Logistic Regression

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based on the slides of Nick Rouzzi and Vibhav Gogate

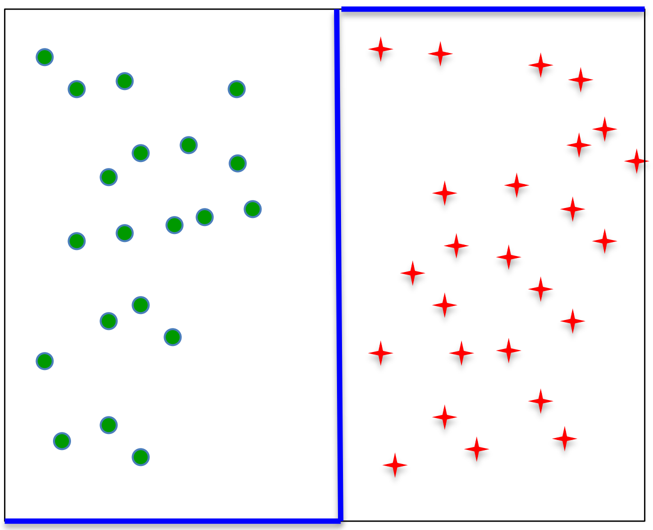
Last Time

- Supervised learning via naive Bayes
  - Use MLE to estimate a distribution $p(x, y) = p(y)p(x|y)$
  - Classify by looking at the conditional distribution, $p(y|x)$
- Today: logistic regression

Logistic Regression

- Learn $p(Y|X)$ directly from the data
  - Assume a particular functional form, e.g., a linear classifier $p(Y = 1|x) = 1$ on one side and 0 on the other
  - Not differentiable…
    - Makes it difficult to learn
    - Can't handle noisy labels

$p(Y = 1|x) = 0$



$p(Y = 1|x) = 1$

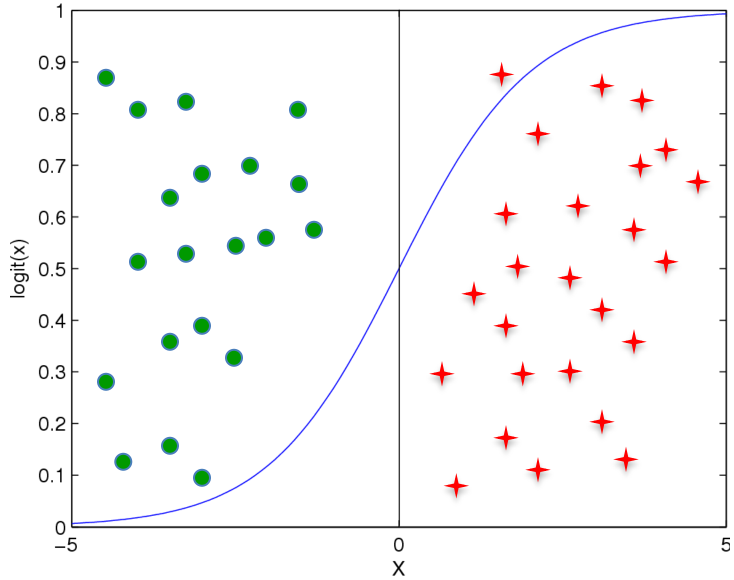
Logistic Regression

$$\frac{e^{100}}{1+e^{100}} \approx 1$$

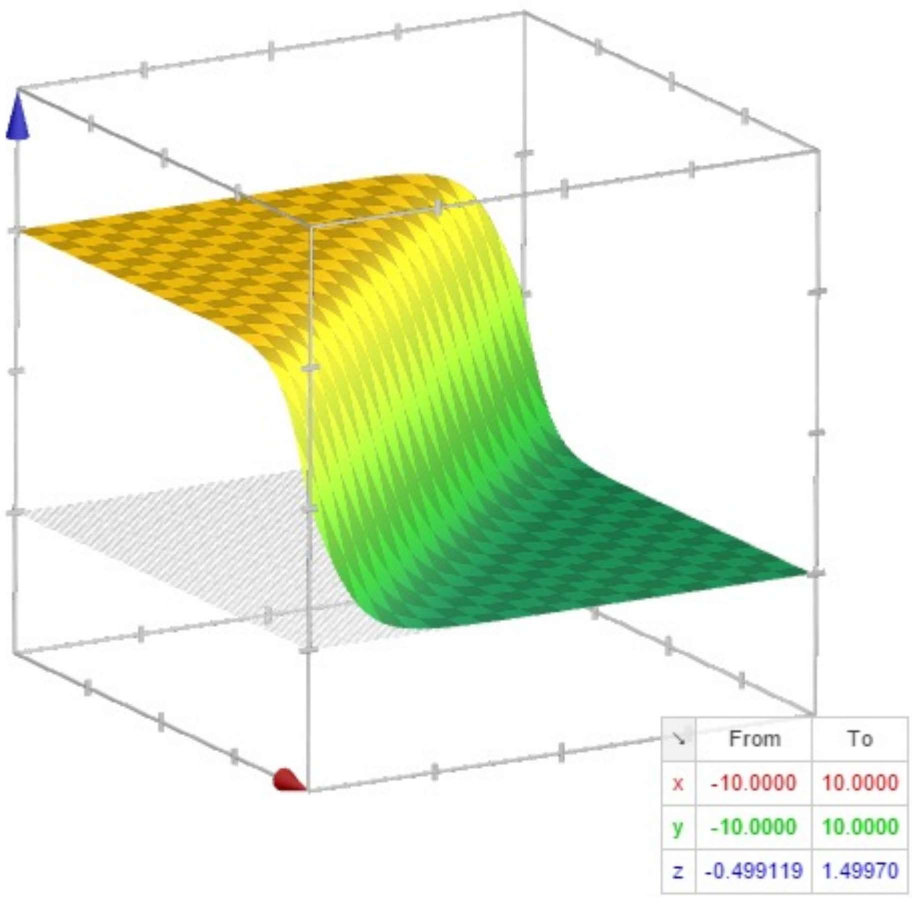
- Learn $p(y|x)$ directly from the data
  - Assume a particular functional form

$$p(Y = -1|x) = \frac{1}{1 + \exp(w^T x + b)}$$

$$p(Y = 1|x) = \frac{\exp(w^T x + b)}{1 + \exp(w^T x + b)}$$



Logistic Function in m Dimensions



$$p(Y = -1|x) = \frac{1}{1 + \exp(w^T x + b)}$$

Can be applied to discrete and continuous features

Functional Form: Two classes

- Given some w and b, we can classify a new point x by assigning the label 1 if $p(Y = 1|x) > p(Y = -1|x)$ and -1 otherwise
  - This leads to a linear classification rule:
    - Classify as a 1 if $w^T x + b > 0$
    - Classify as a -1 if $w^T x + b < 0$

Learning the Weights

- To learn the weights, we maximize the conditional likelihood

$$(w^*, b^*) = \arg\max_{w,b} \prod_{i=1}^{N} p(y^{(i)}|x^{(i)}, w, b)$$

- This is the not the same strategy that we used in the case of naive Bayes
  - For naive Bayes, we maximized the log-likelihood

$p(x_1, x_2, \ldots x_m, y_1, \ldots y_m) \leftarrow$ Likelihood

$p(y_1 \ldots y_m \mid x_1 \ldots x_m) \leftarrow$ Cond. Lik.

Generative vs. Discriminative Classifiers

Generative classifier:
(e.g., Naïve Bayes)

- Assume some **functional form** for $p(x|y), p(y)$
- Estimate parameters of $p(x|y)$, $p(y)$ directly from training data
- Use Bayes rule to calculate $p(y|x)$
- This is a **generative model**
  - **Indirect** computation of $p(Y|X)$ through Bayes rule
  - As a result, **can also generate a sample of the data**, $p(x) = \sum_y p(y)p(x|y)$

Discriminative classifiers:
(e.g., Logistic Regression)

- Assume some **functional form for** $p(y|x)$
- Estimate parameters of $p(y|x)$ directly from training data
- This is a discriminative model
  - Directly learn $p(y|x)$
  - But **cannot obtain a sample of the data** as $p(x)$ is not available
  - Useful for discriminating labels

Learning the Weights

$$\ell(w,b) = \ln \prod_{i=1}^{N} p(y^{(i)}|x^{(i)},w,b)$$

$$= \sum_{i=1}^{N} \ln p(y^{(i)}|x^{(i)},w,b)$$

Learning the Weights

$$\begin{aligned}\ell(w,b) \quad &= \ln \prod_{i=1}^{N} p(y^{(i)}|x^{(i)},w,b) \\ &= \sum_{i=1}^{N} \ln p(y^{(i)}|x^{(i)},w,b) \\ &= \sum_{i=1}^{N} \frac{y^{(i)}+1}{2} \ln p(Y=1|x^{(i)},w,b) + \left(1-\frac{y^{(i)}+1}{2}\right) \ln p(Y=-1|x^{(i)},w,b)\end{aligned}$$

Learning the Weights

$$
\begin{aligned}
\ell(w,b) &= \ln \prod_{i=1}^{N} p(y^{(i)}|x^{(i)},w,b) \\
&= \sum_{i=1}^{N} \ln p(y^{(i)}|x^{(i)},w,b) \\
&= \sum_{i=1}^{N} \frac{y^{(i)}+1}{2} \ln p(Y=1|x^{(i)},w,b) + \left(1-\frac{y^{(i)}+1}{2}\right) \ln p(Y=-1|x^{(i)},w,b) \\
&= \sum_{i=1}^{N} \frac{y^{(i)}+1}{2} \ln \frac{p\left(Y=1\middle|x^{(i)},w,b\right)}{p\left(Y=-1\middle|x^{(i)},w,b\right)} + \ln p(Y=-1|x^{(i)},w,b)
\end{aligned}
$$

Learning the Weights

$$p(y=1|x) = \frac{\exp(w^T x + b)}{1 + \exp(w^T x + b)}$$

$$p(y=-1|x) = \frac{1}{1 + \exp(w^T x + b)}$$

$$
\begin{aligned}
\ell(w,b) &= \ln \prod_{i=1}^{N} p(y^{(i)}|x^{(i)}, w, b) \\
&= \sum_{i=1}^{N} \ln p(y^{(i)}|x^{(i)}, w, b) \\
&= \sum_{i=1}^{N} \frac{y^{(i)}+1}{2} \ln p(Y=1|x^{(i)}, w, b) + \left(1 - \frac{y^{(i)}+1}{2}\right) \ln p(Y=-1|x^{(i)}, w, b) \\
&= \sum_{i=1}^{N} \frac{y^{(i)}+1}{2} \ln \frac{p\left(Y=1\middle|x^{(i)}, w, b\right)}{p\left(Y=-1\middle|x^{(i)}, w, b\right)} + \ln p(Y=-1|x^{(i)}, w, b) \\
&= \sum_{i=1}^{N} \frac{y^{(i)}+1}{2} \left(w^T x^{(i)} + b\right) - \ln\left(1 + \exp\left(w^T x^{(i)} + b\right)\right)
\end{aligned}
$$

Learning the Weights

$$
\begin{aligned}
\ell(w,b) &= \ln \prod_{i=1}^{N} p(y^{(i)}|x^{(i)},w,b) \\
&= \sum_{i=1}^{N} \ln p(y^{(i)}|x^{(i)},w,b) \\
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&= \sum_{i=1}^{N} \frac{y^{(i)}+1}{2} \left(w^T x^{(i)} + b\right) - \ln\left(1+\exp\left(w^T x^{(i)} + b\right)\right)
\end{aligned}
$$

This is concave in w and b: take derivatives and solve!

Learning the Weights

$$
\begin{aligned}
\ell(w,b) &= \ln \prod_{i=1}^{N} p(y^{(i)}|x^{(i)},w,b) \\
&= \sum_{i=1}^{N} \ln p(y^{(i)}|x^{(i)},w,b) \\
&= \sum_{i=1}^{N} \frac{y^{(i)}+1}{2} \ln p(Y=1|x^{(i)},w,b) + \left(1-\frac{y^{(i)}+1}{2}\right) \ln p(Y=-1|x^{(i)},w,b) \\
&= \sum_{i=1}^{N} \frac{y^{(i)}+1}{2} \ln \frac{p\left(Y=1\middle|x^{(i)},w,b\right)}{p\left(Y=-1\middle|x^{(i)},w,b\right)} + \ln p(Y=-1|x^{(i)},w,b) \\
&= \sum_{i=1}^{N} \frac{y^{(i)}+1}{2}\left(w^T x^{(i)}+b\right) - \ln\left(1+\exp\left(w^T x^{(i)}+b\right)\right)
\end{aligned}
$$

No closed form solution ☹

Likelihood Maximization

$$\ell(w, b) = \sum_{i=1}^{N} \frac{y^{(i)} + 1}{2} \left(w^T x^{(i)} + b \right) - \ln\left(1 + \exp\left(w^T x^{(i)} + b\right)\right)$$

The above is the Likelihood
which we maximize!

Learning the Weights

- Can apply gradient ascent to maximize the conditional likelihood

$$\nabla_b \ell = \frac{\partial \ell}{\partial b} = \sum_{i=1}^{N} \left[\frac{y^{(i)} + 1}{2} - p(Y = 1 | x^{(i)}, w, b) \right]$$

$$\nabla_w \ell = \frac{\partial \ell}{\partial w_j} = \sum_{i=1}^{N} x_j^{(i)} \left[\frac{y^{(i)} + 1}{2} - p(Y = 1 | x^{(i)}, w, b) \right]$$

$$w^{t+1} = w^{t} + \alpha \left[\nabla_w \ell \right]_{w = w^t}$$

$$b^{t+1} = b^{t} + \alpha \left[\nabla_b \ell \right]_{b = b^t}$$

Priors

- Can define priors on the weights to prevent overfitting
  - Normal distribution, zero mean, identity covariance

$$p(w) = \prod_j \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{w_j^2}{2\sigma^2}\right)$$

  - "Pushes" parameters towards zero
- Regularization
  - Helps avoid very large weights and overfitting

Priors as Regularization

- The log-MAP objective with this Gaussian prior is then

$$\ln \prod_{i=1}^{N} p\left(y^{(i)} \middle| x^{(i)}, w, b\right) p(w) p(b) = \left[\sum_{i}^{N} \ln p\left(y^{(i)} \middle| x^{(i)}, w, b\right)\right] - \frac{\lambda}{2} \|w\|_2^2$$

  - Quadratic penalty: drives weights towards zero
  - Adds a negative linear term to the gradients
  - Different priors can produce different kinds of regularization

Priors as Regularization

- The log-MAP objective with this Gaussian prior is then

$\mathcal{L}(w)$

$$\ln \prod_{i=1}^{N} p\left(y^{(i)} \middle| x^{(i)}, w, b\right) p(w) p(b) = \left[\sum_{i}^{N} \ln p\left(y^{(i)} \middle| x^{(i)}, w, b\right)\right] - \frac{\lambda}{2} \|w\|_2^2$$

  - Quadratic penalty: drives weights towards zero
  - Adds a negative linear term to the gradients
  - Different priors can produce different kinds of regularization

ℓ_2 regularizer

L2 vs L1 Regularization

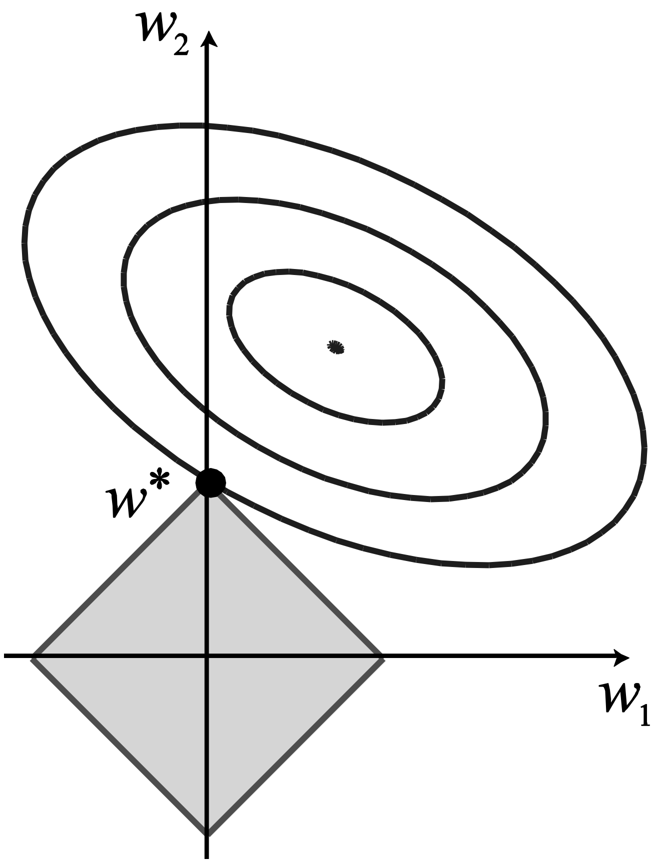
The Likelihood with L2 Regularization:

$$\ln \prod_{i=1}^{N} p\left(y^{(i)} \middle| x^{(i)}, w, b\right) p(w) p(b) = \left[\sum_{i}^{N} \ln p\left(y^{(i)} \middle| x^{(i)}, w, b\right)\right] - \frac{\lambda}{2} \|w\|_2^2$$

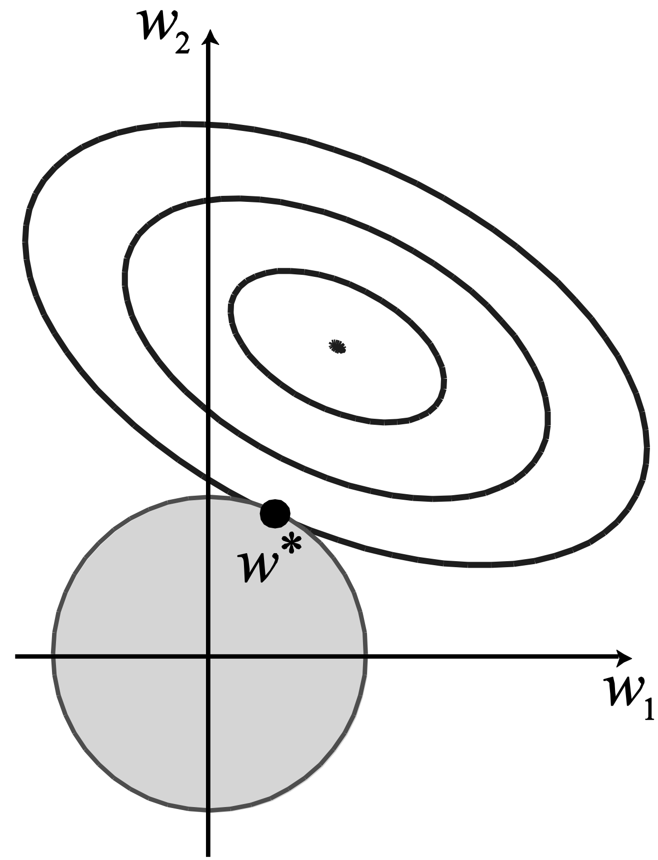
Alternate formulation is L1 Regularization:

$\ln \prod_{i=1}^{N} p\left(y^{(i)} \middle| x^{(i)}, w, b\right) p(w) p(b) = \left[\sum_{i}^{N} \ln p\left(y^{(i)} \middle| x^{(i)}, w, b\right)\right] - \frac{\lambda}{2} \|w\|_1$

Regularization



ℓ_1



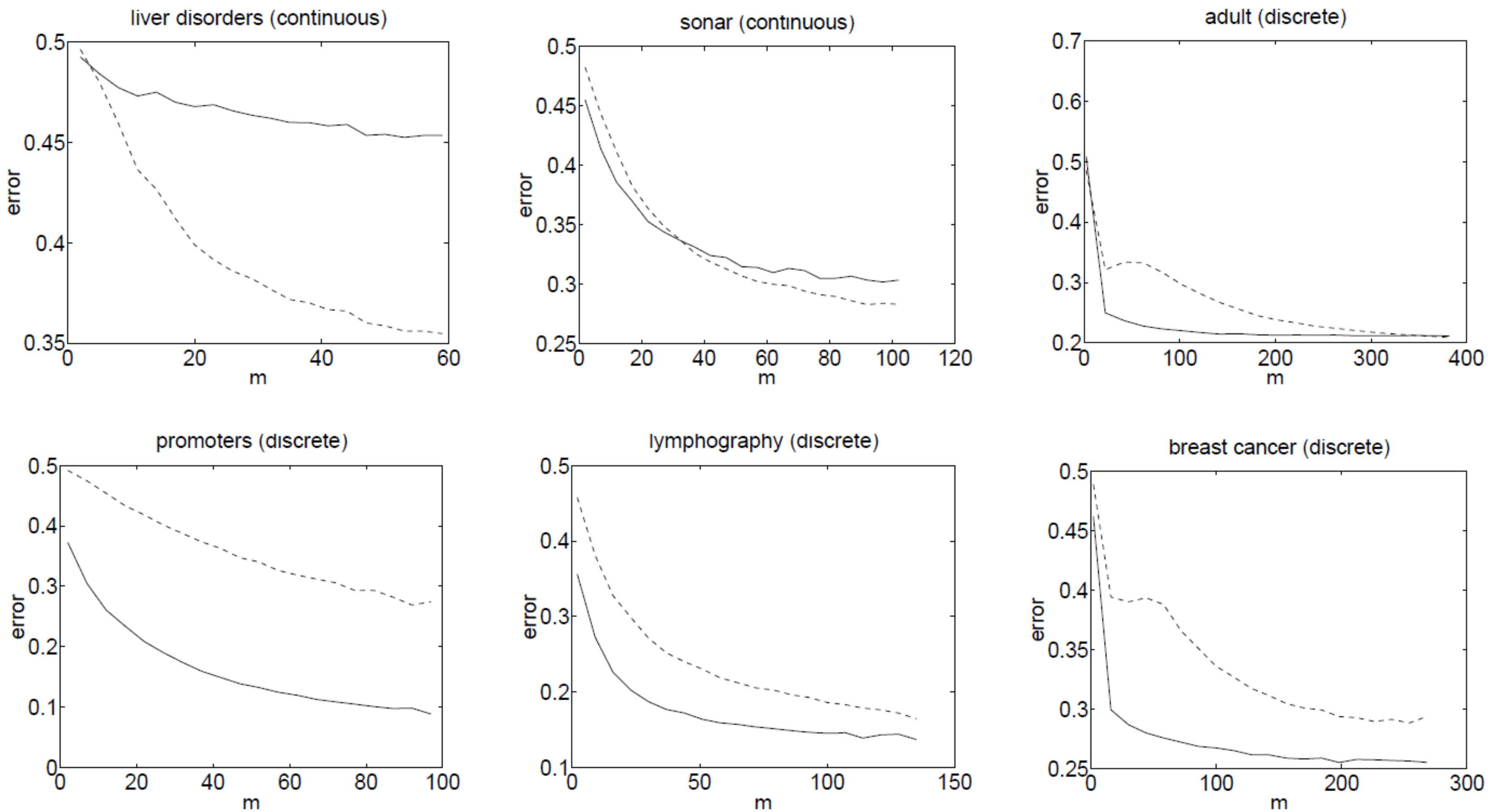
ℓ_2

Naïve Bayes vs. Logistic Regression

- Non-asymptotic analysis (for Gaussian NB)
  - Convergence rate of parameter estimates as size of training data tends to infinity (n = # of attributes in X)
    - Naïve Bayes needs $O(\log n)$ samples
      - NB converges quickly to its (perhaps less helpful) asymptotic estimates
    - Logistic Regression needs $O(n)$ samples
      - LR converges more slowly but makes no independence assumptions (typically less biased)

[Ng & Jordan, 2002]

NB vs. LR (on UCI datasets)



Naïve bayes

Logistic Regression

Sample size m

[Ng & Jordan, 2002]

LR in General

$p(Y=+1|x) = \frac{\exp(w^T x + b)}{1 + \ldots}$

$p(Y=-1|x) = \frac{1}{1 + \ldots}$

- Suppose that $y \in \{1, \ldots, R\}$, i.e., that there are R different class labels
- Can define a collection of weights and biases as follows
  - Choose a vector of biases and a matrix of weights such that for $y \neq R$

$$p(Y = k|x) = \frac{\exp(b_k + \sum_i w_{ki} x_i)}{1 + \sum_{j<R} \exp\left(b_j + \sum_i w_{ji} x_i\right)}$$

$k = 1, 2, \ldots, R-1$

and

$$p(Y = R|x) = \frac{1}{1 + \sum_{j<R} \exp\left(b_j + \sum_i w_{ji} x_i\right)}$$