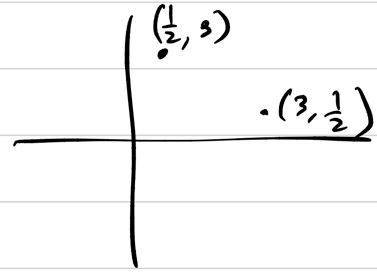
Chapter 3 Vectors

Vector is a n-tuple of numbers:

$n=2$. $(\frac{1}{2}, 3)$ $(3, \frac{1}{2})$ are different vectors
i.e order matters

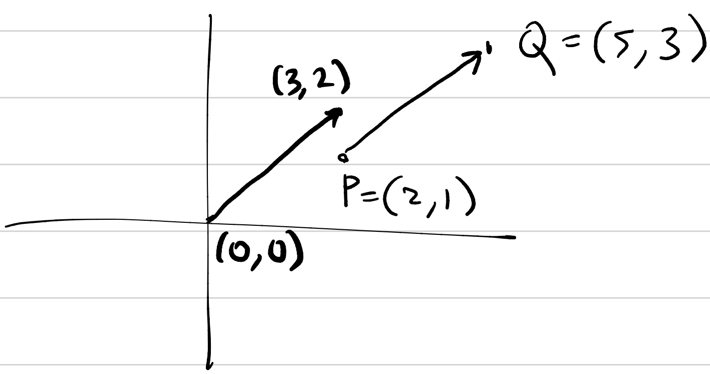
$n=3$ $(-1, 4, 5)$

$n=5$ $(1, \frac{1}{2}, \sqrt{2}, 4, 7)$



Another approach/definition of a vector is "geometric vector": arrow starting at a point P and ending at another point Q

$\overrightarrow{PQ}$



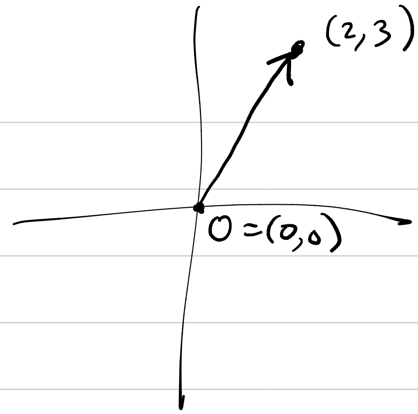
corresponds to 2-tuple

$$\overrightarrow{PQ} = Q - P = (5,3)-(2,1) = (5-2, 3-1) = (3, 2)$$

How to relate/reconcile these different definitions of vector?

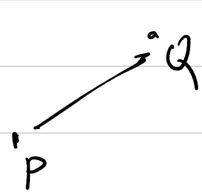
$\left\{ \begin{array}{l} \text{given a vector} \\ \text{as an } n\text{-tuple} \\ (x_1, x_2, \ldots x_n) \end{array} \right\} \longrightarrow \left\{ \begin{array}{l} \text{arrow starting} \\ \text{at origin } O=(0,0,\ldots 0) \\ \text{and ending at the point} \\ (x_1, x_2, \ldots x_n) \end{array} \right.$

$(2,3) \longmapsto$
vector



Other way, from geometric vectors $\overrightarrow{PQ}$ to vectors (n-tuple)

$\overrightarrow{PQ}$ arrow



$P = (p_1, p_2, \ldots p_n)$

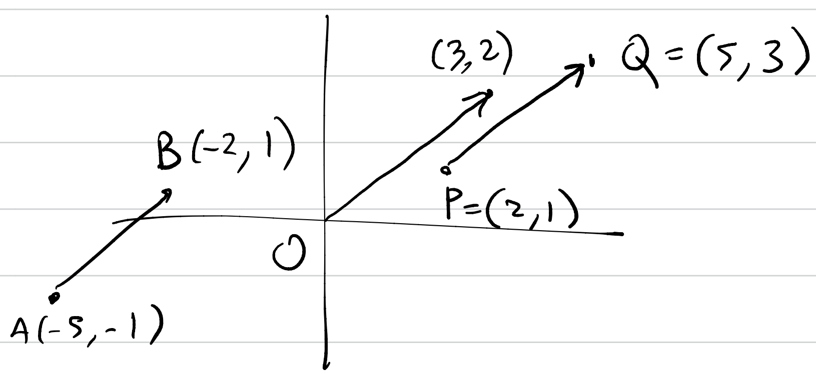
$Q = (q_1, q_2, \ldots q_n)$

$\longrightarrow$ n-tuple

$Q - P =$

$(q_1 - p_1, q_2 - p_2, \ldots, q_n - p_n)$

$\overrightarrow{PQ}$



corresponds to 2-tuple

$Q - P = (5,3) - (2,1)$

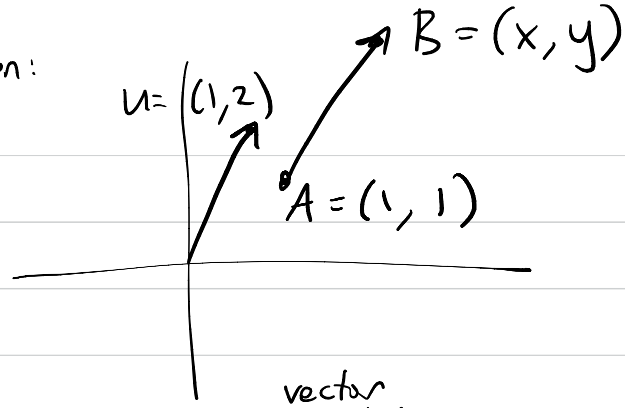
$= (5-2, 3-1)$

$= (3, 2)$

Two vectors $\overrightarrow{PQ}$ & $\overrightarrow{AB}$ are <u>equivalent</u> if they have the same magnitude & direction, i.e $Q - P = B - A$.

5. Find the terminal (ending) point of the vector that is equivalent to $u = (1, 2)$ & whose initial (starting) point is $A(1,1)$

Solution:



$$\overrightarrow{AB} \xrightarrow[\text{as a tuple}]{\text{vector}} B - A = (1,2)$$

$$(x-1,\ \underbrace{y-1}) = (1,2)$$

$x - 1 = 1 \qquad y - 1 = 2$

$x = 2 \qquad y = 3$

Answer $(2,3)$

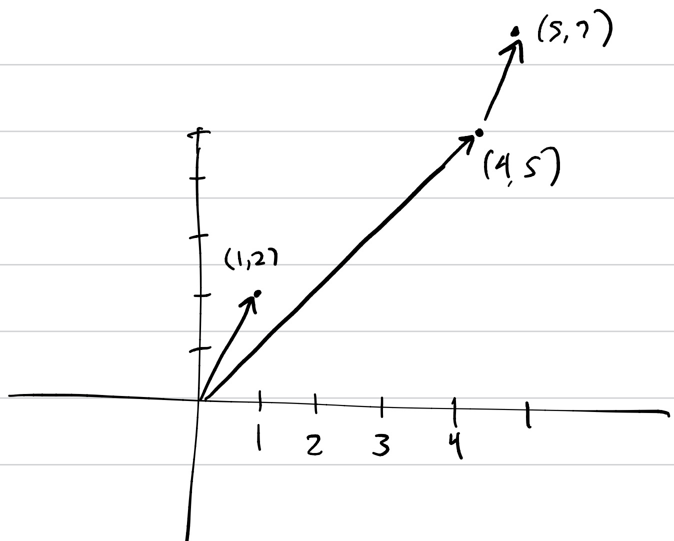
Two operations you can do to vectors:

1) add them:

$$\underbrace{(u_1, u_2, \ldots u_n)}_{\vec{u}} + \underbrace{(v_1, v_2, \ldots v_n)}_{\vec{v}} = \underbrace{(u_1+v_1,\ u_2+v_2, \ldots, u_n+v_n)}_{\vec{u}+\vec{v}}$$

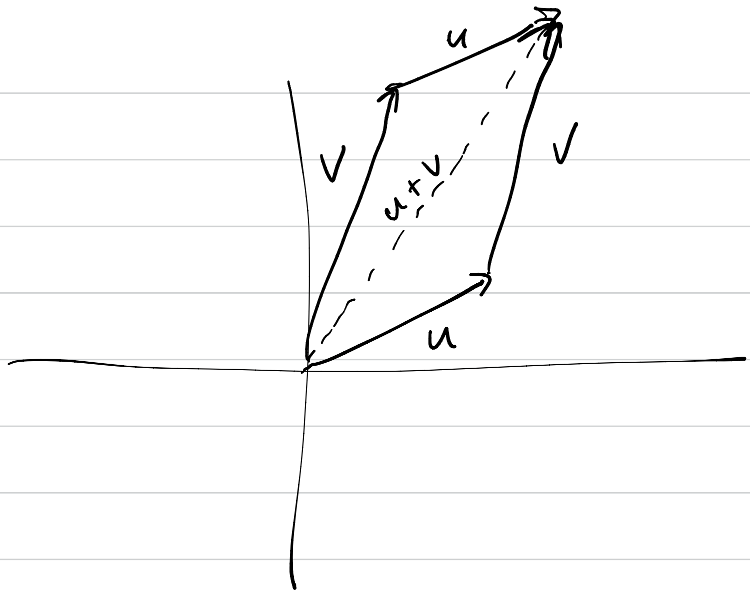
(definition)

$$(4,5) + (1,2) = (5,7)$$



"tip-to-tail"

method of addition of (geometric) vector



parallelogram law
of vector addition

2) multiplying a vector by a scalar k

a number

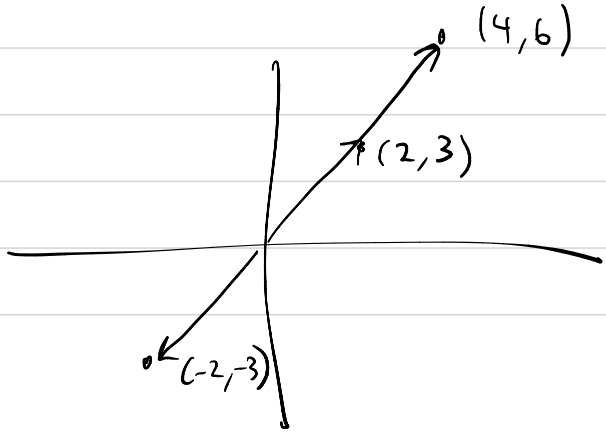
$\vec{v} = (v_1, v_2, \ldots v_n)$

$$k \cdot \vec{v} = (kv_1, kv_2, \ldots, kv_n).$$

scalar vector

Example: $k = \frac{1}{2}$, $\vec{v} = (4, 6)$

$$\tfrac{1}{2}\vec{v} = k \cdot \vec{v} = \left(\tfrac{1}{2} \cdot 4, \tfrac{1}{2} \cdot 6\right) = (2, 3)$$



$k = -\frac{1}{2}$

$kv = (-2, -3)$

Given vectors u, v, w, a linear combination of them is a vector of the form

$$k_1 \cdot u + k_2 \cdot v + k_3 \cdot w$$

Anton
3.1 #14

$\vec{u} = (-3, 2, 1, 0)$

$\vec{v} = (4, 7, -3, 2)$

$\vec{w} = (5, -2, 8, 1)$

Find the components of the vector $\vec{x}$ that satisfy the equation

$$2\vec{u} - \vec{v} + \vec{x} = 7\vec{x} + \vec{w}$$

Solution: $2\vec{u} - \vec{v} = 6\vec{x} + \vec{w}$

$2\vec{u} - \vec{v} - \vec{w} = 6\vec{x}$

$\frac{1}{6}(2\vec{u} - \vec{v} - \vec{w}) = \vec{x}$

Now just plug in numbers for $\vec{u}, \vec{v}, \vec{w}$

$\vec{x} = \frac{1}{6}\left((-6, 4, 2, 0) - (4, 7, -3, 2) - (5, -2, 8, 1)\right)$

$= \frac{1}{6}\left((-6, 4, 2, 0) + (-4, -7, 3, -2) + (-5, 2, -8, -1)\right)$

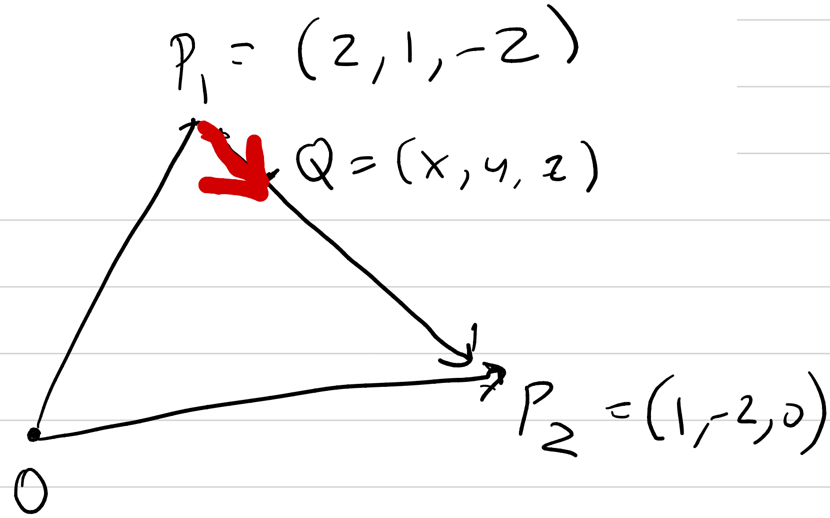
$= \frac{1}{6}\left((-15, -1, -3, -3)\right)$

$= \boxed{\left(-\frac{15}{6}, -\frac{1}{6}, -\frac{3}{6}, -\frac{3}{6}\right)}$

Exercise 4.1.14 Given $P_1(2, 1, -2)$ and $P_2(1, -2, 0)$. Find the coordinates of the point : Q

a. $\frac{1}{5}$ the way from P_1 to P_2

b. $\frac{1}{4}$ the way from P_2 to P_1



idea: $\overrightarrow{P_1Q} = \frac{1}{5} \cdot \overrightarrow{P_1P_2}$

$$\underbrace{Q}_{(x,y,z)} - \underbrace{P_1}_{(2,1,-2)} = \frac{1}{5}\left(P_2 - P_1\right)$$

$$(x-2,\ y-1,\ z+2) = \frac{1}{5}\big((1,-2,0) - (2,1,-2)\big) = \frac{1}{5}(-1,-3,2)$$

$$Q = \frac{1}{5}(P_2 - P_1) + P_1$$

$$= \frac{1}{5}P_2 - \frac{1}{5}P_1 + \underbrace{P_1}_{\frac{5}{5}P_1}$$

$$Q = \frac{1}{5}P_2 + \frac{4}{5}P_1$$

$$(x-2,\ y-1,\ z+2) = \left(-\frac{1}{5},\ \frac{-3}{5},\ \frac{2}{5}\right)$$

$$x-2 = -\frac{1}{5} \quad \Rightarrow \quad x = -\frac{1}{5} + \underbrace{2}_{\frac{10}{5}} = \boxed{\frac{9}{5}}$$

$$y-1 = -\frac{3}{5} \quad \Rightarrow \quad y = -\frac{3}{5} + 1 = \boxed{\frac{2}{5}}$$

$$z+2 = \frac{2}{5} \qquad z = \frac{2}{5} - 2 = \frac{2-10}{5} = \boxed{\frac{-8}{5}}$$

$$(x,y,z) = \left(\frac{9}{5},\ \frac{2}{5},\ \frac{-8}{5}\right)$$

$= \frac{1}{5}(1,-2,0) + \frac{4}{5}(2,1,-2)$

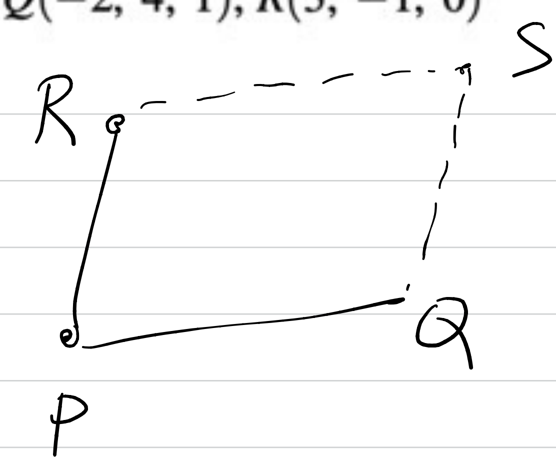
$= \left(\frac{1}{5}, -\frac{2}{5}, 0\right) + \left(\frac{8}{5}, \frac{4}{5}, -\frac{8}{5}\right)$

$$\boxed{Q = \left(\frac{9}{5}, \frac{2}{5}, -\frac{8}{5}\right)}$$

Exercise 4.1.20 Let P, Q, and R be the vertices of a parallelogram with adjacent sides PQ and PR. In each case, find the other vertex S.

a. $P(3, -1, -1)$, $Q(1, -2, 0)$, $R(1, -1, 2)$

b. $P(2, 0, -1)$, $Q(-2, 4, 1)$, $R(3, -1, 0)$



$\overrightarrow{PS} = \overrightarrow{PQ} + \overrightarrow{QS} = \overrightarrow{PQ} + \overrightarrow{PR}$ by Parellogram law

$\overrightarrow{PS} = \overrightarrow{PQ} + \overrightarrow{PR}$ geometric vectors

$S - P = (Q - P) + (R - P)$ n-tuple vectors

$S - P = Q + R - 2P$

$S = Q + R - 2P + P$

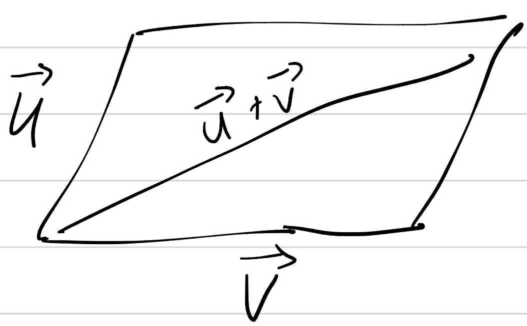
$S = Q + R - P$

now plug in values of P, Q, R

$= (1,-2,0) + (1,-1,2) - (3,-1,-1)$

$= (1,-2,0) + (1,-1,2) + (-3,1,1)$

$\boxed{S = (-1,-2,3)}$



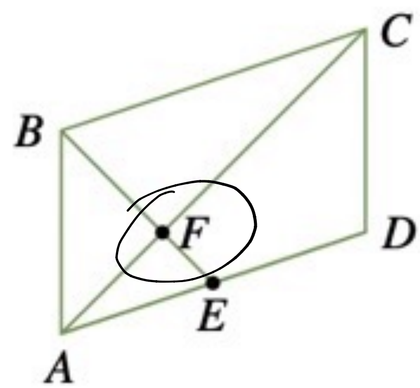
parallelogram

diagonal vector = side + side

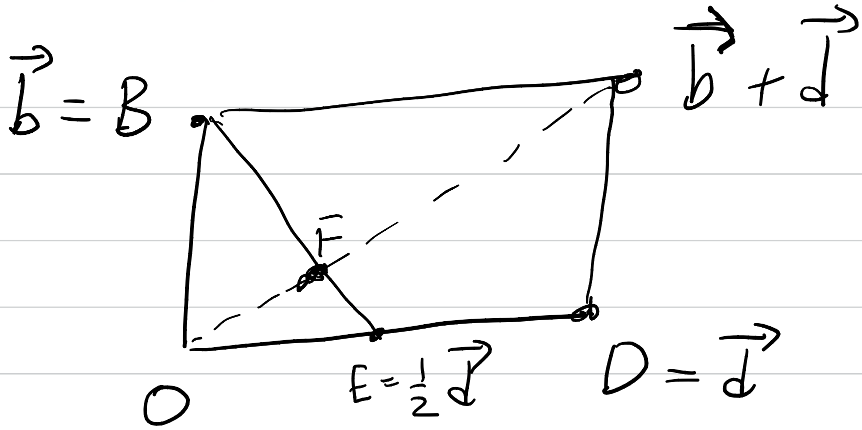
$\vec{u}+\vec{v} \quad \vec{u} \quad \vec{v}$

But caution! We're not saying length of the diagonal is the sum of length of two sides.

Exercise 4.1.33 Consider the parallelogram $ABCD$ (see diagram), and let E be the midpoint of side AD.



Show that BE and AC trisect each other; that is, show that the intersection point is one-third of the way from E to B and from A to C. [*Hint*: If F is one-third of the way from A to C, show that $2\overrightarrow{EF} = \overrightarrow{FB}$ and argue as in Example 4.1.2.]



$\vec{F} = k_1(\vec{b} + \vec{d})$ for k_1 some scalar k_1

($\vec{F}$ is on diagonal)

$\vec{F}$ is on $\overrightarrow{BE}$ means

$\overrightarrow{EF} = k_2 \overrightarrow{EB}$ for some scalar k_2

$$F - E = k_2(\vec{b} - E)$$

$$F = k_2(\vec{b} - E) + E$$

$$F = k_2(\vec{b}) - k_2 E + E$$

$$= k_2(\vec{b}) + (1 - k_2)E$$

$$\vec{F} = k_2(\vec{b}) + (1 - k_2)\frac{1}{2}\vec{d}$$

also

$$F = k_1\vec{b} + k_1\vec{d} \quad \text{(from before)}$$

$$k_1 = k_2 \qquad k_1 = (1 - k_2)\frac{1}{2}$$

2 equations, 2 variables!

$$2k_1 = 1 - k_2$$

since $k_1 = k_2$

$$2k_1 = 1 - k_1$$

$$3k_1 = 1$$

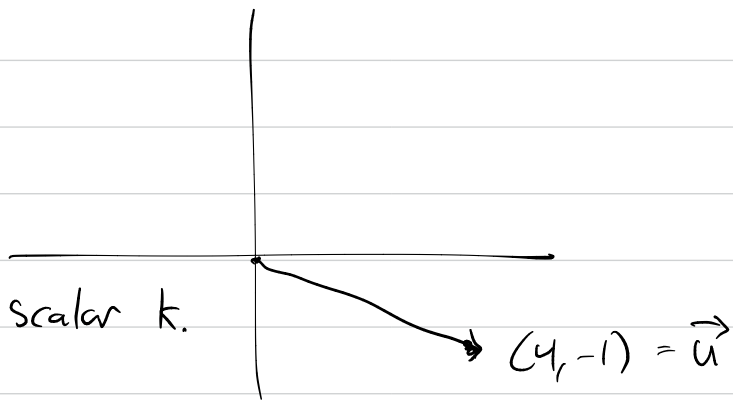
$$k_1 = \frac{1}{3}, \quad k_2 = k_1 = \frac{1}{3}$$

16. For what values of t is the given vector parallel to $\vec{u} = (4,-1)$

a) $(8t,-2)$
b) $(8t,2t)$
c) $(1,t^2)$

a) For what value of t is $\vec{v} = (8t,-2)$ parallel to $\vec{u} = (4,-1)$?

Same direction, but maybe not same magnitude



So $\vec{v} = k\vec{u}$ for some scalar k.

$\vec{v} = 2\vec{u}$

$(8t,-2) = k(4,-1)$
$(8t,-2) = (4k,-k)$

so $-2 = -k$
so $2 = k$

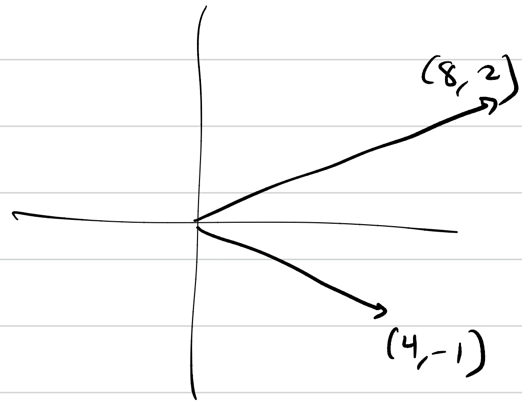
$(8t,-2) = k \cdot \vec{u} = 2 \cdot \vec{u} = (8,-2)$

$8t = 8$

$\boxed{t = 1}$

$t \cdot (8,2)$

b) $(8t, 2t) = k \cdot (4,-1)$



$(8t, 2t) = (4k, -k)$

$8t = 4k$ | $2t = -k$

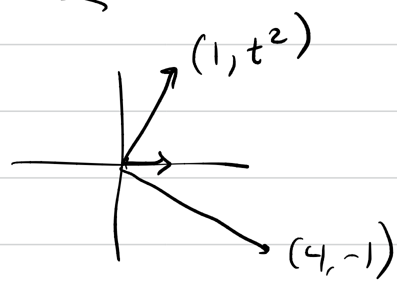
$t = \frac{4k}{8}$ | $t = -\frac{1}{2}k$

$t = \frac{1}{2}k$

$\frac{1}{2}k = -\frac{1}{2}k \Rightarrow \frac{1}{2}k + \frac{1}{2}k = 0$

$\cancel{k = 0}$

no such value of t exists.



c) $(1, t^2) = k(4,-1)$

no such value of t exists

23. Let P be the point $(2, 3, -2)$ and Q the point $(7, -4, 1)$.

a. Find the midpoint of the line segment connecting the points P and Q.

Answer

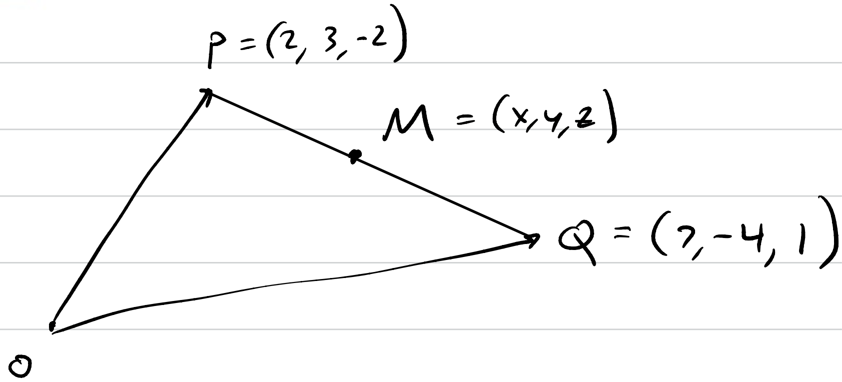
Solution

b. Find the point on the line segment connecting the points P and Q that is $\frac{3}{4}$ of the way from P to Q.

Answer

Solution

a)



Insight: $\overrightarrow{PM} = \frac{1}{2}\overrightarrow{PQ}$

$$M - P = \frac{1}{2}(Q - P)$$

$$(x, y, z) - (2, 3, -2) = \frac{1}{2}\left((7, -4, 1) - (2, 3, -2)\right)$$

$$(x, y, z) - (2, 3, -2) = \frac{1}{2}(5, -7, 3)$$

$$(x, y, z) = \frac{1}{2}(5, -7, 3) + (2, 3, -2)$$

$$= (2.5, -3.5, \underline{1.5}) + (2, 3, -2)$$

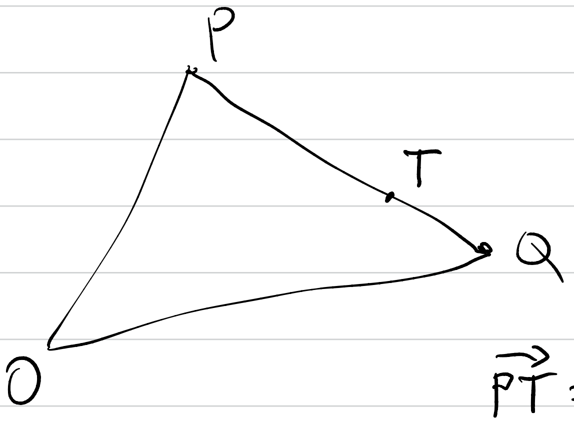
$$M = \frac{1}{2}(Q - P) + P \qquad = \boxed{(4.5, -0.5, -0.5)}$$

$$= \frac{1}{2}Q - \frac{1}{2}P + P \qquad = \left(\frac{9}{2}, -\frac{1}{2}, -\frac{1}{2}\right)$$

$$= \frac{1}{2}Q + \frac{1}{2}P$$

$$M = \frac{1}{2}(P + Q) = \frac{1}{2}\left((2, 3, -2) + (7, -4, 1)\right) = \frac{1}{2}(9, -1, -1)$$

b) Find point $\frac{3}{4}$ths of the way from P to Q



$$\overrightarrow{PT} = \frac{3}{4}\overrightarrow{PQ}$$

$$T - P = \frac{3}{4}(Q - P)$$

$$T - P = \frac{3}{4}Q - \frac{3}{4}P$$

$$T = \frac{3}{4}Q - \frac{3}{4}P + P$$

$$T = \frac{3}{4}Q + \frac{1}{4}P$$

$$= \frac{3}{4}(7, -4, 1) + \frac{1}{4}(2, 3, -2)$$

3.2 Norm, dot product, distance in $\mathbb{R}^n$

$\mathbb{R}^n = \{ (x_1, x_2, \dots, x_n) \mid x_i \in \mathbb{R} \}$

$\mathbb{R}^2, \mathbb{R}^3, \dots$

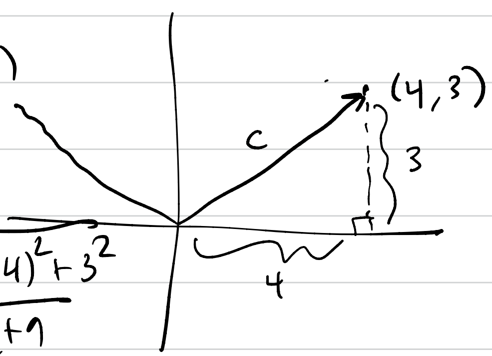
Norm / length of a vector $v = (v_1, v_2, \dots, v_n)$
magnitude

is $\|v\| = \sqrt{v_1^2 + v_2^2 + \dots + v_n^2} \neq v_1 + v_2 + \dots + v_n$

or $|v|$

(this is not the same as $v_1 + v_2 + \dots + v_n$!)

$w = (-4, 3)$



$v = (4, 3)$

$\|v\| = \sqrt{4^2 + 3^2}$

$= \sqrt{16 + 9} = \sqrt{25}$

$\|v\| = 5$

$\|w\| = \sqrt{(-4)^2 + 3^2}$

$= \sqrt{16 + 9}$

$= \sqrt{25}$

$= 5$

$c^2 = 4^2 + 3^2$

$c^2 = 25$

$c = 5$

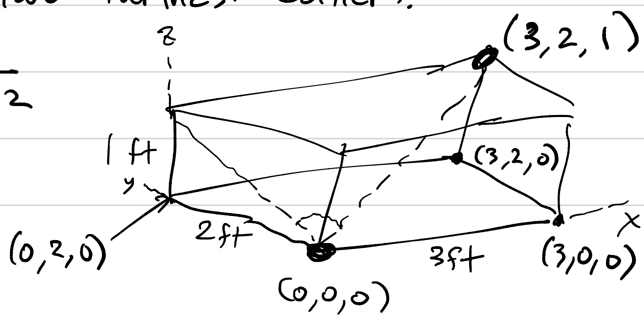
A box has dimensions 2 ft (length), 3 ft (wide), 1 ft (height)

Find the distance between two furthest corners.

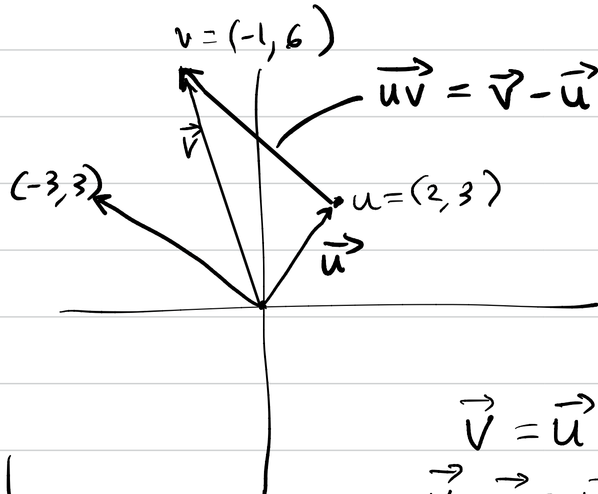
Answer: $\|(3,2,1)\| = \sqrt{3^2 + 2^2 + 1^2}$

$= \sqrt{9 + 4 + 1}$

$= \boxed{\sqrt{14}}$ ft



Find the distance between the points $u = (2, 3)$ $v = (-1, 6)$



$\vec{v} = \vec{u} + \vec{uv}$

$\vec{v} - \vec{u} = \vec{uv}$

$$\|\vec{uv}\| = \|\vec{v} - \vec{u}\|$$

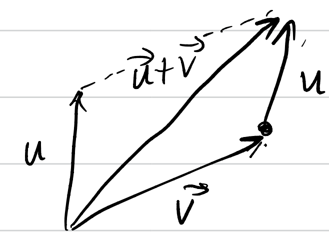
$$= \|(-1, 6) - (2, 3)\|$$

$$= \|(-1-2, 6-3)\|$$

$$= \|(-3, 3)\| = \sqrt{(-3)^2 + 3^2} = \sqrt{9+9} = \sqrt{18}$$

Distance formula: $P_1 = (x_1, y_1)$ $P_2 = (x_2, y_2)$

$$\|P_1 P_2\| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$



Dot product of vectors $\vec{u}, \vec{v} \in \mathbb{R}^n$

$\vec{u} = (u_1, u_2, \dots, u_n)$
$\vec{v} = (v_1, v_2, \dots, v_n)$

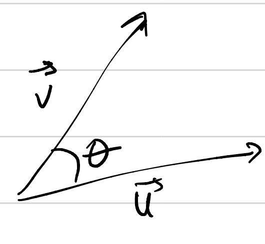
$$\underbrace{\vec{u} \cdot \vec{v}}_{\substack{\text{number} \\ \text{not a vector}}} = u_1 v_1 + u_2 v_2 + \dots + u_n v_n \in \mathbb{R}$$

"in"

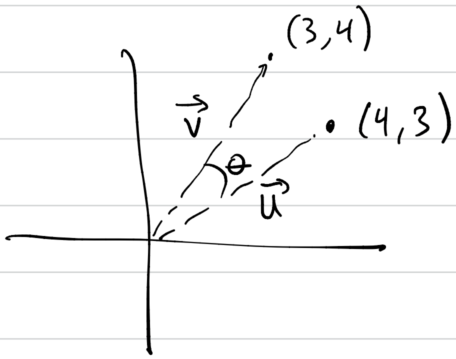
Key fact about dot product is

$$\underbrace{\vec{u} \cdot \vec{v}}_{\text{number}} = \|\vec{u}\| \, \|\vec{v}\| \cos\theta$$

angle between the two vectors $\vec{u}, \vec{v}$



Example Find angle between



$\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2$ definition
$= 4 \cdot 3 + 3 \cdot 4$
$\vec{u} \cdot \vec{v} = 12 + 12 = 24$

Also $\vec{u} \cdot \vec{v} = \|u\| \, \|v\| \cos\theta$
$24 = \left(\sqrt{4^2+3^2}\right)\left(\sqrt{3^2+4^2}\right) \cos\theta$
$24 = 5 \cdot 5 \cos\theta$

$\frac{24}{25} = \cos\theta$

$\cos^{-1}\left(\frac{24}{25}\right) = \theta$

arccos

$\boxed{16.3° = \theta}$

12. Find the Euclidean distance & the angle between the vectors

$u = (1, 2, -3, 0) \quad v = (5, 1, 2, -2)$.

Solution $\|u - v\| = \|(1-5,\ 2-1,\ -3-2,\ 0--2)\|$

$= \|(-4, 1, -5, 2)\|$

$= \sqrt{(-4)^2 + 1^2 + (-5)^2 + 2^2}$

$= \sqrt{16 + 1 + 25 + 4}$

$= \boxed{\sqrt{46}}$ Euclidean distance

$\vec{u} \cdot \vec{v} = \|\vec{u}\| \cdot \|\vec{v}\| \cdot \cos\theta$

$1 \cdot 5 + 2 \cdot 1 + (-3)2 + 0(-2) =$

$5 + 2 + -6 + 0$

$1 = \sqrt{1^2 + 2^2 + (-3)^2 + 0^2}\sqrt{5^2 + 1^2 + 2^2 + (-2)^2}$ $\cos\theta$

$1 = \sqrt{1+4+9}\ \sqrt{25+1+4+4}\ \cos\theta$

$1 = \sqrt{14}\ \sqrt{34}\ \cos\theta$

$\dfrac{1}{\sqrt{14}\sqrt{34}} = \cos\theta$

$\cos^{-1}$

$\boxed{\theta = \arccos\left(\dfrac{1}{\sqrt{14}\sqrt{34}}\right)}$

$\approx 87.37°$ or 1.5 radians

Example $V = (v_1, v_2, \ldots, v_n)$

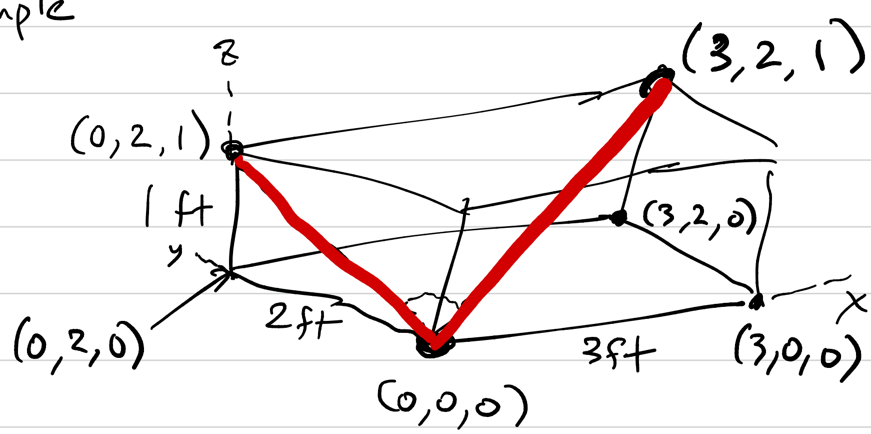
$$V \cdot V = v_1 \cdot v_1 + \cdots + v_n \cdot v_n$$

$$V \cdot V = v_1^2 + \cdots + v_n^2 = \|V\|^2$$

$$\boxed{V \cdot V = \|V\|^2}$$

$$V \cdot V = \|V\| \cdot \|V\| \underbrace{\cos 0}_{1} = \|V\|^2$$

Example



Find the angle between the two diagonals in red.

use dot product

Solution: $\vec{u} = (3, 2, 1)$
$\vec{v} = (0, 2, 1)$

$$\vec{u} \cdot \vec{v} = \|u\| \, \|v\| \cos\theta$$

$$3 \cdot 0 + 2 \cdot 2 + 1 \cdot 1 = \sqrt{3^2 + 2^2 + 1^2} \sqrt{0^2 + 2^2 + 1^2} \cos\theta$$

$$4 + 1 = \sqrt{9 + 4 + 1} \sqrt{5} \cos\theta$$

$$5 = \sqrt{14} \sqrt{5} \cos\theta$$

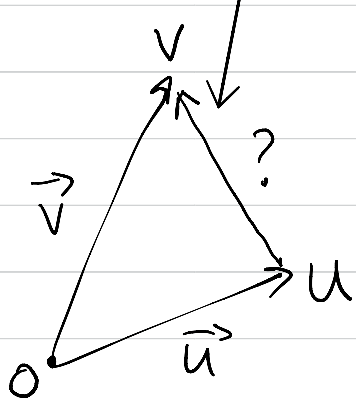
$$\frac{5}{\sqrt{70}} = \cos\theta$$

$$\arccos\left(\frac{5}{\sqrt{70}}\right) = \theta \longrightarrow \theta = 53.3 \text{ degrees}$$

3.2 11b Find the Euclidean distance between u & v & the cosine of angle between them. State whether angle is acute/obtuse/90°.

$\vec{u} = (0, -2, -1, 1)$

$\vec{v} = (-3, 2, 4, 4)$



$\vec{v} - \vec{u} = (-3,\ 2-(-2),\ 4-(-1),\ 4-1)$

$= (-3,\ 4,\ 5,\ 3)$

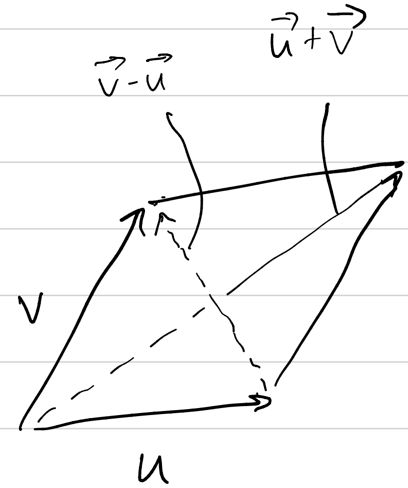
Find the $\|\overrightarrow{UV}\|$

$= \|\vec{v} - \vec{u}\|$

$= \sqrt{(-3)^2 + (4)^2 + (5)^2 + (3)^2}$

$= \sqrt{9 + 16 + 25 + 9}$

$= \boxed{\sqrt{59}}$



$u \cdot v = \|u\| \|v\| \cos\theta$

$0 - 4 - 4 + 4 = \sqrt{4+1+1}\ \sqrt{9+4+16+16}\ \cos\theta$

$-4 = \sqrt{6}\ \sqrt{45}\ \cos\theta$

$\boxed{\dfrac{-4}{\sqrt{6}\sqrt{45}} = \cos\theta}$

Since $\cos\theta$ is negative, θ is obtuse.

If $\theta = 90°$, $\cos\theta = 0$ (and conversely)

3.3 $\underbrace{\text{Orthogonality}}_{\text{perpendicular}}$

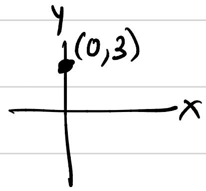
non-zero $\vec{u}, \vec{v}$ are orthogonal (i.e perpendicular) if and only if $\vec{u} \cdot \vec{v} = 0$.

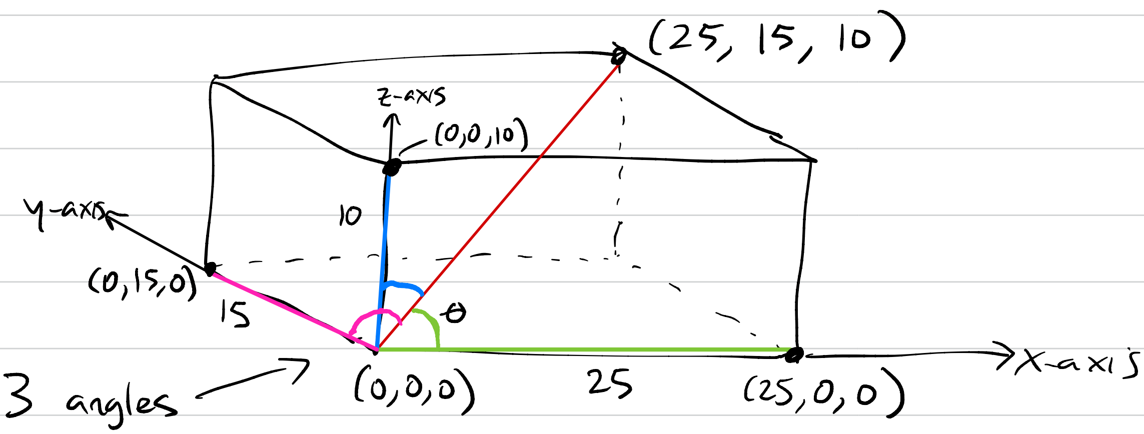
$$\|u\| \|v\| \cos\theta = 0$$

$$\cos\theta = 0$$

$$\theta = 90°$$

25. Estimate to the nearest degree the three angles that the diagonal of a box with dimensions 10 cm x 15 cm x 25 cm makes with the edges of the box.





asked to find these 3 angles

Find the angle between $\vec{u} = (25,0,0)$ & $\vec{v} = (25, 15, 10)$.

Solution: $\vec{u} \cdot \vec{v} = \|u\| \cdot \|v\| \cos\theta$ is the formula to use.

$$25 \cdot 25 + 0 + 0 = \|\sqrt{25^2}\| \, \|\sqrt{25^2 + 15^2 + 10^2}\| \cos\theta$$

$$\cancel{25} \cdot 25 = \cancel{25} \cdot \sqrt{950} \cos\theta$$

$$\begin{array}{r} 625 \\ 225 \\ 100 \\ \hline 950 \end{array}$$

$$\frac{25}{\sqrt{950}} = \cos\theta$$

$$\cos^{-1}\left(\frac{25}{\sqrt{950}}\right) = \theta$$

$$35.8^\circ = \theta$$

(or rounded 36°)

7 Let $\vec{v} = (-2, 3, 0, 6)$ Find all scalars k such that $\|k\vec{v}\| = 5$

Solution. $k\vec{v} = (-2k, 3k, 0, 6k)$

$$\|k\vec{v}\| = \sqrt{(-2k)^2 + (3k)^2 + 0^2 + (6k)^2} = 5$$

$$\sqrt{4k^2 + 9k^2 + 0 + 36k^2} = 5$$

$$\sqrt{49k^2} = 5$$

$$\sqrt{49}\sqrt{k^2}$$

$$7k$$

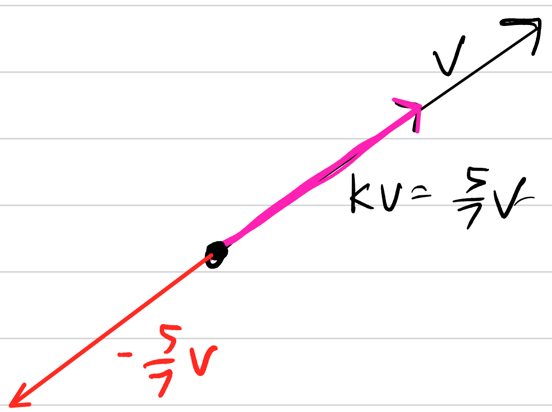
$$49k^2 = 5^2$$

$$49k^2 = 25$$

$$k^2 = \frac{25}{49}$$

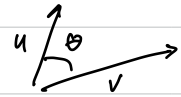
$$k = \pm\sqrt{\frac{25}{49}}$$

$$\boxed{k = \pm\frac{5}{7}}$$



$\vec{u}, \vec{v} \in \mathbb{R}^3$ (or $\mathbb{R}^2$)

$\vec{u} = (u_1, u_2, u_3)$ $\qquad$ $\vec{v} = (v_1, v_2, v_3)$

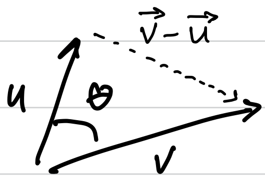


Dot Product $\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 + u_3 v_3$

Theorem. Let θ be the angle between $\vec{u}$ & $\vec{v}$. Then

$$\vec{u} \cdot \vec{v} = \|\vec{u}\| \cdot \|\vec{v}\| \cdot \cos\theta$$

Proof. Apply Law of cosines to the triangle



$$\|\vec{v} - \vec{u}\|^2 = \|u\|^2 + \|v\|^2 - 2\|u\|\|v\|\cos\theta$$

$\vec{v} - \vec{u} = (v_1 - u_1, v_2 - u_2, v_3 - u_3)$

$$(v_1-u_1)^2 + (v_2-u_2)^2 + (v_3-u_3)^2 = (u_1^2 + u_2^2 + u_3^2) + (v_1^2 + v_2^2 + v_3^2) - 2\|u\|\|v\|\cos\theta$$

$$(v_1^2 - 2v_1u_1 + u_1^2) + (v_2^2 - 2v_2u_2 + u_2^2) + (v_3^2 - 2v_3u_3 + u_3^2) =$$

The terms that are squared cancel, left with

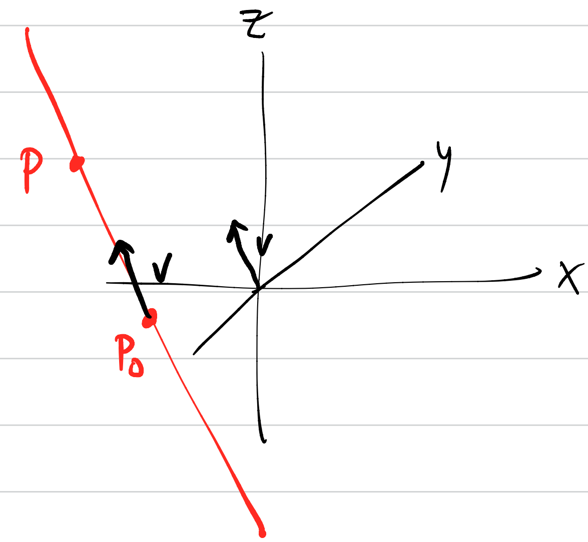
$$-2v_1u_1 - 2v_2u_2 - 2v_3u_3 = -2\|u\|\|v\|\cos\theta$$

$$-\cancel{2}(v_1u_1 + v_2u_2 + v_3u_3) = -\cancel{2}\|u\|\|v\|\cos\theta$$

$$\vec{u} \cdot \vec{v} = \|u\|\|v\|\cos\theta$$

$\square$

3.4 Geometry of lines & planes



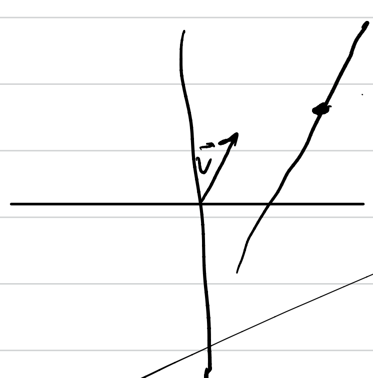
line

$$\overrightarrow{P_0P} = t \cdot \vec{v}$$ $\vec{v}$ = direction vector of line

$$P - P_0 = t \cdot \vec{v}$$

$$\boxed{\vec{P} = t \cdot \vec{v} + P_0 = P_0 + t \cdot \vec{v}}$$ vector equation of a line

Example Find the vector and parametric equations of the line containing the point $(1,2,2)$ & in direction of $\vec{v} = (-1, 0, 3)$



Solution $\boxed{\vec{P} = (1,2,2) + t(-1,0,3)}$

vector form of equation of a line

$\vec{P} = (x,y,z)$

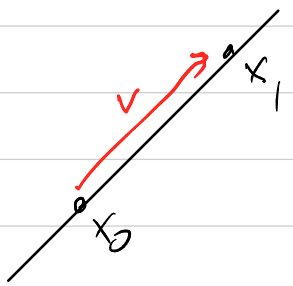
Parametric form of equation of line

$$x = 1 - t$$
$$y = 2 + 0t = 2$$
$$z = 2 + 3t$$

Given two points x_0 & x_1 in $\mathbb{R}^3$

e.g $\quad x_0 = (1,0,2) \quad x_1 = (-1,2,3)$

what is an equation of the line through them?



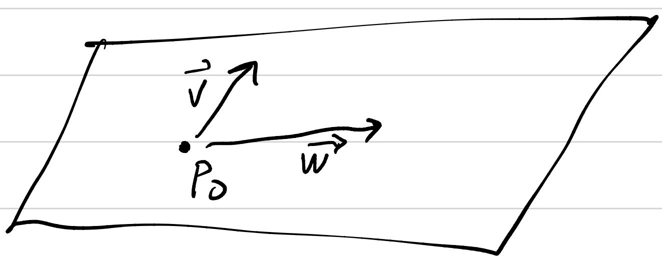
A direction vector for this line is

$$v = \overrightarrow{x_0 x_1} = x_1 - x_0$$

$$= (-1,2,3) - (1,0,2)$$

$$\vec{v} = (-2,2,1)$$

So line is $\boxed{\vec{P} = (1,0,2) + t(-2,2,1)}$ is a vector form of equation of a line.



Plane $\quad P = P_0 + t\vec{v} + s\vec{w} \quad$ vector equation of plane

Example Plane passing through
$P_0 = (1,0,1)$
& parallel to $\vec{v} = (2,1,1) \quad \vec{w} = (0,-1,-1)$

has vector equation $\boxed{P = (1,0,1) + t(2,1,1) + s(0,-1,-1)}$

or

$$\boxed{\begin{aligned} x &= 1 + 2t \\ y &= 0 + t - s \\ z &= 1 + t - s \end{aligned} \quad \text{parametric form}}$$

so take $t=2,\ s=3$ then

$$x = 1+2\cdot 2 = 5$$
$$y = t-s = 2-3 = -1$$
$$z = 1+2-3 = 0$$

so $(x,y,z) = (5,-1,0)$ is one point on this plane.

Last time: equation of plane
$$ax+by+cz+d=0$$
has normal vector $\vec{n} = (a,b,c)$

Example $\quad x+2y+3z+5=0$

$$2y = -5-x-3z$$
$$y = -\frac{5}{2} - \frac{x}{2} - \frac{3z}{2}$$

$$x = -5-2y-3z$$
$$y = \quad y$$
$$z = \quad z$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -5 \\ 0 \\ 0 \end{pmatrix} + y\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} + z\begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix}$$

3.4

22. Find a homogeneous linear system of two equations in 3 variables consisting of those vectors in $\mathbb{R}^3$ that are <u>orthogonal</u> to $\vec{a} = (-3,2,1)$ & $\vec{b} = (0,-2,-2)$

perpendicular $\longleftrightarrow$ dot product $=0$

$\vec{v} = (x,y,z)$

$$\vec{a}\cdot\vec{v} = 0 \Rightarrow (-3,2,1)\cdot(x,y,z) = 0 \Rightarrow \boxed{-3x+2y+z=0}$$
$$\vec{b}\cdot\vec{v} = 0 \Rightarrow (0,-2,-2)\cdot(x,y,z) = 0 \Rightarrow \boxed{-2y-2z=0}$$

perpendicular/orthogonal

Equation of plane with normal vector $\vec{n} = (a, b, c)$ is

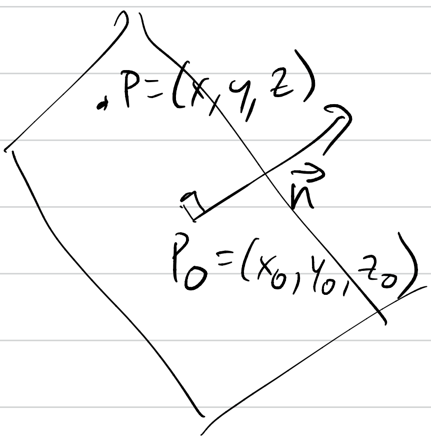
$$ax + by + cz + d = 0$$

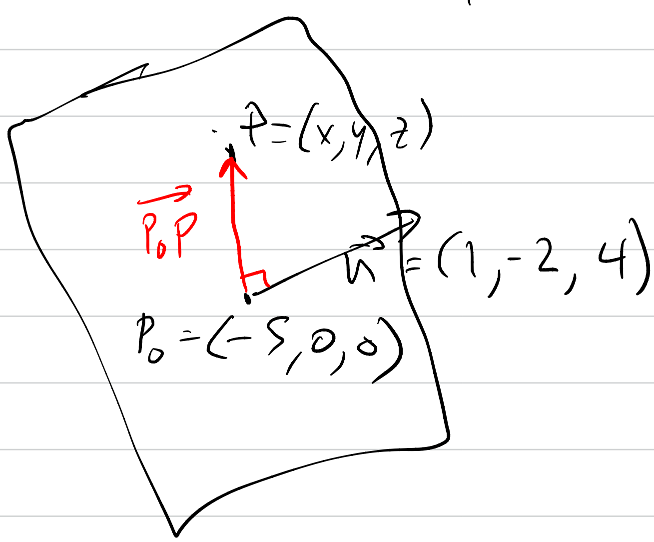
For example the planes with normal vector $\vec{n} = (1, -2, 4)$ is

$$x - 2y + 4z + d = 0$$

where d is any fixed number. e.g $d = 5$.

$x - 2y + 4z + 5 = 0$ is the equation of a plane.





What is the equation of the plane passing through $(-5, 0, 0)$ & with normal vector $\vec{n} = (1, -2, 4)$?

$\overrightarrow{P_0P}$ is perpendicular to $\vec{n} = (1, -2, 4)$

$$\overrightarrow{P_0P} \cdot \vec{n} = 0$$

$$(P - P_0) \cdot \vec{n} = 0$$

$$\left[(x,y,z) - (-5,0,0)\right] \cdot (1,-2,4) = 0$$

$$(x+5, y, z) \cdot (1,-2,4) = 0$$

$$(x+5) \cdot 1 + y \cdot (-2) + z \cdot 4 = 0$$

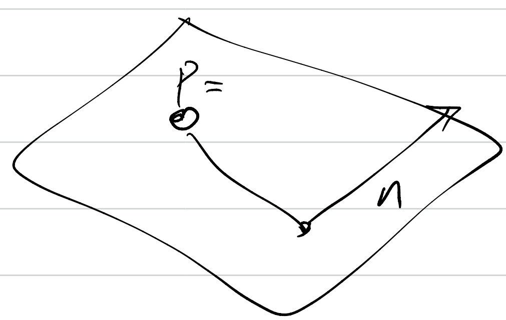
$$1 \cdot x - 2y + 4z + 5 = 0$$

Notice that the coefficients of x, y, z, in this case $1, -2, 4$ give ~~the~~ a normal vector

Example Find equation of plane through $P = (2,-3,5)$ and parallel to $3x - 2y - z = 0$

Solution Normal vector is $\vec{n} = (3,-2,-1)$

$$P_0 = (x,y,z)$$



$$\overrightarrow{P_0P} \cdot \vec{n} = 0$$

$$(P - P_0) \cdot \vec{n} = 0$$

$$\left((2,-3,5) - (x,y,z)\right) \cdot (3,-2,-1) = 0$$

$$(2-x, -3-y, 5-z) \cdot (3,-2,-1) = 0$$

$$3(2-x) + (-2)(-3-y) + (-1)(5-z) = 0$$

$$6 - 3x + 6 + 2y - 5 + z = 0$$

$$\boxed{-3x + 2y + z + 7 = 0}$$

or $\boxed{3x - 2y - z - 7 = 0}$

2nd way: we know the equation of any plane parallel to

$$3x - 2y - z = 0$$

is of the form $3x - 2y - z + d = 0$ for some number d.

(since they have same normal vector $(3, -2, -1)$)
So we just have to find out what d is

We know $P = (2, -3, 5)$ is on this plane, so

$$3(2) - 2(-3) - 5 + d = 0$$

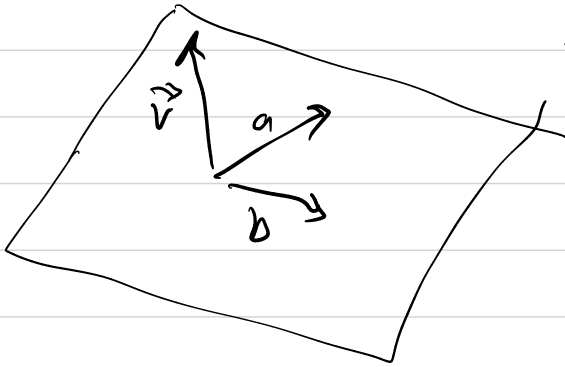
$$6 + 6 - 5 + d = 0$$

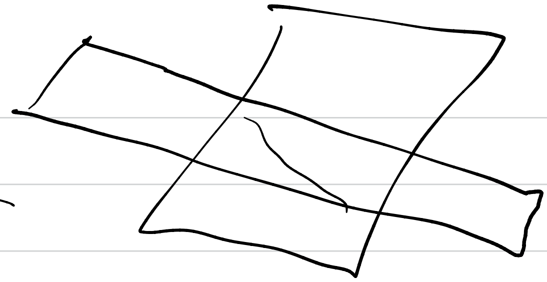
$$12 - 5 + d = 0$$

$$7 + d = 0$$

$$d = -7$$

so answer is $\boxed{3x - 2y - z - 7 = 0}$





3.5 Cross product (only in $\mathbb{R}^3$)

$\vec{v} = (v_1, v_2, v_3) = v_1 i + v_2 j + v_3 k$

$\vec{w} = (w_1, w_2, w_3)$

Cross product $\underbrace{\vec{v} \times \vec{w}}_{\text{a vector in } \mathbb{R}^3} = (\quad , \quad , \quad)$ given by the following formula

$$\vec{v} \times \vec{w} = \underbrace{\begin{vmatrix} i & j & k \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}}_{\text{determinant}} = i\left(\underbrace{\begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix}}_{v_2 w_3 - w_2 v_3}\right) - j\left(\begin{vmatrix} v_1 & v_3 \\ w_1 & w_3 \end{vmatrix}\right) + k\left(\begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix}\right)$$

Example Find cross product of $\vec{a} = (-3, 2, 1)$ $\vec{b} = (0, -2, -2)$

$$\vec{a} \times \vec{b} = \begin{vmatrix} i & j & k \\ -3 & 2 & 1 \\ 0 & -2 & -2 \end{vmatrix} = i\left(\begin{vmatrix} 2 & 1 \\ -2 & -2 \end{vmatrix}\right) - j\left(\begin{vmatrix} -3 & 1 \\ 0 & -2 \end{vmatrix}\right) + k\begin{vmatrix} -3 & 2 \\ 0 & -2 \end{vmatrix}$$

$$= i(-4+2) - j(6-0) + k(6-0)$$

$$= -2i - 6j + 6k$$

$$\vec{a} \times \vec{b} = \boxed{(-2, -6, 6)}$$

Fact : $\vec{a} \times \vec{b}$ is orthogonal to $\vec{a}$ & to $\vec{b}$ for any vectors a and b

(Proof: compute $(\vec{a} \times \vec{b}) \cdot \vec{a} = 0$

in above example, $\underset{a \times b}{(-2, -6, 6)} \cdot (-3, 2, 1) = (-2)(-3) + (-6)(2) + 6 \cdot 1$

$$= 6 - 12 + 6 = 0$$

Lagrange's identity

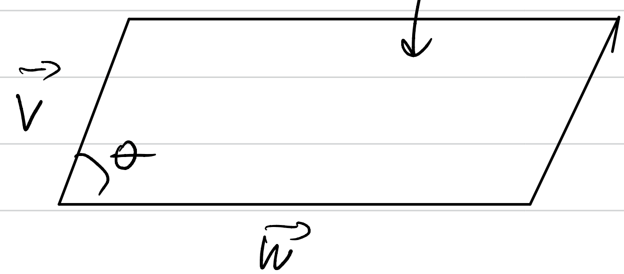
$$\|v \times w\|^2 = \|v\|^2 \|w\|^2 - (v \cdot w)^2$$

So $\|v \times w\|^2 = \|v\|^2 \|w\|^2 - \|v\|^2 \cdot \|w\|^2 \cos^2\theta$

$$= \|v\|^2 \|w\|^2 (1 - \cos^2\theta)$$

$$= \|v\|^2 \|w\| \sin^2\theta$$

$\|v \times w\| = \|v\| \|w\| |\sin\theta|$ Area of the parallelogram



So norm of cross product of $\vec{v}$ & $\vec{w}$ is area of parallelogram v, w make.

In Exercises 11–12, find the area of the parallelogram with the given vertices.

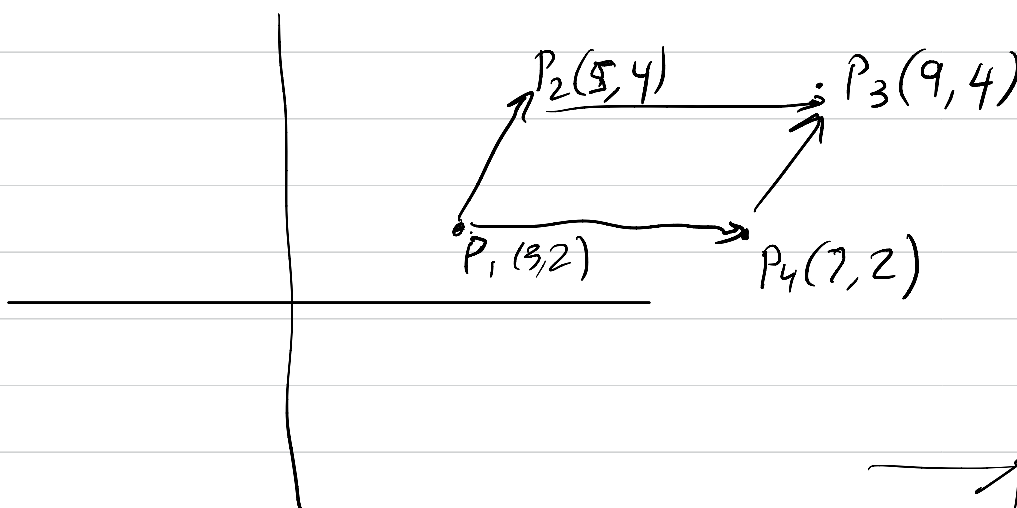
11. $P_1(1, 2), P_2(4, 4), P_3(7, 5), P_4(4, 3)$

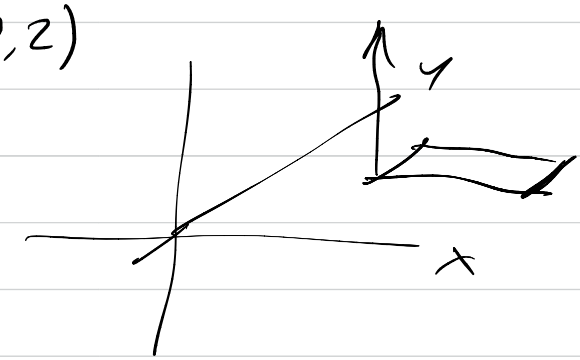
Answer

Solution

12. $P_1(3, 2), P_2(5, 4), P_3(9, 4), P_4(7, 2)$

12.





Solution: Find $\| \overrightarrow{P_1P_2} \times \overrightarrow{P_1P_4} \|$

$$\overrightarrow{P_1P_2} = P_2 - P_1 = (5,4) - (3,2) = (2,2)$$

$$\overrightarrow{P_1P_4} = P_4 - P_1 = (7,2) - (3,2) = (4,0)$$

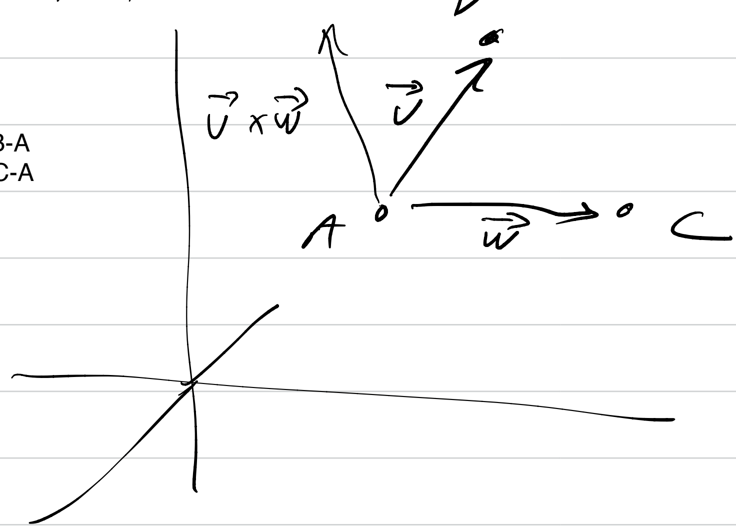
$$\begin{vmatrix} i & j & k \\ 2 & 2 & 0 \\ 4 & 0 & 0 \end{vmatrix} = i(0) - j(0) + k(0-8)$$

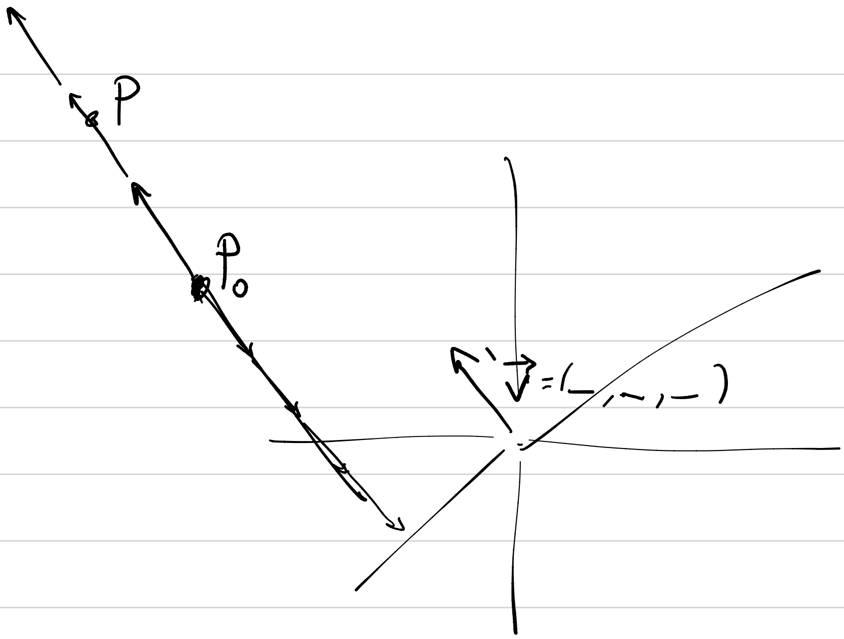
$$= -8k = (0,0,-8)$$

$$\|(0,0,-8)\| = \sqrt{0^2 + 0^2 + (-8)^2} = \boxed{8}$$ Area of parallelogram

A, B, C

v=AB=B-A
w=AC=C-A





$$\overrightarrow{P_0P} = t \cdot \vec{v}$$

scalar

$$\vec{P} - \vec{P_0} = t\vec{v}$$

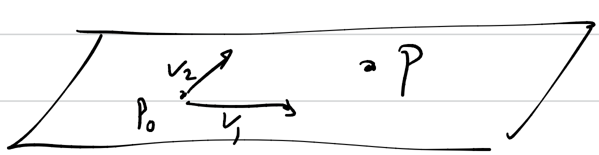
$$\boxed{\vec{P} = \vec{P_0} + t\vec{v}}$$

vector form of the equation of a line through P_0 & in the direction of $\vec{v}$

Planes: Same idea as for lines but <u>two</u> direction vectors

$$P = P_0 + s \cdot \vec{v_1} + t \cdot \vec{v_2} \qquad \underbrace{s, t \in \mathbb{R}}_{\text{scalars}}$$

vectors   vector



Exercise 4.2.21 Find the equation of *all* planes:

a. Perpendicular to the line $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} + t\begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$.

b. Perpendicular to the line $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + t\begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix}$.

c. Containing the origin.

d. Containing $P(3, 2, -4)$.

e. Containing $P(1, 1, -1)$ and $Q(0, 1, 1)$.

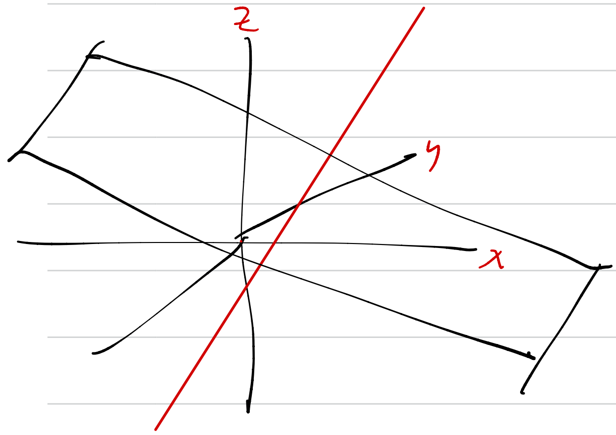
f. Containing $P(2, -1, 1)$ and $Q(1, 0, 0)$.

g. Containing the line $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$.

h. Containing the line $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix} + t\begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix}$.

direction vector $\longrightarrow$ $\begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$ (same as $(2,1,3)$)

because that's the vector that has a t in front
↑
scalar



$ax+by+cz+d=0$
equation of plane.
normal vector to this plane is (a,b,c)

a) So $(a,b,c) = (2,1,3)$ (or some scalar multiple)

$$\boxed{2x+1y+3z+d=0}$$

d can be any number.

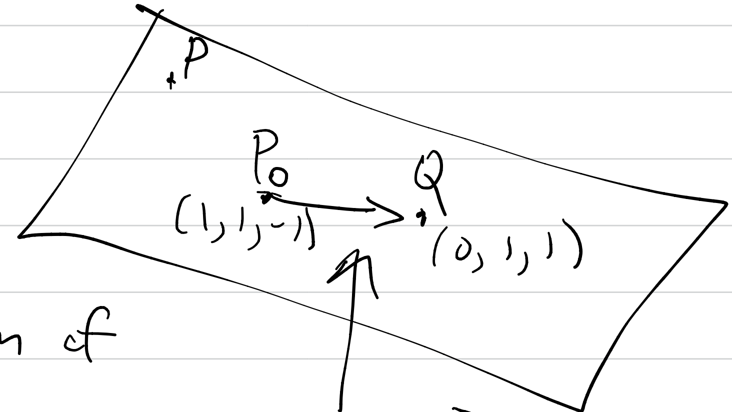
c) $ax+by+cz+d=0$ origin $=(0,0,0)$

$$\underbrace{a\cdot 0 + b\cdot(0) + c\cdot(0)}_{0} + d = 0$$

$d=0$

$$\boxed{ax + by + cz = 0}$$

e)



Let try to find vector form of equation of all planes

$$\vec{v}_1 = \overrightarrow{P_0Q}$$
$$= Q - P$$
$$= (0,1,1) - (1,1,-1)$$
$$\vec{v}_1 = (-1,0,2)$$

vector form of eqn of plane

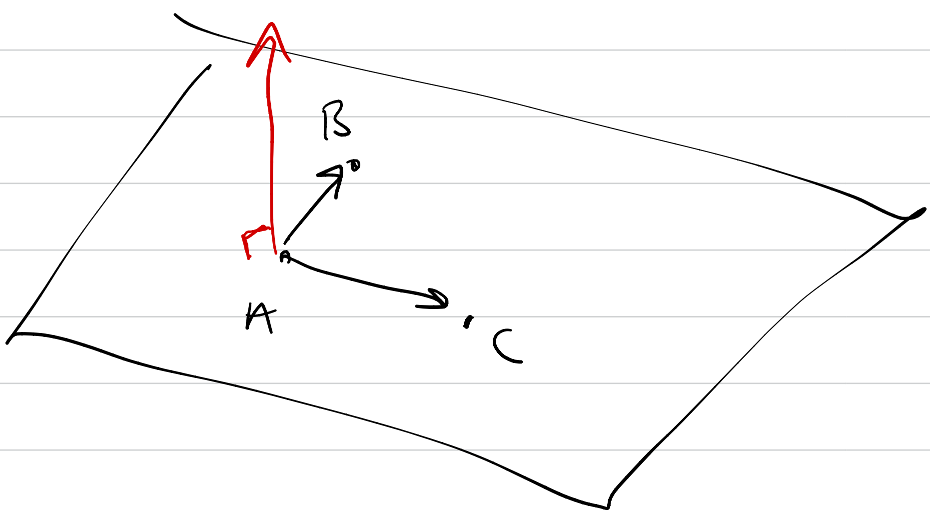
$$P = P_0 + s \cdot \vec{v}_1 + t \cdot \vec{v}_2$$

$$\boxed{P = (1,1,-1) + s(-1,0,2) + t \cdot \vec{v}_2}$$

Exercise 4.2.14 Find an equation of each of the following planes. → Find the normal vector first

a. Passing through $A(2, 1, 3)$, $B(3, -1, 5)$, and $C(1, 2, -3)$.

b. Passing through $A(1, -1, 6)$, $B(0, 0, 1)$, and $C(4, 7, -11)$.

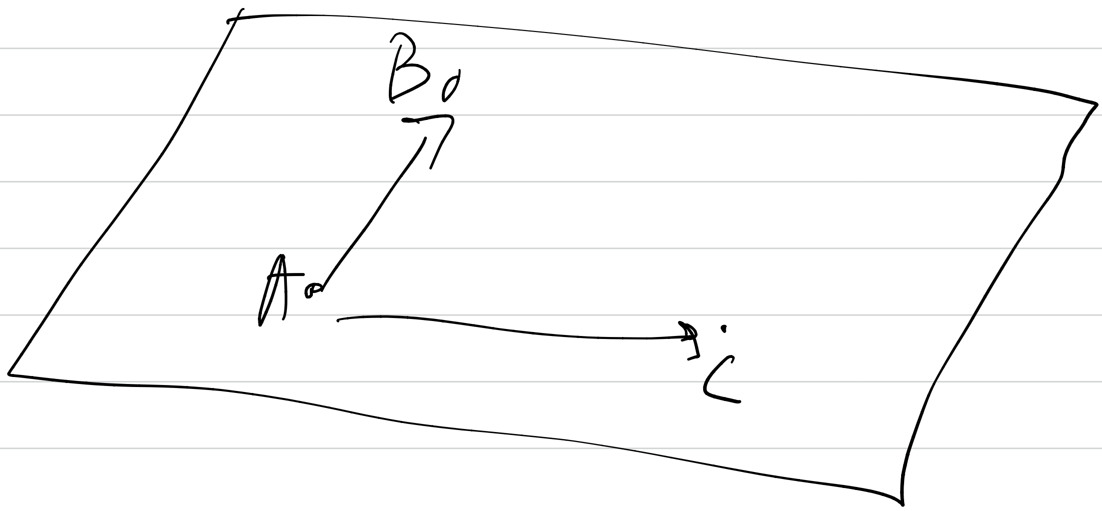


We need to find a vector perpendicular to $\overrightarrow{AB}$ & $\overrightarrow{AC}$

(because that will be normal vector (a, b, c) in $\underbrace{ax + by + cz + d = 0}_{\text{eqn of plane}}$)

This is what cross product is built for!

$A = (2, 1, 3) \qquad B = (3, -1, 5) \qquad C = (1, 2, -3)$



$$\vec{AB} = B - A = (3, -1, 5) - (2, 1, 3)$$

$$\vec{AB} = (1, -2, 2)$$

$$\vec{AC} = C - A = (1, 2, -3) - (2, 1, 3)$$

$$\vec{AC} = (-1, 1, -6)$$

$$\vec{AB} \times \vec{AC} = \begin{vmatrix} i & j & k \\ 1 & -2 & 2 \\ -1 & 1 & -6 \end{vmatrix}$$

$$= i(12 - 2) - j(-6 - (-2)) + k(1 - 2)$$

$= \hat{i}(10) - j(-4) + k(-1)$

$= (10, 4, -1)$ is cross product

So equation of our plane is

$10x + 4y - 1 \cdot z + d = 0$

To find d, substitute any of the points

Let's plug in $A = (2,1,3)$

$$
\begin{array}{ccccccc}
10x & + & 4y & - & z & + d & = 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
10 \cdot 2 & + & 4 \cdot 1 & - & 3 & + d & = 0
\end{array}
$$

$20 + 4 - 3 + d = 0$

$21 + d = 0$

$d = -21$

$$\boxed{10x + 4y - z - 21 = 0}$$

Is $B = (3, -1, 5)$ on this plane?

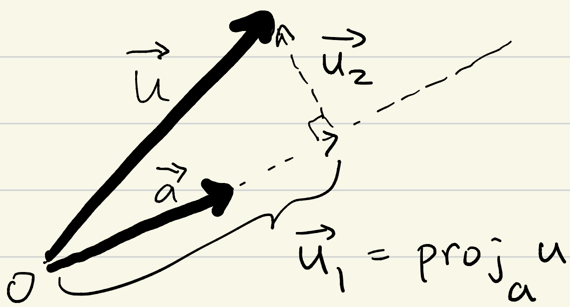
$10 \cdot 3 + 4(-1) - 5 - 21 = 0$

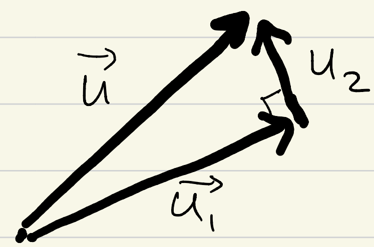
$30 - 4 - 5 - 21 = 0$ ✓

$\vec{u}, \vec{a}$ two vectors ($\vec{a}$ non-zero)

$\vec{u}$ can be expressed uniquely as $\vec{u} = \vec{u}_1 + \vec{u}_2$
where
- $\vec{u}_1$ is parallel to $\vec{a}$ ($\vec{u}_1$ is the orthogonal proj. of $\vec{u}$ on $\vec{a}$)
- $\vec{u}_2$ is perpendicular to $\vec{a}$.

short for projection





Formulas: $\vec{u}_1 = \text{proj}_{\vec{a}} \vec{u} = \underbrace{\left(\dfrac{\vec{u} \cdot \vec{a}}{\|a\|^2}\right)}_{\text{number}} \vec{a}$

$$\vec{u}_2 = \vec{u} - \vec{u}_1$$

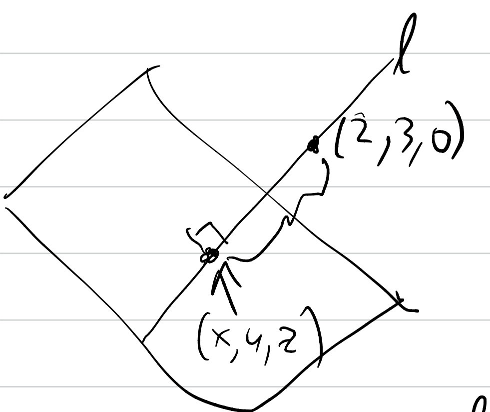
Proof. $\vec{u}_1 = k\vec{a}$ for some scalar k, since u_1 is parallel to $\vec{a}$.

Since $\vec{u} = \vec{u}_1 + \vec{u}_2$, $\vec{u} - \vec{u}_1 = \vec{u}_2$

Since $\vec{u}_2$ is perpendicular to $\vec{a}$, we have $\vec{u}_2 \cdot \vec{a} = 0$.

$$\vec{u}_2 \cdot \vec{a} = 0$$
$$(\vec{u} - \vec{u}_1) \cdot \vec{a} = 0$$
$$(\vec{u} - k\vec{a}) \cdot \vec{a} = 0$$
$$\vec{u} \cdot \vec{a} - k\underbrace{\vec{a} \cdot \vec{a}}_{\|a\|^2} = 0$$

3. Find the shortest distance from the point $P=(2,3,0)$ & the plane $5x+y+z=1$.



What point on the plane $5x+y+z=1$ is closest to $(2,3,0)$?

Plane has normal $\vec{n}=(5,1,1)$

$$l=(2,3,0)+t(5,1,1)$$

equation of line l

$$l=(2,3,0)+(5t,t,t)$$

$$l=(2+5t,3+t,t)$$

$$\left.\begin{array}{l} x=2+5t \\ y=3+t \\ z=t \end{array}\right\}$$ is equation(s) of line l.

$5x+y+z=1$ equation of plane

$$5(2+5t)+3+t+t=1$$

$$10+25t+3+2t=1$$

$$27t+13=1$$

$$27t=1-13$$

$$27t = -12$$

$$t = \frac{-12}{27} = \frac{-\cancel{3}\cdot 4}{\cancel{3}\cdot 9} = -\frac{4}{9}$$

$$\begin{aligned} x &= 2+5t \\ y &= 3+t \\ z &= t \end{aligned} \longrightarrow \begin{aligned} x &= 2-5\left(\frac{4}{9}\right) = \frac{18}{9} - \frac{20}{9} = \frac{-2}{9} \\ y &= 3 - \frac{4}{9} = \frac{27}{9} - \frac{4}{9} = \frac{23}{9} \\ z &= -\frac{4}{9} \end{aligned}$$

So $\left(-\frac{2}{9}, \frac{23}{9}, \frac{-4}{9}\right)$ is the point on the plane closest to $P = (2,3,0)$

How to find the shortest distance?

- Find the distance between $P_1 = \left(-\frac{2}{9}, \frac{23}{9}, \frac{-4}{9}\right)$ and $P_0 = (2,3,0)$.

$$\overrightarrow{P_0P_1} = P_1 - P_0 = \left(\frac{-20}{9}, \frac{-4}{9}, \frac{-4}{9}\right)$$

$$= \underbrace{(5,1,1)}_{\vec{n}} \cdot \underbrace{\left(\frac{-4}{9}\right)}_{t}$$ scalar product

$$\|P_0P_1\| = \sqrt{\left(-\frac{20}{9}\right)^2 + \left(-\frac{4}{9}\right)^2 + \left(-\frac{4}{9}\right)^2}$$

$$= \sqrt{\left(\frac{-4}{9}\right)^2 \left(5^2 + 1^2 + 1^2\right)}$$

$$= \frac{4}{9}\sqrt{25+1+1} = \frac{4}{9}\sqrt{27} = \frac{4}{9}\sqrt{3\cdot 3^2}$$

$$= \frac{4}{9}\cdot 3\sqrt{3}$$

$$= \boxed{\frac{4}{3}\sqrt{3}}$$

2^{nd} way: Formula for distance between $P_0 = (x_0, y_0, z_0)$ & the plane $ax + by + cz + d = 0$ is

$$\boxed{\left|\frac{ax_0 + by_0 + cz_0 + d}{\sqrt{a^2 + b^2 + c^2}}\right|}$$

$a = 5,\ b = 1,\ c = 1\quad d = -1$

$(x_0, y_0, z_0) = (2, 3, 0)$

So distance is $\left| \dfrac{5 \cdot 2 + 1 \cdot 3 + 1 \cdot 0 - 1}{\sqrt{5^2 + 1^2 + 1^2}} \right|$

$$= \frac{12}{\sqrt{27}} = \frac{12\sqrt{27}}{\sqrt{27} \cdot \sqrt{27}}$$

$$= \frac{12}{27} \cdot 3\sqrt{3}$$

$$= \frac{4}{9} \cdot 3\sqrt{3}$$

$$= \frac{4}{3}\sqrt{3}.$$

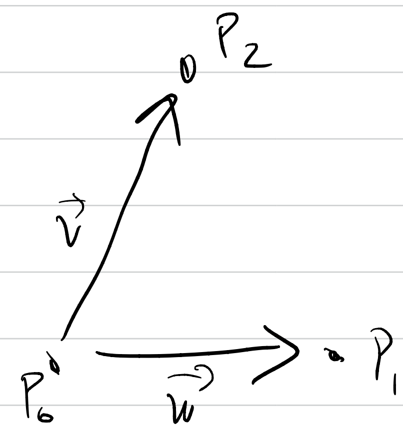
$ax + by + cz + d = 0$

2. Find equation of plane passing through

$P_0 = (2, 1, 3)$

$P_1 = (3, -1, 5)$

$P_2 = (1, 2, -3)$



$P = P_0 + t\vec{v} + s\vec{w}$ ⟵ parametric form

$= (2,1,3) + t(\overrightarrow{P_0P_2}) + s(\overrightarrow{P_0P_1})$

letter s

$= (2,1,3) + t(P_2 - P_0) + s(P_1 - P_0)$

$= (2,1,3) + t(1-2, 2-1, -3-3) + s(3-2, -1-1, 5-3)$

$$\boxed{P = (2,1,3) + t(-1, 1, -6) + s(1, -2, 2)}$$

⟵ vector equation of plane.

$$\boxed{\begin{aligned} x &= 2 - t + s \\ y &= 1 + t - 2s \\ z &= 3 - 6t + 2s \end{aligned}}$$

parametric form of equation of plan

What is the equation of the plane in the form

$ax + by + cz + d = 0$?

$P_0 = (2, 1, 3)$
$P_1 = (3, -1, 5)$ are on the plane.
$P_2 = (1, 2, -3)$

$2a + b + 3c + d = 0$

$3a - b + 5c + d = 0$

$a + 2b - 3c + d = 0$

$$\left[\begin{array}{cccc|c} 2 & 1 & 3 & 1 & 0 \\ 3 & -1 & 5 & 1 & 0 \\ 1 & 2 & -3 & 1 & 0 \end{array}\right]$$

Now find RREF;
I did the first couple of steps
by hand but then
decided to use SageMath

optional to write since

$$\begin{bmatrix} 1 & 2 & -3 & 1 \\ 3 & -1 & 5 & 1 \\ 2 & 1 & 3 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & -3 & 1 \\ 0 & -7 & 14 & -2 \\ 2 & 1 & 3 & 1 \end{bmatrix}$$

$$A.\mathrm{rref}() = \left[\begin{array}{cccc|c} 1 & 0 & 0 & \frac{10}{21} & 0 \\ 0 & 1 & 0 & \frac{4}{21} & 0 \\ 0 & 0 & 1 & \frac{-1}{21} & 0 \end{array}\right]$$

via Sagemath

a, b, c pivot variable

$$a + \frac{10}{21}d = 0 \Rightarrow a = \frac{-10}{21}d$$

$$b + \frac{4}{21}d = 0 \Rightarrow b = -\frac{4}{21}d$$

$$c - \frac{1}{21}d = 0 \qquad c = \frac{1}{21}d$$

$$ax + by + cz + d = 0$$

$$\left(\frac{-10}{21}d\right)x + \left(\frac{-4}{21}d\right)y + \frac{1d}{21}\cdot z + d = 0$$

$$-\frac{10}{21}x + \frac{-4}{21}y + \frac{z}{21} + 1 = 0$$ (divided through by d)

$$-10x - 4y + z + 21 = 0$$

$\Rightarrow P_0 = (2,1,3)$

$$-10\cdot 2 - 4\cdot 1 + 3 + 21 \stackrel{?}{=} 0$$

$$-20 - 4 + 3 + 21 = 0$$

$$\vec{u}\cdot\vec{a} = k\cdot\|a\|^2$$

$$\frac{\vec{u}\cdot\vec{a}}{\|a\|^2} = k$$

so $\vec{u}_1 = k\vec{a} = \left(\dfrac{\vec{u}\cdot\vec{a}}{\|a\|^2}\right)\vec{a}$

$$\vec{u}_2 = \vec{u} - \vec{u}_1 .$$

□

Exercise 4.2.11 In each case, write $\mathbf{u} = \mathbf{u}_1 + \mathbf{u}_2$, where $\mathbf{u}_1$ is parallel to $\mathbf{v}$ and $\mathbf{u}_2$ is orthogonal to $\mathbf{v}$.

a. $\mathbf{u} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}$

b. $\mathbf{u} = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} -2 \\ 1 \\ 4 \end{bmatrix}$

Solution. a) $\vec{u}_1 = \left(\dfrac{u\cdot v}{\|v\|^2}\right)\cdot\vec{v}$

$$= \frac{2\cdot 1 + (-1)(-1) + 1\cdot 3}{1^2 + (-1)^2 + (3)^2}\cdot\vec{v}$$

$$\vec{u}_1 = \frac{6}{11}\cdot\vec{v} = \frac{6}{11}(1,-1,3) = \boxed{\left(\frac{6}{11}, \frac{-6}{11}, \frac{18}{11}\right).}$$

$$\vec{u}_2 = \vec{u} - \vec{u}_1 = (2,-1,1) - \left(\frac{6}{11}, \frac{-6}{11}, \frac{18}{11}\right)$$

$$= \left(\frac{22}{11}, \frac{-11}{11}, \frac{11}{11}\right) - \left(\frac{6}{11}, \frac{-6}{11}, \frac{18}{11}\right)$$

$$= \boxed{\left(\frac{16}{11}, \frac{-5}{11}, \frac{-7}{11}\right)}$$

$5 = -11+6$ $\qquad$ $11-18=-7$

Two vectors u, v are orthogonal/perpendicular if

$\vec{u} \cdot \vec{v} = 0$

Example:

Anton 1d: $\vec{u} = (5, -4, 0, 3)$

$\vec{v} = (-4, 1, -3, 7)$

Are $\vec{u}$ & $\vec{v}$ orthogonal?

Solution: $\vec{u} \cdot \vec{v} = 5(-4) + (-4)\cdot 1 + 0\cdot(-3) + 3\cdot 7$

$= -20 - 4 + 0 + 21$

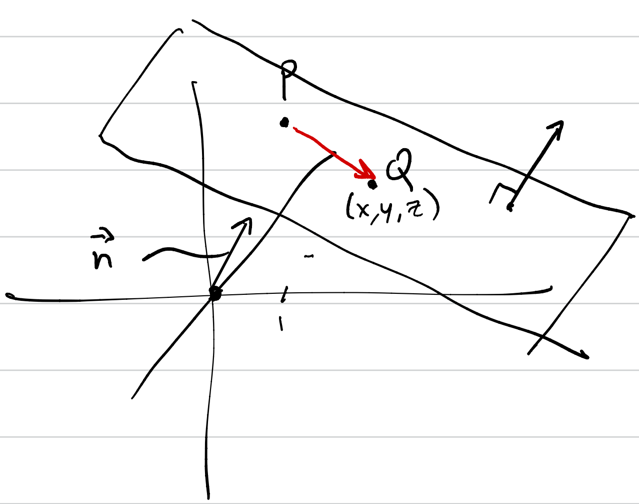
$= 1 - 4 = \underbrace{-3}_{\text{not zero}}$

so u, v are <u>not</u> <u>orthogonal</u>.

Anton 3-6

Find the point-normal form of the equation of the plane passing through the given point P having $\vec{n}$ as a normal = perpendicular

4: $P = (1, 1, 4)$ $\qquad \vec{n} = (1, 9, 8)$



let $Q = (x, y, z)$.

$\vec{PQ} \cdot \vec{n} = 0$

$\downarrow$

$(Q - P) \cdot \vec{n}$

$\big((x, y, z) - (1, 1, 4)\big) \cdot (1, 9, 8) = 0$

$(x-1, y-1, z-4) \cdot (1, 9, 8) =$

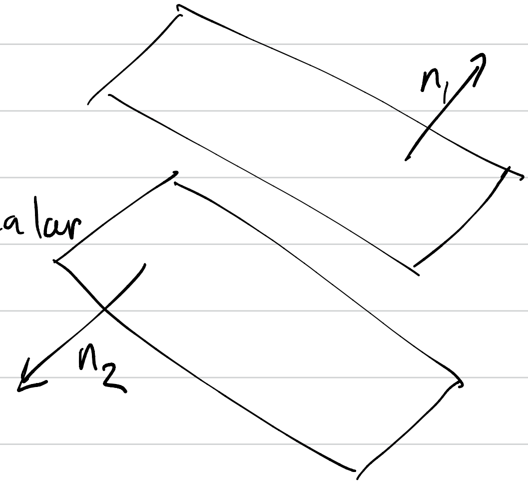
$$\boxed{1(x-1) + 9(y-1) + 8(z-4) = 0}$$

$$x - 1 + 9y - 9 + 8z - 32 = 0$$
$$x + 9y + 8z - 42 = 0$$

The normal vector to a plane allows one to determine if two planes are

- parallel

if n_1 & n_2 are scalar multiples of each other.

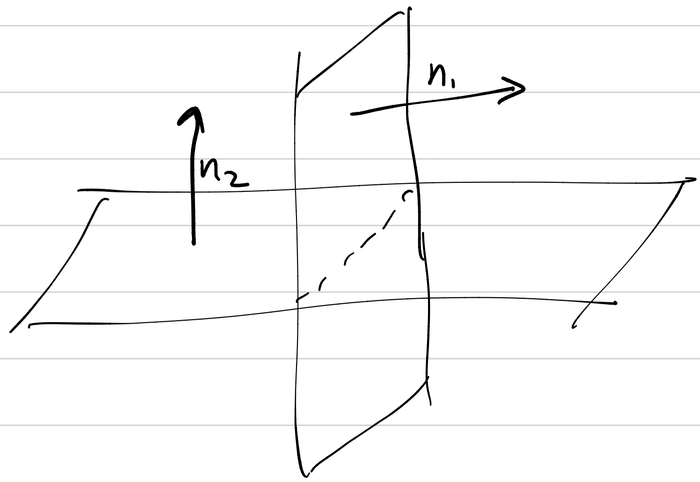


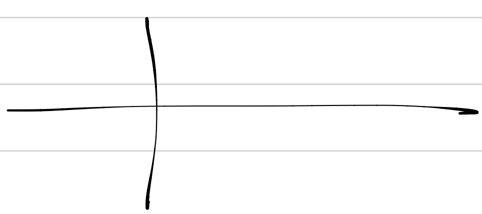
(3-D analog of parallel lines)

- perpendicular

if $\vec{n}_1 \cdot \vec{n}_2 = 0$

(i.e n_1 & n_2 are perpendicular).

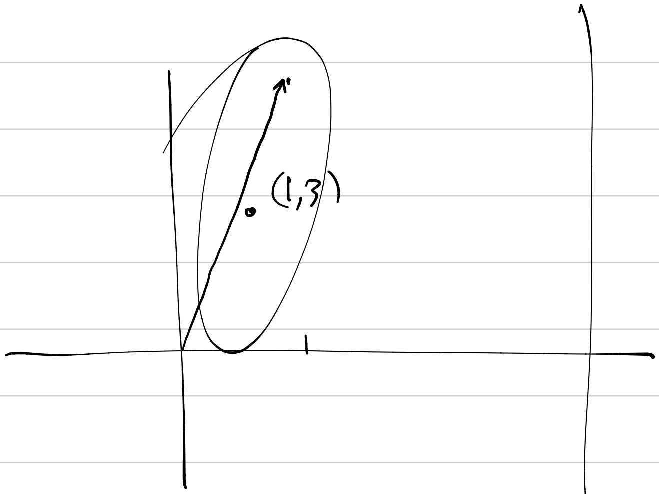


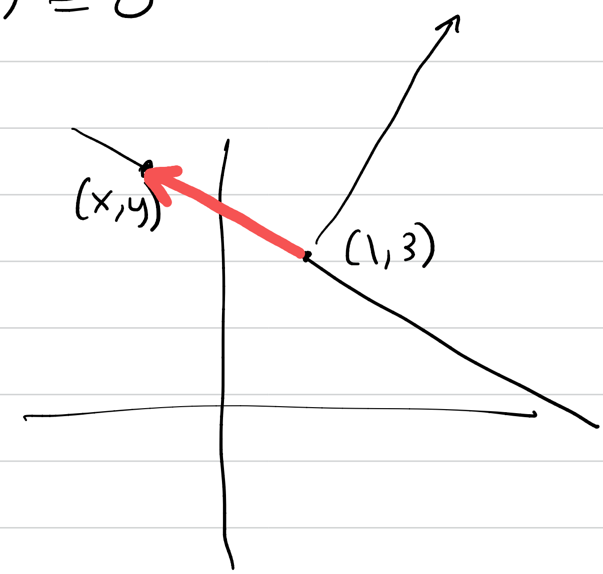


Example: Line through $(1,3)$ with normal vector $(2,5)$ has what equation?

Solution $\quad ((x,y) - (1,3)) \cdot (2,5) = 0$

$$(x-1, y-3) \cdot (2,5) = 0$$





Line L_1: $2(x-1) + 5(y-3) = 0$

$2x - 2 + 5y - 15 = 0 \Rightarrow 2x + 5y - 17 = 0$

Suppose we had another line 2: $\underbrace{4x + 10y}_{\text{normal vector } (4,10)} = 7$

Are L_1 & L_2 parallel?

normal vector of line 1 is $n_1 = (2,5)$

" line 2 $n_2 = (4,10)$

Are n_1 & n_2 scalar multiples of each other?

is there a number k $\quad kn_1 = n_2$

$$k(2,5) = (4,10)$$

$(2k, 5k) = (4, 10)$

$2k = 4$ and $5k = 10$

$k = 2$ $\quad$ $k = 2$

so $k = 2$ works.

Anton 9. Determine if planes

$2y = 8x - 4z + 5$ & $x = \frac{1}{2}z + \frac{1}{4}y$

are parallel.

Solution: Determine normal vectors.

$0 = 8x - 2y - 4z + 5$ $\qquad$ $0 = -x + \frac{1}{4}y + \frac{1}{2}z$

$\vec{n_1} = (8, -2, -4)$ $\qquad$ $\vec{n_2} = (-1, \frac{1}{4}, \frac{1}{2})$

$k \cdot \vec{n_1} = \vec{n_2}$

$k(8, -2, -4) = (-1, \frac{1}{4}, \frac{1}{2})$

$(8k, -2k, -4k) = (-1, \frac{1}{4}, \frac{1}{2})$

$8k = -1$ $\qquad$ $-2k = \frac{1}{4}$ $\qquad$ $-4k = \frac{1}{2}$

$k = \frac{-1}{8}$ $\qquad$ $k = \frac{-1}{8}$ $\qquad$ $k = \frac{-1}{8}$

All these values of k agree, so the planes are parallel.

Exercise 4.2.12 Calculate the distance from the point P to the line in each case and find the point Q on the line closest to P.

a. $P(3,\ 2-1)$

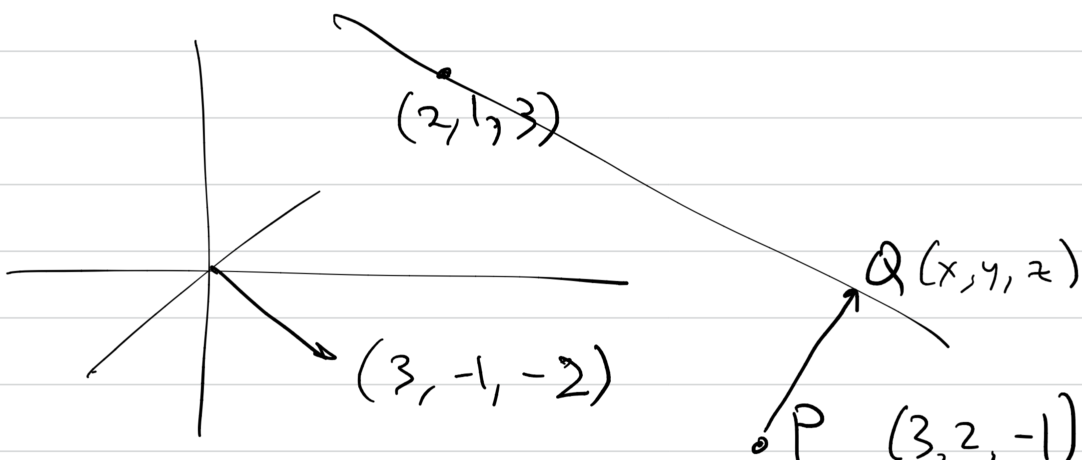
direction vector of the line

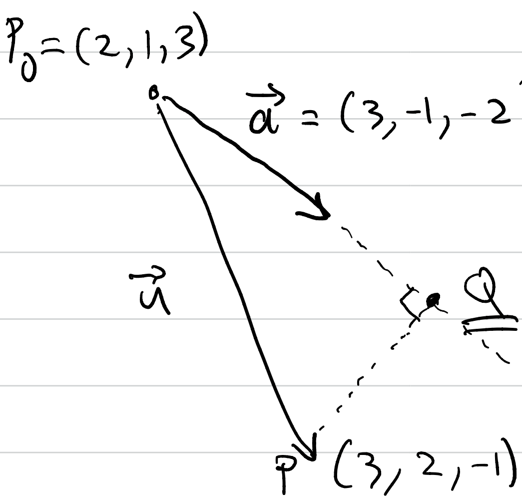
$$\text{line: } \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} + t \begin{bmatrix} 3 \\ -1 \\ -2 \end{bmatrix}$$

b. $P(1,\ -1,\ 3)$

$$\text{line: } \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + t \begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix}$$

a)





$$P_0Q = \text{proj}_{\vec{a}} \overrightarrow{P_0P}$$

$$= \left(\frac{\vec{u} \cdot \vec{a}}{\|\vec{a}\|^2} \right) \vec{a}$$

$$\vec{u} = \overrightarrow{P_0P}$$

$$= P - P_0$$

$$= (3, 2, -1) - (2, 1, 3)$$

$$\vec{u} = (1, 1, -4)$$

$$\vec{a} = (3, -1, -2)$$

$$\overrightarrow{P_0Q} = \left(\frac{3 - 1 + 8}{3^2 + 1^2 + 2^2} \right) \vec{a}$$

$$\overrightarrow{P_0Q} = \left(\frac{10}{14} \right) \vec{a}$$

$$Q - P_0 = \frac{5}{7} \vec{a}$$

$$Q = \frac{5}{7} \vec{a} + P_0$$

$$= \frac{5}{7}(3, -1, -2) + (2, 1, 3)$$

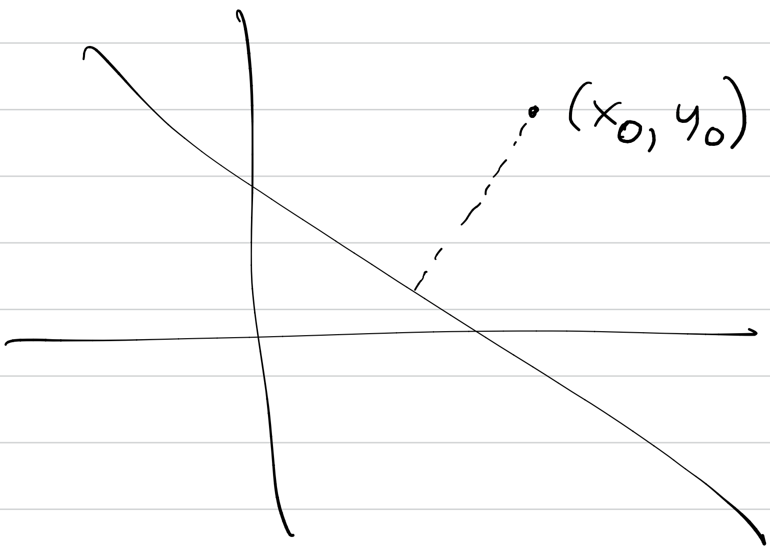
$$Q = \left(\frac{15}{7}, \frac{-5}{7}, \frac{-10}{7} \right) + \left(\frac{14}{7}, \frac{7}{7}, \frac{21}{7} \right)$$

$$= \left(\frac{29}{7}, \frac{2}{7}, \frac{11}{7} \right)$$

distance $PQ = |Q - P|$ ...

Formula for distance $\overset{D}{\vee}$ between a point (x_0, y_0)

& the line $ax + by + c = 0$



$$D = \left| \frac{ax_0 + by_0 + c}{\sqrt{a^2 + b^2}} \right|$$

Anton

22. Find distance between point $\overset{(x_0, y_0)}{(-1, 4)}$ & line

$x - 3y + 2 = 0.$

Solution: $D = \dfrac{1(-1) + -3 \cdot 4 + 2}{\sqrt{1^2 + (-3)^2}}$ $a = 1$, $b = -3$

$$= \left| \frac{-1 - 12 + 2}{\sqrt{10}} \right| = \left| \frac{-11}{\sqrt{10}} \right| = \boxed{\frac{11}{\sqrt{10}}}$$

Anton 16 : Find vector component

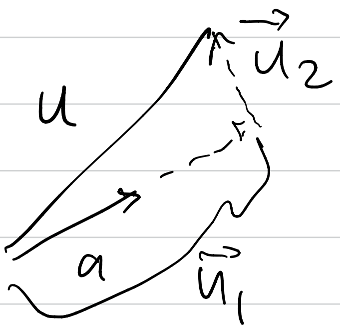
of u along a

u_1

$$u_1 = \left(\frac{\vec{u}\cdot\vec{a}}{\|a\|^2}\right)\vec{a}$$

• vector component of u orthogonal to a

$$\underbrace{\text{vector component of } u \text{ orthogonal to } a}_{u_2 = u - u_1}$$



where $\vec{u} = (-1,-2)$ $\vec{a} = (-2,3)$

Solution:

$$u_1 = \left(\frac{2-6}{(-2)^2+3^2}\right)\vec{a}$$

$$u_1 = \left(\frac{-4}{13}\right)\vec{a}$$

$$= \left(-\frac{4}{13}\right)(-2,3)$$

$$\boxed{u_1 = \left(\frac{8}{13}, -\frac{12}{13}\right)}$$

vector component of u along a

$$\vec{u_2} = u - u_1 = (-1,-2) - \left(\frac{8}{13}, \frac{-12}{13}\right)$$

$$= \left(\frac{-13}{13}, \frac{-26}{13}\right) - \left(\frac{8}{13}, -\frac{12}{13}\right)$$

$$= \left(-\frac{21}{13}, \frac{-26+12}{13}\right)$$

$$\boxed{u_2 = \left(-\frac{21}{13}, -\frac{14}{13}\right)}$$

Shortest distance $\overset{D}{\vee}$ from a point (x_0, y_0, z_0) to the plane $ax+by+cz+d=0$

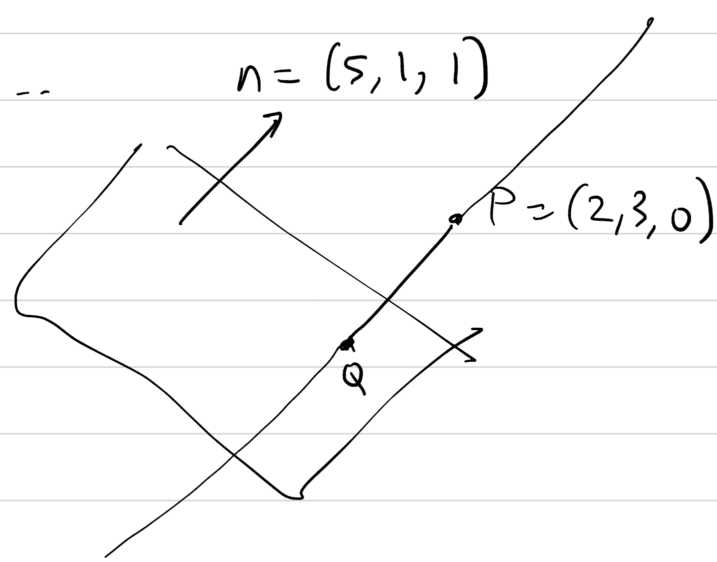
is $$D = \left|\frac{ax_0+by_0+cz_0+d}{\sqrt{a^2+b^2+c^2}}\right|$$

Exercise 4.2.16 In each case, find the shortest distance from the point P to the plane and find the point Q on the plane closest to P.

a. $P(2, 3, 0)$; plane with equation $5x+y+z=1$.

b. $P(3, 1, -1)$; plane with equation $2x+y-z=6$.

a) use formula to find distance ..



Line through P in direction $\vec{n}$

is

$\vec{x} = P + t\vec{n}$

Q is the intersection of this line & plane!

$$\vec{x} = P + tn$$

$$(x,y,z) = (2,3,0) + t(5,1,1)$$

$$= (2,3,0) + (5t,t,t)$$

$$(x,y,z) = (5t+2,\ 3+t,\ t)$$

are the points on the line

The point Q is on the plane $5x+y+z=1$

plug in $x=5t+2$, $y=3+t$, $z=t$ into

$$5(5t+2) + (3+t) + t = 1$$

$$25t + 10 + 3 + t + t = 1$$

$$27t + 13 = 1$$

$$27t = 1-13 = -12$$

$t = \dfrac{-12}{27}$ corresponds to the point Q.

$$\boxed{\begin{aligned} Q:\quad & x = 5t+2 = 5\left(-\tfrac{12}{27}\right)+2 \\ & y = 3+t = 3+\tfrac{-12}{27} \\ & z = t = \tfrac{-12}{27} \end{aligned}}$$

Nicholson 4.2.21f

Find <u>all</u> planes containing $P_0(2,-1,1)$ & $Q(1,0,0)$

Solution: $ax+by+cz+d=0$

plug in $(x,y,z)=(2,-1,1) \rightarrow$ $\boxed{\begin{array}{l} 2a-b+c+d=0 \\ a \qquad\qquad\ \ +d=0 \end{array}}$

plug in $(x,y,z)=(1,0,0) \rightarrow$

"This system we can solve by inspection", in general you might have to row reduce to find free variables, etc

$a+d=0 \Rightarrow d=-a$

$$2a-b+c+d=0 \Rightarrow c=-2a+b-d$$
$$=-2a+b-(-a)$$
$$=-2a+b+a$$
$$c=-a+b$$

So we've expressed c & d in terms of a & b ... a & b turn out to be free variables. (But it's also possible to express a & b in terms of free variables c & d : $a=-d$
$$b=2a+c+d$$
$$=2(-d)+c+d$$
$$b=c-d$$
)

So $ax+by+cz+d=0$ becomes

$\boxed{ax+by+(-a+b)z-a=0}$ which is equivalent to the textbook's answer.