CS 4375
Support Vector Machines

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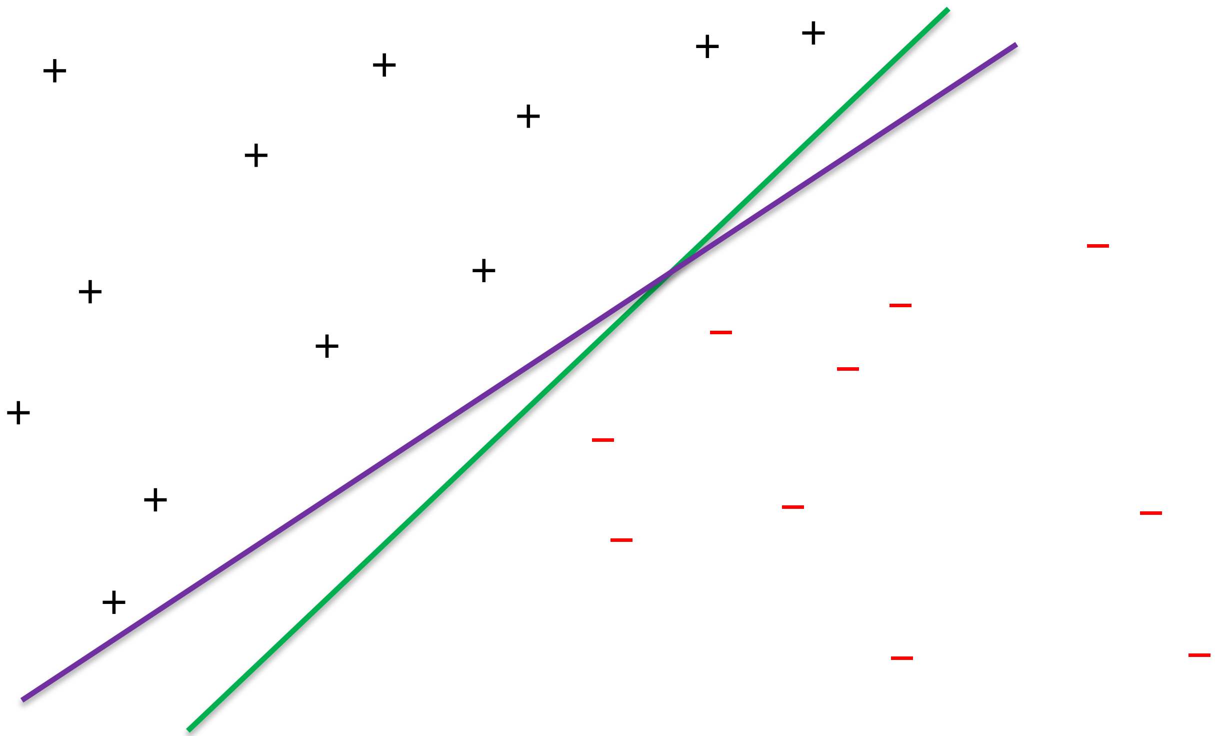
Slides adapted from Nicholas Ruozzi, David Sontag and Vibhav Gogate

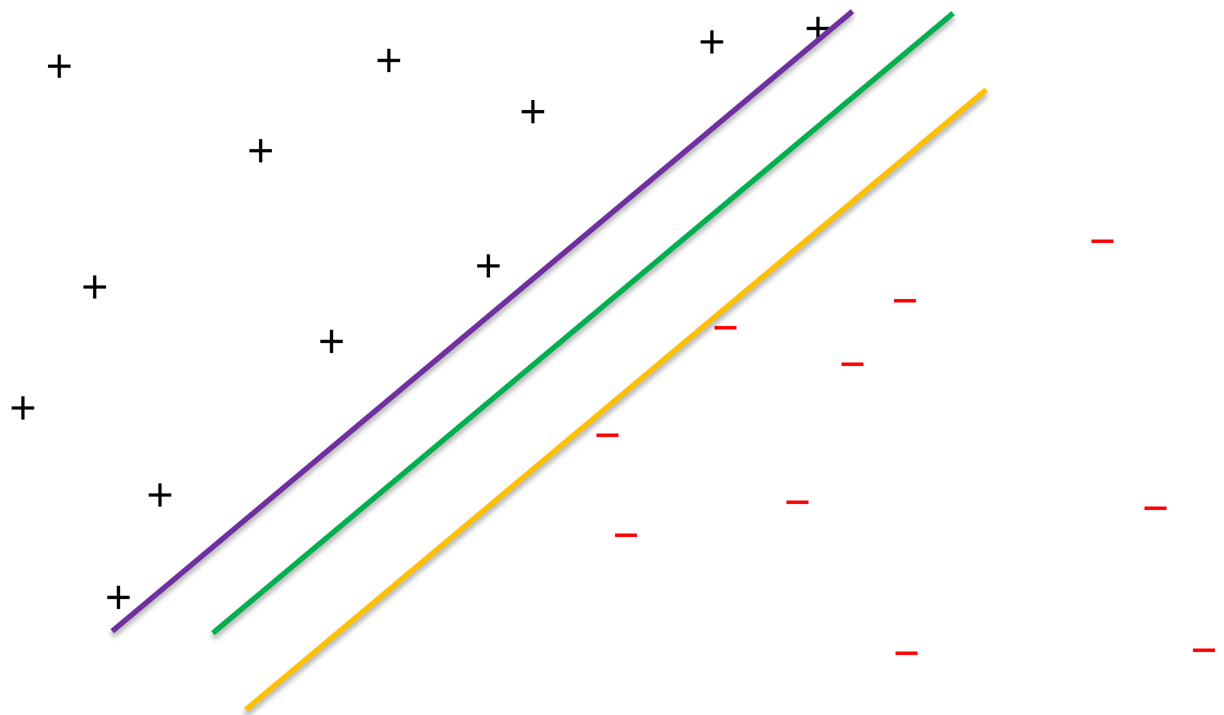
Support Vector Machines

- How can we decide between perfect classifiers?



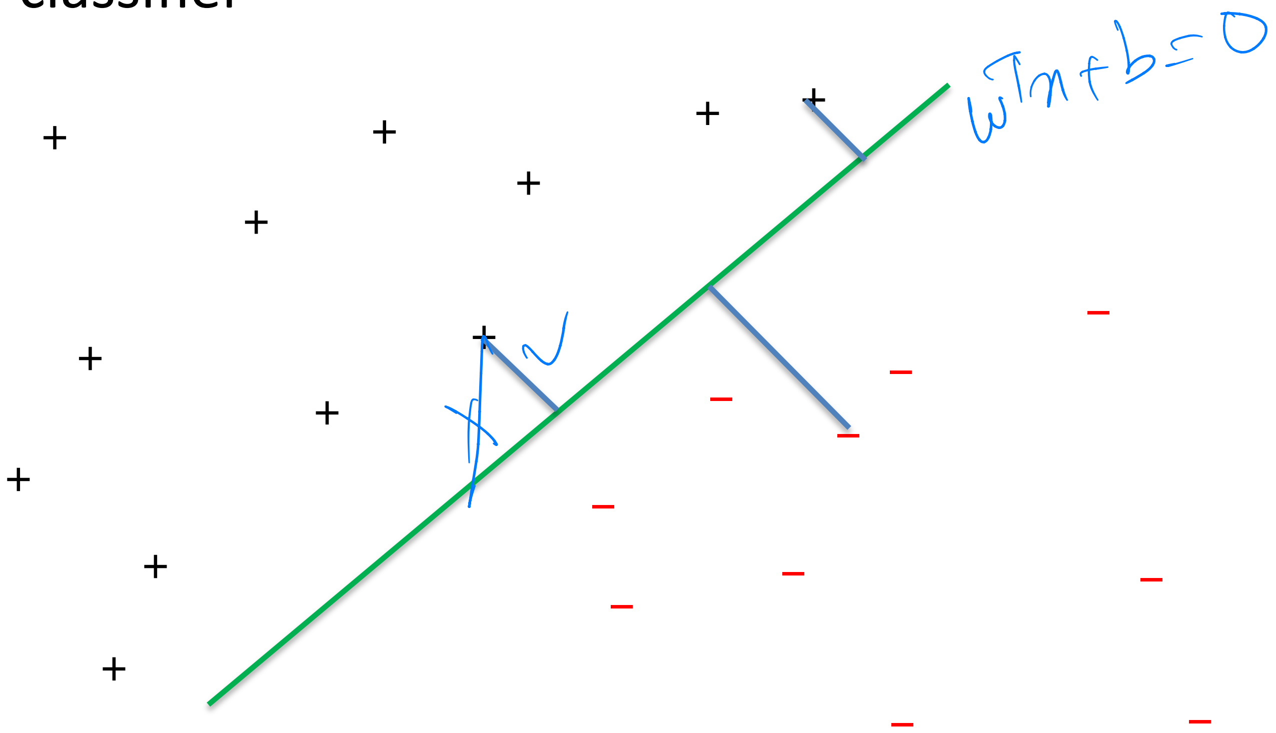
Support Vector Machines

- How can we decide between perfect classifiers?



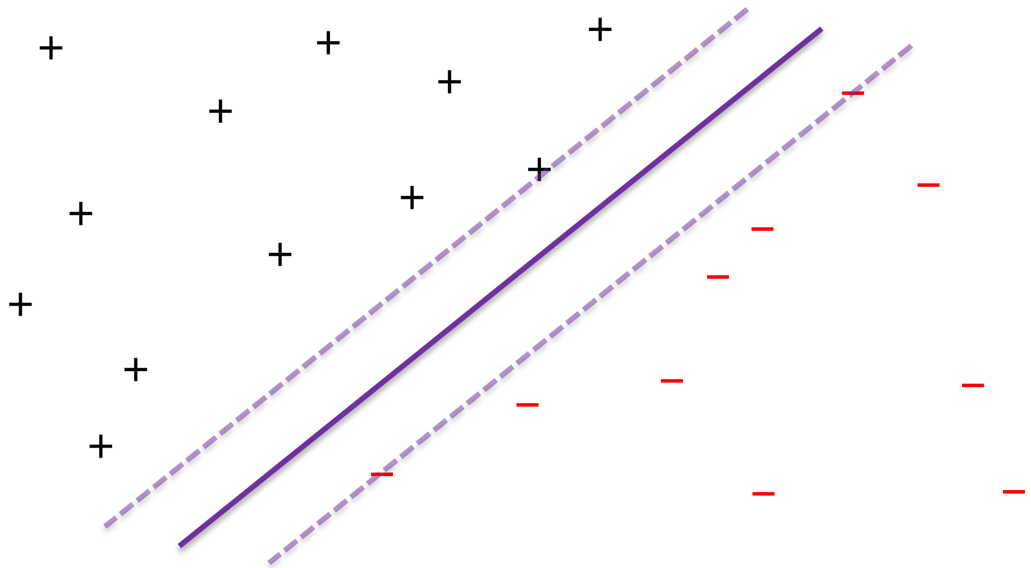
Support Vector Machines

- Define the margin to be the distance of the closest data point to the classifier



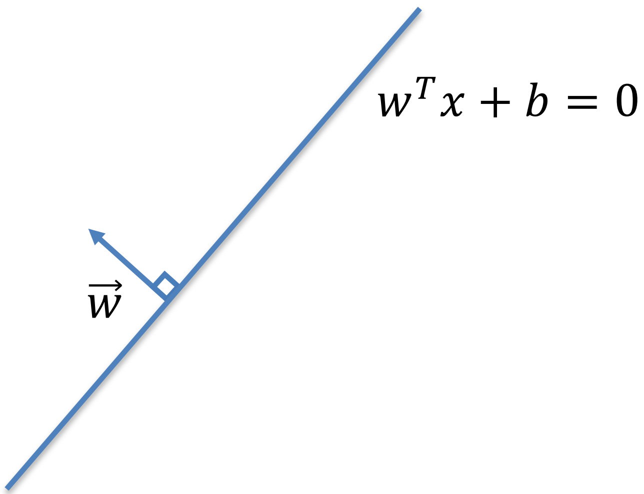
Support Vector Machines

- Support vector machines (SVMs)



- Choose the classifier with the largest margin
  - Has good practical and theoretical performance

Some Geometry



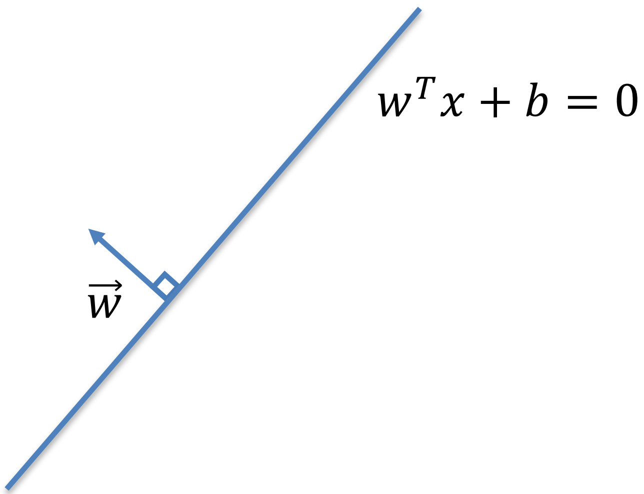
- In n dimensions, a hyperplane is a solution to the equation

$$w^T x + b = 0$$

with $w \in \mathbb{R}^n, b \in \mathbb{R}$

- The vector w is sometimes called the normal vector of the hyperplane

Some Geometry



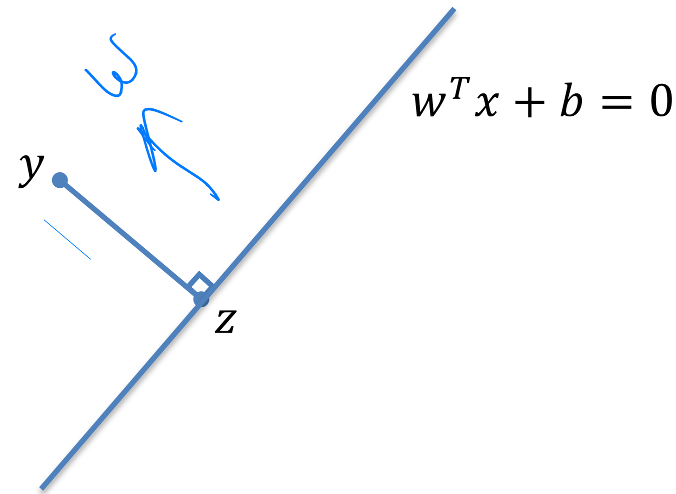
- In n dimensions, a hyperplane is a solution to the equation

$$w^T x + b = 0$$

- Note that this equation is scale invariant for any scalar c

$$c \cdot (w^T x + b) = 0$$

Some Geometry



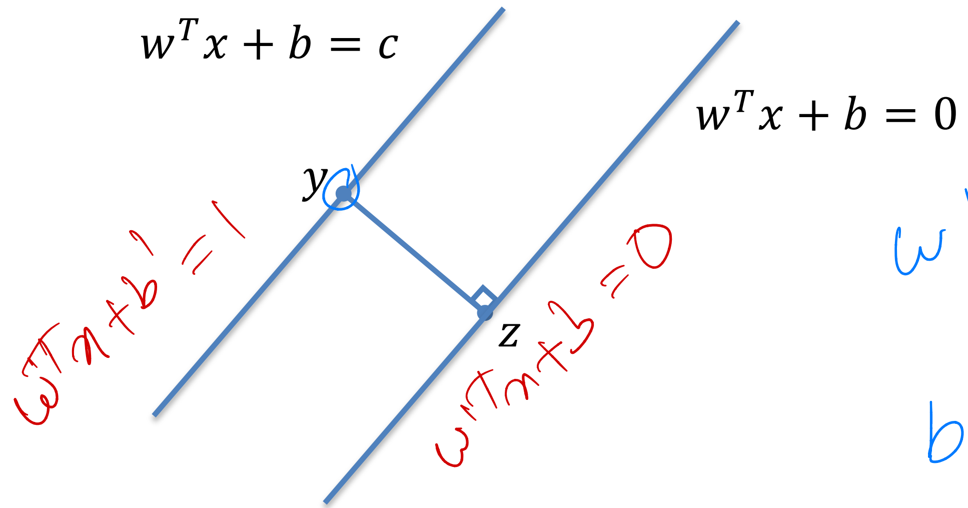
$$\frac{y-z}{\|y-z\|} = \frac{w}{\|w\|}$$

- The distance between a point y and a hyperplane $w^T + b = 0$ is the length of the segment perpendicular to the line to the point y
- The vector from y to z is given by

$$y - z = \|y - z\| \frac{w}{\|w\|}$$

Scale Invariance

$c = w^T y + b$

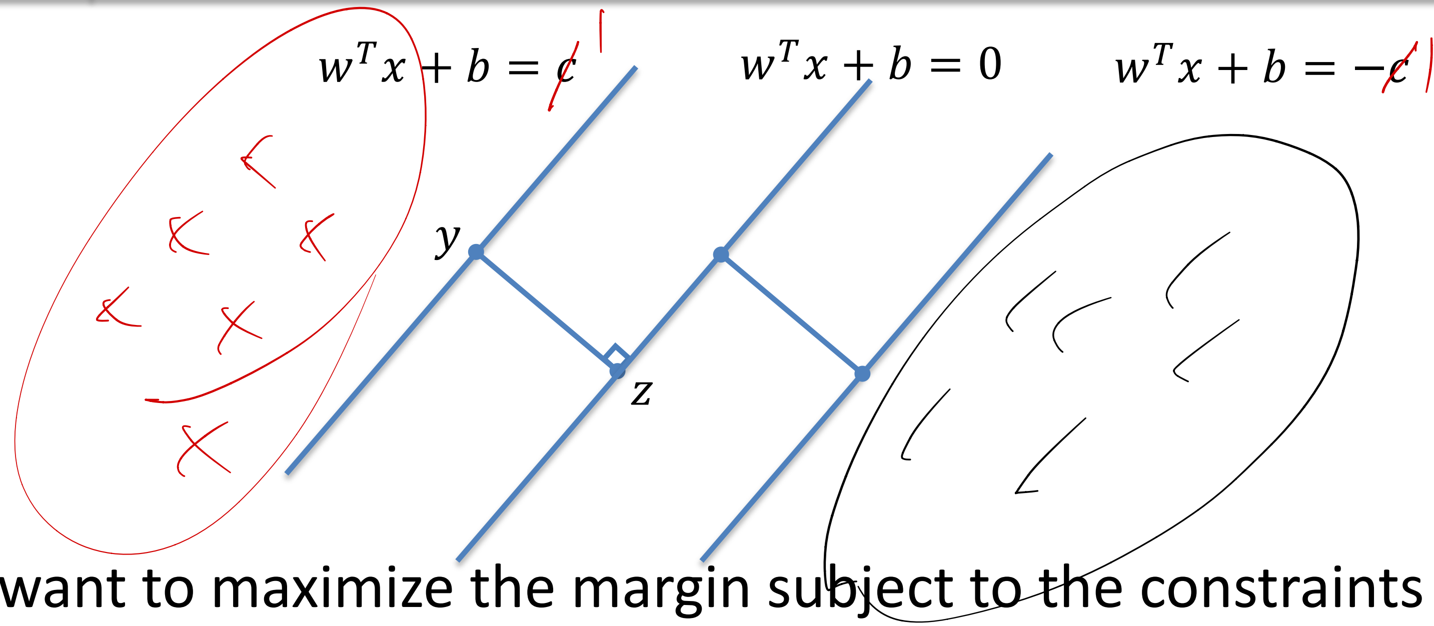


$w' = \frac{w}{w^T y + b}$

$b' = \frac{b}{w^T y + b}$

- By scale invariance, we can assume that $c = 1$
- The maximum margin is always attained by choosing $w^T x + b = 0$ so that it is equidistant from the closest data point classified as +1 and the closest data point classified as -1

Scale Invariance



- We want to maximize the margin subject to the constraints that

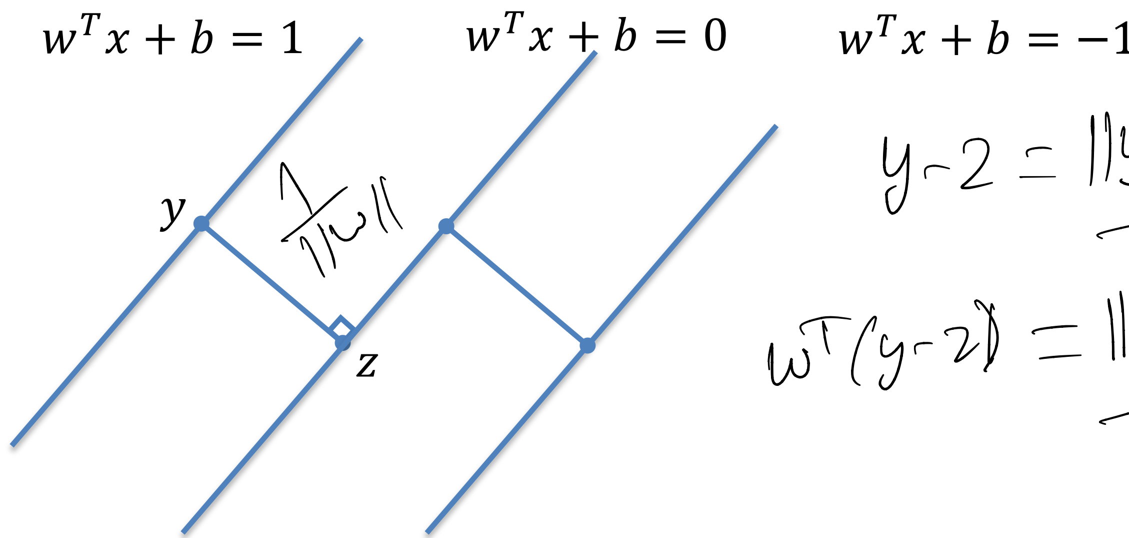
$$y^{(i)}\left(w^T x^{(i)} + b\right) \geq 1$$

- But how do we compute the size of the margin?

$w^T x^{(i)} + b \geq 1$

$w^T x^{(i)} + b \leq -1$

Some Geometry



$$y - z = \frac{\|y-z\| w}{\|w\|}$$

$$w^T(y-z) = \frac{\|y-z\| \|w\|^2}{\cancel{\|w\|}}$$

Putting it all together

$$y - z = \|y - z\| \frac{w}{\|w\|}$$

and

$$w^T y + b = 1$$
$$w^T z + b = 0$$

$$\Rightarrow$$

$$w^T(y - z) = 1$$

and

$$w^T(y - z) = \|y - z\| \|w\|$$

which gives

$$\|y - z\| = 1/\|w\|$$

SVMs

- This analysis yields the following optimization problem

$$\max_{w,b} \frac{1}{||w||}$$ ← margin

such that

$$y^{(i)}(w^T x^{(i)} + b) \geq 1, \text{for all } i$$ ← Constraints

- Or, equivalently,

$$\min_{w,b} ||w||^2$$

such that

$$y^{(i)}(w^T x^{(i)} + b) \geq 1, \text{for all } i$$

$$\max_{w,b} \frac{1}{||w||} \equiv \min_{w,b} ||w|| \equiv \min_{w,b} ||w||^2$$

SVMs

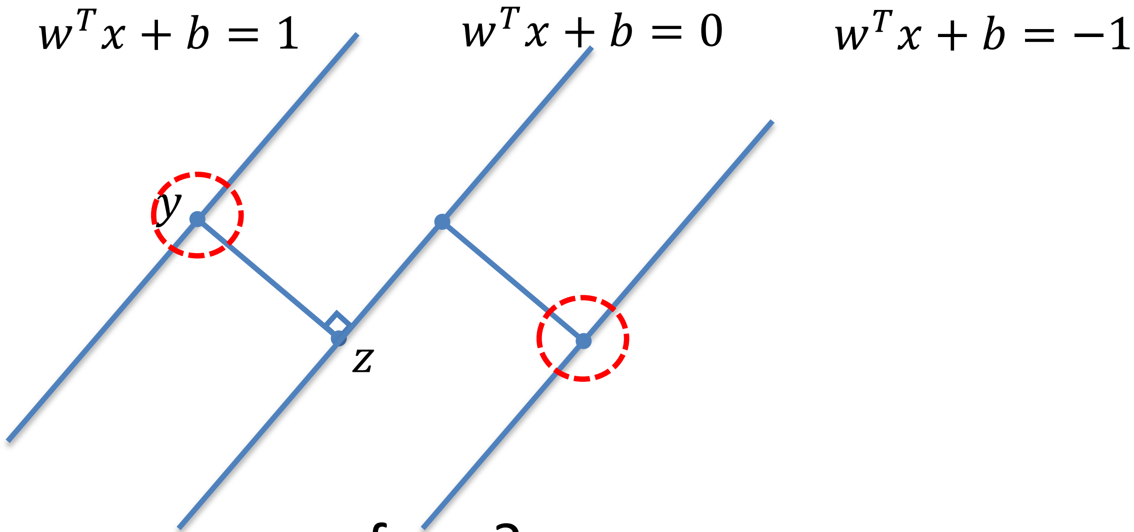
$$\min_{w,b} \|w\|^2$$

such that

$$y^{(i)}\left(w^T x^{(i)} + b\right) \geq 1, \text{for all } i$$

- This is a standard quadratic programming problem
  - Falls into the class of convex optimization problems
  - Can be solved with many specialized optimization tools (e.g., quadprog() in MATLAB)

SVMs



- Where does the name come from?
  - The set of all data points such that $y^{(i)}(w^T x^{(i)} + b) = 1$ are called support vectors
  - The SVM classifier is completely determined by the support vectors (you could delete the rest of the data and get the same answer)

SVMs

- What if the data isn't linearly separable?

  – Poly Features

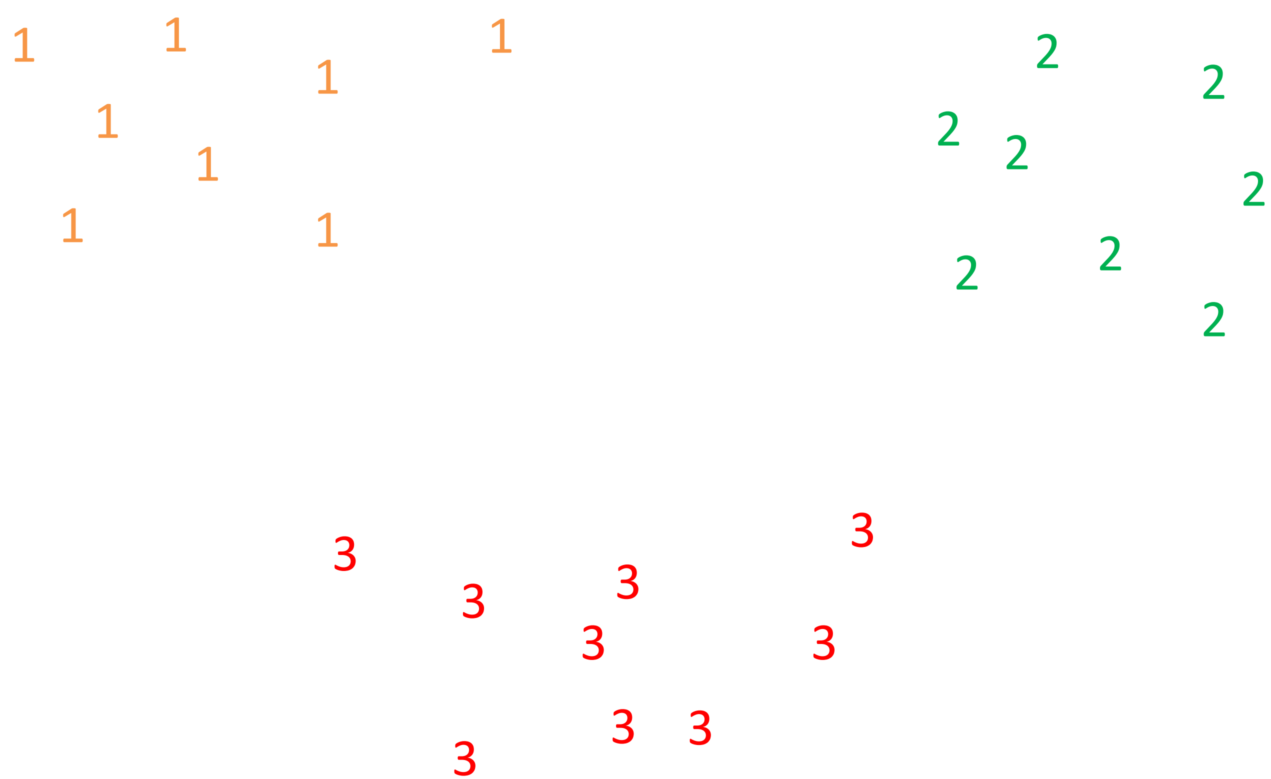
  – Extensions to handle non-sep [Lec 5]

- What if we want to do more than just binary classification (i.e., if $y \in \{1,2,3\}$)?

SVMs

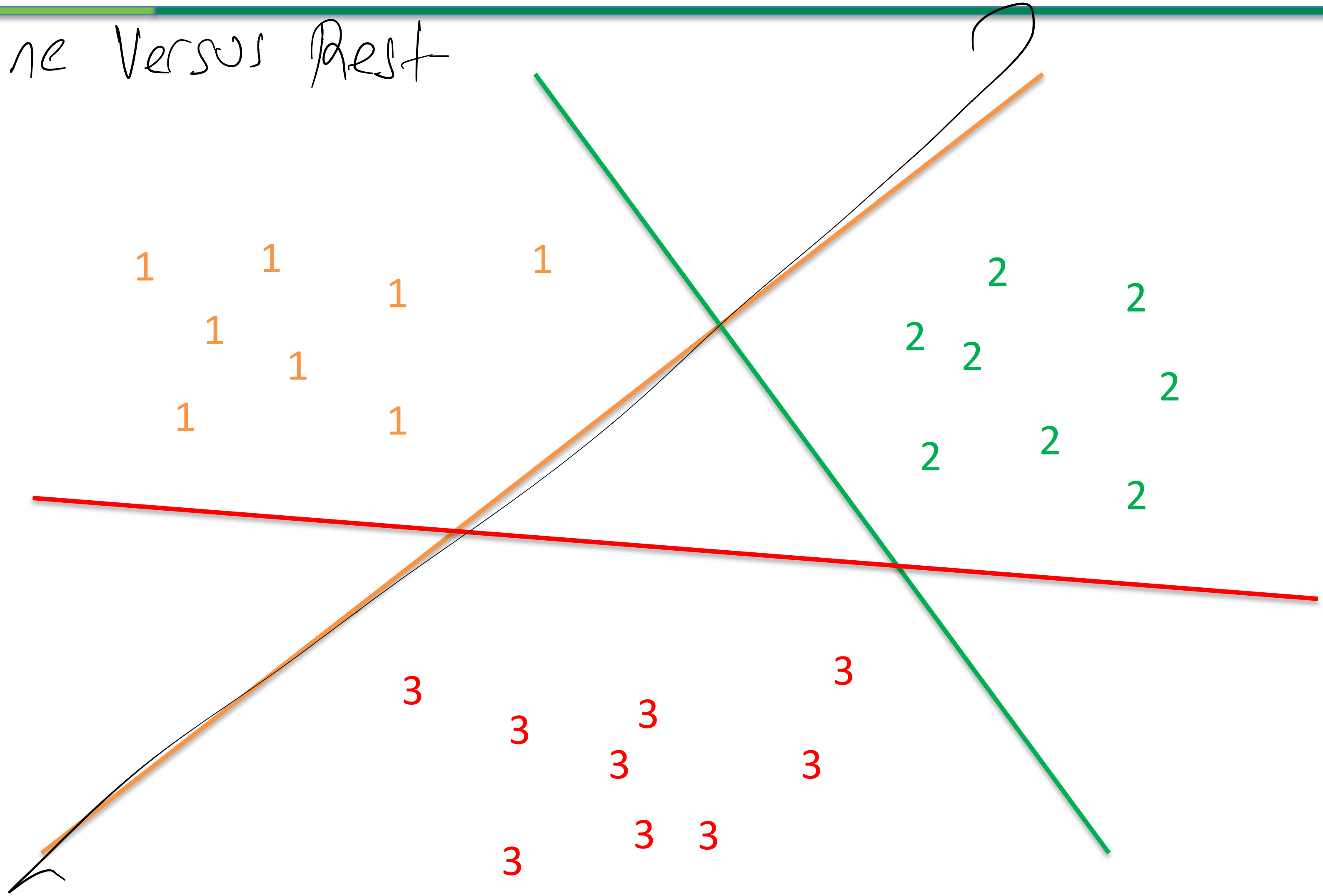
- What if the data isn't linearly separable?
  - Use feature vectors → Poly Features
  - Relax the constraints (coming soon)
- What if we want to do more than just binary classification (i.e., if $y \in \{1,2,3\}$)?

Multiclass Classification



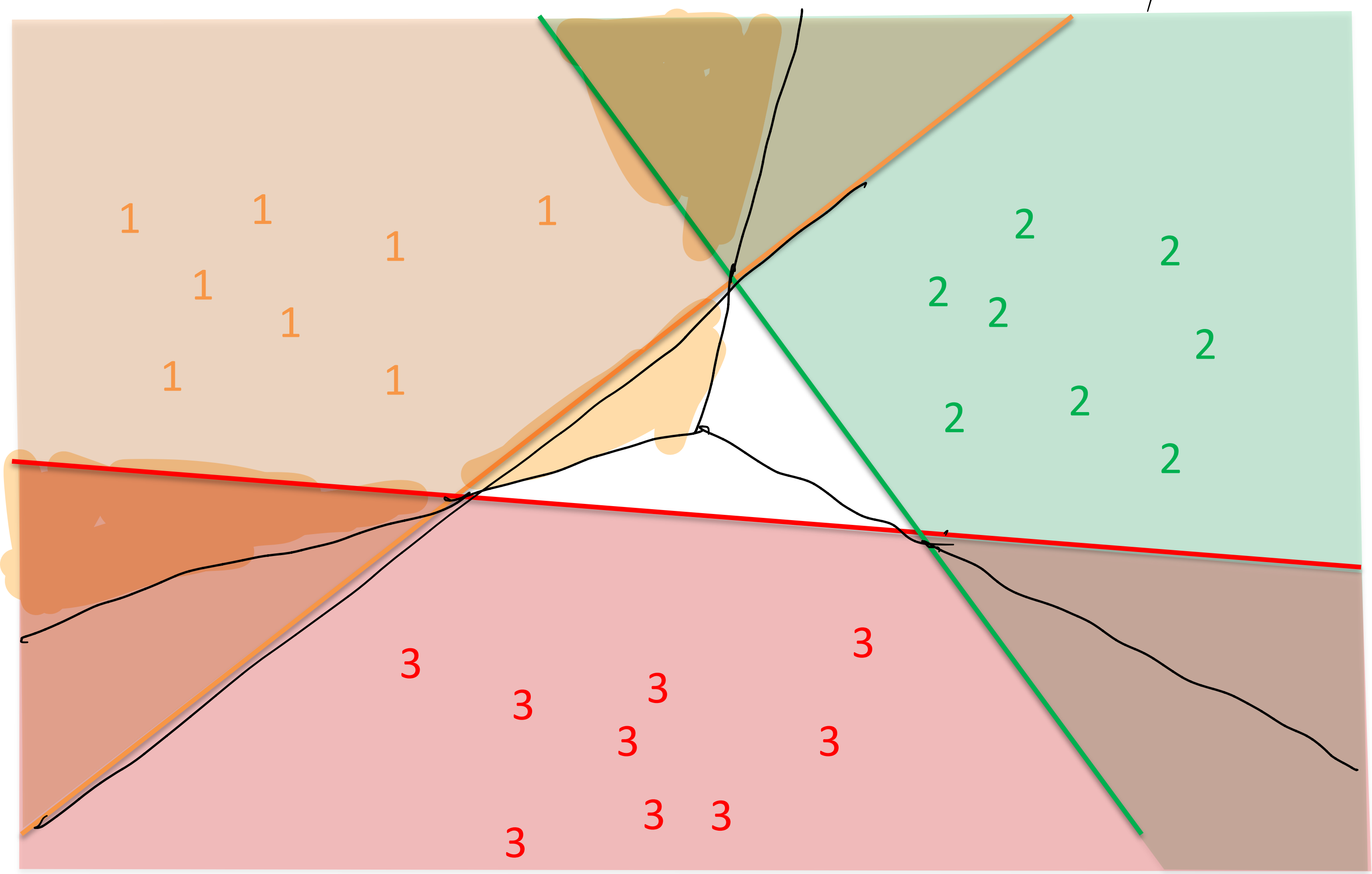
One-Versus-All SVMs

One Versus Rest



One-Versus-All SVMs

$w^{(1)}, b^{(1)}, w^{(2)}, b^{(2)}, w^{(3)}, b^{(3)}$



Regions correctly classified by exactly one classifier

Multi-Class Classification

Model 1: $w^{(1)T}x + b^{(1)}$

Model 2: $w^{(2)T}x + b^{(2)}$

Model 3: $w^{(3)T}x + b^{(3)}$

Test Example x_t

$$\underset{i}{\operatorname{argmax}} \; w^{(i)T}x_t + b^{(i)}$$

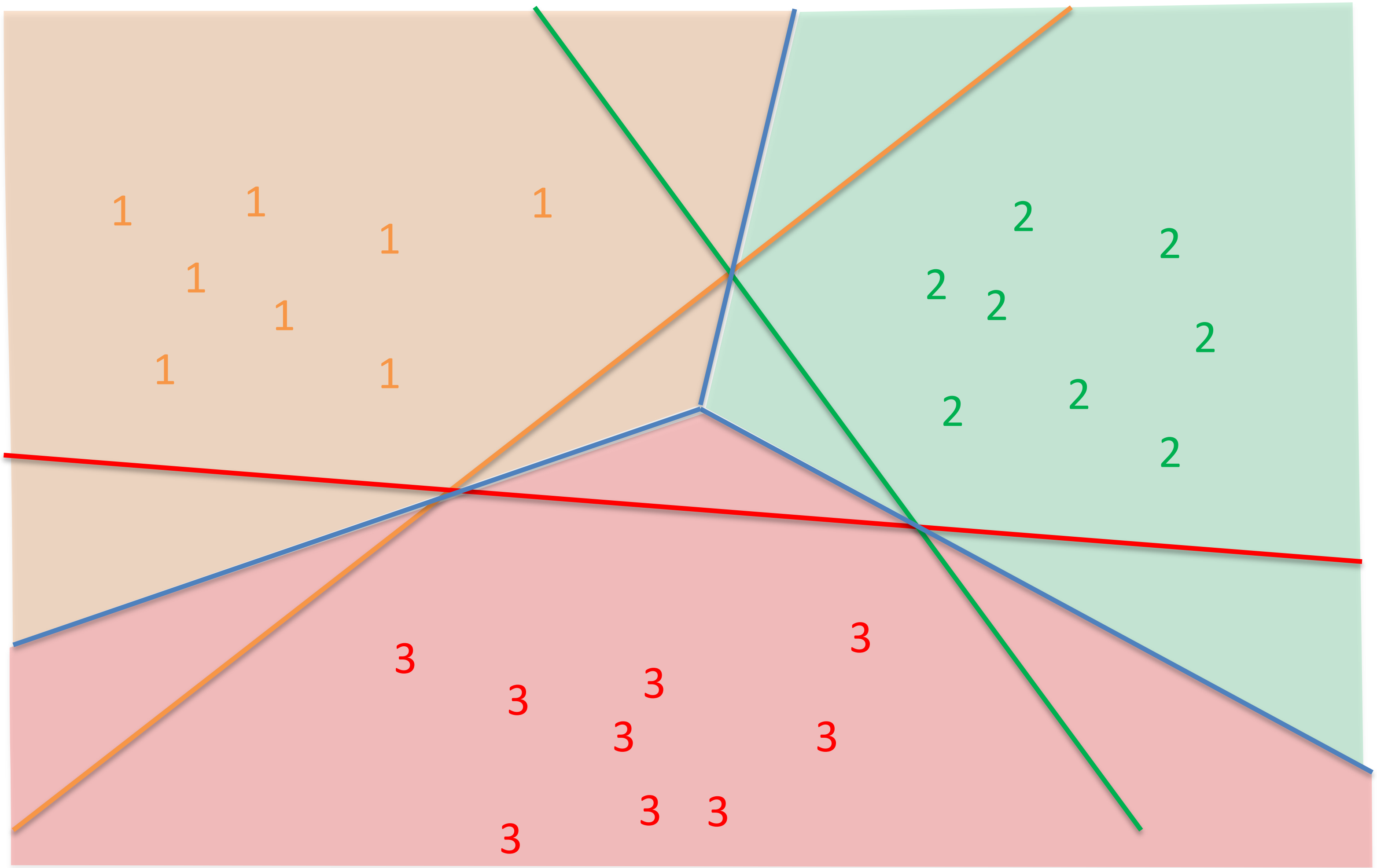
One-Versus-All SVMs

- Compute a classifier for each label versus the remaining labels (i.e., and SVM with the selected label as plus and the remaining labels changed to minuses)
- Let $f^k(x) = w^{(k)^T} x + b^{(k)}$ be the classifier for the k^{th} label
- For a new datapoint x, classify it as

$$k' \in \text{argmax}_k f^k(x)$$

- Drawbacks:
  - If there are L possible labels, requires learning L classifiers over the entire data set

One-Versus-All SVMs

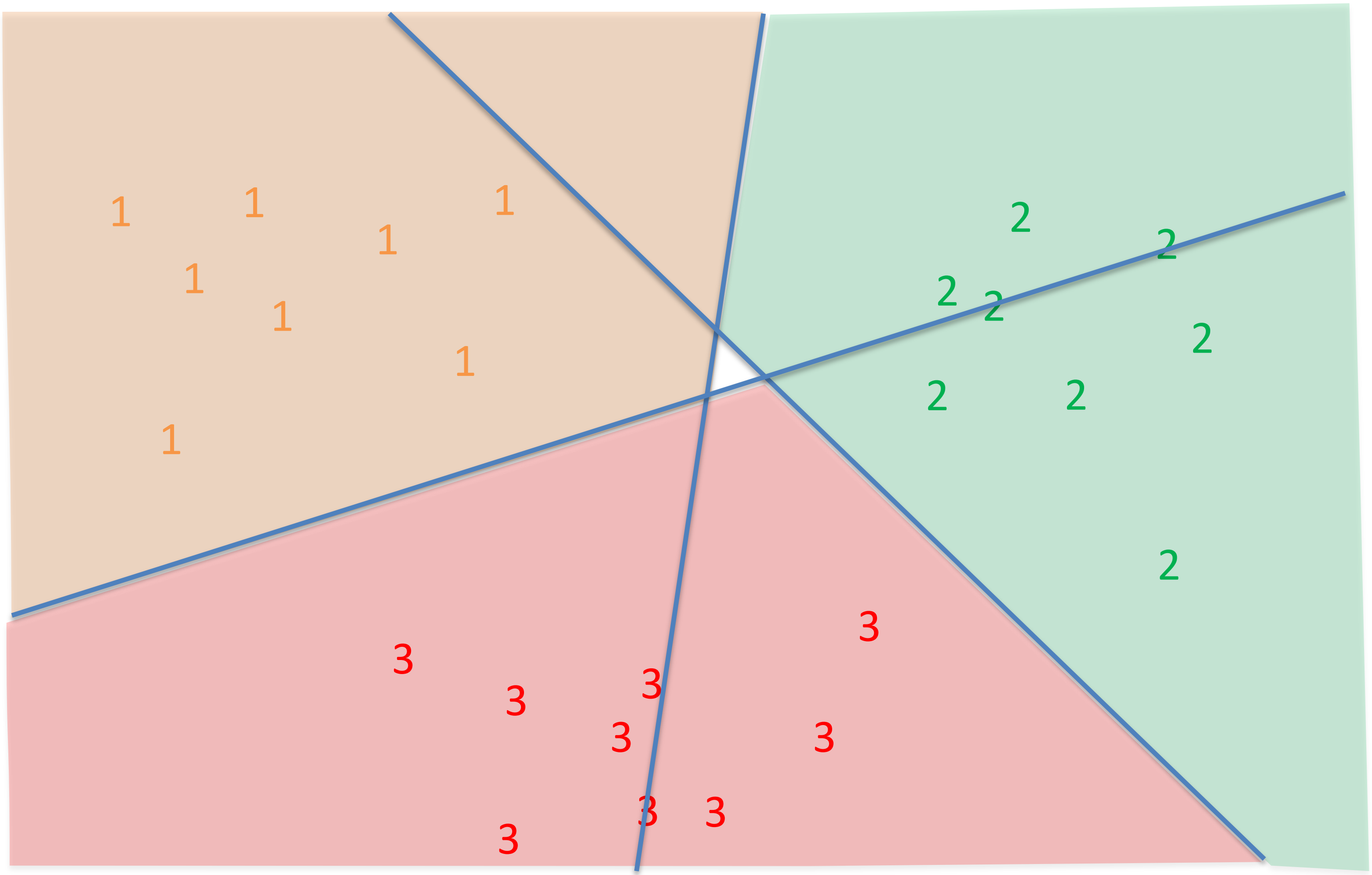


Regions in which points are classified by highest value of $w^T x + b$

One-Versus-One SVMs

- Alternative strategy is to construct a classifier for all possible pairs of labels
- Given a new data point, can classify it by majority vote (i.e., find the most common label among all of the possible classifiers)
- If there are L labels, requires computing $\binom{L}{2}$ different classifiers each of which uses only a fraction of the data
- Drawbacks: Can overfit if some pairs of labels do not have a significant amount of data (plus it can be computationally expensive)

One-Versus-One SVMs



Regions determined by majority vote over the classifiers