1 Systems of Equations and RREF (reduced row echelon form)

1. Solve the system of equations

$$\begin{aligned} 2x + 3y &= 7 \\ 3x + 4y &= 8 \end{aligned}$$

by creating the augmented matrix and finding the RREF of it, and so on. How many solutions (x, y) are there (a pair of numbers (x_0, y_0) counts as one solution)? Then plot the two lines (e.g. go to desmos.com) and confirm your answer,

2. Solve the system of equations

$$\begin{aligned} 2x + 3y &= 7 \\ 4x + 6y &= 8 \end{aligned}$$

by creating the augmented matrix and finding the RREF of it, and so on. How many solutions (x, y) are there (a pair of numbers (x_0, y_0) counts as one solution)? Then plot the two lines (e.g. go to desmos.com) and confirm your answer,

3. Solve the system of equations

$$\begin{aligned} 2x + 3y &= 7 \\ 4x + 6y &= 14 \end{aligned}$$

by creating the augmented matrix and finding the RREF of it, and so on. How many solutions (x, y) are there (a pair of numbers (x_0, y_0) counts as one solution)?Then plot the two lines (e.g. go to desmos.com) and confirm your answer,

4. Give 2 examples of matrices in reduced row echelon form (RREF), and 2 example of matrices that are in row echelon form but not reduced row echelon form (see the next few lines for a review).

   Recall: A matrix is in row echelon form if conditions 1-3 below hold; it is in reduced row echelon form if conditions 1-4 hold:

   1. all rows consisting of only zeroes are at the bottom.
   2. the leading coefficient (i.e. first non-zero entry, also called the pivot) of each non-zero row is 1 (also called pivot 1's)
   3. the leading coefficient of a nonzero row is always strictly to the right of the leading coefficient of the row above it. So all the entries in a column below a pivot 1 are zero.
   4. Each column containing a pivot 1 has zeros in all its other entries.

   The following is an example of a 3×5 matrix in row echelon form, which is not in reduced row echelon form (taken from Wikipedia):

$$\begin{bmatrix} 1 & a_0 & a_1 & a_2 & a_3 \\ 0 & 0 & 2 & a_4 & a_5 \\ 0 & 0 & 0 & 1 & a_6 \end{bmatrix}$$

If we take the above matrix do a few elementary row operations, we get the following matrix which is in *reduced* row echelon form.

$$\begin{bmatrix} 1 & a_0 & 0 & 0 & a_3 \\ 0 & 0 & 1 & 0 & a_5 \\ 0 & 0 & 0 & 1 & a_6 \end{bmatrix}$$

The following matrix is also in reduced row echelon form:

$$\begin{bmatrix} 0 & 1 & 2 & 0 & -1 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

5. Why do we care about putting matrices into RREF?

6. Make up a matrix and find the RREF by hand and by Python

7. Create a system of 2 equations in 3 variables, and solve it by finding the RREF of the augmented matrix. Does it have no solutions, exactly one solution, or infinitely many solutions?

8. Find all solutions to the following system of equations:

$$2x_1 + x_2 + x_3 = 2$$

$$x_1 - x_3 = 2$$

$$4x_1 + 3x_2 + 5x_3 = 2$$

using the following output of SageMath:

```
A=matrix(QQ, 3, 4, [2, 1, 1, 2, 1, 0, -1, 2, 4, 3, 5, 2])
A

    [ 2  1  1  2]
    [ 1  0 -1  2]
    [ 4  3  5  2]

A.rref()

    [ 1  0 -1  2]
    [ 0  1  3 -2]
    [ 0  0  0  0]
```

This SageMath code is saying that if

$$\mathbf{A} = \begin{bmatrix} 2 & 1 & 1 & 2 \\ 1 & 0 & -1 & 2 \\ 4 & 3 & 5 & 2 \end{bmatrix}$$

then the reduced row echelon form is

$$\mathbf{R} = \begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & 3 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Are there infintely many solutions, one solution, or no solutions?

9. I have a certain number of nickels, dimes, and quarters. I have a total of 10 coins and their total value is \$1.05. How many of each coin type could I have? There are two possible scenarios/answers, find them both. (Use linear algebra and RREF, not guess and check).

10. (a) Suppose we have a system $\mathbf{Ax} = \mathbf{b}$ of 3 linear equations in the variables $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = (x_1, x_2, x_3)$, and suppose the RREF of the associated augmented matrix $[\mathbf{A}|\mathbf{b}]$ is

$$\begin{bmatrix} 1 & 3 & 0 & 5 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Based on this information, what are all solutions for (x_1, x_2, x_3)? Are there infinitely many solutions, only one solution, or no solutions?

(b) Suppose we have a system $\mathbf{Ax} = \mathbf{b}$ of 3 linear equations in the variables $\mathbf{x} = (x_1, x_2, x_3)$, and suppose the RREF of the associated augmented matrix $[\mathbf{A}|\mathbf{b}]$ is

$$\begin{bmatrix} 1 & 3 & 0 & 5 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Based on this information, what are all solutions for (x_1, x_2, x_3)? Are there infinitely many solutions, only one solution, or no solutions?

(c) Suppose we have a system $\mathbf{Ax} = \mathbf{b}$ of 3 linear equations in the variables $\mathbf{x} = (x_1, x_2, x_3)$, and suppose the RREF of the associated augmented matrix $[\mathbf{A}|\mathbf{b}]$ is

$$\begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Based on this information, what are all solutions for (x_1, x_2, x_3)? Are there infinitely many solutions, only one solution, or no solutions?

11. (Spring 2022 Final Exam) Choose 3 points $p_i = (x_i, y_i)$ for $i = 1, 2, 3$ in $\mathbb{R}^2$ that are not on the same line (i.e. not *collinear*).

(a) Suppose we want to find numbers a, b, c such that the graph of $y = ax^2 + bx + c$ (a parabola) passes through your 3 points. This question can be translated to solving a matrix equation $\boldsymbol{X\beta} = \boldsymbol{y}$ where $\boldsymbol{\beta}$ and $\boldsymbol{y}$ are 3×1 column vectors, what are $\boldsymbol{X}$, $\boldsymbol{\beta}$, $\boldsymbol{y}$ in your example?

(b) We have learned two ways to solve the previous part (hint: one way starts with R, the other with I). Show both ways. Don't do the arithmetic calculations involved by hand, but instead show to use Python to do the calculations, and confirm they give the same answer. Plot your points and the parabola you found (using e.g. Desmos/Geogebra).

(c) Show how to use linear algebra to find *all* degree 4 polynomials $y = \beta_4 x^4 + \beta_3 x^3 + \beta_2 x^2 + \beta_1 x + \beta_0$ that pass through your three points (there will be infinitely many such polynomials, and use parameters to describe all possibiities). Illustrate in Desmos/Geogebra using sliders.

(d) Pick a 4th point $p_4 = (x_4, y_4)$ that is not on the parabola in part 1 (the one through your three points p_1, p_2, p_3). Try to solve $\boldsymbol{X\beta} = \boldsymbol{y}$ where $\boldsymbol{\beta}$ and $\boldsymbol{y}$ are 3×1 column vectors via the RREF process. What happens?

12. **Problem 1**. In class we associated to any matrix M a linear transformation T_M, and this question asks you to explain how: Specifically, make up a $m \times n$ matrix M (having at least a few entries that are not 0 or 1), with $m, n \geq 2$ and $m \neq n$.

    1. identify/state what the domain and codomain/target of T_M are for your choice of M.
    2. Let e be one of the standard basis vectors in the domain of T_M. What is the value of $T_M(e)$?
    3. make up a vector v in the domain of T_M that is not a standard basis vector and calculate $T_M(v)$. In doing so, you are taking linear combinations of what?
    4. make up a vector w in the codomain/target of T_M (and such that w is not a column of M as otherwise the question is too simple) and show how to determine if w is in the image of T_M, i.e. determine if there is a vector x such that $T_M(x) = w$.

13. **Problem 2.**. Not all transformations are linear transformations.

    1. What two properties/equations must a linear transformation satisfy?
    2. Pick two non-zero numbers $a, b \in \mathbb{R}$ and let $T_{a,b} : \mathbb{R}^2 \to \mathbb{R}^2$ be translation by (a, b), i.e. $T_{a,b}(x, y) = (x + a, y + b)$ for $(x, y) \in \mathbb{R}^2$. Show that $T_{a,b}$ is not a linear transformation. (You can do this for example by computing $T_{a,b}$ on various vectors in $\mathbb{R}^2$ and showing one of the required properties isn't satisfied)
    3. Make up a 2x2 matrix M and numbers $a, b \in \mathbb{R}$. What does the composite $T_{a,b} \circ T_M$ do to a point $(x, y) \in \mathbb{R}^2$? Same question for $T_M \circ T_{a,b}$ (the order has been reversed).

14.
    1. Give an example of two vectors that together do *not* span all of $\mathbb{R}^2$.
    2. Given any three vectors in $\mathbb{R}^3$, explain how to show they are linearly dependent. (You should pick three vectors in $\mathbb{R}^2$, but
    3. Pick two vectors v_1, v_2 in $\mathbb{R}^2$ that span all of $\mathbb{R}^2$ (besides the standard basis vectors e_1 and e_2).
       (a) Explain by giving an example a linear combination of v_1 and v_2 could be.
       (b) Complete the following sentence: The fact that v_1 and v_2 span all of $\mathbb{R}^2$, means (by definition of span) that for every vector $b \in \mathbb{R}^2$, ...?
       (c) Suppose $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation such that $T(v_1) = e_1$ and $T(v_2) = e_2$. Every linear transformation comes from a matrix, and so find the matrix associated to T.

Problem 4

1. Explain how to multiply two matrices, by hand and in Python. (For the by hand part, you don't have to compute every single entry of the product, one or two is enough to convince me you know how to multiply matrices).
2. Illustrate by example in Python that matrix multiplication does not need to be commutative, but is associative (at least for the matrices you multiplied).
3. Let M_1, M_2 be two matrices such that the product $M_1 M_2$ is defined. Explain what the composite $T_{M_1} \circ T_{M_2}$ is - for example calculate $(T_{M_1} \circ T_{M_2})(e_1)$.

Problem 5 This question is simply about understanding the definition of matrix inverse. Let M be an $n \times n$ square matrix.

1. Suppose you don't know how to find the inverse of a matrix (in fact we haven't discussed that yet). But suppose someone gives you an $n \times n$ matrix N and claims that this matrix N is the inverse of M. What could you do to check their claim?
2. Suppose you want to find the inverse of the following matrix

$$M = \begin{bmatrix} -7 & 16 & 0 \\ -4 & 9 & 0 \\ 3 & -8 & 1 \end{bmatrix}$$

   and someone tells you the answer is

$$M^{-1} = \begin{bmatrix} 9 & -16 & 0 \\ 4 & 7 & 0 \\ x & -8 & 1 \end{bmatrix}$$

   but won't tell you the value of the (3,1) entry x. Explain how you can figure out the value of x by using only the definition of matrix inverse (and some basic algebra).
3. Let A and B be invertible $n \times n$ matrices. You wonder what the inverse of AB is, in terms of inverses of A and B, but have no idea. Someone tells you that the inverse of AB is $B^{-1}A^{-1}$. Explain why this fact - inverse of AB is $B^{-1}A^{-1}$ - is now obviously true. And in Python, show an example using 2x2 matrices that $A^{-1}B^{-1}$ need not be the inverse of AB (picking random values of A and B will usually work).

15. **Problem 6** Make up a linear system of two equations in two variables, whose coefficient matrix A is invertible but not the identity. matrix. We've learned how to solve it in two ways - one via the augmented matrix $[A \mid b]$ and another via expressing it as $Ax = b$ and using inverse of A. Show how to use both ways to your system of equations.

16.

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 0 \\ -1 & 2 & 1 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 2 & 3 \\ 1 & 0 \\ -1 & 2 \end{bmatrix}$$

Calculate the following by hand and then verify your answers in Python

(a) Calculate $\mathbf{AB}$ and $2\mathbf{A}$ and A^T by hand and by Python (in Python you have to use @ to indicate matrix multiplication, not $\mathbf{A} * \mathbf{B}$ not $\mathbf{AB}$. Also, the transpose of $\mathbf{A}$ is given by `A.T`

(b) Without multiplying it out, how can you quickly tell **BA** has no chance to equal **AB** (what are the sizes of the two products?)

(c) Why is $\mathbf{A} + \mathbf{B}$ not defined?

17. (a) Multiply out $\begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}$

(b) Add $\begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}$

(c) What is $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^n$ when multiplied out? (here we are raised the matrix to the n-th power, i.e. multiplying itself n times).

(d) Multiply out $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ (do you see how this relates to the formula for inverse of 2 x 2 matrix)

(e) Simplify $\frac{1}{2} \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}$

18. Create two random 3×3 integer matrices A and B in Python.

(a) Compute AB and BA in Python (recall you need to use @ for multiplication, not `A * B`). Are they equal?

(b) If instead B is a 3×4 matrix (3 rows and 4 columns), which of AB or BA is not even defined? What error does Python give if you ask it to compute that product?

19. (Fall 2021 Exam 1)

Make up matrices **A** and **B** such that **A** is a 2 x 4 matrix and **B** is a 4 x 1 matrix, and none of the entries of either matrix is a zero.

$$\mathbf{A} =$$

$$\mathbf{B} =$$

Compute by hand **AB** and **BA** (if defined). If one of the products (**AB** or **BA**) is undefined, explain why.

20. (a) (2 points) Create a system of two linear equations in three variables x, y, z. Solve it via finding the RREF, etc.

(b) (5 points) If the system of equations you made up in part (a) is written as a single matrix equation $\mathbf{Ax} = \mathbf{b}$, what are the entries of each of the matrices $\mathbf{A}, \mathbf{x}, \mathbf{b}$?

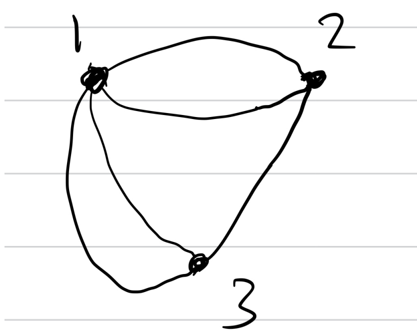
$\mathbf{A} =$ $\qquad\qquad$ $\mathbf{x} =$ $\qquad\qquad$ $\mathbf{b} =$

21. For the graph below

(a) Find the adjacency matrix A (by definition, this is the matrix whose (i, j) entry is the number of edges from i to j)

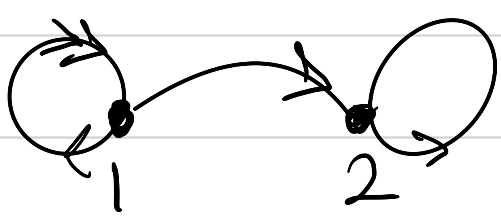
(b) find the number of paths from location 1 to location 2 that use exactly 4 steps/edges. (Use Python)

(c) Find the number of paths from location 2 to location 3 that take 3 steps or less. (Use



Python)

22. For the directed graph (meaning edges have direction, e.g. are one way streets) below,
    (a) Find the adjacency matrix A (by definition, this is the matrix whose (i, j) entry is the number of edges from i to j)
    (b) find the number of paths from location 1 to location 2 that take exactly n steps - do this first by counting all the possible paths for small values of n (e.g. $n = 1, 2, 3..$) to see the pattern, and then verify this by matrix multiplication.



23. (Fall 2021 Exam 1) Make up a graph having 3 vertices (labelled with the integers $1, 2, 3$) and 7 edges. Use Python to compute a matrix whose (i, j)-entry is the number of walks that require exactly 5 steps from vertex i to vertex j. (A walk is just a path but vertices and edges are allowed to be repeated). Find the number of walks from vertex 1 to vertex 3 that require exactly 5 steps. (Submit a picture of your graph as well as a screenshot of the Python calculation)

24. Make up a directed graph with some number of vertices (i.e. the dots/locations) and some number of edges and find the number of paths of length 4 from vertex 1 back to itself.

25. **A** is some 3x3 matrix (I won't tell you what it is) and let $\mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $\mathbf{y} = \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}$. Let $[\mathbf{x}\ \mathbf{y}]$

denote the 3 x 2 matrix whose columns are $\mathbf{x}$ and $\mathbf{y}$, i.e. $\begin{bmatrix}\mathbf{x}\ \mathbf{y}\end{bmatrix} = \begin{bmatrix}1 & 0\\ 2 & -1\\ 3 & 2\end{bmatrix}$. If $\mathbf{A}\mathbf{x} = \begin{bmatrix}-2\\ 20\\ 6\end{bmatrix}$ and $\mathbf{A}\mathbf{y} = \begin{bmatrix}-3\\ 6\\ 1\end{bmatrix}$, what is the value of the matrix multiplication $\mathbf{A}\begin{bmatrix}\mathbf{x}\ \mathbf{y}\end{bmatrix}$? (This doesn't require any computation to solve, just requires thinking about how matrix multiplication works. Your answer should be a matrix - what size?- of numbers).

2 Inverses of matrices

26. Let $\mathbf{A}$ be the following matrix:

$$\mathbf{A} = \begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix}$$

(a) (5 points) Use the formula for inverse of a 2x2 matrix to find $\mathbf{A}^{-1}$.

(b) Use elementary row operations on the matrix $[\mathbf{A} \mid \mathbf{I}]$ (where $\mathbf{I}$ is the 2×2 identity matrix) to find $\mathbf{A}^{-1}$. You can do this either 'by hand' or by asking Python to rref $[A \mid I]$ - but not just by asking Python `A.inverse()`!

(c) Let $\mathbf{x} = \begin{bmatrix}x_1\\ x_2\end{bmatrix}$ The equation $\mathbf{A}\mathbf{x} = \begin{bmatrix}2\\ 3\end{bmatrix}$ represents what system of two equations in two variables?

(d) Solve $\mathbf{A}\mathbf{x} = \begin{bmatrix}2\\ 3\end{bmatrix}$ for x_1, x_2.

(e) Find $\det(\mathbf{A})$ using any method.

(f) Let $\mathbf{x} = \begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}$. Recall $\mathbf{e}_1 = \begin{bmatrix}1\\ 0\\ 0\end{bmatrix}$. Use Cramer's rule to solve the system of equations $\mathbf{A}\mathbf{x} = \mathbf{e}_1$ for $\mathbf{x}$.

27. I have a 3×3 matrix $\mathbf{A}$ that I am not going to tell you what $\mathbf{A}$ is, but I will tell you that

$$\mathbf{A}^{-1} = \begin{bmatrix} 2 & 0 & 2 \\ 1 & 2 & 0 \\ 4 & 0 & 3 \end{bmatrix}$$

Let $\mathbf{x} = \begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}$. Solve the system of equations $\mathbf{A}\mathbf{x} = \begin{bmatrix}1\\ 2\\ 3\end{bmatrix}$ for x_1, x_2, x_3. (Your answers should be specific numbers that x_1, x_2, x_3 must equal.)

28. $\mathbf{A}$ is some 3x3 matrix (I won't tell you what it is) and let $\mathbf{x} = \begin{bmatrix}1\\ 2\\ 3\end{bmatrix}$ and $\mathbf{y} = \begin{bmatrix}0\\ -1\\ 2\end{bmatrix}$ and $\mathbf{z} = \begin{bmatrix}0\\ 0\\ 1\end{bmatrix}$.

If $\mathbf{Ax} = \begin{bmatrix} -2 \\ 20 \\ 6 \end{bmatrix}$ and $\mathbf{Ay} = \begin{bmatrix} -3 \\ 6 \\ 1 \end{bmatrix}$, and $\mathbf{Az} = \begin{bmatrix} -1 \\ 5 \\ 1 \end{bmatrix}$

(a) Find the product $\mathbf{A}\begin{bmatrix}\mathbf{x}\ \mathbf{y}\ \mathbf{z}\end{bmatrix}$ (this doesn't require much work, just knowing how matrix multipllicaiton works) where $\begin{bmatrix}\mathbf{x}\ \mathbf{y}\ \mathbf{z}\end{bmatrix}$ is the 3x3 matrix whose columns are $\mathbf{x}, \mathbf{y}, \mathbf{z}$.

(b) what exactly is $\mathbf{A}$? (use the previous part to start)

29. For which values of k is the matrix below not invertible (i.e. does not have an inverse)?

$$\begin{bmatrix} k-1 & 1 \\ 2 & k \end{bmatrix}$$

30. (a) Find the inverse and the determinant of the following matrix (your answer will involve the variable k)

$$\begin{bmatrix} 2 & 4 \\ 3 & k \end{bmatrix}$$

(b) The system of equations

$$\begin{aligned} 2x + 4y &= 1 \\ 3x + ky &= 2 \end{aligned}$$

has a unique solution (x, y) for all real numbers k except one - which one?

(c) Let $k = 4$. Use the inverse of the coefficient matrix to find x, y (in the above system of equations).

(d) Let $k = 4$. Find the reduced row echelon form (RREF) of the augmented matrix by hand to find x, y (in the above system of equations).

(e) When $k = 6$ determine if the system has infinitely many solutions or no solutions.

(f) What can I change the '1' in $2x + 4y = 1$ to to make the answer in previous part switch (from infinitely many to no solutions, or vice-versa)?

31. Consider the system of equations

$$\begin{aligned} 2x + 6y &= b_1 \\ x + 3y &= b_2 \end{aligned}$$

(a) What is the determinant of the coefficient matrix? What does that tell you about the number of solutions?

(b) For what condition on b_1, b_2 will this system be consistent? (Show using row reduction how you got your answer).

3 Elementary matrices

32. What are elementary matrices? Give some examples. If E is an elementary $n \times n$ matrix, and A is an $n \times m$ matrix, describe in words how to form EA from A (without having to calculate each entry).

33. Let $E = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$.

    (a) What elementary row operation did I do to the identity matrix $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ to get E?

    (b) Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Multiply out EA. What elementary row operation applied to A gives EA?

34. Let $E = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$.

    (a) What elementary row operation did I do to the identity matrix $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ to get E?

    (b) Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Multiply out EA. What elementary row operation applied to A gives EA?

35. Let $E = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

    (a) What elementary row operation did I do to the identity matrix $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ to get E?

    (b) Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Multiply out EA. What elementary row operation applied to A gives EA?

36. (a) Suppose we start with a matrix A and replace its first row by adding to it 2 times the second row ($R1 \to R1 + 2R2$) to get a matrix B. What elementary row operation should we do to B to get back A?

    (b) Let $E = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ and let $E' = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$. Compute EE' and $E'E$.

    (c) What is the inverse of E? (Do you see how it relates to part a?)

4 Linear Transformations and matrices

37. Let $\mathbf{A} = \begin{bmatrix} -1 & 0 \\ 3 & 5 \\ 2 & 4 \end{bmatrix}$. Let $T_{\mathbf{A}} : \mathbb{R}^2 \to \mathbb{R}^3$ be the linear transformation $T_{\mathbf{A}}(\mathbf{x}) = \mathbf{A}\mathbf{x}$ given by left multiplication by $\mathbf{A}$, where $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$. To save space, we also write (x_1, x_2) to mean the column matrix $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$. Find

(a) $T_{\mathbf{A}}(1,0)$

(b) $T_{\mathbf{A}}(0,1)$

(c) $T_{\mathbf{A}}(1,1)$

(d) $T_{\mathbf{A}}(x_1,x_2)$

38. Let T be the transformation defined by the formula

$$T\left(x_1,x_2\right) = \left(-x_1+2x_2, x_1, 3x_1+x_2\right)$$

or using column vectors we write T as

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} -x_1+2x_2 \\ x_1 \\ 3x_1+x_2 \end{bmatrix}$$

(a) Fill in below the correct numbers in superscripts on the $\mathbb{R}$, i.e in the blanks (i.e. I'm asking for the domain and codomain/target of T):

$$T : \mathbb{R}^{-} \longrightarrow \mathbb{R}^{-}$$

(b) Find $T(\mathbf{e}_1)$ and $T(\mathbf{e}_2)$, where $\mathbf{e}_1, \mathbf{e}_2$ are the standard basis vectors in $\mathbb{R}^2$.

(c) Find the matrix associated to the transformation T.

39. A chemist plans to build toy models of various molecules with a bunch of (toy) atoms of various type. Recall from chemistry/biology class or google, that a water molecule is H_2O (so contains two hydrogen atoms and one oxygen atom), sugar (glucose) is $C_6H_{12}O_6$, and caffeine molecule is $C_8H_{10}N_4O_2$.

Form the triple of numbers (c,s,w) where
c denote the number of (toy) caffeine molecules,
s the number of sugar molecules,
w be the number of (toy) water molecules, .
Similarly let a quadruple of numbers (C,H,N,O) denote the number of carbon, hydrogen, nitrogen, and oxygen atoms appearing in total in a combination of c caffiene, s sugar, and w water molecules.

(a) C,H,N,O can be expressed in terms of c,s,w using matrix multiplication as below; what are the entries in the blank matrix:

$$\begin{bmatrix} & \cdots & \\ & & \\ & & \\ & & \end{bmatrix} \begin{bmatrix} c \\ s \\ w \end{bmatrix} = \begin{bmatrix} C \\ H \\ N \\ O \end{bmatrix}$$

(b) Suppose you have some number of caffeine, sugar, and water molecules. Make up some nonzero numbers for each type of molecule and find the total number of each type of atom. For example suppose $(c,s,w) = (2,3,1)$.

(c) If you have some number of caffeine, sugar, and water molecules so that $(C,H,N,O) = (54,94,12,38)$, how many of each molecule do you have?

40. A climate scientist is studying carbon dioxide$\mathbf{CO_2}$ and methane $\mathbf{CH_4}$.
Form a pair of numbers (c, m) where

- c is the number of carbon dioxide molecules
- m be the number of methane molecules,

Similarly let a triple of numbers (C, H, O) denote the number of carbon, hydrogen, oxygen atoms appearing in total in a combination of c carbon dioxide and m methane molecules.

(a) (10 points) C, H, O can be expressed in terms of c, m using matrix multiplication as below; what are the entries in the blank matrix:

$$\begin{bmatrix} & \\ & \\ & \end{bmatrix} \begin{bmatrix} c \\ m \end{bmatrix} = \begin{bmatrix} C \\ H \\ O \end{bmatrix}$$

(b) (5 points) Suppose you have some number of carbon dioxide and methane molecules. Make up some nonzero numbers (c, m) for each type of molecule and find the total number of each type of atom, i.e. find (C, H, O).

41. Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be a linear transformation. That implies T is T_A (left multiplication by A) for some matrix A (i.e. $T(\mathbf{x}) = T_A(\mathbf{x}) = A\mathbf{x}$)

(a) What are the dimensions of A? (Dimensions means number of rows and columns)

(b) Write down the standard basis vector $\mathbf{e}_2$ for $\mathbb{R}^2$ and for $\mathbb{R}^3$ (so there are two different answers)

(c) If $T(\mathbf{e}_1) = (1, 2)$, $T(\mathbf{e}_2) = (0, 0)$ and $T(\mathbf{e}_3) = (1, 0)$, find the matrix A.

(d) Find $T(1, 2, 0)$.

42. Recall that the rotation matrix R_θ associated to counterclock-wise rotation by an angle of θ is given by

$$R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

In particular, the rotation matrix for $\theta = 30^\circ = \pi/6$ radians is given by

$$R_{30^\circ} = \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$$

1. When the point at $(2, 4)$ is rotated counterclockwise about the origin by 30°, what are the coordinates of the new location?
2. When the line $y = 2x$ is rotated counterclockwise about the origin by 30°, what is the equation of the resulting line?

43. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation that sends the points $(0, 0), (1, 0), (0, 1)$ to $(0, 0), (1, 2), (-1, 3)$ respectively (meaning $T(1, 0) = (1, 2)$, etc).

(a) What is $T(1, 1)$?

(b) What shape does T turn the square with vertices $(0, 0), (1, 0), (0, 1), (1, 1)$ into?

(c) What is the matrix of T?

44. Recall that the rotation matrix R_θ associated to counterclock-wise rotation by an angle of θ (about $(0,0)$ is given by

$$R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

A rotation by some angle θ_1 followed by a rotation by some angle θ_2 is obviously the same as rotation by an angle of $\theta_1 + \theta_2$. Since composition of linear transformation corresponds to multiplication of their matrices, we have

$$R_{\theta_1+\theta_2} = R_{\theta_2} R_{\theta_1}$$

The left hand side of this equation is

$$\begin{bmatrix} \cos(\theta_1+\theta_2) & -\sin(\theta_1+\theta_2) \\ \sin(\theta_1+\theta_2) & \cos(\theta_1+\theta_2) \end{bmatrix}$$

Multiply out the right hand side $R_{\theta_2} R_{\theta_1}$ to get the trigonometric identities for $\cos(\theta_1+\theta_2)$ and $\sin(\theta_1+\theta_2)$.

45. Make up a random 2 x 3 matrix A and a 3 x 2 matrix B. Find the matrix associated to the linear transformation $T_A \circ T_B$, (which is the linear transformation T_B followed by T_A).

46. Find the determinant of the given 4 x 4 matrices by hand

(a)

$$\begin{bmatrix} 1 & 3 & 0 & 4 \\ 0 & 2 & 4 & 5 \\ 0 & 0 & 3 & 2 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

(hint: the above matrix is upper triangular)

(b)

$$\begin{bmatrix} 1 & 3 & 0 & 4 \\ 2 & 2 & 4 & 5 \\ 0 & 0 & 3 & 2 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

(c)

$$\begin{bmatrix} 1 & 3 & 0 & 4 \\ 2 & 2 & 4 & 5 \\ 3 & 5 & 3 & 2 \\ 2 & 2 & 4 & 5 \end{bmatrix}$$

47. Create a random 3 x 3 matrix in Python, for example

```
import numpy as np
A=np.random.randint(low=0, high=5, size=(3,3))
```

and compute its determinant by hand and verify the answer using Python `A.det()`

48. Create two random 3×3 integer matrices A and B in Python using the command `random_matrix(QQ,3,3)` twice. Check whether any of the following identities hold or not (use `A.det()` to find the determinant of the matrix A):

    $\det(AB) = \det(A)\det(B)$ (this equation is always true)
    $\det(A^{-1}) = 1/\det(A)$ (this equation is always true when A is invertible so A^{-1} is defined)
    $\det(A + B) = \det(A) + \det(B)$ (this equation is not always true)

49. (a) Find k such that the following matrix does not have an inverse: $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 7 \\ k & 2 & 5 \end{bmatrix}$

    (b) For the matrix A in the previous part, and if $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, for what values of k does $A\mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ have a unique solution?

50. Given that the determinant $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 3$, find the value of the following determinant: $\begin{vmatrix} 3d & 3e & 3f \\ a & b & c \\ g-2a & h-2b & i-2c \end{vmatrix}$

51. Create a random 3 x 3 matrix in Python and compute its determinant and adjoint and verify using Python $(\text{adj}A)A = \det(A)I_n$.

52. . The adjoint of A, denoted below by adj A, satisfies the property $(\text{adj}A)A = \det(A)I_n$.

53. (Spring 2022 Final exam) This question is about the inverse of a matrix square matrix A.

    (a) What is the definition of the inverse of an $n \times n$ matrix A?

    (b) What is the notation we use for the inverse of an $n \times n$ matrix A?

    (c) What is a number we can compute that tells us if a square matrix A has an inverse?

    (d) Give an example of a 2×2 matrix that does *not* have an inverse.

    (e) To find the inverse of a 2×2 matrix, it is fastest to use the formula. What is the formula?

    (f) Give an example of a 2×2 matrix A whose entries are all non-zero and that has inverse A^{-1} and find it.

    (g) for $n \times n$ matrices with $n \geq 3$, there is a formula for the inverse that extends the one in the $2x2$ case, namely
    $$A^{-1} = \frac{1}{\det A}(adjA)$$
    involving something called the adjoint matrix $adj(A)$, but calculating that is very computationally involved as it involves calculating n^2 determinants of minors of A. Instead, an efficient process to find the inverse of A is to

    1. form the augmented $[A \mid I_n]$ (where I_n is the $n \times n$ identity matrix) and

2. find the RREF of $[A \mid I_n]$
3. if the RREF we get is of the form $[I_n \mid B]$, then A is invertible and its inverse will be B.

Then B will be the inverse of A. Explain why this RREF process gives the inverse of A (you do not have to say anything about adjoint matrix $adj(A)$, which is not involved).

(h) For your 2×2 matrix above, find the inverse by RREF-ing $[A \mid I_2]$ and confirm you get the same answer as in earlier part.

54. If A is a square matrix with integer entries, what must be true about A for A^{-1} to also has integer entries? (Hint: use the fact that determinant is multiplicative, i.e. $\det(AB) = \det(A)\det(B)$)

55. (Summer 2021 Exam 2) What is the relationship between the determinant of an n x n matrix and the determinant of its inverse? Illustrate this relationship/equation in Python for $4x4$ matrices. (Look up how to create a random $4x4$ matrix. If the random matrix Python creates happens to have determinant 0, then run the line again until you get a matrix with non-zero determinant). Submit a screenshot of your Python code.

56. Consider the system of equations in the variables x, y; and treat b as a unspecified constant.

$$\begin{aligned} 2x + 4y &= 1 \\ x + 2y &= b \end{aligned}$$

(a) (5 points) Give a value of b that will give infinitely many solutions (x, y).

(b) (5 points) Is there a value of b will give exactly one solution (x, y)?

(c) (5 points) Give a value of b will results in no solutions (x, y)?

57. For all the parts of this question, let P_1 be the plane in $\mathbb{R}^3$ given by the equation

$$x + 2y + 2z + 2 = 0$$

(a) Give an equation of a plane in $\mathbb{R}^3$ that is parallel to P_1. (You only have to give the equation of one such plane, you do not have to give equations of all possible planes).

(b) Give an equation of a plane P_2 in $\mathbb{R}^3$ that is *not* parallel to P_1. (Again, you only have to give the equation of one such plane, you do not have to give equations of all possible planes). Let's refer to this plane as P_2 (for the next part).

(c) Planes P_1 and P_2 intersect in a line. Find a parametric equation for this line.

(d) Find the angle between planes P_1 and P_2. (By definition this is the angle between their normal vectors, and giving either one of the two possible angles is acceptable).

58. Consider the system of equations

$$2x_1 + px_2 = 1$$

$$qx_1 + 6x_2 = 2$$

where p, q are constants.

(a) Assuming the system has a unique solution for x_1 and x_2, use Cramer's rule to solve for x_1, x_2, expressing your answers in terms of p, q.

(b) Under what conditions on p, q will there not be a unique solution (so there is either no solution or infinitely many solutions)?

(c) Under what conditions on p, q will there be infinitely many solutions?

(d) Under what conditions on p, q will there be no solutions?

59. Suppose a, b, c are numbers such that if we let $A = \begin{bmatrix} a & 2 & 3 \\ b & 3 & 7 \\ c & 2 & 5 \end{bmatrix}$ then $\det A = 5$. Use Cramer's Rule to find the value of x in the solution to the following system of equations:

$$\begin{aligned} ax + 2y + 3z &= 2 \\ bx + 3y + 7z &= 1 \\ cx + 2y + 5z &= 0 \end{aligned}$$

60. (Summer 2021 Exam 2) The general linear system of two equations in two variables x, y is of the form

$$\begin{aligned} ax + by &= c \\ dx + ey &= f \end{aligned}$$

where a, b, c, d, e, f are treated as unspecified but fixed numbers. Cramer's rule says that the solutions for x and y are given by what expressions or formulas involving a, b, c, d, e, f?

$$x = \quad , \qquad y =$$

61. Consider the two planes in $\mathbb{R}^3$ given by

$$2x + 4y + 6z + 6 = 0$$

$$x + 5y + 8z = 1$$

1. Give normal vectors for the planes.
2. Determine whether the two planes are perpendicular
3. Consider the numbers 5 and 8 in equation of the second plane $x + 5y + 8z = 1$. What numbers should they be replaced with to make it parallel to the first plane? (i.e. find a, b such that the plane $x + ay + bx = 1$ is parallel to $2x + 4y + 6z + 6 = 0$).

62. Make up two planes in $\mathbb{R}^3$ that are perpendicular.

63. Consider the two planes in $\mathbb{R}^3$ given by

$$3x + 4y + 26z = 25$$

$$2x + 4y + 20z = 22$$

The two planes intersect in a line. Find a point on this line and a direction vector for this line.

64. The two planes in $\mathbb{R}^3$

$$\begin{aligned} x + 2y - 3z &= 1 \\ 2x + 7y + 2z &= 0 \end{aligned}$$

intersect in a line. Find a parametric equation for the line.

65. Give an example of two planes in $\mathbb{R}^3$ that do not intersect.

66. (10 points) Find the angle between the two planes

$$\begin{aligned} x + 2y - 3z &= 1 \\ 3x + 7y + 2z &= 0 \end{aligned}$$

(By definition this is the angle between their normal vectors, and giving either one of the two possible angles is acceptable).

67. (Spring 2022 Final Exam) Pick a non-zero vector $\mathbf{v} \in \mathbb{R}^3$.
    (a) What are all vectors $\mathbf{w} \in \mathbb{R}^3$ that are orthogonal to $\mathbf{v}$? For the remaining parts of this problem, pick one such vector $\mathbf{w}$ orthogonal to $\mathbf{v}$.
    (b) What is the/a parametric equation of the plane P spanned by $\mathbf{v}, \mathbf{w}$?
    (c) Find the/an equation of the plane P spanned by $\mathbf{v}, \mathbf{w}$ of the form $ax + by + cz + d = 0$.
    (d) The *cross product* of $\mathbf{v} = (v_1, v_2, v_3)$ and $\mathbf{w} = (w_1, w_2, w_3)$ is a vector $\mathbf{v} \times \mathbf{w} \in \mathbb{R}^3$ whose x, y, z (also known as i, j, k) components can be remembered by the following notational trick:

$$\mathbf{v} \times \mathbf{w} = \det \begin{bmatrix} i & j & k \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{bmatrix}$$

    and the cross product $\mathbf{w} \times \mathbf{w}$ will always be a vector orthogonal to $\mathbf{v}$ and $\mathbf{w}$. Compute the cross product $\mathbf{v} \times \mathbf{w}$, and compare it with the normal (a, b, c) vector to the plane you found in the previous part...what should you expect?
    (e) Consider the set of 3 vectors $\mathbf{v}, 3\mathbf{v}, \mathbf{w}$. Is this set linearly dependent or independent? Why?
    (f) Pick a point Q not on your plane. How do you know it is not on the plane? (easy)
    (g) Express the orthogonal projection of $\mathbf{OQ}$ onto the plane P as a linear combination of $\mathbf{v}$ and $\mathbf{w}$. Here's a 4 min youtube video (made by someone else) doing an example - `https://youtu.be/rUP1HINVoy4`

68. The line through $(1, 2, 3)$ with direction vector $(-2, 4, 2)$ intersects the plane $-x + y + z = 4$ at what point? Also, give a parametric equation for this line.

69. (Summer 2021 Exam 1) Find the point on the plane $-x + 2y + 2z = 4$ closest to the point $(1, 2, 3)$.

70. (Summer 2021 Exam 2) For this question, consider the plane in $\mathbb{R}^3$ given by the equation

$$x + y - 2z = 4$$

Let's call this plane P.

(a) Give a point on the plane P

(b) Give a point Q not on this plane P

(c) Find the closest point on the plane P to the point Q you chose in part b.

(d) Give an example of an equation of a plane that is perpendicular to the plane P, and briefly explain or show work indicating the planes are perpendicular.

71. Let L be the line in $\mathbb{R}^3$ through the origin and with direction vector $(1, 1, 2)$. Find the point on this line L closest to the point $(1, 2, 3)$.

72. Consider the following three points in $\mathbb{R}^3$: $A = (1, 1, 1)$, $B = (1, 0, 2)$, $C = (-1, 1, 3)$

    1. Find the cosine of the angle between $\overrightarrow{AB}$ and $\overrightarrow{AC}$
    2. find the equation of the plane through these three A, B, C
    3. Find the area of the triangle formed by the points A, B, C.

73. Make up two vectors nonzero vectors $v_1 = (x_1, y_1)$ and $v_2 = (x_2, y_2)$ in $\mathbb{R}^2$ that are in different directions (i.e. one is not a scalar multiple of the other, or equivalently x_1/y_1 and x_2/y_2 are different). Show how to

    1. use the dot product to calculate the angle between these two vectors.
    2. use the determinant to calculate the area of the parallelogram spanned by v_1, v_2
    3. use geogebra (or python) to plot the parallelogram in the previous part, and see if your answer for the angle looks reasonable

74. Make up a non zero vector $v = (a, b, c)$ in $\mathbb{R}^3$ and a point $p = (x_0, y_0, z_0) \in \mathbb{R}^3$. (I suggest you reverse engineer to make the final answers work out nicely - p can be anything, but then choose some point with integer coordinates that q will end up being, then choose v based on these choices of p and q.).

    1. Give an equation of a plane P with normal vector v_1 such that the plane P does *not* pass through the point p (infinitely many possible answers, just give one)
    2. Find the equation of the line ℓ through p with direction vector v_1.
    3. Find the point q where your line ℓ intersects the plane P
    4. Find the distance $\|pq\|$ in two ways - first via the distance formula and secondly via the formula $|\frac{ax_0+by_0+cz_0+d}{\sqrt{a^2+b^2+c^2}}|$ if your plane has equation $ax + by + cz + d = 0$.
    5. Illustrate the above in Geogebra

75. Let $\ell_1(t) = p_1 + tv_1$ and $\ell_2(s) = p_2 + sv_1$ be the parametric equations of two lines in $\mathbb{R}^3$. Pick some values for p_1, p_2, v_1, v_2 (each one of these is a triple of numbers) and explain how to use linear algebra RREF to determine whether these two lines intersect.

76. Let e_1, e_2 be the standard basis vectors of $\mathbb{R}^2$. Make up 3 nonzero vectors v_1, v_2, v_3 in $\mathbb{R}^2$.

    1. Show how express v_1 as a linear combination of e_1 and e_2 (this is very easy)
    2. Show how express e_1 as a linear combination of v_1 and v_2 (or explain if this is not possible for your choice of v_1, v_2).

3. State the definition vectors for a set of $\{w_1, w_2, \dots, w_n\}$ (in any $\mathbb{R}^m$) to be linearly dependent vs linearly independent, and use this definition to determine whether v_1, v_2, v_3 are linearly dependent or linearly independent.

77. (Fall 2021 Final Exam)
    (a) Make up an equation of a plane P_1 in $\mathbb{R}^3$ passing though the point $(1, 2, 0)$.
    (b) Give a parametric equation for a line l in $\mathbb{R}^3$ that is perpendicular to your plane P_1 in the previous part (many possible correct answers).
    (c) Find the point of intersection of your line l and the plane P_1 from the previous two parts.

78. (Summer 2021 Exam 2) Imagine a xyz-coordinate system is superimposed on the airspace around an airport, with the control tower at the origin $(0, 0, 0)$. In this coordinate system, one incoming airplane A is located at coordinates $(1, 1, 1)$ (units are kilometers) for a brief moment in time.
    (a) (5 points) At that moment, how far away is airplane A from the control tower?
    (b) (0 points) Make up coordinates (p, q, r) for another airplane B (for use in the next part).

    $$p = \qquad q = \qquad r =$$

    To eliminate some easy scenarios, choose numbers p, q, and r such that p, q and r are not all equal to the same number, and such that $p + q + r$ is not equal to 0
    (c) (10 points) For an observer in the control tower, find the angle between the lines of sight to airplanes A and B.

79. Recall that a vector space V over $\mathbb{R}$ is just a set V with two operations:
    1. there is some rule (called vector addition $+$) that takes any two elements $\mathbf{v}, \mathbf{w}$ of V to get another element (denoted $\mathbf{v} + \mathbf{w}$) of V
    2. there is some rule (called scalar multiplication) that takes any real number $r \in \mathbb{R}$ and any element $\mathbf{v}$ of V to get another element (denoted $r\mathbf{v}$) of V

    Give some examples of vector spaces.

80. What is the standard basis of $\mathbb{R}^2$? What is an example of a basis of $\mathbb{R}^2$ other than the standard basis? Give two vectors in $\mathbb{R}^2$ that do not form a basis.

81. Let $\mathbf{v}_1 = (1, -1, 0, 0)$, $\mathbf{v}_2 = (1, 2, -2, 1)$, $\mathbf{v}_3 = (1, -2, 1, 0)$.
    (a) (5 points) Determine whether $\mathbf{w} = (1, 0, 0, 1)$ is in the span of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$, and if so, express $\mathbf{w}$ as a **linear combination** of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$.
    (b) (5 points) Determine whether $\mathbf{w} = (1, 0, 0, 2)$ is in the span of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$, and if so, express $\mathbf{w}$ as a **linear combination** of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$.

82. (Summer 2021 Exam 2) For this question, make up a random vector $\mathbf{v} = (p, q, r)$ in $\mathbb{R}^3$ other than the origin $(0, 0, 0)$.
    (a) (5 points) Express $\mathbf{v}$ as a linear combination of the standard basis vectors of $\mathbb{R}^3$.
    (b) (10 points) Show how to determine whether the collection of vectors $(1, 2, 1), (-1, 0, 3)$ and $\mathbf{v}$ form a basis of $\mathbb{R}^3$.

83. Let $\mathbf{v}_1 = (1, -1, 0, 0)$, $\mathbf{v}_2 = (1, 2, -2, 1)$, $\mathbf{v}_3 = (-1, -8, 6, -3)$.
    (a) Without doing any work, does $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ form a basis for $\mathbb{R}^4$?
    (b) Determine whether $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are **linearly independent** or not (and show work justifying your answer, of course).

84. For this question, make up a random vector $\mathbf{v} = (p, q, r)$ in $\mathbb{R}^3$ other than the origin $(0, 0, 0)$.
    (a) (5 points) Express $\mathbf{v}$ as a linear combination of the standard basis vectors of $\mathbb{R}^3$.
    (b) (10 points) Show how to determine whether the collection of vectors $(1, 2, 1), (-1, 0, 3)$ and $\mathbf{v}$ form a basis of $\mathbb{R}^3$.

85. a) In a Colab notebook, code up three functions that do the elementary row operations of a matrix (you are free to use the code I made).
    b) Make a system of 3 linear equations in 4 variables and explain how to use the functions you coded in the previous part to find all solutions to your system of equations via the Gauss-Jordan/RREF process. (Note that in python you should enter at least one of your entries of your matrix to be a float/decimal, e.g. 1.0 instead of 1 otherwise Python may round any decimals encountered in the RREF process to integers, which we don't want. Also this problem is about explaining how to get the RREF form, so just for this problem, do not use the sympy.Matrix's `rref()` method as shown in the image below unless you want to check your answer.)

    All of the following problems require finding RREF of a matrix at some point and you should use the following commands to find RREF in Python:

```
[1] import sympy

[2] A=sympy.Matrix([1, 2, 1, 0, 3, 5, 0, 1]).reshape(2,4)

[3] A

    ⎡1  2  1  0⎤
    ⎣3  5  0  1⎦

    A.rref()

    (Matrix([
     [1, 0, -5,  2],
     [0, 1,  3, -1]]), (0, 1))
```

86. Make up a 2×3 matrix A and show how to find all right inverses of A, i.e. find all 3×2 matrices B such that $AB = I$ where I is the identity matrix of the appropriate size. First figure out what must be the dimensions (# of rows x # cols) of B? Of I? Hint: if we split up B into column as $B = [b_1\ b_2]$, then $AB = [Ab1\ Ab_2]$ are the columns of AB.

87. (linear independence, span, basis)
    a) Let V be a vector space. What does it mean for a set of vectors $\{v_1, v_2, \ldots, v_n\}$ in V to:
        i) be linearly independent?
        ii) span V?
        iii) be a basis of V

b) Explain why the standard basis vectors $\{e_1, e_2, e_3\}$ of $\mathbb{R}^3$ is indeed a basis (as it's name suggests) by showing the collection of vectors $\{e_1, e_2, e_3\}$
   i) is linearly independent
   ii) spans $\mathbb{R}^3$?

88. (dimension)
   a Let V be a vector space. What is the definition of dimension of V?
   It is a general theorem that there are exactly n vectors in a basis of $\mathbb{R}^n$. For parts b and c below, let's pretend we didn't know this, and show by example that for a basis of $\mathbb{R}^3$, 4 vectors are too many (part b) and 2 vectors are too few (part c) (and so 3 is 'just right' to form a basis)
   b Make up four non-zero vectors in $\mathbb{R}^3$ and show that they are not linearly independent by setting up an appropriate system of equations and solving it via finding RREF
   c Make up two non-zero vectors in $\mathbb{R}^3$ and show that they do not span $\mathbb{R}^3$ by setting up an appropriate system of equations.

89. Make up a 3×4 matrix A, and also denote by A the associated linear transformation $A : \mathbb{R}^4 \to \mathbb{R}^3$.
   1. Find a basis of the nullspace/kernel (those two are the same thing) of A.
   2. Find a basis for the image/range/column space (these three are the same thing) of A.

90. (Coordinates in different bases)
   Let
   $$\mathbf{v}_1 = (4, 1, -1)^T, \quad \mathbf{v}_2 = (-5, -1, 2)^T, \quad \mathbf{v}_3 = -(4, 0, 5)^T$$
   be 3 vectors in $\mathbb{R}^3$. The vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ have been chosen so that they form a basis $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ of $\mathbb{R}^3$. Let $\mathcal{S} = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ be the standard basis of $\mathbb{R}^3$.
   1. If the vector $\mathbf{v} \in \mathbb{R}^3$ has coordinates $[\mathbf{v}]_{\mathcal{B}} = (1, 2, 3)$ in the basis $\mathcal{B}$, what are its coordinates $[\mathbf{v}]_{\mathcal{S}}$ in the standard basis?
   2. Make up a vector $\mathbf{v} = (a, b, c)^T = (a, b, c)^T_{[\mathcal{S}]}$ in $\mathbb{R}^3$. Show how to find the coordinates of $\mathbf{v}$ in the basis $\mathcal{B}$

91. Consider the system of equations:
   $$\begin{aligned} 2x + 4y - 6z &= 2 \\ x + 2y - 4z &= 1 \end{aligned}$$
   (a) If we write this system as a single matrix equation $\mathbf{A}\mathbf{x} = \mathbf{b}$ as done throughout the semester, what are $\mathbf{A}, \mathbf{b}, \mathbf{x}$?
   (b) (10 points) Show the process we used to find all solutions to the given system of equations. Using Python for some computations is fine, just attach screenshot

(c) (5 points) What is the geometric shape of the solution set? (e.g. is it empty, a point, a line, a plane?)

(d) (5 points) Find $\det \mathbf{A}$.

(e) What is the rank of $\mathbf{A}$?

92. (Final exam Summer 2021) The following system of equations in the variables x, y, z has a **unique** solution (x, y, z) for what value(s) of a?

$$\begin{aligned} ax + 2y \quad\quad &= 1 \\ 2x + 3y - 5z &= 1 \\ x - 2y + 4z &= 0 \end{aligned}$$

(You do not actually have to find the solutions for x, y, z. Also, "unique" means *exactly one* triple of numbers (x, y, z) satisfies the system of equations).

93. Make up three vectors in $\mathbb{R}^3$ that are linearly dependent. Then put them as columns in a 3x3 matrix. What is the determinant of this matrix?

94. (a) Define what it means for vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ to be a **basis** of a vector space V.

(b) Let $\mathbf{v}_1 = (3, -1, 2)$, $\mathbf{v}_2 = (1, 2, -2)$, $\mathbf{v}_3 = (-1, -8, c)$. Determine the values of c that make $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ a **basis** of $\mathbb{R}^3$,

95. Let

$$\mathbf{v}_1 = (4, 1, -1), \quad \mathbf{v}_2 = (-5, -1, 2), \quad \mathbf{v}_3 = -4, 0, 5$$

The $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ have been chosen so that they are linearly independent and hence form a basis $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ of $\mathbb{R}^3$. Let $\mathcal{S} = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ be the standard basis of $\mathbb{R}^3$.

(a) What are the change of basis matrices $P_{\mathcal{B}\to\mathcal{S}}$ and $P_{\mathcal{S}\to\mathcal{B}}$?

(b) If the vector $\mathbf{v} \in \mathbb{R}^3$ has coordinates $[\mathbf{v}]_{\mathcal{B}} = (1, 1, 0)$ in the basis $\mathcal{B}$, what are its coordinates $[\mathbf{v}]_{\mathcal{S}}$ in the standard basis?

(c) What are the coordinates of $\mathbf{e}_1 = (1, 0, 0)$ in the basis $\mathcal{B}$?

96. In parts 1 and 2 of this question (on the next page), we give a matrix $\mathbf{A}$ and the RREF of $\mathbf{A}$, from which you can easily read off the row space, null space, and column space. For the each of the matrices $\mathbf{A}$ on the next page, answer the following questions:

(a) Which columns are the pivot columns of $\mathbf{A}$?

(b) Find a basis for the row space of $\mathbf{A}$. What is the dimension of this row space?

(c) Find a basis for the null space of $\mathbf{A}$. What is the dimension of this null space?

(d) Find a basis for the column space of $\mathbf{A}$. What is the dimension of this column space?

(e) Find a basis for the row space of $\mathbf{A}$ that consists of rows of $\mathbf{A}$.

Part 1).

```
A

[ 1  1 -2 -1 -3 -2 -4]
[ 0  1  3 -1 -1  1 -2]
[ 0  2  6 -1  0  2  1]
[ 1  2  1 -1 -1  0  1]
```

```
A.rref()

[ 1  0 -5  0  0 -1  2]
[ 0  1  3  0  0  0  1]
[ 0  0  0  1  0 -2  1]
[ 0  0  0  0  1  1  2]
  1  2  3  4  5  6  7
```

Part 2).

```
A

[ 1  1  0  6  3  7 -7]
[-1  0 -4 -4 -1 -3  4]
[ 0  0  0  0  1  2 -2]
[ 1  0  4  4  0  1 -2]
[ 1  1  0  6  2  5 -5]
```

```
A.rref()

[ 1  0  4  4  0  1 -2]
[ 0  1 -4  2  0  0  1]
[ 0  0  0  0  1  2 -2]
[ 0  0  0  0  0  0  0]
[ 0  0  0  0  0  0  0]
  1  2  3  4  5  6  7
```

97. Consider the system of equations:

$$
\begin{aligned}
2x + 4y - 6z &= 2 \\
x + 2y - 3z &= 1 \\
3x + 7y + 2z &= 0
\end{aligned}
$$

(a) (5 points) If we write this system as a single matrix equation $\mathbf{A}\mathbf{x} = \mathbf{b}$ as done throughout the semester, what are $\mathbf{A}, \mathbf{b}, \mathbf{x}$?

(b) (5 points) Row reduce the augmented matrix $[\mathbf{A} \mid \mathbf{b}]$ by hand or by Python

(c) (5 points) Use the rref from the previous part to find all solutions to the given system of equations.

(d) (5 points) What is the geometric shape of the solution set? (e.g. is it empty, a point, a line, a plane?)

(e) (5 points) Find det **A**.

(f) (5 points) What is the rank of **A**? What is the definition of rank of a matrix?

98. (Summer 2021 Final Exam)

(a) (5 points) Give an example of a 2x4 matrix A of rank 1. Briefly explain why your matrix has rank 1.

$$A =$$

(b) (10 points) Show how to find a basis for the null space of the matrix A you created in the previous part. (It is fine if you chose a matrix that makes this part relatively easier).

(c) (5 points) What is the dimension of the null space of A?

99. (Fall 2020 Final Exam) Consider the following output of SageMath.

```
A=matrix(QQ, 3, 4, [2, 1, 1, 2, 1, 0, -1, 2, 4, 3, 5, 2])
A
```

```
[ 2  1  1  2]
[ 1  0 -1  2]
[ 4  3  5  2]
```

```
A.rref()
```

```
[ 1  0 -1  2]
[ 0  1  3 -2]
[ 0  0  0  0]
```

(a) If the matrix A is the augmented matrix associated to a system of equations (via the process we have been using all semester long), what is that system of equations?

(b) Use the output of A.rref() given above the find all solutions to the system of equations in part a.

(c) The null space of A is the set of solutions to a slightly *different* system of equations. What is that system of equations?

(d) What is the rank of A?

(e) Show how to find a basis for each of the following vector spaces: the null space, the row space, and the column space, of A

100. **Problem** Make up a 2x2 matrix M, all of whose entries are non-zero. Calculate its determinant using the ad-bc formula, and then use elementary school geometry formulas for areas of triangles and trapezoids to verify that the determinant calculates the (signed) area of the appropriate

101. **Problem** In class there was a nice question about inverses of non-square matrices. The good news is that the RREF process we discussed to find the inverse of a square matrix works the same in that case too. So create a 2x3 matrix A, not having any zeros (to eliminate easy cases), and such that row 2 is not a scalar multiple of row 1 (this will guarantee right inverses will exist). Explain how to find all 3x2 matrices B such that $AB = I_2$ (B is the right inverse of A, and here I_2 denotes the 2x2 identity matrix.)

   Hint: The first step is to conceptualize the unknown matrix B as having two columns $B = [v_1\ v_2]$, where each $v_1, v_2 \in \mathbb{R}^3$. Second step is to go back to the definition of the matrix multiplication $A[v_1\ v_2]$. Third step is relate answer in previous step to requirement $AB = I_2$. Fourth step is to use Day 1 RREF technique to find v_1, v_2.)

   Problem 2.. Not all transformations are linear transformations and so cannot be represented by a matrix. For example, previously we have seen that translation $T_{a,b} : \mathbb{R}^2 \to \mathbb{R}^2$ given by (a, b), i.e. $T_{a,b}(x, y) = (x + a, y + b)$ for $(x, y) \in \mathbb{R}^2$. Show that $T_{a,b}$ is not a linear transformation However, translations and linear transformations are useful, and the following trick to convert a translation to a linear transformation is used in 3D graphics, machine learning, etc. Insert $\mathbb{R}^2$ inside $\mathbb{R}^3$ via $i : \mathbb{R}^2 \to \mathbb{R}^3$ by $i(x, y) = (x, y, 1)$, so geometrically we are bumping the usual xy-plane $z = 0$ up 1 unit in the z-direction to $z = 1$. Find a linear transformation $\overline{T} : \mathbb{R}^3 \to \mathbb{R}^3$ that acts as translation $T_{a,b}$ on the inserted copy $i(\mathbb{R}^2)$, in other words on the embedded xy-plane $z = 1$. such that

$$\overline{T} \circ i = i \circ T_{a,b}$$

   on $\mathbb{R}^2$ i.e.

$$\overline{T} \circ i(x, y) = \overline{T}(x, y, 1)$$

   =(x+a, y+b, 1) and

$$i \circ T_{a,b}(x, y) = i(x + a, y + b) = (x + a, y + b, 1)$$

   In other words, give an appropriate formula

$$\overline{T}(x, y, z) = M \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

   And find the associated 3x3 matrix M such that $T = T_M$. (The answer is basically given in `https://youtu.be/Do_vEjd6gF0` which explains why computer graphics developers use 4x4 matrices to model 3 dimensional transformations; this question asks simpler 2d situation).

102. (Fall 2021 Exam 3) Make up some values for two different non-zero numbers x and y

$$x =$$

$$y =$$

   and use these values to set

$$\mathbf{v}_1 = (x, y)$$

   to use for this problem.

   (a) (5 points) Express $\mathbf{v}_1$ as a linear combination of the standard basis vectors $\mathbf{e}_1, \mathbf{e}_2$ (just the answer is fine, not necessary to show work).

(b) (5 points) Give an example of a non-zero vector $\mathbf{w} \in \mathbb{R}^2$ such that $\mathbf{v}_1$ and $\mathbf{w}$ are linearly dependent (just the answer is fine, not necessary to show work).

(c) (5 points) Give an example of a non-zero vector $\mathbf{v}_2 \in \mathbb{R}^2$ such that $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly INdependent, and show how to determine they are linearly independent.

(d) (5 points) Express (y, x) (here x and y are the numbers you chose at the beginning) as a linear combination of the two vectors $\mathbf{v}_1$ and $(0, 1)$, if possible. (Show how you got your answer).

(e) (5 points) Create a non-zero 3 x 2 matrix $\mathbf{M}$ having $(1, 1)$ in the null space of $\mathbf{M}$. Explain in a sentence why your matrix has $(1, 1)$ in its null space.

103. (Fall 2021 Exam 3) Make up a 3 x 3 matrix $\mathbf{A}$, with none of the entries equal to 0, and such that the second row is twice the first row.

(a) (10 points) Show how to find the null space of $\mathbf{A}$, and give a basis for the null space. (Use Python/cocalc for this part and and attach a screenshot of the computation)

(b) (5 points) Give an example of a vector in the null space of $\mathbf{A}$, other than the zero vector and the vectors in your basis in the previous part.

(c) (10 points) Show how to determine if $(1, 1, 1)$ is in the column space of $\mathbf{A}$. (Use Python/cocalc for this part and and attach a screenshot of the computation)

(d) (5 points) Find a basis for the row space of $\mathbf{A}$.

(e) (5 points) What is the rank of $\mathbf{A}$?

104. (Fall 2021 Exam 3) Make up two non-zero vectors $\mathbf{v}$ and $\mathbf{w}$ in $\mathbb{R}^3$.

$$\mathbf{v} =$$

$$\mathbf{w} =$$

(a) (5 points) Give an example of a vector that is a linear combination of $\mathbf{v}$ and $\mathbf{w}$.

(b) (10 points) Let $\mathbf{u} = \mathbf{v} + \mathbf{w}$. Determine whether $\mathbf{u}, \mathbf{v}, \mathbf{w}$ form a basis for $\mathbb{R}^3$ and explain/justify your answer.

105. (a) (5 points) Make up a matrix with $(1, 1, 1)$ in both its nullspace and column space (and explain why your matrix satisfies these requirements).

(b) (5 points) Show how to find a basis of the nullspace of your matrix from the previous part.

(c) (5 points) Show how to find a basis of the column space of your matrix from the previous part.

(d) (5 points) Make up a matrix with $(1, 1, 1)$ in both its row space and column space (and explain why your matrix satisfies these requirements).

(e) (5 points) Explain why there can't be a matrix with $(1, 1, 1)$ in both its nullspace and row space

106. (5 points) Make up a number λ and then make up a matrix having $(1, 1, 1)$ as an eigenvector with eigenvalue λ

107. Given a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$, the kernel of T is the set

$$Ker(T) = \{v \in \_\_\_\_\_ \mid \_\_\_\_\_\}$$

The image of T is

$$Im(T) = \{w \in \_\_\_\_\_ \mid \_\_\_\_\_\}$$

$$Ker(M) = \{v \in \_\_\_\_\_ \mid \_\_\_\_\_\}$$

The column space of M is

$$Col(M) = \{w \in \_\_\_\_\_ \mid \_\_\_\_\_\}$$

5 Eigenvalue and eigenvectors

108. Recall that one condition in the definition of a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ was that for all vectors $v \in \mathbb{R}^n$ and all scalars $c \in \mathbb{R}$, the vector $T(cv)$ must equal the vector $cT(v)$. We will try to identify what this condition means geometrically, and to make things simple let's let $n = 2$.

    1. Make up a non-zero vector $v \in \mathbb{R}^2$. Explain what the "set builder" notation (fundamental and universal in math):
    $$S = \{cv \mid c \in \mathbb{R}\}$$
    What does the set S looks like when plotted in the xy-plane $\mathbb{R}^2$ (draw a sketch, hand-drawn is fine).. Give a few elements/points/vectors (these all mean the exact same thing here) in S and a few that are not in S. Is $(0, 0) \in S$?
    2. I haven't given you what T is, so you can't specify what $T(v)$ is. If $T(v) = 0$ (where by 0 we really mean the zero *vector* $(0, 0) \in \mathbb{R}^2$), what does the set
    $$S' = \{cT(v) \mid c \in \mathbb{R}\}$$
    look like when plotted? What does S' look like if $T(v)$ is not zero (vector).
    3. Explain why a linear transformation, whenever it takes as input the set of points forming a line through the origin the output is a set of points forming a line through the origin.
    4. If v is an eigenvector for T, what is the relationship between the sets S and S' defined above.

109. (20 points) For this problem, create a matrix by filling in the three empty blanks in the 2x2 matrix

$$\mathbf{A} = \begin{bmatrix} \_ & \_ \\ 0 & \_ \end{bmatrix}$$

with three *different* numbers. Then for this choice of $\mathbf{A}$, find

(a) the characteristic polynomial of $\mathbf{A}$

(b) the eigenvalues of $\mathbf{A}$

(c) eigenvectors for each eigenvalue of $\mathbf{A}$

(d) and matrices P and D such that D is diagonal and P is invertible and $\mathbf{A} = PDP^{-1}$.

110. (Summer 2021 Final Exam) Let $A = \begin{bmatrix} -7 & 10 \\ -5 & 8 \end{bmatrix}$

   (a) (5 points) Explain why $v = (2, 1)$ is an eigenvector for A, and find the corresponding eigenvalue.

   (b) (10 points) Find the other eigenvalue and a corresponding eigenvector.

   (c) (5 points) Diagonalize A, meaning factor $A = PDP^{-1}$ where D is a diagonal matrix.

   (d) (5 points) Use the previous part to find A^n for any positive integer n.

111. Show that $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ have the same characteristic polynomial, but one is obviously diagonalizable and explain why the other is not diagonalizable.

112. Use Python to create some random 3x3 matrices, and use Python to compute its characteristic polynomial and to diagonalize the matrix (if possible).

113. Look up an explanation of the fact that a matrix is diagonalizable if and only if its minimal polynomial has distinct roots.

114. Create a 2 x 2 matrix that has $(1, 2)$ as an eigenvector.

115. Create two different 3 x 3 matrix with eigenvalues $1, 1, 2$ (hint: make them upper triangular, or start with a diagonal matrix D and create PDP^{-1} for any invertible matrix P) .

116. (a) Given a square matrix $\mathbf{A}$ having an eigenvector with eigenvalue of 3, this quickly translates to what matrix has a null space bigger than just the 0 vector $\mathbf{0}$

   (b) Create a 2x2 matrix having 0 as an eigenvalue and with none of its four entries being 0. Explain briefly why your matrix has 0 as an eigenvalue. (Hint: start with $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and if it has an eigenvalue of 0 then what must be true?)

117. .

   (a) Create a 2x2 matrix A that has $v = (2, 3)$ as one of its eigenvectors (you are free to choose the value of the associated eigenvalue, and free to use any method we discussed). Let's call this matrix A for the remaining parts of this problem.

   (b) Find the characteristic polynomial of A.

   (c) Find the eigenvalues of the matrix A and corresponding eigenvectors.

   (d) Diagonalize A, meaning find matrices P and D such that $A = PDP^{-1}$ and D is a diagonal matrix. (Just giving P and D is fine, you do not have to multiply out PDP^{-1} and verify it equals A. If your A can't be diagonalized, explain why).

   (e) What is A^n? Express your answer as a function of the positive integer n.

118. For this problem, fill in the three empty blanks in the 2x2 matrix

$$\mathbf{A} = \begin{bmatrix} \_ & \_ \\ 0 & \_ \end{bmatrix}$$

with three *different* numbers. Then for this choice of **A**, find

(a) the characteristic polynomial of **A**

(b) the eigenvalues of **A**

(c) eigenvectors for each eigenvalue of **A**

(d) and matrices P and D such that $\mathbf{A} = PDP^{-1}$.

119. For this problem, fill in the three empty blanks in the 3x3 matrix $\mathbf{A} = \begin{bmatrix} 2 & 0 & \_ \\ 0 & 2 & \_ \\ 0 & 0 & \_ \end{bmatrix}$ with some non-zero numbers. Then find two eigenvectors of **A** having an eigenvalue of 2.

120. In Python to create a 3 x 3 matrix **A** with first row $(1, 1, 2)$. Then use Python to find the characteristic polynomial of **A** and the eigenvalues of **A** (recall that you can use tab completion to find the relevant commands, which I don't remember)

121. Let A be an $n \times n$ matrix.

(a) What are the definitions of eigenvector and eigenvalues of A?

(b) Let P be an $n \times n$ invertible matrix, and D a diagonal matrix (meaning, one whose entries are all zero except possibly on the diagonals). If

$$A = PDP^{-1}$$

then explain why the columns of P are eigenvectors of A and the diagonal entries of D are the corresponding eigenvalues of A.

(c) Look up and share something interesting about the Fibonacci numbers, defined as $f_1 = 1, f_2 = 1$ and

$$f_n = f_{n-1} + f_{n-2}$$

(d) Explain how matrix diagonalization can be used to show Binet's formula (I think I have this right)

$$f_n = \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2} \right)^n - \frac{1}{\sqrt{5}} \left(\frac{1-\sqrt{5}}{2} \right)^n$$

(e) Use Python to compute the 100th Fibonacci number. Hint: don't use recursion - instead use *iteration* - build a list whose nth item is f_n (or you could also use Binet formula).

122. Consider the system of equations in the variables x, y; and treat d as a unspecified constant.

$$\begin{aligned} 2x + 4y &= 0 \\ x + dy &= 0 \end{aligned}$$

(a) (5 points) Give a value of d that will give infinitely many solutions (x, y). For this value of d what are all the solutions?

(b) (5 points) Give a value of d that will give exactly one solution (x, y). What is this one solution?

6 Markov Chains

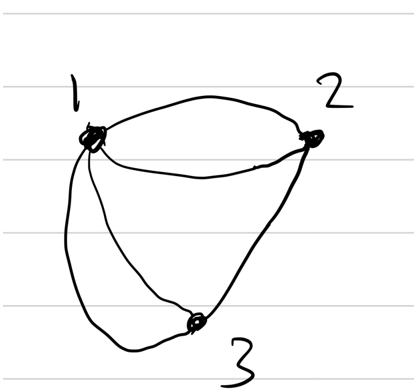
123. Suppose we are in some Markov situation where there are three possible conditions/states: call them $1, 2, 3$, and as usual let state vectors $v(t)$ record the probability that at time t you are in states 1, 2, or 3. respectively. So if $v(10) = (0.2, 0.7, 0.1)$ that means at time $t = 10$ there is a 20% probability one is in state 1, 70% probability one is in state 2, and 10% probability one is in state 3. Suppose the transition matrix P is the following matrix, in which each column which adds to 1 (such a matrix is called a left stochastic matrix).

$$P = \begin{bmatrix} 0.2 & 0.1 & 0.4 \\ 0.3 & 0.2 & 0.3 \\ 0.5 & 0.7 & 0.3 \end{bmatrix}$$

(a) What does the the $(3, 2)$ entry 0.7 in the above matrix translate to in words (or in a story problem)

(b) What is the steady state vector?

124. . (Summer 2021 Final Exam)



A president has 3 secure locations (vertices labeled 1, 2, 3) that are connected by various daily flights (the edges of the graph) as shown above. Each day, the president is moved to a different location using a randomly chosen flight; each flight out of a location is equally likely to be chosen, independent of its destination.

(a) Suppose we know that the president is at location 3. What is the state vector we associated to this scenario?

(b) (10 points) Find the transition matrix P we associated to the situation.

(c) (5 points) Suppose the president is at location 2 on Sunday morning (before taking a flight for the day). Find the probability that the president is at location 1 at the beginning of Friday of the same week (before taking a flight for that day). Use Python and attach the screenshot.

(d) (10 points) In the long run, find the probabilities that the president spends in each location (i.e. the steady state vector).

125. (30 points) On any given Sunday, the local football team may win or lose (let's assume no ties), and the outcome influences whether they will win or lost the next game. The season has 16 games numbered 1 through 16. For some unexplained reason, it turns out that

- if the team wins a game, the probability they win the next game is 0.6.
- If the team loses a game, the probability they lose the next game is 0.7.

Using this information, answer the following questions:

(a) If the team wins a game, what is the probability they lose the next game?

(b) Find the two matrix products

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

(Your two answers will involve a, b, c and/or d.This calculation has nothing to do with the above situation, but remembering this fact I feel is the key to remembering how to make the transition matrix in part d below).

Suppose we model the situation via a Markov model with the state vector $p_n = \begin{bmatrix} w_n \\ l_n \end{bmatrix}$ of game n is a 2×1 column vector where w_n is the probability the team wins game n, and l_n is the probability the team loses game n. Here n could be any integer 1 through 16.

(c) The state vector $p_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ corresponds to what scenario? The state vector $p_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ corresponds to what scenario?

(d) Find the 2 x 2 transition matrix P associated to this Markov story problem, i.e. find P such that $p_{n+1} = Pp_n$ for all $n \geq 1$.

$$\underbrace{\begin{bmatrix} w_{n+1} \\ l_{n+1} \end{bmatrix}}_{p_{n+1}} = \underbrace{\begin{bmatrix} - & - \\ - & - \end{bmatrix}}_{P} \underbrace{\begin{bmatrix} w_n \\ l_n \end{bmatrix}}_{p_n}$$

(e) Perhaps surprisingly, the probabilities w_n and l_n stabilize pretty quickly as n increases, and are independent of starting values w_1 and l_1. Find to two decimal places the probabilities w_n and l_n when n is any number larger than say 10. At the end of the 16 game season, how many games does this (not necessarily realistic) model predict the team will win? (I would use Python for this part, include a screenshot if you do).

7 Gram-Schmidt

126. (10 points) Apply the Gram-Schmidt algorithm to the pair of vectors $v_1 = (1, 1, 1)$ and $v_2 = (0, 1, 2)$ to return an orthonormal pair of vectors (q_1, q_2). (For the standard Euclidean inner product, i.e. dot product).

127. Code the Gram-Schmidt algorithm described above in Python "from scratch", meaning don't use the built-in gram schmidt in np.linalg for this question. Show it in action and plot results in Geogebra/Desmos. (In case it's useful, here are my notes on the Gram-Schmidt algorithm from a previous semester :
`https://app.box.com/s/fyum7nrwvfug0n1wvzglledze313soo0`)

8 Symmetric Matrices

128. Give an example of a symmetric matrix? What is the main fact about the eigenvalues and eigenvectors of a real symmetric matrix? Illustrate it in Python.

129. (Fall 2020 Final Exam) Let $v_1 = (1,1,1)$ and $v_2 = (1,1,-2)$.
   (a) Show that $\{v_1, v_2\}$ forms an orthogonal set of vectors. Is it an orthonormal set?
   (b) Show how to find a vector v_3 so that $\{v_1, v_2, v_3\}$ form an orthogonal set.
   (c) What is the definition of an orthogonal matrix P? Use the answer in the previous part to form a $3x3$ orthogonal matrix P whose three columns are scalar multiples of v_1, v_2, v_3 respectively.

9 Singular Value Decomposition

130. (Fall 2020 Final Exam)
   (a) Show how to find by hand the singular values of the matrix $A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}$. (The characteristic polynomial that eventually arises in this example doesn't factor nicely and so use the quadratic formula).
   (b) Show how to use Python or python the find the singular values of $A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ and include a photo/screenshot of your calculation.

131. Make up a 3×4 matrix A, and also denote by A the associated linear transformation $A : \mathbb{R}^4 \to \mathbb{R}^3$.
   1. Make up a vector $v \in \mathbb{R}^?$ in an appropriate dimensional space and compute $A(v)$.
   2. Find a basis of the nullspace/kernel (those two are the same thing) of A.
   3. Find a basis for the image/range/column space (these three are the same thing) of A.
   4. Find a basis for the row space of A
   5. What is the definition of **rank** of a matrix, and what is the value of the rank of you matrix A?

   (You should know how to answer the above questions both with Python and by hand w/o a calculator, although in the write up you can show just one of the two ways).

132. 1. Explain how the following image from our text (Linear Algebra Done Wrong) is trying to show that a rotation is a linear transformation.

3. Linear Transformations. Matrix–vector multiplication 13

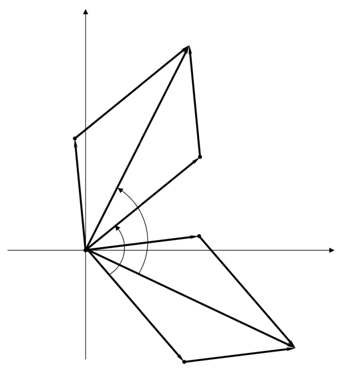


Figure 1. Rotation

2. Pick some $p(x) \in P_2$, and let $M : P_2 \to P_4$ be the map defined by $M(q(x)) = p(x)q(x)$, which takes a vector $q(x)$ in P_2 as input and multiplies it by $p(x)$. Determine if M is a linear transformation and if so, find its associated matrix with respect to standard bases $\{1, x, x^2\}$ of P_2 and $\{1, x, x^2, x^3, x^4\}$ of P_4.

3. Find the matrix of the derivative $\frac{d}{dx} : P_3 \to P_2$ with respect to standard bases of P_3 and P_2.

133. a) Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be the linear transformation $T(x, y, z) = (x - y, z)$ (to save space, we are writing row vectors but we really mean column vectors). Find the matrix we associate to T.

b) Let $S : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation associated to the matrix $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$. Find

   a) the expression for the composite $(S \circ T)(x, y, z)$
   b) the matrix we associate to $S \circ T : \mathbb{R}^3 \to \mathbb{R}^2$ (where T is the linear transformation from the previous part).

134. Let A be an $n \times n$ matrix.

1. What are the definitions of eigenvector and eigenvalues of A?

2. Let P be an $n \times n$ invertible matrix, and D a diagonal matrix (meaning, one whose entries are all zero except possibly on the diagonals). If

$$A = PDP^{-1}$$

then explain why the columns of P are eigenvectors of A and the diagonal entries of D are the corresponding eigenvalues of A.

3. Create a 2×2 matrix P with determinant 1 or -1, and also make up a 2×2 diagonal matrix $D = \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}$. State the formula for the inverse of a 2×2 matrix and use it to find P^{-1} and calculate $A = PDP^{-1}$ (by hand, and confirm your calculation using Python; the command for inverse of P in numpy (abbrev. as np) is `np.linalg.inv(P)`).

4. Pretend you were given the matrix A that you created in previous part, and you didn't know P and D. Explain how you would find eigenvectors, eigenvalues, and a diagonalization of A. (The eigenvectors are not unique, so P does not have to be the same as in previous part).

135. Fill in the six boxes (3 under example, 3 under nonexample) in the following table (on next page, sorry), with each one of boxes containing a collections of vectors in $\mathbb{R}^3$ - none of which are the standard basis vectors - satisfying the condition in the row and column label it belongs to.

Table 1:

	example	non-example
linear independent		
span		
basis		

136. 1. Make up two vectors v_1, v_2 in $\mathbb{R}^3$ and show how to find the angle between them. Then plot the vectors in Geogebra - and have Geogebra calculate angle between them. (Include screenshot in your submission to Brightspace).

2. Create two orthogonal vectors v_1 and v_2 in $\mathbb{R}^3$ (other than the standard basis vectors) and plot them in Geogebra - check they look orthogonal. (Include screenshot in your submission to Brightspace).

3. Make up 3 linearly vectors v_1, v_2, v_3 (other than the standard basis vectors) in $\mathbb{R}^3$ and explain how to find the orthogonal projection of v_3 onto the plane spanned by v_1, v_2 (and show how to do the required calculations in Python).

137. 1. Explain why the least squares approximate solution to $Ax = b$ is

$$\hat{x} = (A^T A)^{-1} A^T b$$

2. In python, generate some "fake" data (x_i, y_i) so that the y-values are roughly a linear combination of some functions of x (e.g.

$$y = a_1 \cos(x) + a_2 x^2 + a_3 + a_4 \sin(x) + \epsilon$$

for some scalars a_1, a_2, a_3, a_4 that you make up), and add some random noise ϵ to mimic the fact that data in real life doesn't usually perfectly fit any formula. Then pretend you don't know what values a_1, a_2, a_3, a_4 created the data (but you do know the functions you took linear combinations of). Use the least squares formula above to find the linear combination of the chosen functions that best approximates your data.

`https://colab.research.google.com/drive/1zhX6ylLuSR5jysKXSNveqLchh90XxE8P?usp=sharing`

For example I did this in the colab notebook above (which I'll go through in class and which you can use as a template) for the linear combination

$$\mathtt{y} = 3 * \mathtt{np.sin}(\mathtt{x}) + 5 * \mathtt{np.cos}(\mathtt{x}) + 7 + \mathtt{np.random.random}(100)$$

use different functions and linear combinations

(If you want a reference for this least squares material see Chapter 5 Section 4 of our text (e.g. p.137 and ff) or `https://textbooks.math.gatech.edu/ila/least-squares.html`)

138. 1. Make up some 2x2 matrix and compute their inverses by row reducing [A — I] and then confirm your answer in python

2. Make up a 3x3 matrix and compute its determinant via expansion by minors and check you answer in python.

MA 3160 Spring 2023 Exam 2

139. (Matrix Multiplication)

a) Let A be an $m \times n$ matrix and $\mathbf{b} \in \mathbb{R}^n$ an n-dimensional column vector. How did we define matrix-vector multiplication $A\mathbf{b}$? (You can use numbers to illustrate, but make sure you mention the term linear combination)

b) Let A and B be two matrices. What must be true in order for AB to exist?

c) If $B = [v_1 \mid v_2 \mid \cdots \mid v_n]$ where the v_i are the columns of B, and if AB is defined, how did we define the matrix-matrix multiplication AB in terms of matrix-vector multiplications?

d) If $(AB)_{i,j}$ denotes the (i, j) entry of AB (meaning the number in row i and column j of AB), in a few words describe how to find it. (Answer I'm looking for involves dot product of row/col of A and B).

e) If A^T denotes the transpose of A, then by definition

$$(A^T)_{i,j} = A_{j,i}$$

and the rows of A become the columns of A^T. Use this along with the previous part to explain **why** $(AB)^T = B^T A^T$, by showing that the (i, j) entries on both sides are the same.

f) Make up two 3 x 3 matrices A, B. Show how to compute by hand the $(2, 3)$ entry of AB. Then show how to compute AB in python, using the sympy library, where the matrix constructor is `sympy.Matrix([])` and via the numpy library, where matrix constructor is `np.array([ ])`

140. (Matrix Inverses)

a) Let A be a suppose we wanted to find a 3x3 matrix C such that $AC = I_3$ where I_3 is the 3x3 identity matrix. Explain **why** (not how to, that's the next part) finding the RREF of $[A \mid I_3]$ will lead to finding C. (hint: break $C = [v_1 \mid v_2 \mid v_3]$ into columns and use the definition of AC in terms of matrix-vector multiplication).

b) Make up a 3×3 matrix A, and do the previous step using Python to find C (if such a C exists).

c) Use Python's built-in command `numpy.linalg.inv(A)` to find the inverse of A. (If A is a `sympy.Matrix` object, you can convert it to numpy object by $A.numpy()$ as I discovered in class on day.) It should agree with C that you found by RREF-ing $[A \mid C]$

141. (Projections)

a) Given three (column) vectors $\mathbf{a_1}, \mathbf{a_2}, \mathbf{b}$ in $\mathbb{R}^n$, let $S = \text{Span}(\mathbf{a_1}, \mathbf{v_2})$. Let $A = [\mathbf{a_1} \mid \mathbf{a_2}]$ be the matrix whose columns are $\mathbf{a_1}, \mathbf{a_2}$. Explain **why**

$$\text{Proj}_S\mathbf{b} = A(A^T A)^{-1} A\mathbf{b}$$

b) Show how to use Python compute $\text{Proj}_S\mathbf{b}$

c) Illustrate the above in Geogebra

142. (Planes)

a) Make up specific vectors $\mathbf{a_1}, \mathbf{a_2}$ in $\mathbb{R}^3$ and compute cross product $\mathbf{a_1} \times \mathbf{a_2}$ to get a vector orthogonal to $\mathbf{a_1}, \mathbf{a_2}$,

b) Explain how to find the xyz-equation of the plane passing through the point $(0, 0, 7)$ and parallel to the vectors $\mathbf{a_1}, \mathbf{a_2}$

c) Illustrate the above in Geogebra.

MA 3160 Final exam Spring 2023

The final exam is an oral exam over Zoom, here are instructions: schedule a Zoom appointment at www.calendly.com/morey-ow to discuss the following questions to guide to conversation. The Zoom meeting will take about 90-120 minutes and schedule it to take place anytime May 4- May 18 11:59pm (weekends and evenings are fine).

143. I will give you a 3×3 matrix A, and (to save you some work) I will give you its *characteristic polynomial* $\det(A - \lambda I)$ in factored form and ask you to identify the eigenvalues of A and to find the eigenspaces associated with each eigenvalue.

For example, let

$$A = \begin{bmatrix} 7 & -20 & -5 \\ 5 & -18 & -5 \\ -10 & 40 & 12 \end{bmatrix}$$

which has characteristic polynomial $-(\lambda - 2)^2(\lambda + 3)$. Show how to find (entirely by hand) the $\lambda = 2$-eigenspace E_2, i.e. all eigenvectors with eigenvalue 2:

$$E_2 = \{\mathbf{v} \in \mathbb{R}^3 \mid Av = 2v\}$$

144. Explain matrix multiplication (what it is, when it is possible, how to view it as capturing linear combinations of columns of A)

145. (a) What are two formulas for the dot product of vectors in $\mathbb{R}^3$? Given two vectors in $\mathbb{R}^3$, show how to find the angle between them.

(b) Let $\mathbf{v}_1, \mathbf{v}_2$ be two column vectors in $\mathbb{R}^3$. Explain why $\mathbf{v}_1^T \mathbf{v}_2 = 0$ means that $\mathbf{v}_1, \mathbf{v}_2$ are perpendicular.

(c) Let A be a matrix. What does it mean to *diagonalize* A? What does it mean to *orthogonally diagonalize* A? How does one go about diagonalizing a square matrix (when possible)?

(d) Let A be a symmetric matrix, with two eigenvectors $\mathbf{v}_1, \mathbf{v}_2$ with distinct eigenvalues λ_1, λ_2. Explain why $\mathbf{v}_1$ and $\mathbf{v}_2$ have to be orthogonal (hint:fiddle around with $(A\mathbf{v}_1)^T \mathbf{v}_2$)

146. I will make up a small matrix A, and ask you to explain how to find its singular value decomposition, using Python only for finding eigenvectors and eigenvectors of various matrices.

147. (a) Given a vector $\mathbf{b} \in \mathbb{R}^3$ and a plane $\mathbf{P} = \{s\mathbf{v_1} + t\mathbf{v_2} \mid s, t \in \mathbb{R}\}$ through the origin in $\mathbb{R}^3$, explain how to figure out to find the projection of $\mathbf{b}$ onto $\mathbf{P}$?

(b) Explain why the least squares approximate solution to $Ax = b$ is

$$\hat{x} = (A^T A)^{-1} A^T b$$

(c) I will make up a few (three or four) data points (x_i, y_i) and ask you to explain how to use the previous part to find the least-squares line of best fit through them.

148. I have a certain number of nickels, dimes, and quarters. I have a total of say 10 coins and their total value is $1.05. How many of each coin type could I have? There are two possible scenarios/answers, find them both. (Use linear algebra and RREF, not guess and check).

149. (Optional) I was initially going to ask you to compress an image using Singular Value Decomposition. That might be too much to ask, so I did this here https://colab.research.google.com/drive/1nqCyk-6MCZRaU1rwMDg4UuwBw0vGqB6e?usp=sharing.
Explain what is going on in this notebook. Feel free to make a copy of this notebook and add cells in between to explore the shape of various matrices etc.