Chapter 5 Eigenvalues & eigenvectors

$n \times n$ matrix A

An eigenvector of A is a $n \times 1$ (column) vector v, $v \neq 0$, such that

$$Av = \lambda v \quad \text{for some scalar } \lambda$$

eigenvector

eigenvalue

eigen = own/itself in german

Example: $A = \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix}$ $\quad v = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$$Av = \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3\cdot 1 + 0\cdot 2 \\ 8\cdot 1 - 2\cdot 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix} = 3\begin{bmatrix} 1 \\ 2 \end{bmatrix} = 3v$$

eigenvalue associated to eigenvector v.

$$Av = \lambda v$$

$$Av - \lambda v = 0$$

$$(A-\lambda I)^{-1}\underbrace{(A - \lambda I)}_{\text{square matrix}} v = 0 \; (A-\lambda I)^{-1} \qquad v \neq 0$$

v: nonzero vector

$$3v = 3\begin{bmatrix} 1 & \\ & 1 \end{bmatrix} v$$

If $A - \lambda I$ had an inverse, then multiplying both sides

$$v = (A - \lambda I)^{-1} 0 = 0 \quad \text{but } v \neq 0.$$

So $A - \lambda I$ does not have an inverse! So $\boxed{\det(A - \lambda I) = 0}$

Inverse of $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \frac{1}{\det} \cdot \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

Example: $A = \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix}$ Find eigenvectors & eigenvalues.

$$\det(\underbrace{A - \lambda I}) = 0$$

$$\det\left(\begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix} - \begin{bmatrix} \lambda & \\ & \lambda \end{bmatrix}\right) = 0$$

$$\det\begin{pmatrix} 3-\lambda & 0 \\ 8 & -1-\lambda \end{pmatrix} = 0$$

$$(3-\lambda)(-1-\lambda) - 8 \cdot 0 = 0$$

$$-3 - 3\lambda + \lambda + \lambda^2 = 0$$

characteristic equation

$$\lambda^2 - 2\lambda - 3 = 0$$

$$(\lambda - 3)(\lambda + 1) = 0$$

$\lambda - 3 = 0$ or $\lambda + 1 = 0$

$\lambda = 3$ $\qquad$ $\lambda = -1$

$\lambda = 3$ or -1 are the eigenvalues

Case 1

$\lambda = 3$ $\quad$ Goal: find eigenvector v

$$(A - 3I)v = 0$$

$$A - 3I = \begin{bmatrix} 3-3 & 0 \\ 8 & -1-3 \end{bmatrix} v = 0$$

$$\begin{bmatrix} 0 & 0 \\ 8 & -4 \end{bmatrix}\begin{bmatrix} \ \\ \ \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\downarrow$ RREF $\quad \uparrow$ v

RREF

$$\begin{bmatrix} 0 & 0 \\ 1 & -\frac{4}{8} \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -\frac{4}{8} \\ 0 & 0 \end{bmatrix} v = 0$$

$$v = \begin{bmatrix} x \\ y \end{bmatrix}$$

$$x - \frac{4}{8} y = 0$$

Case 2 $\quad \lambda = -1$

$$A = \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix}$$

Find eigenvector v.

$A - \lambda I$

$\downarrow$

$$\begin{bmatrix} 3 - -1 & 0 \\ 8 & -1 - -1 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 0 \\ 8 & 0 \end{bmatrix} v = 0$$

$$\begin{bmatrix} 1 & 0 \\ 8 & 0 \end{bmatrix} v = 0$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} v = 0$$

RREF

$$x = \frac{4}{8} y = \frac{1}{2} y$$

$$y = y$$

$$v = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{1}{2} y \\ y \end{bmatrix} = $$

$$= \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix} y$$

$y = 1 \quad v = \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}$

is an eigenvector of A.

$y = 2 \quad \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix} 2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$x_1 + 0x_2 = 0$$

$$x_1 = 0 \qquad \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} x_2$$

$$x_2 = x_2$$

$v = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is eigenvector with eigenvalue -1.

Given A, can we diagonalize it?

find an invertible matrix P such that $P^{-1}AP = D$ (diagonal)

$$D = \begin{bmatrix} * & 0 \\ 0 & * \end{bmatrix}$$

Answer: Let $v_1, v_2, \ldots v_n$ be eigenvectors of A with eigenvalues $\lambda_1, \lambda_2, \ldots \lambda_n$ (assume $v_1, v_2, \ldots v_n$ form a basis of $\mathbb{R}^n$)

$$P = \begin{bmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{bmatrix} \qquad D = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$$

P = put the eigenvectors together

$$P^{-1}AP = D$$

$$AP = PD$$

$$A\begin{bmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{bmatrix} = \begin{bmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{bmatrix}\begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$$

$$\begin{bmatrix} | & & | \\ Av_1 & & Av_n \\ | & & | \end{bmatrix} = \begin{bmatrix} | & & | \\ \lambda_1 v_1 & \cdots & \lambda_n v_n \\ | & & | \end{bmatrix}$$

So columns of P must be eigenvectors of A.

Example Diagonalize $A = \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix}$

Solution: $P = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$

eigenvectors of A were $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$D = \begin{bmatrix} 3 & \\ & -1 \end{bmatrix}$

or: $P = \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}$

$D = \begin{bmatrix} -1 & \\ & 3 \end{bmatrix}$

Eigenvalues with multiplicity

$A = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}$ Find eigenvectors & eigenvalues

$A - \lambda I = \begin{bmatrix} 3-\lambda & 1 \\ 0 & 3-\lambda \end{bmatrix}$

$\det \begin{bmatrix} 3-\lambda & 1 \\ 0 & 3-\lambda \end{bmatrix} = 0$

$(3-\lambda)^2 = 0$ So $\lambda = 3$ is eigenvalue with algebraic multiplicity 2.

Next find eigenvector(s). $\rightarrow$ $(A-3I)v = 0$

$A - 3I = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$

$x_1 = x_1$
$x_2 = 0$ so $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} x_1$

so $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is (only) eigenvector up to scalar

A can't be diagonalized because

$P = \begin{bmatrix} v_1 & v_2 \end{bmatrix} = \begin{bmatrix} 1 & X \\ 0 & \end{bmatrix}$

not enough eigenvectors

λ-eigenspace $= \{ v \mid (A-\lambda I)v = 0 \}$ is space of vectors w/ eigenvalue λ

Example: $A = \begin{bmatrix} 3 & \\ & 3 \end{bmatrix}$ is obviously diagonalizable. What are eigenvalues & eigenvectors?

$$A - \lambda I = \begin{bmatrix} 3-\lambda & 0 \\ 0 & 3-\lambda \end{bmatrix}$$

$$\det = 0$$

$$(3-\lambda)^2 = 0$$ $\lambda = 3$ only eigenvalue alg mult $= 2$.

Next find eigenvectors.

$$A - 3I = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

No pivots so all variables x_1 & x_2 are free

$$\begin{matrix} x_1 = x_1 \\ x_2 = x_2 \end{matrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} x_1 + \begin{bmatrix} 0 \\ 1 \end{bmatrix} x_2$$

so $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ & $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is a basis for 3-eigenspace.

$$P = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad D = \begin{pmatrix} 3 & \\ & 3 \end{pmatrix}$$

Fact: If the eigenvalues of A are distinct then the eigenvectors span $\mathbb{R}^n$ (i.e enough eigenvectors to make P so A is diagonalizable)

5.4 Dynamical Systems & Markov Chains

Example (Nicholson et al Open Linear Algebra with Applications, 2019A, p.141 #2.9.4)

Assume there are three social classes: upper, middle, & lower, and social mobility behaves as follows

1. Of the children of the upper class parents,
   70% remain upper class
   10% become middle class
   20% become lower class

2. Of the children of the middle class parents,
   80% become upper class
   10% remain middle class
   10% become lower class

3. Of the children of the lower class parents
   60% become upper class
   30% become middle class
   10% remain lower class

a. Find the probability that the grand-child of lower class parents becomes upper class

b. After a long time (e.g 20 generations), what will the break down of society be in terms of upper, middle, lower class (percentage-wise)?

Solution: There are three "states" in this problem: being in upper, middle, or lower class.

Let $t = 0, 1, 2, \ldots$ count the generations

Let $x_1(t)$ be the probability that a person in generation t is in the upper class

$x_2(t)$ 〃 middle class

$x_3(t)$ 〃 lower class

$$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} \quad \text{"state vector"}$$

$$x_1(t) + x_2(t) + x_3(t) = 1 \quad \text{since probabilities add to 1}$$

so in part a, we start with a person we know is in the lower class and so we start with

$$x(0) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

In part b, we don't know what $x_1(0)$, $x_2(0)$, $x_3(0)$ are, but it won't matter.

Key observation:

At generation $t+1$, the probability that a person is in the upper class is

$$x_1(t+1) = .7 \cdot x_1(t) + .8\, x_2(t) + .6\, x_3(t)$$

70% of upper class has upper class kids

80% of middle class parents ($x_2(t)$) kids who become upper class

60% of lower class ($x_3(t)$) has kids who become upper class

In other words, the probability of being in a certain state (upper class) at time $t+1$ is a <u>linear combination</u> of the probabilities (of being in various states) at time t.

$$\begin{bmatrix} x_1(t+1) \\ \\ \\ \end{bmatrix} = \begin{bmatrix} .7 & .8 & .6 \\ & & \\ & & \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$$

Completing the matrix from the information provided gives

$$\begin{bmatrix} x_1(t+1) \\ \\ \\ \end{bmatrix} = \begin{bmatrix} .7 & .8 & .6 \\ .1 & .1 & .3 \\ .2 & .1 & .1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$$

"Of the children of the upper class parents,"
70% remain upper class
10% become middle class
20% become lower class

Note each column of this matrix adds to 1, and all entries are non-negative.
(Rows do not have to add to one.)
"Stochastic matrix"

Let $P = \begin{bmatrix} .7 & .8 & .6 \\ .1 & .1 & .3 \\ .2 & .1 & .1 \end{bmatrix}$ & $X(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$ "state vector"

"transition matrix"

$$\boxed{x(t+1) = P \cdot x(t)}$$

"Markov chain" is any situation where state vector $x(t+1)$ at time $t+1$ is a stochastic matrix times a state vector at time t.

Note $x(2) = P \cdot x(1) = P(P \cdot x(0)) = P^2 \cdot x(0)$.

$x(3) = P \cdot x(2) = P(P^2 x(0)) = P^3 x(0)$.

In general, $x(k) = P^k x(0)$

a. Find the probability that the grand-child of lower class parents becomes upper class

t	parents
$t+1$	children
$t+2$	grand children

$X(t) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ ← lower class is x_3

parent's state vector

$x(t+1) = P\,x(0) = \begin{bmatrix} .7 & .8 & .6 \\ . & .1 & .3 \\ . & .1 & .1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} .6 \\ .3 \\ .1 \end{bmatrix}$

children's state vector

$x(t+2) = P \cdot x(1) = \begin{bmatrix} .7 & .8 & .6 \\ .1 & .1 & .3 \\ .2 & .1 & .1 \end{bmatrix} \begin{bmatrix} .6 \\ .3 \\ .1 \end{bmatrix} = \begin{bmatrix} .72 \\ .12 \\ .16 \end{bmatrix}$ ← $\underbrace{.7 \cdot .6}_{.42} + \underbrace{.8 \cdot .3}_{.24} + \underbrace{.6 \cdot .1}_{.06}$

grandchildrens state vector

Part a is asking for $x_1(t+2) = .72 = \boxed{72\%}$

↖ upper class state

After a long time (e.g 20 generations), what will the break down of society be?

In SageMath, computing powers of P, namely $P, P^2, P^3, \ldots$

shows these powers stabilize to $\begin{bmatrix} .696 & .696 & .696 \\ .134 & .134 & .134 \\ .170 & .170 & .170 \end{bmatrix}$

So multiplying $\begin{bmatrix} .696 & .696 & .696 \\ .134 & .134 & .134 \\ .170 & .170 & .170 \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{bmatrix}$

for any $x(0) = \begin{bmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{bmatrix}$ such that $x_1(0) + x_2(0) + x_3(0) = 1$

will give $\boxed{\begin{bmatrix} .696 \\ .134 \\ .170 \end{bmatrix}}$

How to find answer without Sage Math?

If $x(t) \to x$ for some $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ (& $x_1 + x_2 + x_3 = 1$)
as $t \to \infty$

then $x(t+1) = P \cdot x(t)$ becomes

$$x = P \cdot x$$

i.e x is an eigenvector for P with eigenvalue 1.

$$0 = Px - x$$
$$0 = (P - I)x$$

so x is in the nullspace of $P - I = \begin{bmatrix} -.3 & .8 & .6 \\ .1 & -.9 & .3 \\ .2 & .1 & -.9 \end{bmatrix}$.

To find nullspace, row reduce $P - I$

to get $\begin{bmatrix} 1 & 0 & -78/19 \\ 0 & 1 & -15/19 \\ 0 & 0 & 0 \end{bmatrix}$

Hence $x_1 - \frac{78}{19} x_3 = 0$ so $x_1 = \frac{78}{19} x_3$

$x_2 - \frac{15}{19} x_3 = 0$ $\quad x_2 = \frac{15}{19} x_3$

$x_3 = x_3$

(so nullspace is $\left\{ x_3 \cdot \begin{bmatrix} 78/19 \\ 15/19 \\ 1 \end{bmatrix} \,\middle|\, x_3 \in \mathbb{R} \right\}$)

Also need $x_1 + x_2 + x_3 = 1$

So $\left(\frac{78}{19} + \frac{15}{19} + 1\right) x_3 = 1$

$$\frac{112}{19} \cdot x_3 = 1$$

$$x_3 = \frac{19}{112} = 0.1694 \approx 0.170$$

$$x_1 = \frac{78}{19} x_3 = \frac{78}{112} = 0.694$$

$$x_2 = \frac{15}{19} x_3 = \frac{15}{112} = 0.134$$

So the eventual breakdown of society will be

69% upper class (x_1)

13% middle class (x_2)

17% lower class (x_3)

5.5 Stochastic Matrices

$x(t) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\qquad$ $x(t+1) = \begin{bmatrix} 0.95 \\ 0.05 \end{bmatrix}$

13. On a given day the air quality in a certain city is either good or bad. Records show that when the air quality is good on one day, then there is a 95% chance that it will be good the next day, when air quality is bad on one day, then there is a 45% chance that it will be bad the next day.
    a. Find a transition matrix for this phenomenon.
    b. If the air quality is good today, what is the probability that it will be good two days from now?
    c. If the air quality is bad today, what is the probability that it will be bad three days from now?
    d. If there is a 20% chance that the air quality will be good today, what is the probability that it will be good tomorrow?

$$x(t) = \begin{bmatrix} \text{prob. air quality good} \\ \text{prob. air quality bad} \end{bmatrix}$$

2×1

P

transition matrix

$$\underset{\substack{2\times 1 \\ \text{column vector}}}{x(t+1)} = \underset{2\times2}{P} \; \underset{2\times1}{x(t)}$$

$$\begin{bmatrix} 0.95 \\ 0.5 \end{bmatrix} = \underset{2\times 2}{\begin{bmatrix} \; & \; \\ \; & \; \end{bmatrix}} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

so first column of P is

$$P = \begin{bmatrix} 0.95 & ? \\ 0.05 & ? \end{bmatrix}$$

"When air quality is bad one day, there is 45% chance it's bad the next day"

if $\underset{\text{bad air day}}{x(t) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}}$ then $x(t+1) = \begin{bmatrix} .55 \\ .45 \end{bmatrix}$ $\quad 1 - 0.45$

$$\begin{bmatrix} 0.55 \\ 0.45 \end{bmatrix} = P \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

so $\boxed{P = \begin{bmatrix} 0.95 & 0.55 \\ 0.05 & 0.45 \end{bmatrix}}$ is transition matrix

b) If $\underbrace{\text{air quality is good today}}_{x(t) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}}$, what is probability that it is good two days from now

$$X(t+2) = P\,x(t+1)$$
$$= P \cdot P\,x(t)$$
$$= P^2 x(t)$$
$$= P^2 \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$
$$=$$

$$P^2 = \begin{bmatrix} 0.95 & 0.55 \\ 0.05 & 0.45 \end{bmatrix} \begin{bmatrix} 0.95 & 0.55 \\ 0.05 & 0.45 \end{bmatrix} = \begin{bmatrix} 0.93 & 0.77 \\ 0.07 & 0.23 \end{bmatrix}$$

```
P=matrix(RDF, 2, 2, [0.95, 0.55, 0.05, 0.45])

P

[0.95 0.55]
[0.05 0.45]

P^2

[0.9299999999999999                0.77]
[                0.07              0.23]
```

So $P^2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.93 \\ 0.07 \end{bmatrix}$

So good air quality probability is $\boxed{0.93}$

c) If air quality is bad today, what is probability it is bad 3 days from now?

$$x(t+3) = P^3 x(t) = P^3 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.922 & 0.858 \\ 0.078 & 0.142 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

bad air quality day

$$= \begin{bmatrix} 0.858 \\ 0.142 \end{bmatrix}$$

So answer is $\boxed{0.142}$

d) If 20% chance air quality is good today, what is probability that it is good tomorrow

$$x(t) = \begin{bmatrix} .2 \\ .8 \end{bmatrix}$$

$$x(t+1) = P x(t)$$

$$= \begin{bmatrix} 0.95 & 0.55 \\ 0.05 & 0.45 \end{bmatrix} \cdot \begin{bmatrix} .2 \\ .8 \end{bmatrix}$$

$$= \begin{bmatrix} .63 \\ .37 \end{bmatrix}$$

So answer is $\boxed{0.63}$