# Anton 6.3 Gram-Schmidt process and $A = QR$ matrix factorization

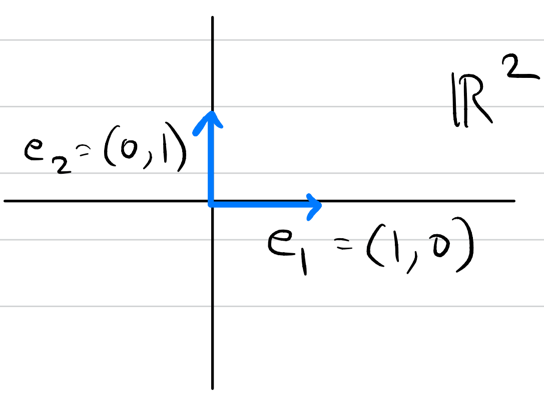
Let $V$ be a vector space with inner product $\langle\,,\,\rangle$
(e.g $V = \mathbb{R}^n$ with dot product also called Euclidean product)

Definition: A set $S = \{q_1, q_2, \ldots, q_n\}$ of vectors is an <u>orthogonal set</u> (or just orthogonal) if $\langle q_i, q_j \rangle = 0$ for all $i \neq j$

<u>orthonormal set</u> in addition to $\langle q_i, q_j \rangle = 0$ for $i \neq j$, each $q_i$ is a unit vector i.e. $\langle q_i, q_i \rangle = 1$ for all $i$.

(Once you have an orthogonal set, it is easy to create an <u>orthonormal</u> set - just divide each vector by its norm to create a unit vector).

Standard basis $\{e_1, e_2, \ldots e_n\}$ of $\mathbb{R}^n$ is an orthonormal basis for $\langle\,,\,\rangle$ = dot product

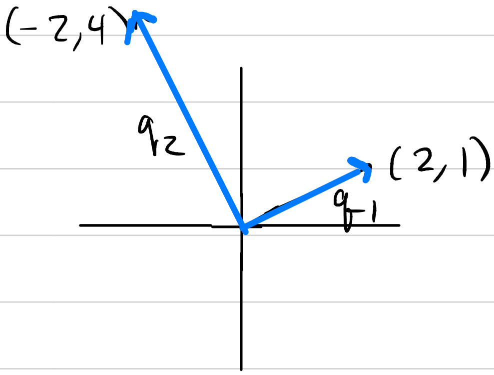



$\langle e_1, e_2 \rangle = 1 \cdot 0 + 0 \cdot 1 = 0$

$\langle e_1, e_1 \rangle = 1^2 + 0^2 = 1$

$\langle e_2, e_2 \rangle = 0^2 + 1^2 = 1$

Example $\quad \mathbb{R}^2 \quad \langle\, , \rangle$ = dot product




$q_1 = (2,1) \qquad q_2 = (-2,4)$

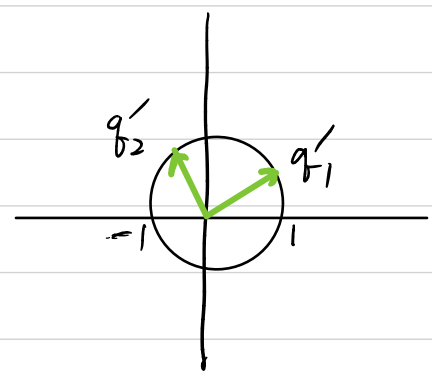
$\{q_1, q_2\}$ are orthogonal but not orthonormal

$\langle q_1, q_2 \rangle = 2(-2) + 1 \cdot 4 = 0$

$\langle q_1, q_1 \rangle = 2^2 + 1^2 = 5 \Rightarrow \|q_1\| = \sqrt{5}$

$\langle q_2, q_2 \rangle = (-2)^2 + 4^2 = 20 \Rightarrow \|q_2\| = \sqrt{20}$

Let's normalize $q_1$ & $q_2$ to get an orthonormal set $\{q_1', q_2'\}$




$q_1' = \frac{1}{\sqrt{5}} q_1 = \left(\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right)$

& $q_2' = \frac{1}{\sqrt{20}}(-2,4) = \left(\frac{-2}{\sqrt{20}}, \frac{4}{\sqrt{20}}\right)$

$\{q_1', q_2'\}$ are orthonormal

$$\|q_1'\|^2 = \left(\frac{2}{\sqrt{5}}\right)^2 + \left(\frac{1}{\sqrt{5}}\right)^2 = \frac{2^2}{5} + \frac{1^2}{5} = \frac{5}{5} = 1.$$

similarly $\|q_2'\|^2 = 1$

# Why is an orthogonal/orthonormal basis better?

It is easy to find coordinates of a vector $v$ with respect to an orthonormal basis $\{q_1, q_2, \dots, q_n\}$, by using the inner product (instead of solving system of linear equations)

**Fact:** If $\{q_1, \dots, q_n\}$ is an orthogonal set and $v \in \text{Span}\{q_1, \dots, q_n\}$

then $$v = \frac{\langle v, q_1\rangle}{\|q_1\|^2} q_1 + \frac{\langle v, q_2\rangle}{\|q_2\|^2} q_2 + \dots + \frac{\langle v, q_n\rangle}{\|q_n\|^2} q_n$$

If $\{q_1, \dots, q_n\}$ is an ortho<u>normal</u> set and $v \in \text{Span}\{q_1, \dots, q_n\}$

then $$v = \langle v, q_1\rangle q_1 + \langle v, q_2\rangle q_2 + \dots + \langle v, q_n\rangle q_n$$

Proof: $v = c_1 q_1 + c_2 q_2 + \dots + c_n q_n$ for some numbers $c_1, c_2, \dots, c_n$

We have to find what $c_1, c_2, \dots, c_n$ are

$$\begin{aligned}
\langle v, q_1\rangle &= \langle c_1 q_1 + c_2 q_2 + \dots + c_n q_n, q_1\rangle \\
&= \langle c_1 q_1, q_1\rangle + \langle c_2 q_2, q_1\rangle + \dots + \langle c_n q_n, q_1\rangle \\
&= c_1 \underbrace{\langle q_1, q_1\rangle}_{\|q_1\|^2} + c_2 \underbrace{\langle q_2, q_1\rangle}_{0} + \dots + c_n \underbrace{\langle q_n, q_1\rangle}_{0}
\end{aligned}$$

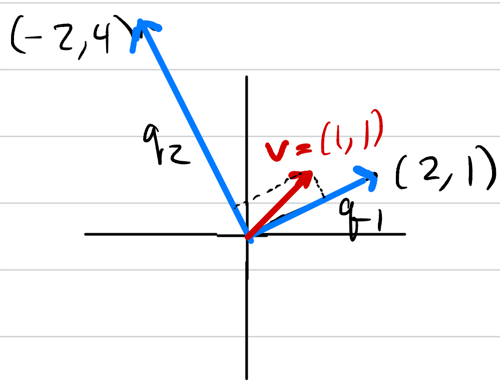
(by orthogonal)

$$\langle v, q_1\rangle = c_1 \|q_1\|^2 \qquad \text{solve for } c_1$$

$$\frac{\langle v, q_1\rangle}{\|q_1\|^2} = c_1 ,$$ is the first coordinate of $v$ in basis $\{q_1, q_2, \cdots q_n\}$.

similarly $c_2 = \dfrac{\langle v, q_2\rangle}{\|q_2\|^2}$, etc. $\square$

Example. Express $V = (1,1)$ as a linear combination of $q_1$ & $q_2$ from before



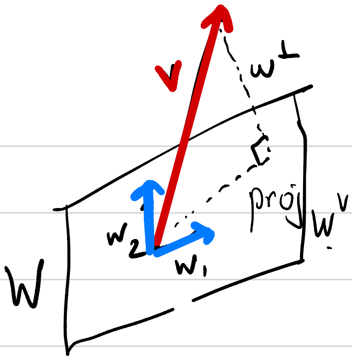

Solution: $\{q_1, q_2\}$ is orthogonal (but not orthonormal) so we use formula

$$V = \frac{\langle v, q_1\rangle}{\|q_1\|^2} q_1 + \frac{\langle V, q_2\rangle}{\|q_2\|^2} q_2$$

$$V = \frac{(1,1)\cdot(2,1)}{2^2+1^2} q_1 + \frac{(1,1)\cdot(-2,4)}{(-2)^2+4^2} q_2$$

$$= \frac{1\cdot 2 + 1\cdot 1}{5} q_1 + \frac{1\cdot(-2) + 1\cdot(4)}{20} q_2$$

$$V = \boxed{\frac{3}{5} q_1 + \frac{1}{10} q_2}$$




Orthogonal

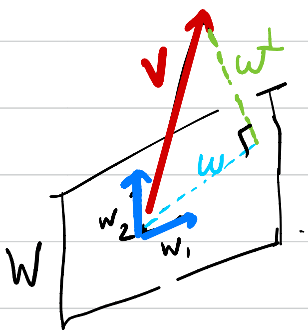
# Projection of a vector $\mathbf{v}$ onto a subspace $W = \text{Span}\{w_1, w_2, \ldots w_r\}$ where

$w_1, \ldots w_r$ is an orthogonal set of vectors,

Let $W^\perp = \{v \in V \mid \langle v, w \rangle = 0 \text{ for every } w \in W\}$ "orthogonal complement."

For $\mathbf{v} \in V$ we want to express $\mathbf{v} = \underbrace{w}_{\substack{\text{proj}_W v \\ \text{in } W}} + w^\perp$

in $W^\perp$




The formulas are:

$$\boxed{\text{proj}_W v = w = \frac{\langle v, w_1 \rangle}{\|w_1\|^2} w_1 + \frac{\langle v, w_2 \rangle}{\|w_2\|^2} w_2 + \cdots + \frac{\langle v, w_r \rangle}{\|w_r\|^2} w_r}$$

$$\boxed{w^\perp = v - \frac{\langle v, w_1 \rangle}{\|w_1\|^2} w_1 - \frac{\langle v, w_2 \rangle}{\|w_2\|^2} w_2 - \cdots - \frac{\langle v, w_r \rangle}{\|w_r\|^2} w_r}$$

If $w_1, w_2, \ldots, w_r$ are ortho<u>normal</u>, the formulas become

$$\boxed{\text{proj}_W v = w = \langle v, w_1 \rangle w_1 + \langle v, w_2 \rangle w_2 + \cdots + \langle v, w_r \rangle w_r}$$

$$\boxed{w^\perp = v - \langle v, w_1 \rangle w_1 - \langle v, w_2 \rangle w_2 - \cdots - \langle v, w_r \rangle w_r}$$

Why?

$v = w + w^{\perp}$

$v = \underbrace{(c_1 w_1 + c_2 w_2 + \cdots + c_r w_r)}_{w \in W \text{ is a linear combination of } w_1, \ldots, w_r.} + w^{\perp}$ for some $c_1, c_2, \ldots c_r \in \mathbb{R}$

$\langle v, w_1 \rangle = \langle c_1 w_1 + \cdots + c_r w_r + w^{\perp}, w_1 \rangle$

$= \langle c_1 w_1, w_1 \rangle + \underbrace{\langle c_2 w_2, w_1 \rangle}_{=0} + \cdots + \underbrace{\langle c_r w_r, w_1 \rangle}_{=0} + \underbrace{\langle w^{\perp}, w_1 \rangle}_{=0}$

$\langle v, w_1 \rangle = c_1 \langle w_1, w_1 \rangle$

$\frac{\langle v, w_1 \rangle}{\|w_1\|^2} = c_1 \qquad \left( \|w_1\|^2 = \langle w_1, w_1 \rangle \right)$

Similarly $c_2 = \frac{\langle v, w_2 \rangle}{\|w_2\|^2}, \ldots$ .

So $v = \underbrace{c_1 w_1 + \cdots \; c_r w_r}_{w} + w^{\perp}$

$v = \frac{\langle v, w_1 \rangle}{\|w_1\|^2} w_1 + \cdots + \frac{\langle v, w_r \rangle}{\|w_r\|^2} w_r + w^{\perp}$

So $\boxed{w^{\perp} = v - \frac{\langle v, w_1 \rangle}{\|w_1\|^2} w_1 - \frac{\langle v, w_2 \rangle}{\|w_2\|^2} w_2 - \cdots - \frac{\langle v, w_r \rangle}{\|w_r\|^2} w_r}$

$$\boxed{w = \frac{\langle v, w_1 \rangle}{\|w_1\|^2} w_1 + \frac{\langle v, w_2 \rangle}{\|w_2\|^2} w_2 + \cdots + \frac{\langle v, w_r \rangle}{\|w_r\|^2} w_r}$$

Anton 6.3 #23

Find the orthogonal projection of $b = (1,2,0,-2)$ on the subspace spanned by $v_1 = (1,1,1,1)$ & $v_1 = (1,1,-1,-1)$

Solution $\text{proj}_V b = \dfrac{\langle b, v_1 \rangle}{\|v_1\|^2} v_1 + \dfrac{\langle b, v_2 \rangle}{\|v_2\|^2} v_2$

($V = \text{span}\{v_1, v_2\}$)

$v_1 = (1,1,1,1)$ $\quad \|v_1\|^2 = 1^2 + 1^2 + 1^2 + 1^2 = 4$

$b = (1,2,0,-2)$

$\langle b, v_1 \rangle = 1 \cdot 1 + 2 \cdot 1 + 0 \cdot 1 + (-2) \cdot 1$

$= 1 + 2 + 0 - 2$

$= 1$

$v_2 = (1,1,-1,-1)$ $\quad \|v_2\|^2 = 4$

$b = (1,2,0,-2)$

$\langle b, v_2 \rangle = 1 \cdot 1 + 2 \cdot 1 + 0(-1) + (-2)(-1)$

$= 1 + 2 + 0 + 2$

$= 5$

$$\text{proj}_V b = \frac{\langle b, v_1 \rangle}{\|v_1\|^2} v_1 + \frac{\langle b, v_2 \rangle}{\|v_2\|^2} v_2$$

$$= \frac{1}{4} v_1 + \frac{5}{4} v_2$$

$$= \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right) + \left(\frac{5}{4}, \frac{5}{4}, -\frac{5}{4}, -\frac{5}{4}\right)$$

$$\text{proj}_V b = \boxed{\left(\frac{3}{2}, \frac{3}{2}, -1, -1\right)}$$

Now Gram-Schmidt is easy. Here is what Gram-Schmidt does:

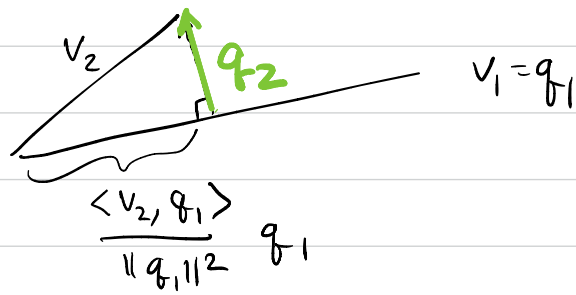
$$\boxed{\text{Given a basis } \{v_1, v_2, \dots v_n\} \text{ of } V \text{ find an orthogonal/orthonormal basis } \{q_1, q_2, \dots, q_n\} \text{ of } V.}$$

Key formula
$$\boxed{w^\perp = v - \frac{\langle v, w_1\rangle}{\|w_1\|^2} w_1 - \frac{\langle v, w_2\rangle}{\|w_2\|^2} w_2 - \dots - \frac{\langle v, w_r\rangle}{\|w_r\|^2} w_r}$$

1. Set $q_1 = v_1$ (or any scalar multiple)
2. Set $q_2 = v_2 - \dfrac{\langle v_2, q_1\rangle}{\|q_1\|^2} q_1$

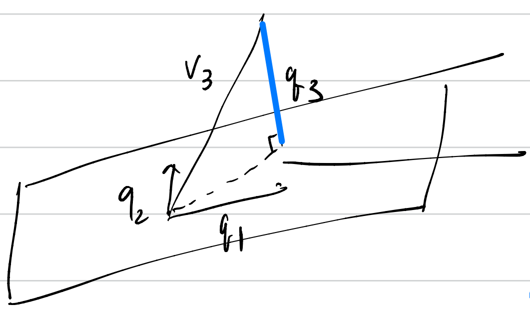
$q_2$ is orthogonal to $q_1$

not orthogonal




Can replace $q_2$ by a scalar multiple of it (eg to clear denominator)

3. set $q_3 = v_3 - \dfrac{\langle v_3, q_1\rangle}{\|q_1\|^2} q_1 - \dfrac{\langle v_3, q_2\rangle}{\|q_2\|^2} q_2$




$$\frac{\langle v_3, q_1\rangle}{\|q_1\|^2} q_1 + \frac{\langle v_3, q_2\rangle}{\|q_2\|^2} q_2 = \operatorname{proj}_{q_1, q_2} v_3$$

$q_3$ is orthogonal to $q_1, q_2$

4. And so on.

$$q_i = v_i - \frac{\langle v_i, q_1\rangle}{\|q_1\|^2} q_1 - \frac{\langle v_i, q_2\rangle}{\|q_2\|^2} q_2 - \dots - \frac{\langle v, q_{i-1}\rangle}{\|q_{i-1}\|^2} q$$

$q_i$ is orthogonal to $q_1, q_2, \dots, q_{i-1}$.
(can replace $q_i$ by any scalar multiple of it)

5. Get orthogonal basis $\{q_1, q_2, \dots q_n\}$
To get orthonormal basis (each vector a unit vector

Anton 6.3

31. Let $R^4$ have the Euclidean inner product. Use the Gram–Schmidt process to transform the basis $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4\}$ into an orthonormal basis.

$$\mathbf{u}_1 = (0,2,1,0), \quad \mathbf{u}_2 = (1,-1,0,0),$$
$$\mathbf{u}_3 = (1,2,0,-1), \quad \mathbf{u}_4 = (1,0,0,1)$$

Solution. Set $\boxed{q_1 = (0,2,1,0)}$ $(=u_1)$ $\qquad \|q_1\|^2 = 0^2+2^2+1^2+0^2 = 5$

Initially set $q_2 = u_2 - \dfrac{\langle u_2, q_1\rangle}{\|q_1\|^2} q_1$ $\qquad u_2 = (1,-1,0,0)$, $q_1 = (0,2,1,0)$

$= (1,-1,0,0) - \dfrac{-2}{5} q_1$ $\qquad \langle u_2, q_1\rangle = 0 \;\; -2 \;\; 0 \;\; 0 = -2$

Scale through by 5 to clear denominator

$5q_2 = (5,-5,0,0) + 2(q_1)$

$= (5,-5,0,0) + 2(0,2,1,0)$

rename $5q_2$ as $q_2$ $\qquad q_2 = (5,-5,0,0) + (0,4,2,0)$

$\boxed{q_2 = (5,-1,2,0)}$ $\qquad \|q_2\|^2 = 5^2 + 1^2 + 2^2 = 30$

$q_3 = u_3 - \dfrac{\langle u_3, q_1\rangle}{\|q_1\|^2} q_1 - \dfrac{\langle u_3, q_2\rangle}{\|q_2\|^2} q_2$ $\qquad u_3 = (1,2,0,-1)$, $q_1 = (0,2,1,0)$

$\langle u_3, q_1\rangle = 0 \;\; 4 \;\; 0 \;\; 0 \; \boxed{4}$

$= (1,2,0,-1) - \dfrac{4}{5} q_1 - \dfrac{3}{30} q_2$ $\qquad u_3 = (1,2,0,-1)$

$= (1,2,0,-1) - \frac{4}{5} q_1 - \frac{1}{10} q_2$ $\qquad q_2 = (5,-1,2,0)$

$= 5 \;\; -2 \;\; 0 \;\; 0$

Rescale by 10 $\qquad \langle u_3, q_2\rangle = \boxed{3}$ (sum)

$= (10,20,0,-10) - 8q_1 - q_2$

Rescale by 10

$$= (10, 20, 0, -10) - 8q_1 - q_2$$

$$= (10, 20, 0, -10) - 8(0,2,1,0) - (5,-1,2,0)$$

$$= (5, 5, -10, -10)$$

Rescale by $\frac{1}{5}$

$$\boxed{q_3 = (1, 1, -2, -2)}$$

$$\|q_3\|^2 = 1^2 + 1^2 + (-2)^2 + (-2)^2 = 10$$

$$q_4 = u_4 - \frac{\overset{\langle u_4, q_1\rangle}{0}}{5} q_1 - \frac{\overset{\langle u_4, q_2\rangle}{5}}{30} q_2 - \frac{\overset{\langle u_4, q_3\rangle}{-1}}{10} q_3$$

$$= (1,0,0,1) - \frac{5}{30}(5,-1,2,0) + \frac{1}{10}(1,1,-2,-2)$$

Rescale by 30

$$= (30,0,0,30) - 5(5,-1,2,0) + 3(1,1,-2,-2)$$

$$= (8, 8, -16, 24)$$

Rescale by 8

$$\boxed{q_4 = (1, 1, -2, 3)}$$

$$\|q_4\|^2 = 1^2 + 1^2 + (-2)^2 + 3^2 = 15$$

So an orthogonal basis of $\mathbb{R}^4$ that Gram-schmidt gives (up to scaling)

$q_1 = (0, 2, 1, 0)$ $\qquad \|q_1\|^2 = 5$

$q_2 = (5, -1, 2, 0)$ $\qquad \|q_2\|^2 = 30$

$q_3 = (1, 1, -2, -2)$ $\qquad \|q_3\|^2 = 10$

$q_4 = (1, 1, -2, 3)$ $\qquad \|q_4\|^2 = 15$

So the orthonormal basis Gram-Schmidt yields is found by replacing each $q_i$ by $\frac{q_i}{\|q_i\|}$ ← not squared (which is a unit vector)

$$\boxed{\begin{aligned} q_1 &= \left(0, \tfrac{2}{\sqrt{5}}, \tfrac{1}{\sqrt{5}}, 0\right) \\ q_2 &= \left(\tfrac{5}{\sqrt{30}}, -\tfrac{1}{\sqrt{30}}, \tfrac{2}{\sqrt{30}}, 0\right) \\ q_3 &= \left(\tfrac{1}{\sqrt{10}}, \tfrac{1}{\sqrt{10}}, -\tfrac{2}{\sqrt{10}}, -\tfrac{2}{\sqrt{10}}\right) \\ q_4 &= \left(\tfrac{1}{\sqrt{15}}, \tfrac{1}{\sqrt{15}}, -\tfrac{2}{\sqrt{15}}, \tfrac{3}{\sqrt{15}}\right) \end{aligned}}$$

# Sage Math

A.gram_schmidt()[0] applies Gram-Schmidt to the

(underscore)

**rows** (not columns) of A, and returns matrix G whose rows are orthogonal (not orthonormal)

Previous example in SageMath:

```
A=matrix(QQ, 4, 4, [0,2,1,0, 1, -1, 0, 0, 1, 2, 0, -1, 1, 0, 0, 1])
```

```
A
```

```
[ 0  2  1  0]
[ 1 -1  0  0]
[ 1  2  0 -1]
[ 1  0  0  1]
```

← rows of A are given vectors

```
A.gram_schmidt()
```

```
([    0     2     1     0]
[    1  -1/5   2/5     0]
[  1/2   1/2    -1    -1]
[ 4/15  4/15 -8/15   4/5], [   1    0    0    0]
[-2/5    1    0    0]
[ 4/5  1/2    1    0]
[   0  5/6 -1/5    1])
```

?? a list of two matrices

```
typeset_mode()
```

← toggle to make output easier to read

```
A.gram_schmidt()
```

returns a pair of matrices $(G, L)$

with $A = LG$

this is the matrix whose rows are the output of gram-schmidt

Answer

$$\left(\underbrace{\boxed{\begin{pmatrix} 0 & 2 & 1 & 0 \\ 1 & -\frac{1}{5} & \frac{2}{5} & 0 \\ \frac{1}{2} & \frac{1}{2} & -1 & -1 \\ \frac{4}{15} & \frac{4}{15} & -\frac{8}{15} & \frac{4}{5} \end{pmatrix}}}_{G}, \underbrace{\begin{pmatrix} 1 & 0 & 0 & 0 \\ -\frac{2}{5} & 1 & 0 & 0 \\ \frac{4}{5} & \frac{1}{2} & 1 & 0 \\ 0 & \frac{5}{6} & -\frac{1}{5} & 1 \end{pmatrix}}_{L \text{ lower } \triangle}\right)$$

row $2 = (1, -\frac{1}{5}, \frac{2}{5}, 0)$

our answer by hand $= (5, -1, 2, 0)$ is a scalar multiple

If you need orthonormal output of Gram-Schmidt

use A=matrix( ~~QQ~~ , - - - )

RDF — real decimal field

QQbar

```
A=matrix(QQbar, 4, 4, [0,2,1,0, 1, -1, 0, 0, 1, 2, 0, -1, 1, 0, 0, 1])
```

```
A
```

$$\begin{pmatrix} 0 & 2 & 1 & 0 \\ 1 & -1 & 0 & 0 \\ 1 & 2 & 0 & -1 \\ 1 & 0 & 0 & 1 \end{pmatrix}$$

```
A.gram_schmidt(orthonormal=True)
```

$$\left( \begin{pmatrix} 0 & 0.8944271909999159? & 0.4472135954999580? & 0 \\ 0.912870929175277? & -0.1825741858350554? & 0.3651483716701108? & 0 \\ 0.3162277660168379? & 0.3162277660168379? & -0.6324555320336758? & -0.6324555320336758? \\ 0.2581988897471611? & 0.2581988897471611? & -0.516397779494323? & 0.774596669241484? \end{pmatrix}, \begin{pmatrix} 2.236067977499790? & 0 \\ -0.8944271909999159? & 1.095445115010333? \\ 1.788854381999832? & 0.5477225575051661? \\ 0 & 0.912870929175277? \end{pmatrix} \right.$$

```
G=A.gram_schmidt(orthonormal=True)[0]
G
```

$$\begin{pmatrix} 0 & 0.8944271909999159? & 0.4472135954999580? & 0 \\ 0.912870929175277? & -0.1825741858350554? & 0.3651483716701108? & 0 \\ 0.3162277660168379? & 0.3162277660168379? & -0.6324555320336758? & -0.6324555320336758? \\ 0.2581988897471611? & 0.2581988897471611? & -0.516397779494323? & 0.774596669241484? \end{pmatrix}$$

```
G.T*G
```

T = transpose, could also write G.transpose()

$$\begin{pmatrix} 1.000000000000000? & 0.?e-17 & 0.?e-17 & 0.?e-17 \\ 0.?e-17 & 1.000000000000000? & 0.?e-17 & 0.?e-17 \\ 0.?e-17 & 0.?e-17 & 1.000000000000000? & 0.?e-17 \\ 0.?e-17 & 0.?e-17 & 0.?e-17 & 1.000000000000000? \end{pmatrix}$$

$10^{-17} \approx 0$

question marks mean decimal approximation

Prof Robert Lipshitz (U. Oregon) came up with the following "better gram-schmidt" function

Define a new function "better_gs" that takes a

- takes a list of vectors
- makes a matrix A whose <u>rows</u> are those vectors
- applies built in gram-schmidt to get orthogonal rows
- scales each row by norm to get orthonormal rows

```
def better_gs(vectors):
    A=matrix(vectors)
    G=A.gram_schmidt()[0]
    return [v/v.norm() for v in G.rows()]
```

```
v1=vector([0, 2, 1, 0])
v2=vector([1, -1, 0, 0])
v3=vector([1,2,0,-1])
v4=vector([1,0,0,1])
```

```
v1
```

$$(0, 2, 1, 0)$$

```
v1.norm()
```

$$\sqrt{5}$$

```
better_gs([v1, v2, v3, v4])
```

$$\left[\left(0, \frac{2}{5}\sqrt{5}, \frac{1}{5}\sqrt{5}, 0\right), \left(\frac{5}{6}\sqrt{\frac{6}{5}}, -\frac{1}{6}\sqrt{\frac{6}{5}}, \frac{1}{3}\sqrt{\frac{6}{5}}, 0\right), \left(\frac{1}{5}\sqrt{\frac{5}{2}}, \frac{1}{5}\sqrt{\frac{5}{2}}, -\frac{2}{5}\sqrt{\frac{5}{2}}, -\frac{2}{5}\sqrt{\frac{5}{2}}\right), \left(\sqrt{\frac{1}{15}}, \sqrt{\frac{1}{15}}, -2\sqrt{\frac{1}{15}}, 3\sqrt{\frac{1}{15}}\right)\right]$$

QR factorization. This will follow from Gram-Schmidt

In recent years a numerical algorithm based on the Gram–Schmidt process, and known as ***QR-decomposition***, has assumed growing importance as the mathematical foundation for a wide variety of numerical algorithms, including those for computing eigenvalues of large matrices. The technical aspects of such algorithms are discussed in books that specialize in the numerical aspects of linear algebra. However, we will discuss some of the underlying ideas here

Anton 6.3

Given an $n \times n$ matrix $A$ of rank $n$, we can write it as

$$A = QR$$

orthogonal matrix

upper triangular matrix

An orthogonal matrix $Q$ is a $n \times n$ matrix $Q$ such that $\boxed{Q^T Q = I_n}$, equivalently $\boxed{Q^T = Q^{-1}}$, and so for orthogonal

This condition shows the columns (& rows) of an orthogonal matrix are not just orthogonal, but orthonormal

Here's how to find the $A = QR$ factorization using Gram-Schmidt

If $v_1, \dots v_n$ are the columns of $A$, apply Gram-Schmidt to get an orthonormal (not just orthogonal) basis $q_1, \dots q_n$.

Set $Q$ to be the matrix whose colums are $q_i$

$$Q = [q_1 \; q_2 \; \dots \; q_n]$$

For $i = 1, 2, \dots, n$

key insight: $v_1, v_2, \dots v_i$ is in the span of $q_1, q_2, \dots q_i$ (think about the Gram schmid process)

$v_1 = \langle v_1, q_1 \rangle q_1$

$v_2 = \langle v_2, q_1 \rangle q_1 + \langle v_2, q_2 \rangle q_2$

$v_3 = \langle v_3, q_1 \rangle q_1 + \langle v_3, q_2 \rangle q_2 + \langle v_3, q_3 \rangle q_3$

so $A = Q\,R$ where

$$\underbrace{\begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} q_1 & \cdots & q_n \end{bmatrix}}_{Q} \underbrace{\begin{bmatrix} \langle v_1, q_1\rangle & \langle v_2, q_1\rangle & \cdots & \langle v_n, q_1\rangle \\ & \langle v_2, q_2\rangle & & \langle v_n, q_2\rangle \\ & & & \vdots \\ & & & \langle v_n, q_n\rangle \end{bmatrix}}_{\substack{R \\ \text{upper triangular}}}$$

$$A = Q\,R$$

$$A^T = (Q\,R)^T = R^T Q^T$$