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Artificial Intelligence Research

Exploration-Exploitation in Reinforcement Learning

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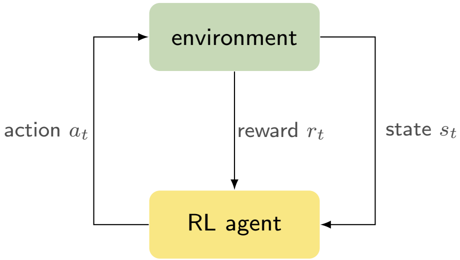
Reinforcement Learning



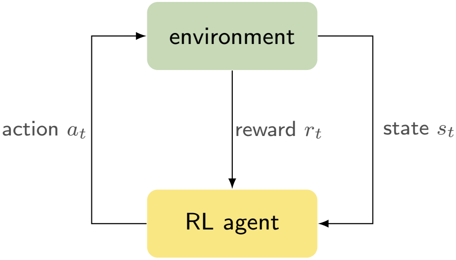
"Reinforcement learning is learning how to map states to actions so as to maximize a numerical reward signal in an unknown and uncertain environment.

In the most interesting and challenging cases, actions affect not only the immediate reward but also the next situation and all subsequent rewards (delayed reward).

The agent is not told which actions to take but it must discover which actions yield the most reward by trying them (trial-and-error)."

— Sutton and Barto [1998]

Reinforcement Learning



"**Reinforcement learning** is learning how to map states to actions so as to **maximize** a numerical **reward** signal in an unknown and **uncertain** environment.

In the most interesting and challenging cases, **actions** affect not only the immediate reward but also the **next situation** and all subsequent rewards (**delayed reward**).

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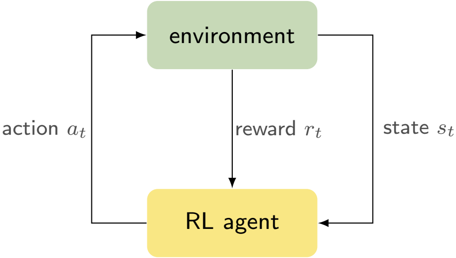
Reinforcement Learning



Exploitation

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Exploration

— Sutton and Barto [1998]

Disclaimer: the Real Title

Regret Minimization in

Infinite-Horizon

Finite Markov Decision Processes

Organization

Website
`https://rlgammazero.github.io`

Markov Decision Process

A discrete-time finite Markov decision process (MDP) is a tuple $M = \langle \mathcal{S}, \mathcal{A}, r, p \rangle$

- State space $\mathcal{S}$, $|\mathcal{S}| = S < \infty$
- Action space $\mathcal{A}$, $|\mathcal{A}| = A < \infty$
- Transition distribution $p(\cdot|s,a) \in \Delta(\mathcal{S})$
- Reward distribution with expectation $r(s,a) \in [0, r_{\max}]$

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☞ The process generates history $H_t = (s_1, a_1, \ldots, s_{t-1}, a_{t-1}, s_t)$, with $s_{t+1} \sim p(\cdot|s_t, a_t)$

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In (contextual) bandit, actions do not influence the evolution of states

Policies

An agent acts according to a *policy*

	stationary	history-dependent
deterministic	$\pi : \mathcal{S} \to \mathcal{A}$	$\pi_t : \mathcal{H}_t \to \mathcal{A}$
stochastic	$\pi : \mathcal{S} \to \Delta(\mathcal{A})$	$\pi_t : \mathcal{H}_t \to \Delta(\mathcal{A})$

Classification

An MDP M is

- *ergodic* if it is possible to go from any state to any other state under *any* deterministic stationary policy

$$\forall s, s', \forall \pi : \mathcal{S} \rightarrow \mathcal{A}, \exists t < \infty, \text{ s.t. } \mathbb{P}_\pi^M\big(s_t = s' | s_0 = s\big) > 0$$

- *communicating* if it is possible to go from any state to any other state under *a specific* deterministic stationary policy

$$\forall s, s', \exists \pi : \mathcal{S} \rightarrow \mathcal{A}, \exists t < \infty, \text{ s.t. } \mathbb{P}_\pi^M\big(s_t = s' | s_0 = s\big) > 0$$

☞ A communicating MDP has *finite diameter*

$$D_M = \max_{s,s' \in \mathcal{S}} \min_{\pi : \mathcal{S} \rightarrow \mathcal{A}} \mathbb{E}\big[T_\pi^M(s, s')\big]$$

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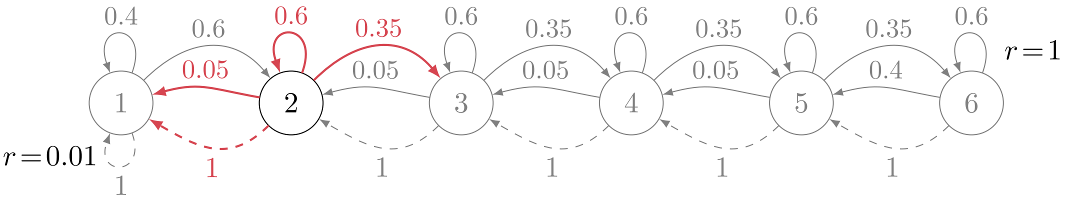
👉 A communicating MDP has *finite diameter*

$$D_M = \max_{s,s' \in \mathcal{S}} \underbrace{\min_{\pi:\mathcal{S}\to\mathcal{A}} \mathbb{E}\big[T_{\pi}^{M}(s,s')\big]}_{\text{shortest path}}$$

River Swim: Markov Decision Processes

Strehl and Littman [2008]



- $\mathcal{S} = \{1, 2, 3, 4, 5, 6\}$, $\mathcal{A} = \{L, R\}$
- $\pi_L(s) = L$, $\pi_R(s) = R$
- $M \oplus \pi_R$ is *ergodic* but $M \oplus \pi_L$ is *not ergodic*
- $T^M_{\pi_L}(6, 1) = 5, \quad D_M = \mathbb{E}\big[T^M_{\pi_R}(1, 6)\big] \approx 14.7$

Gain and Bias

Gain of a deterministic stationary policy π

$$g^\pi_M(s) = \lim_{T\to\infty} \mathbb{E}\left[\frac{1}{T}\sum_{t=1}^{T} r(s_t,a_t)\middle| s_0 = s, a_t = \pi(s_t)\right]$$

Bias of a deterministic stationary policy π

$$h^\pi_M(s) := C\text{-}\lim_{T\to\infty} \mathbb{E}\left[\sum_{t=1}^{T} \big(r(s_t,a_t) - g^\pi_M(s_t)\big)\middle| s_0 = s, a_t = \pi(s_t)\right]$$

Span of the bias function

$$\mathrm{sp}\big(h^\pi_M\big) = \max_s h^\pi_M(s) - \min_s h^\pi_M(s)$$

Bellman operators

$$= \sum_{s'} p(s'|s,a)h(s')$$

Bellman operator $L_M^a : \mathbb{R}^S \to \mathbb{R}^S$

$$L_M^a h(s) = r(s,a) + \ p(\cdot|s,a)^\mathsf{T} h$$

Optimal Bellman operator $L_M^\star : \mathbb{R}^S \to \mathbb{R}^S$

$$L_M^\star h(s) = \max_{a \in \mathcal{A}} \left\{ r(s,a) + p(\cdot|s,a)^\mathsf{T} h \right\}$$

Optimality gap of action a at s

$$\delta_M^\star(s,a) \ = \ L_M^\star h_M^\star(s) - L_M^a h_M^\star(s)$$

a.k.a. advantage function

Optimality

Optimal policy and *optimal gain*

$$\pi^\star_M \in \arg\max_\pi g^\pi_M(s) \qquad g^\star_M = g^{\pi^\star}_M(s) \;\; \forall s \in \mathcal{S}$$

Optimality equation

$$h^\star_M(s) + g^\star_M = L^\star_M h^\star_M(s)$$

Greedy policy w.r.t. $h^\star_M$ is optimal

$$\pi^\star_M(s) \in \arg\max_{a \in \mathcal{A}} \left\{ r(s,a) + p(\cdot|s,a)^\mathsf{T} h^\star_M \right\}$$

Set of optimal actions in state s

$$\Pi^\star_M(s) = \arg\max_{a \in \mathcal{A}} \left\{ r(s,a) + p(\cdot|s,a)^\mathsf{T} h^\star_M \right\}$$

Optimality

deterministic stationary

Optimal policy and *optimal gain*

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Optimality

deterministic stationary

constant gain*

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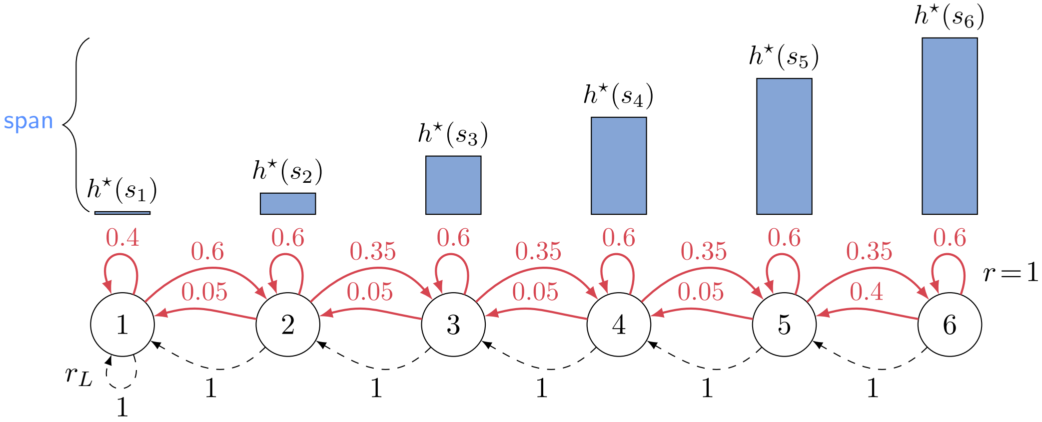
*In communicating MDPs

River Swim: Optimality



- $\pi^\star = \pi_R$
- If $r_L = 0.01$, $g^\star \approx 0.43$, $\mathrm{sp}(h^\star) \approx 6.4$

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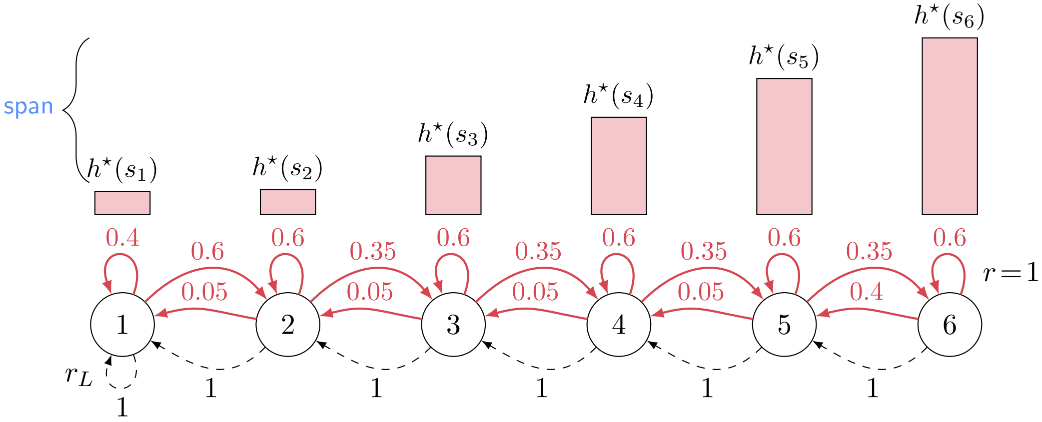

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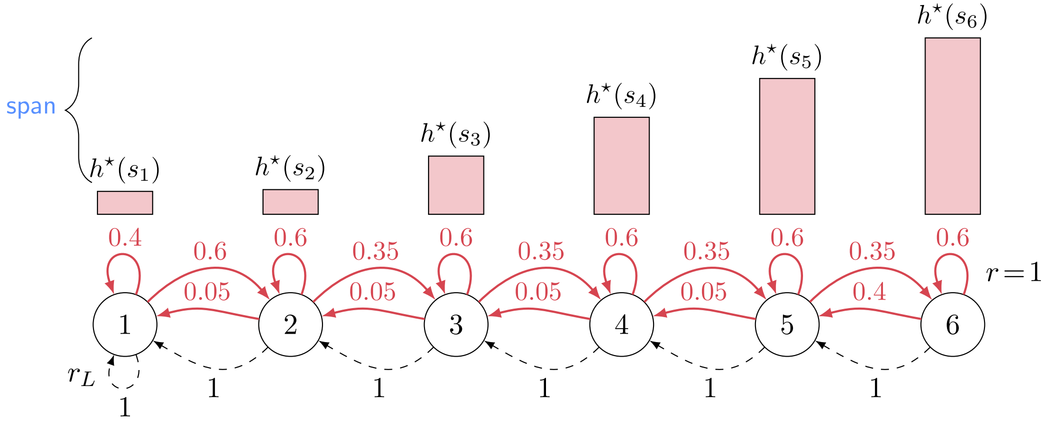

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D is constant

Value Iteration

initialize $v_0(s) = 0 \;\; \forall s \in \mathcal{S}$, $n = 0$, ε

repeat

  for $s \in \mathcal{S}$ **do**

  $$v_{n+1}(s) = L_M^\star v_n(s) = \max_{a \in \mathcal{A}} \left\{ r(s,a) + p(\cdot|s,a)^\mathsf{T} v_n \right\}$$

  end

  $n = n + 1$

until *sp*$(v_{n+1} - v_n) < \varepsilon$

return greedy policy

$$\pi_\varepsilon(s) = \arg\max_{a \in \mathcal{A}} L_M^a v_n(s) = \arg\max_{a \in \mathcal{A}} \left\{ r(s,a) + p(\cdot|s,a)^\mathsf{T} v_n \right\}$$

Value Iteration

Theorem (Thm. 8.5.5 [Puterman, 1994])

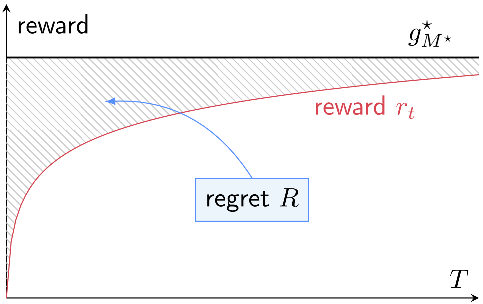
In any communicating MDP M, value iteration is such that

- *convergence: for any ε, there exists n_ϵ s.t. the stopping condition is met*
- *optimality: policy π_ε is ϵ-optimal*

$$g_M^{\pi_\varepsilon}(s) \geq g_M^\star - \varepsilon$$

Regret Minimization

Regret Minimization



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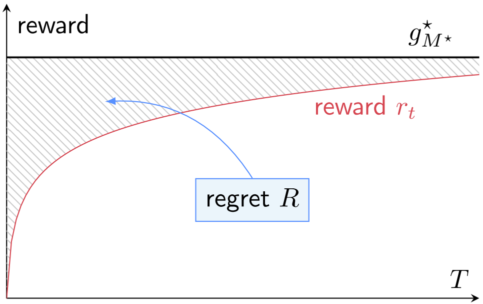


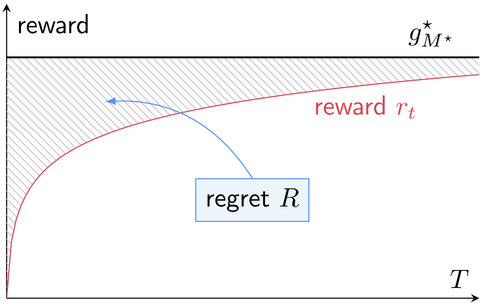
unknown true MDP $M^\star = \langle \mathcal{S}, \mathcal{A}, r, p \rangle$

algorithm $\mathfrak{A} = \{\pi_t\}$

$$R(T,\ M^\star,\ \mathfrak{A}) = T g^\star_{M^\star} - \sum_{t=1}^{T} r_t$$

reward obtained by $\mathfrak{A}$

Regret Minimization



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Expected regret w.r.t. randomness of s_t, r_t, and (possibly) $\mathfrak{A}$

$$\overline{R}(T, M^\star, \mathfrak{A}) = \mathbb{E}\big[R(T, M^\star, \mathfrak{A})\big]$$

Problem-Dependent Lower Bound

Let $M = \langle \mathcal{S}, \mathcal{A}, r, p \rangle$ and $M' = \langle \mathcal{S}, \mathcal{A}, r, p' \rangle$

- *Difference* between M and M' at s, a (w.l.o.g. assuming reward known)

$$\mathsf{KL}_{M,M'}(s,a) = \mathsf{KL}\big(p(\cdot|s,a) \| p'(\cdot|s,a)\big)$$

- *Set of alternative* (confusing) models w.r.t. M

same everywhere but in (s,a)

$$\mathcal{M}^{\mathsf{alt}}_M(s,a) = \Big\{ M' : p'(\cdot|s',a') = p(\cdot|s',a'), \text{ for all } (s',a') \neq (s,a), \\ a \notin \Pi^\star_M(s) \ , \ a \in \Pi^\star_{M'}(s) \Big\}$$

sub-optimal in M

optimal in M'

Problem-Dependent Lower Bound

Theorem (Thm. 1 Burnetas and Katehakis [1997], Thm. 2 Ok et al. [2018])

Let $\mathfrak{A}$ be s.t. $\overline{R}(T, M, \mathfrak{A}) = o(T^\alpha)$ for all $\alpha > 0$ and ergodic MDP M. For any ergodic MDP $M^\star$ with $r_{\max} = 1$, the expected regret is lower bounded as

$$\liminf_{T \to \infty} \frac{\overline{R}(T, M^\star, \mathfrak{A})}{\log T} \geq K_{M^\star}$$

where

cumulative regret

$$K_{M^\star} = \inf_{\eta \geq 0} \sum_{s,a} \eta(s,a) \delta^\star_{M^\star}(s,a)$$

$$\text{s.t.} \sum_{s,a} \eta(s,a) KL_{M^\star, M}(s,a) \geq 1 \quad \forall M \in \mathcal{M}^{alt}_{M^\star}(s,a)$$

"evidence" of difference between $M^\star$ and M

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"evidence" of difference between $M^\star$ and M

Similar to [Lai and Robbins, 1985] for MAB but alternative models and regret are different.

Problem-Dependent Lower Bound

Theorem (Thm. 1 Burnetas and Katehakis [1997], Thm. 2 Ok et al. [2018])

Let $\mathfrak{A}$ *be s.t.* $\overline{R}(T, M, \mathfrak{A}) = o(T^\alpha)$ *for all* $\alpha > 0$ *and ergodic MDP* M. *For any ergodic MDP* $M^\star$ *with* $r_{\max} = 1$, *the expected regret is lower bounded as*

$$\liminf_{T\to\infty} \frac{\overline{R}(T, M^\star, \mathfrak{A})}{\log T} \geq K_{M^\star}$$

where

$$K_{M^\star} \leq 2 \frac{(C+1)^2}{\min_{s,a} \delta_{M^\star}(s,a)} SA \qquad C = sp(h^\star_{M^\star})$$

Minimax Lower Bound

Theorem (Thm. 5 Jaksch et al. [2010])

For any communicating MDP $M^\star$ *with* $r_{\max} = 1,\ S, A \geq 10,\ D \geq 20 \log_A S$*, any algorithm* $\mathfrak{A}$ *at any time* $T \geq DSA$ *suffers a regret*

$$\sup_{M^\star} \overline{R}(T, M^\star, \mathfrak{A}) \geq 0.015\sqrt{DSAT}$$

Minimax Lower Bound

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For any communicating MDP $M^\star$ *with* $r_{\max} = 1$, $S, A \geq 10$, $D \geq 20 \log_A S$, *any algorithm* $\mathfrak{A}$ *at any time* $T \geq DSA$ *suffers a regret*

$$\sup_{M^\star} \overline{R}(T, M^\star, \mathfrak{A}) \geq 0.015\sqrt{DSAT}$$

In MAB $\Omega(\sqrt{AT})$ since $D = 1$ and $S = 1$.

Open Questions

C could be arbitrarily large
($C = \infty$ for non ergodic)

1. *Asymptotic* regime and *ergodicity* assumption

$$\mathbb{P}^{\pi}_{M}\left[N_T(s) \geq \rho T\right] \geq 1 - C \exp(-\rho T/2)$$

[Prop.2 Burnetas and Katehakis [1997]]

2. *Span vs. diameter*

$D = 2\mathrm{sp}(h^\star)$ in the proof

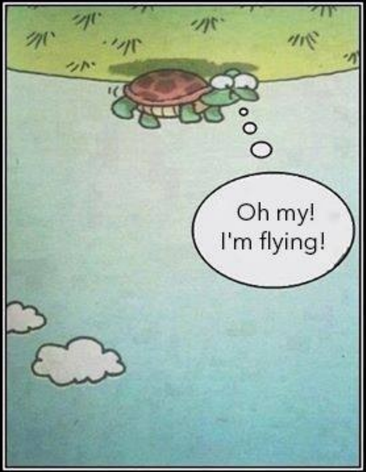
$$\overline{R}(T, M^\star, \mathfrak{A}) \geq 0.015\sqrt{D\, SAT}$$

3. *Number of states vs branching factor* $\Gamma = \max_{s,a} |\mathrm{supp}(p(\cdot|s,a))|$

$$\overline{R}(T, M^\star, \mathfrak{A}) \geq 0.015\sqrt{D\, S\, AT}$$

$\Gamma = 2$ in the proof

The Optimism Principle: Intuition



OPTIMISM
It's the best way to see life.

The Optimism Principle: Intuition

Exploration vs. Exploitation

The Optimism Principle: Intuition

Exploration vs. Exploitation

Optimism in Face of Uncertainty

When you are uncertain, consider the **best possible world (reward-wise)**

The Optimism Principle: Intuition

Exploration vs. Exploitation

Optimism in Face of Uncertainty

When you are uncertain, consider the **best possible world (reward-wise)**

If the best possible world is **correct**

$\Longrightarrow$ **no regret**

Exploitation

If the best possible world is **wrong**

$\Longrightarrow$ **learn useful information**

Exploration

The Optimism Principle: Intuition

Exploration vs. Exploitation

Optimism in **gain**

Optimism in Face of Uncertainty

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If the best possible world is **correct**

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Exploitation

If the best possible world is **wrong**

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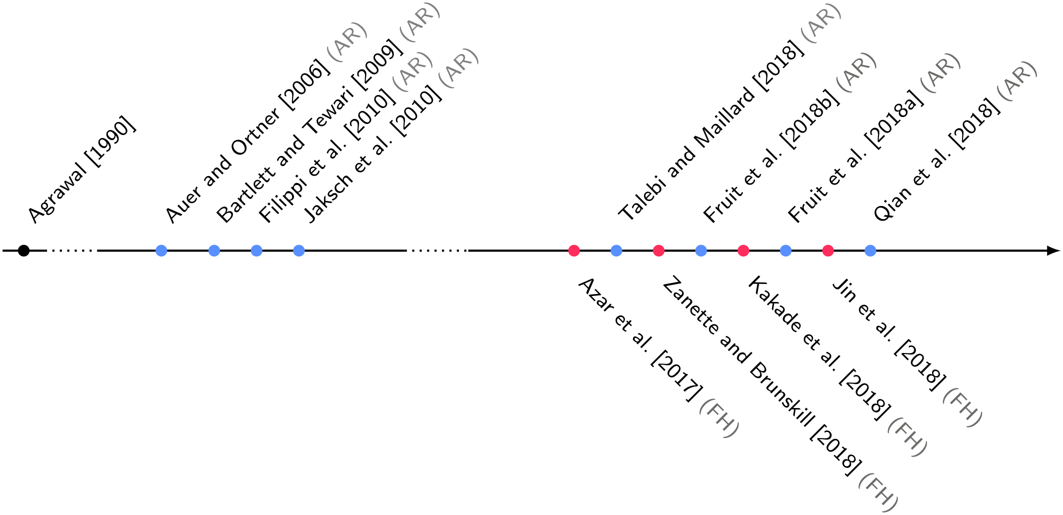
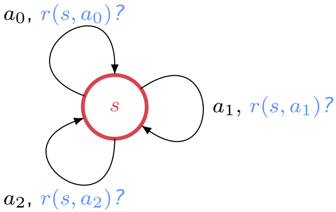
Exploration

History: OFU for Regret Minimization in RL

FH: finite-horizon
AR: average reward



Gain Optimism: Example



- Deterministic *policies*:
  - $\pi_0(s) = a_0$
  - $\pi_1(s) = a_1$
  - $\pi_2(s) = a_2$

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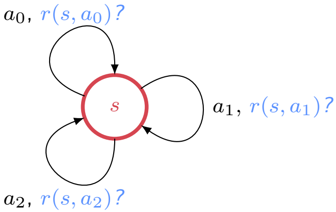

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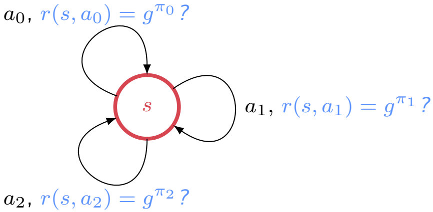
- Optimism

$$\widetilde{\pi} = \arg\max_{\pi_i} \mathsf{UCB}(g^{\pi_i})$$

Gain Optimism: Example



- Deterministic *policies*:
  - $\pi_0(s) = a_0$
  - $\pi_1(s) = a_1$
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- Reward $r(s, a_i)$ = *gain* g^{π_i}
- Optimism

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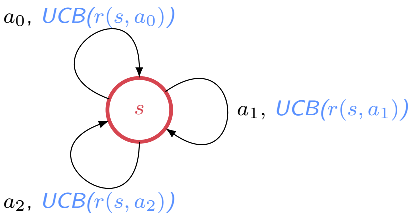
Gain Optimism: Example



- Deterministic *policies*:
  - $\pi_0(s) = a_0$
  - $\pi_1(s) = a_1$
  - $\pi_2(s) = a_2$
- Reward $r(s, a_i)$ = *gain* g^{π_i}
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$$\text{UCB}(g^{\pi_i}) = \text{UCB}(r(s, a_i))$$

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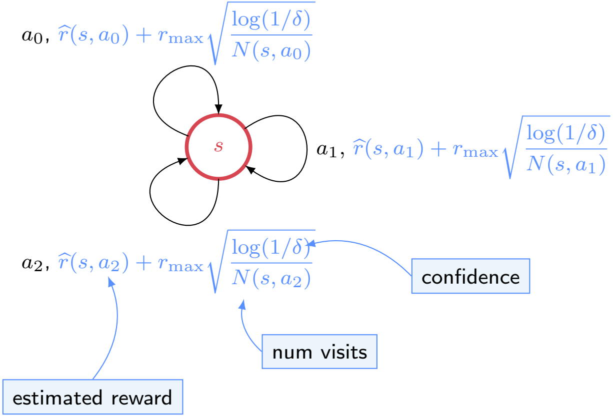

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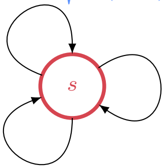
Gain Optimism: Example

$$a_0,\ \widehat{r}(s,a_0) + r_{\max}\sqrt{\frac{\log(1/\delta)}{N(s,a_0)}}$$



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$$a_2,\ \widehat{r}(s,a_2) + r_{\max}\sqrt{\frac{\log(1/\delta)}{N(s,a_2)}}$$

confidence

num visits

estimated reward

- Deterministic *policies*:
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☞ UCB algorithm (Bandit)

Gain Optimism: Implementation

Tentative algorithm

Observe s_1
for $t = 1, 2, \dots$ **do**
  Compute $\pi_t \leftarrow \arg\max_{\pi} \mathit{UCB}_t(g^\pi)$
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Gain Optimism: Implementation

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end

3 major issues:

- *Upper confidence bounds*: construct $\text{UCB}_t(g^{\pi})$ with unknown dynamics
- *Computational complexity*: exponential number of policies
- *Frequent policy update*: inefficient exploration

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Bounded Parameter MDP: Definition

Bounded parameter MDP [Strehl and Littman, 2008]

$$\mathcal{M}_t = \Big\{ \langle \mathcal{S}, \mathcal{A}, r, p \rangle : \; r(s,a) \in B_t^r(s,a), \; p(\cdot|s,a) \in B_t^p(s,a), \forall (s,a) \in \mathcal{S} \times \mathcal{A} \Big\}$$

Compact *confidence sets*

$$B_t^r(s,a) := \Big[\widehat{r}_t(s,a) - \beta_t^r(s,a), \; \widehat{r}_t(s,a) + \beta_t^r(s,a) \Big]$$

$$B_t^p(s,a) := \Big\{ p(\cdot|s,a) \in \Delta(\mathcal{S}) : \; \|p(\cdot|s,a) - \widehat{p}_t(\cdot|s,a)\|_1 \leq \; \beta_t^p(s,a) \Big\}$$

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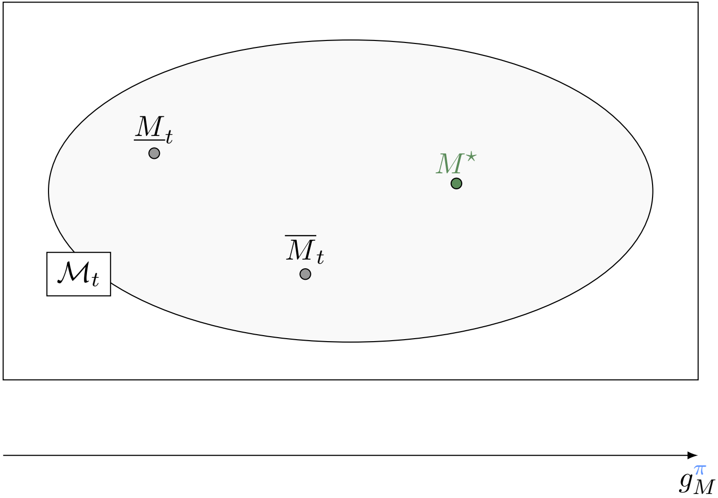
Confidence bounds based on [Hoeffding, 1963] and [Weissman et al., 2003]

$$\beta_t^r(s,a) \propto \sqrt{\frac{\log(N_t(s,a)/\delta)}{N_t(s,a)}}$$

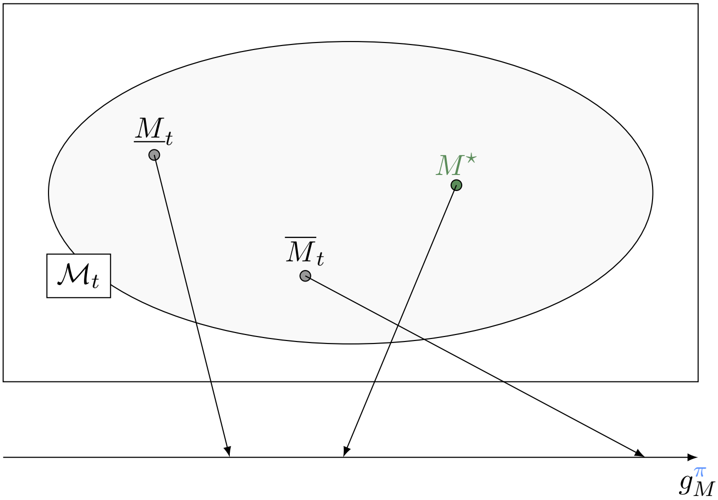
$$\beta_t^p(s,a) \propto \sqrt{\frac{S \log(N_t(s,a)/\delta)}{N_t(s,a)}}$$

Bounded Parameter MDP: Optimism



Fix a *policy* π

Bounded Parameter MDP: Optimism



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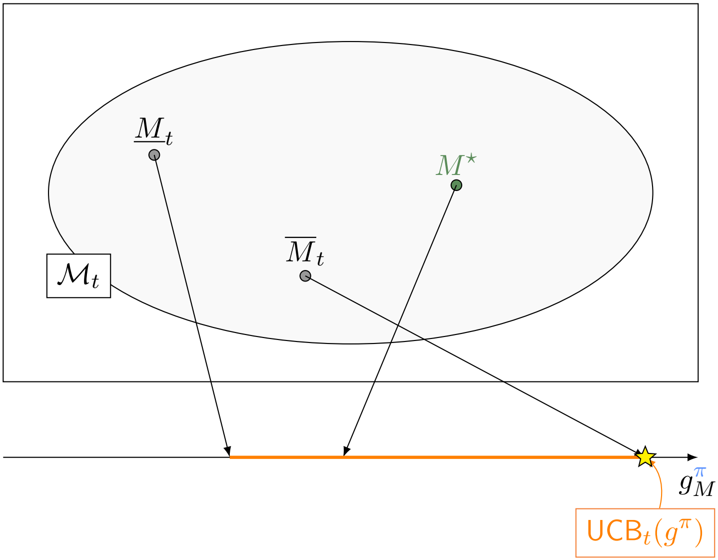

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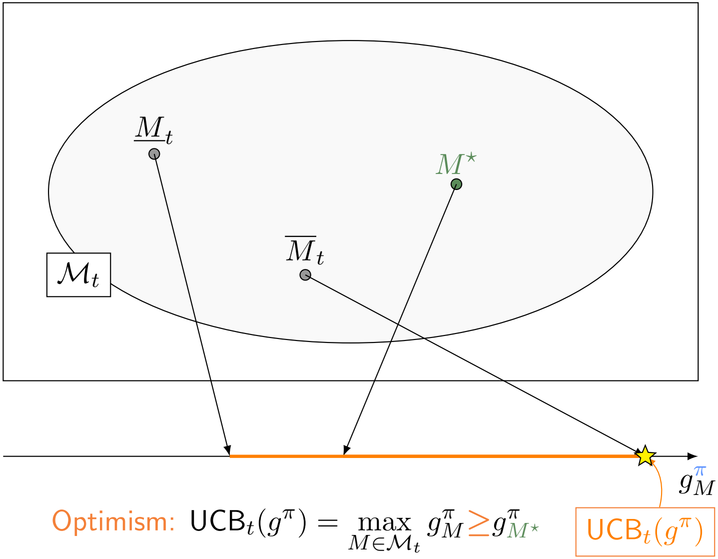

Fix a *policy* π

Optimism: $\mathrm{UCB}_t(g^\pi) = \max_{M \in \mathcal{M}_t} g_M^\pi \geq g_{M^\star}^\pi$

Gain Optimism: Implementation

Tentative algorithm

Observe state s_1
for $t = 1, 2, \dots$ **do**
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3 major issues:

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Extended MDP

[Strehl and Littman, 2008, Jaksch et al., 2010]

Theorem (Bounded parameter MDP $\iff$ Extended MDP)

Let $\mathcal{M}_t^+ := \langle \mathcal{S}, \mathcal{A}_t^+, r^+, p^+ \rangle$ *be an extended MDP such that*

$$\mathcal{A}_t^+(s) = \mathcal{A}(s) \times B_t^r(s,a) \times B_t^p(s,a)$$

Continuous **compact** action space

with $a^+ = (a, r, p) \in \mathcal{A}_t^+(s), \;\; r^+(s, a^+) = r, \;\; p^+(\cdot|s, a^+) = p.$

Then the optimal gain of $\mathcal{M}_t^+$ *satisfies*

$$g^*_{\mathcal{M}_t^+} := \max_\pi \left\{ \max_{M \in \mathcal{M}_t} g_M^\pi \right\}$$

Let $\pi_t^+ = \arg\max_\pi g^\pi_{\mathcal{M}_t^+}$, *then*

$$\pi_t = \arg\max_\pi \left\{ \max_{M \in \mathcal{M}_t} g_M^\pi \right\} \text{ s.t. } \pi_t(s) = \pi_t^+(s)[a]$$

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ompact

pace

with $a^+ =$

Abuse of notation: $\mathcal{M}_t$ denotes the extended MDP

Then the optimal gain of $\mathcal{M}_t^+$ satisfies

$$g^*_{\mathcal{M}_t^+} := \max_\pi \left\{ \max_{M \in \mathcal{M}_t} g^\pi_M \right\}$$

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Extended Value Iteration

Value iteration on $\mathcal{M}_t$

$$v_{n+1}(s) = \mathcal{L}_t v_n(s) = \max_{(a,r,p)\in\mathcal{A}(s)\times B_t^r(s,a)\times B_t^p(s,a)} \left\{ r + p^\mathsf{T} v_n \right\}$$

$$= \max_{a\in\mathcal{A}(s)} \left\{ \max_{r\in B_t^r(s,a)} r + \max_{p\in B_t^p(s,a)} p^\mathsf{T} v_n \right\}$$

$$= \max_{a\in\mathcal{A}(s)} \left\{ \widehat{r}_t(s,a) + \beta_t^r(s,a) + \max_{p\in B_t^p(s,a)} p^\mathsf{T} v_n \right\}$$

$\pi_t =$ *Greedy policy* w.r.t. v_n

Gain Optimism: Implementation

Tentative algorithm

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  - How to efficiently *compute* $\max_{M \in \mathcal{M}_t} g^{\pi}_M$ for every π? ✔
- *Computational complexity*: exponential number of policies ✔
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Optimism: Frequency of Policy Updates

Proposition [Ortner, 2010]

There exists an MDP s.t.

$$\Omega(T) \text{ number of policy updates} \implies \textit{linear regret}.$$

$\implies$ $o(T)$ number of policy updates

Final Algorithm: UCRL2

Initialize $t \leftarrow 1$
Observe state s_1
Initialize empirical means $\widehat{r}_1 = r_{\max}$ and $\widehat{p}_1 = (1/S, \dots, 1/S)^\mathsf{T}$
Initialize visit counts $N_1 = 0$
for *episodes* $k = 1, 2, \dots$ **do**

  Set $t_k \leftarrow t$
  Build extended MDP $\mathcal{M}_k := \mathcal{M}_{t_k}$
  Using EVI, compute *optimistic policy* π_k and $(h_k, g_k) \in \mathbb{R}^S \times [0, r_{\max}]$ such that

$$\boxed{\mathcal{L}_{\mathcal{M}_k} h_k = \mathcal{L}^{\pi_k}_{\mathcal{M}_k} h_k = h_k + g_k e} \quad \text{with} \quad g_k = g^\star_{\mathcal{M}_k} \geq g^\star_{M^\star}$$

  while $N_t(s_t, a_t) < \max\{1,\ N_{t_k}(s_t, a_t)\}$ **do**

    Take action $a_t = \pi_k(s_t)$
    Observe reward r_t and next state s_{t+1}
    Compute new empirical means $\widehat{r}_{t+1}(s_t, a_t)$ and $\widehat{p}_{t+1}(\cdot|s_t, a_t)$
    Compute new visit count $N_{t+1}(s_t, a_t)$
    $t \leftarrow t + 1$

  end

end

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Optimism

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Bellman equation in $\mathcal{M}_k$

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    $t \leftarrow t + 1$
  end
end

Bellman equation in $\mathcal{M}_k$

Optimism

Stopping condition of an episode

UCRL2: Regret Guarantees

Theorem (Thm.2 of [Jaksch et al., 2010])

There exists a numerical constant $\beta > 0$ such that in any communicating MDP $M^\star = \langle \mathcal{S}, \mathcal{A}, r, p \rangle$, with probability at least $1-\delta$, UCRL2 suffers a regret bounded as

$$\forall T \geq 1, \;\; R(T, M^\star, \text{UCRL2}) \leq \beta \cdot r_{\max} DS \sqrt{AT \log\left(\frac{T}{\delta}\right)}$$

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Comparison to lower bound

$$\overline{R}(T, M^\star, \text{UCRL}) \geq 0.015\sqrt{DSAT}$$

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$$\overline{R}(T, M^\star, \text{UCRL}) \geq 0.015 \sqrt{DSAT}$$

- Can the gap between upper and lower bound be closed? ☞ More on this later

UCRL2: Regret Guarantees (cont'd.)

Theorem (Thm.4 of [Jaksch et al., 2010])

There exists a numerical constant $\beta > 0$ such that in any ergodic MDP $M^\star = \langle \mathcal{S}, \mathcal{A}, r, p \rangle$, for all $T \geq 1$, UCRL2 (with $\delta = 1/T$) suffers a regret bounded as

$$\overline{R}(T, M^\star, \text{UCRL2}) \leq \beta \cdot r_{\max} \frac{D^2 S^2 A \log(T)}{\delta_g^\star} + \textit{Big constant independent of } T$$

with

- $\delta_g^\star := g_{M^\star}^\star - \max_{s \in \mathcal{S}, \pi} \left\{ g_{M^\star}^\pi(s) < g_M^\star \right\}$ $\sim$ *"gap in gain"*

UCRL2: Regret Guarantees (cont'd.)

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with

- $\delta_g^\star := g_{M^\star}^\star - \max_{s \in \mathcal{S}, \pi} \left\{ g_{M^\star}^\pi(s) < g_M^\star \right\} \quad \sim$ *"gap in gain"*

Comparison to lower bound

$$\liminf_{T \to \infty} \frac{\overline{R}(T, M^\star, \mathfrak{A})}{\log T} \geq K_{M^\star}, \text{ with } K_{M^\star} \lesssim \frac{D^2 S A}{\min_{s,a} \delta_{M^\star}^\star(s,a)}$$

UCRL2: Regret Guarantees (cont'd.)

Theorem (Thm.4 of [Jaksch et al., 2010])

There exists a numerical constant $\beta > 0$ such that in any ergodic MDP $M^\star = \langle \mathcal{S}, \mathcal{A}, r, p \rangle$, for all $T \geq 1$, UCRL2 (with $\delta = 1/T$) suffers a regret bounded as

$$\overline{R}(T, M^\star, \text{UCRL2}) \leq \beta \cdot r_{\max} \frac{D^2 S^2 A \log(T)}{\delta_g^\star} + \textit{Big constant independent of } T$$

with

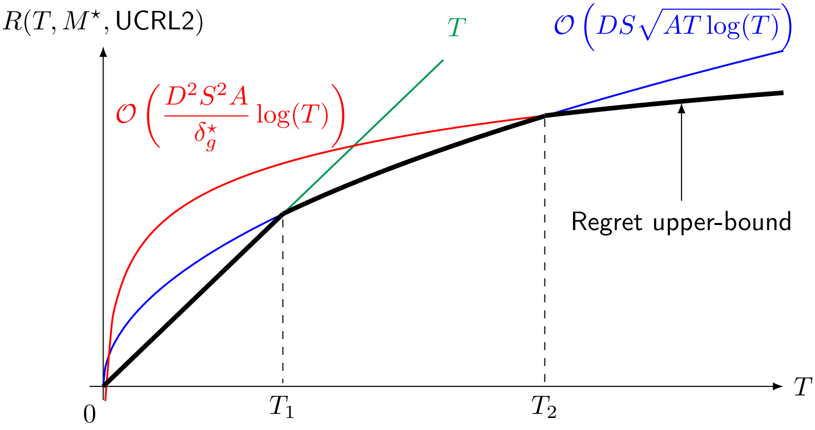
- $\delta_g^\star := g_{M^\star}^\star - \max_{s \in \mathcal{S}, \pi} \left\{ g_{M^\star}^\pi(s) < g_M^\star \right\} \quad \sim$ "gap in gain"

how do they compare?

Comparison to lower bound

$$\liminf_{T \to \infty} \frac{\overline{R}(T, M^\star, \mathfrak{A})}{\log T} \geq K_{M^\star}, \text{ with } K_{M^\star} \lesssim \frac{D^2 S A}{\min_{s,a} \delta_{M^\star}^\star(s,a)}$$

Qualitative Regret Shape



*illustrative plot

Regret Bound of UCRL2: Proof Sketch

$$\boxed{1}\ R(T, M^\star, \text{UCRL2}) = \sum_{k=1}^{m} \sum_{t=t_k}^{t_{k+1}-1} g^\star_{M^\star} - r(s_t, a_t) \leq \sum_{k=1}^{m} \sum_{t=t_k}^{t_{k+1}-1} g_k - r(s_t, a_t)$$

Split in episodes

Optimism: $g_k \geq g^\star_{M^\star}$

Regret Bound of UCRL2: Proof Sketch

1 $$R(T, M^\star, \text{UCRL2}) = \sum_{k=1}^{m}\sum_{t=t_k}^{t_{k+1}-1} g^\star_{M^\star} - r(s_t,a_t) \leq \sum_{k=1}^{m}\sum_{t=t_k}^{t_{k+1}-1} g_k - r(s_t,a_t)$$

2 $$\sum_{k=1}^{m}\sum_{t=t_k}^{t_{k+1}-1} g_k = \sum_{k=1}^{m}\sum_{t=t_k}^{t_{k+1}-1} r_k(s_t,a_t) + p_k(\cdot|s_t,a_t)^\mathsf{T} h_k - h_k(s_t)$$

Bellman equation ($a_t = \pi_k(s_t)$):

$$L^{\pi_k}_{\mathcal{M}_k} h_k(s_t) = h_k(s_t) + g_k$$

Regret Bound of UCRL2: Proof Sketch

1-2 $$R(T, M^\star, \mathsf{UCRL2}) \leq \sum_{k=1}^{m} \sum_{t=t_k}^{t_{k+1}-1} r_k(s_t, a_t) - r(s_t, a_t) + p_k(\cdot|s_t, a_t)^\mathsf{T} h_k - h_k(s_t)$$

Regret Bound of UCRL2: Proof Sketch

1-2 $$R(T, M^\star, \text{UCRL2}) \leq \sum_{k=1}^{m} \sum_{t=t_k}^{t_{k+1}-1} r_k(s_t, a_t) - r(s_t, a_t) + p_k(\cdot|s_t, a_t)^\mathsf{T} h_k - h_k(s_t)$$

Assumption: true reward is known $r = r_k$

Regret Bound of UCRL2: Proof Sketch

1-2 $R(T, M^\star, \textsf{UCRL2}) \leq \sum_{k=1}^{m} \sum_{t=t_k}^{t_{k+1}-1} p_k(\cdot|s_t, a_t)^\mathsf{T} h_k - h_k(s_t)$

Regret Bound of UCRL2: Proof Sketch

1-2 $R(T, M^\star, \mathsf{UCRL2}) \leq \sum_{k=1}^{m} \sum_{t=t_k}^{t_{k+1}-1} p_k(\cdot|s_t, a_t)^\mathsf{T} h_k - h_k(s_t)$

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Regret Bound of UCRL2: Proof Sketch

1-2 $$R(T, M^\star, \textsf{UCRL2}) \leq \sum_{k=1}^{m} \sum_{t=t_k}^{t_{k+1}-1} p_k(\cdot|s_t,a_t)^\mathsf{T} h_k - h_k(s_t)$$

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4 $$\sum_{k=1}^{m} \sum_{t=t_k}^{t_{k+1}-1} p(\cdot|s_t,a_t)^\mathsf{T} h_k - h_k(s_t) = \sum_{k=1}^{m} \sum_{t=t_k}^{t_{k+1}-1} p(\cdot|s_t,a_t)^\mathsf{T} h_k - h_k(s_{t+1}) + \sum_{k=1}^{m} \sum_{t=t_k}^{t_{k+1}-1} h_k(s_{t+1}) - h_k(s_t)$$

Regret Bound of UCRL2: Proof Sketch

1-2 $$R(T, M^\star, \text{UCRL2}) \leq \sum_{k=1}^{m} \sum_{t=t_k}^{t_{k+1}-1} p_k(\cdot|s_t, a_t)^\mathsf{T} h_k - h_k(s_t)$$

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$$+ \sum_{k=1}^{m} \sum_{t=t_k}^{t_{k+1}-1} h_k(s_{t+1}) - h_k(s_t)$$

Martingale Difference Sequence
(Azuma's inequality)

Regret Bound of UCRL2: Proof Sketch

1-2 $$R(T, M^\star, \text{UCRL2}) \leq \sum_{k=1}^{m} \sum_{t=t_k}^{t_{k+1}-1} p_k(\cdot|s_t, a_t)^\mathsf{T} h_k - h_k(s_t)$$

3 $$p_k(\cdot|s_t, a_t)^\mathsf{T} h_k - h_k(s_t) = \Big(p_k(\cdot|s_t, a_t) - p(\cdot|s_t, a_t)\Big)^\mathsf{T} h_k + p(\cdot|s_t, a_t)^\mathsf{T} h_k - h_k(s_t)$$

4 $$\sum_{k=1}^{m} \sum_{t=t_k}^{t_{k+1}-1} p(\cdot|s_t, a_t)^\mathsf{T} h_k - h_k(s_t) \lesssim \sup_k \{\mathsf{sp}(h_k)\} \sqrt{T \log(T/\delta)} + \sum_{k=1}^{m} \sum_{t=t_k}^{t_{k+1}-1} h_k(s_{t+1}) - h_k(s_t)$$

Regret Bound of UCRL2: Proof Sketch

1-2 $$R(T, M^\star, \text{UCRL2}) \leq \sum_{k=1}^{m} \sum_{t=t_k}^{t_{k+1}-1} p_k(\cdot|s_t, a_t)^\mathsf{T} h_k - h_k(s_t)$$

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4 $$\sum_{k=1}^{m} \sum_{t=t_k}^{t_{k+1}-1} p(\cdot|s_t, a_t)^\mathsf{T} h_k - h_k(s_t) \lesssim \sup_k \{\mathsf{sp}(h_k)\} \sqrt{T \log(T/\delta)}$$

$$+ \sum_{k=1}^{m} \sum_{t=t_k}^{t_{k+1}-1} h_k(s_{t+1}) - h_k(s_t) \longleftarrow \text{Telescopic sum}$$

Regret Bound of UCRL2: Proof Sketch

$$\boxed{1\text{-}2}\quad R(T, M^\star, \mathsf{UCRL2}) \leq \sum_{k=1}^{m} \sum_{t=t_k}^{t_{k+1}-1} p_k(\cdot|s_t,a_t)^\mathsf{T} h_k - h_k(s_t)$$

$$\boxed{3}\quad p_k(\cdot|s_t,a_t)^\mathsf{T} h_k - h_k(s_t) = \Big(p_k(\cdot|s_t,a_t) - p(\cdot|s_t,a_t)\Big)^\mathsf{T} h_k + p(\cdot|s_t,a_t)^\mathsf{T} h_k - h_k(s_t)$$

$$\boxed{4}\quad \sum_{k=1}^{m} \sum_{t=t_k}^{t_{k+1}-1} p(\cdot|s_t,a_t)^\mathsf{T} h_k - h_k(s_t) \lesssim \sup_k \{\mathsf{sp}(h_k)\} \sqrt{T \log(T/\delta)}$$

$$+ \; m \sup_k \{\mathsf{sp}(h_k)\}$$

Number of episodes
(stopping condition)

Regret Bound of UCRL2: Proof Sketch

1-2 $R(T, M^\star, \mathsf{UCRL2}) \leq \sum_{k=1}^{m} \sum_{t=t_k}^{t_{k+1}-1} p_k(\cdot|s_t,a_t)^\mathsf{T} h_k - h_k(s_t)$

3 $p_k(\cdot|s_t,a_t)^\mathsf{T} h_k - h_k(s_t) = \Big(p_k(\cdot|s_t,a_t) - p(\cdot|s_t,a_t)\Big)^\mathsf{T} h_k + p(\cdot|s_t,a_t)^\mathsf{T} h_k - h_k(s_t)$

4 $\sum_{k=1}^{m} \sum_{t=t_k}^{t_{k+1}-1} p(\cdot|s_t,a_t)^\mathsf{T} h_k - h_k(s_t) \lesssim \sup_k \{\mathsf{sp}(h_k)\} \sqrt{T \log(T/\delta)}$

$+ SA\log(T) \sup_k \{\mathsf{sp}(h_k)\}$

Regret Bound of UCRL2: Proof Sketch

1-2 $$R(T, M^\star, \text{UCRL2}) \leq \sum_{k=1}^{m} \sum_{t=t_k}^{t_{k+1}-1} p_k(\cdot|s_t,a_t)^\mathsf{T} h_k - h_k(s_t)$$

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$\boxed{\mathrm{sp}(h_k) \leq r_{\max} D}$ [Bartlett and Tewari, 2009, Jaksch et al., 2010]

Regret Bound of UCRL2: Proof Sketch

1-2 $R(T, M^\star, \text{UCRL2}) \leq \sum_{k=1}^{m} \sum_{t=t_k}^{t_{k+1}-1} p_k(\cdot|s_t, a_t)^\mathsf{T} h_k - h_k(s_t)$

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$\mathrm{sp}(h_k) \leq r_{\max} D$ [Bartlett and Tewari, 2009, Jaksch et al., 2010]

Regret Bound of UCRL2: Proof Sketch

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5 $$\sum_{k=1}^{m} \sum_{t=t_k}^{t_{k+1}-1} \Big(p_k(\cdot|s_t,a_t) - p(\cdot|s_t,a_t)\Big)^\mathsf{T} h_k = \sum_{k=1}^{m} \sum_{t=t_k}^{t_{k+1}-1} \underbrace{\Big(p_k(\cdot|s_t,a_t) - \widehat{p}_k(\cdot|s_t,a_t)\Big)^\mathsf{T} h_k}_{\leq \textsf{sp}(h_k)\beta_k^p(s,a)} + \underbrace{\Big(\widehat{p}_k(\cdot|s_t,a_t) - p(\cdot|s_t,a_t)\Big)^\mathsf{T} h_k}_{\leq \textsf{sp}(h_k)\beta_k^p(s,a)}$$

Regret Bound of UCRL2: Proof Sketch

1-2 $$R(T, M^\star, \text{UCRL2}) \leq \sum_{k=1}^{m} \sum_{t=t_k}^{t_{k+1}-1} p_k(\cdot|s_t,a_t)^\mathsf{T} h_k - h_k(s_t)$$

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Regret Bound of UCRL2: Proof Sketch

1-2 $R(T, M^\star, \textsf{UCRL2}) \leq \sum_{k=1}^{m} \sum_{t=t_k}^{t_{k+1}-1} p_k(\cdot|s_t,a_t)^\mathsf{T} h_k - h_k(s_t)$

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5 $\sum_{k=1}^{m} \sum_{t=t_k}^{t_{k+1}-1} \Big(p_k(\cdot|s_t,a_t) - p(\cdot|s_t,a_t)\Big)^\mathsf{T} h_k \lesssim r_{\max} DS \sqrt{AT \log(T/\delta)}$

Refined Confidence Bounds

- UCRL2 with *Bernstein bounds* (instead of Hoeffding/Weissman):
  - see tutorial website

$$R(T, M^\star, \text{UCRL2B}) = \mathcal{O}\left(\sqrt{D\Gamma SAT \log\left(\frac{T}{\delta}\right) \log(T)}\right)$$

- Still not matching the lower bound!
- For most MPDs: $\Gamma \ll S$

Refined Confidence Bounds

- UCRL2 with *Bernstein bounds* (instead of Hoeffding/Weissman):
  - see tutorial website

$$R(T, M^\star, \text{UCRL2B}) = \mathcal{O}\left(\sqrt{D\Gamma SAT \log\left(\frac{T}{\delta}\right)} \log\left(T\right)\right)$$

  - Still not matching the lower bound!
  - For most MPDs: $\Gamma \ll S$

- *Kullback-Leibler* UCRL [Filippi et al., 2010, Talebi and Maillard, 2018]:

$$R(T, M^\star, \text{UCRL-KL}) = \mathcal{O}\left(\sqrt{\underbrace{\sum_{s,a} \mathbb{V}_{X \sim p^\star(\cdot|s,a)}\left(h^\star_{M^\star}(X)\right)}_{\leq D^2 SA} ST \log\left(\frac{T}{\delta}\right)} + D\sqrt{T}\right)$$

  - Only for ergodic MDPs!

Infinite Diameter (weakly communicating MDPs)

- *Known* bound on the optimal bias span $C \geq \text{sp}(h^\star_{M^\star})$ [Bartlett and Tewari, 2009, Fruit et al., 2018b]

$$R(T, M^\star, \textsf{SCAL}) = \mathcal{O}\left(\sqrt{C\Gamma SAT \log\left(\frac{T}{\delta}\right)\log(T)}\right)$$

Requires prior knowledge!

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$$R(T, M^\star, \text{SCAL}) = \mathcal{O}\left(\sqrt{C\Gamma SAT \log\left(\frac{T}{\delta}\right)\log(T)}\right)$$

Requires prior knowledge!

- No prior knowledge: TUCRL [Fruit et al., 2018a]:

$$R(T, M^\star, \text{SCAL}) = \mathcal{O}\left(\sqrt{D_{\text{com}}S_{\text{com}}\Gamma AT \log\left(\frac{T}{\delta}\right)\log(T)}\right)$$

Never achieves *logarithmic* regret! Intrinsic limitation of the setting!

Open Questions

1. *Tightness of minimax $\mathcal{O}(\sqrt{T})$ regret bounds for infinite horizon problems*
   - Dependency on Γ: regret + sample complexity bounds?
   - Analysis not tight *vs.* change in the algorithm?
   - Lower bound not tight?

2. *Finite time logarithmic upper and lower regret bounds*
   - Non-asymptotic lower bounds
   - Tighter analysis of UCRL-like algorithms? New algorithms?

Posterior Sampling

a.k.a. Thompson Sampling [Thompson, 1933]

Keep Bayesian posterior for the *unknown* MDP

A sample from the posterior is used as an estimate of the unknown MDP

Exploration

Few samples $\implies$ uncertainty in the estimate

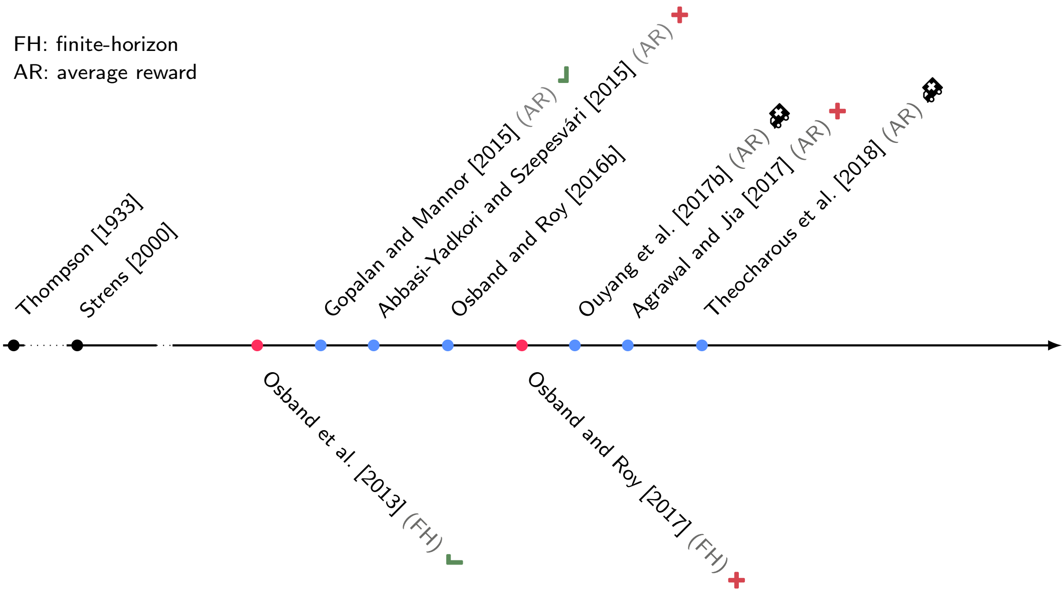
More samples $\implies$ posterior concentrates on the true MDP

Exploitation

Set of MDPs

History: PS for Regret Minimization in RL

FH: finite-horizon
AR: average reward



Posterior Sampling

$t \leftarrow 1$
for *episode* $k = 1, 2, \ldots$ **do**
  $t_k \leftarrow t$
  $M_k \sim \mu_{t_k}$
  $\pi_k \in \arg\max_{\pi}\{g^{\pi}_{M_k}\}$
  while *not enough knowledge* **do**
    Take action $a_t \sim \pi_k(\cdot|s_t)$
    Observe reward r_t and next state s_{t+1}
    Compute μ_{t+1} based on μ_t and (s_t, a_t, r_t, s_{t+1})
    $t \leftarrow t + 1$
  end
end

Posterior Sampling

$t \leftarrow 1$
for *episode* $k = 1, 2, \ldots$ **do**
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    Compute μ_{t+1} based on μ_t and (s_t, a_t, r_t, s_{t+1})
    $t \leftarrow t+1$
  end
end

Prior distribution:

$$\forall\Theta,\quad \mathbb{P}(M^* \in \Theta) = \mu_1(\Theta)$$

Posterior distribution:

$$\forall\Theta,\quad \mathbb{P}(M^* \in \Theta | H_t, \mu_1) = \mu_t(\Theta)$$

Priors

- Dirichlet (transitions)
- Beta, Normal-Gamma, etc. (rewards)

Bayesian Regret

$$R^B(T, \mu_1, \mathfrak{A}) = \mathbb{E}_{M^\star \sim \mu_1}\left[\underbrace{\overline{R}(T, M^\star, \mathfrak{A})}_{:= \mathbb{E}\left[R(T, M^\star, \mathfrak{A})\right]} \right] = \mathbb{E}\left[\sum_{t=1}^{T} g^\star_{M^\star} - r(s_t, a_t)\right]$$

TSDE: Thompson Sampling with Dynamic Episodes

[Ouyang et al., 2017b]

Episode length $l_k = t_{k+1} - t_k$ is *dynamically determined* by

1. Doubling of visits (stochastic)
2. Increasing length of previous episode by one (deterministic)

$$t_{k+1} = \min\left\{ t > t_k \;:\; \underbrace{\exists (s,a), N_t(s,a) > 2N_{t_k}(s,a)}_{(ST1)} \text{ or } \underbrace{t > t_k + \; l_{k-1}}_{(ST2)} \right\}$$

(ST2) is $\sigma(H_{t_k})$-measurable

$l_k \leq l_{k-1} + 1$

TSDE: Regret Guarantees

Theorem ([Ouyang et al., 2017b])

There exists a numerical constant $\beta > 0$ such that for any prior μ_1 whose support is a subset of communicating MDPs, TSDE *suffers a regret bounded as*

$$\forall T \geq 1, \quad R^B(T, \mu_1, \text{TSDE}) \leq \beta \cdot \left(CS\sqrt{AT\log(AT)} \right)$$

where

$$\mu_1 \quad \textit{is such that} \quad \sup_{M^\star \sim \mu_1} \left\{ sp(h^\star_{M^\star}) \right\} \leq C < +\infty \qquad \text{(ASM-SP)}$$

Proof Step 1: Regret Decomposition

- The support of the prior μ_1 is a subset of communicating MDPs
  M_k is communicating and optimality equation (i.e., constant gain)

$$R^B(T, \mu_1, \mathsf{TSDE}) \leq \underbrace{T\mathbb{E}\left[g^\star_{M^\star}\right] - \mathbb{E}\left[\sum_{k=1}^{k_T} l_k \, g^\star_{M_k}\right]}_{R_g}$$

$$+ \; \mathbb{E}\left[\sum_{k=1}^{k_T} \sum_{t=t_k}^{t_{k+1}-1} \left(h_k(s_t) - h_k(s_{t+1})\right)\right]$$

$$+ \; \mathbb{E}\left[\sum_{k=1}^{k_T} \sum_{t=t_k}^{t_{k+1}-1} \left(h_k(s_{t+1}) - p_k(\cdot|s_t, a_t)^\mathsf{T} h_k\right) + r_k(s_t, a_t) - r(s_t, a_t)\right]$$

Proof Step 1: Regret Decomposition

- The support of the prior μ_1 is a subset of communicating MDPs
  M_k is communicating and optimality equation (i.e., constant gain)

$$R^B(T, \mu_1, \mathsf{TSDE}) \leq \underbrace{T\mathbb{E}\left[g^\star_{M^\star}\right] - \mathbb{E}\left[\sum_{k=1}^{k_T} l_k \, g^\star_{M_k}\right]}_{R_g}$$

$$+ \; \mathbb{E}\left[\sum_{k=1}^{k_T} \sum_{t=t_k}^{t_{k+1}-1} \left(h_k(s_t) - h_k(s_{t+1})\right)\right]$$

Telescopic sum
+ span bound (ASM-SP)†

$$+ \; \mathbb{E}\left[\sum_{k=1}^{k_T} \sum_{t=t_k}^{t_{k+1}-1} \left(h_k(s_{t+1}) - p_k(\cdot|s_t, a_t)^\mathsf{T} h_k\right) + r_k(s_t, a_t) - r(s_t, a_t)\right]$$

† as in UCRL2

Proof Step 1: Regret Decomposition

- The support of the prior μ_1 is a subset of communicating MDPs
  M_k is communicating and optimality equation (i.e., constant gain)

$$R^B(T, \mu_1, \mathsf{TSDE}) \leq \underbrace{T\mathbb{E}\left[g^\star_{M^\star}\right] - \mathbb{E}\left[\sum_{k=1}^{k_T} l_k \, g^\star_{M_k}\right]}_{R_g}$$

$$+ \; \mathbb{E}\left[\sum_{k=1}^{k_T} \sum_{t=t_k}^{t_{k+1}-1} \left(h_k(s_t) - h_k(s_{t+1})\right)\right]$$

$$+ \; \mathbb{E}\left[\sum_{k=1}^{k_T} \sum_{t=t_k}^{t_{k+1}-1} \left(h_k(s_{t+1}) - p_k(\cdot|s_t, a_t)^\mathsf{T} h_k\right) + r_k(s_t, a_t) - r(s_t, a_t)\right]$$

Telescopic sum
+ span bound (ASM-SP)†

Confidence sets†

† as in UCRL2

Proof Step 2: Bounding R_g

Thompson Sampling Lemma [Osband et al., 2013, Ouyang et al., 2017b]

Let t_k be an almost surely finite $\sigma(H_{t_k})$-stopping time. For any measurable function f and $\sigma(H_{t_k})$-measurable variable X

$$\mathbb{E}[f(M_k, X)|H_{t_k}] = \mathbb{E}[f(M^\star, X)|H_{t_k}]$$

Proof Step 2: Bounding R_g

$$R_g = \mathbb{E}\left[\sum_{t=1}^{k_T} l_k \quad g^\star_{M^\star}\right] - \mathbb{E}\left[\sum_{k=1}^{k_T} l_k \quad g^\star_{M_k}\right]$$

Proof Step 2: Bounding R_g

random duration of episode k
not $\sigma(H_{t_k})$-measurable

$$R_g = \mathbb{E}\left[\sum_{t=1}^{k_T} l_k \;\; g^\star_{M^\star}\right] - \mathbb{E}\left[\sum_{k=1}^{k_T} l_k \;\; g^\star_{M_k}\right]$$

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$$\leq \mathbb{E}\left[\sum_{k=1}^{k_T} (l_{k-1}+1)\left(g^\star_{M^\star} - g^\star_{M_k}\right)\right] + \mathbb{E}\left[\sum_{k=1}^{k_T} (l_{k-1}+1-l_k) g^\star_{M_k}\right] \qquad \left(\begin{array}{c}\text{by (ST2)}\\ l_k \leq l_{k-1}+1\end{array}\right)$$

Proof Step 2: Bounding R_g

random duration of episode k
not $\sigma(H_{t_k})$-measurable

$$R_g = \mathbb{E}\left[\sum_{t=1}^{k_T} l_k \;\; g^\star_{M^\star}\right] - \mathbb{E}\left[\sum_{k=1}^{k_T} l_k \;\; g^\star_{M_k}\right]$$

$$\leq \mathbb{E}\left[\sum_{k=1}^{k_T}(l_{k-1}+1)\left(g^\star_{M^\star} - g^\star_{M_k}\right)\right] + \mathbb{E}\left[\sum_{k=1}^{k_T}(l_{k-1}+1-l_k)g^\star_{M_k}\right] \qquad \left(\begin{array}{c}\text{by (ST2)}\\ l_k \leq l_{k-1}+1\end{array}\right)$$

$$\leq \qquad 0 \qquad + \; r_{\max}\mathbb{E}[k_T]$$

t_k is a stopping time
$(l_{k-1}+1)$ is $\sigma(H_{t_k})$-measurable
$\implies$ use TS lemma

Proof Step 2: Bounding R_g

random duration of episode k
not $\sigma(H_{t_k})$-measurable

$$
\begin{aligned}
R_g &= \mathbb{E}\left[\sum_{t=1}^{k_T} l_k \;\; g^\star_{M^\star}\right] - \mathbb{E}\left[\sum_{k=1}^{k_T} l_k \;\; g^\star_{M_k}\right] \\
&\leq \mathbb{E}\left[\sum_{k=1}^{k_T} (l_{k-1}+1)\left(g^\star_{M^\star} - g^\star_{M_k}\right)\right] + \mathbb{E}\left[\sum_{k=1}^{k_T} (l_{k-1}+1-l_k) g^\star_{M_k}\right] \qquad \left(\begin{array}{c}\text{by (ST2)} \\ l_k \leq l_{k-1}+1\end{array}\right) \\
&\leq \qquad 0 \qquad + \; r_{\max}\mathbb{E}[k_T]
\end{aligned}
$$

t_k is a stopping time
$(l_{k-1}+1)$ is $\sigma(H_{t_k})$-measurable
$\implies$ use TS lemma

$$\sum_{k=1}^{k_T} l_{k-1} = l_0 + \sum_{k=1}^{k_T-1} l_k \leq T$$

$g^{\pi_k}_{M_k} \in [0, r_{\max}], \forall k$

Proof Step 2: Bounding R_g

random duration of episode k
not $\sigma(H_{t_k})$-measurable

$$R_g = \mathbb{E}\left[\sum_{t=1}^{k_T} l_k \;\; g^\star_{M^\star}\right] - \mathbb{E}\left[\sum_{k=1}^{k_T} l_k \;\; g^\star_{M_k}\right]$$

$$\leq \mathbb{E}\left[\sum_{k=1}^{k_T}(l_{k-1}+1)\left(g^\star_{M^\star} - g^\star_{M_k}\right)\right] + \mathbb{E}\left[\sum_{k=1}^{k_T}(l_{k-1}+1-l_k)g^\star_{M_k}\right] \qquad \left(\begin{array}{c}\text{by (ST2)}\\ l_k \leq l_{k-1}+1\end{array}\right)$$

$$\leq \qquad 0 \qquad + \; r_{\max}\mathbb{E}[k_T]$$

$$\leq r_{\max}\sqrt{2SAT\log(T)}$$

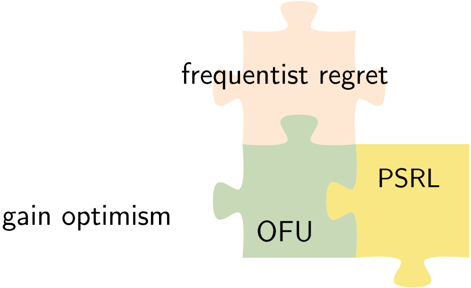
([Ouyang et al., 2017b] Lem. 1)

t_k is a stopping time
$(l_{k-1}+1)$ is $\sigma(H_{t_k})$-measurable
$\implies$ use TS lemma

$$\sum_{k=1}^{k_T} l_{k-1} = l_0 + \sum_{k=1}^{k_T-1} l_k \leq T$$
$g^{\pi_k}_{M_k} \in [0, r_{\max}], \forall k$

OPT-PSRL: Optimistic Posterior Sampling

[Agrawal and Jia, 2017]



1. Sample posterior $\psi = \widetilde{O}(S)$ times

$$p^i_{sa} \sim \mu_{t_k}(s,a), \quad i = 1, \dots, \psi$$

$\mathcal{M}_k$ is an *discrete extended* MDP

$$\widetilde{p}(\cdot, s, a^i) = p^i_{s,a}, \qquad a^i \in \mathcal{A} \times \{1, \dots, \psi\}$$

2. Solve $\mathcal{M}_k$ for π_k

$$g^\star_{M_k} \geq g^\star_{M^\star} - \widetilde{O}\left(D\sqrt{SA/T}\right)$$

OPT-PSRL: Regret Guarantees

Theorem ([Agrawal and Jia, 2017])

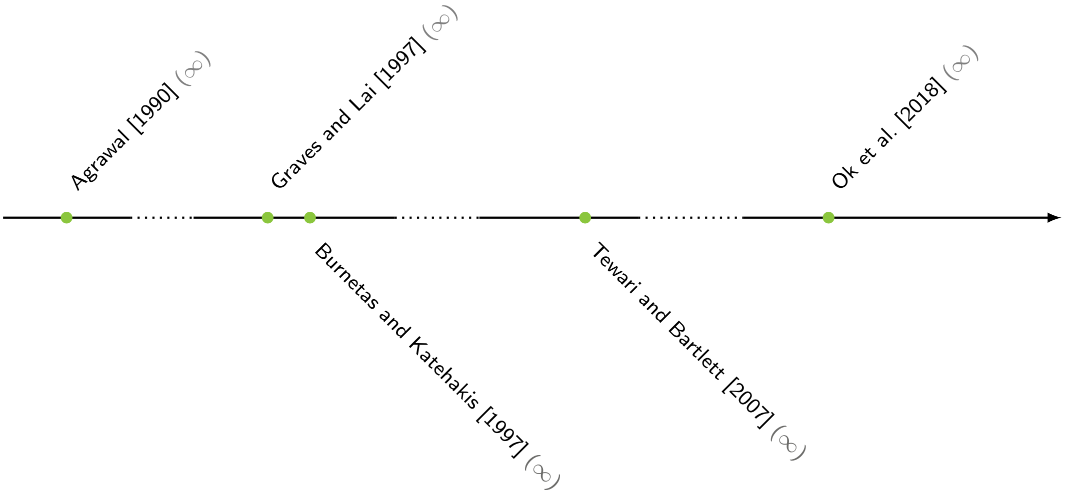
There exists a numerical constant $\alpha, \beta > 0$ *such that in any communicating MDP* $M^\star$*, with probability at least* $1-\delta$ *and for any* $T \geq \alpha D A \log^2(T/\delta)$*, Opt-PSRL suffers a regret bounded as:*

$$R(T, M^\star, \text{Opt-PSRL}) \leq \beta r_{\max} \cdot \left(DS\sqrt{AT \log\left(\frac{T}{\delta}\right)} + poly(S,A) D T^{1/4} \log\left(\frac{T}{\delta}\right)\right)$$

Open Questions

1. *The nature of bounded bias span assumption (Asm. ASM-SP)*
   - Used in [Ouyang et al., 2017b, Theocharous et al., 2018]
   - $\mathrm{supp}(\mu_1)$ is continuous, then $\sup_{M^\star \sim \mu_1} \{\mathrm{sp}(h^\star_{M^\star})\} = +\infty$ [e.g., Fruit et al. [2018a]]
2. *Statistical efficiency of* PSRL
   - Claimed efficient Bayesian or frequentist $\widetilde{O}(D\sqrt{SAT})$ regret bound
   - Not supported by proofs, incorrect Lem. C.1 [Osband and Roy, 2016a] and Lem. C.2 [Agrawal and Jia, 2017] [i see tutorial website]

History: Asymptotic Regret Minimization



Asymptotic Lower-Bound

Theorem (Thm. 2, [Burnetas and Katehakis, 1997])

Any algorithm $\mathfrak{A}$ *s.t.* $\overline{R}(T, M, \mathfrak{A}) = o(T^\alpha)$ *for all* $\alpha > 0$ *and ergodic MDP* M *should satisfy*

$$\forall (s,a) : \mathcal{M}^{alt}_{M^\star}(s,a), \quad \liminf_{T\to\infty} \boxed{\frac{\mathbb{E}[N_T(s,a)]}{\log T} \geq \frac{1}{\inf_{M \in \mathcal{M}^{alt}_{M^\star}(s,a)} KL_{M^\star,M}(s,a)}}$$

👉 Should be satisfied by optimal algorithms

necessary to be uniformly good on all the possible *alternative* models

BKIA: Burnetas-Katehakis Index Algorithm

[Burnetas and Katehakis, 1997]

for $t = 1, \dots, T$ **do**

$D_t(s) \leftarrow \{a \in \mathcal{A}(s) : N_t(s,a) \geq \log^2(N_t(s))\}$

$(g_t, h_t) \leftarrow$ solve $\widehat{M}_t = \langle \mathcal{S}, D_t, \widehat{p}_t, r \rangle$

if $\exists a \in \Pi^\star_{\widehat{M}_t}(s_t),\ N_t(s_t,a) \geq \log^2(N_t(s_t)+1)$ **then**

$a_t \in \arg\max_{a \in \mathcal{A}(s_t)} \{b_t(s,a;h_t)\}$

else

$a_t \in \arg\min_{a \in \Pi^\star_{\widehat{M}_t}(s_t)} \{N_t(s,a)\}$

end

Observe reward r_t and next state s_{t+1}

end

A Solve empirical MDP $\widehat{M}_t$ on a restricted action set

B Select maximum index action

C Force exploration of "underestimated" actions

BKIA: Interpretation

B *Exploration & Exploitation*

$$a_t \in \arg\max_{a \in \mathcal{A}} \{b_t(s_t, a)\}$$

Optimistic greedy

$$b_t(s,a) = \sup_{q \in \Delta(\mathcal{S})} \left\{ L_q^a h^\star_{\widehat{M}_t}(s) \ : \ N_t(s,a)\,\mathsf{KL}(\widehat{p}_t(\cdot|s_t,a)\|q) \leq \log(t) \right\}$$

related to $$- \inf_{M \in \mathcal{M}^{\mathsf{alt}}_{\widehat{M}_t}(s,a)} \left\{ \delta^\star_{\widehat{M}_t}(s,a) \ : \ N_t(s,a)\,\mathsf{KL}_{\widehat{M}_t,M}(s,a) \leq \log(t) \right\}$$

A not so explicit way of controlling the lower bound

BKIA: Interpretation

B *Exploration & Exploitation*

$$a_t \in \arg\max_{a \in \mathcal{A}} \{b_t(s_t, a)\}$$

Optimistic greedy

$$b_t(s,a) = \sup_{q \in \Delta(\mathcal{S})} \left\{ L_q^a h^\star_{\widehat{M}_t}(s) \;:\; N_t(s,a)\,\mathsf{KL}(\widehat{p}_t(\cdot|s_t,a)\|q) \leq \log(t) \right\}$$

related to $-\inf_{M \in \mathcal{M}^{\mathsf{alt}}_{\widehat{M}_t}(s,a)} \left\{ \delta^\star_{\widehat{M}_t}(s,a) \;:\; N_t(s,a)\,\mathsf{KL}_{\widehat{M}_t,M}(s,a) \leq \log(t) \right\}$

A not so explicit way of controlling the lower bound

Computing b_t is similar to KL-UCB [Garivier and Cappé, 2011] for MAB.

BKIA: Interpretation

C *Forced Exploration*

$$\text{when} \quad \forall a \in \Pi^\star_{\widehat{M}_t}(s_t),\ N_t(s_t,a) < \log^2(N_t(s_t)+1)$$

- BKIA prevents that *all* optimal actions *will become* under-explored

$$\implies a_t \in \Pi^\star_{\widehat{M}_t}(s_t)$$

Asymptotic monotonic property

$$\mathbb{P}\left(g^\star_{M^\star(D_{t+1})} \geq g^\star_{M^\star(D_t)}\right) = 1 - o\left(\frac{1}{t}\right) \quad \text{as } t \to \infty$$

BKIA: Regret Guarantees

Theorem (Thm. 1, [Burnetas and Katehakis, 1997])

For any ergodic MDP $M^\star$, the expected regret of BKIA is upper bounded as

$$\limsup_{T\to\infty} \frac{\overline{R}(T, M^\star, BKIA)}{\log T} \leq K^\star_{M^\star}$$

BKIA: Regret Guarantees

Theorem (Thm. 1, [Burnetas and Katehakis, 1997])

For any ergodic MDP $M^\star$, the expected regret of BKIA is upper bounded as

$$\limsup_{T\to\infty} \frac{\overline{R}(T, M^\star, BKIA)}{\log T} \leq K^\star_{M^\star}$$

👉 OLP [Tewari and Bartlett, 2007] replaces the KL constraint with an L_1

BKIA: Regret Proof

By [Prop. 1, [Burnetas and Katehakis, 1997]]

$$\overline{R}(T, M^\star, \mathfrak{A}) = \sum_s \sum_{a \notin \Pi^\star_{M^\star}(s)} \mathbb{E}\left[N_T(s,a)\right] \delta^\star_{M^\star}(s,a) + O(1), \qquad \text{as } T \to +\infty$$

We define W_T^1 s.t.

$$\mathbb{E}[N_T(s,a)] \leq \mathbb{E}[W_T^1(s,a,\varepsilon)] + o(\log T)$$

Ergodicity of MDP (g and h continuity) about $h^\star_{\widehat{M}_t} \to h^\star_{M^\star}$

BKIA: Regret Proof

Event

$$E_t^1 = \left\{ \|h^\star_{\widehat{M}_t} - h^\star_{M^\star}\|_\infty \leq \varepsilon \ \wedge \ \Pi^\star_{\widehat{M}_t}(s) \subseteq \Pi^\star_{M^\star}(s), \forall s \right\}$$

$\widehat{M}_t \approx M^\star$

$$E_t^2 = \left\{ b_t(s,a) < L^\star_{M^\star} h^\star_{M^\star}(s) - 2\varepsilon \right\}$$

BKIA: Regret Proof

Event

$$E_t^1 = \left\{ \|h^\star_{\widehat{M}_t} - h^\star_{M^\star}\|_\infty \leq \varepsilon \ \wedge \ \Pi^\star_{\widehat{M}_t}(s) \subseteq \Pi^\star_{M^\star}(s), \forall s \right\}$$

$\widehat{M}_t \approx M^\star$

$$E_t^2 = \left\{ b_t(s,a) < L^\star_{M^\star} h^\star_{M^\star}(s) - 2\varepsilon \right\}$$

$$W_T^1(s,a,\varepsilon) = \sum_{t=1}^{T} \mathbb{1}\left(s_t, a_t = s, a\right) \times \ \mathbb{1}\left(E_t^1 \wedge E_t^2\right)$$

? One Step Optimism

$$\forall (s,a) : \mathcal{M}^{\mathsf{alt}}_{M^\star}(s,a) \neq \emptyset$$

$$\lim_{\varepsilon \to 0} \limsup_{T \to \infty} \frac{\mathbb{E}[W_T^1(s,a,\varepsilon)]}{\log T} \leq \frac{1}{\inf_{M \in \mathcal{M}^{\mathsf{alt}}_{M^\star}(s,a)} \mathsf{KL}_{M^\star, M}(s,a)}$$

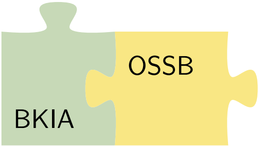
DEL: Directed Exploration Learning
[Ok et al., 2018]

- DEL exploits the same idea of BKIA

Explore suboptimal actions no more than what prescribed by the lower bound

- Exploration rate of sub-optimal action is *directed by the lower bound*

$$\text{target} \quad \eta_t(s,a) \approx \mathbb{E}\big[N_T(s,a)\big]$$



$\approx$ DEL

OSSB [Combes et al., 2017] asymptotic optimal algorithm for structured bandit

DEL

for $t = 1, \dots, T$ **do**

$D_t(s) \leftarrow \{a \in \mathcal{A}(s) : N_t(s,a) \geq \log^2(N_t(s))\}$
$(g_t, h_t) \leftarrow$ solve $\widehat{M}_t = \langle \mathcal{S}, D_t, \widehat{p}_t, r \rangle$

if $\forall a \in \Pi^\star_{\widehat{M}_t}(s_t),\ N_t(s_t,a) < \log^2(N_t(s_t)+1)$ **then**

$a_t \in \underset{a \in \Pi^\star_{\widehat{M}_t}(s_t)}{\arg\min} \{N_t(s,a)\}$

else if $C^{xpt}(H_t)$ **then**

B1 *exploit* ($a_t \in \Pi^\star_{\widehat{M}_t}(s_t)$)

else

B2 *explore*

end

Observe reward r_t and next state s_{t+1}

end

A Solve empirical MDP $\widehat{M}_t$ on a restricted action set

C Force exploration of "underestimated" actions

BKIA automatically trade-off exploration and exploitation $B1 + B2 \approx B_{\text{BKIA}}$

DEL: Exploration

B2 Directly *optimize the lower bound* on the estimated MDP $\widehat{M}_t$

$$\eta_t = \arg\inf_{\eta \in \mathbb{R}^{S\times A}} \sum_{s,a} \eta(s,a)\delta^{\star}_{\widehat{M}_t}(s,a)$$
$$\text{s.t.} \sum_{s,a} \eta(s,a)\mathsf{KL}_{\widehat{M}_t,M}(s,a) \geq 1 \quad \forall M \in \mathcal{M}^{\mathsf{alt}}_{\widehat{M}_t}(s,a)$$

$$a_t \in \arg\min_{\mathcal{A}:\, N_t(s_t,a) \leq \eta_t(s_t,a)\gamma_t} \{N_t(s_t,a)\}$$

* $\gamma_t = (1+\gamma)(1+\log t)$

DEL: Exploration

B2 Directly *optimize the lower bound* on the estimated MDP $\widehat{M}_t$

$$\eta_t = \arg\inf_{\eta \in \mathbb{R}^{S\times A}} \sum_{s,a} \eta(s,a)\delta^\star_{\widehat{M}_t}(s,a)$$

$$\text{s.t.} \sum_{s,a} \eta(s,a)\mathsf{KL}_{\widehat{M}_t,M}(s,a) \geq 1 \quad \forall M \in \mathcal{M}^{\mathsf{alt}}_{\widehat{M}_t}(s,a)$$

$$a_t \in \arg\min_{\mathcal{A}:\, N_t(s_t,a) \leq \eta_t(s_t,a)\gamma_t} \{N_t(s_t,a)\}$$

* $\gamma_t = (1+\gamma)(1+\log t)$

Lower bound sets the desired number of visits

$$\eta_t(s_t,a) \approx \mathbb{E}_{\widehat{M}_t}\Big[N_T(s_t,a)\Big] \approx \mathbb{E}_{M^\star}\Big[N_T(s_t,a)\Big]$$

then track it (in one step)

η_t computed on $\widehat{M}_t$ and not $M^\star$ (wrong target)

DEL: Regret Guarantees

Theorem (Thm. 4, [Ok et al., 2018])

For any ergodic MDP $M^\star$ and under some technical conditions, for any $\gamma > 0$, the expected regret of DEL(γ) is upper bounded as

$$\limsup_{T\to\infty} \frac{\overline{R}(T, M^\star, \mathsf{DEL}(\gamma))}{\log T} \leq (1+\gamma) K^\star_{M^\star}$$

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$$\limsup_{T\to\infty} \frac{\overline{R}(T, M^\star, \mathsf{DEL}(\gamma))}{\log T} \leq (1+\gamma) K^\star_{M^\star}$$

👉 DEL works for MDPs with structure (e.g., Lipschitz continuity)

Open Questions

- *The role of forced exploration*
  - Why do we need to force exploration?
  - Is it due to the lack of long-term optimism?
  - Is it really required at algorithmic level?
- *Finite Time Analysis*
- *Refined lower bound*
  - Current lower bound is derived from a bandit perspective

Markov Decision Process

A discrete-time finite Markov decision process (MDP) is a tuple $M = \langle \mathcal{S}, \mathcal{A}, r, p \rangle$

- State space $\mathcal{S}$, $|\mathcal{S}| = S < \infty$
- Action space $\mathcal{A}$, $|\mathcal{A}| = A < \infty$

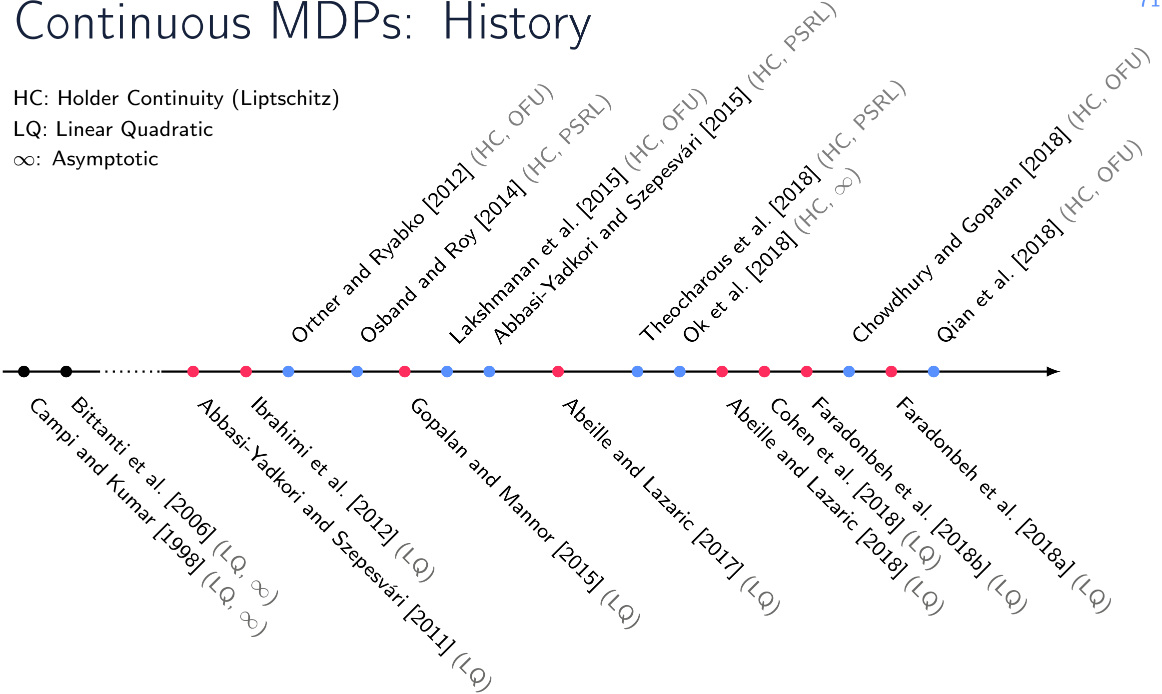
finite

- Transition distribution $p(\cdot|s,a) \in \Delta(\mathcal{S})$
- Reward distribution with expectation $r(s,a) \in [0, r_{\max}]$

The process generates history $H_t = (s_1, a_1, \ldots, s_{t-1}, a_{t-1}, s_t)$, with $s_{t+1} \sim p(\cdot|s_t, a_t)$

Continuous MDPs: History

HC: Holder Continuity (Liptschitz)
LQ: Linear Quadratic
∞: Asymptotic



Hölder Continuity

$\mathcal{S}$ continuous
$\mathcal{A}$ discrete

$L, \alpha > 0$ s.t. $\forall s, s' \in \mathcal{S}, a \in \mathcal{A}$:

$$|r(s,a) - r(s',a)| \leq r_{\max} L |s - s'|^\alpha$$

$$\|p(\cdot|s,a) - p(\cdot|s',a)\|_1 \leq L|s - s'|^\alpha$$

HC1 Asm.

$$\mathrm{sp}(h^\star_{M^\star}) \leq C$$

HC2 Asm.

Hölder Continuity

$\mathcal{S}$ continuous
$\mathcal{A}$ discrete

$L, \alpha > 0$ s.t. $\forall s, s' \in \mathcal{S}, a \in \mathcal{A}$:

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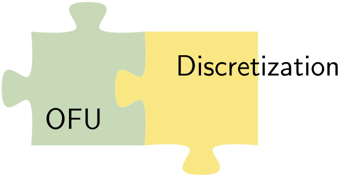
HC1 Asm.

$$\mathrm{sp}(h^{\star}_{M^{\star}}) \leq C$$

HC2 Asm.



[Ortner and Ryabko, 2012, Lakshmanan et al., 2015, Qian et al., 2018]

L, α, C, T known in advance

OFU: Hölder Continuity

Theorem (Ortner and Ryabko [2012], Lakshmanan et al. [2015], Qian et al. [2018])

For any MDP M satisfying Asm. (HC1) and (HC2), with probability at least $1-\delta$ it holds that for any $T \geq 1$, the regret of UCCRL *and* SCCAL$^+$ *is bounded as*

$$R(T, M^\star, \{\mathsf{UCCRL}, \mathsf{SCCAL}^+\}) \leq \beta \cdot CL\sqrt{A \log\left(\frac{T}{\delta}\right)}T^{(2+\alpha)/(2+2\alpha)}$$

If the transition function is κ-times smoothly differentiable ($\gamma = \alpha + \kappa$)

$$R(T, M^\star, \mathsf{UCCRL\text{-}KD}) \leq \beta \cdot CL\sqrt{A \log\left(\frac{T}{\delta}\right)}T^{(\gamma+2\alpha+\alpha\gamma)/(\gamma+\alpha+2\alpha\gamma)}$$

OFU: Liptschitz Continuity ($\alpha = 1$)

Theorem (Ok et al. [2018])

For any MDP M satisfying Asm. (HC1) and (HC2) with $\alpha = 1$ the regret of DEL *is bounded as*

$$\limsup_{T\to\infty} \frac{\overline{R}(T, M^\star, \text{DEL})}{\log T} \leq S_L A \frac{(C+1)^3}{(\min_{s,a} \delta^\star_{M^\star}(s,a))^2}$$

with

$$S_L = \min\{S, \frac{8L(C+1)}{\min_{s,a} \delta^\star_{M^\star}(s,a)} + 1\}$$

Comparison

$$R(T, M^\star, \{\text{UCCRL}, \text{SCCAL}^+\}) = \widetilde{O}(T^{3/4})$$

$$R(T, M^\star, \text{UCCRL-KD}) = \widetilde{O}(T^{2/3}) \text{ as } \kappa \to \infty$$

Linear Quadratic Systems

$$
\begin{aligned}
\max_{\pi} \quad & \lim_{T\to\infty} \frac{1}{T}\mathbb{E}\left[\sum_{t=1}^{T} r(s_t, a_t)\right] \\
\text{s.t.} \quad & s_{t+1} = f(s_t, a_t, \epsilon_{t+1}) \\
& a_t \sim \pi(s_t)
\end{aligned}
$$

Linear Quadratic Systems

Quadratic Reward

$$\max_{\pi} \quad \lim_{T\to\infty} \frac{1}{T}\mathbb{E}\left[\sum_{t=1}^{T} -\left(s_t^\mathsf{T} Q s_t + a_t^\mathsf{T} R a_t\right)\right]$$

$$\text{s.t.} \quad s_{t+1} = As_t + Ba_t + \epsilon_{t+1}$$

$$a_t \sim \pi(s_t)$$

Linear Dynamics

LQ system $M = \langle A, B, Q, R \rangle$

Linear Quadratic Systems: Optimal Policy

solution of Discrete Algebraic Riccati Equation (DARE)

- *Optimal policy*

$$\pi^\star_M(s) = \Sigma^\star_M s$$
$$\Sigma^\star_M = -(R + B^\mathsf{T} P_M B)^{-1}(B^\mathsf{T} P_M A)$$

- *Optimal gain*

$$g^\star_M = Tr(P_M)$$

Linear Quadratic Systems: Optimal Policy

solution of Discrete Algebraic Riccati Equation (DARE)

- *Optimal policy*

$$\pi^\star_M(s) = \Sigma^\star_M s$$
$$\Sigma^\star_M = -(R + B^\mathsf{T} P_M B)^{-1}(B^\mathsf{T} P_M A)$$

- *Optimal gain*

$$g^\star_M = Tr(P_M)$$

if (A, B) are controllable, $\Sigma^\star_M$ makes the system *stable*

OFU-LQ

[Abbasi-Yadkori and Szepesvári, 2011]

assume Q and R are known

Optimism in LQ

- Estimation

Regularized Least Squares

$$\widehat{M}_t = \langle \widehat{A}_t, \widehat{B}_t, Q, R \rangle \qquad \text{where} \qquad (\widehat{A}_t, \widehat{B}_t) = \widehat{\theta}_t \leftarrow H_t$$

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Statistically admissible models

$$B_t^{\mathsf{RLS}} = \left\{\theta \ : \ Tr\Big((\theta - \widehat{\theta}_t)^\mathsf{T} V_t (\theta - \widehat{\theta}_t)\Big) \leq \beta_t \right\}$$

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so that θ_t is controllable

- Planning

$$\theta_t = \arg\max_{\theta \in \Theta \cap B_t^{\mathsf{RLS}}} \{g_\theta^\star\}$$

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Hard non-convex optimization problem

OFU-LQ: Regret

Theorem ([Abbasi-Yadkori and Szepesvári, 2011])

For any $\delta \in]0,1[$, for any time T, with probability at least $1-\delta$, the regret of OFU-LQ *algorithm is bounded as*

$$R(T, M^\star, \text{OFU-LQ}) = \widetilde{O}\big(\sqrt{T \log(1/\delta)}\big)$$

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major challenge

$$\Sigma^\star_{M^\star} \to M^\star \quad \text{stable controller} \checkmark$$
$$\Sigma_t \to M^\star \quad ???$$

central to the proof is how to control $\|s_t\|$

Open Question

Hölder continuity

1. *Posterior Sampling*
   - [Theocharous et al., 2018] proved $\widetilde{O}(C\sqrt{T})$
     - Under Asm. ASM-SP and Hölder continuity
     - Only for system parametrized by 1-dimensional parameter
2. Matching Lower Bound

LQ Systems

1. *Posterior Sampling*
   - [Ouyang et al., 2017a] prove $\widetilde{O}(\sqrt{T})$ Bayesian regret under restrictive assumptions
   - [Abeille and Lazaric, 2017, 2018] proved $\widetilde{O}(\sqrt{T})$ regret for PSRL with rejection sampling but only for 1-dimensional systems
2. *Efficient* OFU: many recent advances [Faradonbeh et al., 2018a, Cohen et al., 2019]

Other Settings

- Non-realizable approximated MDP (e.g. [Jiang et al., 2017])
- Non-stationary/adversarial environments (e.g. [Even-Dar et al., 2009, Neu et al., 2014])
- MDPs with arbitrary structure (e.g. [Gopalan and Mannor, 2015])
- Hierarchical exploration (e.g. [Fruit and Lazaric, 2017, Fruit et al., 2017])
- Low-exploration MDPs (e.g. [Zanette and Brunskill, 2018])
- Active/unsupervised exploration (e.g. [Lim and Auer, 2012, Hazan et al., 2018, Tarbouriech and Lazaric, 2019])
- Partially observable MDPs and beyond (e.g. [Jiang et al., 2017, Azizzadenesheli et al., 2016])

Summary

Alg.	Asymptotic (ergodic)	Finite-time (comm.)
Lower bound	$\frac{C^2SA}{\min_{s,a}\delta^\star_{M^\star}(s,a)}\ln(T)$	$\sqrt{DSAT}$
UCRL2B	$\frac{D^2S^2A}{\delta^\star_g}\ln(T)$	$\sqrt{DS\Gamma AT\ln(T)}$
SCAL	$\frac{C^2S^2A}{\delta^\star_g}\ln(T)$	$\sqrt{CS\Gamma AT\ln(T)}$
TSDE	?	$CS\sqrt{AT\ln(T)}$
BKIA/DEL	$\frac{C^2SA}{\min_{s,a}\delta^\star_{M^\star}(s,a)}\ln(T)$	?

- $\Gamma = \max_{s,a} |\mathrm{supp}(p(\cdot|s,a))|$
- $D_M = \max_{s,s'\in\mathcal{S}} \min_{\pi:\mathcal{S}\to\mathcal{A}} \mathbb{E}[T^M_\pi(s,s')]$
- $C \geq \mathrm{sp}(h^\star)$

- $\delta^\star_M(s,a) \;=\; L^\star_M h^\star_M(s) - L^a_M h^\star_M(s)$
- $\delta^\star_g := g^\star_M - \max_{s\in\mathcal{S},\pi}\left\{g^\pi_{M^\star}(s) < g^\star_M\right\}$

Open Question: Summary

Alg.	Asymptotic (ergodic)	Finite-time (comm.)
Lower bound	$\frac{C^2SA}{\min_{s,a}\delta^\star_{M^\star}(s,a)}\ln(T)$	$\sqrt{DSAT}$
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TSDE	?	$CS\sqrt{AT\ln(T)}$ (Bayes)
BKIA/DEL	$\frac{C^2SA}{\min_{s,a}\delta^\star_{M^\star}(s,a)}\ln(T)$	?

Closing the gap between upper and lower bounds and settings (ergodic/asymptotic vs communicating/worst-case)

Many lessons learned from bandit but need to deal with dynamical nature of the problem.

Open Questions

- Unifying finite-horizon, infinite-horizon regret and discounted PAC-MDP guarantees (e.g. [Dann et al., 2017])
- Model-based vs model-free (e.g. [Jin et al., 2018, Szepesvari et al., 2019])
- Scalable exp-exp (e.g. Bellemare et al. [2016], Tang and Agrawal [2018], Fortunato et al. [2017])

Open Questions: Model-free vs Model-based

Model-based exploration

- 👍 sample efficient (regret $O(\sqrt{T})$)
- 👎 solves an MDP at each episode ($O(S^2 A)$)
- 👎 difficult to extend to function approximation

Model-free exploration

- 👎 sample inefficient (regret $O(T^{2/3})$?)
- 👍 simple update at each step ($O(1)$)
- 👍 easy to extend to function approximation

Open Questions: Model-free vs Model-based

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Model-free exploration

- sample inefficient (regret $O(T^{2/3})$?)
- simple update at each step ($O(1)$)
- easy to extend to function approximation

Sample and computationally efficient exploration algorithm? (see Jin et al. [2018])

Resources

Reinforcement Learning

- Books
  - Martin L. Puterman. *Markov Decision Processes: Discrete Stochastic Dynamic Programming*. John Wiley & Sons, Inc., New York, NY, USA, 1994
  - Richard S Sutton and Andrew G Barto. *Reinforcement learning: An introduction*, volume 1. MIT press Cambridge, 1998
  - Dimitri P. Bertsekas. *Dynamic Programming and Optimal Control, Vol. II*. Athena Scientific, 3rd edition, 2007
  - Csaba Szepesvari. *Algorithms for Reinforcement Learning*. Morgan and Claypool Publishers, 2010
- Courses (with good references for exploration)
  - Nan Jiang. Cs598 statistical reinforcement learning. http://nanjiang.cs.illinois.edu/cs598/
  - Emma Brunskill. Cs234 reinforcement learning winter 2019. http://web.stanford.edu/class/cs234/index.html
  - Alessandro Lazaric. Mva reinforcement learning. http://chercheurs.lille.inria.fr/~lazaric/Webpage/Teaching.html
  - Alexandre Proutiere. Reinforcement learning: A graduate course. http://www.it.uu.se/research/systems_and_control/education/2017/relearn/

Resources

Exploration-Exploitation and Regret Minimization

- Books

  - Sébastien Bubeck and Nicolò Cesa-Bianchi. Regret analysis of stochastic and nonstochastic multi-armed bandit problems.
    Foundations and Trends® in Machine Learning, 5(1):1–122, 2012
  - Tor Lattimore and Csaba Szepesvári. Bandit algorithms.
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Thank you!

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Artificial Intelligence Research

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