CS 4375: Extra Topic (Reading Material)
Lagrange Multipliers
& Duality of SVMs

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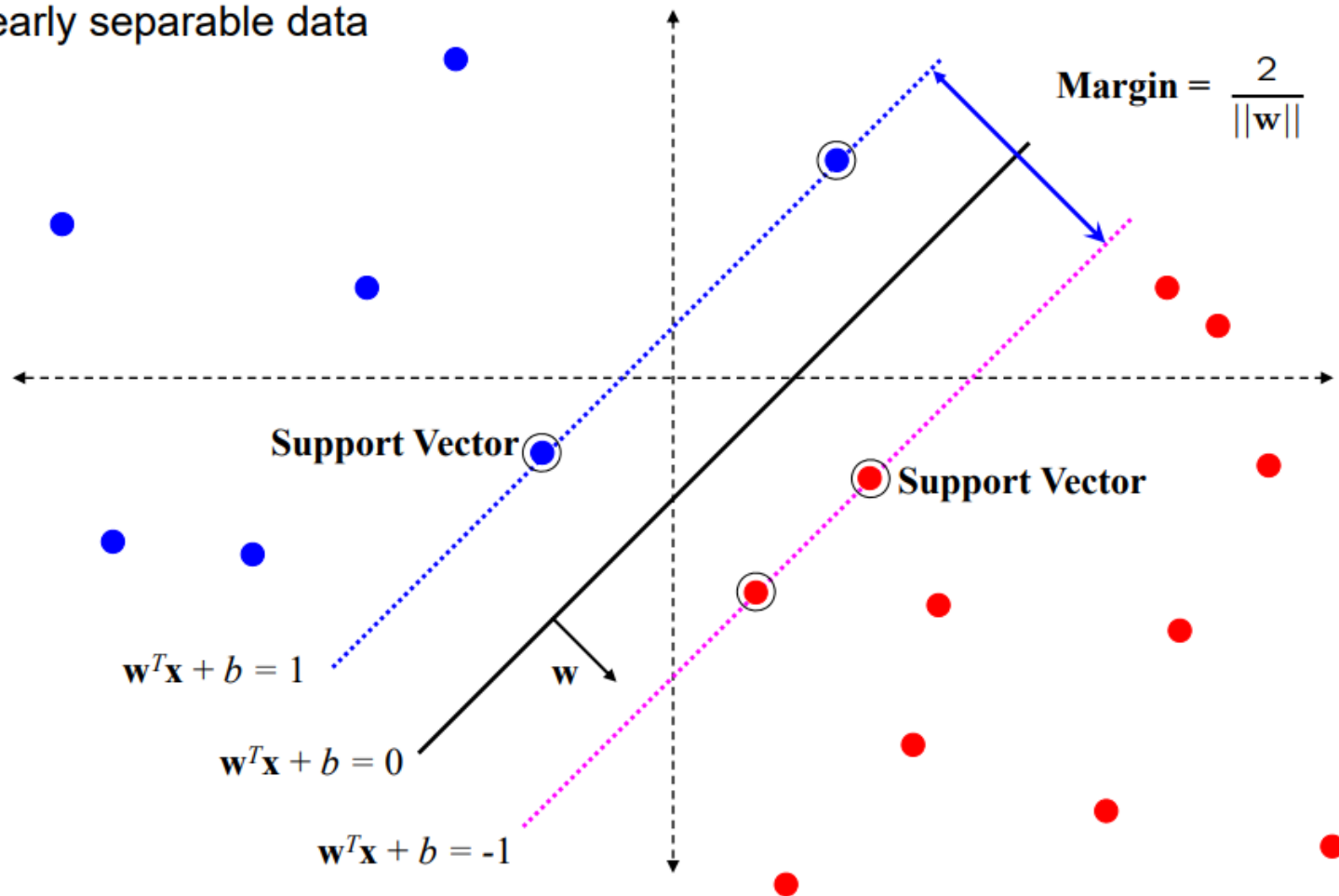Most Slides are from Nick Rouzzi! Thanks Nick!!

The Strategy So Far…

- Choose hypothesis space

- Construct loss function (ideally convex)

- Minimize loss to "learn" correct parameters

Recap: SVM

linearly separable data

$$\text{Margin} = \frac{2}{\|\mathbf{w}\|}$$

Support Vector

Support Vector

$\mathbf{w}^T\mathbf{x} + b = 1$

$\mathbf{w}$

$\mathbf{w}^T\mathbf{x} + b = 0$

$\mathbf{w}^T\mathbf{x} + b = -1$

SVM Optimization Problem

- Recall: The SVM optimization problem:

$$\min_{w,b} \|w\|^2$$

such that

$$y^{(i)}\big(w^T x^{(i)} + b\big) \geq 1, \text{for all } i$$

- This is a standard quadratic programming problem

  - Falls into the class of convex optimization problems

  - Can be solved with many specialized optimization tools (e.g., quadprog() in MATLAB)

Constrained Optimization

A mathematical detour, we'll come back to SVMs soon!

$$\min_{x \in \mathbb{R}^n} f_0(x)$$

subject to:

$$f_i(x) \leq 0, \qquad i = 1, \dots, m$$
$$h_i(x) = 0, \qquad i = 1, \dots, p$$

Constrained Optimization

$$\min_{x \in \mathbb{R}^n} f_0(x)$$

f_0 is not necessarily convex

subject to:

$$f_i(x) \leq 0, \qquad i = 1, \ldots, m$$
$$h_i(x) = 0, \qquad i = 1, \ldots, p$$

General Optimization

$$\min_{x \in \mathbb{R}^n} f_0(x)$$

subject to:

Constraints do not need to be linear

$$f_i(x) \leq 0, \qquad i = 1, \ldots, m$$
$$h_i(x) = 0, \qquad i = 1, \ldots, p$$

Example

$$\min_{x \in \mathbb{R}^3} x_1 \log x_1 + x_2 \log x_2$$

subject to:

$$x_1 + x_2 = 1$$
$$x_1 \geq 0$$
$$x_2 \geq 0$$

Example

$$\min_{x \in \mathbb{R}^3} x_1 \log x_1 + x_2 \log x_2$$

subject to:

$$1 - x_1 - x_2 = 0$$
$$-x_1 \leq 0$$
$$-x_2 \leq 0$$

Lagrangian

$$L(x, \lambda, \nu) = f_0(x) + \sum_{i=1}^{m} \lambda_i f_i(x) + \sum_{i=1}^{p} \nu_i h_i(x)$$

- Incorporate constraints into a new objective function

- $\lambda \geq 0$ and ν are vectors of ***Lagrange multipliers***

- The Lagrange multipliers can be thought of as enforcing soft constraints

Example

$$\min_{x \in \mathbb{R}^3} x_1 \log x_1 + x_2 \log x_2$$

subject to:

$$1 - x_1 - x_2 = 0$$
$$-x_1 \leq 0$$
$$-x_2 \leq 0$$

$$L(x_1, x_2, \nu_1, \lambda_1, \lambda_2)$$
$$= x_1 \log x_1 + x_2 \log x_2 + \nu_1 \cdot (1 - x_1 - x_2) - \lambda_1 x_1 - \lambda_2 x_2$$

Duality

- Construct a dual function by minimizing the Lagrangian over the primal variables

$$g(\lambda, \nu) = \inf_{x} L(x, \lambda, \nu)$$

- $g(\lambda, \nu) = -\infty$ whenever the Lagrangian is not bounded from below for a fixed λ and ν

Example

$$\min_{x \in \mathbb{R}^2} x_1 \log x_1 + x_2 \log x_2$$

subject to:

$$1 - x_1 - x_2 = 0$$
$$-x_1 \leq 0$$
$$-x_2 \leq 0$$

Example

$$\min_{x \in \mathbb{R}^3} x_1 \log x_1 + x_2 \log x_2$$

subject to:

$$1 - x_1 - x_2 = 0$$
$$-x_1 \leq 0$$
$$-x_2 \leq 0$$

$$L(x_1, x_2, \nu_1, \lambda_1, \lambda_2)$$
$$= x_1 \log x_1 + x_2 \log x_2 + \nu_1 \cdot (1 - x_1 - x_2) - \lambda_1 x_1 - \lambda_2 x_2$$

Example

$$\min_{x \in \mathbb{R}^3} x_1 \log x_1 + x_2 \log x_2$$

subject to:

$$1 - x_1 - x_2 = 0$$
$$-x_1 \leq 0$$
$$-x_2 \leq 0$$

The Primal Problem

$$\min_{x \in \mathbb{R}^n} f_0(x)$$

subject to:

$$f_i(x) \leq 0, \qquad i = 1, \ldots, m$$
$$h_i(x) = 0, \qquad i = 1, \ldots, p$$

Equivalently,

$$\inf_{x} \sup_{\lambda \geq 0, \nu} L(x, \lambda, \nu)$$

Why are these equivalent?

The Primal Problem

$$\min_{x \in \mathbb{R}^n} f_0(x)$$

subject to:

$$f_i(x) \leq 0, \qquad i = 1, \ldots, m$$
$$h_i(x) = 0, \qquad i = 1, \ldots, p$$

Equivalently,

$$\inf_{x} \sup_{\lambda \geq 0, \nu} L(x, \lambda, \nu)$$

$$\sup_{\lambda \geq 0, \nu} \left[f_0(x) + \sum_{i=1}^{m} \lambda_i f_i(x) + \sum_{i=1}^{p} \nu_i h_i(x) \right] = \infty$$

whenever x violates the constraints

The Dual Problem

$$\sup_{\lambda \geq 0, \nu} g(\lambda, \nu)$$

Equivalently,

$$\sup_{\lambda \geq 0, \nu} \inf_{x} L(x, \lambda, \nu)$$

- The dual problem is always concave, even if the primal problem is not convex

  - For each x, $L(x, \lambda, \nu)$ is a linear function in λ and ν

  - Minimum (or infimum) of linear functions is concave!

Primal vs. Dual

$$\sup_{\lambda \geq 0, \nu} \inf_{x} L(x, \lambda, \nu) \leq \inf_{x} \sup_{\lambda \geq 0, \nu} L(x, \lambda, \nu)$$

- Why?

  - $g(\lambda, \nu) \leq L(x, \lambda, \nu)$ for all x

  - $L(x', \lambda, \nu) \leq f_0(x')$ for any feasible x', $\lambda \geq 0$

    - x is feasible if it satisfies all of the constraints

  - Let x^* be the optimal solution to the primal problem and $\lambda \geq 0$

  $$g(\lambda, \nu) \leq L(x^*, \lambda, \nu) \leq f_0(x^*)$$

Example: Solving the Dual Problem

$$\min_{x \in \mathbb{R}^3} x_1 \log x_1 + x_2 \log x_2$$

subject to:

$$1 - x_1 - x_2 = 0$$
$$-x_1 \leq 0$$
$$-x_2 \leq 0$$

More Examples

- Minimize $x^2 + y^2$ subject to $x + y \geq 1$

- Given a point $z \in \mathbb{R}^n$ and a hyperplane $w^T x + b = 0$, find the projection of the point z onto the hyperplane

Duality

- Under certain conditions, the two optimization problems are equivalent

$$\sup_{\lambda \geq 0, \nu} \inf_{x} L(x, \lambda, \nu) = \inf_{x} \sup_{\lambda \geq 0, \nu} L(x, \lambda, \nu)$$

  - This is called strong duality

- If the inequality is strict, then we say that there is a duality gap

  - Size of gap measured by the difference between the two sides of the inequality

Slater's Condition

For any optimization problem of the form

$$\min_{x \in \mathbb{R}^n} f_0(x)$$

subject to:

$$f_i(x) \leq 0, \qquad i = 1, \ldots, m$$
$$Ax = b$$

where $f_0, \ldots, f_m$ are convex functions, strong duality holds if there exists an x such that

$$f_i(x) < 0, \qquad i = 1, \ldots, m$$
$$Ax = b$$

Dual SVM

$$\min_{w} \frac{1}{2} \|w\|^2$$

such that

$$y_i\left(w^T x^{(i)} + b\right) \geq 1, \text{for all } i$$

- Note that Slater's condition holds as long as the data is linearly separable

Dual SVM

$$L(w, b, \lambda) = \frac{1}{2} w^T w + \sum_i \lambda_i (1 - y_i(w^T x^{(i)} + b))$$

Convex in w, so take derivatives to form the dual

$$\frac{\partial L}{\partial w_k} = w_k + \sum_i -\lambda_i y_i x_k^{(i)} = 0$$

$$\frac{\partial L}{\partial b} = \sum_i -\lambda_i y_i = 0$$

Dual SVM

$$L(w, b, \lambda) = \frac{1}{2} w^T w + \sum_i \lambda_i (1 - y_i (w^T x^{(i)} + b))$$

Convex in w, so take derivatives to form the dual

$$w = \sum_i \lambda_i y_i x^{(i)}$$

$$\sum_i \lambda_i y_i = 0$$

Dual SVM

$$\max_{\lambda \geq 0} -\frac{1}{2} \sum_i \sum_j \lambda_i \lambda_j y_i y_j x^{(i)^T} x^{(j)} + \sum_i \lambda_i$$

such that

$$\sum_i \lambda_i y_i = 0$$

- By strong duality, solving this problem is equivalent to solving the primal problem

  - Given the optimal λ, we can easily construct w (b can be found by complementary slackness…)

Complementary Slackness

- Suppose that there is zero duality gap

- Let x^* be an optimum of the primal and (λ^*, ν^*) be an optimum of the dual

$$f_0(x^*) = g(\lambda^*, \nu^*)$$

$$= \inf_x \left[f_0(x) + \sum_{i=1}^{m} \lambda_i^* f_i(x) + \sum_{i=1}^{p} \nu_i^* h_i(x) \right]$$

$$\leq f_0(x^*) + \sum_{i=1}^{m} \lambda_i^* f_i(x^*) + \sum_{i=1}^{p} \nu_i^* h_i(x^*)$$

$$= f_0(x^*) + \sum_{i=1}^{m} \lambda_i^* f_i(x^*)$$

$$\leq f_0(x^*)$$

Complementary Slackness

- This means that

$$\sum_{i=1}^{m} \lambda_i^* f_i(x^*) = 0$$

- As $\lambda \geq 0$ and $f_i(x_i^*) \leq 0$, this can only happen if $\lambda_i^* f_i(x^*) = 0$ for all i

- Put another way,

  - If $f_i(x^*) < 0$ (i.e., the constraint is not tight), then $\lambda_i^* = 0$

  - If $\lambda_i^* > 0$, then $f_i(x^*) = 0$

  - ONLY applies when there is no duality gap

Dual SVM (Obtaining b)

$$\max_{\lambda \geq 0} -\frac{1}{2} \sum_i \sum_j \lambda_i \lambda_j y_i y_j x^{(i)^T} x^{(j)} + \sum_i \lambda_i$$

such that

$$\sum_i \lambda_i y_i = 0$$

- By complementary slackness, $\lambda_i^* > 0$ means that $x^{(i)}$ is a support vector (can then solve for b using w)

- In particular,

$$b = y_i - w.x_i$$

for any i where $\lambda_i > 0$

Dual SVM

$$\max_{\lambda \geq 0} -\frac{1}{2} \sum_i \sum_j \lambda_i \lambda_j y_i y_j x^{(i)^T} x^{(j)} + \sum_i \lambda_i$$

such that

$$\sum_i \lambda_i y_i = 0$$

- Takes $O(n^2)$ time just to evaluate the objective function

  - Active area of research to try to speed this up