Lecture 5
SVMs with Slack
(Not Linearly Separable)

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Slides adapted from Nick Rouzzi

Recap SVMs

$$\min_{w,b} ||w||^2 \longleftarrow \frac{1}{||w||} \text{ (margin)}$$

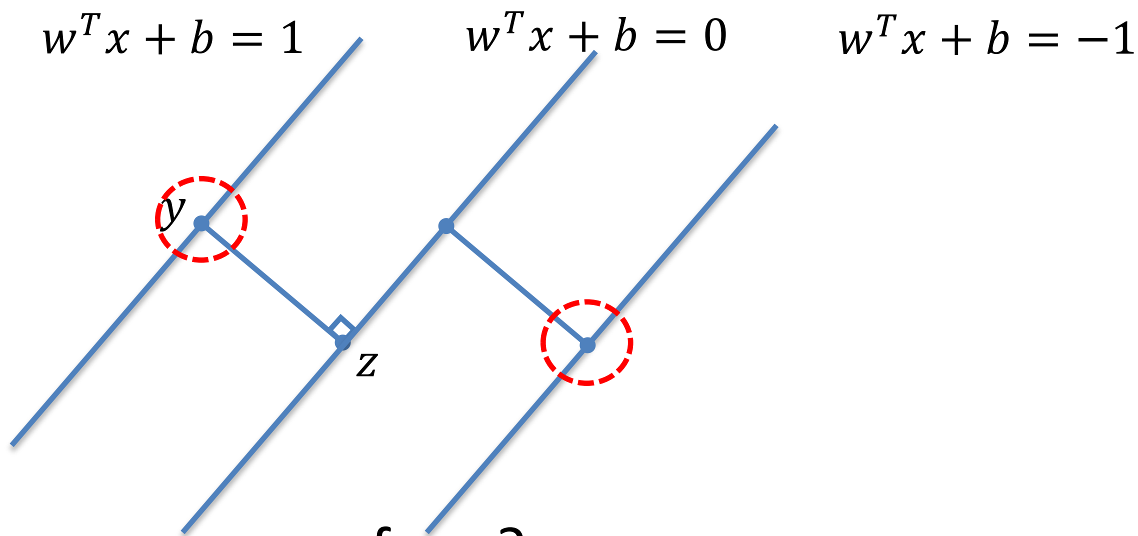
Max margin

such that

$$y^{(i)}\left(w^T x^{(i)} + b\right) \geq 1, \text{for all } i$$

- This is a standard quadratic programming problem
  - Falls into the class of convex optimization problems
  - Can be solved with many specialized optimization tools (e.g., quadprog() in MATLAB)

Recap SVMs



- Where does the name come from?
  - The set of all data points such that $y^{(i)}(w^T x^{(i)} + b) = 1$ are called support vectors
  - The SVM classifier is completely determined by the support vectors (you could delete the rest of the data and get the same answer)

Dual SVM = Original SVM Formulation

$$\max_{\lambda \geq 0} -\frac{1}{2}\sum_i \sum_j \lambda_i \lambda_j y_i y_j \underbrace{\left(x^{(i)}\right)^T \left(x^{(j)}\right)} + \sum_i \lambda_i$$

such that

$$\sum_i \lambda_i y_i = 0$$

λ^*

$\hookrightarrow (w^*, b^*)$

- This is the same as original SVM formulation!
- The dual formulation only depends on inner products between the data points

Dual SVM

$$\max_{\lambda \geq 0} -\frac{1}{2}\sum_i \sum_j \lambda_i \lambda_j y_i y_j \Phi\left(x^{(i)}\right)^T \Phi\left(x^{(j)}\right) + \sum_i \lambda_i$$

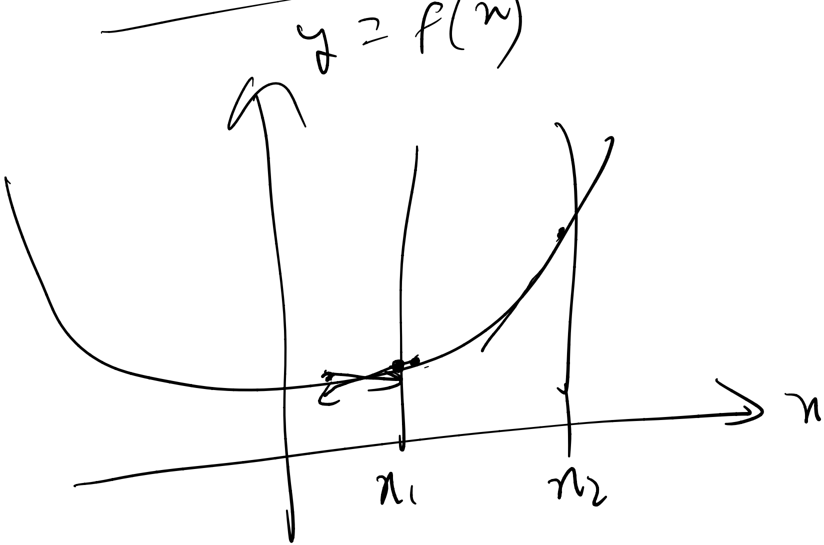
such that

$$\sum_i \lambda_i y_i = 0$$

- The dual formulation only depends on inner products between the data points
    - Same thing is true if we use feature vectors instead

Projected GD

$$\min f(x)$$
$$\text{s.t.} \quad x \geq x_1$$
$$x \leq x_2$$



The Kernel Trick

- For some feature vectors, we can compute the inner products quickly, even if the feature vectors are very large

- This is best illustrated by example

  - Let $\phi(x_1, x_2) = \begin{bmatrix} x_1 x_2 \\ x_2 x_1 \\ x_1^2 \\ x_2^2 \end{bmatrix}$

  - $\phi(x_1, x_2)^T \phi(z_1, z_2) = x_1^2 z_1^2 + 2x_1 x_2 z_1 z_2 + x_2^2 z_2^2$
    $$= (x_1 z_1 + x_2 z_2)^2$$
    $$= (x^T z)^2$$

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$$= (x_1 z_1 + x_2 z_2)^2$$

$$= (x^T z)^2$$

Reduces to a dot product in the original space

The Kernel Trick

- The same idea can be applied for the feature vector ϕ of all polynomials of degree (exactly) d

$k(x, y) = (x^T y)^d$

  - $\phi(x)^T \phi(z) = (x^T z)^d$

- More generally, a kernel is a function $k(x, z) = \phi(x)^T \phi(z)$ for some feature map ϕ

- Rewrite the dual objective

$$\max_{\lambda \geq 0, \sum_i \lambda_i y_i = 0} -\frac{1}{2} \sum_i \sum_j \lambda_i \lambda_j y_i y_j \boxed{k(x^{(i)}, x^{(j)})} + \sum_i \lambda_i$$

Examples of Kernels

- Polynomial kernel of degree exactly d
  - $k(x,z) = (x^T z)^d$
- General polynomial kernel of degree d for some c
  - $k(x,z) = (x^T z + c)^d$
- Gaussian kernel for some σ
  - $k(x,z) = \exp\left(\frac{-\|x-z\|^2}{2\sigma^2}\right)$
  - The corresponding ϕ is infinite dimensional!
- So many more…

Gaussian Kernels

- Consider the Gaussian kernel

$$\exp\left(\frac{-\|x-z\|^2}{2\sigma^2}\right) = \exp\left(\frac{-(x-z)^T(x-z)}{2\sigma^2}\right)$$

$$= \exp\left(\frac{-\|x\|^2 + 2x^Tz - \|z\|^2}{2\sigma^2}\right)$$

$$= \exp\left(-\frac{\|x\|^2}{2\sigma^2}\right)\exp\left(-\frac{\|z\|^2}{2\sigma^2}\right)\exp\left(\frac{x^Tz}{\sigma^2}\right)$$

- Use the Taylor expansion for $\exp()$

$$\exp\left(\frac{x^Tz}{\sigma^2}\right) = \sum_{n=0}^{\infty}\frac{(x^Tz)^n}{\sigma^{2n}n!}$$

Gaussian Kernels

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$$= \exp\left(-\frac{\|x\|^2}{2\sigma^2}\right)\exp\left(-\frac{\|z\|^2}{2\sigma^2}\right)\exp\left(\frac{x^Tz}{\sigma^2}\right)$$

- Use the Taylor expansion for exp()

$$\exp\left(\frac{x^Tz}{\sigma^2}\right) = \sum_{n=0}^{\infty}\frac{(x^Tz)^n}{\sigma^{2n}n!}$$

Polynomial kernels of every degree!

Kernels

- Bigger feature space increases the possibility of overfitting
  - Large margin solutions may still generalize reasonably well
- Alternative: add “penalties” to the objective to disincentivize complicated solutions

SVMs with Slack (Remove Linear Separability)

- Allow misclassification
  - Penalize misclassification linearly (just like in the perceptron algorithm)
    - Again, easier to work with than counting misclassifications
    - Objective stays convex
  - Will let us handle data that isn't linearly separable!
  - Idea: Take the constraints into the main objective
    - The objective function then becomes exactly like what we have seen in Perceptron/Linear Regression

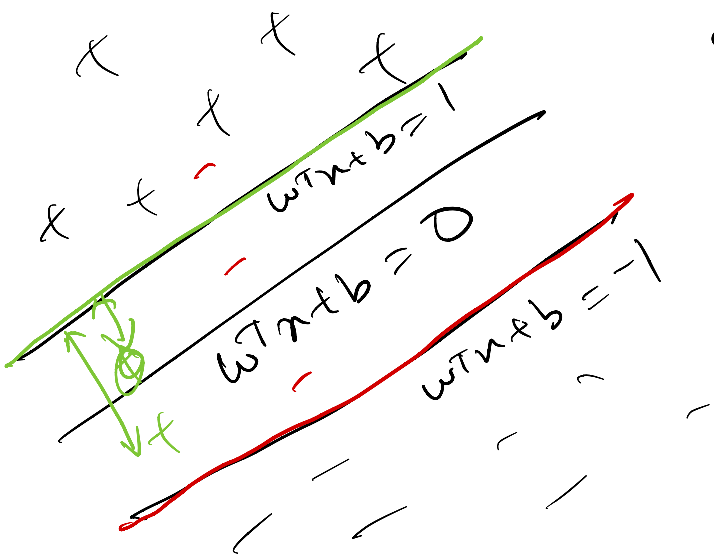
SVMs with Slack

$$\min_{w,b,\xi} \frac{1}{2}\|w\|^2 + c\sum_i \xi_i$$

such that

$$y_i\left(w^T x^{(i)} + b\right) \geq 1 - \xi_i, \text{for all } i$$

$$\xi_i \geq 0, \text{for all } i$$



SVMs with Slack

$$\min_{w,b,\xi} \frac{1}{2} \|w\|^2 + c \sum_i \xi_i$$

such that

$$y_i(w^T x^{(i)} + b) \geq 1 - \xi_i, \text{for all } i$$

$$\xi_i \geq 0, \text{for all } i$$

Potentially allows some points to be misclassified/inside the margin

SVMs with Slack

$$\min_{w,b,\xi} \frac{1}{2}\|w\|^2 + c\sum_i \xi_i$$

Constant c determines degree to which slack is penalized

such that

$$y_i\left(w^T x^{(i)} + b\right) \geq 1 - \xi_i, \text{for all } i$$

$$\xi_i \geq 0, \text{for all } i$$

SVMs with Slack

$$\min_{w,b,\xi} \frac{1}{2}\|w\|^2 + c\sum_i \xi_i$$

such that

$$y_i\left(w^T x^{(i)} + b\right) \geq 1 - \xi_i, \text{for all } i$$

$$\xi_i \geq 0, \text{for all } i$$

- How does this objective change with c?

SVMs with Slack

$$\min_{w,b,\xi} \frac{1}{2}\|w\|^2 + c\sum_i \xi_i$$

such that

$$y_i\left(w^T x^{(i)} + b\right) \geq 1 - \xi_i, \text{for all } i$$

$$\xi_i \geq 0, \text{for all } i$$

- How does this objective change with c?
  - As $c \rightarrow \infty$, requires a perfect classifier
  - As $c \rightarrow 0$, allows arbitrary classifiers (i.e., ignores the data)

SVMs with Slack

$$\min_{w,b,\xi} \frac{1}{2}\|w\|^2 + c\sum_i \xi_i$$

such that

$$y_i\left(w^T x^{(i)} + b\right) \geq 1 - \xi_i, \text{for all } i$$

$$\xi_i \geq 0, \text{for all } i$$

- How should we pick c?

SVMs with Slack

$$\min_{w,b,\xi} \frac{1}{2} \|w\|^2 + c \sum_i \xi_i$$

such that

$$y_i \left(w^T x^{(i)} + b \right) \geq 1 - \xi_i, \text{for all } i$$

$$\xi_i \geq 0, \text{for all } i$$

- How should we pick c?
  - Divide the data into three pieces training, testing, and validation
  - Use the validation set to tune the value of the hyperparameter c

Evaluation Methodology

- General learning strategy
  - Build a classifier using the training data ← Model Parameters
  - Select hyperparameters using validation data ← C
  - Evaluate the chosen model with the selected hyperparameters on the test data

How can we tell if we overfit the training data?

ML in Practice

- Gather Data + Labels
- Select feature vectors
- Randomly split into three groups
  - Training set
  - Validation set
  - Test set

80%

10%

10%

- Experimentation cycle
  - Select a “good” hypothesis from the hypothesis space
  - Tune hyper-parameters using validation set
  - Compute accuracy on test set (fraction of correctly classified instances)

SVMs with Slack

$$\min_{w,b,\xi} \frac{1}{2}\|w\|^2 + c\sum_i \xi_i$$

such that

$$y_i\left(w^T x^{(i)} + b\right) \geq 1 - \xi_i, \text{for all } i$$

$$\xi_i \geq 0, \text{for all } i$$

- What is the optimal value of ξ for fixed w and b?

SVMs with Slack

$$\min_{w,b,\xi} \frac{1}{2}\|w\|^2 + c\sum_i \xi_i$$

such that

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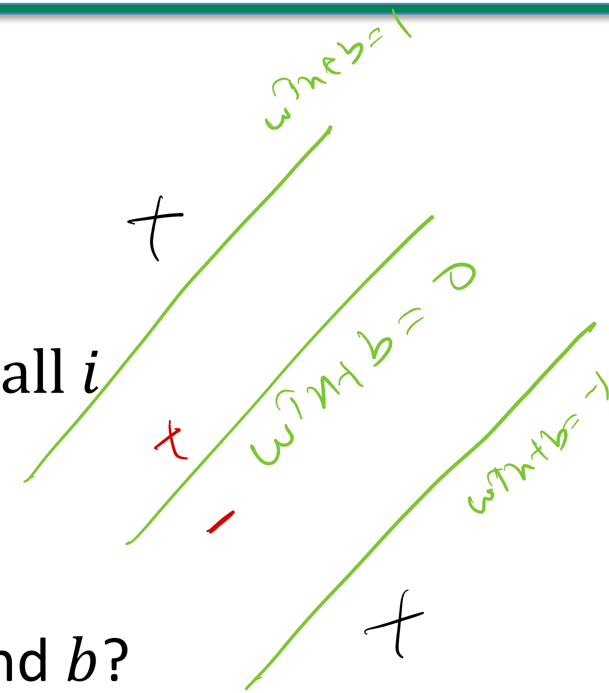


- What is the optimal value of ξ for fixed w and b?
  - If $y_i\left(w^T x^{(i)} + b\right) \geq 1$, then $\xi_i = 0$
  - If $y_i\left(w^T x^{(i)} + b\right) < 1$, then $\xi_i = 1 - y_i\left(w^T x^{(i)} + b\right)$

$\xi \to$ Closed form sol^n?

SVMs with Slack

$$\min_{w,b,\xi} \frac{1}{2}\|w\|^2 + c\sum_i \xi_i$$

such that

$$y_i\left(w^T x^{(i)} + b\right) \geq 1 - \xi_i, \text{for all } i$$

$$\xi_i \geq 0, \text{for all } i$$

- We can formulate this slightly differently
  - $\xi_i = \max\{0, 1 - y_i\left(w^T x^{(i)} + b\right)\}$
  - Does this look familiar?
  - Hinge loss provides an upper bound on Hamming loss

Hinge Loss Formulation

- Obtain a new objective by substituting in for ξ

$$\min_{w,b} \frac{1}{2}||w||^2 + c\sum_i \max\{0, 1 - y_i\left(w^T x^{(i)} + b\right)\}$$

Can minimize with gradient descent!

Hinge Loss Formulation

- Obtain a new objective by substituting in for ξ

$$\min_{w,b} \underbrace{\frac{1}{2}||w||^2}_{\text{Penalty to prevent overfitting}} + c \underbrace{\sum_i \max\{0, 1 - y_i\left(w^T x^{(i)} + b\right)\}}_{\text{Hinge loss}}$$

Penalty to prevent overfitting

"Avoids overfitting to the Data"

Hinge loss

"Fit well to the Data"

REGULARIZATION!

- Until now, we have seen the following optimization problems:

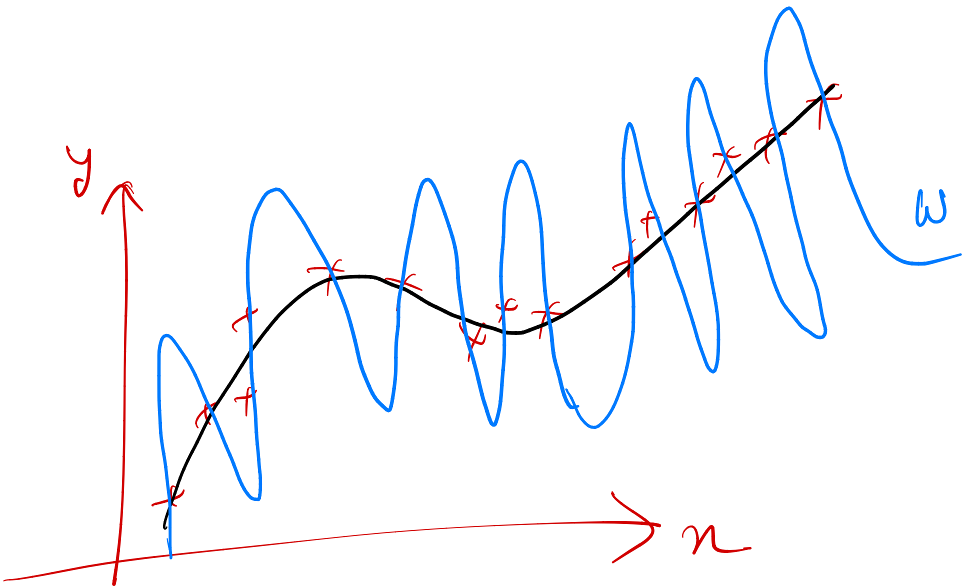
$$\min_{w,b} \sum_i L\left(f\left(x^{(i)}, w, b\right), y_i\right) \leftarrow \text{Fit well to the Train Data.}$$

- In the case of Linear regression, L was the squared loss
- In Perceptron, L was Perceptron Loss
- The regularized version of this is:

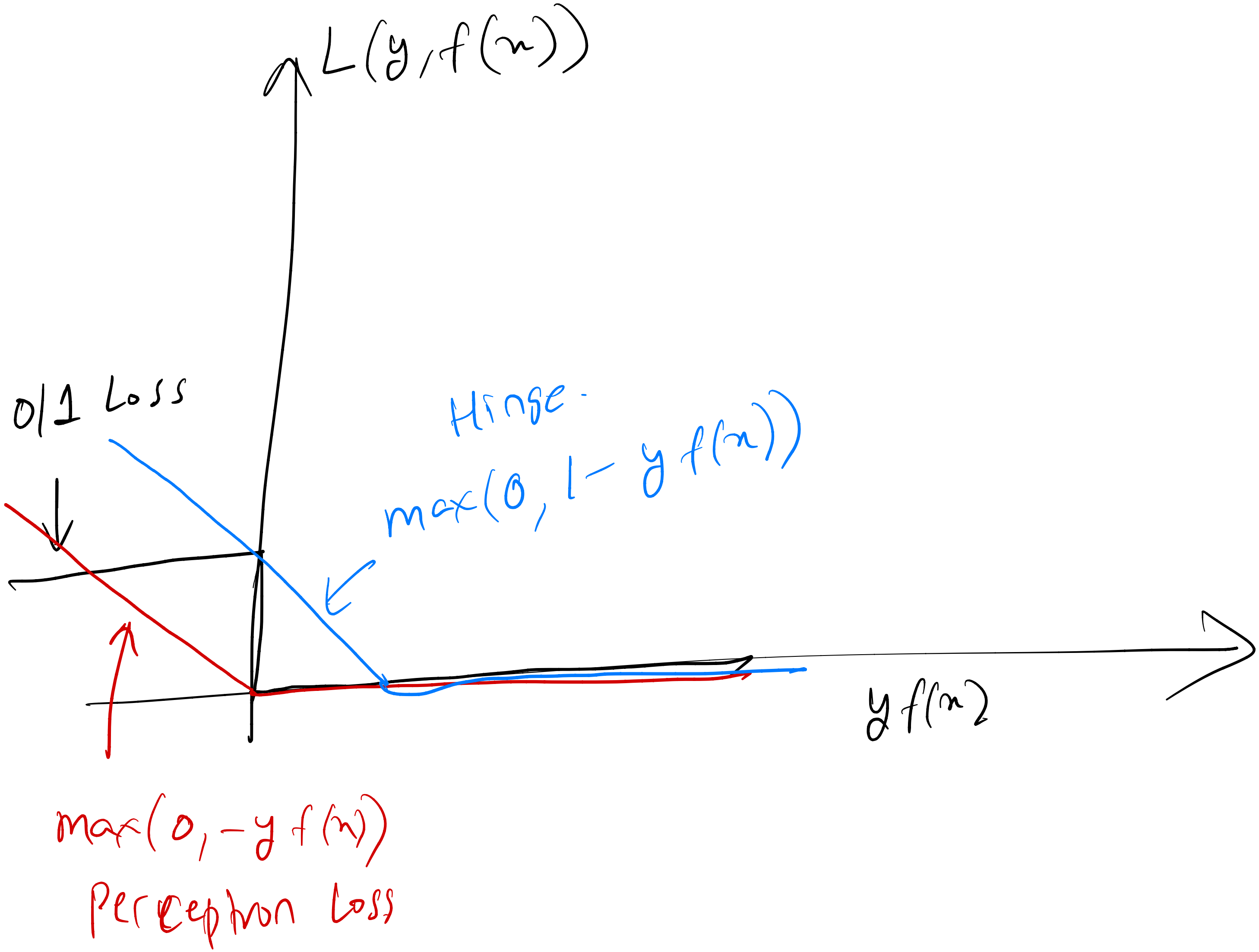
$$\min_{w,b} \frac{1}{2} ||w||^2 + c \sum_i L\left(f\left(x^{(i)}, w, b\right), y_i\right)$$

- c is a hyper-parameter (again, to be tunes on validation set) why does $\|w\|^2$ prevent overfitting?

$\|w\| >>> \|w\|$



Perceptron vs Hinge vs Square vs Zero-One Loss



Imbalanced Data

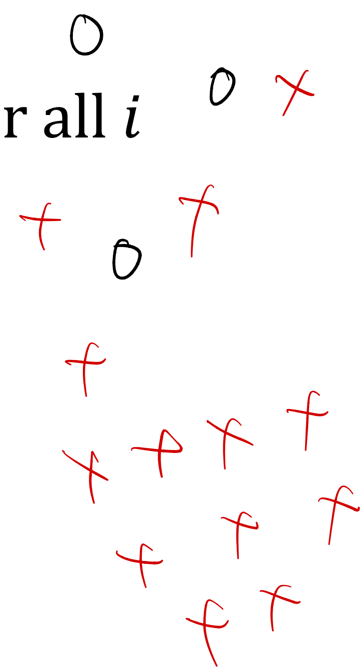
- If the data is imbalanced (i.e., more positive examples than negative examples), may want to evenly distribute the error between the two classes

$$\min_{w,b,\xi} \frac{1}{2}\|w\|^2 + \frac{c}{N_+}\sum_{i:y_i=1}\xi_i + \frac{c}{N_-}\sum_{i:y_i=-1}\xi_i$$

such that

$$y_i\left(w^T x^{(i)} + b\right) \geq 1 - \xi_i, \text{for all } i$$

$$\xi_i \geq 0, \text{for all } i$$



Generalization

- We argued, intuitively, that SVMs generalize better than the perceptron algorithm
  - How can we make this precise?

Margin $\longrightarrow$ Generalization
(Test set performance)

Roadmap

- Where are we headed?
  - Non-Parametric Methods
    - k nearest neighbor
    - Decision trees
  - Probabilistic Methods
    - Bayesian Methods
    - Naïve Bayes
    - Logistic Regression
  - Unsupervised Learning
    - Clustering