

HANDOUT #1. BIOELECTRIC CURRENTS AND VOLTAGES

A. Bioelectric currents

Ohm's Law states that the flow of current I through a conductor is proportional to the voltage (or electrical potential) difference V across the conductor, with a proportionality factor given by the conductance G of the conductor, where G is the reciprocal of the resistance R . Units of I , V , G and R are: amp (A), volt (V), siemens (S) and ohms (Ω), respectively.

$$I = GV = \frac{1}{R}V \quad (1)$$

Note that the usual context of (1) is a two-terminal circuit element, where current is flowing through a wire and voltage is developed across the two terminals. In an electrolyte occupying a volume, current can flow in three-dimensions, and voltage (we'll use the term electrical potential, or potential for short) is spatially distributed throughout the volume. The analog to Ohm's law is where current is carried by positively or negatively charged ions that are driven by an electric field \mathbf{E} , where \mathbf{E} is the negative of the spatial gradient of an electrical potential Φ .

$$\mathbf{J} = \sigma \mathbf{E} \quad (2)$$

For simplicity in this handout, all gradients in the system are considered to be along the x -direction, so that,

$$\mathbf{E} = -\frac{d\Phi}{dx} \quad (3)$$

Units of \mathbf{J} , \mathbf{E} , σ , and Φ are: amp/cm², volt/cm, siemens/cm, and volt, respectively. \mathbf{J} arising in this manner is referred to as drift current and has units of current density. Note that in these notes, a bolded variable indicates it is a vector quantity (having magnitude and direction), and a non-bolded variable indicates it is a scalar or the magnitude of the vector quantity.

σ is the conductivity of the solution, and can be dissected further by adopting a different starting point, namely that of ion flux (moles/cross-sectional area per unit time) instead of current density (amp/cross-sectional area, or coulomb/cross-sectional area per unit time). We know from basic physics that ions are driven by an external electric field \mathbf{E} acting on their charge (producing Lorentz forces), and hence positive and negative ions will move in opposite directions for the same field. If \mathbf{j} is defined to be the ion flux, then for a given ion,

$$j_{drift} = (\dots)\left(\frac{z}{|z|}\right)\left(-\frac{d\Phi}{dx}\right) \quad (4)$$

where z is the ion valence and the term in parentheses is yet to be determined. From fundamental principles, ion flux will also scale with the ion concentration. Inserting this information into (4) leaves us with a final residual term which we call u and whose dimension we can further deduce, given that we know the units of j (moles/cross-sectional area per unit time), c (moles/unit volume), and E (volts/unit distance).

$$\underbrace{j_{drift}}_{\frac{\text{mol/cm}^2}{\text{s}}} = \underbrace{u}_{\frac{\text{cm/s}}{\text{V/cm}}} \underbrace{(c)}_{\frac{\text{mol}}{\text{cm}^3}} \underbrace{\left(\frac{z}{|z|}\right)}_{\text{unitless}} \underbrace{\left(-\frac{d\Phi}{dx}\right)}_{\frac{\text{V}}{\text{cm}}} \quad (5)$$

u is the ion mobility, and tells us how fast the ion will move (cm/sec) when subjected to a potential gradient (V/cm); i.e., velocity per unit field.¹

Conversion of ion flux j to current density J is simply a matter of multiplying j by zF , where F is Faraday's constant (total charge per mole).

$$J_{drift} = zFj_{drift} = -u|z|Fc \frac{d\Phi}{dx} \quad (6)$$

Comparing (6) to (2) and (3), we can identify the conductivity to be,

$$\sigma = u|z|Fc \quad (7)$$

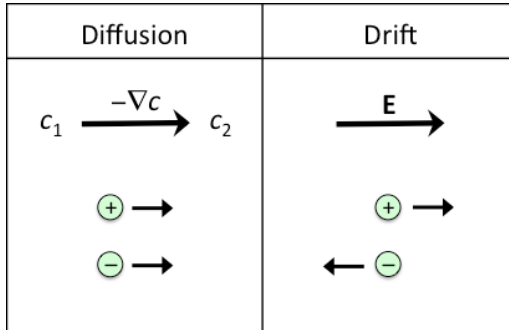
Thus, conductivity depends on the ionic mobility, ionic charge, and ionic concentration.

At this point, we acknowledge that ion flux can also occur by diffusion. Net ion movement is driven by a concentration gradient (Fick's law of diffusion), regardless of their charge. Here, we assume that the gradient is along the x -direction.

$$\underbrace{j_{diff}}_{\frac{\text{mol/cm}^2}{\text{s}}} = \underbrace{D}_{\frac{\text{cm}^2}{\text{s}}} \underbrace{\left(-\frac{dc}{dx}\right)}_{\frac{\text{mol/cm}^3}{\text{cm}}} \quad (8)$$

D is the diffusion coefficient (or diffusivity). Fig. 1 illustrates conceptually the difference between ion flux by diffusion and ion flux by drift.

Figure 1



Multiplying the molar flux j_{diff} by zF (total charge per mole) gives us the diffusional current density J_{diff} carried by the ion.

¹ At higher concentrations c should be replaced in (5) by the ionic activity.

$$J_{diff} = zFj_{diff} = -DzF \frac{dc}{dx} \quad (9)$$

Thus, the total molar flux and total current density are the sum of diffusion and drift components:

$$j = \underbrace{-D \frac{dc}{dx}}_{\text{diffusion}} - \underbrace{uc \frac{z}{|z|} \frac{d\Phi}{dx}}_{\text{drift}} \quad (10)$$

$$J = \underbrace{-DzF \frac{dc}{dx}}_{\text{diffusion}} - \underbrace{u|z|Fc \frac{d\Phi}{dx}}_{\text{drift}} \quad (11)$$

We can generalize (11) for **multiple ionic species**:

$$J_k = -D_k z_k F \frac{dc_k}{dx} - u_k |z_k| F c_k \frac{d\Phi}{dx} \quad (k=1, \dots, n) \quad (12)$$

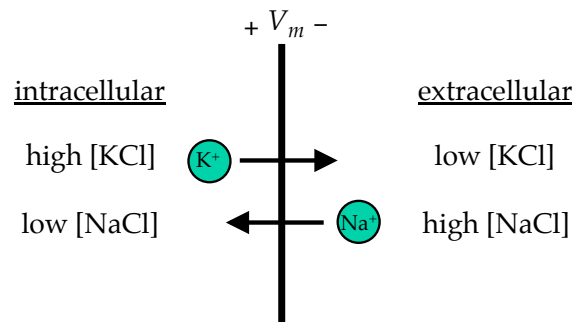
and even further, to **two or three-dimensions**:

$$J_k = -D_k z_k F \nabla c_k - u_k |z_k| F c_k \nabla \Phi \quad (13)$$

B. Bioelectric voltage sources

We now consider the genesis of bioelectrical potentials. For illustration, let's consider a cellular membrane in which ionic concentrations are different on both sides, and ion fluxes occur only in one dimension (along x).

Figure 2

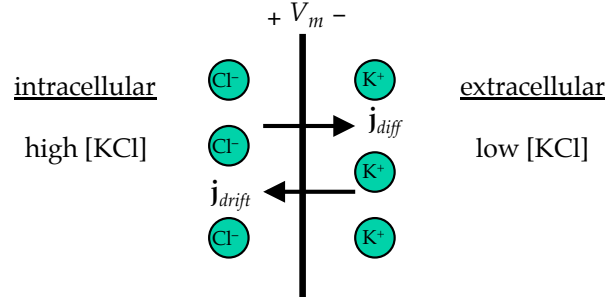


The transmembrane potential V_m will depend on the permeability of the membrane to the various ionic species. If the membrane is permeable only to K^+ , then V_m is typically around -72 mV for nerve axon. On the other hand, if it is permeable only to Na^+ , then V_m is around +55 mV. And if permeable to both K^+ and Na^+ , then V_m will be somewhere in between the two extreme values. How does this come about?

We can intuitively understand the genesis for V_m in the case of single ion permeability, say to K^+ . By Fick's law, K^+ will flow across the membrane from left to right, but negative Cl^- counterions will be left behind because they cannot cross the membrane. Thus, a charge separation will develop, which in turn generates a growing electric field oriented in a

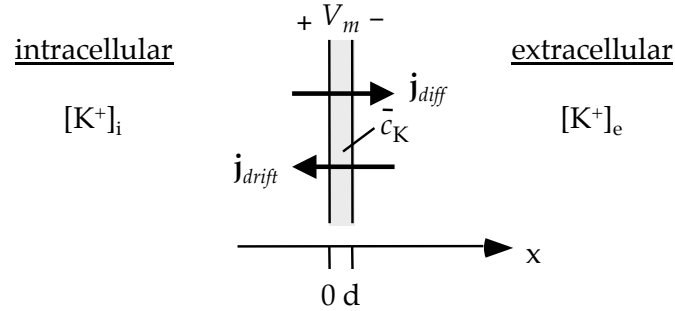
direction opposite to the concentration gradient. Thus, the diffusional flux (j_{diff}) will be counteracted by an electric field-mediated drift (j_{drift}), until the two become equal, and equilibrium is reached.² V_m is then the Nernst potential for K^+ .

Figure 3



We can now translate the descriptive arguments into a mathematical representation. Let us consider the cell membrane to be uniform and planar of infinite extent (in the directions orthogonal to x), with thickness d , having an *intramembrane* concentration \bar{c}_K of K^+ , and concentrations of $[K^+]_i$ and $[K^+]_e$ in the bulk solutions.

Figure 3



It is commonly assumed that \bar{c}_K at the surfaces of the membrane is proportional to the bulk concentrations $[K^+]_i$ and $[K^+]_e$ by a factor called the partition coefficient β :

$$\bar{c}_K(0) = \beta_K [K^+]_i \quad (14a)$$

$$\bar{c}_K(d) = \beta_K [K^+]_e \quad (14b)$$

As we will see, β_K will cancel out in the end, so we could have made the simpler (although incorrect) assumption that the c_K s at the surfaces of the membrane are equal to the those in the bulk.

In general, \bar{c}_K and Φ may be nonlinear functions of x . Since j is considered to flow normal to the plane of the membrane, we can apply (10) to the interior of the membrane ($0 < x < d$) with $z = 1$, where the net flux across the membrane is zero when equilibrium is reached.

² This is the Donnan equilibrium.

$$j_K = \underbrace{-D_K \frac{d\bar{c}_K}{dx}}_{\text{diffusion}} - \underbrace{u_K \bar{c}_K \frac{d\Phi}{dx}}_{\text{drift}} = 0 \quad (15a)$$

$$\Rightarrow \underbrace{D_K \frac{d\bar{c}_K}{dx}}_{\text{diffusion}} = \underbrace{-u_K \bar{c}_K \frac{d\Phi}{dx}}_{\text{drift}} \quad (15b)$$

The Einstein relation relates the diffusion coefficient of an ion to its mobility:

$$D = \frac{uRT}{|z|F} \quad (16)$$

Inserting (16) into (15b) with $z = 1$ gives:

$$\underbrace{\frac{u_K RT}{F} \frac{d\bar{c}_K}{dx}}_{\text{diffusion}} = \underbrace{-u_K \bar{c}_K \frac{d\Phi}{dx}}_{\text{drift}} \quad (17)$$

from which,

$$\frac{1}{\bar{c}_K} \frac{d\bar{c}_K}{dx} = -\frac{F}{RT} \frac{d\Phi}{dx} \quad (18)$$

Integrating (18) over x from 0 to d ,

$$\ln \frac{\bar{c}_K(d)}{\bar{c}_K(0)} = \frac{FV_m}{RT} \quad (19a)$$

$$V_m = \frac{RT}{F} \ln \frac{\bar{c}_K(d)}{\bar{c}_K(0)} \quad (19b)$$

where V_m is the transmembrane potential.³ Applying (14) to (19b), we finally obtain the *Nernst equation*,

$$V_m = \frac{RT}{F} \ln \frac{[K]_e}{[K]_i} \quad (20)$$

RT/F is 26.7 mV at physiological temperatures (37°C), and 25 mV at room temperature (20°C). If (20) is written as \log_{10} instead of \ln , then RT/F is replaced by $2.3RT/F$ and has values of 61 and 58 mV at 37°C and 20°C, respectively.

If the membrane is instead exclusively permeable to Na^+ , then (19) becomes the Nernst potential for Na^+ , around +55 mV for nerve axon. For permeability to both K^+ and Na^+ , one can write the individual fluxes for K^+ and Na^+ from (10):

$$j_K = -D_K \frac{d\bar{c}_K}{dx} - u_K \bar{c}_K \frac{d\Phi}{dx} \quad (21a)$$

$$j_{\text{Na}} = -D_{\text{Na}} \frac{d\bar{c}_{\text{Na}}}{dx} - u_{\text{Na}} \bar{c}_{\text{Na}} \frac{d\Phi}{dx} \quad (21b)$$

³ In obtaining (19b), were any assumptions made regarding the functional dependence of \bar{c}_K or Φ on x ?

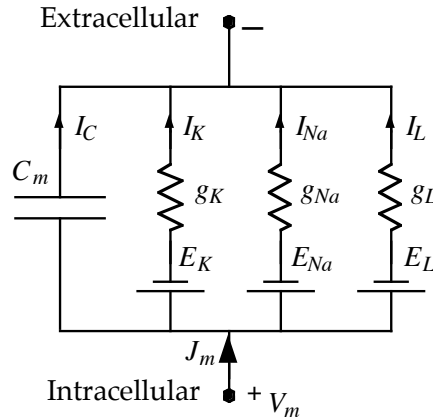
However, unlike before for the Donnan equilibrium, neither j_K nor j_{Na} is zero. Fortunately, it is possible to solve each equation in closed form if one assumes a **constant field** within the membrane. The derivation can be found in the **Appendix**, and the result is an expression for V_m known as the Goldman-Hodgin-Katz equation.

$$V_m = \frac{RT}{F} \ln \frac{P_K[K]_e + P_{Na}[Na]_e}{P_K[K]_i + P_{Na}[Na]_i} \quad (22)$$

where P_K and P_{Na} are the membrane permeabilities for K^+ and Na^+ , respectively, as defined in the Appendix. (22) can be expanded to include more ionic species (such as Cl^-), which will introduce additional terms in the numerator and denominator of the right-hand side. As expected, we see that (22) reduces to the Nernst potential for K^+ (19) when $P_K \gg P_{Na}$, and to the Nernst potential for Na^+ when $P_K \ll P_{Na}$. Intermediate weights of P_K and P_{Na} will result in intermediate values of V_m .

An alternate expression for V_m which does not rely on a constant field assumption is the 1952 Hodgkin-Huxley (HH) formulation of an excitable membrane. The equivalent circuit for nerve axon is shown in Fig. 4.

Figure 4



Each pathway for ionic current is represented by an equivalent circuit consisting of a time-varying resistance (conductance g) in series with a battery having a strength equal to the Nernst potential for that ionic species. In steady-state,

$$V_m = \frac{g_K}{g_K + g_{Na} + g_L} \frac{RT}{F} \ln \frac{[K]_e}{[K]_i} + \frac{g_{Na}}{g_K + g_{Na} + g_L} \frac{RT}{F} \ln \frac{[Na]_e}{[Na]_i} + \frac{g_L}{g_K + g_{Na} + g_L} \frac{RT}{F} \ln \frac{[L]_e}{[L]_i} \quad (23)$$

As expected, when $g_K \gg g_{Na}$ and g_L , V_m approaches the Nernst potential for K^+ . Similarly, V_m approaches the Nernst potential for Na^+ when $g_{Na} \gg g_K$ and g_L .

Appendix

A constant field within the membrane is represented as,

$$\frac{d\Phi}{dx} = -\frac{V_m}{d} \quad (24)$$

Under this condition, the solutions to (20) become,

$$j_K = -\frac{D_K F V_m}{RT d} \frac{\bar{c}_K(d) - \bar{c}_K(0) e^{-F V_m / RT}}{1 - e^{-F V_m / RT}} \quad (25a)$$

$$j_{Na} = -\frac{D_{Na} F V_m}{RT d} \frac{\bar{c}_{Na}(d) - \bar{c}_{Na}(0) e^{-F V_m / RT}}{1 - e^{-F V_m / RT}} \quad (25b)$$

Again, taking into account the partition coefficients for K^+ and Na^+ at the membrane, and multiplying (25) by zF to get the membrane currents, (24) becomes,

$$J_K = -\frac{P_K F^2 V_m}{RT} \frac{[K]_e - [K]_i e^{-F V_m / RT}}{1 - e^{-F V_m / RT}} \quad (26a)$$

$$J_{Na} = -\frac{P_{Na} F^2 V_m}{RT} \frac{[Na]_e - [Na]_i e^{-F V_m / RT}}{1 - e^{-F V_m / RT}} \quad (26b)$$

where the permeabilities P_K and P_{Na} are defined as,

$$P_K = \frac{D_K \beta_K}{d} \quad (27a)$$

$$P_{Na} = \frac{D_{Na} \beta_{Na}}{d} \quad (27b)$$

Finally, at steady-state (quasi-equilibrium) the sum of the two ionic currents must be zero,

$$J_K + J_{Na} = 0 \quad (28)$$

which after some manipulation results in,

$$V_m = \frac{RT}{F} \ln \frac{P_K [K]_e + P_{Na} [Na]_e}{P_K [K]_i + P_{Na} [Na]_i} \quad (29)$$