First half of linear algebra : solving systems of linear equations
Second half " : factoring matrices

Solving systems of linear equations :

$\mathbb{R}$ = set of real numbers

Find all solutions $(w,x,y,z) \in \mathbb{R}^4$ to the system of linear equations

$$\begin{aligned} 2w + 3x + 4y - z &= 1 \\ x + y + z &= 2 \\ 2w + \quad + 2y + 2z &= 4 \end{aligned}$$

Key idea: we can repeatedly apply the following three operations, to simplify the given equations, in a way that does not change the set of solutions :

3 Elementary row operations :

E 1. multiply any equation by a nonzero number (called a scalar)
E 2. interchange the positions of any two equations
E 3. replace an equation by adding to it a scalar multiple of another equation

Let's start with a simpler example to illustrate :

Find all solutions $(x,y) \in \mathbb{R}^2$ to

$$\begin{aligned} 3x + 2y &= 7 \\ 2x + 4y &= 8 \end{aligned}$$

3 Elementary row operations:

E 1. multiply any equation by a nonzero number (called a scalar)

E 2. interchange the positions of any two equations

E 3. replace an equation by adding to it a scalar multiple of another equation

(not a standard numbering so don't memorize the numbers)

There are many different orders in which these row ops can be applied, but eventually end result (reduced row echelon form) will be identical.

explained shortly

Given:

$$3x + 2y = 7$$
$$2x + 4y = 8$$

$$\tfrac{1}{2}(2x+4y) = \tfrac{1}{2}(8)$$
$$1x + 2y = 4$$

(E1)

$$3x + 2y = 7$$
$$1 \cdot x + 2y = 4$$

pivot 1

(E2)

$$1 \cdot x + 2y = 4$$
$$3x + 2y = 7$$

$$-3(x+2y) = -3 \cdot 4 \rightarrow -3x - 6y = -12$$

$$\begin{array}{r} 3x + 2y = 7 \\ \hline 0x - 4y = -5 \end{array}$$

(add)

(E3)

$$x + 2y = 4$$
$$-4y = -5$$

$$\tfrac{1}{-4}(-4y) = \tfrac{1}{-4}(-5)$$

(E1)

$$x + 2y = 4$$
$$1 \cdot y = \tfrac{5}{4}$$

another pivot 1

$$-2y = -2 \cdot \tfrac{5}{4} \rightarrow -2y = -\tfrac{10}{4}$$

$$\begin{array}{r} x + 2y = 4 \\ \hline x + 0y = \tfrac{6}{4} = \tfrac{3}{2} \end{array}$$

add

(E 3)

$$x = 3/2$$
$$y = \tfrac{5}{4}$$

Done! $(x,y) = (3/2, 5/4)$ is only solution.

Use matrices to avoid writing x & y's & = all the time

$$3x + 2y = 7$$
$$2x + 4y = 8$$

$\downarrow$ "augmented matrix"

$$\left[\begin{array}{cc|c} 3 & 2 & 7 \\ 2 & 4 & 8 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 3 & 2 & 7 \\ 1 & 2 & 4 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & 2 & 4 \\ 3 & 2 & 7 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & 2 & 4 \\ 0 & -4 & -5 \end{array}\right]$$

$$\begin{array}{cc|c} -3 & -6 & -12 \\ 3 & 2 & 7 \\ \hline 0 & -4 & -5 \end{array}$$

$$\left[\begin{array}{cc|c} 1 & 2 & 4 \\ 0 & 1 & 5/4 \end{array}\right]$$

<—This matrix is in row echelon form
but not reduced row echelon form because of the 2

$$\begin{array}{cc|c} 0 & -2 & -10/4 \\ 1 & 2 & 4 \\ \hline 1 & 0 & 3/2 \end{array}$$

$$\left[\begin{array}{cc|c} 1 & 0 & 3/2 \\ 0 & 1 & 5/4 \end{array}\right]$$

<—This matrix is in reduced row echelon form

$$x = 3/2$$
$$y = 5/4$$

is only solution!

Here is what it means for a matrix to be in row echelon form:

$$\begin{bmatrix} 0 & 1 & 2 & 4 & -2 \\ 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

① First nonzero entry in each row is a 1

② If there are rows consisting entirely of zeroes, we move them to the bottom of the matrix

③ In any two successive rows that do not consist entirely of zeroes the leading 1 in the lower row occurs further to the right than the leading 1 in the higher row.

A matrix is in reduced row echelon form

$$\begin{bmatrix} 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

④ if in addition to conditions 1,2,3, each column that contains a leading 1 has zeroes everywhere else in the column.

Gaussian elimination is this recipe for starting with any matrix & getting it into row echelon form by doing elementary row operations

Gauss-Jordan is recipe to get reduced row echelon form.

$\mathbb{R}$ = set of real numbers

Find all solutions $(w,x,y,z) \in \mathbb{R}^4$ to the system of linear equations

$$2w + 3x + 4y - z = 1$$
$$x + y + z = 2$$
$$2w + \quad + 2y + 2z = 4$$

Solution: First form augmented matrix $\left[\begin{array}{cccc|c} 2 & 3 & 4 & -1 & 1 \\ 0 & 1 & 1 & 1 & 2 \\ 2 & 0 & 2 & 2 & 4 \end{array}\right]$

goal: get to a matrix in reduced row ech form,

for example $\left[\begin{array}{cccc|c} 1 & 0 & 0 & * & * \\ 0 & 1 & 0 & * & * \\ 0 & 0 & 1 & * & * \end{array}\right]$

using Elementary row ops:

E1: scale a row (by nonzero)

E2: interchange two rows

E3: replace a row by adding to it a multiple of another

These do not change the solutions (w,x,y,z)!

E1 $\left[\begin{array}{cccc|c} 2 & 3 & 4 & -1 & 1 \\ 0 & 1 & 1 & 1 & 2 \\ 1 & 0 & 1 & 1 & 2 \end{array}\right]$

E2 $\left[\begin{array}{cccc|c} 1 & 0 & 1 & 1 & 2 \\ 0 & 1 & 1 & 1 & 2 \\ 2 & 3 & 4 & -1 & 1 \end{array}\right]$

$(-2)R1$

$$\begin{array}{rrrr|r} -2 & 0 & -2 & -2 & -4 \\ 2 & 3 & 4 & -1 & 1 \\ \hline 0 & 3 & 2 & -3 & -3 \end{array}$$

R3

E3 $\left[\begin{array}{cccc|c} 1 & 0 & 1 & 1 & 2 \\ 0 & 1 & 1 & 1 & 2 \\ 0 & 3 & 2 & -3 & -3 \end{array}\right]$

(no changes, just copied from previous slide)

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 1 & 2 \\ 0 & 1 & 1 & 1 & 2 \\ 0 & 3 & 2 & -3 & -3 \end{array}\right]$$

$\xrightarrow{(-3)R2}$

add

$$\begin{array}{ccccc} 0 & -3 & -3 & -3 & -6 \\ 0 & 3 & 2 & -3 & -3 \\ \hline 0 & 0 & -1 & -6 & -9 \end{array}$$

E3

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 1 & 2 \\ 0 & 1 & 1 & 1 & 2 \\ 0 & 0 & -1 & -6 & -9 \end{array}\right]$$

E3

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & -5 & -7 \\ 0 & 1 & 1 & 1 & 2 \\ 0 & 0 & -1 & -6 & -9 \end{array}\right]$$

R1 replaced by adding to it R3

E3

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & -5 & -7 \\ 0 & 1 & 0 & -5 & -7 \\ 0 & 0 & -1 & -6 & -9 \end{array}\right]$$

R2 replaced by adding to it R3

E1

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & -5 & -7 \\ 0 & 1 & 0 & -5 & -7 \\ 0 & 0 & 1 & 6 & 9 \end{array}\right]$$

now matrix is in <u>reduced</u> <u>row echelon</u> <u>form</u>

and we can easily find the answer

$$w - 5z = -7$$
$$x + -5z = -7$$
$$y + 6z = 9$$

$$\begin{aligned} w \quad\quad\quad\quad\quad - 5z &= -7 \\ x \quad\quad\quad - 5z &= -7 \\ y + 6z &= 9 \end{aligned}$$

Answer:
$$\boxed{\begin{aligned} w &= -7 + 5z \\ x &= -7 + 5z \\ y &= 9 - 6z \\ z &= z \end{aligned}}$$

$$\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -7 \\ -7 \\ 9 \\ 0 \end{bmatrix} + z \cdot \begin{bmatrix} 5 \\ 5 \\ -6 \\ 1 \end{bmatrix}$$

z is a "free variable", it can be anything, and then all solutions (w, x, y, z) are given by boxed answer above

(For example if $z = 0$, we get a solution $(-7, -7, 9, 0)$
if $z = 2$ we get a solution $(3, 3, -3, 2)$)

$-7 + 5 \cdot (2)$

Exercise 1.1.10 Find the solution of each of the following systems of linear equations using augmented matrices.

a.
$$\begin{aligned} x + y + 2z &= -1 \\ 2x + y + 3z &= 0 \\ -2y + z &= 2 \end{aligned}$$

b.
$$\begin{aligned} 2x + y + z &= -1 \\ x + 2y + z &= 0 \\ 3x \quad - 2z &= 5 \end{aligned}$$

a)
$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & -1 \\ 2 & 1 & 3 & 0 \\ 0 & -2 & 1 & 2 \end{array}\right]$$

Augmented matrix

$$\begin{array}{cccc|l} 2 & 1 & 3 & 0 & R2 \\ -2 & -2 & -4 & 2 & -2R1 \\ \hline 0 & -1 & -1 & 2 & \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & -1 \\ 0 & -1 & -1 & 2 \\ 0 & -2 & 1 & 2 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & -1 \\ 0 & 1 & 1 & -2 \\ 0 & -2 & 1 & 2 \end{array}\right] \begin{array}{l} \\ R2 \\ \\ \end{array}$$

$$\begin{array}{cccc|l} 0 & -2 & 1 & 2 & R3 \\ 0 & 2 & 2 & -4 & 2R2 \\ \hline 0 & 0 & 3 & -2 & \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & -1 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 3 & -2 \end{array}\right]$$

$$\begin{array}{cccc|l} 1 & 1 & 2 & -1 & R1 \\ 0 & -1 & -1 & 2 & -1R2 \\ \hline 1 & 0 & 1 & 1 & \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 3 & -2 \end{array}\right]$$

done w/ 1st & 2nd column

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 1 & -2/3 \end{array}\right] \longrightarrow$$

row echelon form

$z = -2/3$ R3

$y + z = -2$ R2

back substitutio

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 5/3 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 1 & -2/3 \end{array}\right]$$

Replace R1 by adding to it $-1 \cdot R3$

1	0	1	1	R1
0	0	-1	2/3	$-1 \cdot R3$
1	0	0	5/3	

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 5/3 \\ 0 & 1 & 0 & -4/3 \\ 0 & 0 & 1 & -2/3 \end{array}\right]$$

Replace R2 by adding to it $-1 \cdot R3$

0	1	1	-2	R2
0	0	-1	2/3	$-1 \cdot R3$
0	1	0	-4/3	

reduced row echelon form

rref

translates to equations

$$\boxed{\begin{array}{lll} x & & = 5/3 \\ & y & = -4/3 \\ & \quad z & = -2/3 \end{array}}$$

$$(x, y, z) = \left(5/3, -4/3, -2/3\right)$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5/3 \\ -4/3 \\ -2/3 \end{pmatrix}$$

Here's how to do the previous problem in SageMath (go to cocalc.com)

```
A=matrix(QQ, 3, 4, [1, 1, 2, -1, 2, 1, 3, 0, 0, -2, 1, 2])
A
```

```
[ 1  1  2 -1]
[ 2  1  3  0]
[ 0 -2  1  2]
```

```
A.rref()
```

```
[   1    0    0  5/3]
[   0    1    0 -4/3]
[   0    0    1 -2/3]
```

(notes continue on next page)

Exercise 1.1.18 A zookeeper wants to give an animal 42 mg of vitamin A and 65 mg of vitamin D per day. He has two supplements: the first contains 10% vitamin A and 25% vitamin D; the second contains 20% vitamin A and 25% vitamin D. How much of each supplement should he give the animal each day?

	vitamin A	vitamin D
	42 mg	65 mg
supplement 1	10%	25%
supplement 2	20%	25%

Answer will look like some number of milligrams of supp 1 & some number of mg of supp 2

x $\qquad$ y

$$0.1x + 0.2y = 42 \quad \text{vitamin A}$$
$$.25x + .25y = 65 \quad \text{vitamin D}$$

Lets solve using SageMath instead of by hand

```
B=matrix(QQ, 2, 3, [0.1, 0.2, 42, .25, .25, 65])
```

```
B
```

```
[1/10  1/5   42]
[ 1/4  1/4   65]
```

```
B.rref()
```

```
[  1    0 100]
[  0    1 160]
```

so $\boxed{x = 100 \quad y = 160}$ are the answers

i.e 100 mg of supplement 1

& 160 mg of supplement 2

Linear Algebra: solving linear equation

$$\begin{aligned} 2x + 3y &= 7 \\ 2x + 2y &= 8 \end{aligned} \quad \longleftrightarrow \quad \left[\begin{array}{cc|c} 2 & 3 & 7 \\ 2 & 2 & 8 \end{array}\right]$$

Solve this equation by finding the $\underbrace{\text{reduced row echelon form}}_{\text{RREF}}$ of the matrix.

3 elementary row operations

1) multiply any row/equation by a nonzero scalar (number)
2) interchange/swap two rows
3) replace any row by adding to it a scalar multiple of another row.

$$\begin{aligned} 2x + 3y &= 7 \\ 2x + 2y &= 8 \end{aligned} \qquad \left[\begin{array}{cc|c} 2 & 3 & 7 \\ 2 & 2 & 8 \end{array}\right]$$

multiplied row 2 by 1/2

$$\left[\begin{array}{cc|c} 2 & 3 & 7 \\ 1 & 1 & 4 \end{array}\right]$$

pivot 1

$$\begin{aligned} x + y &= 4 \\ 2x + 3y &= 7 \end{aligned}$$

interchanged R1 and R2

$$\left[\begin{array}{cc|c} 1 & 1 & 4 \\ 2 & 3 & 7 \end{array}\right]$$

$$x + y = 4$$
$$y = -1$$

$R2 \to R2 + (-2)R1$ $\quad \begin{bmatrix} 1 & 1 & 4 \\ 0 & 1 & -1 \end{bmatrix}$

$$\begin{array}{rrrl} 2 & 3 & 7 & R2 \\ -2 & -2 & -8 & (-2)R1 \\ \hline 0 & 1 & -1 & \end{array}$$

first pivot 1

$$\left[\begin{array}{cc|c} 1 & 1 & 4 \\ 0 & 1 & -1 \end{array}\right]$$

$R1 \to R1 + (-1)R2$

$$\begin{array}{rrr} 1 & 1 & 4 \\ 0 & -1 & 1 \\ \hline 1 & 0 & 5 \end{array}$$

$$\boxed{\begin{aligned} x \quad &= 5 \\ y &= -1 \end{aligned}}$$

$$\left[\begin{array}{cc|c} 1 & 0 & 5 \\ 0 & 1 & -1 \end{array}\right]$$

RREF

Suppose RREF was

$$\begin{array}{c} \left[\begin{array}{cccc|c} 1 & 2 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 1 & 5 \end{array}\right] \\ \begin{array}{cccc} w & x & y & z \end{array} \end{array} \longleftrightarrow$$

What are the equations & what are their solutions?

Answer:

$$w + 2x = 3$$
$$y = 4$$
$$z = 5$$

$$w = 3 - 2x$$
$$x = x$$
$$y = 4$$
$$z = 5$$

x is free variable. A free variable is a variable that does NOT correspond to column containing a pivot 1.
We solve for pivot variables in terms of free variables,
The free variables are free to be any value.

17 Solve the following system of 2 equations in 4 variables:

$$3x_1 + x_2 + x_3 + x_4 = 0$$

$$5x - x_2 + x_3 - x_4 = 0$$

Solution:

$$A = \left[\begin{array}{cccc|c} 3 & 1 & 1 & 1 & 0 \\ 5 & -1 & 1 & -1 & 0 \end{array}\right]$$

$$\mathrm{rref}(A) = \begin{array}{c} \left[\begin{array}{cccc|c} 1 & 0 & 0.25 & 0 & 0 \\ 0 & 1 & 0.25 & 1 & 0 \end{array}\right] \\ \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array} \end{array}$$

by calculator or computer

$$x_1 + 0.25x_3 = 0$$

$$x_2 + 0.25x_3 + x_4 = 0$$

Solve for the pivot variables (corresponding to leading 1's in each row)

Answer:

$$\boxed{\begin{aligned} x_1 &= -0.25x_3 \\ x_2 &= -0.25x_3 - x_4 \\ x_3 &= x_3 \\ x_4 &= x_4 \end{aligned}}$$

x_3, x_4 are free variables

$$\boxed{\begin{aligned} x_1 &= -0.25s \\ x_2 &= -0.25s - t \\ x_3 &= s \\ x_4 &= t \end{aligned}}$$

Let's Find RREF(A) by hand

$$A = \left[\begin{array}{cccc|c} 3 & 1 & 1 & 1 & 0 \\ 5 & -1 & 1 & -1 & 0 \end{array}\right]$$

$$\left[\begin{array}{cccc|c} 1 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\ 5 & -1 & 1 & -1 & 0 \end{array}\right]$$

$R1 \to \frac{1}{3} R1$

$R2 \to R2 + (-5) R1$

$$\begin{array}{lccccc} R2 & 5 & -1 & 1 & -1 & 0 \\ (-5)R1 & -5 & -\frac{5}{3} & -\frac{5}{3} & -\frac{5}{3} & 0 \\ \hline & 0 & -\frac{8}{3} & -\frac{2}{3} & -\frac{8}{3} & 0 \end{array}$$

$$\begin{bmatrix} 1 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & -\frac{8}{3} & -\frac{2}{3} & -\frac{8}{3} & 0 \end{bmatrix}$$

$R2 \to \frac{R2}{-8/3} = \frac{-3}{8} \cdot R2$

$$\begin{bmatrix} 1 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 1 & \frac{1}{4} & 1 & 0 \end{bmatrix}$$

$R1 \to R1 + (-\frac{1}{3}) R2$

$$\begin{array}{ccccc} 1 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & -\frac{1}{3} & -\frac{1}{12} & -\frac{1}{3} & 0 \\ \hline 1 & 0 & \frac{1}{4} & 0 & 0 \end{array}$$

$$\boxed{\text{RREF} = \begin{bmatrix} 1 & 0 & \frac{1}{4} & 0 & 0 \\ 0 & 1 & \frac{1}{4} & 1 & 0 \end{bmatrix}}$$

9/4/2014

3b Solve the system whose augmented matrix is:

Section 1.2

$$\begin{bmatrix} 1 & 0 & 8 & -5 & 6 \\ 0 & 1 & 4 & -9 & 3 \\ 0 & 0 & 1 & 1 & 2 \end{bmatrix}$$

Notice This matrix row echelon form, but not <u>reduced</u> row echelon form because pivot 1's (highlighted) have zeroes below them but not above them.

Let the variables be x_1 x_2 x_3 x_4

$$x_1 + 8x_3 - 5x_4 = 6$$
$$x_2 + 4x_3 - 9x_4 = 3$$
$$x_3 + x_4 = 2$$

$$\boxed{x_3 = 2 - x_4}$$

$$x_2 = 3 - 4x_3 + 9x_4$$
$$= 3 - 4(2 - x_4) + 9x_4$$
$$= 3 - 8 + 4x_4 + 9x_4$$
$$\boxed{x_2 = -5 + 13x_4}$$

$$x_1 = 6 - 8x_3 + 5x_4$$
$$= 6 - 8(2 - x_4) + 5x_4$$
$$= 6 - 16 + 8x_4 + 5x_4$$
$$\boxed{x_1 = -10 + 13x_4}$$

x_4 is "free variable" while x_1, x_2, x_3 are "leading variable"

7b. $6x_1 - x_2 + 3x_3 = 4$

$5x_2 - x_3 = 1$ Find solutions.

Solution

Form augmented matrix & find its rref (reduced row echelon form)

$$\left[\begin{array}{ccc|c} 6 & -1 & 3 & 4 \\ 0 & 5 & -1 & 1 \end{array}\right]$$

divide row 1 by 6

$$\left[\begin{array}{ccc|c} 1 & -\frac{1}{6} & \frac{3}{6} & \frac{4}{6} \\ 0 & 5 & -1 & 1 \end{array}\right]$$

divide R2 by 5

multiply by $\frac{1}{5}$

$$\left[\begin{array}{ccc|c} 1 & -\frac{1}{6} & \frac{1}{2} & \frac{2}{3} \\ 0 & 1 & -\frac{1}{5} & \frac{1}{5} \end{array}\right]$$

$R1 \to R1 + \frac{1}{6}R2$

$R1$	1	$-\frac{1}{6}$	$\frac{1}{2} = \frac{15}{30}$	$\frac{2}{3} = \frac{20}{30}$
$\frac{1}{6}R2$	0	$\frac{1}{6}$	$\frac{-1}{30}$	$\frac{1}{30}$
	1	0	$\frac{14}{30}$	$\frac{21}{30}$

$$\left[\begin{array}{ccc|c} 1 & 0 & \frac{14}{30} & \frac{21}{30} \\ 0 & 1 & -\frac{1}{5} & \frac{1}{5} \end{array}\right]$$

is RREF.

$$x_1 + \quad + \frac{14}{30}x_3 = \frac{21}{30}$$

$$x_2 - \frac{1}{5}x_3 = \frac{1}{5}$$

Solve for pivot variables (x_1, x_2)

$$x_1 = \frac{21}{30} - \frac{14}{30} x_3$$

x_3 is not a pivot, so it's free.

$$x_2 = \frac{1}{5} + \frac{1}{5} x_3$$

$$x_3 = x_3$$

<-- it's very easy to forget the simple equation associated to a free variable

or:

$$\boxed{\begin{aligned} x_1 &= \frac{21}{30} - \frac{14}{30} t \\ x_2 &= \frac{1}{5} + \frac{1}{5} t \\ x_3 &= t \end{aligned}}$$

5. $x_1 + x_2 + 2x_3 = 8$
$-x_1 - 2x_2 + 3x_3 = 1$
$3x_1 - 7x_2 + 4x_3 = 10$

Solve this by Gaussian Elimination

Solution:

(1) $\begin{bmatrix} 1 & 1 & 2 & 8 \\ -1 & -2 & 3 & 1 \\ 3 & -7 & 4 & 10 \end{bmatrix}$ "Augmented matrix"

Text

we want to replace 2^{nd} row so -1 is a 0
Rows 1 & 3 stay the same for now so copy those down first

$\begin{bmatrix} 1 & 1 & 2 & 8 \\ 0 & -1 & 5 & 9 \\ 3 & -7 & 4 & 10 \end{bmatrix}$

$$\begin{array}{rrrr} 1 & 1 & 2 & 8 \\ -1 & -2 & 3 & 1 \\ \hline 0 & -1 & 5 & 9 \end{array}$$

$\begin{bmatrix} 1 & 1 & 2 & 8 \\ 0 & -1 & 5 & 9 \\ 0 & -10 & -2 & -14 \end{bmatrix}$ $\xrightarrow{(-3)}$

$$\begin{array}{rrrr} -3 & -3 & -6 & -24 \\ 3 & -7 & 4 & 10 \\ \hline 0 & -10 & -2 & -14 \end{array}$$

$\begin{bmatrix} 1 & 1 & 2 & 8 \\ 0 & 1 & -5 & -9 \\ 0 & -10 & -2 & -14 \end{bmatrix}$

--

$$\begin{bmatrix} 1 & 1 & 2 & 8 \\ 0 & 1 & -5 & -9 \\ 0 & -10 & -2 & -14 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 2 & 8 \\ 0 & 1 & -5 & -9 \\ 0 & 0 & -52 & -104 \end{bmatrix}$$

$\cdot 10 \longrightarrow$

$$\begin{array}{cccc} 0 & 10 & -50 & -90 \\ 0 & -10 & -2 & -14 \\ \hline 0 & 0 & -52 & -104 \end{array}$$

$$\begin{bmatrix} 1 & 1 & 2 & 8 \\ 0 & 1 & -5 & -9 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

$\longleftarrow$ divided entire row by -52

$\frac{-104}{-52} = 2$

$$x_1 + x_2 + 2x_3 = 8$$
$$x_2 - 5x_3 = -9$$
$$\boxed{x_3 = 2}$$

$$x_2 = -9 + 5x_3 = -9 + 10 = 1$$
$$\boxed{x_2 = 1}$$

$$x_1 = 8 - x_2 - 2x_3$$
$$= 8 - 1 - 2\cdot 2 = 8 - 1 - 4 = 8 - 5 = 3$$
$$\boxed{x_1 = 3}$$

Section 1.1
\# 26 p.10

Parabola $y = ax^2 + bx + c$ passes through the points $(1,1)$ $(2,4)$, $(-1,1)$. Find a, b, c.

$$y = ax^2 + bx + c$$

$(1, 1) \rightarrow \quad 1 = a \cdot 1^2 + b \cdot 1 + c$

$\uparrow \ \uparrow$
$x \ \ y$

$$\boxed{1 = a + b + c}$$

$(2, 4) \quad \boxed{4 = a \cdot 4 + 2b + c}$

$(-1, 1) \quad 1 = a(-1)^2 - b + c$

$$\boxed{1 = a - b + c}$$

3 linear equations in a, b, c. I leave it to you to solve from here.

1.3 Matrices & Matrix operations

$$A = \begin{bmatrix} 3 & 4 & 1 \\ 2 & 5 & 7 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$$

row 2 column 1

2 rows & 3 columns

so we say A is a 2×3 matrix

rows column

Addition & subtraction of matrices is easy - it's just done component wise:

$$\begin{bmatrix} 3 & 4 & 1 \\ 2 & 5 & 7 \end{bmatrix} + \begin{bmatrix} -1 & 0 & 2 \\ 0 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 4 & 3 \\ 2 & 6 & 10 \end{bmatrix}$$

$2 \times 3 \qquad 2 \times 3 \qquad 2 \times 3$

Matrix multiplication is not as simple

Motivation:

$$5x + 3y + 6z = 1$$
$$3x + y + z = 3$$
$$x + y + z = 7$$

$$\downarrow$$

$$\begin{bmatrix} 5 & 3 & 6 \\ 3 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5x+3y+6z \\ 3x+y+z \\ x+y+z \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 7 \end{bmatrix}$$

$$3 \times \underbrace{\boxed{3 \qquad\qquad 3}}_{} \times 1 \qquad\qquad 3 \times 1$$

columns rows

Examples of matrix multiplication

$$A \to \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 3 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ 0 & 2 \\ 1 & 1 \end{bmatrix} \leftarrow B \quad = \begin{bmatrix} 2 & \\ 4 & \end{bmatrix} \leftarrow C \quad = C$$

$$2 \times \boxed{3 \qquad\qquad 3} \times 2 \qquad\qquad 2 \times 2$$

ij^{th} entry of C is (row i of A) dot (column j of B)

1,1 entry of C is (row 1 of A) dot (column 1 of B)

$$(2\ 1\ 0) \cdot \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = 2 + 1\cdot 0 + 0 \cdot 1 = 2$$

2,1 entry of C is (row 2 of A) dot (column 1 B) $= 1\cdot 1 + 2 \cdot 0 + 3 \cdot 1 = 4$

$$\begin{array}{ccccc} A & \cdot & B & & C \\ \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 3 \end{bmatrix} & & \begin{bmatrix} 1 & 1 \\ 0 & 2 \\ 1 & 1 \end{bmatrix} & = & \begin{bmatrix} 2 & 4 \\ 4 & 8 \end{bmatrix} \\ 2 \times 3 & & 3 \times 2 & & \end{array}$$

2,1 entry of C = (row 2 of A) dot (column 1 of B)

$$= (1\ 2\ 3) \cdot \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = 1 \cdot 1 + 2 \cdot 0 + 3 \cdot 1$$

$$= 1 + 0 + 3$$

$$= 4$$

Scalar multiplication:
(number)

$$5 \cdot \begin{bmatrix} 1 & 2 \\ 4 & -1 \end{bmatrix} = \begin{bmatrix} 5 & 10 \\ 20 & -5 \end{bmatrix}$$

scalar · matrix

Transpose: of a matrix A is denote A^T

$$(A^T)_{ij} = A_{ji} \leftarrow \text{flipped}$$

ij^{th} entry of A^T $\qquad$ ji^{th} entry of A

Example $A = \begin{bmatrix} 1 & 0 & 2 \\ 5 & 1 & -1 \end{bmatrix}$ 2×3

$A^T = \begin{bmatrix} 1 & 5 \\ 0 & 1 \\ 2 & -1 \end{bmatrix}$ 3×2

$(A^T)_{21} = A_{12} = 0$

row 1 column 2

Section 1.4 Inverses, Algebraic properties

Matrix multiplication is associative: $(AB)C = A(BC)$

Matrix multiplication is NOT commutative: $AB \neq BA$

$A = 2 \times 3$ $B = 3 \times 2$

$AB = 2 \times 2$

$BA = 3 \times 3$

3×2 2×3

$I_n = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix}$ $n \times n$

Identity matrix

$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ 2×2

$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ 3×3

Reason it's called the identity matrix:

if A is an $n \times n$ matrix, then:

$$I_n A = A$$
$$A I_n = A$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

If A is an $n \times n$ matrix, an inverse of A is an $n \times n$ matrix such that

$$AA^{-1} = I_n$$
$$A^{-1}A = I_n$$

Formula for inverse of 2×2 matrix:

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then $A^{-1} = \begin{bmatrix} \frac{d}{\Delta} & \frac{-b}{\Delta} \\ \frac{-c}{\Delta} & \frac{a}{\Delta} \end{bmatrix} = \frac{1}{\Delta} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

where $\Delta = ad - bc$ (determinant

$$A = \begin{bmatrix} 8 & 5 \\ 2 & 1 \end{bmatrix}, \quad \Delta = 8 - 10 = -2$$

$$A^{-1} = \frac{1}{-2} \begin{bmatrix} 1 & -5 \\ -2 & 8 \end{bmatrix} = \begin{bmatrix} -1/2 & 5/2 \\ 1 & -4 \end{bmatrix}$$

$$\begin{aligned} 2x+3y&=7 \\ 5x+8y&=9 \end{aligned} \quad \longleftrightarrow \quad \left[\begin{array}{cc|c} 2 & 3 & 7 \\ 5 & 8 & 9 \end{array}\right]$$

$$\begin{bmatrix} 2 & 3 \\ 5 & 8 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7 \\ 9 \end{bmatrix}$$

$$\begin{bmatrix} 2x+3y \\ 5x+8y \end{bmatrix} = \begin{bmatrix} 7 \\ 9 \end{bmatrix}$$

$$2\times 1 \quad = \quad 2\times 1.$$

Matrix multiplication

$$\underset{2\times \boxed{3}}{\begin{bmatrix} 1 & 0 & 1 \\ 1 & 2 & 3 \end{bmatrix}} \underset{\boxed{3}\times 1}{\begin{bmatrix} 2 \\ 5 \\ 7 \end{bmatrix}} = \underset{\substack{2\times 1 \\ \text{matrix}}}{\begin{bmatrix} 1\cdot 2+0\cdot 5+1\cdot 7 \\ 1\cdot 2+2\cdot 5+3\cdot 7 \end{bmatrix}} \begin{array}{l} \text{row 1 column 1 spot} \\ \text{row 2 column 1 spot} \end{array}$$

$$= \begin{bmatrix} 9 \\ 33 \end{bmatrix}$$

Definition of dot product of two vectors

dot product: $(a,b,c)\cdot(x,y,z) = ax+by+cz$

$$A = \begin{bmatrix} 3 & -2 & 7 \\ 6 & 5 & 4 \\ 0 & 4 & 9 \end{bmatrix} \qquad B = \begin{bmatrix} 6 & -2 & 4 \\ 0 & 1 & 3 \\ 7 & 7 & 5 \end{bmatrix}$$

7a) Find the first row of AB

row1 col1

row1 · col 2

$$\underset{3 \times 3}{A} \quad \underset{3 \times 3}{B} = \begin{bmatrix} \boxed{67} & 41 & \_ \\ \_ & \_ & \_ \\ \_ & \_ & \_ \end{bmatrix}$$

3×3

rows columns

$$\underset{A}{\text{row1}} \cdot \underset{B}{\text{col1}} = (3,-2,7)\cdot(6,0,7)$$
$$= 3\cdot 6 + -2\cdot 0 + 7\cdot 7$$
$$= 18 + 0 + 49$$
$$= 67$$

$$\underset{A}{\text{row1}} \cdot \underset{B}{\text{col2}} = (3,-2,7)\cdot(-2,1,7)$$
$$= 3\cdot(-2) + (-2)(1) + 7\cdot 7$$
$$= -6 - 2 + 49$$
$$= 41$$

9a) Express each column vector of AA as a linear combination of column vectors of A

$$A = \begin{bmatrix} 3 & -2 & 7 \\ 6 & 5 & 4 \\ 0 & 4 & 9 \end{bmatrix}$$

Solution:

$$AA = \begin{bmatrix} 3 & -2 & 7 \\ 6 & 5 & 4 \\ 0 & 4 & 9 \end{bmatrix} \begin{bmatrix} 3 & -2 & 7 \\ 6 & 5 & 4 \\ 0 & 4 & 9 \end{bmatrix}$$

$$= \begin{bmatrix} | & | & | \\ c_1 & c_2 & c_3 \\ | & | & | \end{bmatrix}$$

$$c_1 = 3\begin{bmatrix} 3 \\ 6 \\ 0 \end{bmatrix} + 6\begin{bmatrix} -2 \\ 5 \\ 4 \end{bmatrix} + 0\begin{bmatrix} 7 \\ 4 \\ 9 \end{bmatrix}$$

$$= \begin{bmatrix} 3 \cdot 3 \\ 3 \cdot 6 \\ 3 \cdot 0 \end{bmatrix} + \begin{bmatrix} -12 \\ 30 \\ 24 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 9 \\ 18 \\ 0 \end{bmatrix} + \begin{bmatrix} -12 \\ 30 \\ 24 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 9-12+0 \\ 18+30+0 \\ 0+24+0 \end{bmatrix} = \begin{bmatrix} -3 \\ 48 \\ 24 \end{bmatrix}$$

Scalar multiplication

$$\underset{\text{scalar}}{\overset{\uparrow}{7}} \underbrace{\begin{bmatrix} a & b \\ c & d \end{bmatrix}}_{\text{matrix}} = \begin{bmatrix} 7a & 7b \\ 7c & 7d \end{bmatrix}$$

Matrix addition

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} w & x \\ y & z \end{bmatrix} = \begin{bmatrix} a+w & b+x \\ c+y & d+z \end{bmatrix}$$

$$\begin{aligned} 2x + 3y &= 7 \\ 5x + 7y &= 8 \end{aligned} \iff \begin{bmatrix} 2 & 3 \\ 5 & 7 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7 \\ 8 \end{bmatrix}$$

Regular algebra: solve for X

$$5X = 10$$

$$\frac{1}{5}\, 5X = \frac{1}{5}\, 10$$

$$1 \cdot X = 2$$

2x2 Identity matrix:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

3x3 Identity matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

If A is an $\underbrace{n \times n}_{\text{square matrix}}$ matrix, then sometimes

there exists a matrix inverse of A, denoted $\underbrace{A^{-1}}_{n \times n \text{ matrix}}$

such that $A^{-1}A = I_n$ and $AA^{-1} = I_n$

$n \times n$ identity matrix

Formula for inverse of 2×2 matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

then $A^{-1} = \dfrac{1}{\Delta} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ where $\Delta = ad - bc$ is determinant of A

Example $A = \begin{bmatrix} 2 & 3 \\ 5 & 7 \end{bmatrix}$ Find A^{-1}

Solution $A^{-1} = \dfrac{1}{\Delta} \begin{bmatrix} 7 & -3 \\ -5 & 2 \end{bmatrix}$

$$\Delta = 2 \cdot 7 - 3 \cdot 5$$
$$= 14 - 15$$
$$= -1$$

$$= \frac{1}{-1} \begin{bmatrix} 7 & -3 \\ -5 & 2 \end{bmatrix}$$

$$A^{-1} = -1\begin{bmatrix} 7 & -3 \\ -5 & 2 \end{bmatrix} = \begin{bmatrix} -7 & 3 \\ 5 & -2 \end{bmatrix}$$

Example:

$$A = \begin{bmatrix} 2 & 3 \\ 5 & 7 \end{bmatrix} \qquad A\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7 \\ 8 \end{bmatrix} \qquad \text{solve for } \begin{bmatrix} x \\ y \end{bmatrix}$$

Solution:

$$A^{-1}A\begin{bmatrix} x \\ y \end{bmatrix} = A^{-1}\begin{bmatrix} 7 \\ 8 \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{A^{-1}A}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -7 & 3 \\ 5 & -2 \end{bmatrix}\begin{bmatrix} 7 \\ 8 \end{bmatrix}$$

$$2\times\boxed{2 \qquad 2}\times 1 \qquad\qquad 2\times\boxed{2 \qquad 2}\times 1$$

$$\underset{2\times 1}{\begin{bmatrix} x \\ y \end{bmatrix}} = \underset{2\times 1}{\begin{bmatrix} -7\cdot 7 + 3\cdot 8 \\ 5\cdot 7 - 2\cdot 8 \end{bmatrix}} = \begin{bmatrix} -49+24 \\ 35-16 \end{bmatrix}$$

$$= \begin{bmatrix} -25 \\ 19 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -25 \\ 19 \end{bmatrix}$$

26. $-x_1 + 5x_2 = 4$

$-x_1 - 3x_2 = 1$

solve x_1 & x_2.

Solution

$$\begin{bmatrix} -1 & 5 \\ -1 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

coefficient matrix

$$A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

(where $A = \begin{bmatrix} -1 & 5 \\ -1 & -3 \end{bmatrix}$)

multiply both sides by A^{-1} on the left.

$$\underbrace{A^{-1} A}_{I_2} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = A^{-1} \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = A^{-1} \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

=

$$A^{-1} = \frac{1}{\Delta} \begin{bmatrix} -3 & -5 \\ 1 & -1 \end{bmatrix} = \frac{1}{8} \begin{bmatrix} -3 & -5 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} -3/8 & -5/8 \\ 1/8 & -1/8 \end{bmatrix}$$

$$\Delta = (-1)(-3) - (-1)(5)$$
$$= 3 - (-5)$$
$$= 8$$

9/11/2014

1.5 Elementary matrices and finding A^{-1}

Recall:
Elementary row operations
1. Switch two rows
2. Replace a row by adding to it a constant times another row
3. multiply a row by a nonzero constant.

Each of these operations has a corresponding or associated <u>elementary matrix</u> defined by performing the operation on the identity matrix $I_m = \begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{bmatrix}$ for some m.

$m \times m$

Example: Let's fix $m = 3$ $\quad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

An Elementary matrix corresponding to switching first two rows is:

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

An example of elementary matrix corresponding to replacing R2 by R2+6R3

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 6 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{array}{ccc|l} 0 & 1 & 0 & R2 \\ 0 & 0 & 6 & 6R3 \\ \hline 0 & 1 & 6 & \end{array}$$

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is an elementary matrix corresponding to multiplying row 2 of $I_3 = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix}$ by 6.

Simple fact: Given some $m \times n$ matrix A, performing an elementary row operation on A results in the matrix EA (where E is the elementary associated to the chosen row operation)

Examples: $A = \begin{bmatrix} 1 & 2 \\ 3 & 0 \\ 4 & 5 \end{bmatrix}$

3×2
row columns

Switching the first two rows of A gives $\begin{bmatrix} 3 & 0 \\ 1 & 2 \\ 4 & 5 \end{bmatrix}$

$$EA = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 0 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 1 & 2 \\ 4 & 5 \end{bmatrix}$$

$3 \times \boxed{3 \quad 3} \times 2$ $\qquad$ 3×2

2^{nd} example: Start with A, replace $R2$ by $R2 + 6R3$ will give $\begin{bmatrix} 1 & 2 \\ 27 & 30 \\ 4 & 5 \end{bmatrix}$

$$\underset{E}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 6 \\ 0 & 0 & 1 \end{bmatrix}} \underset{A}{\begin{bmatrix} 1 & 2 \\ 3 & 0 \\ 4 & 5 \end{bmatrix}} = \begin{bmatrix} 1 & 2 \\ 27 & 30 \\ 4 & 5 \end{bmatrix}$$

$3 \times \boxed{3 \quad 3} \times 2$ $\qquad$ 3×2

$R2: \quad 3 \quad 0$
$6R3: \quad 24 \quad 30$

$\quad 27 \quad 30$

Suppose A is an invertible matrix, say A is $m \times m$.
We can do a bunch of elementary row operations to A to get the matrix A in reduced row echelon form

0 above (and below) the pivot 1's.

If A is invertible, this reduced row echelon matrix will be the Identity matrix $I_m = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$$I_m = \boxed{E_k \cdots E_3 E_2 E_1} A$$

start with A

did elementary row operation to A

ended up with identity matrix

$$I_m = \boxed{} A$$

$$I_m = A^{-1} A$$

$$A^{-1} = E_k \cdots E_3 E_2 E_1$$

$$A^{-1} = E_k \cdots E_3 E_2 E_1 \cdot I_m$$

Algorithm to find inverse of A:

Make $[A \mid I_m] \rightsquigarrow [I_m \mid A^{-1}]$

& perform elementary row operations to get A to I_m

Example: $A = \begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix}$ Find A^{-1}

Solution: $\left[\underbrace{\begin{matrix} 3 & 1 \\ 5 & 2 \end{matrix}}_{A} \middle| \underbrace{\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}}_{I_2}\right]$

We have to find RREF of this 2 x 4 matrix.
You can do any of the elementary row operations
and in any order, you don't have to choose
the ones I chose below:

multiply R1 by 1/3

$$\left[\begin{matrix} 1 & 1/3 \\ 5 & 2 \end{matrix} \middle| \begin{matrix} 1/3 & 0 \\ 0 & 1 \end{matrix}\right] \xrightarrow{(-5)}$$

$$\begin{array}{cccc} -5 & -5/3 & -5/3 & 0 \\ 5 & 2 & 0 & 1 \\ \hline 0 & 1/3 & -5/3 & 1 \end{array}$$

$$\begin{bmatrix} 1 & 1/3 & 1/3 & 0 \\ 0 & 1/3 & -5/3 & 1 \end{bmatrix} \xrightarrow{-1}$$

$$\begin{array}{cccc} 1 & 1/3 & 1/3 & 0 \\ 0 & -1/3 & 5/3 & -1 \\ \hline 1 & 0 & 2 & -1 \end{array}$$

$$\begin{bmatrix} 1 & 0 & 2 & -1 \\ 0 & 1/3 & -5/3 & 1 \end{bmatrix} \qquad R1 \rightarrow R1 + (-1)R_2$$

$$\left[\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} \middle| \underbrace{\begin{matrix} 2 & -1 \\ -5 & 3 \end{matrix}}_{A^{-1}}\right] \qquad (3R_2)$$

Answer

p. 59 #15

Find A^{-1} where $A = \begin{bmatrix} 2 & 6 & 6 \\ 2 & 7 & 6 \\ 2 & 7 & 7 \end{bmatrix}$

Solution: $\begin{array}{ccc|ccc} 2 & 6 & 6 & 1 & 0 & 0 \\ 2 & 7 & 6 & 0 & 1 & 0 \\ 2 & 7 & 7 & 0 & 0 & 1 \end{array} \xrightarrow{-1}$ $\begin{array}{cccccc} -2 & -6 & -6 & -1 & 0 & 0 \\ 2 & 7 & 6 & 0 & 1 & 0 \\ \hline 0 & 1 & 0 & -1 & 1 & 0 \end{array}$

$$\left[\begin{array}{ccc|ccc} 2 & 6 & 6 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 2 & 7 & 7 & 0 & 0 & 1 \end{array}\right] \quad R2 \to R2 + (-1)R_1$$

(-1) $\begin{array}{cccccc} -2 & -6 & -6 & -1 & 0 & 0 \\ 2 & 7 & 7 & 0 & 0 & 1 \\ \hline 0 & 1 & 1 & -1 & 0 & 1 \end{array}$

$$\left[\begin{array}{ccc|ccc} 2 & 6 & 6 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 1 & 1 & -1 & 0 & 1 \end{array}\right] \quad R3 \to R3 + (-1)R1$$

-1 $\begin{array}{cccccc} 0 & -1 & 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & -1 & 0 & 1 \\ \hline 0 & 0 & 1 & 0 & -1 & 1 \end{array}$

$$\left[\begin{array}{ccc|ccc} 2 & 6 & 6 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}\right] \quad R3 \to R3 + (-1)R_2$$

$$\left[\begin{array}{cccccc} 1 & 3 & 3 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}\right] \xrightarrow{(-3)} \begin{array}{cccccc} 1 & 3 & 3 & \frac{1}{2} & 0 & 0 \\ 0 & -3 & 0 & 3 & -3 & 0 \\ \hline 1 & 0 & 3 & \frac{7}{2} & -3 & 0 \end{array}$$

$$\left[\begin{array}{cccccc} 1 & 0 & 3 & 7/2 & -3 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}\right] \xrightarrow{(-3)}$$

$$\begin{array}{cccccc} 1 & 0 & 3 & 7/2 & -3 & 0 \\ 0 & 0 & -3 & 0 & 3 & -3 \\ \hline 1 & 0 & 0 & 7/2 & 0 & -3 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 7/2 & 0 & -3 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}\right]$$

A^{-1}

$$A^{-1} = \begin{bmatrix} 7/2 & 0 & -3 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

Here’s how to find the inverse in SageMath using the builit-in method inverse():

```
A=matrix(QQ, 3, 3, [2, 6, 6, 2, 7, 6, 2, 7, 7])
```

```
A
```

```
[2 6 6]
[2 7 6]
[2 7 7]
```

```
A.inverse()
```

```
[7/2    0   -3]
[ -1    1    0]
[  0   -1    1]
```

Illustrating the long way to compute inverse:

```
identity_matrix(3)
```

```
[1 0 0]
[0 1 0]
[0 0 1]
```

```
B=A.augment(identity_matrix(3), subdivide="true")
B
```

```
[2 6 6|1 0 0]
[2 7 6|0 1 0]
[2 7 7|0 0 1]
```

```
B.rref()
```

```
[   1    0    0|7/2    0   -3]
[   0    1    0| -1    1    0]
[   0    0    1|  0   -1    1]
```

this 3x3 matrix is the inverse of A from the previous page

Example Identify which elementary row operation E corresponds to & Compute EA

a. $E = \begin{bmatrix} -6 & 0 \\ 0 & 1 \end{bmatrix}, A = \begin{bmatrix} -1 & -2 & 5 & -1 \\ 3 & -6 & -6 & -6 \end{bmatrix}$

Solution $I = \begin{bmatrix} 1 & \\ & 1 \end{bmatrix}$ we multiplied row 1 by -6. to get E.

$$EA = \underset{2\times 2}{\begin{bmatrix} -6 & 0 \\ 0 & 1 \end{bmatrix}} \underset{2\times 4}{\begin{bmatrix} -1 & -2 & 5 & -1 \\ 3 & -6 & -6 & -6 \end{bmatrix}}$$

$$= \underset{2\times 4}{\begin{bmatrix} +6 & 12 & -30 & 6 \\ 3 & -6 & -6 & -6 \end{bmatrix}}$$

Find solutions (x,y) to the system

$$2x + 3y = 7$$
$$4x + 5y = 8$$

$\longleftrightarrow$ convert to matrix form

$$\left[\begin{array}{cc|c} 2 & 3 & 7 \\ 4 & 5 & 8 \end{array}\right]$$

(We've learned two ways to do this)

or

$$\begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7 \\ 8 \end{bmatrix}$$

Solution 1 Find Reduced Row Echelon Form RREF

$$\left[\begin{array}{cc|c} 2 & 3 & 7 \\ 4 & 5 & 8 \end{array}\right]$$

Augmented

second way: $\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}$ $\quad \vec{b} = \begin{bmatrix} 7 \\ 8 \end{bmatrix}$

$$A\vec{x} = \vec{b}$$

$$A^{-1}A\vec{x} = A^{-1}\vec{b}$$

$$\vec{x} = A^{-1}\vec{b}$$

$$\left[\begin{array}{cc|c} 2 & 3 & 7 \\ 0 & -1 & -6 \end{array}\right]$$

$$\begin{array}{rrrr} R2 & 4 & 5 & 8 \\ -2R1 & -4 & -6 & -14 \\ \hline & 0 & -1 & -6 \end{array}$$

$$\left[\begin{array}{cc|c} 2 & 3 & 7 \\ 0 & 1 & 6 \end{array}\right]$$

$$\begin{bmatrix} 2 & 0 & -11 \\ 0 & 1 & 6 \end{bmatrix}$$

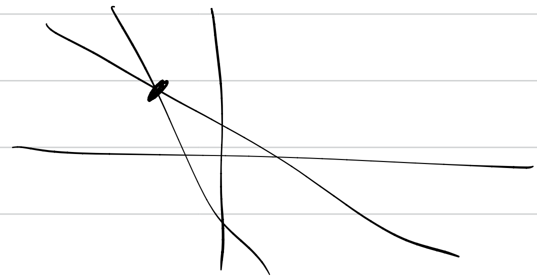
$$\begin{array}{rrrr} 2 & 3 & 7 & R1 \\ 0 & -3 & -18 & -3R2 \\ \hline 2 & 0 & -11 & \end{array}$$

$$\left[\begin{array}{cc|c} 1 & 0 & -11/2 \\ 0 & 1 & 6 \end{array}\right] \text{ is RREF}$$

$x + 0y = -11/2 = -5.5$

$0x + 1 \cdot y = 6$

$$\boxed{\begin{array}{l} x = -5.5 \\ y = 6 \end{array}}$$



Solution 2:

(only works if A is a square matrix & has an inverse)

$$A\vec{x} = \vec{b}$$

$$\vec{x} = A^{-1}\vec{b}$$

$$A = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$$

Formula for inverse of 2×2 matrix:

if $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then

$$A^{-1} = \frac{1}{\Delta} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

determinant $\Delta = ad - bc$

If determinant $\Delta = 0$, then no inverse!

$$\Delta = 2 \cdot 5 - 4 \cdot 3 = 10 - 12 = -2$$

$$A^{-1} = \frac{1}{-2} \begin{bmatrix} 5 & -3 \\ -4 & 2 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} -5/2 & 3/2 \\ 2 & -1 \end{bmatrix}$$

$$\vec{x} = A^{-1}\vec{b} \qquad \vec{b} = \begin{bmatrix} 7 \\ 8 \end{bmatrix}$$

$$\vec{x} = \begin{bmatrix} x \\ y \end{bmatrix} = \underbrace{\begin{bmatrix} -5/2 & 3/2 \\ 2 & -1 \end{bmatrix}}_{2 \times 2} \underbrace{\begin{bmatrix} 7 \\ 8 \end{bmatrix}}_{2 \times 1} = \underbrace{\begin{bmatrix} -\frac{5}{2} \cdot 7 + \frac{3}{2} \cdot 8 \\ 2 \cdot 7 + (-1) 8 \end{bmatrix}}_{2 \times 1} = \begin{bmatrix} -5.5 \\ 6 \end{bmatrix}$$

same answer as before

1.6 More on Linear systems & Invertible matrices.

System of m linear equations in n variables $x_1, x_2, \dots x_n$

can be written compactly as

$$\boxed{A\vec{x} = \vec{b}} \qquad \text{where } \vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

$n \times 1$

$A = m \times n$ matrix

$\vec{x} = n \times 1$ matrix

$\vec{b} = m \times 1$ matrix

Given $A, \vec{b}$ the goal is to somehow "solve" for $\vec{x}$

Three possibilities:

1. No such $\vec{x}$ exists (i.e system $A\vec{x} = \vec{b}$ is inconsistent)

2. There is exactly one solution $\vec{x}$
3. There are infinitely many solutions $\vec{x}$

] system is consistent

Example
$$x - 3y = 1$$
$$x - 3y = 2$$
inconsistent

Let's see what happens when we do row reduction to augmented matrix

$$\left[\begin{array}{cc|c} 1 & -3 & 1 \\ 1 & -3 & 2 \end{array}\right] \xrightarrow{-1} \begin{array}{ccc} -1 & 3 & -1 \\ 1 & -3 & 2 \\ \hline 0 & 0 & 1 \end{array}$$

$$\downarrow$$

$$\left[\begin{array}{cc|c} 1 & -3 & 1 \\ 0 & 0 & 1 \end{array}\right] \longrightarrow 0x + 0y = 1$$

tells us system is inconsistent.

Example: $\begin{aligned} x - 3y &= 1 \\ 2x - 6y &= 2 \end{aligned}$ has infinitely many solutions

$$\left[\begin{array}{cc|c} 1 & -3 & 1 \\ 2 & -6 & 2 \end{array}\right] \xrightarrow{(-2)} \begin{array}{ccc} -2 & 6 & -2 \\ 2 & -6 & 2 \\ \hline 0 & 0 & 0 \end{array}$$

$$\left[\begin{array}{cc|c} 1 & -3 & 1 \\ 0 & 0 & 0 \end{array}\right] \longrightarrow \underline{x} - 3y = 1 \quad \begin{bmatrix} x \\ y \end{bmatrix} = \begin{matrix} 3y + 1 \\ y + 0 \end{matrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix} y + \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

p66

15. Determine conditions on b_1, b_2, b_3 that make the following system consistent:

$$\begin{aligned} x_1 - 2x_2 + 5x_3 &= b_1 \\ 4x_1 - 5x_2 + 8x_3 &= b_2 \\ -3x_1 + 3x_2 - 3x_3 &= b_3 \end{aligned} \xrightarrow{\text{matrix}} \left[\begin{array}{ccc|c} 1 & -2 & 5 & b_1 \\ 4 & -5 & 8 & b_2 \\ -3 & 3 & -3 & b_3 \end{array}\right]$$

$$\begin{array}{cccc} 1 & -2 & 5 & b_1 \\ 4 & -5 & 8 & b_2 \\ -3 & 3 & -3 & b_3 \end{array} \qquad \xrightarrow{-4} \qquad \begin{array}{cccc} -4 & 8 & -20 & -4b_1 \\ 4 & -5 & 8 & b_2 \\ \hline 0 & 3 & -12 & -4b_1+b_2 \end{array}$$

$$\begin{array}{cccc} 1 & -2 & 5 & b_1 \\ 0 & 3 & -12 & -4b_1+b_2 \\ -3 & 3 & -3 & b_3 \end{array} \qquad \xrightarrow{+3} \qquad \begin{array}{cccc} 3 & -6 & 15 & 3b_1 \\ -3 & 3 & -3 & b_3 \\ \hline 0 & -3 & 12 & 3b_1+b_3 \end{array}$$

$$\begin{array}{cccc} 1 & -2 & 5 & b_1 \\ 0 & 3 & -12 & -4b_1+b_2 \\ 0 & -3 & 12 & 3b_1+b_3 \end{array}$$

$$\left.\begin{array}{ccc|c} 1 & -2 & 5 & b_1 \\ 0 & 3 & -12 & -4b_1+b_2 \\ 0 & 0 & 0 & -b_1+b_2+b_3 \end{array}\right.$$

$$\left.\begin{array}{ccc|c} 1 & -2 & 5 & b_1 \\ 0 & 1 & -4 & -\frac{4}{3}b_1+\frac{b_2}{3} \\ 0 & 0 & 0 & -b_1+b_2+b_3 \end{array}\right. \quad \Longleftrightarrow \quad \underbrace{0x_1+0x_2+0x_3}_{0} = -b_1+b_2+b_3$$

So we need $\boxed{-b_1+b_2+b_3=0}$ for the system to be consistent, ie have at least one solution.

$$-b_1 + b_2 + b_3 = 0$$

row 2 says

$$y - 4z = -\frac{4}{3}b_1 + \frac{b_2}{3}$$

$$\boxed{y = 4z + -\frac{4}{3}b_1 + \frac{b_2}{3}}$$

row 1 says

$$x - 2y + 5z = b_1$$

$$\begin{aligned} x &= 2y - 5z + b_1 \\ &= 2\left(4z - \frac{4}{3}b_1 + \frac{b_2}{3}\right) - 5z + b_1 \\ &= 8z - \frac{8}{3}b_1 + \frac{2}{3}b_2 - 5z + b_1 \end{aligned}$$

$$\boxed{x = 3z - \frac{5}{3}b_1 + \frac{2}{3}b_2}$$

Theorem: If A is an $n \times n$ matrix then the following are equivalent:

a) A is invertible (i.e A^{-1} exists)

b) $A\vec{x} = \vec{0}$ has only $\vec{x} = \vec{0}$ as a solution. $\vec{0} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$

c) The reduced row echelon form of A is $I_n = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix}$

d) A is expressible as a product of elementary matrices

e) $A\vec{x} = \vec{b}$ is consistent for every $n \times 1$ matrix $\vec{b}$

f) $A\vec{x} = \vec{b}$ has exactly one solution for every $n \times 1$ matrix $\vec{b}$.

Proofs: $a \Rightarrow b$: Assume A is invertible, we have to show b) holds:

suppose $A\vec{x} = \vec{0}$

$$A^{-1}A\vec{x} = A^{-1}\vec{0}$$

$$\vec{x} = A^{-1}\vec{0}$$

$$\vec{x} = \vec{0} \quad \checkmark$$

$b \Rightarrow c$ row reducing A, if the end result is not I_n you would have a row of 0's & that will enable you to find more that one solution to $A\vec{x} = 0$ besides $\vec{x} = 0$.

Solve system by finding inverse of coefficient matrix.

d) $$\begin{aligned} x + 4y + 2z &= 1 \\ 2x + 3y + 3z &= -1 \\ 4x + y + 4z &= 0 \end{aligned}$$

$\rightarrow A \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$

3×1 column vector

Solution

$$A = \begin{bmatrix} 1 & 4 & 2 \\ 2 & 3 & 3 \\ 4 & 1 & 4 \end{bmatrix}$$

coefficient matrix

Find A^{-1}.

rref of $[A \mid I_3]$

find reduced row echelon form

$$\left[\begin{array}{ccc|ccc} 1 & 4 & 2 & 1 & 0 & 0 \\ 2 & 3 & 3 & 0 & 1 & 0 \\ 4 & 1 & 4 & 0 & 0 & 1 \end{array}\right]$$

or

lazy: use cocalc/sagemath

$$A^{-1} = \begin{bmatrix} 9/5 & -14/5 & 6/5 \\ 4/5 & -4/5 & 1/5 \\ -2 & 3 & -1 \end{bmatrix}$$

$$A \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

3×1
column vector

$$A\vec{x} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

$$\vec{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

high school algebra:

$$2x = 6$$

$$\frac{1}{2} \cdot 2x = \frac{1}{2} \cdot 6$$

$$x = 3$$

to solve, we multiplied both sides of $2x=1$ by $\frac{1}{2}$, which is inverse of 2

same idea to solve $A\vec{x} = b$ for $\vec{x}$

multiply both sides by inverse of A

$$\underbrace{A^{-1}A}\vec{x} = A^{-1}\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

$$I_3 \vec{x} = A^{-1}\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

$$\underset{3\times 3}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \cdot \underset{3\times 1}{\begin{bmatrix} x \\ y \\ z \end{bmatrix}} = \underset{(3 \times 3)}{\begin{bmatrix} 9/5 & -14/5 & 6/5 \\ 4/5 & -4/5 & 1/5 \\ -2 & 3 & -1 \end{bmatrix}} \cdot \underset{(3 \times 1)}{\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}}$$

$$\underset{3 \times 1}{\begin{bmatrix} x \\ y \\ z \end{bmatrix}} = \underset{3 \times 1}{\begin{bmatrix} \frac{9}{5}\cdot 1 + \left(-\frac{14}{5}\right)(-1) + \frac{6}{5}\cdot 0 \\ \frac{4}{5}\cdot 1 + \left(-\frac{4}{5}\right)(-1) + 0 \\ -2\cdot 1 + 3(-1) + 0 \end{bmatrix}} = \underset{3 \times 1}{\boxed{\begin{bmatrix} \frac{23}{5} \\ \frac{8}{5} \\ -5 \end{bmatrix}}}$$

$$x = \frac{23}{5}, \quad y = \frac{8}{5}, \quad z = -5$$

Given $A^{-1} = \begin{bmatrix} 1 & -1 & 3 \\ 2 & 0 & 5 \\ -1 & 1 & 0 \end{bmatrix}$

b. Find a matrix B such that

$$AB = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

c. Find a matrix C such that

$$CA = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 1 & 1 \end{bmatrix}.$$

Solution

b)

$$AB = \begin{bmatrix} & & \end{bmatrix}$$

$$\underbrace{A^{-1}A}_{I_3} B = A^{-1}\begin{bmatrix} & & \end{bmatrix}$$

$$I_3 B =$$

$$B = \begin{bmatrix} 1 & -1 & 3 \\ 2 & 0 & 5 \\ -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

$3 \times \boxed{3 \qquad\qquad 3} \times 3$

row 2 col 3

$$= \begin{bmatrix} (1\cdot 1 + -1\cdot 0 + 3\cdot 1) & & \\ & & \underset{4 \quad 0 \quad 0}{(2\cdot 2+0\cdot 1+5\cdot 0)} \\ & & \end{bmatrix} = \begin{bmatrix} 4 & & \\ & & \boxed{4} \\ & & \end{bmatrix}$$

I leave it to you to compute the remaining entries (or use SageMath)

3×3 $\qquad\qquad$ 3×3

c) $CA = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 1 & 1 \end{bmatrix}$

Solution: multiply both sides on the right by A^{-1}

$$CA \cdot A^{-1} = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 1 & 1 \end{bmatrix} A^{-1}$$

$2 \times \boxed{3 \qquad 3} \times 3$

I leave it to you to do this tedious matrix multiplication (or use SageMath to save time)

Exercise 4.4 (p. 34, 1.13). Solve the following system of equations for x and y (which is *not a linear* system of equations in x, y).

set $a = x^2$, $b = xy$, $c = y^2$

$$\begin{aligned} \boxed{x^2 + xy - y^2 = 1} &\longleftrightarrow a + b - c = 1 \\ 2x^2 - xy + 3y^2 = 13 &\longleftrightarrow 2a - b + 3c = 13 \\ x^2 + 3xy + 2y^2 = 0 &\longleftrightarrow a + 3b + 2c = 0 \end{aligned}$$

[*Hint*: These equations are linear in the new variables $x_1 = x^2$, $x_2 = xy$, and $x_3 = y^2$.]

trick!

$$\begin{aligned} x_1 + x_2 - x_3 &= 1 \\ 2x_1 - x_2 + 3x_3 &= 13 \\ x_1 + 3x_2 + 2x_3 &= 0 \end{aligned}$$

for no good reason I used x_1, x_2, x_3 in place of a,b,c …

is a linear system

$$\begin{bmatrix} 1 & 1 & -1 \\ 2 & -1 & 3 \\ 1 & 3 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 13 \\ 0 \end{bmatrix}$$

Solve for x_1, x_2, x_3. by finding inverse of coefficient matrix A

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = A^{-1} \cdot \begin{bmatrix} 1 \\ 13 \\ 0 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 1 & -1 \\ 2 & -1 & 3 \\ 1 & 3 & 2 \end{bmatrix}$$

found A^{-1} via SageMath

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ -2 \\ 1 \end{bmatrix}$$

(Cocalc.)

$$A^{-1} \cdot \begin{bmatrix} 1 \\ 13 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} x^2 \\ xy \\ y^2 \end{bmatrix} = \begin{bmatrix} 4 \\ -2 \\ 1 \end{bmatrix}$$

$x^2 = 4$ $\quad xy = -2$ $\quad y^2 = 1$

$x = 2$ $\quad y = 1$

or $x = -2$ $\quad$ or $y = -1$

$$\boxed{(x,y) = (2,-1) \text{ or } (-2,1)}$$

Now an example with coefficient matrix A w/ determinant 0

$$2x + 3y = b_1$$
$$4x + 6y = b_2$$

Find conditions on the b_1 & b_2 so that the system of equations above is consistent (i.e has at least one solution (x,y)).

Note $\begin{bmatrix} 2 & 3 \\ 4 & 6 \end{bmatrix}$ has determinant $2\cdot 6 - 3\cdot 4 = 12 - 12 = 0$.

So it has no inverse.

We have to find RREF.

$$\left[\begin{array}{cc|c} 2 & 3 & b_1 \\ 4 & 6 & b_2 \end{array}\right]$$ ← augmented matrix

$$\left[\begin{array}{cc|c} 2 & 3 & b_1 \\ 0 & 0 & b_2 - 2b_1 \end{array}\right]$$ $R2 \leftrightarrow R2 - 2\cdot R1$

$$\begin{array}{rccc} R2 & 4 & 6 & b_2 \\ -2R1 & -4 & -6 & -2b_1 \\ \hline & 0 & 0 & b_2 - 2b_1 \end{array}$$

$$0x + 0y = b_2 - 2b_1$$

so $b_2 - 2b_1 = 0$ for the system to be consistent! (i.e have a solution)

If $b_2 - 2b_1 \neq 0$, there are <u>no</u> solutions!

If $b_2 - 2b_1 = 0$

$$\begin{bmatrix} 2 & 3 & b_1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 1 & 3/2 & b_1/2 \\ 0 & 0 & 0 \end{array}\right]$$

$$x + \frac{3}{2}y = \frac{b_1}{2}$$

$$x = \frac{b_1}{2} - \frac{3}{2}y$$

$$y = y$$

y is free variable

OR

$$x = \frac{b_1}{2} - \frac{3}{2}t$$

$$y = t$$

for any t

Infinitely many solutions!

9/18/2014

1.8 Matrix Transformations

Transformation = function = map

take something as input, give an output

In this class, the inputs will be n-tuples

$(x_1, x_2, \ldots, x_n)$

output will be m-tuple $(w_1, w_2, \ldots w_m)$

$\mathbb{R}$ = denotes the set of real numbers
(book uses just R)

$\mathbb{R}^n$ is the set of n-tuples of real numbers

example: with $n=3$ $\quad \mathbb{R}^3 = \{(x_1, x_2, x_3) \mid x_i \in \mathbb{R}\}$
$i = 1, 2, 3$

$(1, 0, -1) \in \mathbb{R}^3$

element of

$(0, 1, -1)$ is another element of $\mathbb{R}^3$

Very often we will write n-tuples as column vectors

$\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$ is same as $(0, 1, -1)$

Given $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3$

and a matrix $A = \begin{bmatrix} 0 & 1 & 1 \\ 2 & 1 & 0 \end{bmatrix}$ (for example)

the matrix transformation T_A defined by A is

$$\vec{w} = T_A(\vec{x}) = A\vec{x}$$

T_A:R^3 —> R^2

$\mathbb{R}^3 \to \mathbb{R}^2$

where $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

so $\vec{w} = \underbrace{\begin{bmatrix} 0 & 1 & 1 \\ 2 & 1 & 0 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{\vec{x}} = \underbrace{\begin{bmatrix} 0 \cdot x_1 + 1 \cdot x_2 + 1 \cdot x_3 \\ 2x_1 + x_2 + 0 \cdot x_3 \end{bmatrix}}_{A\vec{x}} = \begin{bmatrix} x_2 + x_3 \\ 2x_1 + x_2 \end{bmatrix}$

$\vec{w} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$

$2 \times \boxed{3 \quad\quad 3} \times 1$ $\quad$ 2×1

$$\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} x_2 + x_3 \\ 2x_1 + x_2 \end{bmatrix}$$

$$w_1 = x_2 + x_3$$
$$w_2 = 2x_1 + x_2$$

More concretely if $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ so $T_A\left(\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}\right) = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$

$\left(A = \begin{bmatrix} 0 & 1 & 1 \\ 2 & 1 & 0 \end{bmatrix} \right)$

We get same answer via

$T_A(\vec{x}) = A\vec{x}$

$$T_A\left(\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}\right) = \underbrace{\begin{bmatrix} 0 & 1 & 1 \\ 2 & 1 & 0 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}}_{\vec{x}} = \begin{bmatrix} 0\cdot 1 + 1\cdot 0 + 1(-1) \\ 2\cdot 1 + 1\cdot 0 + 0(-1) \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

$2 \times \boxed{3 \quad 3} \times 1$

$\updownarrow$

$(-1, 2)$

$T_A : \mathbb{R}^{\boxed{3}} \longrightarrow \mathbb{R}^{\boxed{2}}$

domain input

codomain output

A is a 2×3 matrix

In general, if A is an $m \times n$ matrix

$\uparrow$ rows $\uparrow$ columns

then $T_A : \mathbb{R}^n \longrightarrow \mathbb{R}^m$

p. 82 #2 A is 4×5 matrix. Find the domain & codomain of $T_A(\vec{x}) = A\vec{x}$

Answer: domain $\mathbb{R}^5$

codomain $\mathbb{R}^4$

going backwards: what is the matrix A associated to

$$w_1 = x_2 + x_3$$
$$w_2 = 2x_1 + x_2$$

Domain requires x_1, x_2, x_3 so domain $\mathbb{R}^3$

codomain/output is w_1, w_2 so codomain is $\mathbb{R}^2$

so $T_A : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$

so our matrix A is 2×3 matrix

reading off the coefficients of $w_1 = 0x_1 + 1 \cdot x_2 + 1 \cdot x_3$

gives $A = \begin{bmatrix} 0 & 1 & 1 \\ 2 & 1 & 0 \end{bmatrix}$

$\mathbb{R}^3$ standard basis vectors:

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \qquad e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

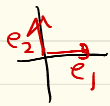
$\Updownarrow$

$(1,0,0)$

($\mathbb{R}^4$ has 4 standard basis vectors e_1, e_2, e_3, e_4)

$\mathbb{R}^2$ $\quad e_1 = (1,0)$

$\quad e_2 = (0,1)$



Every vector in $\mathbb{R}^3$ is a sum of some multiples of e_1, e_2, e_3:

Example: $\begin{bmatrix} 2 \\ 5 \\ -7 \end{bmatrix} \in \mathbb{R}^3$

$$\begin{bmatrix} 2 \\ 5 \\ -7 \end{bmatrix} = 2e_1 + 5e_2 + (-7)e_3$$

$$= 2\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 5\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + (-7)\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 5 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ -7 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ 5 \\ -7 \end{bmatrix}$$

Let A be a matrix. Then $T_A(e_i)$ is the i^{th} column of A: (this fact is used very often!!)

Why? $T_A(e_i) = Ae_i = \underbrace{\begin{bmatrix} | & | & | & | & | \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix}}_{e_i}$

Given $(x_1, x_2, x_3) \in \mathbb{R}^3$ as input,

Let $w_1 = x_1 + x_3$

$w_2 = x_1 + x_2 + x_3$

$$\mathbb{R}^3 \xrightarrow{T} \mathbb{R}^2$$

$$(x_1, x_2, x_3) \longmapsto (w_1, w_2)$$

Find the matrix associated to this transformation

$$T(e_1) = T\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) = \begin{bmatrix}1\\1\end{bmatrix} \begin{array}{l} \leftarrow 1+0 \\ \leftarrow 1+0+0 \end{array}$$

$$T(e_2) = T\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right) = \begin{bmatrix}0+0\\0+1+0\end{bmatrix} = \begin{bmatrix}0\\1\end{bmatrix}$$

$$T(e_3) = T\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right) = \begin{bmatrix}0+1\\0+0+1\end{bmatrix} = \begin{bmatrix}1\\1\end{bmatrix}$$

so matrix $A = \begin{bmatrix}T(e_1) & T(e_2) & T(e_3)\end{bmatrix} = \begin{bmatrix}1 & 0 & 1\\1 & 1 & 1\end{bmatrix}$

Exercise 5.1. Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 9 & 7 & 2 \\ 0 & 6 & 2 \end{bmatrix}$ and let $T_A : \mathbb{R}^3 \to \mathbb{R}^3$ be given by $T_A(x) = Ax$ where $x \in \mathbb{R}^3$ is treated as a 3×1 column vector.

1. Find $T(1,0,0)$, $T(0,1,0)$, and $T(0,0,1)$ (these can be read off immediately from the matrix).
2. Find $T(1,1,1)$. (Answer:$(6,18,8)$)

Solution 1. $T_A(1,0,0) = T_A\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) = \begin{bmatrix}1\\9\\0\end{bmatrix}$ or $(1,9,0)$

$$\left(\begin{bmatrix} 1 & 2 & 3 \\ 9 & 7 & 2 \\ 0 & 6 & 2 \end{bmatrix}\underset{\uparrow \atop e_1}{\begin{bmatrix}1\\0\\0\end{bmatrix}} = \begin{bmatrix}1\\9\\0\end{bmatrix}\right)$$

2. $T_A(1,1,1) = \begin{bmatrix} 1 & 2 & 3 \\ 9 & 7 & 2 \\ 0 & 6 & 2 \end{bmatrix}\begin{bmatrix}1\\1\\1\end{bmatrix} = \begin{bmatrix}6\\18\\8\end{bmatrix}$ $1+2+3$

$3 \times \boxed{3 \quad 3} \times 1 \qquad 3 \times 1$

OR: $(1,1,1) = 1 \cdot e_1 + 1 \cdot e_2 + 1 \cdot e_3$

$$\begin{aligned} T_A(1,1,1) &= T(e_1 + e_2 + e_3) \\ &= T(e_1) + T(e_2) + T(e_3) \\ &= \begin{bmatrix}1\\9\\0\end{bmatrix} + \begin{bmatrix}2\\7\\6\end{bmatrix} + \begin{bmatrix}3\\2\\2\end{bmatrix} \\ &= \begin{bmatrix}6\\18\\8\end{bmatrix} \end{aligned}$$

Exercise 5.2. Let $T : \mathbb{R}^3 \to \mathbb{R}$ be given by $T(x, y, z) = (2x + y - 3z, x + y)$. This is shorthand for

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 2x + y - 3z \\ x + y \end{bmatrix}$$

Then T is a linear transformation, and hence corresponds to multiplying elements of $\mathbb{R}^3$ (entered as $3x1$ column vectors) on the left by a $2x3$ matrix A. Find A

Solution: Just look at where standard basis vectors go!!

$$T(e_1) = T\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 2\cdot 1 + 0 - 3\cdot 0 \\ 1+0 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 2 & ? & ? \\ 1 & ? & ? \end{bmatrix}$$

leave it to you to find the remaining two columns

For any matrix A, the transformation $T_A : \mathbb{R}^n \to \mathbb{R}^m$ is a <u>linear transformation</u>, meaning:

① For every $u, v \in \mathbb{R}^n$
"in"

$$T_A(u+v) = T_A(u) + T_A(v)$$

(because: $A(u+v) = Au + Av$)

② For every $k \in \mathbb{R}$, $v \in \mathbb{R}^n$

$$T_A(kv) = k\,T_A(v).$$

(because: $A(kv) = kAv$)

Back to $A = \begin{bmatrix} 2 & 3 & 1 \\ -1 & 3 & 0 \end{bmatrix}$.

What is $T_A(3e_1 + 4e_2 - e_3)$?

$$\begin{aligned} T_A(3e_1+4e_2-e_3) &= T_A(3e_1) + T_A(4e_2) + T_A(-e_3) \\ &= 3T_A(e_1) + 4T_A(e_2) + (-1)T_A(e_3) \\ &= 3\begin{bmatrix} 2 \\ -1 \end{bmatrix} + 4\begin{bmatrix} 3 \\ 3 \end{bmatrix} + (-1)\begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} 6 \\ -3 \end{bmatrix} + \begin{bmatrix} 12 \\ 12 \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 17 \\ 9 \end{bmatrix} \end{aligned}$$

Easier way:

$$3e_1 + 4e_2 - e_3 = \begin{bmatrix} 3 \\ 4 \\ -1 \end{bmatrix}$$

What is $T_A\left(\begin{bmatrix} 3 \\ 4 \\ -1 \end{bmatrix}\right)$? $\quad A = \begin{bmatrix} 2 & 3 & 1 \\ -1 & 3 & 0 \end{bmatrix}$

$$T_A\left(\begin{bmatrix} 3 \\ 4 \\ -1 \end{bmatrix}\right) = A \cdot \begin{bmatrix} 3 \\ 4 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 & 3 & 1 \\ -1 & 3 & 0 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \cdot 3 + 3 \cdot 4 + 1 \cdot (-1) \\ (-1) \cdot 3 + 3 \cdot 4 + 0(-1) \end{bmatrix}$$

$$= \begin{bmatrix} 17 \\ 9 \end{bmatrix}$$

Let $T: \mathbb{R}^3 \to \mathbb{R}^2$ linear transformation so that $T(1,0,-1) = (2,3)$
$T(2,1,3) = (-1,0)$

Find $T(8,3,7)$

Solution:

Step 1. Express $(8,3,7) = \underline{a}\,\underline{(1,0,-1)} + \underline{b}\,(2,1,3)$
Find a, b.

$$\begin{bmatrix} 8 \\ 3 \\ 7 \end{bmatrix} = a\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + b\begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$$

One way:

$$\begin{bmatrix} 8 \\ 3 \\ 7 \end{bmatrix} = \begin{bmatrix} a \\ 0 \\ -a \end{bmatrix} + \begin{bmatrix} 2b \\ b \\ 3b \end{bmatrix}$$

$$\begin{bmatrix} 8 \\ 3 \\ 7 \end{bmatrix} = \begin{bmatrix} a+2b \\ b \\ -a+3b \end{bmatrix}$$

$b = 3$

$8 = a + 2\cdot b$

$8 = a + 2\cdot 3$

$8 = a + 6$

$2 = a$

so $(8,3,7) = 2(1,0,-1) + 3(2,1,3)$

Step 2 (Since T is a linear transformation)

$T(8,3,7) = T\big(2(1,0,-1) + 3(2,1,3)\big)$

$= T(\qquad) + T(\qquad)$

$$= T(2\cdot(1,0,-1)) + T(3(2,1,3))$$

$$= 2\underbrace{T((1,0,-1))} + 3\,\underbrace{T(2,1,3)}$$

$$= 2\cdot(2,3) + 3\cdot(-1,0)$$

given as part of the problem

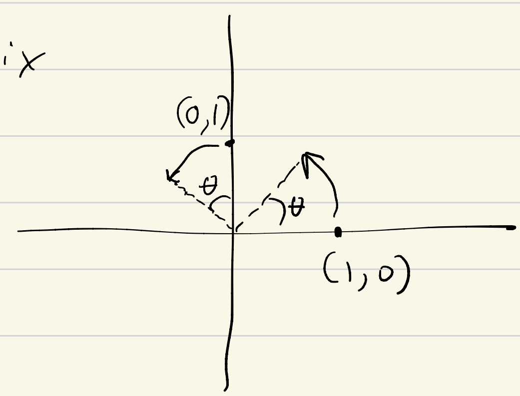
$$= (4,6) + (-3,0)$$

$$= (4-3,\ 6+0)$$

$$= \boxed{(1,6)}$$

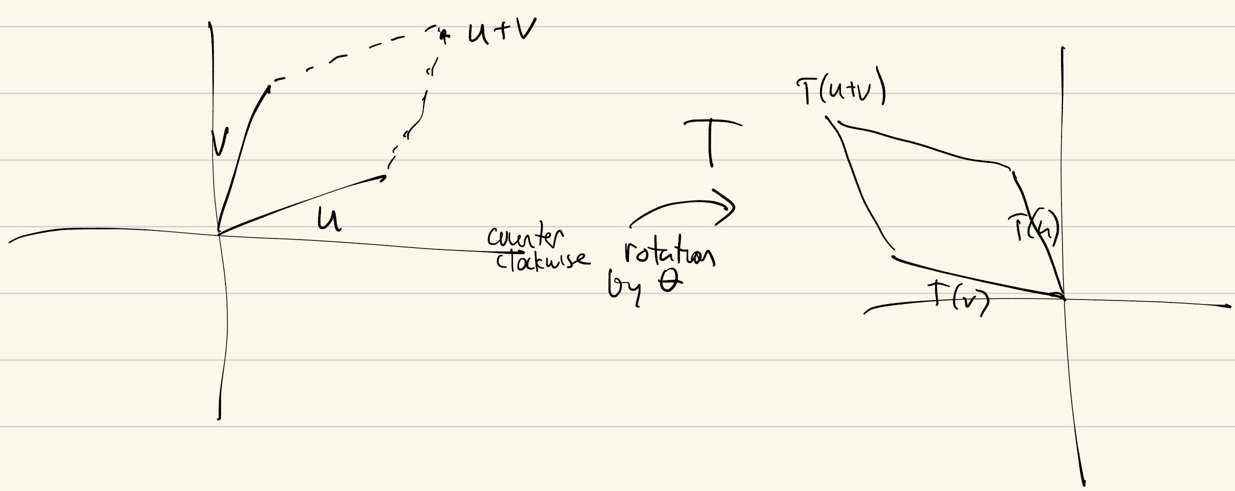
$$T(8,3,7) = (1,6)$$

Rotation matrix $\qquad \mathbb{R}^2$



$T:\mathbb{R}^2 \to \mathbb{R}^2$ of rotating by an angle of θ around the origin is a linear transformation

$$T(u+v) = T(u) + T(v)$$



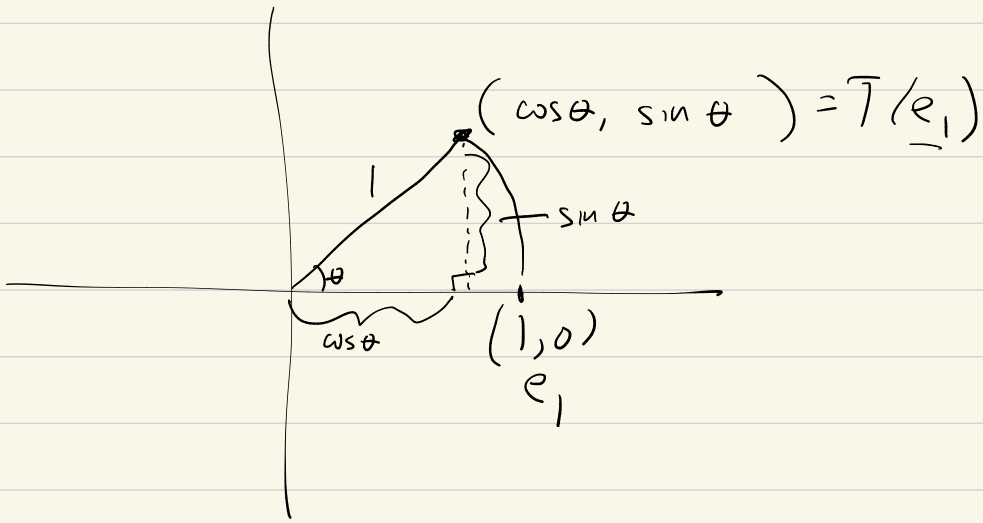
So associated matrix has columns $\qquad T: \mathbb{R}^2 \to \mathbb{R}^2$

$$R_\theta = \begin{bmatrix} T(e_1) & T(e_2) \end{bmatrix} = \boxed{\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}}$$

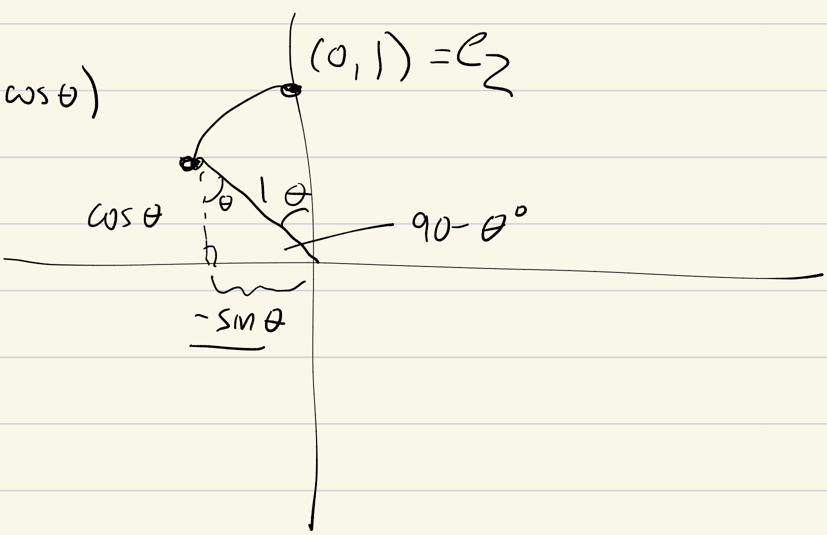
1st column $\qquad$ 2nd column

matrix for

why?



$T(e_2) = (-\sin\theta, \cos\theta)$



9/23

$A = \begin{bmatrix} 2 & 3 & 1 \\ -1 & 3 & 0 \end{bmatrix}$ 2×3 matrix

so $T_A : \mathbb{R}^3 \to \mathbb{R}^2$

$\mathbb{R}^3$ has $e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

What is $T_A(e_1)$?

$T_A(e_2)$?

$T_A(e_3)$?

$$T_A(e_1) = A \cdot e_1 = \begin{bmatrix} 2 & 3 & 1 \\ -1 & 3 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2\cdot 1 + 3\cdot 0 + 1\cdot 0 \\ -1\cdot 1 + 3\cdot 0 + 0\cdot 0 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

$2 \times \boxed{3 \quad 3} \times 1 \qquad 2 \times 1$

so $T_A(e_1) = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

$T_A(e_2) = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$

$T_A(e_3) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Point: it is easy to look at the matrix and figure out $T_A(e_1), \ldots T_A(e_n)$

Answer: $T_A(e_i)$ is the i^{th} column of A.

recall what a function is: $f: \underset{\text{(input)}}{\overset{\text{domain}}{\mathbb{R}}} \longrightarrow \underset{\text{(output)}}{\overset{\text{codomain}}{\mathbb{R}}}$

$\mathbb{R}$ = real numbers

$f(x) = x^2$ not a linear function

$f(4) = (4)^2 = 16$

Functions in Linear algebra: $T: \mathbb{R}^n \longrightarrow \mathbb{R}^m$

$\mathbb{R}^2 = \{ (x, y) \mid x, y \text{ are real numbers} \}$ R^2 is a "set", which roughly means it's a container of elements

examples of elements of R^2 $(1, 3)$, $(-\frac{1}{2}, 2)$, $(\pi, 5) \neq (5, \pi)$

$\pi \approx 3.1415$

$\mathbb{R}^3$ = set of ordered triples

(x, y, z) or (x_1, x_2, x_3)

$(1, 0, -2)$

$\mathbb{R}^n$ = set of n-tuples $\underbrace{(x_1, x_2, \dots, x_n)}_{\text{row vector}} \longleftrightarrow \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}}_{\text{column vector}}$

Let $A = \begin{bmatrix} 1 & 1 & 2 \\ 3 & 0 & 0 \end{bmatrix}$ 2×3 matrix (rows × columns)

Define the function $T_A: \mathbb{R}^3 \longrightarrow \mathbb{R}^2$

as matrix multiplication by A:

$(x_1, x_2, x_3) \xmapsto{T_A} \underset{A}{\begin{bmatrix} 1 & 1 & 2 \\ 3 & 0 & 0 \end{bmatrix}} \underset{\vec{x}}{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}} = \underset{\text{output}}{\begin{bmatrix} 1 \cdot x_1 + 1 \cdot x_2 + 2x_3 \\ 3x_1 \end{bmatrix}}$

input

treat as column vector

$2 \times \boxed{3 \quad\quad 3} \times 1$ $\quad 2 \times 1$

matrix multiplication by A (on the left)

$= \begin{bmatrix} x_1 + x_2 + 2x_3 \\ 3x_1 \end{bmatrix}$

= output

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 3 & 0 & 0 \end{bmatrix}$$

$$T_A : \mathbb{R}^3 \to \mathbb{R}^2$$

$$\underset{\text{input}}{(x_1, x_2, x_3)} \to (\underbrace{x_1 + x_2 + 2x_3}_{\text{first output}}, \underbrace{3x_1}_{\text{2nd output}})$$

$$T_A(1,0,0) = (1+0+2\cdot 0, 3\cdot 1) = (1, 3) \leftrightarrow \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$e_1 = (1,0,0)$
$e_2 = (0,1,0)$ $\qquad e_1, e_2, e_3$ standard basis vectors for $\mathbb{R}^3$
$e_3 = (0,0,1)$

(In $\mathbb{R}^2$ $\qquad e_1 = (1,0) \leftrightarrow \begin{bmatrix} 1 \\ 0 \end{bmatrix}$
$\qquad e_2 = (0,1) \leftrightarrow \begin{bmatrix} 0 \\ 1 \end{bmatrix}$)

$$\underset{(0,1,0)}{T_A(e_2)} = A e_2 = \underbrace{\begin{bmatrix} 1 & 1 & 2 \\ 3 & 0 & 0 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}}_{\substack{e_2 \text{ as a} \\ \text{column vector}}} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

General Fact: For A an $m \times n$ matrix

$T_A : \mathbb{R}^n \to \mathbb{R}^m$ and

$T_A(e_i)$ = column i of A

16. Find the standard matrix for the transformation $T: R^4 \to R^2$ defined by

$$\begin{aligned} w_1 &= 2x_1 + 3x_2 - 5x_3 - x_4 \\ w_2 &= x_1 - 5x_2 + 2x_3 - 3x_4 \end{aligned}$$

and then compute $T(1, -1, 2, 4)$ by directly substituting in the equations and then by matrix multiplication.

Solution: $T = T_A$ for some 2×4 matrix A

Find matrix A.

Just compute T applied to e_1, e_2, e_3, e_4.

$$A = \begin{bmatrix} | & & & \\ T(e_1) & T(e_2) & T(e_3) & T(e_4) \\ | & & & \end{bmatrix}$$

first column

$e_1 = (1,0,0,0) \leftrightarrow \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

$$T(e_1) = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 2\cdot 1 + 3\cdot 0 - 5\cdot 0 - 0 \\ 1 - 5\cdot 0 + 2\cdot 0 - 3\cdot 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$T(e_2) = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 3 \\ -5 \end{bmatrix}$$

$$T(e_3) = \begin{bmatrix} -5 \\ 2 \end{bmatrix}$$

$$T(e_4) = \begin{bmatrix} -1 \\ -3 \end{bmatrix}$$

$$\boxed{A = \begin{bmatrix} 2 & 3 & -5 & -1 \\ 1 & -5 & 2 & -3 \end{bmatrix}}$$

$$T(1,-1,2,4) = \begin{bmatrix} 2 & 3 & -5 & -1 \\ 1 & -5 & 2 & -3 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 2 \\ 4 \end{bmatrix}$$

$2 \times \boxed{4 \quad\quad 4} \times 1$

$$= \begin{bmatrix} 2\cdot 1 + 3(-1) + (-5)2 + (-1)4 \\ 1\cdot 1 + (-5)(-1) + (2)(2) + (-3)(4) \end{bmatrix}$$

2×1

$$= \begin{bmatrix} 2 - 3 - 10 - 4 \\ 1 + 5 + 4 - 12 \end{bmatrix} = \begin{bmatrix} -15 \\ -2 \end{bmatrix}$$

$$= (-15, -2)$$

Linear Transformation: $T: \mathbb{R}^n \to \mathbb{R}^m$

satisfies 1) $T(v+w) = T(v) + T(w)$

$v = (v_1, v_2, \ldots, v_n)$ "vectors" $\Leftrightarrow$ "n-tuples"

$(v+w)^2 \neq v^2 + w^2$

$w = (w_1, w_2, \ldots, w_n)$

$v+w = (v_1+w_1, v_2+w_2, \ldots v_n+w_n)$

2) $T(r \cdot v) = rT(v)$ $\quad r \in \mathbb{R}$, $v \in \mathbb{R}^n$

$v = (v_1, v_2, \ldots, v_n)$

$r \cdot v = (rv_1, rv_2, \ldots, r \cdot v_n)$

$\mathbb{R}^2$: $\quad e_1 = (1,0) \leftrightarrow \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$e_2 = (0,1) \leftrightarrow \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ $\qquad$ standard basis

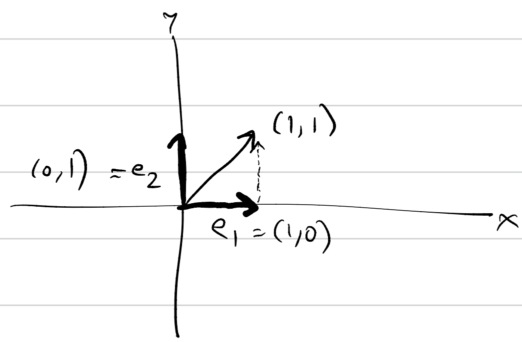
$$(x_1, x_2) \mapsto \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = x_1 e_1 + x_2 e_2$$

$$= \begin{bmatrix} x_1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ x_2 \end{bmatrix}$$

$$= \begin{bmatrix} x_1 + 0 \\ 0 + x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$(2,-3) = 2e_1 + (-3)e_2$$

$$(1,1) = 1 \cdot e_1 + 1 \cdot e_2$$



$T: \mathbb{R}^n \to \mathbb{R}^m$

If T is a linear transformation, & you know $T(e_1), T(e_2), \ldots, T(e_n)$

then you can find $T(\text{any vector})$

Example $\quad T: \mathbb{R}^2 \to \mathbb{R}^3$ is linear transformation

$T(e_1) = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} \qquad T(e_2) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

What is $T\left(\begin{bmatrix} 2 \\ -3 \end{bmatrix}\right)$?

Answer: $T = T_A$ where

$$A = \begin{bmatrix} | & | \\ T(e_1) & T(e_2) \\ | & | \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 0 \\ 4 & 1 \end{bmatrix}$$

$$T_A(2,-3) = \begin{bmatrix} 1 & 0 \\ 2 & 0 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ -3 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 2\cdot 4 + 1\cdot(-3) \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 5 \end{bmatrix}$$

$$(2,-3) = 2e_1 - 3e_2$$

$$\begin{aligned} T(2,-3) &= T(2e_1 + -3e_2) \\ &= T(2e_1) + T(-3e_2) \\ &= 2T(e_1) - 3T(e_2) \\ &= 2\begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} - 3\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 2 \\ 4 \\ 8 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 5 \end{bmatrix} \end{aligned}$$

1.9 Composition of Matrix Transformations

Simple, but a bit technical/dry for a first course

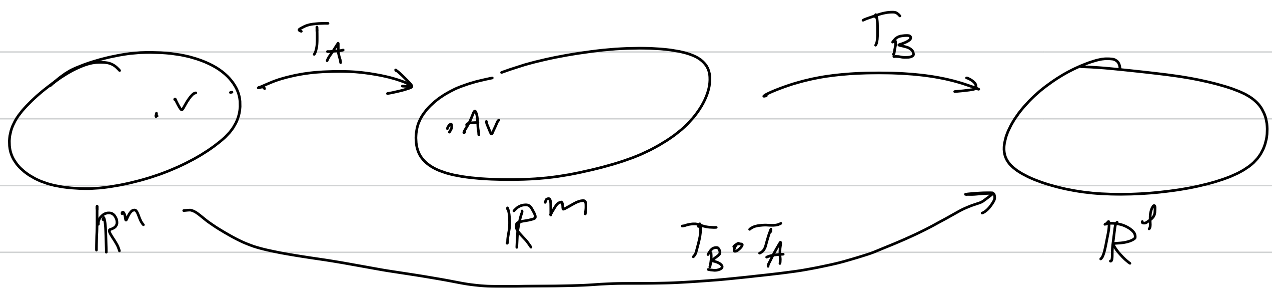
Why is matrix multiplication so strange?

Short Answer: $\underset{l\times m}{B}\ \underset{m\times n}{A}$ is the matrix associated to the composition $T_B \circ T_A : \mathbb{R}^n \to \mathbb{R}^l$

Let A be an $m\times n$ matrix.

$T_A : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation

$T_A(v) = Av$



Let B be an $l\times m$ matrix. $T_B : \mathbb{R}^m \to \mathbb{R}^l$ is a linear transformation.

Theorem $T_{BA} = T_B \circ T_A$

BA matrix multiplication

Proof. $T_B \circ T_A$ is a linear transformation:

$$(T_B\circ T_A)(u+v) = T_B(T_A(u+v)) = T_B(T_A u + T_A v)$$

$$= T_B(T_A u) + T_B(T_A)v$$

$$= (T_B\circ T_A)u + (T_B \circ T_A)v$$

$$(T_B\circ T_A)(kv) = T_B(T_A(kv)) = T_B(kT_A(v)) = kT_B(T_A v)$$

$$= k(T_B\circ T_A)v.$$

So there is an $l\times n$ matrix M such that $T_M = T_B\circ T_A$.
because every linear transformation is multiplication by a matrix.

We want to show $M = BA$

The ith column of M is $T_M(e_i)$

$$T_M(e_i) = (T_B \circ T_A)(e_i) = T_B(T_A e_i) = T_B \begin{bmatrix} | \\ A_i \\ | \end{bmatrix} = B \begin{bmatrix} | \\ A_i \\ | \end{bmatrix}$$

ith column of A

$$= \underset{l \times m}{\begin{bmatrix} - & B_1 & - \\ - & B_2 & - \\ & \vdots & \\ - & B_l & - \end{bmatrix}} \underset{m \times 1}{\begin{bmatrix} | \\ A_i \\ | \end{bmatrix}}$$

$$\text{ith column of } M = \underset{l \times 1}{\begin{bmatrix} B_1 \cdot A_i \\ B_2 \cdot A_i \\ \vdots \\ B_l \cdot A_i \end{bmatrix}}$$

Meanwhile, by the strange definition of matrix multiplication BA, the ith column of BA is found by <u>dot</u> multiplying each of the rows of B by column i of A. In other words

$$\text{the ith column of } BA = \underset{l \times 1}{\begin{bmatrix} B_1 \bullet A_i \\ B_2 \bullet A_i \\ \vdots \\ B_l \bullet A_i \end{bmatrix}}$$

dot product

So the ith column of M agrees with the ith column of BA, for every $i = 1, 2, \ldots n$, and hence $M = BA$!

Chapter 2 Determinants:

(Determinants only apply to <u>square</u> matrices, i.e. of size: 2×2, 3×3, 4×4, ..., $n\times n$, ...)

Recall in 2×2 case: $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then

$$A^{-1} = \frac{1}{\Delta}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \quad \text{where } \Delta = ad - bc$$

determinant

$$\begin{matrix} a & b \\ c & d \end{matrix}$$

Determinant determines if a matrix has an inverse (if determinant is 0, then matrix does not have inverse if determinant is nonzero, then matrix has an inverse.)

2.1 Let A be an $n\times n$ matrix.

The <u>minor</u> M_{ij} is the determinant of the $(n-1)\times(n-1)$ submatrix formed by deleting row i and column j.

The <u>cofactor</u> $C_{ij} = (-1)^{i+j} M_{ij}$

Example: 3x3 matrix $\begin{bmatrix} 3 & 1 & -4 \\ 2 & 5 & 6 \\ 1 & 4 & 8 \end{bmatrix}$

$M_{1,1}$ $\begin{matrix} \cancel{3} & \cancel{1} & \cancel{-4} \\ \cancel{2} & 5 & 6 \\ \cancel{1} & 4 & 8 \end{matrix}$ $\boxed{\begin{matrix} 5 & 6 \\ 4 & 8 \end{matrix}} \xrightarrow{\text{determinant}}$

$$= 5 \cdot 8 - 4 \cdot 6$$
$$= 40 - 24$$
$$M_{1,1} = 16$$

$M_{2,3}$ $\begin{matrix} 3 & 1 & \cancel{4} \\ \cancel{2} & \cancel{5} & \cancel{6} \\ 1 & 4 & \cancel{8} \end{matrix}$ $\begin{vmatrix} 3 & 1 \\ 1 & 4 \end{vmatrix} = 3 \cdot 4 - 1 \cdot 1$

$$= 12 - 1 = 11$$

absolute
value sign
means determinant

$$C_{2,3} = (-1)^{2+3} M_{2,3} = -1 \cdot 11 = -11$$

$$C_{1,1} = (-1)^{1+1} M_{1,1} = 1 \cdot M_{1,1} = 16.$$

Definition (of determinant, via cofactor expansion)

Take any row or column of the square (n x n) matrix

For each entry in the selected row or column, multiply it by its cofactor. Then add up all these products and the number you get is called the determinant.

Find the determinant of $A = \begin{matrix} 3 & 1 & -4 \\ 2 & 5 & 6 \\ 1 & 4 & 8 \end{matrix}$

Solution: Pick any row or column, and "expand by cofactors along that row or column":

Let's pick row 1:

$$\text{Determinant} = 3C_{1,1} + 1\cdot C_{1,2} + -4C_{1,3}$$

$$= 3\cdot(-1)^2 \underbrace{\begin{vmatrix} 5 & 6 \\ 4 & 8 \end{vmatrix}}_{M_{1,1}} + 1\cdot(-1)^3 \underbrace{\begin{vmatrix} 2 & 6 \\ 1 & 8 \end{vmatrix}}_{M_{1,2}} - 4(-1)^4 \underbrace{\begin{vmatrix} 2 & 5 \\ 1 & 4 \end{vmatrix}}_{M_{1,3}}$$

$$= 3\cdot 16 - 1\cdot 10 - 4\cdot 3$$

$$= 48 - 10 - 12$$

$$= 26$$

Arrow technique (only for 3x3! Not for 4x4, or higher)

$$A = \begin{bmatrix} 3 & 1 & -4 \\ 2 & 5 & 6 \\ 1 & 4 & 8 \end{bmatrix}$$

Find det(A) using arrow technique

Solution

$$A = \begin{matrix} 3 & 1 & -4 & 3 & 1 \\ 2 & 5 & 6 & 2 & 5 \\ 1 & 4 & 8 & 1 & 4 \end{matrix}$$

Copy first two columns (in purple)

$$\det(A) = 3\cdot5\cdot8 + 1\cdot6\cdot1 + (-4)2\cdot4 - \left(1\cdot5(-4) + 4\cdot6\cdot3 + 8\cdot2\cdot1\right)$$

$$= \underbrace{(120 + 6 + -32)}_{94} - \underbrace{(-20 + 72 + 16)}_{68}$$

$$= 94 - 68$$

$$= 26$$

Anton presents definition of determinant via expansion by minors, which is a strange/complicated procedure, but works for an introduction.

A more abstract viewpoint on the determinant (of an $n \times n$ matrix is) that the determinant is the unique function whose

- input is an $n \times n$ matrix
- output is a single number

satisfying the following properties:

1) determinant is "multilinear" in the rows:

a) if a row is multiplied by a scalar k, the determinant is multiplied by k

e.g $\det \begin{bmatrix} 2 & 4 & 6 \\ 1 & 1 & 1 \\ 3 & 2 & 7 \end{bmatrix} = 2 \cdot \det \begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \\ 3 & 2 & 7 \end{bmatrix}$

b) if a row is expressed as a sum of two vectors, the determinant is the sum of determinants of the two matries where that row is replaced by each of the vectors (but other rows are unchanged)

e.g $(2,4,6) = (2,0,0) + (0,4,6)$

so $\det \begin{bmatrix} 2 & 4 & 6 \\ 1 & 1 & 1 \\ 3 & 2 & 7 \end{bmatrix} = \det \begin{bmatrix} 2 & 0 & 0 \\ 1 & 1 & 1 \\ 3 & 2 & 7 \end{bmatrix} + \det \begin{bmatrix} 0 & 4 & 6 \\ 1 & 1 & 1 \\ 3 & 2 & 7 \end{bmatrix}$

2) Interchanging two different rows multiplies determinant by -1

e.g $\det \begin{bmatrix} 2 & 4 & 6 \\ 1 & 1 & 1 \\ 3 & 2 & 7 \end{bmatrix} = (-1) \det \begin{bmatrix} 1 & 1 & 1 \\ 2 & 4 & 6 \\ 3 & 2 & 7 \end{bmatrix}$

3) $\det(I_n) = 1$ (where $I_n = \begin{bmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{bmatrix}$ is $n \times n$ identity matrix

Note One consequence of condition "2) interchanging two rows multiplies determinant by (-1)" is determinant of matrix with two identical rows is 0:

e.g if $D = \begin{bmatrix} 2 & 4 & 6 \\ 1 & 1 & 1 \\ 2 & 4 & 6 \end{bmatrix}$ then $\det D = (-1) \det D$

by swapping rows 1 & 3, but that leaves the matrix D unchanged

& solving $x = -x$ $(x = \det D)$

$$2x = 0$$

$$x = 0$$

so $\det D = 0$

(properties of the determinant listed three or four pages ago)

Example Use properties 1, 2, 3 to show $\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$.

Solution: Since $(a,b) = (a,0) + (0,b)$ & $(c,d) = (c,0) + (0,d)$

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \underbrace{\det \begin{bmatrix} a & 0 \\ c & d \end{bmatrix}}_{} + \underbrace{\det \begin{bmatrix} 0 & b \\ c & d \end{bmatrix}}_{}$$

$$= \det \begin{bmatrix} a & 0 \\ c & 0 \end{bmatrix} + \det \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} + \det \begin{bmatrix} 0 & b \\ c & 0 \end{bmatrix} + \det \begin{bmatrix} 0 & b \\ 0 & d \end{bmatrix}$$

$$= ac \underbrace{\det \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}}_{=0} + ad \underbrace{\det \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{=1} + bc \det \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + bd \underbrace{\det \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}}_{=0}$$

swap rows

$$= 0 + ad + bc(-1) \underbrace{\det \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{=1} + 0$$

$$= ad - bc$$

Chapter 2, Determinants

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Determinant: $\Delta = ad - bc$

$\Delta \neq 0$ if and only if A has an inverse

$$A^{-1} = \frac{1}{\Delta} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} d/\Delta & -b/\Delta \\ -c/\Delta & a/\Delta \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & -2 \\ 5 & 3 \end{bmatrix}$$

$\det A = 1 \cdot 3 - (-2)(5)$

$= 3 + 10 = 13 \neq 0$

so A has an inverse

$$\begin{bmatrix} -1 & 3 & 1 \\ 2 & 5 & 3 \\ 1 & -2 & 1 \end{bmatrix}$$

3×3 matrix

Here's how to find the determinant of this matrix.

① Pick a row or column. (Amazing fact: we'll get the same answer no matter what row or column you pick)

② "Expand by cofactors along that row/column"

Pick first column $\begin{matrix} -1 \\ 2 \\ 1 \end{matrix}$

$$\begin{bmatrix} + & - & + & - \\ - & + & - & + \\ + & & (-1)^{i+j} & \\ - & & & \end{bmatrix}$$

$$\begin{bmatrix} + & - \\ - & + \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$a(d) - c(b)$

$$\begin{bmatrix} -1 & 3 & 1 \\ 2 & 5 & 3 \\ 1 & -2 & 1 \end{bmatrix} \qquad \begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

$$= (-1)\begin{vmatrix} 5 & 3 \\ -2 & 1 \end{vmatrix} - 2\begin{vmatrix} 3 & 1 \\ -2 & 1 \end{vmatrix} + 1\begin{vmatrix} 3 & 1 \\ 5 & 3 \end{vmatrix}$$

$$= -1\left(5\cdot 1 - (-2)(3)\right) - 2\left(3\cdot 1 - (-2)\cdot 1\right) + 1(9-5)$$

$$= -1(11) - 2(5) + 1\cdot 4$$

$$= -11 - 10 + 4 = -17$$

$$\begin{bmatrix} -1 & 3 & 1 \\ 2 & 5 & 3 \\ 1 & -2 & 1 \end{bmatrix} \qquad \begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

Let's compute determinant again, expand by cofactors along 2^{nd} row

$$\det = \underset{(-1)^{2+1}}{\underset{\uparrow}{-}} 2 \begin{vmatrix} 3 & 1 \\ -2 & 1 \end{vmatrix} + 5 \begin{vmatrix} -1 & 1 \\ 1 & 1 \end{vmatrix} - 3 \begin{vmatrix} -1 & 3 \\ 1 & -2 \end{vmatrix}$$

$$= -2\left|3 - -2\right| + 5(-1-1) - 3(2-3)$$

$$= -2\cdot 5 + 5(-2) - 3(-1)$$

$$= -10 - 10 + 3 = -20 + 3 = \boxed{-17}$$

$$\det\begin{bmatrix} a & b \\ c & d \end{bmatrix} = a(d) - c(b) \quad \checkmark$$

$$\begin{bmatrix} -1 & 3 & 1 \\ 2 & 5 & 3 \\ 1 & -2 & 1 \end{bmatrix} \begin{matrix} -1 & 3 \\ 2 & 5 \\ 1 & -2 \end{matrix}$$

$$= -1\cdot 5\cdot 1 + 3\cdot 3\cdot 1 + 1\cdot 2\cdot(-2) - \left[1\cdot 5\cdot 1 + (-2)(3)(-1) + 1\cdot 2\cdot 3\right]$$

$$= -5 + 9 - 4 - \left[5 + 6 + 6\right]$$

$$= 0 - 17 = -17$$

This trick only works for 3×3 matrices

$$\det\begin{bmatrix} -1 & 3 & 1 \\ 0 & 5 & 3 \\ 0 & 0 & 1 \end{bmatrix} = -1\begin{vmatrix} 5 & 3 \\ 0 & 1 \end{vmatrix} - 0\cdot ? + 0\cdot ?$$

upper triangular matrix

$$= -1\cdot 5\cdot 1$$

product of diagonal entries

$$\begin{bmatrix} -1 & 3 & 1 \\ & 5 & 3 \\ & & 1 \end{bmatrix}$$

2.2 Determinants via Row reduction

Three elementary row operations (& effect on determinants)

1) switching two rows: changes determinant by a sign (ie multiplies it by -1)

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad \begin{bmatrix} c & d \\ a & b \end{bmatrix}$$

$$ad-bc \qquad bc-ad$$

2) multiplying a row by a nonzero scalar k : multiplies the determinant by k

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad \begin{bmatrix} ka & kb \\ c & d \end{bmatrix}$$

$$ad-bc \qquad kad-kbc = k(ad-bc)$$

3) replacing a row by adding to it a scalar multiple of another row : No change in determinant!

All three properties also hold for when applied to columns in place of rows

$$\begin{vmatrix} -1 & 3 & 1 \\ 2 & 5 & 3 \\ 1 & -2 & 1 \end{vmatrix} = - \begin{vmatrix} 1 & -2 & 1 \\ 2 & 5 & 3 \\ -1 & 3 & 1 \end{vmatrix}$$

(interchanged row 1 & row 3)

$$= - \begin{vmatrix} 1 & -2 & 1 \\ 0 & 9 & 1 \\ -1 & 3 & 1 \end{vmatrix}$$

$$\begin{array}{ccc} -2 & 4 & -2 \\ 2 & 5 & 3 \\ \hline 0 & 9 & 1 \end{array}$$

$$= - \begin{vmatrix} 1 & -2 & 1 \\ 0 & 9 & 1 \\ 0 & 1 & 2 \end{vmatrix}$$

$$\begin{array}{ccc} 1 & -2 & 1 \\ -1 & 3 & 1 \\ \hline 0 & 1 & 2 \end{array}$$

interchanged R2 & R3

$$= \cancel{-}\,+ \begin{vmatrix} 1 & -2 & 1 \\ 0 & 1 & 2 \\ 0 & 9 & 1 \end{vmatrix}$$

$-9R2$

$$= \begin{vmatrix} 1 & -2 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & -17 \end{vmatrix}$$

$$\begin{array}{ccc} 0 & -9 & -18 \\ 0 & 9 & 1 \\ \hline 0 & 0 & -17 \end{array}$$

$= 1 \cdot 1 \cdot (-17)$ (product of diagonal entries)

$= -17$ ✓

Minor M_{ij} = determinant of matrix that remains when you delete row i & col j.

Cofactor $C_{ij} = (-1)^{i+j} M_{ij}$

can then form matrix M of minors and C of factors, whose i,j entry is M_ij and C_ij

Example $A = \begin{bmatrix} 1 & -2 & 3 \\ 6 & 7 & -1 \\ -3 & 1 & 4 \end{bmatrix}$ $\qquad \begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$

Find all minors & cofactors

$M_{1,1} = \begin{vmatrix} 7 & -1 \\ 1 & 4 \end{vmatrix} = 28 - (1 \cdot -1) = 29$

$C_{1,1} = M_{1,1} \cdot (-1)^{1+1} = M_{1,1} \cdot 1$

Minor matrix $= \begin{bmatrix} 29 & & \\ & & \\ & & \end{bmatrix}$

I leave it to you to complete the tedious process of filling the remaining 8 entries

Cofactor matrix $= \begin{bmatrix} 29 & & \\ & & \\ & & \end{bmatrix}$

$$A = \begin{bmatrix} 1 & -2 & 3 \\ 6 & 7 & -1 \\ -3 & 1 & 4 \end{bmatrix}$$

adjoint matrix of A

is denoted $\operatorname{adj}(A)$ & is by definition the <u>transpose</u> of the cofactor matrix

Important fact

$$\underbrace{A}_{n \times n} \cdot \underbrace{\operatorname{adj}(A)}_{n \times n} = \underbrace{(\det A) I_n}_{n \times n} = \begin{bmatrix} \det A & & \\ & \det A & \\ & & \ddots \ \det A \end{bmatrix}$$

Illustration in 2×2 case:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad \text{Minors} = \begin{bmatrix} d & c \\ b & a \end{bmatrix}$$

$$c \cdot (-1)^{1+2} = c \cdot (-1)^3 = -c$$

$$\begin{bmatrix} + & - \\ - & + \end{bmatrix} \qquad \text{Cofactor} = \begin{bmatrix} d & -c \\ -b & a \end{bmatrix}$$

$$\text{Adjoint matrix} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$A \cdot \operatorname{adj}(A) = (\det A)\, I_n$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} ad-bc & a(-b)+ba \\ cd-dc & c(-b)+da \end{bmatrix}$$

$\underline{2} \times \boxed{2 \quad\quad 2} \times \underline{2}$ $\quad\quad$ 2×2

$$= \begin{bmatrix} ad-bc & 0 \\ 0 & ad-bc \end{bmatrix}$$

$$= (ad-bc) \cdot I_2 \quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= (\det A) \cdot I_2$$

In Exercises 9–14, evaluate the determinant of the matrix by first reducing the matrix to row echelon form and then usin some combination of row operations and cofactor expansion.

9. $\begin{bmatrix} 3 & -6 & 9 \\ -2 & 7 & -2 \\ 0 & 1 & 5 \end{bmatrix}$

$$\begin{vmatrix} 3 & -6 & 9 \\ -2 & 7 & -2 \\ 0 & 1 & 5 \end{vmatrix} = 3 \begin{vmatrix} 1 & -2 & 3 \\ -2 & 7 & -2 \\ 0 & 1 & 5 \end{vmatrix}$$

$$= 3 \begin{vmatrix} 1 & -2 & 3 \\ 0 & 3 & 4 \\ 0 & 1 & 5 \end{vmatrix} \quad R2 \Leftrightarrow R2+2R1$$

$$\begin{array}{ccc} -2 & 7 & -2 \\ 2 & -4 & 6 \\ \hline 0 & 3 & 4 \end{array}$$

$$= -3 \begin{vmatrix} 1 & -2 & 3 \\ 0 & 1 & 5 \\ 0 & 3 & 4 \end{vmatrix}$$

$$\begin{array}{cccc} R3 & 0 & 3 & 4 \\ -3R2 & 0 & -3 & -15 \\ \hline & 0 & 0 & -11 \end{array}$$

$$= -3 \begin{vmatrix} 1 & -2 & 3 \\ 0 & 1 & 5 \\ 0 & 0 & -11 \end{vmatrix}$$

$$= -3(-11) = \boxed{33}$$

Expansion by minors along row 3

$$\begin{vmatrix} 3 & -6 & 9 \\ -2 & 7 & -2 \\ 0 & 1 & 5 \end{vmatrix} = 0? \; - 1 \cdot \begin{vmatrix} 3 & 9 \\ -2 & -2 \end{vmatrix} + 5 \begin{vmatrix} 3 & -6 \\ -2 & 7 \end{vmatrix}$$

$= -1(-6+18) + 5(21-12)$

$= -1(12) + 5(9)$

$= -12 + 45 = 33$

In [Exercises 29–30](#), show that det(*A*) = 0 *without directly evaluating the determinant.*

29. $A = \begin{bmatrix} -2 & 8 & 1 & 4 \\ 3 & 2 & 5 & 1 \\ 1 & 10 & 6 & 5 \\ 4 & -6 & 4 & -3 \end{bmatrix}$

Solution

row 3 is row 1 + row 2

$R3 = R1 + R2$

$R3 - R1 - R2 = 0$

$|A| = \begin{vmatrix} -2 & 8 & 1 & 4 \\ 3 & 2 & 5 & 1 \\ & & & \\ 4 & -6 & 4 & -3 \end{vmatrix}$

$R3 \to$

$R3 - R1$

Jessica found a better way:

C2=2* C4

so det $A = 0$

C2-2*C4 = 0 so det $A = 0$

Lemma. Inverse of an elementary matrix is an elementary matrix

Proof: Each elementary row operation is reversible, by an elementary row operation

- If row i & row j were swapped, swap row i & row j again to get to the original matrix

$$E = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad E^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

- If row i was multiplied by a nonzero scalar s, multiply row i by the nonzero scalar $\frac{1}{s}$ to get the original matrix

$$E = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \qquad E^{-1} = \begin{bmatrix} 1/2 & 0 \\ 0 & 1 \end{bmatrix}$$

- If row i is replaced by adding to it s(row j), replace row i by adding to it (-s)(row j) to get the original matrix

$$E = \begin{bmatrix} 1 & 2 \\ & 1 \end{bmatrix} \qquad E^{-1} = \begin{bmatrix} 1 & -2 \\ & 1 \end{bmatrix}$$

Lemma Every invertible matrix can be written as a product of elementary matrices

Proof $E_r \cdots E_1 \cdot A = I$ by row reduction

(for some elementary matrices E_1, … E_r corresponding to elementary row operations starting from A)

$$\underbrace{E_r^{-1}(E_r}_{I} E_{r-1} \cdots A) = E_r^{-1}$$

$$E_{r-1} E_{r-2} \cdots A = E_r^{-1}$$

$$E_{r-2} \cdots A = E_{r-1}^{-1} E_r^{-1}$$

$A = E_1^{-1} E_2^{-1} \cdots E_r^{-1}$ is a product of elementary matrices.

Theorem For A, B $n \times n$ matrices, $\boxed{\det(AB) = \det(A)\det B}$

(assuming $\det(EM) = \det(E)\det(M)$ for any elementary matrix E).

Proof. If A is not invertible, then $\det A = 0$, and AB is also not invertible so $\det(AB) = 0$. So both sides of $\det(AB) = \det(A)\det(B)$ are 0.

So assume A is invertible. Write $A = E_1 \cdots E_r$ as a product of elementary matrices.

Then
$$
\begin{aligned}
\det(AB) &= \det(E_1 \cdots E_r B) \\
&= \det(E_1)\det(E_2 \cdots E_r B) \\
&\vdots \\
&= \det(E_1)\det(E_2)\cdots\det(E_r)\det(B) \\
&= \det(E_1E_2)\det(E_3)\cdots\det(E_r)\det(B) \\
&\vdots \\
&= \det(E_1E_2\cdots E_r)\det(B) \\
&= \det(A)\det(B)
\end{aligned}
$$

"Formula" for inverse of an $n\times n$ matrix A

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$A^{-1} = \frac{1}{\Delta} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ determinant $\Delta = ad - bc$

$\Delta \neq 0$ if and only if A^{-1} exists

Generalize this to any $n\times n$ matrices. First we need to find "adjoint" matrix.

A $n\times n$ matrix

$A^{-1} = \frac{1}{\Delta} \operatorname{adj}(A)$ Δ = determinant of A

adjoint matrix of A.

How to find adjoint matrix of A

0. In SageMath, A.adjugate()

1. Find the matrix of minors

$$\begin{bmatrix} M_{11} & M_{12} & \cdots \\ M_{21} & \vdots & \\ \vdots & \vdots & \end{bmatrix}$$

$M_{i,j}$ = determinant of $(n-1)\times(n-1)$ matrix formed by deleting row i & col j from A.

2. Adjust signs on matrix of minors

$\underbrace{\text{put "}-\text{" sign}}$ on entries where $i+j$ = odd
(i,j)

multiply of (-1)

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix} \qquad \begin{bmatrix} + & - \\ - & + \end{bmatrix} \qquad \begin{bmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{bmatrix}$$

Result is called cofactor Matrix C

3. Take the transpose of cofactor matrix (flip along the diagonal)

$$\mathrm{adj}(A) = C^T$$

$$[\mathrm{adj}(A)]_{\underline{\underline{i,j}}} = C_{\underline{\underline{j,i}}}$$

$$\begin{bmatrix} 2 & 3 & 5 \\ 7 & 5 & 4 \\ -1 & -3 & 8 \end{bmatrix}^T = \begin{bmatrix} 2 & 7 & -1 \\ 3 & 5 & -3 \\ 5 & 4 & 8 \end{bmatrix}$$

2.3 Properties of the determinant

$\det(AB) = \det(A)\det(B)$.

$\det(A+B)$ no real formula

Cramer Rule:

A is an $n \times n$ matrix

$$A\vec{x} = \vec{b}$$

system of n equations in n variable

$x_1, x_2, \ldots x_n$

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \qquad b = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$$

Cramer Rule: Solutions to $A\vec{x} = b$ are

Let A_i be matrix A but replace column i with $\vec{b}$

Then
$$x_i = \frac{\det(A_i)}{\det(A)}.$$

for $i = 1, 2, \dots n$.

Example Use Cramer's Rule to solve

$$2x + 4y = 7$$
$$x + 3y = 5$$

Solution: $A = \begin{bmatrix} 2 & 4 \\ 1 & 3 \end{bmatrix}$ $\vec{x} = \begin{bmatrix} x \\ y \end{bmatrix}$ $\vec{b} = \begin{bmatrix} 7 \\ 5 \end{bmatrix}$

$A_1 = \begin{bmatrix} 7 & 4 \\ 5 & 3 \end{bmatrix}$ $A_2 = \begin{bmatrix} 2 & 7 \\ 1 & 5 \end{bmatrix}$

$$\det(A) = 2\cdot 3 - 4\cdot 1 = 6 - 4 = 2.$$

$$x = x_1 = \frac{\det(A_1)}{\det(A)} = \frac{7\cdot 3 - 4\cdot 5}{2} = \frac{21-20}{2} = \frac{1}{2}$$

$$A_2 = \begin{bmatrix} 2 & 7 \\ 1 & 5 \end{bmatrix}$$

$$y = x_2 = \frac{\det(A_2)}{\det(A)} = \frac{2\cdot5-1\cdot7}{2} = \frac{10-7}{2} = \frac{3}{2}$$

$$\boxed{x = \frac{1}{2}, \quad y = \frac{3}{2}}$$

31. Use Cramer's rule to solve for the unknown y without solving for the unknowns x, z, and w.

$$\begin{aligned} 4x + y + z + w &= 6 \\ 3x + 7y - z + w &= 1 \\ 7x + 3y - 5z + 8w &= -3 \\ x + y + z + 2w &= 3 \end{aligned}$$

Let me solve for z instead.

$$A = \begin{bmatrix} 4 & 1 & 1 & 1 \\ 3 & 7 & -1 & 1 \\ 7 & 3 & -5 & 8 \\ 1 & 1 & 1 & 2 \end{bmatrix} \qquad b = \begin{matrix} 6 \\ 1 \\ -3 \\ 3 \end{matrix}$$

$z = x_3$ $\quad (x = x_1,\ y = x_2,\ z = x_3,\ w = x_4)$

Cramer rule:
$$x_3 = \frac{\det(A_3)}{\det(A)} = \frac{\begin{vmatrix} 4 & 1 & 6 & 1 \\ 3 & 7 & 1 & 1 \\ 7 & 3 & -3 & 8 \\ 1 & 1 & 3 & 2 \end{vmatrix}}{\det(A)}$$

Exercise 3.2.28 If $A^{-1} = \begin{bmatrix} 3 & 0 & 1 \\ 0 & 2 & 3 \\ 3 & 1 & -1 \end{bmatrix}$ find adj A.

$$A^{-1} = \frac{1}{\Delta} \operatorname{adj}(A)$$

$$(\Delta) A^{-1} = \operatorname{adj}(A)$$

$\det(A)$. But we don't know what A is, so how can we find $\det(A)$?

$$\det(AB) = \det(A)\det(B)$$

$$\det(AA^{-1}) = \det(A)\det(A^{-1})$$

$$\det(I_3) = \det(A)\det(A^{-1})$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\det(I_3) = 1$$

$$1 = \det(A)\det(A^{-1})$$

$$\det(A^{-1}) = \frac{1}{\det(A)}$$

or $$\det(A) = \frac{1}{\det(A^{-1})}$$

$$A^{-1} = \begin{bmatrix} 3 & 0 & 1 \\ 0 & 2 & 3 \\ 3 & 1 & -1 \end{bmatrix}$$

$$\det(A^{-1}) = 3\begin{vmatrix} 2 & 3 \\ 1 & -1 \end{vmatrix} - 0\begin{vmatrix} & \\ & \end{vmatrix} + 3\begin{vmatrix} 0 & 1 \\ 2 & 3 \end{vmatrix}$$

$$= 3(-2-3) + 3(-2)$$

$$= 3(-5) - 6$$

$$= -21$$

$$\det(A) = -\frac{1}{21}$$

$$\operatorname{adj}(A) = \det(A) \cdot A^{-1}$$

$$= -\frac{1}{21}\begin{bmatrix} 3 & 0 & 1 \\ 0 & 2 & 3 \\ 3 & 1 & -1 \end{bmatrix} = \begin{bmatrix} -1/7 & 0 & -1/21 \\ 0 & -\frac{2}{21} & -1/7 \\ -1/7 & -1/21 & 1/21 \end{bmatrix}$$

Confirmed in SageMath

Exercise 3.2.6 Let $A = \begin{bmatrix} a & b & c \\ p & q & r \\ u & v & w \end{bmatrix}$ and assume that $\det A = 3$. Compute:

a. $\det(2B^{-1})$ where $B = \begin{bmatrix} 4u & 2a & -p \\ 4v & 2b & -q \\ 4w & 2c & -r \end{bmatrix}$

$$\det(2B^{-1}) = 2^3 \det(B^{-1}) = 8 \cdot \frac{1}{\det(B)}$$

because 3x3 matrix & every entry (and thus every row) of B^{-1} is multiplied by 2 in $2B^{-1}$

Let's find det(B).

$$\begin{vmatrix} 4u & 2a & -p \\ 4v & 2b & -q \\ 4w & 2c & -r \end{vmatrix} = 4 \begin{vmatrix} u & 2a & -p \\ v & 2b & -q \\ w & 2c & -r \end{vmatrix}$$

$$= 4(2)(-1) \begin{vmatrix} u & a & p \\ v & b & q \\ w & c & r \end{vmatrix}$$

$\det(A) = \det(A^T)$

$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \xrightarrow{\det} ad - bc$

$A^T = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \xrightarrow{\det} ad - bc$

$$= -8 \begin{vmatrix} u & v & w \\ a & b & c \\ p & q & r \end{vmatrix}$$

$$= -8(-1) \begin{vmatrix} a & b & c \\ u & v & w \\ p & q & r \end{vmatrix}$$ interchanged R1 & R2

$$= 8(-1) \begin{vmatrix} a & b & c \\ p & q & r \\ u & v & w \end{vmatrix}.$$

$$= -8 \det(A) = -8 \cdot 3 = -24$$

given

Final answer: $\dfrac{8}{\det(B)} = \dfrac{8}{-24} = \boxed{-\dfrac{1}{3}}$

Proof of Cramer's rule:

$$A\vec{x} = \vec{b}$$

$$\underbrace{A^{-1}A}_{I} \cdot \vec{x} = A^{-1}\vec{b}$$

$$\vec{x} = A^{-1}\vec{b}$$

$$\vec{x} = \frac{1}{\det(A)} \cdot \underbrace{\operatorname{adj}(A)}_{n\times n}\underbrace{\vec{b}}_{n\times 1} \qquad \left(\text{since } A^{-1} = \frac{1}{\det(A)} \cdot \operatorname{adj}(A)\right)$$

so $$x_j = \frac{1}{\det(A)} \left(\operatorname{adj}(A)\vec{b}\right)_{j,1}$$

$$= \frac{1}{\det(A)} \sum_{k=1}^{n} \operatorname{adj}(A)_{j,k} \cdot b_k$$

$$= \frac{1}{\det(A)} \sum_{k=1}^{n} \operatorname{cof}(A)_{k,j} \cdot b_k$$

since adjoint matrix is transpose of cofactor matrix and vice versa

$$= \frac{1}{\det(A)} \sum_{k=1}^{n} (-1)^{k+j} M_{k,j}\, b_k$$

$$= \frac{1}{\det(A)} \Big(\underbrace{(-1)^{1+j} M_{1,j}\, b_1 + (-1)^{2+j} M_{2,j}\, b_2 + \dots + (-1)^{n+j} M_{n,j}\, b_n}_{\text{expansion by minors along column } j \text{ of } A_j} \Big)$$

$$A_j = \begin{array}{c} \begin{pmatrix} & b_1 & \\ & \vdots & \\ & b_n & \end{pmatrix} \\ \text{column } j \end{array}$$

$$x_j = \frac{1}{\det(A)} \cdot \det(A_j)$$

$\square$