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How to solve an MDP incrementally: Reinforcement Learning

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Acknowledgments

Special thanks to Alessandro Lazaric for providing these slides from the RL class we teach in Paris.

The value functions can be represented **exactly** (e.g., tabular setting).

Learning From Samples

- Dynamic programming algorithms require an *explicit* definition of
  - transition probabilities $p(\cdot|s,a)$
  - reward function $r(s,a)$

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  - transition probabilities $p(\cdot|s,a)$
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- This knowledge is often *unavailable* (i.e., wind intensity, human-computer-interaction).
- *Can we relax this assumption?*

Setting

- *Learning with generative model.* A *black-box simulator* f of the environment is available. Given (s, a),

$$f(s, a) = \{s', r\} \text{ with } s' \sim p(\cdot|s, a), r = r(s, a).$$

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- *Episodic learning.* Multiple *trajectories* can be repeatedly generated from some initial states and terminating when a *reset* condition is achieved:

$$\left(s_{0,i}, s_{1,i}, \ldots, s_{T_i,i}\right)_{i=1}^{n}$$

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- *Online learning.* At each time t the agent is at state s_t, it takes action a_t, it observes a transition to state s_{t+1}, and it receives a reward r_t. We *assume* that $s_{t+1} \sim p(\cdot|s_t, a_t)$ and $r_t = r(s_t, a_t)$ (i.e., MDP assumption). No *reset*.

RL Interaction Protocol

Let $\mathcal{F}_t$ be the filtration generated up to time t

$$\mathcal{F}_t = \sigma(s_0, a_0, r_0, \ldots, s_{t-1}, a_{t-1}, r_{t-1}, s_t)$$

At each time t

- select an action $a_t \sim \pi(s_t)$

$$\pi_t \text{ is } \mathcal{F}_t - \text{measurable}$$

- observe the reward and next state

$$r_t \sim \nu(s_t, a_t), \quad s_{t+1} \sim p(\cdot|s_t, a_t)$$

Reinforcement Learning Algorithms

Reinforcement Learning Algorithms

Policy Evaluation

Policy Learning

Outline

Policy Evaluation

Fixed policy π

For $i = 1, \ldots, n$

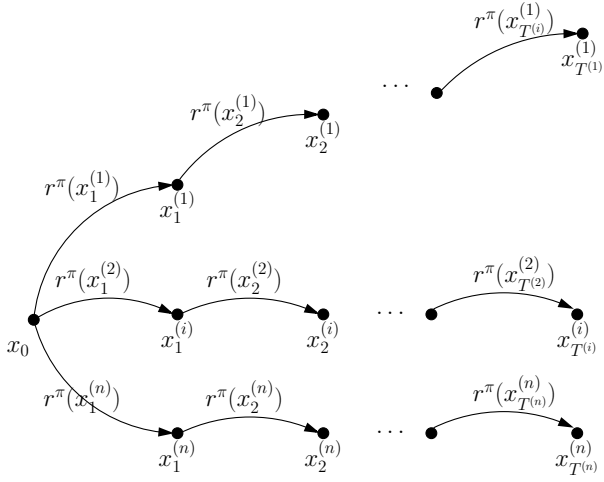
1. Set $t = 0$
2. Set initial state s_0
3. **While** ($s_{t,i}$ not terminal) *[execute one trajectory]*
   1. Take action $a_{t,i} = \pi(s_{t,i})$
   2. Observe **next state** $s_{t+1,i}$ and **reward** $r_{t,i} = r(s_{t,i}, a_{t,i}) \sim \nu(s_{t,i}, a_{t,i})$
   3. Set $t = t + 1$

   EndWhile

EndFor

Return: Estimate of the value function $\widehat{V}^{\pi}(\cdot)$

The RL Interaction Protocol



Monte-Carlo Estimation

Cumulative sum of rewards

$$V^{\pi}(s) = \mathbb{E}\left[\sum_{t=0}^{T} \gamma^t r(s_t, a_t) | s_0 = s; \pi\right]$$

- Given n trajectories under π starting from $s_{0,i} = s$

$$h^i = (s_{0,i}, a_{0,i}, r_{0,i}, \ldots, s_{T_i,i})$$

we can estimate the value function as

$$\widehat{V}_n^{\pi}(s_0) = \frac{1}{n}\sum_{i=1}^{n}\Bigg[\underbrace{\sum_{t=0}^{T_i}\gamma^t r_{t,i}}_{:=\widehat{R}_i(s_0)}\Bigg]$$

Monte-Carlo Approximation: Properties

Theorem

The returns used in the Monte-Carlo estimation starting from an initial state s_0 are unbiased estimators of V^π

$$\mathbb{E}[\widehat{R}_i(s_0)] = \mathbb{E}\Big[r_0 + \gamma r_{1,i} + \cdots + \gamma^{T_i} r_{T_i,i}\Big] = V^\pi(s_0).$$

Furthermore, the Monte-Carlo estimator converges to the value function

$$\widehat{V}_n^\pi(s_0) \xrightarrow{a.s.} V^\pi(s_0).$$

- It applies to any state s used as the *beginning* of a trajectory (sub-trajectories could be used in practice)
- Finite-sample guarantees are possible (after n trajectories)

Monte-Carlo Approximation: Extensions

Non-episodic problems

- Interrupt trajectories after H steps

$$\widehat{R}_i(s_0) = \sum_{t=0}^{H} \gamma^t r_{t,i}$$

- Every return is ignoring a term $\sum_{t=H+1}^{\infty} \gamma^t r_{t,i}$

Monte-Carlo Approximation: Properties

Theorem

*The Monte-Carlo estimator computed over H steps converges to a **biased** value function*

$$\widehat{V}_n^{\pi}(s_0) \xrightarrow{a.s.} \overline{V}_H^{\pi}(s_0),$$

such that

$$\left|\overline{V}_H^{\pi}(s_0) - V^{\pi}(s_0)\right| \leq \gamma^H \frac{r_{\max}}{1-\gamma}$$

Monte-Carlo: an Incremental Implementation

General Monte Carlo estimate. Let Z_i be i.i.d. with mean μ, then

$$\widehat{\mu}_n = \frac{1}{n}\sum_{i=1}^{n} Z_i$$

Incremental update

$$\begin{aligned}\widehat{\mu}_n &= \frac{n-1}{n}\widehat{\mu}_{n-1} + \frac{1}{n}Z_n \\ &= \widehat{\mu}_{n-1} + \alpha(n)(Z_n - \widehat{\mu}_{n-1})\end{aligned}$$

with $\alpha(n) = \frac{1}{n}$

Can we use different learning rates?

Stochastic Approximation

Goal : Find the solution to $\phi(x^\star) = 0$ based on access to noisy function evaluations, i.e. for every x, one can observe a random value

$$Y = \phi(x) + \epsilon$$

where ϵ has a zero mean (conditionally to previous queries)

Stochastic Approximation

Robbins-Monro algorithm

Given an initial x_0, for all $n \geq 1$

- observe $Y_n = \phi(x_{n-1}) + \epsilon_n$
- update $x_n = x_{n-1} + \alpha_n Y_n$

☞ Special case: estimate a mean μ based on iid samples Z_i

$$\phi(x) = \mu - x$$

with $Y_n = Z_n - \widehat{\mu}_{n-1}$, the RM update is indeed

$$\widehat{\mu}_n = \widehat{\mu}_{n-1} + \alpha_n (Z_n - \widehat{\mu}_{n-1})$$

Stochastic Approximation

Robbins-Monro Convergence

Let $\phi : \mathcal{I} \subseteq \mathbb{R} \to \mathbb{R}$. Under the following assumptions

- ϕ is continuous and $\forall x \neq x^\star$, $(x - x^\star)\phi(x) < 0$
- there exists $C > 0$ such that $\mathbb{E}[Y_n | x_{n-1}] \leq C(1 + x_{n-1}^2)$
- the sequence $\{\alpha_n\}$ satisfy

$$\sum_{n=1}^{\infty} \alpha_n = \infty \quad \text{and} \quad \sum_{n=1}^{\infty} \alpha_n^2 < \infty \tag{1}$$

the Robbins-Monro algorithm is such that $x_n \xrightarrow{a.s.} x^\star$

☞ the mean estimator converges to μ for any sequence $\{\alpha_n\}$ satisfying (1)

Monte-Carlo for Value Function

Estimated value function

$$\widehat{V}_n^{\pi}(s_0) = \frac{1}{n}\sum_{i=1}^{n}\widehat{R}_i(s_0) = \frac{n-1}{n}\widehat{V}_{n-1}^{\pi}(s_0) + \frac{1}{n}\widehat{R}_n(s_0)$$
$$\approx \big(1-\alpha(n)\big)\widehat{V}_{n-1}^{\pi}(s_0) + \alpha(n)\widehat{R}_n(s_0)$$

- Incremental Monte-Carlo estimation converges to V^{π}

$$\widehat{V}_n^{\pi}(s_0) \xrightarrow{a.s.} V^{\pi}(s_0).$$

  for a wide range of choice of learning rate schemes.
- This scheme is often referred to as *TD(1)*.

Outline

Robbins-Monro for fixed points

Goal : Find the solution to $x^\star = T(x^\star)$ based on access to noisy evaluations of $T(x)$.

Stochastic approximation for a fixed point

Given an initial x_0, for all $n \geq 1$

- a noisy evaluation Z_n: $\mathbb{E}[Z_n | x_{n-1}] = T(x_{n-1})$
- update $x_n = x_{n-1} + \alpha_n (Z_n - x_{n-1})$

👉 corresponds to the Robbins-Monro algorithm with

$$\phi(x) = T(x) - x \quad \text{and} \quad Y_n = Z_n - x_{n-1}$$

State Value Function

Fixed point of Bellman equation

$$V^{\pi}(s) = r^{\pi}(s) + \gamma \sum_{s'} p^{\pi}(s'|s) V^{\pi}(s')$$

Temporal-Difference TD(0) Estimation

At each step t, observe s_t, r_t, s_{t+1} following π and define

$$Z_t = r_t + \gamma \widehat{V}^{\pi}(s_{t+1}) \qquad [\text{noisy evaluations of } \mathcal{T}^{\pi}\widehat{V}(s_t)]$$

Then, the "Robbins-Monro" update is

$$\begin{aligned}\widehat{V}^{\pi}(s_t) &= \widehat{V}^{\pi}(s_t) + \alpha\Big(Z_t - \widehat{V}^{\pi}(s_t)\Big) \\ &= (1-\alpha)\widehat{V}^{\pi}(s_t) + \alpha Z_t\end{aligned}$$

Temporal-Difference TD(0) Estimation

Interpretation: temporal-difference error

The Robbins-Monro update rewrites

$$\widehat{V}^{\pi}(s_t) = \widehat{V}^{\pi}(s_t) + \alpha_t \delta_t$$

defining the **Temporal difference error** of estimate $\widehat{V}^{\pi}$ w.r.t. transition (s_t, r_t, s_{t+1}):

$$\delta_t = r_t + \gamma \widehat{V}^{\pi}(s_{t+1}) - \widehat{V}^{\pi}(s_t)$$

- Bellman error for function $\widehat{V}$ at state s

$$\mathcal{B}^{\pi}(\widehat{V}; s) = \mathcal{T}^{\pi}\widehat{V}(s) - \widehat{V}(s) = r^{\pi}(s) + \gamma \sum_{s'} p^{\pi}(s'|s)\widehat{V}(s') - \widehat{V}(s) \qquad [\mathcal{B}^{\pi}(V^{\pi}; s) = 0]$$

- Conditioned on s_t, δ_t is an *unbiased* estimate of $\mathcal{B}^{\pi}$

$$\mathbb{E}_{r_t, s_{t+1}}\left[\delta_t \middle| s_t\right] = r^{\pi}(s_t) + \gamma \mathbb{E}_{s_{t+1}|s_t}\left[\widehat{V}^{\pi}(s_{t+1})\right] - \widehat{V}^{\pi}(s_t) = \mathcal{B}^{\pi}(\widehat{V}^{\pi}, s_t)$$

- Expected dynamics of TD(0)

$$\overline{V}^{\pi}(s_t) = \overline{V}^{\pi}(s_t) + \alpha(s_t)\mathcal{B}^{\pi}(\overline{V}^{\pi}; s_t)$$

Temporal-Difference TD(0) Estimation

Interpretation: moving average

- Mix between old and new estimate of $V^{\pi}(s_t)$

  old estimate $\widehat{V}^{\pi}(s_t)$ new estimate $r_t + \gamma \widehat{V}^{\pi}(s_{t+1})$

- **Weighted average**

$$\widehat{V}^{\pi}(s_t) = (1-\alpha)\widehat{V}^{\pi}(s_t) + \alpha\Big(r_t + \gamma \widehat{V}^{\pi}(s_{t+1})\Big)$$

Temporal Difference $TD(0)$

Input: π, T, s_0, V_0
$V = V_0$, $s = s_0$
$N = 0_S$
for $t = 1, \ldots, T$ **do**
  N(s) = N(s) + 1
  Execute action $a \sim \pi(s)$
  Observe r and next state s'
  Update

$$\widehat{V}(s) = \widehat{V}(s) + \alpha_{N(s)}(r + \gamma \widehat{V}(s') - \widehat{V}(s))$$

  $s = s'$
end
return V

Possible variants

- instead of a very long trajectory, smaller trajectories with restarts
- update V for all states only at the end of each trajectory (offline)
- for each state s, update V(s) at most once per trajectory (first visit)

Temporal-Difference TD(0): Properties

Theorem

Let TD(0) run with learning rate $\alpha(N_t(s_t))$ where is the number of visits to the state s_t. If all states are visited infinitely often and the learning rate is set such that

$$\sum_{t=0}^{\infty} \alpha(t) = \infty \qquad \sum_{t=0}^{\infty} \alpha(t)^2 < \infty \qquad \textit{[Robbins Monro's condition]}$$

then for any state $s \in \mathcal{S}$

$$\widehat{V}^{\pi}(s) \xrightarrow{a.s.} V^{\pi}(s).$$

☞ standard choise is $\alpha_t(s) = \frac{1}{i^{\beta}}$ for $\beta \in (1/2, 1]$

$$\widehat{V}_t(s) = \widehat{V}_{t-1}(s) + \frac{1}{N_t(s)^{\beta}}(r + \gamma \widehat{V}_{t-1}(s') - \widehat{V}_{t-1}(s))$$

where $N_t(s)$ is the number of visits to state s before t

From Monte-Carlo to Temporal Differences

Temporal difference $\delta_t = r_t + \gamma \widehat{V}^{\pi}(s_{t+1}) - \widehat{V}^{\pi}(s_t)$

Update after the n-th trajectory

$$
\begin{aligned}
\widehat{V}^{\pi}_{n+1}(s_0) &= (1-\alpha_{n+1})\widehat{V}^{\pi}_{n}(s_0) + \alpha_{n+1}\widehat{R}_{n+1}(s_0) \\
&= \widehat{V}^{\pi}_{n}(s_0) + \alpha_{n+1}\Big(\widehat{R}_{n+1}(s_0) - \widehat{V}^{\pi}_{n}(s_0)\Big) \quad \text{[MC]} \\
&= \widehat{V}^{\pi}_{n}(s_0) + \alpha_{n+1}\Big(r_{0,n} + \gamma r_{1,n} + \gamma^2 r_{2,n} + \gamma^3 r_{3,n} + \cdots - \widehat{V}^{\pi}_{n}(s_0)\Big) \\
&= \widehat{V}^{\pi}_{n}(s_0) + \alpha_{n+1}\Big(r_{0,n} + \gamma\widehat{V}_{n}(s_{1,n}) - \widehat{V}^{\pi}_{n}(s_0) - \gamma\widehat{V}^{\pi}_{n}(s_{1,n}) + \gamma r_{1,n} + \gamma^2 r_{2,n} + \gamma^3 r_{3,n} + \cdots\Big) \\
&= \widehat{V}^{\pi}_{n}(s_0) + \alpha_{n+1}\Big(\delta_{0,n} - \gamma\widehat{V}^{\pi}_{n}(s_{1,n}) + \gamma r_{1,n} + \gamma^2 r_{2,n} + \gamma^3 r_{3,n} + \cdots\Big) \\
&= \widehat{V}^{\pi}_{n}(s_0) + \alpha_{n+1}\Big(\delta_{0,n} + \gamma r_{1,n} + \gamma^2\widehat{V}^{\pi}_{n}(s_{2,n}) - \gamma\widehat{V}^{\pi}_{n}(s_{1,n}) - \gamma^2\widehat{V}^{\pi}_{n}(s_{2,n}) + \gamma^2 r_{2,n} + \gamma^3 r_{3,n} + \cdot \\
&= \widehat{V}^{\pi}_{n}(s_0) + \alpha_{n+1}\Big(\delta_{0,n} + \gamma\delta_{1,n} - \gamma^2\widehat{V}^{\pi}_{n}(s_{2,n}) + \gamma^2 r_{2,n} + \gamma^3 r_{3,n} + \cdots\Big) \\
&= \widehat{V}^{\pi}_{n}(s_0) + \alpha_{n+1}\Big(\delta_{0,n} + \gamma\delta_{1,n} + \gamma^2\delta_{2,n} + \ldots + \gamma^{T_n-1}\delta_{T_n,n}\Big)
\end{aligned}
$$

Comparison between Inc-MC and TD(0)

Temporal difference $\delta_t = r_t + \gamma \widehat{V}^\pi(s_{t+1}) - \widehat{V}^\pi(s_t)$

- Incremental Monte-Carlo

$$\widehat{V}^\pi(s_0) = \widehat{V}^\pi(s_0) + \alpha[\delta_0 + \gamma\delta_1 + \cdots + \gamma^{T-1}\delta_T].$$

$\Rightarrow$ *No bias*, *large variance* *[long trajectory]*

TD(0)

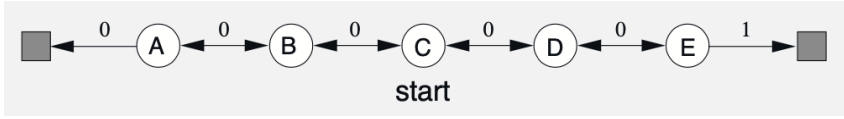
- TD(0)

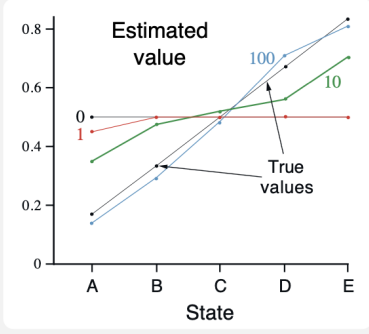
$$\widehat{V}^\pi(s_0) = \widehat{V}^\pi(s_0) + \alpha\delta_0.$$

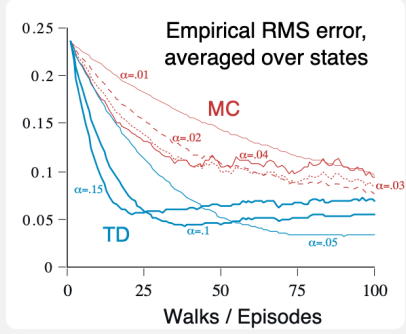
$\Rightarrow$ *Large bias* *["bootstrapping" on wrong values]*, *small variance*

Example: Random Walk

[1]







From Monte Carlo to Temporal Difference

Limitation of naive Monte-Carlo :

- performing a full trajectory is needed before the update
- we only update the value of the initial state s_0

Extension

- update the values of multiple states after each trajectory
- online updates, after each transition

Every Visit Monte Carlo (a.k.a. TD(1))

After the ith trajectory, instead of updating only s_0, for all $k = T_i - 1$ down to 0

$$\widehat{V}(s_{k,i}) = \widehat{V}(s_{k,i}) + \alpha_i(s_{k,i}) \left(\sum_{t=k}^{T_i} \gamma^{t-k} r_{t,i} - \widehat{V}(s_{k,i}) \right)$$

Remarks

- exact formula for episodic settings
- multiple updates of states visited more then once in the trajectory
- **first visit** variant: update $s_{k,i}$ only if $s_{k,i} \notin \{s_{0,i}, \ldots, s_{k-1,i}\}$

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The $\mathcal{T}_\lambda^\pi$ Bellman operator

Definition

Given $\lambda < 1$, then the Bellman operator $\mathcal{T}_\lambda^\pi$ is

$$\mathcal{T}_\lambda^\pi = (1-\lambda) \sum_{m \geq 0} \lambda^m (\mathcal{T}^\pi)^{m+1}.$$

The $\mathcal{T}_\lambda^\pi$ Bellman operator

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Remark: convex combination of the m-step Bellman operators $(\mathcal{T}^\pi)^m$ weighted by a sequences of coefficients defined as a function of a λ.

Temporal Difference $TD(\lambda)$

Idea: use the whole series of temporal differences to update $\widehat{V}^{\pi}$

- *Temporal difference* of a function $\widehat{V}^{\pi}$ for a transition $\langle s_t, r_t, s_{t+1} \rangle$

$$\delta_t = r_t + \gamma \widehat{V}^{\pi}(s_{t+1}) - \widehat{V}^{\pi}(s_t)$$

- Estimated value function

$$\widehat{V}^{\pi}(s_t) = \widehat{V}^{\pi}(s_t) + \alpha(s_t) \sum_{\tau=t}^{T} (\gamma\lambda)^{\tau-t} \delta_{\tau}$$

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$\Rightarrow$ Still requires the whole trajectory before updating...

Temporal Difference $TD(\lambda)$: Eligibility Traces

- *Eligibility* traces $z \in \mathbb{R}^S$
- For every transition $s_t \rightarrow s_{t+1}$

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$$\delta_t = r^\pi(s_t) + \gamma \widehat{V}^\pi(s_{t+1}) - \widehat{V}^\pi(s_t)$$

  2. Update the eligibility traces

$$z(s) = \begin{cases} \lambda z(s) & \text{if } s \neq s_t \\ 1 + \lambda z(s) & \text{if } s = s_t \\ 0 & \text{if } s_t = s_0 \text{ (reset the traces)} \end{cases}$$

Temporal Difference $TD(\lambda)$: Eligibility Traces

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  3. For all state $s \in S$ *[all states are updated at each step]*

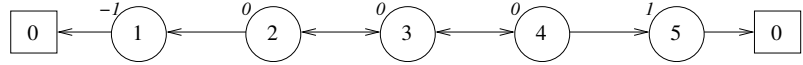
$$\widehat{V}^\pi(s) \leftarrow \widehat{V}^\pi(s) + \alpha(s) z(s) \delta_t.$$

Sensitivity to λ

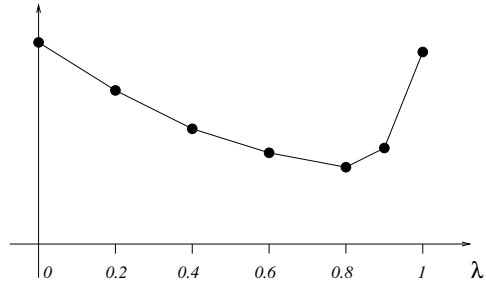
- $\lambda < 1$: *smaller variance* w.r.t. $\lambda = 1$ ($\approx$ inc-MC).
- $\lambda > 0$: *faster propagation* of rewards w.r.t. $\lambda = 0$.

Example: Sensitivity to λ

Linear chain example



Error $\sum_{s \in S} (\widehat{V}^{\pi}(s) - V^{\pi}(s))^2$ after $n = 100$ trajectories



Reinforcement Learning Algorithms

Tools

Policy Evaluation

Policy Learning

Policy Learning

Learn optimal policy π^*

For $i = 1, \ldots, n$

1. Set $t = 0$
2. Set initial state s_0
3. **While** ($s_{t,i}$ not terminal) *[execute one trajectory]*
   1. **Take action** $a_{t,i}$
   2. Observe next state $s_{t+1,i}$ and reward $r_{t,i} = r(s_{t,i}, a_{t,i})$
   3. Set $t = t + 1$

   EndWhile

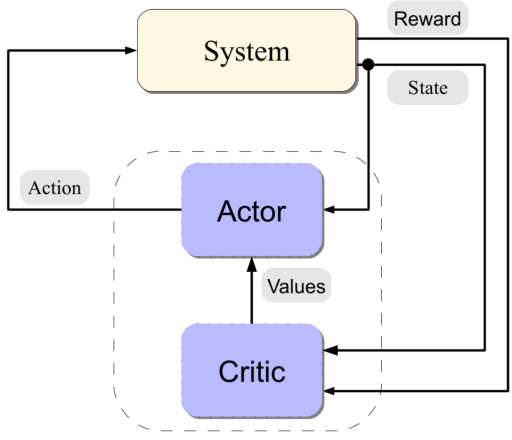
EndFor

Return: Estimate of the optimal policy $\widehat{\pi}^*$

Outline

Actor-Critic Methods

- Critic: estimate the value function of the policy given by the actor
- Actor: change the policy to improve the value given by the critic



Policy Iteration

1. Let π_0 be *any* stationary policy
2. At each iteration $k = 1, 2, \dots, K$
   - *Policy evaluation* given π_k, compute Q^{π_k}.
   - *Policy improvement*: compute the *greedy* policy

$$\pi_{k+1}(s) \in \arg\max_{a \in A} Q_k^{\pi}(s, a)$$

3. Return the last policy π_K

Policy Iteration is an instance of actor-critic methods:

- Actor: acts greedily (i.e., $\pi_{k+1} = \text{greedy}(Q^{\pi_k})$)
- Critic: compute Q given the policy π_k

Generalized Policy Iteration

More generally:

- Actor: *policy improvement* (even approximate)
- Critic: *policy evaluation*

[1] refers to actor/critic methods as **generalized policy iteration** (GPI)

👉 many possible schemas!

SARSA

Actor: the goal is to move the policy toward the greedy policy w.r.t. to the critic value

Possible strategies

- greedy (does not explore enough!)
- *ϵ-greedy* policy $\pi_Q(a|s)$

$$a = \mathcal{U}(\mathcal{A}) \qquad \text{with probability } \epsilon$$
$$a = \arg\max_{\overline{a}} Q(s, \overline{a}) \qquad \text{with probability } 1 - \epsilon$$

- *soft-max (random) exploratory* policy with temperature τ

$$\pi_Q(a|s) = \frac{\exp(Q(s,a)/\tau)}{\sum_{a'} \exp(Q(s,a')/\tau)}$$

The higher $Q(x, a)$, the more probability to take action a in state s

SARSA

Critic: uses TD to update the estimated Q-function

- Compute the temporal difference on the trajectory $\langle s_t, a_t, r_t, s_{t+1}, a_{t+1} \rangle$ (with actions chosen according to $\pi_Q(a|s)$)

$$\delta_t = r_t + \gamma \widehat{Q}(s_{t+1}, a_{t+1}) - \widehat{Q}(s_t, a_t)$$

SARSA

Critic: uses TD to update the estimated Q-function

- Compute the temporal difference on the trajectory $\langle s_t, a_t, r_t, s_{t+1}, a_{t+1} \rangle$ (with actions chosen according to $\pi_Q(a|s)$)

$$\delta_t = r_t + \gamma \widehat{Q}(s_{t+1}, a_{t+1}) - \widehat{Q}(s_t, a_t)$$

- Update the estimate of Q as

$$\widehat{Q}(s_t, a_t) = \widehat{Q}(s_t, a_t) + \alpha(s_t, a_t)\delta_t$$

State Action Reward State Action (SARSA) update

SARSA

Input: T, s_0, Q_0
$Q = Q_0$, $s = s_0$, $a = \mathcal{U}(\mathcal{A})$
$N = 0_{S \times A}$
for $t = 1, \ldots, T$ **do**
  N(s,a) = N(s,a) + 1
  Execute action a
  Observe r and next state s'
  $a' \sim \pi_Q(\cdot|s')$
  Update

$$\widehat{Q}(s,a) = \widehat{Q}(s,a) + \alpha_{N(s,a)}(r + \gamma \widehat{Q}(s', a') - \widehat{Q}(s,a))$$

  $s = s'$, $a = a'$
end
return $\widehat{Q}$

Examples

- $\pi_{t,Q}(a|s) = \dfrac{\exp(Q(s,a)/\tau_t}{\sum_b \exp(Q(s,b)/\tau_t)}$
- $\pi_{t,Q}(a|s) = (1-\epsilon_t)\mathbb{1}\left(a_t = \arg\max_b Q(s,b)\right) + \epsilon_t/A$

SARSA: Properties (informal)

- The TD updates make $\widehat{Q}$ converge to Q^{π}
- The update of π_Q allows to improve the policy
- A decreasing temperature allows to become more and more greedy
  $\Rightarrow$ If $\tau \to 0$ with a proper rate, then $\widehat{Q} \to Q^*$ and $\pi_Q \to \pi^*$
- Similarly for ϵ-greedy

SARSA: Limitations

The actions a_t need to be selected according to the current Q

$\Rightarrow$ **On-policy learning**

Outline

The Optimal Bellman Equation

Proposition

The optimal value function Q^* (i.e., $Q^* = \max_\pi Q^\pi$) is the solution to the optimal Bellman equation:

$$Q^*(s,a) = r(s,a) + \gamma \sum_{s'} p(s'|s,a) \max_{a' \in A} Q^*(s',a')\Big].$$

Q-Learning

Idea: use TD for the optimal Bellman operator

- Compute the (optimal) temporal difference on the trajectory $\langle s_t, a_t, r_t, s_{t+1} \rangle$ (with actions chosen **arbitrarily**!)

$$\delta_t = r_t + \gamma \max_{a'} \widehat{Q}(s_{t+1}, a') - \widehat{Q}(s_t, a_t)$$

- Update the estimate of Q as

$$\widehat{Q}(x_t, a_t) = \widehat{Q}(s_t, a_t) + \alpha(s_t, a_t)\delta_t$$

Q-Learning

Input: T, s_0, Q_0
$Q = Q_0$, $s = s_0$
$N = 0_{S \times A}$
for $t = 1, \ldots, T$ **do**
  Execute action $a \sim \pi_{t,Q}(s)$
  Observe r and next state s'
  N(s,a) = N(s,a) + 1
  Update

$$\widehat{Q}(s,a) = \widehat{Q}(s,a) + \alpha_{N(s,a)}(r + \gamma \max_{a'} \widehat{Q}(s', a') - \widehat{Q}(s,a))$$

  $s = s'$
end
return $\widehat{Q}$

Examples

- $\pi_{t,Q}(a|s) = \dfrac{\exp(Q(s,a)/\tau_t}{\sum_b \exp(Q(s,b)/\tau_t)}$
- $\pi_{t,Q}(a|s) = (1-\epsilon_t)\mathbb{1}\left(a_t = \arg\max_b Q(s,b)\right) + \epsilon_t/A$

Q-Learning: Properties

Proposition

If the learning rate satisfies the Robbins-Monro conditions in all states $s, a \in S \times A$

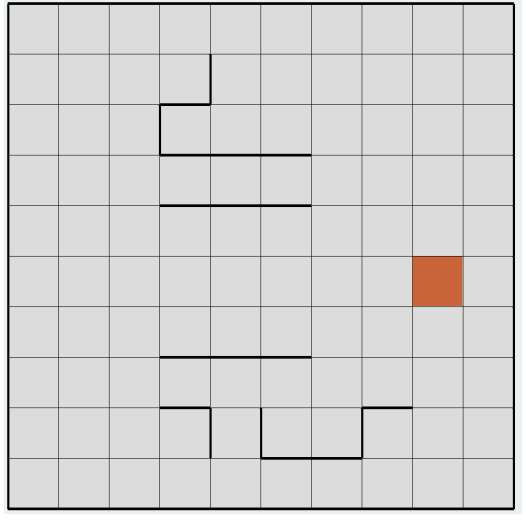
$$\sum_{i=0}^{\infty} \alpha_i(s,a) = \infty, \qquad \sum_{i=0}^{\infty} \alpha_i^2(s,a) < \infty,$$

and all state-action pairs are tried infinitely often, then for all $s, a \in S \times A$

$$\widehat{Q}(s,a) \xrightarrow{a.s.} Q^*(s,a)$$

Remark: this is another example of R-M algorithm
Remark: "infinitely often" requires a steady exploration policy

The Grid-World Problem



Q-Learning vs. SARSA

Update rule

- *Q-Learning*

$$\widehat{Q}(s_t, a_t) = \widehat{Q}(s_t, a_t) + \alpha(r_t + \gamma \max_{a'} \widehat{Q}(s_{t+1}, a') - \widehat{Q}(s_t, a_t))$$

- *SARSA*

$$\widehat{Q}(s_t, a_t) = \widehat{Q}(s_t, a_t) + \alpha(r_t + \gamma \widehat{Q}(s_{t+1}, a_{t+1}) - \widehat{Q}(s_t, a_t))$$

Both aim at learning the target policy $\pi^\star(s) = \arg\max_a Q^\star(s, a)$

- Q-learning can use any behavioral policy (*off-policy learning*)
- for SARSA the behavioral policy should be close to the estimated target policy (*on-policy learning*)

Summary

- Policy evaluation: Monte-Carlo and Temporal Difference (definition, pros and cons)
- Policy learning: SARSA and Q-learning (definition, guarantees)

Thank you!

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Bibliography

[1] Richard S. Sutton and Andrew G. Barto. *Reinforcement Learning: An Introduction*. Bradford Book, MIT Press, Cambridge, MA, 1998.