



Bayesian Methods

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based on the slides of Vibhav Gogate and Nick Rouzzi

Binary Variables



$$X = \# \text{ Heads}$$

$$X = \{0, 1\}$$

- Coin flipping: heads=1, tails=0 with bias μ

$$p(X = 1 | \mu) = \mu$$

- Bernoulli Distribution $p(X = 0 | \mu) = 1 - \mu$

$$\text{Bern}(x | \mu) = \mu^x \cdot (1 - \mu)^{1-x}$$

$$E[X] = \mu = (1 - \mu) \cdot 0 + \mu \cdot 1$$

$$\text{var}(X) = \mu \cdot (1 - \mu)$$

$$E[X] = \sum_{x \in X} x \cdot p(X = x)$$

$$\begin{aligned} \text{Var}(X) &= \sum_x (x - \mu)^2 p(X = x) \\ &= E[(X - \mu)^2] \end{aligned}$$

$$\text{Var}(X) = \frac{\text{Variance}}{\sum_x (x - \mu)^2 P(X=x)}$$

$$= E[X - \mu]^2 \rightarrow \mu = E[X]$$

$$= E[X^2] - 2\mu \underbrace{E[X]}_{\mu} + \mu^2$$

$$= E[X^2] - \mu^2 = E[X^2] - (E[X])^2$$

Coin Flip case: $E[X^2] = \mu \cdot 1^2 + (1-\mu) \cdot 0^2 = \mu$

$$(E[X])^2 = \mu^2$$

$$\Rightarrow \text{Var}(X) = \mu - \mu^2 = \mu(1-\mu)$$

Binary Variables



- N coin flips: X_1, \dots, X_N # Heads in N Coin flips

Each X_i

$$p(\sum_i X_i = m | N, \mu) = \binom{N}{m} \mu^m (1 - \mu)^{N-m}$$

↳ Independent

↳ Identically Distributed

- Binomial Distribution

$$\text{Bin}(m | N, \mu) = \binom{N}{m} \mu^m (1 - \mu)^{N-m}$$

$m=2$
 $N=3$
HHT
HTH
THH

$$E \left[\sum_{i=1}^N X_i \right] = N\mu$$

$$\text{var} \left[\sum_i X_i \right] = N\mu(1 - \mu)$$

$$\sum_{m=0}^N \binom{N}{m} \mu^m (1 - \mu)^{N-m} = 1$$

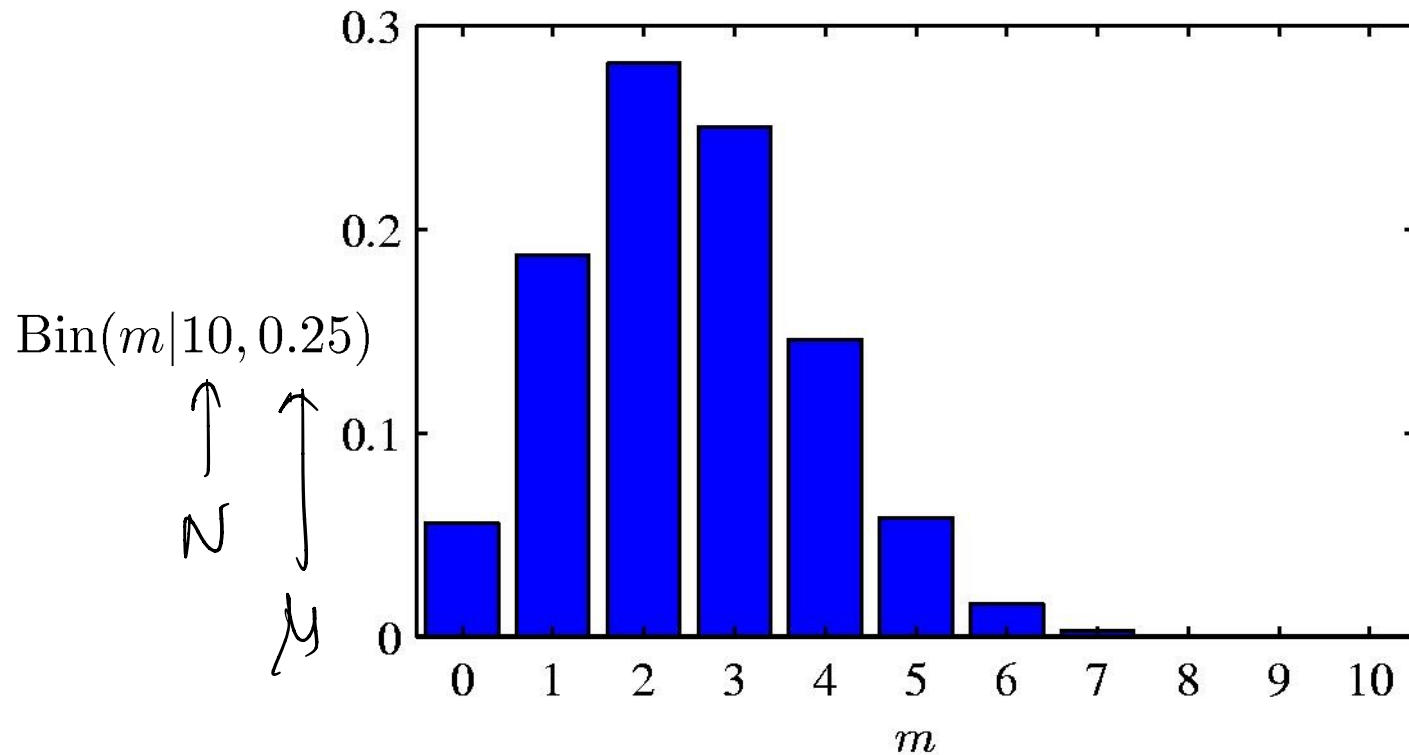
Key: X_i 's are independent & identically distributed

$$\rightarrow (a+b)^N = \sum_{m=0}^N \binom{N}{m} a^m b^{N-m}$$

$$\sum_{m=0}^N \binom{N}{m} y^m (1-y)^{N-m} = 1$$

Binomial Expansion

Binomial Distribution



$$E[X] = 2.5 = 10 \times 0.25$$

Estimating the Bias of a Coin



- Suppose that we have a coin, and we would like to figure out what the probability is that it will flip up heads
 - How should we estimate the bias?

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Estimating the Bias of a Coin



- Suppose that we have a coin, and we would like to figure out what the probability is that it will flip up heads
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- With these coin flips, our estimate of the bias is: $3/5$
 - Why is this a good estimate?

Coin Flipping – Binomial Distribution



- $P(\text{Heads}) = \theta, P(\text{Tails}) = 1 - \theta$
- Flips are i.i.d.
 - Independent events
 - Identically distributed according to Binomial distribution
- Our training data consists of α_H heads and α_T tails

$$p(D|\theta) = \theta^{\alpha_H} \cdot (1 - \theta)^{\alpha_T}$$

$$p(D|\theta) = \prod_{i=1}^{|D|} p(D_i|\theta) \quad \dots \quad \underline{\text{Independence}}$$

$$= \theta^{\alpha_H} (1-\theta)^{\alpha_T}$$

$$p(D_i|\theta) = \theta^{I(D_i=H)} (1-\theta)^{I(D_i=T)}$$

Maximum Likelihood Estimation (MLE)

- **Data:** Observed set of α_H heads and α_T tails
- **Hypothesis:** Coin flips follow a Bernoulli distribution
- **Learning:** Find the “best” θ
- **MLE:** Choose θ to maximize probability of D given θ

$$\begin{aligned}\hat{\theta} &= \arg \max_{\theta} P(\mathcal{D} \mid \theta) \\ &= \arg \max_{\theta} \ln P(\mathcal{D} \mid \theta)\end{aligned}$$

First Parameter Learning Algorithm



$$\begin{aligned}\hat{\theta} &= \arg \max_{\theta} \ln P(\mathcal{D} \mid \theta) \\ &= \arg \max_{\theta} \ln \theta^{\alpha_H} (1 - \theta)^{\alpha_T}\end{aligned}$$

Set derivative to zero, and solve!

$$\begin{aligned}\frac{d}{d\theta} \ln P(\mathcal{D} \mid \theta) &= \frac{d}{d\theta} [\ln \theta^{\alpha_H} (1 - \theta)^{\alpha_T}] \\ &= \frac{d}{d\theta} [\alpha_H \ln \theta + \alpha_T \ln(1 - \theta)] \\ &= \alpha_H \frac{d}{d\theta} \ln \theta + \alpha_T \frac{d}{d\theta} \ln(1 - \theta) \\ &= \frac{\alpha_H}{\theta} - \frac{\alpha_T}{1 - \theta} = 0\end{aligned}$$

First Parameter Learning Algorithm



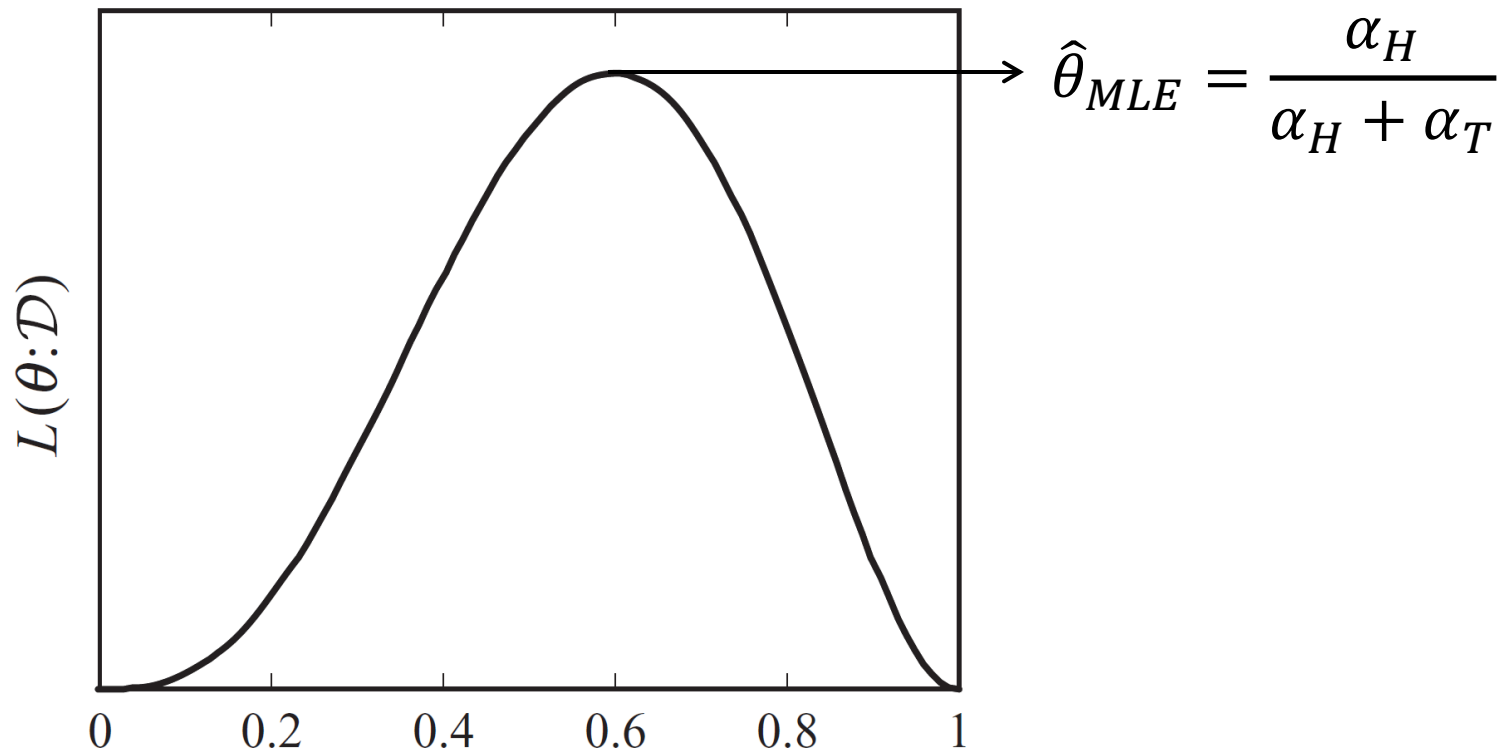
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$$\boxed{\hat{\theta}_{MLE} = \frac{\alpha_H}{\alpha_H + \alpha_T}}$$

Coin Flip MLE





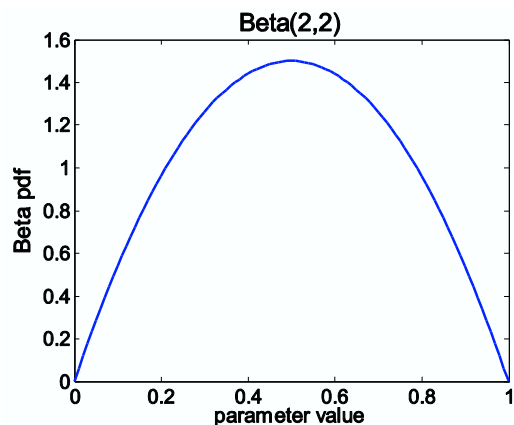
- Suppose we have 5 coin flips all of which are heads
 - Our estimate of the bias is?



- Suppose we have 5 coin flips all of which are heads
 - MLE would give $\theta_{MLE} = 1$
 - This event occurs with probability $\frac{1}{2^5} = \frac{1}{32}$ for a fair coin
 - Are we willing to commit to such a strong conclusion with such little evidence?

- Priors are a Bayesian mechanism that allow us to take into account “prior” knowledge about our belief in the outcome
- Rather than estimating a single θ , consider a distribution over possible values of θ given the data
 - Update our prior after seeing data

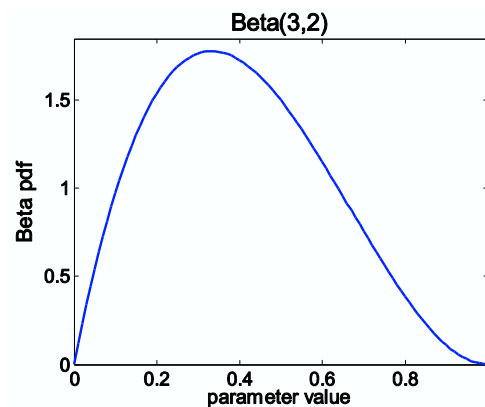
Our best guess in the absence of any data



Observe flips
e.g.: {tails, tails}



Our estimate after we see some data

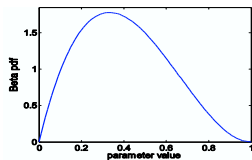


Bayesian Learning



Apply Bayes rule:

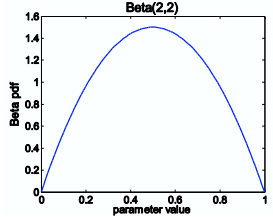
Posterior



Posterior

Data Likelihood

Prior



Prior

$$p(\theta|D) = \frac{p(D|\theta)p(\theta)}{p(D)}$$

Normalization

- Or equivalently: $p(\theta|D) \propto p(D|\theta)p(\theta)$
- For uniform priors this reduces to the MLE objective

$$p(\theta) \propto 1 \quad \Rightarrow \quad p(\theta|D) \propto p(D|\theta)$$

- How do we pick a good prior distribution?
 - Could represent expert domain knowledge
 - Statisticians choose them to make the posterior distribution “nice” (conjugate priors)
- What is a good prior for the bias in the coin flipping problem?

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 - Could represent expert domain knowledge
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 - Truncated Gaussian (tough to work with)
 - Beta distribution (works well for binary random variables)

Coin Flips with Beta Distribution

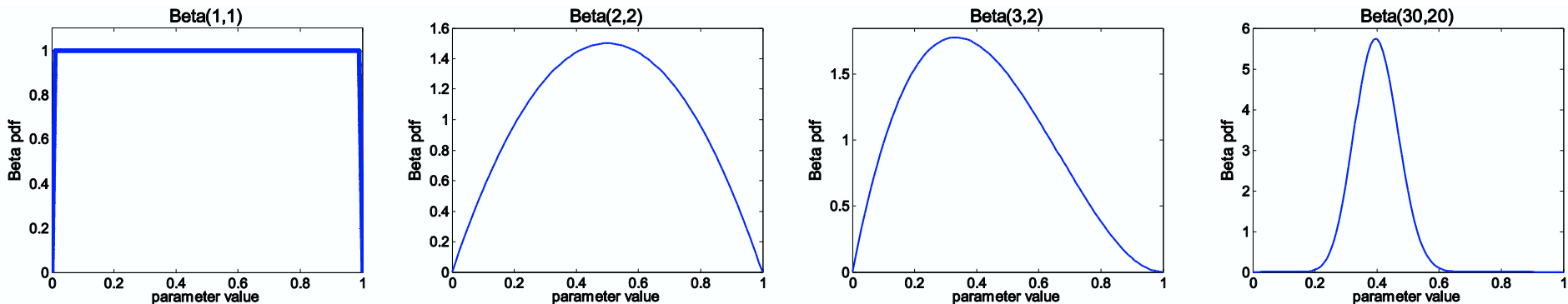


Likelihood function:

$$P(\mathcal{D} \mid \theta) = \theta^{\alpha_H} (1 - \theta)^{\alpha_T}$$

Prior:

$$P(\theta) = \frac{\theta^{\beta_H-1} (1 - \theta)^{\beta_T-1}}{B(\beta_H, \beta_T)} \sim \text{Beta}(\beta_H, \beta_T)$$



$$\begin{aligned} P(\theta \mid \mathcal{D}) &\propto \theta^{\alpha_H} (1 - \theta)^{\alpha_T} \theta^{\beta_H-1} (1 - \theta)^{\beta_T-1} \\ &= \theta^{\alpha_H+\beta_H-1} (1 - \theta)^{\alpha_T+\beta_T-1} \\ &= \text{Beta}(\alpha_H+\beta_H, \alpha_T+\beta_T) \end{aligned}$$

- Choosing θ to maximize the posterior distribution is called maximum a posteriori (MAP) estimation

$$\theta_{MAP} = \arg \max_{\theta} p(\theta|D)$$

- The only difference between θ_{MLE} and θ_{MAP} is that one assumes a uniform prior (MLE) and the other allows an arbitrary prior



- Suppose we have 5 coin flips all of which are heads
 - MLE would give $\theta_{MLE} = 1$
 - MLE with a $Beta(2,2)$ prior gives $\theta_{MAP} = \frac{6}{7} \approx .857$
 - As we see more data, the effect of the prior diminishes
 - $\theta_{MAP} = \frac{\alpha_H + \beta_H - 1}{\alpha_H + \beta_H + \alpha_T + \beta_T - 2} \approx \frac{\alpha_H}{\alpha_H + \alpha_T}$ for large # of observations

- How many coin flips do we need in order to guarantee that our learned parameter does not differ too much from the true parameter (with high probability)?
- Can use Chernoff bound
 - Suppose Y_1, \dots, Y_N are i.i.d. random variables taking values in $\{0, 1\}$ such that $E_p[Y_i] = y$. For $\epsilon > 0$,

$$p\left(\left|y - \frac{1}{N} \sum_i Y_i\right| \geq \epsilon\right) \leq 2e^{-2N\epsilon^2}$$

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 - For the coin flipping problem with X_1, \dots, X_n iid coin flips and $\epsilon > 0$,

$$p\left(\left|\theta_{true} - \frac{1}{N} \sum_i X_i\right| \geq \epsilon\right) \leq 2e^{-2N\epsilon^2}$$

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$$\delta \geq 2e^{-2N\epsilon^2} \Rightarrow N \geq \frac{1}{2\epsilon^2} \ln \frac{2}{\delta}$$