Notation.

Let $\langle \, , \, \rangle$ denote dot product on $\mathbb{R}^n$. For column vectors

$\underbrace{x, y}_{n \times 1 \text{ matrices}} \in \mathbb{R}^n$,

$$\langle x, y \rangle = \underset{n\times 1}{x} \cdot \underset{n \times 1}{y}$$

dot product, not matrix multiplication

$$x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \quad y = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$$

$n \times 1$ $\quad$ $n \times 1$

$$= x_1 y_1 + x_2 y_2 + \dots + x_n y_n$$

we'll need today Easy fact: $\underbrace{\langle x, y \rangle}_{\text{dot product}} = \underbrace{x^T y}_{\text{matrix multiplication, not dot product}}$

$$\underbrace{[x_1 \; \dots \; x_n]}_{x} \underbrace{\begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}}_{y} = x_1 y_1 + x_2 y_2 + \dots x_n y_n$$

$1 \times n$ $\quad n \times 1 \quad = 1 \times 1$ matrix

Example

$$x = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad y = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

$$\langle x, y \rangle = 1 \cdot 3 + 2 \cdot 4 = 3 + 8 = 11$$

$$x^T y = \underset{1 \times 2}{[1 \;\; 2]} \underset{2 \times 1}{\begin{bmatrix} 3 \\ 4 \end{bmatrix}} = 3 + 8 = 11$$

so $\underbrace{\langle x, y \rangle}_{\text{dot product}} = \underbrace{x^T y}_{\text{matrix mult of column vectors } x, y}$

Recall transpose of a matrix: rows become colums and vice versa

$$(A^T)_{ij} = A_{ji}$$

flip order

Example: $A = \begin{bmatrix} 1 & 2 & 4 \\ 9 & 7 & 3 \end{bmatrix}$ $\quad A^T = \begin{bmatrix} 1 & 9 \\ 2 & 7 \\ 4 & 3 \end{bmatrix}$

2×3 $\quad 3 \times 2$

```
A=matrix(QQ, 2, 3, [1, 2, 4, 9, 7, 3])
A
```

```
[1 2 4]
[9 7 3]
```

```
A*(A.transpose())
```

```
[ 21  35]
[ 35 139]
```

$= AA^T$

symmetric matrices

```
(A.transpose())*A
```

```
[82 65 31]
[65 53 29]
[31 29 25]
```

$= A^TA$

Fact: $(AB)^T = B^TA^T$ (not A^TB^T)

$$(A^TA)^T = A^T(A^T)^T = A^TA$$

7.1 Orthogonal Matrices

Definition. A $n \times n$ square matrix is <u>orthogonal</u> if

$$\boxed{I_n = A^T A}$$

$I = \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix}$ identity matrix

The columns (and rows) of an orthogonal matrix form an <u>orthonormal</u> (not just orthogonal) basis

Proof:

If $A = \begin{bmatrix} | & | & & | \\ c_1 & c_2 & \cdots & c_n \\ | & | & & | \end{bmatrix}$ is an orthogonal matrix with columns $c_1, \ldots, c_n$

$$I = A^T A$$

$$\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix} = \begin{bmatrix} - & c_1^T & - \\ - & c_2^T & - \\ & \vdots & \\ - & c_n^T & - \end{bmatrix} \begin{bmatrix} | & | & & | \\ c_1 & c_2 & \cdots & c_n \\ | & | & & | \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & \\ & 1 & \\ & & \ddots \; 1 \end{bmatrix} = \begin{bmatrix} c_1^T c_1 & c_1^T c_2 & \cdots \\ \vdots & & \\ & & c_n^T c_n \end{bmatrix}$$

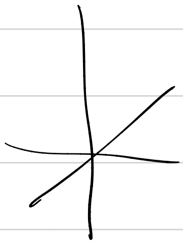
$$1 = c_1^T c_1 = \langle c_1, c_1 \rangle \qquad 0 = c_1^T c_2 = \langle c_1, c_2 \rangle$$

and so on.

So the columns of A form orthonormal basis

Example of orthogonal matrix: (from Anton 7.1)

$$A = \begin{bmatrix} \frac{3}{7} & \frac{2}{7} & \frac{6}{7} \\ -\frac{6}{7} & \frac{3}{7} & \frac{2}{7} \\ \frac{2}{7} & \frac{6}{7} & -\frac{3}{7} \end{bmatrix}$$



$$\frac{3}{7}\cdot\frac{3}{7} + \left(-\frac{6}{7}\right)\left(-\frac{6}{7}\right) + \left(\frac{2}{7}\right)\left(\frac{2}{7}\right) = 1$$

$$A^T A = \begin{bmatrix} \frac{3}{7} & -\frac{6}{7} & \frac{2}{7} \\ \frac{2}{7} & \frac{3}{7} & \frac{6}{7} \\ \frac{6}{7} & \frac{2}{7} & -\frac{3}{7} \end{bmatrix} \begin{bmatrix} \frac{3}{7} & \frac{2}{7} & \frac{6}{7} \\ -\frac{6}{7} & \frac{3}{7} & \frac{2}{7} \\ \frac{2}{7} & \frac{6}{7} & -\frac{3}{7} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

If A is orthogonal,

$$\boxed{A^T A = I}$$

$$(A^T A) A^{-1} = A^{-1}$$

$$\boxed{A^T = A^{-1}}$$

so the inverse of A is easy to compute - it's just its transpose

Also $\quad A A^T = A A^{-1} = I \quad$ so $\quad \boxed{A A^T = I}$

$A^T = A^{-1}$

$I_n = A^T A \iff$ columns of A are orthonormal

$I_n = A A^T \iff$ rows of A are orthonormal

$$A = \begin{bmatrix} - & r_1 & - \\ & \vdots & \\ - & r_n & - \end{bmatrix} \qquad A^T = \begin{bmatrix} | & & | \\ r_1^T & \cdots & r_n^T \\ | & & | \end{bmatrix}$$

$$\begin{aligned} I_n &= AA^T \\ &= \begin{bmatrix} - & r_1 & - \\ & \vdots & \\ - & r_n & - \end{bmatrix} \begin{bmatrix} | & & | \\ r_1^T & \cdots & r_n^T \\ | & & | \end{bmatrix} \end{aligned}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix} = \begin{bmatrix} r_1 r_1^T & r_1 r_2^T & \cdots \\ \vdots & & \end{bmatrix}$$

Theorem: If A is orthogonal, then multiplication by A preserves dot products (and norms) as follows:

$\left(\begin{array}{c} x, y \in \mathbb{R}^n \\ n\times 1 \text{ column vectors} \end{array} \right)$

$$\langle Ax, Ay \rangle = \langle x, y \rangle$$

$$\|Ax\| = \|x\|$$

Proof.

$$\langle Ax, Ay \rangle = \underbrace{(Ax)^T}_{1\times n} \underbrace{Ay}_{n\times 1} = x^T \underbrace{A^T A}_{I} y = x^T y = \langle x, y \rangle$$

matrix mult

$$\|Ax\|^2 = \langle Ax, Ax \rangle = \langle x, x \rangle = \|x\|^2$$

$\square$

7.2 Symmetric matrices, Orthogonal Diagonalization

Definition: A (square) matrix A is <u>symmetric</u> if

$$A = A^T$$

Example: $A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ is symmetric

Example: If B is <u>any</u> $m \times n$ matrix, then $\underset{n\times m}{B^T}\underset{m\times n}{B}$

and BB^T are symmetric matrices, since

using

$$(B^TB)^T = B^T(B^T)^T = B^TB \qquad \left((XY)^T = Y^TX^T \right)$$

$$(BB^T)^T = (B^T)^TB^T = BB^T$$

Here are the important results on symmetric matrices.

Theorem The following are equivalent

a) A is an $n \times n$ symmetric matrix

b) A has an orthonormal basis $v_1, \dots v_n$ of eigenvectors: so if $P = \begin{bmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{bmatrix}$

then
- P is orthogonal ($P^T P = I \Leftrightarrow P^{-1} = P^T$)
- $P^{-1} A P = D$ for $D = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}$

diagonal matrix of eigenvalues

(We say A is orthogonally diagonalizable if conditions in (b) hold)

$$A = PDP^{-1} \Longleftrightarrow AP = PD$$

Theorem. Let A be a symmetric matrix with entries in $\mathbb{R}$ (as opposed to complex numbers)

a) The eigenvalues of A are all real numbers

b) Eigenvectors with different eigenvalues are othogonal

Example (Nicholson 8.2.4)

Let $A = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ -1 & 2 & 5 \end{bmatrix}$ note A is a symmetric matrix.

Find an "orthogonal diagonalization" of A, i.e an orthogonal matrix P and a diagonal matrix D such that

$$P^{-1} A P = D$$

or equivalently since $P^{-1} = P^T$

$$P^T A P = D$$

Solution: Same idea/process as in Chapter 5 of Anton:

just remember this

$P = \begin{bmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{bmatrix}$ $v_1, \ldots v_n$ eigenvectors of A

$D = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$ $\lambda_1, \ldots \lambda_n$ eigenvalues of A

but now just have to make $\{v_1, \ldots v_n\}$ an orthonormal basis (so P will be orthogonal). For symmetric matrices this additional requirement is possible.

Step 1. Find the eigenvalues λ, by finding the characteristic polynomial

$A = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ -1 & 2 & 5 \end{bmatrix}$ $0 = \det(A - \lambda I) = \det \begin{vmatrix} 1-\lambda & 0 & -1 \\ 0 & 1-\lambda & 2 \\ -1 & 2 & 5-\lambda \end{vmatrix}$

some books use $\det(\lambda I - A) = (-1)^3 \det(A - \lambda I)$

(Reminder of why $\det(A-\lambda I)=0$: $\quad Av=\lambda v \quad$ v eigenvector $v\neq\vec{0}$

$$Av-\lambda v=0$$
$$(A-\lambda I)v=0$$

so $A-\lambda I$ can't be invertible because if it was then $v=\vec{0}$ but eigenvectors can't be $\vec{0}$)

$$0=\det(A-\lambda I) = \det\begin{vmatrix} 1-\lambda & 0 & -1 \\ 0 & 1-\lambda & 2 \\ -1 & 2 & 5-\lambda \end{vmatrix}$$

Expanding by minors along top row

$$0 = (1-\lambda)\left[(1-\lambda)(5-\lambda)-2\cdot 2\right] - 0(\quad) + (-1)\left[0\cdot 2-(-1)(1-\lambda)\right]$$

$$= (1-\lambda)\left[(1-\lambda)(5-\lambda)-4\right] - (1-\lambda)$$

$$= (1-\lambda)\left[5-6\lambda+\lambda^2-1\right]$$

$$= (1-\lambda)\left[\lambda^2-6\lambda\right]$$ $\Leftarrow$ "characteristic polynomial"

$$= (1-\lambda)(\lambda-6)\lambda$$ so $\lambda=1,6,0$ are eigenvalues

all of multiplicity one so each eigenspace will be one dimensional & we won't have to run Gram-Schmidt

```
A=matrix(QQ, 3, 3, [1, 0, -1, 0, 1, 2, -1, 2, 5])
A
```

```
[ 1  0 -1]
[ 0  1  2]
[-1  2  5]
```

```
A.characteristic_polynomial()
```

A.charpoly()

```
x^3 - 7*x^2 + 6*x
```

```
A.characteristic_polynomial().factor()
```

```
(x - 6) * (x - 1) * x
```

```
A.eigenvalues()
```

```
[6, 1, 0]
```

<u>Step 2</u> Find an orthonormal basis for the eigenspace E_λ for <u>each</u> eigenvalue λ

$$E_\lambda = \{ v \in \mathbb{R}^n \mid \underbrace{Av = \lambda v}_{(A-\lambda I)v = 0} \}$$

$$E_\lambda = \text{null space of } \underbrace{A - \lambda I}_{n \times n \text{ matrix}}$$

$\underline{\lambda = 6 :}$

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ -1 & 2 & 5 \end{bmatrix} \qquad A - 6I = \begin{bmatrix} 1-6 & 0 & -1 \\ 0 & 1-6 & 2 \\ -1 & 2 & 5-6 \end{bmatrix}$$

$$= \begin{bmatrix} -5 & 0 & -1 \\ 0 & -5 & 2 \\ -1 & 2 & -1 \end{bmatrix}$$

Find null space of this matrix by row reducing it.

Could do by hand

```
A-6*identity_matrix(3)
```

$A - 6I =$

```
[-5  0 -1]
[ 0 -5  2]
[-1  2 -1]
```

```
(A-6*identity_matrix(3)).rref()
```

```
[   1    0  1/5]
[   0    1 -2/5]
[   0    0    0]
```

So if (x_1, x_2, x_3) are the coordinates and $(A-6I)\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$

$x_1 + 0x_2 + 1/5\, x_3 = 0$ $\qquad x_1 = -\frac{1}{5} x_3$

$x_2 - 2/5\, x_3 = 0$ $\qquad x_2 = \frac{2}{5} x_3$

x_3 is free variable $\qquad x_3 = x_3$

$$x_1 = -\frac{1}{5} x_3$$
$$x_2 = \frac{2}{5} x_3 \quad = \begin{bmatrix} -\frac{1}{5} \\ \frac{2}{5} \\ 1 \end{bmatrix} x_3$$
$$x_3 = x_3$$

so E_6-eigenspace is 1-dimensional and spanned by

$\begin{bmatrix} -\frac{1}{5} \\ \frac{2}{5} \\ 1 \end{bmatrix}$, can rescale it by 5 to $v_1 = \begin{bmatrix} -1 \\ 2 \\ 5 \end{bmatrix}$ (i.e. choose $x_3 = 5$)

recall $P = \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix}$ so if all we want is to diagonalize A,

can start with $P = \begin{bmatrix} -1 & & \\ 2 & & \\ 5 & & \end{bmatrix}$

But we want to orthogonally diagonalize – i.e. P must be an orthogonal matrix – so columns must be orthonormal. So we need to normalize/rescale v_1 to have norm 1:

$v_1 = \begin{bmatrix} -1 \\ 2 \\ 5 \end{bmatrix}$ $\qquad \|v_1\|^2 = (-1)^2 + 2^2 + 5^2 = 1+4+25=30$

$\qquad \|v_1\| = \sqrt{30}$

unit eigenvector $\frac{1}{\sqrt{30}} v_1 = \begin{bmatrix} -\frac{1}{\sqrt{30}} \\ \frac{2}{\sqrt{30}} \\ 5/\sqrt{30} \end{bmatrix}$

$(\lambda=6)$

so $P = \begin{bmatrix} -\frac{1}{\sqrt{30}} & & \\ \frac{2}{\sqrt{30}} & & \\ 5/\sqrt{30} & & \end{bmatrix}$

or just write it $P = \begin{bmatrix} \frac{1}{\sqrt{30}} v_1 & & \end{bmatrix}$ where $v_1 = \begin{bmatrix} -1 \\ 2 \\ 5 \end{bmatrix}$

$\underline{\lambda = 1}$ Repeat same process

```
A-1*identity_matrix(3)
```

```
[ 0  0 -1]
[ 0  0  2]
[-1  2  4]
```

$= A - 1 \cdot I$

```
(A-1*identity_matrix(3)).rref()
```

```
[ 1 -2  0]
[ 0  0  1]
[ 0  0  0]
```

pivot variables x_1, x_3

free variables: x_2

$x_1 - 2x_2 + 0x_3 = 0$

$x_3 = 0$

Solve for pivot variables

$x_1 = 2x_2$

$x_2 = x_2$

$x_3 = 0$

$= \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} x_2$

so $v_2 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$ spans $E_{\lambda=1}$ eigenspace.

$\|v_2\|^2 = 2^2 + 1^2 + 0^2 = 5$

so $\frac{1}{\sqrt{5}} v_2$ is unit eigenvector

$$P = \begin{bmatrix} \frac{1}{\sqrt{30}} v_1 & \frac{1}{\sqrt{5}} v_2 & \_\_\_ \end{bmatrix} \qquad v_1 = \begin{bmatrix} -1 \\ 2 \\ 5 \end{bmatrix} \quad v_2 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

Notice v_1 & v_2 (and therefore $\frac{1}{\sqrt{30}} v_1$ & $\frac{1}{\sqrt{5}} v_2$) are already orthogonal:

$$\begin{aligned} \langle v_1, v_2 \rangle &= (-1)(2) + (2)(1) + (5)(0) \\ &= -2 + 2 + 0 \\ &= 0 \end{aligned}$$

and this is not an accident – the theorem says that for <u>symmetric matrices</u>, eigenvectors with <u>different</u> eigenvalues are <u>orthogonal</u>.

<u>$\lambda = 0$</u> Repeat

$A - \lambda I = A - 0I = A$ so we just need A.rref()

```
A.rref()
```

```
[ 1  0 -1]
[ 0  1  2]
[ 0  0  0]
```

$\rightarrow x_1 + 0x_2 - x_3 = 0$

$\rightarrow x_2 + 2x_3 = 0$

$x_1 = x_3$

$x_2 = -2x_3 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} x_3$

$x_3 = x_3$

x_1, x_2 pivots

x_3 free

$$v_3 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \qquad \|v_3\|^2 = (-1)^2 + 2^2 + 1^2 = 1+4+1 = 6$$

so $\frac{1}{\sqrt{6}} v_3$ is unit eigenvector

$$\boxed{P = \begin{bmatrix} \frac{1}{\sqrt{30}} v_1 & \frac{1}{\sqrt{5}} v_2 & \frac{1}{\sqrt{6}} v_3 \end{bmatrix}} \quad v_1 = \begin{bmatrix} -1 \\ 2 \\ 5 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

$\lambda = 6 \qquad \lambda = 1 \qquad \lambda = 0$

once again notice v_3 is already orthogonal to v_1 & v_2:

$$\boxed{D = \begin{bmatrix} 6 & & \\ & 1 & \\ & & 0 \end{bmatrix}}$$

$$\langle v_1, v_3 \rangle = (-1)\cdot(1) + (2)(-2) + (5)(1)$$
$$= -1 - 4 + 5$$
$$= 0$$

$$\langle v_2, v_3 \rangle = 2\cdot 1 + 1\cdot 2$$

$P^{-1}AP = D$ & P is orthogonal.

A two or three step solution in SageMath

Here again is the original problem

Let $A = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ -1 & 2 & 5 \end{bmatrix}$ note A is a symmetric matrix.

Find an "orthogonal diagonalization" of A, i.e. an orthogonal matrix P and a diagonal matrix D such that

$$P^{-1} A P = D$$

or equivalently since $P^{-1} = P^T$

$$P^T A P = D$$

almost P but columns need to be normalized

```
A.eigenmatrix_right()
```

$$\left(\underbrace{\boxed{\begin{pmatrix} 6 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}}}_{D}, \begin{pmatrix} 1 & 1 & 1 \\ -2 & \frac{1}{2} & -2 \\ -5 & 0 & 1 \end{pmatrix} \right)$$

diagonal entries are eigenvalues

columns are eigenvectors

```
B=A.eigenmatrix_right()[1]
```

```
B
```

$$\begin{pmatrix} 1 & 1 & 1 \\ -2 & \frac{1}{2} & -2 \\ -5 & 0 & 1 \end{pmatrix}$$

```
def normalize_columns(M):
    return matrix([v/v.norm() for v in M.columns()]).transpose()
```

converts rows to columns

makes a list of vectors that normalizes each

```
P=normalize_columns(B)
P
```

$$P \longrightarrow \boxed{\begin{pmatrix} \frac{1}{30}\sqrt{30} & \frac{2}{5}\sqrt{5} & \frac{1}{6}\sqrt{6} \\ -\frac{1}{15}\sqrt{30} & \frac{1}{5}\sqrt{5} & -\frac{1}{3}\sqrt{6} \\ -\frac{1}{6}\sqrt{30} & 0 & \frac{1}{6}\sqrt{6} \end{pmatrix}}$$

```
P.T*A*P
```

$P^T A P = D$

$$\begin{pmatrix} 6 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Another example : $\underbrace{\text{Orthogonally diagonalize}}_{\text{find } P \,\&\, D}$ $A = \begin{pmatrix} 8 & -2 & 2 \\ -2 & 5 & 4 \\ 2 & 4 & 5 \end{pmatrix}$

```
A
```

$$\begin{pmatrix} 8 & -2 & 2 \\ -2 & 5 & 4 \\ 2 & 4 & 5 \end{pmatrix}$$

```
A.charpoly().factor()
```

multiplicity of $\lambda = 9$ eigenvalue is 2

$$x \cdot (x-9)^2$$

```
A.eigenmatrix_right()
```

$$\left(\begin{pmatrix} 0 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{pmatrix}, \begin{array}{c} \begin{array}{ccc} v_1 & v_2 & v_3 \end{array} \\ \begin{pmatrix} 1 & 1 & 0 \\ 2 & 0 & 1 \\ -2 & \frac{1}{2} & 1 \end{pmatrix} \end{array} \right)$$

v_2, v_3 are not orthogonal

$\langle v_2, v_3 \rangle = \frac{1}{2}$

so we need to apply gram-schmidt

```
def gs_columns(M):
    N=M.transpose().gram_schmidt()[0].transpose() # makes the columns of M orthogonal,
                                                    #gram_schmidt returns a pair of matrices, we want the first one, so use [0]
                                                   #we have to inserts some transposes in gram_schmidt orthoginalizes the rows
    return matrix([v/v.norm() for v in N.columns()]).transpose() # N has orthogonal columns we need to make columns  orthonorma
```

```
gs_columns(A.eigenmatrix_right()[1])
```

$$P = \begin{pmatrix} \frac{1}{3} & \frac{2}{5}\sqrt{5} & -\frac{2}{3}\sqrt{\frac{1}{5}} \\ \frac{2}{3} & 0 & \frac{5}{3}\sqrt{\frac{1}{5}} \\ -\frac{2}{3} & \frac{1}{5}\sqrt{5} & \frac{4}{3}\sqrt{\frac{1}{5}} \end{pmatrix}$$

Check $P^t A P = D$

```
P=gs_columns(B)
P
```

$$P = \begin{pmatrix} \frac{1}{3} & \frac{2}{5}\sqrt{5} & -\frac{2}{3}\sqrt{\frac{1}{5}} \\ \frac{2}{3} & 0 & \frac{5}{3}\sqrt{\frac{1}{5}} \\ -\frac{2}{3} & \frac{1}{5}\sqrt{5} & \frac{4}{3}\sqrt{\frac{1}{5}} \end{pmatrix}$$

```
P.T*A*P
```

✓ $D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{pmatrix}$

Proofs of the earlier theorems

<u>Theorem</u> The following are equivalent

a) A is an $n \times n$ symmetric matrix

b) A has an <u>orthonormal</u> basis $v_1, \dots v_n$ of <u>eigenvectors</u>: so if $P = [v_1 \cdots v_n]$

then
- P is orthogonal ($P^T P = I \Leftrightarrow P^{-1} = P^T$)
- $P^{-1} A P = D$ for $D = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}$

diagonal matrix of eigenvalues

(we say A is <u>orthogonally diagonalizable</u> if conditions in (b) hold)

Proof of theorem: $b \Rightarrow a$ is the easy direction

Assume $P^{-1} A P = D$ with $P^{-1} = P^T$ (i.e P is orthogonal)

$$A = PDP^{-1}$$

$$A = PDP^T$$

so $A^T = (PDP^T)^T = (P^T)^T D^T \cdot P^T = PDP^T = A$

so A is symmetric.

$a \Rightarrow b$ See Nicholson 8.2.2 p.426

Proof of earlier theorem

Theorem. Let A be a symmetric matrix with entries in $\mathbb{R}$ (as opposed to complex numbers)

a) The eigenvalues of A are all real numbers

b) Eigenvectors with different eigenvalues are othogonal

Proof: a) suppose $Ax = \lambda x$ for $\lambda \in \mathbb{C}$, $x \in \mathbb{C}^n$
Compute $\bar{x}^T A x$ in two ways:

1) $\bar{x}^T (Ax) = \bar{x}^T \lambda x = \lambda \bar{x}^T x$

2) $\bar{x}^T A x \underset{\text{since } A^T = A}{=} (A\bar{x})^T x = (\bar{A}\bar{x})^T x = \bar{\lambda} \bar{x}^T x$

since $A = \bar{A}$

So $\lambda = \bar{\lambda}$ so λ is real.

b) suppose x, y are eigenvectors with distinct eigen-values:

$$Ax = \lambda_1 x \quad \& \quad Ay = \lambda_2 y \quad \text{with } \lambda_1 \neq \lambda_2$$

$$(\lambda_1 x)^T y = (Ax)^T y = x^T A^T y \underset{A^T = A}{=} x^T A y = x^T(\lambda_2 y) = \lambda_2 x^T y$$

since A is symmetric

so $(\lambda_1 - \lambda_2) x^T y = 0$

$$x^T y = 0$$

$a \Rightarrow b$ is harder. Assume A is symmetric.

We first show A is diagonalizable - i.e we can find enough eigenvectors for each eigenvalue λ_0

Let $E_{\lambda_0} = \{ v \in \mathbb{R}^n \mid (A - \lambda_0) v = 0 \}$ be the λ_0-eigen space.

Let k be its dimension, so by Gram-Schmidt we can find an orthonormal basis

$$u_1, u_2, \ldots u_k \text{ of } E_{\lambda_0}.$$

Extend to an ortho<u>normal</u> basis (by Gram-Schmidt)

$$u_1, u_2, \ldots u_n, \; u_{k+1}, \ldots u_n \text{ of } \mathbb{R}^n$$

Let $P = \begin{bmatrix} | & | & & | \\ u_1 & u_2 & \cdots & u_n \\ | & | & & | \end{bmatrix}$.

P is orthogonal since since the $u_1, \ldots u_n$ are orthonormal

Then
$$AP = A\begin{bmatrix} u_1 & u_2 & \cdots & u_n \end{bmatrix}$$

$$= \begin{bmatrix} Au_1 & Au_2 & \cdots & Au_n \end{bmatrix}$$

$$= \begin{bmatrix} \lambda_0 u_1 & \lambda_0 u_2 & \cdots & \lambda_0 u_k & Au_{k+1} & \cdots & Au_n \end{bmatrix}$$

$$AP = \begin{bmatrix} u_1 & u_2 & \cdots & u_n \end{bmatrix} \begin{array}{c} \overbrace{\hphantom{\lambda_0 \ddots \lambda_0}}^{k} \; \overbrace{\hphantom{X}}^{n-k} \\ \left[\begin{array}{ccc|c} \lambda_0 & & & \\ & \ddots & & X \\ & & \lambda_0 & \\ \hline & & & Y \end{array}\right] \end{array} \begin{array}{l} \}\, k \\ \}\, n-k \end{array}$$

$$AP = P \left[\begin{array}{c|c} \lambda_0 I_k & X \\ \hline 0 & Y \end{array}\right]$$

X $k \times (n-k)$ matrix

Y $(n-k) \times (n-k)$ matrix

$$P^{-1}AP = \left[\begin{array}{c|c} \lambda_0 I_k & X \\ \hline 0 & Y \end{array}\right]$$

(since $P^{-1} = P^T$ since P is orthogonal)

$$P^TAP = \left[\begin{array}{c|c} \lambda_0 I_k & X \\ \hline 0 & Y \end{array}\right]$$

Now $P^T A P$ is symmetric since A is symmetric

$$(P^T A P)^T = P^T A^T (P^T)^T = P^T A P$$

So in $$P^T A P = \left[\begin{array}{c|c} \lambda_0 I_k & X \\ \hline 0 & Y \end{array}\right]$$

RHS must be symmetric so $X = 0$ & $Y = Y^T$.

Now use induction on dimension/size

7.2 # 7 $A = \begin{bmatrix} 6 & 2\sqrt{3} \\ 2\sqrt{3} & 7 \end{bmatrix}$

Find orthogonal P such that $P^{-1}AP$ is diagonal. D

Remember: from Ch 5

$P =$ columns are eigenvectors

$D =$ diagonal entries are eigenvalues

$$A - \lambda I = \begin{bmatrix} 6-\lambda & 2\sqrt{3} \\ 2\sqrt{3} & 7-\lambda \end{bmatrix}$$

$$0 = \det(A - \lambda I) = (6-\lambda)(7-\lambda) - 2\sqrt{3}\cdot 2\cdot\sqrt{3}$$

$$= 42 - 13\lambda + \lambda^2 - 12$$

$$0 = \lambda^2 - 13\lambda + 30$$

$$0 = (\lambda-3)(\lambda-10)$$

$\lambda - 3 = 0$ $\lambda = 10$

$\lambda = 3$

eigenvalues

$D = \begin{bmatrix} 3 & \\ & 10 \end{bmatrix}$

$\lambda = 3$ Find eigenvector(s)

$$A - 3I = \begin{bmatrix} 6-3 & 2\sqrt{3} \\ 2\sqrt{3} & 7-3 \end{bmatrix} = \begin{bmatrix} 3 & \sqrt{12} \\ \sqrt{12} & 4 \end{bmatrix} v = 0$$

Find RREF

$$\begin{bmatrix} 3 & \sqrt{12} \\ 1 & \frac{4}{\sqrt{12}} \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 \\ 1 & \frac{4}{\sqrt{12}} \end{bmatrix}$$

$$\frac{4}{\sqrt{12}} = \frac{4}{2\sqrt{3}} = \frac{2}{\sqrt{3}} = \frac{2}{3}\sqrt{3}$$

$$\begin{bmatrix} 1 & \frac{2}{3}\sqrt{3} \\ 0 & 0 \end{bmatrix}$$

$x_1 + \frac{2}{3}\sqrt{3}\, x_2 = 0$

x_2 free variable

$$\begin{aligned} x_1 &= -\tfrac{2}{3}\sqrt{3}\, x_2 \\ x_2 &= x_2 \end{aligned} = \begin{bmatrix} -\frac{2}{3}\sqrt{3} \\ 1 \end{bmatrix} x_2$$

so this is an eigenvector. but not of norm 1

$$\text{norm}^2 = \left(-\tfrac{2}{3}\sqrt{3}\right)^2 + 1$$

$$= \frac{4}{9} \cdot 3 + 1$$

$$= \frac{4}{3} + 1 = \frac{7}{3}$$

$$v = \begin{bmatrix} -\frac{2}{3}\sqrt{3} \\ 1 \end{bmatrix} \qquad \|v\| = \frac{\sqrt{7}}{\sqrt{3}}$$

$$\frac{v}{\|v\|} = \begin{bmatrix} -\frac{2}{\sqrt{7}} \\ \frac{\sqrt{3}}{\sqrt{7}} \end{bmatrix}$$ is unit eigenvector.

Do the same process for $\lambda = 10$.

$$\rightarrow \begin{bmatrix} \frac{\sqrt{3}}{\sqrt{7}} \\ \frac{2}{\sqrt{7}} \end{bmatrix}$$

$$P = \begin{bmatrix} -\frac{2}{\sqrt{7}} & \frac{\sqrt{3}}{\sqrt{7}} \\ \frac{\sqrt{3}}{\sqrt{7}} & \frac{2}{\sqrt{7}} \end{bmatrix}$$

We are going to explain <u>Spectral</u> <u>Decomposition</u> of a symmetric matrix.

Let A be a <u>symmetric matrix.</u>

So it has an "<u>orthogonal diagonalization</u>":

$$P^{-1}AP = D \quad \left(\text{or} \quad P^T A P = D\right) \quad \left(P^{-1} = P^T \text{ since } P \text{ is orthogonal}\right)$$

$$AP = PD$$

$$A = PDP^{-1}$$

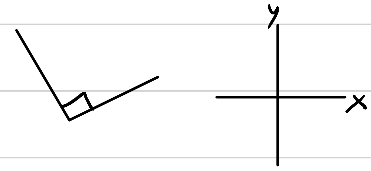
$$\boxed{A = PDP^T}$$

eigenvalues

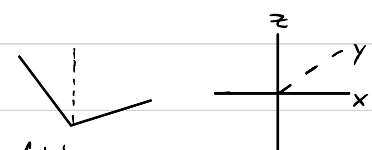
$$D = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} \qquad P = \begin{bmatrix} | & & | \\ u_1 & \cdots & u_n \\ | & & | \end{bmatrix}$$

columns $u_1, u_2, \ldots u_n$ are the eigenvectors and form an <u>orthonormal</u> <u>basis:</u>

2D:



3D:



(like x, y, z coordinate axis)

We saw that having an orthonormal basis $u_1, u_2, \ldots, u_n$ made various formulas simplify since $\langle u_i, u_j \rangle = 0$ ($i \neq j$) and $\langle u_i, u_i \rangle = 1$ led to terms "dissapearing".

Given any vector $v \in \mathbb{R}^n$

$$v = \langle v, u_1 \rangle u_1 + \langle v, u_2 \rangle u_2 + \cdots + \langle v, u_n \rangle u_n$$

Anton 7.2 Example $A = \begin{bmatrix} 1 & 2 \\ 2 & -2 \end{bmatrix}$ symmetric matrix

```
A=matrix(QQ, 2, 2, [1, 2, 2, -2])
```

Let's find orthogonal diagonalization of A.

```
A
```

$$\begin{pmatrix} 1 & 2 \\ 2 & -2 \end{pmatrix}$$

```
A.eigenmatrix_right()
```

$$\left(\underbrace{\begin{pmatrix} 2 & 0 \\ 0 & -3 \end{pmatrix}}_{D}, \begin{pmatrix} 1 & 1 \\ \frac{1}{2} & -2 \end{pmatrix} \right)$$

orthogonal eigenvectors but not orthonormal

```
def normalize_columns(M):  #given a matrix M, returns columns of M normalized
    return matrix([v/v.norm() for v in M.columns()]).transpose()
```

```
normalize_columns(A.eigenmatrix_right()[1])  #the [1] picks out the second matrix in A.eigenmatrix_right()
```

$$\boxed{P = \begin{pmatrix} \frac{2}{5}\sqrt{5} & \frac{1}{5}\sqrt{5} \\ \frac{1}{5}\sqrt{5} & -\frac{2}{5}\sqrt{5} \end{pmatrix}}$$

columns u_1 u_2

now The eigenvectors are orthonormal

$$A = \begin{bmatrix} 1 & 2 \\ 2 & -2 \end{bmatrix} = PDP^T$$

$$= \begin{bmatrix} u_1 & u_2 \end{bmatrix} \begin{bmatrix} 2 & \\ & -3 \end{bmatrix} \begin{bmatrix} u_1^T \\ u_2^T \end{bmatrix}$$

$$= \begin{bmatrix} | & | \\ 2u_1 & -3u_2 \\ | & | \end{bmatrix} \begin{bmatrix} - & u_1^T & - \\ - & u_2^T & - \end{bmatrix}$$

Spectral Decomposition of A $\rightarrow$ $\boxed{A = 2u_1u_1^T - 3u_2u_2^T}$

$$= 2\begin{bmatrix} \frac{4}{5} & \frac{2}{5} \\ \frac{2}{5} & \frac{1}{5} \end{bmatrix} + 3\begin{bmatrix} \frac{1}{5} & -\frac{2}{5} \\ -\frac{2}{5} & \frac{4}{5} \end{bmatrix}$$

$$u_1 = \begin{bmatrix} \frac{2}{5}\sqrt{5} \\ \frac{1}{5}\sqrt{5} \end{bmatrix} \qquad u_2 = \begin{bmatrix} \frac{1}{5}\sqrt{5} \\ -\frac{2}{5}\sqrt{5} \end{bmatrix}$$

$$\underset{2\times 2}{u_1 u_1^T} = \underset{2\times 1}{\begin{bmatrix} \frac{2}{5}\sqrt{5} \\ \frac{1}{5}\sqrt{5} \end{bmatrix}} \underset{1\times 2}{\begin{bmatrix} \frac{2}{5}\sqrt{5} & \frac{1}{5}\sqrt{5} \end{bmatrix}} = \underset{2\times 2}{\begin{bmatrix} \frac{2}{5}\sqrt{5}\cdot\frac{2}{5}\sqrt{5} & \frac{2}{5}\sqrt{5}\cdot\frac{1}{5}\sqrt{5} \\ \frac{1}{5}\sqrt{5}\cdot\frac{2}{5}\sqrt{5} & \frac{1}{5}\sqrt{5}\cdot\frac{1}{5}\sqrt{5} \end{bmatrix}}$$

$$u_1 u_1^T = \begin{bmatrix} \frac{4}{5} & \frac{2}{5} \\ \frac{2}{5} & \frac{1}{5} \end{bmatrix}$$

Sage code to compute $u_1 u_1^T$:

```
P=normalize_columns(A.eigenmatrix_right()[1])  #the [1] picks out the second matrix in A.eigenmatrix_right()
```

```
P
```

$$\begin{pmatrix} \frac{2}{5}\sqrt{5} & \frac{1}{5}\sqrt{5} \\ \frac{1}{5}\sqrt{5} & -\frac{2}{5}\sqrt{5} \end{pmatrix}$$

```
u1=P.columns()[0]  #[0] picks out the first column
u1  #row vector, no longer a column matrix
u1.column() # gives a 2x1 matrix i.e. column vector/matrix
```

$$\left(\frac{2}{5}\sqrt{5}, \frac{1}{5}\sqrt{5}\right)$$

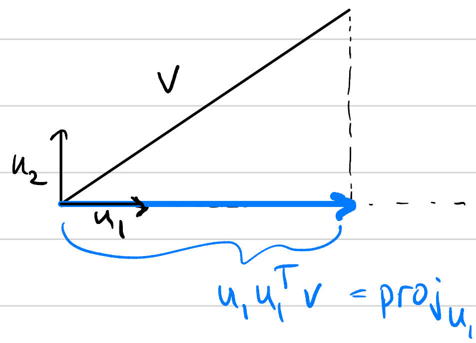
$$\begin{pmatrix} \frac{2}{5}\sqrt{5} \\ \frac{1}{5}\sqrt{5} \end{pmatrix}$$

```
u1.column()*matrix(u1)
```

$$\begin{pmatrix} \frac{4}{5} & \frac{2}{5} \\ \frac{2}{5} & \frac{1}{5} \end{pmatrix}$$

Similar computation $u_2 u_2^T$

Key Insight: multiplication by $u_1 u_1^T$ of any vector v, i.e. $\underbrace{(u_1 u_1^T)}_{2\times 2}\underbrace{v}_{\substack{2\times 1\\ \text{column vector}}}$ gives the projection $\text{proj}_{u_1} v$ of v onto u_1:



Why? Let v be any vector.

Since $\{u_1, u_2\}$ form an orthonormal basis

$$v = \langle v, u_1 \rangle u_1 + \langle v, u_2 \rangle u_2 .$$

Then $$(u_1 u_1^T) v = \underbrace{\langle v, u_1 \rangle}_{\text{scalar}} \underbrace{(u_1 u_1^T)}_{2\times 2} \underbrace{u_1}_{2\times 1} + \underbrace{\langle v, u_2 \rangle}_{\text{scalar}} \underbrace{(u_1 u_1^T)}_{2\times 2} \underbrace{u_2}_{2\times 1}$$

$$u_1 (\underbrace{u_1^T u_1}_{1}) \quad \text{since } \langle u_1, u_1 \rangle = 1 \qquad u_1 (\underbrace{u_1^T u_2}_{0}) \quad \text{since } \langle u_1, u_2 \rangle = 0$$

$$(u_1 u_1^T) v = \langle v, u_1 \rangle u_1$$
$$= \text{proj}_{u_1} v \;!$$

Similarly, $u_2 u_2^T$ is projection onto u_2

Spectral decomposition $\qquad A = \begin{bmatrix} 1 & 2 \\ 2 & -2 \end{bmatrix}$

$$\boxed{A = 2u_1u_1^T - 3u_2u_2^T}$$

So for any vector v, $\quad Av = \underbrace{2(u_1u_1^T)v}_{\substack{2\cdot \text{ projection of } Av \\ \text{onto } u_1}} - \underbrace{3(u_2u_2^T)v}_{\substack{-3\cdot \text{ projection of } Av \\ \text{onto } u_2}}$

example take $v = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$A = 2\begin{bmatrix} \frac{4}{5} & \frac{2}{5} \\ \frac{2}{5} & \frac{1}{5} \end{bmatrix} + (-3)\begin{bmatrix} \frac{1}{5} & -\frac{2}{5} \\ -\frac{2}{5} & \frac{4}{5} \end{bmatrix}$$

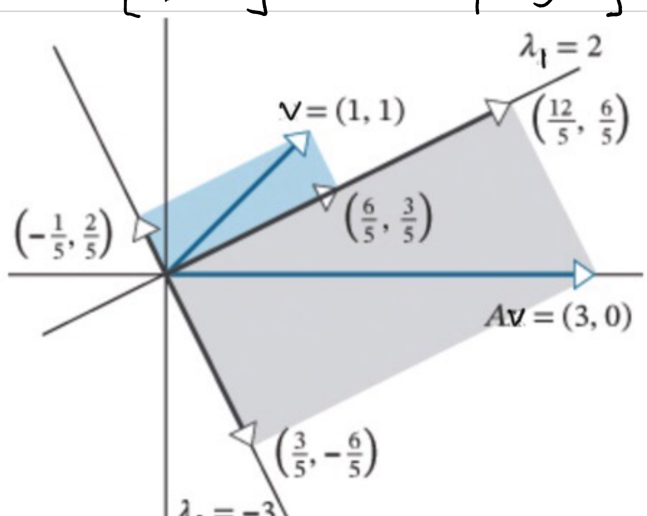
$$Av = \begin{bmatrix} 1 & 2 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = 2\begin{bmatrix} \frac{4}{5} & \frac{2}{5} \\ \frac{2}{5} & \frac{1}{5} \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + (-3)\begin{bmatrix} \frac{1}{5} & -\frac{2}{5} \\ -\frac{2}{5} & \frac{4}{5} \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$Av = \begin{bmatrix} 3 \\ 0 \end{bmatrix} = 2\begin{bmatrix} \frac{6}{5} \\ \frac{3}{5} \end{bmatrix} + (-3)\begin{bmatrix} -\frac{1}{5} \\ \frac{2}{5} \end{bmatrix}$$

$$Av = \begin{bmatrix} \frac{12}{5} \\ \frac{6}{5} \end{bmatrix} + \begin{bmatrix} \frac{3}{5} \\ -\frac{6}{5} \end{bmatrix}$$

illustrates the geometric effect of the eigenvalues

Anton 7.2
Example 2



The 2×2 example generalizes to any $n \times n$ symmetric matrix A.

From an orthogonal diagonalization of A

$A = PDP^T$ $\qquad P = \begin{bmatrix} | & & | \\ u_1 & \cdots & u_n \\ | & & | \end{bmatrix}$ $\qquad D = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$

orthonormal basis of eigenvectors

$$A = \begin{bmatrix} u_1 & \cdots & u_n \end{bmatrix} \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} \begin{bmatrix} - u_1^T - \\ - u_2^T - \\ - u_n^T - \end{bmatrix}$$

$n \times 1$ column vector

$$A = \begin{bmatrix} | & & | \\ \lambda_1 u_1 & \cdots & \lambda_n u_n \\ | & & | \end{bmatrix} \begin{bmatrix} - u_1^T - \\ - u_2^T - \\ - u_n^T - \end{bmatrix}$$

$$\boxed{A = \lambda_1 \underbrace{u_1 u_1^T}_{\substack{n \times n \\ \text{matrix}}} + \cdots + \lambda_n (u_n u_n^T)}$$

Spectral decomposition of A

projection onto u_1