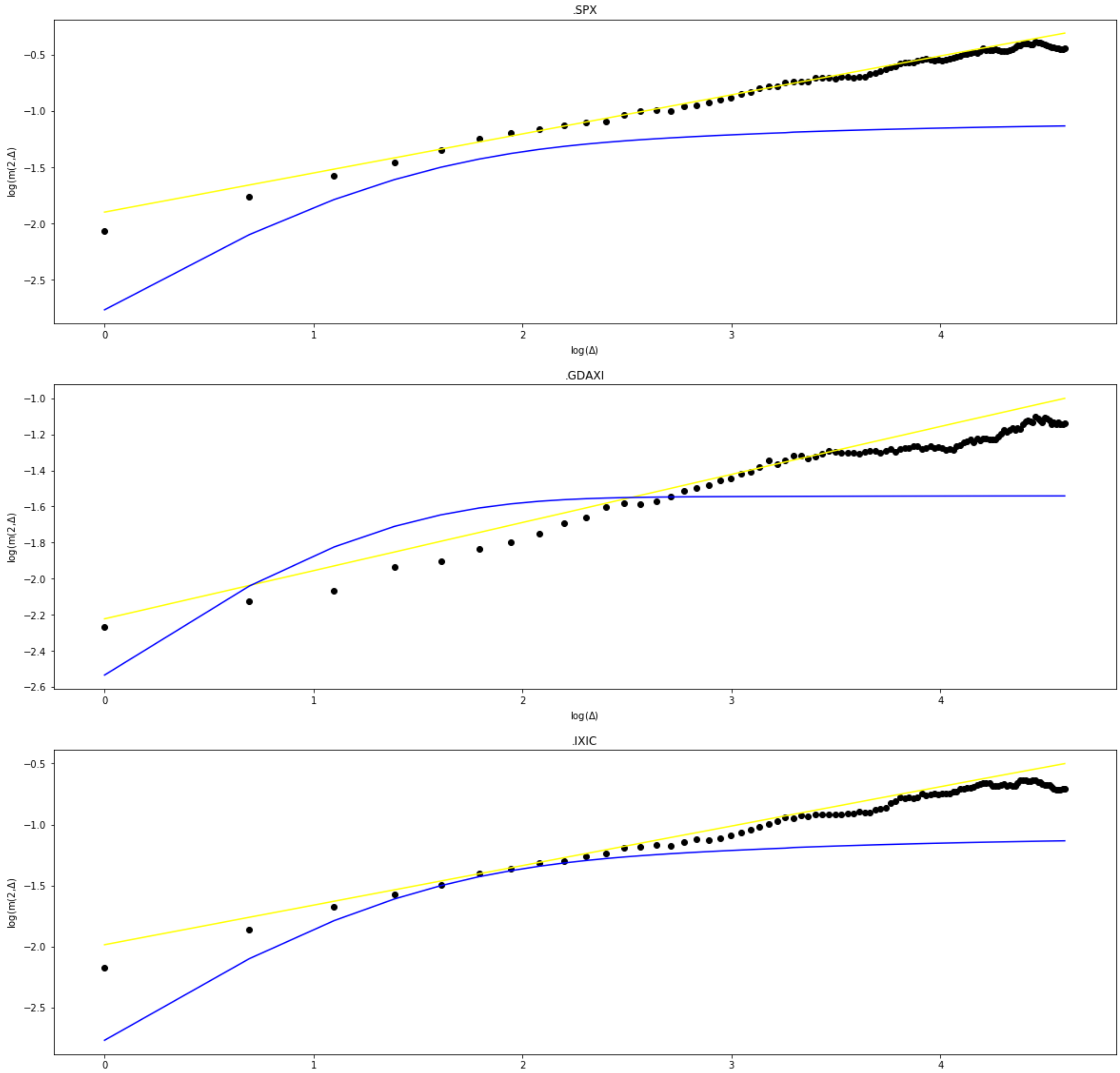

Figure 9 -Long memory models such as the FSV model of Comte and Renault are not compatible with S&P500, DAX et NASDAQ volatility data. Black points are empirical estimates of m(2, Δ); the blue line is the FSV model with α = 0.5 and H=0.53; the yellow line is the RFSV model wit

4.2.1.5 Simulation of Rough Fractional Stochastic Volatility model

We simulated here a fractional Brownian Motion to compare with the volatility of the S&P500 on the time-period defined. We see in both of graphs an alternation of persistent periods of high volatility with low volatility periods. Moreover, on the empirical volatility, we see that on a restrictive time window seems to have the same kind of qualitative properties as the entire sample path. The similarity between different time-scaled data is a very interesting point on the fractality of the model. It is striking to see that with our time-period the similarity and the qualitative properties holds, it confirms the hypothesis, and we have the same computation.

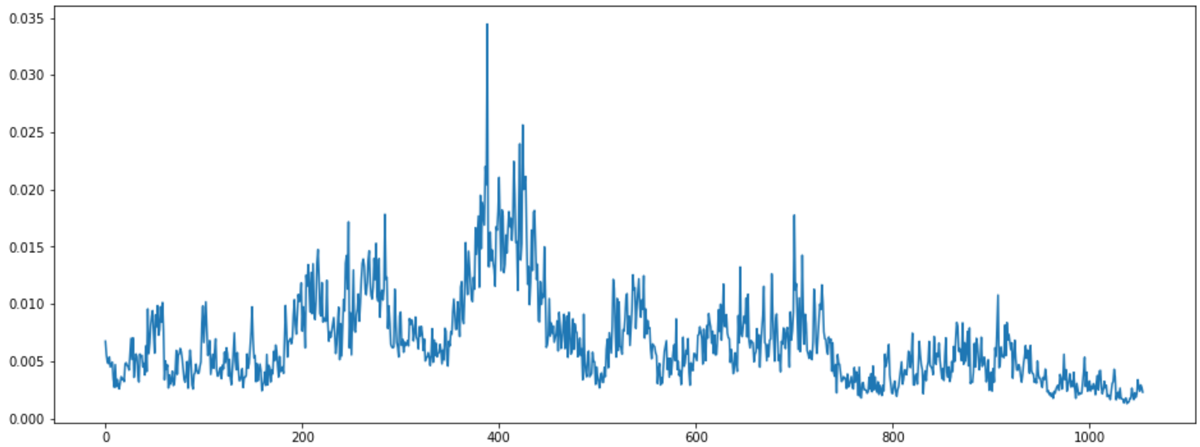

Figure 10 - Simulation of a fractional Brownian Motion

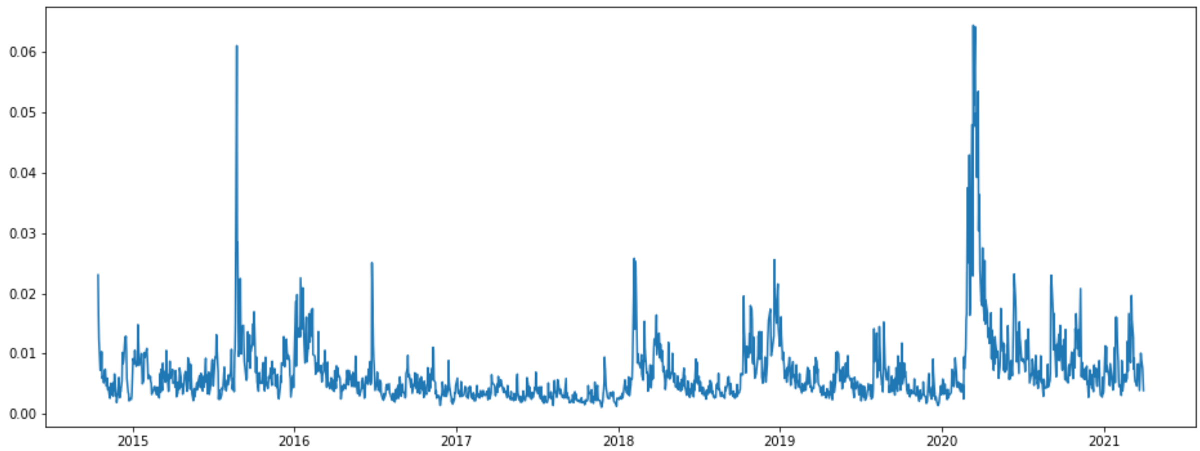

Figure 11 - Realized volatility of the S&P500

Concerning the simulation itself, we used two different ways. The direct one is to download a package and use already coded function. The other way is to code our own simulation of the fBM. Indeed, we studied how to do this. Then, as one can see in the Jupyter Notebook, we manage to simulate a good path for the fractional Brownian Motion through our homemade class.

4.2.1.6 Forecasting

Here, we compute the predictive variance and compare it with the actual variance. We superpose the predictive and the actual to see how far the predictive is, and as one can denote for the three indexes, it seems to be (obviously) more accurate when delta is small. Plus, we must lag more it is difficult to predict the variance. Moreover, whatever the delta is the prediction seems to be very accurate. As it is written the prediction in red line fit extremely well the actual variance in every index whatever it is the DAX, S&P500 or NASDAQ. The delta lag here are 1, 5 and 20 days. These results will be very useful to compare with classical predictive model. As the paper states, we will see how precise the model will be by looking at P-rations of the RFSV model. Furthermore, these forecasts clearly show how really consistent and fractal the model is.

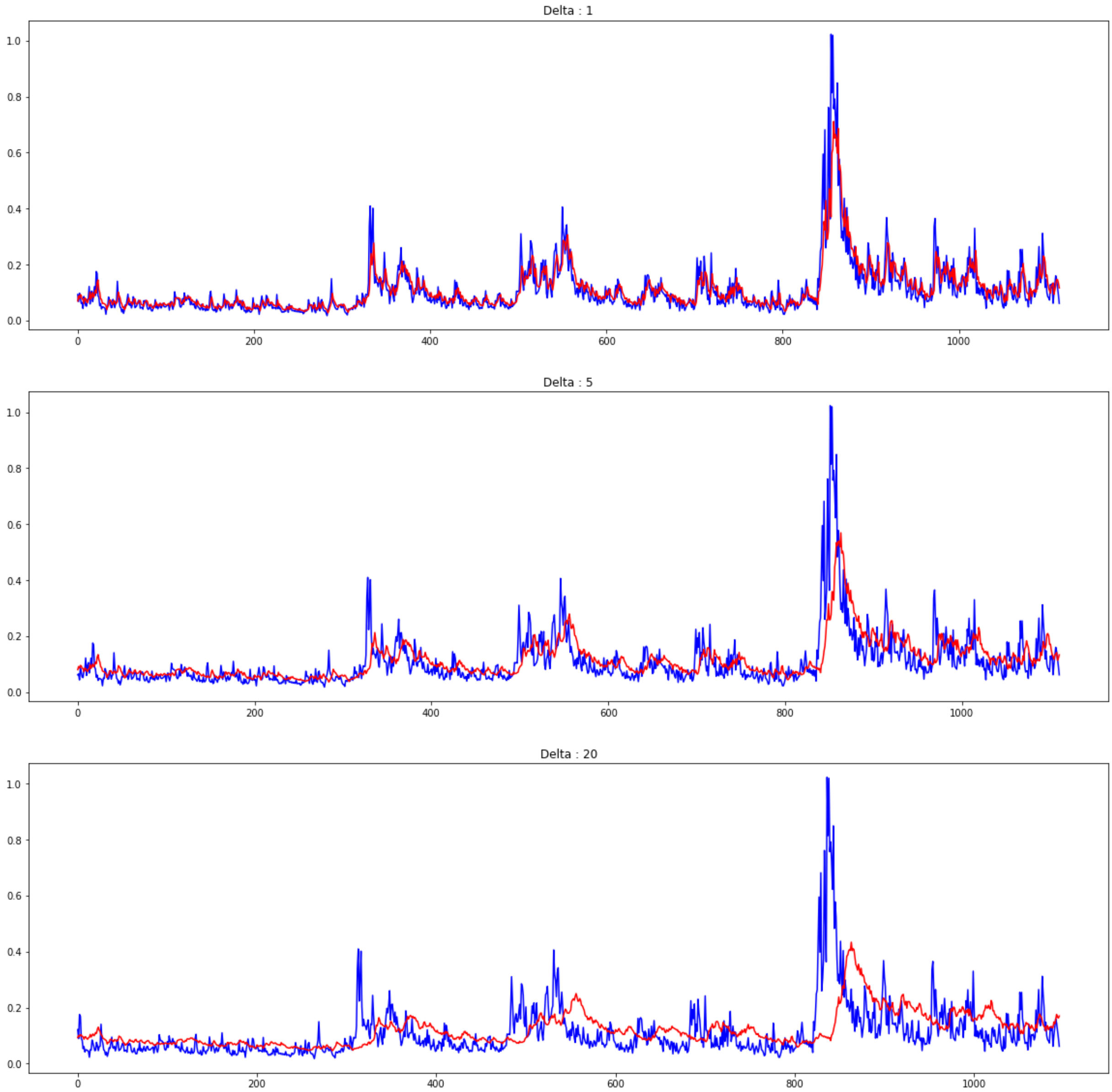

Figure 12 - Predictive variance (red line) vs realized variance (blue line) for S&P500

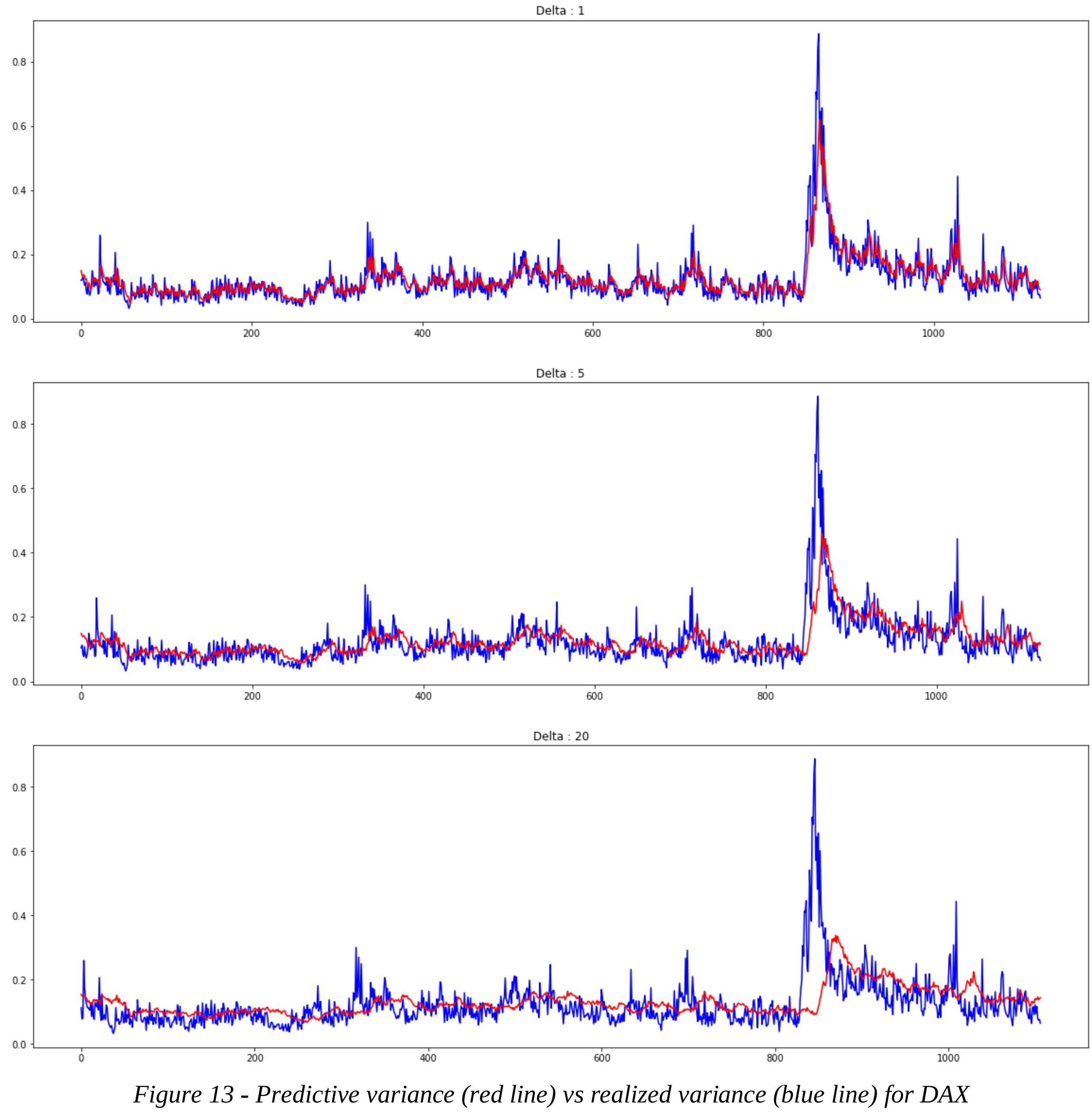

Figure 13 - Predictive variance (red line) vs realized variance (blue line) for DAX

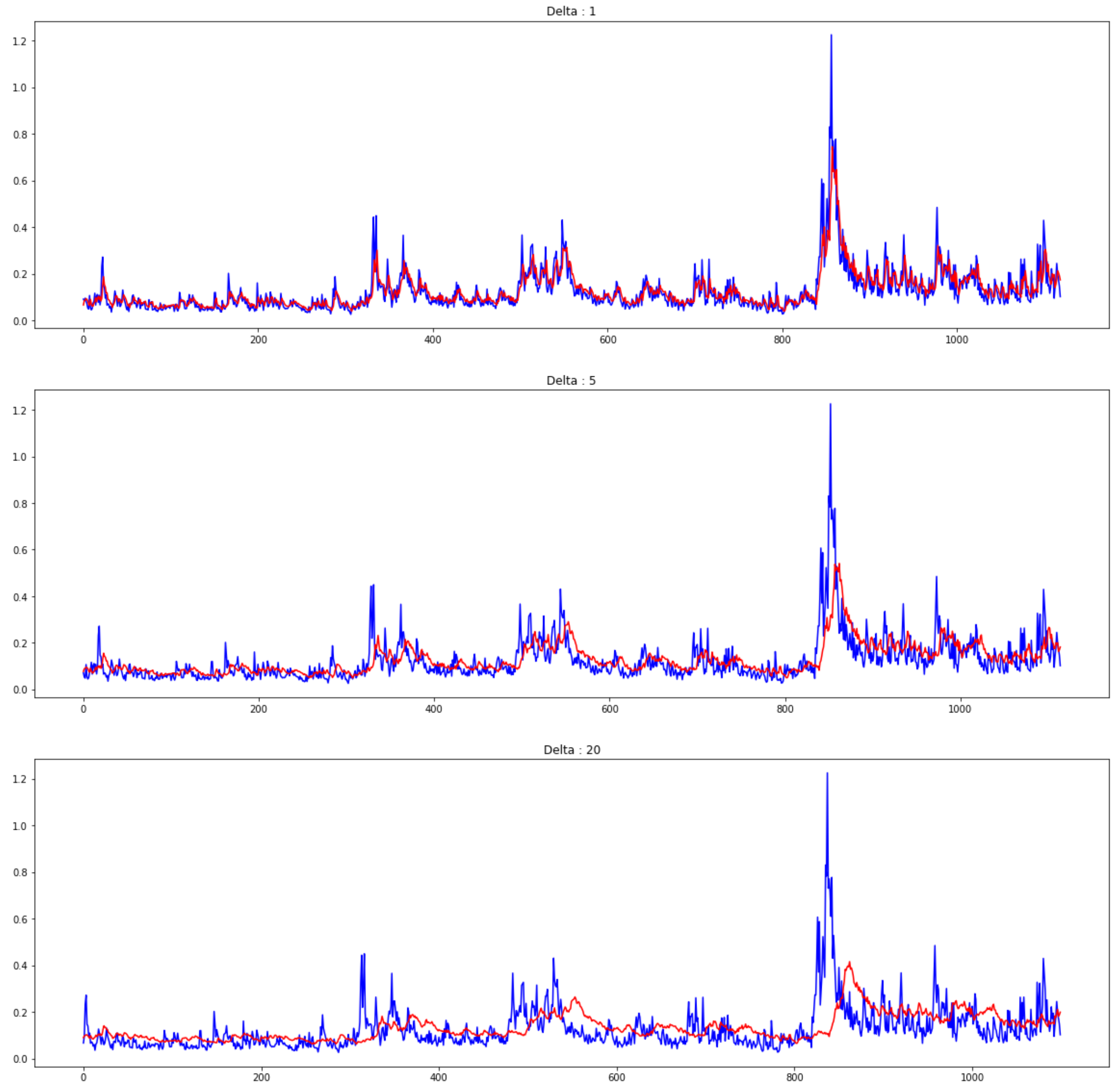

Figure 14 - Predictive variance (red line) vs realized variance (blue line) for NASDAQ

4.2.1.7 P-Ratios

In order to offer more precisions and granularity on predictions, we computed P-ratio for each of our indexes. It allows to better understand how the RFSV perform. Indeed, we were able to retrieve results of the same order magnitude. Unfortunately, AR and HAR are not presented here, further details on this are present in the Discussion part. Here again, we find a proof for consistency and fractality. Indeed, RFSV has a better predictive power to modelized the log-variance. However, it seems that there is a negative effect of increasing the lags. This appears to be logical regarding the forecasts provided before.

S&P500	$\Delta = 1$	0.2956
	$\Delta = 5$	0.5525
	$\Delta = 20$	0.8890
DAX	$\Delta = 1$	0.3641
	$\Delta = 5$	0.5240
	$\Delta = 20$	0.8762
NASDAQ	$\Delta = 1$	0.3317
	$\Delta = 5$	0.5818
	$\Delta = 20$	0.9045

Figure 15 – P-Ratios for the RFSV predictors for log-variance

4.2.2 The COVID-19 effect, zooming for a shorter time-period

We decided to test again for fractality and consistency by doing the same procedure over the COVID-19 period (i.e dec-19 to jan-21). The rationale behind this approach is to investigate if the model resist to abnormal market condition in a backtested environment. All our results are presented in appendix.

It appears that globally, the model has been negatively impacted by the pandemic. However, it seems to be robust since results are not catastrophic neither. For instance, distributions are not Gaussian anymore, which is logical from selected data. Moreover, the comparison graphs of RFSV against FSV shows that the previous hegemony of RFSV is not as clear as before. We encourage lector to compare displayed result in part 4.2.1 and in appendix to have a better understanding of the COVID-19 effect.

5 Discussion and critics

5.1 How does the article relate to the course?

The interesting fact about computing the RFSV is that after finding the log-volatility is behaving like a fractional Brownian motion in order 0.1, a lot of models using Brownian motions have been adjusted to this paper. For instance, the rough Heston model or the rough Bergomi model are model increased by rough volatility.

About the rough Heston model, Rosenbaum and al. have worked on "roughening" the model. Indeed, as we know Heston model is used to modelized volatility but as we see in the paper it is fit better with a fBM. With "roughening" the model, implied volatility ATM skew is better modelized but also, the dynamics of the volatility surface with explicit formula for characteristic function are good.

We can see here the rough Heston model:

$$dS_t = S_t\sqrt{V_t}dW_t$$
$$V_t = V_0 + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\lambda(\theta - V_s)ds \ + \ \frac{\lambda\nu}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\sqrt{V_s}dB_s$$
$$\langle dW_t dB_t\rangle \ = \ \rho dt \ with \ \alpha \in \left(\frac{1}{2}, 1\right)$$

Another model using RFSV is the rough Bergomi model. What is interesting here, is the fact that this model can modelized the stochastic volatility thanks to its power-law ATM volatility skew. It is more interesting when it has been proved that it is consistent with empirical studies by Rosenbaum and al. Moreover, for a more realistic fractional Bergomi model, there is also a two factors rBergomi model. But the fact that Bergomi model has only three parameters which are H, η and ρ are an advantage for a quick calibration but also find the best fit to observed prices.

Here is the rough Bergomi model:

$$dX_t = -\frac{1}{2}V_t dt + \sqrt{V_t}dW_t$$
$$d\xi_t^u = \xi_t^u \eta\sqrt{2\alpha+1}(u-t)^\alpha dB_t$$
$$\langle dW_t dB_t\rangle \ = \ \rho dt \ with \ \alpha = \ H - \frac{1}{2} \in \left(-\frac{1}{2}, 1\right)$$

5.2 Is the result of importance nowadays? Is it still used?

The answer to this question is: "Yes, obviously." With all the "roughening" of old model, this paper is still considered as fundamental in current related literature. Introduction of RFSV is relatively new (2014) and we are already all the application on Heston and Bergomi model.
From O. El Euch's PhD thesis: "Quantitative Finance Under Rough Volatility", one can find a lot of applications. For instance, there are applications for hedging where he approximates by

the rough Heston formula compare to the “smooth” Heston model. However, the problem is that theory is easier than the reality and replication can be perfectly only be done theoretically.

5.3 Precisions and critics

AR and HAR are not presented here even if work has been done on those models. This is due to the fact that the authors were not very precise in this part. Estimation of those two models is globally opaque. Indeed, even if the authors precise that they use R library and Yule-Walker method this is not sufficient to retrieve similar results. Indeed, as presented in the Notebook, we tried to estimate AR using this method. However, we were not able to find coherent results.

Another critic would reside on the problematic itself. Authors are stating that their aim is to modelized volatility with high-frequency data. But they are using daily data, which are not high frequency. However, we assume that the authors consider that the fractality of the volatility is real. Since means that their results should be valid both for daily data and (less than) 1-second data. But this is a strong assumption which demands further research.

As a last criticism, authors are really taking some key equations from other papers without going further. For instance, the forecasting equation is taking from another paper with no more explanation. To have, a firm grasp of the subject, one should open cited references (as we did).

6 Conclusion

Volatility was for a long time perceived as a long memory process; it could even be called "soft" volatility. Following this paper, authors conclude that volatility does not have the property of long memory and we can see that the term "rough" volatility seems much more appropriate. Thanks to several tools and several papers preceding those cited in the appendix, Rosenbaum, Gatheral and Jaisson succeeded in modeling volatility and projecting volatility for “high frequency data”.

The fundation tool that allowed them to model the erratic behavior of volatility, although other authors have tried, such as Dupire's local volatility or Heston's or Hull & White's homogeneous volatility, is the Fractional Brownian motion. After we tried to understand the basic tools, we looked at other demonstrations and gave elements of answer to understand them from previous papers, as for the "Lamperti's transformation", the "Fractional Ornstein-Uhlenbeck Processes" or the "Long Memory In Continuous-Time Stochastic Volatility Models". This allowed us to better understand closed formulas and therefore to take them in order to implement them on Python. To be consistent with our Python results we first tried to decipher and understand the outputs of the authors to understand them. Here the modeling of the fBM for example for the Hurst parameter is less than 1/2 which was not the case with the FSV.

We therefore decided to compare the Python output of the authors with ours but with a different time period to see if, beyond the results on a given time where the economic situation seemed uniform, the results seem to be consistent. The results are in perfect agreement with what was found, which seems that beyond the fact that he managed to find a formula to model the volatility via an fBM, it seems to be able to hold according to whether the time period is restricted or that the time period increases. The consistency confirms that the "rough" volatility does not have the long memory property through this exercise. Moreover, when looking at a stressed period, typically COVID-19, the model is still efficient even if it is impacted.

This modeling and this paper allowed us to discover a much richer universe than the classical volatilities studied in class. We are proud to have been able to understand and model what we have done, but we are also aware that this is only a tiny fraction of what is done in the research field. It is with envy and excitement that we discover other papers directly or indirectly related to this area of finance in other courses or in the professional field, whether in the distant or near future. It goes without saying that it is a very good exercise that has allowed us to reach a certain autonomy in reading a paper, whether it is the theoretical or empirical part.

7 References

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8 Appendix

8.1 Prices of indices – COVID-19 period

Figure 16 - Prices of indices – COVID-19 period

8.2 Realized variance – COVID-19 period:

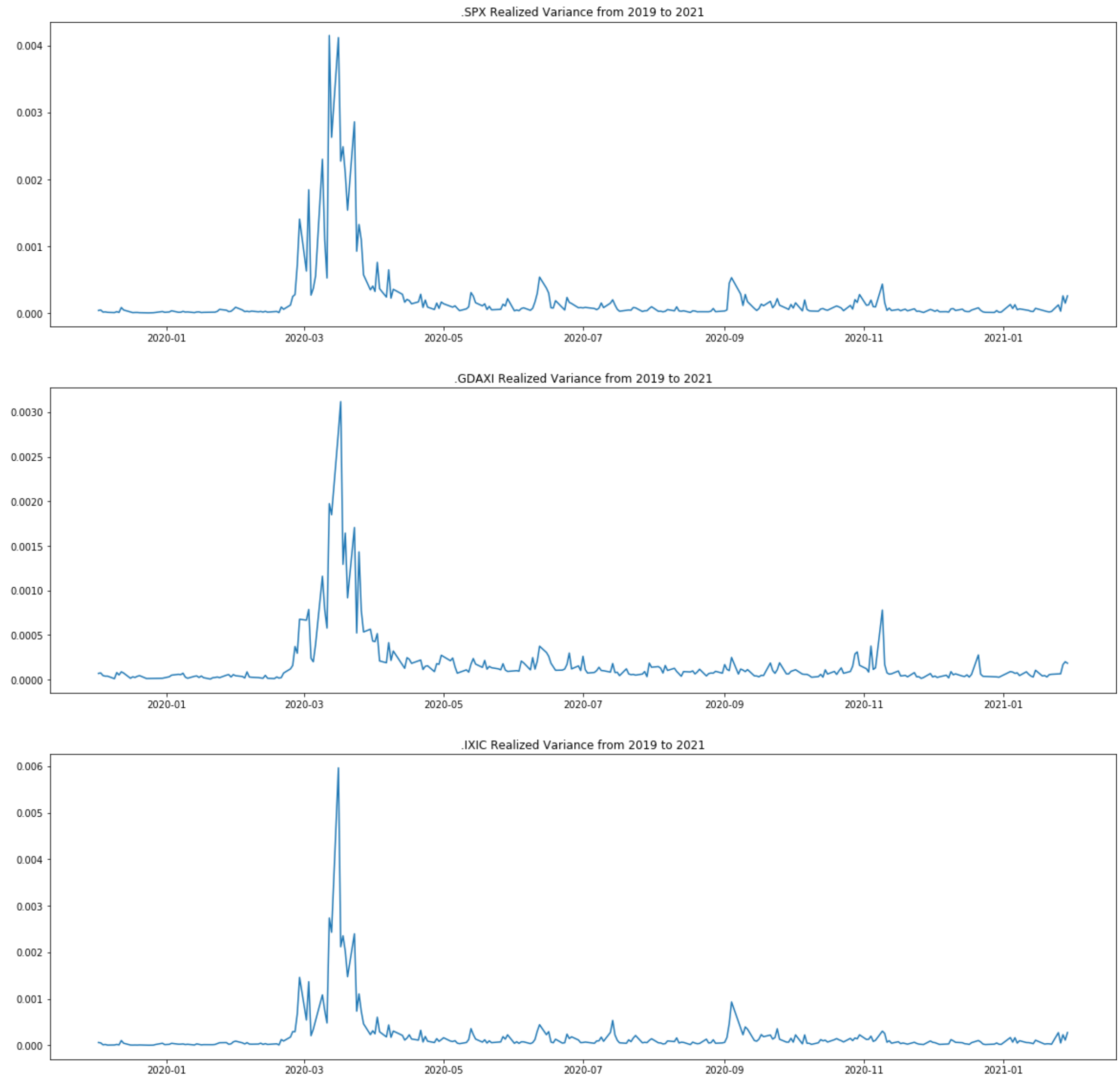

Figure 17- Realized variance – COVID-19 period

8.3 Regression of the log m(q, Δ) as a function of log Δ for – COVID-19 Period

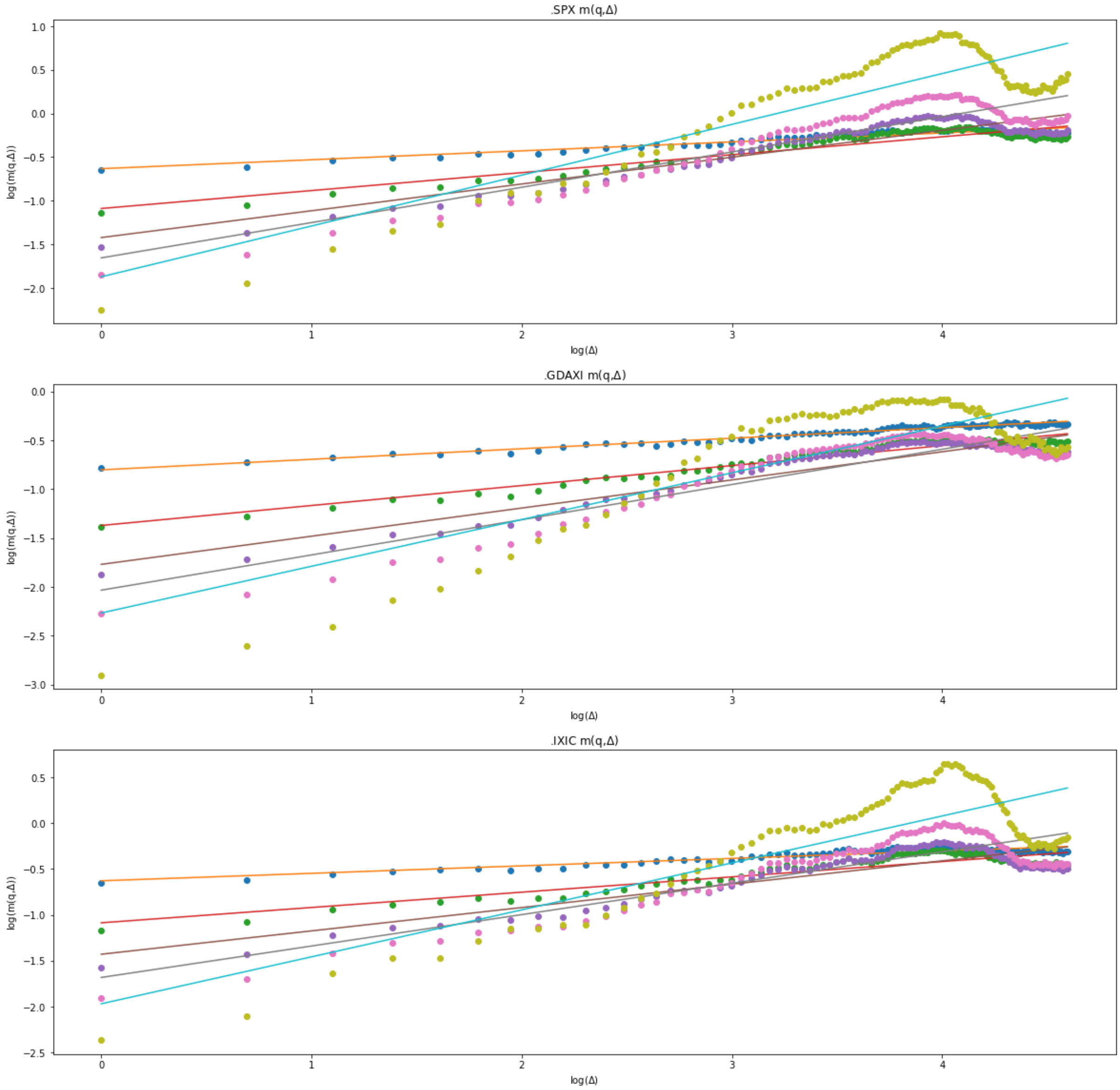

Figure 18 - Regression of the log m(q, Δ) as a function of log Δ for – COVID-19 Period

8.4 Regression ζq (blue) and empirical estimation times q (red)

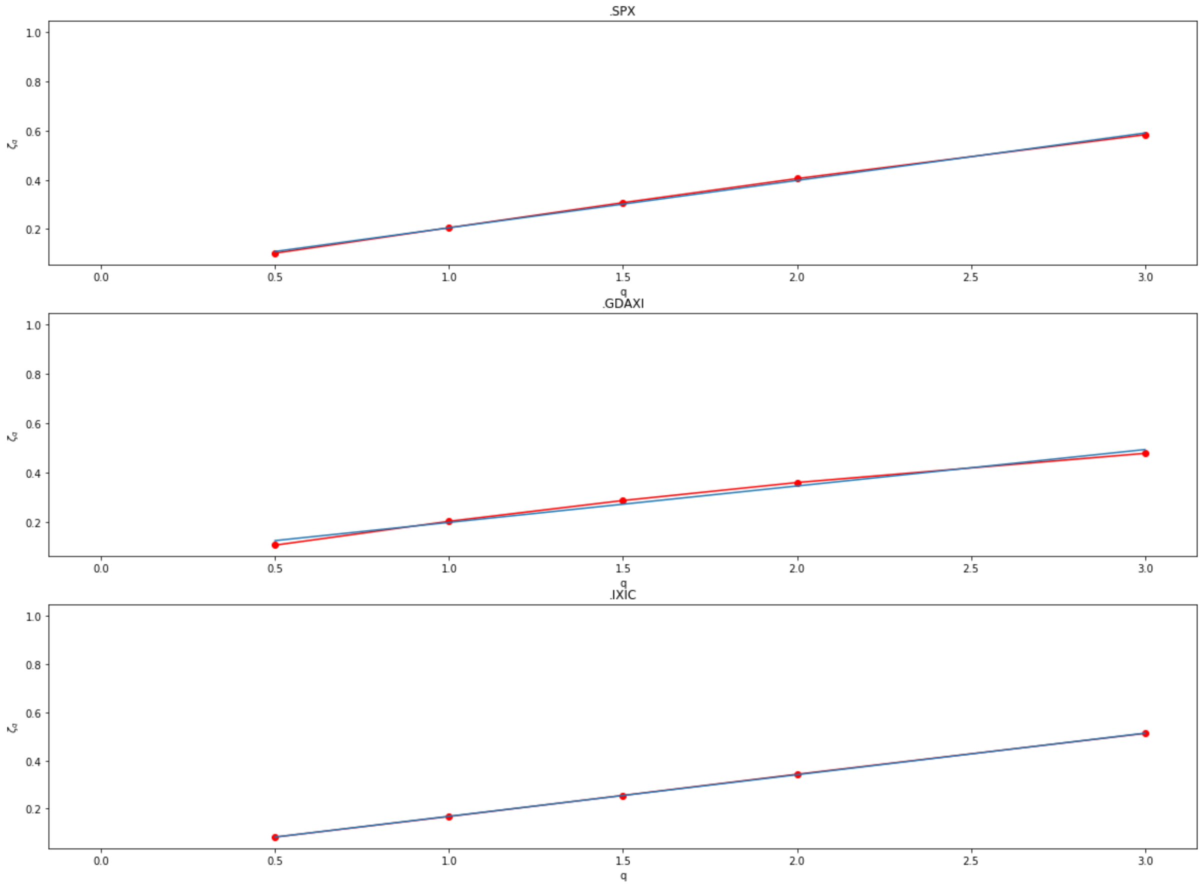

Figure 19 - ζq (blue) and empirical estimation times q (red) - COVID-19 period

8.5 Histograms of the S&P – COVID-19 Period

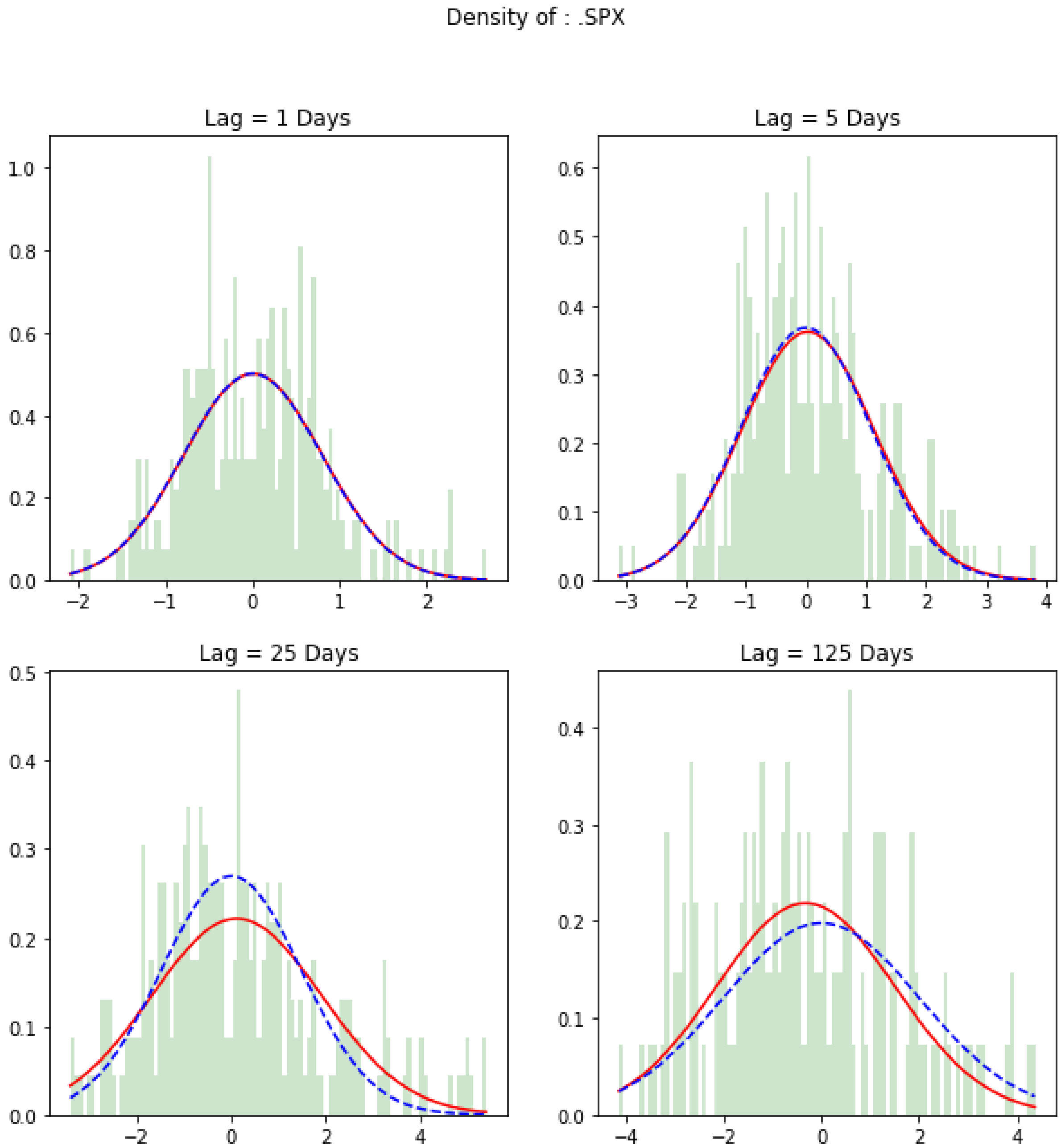

Figure 20 - Histograms of the S&P – COVID-19 Period

8.6 Histograms of the DAX – COVID-19 Period

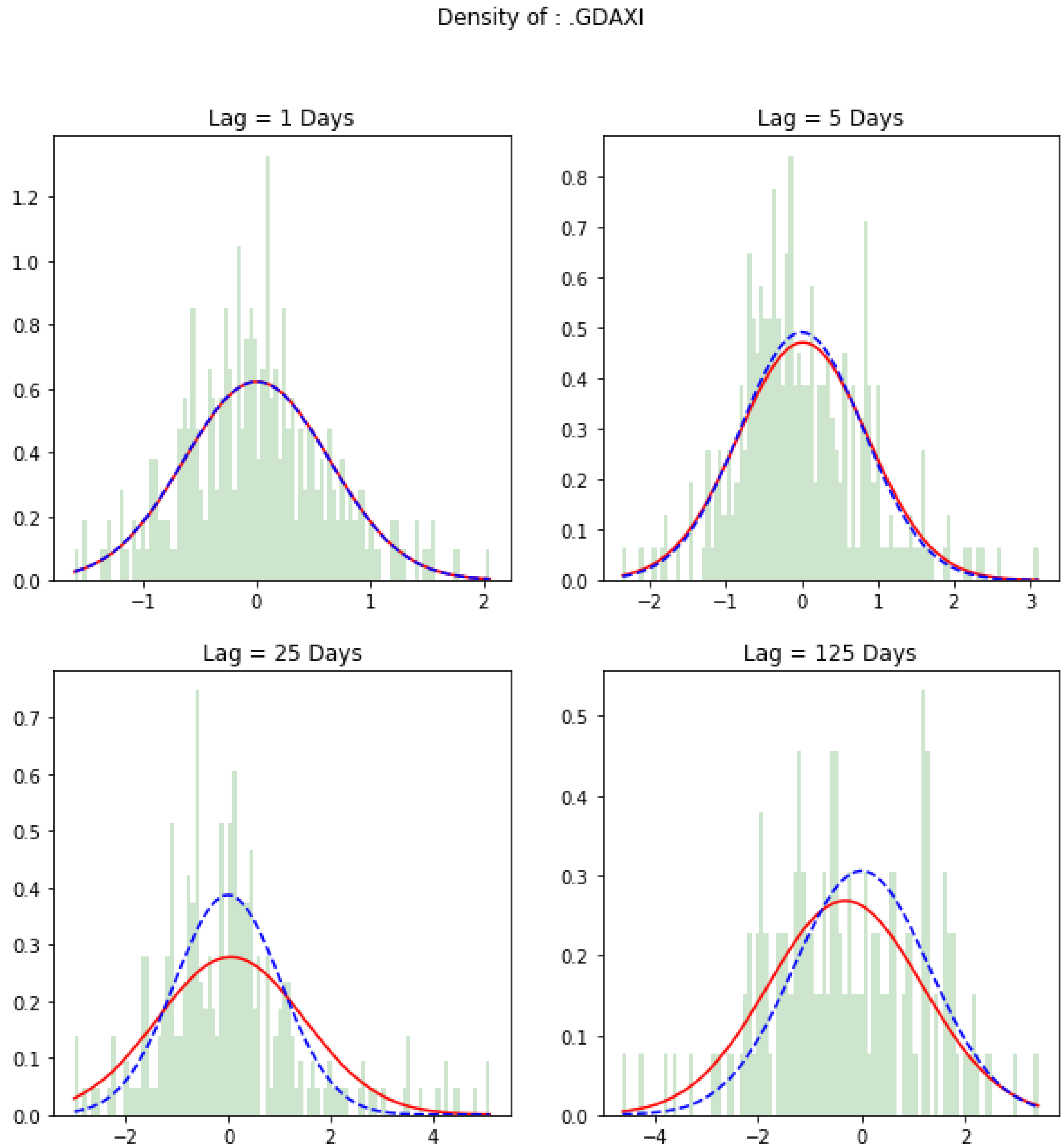

Figure 21 - Histograms of the DAX – COVID-19 Period

8.7 Histograms of the NASDAQ – COVID-19 Period

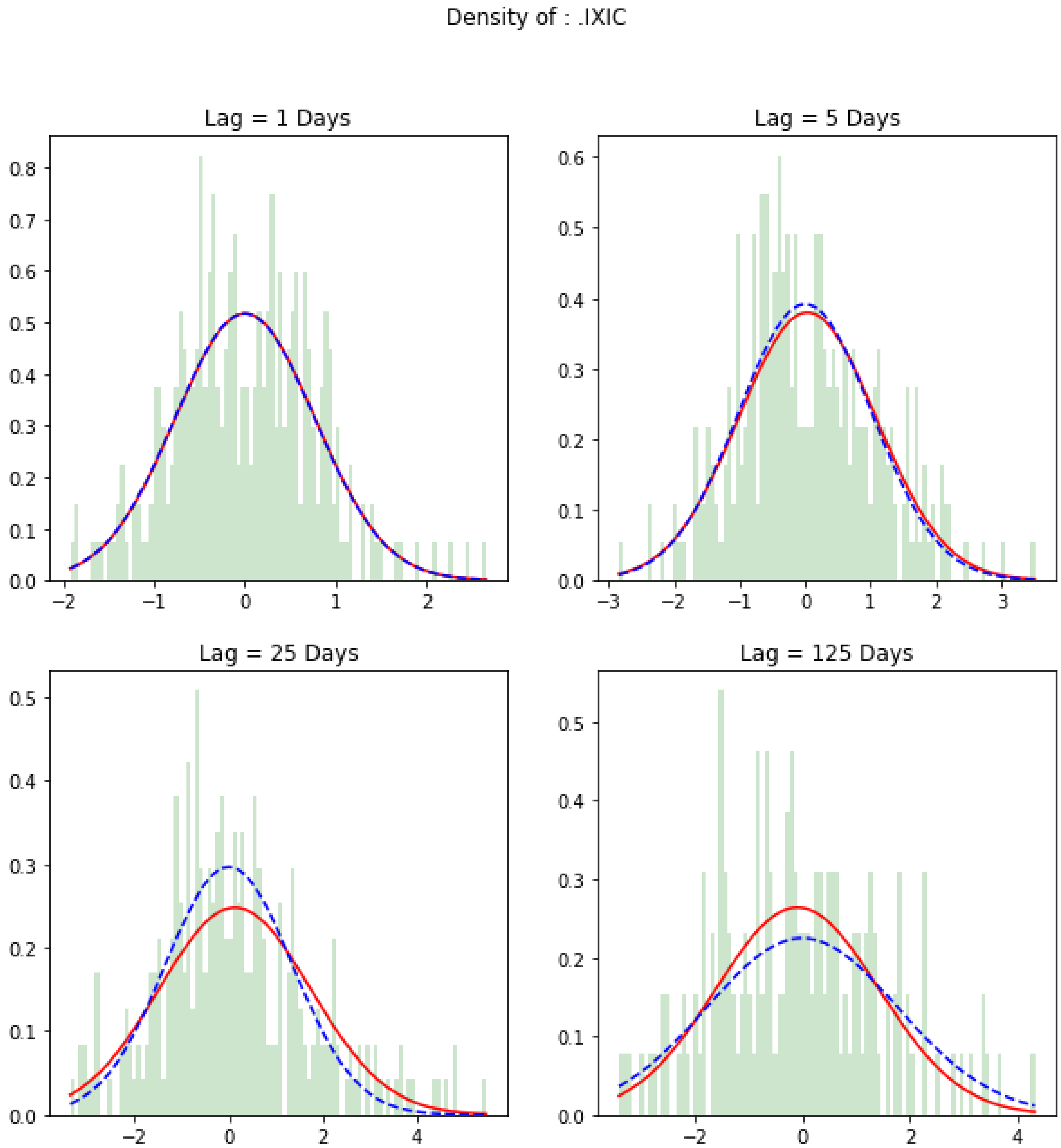

Figure 22 - Histograms of the NASDAQ – COVID-19 Period

8.8 Long memory models – COVID-19 Period

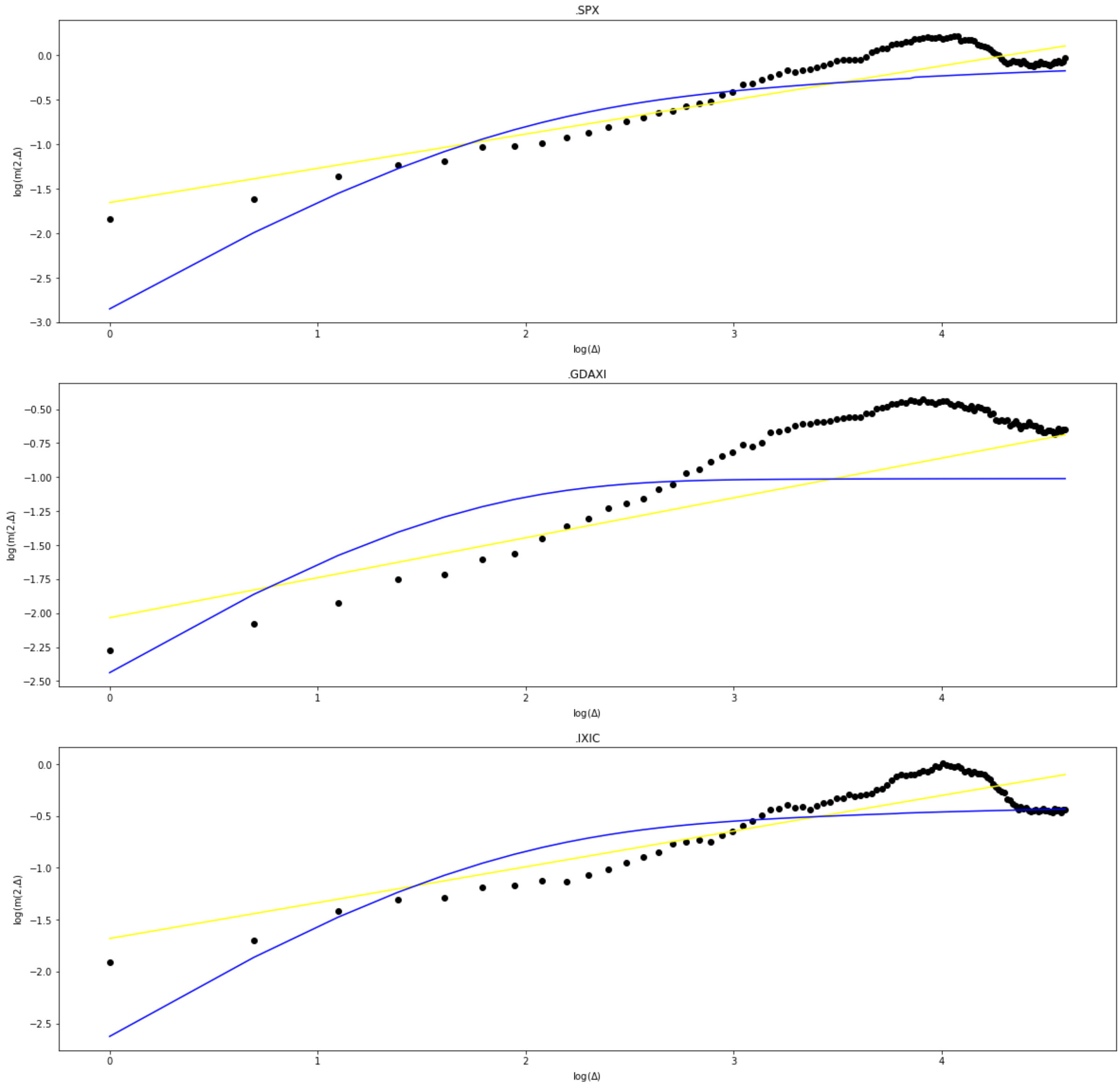

Figure 23 - Long memory models – COVID-19 Period

8.9 Simulation of a fBM vs Realized Variance – COVID-19 Period

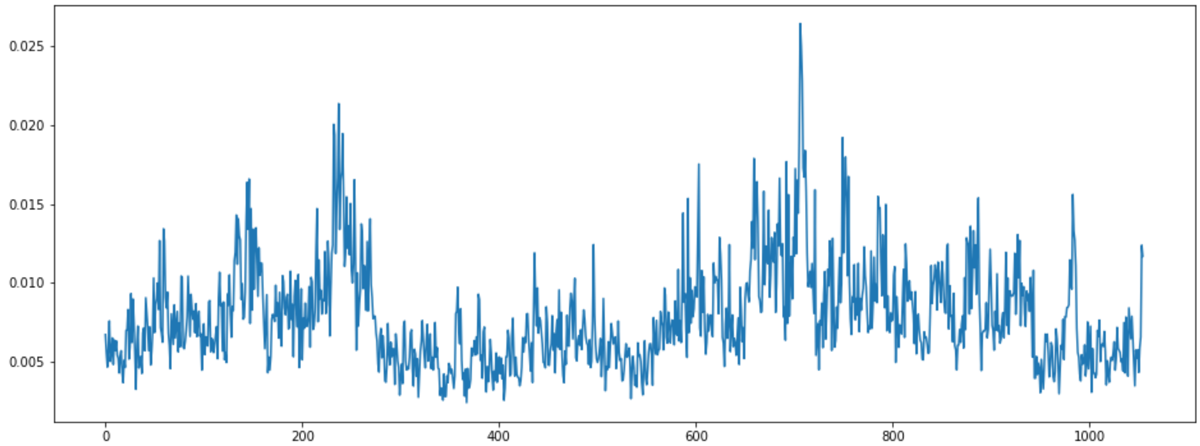

Figure 24 - Simulation of a fBM – COVID-19 Period

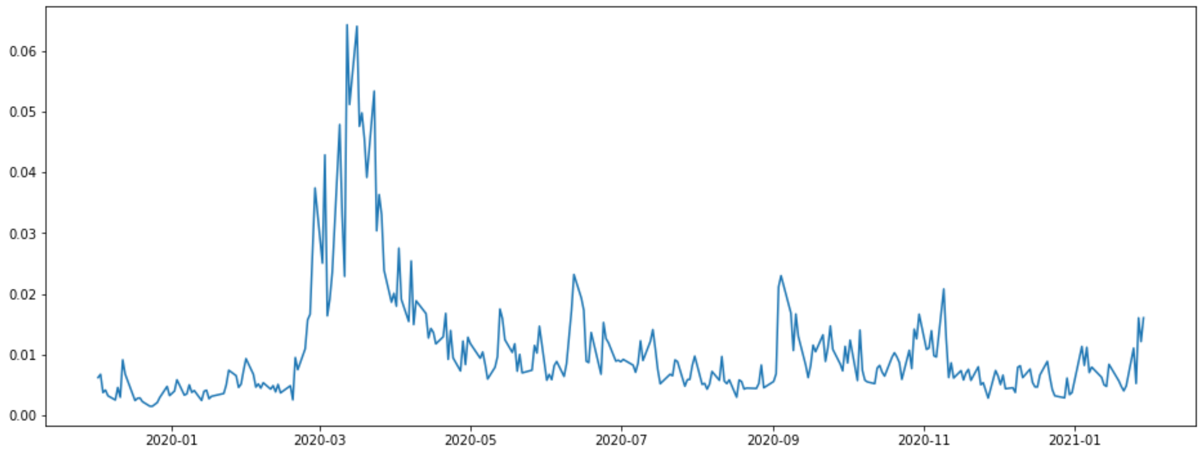

Figure 25 - Realized Variance – COVID-19 Period

8.10 Predictive variance vs realized variance for S&P500 – COVID-19 Period

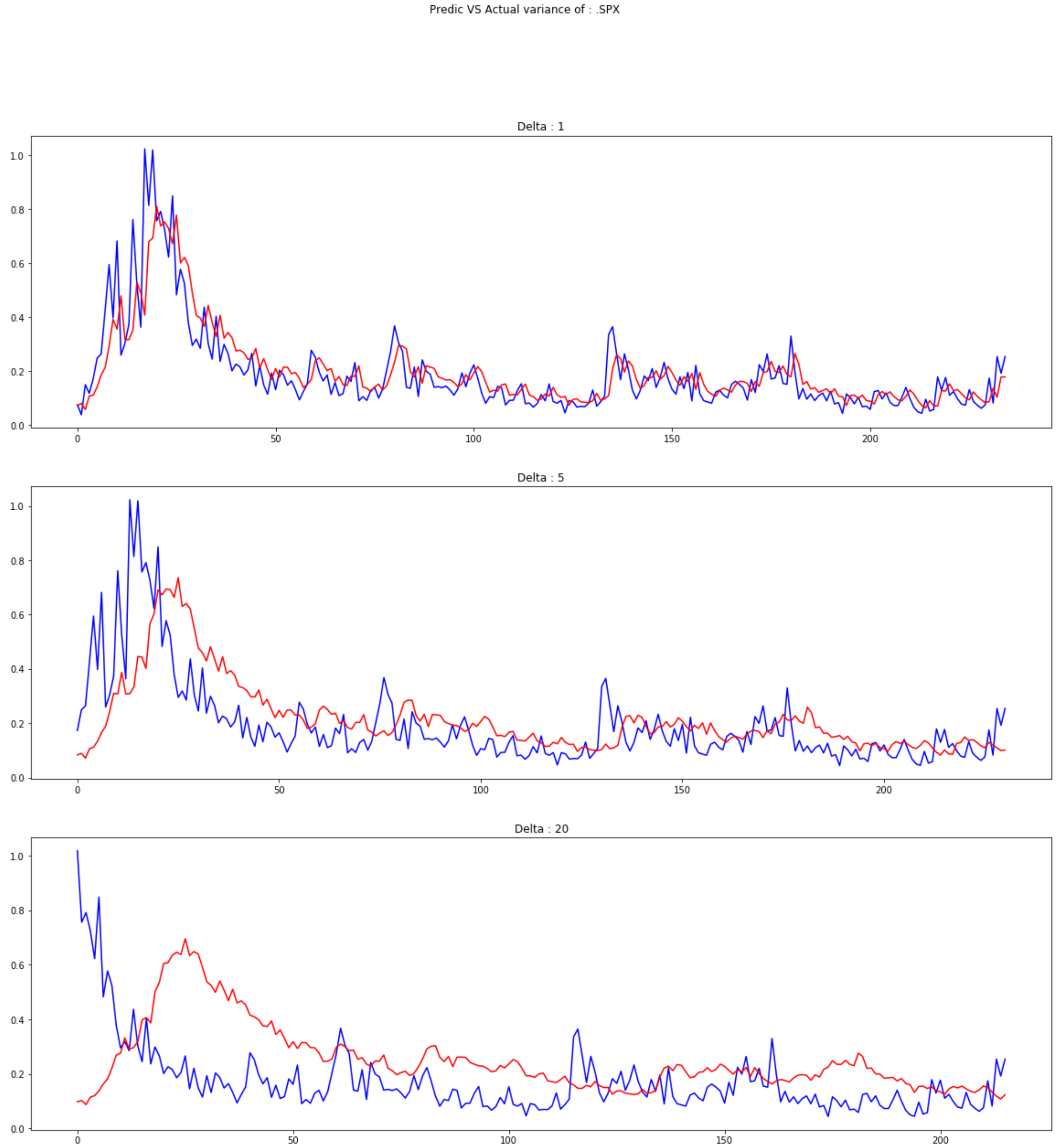

Figure 26 - Predictive variance vs realized variance for S&P500 – COVID-19 Period

8.11 Predictive variance vs realized variance for DAX– COVID-19 Period

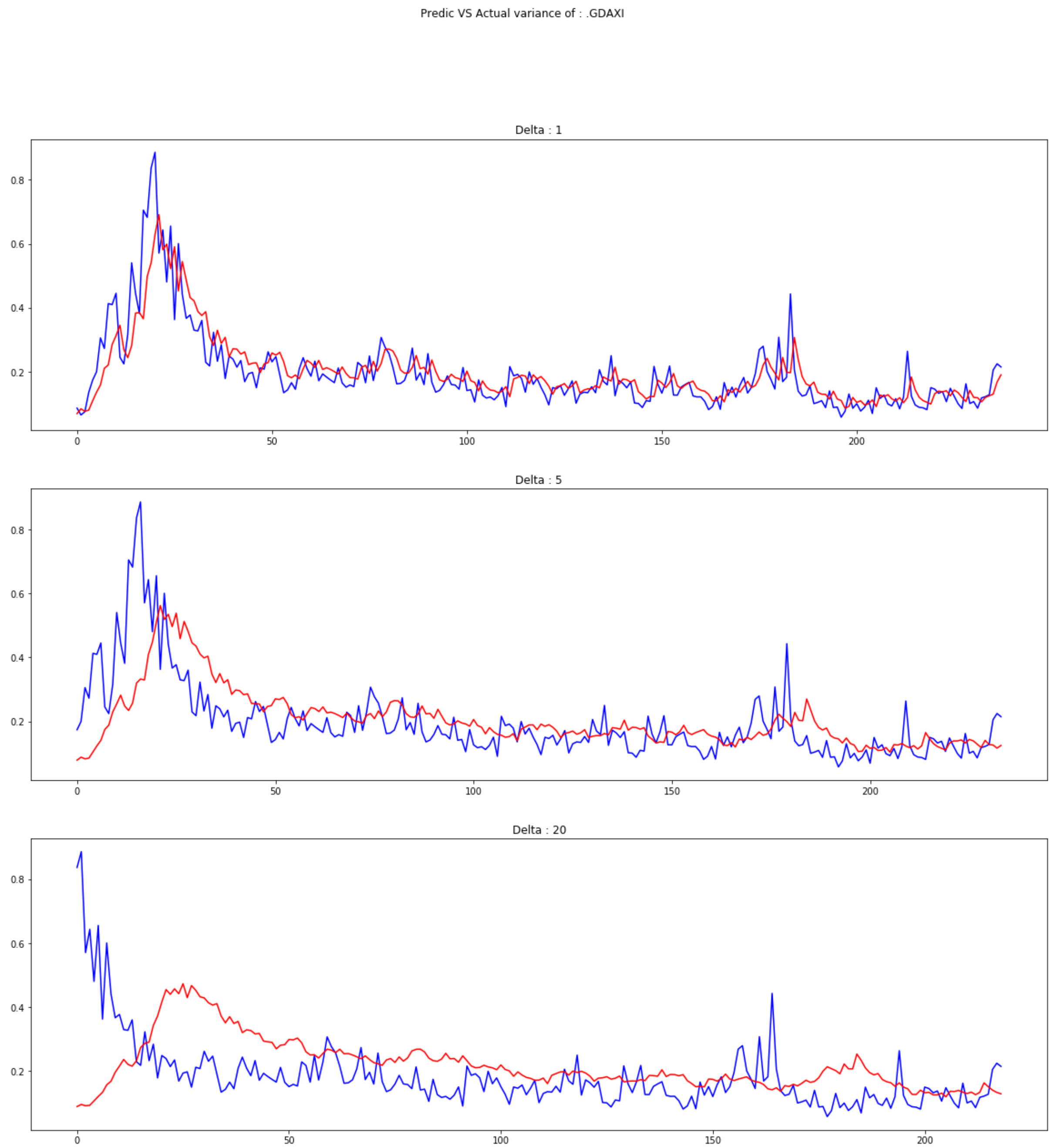

Figure 27 - Predictive variance vs realized variance for DAX– COVID-19 Period

8.12 Predictive variance vs realized variance for NASDAQ – COVID-19 Period

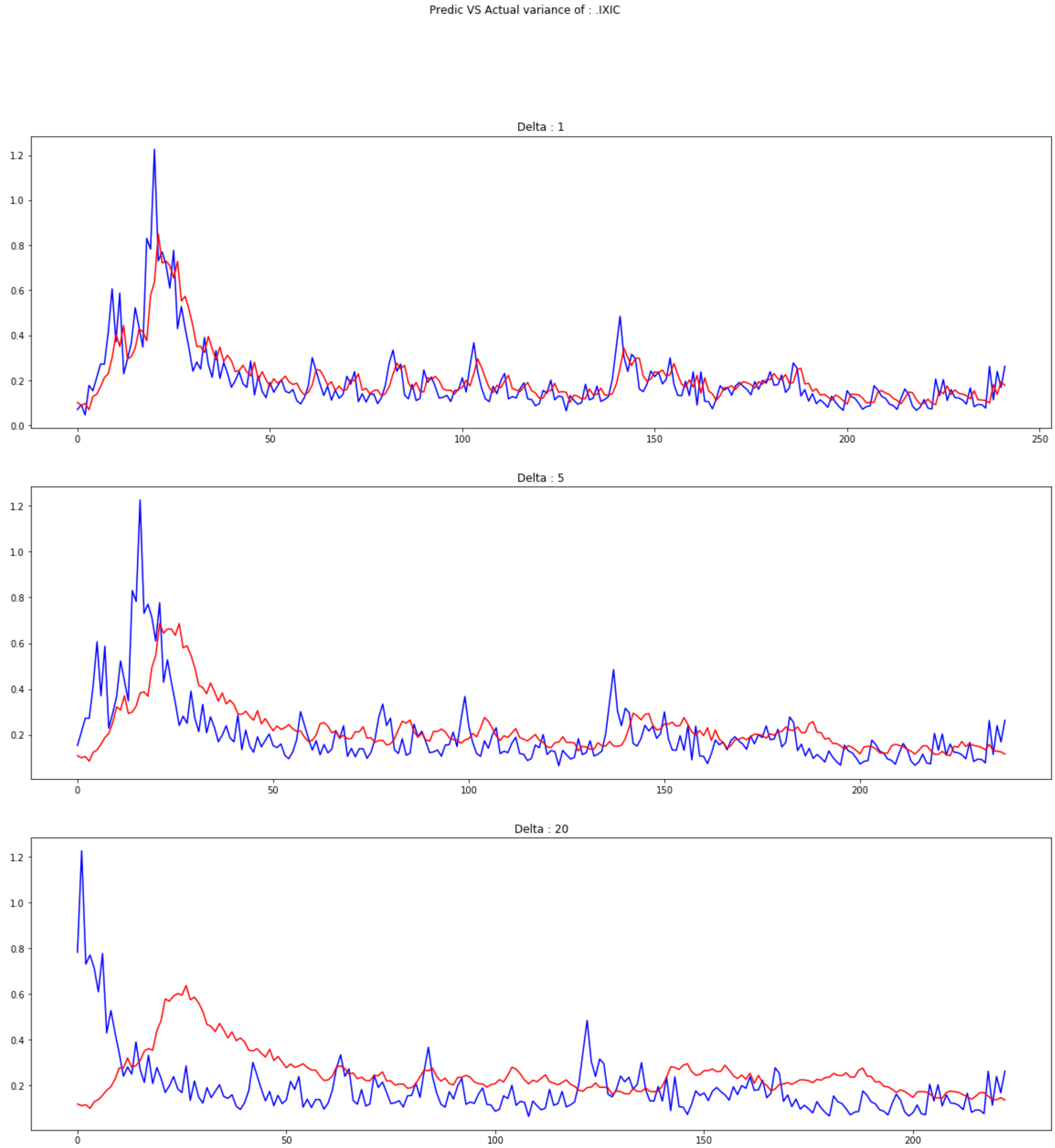

Figure 28 - Predictive variance vs realized variance for NASDAQ – COVID-19 Period

8.13 Autocovariance – COVID-19 Period

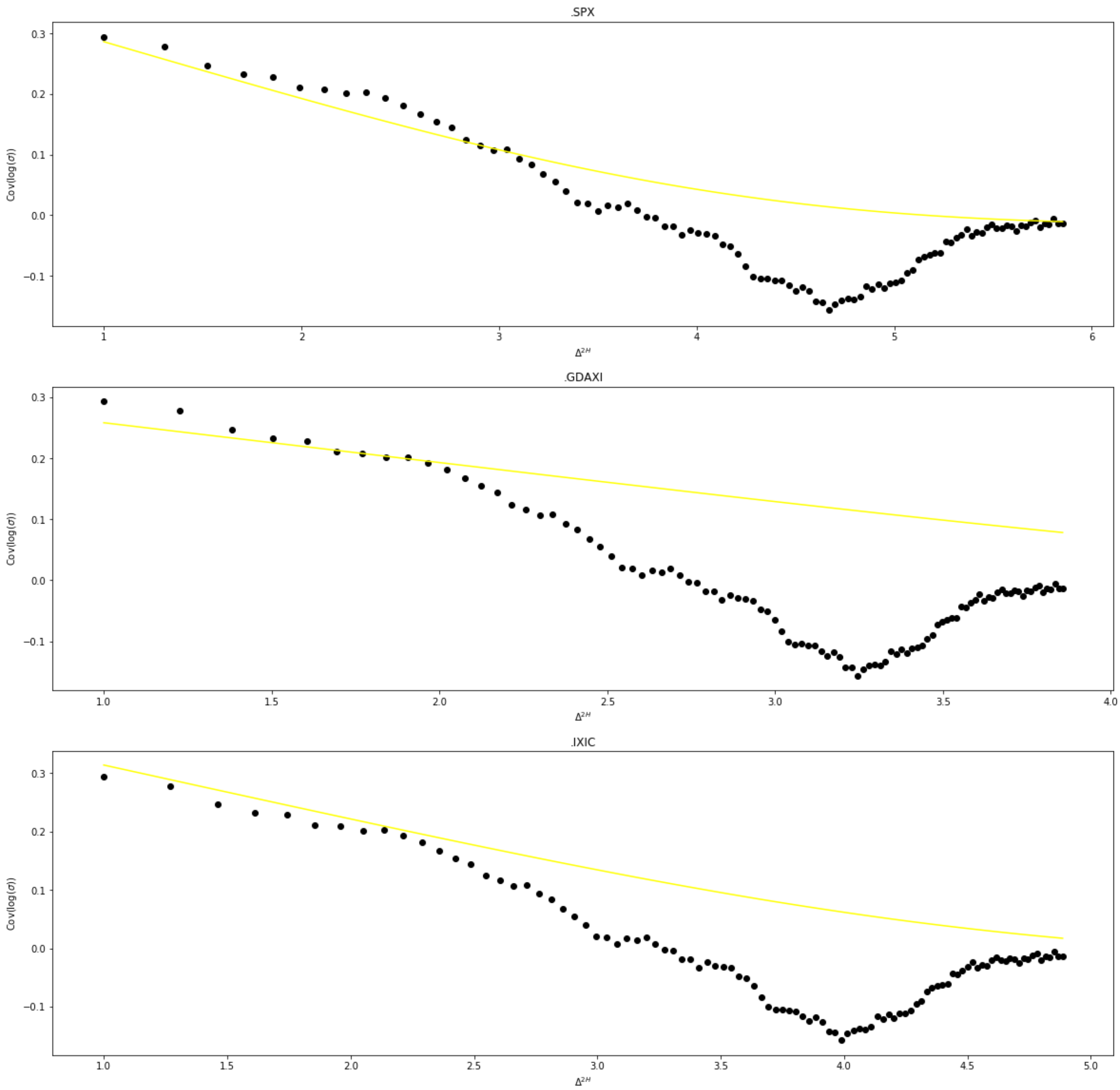

Figure 29 - Autocovariance – COVID-19 Period

8.14 Ratio P – COVID-19 Period

S&P500	Δ = 1	0.3342
	Δ = 5	0.5273
	Δ = 20	0.8682
DAX	Δ = 1	0.3453
	Δ = 5	0.4167
	Δ = 20	0.6961
NASDAQ	Δ = 1	0.4450
	Δ = 5	0.6389
	Δ = 20	0.9180

Figure 30 – P-Ratios for the RFSV predictors for log-variance