Chapter 4.

4.1 Vector Spaces

vectors – first: arrows
second: n-tuples $(x_1, x_2, .. x_n)$
third: elements of a set that has certain properties

vector space

vector space axioms (~10)

Main Example of Vector Space: $\boxed{\mathbb{R}^n}$ (set of n-tuples)

but there are other examples: set $\underbrace{M_{r\times c}}_{\cong \mathbb{R}^{rc}}$ of all $r\times c$ matrices.

$\mathbb{R}^n$: standard basis vectors
$$e_1 = (1, 0, .. 0)$$
$$e_2 = (0, 1, ... 0)$$
$$\vdots$$
$$e_n = (0, 0, ... 1)$$

In Chapter 4 we learn that every vector space has a basis

Definition <u>Vector space</u> V is a set with a $\underbrace{\text{bunch of properties}}_{\text{two operations}}$:

- $+ : V \times V \to V$
  addition $(v, w) \to v + w$

- scalar multiplication:
$$\mathbb{R} \times V \to V$$
$$(r, v) \to rv$$
$$5 \quad v \to 5v$$

Examples: 1) $V = \mathbb{R}^2$

$$+ : \mathbb{R}^2 \times \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$

$$(x_1, y_1)\ (x_2, y_2) \longrightarrow (x_1 + x_2, y_1 + y_2)$$

$$(2,3) + (0,1) = (2,4)$$

scalar mult: $\underset{\mathbb{R}}{5}\underbrace{(2,3)}_{V} = \underbrace{(10,15)}_{V = \mathbb{R}^2}$

$\vec{0} = (0,0)$

$-(1,3) = (-1,-3)$

2) $V = M_{2\times 2}$ 2×2 matrices.

$$\begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 7 \end{bmatrix}$$

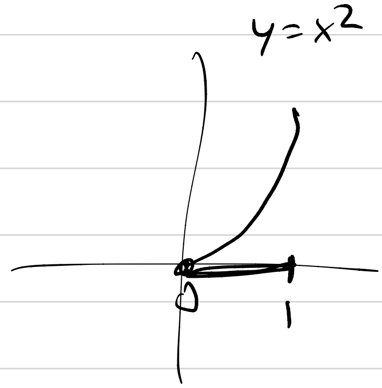
$\vec{0} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$$5\begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 5 & 5 \\ 5 & 15 \end{bmatrix}$$

$-\begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ -1 & -3 \end{bmatrix}$

3) V = Functions from $[0,1]$ to $\mathbb{R}$

$$f : [0,1] \to \mathbb{R}$$



f, g

$$(f+g)(x) = f(x) + g(x)$$

$$(rf)(x) = r\,f(x)$$

$\vec{0} = f(x) = 0$

$f(x) = x^2$

$(-f)(x) = -(x^2)$

$f + -f = x^2 + -x^2$

$= 0$

$f(x) = x^2 \qquad g(x) = x^3$

$$(f+g)(x) = f(x) + g(x) = x^2 + x^3$$

$$(5f)(x) = 5f(x) = 5x^2$$

Vector Space Axioms for a vectors

↳ assumption (not something we need to prove)

Vector Space Axioms

The following definition consists of ten axioms, eight of which are properties of vectors in R^n that were stated in Theorem 3.1.1. It is important to keep in mind that one does not *prove* axioms; rather, they are assumptions that serve as the starting point for proving theorems.

Definition 1

Let V be an arbitrary nonempty set of objects for which two operations are defined: addition and multiplication by numbers called ***scalars***. By ***addition*** we mean a rule for associating with each pair of objects $\mathbf{u}$ and $\mathbf{v}$ in V an object $\mathbf{u} + \mathbf{v}$, called the ***sum*** of $\mathbf{u}$ and $\mathbf{v}$; by ***scalar multiplication*** we mean a rule for associating with each scalar k and each object $\mathbf{u}$ in V an object $k\mathbf{u}$, called the ***scalar multiple*** of $\mathbf{u}$ by k. If the following axioms are satisfied by all objects $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ in V and all scalars k and m, then we call V a ***vector space*** and we call the objects in V ***vectors***.

1. If $\mathbf{u}$ and $\mathbf{v}$ are objects in V, then $\mathbf{u} + \mathbf{v}$ is in V.
2. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
3. $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$
4. There exists an object in v, called the ***zero vector***, that is denoted by $\mathbf{0}$ and has the property that $\mathbf{0} + \mathbf{u} = \mathbf{u} + \mathbf{0} = \mathbf{u}$ for all $\boldsymbol{u}$ in v.
5. For each $\mathbf{u}$ in V, there is an object $-\mathbf{u}$ in V, called a ***negative*** of $\mathbf{u}$, such that $\mathbf{u} + (-\mathbf{u}) = (-\mathbf{u}) + \mathbf{u} = \mathbf{0}$.
6. If k is any scalar and $\mathbf{u}$ is any object in V, then $k\mathbf{u}$ is in V.
7. $k(\mathbf{u} + \mathbf{v}) = k\mathbf{u} + k\mathbf{v}$
8. $(k + m)\mathbf{u} = k\mathbf{u} + m\mathbf{u}$
9. $k(m\mathbf{u}) = (km)(\mathbf{u})$
10. $1\mathbf{u} = \mathbf{u}$

Proposition: If V is a vector space, $v \in V$ then

$$\boxed{0v = \vec{0}}$$

number 0 $\vec{0} \in V$

Example: $0(1,3) = (0,0)$ when $V = \mathbb{R}^2$

Proof: $\underbrace{(0+0)}_{0}v = 0v + 0v$

$$0v = 0v + 0v$$

$$\underbrace{0v + (-0v)}_{\vec{0}} = 0v + \underbrace{0v + (-0v)}_{\vec{0}}$$

$$\vec{0} = 0v + \vec{0}$$

$\vec{0} = 0v$

2. Let V be the set of all ordered pairs of real numbers, and consider the following addition and scalar multiplication operations on $\mathbf{u} = (u_1, u_2)$ and $\mathbf{v} = (v_1, v_2)$:

$V = \mathbb{R}^2$

$$\mathbf{u} + \mathbf{v} = (u_1 + v_1 + 1, u_2 + v_2 + 1), \quad k\mathbf{u} = (ku_1, ku_2)$$

a. Compute $\mathbf{u} + \mathbf{v}$ and $k\mathbf{u}$ for $\mathbf{u} = (0, 4)$, $\mathbf{v} = (1, -3)$, and $k = 2$.

b. Show that $(0, 0) \neq \mathbf{0}$.

c. Show that $(-1, -1) = \mathbf{0}$.

d. Show that Axiom 5 holds by producing a vector $-\mathbf{u}$ such that $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$ for $\mathbf{u} = (u_1, u_2)$.

e. Find two vector space axioms that fail to hold.

a) $u + v = (2, 2)$ $\qquad 2u = (0, 8)$

b) $(0,0) + (1,2) = \cancel{(1,2)}$ $(2,3)$

not the $\vec{0}$

if $(0,0) = \vec{0}$
but $(0,0) + (1,2) = (2,3)$
so $(0,0)$ is not $\vec{0}$ object

c) $(-1,-1) + (x,y) = (x,y)$ $\qquad \vec{0} + v = v$

so $(-1,-1) = \vec{0}$

d) $u = (x,y)$ what is $-u$? Not $(-x,-y)$!

$-u + u = \vec{0}$ $\qquad -u = (a,b)$

$-u + (x,y) = (-1,-1)$

$(a,b) + (x,y) = (-1,-1)$

$(1,1) + (x,y) = (x+2, y+2)$

$(a,b) + (x,y) = (a+x+1, b+y+1) = (-1,-1)$

$a + x + 1 = -1$ $\quad\quad$ $b + y + 1 = -1$

$a = -1 - x - 1$ $\quad\quad$ $b = -1 - 1 - y$

$a = -2 - x$ $\quad\quad$ $b = -2 - y$

if $u = (x, y)$

$-u = (-2 - x, -2 - y)$

e) $k(u+v) = ku + kv$

$2((0,0)+(0,0)) \overset{?}{=} 2 \cdot (0,0) + 2 \cdot (0,0)$

$2(1,1) = (0,0) + (0,0)$

$(2,2) \neq (1,1)$

So NOT a vector space

9) Is the set of matrices of the form $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$

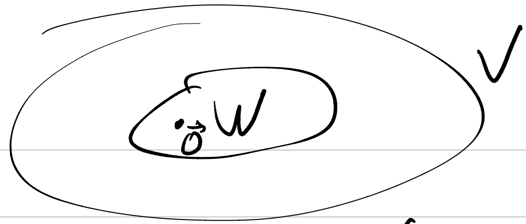
where $a, b \in \mathbb{R}$ is this a vector space with the standard matrix addition & scalar multiplication?

Yes

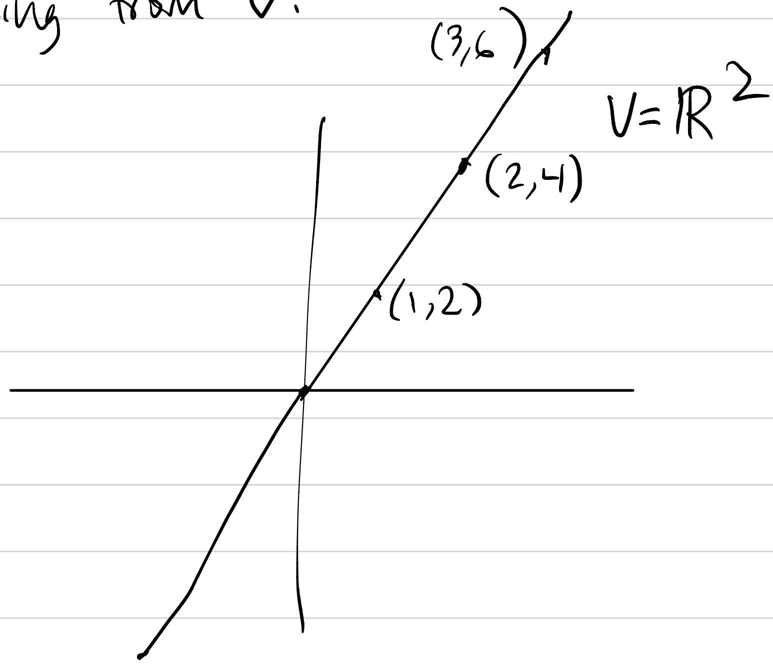
$\begin{bmatrix} a & 1 \\ 1 & b \end{bmatrix}$

don't form a vectorspace

4.2 Subspaces



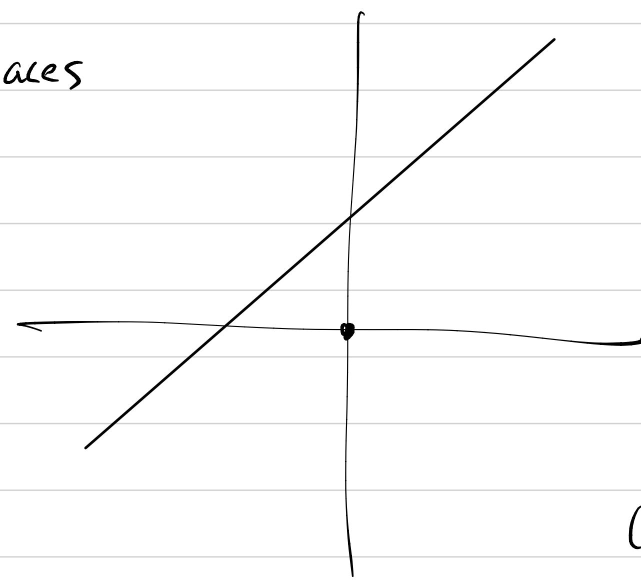
Definition: A <u>subspace</u> W of a vector space V is subset and is a vector space under the addition & scalar multiplication coming from V.

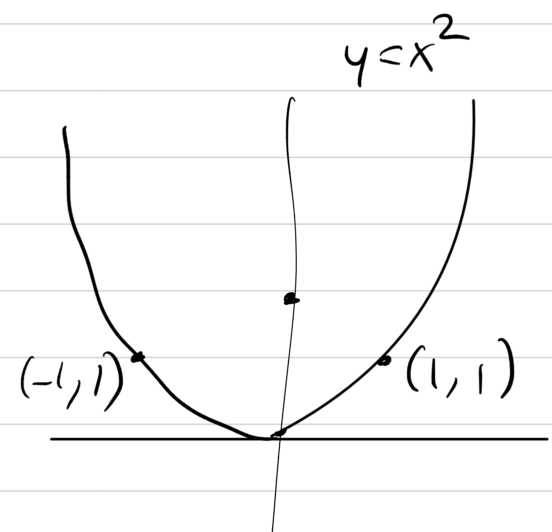


~~$W_1 = \{(0,0)\}$ is a subspace~~

W = any line through the origin is a subspace.

Not a subspaces





$(-1,1)+(1,1) = (0,2)$

Let V be a vector space. Let v_1, v_2, v_3 be some vectors.

The subspace W spanned by v_1, v_2, v_3 is set of all linear combinations of $v_1, v_2\ v_3$

$$W = \{ a_1 v_1 + a_2 v_2 + a_3 v_3 \mid a_1, a_2, a_3 \in \mathbb{R} \}$$

$$2v_1 + 1.1 v_2 + 5 v_3$$

Let A be $m \times n$ matrix. $A : \mathbb{R}^n \to \mathbb{R}^m$
$x \mapsto Ax$

$$\text{m rows} \left[\quad \underbrace{A}_{\text{n columns}} \quad \right] \underbrace{\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}}_{x \in \mathbb{R}^n}$$

kernel of A $= \{ x \in \mathbb{R}^n \mid Ax = \vec{0} \}$
is a subspace of $\mathbb{R}^n$

Example. Find kernel of $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ $A : \mathbb{R}^2 \to \mathbb{R}^2$

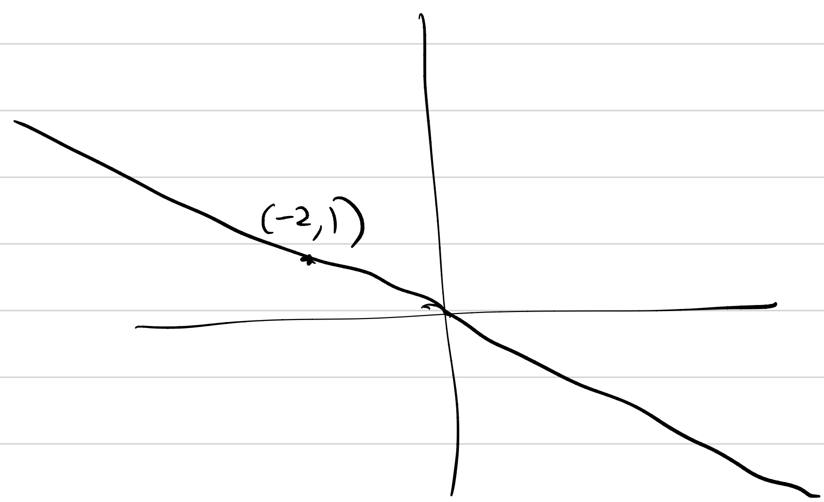
i.e solve $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$ is reduced ref

$$x + 2y = 0$$

$x = -2y$
$y = y$

so $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix} t$



Theorem: Subspace Test: A nonempty subset W of a vector space V is a subspace if & only if the following two conditions are satisfied

1) If $u, v \in W$ then $u+v \in W$
2) If $k \in \mathbb{R}$, $u \in W$, then $ku \in W$

i.e that W is closed under addition & scalar mult.

2. Use subspace test to check if

$W = \{(a,b,c) \in \mathbb{R}^3 \mid b = a+c+1\}$ is a subspace

$(1, \underline{4}, 2)$ is in W

$(0,1,0)$ is in W

$(0,0,0)$ is not in W ~~so~~ so W is not a subspace.

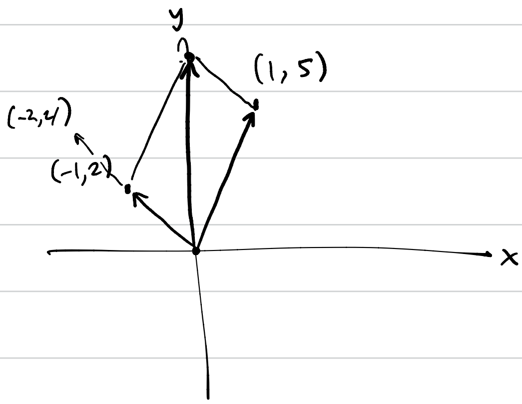
add $(1,5,2)$ not in W

$\mathbb{R}^2$ (or $\mathbb{R}^n$) is a <u>vector space</u>.

$\mathbb{R}^2 = \{(x,y) \mid x,y \in \mathbb{R}\}$

$(1,5)$ & $(-1,2)$

- elements of a vector space are called vectors
- can add vectors
  $(1,5) + (-1,2) = (0,7)$
- multiply vector by a scalar
  $$\underset{\substack{\uparrow \\ \text{scalar}}}{2} \cdot \underset{\substack{\uparrow \\ \text{vector}}}{(-1,2)} = (-2,4)$$



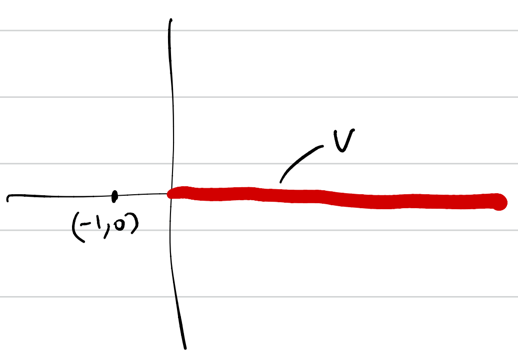
4.1 Anton

Vector space or not?

4. The set V of all pairs of numbers of the form $(x,0)$, with standard operations on $\mathbb{R}^2$, with $x \geq 0$.

$$V = \{(x,0) \mid x \geq 0\}$$

$$= \{(0,0), (1,0), (\tfrac{1}{2},0), (3,0), \ldots\}$$



not in

In $\mathbb{R}^2$ $\quad -(1,0) = (-1,0) \notin V$

So V is not a vector space, fails #5 in Anton's list of axioms of a vector

7'. Pairs of real numbers of real number w/ standard vector addition but scalar multiplication being given by

$$\underline{k} \cdot \underline{(x,y)} = (k^2 x, y)$$

Is this a vector space?

check axiom #7

$$r(\vec{u}+\vec{v}) \stackrel{?}{=} r\vec{u} + r\vec{v}$$

$= r((x,y)+(w,z))$	$r(x,y) + r(w,z)$
$= r((x+w, y+z))$	$(r^2x, y) + (r^2w, z)$
$= (r^2(x+w), y+z)$	$(r^2x+r^2w, y+z)$
$= (r^2x+r^2w, y+z)$	$(r^2x+r^2w, y+z)$

$= \checkmark$

Check axiom 8: $(r_1+r_2)\vec{u} \stackrel{?}{=} r_1\vec{u} + r_2\vec{u}$

$= (r_1+r_2)\cdot(x,y)$	$r_1\cdot(x,y) + r_2(x,y)$
$= ((r_1+r_2)^2x, y)$	$(r_1^2x, y) + (r_2^2x, y)$
$((r_1+r_2)^2x, y)$	$(r_1^2x + r_2^2x, 2y)$

$\neq$ not equal!

$$(r_1+r_2)^2 x \neq r_1^2 x + r_2^2 x$$

$$(r_1^2 + 2r_1r_2 + r_2^2)x$$

$$r_1^2 x + \boxed{2r_1r_2x} + r_2^2 x \stackrel{?}{=} r_1^2 x + r_2^2 x$$

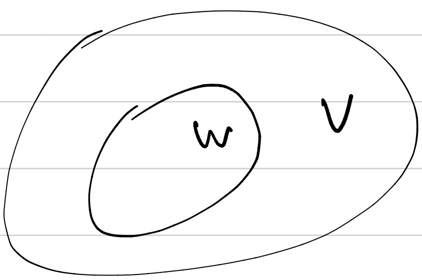
not on right hand side

4.2 Subspaces

Let V be a vector space (e.g $\mathbb{R}^n$)

Definition: A subset W of V is a subspace if

vector space so it has addition & scalar



W with these operations is a vector space.

So W is a subspace. Here are which of the 10 axioms that need to be checked

Axiom 1: W is closed under addition; if $u, v \in W$, then $u + v \in W$

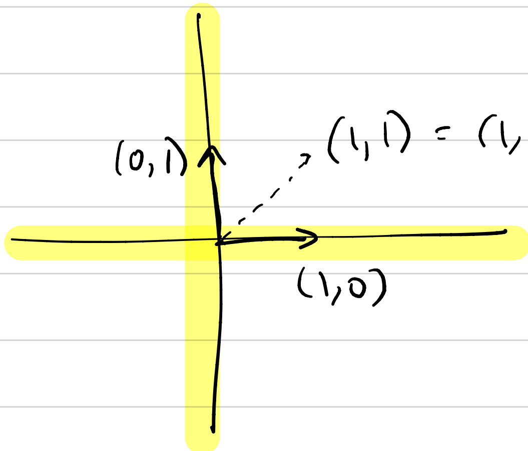
Axiom 4: $\vec{0} \in W$

Axiom 5: If $\vec{u} \in W$ then $-\vec{u} \in W$

Axiom 6: W is closed under scalar multiplication

$V = \mathbb{R}^2$ $\quad W = x\text{-axis} \underset{\text{union}}{\cup} y\text{-axis}.$

is not a subspace



$(1,1) = (1,0) + (0,1)$ is not in W

So W is not a vector space

4.2 $\quad P_3$ = set of all polynomials of degree 3 or less

$$= \{ a_0 + a_1 x + a_2 x^2 + a_3 x^3 \mid a_i \in \mathbb{R} \}$$

$\cong \mathbb{R}^4$

This is a vector space under usual addition of polynomials

Use subspace test (Check axioms 1 & 6)

5. a. W = All polynomials for which $a_0 = 0$.

Solution

$$\vec{u} = a_1 x + a_2 x^2 + a_3 x^3 \in W$$

$a_0 = 0$

$$\vec{v} = b_1 x + b_2 x^2 + b_3 x^3$$

Axiom 1: is $\vec{u} + \vec{v}$ in W? Yes

$$\vec{u} + \vec{v} = (a_1 + b_1)x + (a_2 + b_2)x^2 + (a_3 + b_3)x^3$$

no "x" term so it's in W.

Axiom 6 Is $r \cdot \vec{u}$ in W? Yes

$$r \cdot (a_1 x + a_2 x^2 + a_3 x^3) = r a_1 x + r a_2 x^2 + r a_3 x^3 \in W$$

So W is a subspace

5.1.1 Subspace or not?

a) $U = \{(1, s, t) \mid s, t \in \mathbb{R}\} \subseteq \mathbb{R}^3$

$$\underbrace{(1, s, t)}_{\text{in } U} + \underbrace{(1, s', t')}_{\text{in } U} = \underbrace{(2, s+s', t+t')}_{\text{not in } U} \notin U$$

so not a subspace

4.3 Spanning sets

Let V be a vector space (e.g $\mathbb{R}^n$)

Definition Let $v_1, \dots v_r$ be vectors in V. We say $v_1, \dots v_r$ **span** V if every vector w in V is a linear combination of the $v_1, \dots v_r$:

$$w = c_1 v_1 + c_2 v_2 + \dots + c_r v_r$$

for some scalars $c_1, c_2, \dots c_r \in \mathbb{R}$

Example The standard basis vectors $e_1, \dots, e_n$ span $\mathbb{R}^n$ since for any vector $w = (w_1, \dots, w_n) = \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix}$ (as a column vector)

$$\begin{aligned} w &= w_1 e_1 + \dots + w_n e_n \\ &\quad w_1 \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \dots + w_n \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} w_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \dots \begin{bmatrix} 0 \\ \vdots \\ 0 \\ w_n \end{bmatrix} \\ &= \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix} \end{aligned}$$

Let $v_1, \dots v_r$ be vectors in $\mathbb{R}^n$. Write $v_1, \dots v_n$ as column vectors, (so if $v_1 = (a_1, \dots a_n)$ as an n-tuple, write $v_1 = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$) and form the $n \times r$ matrix A whose **columns** are $v_1, \dots v_r$

$$A = \begin{bmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{bmatrix}$$

not ones but vertical dashes

Then the linear combination $c_1 v_1 + \dots + c_r v_r$, when expressed as a column vector is $A\begin{bmatrix} c_1 \\ \vdots \\ c_r \end{bmatrix}$:

$$c_1 v_1 + \dots + c_r v_r = \underbrace{\boxed{\underset{n \times r}{A} \underset{r \times 1}{\begin{bmatrix} c_1 \\ \vdots \\ c_r \end{bmatrix}}}}_{\substack{n \times 1 \\ \text{column vector}}} = \begin{bmatrix} | & & | \\ v_1 & \cdots & v_r \\ | & & | \end{bmatrix}\begin{bmatrix} c_1 \\ \vdots \\ c_r \end{bmatrix} = c_1\begin{bmatrix} | \\ v_1 \\ | \end{bmatrix} + \dots + c_r\begin{bmatrix} | \\ v_r \\ | \end{bmatrix}$$

Example Let $v_1 = (1,5,0,-1)$, $v_2 = (3,2,-1,0)$, $v_3 = (7,7,6,2)$ be three vectors in $\mathbb{R}^4$.

Express the linear combination $c_1 v_1 + c_2 v_2 + c_3 v_3$ as a column vector

Solution

1) Form the matrix A whose columns are v_1, v_2 & v_3 (as column vectors)

$$A = \begin{bmatrix} 1 & 3 & 7 \\ 5 & 2 & 7 \\ 0 & -1 & 6 \\ -1 & 0 & 2 \end{bmatrix}$$

Answer

2) $$A \cdot \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \underset{4 \times 3}{\begin{bmatrix} 1 & 3 & 7 \\ 5 & 2 & 7 \\ 0 & -1 & 6 \\ -1 & 0 & 2 \end{bmatrix}} \underset{3 \times 1}{\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}} = \underset{4 \times 1}{\begin{bmatrix} c_1 + 3c_2 + 7c_3 \\ 5c_1 + 2c_2 + 7c_3 \\ 0 \cdot c_1 - c_2 + 6c_3 \\ -1 \cdot c_1 + 0c_2 + 2c_3 \end{bmatrix}}$$

Compare:

$$c_1 v_1 + c_2 v_2 + c_3 v_3 = c_1(1,5,0,-1) + c_2(3,2,-1,0) + c_3(7,7,6,2)$$
$$= (c_1, 5c_1, 0, -c_1) + (3c_2, 2c_2, -c_2, 0) + (7c_3, 7c_3, 6c_3, 2c_3)$$
$$= (c_1 + 3c_2 + 7c_3,\ 5c_1 + 2c_2 + 7c_3,\ 0c_1 - c_2 + 6c_3,\ -c_1 + 0c_2 + 2c_3)$$

$v_1 = (2,2,2) \quad v_2 = (0,0,3), \quad v_3 = (0,1,1)$

Linear combination of v_1, v_2, v_3 is

- $c_1 v_1 + c_2 v_2 + c_3 v_3$ where c_1, c_2, c_3 are numbers/scalars
- example: $2v_1 + (-1)v_2 + 3v_3$

$$= 2(2,2,2) + (-1)(0,0,3) + 3(0,1,1)$$
$$= (4,4,4) + (0,0,-3) + (0,3,3)$$
$$= (4,7,4)$$

So $(4,7,4)$ is a linear combination of v_1, v_2, v_3

$(4,7,4)$ is in the span of v_1, v_2, v_3.

Given vectors v_1, v_2, v_3 (or more vectors) their span is

the set of vectors that are linear combinations of $v_1, v_2, v_3, \ldots$

Given vectors $v_1, v_2, v_3, \ldots v_n$ in $\mathbb{R}^m$

- form the matrix whose columns are $\overset{m \times n}{\begin{bmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{bmatrix}} \Bigg\} m \text{ rows}$
- Then $\underbrace{c_1 v_1 + c_2 v_2 + \cdots + c_n v_n}_{\text{linear combination}}$ is $\underset{m \times n}{\begin{bmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{bmatrix}} \underset{n \times 1}{\begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}}$ $m \times 1$

$v_1 = (2,2,2) \quad v_2 = (0,0,3), \quad v_3 = (0,1,1)$

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 2 & 0 & 1 \\ 2 & 3 & 1 \end{bmatrix}$$

To compute linear combination $2v_1 + (-1)v_2 + 3v_3$

$$\begin{bmatrix} 2 & 0 & 0 \\ 2 & 0 & 1 \\ 2 & 3 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \\ 7 \\ 4 \end{bmatrix}$$

$$\underline{3} \times \boxed{3 \qquad\qquad 3} \times \underline{1} \qquad\qquad 3 \times 1$$

Do. v_1, v_2, v_3 span all of $\mathbb{R}^3$ Yes.

$$A \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

↑ could be anything in $\mathbb{R}^3$

A has determinant $-6 \neq 0$
So A^{-1} exist

$$A^{-1} A \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = A^{-1} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$I_3 \cdot \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = A^{-1} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = A^{-1} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Exercise 5.1.2 In each case determine if $\mathbf{x}$ lies in $U = \text{span}\{\mathbf{y},\ \mathbf{z}\}$. If $\mathbf{x}$ is in U, write it as a linear combination of $\mathbf{y}$ and $\mathbf{z}$; if $\mathbf{x}$ is not in U, show why not.

a. $\mathbf{x} = (2,\ -1,\ 0,\ 1)$, $\mathbf{y} = (1,\ 0,\ 0,\ 1)$, and $\mathbf{z} = (0,\ 1,\ 0,\ 1)$.

Is $\vec{x} = c_1 \vec{y} + c_2 \vec{z}$ for some numbers c_1, c_2

$$A = \begin{array}{cc} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 1 & 1 \end{bmatrix} \\ \uparrow \quad \uparrow \\ y \quad z \end{array}$$

$$A \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

$$4 \times \boxed{2 \quad 2} \times 1 = 4 \times 1$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

Find RREF of the augmented matrix

$$\left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 1 & -1 \end{array}\right]$$

$$\begin{array}{cccl} -1 & 0 & -2 & (-1)R_1 \\ 1 & 1 & 1 & R_4 \\ \hline 0 & 1 & -1 & \end{array}$$

$R4 \leftrightarrow R4 + (-1)R_1$

$$\left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right] \quad \text{is RREF}$$

$c_1 + 0c_2 = 2 \Rightarrow c_1 = 2$

$0c_1 + c_2 = -1 \Rightarrow c_2 = -1$

Answer: Yes x is a linear combination of y & z
$x = 2y - z$

Linear independence/dependence:

We say vectors $v_1, v_2, \ldots v_n$ are <u>linearly independent</u>

if the only linear combination that is $\vec{0}$ is the $(0, 0, \ldots 0)$ linear combination:

if $c_1v_1 + c_2v_2 + \ldots c_nv_n = \vec{0}$ then $c_1 = 0, c_2 = 0, \ldots, c_n = 0$.

We say $v_1, v_2, \ldots v_n$ are linearly <u>dependent</u> if

there is some choices $c_1, c_2, \ldots, c_n$, not all zero, with
$c_1v_1 + c_2v_2 + \ldots c_nv_n = 0$

Simple example of linearly dependent vectors!

$$\underbrace{(1,0)}_{e_1}, \underbrace{(0,1)}_{e_2}, \underbrace{(2,2)}_{v}$$

Why? $(2,2) = 2e_1 + 2e_2$

$$0 = 2e_1 + 2e_2 - (2,2)$$

$$0 = 2e_1 + 2e_2 - v$$

so $c_1 = 2, \ c_2 = 2, \ c_3 = -1$ gives $\vec{0}$ linear combination

But e_1, e_2 are linearly independent

Why? Because $c_1e_1 + c_2e_2 = \vec{0} = (0,0)$

(c_1, c_2)

4.4 Anton

3. In each part, determine whether the vectors are linearly independent or are linearly dependent in R^4.

a. $\underbrace{(3, 8, 7, -3)}_{v_1}, \underbrace{(1, 5, 3, -1)}_{v_2}, \underbrace{(2, -1, 2, 6)}_{v_3}, \underbrace{(4, 2, 6, 4)}_{v_4}$

$$c_1v_1 + c_2v_2 + c_3v_3 + c_4v_4 = \vec{0} = (0,0,0,0)$$

Solution:

$$A = \begin{bmatrix} 3 & 1 & 2 & 4 \\ 8 & 5 & -1 & 2 \\ 7 & 3 & 2 & 6 \\ -3 & -1 & 6 & 4 \end{bmatrix}$$

$\uparrow$

v_1

Easy way: Find $\det(A)$ If $\det(A) \neq 0$ then A^{-1} exists so vector are linearly independent

$$A \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$A^{-1} A \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = A^{-1} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

If $\det(A) = 0$, then vectors are linearly dependent.

4.4 Anton

9.

a. Show that the three vectors $\mathbf{v}_1 = (0, 3, 1, -1)$, $\mathbf{v}_2 = (6, 0, 5, 1)$, and $\mathbf{v}_3 = (4, -7, 1, 3)$ form a linearly dependent set in R^4.

Solution

b. Express each vector in part (a) as a linear combination of the other two.

$$A = \begin{bmatrix} 0 & 6 & 4 \\ 3 & 0 & -7 \\ 1 & 5 & 1 \\ -1 & 1 & 3 \end{bmatrix}$$

not a square matrix
(it's a 4 x 3 matrix)
so cannot use determinant

$$A \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Solve for c_1, c_2, c_3

$4 \times \boxed{3 \quad 3} \times 1$ $\quad$ 4×1

```
A
```

```
[ 0  6  4  0]
[ 3  0 -7  0]
[ 1  5  1  0]
[-1  1  3  0]
```

```
A.rref()
```

```
[   1    0 -7/3    0]
[   0    1  2/3    0]
[   0    0    0    0]
[   0    0    0    0]
```

$$c_1 \quad - 7/3\, c_3 = 0 \qquad \Rightarrow \qquad c_1 = 7/3\, c_3$$

$$c_2 + 2/3\, c_3 = 0 \qquad\qquad c_2 = -2/3\, c_3$$

$$c_3 = c_3$$

Take $c_3 = 3$ (or any non zero number)

$c_1 = \frac{7}{3} c_3 = 7$ $\qquad 7v_1 - 2v_2 + 3v_3 = \vec{0}$

$c_2 = -\frac{2}{3} c_3 = -2$ $\qquad 3v_3 = -7v_1 + 2v_2$

$c_3 = 3$ $\qquad v_3 = -\frac{7}{3} v_1 + \frac{2}{3} v_2$

So we have a non-trivial linear combination adding to 0

trivial = all zeroes for c_1, c_2, c_3

So the vectors are linearly dependent.

17. Let $v_1 = (1,-1,0,0)$

$v_2 = (1,2,-2,1)$ $\quad w = (1,0,0,1)$

$v_3 = (1,-2,1,0)$

Is w a linear comb of v_1, v_2, v_3?

$$w \stackrel{?}{=} c_1 v_1 + c_2 v_2 + c_3 v_3$$

form $A = \begin{bmatrix} 1 & 1 & -1 \\ -1 & 2 & -2 \\ 0 & -2 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

columns of A are given vectors

$$A \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = w = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ -1 & 2 & -2 \\ 0 & -2 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ -1 & 2 & -2 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{array}\right]$$ augmented matrix

RREF is

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right] \longrightarrow \begin{array}{l} c_1 = -2 \\ c_2 = 1 \\ c_3 = 2 \end{array}$$

a) Yes, w is in span v_1, v_2, v_3

$$w = -2v_1 + v_2 + 2v_3$$

Same question but now w = (1,0,0,2)

b) $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$ is RREF (via SageMath)

$\rightarrow$ system is inconsistent so no solution

So NO, w is not in the span of v_1, v_2, v_3

vector space

9. Determine whether the following polynomials span P_2

polynomials of degree 2 or less with real coefficients so ax^2+bx+c

$$p_1 = 1 - x + 2x^2 \qquad p_2 = 3 + x$$

$$p_3 = 5 - x + 4x^2 \qquad p_4 = -2 - 2x + 2x^2$$

$$p_1 \Leftrightarrow (1,-1,2) \qquad p_2 \Leftrightarrow (3,1,0)$$

$$p_3 \Leftrightarrow (5,-1,4) \qquad p_4 \Leftrightarrow (-2,-2,2)$$

$P_2 \cong \mathbb{R}^3$

(c,b,a)

So question is, do

$$(1,-1,2) \quad (3,1,0) \quad (5,-1,4) \quad (-2,-2,2)$$

span $\mathbb{R}^3$?

$$\begin{bmatrix} 1 & 3 & 5 & -2 \\ -1 & 1 & -1 & 2 \\ 2 & 0 & 4 & 2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \underbrace{\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}}_{\text{arbitrary element of } \mathbb{R}^3}$$

To solve this, find RREF of augmented matrix

$$\left[\begin{array}{cccc|c} 1 & 3 & 5 & -2 & a_1 \\ -1 & 1 & -1 & -2 & a_2 \\ 2 & 0 & 4 & 2 & a_3 \end{array}\right]$$

$$\left[\begin{array}{cccc|c} 1 & 3 & 5 & -2 & a_1 \\ 0 & 4 & 4 & -4 & a_1+a_2 \\ 0 & -6 & -6 & 6 & a_3-2a_1 \end{array}\right]$$

$2 - -4$

$$\left[\begin{array}{cccc|c} 1 & 3 & 3 & -2 & a_1 \\ 0 & 1 & 1 & -1 & \frac{a_1+a_2}{4} \\ 0 & 1 & 1 & -1 & -\frac{a_3}{6}+\frac{2a_1}{6} \end{array}\right]$$

$$\left[\begin{array}{cccc|c} 1 & 3 & 3 & -2 & a_1 \\ 0 & 1 & 1 & -1 & \frac{a_1+a_2}{4} \\ 0 & 0 & 0 & 0 & \underbrace{-\frac{a_3}{6}+\frac{a_1}{3}-\frac{a_1}{4}-\frac{a_2}{4}}_{\neq 0 \text{ if } a_1=0,\ a_2=0,\ a_3\neq 0} \end{array}\right]$$

For the matrix A given below, find a basis for the kernel of A,
The kernel is the set of vectors x (in R^4 since A is 3 x 4) such that

4.3 # 15 $\qquad Ax = 0$

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

to solve Ax=0 for x, we find rref(A).

rref(A)

$$= \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$c_1 + c_3 = 0$$
$$c_2 + c_4 = 0$$

$$\begin{aligned} c_1 &= -c_3 \\ c_2 &= -c_4 \\ c_3 &= c_3 \\ c_4 &= c_4 \end{aligned} = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} c_3 + \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} c_4$$

So $v_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ & $v_2 = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}$ span W.

b) does $u = (1, 0, -1, 0)$ & $v = (1, 1, -1, 1)$ span W

$u = -v_1$ $\qquad$ $v_1 = -u$

$v = -v_1 - v_2$ $\qquad$ $v_2 = -v_1 - v$

$\qquad\qquad\qquad = u - v$

"Basis": main example

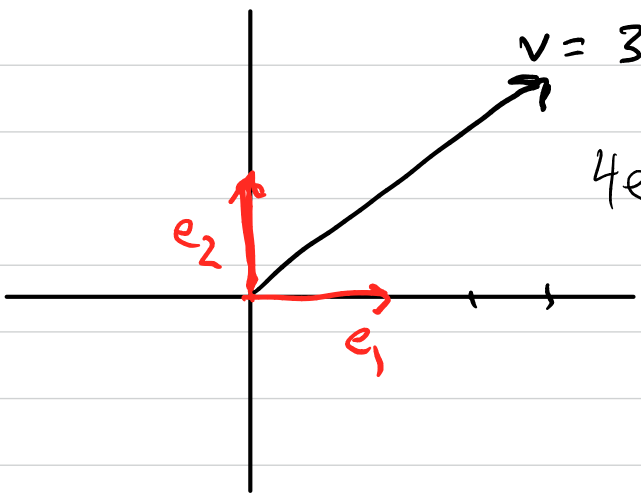
standard basis

In $\mathbb{R}^2$ $\quad e_1 = (1,0) = \begin{bmatrix}1\\0\end{bmatrix}$ $\qquad v = (3,2) = \begin{bmatrix}3\\2\end{bmatrix}$

$e_2 = (0,1) = \begin{bmatrix}0\\1\end{bmatrix}$

$v = 3\vec{e}_1 + 2\vec{e}_2$

so v can be expressed uniquely as a linear combination of e_1 & e_2



$4e_1 + 5e_2$

$= (4,5)$

In $\mathbb{R}^3$, $\quad e_1 = (1,0,0)$
$e_2 = (0,1,0)$
$e_3 = (0,0,1)$ $\quad$ form a basis.

Definition: Let V be a vector space. A set of vectors $\{v_1, v_2, .. v_n\}$ is called a <u>basis</u> of V if the following two conditions hold:

1) • $v_1, v_2, .., v_n$ are **linearly independent**:
(if $c_1v_1 + c_2v_2 + .. + c_nv_n = \vec{0}$ then $c_1 = c_2 = .. = c_n = 0$)

2) • $v_1, v_2, .. v_n$ **span** V:

Every vector $v \in V$ is a linear combination of $v_1, v_2, v_3, .., v_n$.

$v = c_1v_1 + c_2v_2 + .. + c_nv_n$ for some $c_1, c_2, .. c_n$.

These numbers $c_1, c_2, .., c_n$ are called the coordinates of the vector v in the basis $v_1, v_2, .. v_n$

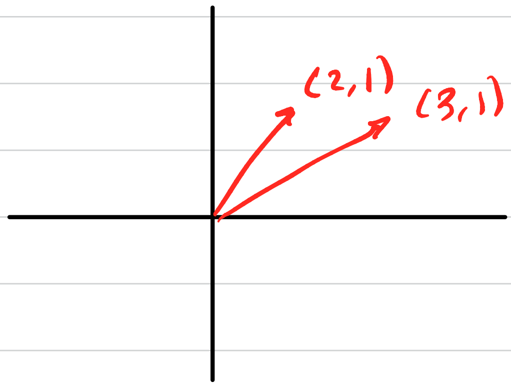
Example: In $\mathbb{R}^n$, the standard basis vectors

$$e_1 = (1, 0, 0 .. 0)$$
$$e_2 = (0, 1, 0, .., 0)$$
$$\vdots$$
$$e_n = (0, 0, .. 0, 1)$$

form a basis of $\mathbb{R}^n$.

$$c_1e_1 + c_2e_2 + .. + c_ne_n = c_1(1,0,..0) + c_2(0,1,0..) + .. + c_n(0,..,1)$$
$$= (c_1, 0, .. 0) + (0, c_2, 0, 0, ..) + .. + (0, .., c_n)$$
$$= (c_1, c_2, .., c_n)$$

In $\mathbb{R}^2$



These vectors $v_1 = (2,1)$ $v_2 = (3,1)$ form a basis of $\mathbb{R}^2$.

Linear independence:

if $c_1 v_1 + c_2 v_2 = \vec{0}$ then $c_1 = c_2 = 0$

$$c_1(2,1) + c_2(3,1) = (0,0)$$

$$(2c_1, c_1) + (3c_2, c_2) = (0,0)$$

$$(2c_1 + 3c_2, c_1 + c_2) = (0,0)$$

$$2c_1 + 3c_2 = 0$$
$$c_1 + c_2 = 0$$

$A = \begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix}$ $\qquad$ $\begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Note the columns of are the column vectors

$A^{-1} = \frac{1}{\det A}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ $\qquad$ $v_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ $v_2 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

$$A^{-1} = \frac{1}{-1}\begin{bmatrix} 1 & -3 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} -1 & 3 \\ 1 & -2 \end{bmatrix}$$

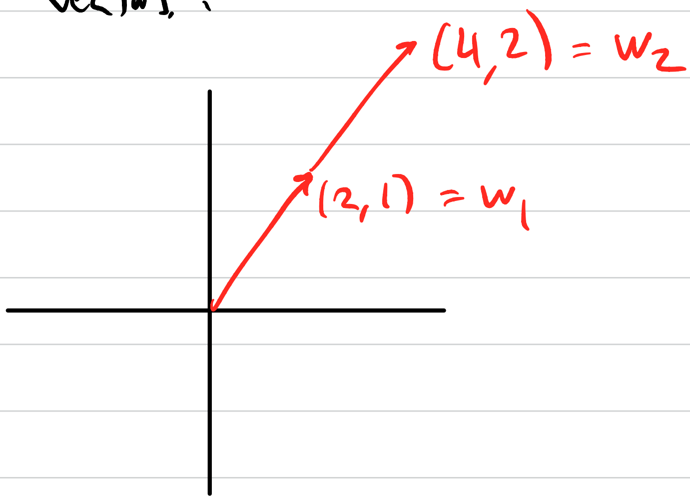
$$A\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$A^{-1}A\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = A^{-1}\begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

So v_1 & v_2 are linearly independent ✓

(Question: What would be an example of linearly dependent vectors?



w_1, w_2 are linearly dependent

$$2w_1 = w_2$$
$$2w_1 - w_2 = \vec{0} = (0,0)$$

so $c_1 = 2$ $c_2 = -1$

gives a linear combination that adds up to 0

Do $v_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ & $v_2 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ span $V = \mathbb{R}^2$?

For any $v = (x,y) \in \mathbb{R}^2$ we have to find

$v = (-5,1)$

scalars c_1 & c_2 such that

$$c_1 v_1 + c_2 v_2 = \begin{bmatrix} x \\ y \end{bmatrix}$$

$$c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

$$A^{-1} A \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = A^{-1} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} -1 & 3 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} -x+3y \\ x-2y \end{bmatrix}$$

if $v = (-5, 1)$
$c_1 = -(-5) + 3 = 8$
$c_2 = -5 - 2 \cdot 1 = -7$
$v = 8 \cdot (2,1) + (-7)(3,1)$

$$\boxed{\text{So if } A^{-1} \text{ exist } (\det A \neq 0) \text{ then columns of } A \text{ form a basis!}}$$

(for any n vectors in $\mathbb{R}^n$)

(Also, if $\det A = 0$, then columns do not form a basis because columns won't be linearly independent).

$v = (16, 8) + (-21, -7)$
$= (16-21, 8-7)$
$= (-5, 1)$ ✓

Anton 4.5

2. I show that the following set of vectors forms a basis for R^3.

$$\{(3,1,-4),(2,5,6),(1,4,8)\}$$

Definition of basis is 2 things
- linear indepence
- span

both of them involve some question linear combination
To deal with linear combinations, form

$$A = \begin{bmatrix} 3 & 2 & 1 \\ 1 & 5 & 4 \\ -4 & 6 & 8 \end{bmatrix}$$

- linear independence: solve

$$A \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

want: only solution is $(c_1, c_2, c_3) = (0,0,0)$

- span solve

$$A \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

arbitrary vector $v = (b_1, b_2, b_3)$ in $\mathbb{R}^3$.

Key fact: if $\det(A) \neq 0$, then A^{-1} exists, and so both linear independence & span hold.

if $\det(A) = 0$, then the vectors are not linearly independent so won't form a basis.

Summary: Throw given vectors into a matrix & compute its determinant

$$A = \begin{bmatrix} 3 & 2 & 1 \\ 1 & 5 & 4 \\ -4 & 6 & 8 \end{bmatrix}$$

$$\det(A) = 3\begin{vmatrix} 5 & 4 \\ 6 & 8 \end{vmatrix} - 2\begin{vmatrix} 1 & 4 \\ -4 & 8 \end{vmatrix} + 1\begin{vmatrix} 1 & 5 \\ -4 & 6 \end{vmatrix}$$

$$= 3(40-24) - 2(8--16) + 1(6--20)$$

$$= 3(16) - 2(24) + (26)$$

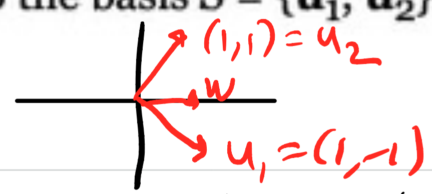
$$= 48 - 48 + 26 = 26 \neq 0$$

so $\det(A) \neq 0$ so the vectors form a basis.

Anton 4.5

12. Find the coordinate vector of w relative to the basis $S = \{\mathbf{u}_1, \mathbf{u}_2\}$ for R^2.

a. $\mathbf{u}_1 = (1, -1)$, $\mathbf{u}_2 = (1, 1)$; $\mathbf{w} = (1, 0)$



$w = (1,0)$ but 1 & 0 are coordinates with respect to standard basis e_1, e_2

$$w = 1\cdot e_1 + 0\cdot e_2 \quad \Big(= 1\cdot\underbrace{(1,0)}_{e_1} + 0\cdot\underbrace{(0,1)}_{e_2}\Big)$$

This question is asking find c_1, c_2 such that

$$w = \underline{c_1} u_1 + \underline{c_2} u_2$$

$w \downarrow$ $\qquad \downarrow$ linear combination $\qquad u_1 = (1,-1) \qquad u_2 = (1,1)$

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = A \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$$A \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = A^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$A^{-1} = \frac{1}{\underset{\uparrow \text{ determinant}}{\Delta}} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

$$= \frac{1}{1--1} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}$$

$$\boxed{c_1 = \frac{1}{2}, \quad c_2 = \frac{1}{2}}$$

coordinates of w relative to basis u_1 & u_2

$S = \{u_1, u_2\}$

$$w = \frac{1}{2} u_1 + \frac{1}{2} u_2$$

$$\frac{1}{2}(1,-1) + \frac{1}{2}(1,1)$$

$$= \left(\frac{1}{2}, -\frac{1}{2}\right) + \left(\frac{1}{2}, \frac{1}{2}\right) = (1,0) \checkmark$$

Anton 4.6 Find a basis for solution space of the homogeneous linear system.

2) $$\begin{aligned} 3x_1 + x_2 + x_3 + x_4 &= 0 \\ 5x_1 - x_2 + x_3 - x_4 &= 0 \end{aligned}$$

$$\begin{bmatrix} 3 & 1 & 1 & 1 \\ 5 & -1 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

In general The set $W = \{ \vec{x} \mid Ax = \vec{0} \}$ forms a vector space.

- $\underbrace{\vec{x}, \vec{y} \in W}$, then $\vec{x} + \vec{y} \in W$

$A\vec{x} = \vec{0}$

$A\vec{y} = \vec{0}$

so $A\vec{x} + A\vec{y} = \vec{0}$

$A(\vec{x} + \vec{y}) = \vec{0}$

- If $\vec{x} \in W$ then $c\vec{x} \in W$

$A\vec{x} = 0$

$cA\vec{x} = 0 \Rightarrow A(c\vec{x}) = 0$

Scalar

row reduce! (the augmented matrix)

```
A=matrix(QQ, 2, 5, [3, 1, 1, 1, 0, 5, -1, 1, -1, 0])
```

```
A
```

```
[ 3  1  1  1  0]
[ 5 -1  1 -1  0]
```

```
A.rref()
```

```
[  1    0 1/4    0    0]
[  0    1 1/4    1    0]
```

$$\begin{cases} x_1 + 0x_2 + \frac{1}{4}x_3 + 0x_4 = 0 \\ x_2 + \frac{1}{4}x_3 + x_4 = 0 \end{cases}$$

$x_1 = -\frac{1}{4}x_3 = -\frac{1}{4}s$

$x_2 = -\frac{1}{4}x_3 - x_4 = -\frac{1}{4}s - t$

$x_3 = s$

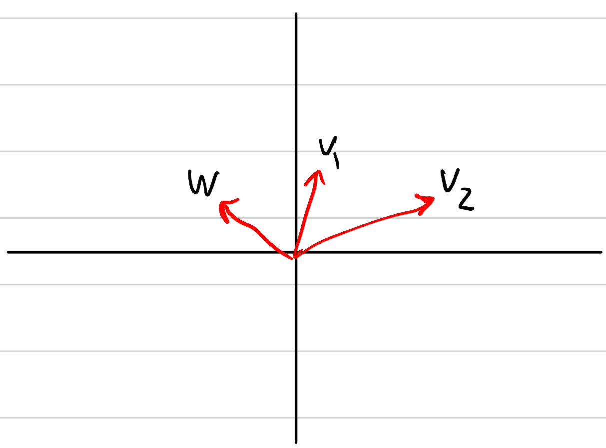
$x_4 = t$

x_1, x_2 are pivot variable so depend on x_3, x_4 which are free

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = s\begin{bmatrix} -1/4 \\ -1/4 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

Answer: $(-1/4, -1/4, 1, 0)$ & $(0, -1, 0, 1)$ form a basis for the solution space.

In $\mathbb{R}^2$, 3 vectors cannot form a basis (but can span $\mathbb{R}^2$)



$w = 3v_1 + (-5)v_2$

$\vec{0} = 3v_1 + (-5)v_2 - w$

So v_1, v_2, w are not linearly independent

So v_1, v_2, w span $\mathbb{R}^2$ but are not linearly independ so not a basis

v_1, v_2 span $\mathbb{R}^2$ & are linearly indep, so do form a basis.

Quiz tomorrow HW #12, #19, #17 Exam 2 on Monday

#19 let $v_1 = (3,-1,2)$
$v_2 = (1,2,-2)$
$v_3 = (-1,-8,c)$

Determine values of c such that v_1, v_2, v_3 form a basis of $\mathbb{R}^3$.

$$A = \begin{bmatrix} 3 & 1 & -1 \\ -1 & 2 & -8 \\ 2 & -2 & c \end{bmatrix}$$

v_1, v_2, v_3 form a basis $\Leftrightarrow \det(A) \neq 0$

expand along first row

$$\det(A) = 3\begin{vmatrix} 2 & -8 \\ -2 & c \end{vmatrix} - 1\begin{vmatrix} -1 & -8 \\ 2 & c \end{vmatrix} - 1\begin{vmatrix} -1 & 2 \\ 2 & -2 \end{vmatrix}$$

$$= 3(2c-16) - 1(-c+16) - 1(2-4)$$

$$= 6c - 48 + c - 16 + 2 \neq 0$$

$$= 7c - 62 \neq 0$$

$$7c \neq 62$$

$$c \neq \frac{62}{7}$$

4.5 Dimension

Theorem. Let V be a vector space that has a basis consisting of a finite number of vectors:

$$B = \{v_1, v_2, \dots, v_n\}$$

Then any other basis of V also has n vectors.

The number n is called the dimension of the vector space.

Example: $\mathbb{R}^2$ has a basis $B = \{e_1, e_2\} = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$

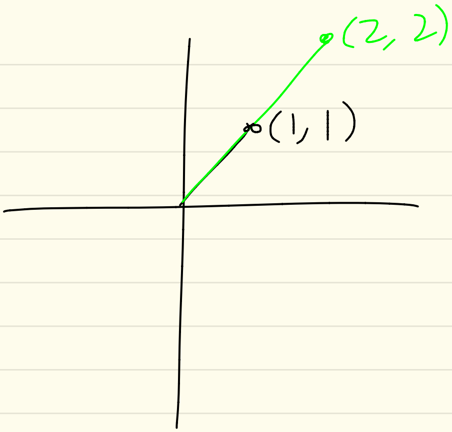
So any basis of $\mathbb{R}^2$ will have to have exactly 2 vectors.

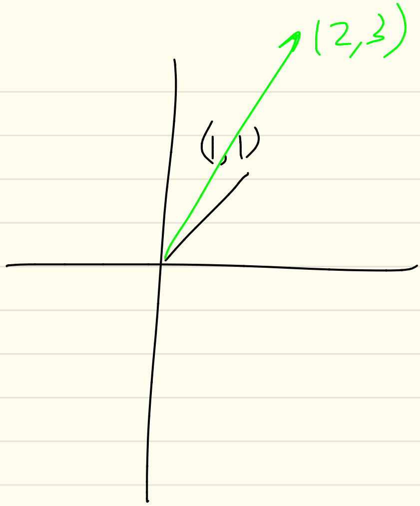
$\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \end{bmatrix} \right\}$ is not a basis because they are not linearly independent!

$$2\begin{bmatrix} 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 2 \\ 2 \end{bmatrix} = 0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

or: form the matrix $\begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} \xrightarrow{\det} 0$

But $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \end{bmatrix} \right\}$ will form a basis because determinant is non-zero. $\begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} \Rightarrow 3 - 2 = 1 \neq 0$





p.228 #2 Find a basis for the solution space of the homogeneous linear system, and find the dimension of that space

zeroes

$$3x_1 + x_2 + x_3 + x_4 = 0$$
$$5x_1 - x_2 + x_3 - x_4 = 0$$

$(x_1, x_2, x_3, x_4) \in \mathbb{R}^4$

Solution: $\left[\begin{array}{cccc|c} 3 & 1 & 1 & 1 & 0 \\ 5 & -1 & 1 & -1 & 0 \end{array}\right]$

$$\begin{array}{cccc} -6 & -2 & -2 & -2 \\ 5 & -1 & 1 & -1 \\ \hline -1 & -3 & -1 & -3 \end{array}$$

$\begin{array}{cccc|c} 3 & 1 & 1 & 1 & 0 \\ 5 & -1 & 1 & -1 & 0 \end{array}$ $\quad \begin{array}{cccc|c} 3 & 1 & 1 & 1 & 0 \\ -1 & -3 & -1 & -3 & 0 \end{array}$ $\quad \left[\begin{array}{cccc|c} 3 & 1 & 1 & 1 & 0 \\ 1 & 3 & 1 & 3 & 0 \end{array}\right]$

$$\left[\begin{array}{cccc|c} 1 & 3 & 1 & 3 & 0 \\ 3 & 1 & 1 & 1 & 0 \end{array}\right]$$

$$\begin{array}{cccc} -3 & -9 & -3 & -9 \\ 3 & 1 & 1 & 1 \\ \hline 0 & -8 & -2 & -8 \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & 3 & 1 & 3 & 0 \\ 0 & -8 & -2 & -8 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cccc|c} 1 & 3 & 1 & 3 & 0 \\ 0 & 4 & 1 & 4 & 0 \end{array}\right]$$

$$\left[\begin{array}{cccc|c} 1 & 3 & 1 & 3 & 0 \\ & 1 & \frac{1}{4} & 1 & 0 \end{array}\right]$$

$$x_2 + \frac{1}{4}x_3 + x_4 = 0$$

$$x_2 = -\frac{1}{4}x_3 - x_4$$

$$x_1 + 3x_2 + x_3 + 3x_4 = 0$$

$$x_1 = -3x_2 - x_3 - 3x_4$$

$$x_1 = -3\left(-\frac{1}{4}x_3 - x_4\right) - x_3 - 3x_4$$

$$x_1 = \frac{3}{4}x_3 + 3x_4 - x_3 - 3x_4$$

$$x_1 = -\frac{1}{4}x_3$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -\frac{1}{4}x_3 \\ -\frac{1}{4}x_3 - x_4 \\ x_3 \\ x_4 \end{bmatrix}$$

$$= x_3 \begin{bmatrix} -\frac{1}{4} \\ -1/4 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

Answer: $\begin{bmatrix} -1/4 \\ -1/4 \\ 1 \\ 0 \end{bmatrix}$ & $\begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}$

form a basis for the solution space. The dimension is 2 since there are two vectors in the basis.

---

p. 228 # 7 Find a basis for the given subspace of $\mathbb{R}^3$ & state its dimension

a) The plane $3x - 2y + 5z = 0$.

$$\left[\begin{array}{ccc|c} 3 & -2 & 5 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & -\frac{2}{3} & \frac{5}{3} & 0 \end{array}\right]$$

$$x - \frac{2}{3}y + \frac{5}{3}z = 0$$

$$x = \frac{2}{3}y - \frac{5}{3}z$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{2}{3}y - \frac{5}{3}z \\ y \\ z \end{bmatrix} = y\begin{bmatrix} 2/3 \\ 1 \\ 0 \end{bmatrix} + z\begin{bmatrix} -5/3 \\ 0 \\ 1 \end{bmatrix}$$

Answer: Basis $= \left\{ \begin{bmatrix} 2/3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -5/3 \\ 0 \\ 1 \end{bmatrix} \right\}$

dimension $= 2$.

c) The line $x = 2t$, $y = -t$, $z = 4t$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2t \\ -t \\ 4t \end{bmatrix} = t\begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}$$

basis $= \left\{ \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix} \right\}$

dimension $= 1$.

d) all vectors of the form (a, b, c) with $b = a + c$.

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} a \\ a+c \\ c \end{bmatrix} = a\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + c\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$$\text{basis} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$$

dimension is 2.

---

p. 228 #12 Find a standard basis vector ($e_1, e_2,$ or e_3) of $\mathbb{R}^3$ that can be added to the set $\{v_1, v_2\}$ to produce a basis for $\mathbb{R}^3$

a) $v_1 = (-1, 2, 3)$, $v_2 = (1, -2, -2)$

$$\begin{vmatrix} -1 & 1 & a \\ 2 & -2 & b \\ 3 & -2 & c \end{vmatrix} = a(-4+6) - b(2-3) + c(2-2)$$

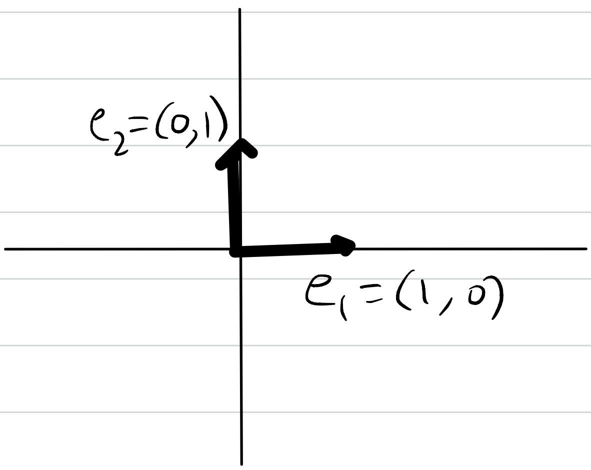
$$= 2a + b \neq 0$$

e_1, e_2 work because $e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \rightarrow 2a+b = 2\cdot1+0 = 2$

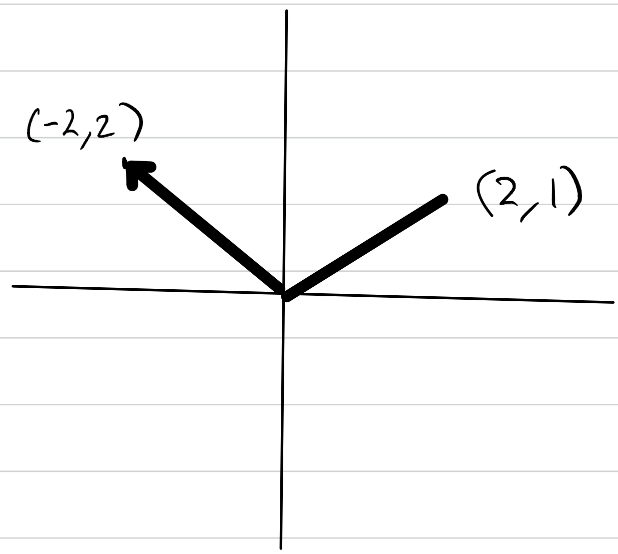
$e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \rightarrow 2a+b = 2\cdot0+1 = 1$

$\mathbb{R}^n$ has dimension n: any basis has n vectors

$\mathbb{R}^2$: any basis has 2 vectors (but not all choices of two vectors v_1, v_2 give a basis)



e_1, e_2 standard basis



$$\begin{bmatrix} 2 & -2 \\ 1 & 2 \end{bmatrix}$$

Section 4.6 Anton

12. Find a standard basis vector for R^3 that can be added to the set $\{\mathbf{v}_1, \mathbf{v}_2\}$ to produce a basis for R^3.

a. $\mathbf{v}_1 = (-1, 2, 3)$, $\mathbf{v}_2 = (1, -2, -2)$

b. $\mathbf{v}_1 = (1, -1, 0)$, $\mathbf{v}_2 = (3, 1, -2)$

13. Find standard basis vectors for R^4 that can be added to the set $\{\mathbf{v}_1, \mathbf{v}_2\}$ to produce a basis for R^4.

$$\mathbf{v}_1 = (1, -4, 2, -3), \quad \mathbf{v}_2 = (-3, 8, -4, 6)$$

13. v_1, v_2 given

$v_1, v_2, \underbrace{e_1, e_2, e_3, e_4}_{\text{we know these form 4 form a basis}}$ 6 vectors, so can't be linearly independent

$e_1 = (1,0,0,0)$

How to figure out this question.

① Form matrix whose columns are those vectors

$$\begin{array}{cc} A = & \begin{bmatrix} 1 & -3 & 1 & 0 & 0 & 0 \\ -4 & 8 & 0 & 1 & 0 & 0 \\ 2 & -4 & 0 & 0 & 1 & 0 \\ -3 & 6 & 0 & 0 & 0 & 1 \end{bmatrix} \\ & \begin{array}{cccccc} v_1 & v_2 & e_1 & e_2 & e_3 & e_4 \end{array} \end{array}$$

4x6, can't take determinant

② A.rref() compute reduced row echelon form

```
A=matrix(QQ, 4, 6, [1, -3, 1, 0, 0, 0, -4, 8, 0, 1, 0, 0, 2, -4, 0, 0, 1, 0, -3, 6, 0, 0, 0, 1])
```

A

$$\begin{array}{cccccc} c1 & c2 & c3 & c4 & c5 & c6 \\ \end{array}$$

```
[ 1 -3  1  0  0  0]
[-4  8  0  1  0  0]
[ 2 -4  0  0  1  0]
[-3  6  0  0  0  1]
```

$$\begin{array}{cccccc} v_1 & v_2 & e_1 & e_2 & e_3 & e_4 \end{array}$$

```
A.rref()
```

```
[   1    0   -2    0    0   -1]
[   0    1   -1    0    0 -1/3]
[   0    0    0    1    0 -4/3]
[   0    0    0    0    1  2/3]
```

3. Identify the pivots (highlighted in green)

Insight: the pivot columns in A.rref() are linearly independent, and because row reduction does not affect solution space, the corresponding columns in A are linearly independent

Columns 1, 2, 4, 5 are linearly independent in A.rref()
so they are linearly independent in A.

So $\boxed{v_1, v_2, e_2, e_3}$ are linearly independent so form a basis of $\mathbb{R}^4$.

So notice e_1 is skipped & in A.
$\Updownarrow$
column 3

In A.rref, col 3 is a linear combination of col 1 & col 2
$\Downarrow$

$$\text{col } 3 = -2(\text{col } 1) - 1(\text{col } 2)$$

Same linear relation holds in A, since row reduction doesn't change solution space let's check the following holds in A (not just A.rref)

$$\text{col } 3 = -2(\text{col } 1) - 1(\text{col } 2)$$

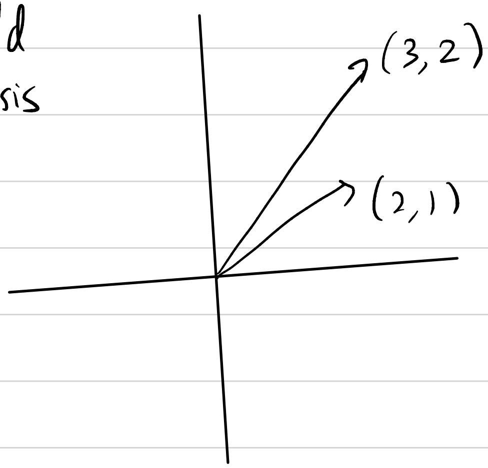
$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \stackrel{?}{=} -2 \begin{pmatrix} 1 \\ -4 \\ 2 \\ -3 \end{pmatrix} - 1 \cdot \begin{pmatrix} -3 \\ 8 \\ -4 \\ 6 \end{pmatrix}$$

$$= \begin{pmatrix} -2 \\ 8 \\ -4 \\ 6 \end{pmatrix} + \begin{pmatrix} 3 \\ -8 \\ 4 \\ -6 \end{pmatrix}$$

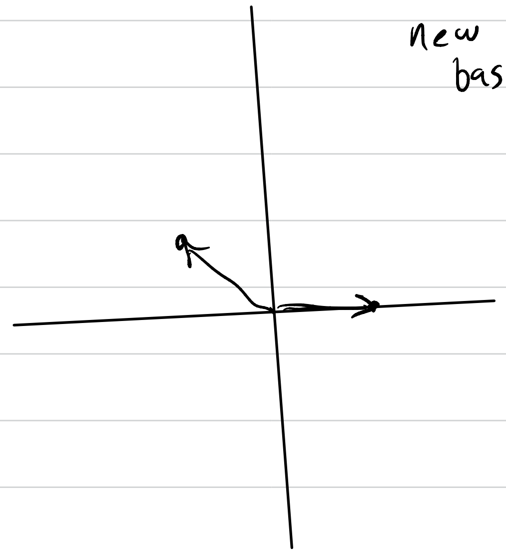
$$\stackrel{\checkmark}{=} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

4.7 Change of basis

old basis



new basis



$$\begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} \qquad \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

$B = \{ \underset{u_1}{(2,1)}, \underset{u_2}{(3,2)} \}$ $\qquad$ $B' = \{ \underbrace{(1,0)}_{u_1'}, \underbrace{(-1,1)}_{u_2'} \}$

$[v]_B = (4,7)$ ← random example of coordinates

↑ coordinates with respect to basis B

means

$$v = 4 \cdot u_1 + 7u_2$$

$$= 4\begin{pmatrix} 2 \\ 1 \end{pmatrix} + 7\begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

$$v = \begin{pmatrix} 8 \\ 4 \end{pmatrix} + \begin{pmatrix} 21 \\ 14 \end{pmatrix} = \begin{pmatrix} 29 \\ 18 \end{pmatrix} = v$$

29 & 18 are the coordinates of v in standard basis e_1, e_2

What are the coordinates of v in the basis B'?

$$v = \boxed{}\, u_1' + \boxed{}\, u_2'$$

i.e what is $[v]_{B'}$

$v = 4u_1 + 7u_2$ (by definition of v)

$$\underset{1\times 2}{v = [u_1 \;\; u_2]} \underset{2\times 1}{\begin{bmatrix} 4 \\ 7 \end{bmatrix}} \qquad\qquad v = [u_1' \;\; u_2'] \begin{bmatrix} ? \\ ? \end{bmatrix}$$

Express u_1 (& u_2) as linear combination of u_1', u_2'.

Recipe:

(1) $$A = \left[\begin{array}{cc|cc} 1 & -1 & 2 & 3 \\ 0 & 1 & 1 & 2 \end{array}\right]$$

new basis | old basis

(2) Row reduce i.e compute A.rref().

In this example we can do this by hand

$$A.\text{rref} = \left[\begin{array}{cc|cc} 1 & 0 & 3 & 5 \\ 0 & 1 & 1 & 2 \end{array}\right]$$

$R1 \leftrightarrow R1 + R2$

1	-1	2	3
0	1	1	2
1	0	3	5

$P_{B\to B'}$ = change of basis matrix from B to B'

transition matrix

$$[v]_{B'} = P_{B \to B'} \cdot [v]_B$$

$$= \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 4 \\ 7 \end{bmatrix}$$

$$[v]_{B'} = \begin{bmatrix} 12 + 35 \\ 4 + 14 \end{bmatrix} = \begin{bmatrix} 47 \\ 18 \end{bmatrix}$$

i.e $v = 47u_1' + 18u_2'$

$$= 47\begin{pmatrix} 1 \\ 0 \end{pmatrix} + 18\begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$\begin{array}{r} \overset{3}{\cancel{4}}7 \\ -18 \\ \hline 29 \end{array}$$

$$= \begin{pmatrix} 47 \\ 0 \end{pmatrix} + \begin{pmatrix} -18 \\ 18 \end{pmatrix}$$

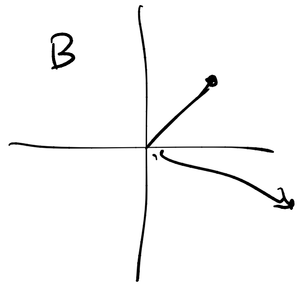
$$v = \begin{pmatrix} 29 \\ 18 \end{pmatrix} \quad \checkmark$$

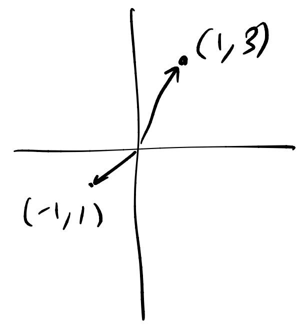
1. Consider the bases $B = \{\mathbf{u}_1, \mathbf{u}_2\}$ and $B' = \{\mathbf{u}_1', \mathbf{u}_2'\}$ for R^2, where

$$\mathbf{u}_1 = \begin{bmatrix} 2 \\ 2 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} 4 \\ -1 \end{bmatrix}, \quad \mathbf{u}_1' = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \quad \mathbf{u}_2' = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$$

a. Find the transition matrix from B' to B.

B





b. Find the transition matrix from B to B'.

c. Compute the coordinate vector $[\mathbf{w}]_B$, where

$$\mathbf{w} = \begin{bmatrix} 3 \\ -5 \end{bmatrix}$$

and use (11) to compute $[\mathbf{w}]_{B'}$.

d. Check your work by computing $[\mathbf{w}]_{B'}$ directly.

b) $P_{B \to B'}$

1 $$\left[\underbrace{\begin{matrix} 1 & -1 \\ 3 & -1 \end{matrix}}_{\substack{\text{new basis} \\ B'}} \;\middle|\; \underbrace{\begin{matrix} 2 & 4 \\ 2 & -1 \end{matrix}}_{\substack{\text{old basis} \\ B}}\right]$$

2 Row reduce

```
A=matrix(QQ, 2, 4, [1, -1, 2, 4, 3, -1, 2, -1])
```

```
A
```

```
[ 1 -1  2  4]
[ 3 -1  2 -1]
```

```
A.rref()
```

```
[    1     0     0  -5/2]
[    0     1    -2 -13/2]
```

$$\boxed{P_{B\to B'} = \begin{bmatrix} 0 & -5/2 \\ -2 & -13/2 \end{bmatrix}}$$

c) $[w]_B = P_{S\to B} [w]_S$

S = standard basis

$B = \{u_1, u_2\}$

$= (P_{B\to S})^{-1} [w]_S$

$= \left(\begin{bmatrix} u_1 & u_2 \end{bmatrix}\right)^{-1} [w]_S$

$= \left(\begin{bmatrix} 3 & 4 \\ 2 & -1 \end{bmatrix}\right)^{-1} [w]_S$

$= \frac{1}{-10}\begin{bmatrix} -1 & -4 \\ -2 & 2 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ -5 \end{bmatrix}$

$[w]_B = \frac{1}{-10}\begin{bmatrix} -3+20 \\ -6-10 \end{bmatrix} = \frac{1}{-10}\begin{bmatrix} 17 \\ -16 \end{bmatrix} = \boxed{\begin{bmatrix} -\frac{17}{10} \\ +16/10 \end{bmatrix}}$

$$[w]_{B'} = P_{B \to B'} \cdot [w]_B$$

part b

$$\begin{bmatrix} 0 & -5/2 \\ -2 & -13/2 \end{bmatrix} \begin{bmatrix} -\frac{17}{10} \\ \frac{16}{10} \end{bmatrix}$$

$$= \begin{bmatrix} -4 \\ -7 \end{bmatrix}$$

4.6 Change of Basis

Let $\mathcal{B} = \{u_1, u_2, \dots u_n\}$

$\mathcal{B}' = \{u_1', u_2', \dots u_n'\}$

be two bases of a vector space V.

Let $B = [u_1 \ u_2 \ \dots \ u_n]$

$B' = [u_1 \ u_2 \ \dots \ u_n]$

Let $v \in V$ vector.

$v = c_1 u_1 + c_2 u_2 + \dots + c_n u_n$

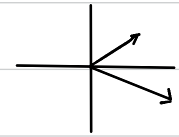
$c_1, c_2, \dots, c_n$ are the coordinates of v in basis $\mathcal{B}$

$$v = [u_1 \ \dots \ u_n] \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = \underline{B \cdot \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}}$$

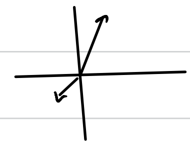
Example

same as $(2, 2)$

$\mathcal{B} = \left\{ \begin{bmatrix} 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ -1 \end{bmatrix} \right\}$



$\mathcal{B}' = \left\{ \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \end{bmatrix} \right\}$



$B = \begin{bmatrix} 2 & 4 \\ 2 & -1 \end{bmatrix}$

$B' = \begin{bmatrix} 1 & -1 \\ 3 & -1 \end{bmatrix}$

Similarly, $\underline{v = B' \cdot \begin{bmatrix} c_1' \\ \vdots \\ c_n' \end{bmatrix}}$

$$B \cdot \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = B' \begin{bmatrix} c_1' \\ \vdots \\ c_n' \end{bmatrix}$$

$$\underbrace{(B')^{-1} B} \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} c_1' \\ \vdots \\ c_n' \end{bmatrix}$$

so $\boxed{P_{B \to B'} = (B')^{-1} B}$ is "change of basis matrix" "transition matrix"

$$P_{B \to B'} \cdot [v]_B = [v]_{B'}$$

coordinates of vector v in basis B, (as a column vector)

Recipe last time: form $[B' \mid B]$ and row reduce. to $[I \mid P_{B' \to B}]$

why does this work? Multiply $(B')^{-1}[B' \mid B] = [I \mid \underline{(B')^{-1}B}]$

(multiplying on left
corresponds to row operations)

$$P_{B \to B'} = (B')^{-1} B$$

If $S = \{e_1, e_2, \dots, e_n\}$ is the standard basis, so matrix

$S = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ in $\mathbb{R}^3$

$$P_{B \to S} = I^{-1} B = B$$

$S = I$ in generneral

identity matrix

$$\boxed{P_{S \to B} = B^{-1}}$$

3. Consider the bases $B = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ and $B' = \{\mathbf{u}'_1, \mathbf{u}'_2, \mathbf{u}'_3\}$ for R^3, where

$$B = \begin{bmatrix} 2 & 2 & 1 \\ 1 & -1 & 2 \\ 1 & 1 & 1 \end{bmatrix}$$

$$\mathbf{u}_1 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}, \quad \mathbf{u}_3 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

$$B' = \begin{bmatrix} 3 & 1 & -1 \\ 1 & 1 & 0 \\ -5 & -3 & 2 \end{bmatrix}$$

$$\mathbf{u}'_1 = \begin{bmatrix} 3 \\ 1 \\ -5 \end{bmatrix}, \quad \mathbf{u}'_2 = \begin{bmatrix} 1 \\ 1 \\ -3 \end{bmatrix}, \quad \mathbf{u}'_3 = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$$

a. Find the transition matrix B to B'.

$$P_{B \to B'} = (B')^{-1} B = \begin{bmatrix} 3 & 2 & 5/2 \\ -2 & -3 & -1/2 \\ 5 & 1 & 6 \end{bmatrix}$$

via sagemath

B'.inverse() * B

must include

b. Compute the coordinate vector $[\mathbf{w}]_B$, where

$$\mathbf{w} = \begin{bmatrix} -5 \\ 8 \\ -5 \end{bmatrix}$$

$$P_{S \to B} = B^{-1}$$

so $$[w_B] = B^{-1} [w]_S = B^{-1} \begin{bmatrix} -5 \\ 8 \\ -5 \end{bmatrix} = \begin{bmatrix} 9 \\ -9 \\ -5 \end{bmatrix}$$

via sagemath.

and use ~~(12)~~ part a to compute $[\mathbf{w}]_{B'}$.

c. Check your work by computing $[\mathbf{w}]_{B'}$ directly.

$$[w]_{B'} = P_{B \to B'} \cdot [w]_B$$

$$= \begin{bmatrix} 3 & 2 & 5/2 \\ -2 & -3 & -1/2 \\ 5 & 1 & 6 \end{bmatrix} \begin{bmatrix} 9 \\ -9 \\ -5 \end{bmatrix}$$

$$= \begin{bmatrix} -7/2 \\ 23/2 \\ 6 \end{bmatrix}$$

$$[w]_B = \begin{bmatrix} 9 \\ -9 \\ -5 \end{bmatrix}$$

$$\begin{tikzcd}
B \arrow[dd, "P\cdot(\ )"'] & \\
& S \arrow[ul, "B^{-1}\cdot w"'] \arrow[dl, "(B')^{-1}\cdot w"] \\
B' &
\end{tikzcd}
\qquad w = \begin{bmatrix} -5 \\ 8 \\ 5 \end{bmatrix}$$

$$[w]_{B'} = P_{B \to B} \cdot [w]_B$$

$$= \begin{bmatrix} -7/2 \\ 23/2 \\ 6 \end{bmatrix}$$

4. Let
$$\mathbf{v}_1 = (4,1,-1), \quad \mathbf{v}_2 = (-5,-1,2), \quad \mathbf{v}_3 = (-4,0,5)$$
The $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ have been chosen so that they are linearly independent and hence form a basis $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ of $\mathbb{R}^3$. Let $\mathcal{S} = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ be the standard basis of $\mathbb{R}^3$.

(a) What are the change of basis matrices $P_{\mathcal{B}\to\mathcal{S}}$ and $P_{\mathcal{S}\to\mathcal{B}}$?

(b) If the vector $\mathbf{v} \in \mathbb{R}^3$ has coordinates $[\mathbf{v}]_{\mathcal{B}} = (1,1,0)$ in the basis $\mathcal{B}$, what are its coordinates $[\mathbf{v}]_{\mathcal{S}}$ in the standard basis?

(c) What are the coordinates of $\mathbf{e}_1 = (1,0,0)$ in the basis $\mathcal{B}$?

$P_{B\to C} = C^{-1}B$ general formula, where B = matrix whose columns are vectors in $\mathcal{B}$

a) $P_{B\to S} = \underset{\substack{\uparrow \\ I}}{S^{-1}} B = B$ $\qquad S = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$

$$\begin{array}{ccc} \uparrow & \uparrow & \uparrow \\ e_1 & e_2 & e_3 \end{array}$$

$P_{S\to B} = B^{-1} S = B^{-1}$ $\qquad B = \begin{bmatrix} 4 & -5 & -4 \\ 1 & -1 & 0 \\ -1 & 2 & 5 \end{bmatrix}$

b) $[v]_S = P_{B\to S} \cdot [v]_B$

$$\downarrow \qquad \downarrow$$

$$B \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$\downarrow$$

$$[v]_S = \begin{bmatrix} 4 & -5 & -4 \\ 1 & -1 & 0 \\ -1 & 2 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 4-5 \\ 1-1 \\ -1+2 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

or: $\quad v = c_1 v_1 + c_2 v_2 + c_3 v_3 \qquad (c_1, c_2, c_3) = (1,1,0)$

$$v = 1\cdot v_1 + 1\cdot v_2 + 0\cdot v_3$$

c) $[v]_B = P_{S \to B} \cdot [v]_S$

$$= B^{-1} \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Compute B^{-1} using Sagemath

```
B=matrix(QQ, 3, 3, [4, -5, -4, 1, -1, 0, -1, 2, 5])
```

```
B
```

```
[ 4 -5 -4]
[ 1 -1  0]
[-1  2  5]
```

```
B.inverse()
```

```
[-5 17 -4]
[-5 16 -4]
[ 1 -3  1]
```

$$[v]_B = \begin{bmatrix} -5 & 17 & -4 \\ -5 & 16 & -4 \\ 1 & -3 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \boxed{\begin{bmatrix} -5 \\ -5 \\ 1 \end{bmatrix}}$$

$$\boxed{[v]_B = (-5, -5, 1)}$$

4.8 Row space, Column space, and Null space

Any $m \times n$ matrix $A = \begin{bmatrix} - r_1 - \\ - r_2 - \\ \vdots \\ - r_m - \end{bmatrix} = \begin{bmatrix} | & | & & | \\ c_1 & c_2 & \cdots & c_n \\ | & | & & | \end{bmatrix}$

m rows, n columns

example 3×4 matrix

$$A = \begin{bmatrix} 0 & 1 & -1 & 2 \\ 7 & 3 & 4 & 5 \\ 3 & 0 & 0 & 1 \end{bmatrix}$$

has 3 vector spaces associated

1) **Row space** $\text{Row}(A)$ (A.row_space() in SageMath)

is span of the rows $r_1, r_2, \dots r_m \in \mathbb{R}^n$ of A

$$\text{Row}(A) = \{ \underbrace{y_1 r_1 + y_2 r_2 + \cdots + y_m r_m}_{\text{linear combination of the rows } r_1, \dots r_m} \mid y_1, y_2, \dots y_m \in \mathbb{R} \}$$

example: $\text{Row}(A) = \text{span}\big((0,1,-1,2), (7,3,4,5), (3,0,0,1) \big)$

so for example $2(0,1,-1,2) + 1(7,3,4,5) = (9,5,2,9) \in \text{Row}(A)$

row space is a subspace of $\mathbb{R}^n$

2) **Column space** $\text{Col}(A)$ (A.column_space() in SageMath)

is span of the columns $c_1, \dots c_n \in \mathbb{R}^m$ of A

$$\text{Col}(A) = \{ \underbrace{x_1 c_1 + x_2 c_2 + \cdots + x_n c_n}_{\text{linear combination of columns}} \mid x_1, x_2, \dots x_n \in \mathbb{R} \}$$

example 3×4 matrix

$$A = \begin{bmatrix} 0 & 1 & -1 & 2 \\ 7 & 3 & 4 & 5 \\ 3 & 0 & 0 & 1 \end{bmatrix}$$

$$\text{col}(A) = \text{span}\left(\begin{bmatrix} 0 \\ 7 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 4 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ 1 \end{bmatrix} \right)$$

subspace of $\mathbb{R}^3$

3) **Null space** Null(A) (A.right_kernel() in SageMath)

is the solutions $x \in \mathbb{R}^n$ to $\underset{m\times n \; n\times 1}{Ax} = \underset{m\times 1}{\begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}}$ $\left(x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}\right.$ written as an $n\times 1$ column vector$\left.\right)$

$$\mathrm{Null}(A) = \{ x \in \mathbb{R}^n \mid Ax = \vec{0} \}$$

$$= \left\{ x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \;\middle|\; x_1 c_1 + x_2 c_2 + \cdots + x_n c_n = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} \right\},$$

example 3×4 matrix

$$A = \begin{bmatrix} 0 & 1 & -1 & 2 \\ 7 & 3 & 4 & 5 \\ 3 & 0 & 0 & 1 \end{bmatrix} \qquad \mathrm{Null}(A) = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \;\middle|\; \begin{bmatrix} 0 & 1 & -1 & 2 \\ 7 & 3 & 4 & 5 \\ 3 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}$$

Goal: Given a matrix A, find a basis of its row space, basis of its column space, and basis of its null space.

Idea: Row reduction is all you need (A.rref() in SageMath)
reduced row echelon form

Proposition. If a matrix R is in row echelon form then

- row vectors with leading 1's form a basis of row(R)
- column vectors with leading 1's of the row vectors form a basis of col(R).

Proof: Analyze positions of the pivot 1's. Example

$$R = \begin{array}{c} \begin{matrix} c_1 & c_2 & c_3 & c_4 & c_5 \end{matrix} \\ \begin{bmatrix} 1 & 2 & 2 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \begin{matrix} r_1 \\ r_2 \\ r_3 \\ r_4 \end{matrix}$$

Rows r_1, r_2, r_3 form a basis for row(R)

Columns c_1, c_3, c_4 form a basis for col(R).

Theorem. Elementary row operations on a matrix A

a) do not change the row space $\text{row}(A)$ of A

b) do not change the null space $\text{null}(A)$

(They usually change the column space $\text{col}(A)$)

Proof. a) • Scaling a row by a non zero scalar does not change $\text{row}(A)$

• Interchanging two rows does not change $\text{row}(A)$

• Let A' be the result of replacing row r_k with $r_k' = r_k + a r_l$ $(l \neq k)$

Then any linear combination of rows of A' will be

$= x_k r_k' +$ (linear comb of rows of A besides r_k)

$= x_k (r_k + a r_l) +$ (linear comb of rows of A besides r_k)

$= x_k r_k + x_k a r_l +$ (linear comb of rows of A besides r_k)

$=$ linear combinination of rows of A

So $\text{row}(A') \subseteq \text{row}(A)$

Conversely since $r_k' = r_k + a r_l$, then $r_k' - a r_l = r_k$

so any linear combination of rows of A

the r_k can be replaced with $r_k' - a r_l$

and so will be a linear combination of rows of A'

So $\text{row}(A) \subseteq \text{row}(A')$.

Hence $\text{row}(A) = \text{row}(A')$.

b) Elementary row operations on A do not change $Null(A)$

Proof: Elementary row operation on A corresponds to multiplying on the left by an elementary matrix E, new matrix is EA.

If $x \in Null(A)$, then $Ax = 0$,

$$EAx = E \cdot 0 = 0$$

so $x \in Null(EA)$.

So $Null(A) \subseteq Null(EA)$.

Conversely,

If $x \in Null(EA)$, $EAx = 0$

$$E^{-1}EAx = 0$$

$$Ax = 0$$

so $x \in Null(A)$.

So $Null(EA) \subseteq Null(A)$.

Hence $Null(A) = Null(EA)$, i.e an elementary row operation does not change the null space of a matrix

Example. Let A be the matrix below. Find a basis for the Null space of A

$$\begin{matrix} 2 & 3 & 0 & 4 & 2 \\ 2 & 3 & 1 & 1 & 0. \end{matrix}$$

Solution. 1) figure out the reduced row echelon form of A, and identify pivots & free variables

$$\begin{matrix} 1 & 3/2 & 0 & 2 & 1 \\ 0 & 0 & 1 & -3 & -2 \end{matrix}$$

pivots variables x_1, x_3

free variables x_2, x_4, x_5

2) Express pivot variables in terms of free variables.

$$\boxed{x_1} + \frac{3}{2}x_2 + 0x_3 + 2x_4 + x_5 = 0$$

$$\boxed{x_3} - 3x_4 - 2x_5 = 0$$

$$x_1 = -\frac{3}{2}x_2 - 2x_4 - x_5$$

$$x_3 = 3x_4 + 2x_5$$

3)
$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -\frac{3}{2}x_2 - 2x_4 - x_5 \\ x_2 \\ 3x_4 + 2x_5 \\ x_4 \\ x_5 \end{bmatrix}$$

Don't forget to include x_2=x_2 and x_4=x_4 and x_5=x_5 since x_2, x_4, x_5 are free variables

$$= \begin{bmatrix} -\frac{3}{2} \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} x_2 + \begin{bmatrix} -2 \\ 0 \\ 3 \\ 1 \\ 0 \end{bmatrix} x_4 \begin{bmatrix} -1 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix} x_5$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -\frac{3}{2} \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} x_2 + \begin{bmatrix} -2 \\ 0 \\ 3 \\ 1 \\ 0 \end{bmatrix} x_4 \quad \begin{bmatrix} -1 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix} x_5$$

So one basis for null space of A is

$$\begin{bmatrix} -\frac{3}{2} \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} -2 \\ 0 \\ 3 \\ 1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} -1 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix}$$

It's formed by taking non-pivot columns of A.rref() & inserting some 0's & 1's.

Example. Let A be the matrix below. Find a basis for the row space of A

Solution: 1) Find reduced row echelon form A.rref().
(Answer) 2) The nonzero rows of A.rref() will be a basis for Row(A) because

- they span row(A) since doing elementary row operations does not change the row space of a matrix

- They are linearly independent by analyzing positions of pivot 1's

- Warning: the corresponding rows of A (as opposed to A.rref()) may or may not form a basis of Row(A)

Example. Let A be the matrix below. Find a basis for the row space of A consisting of rows of A.

Solution #1) Use built in SageMath command A.pivot_rows()

Solution #2) Use fact that our process of finding column space of a matrix returns a basis consisting of columns of the original matrix (not columns of the rref of the matrix)

In transpose of a matrix, rows become columns & vice versa

So apply this procedure to find basis of column space of the transpose of A.

Theorem. Elementary row operations on a matrix A typically change the column space $col(A)$ of A but <u>preserve linear relations between columns of A</u>:

i.e if $A = \begin{bmatrix} | & & | \\ c_1 & \cdots & c_n \\ | & & | \end{bmatrix}$

and

$$x_1 c_1 + x_2 c_2 + \cdots x_n c_n = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} \quad \text{for some } x_1, \ldots x_n \in \mathbb{R},$$

and if $EA = \begin{bmatrix} | & & | \\ c_1' & \cdots & c_n' \\ | & & | \end{bmatrix}$ then

E elementary matrix

$$x_1 c_1' + \cdots + x_n c_n' = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}.$$

Hence elementary row operations preserve linearly independence of sets of columns of a matrix.

Proof: $x_1 c_1 + x_2 c_2 + \cdots x_n c_n = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$ translates to $Ax = 0$

$$\begin{bmatrix} | & & | \\ c_1 & \cdots & c_n \\ | & & | \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ \\ 0 \end{bmatrix}$$

$$Ax = 0 \iff EAx = 0. \qquad \square$$

Example. Let A be the matrix below. Find a basis for the <u>column space of A</u>

Solution: 1) Find reduced row echelon form A.rref().

2. Identify the pivot columns of A.rref()
These form a basis of col(A.rref()) (but not col(A))

(Answer) 3. The corresponding (i.e. same position) columns of A form a basis for col(A)

a) Find a subset of the vectors

$$v_1 = (2,4,-8)$$
$$v_2 = (3,6,-12)$$
$$v_3 = (-1,1,2)$$
$$v_4 = (7,5,-22)$$
$$v_5 = (4,2,-12)$$

that forms a basis for the subspace of $\mathbb{R}^3$ spanned by the vectors. (there is more than one correct basis ; just find one)

b) Express each vector not in the basis as a linear combination of basis vectors.

Solution: Because our procedure for finding a basis for the column space of a matrix A returns a subset of the columns of A (as opposed to A.rref()), we

1) form a matrix A whose columns are $v_1, v_2, \ldots v_5$:

$A =$

2) Find reduced row echelon form of A & identify pivot columns:

3) The columns of A corresponding to the pivot columns give an answer to part(a).

$\boxed{v_1 = (2,4,-8) \ \& \ v_3 = (-1,1,2)}$

b) Express each vector not in the basis as a linear combination of basis vectors.

So we need to express each of v_2, v_4, v_5 as a linear combination of v_1, v_3

Let $w_1, w_2, \ldots w_5$ be the corresponding columns of A.rref().

The key idea is that linear relations among the w_i's also give linear relations among the v_i's.

It is easy to read off from A.rref() the linear relations of the w_i

$$w_2 = \frac{3}{2} w_1$$

$$w_4 = 2w_1 - 3w_3$$

$$w_5 = 1 \cdot w_1 - 2w_3$$

So answers are

$$\boxed{\begin{aligned} v_2 &= \frac{3}{2} v_1 \\ v_4 &= 2v_1 - 3v_3 \\ v_5 &= 1 \cdot v_1 - 2v_3 \end{aligned}}$$

In Exercises 9–10, find bases for the null space and row space of A.

9.

a. $A = \begin{bmatrix} 1 & -1 & 3 \\ 5 & -4 & -4 \\ 7 & -6 & 2 \end{bmatrix}$

b. $A = \begin{bmatrix} 2 & 0 & -1 \\ 4 & 0 & -2 \\ 0 & 0 & 0 \end{bmatrix}$

10.

a. $A = \begin{bmatrix} 1 & 4 & 5 & 2 \\ 2 & 1 & 3 & 0 \\ -1 & 3 & 2 & 2 \end{bmatrix}$

b. $A = \begin{bmatrix} 1 & 4 & 5 & 6 & 9 \\ 3 & -2 & 1 & 4 & -1 \\ -1 & 0 & -1 & -2 & -1 \\ 2 & 3 & 5 & 7 & 8 \end{bmatrix}$

10. Nullspace $A\vec{x} = 0$

A.rref() & easily read off solutions!

```
A
```

```
[ 1  4  5  2]
[ 2  1  3  0]
[-1  3  2  2]
```

```
A.rref()
```

```
[   1     0     1 -2/7]
[   0     1     1  4/7]
[   0     0     0    0]
```

Basis for row space $(1,0,1,-2/7)$ $(0,1,1,4/7)$

(These vectors are linearly independant

$c_1(1,0,1,-2/7) + c_2(0,1,1,4/7) = (0,0,0,0)$

$(c_1, 0, \ldots) + (0, c_2, \ldots)$

$(c_1, c_2, \ldots) = (0,0,0,0)$

$c_1 = 0$ & $c_2 = 0$)

Basis for null space

why? $x_1 + 0x_2 + 1x_3 - 2/7 x_4 = 0$ (first row)

$x_2 + x_3 + 4/7 x_4 = 0$

$$x_1 = -x_3 + \frac{2}{7}x_4$$

$$x_2 = -x_3 - \frac{4}{7}x_4$$

$$x_3 = x_3$$

$$x_4 = x_4$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} x_3 + \begin{bmatrix} 2/7 \\ -4/7 \\ 0 \\ 1 \end{bmatrix} x_4$$

$\begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix}$ & $\begin{bmatrix} 2/7 \\ -4/7 \\ 0 \\ 1 \end{bmatrix}$ form a basis for null space

Basis for column space: Columns of the <u>original</u> matrix that are in pivot columns

not A.rref()

pivot columns are 1 & 2

so column space basis is $\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$ & $\begin{bmatrix} 4 \\ 1 \\ 3 \end{bmatrix}$.

Theorem: dim(row space) = dim(column space)

Proof: Both equal the number of pivots!
dimensions

Definition: The <u>rank</u> of matrix is the dim(row space)
(= dim(column space))

Theorem: If A is $m \times n$ matrix of rank r,

then $\dim(\text{null space}) = n - r$

$$\dim(\text{null space}) + \dim(\text{column space}) = n$$

13.

a. find bases for the row space and column space of the matrix

$$A = \begin{bmatrix} 1 & -2 & 5 & 0 & 3 \\ -2 & 5 & -7 & 0 & -6 \\ -1 & 3 & -2 & 1 & -3 \\ -3 & 8 & -9 & 1 & -9 \end{bmatrix}$$

4×5

b. find a basis for the row space of A that consists entirely of row vectors of A.

Trick: Find a basis for column space of A^T.

transpose (interchange rows & columns)

5×4

$$A^T = \begin{bmatrix} 1 & -2 & -1 & -3 \\ -2 & 5 & 3 & 8 \\ 5 & -7 & -2 & -9 \\ 0 & 0 & 1 & 1 \\ 3 & -6 & -3 & -9 \end{bmatrix}$$

b)

A^T

```
D=matrix(QQ, 5, 4, [1, -2, -1, -3, -2, 5, 3, 8, 5, -7, -2, -9, 0, 0, 1, 1, 3, -6, -3, -9])
```

```
D
```

```
[ 1 -2 -1 -3]
[-2  5  3  8]
[ 5 -7 -2 -9]
[ 0  0  1  1]
[ 3 -6 -3 -9]
```

```
D.rref()
```

```
[1 0 0 0]
[0 1 0 1]
[0 0 1 1]
[0 0 0 0]
[0 0 0 0]
```

pivot columns in A^T.rref() are columns 1, 2, 3

so columns 1, 2, 3 of A^T form a basis of column space of A

so a basis for row space of A is $\boxed{\text{1st, 2nd, 3rd rows of } A.}$

5. In SageMath, create a 4x7 matrix **A** with rank 4 using the following command:

   `A=random_matrix(ZZ,4,7,algorithm='echelonizable',rank=4, upper_bound=8)`

   (The `algorithm='echelonizable'` part creates a matrix whose RREF will have integer entries, and the upper_bound=8 part just makes a matrix who entries have absolute value ≤ 8. These two commands are added just to make the numbers easier to deal with.)

   Type `A` so I can see what **A** is (make sure to submit a photo/screenshot of **A** - otherwise I can't check your answers).

   (a) Which columns are the pivot columns of **A**?

   (b) Find a basis for the row space of **A**. What is the dimension of this row space?

   (c) Find a basis for the null space of **A**. What is the dimension of this null space?

   (d) Find a basis for the column space of **A**. What is the dimension of this column space?

   (e) Find a basis for the row space of **A** that consists of rows of **A**.

```
A=random_matrix(ZZ, 4, 7, algorithm='echelonizable', rank=4, upper_bound=8)
```

```
A
```

```
[ 1  1 -2 -1 -3 -2 -4]
[ 0  1  3 -1 -1  1 -2]
[ 0  2  6 -1  0  2  1]
[ 1  2  1 -1 -1  0  1]
```

```
A.rref()
```

```
[ 1  0 -5  0  0 -1  2]
[ 0  1  3  0  0  0  1]
[ 0  0  0  1  0 -2  1]
[ 0  0  0  0  1  1  2]
```

1 2 3 4 5 6 7

b) $(1,0,-5,0,0,-1,2)$
$(0,1,3,0,0,0,1)$
3rd row
4th row

a) pivot columns are columns 1, 2, 4, 5 (in this example)

b) pivot rows in A.rref form a basis for row space, which in this case is all 4 rows. dimension of row space is 4

c)
$c_1 - 5c_3 - 1c_6 + 2c_7 = 0 \qquad c_1 = 5c_3 + c_6 - 2c_7$

$c_2 + 3c_3 + c_7 = 0 \qquad c_2 = -3c_3 - c_7$

$c_4 - 2c_6 + c_7 = 0 \qquad c_4 = 2c_6 - c_7$

$c_5 + c_6 + 2c_7 = 0 \qquad c_5 = -c_6 - 2c_7$

free variables c_3, c_6, c_7

$c_3 = c_3$
$c_6 = c_6$
$c_7 = c_7$

$$\begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \\ c_6 \\ c_7 \end{bmatrix} = \underbrace{\begin{bmatrix} 5 \\ -3 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} c_3 + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 2 \\ -1 \\ 1 \\ 0 \end{bmatrix} c_6 + \begin{bmatrix} -2 \\ -1 \\ 0 \\ -1 \\ -2 \\ 0 \\ 1 \end{bmatrix} c_7}_{\text{these three vectors are a basis for the null space of } A.}$$

$c_3 = c_3$

these three vectors are a basis for the null space of A.

dimension of null space $3 = 7 - 4$

7: # of colums; 4: rank

d) basis for column space is $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 2 \\ 2 \end{bmatrix} \begin{bmatrix} -1 \\ -1 \\ -1 \\ -1 \end{bmatrix} \begin{bmatrix} -3 \\ -1 \\ 0 \\ 1 \end{bmatrix}$

```
A=random_matrix(ZZ, 4, 7, algorithm='echelonizable', rank=4, upper_bound=8)
```

dim is 4

```
A
```

```
[ 1  1 -2 -1 -3 -2 -4]
[ 0  1  3 -1 -1  1 -2]
[ 0  2  6 -1  0  2  1]
[ 1  2  1 -1 -1  0  1]
```

```
A.rref()
```

```
[ 1  0 -5  0  0 -1  2]
[ 0  1  3  0  0  0  1]
[ 0  0  0  1  0 -2  1]
[ 0  0  0  0  1  1  2]
```

e) Find a basis for row space of A consisting of rows of A.

Solution: Trick: find rref of A^T (transpose of A)

rows of A = columns of A^T

find basis for column space of A^T using A^T.rref()

In this case the question is a bit silly since row space is 4 dimensional, so all 4 rows must be elements of a basis of the row space; But here is the calculation one would do.

```
A

[ 1  1 -2 -1 -3 -2 -4]
[ 0  1  3 -1 -1  1 -2]
[ 0  2  6 -1  0  2  1]
[ 1  2  1 -1 -1  0  1]
```

```
A.transpose()

[ 1  0  0  1]
[ 1  1  2  2]
[-2  3  6  1]
[-1 -1 -1 -1]
[-3 -1  0 -1]
[-2  1  2  0]
[-4 -2  1  1]
```

```
A.transpose().rref()

[1 0 0 0]
[0 1 0 0]
[0 0 1 0]
[0 0 0 1]
[0 0 0 0]
[0 0 0 0]
[0 0 0 0]
```

So answer is all 4 rows of A

$(1,1,-2,-1,-3,-2,-4)$ etc

$(0,1,3 \ldots)$

$(0,2,6 \ldots)$

$(1,2,1, \ldots)$

Same problem (#5) but for a 5x7 matrix of rank 3

```
A=random_matrix(ZZ, 5, 7, algorithm='echelonizable', rank=3, upper_bound=8)
```

```
A
```

$$\begin{bmatrix} 1 & 1 & 0 & 6 & 3 & 7 & -7 \\ -1 & 0 & -4 & -4 & -1 & -3 & 4 \\ 0 & 0 & 0 & 0 & 1 & 2 & -2 \\ 1 & 0 & 4 & 4 & 0 & 1 & -2 \\ 1 & 1 & 0 & 6 & 2 & 5 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_7 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

```
A.rref()
```

$$\begin{array}{c} \begin{bmatrix} 1 & 0 & 4 & 4 & 0 & 1 & -2 \\ 0 & 1 & -4 & 2 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\ \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{matrix} \end{array}$$

$\to x_1 + 4x_3 + 4x_4 + x_6 - 2x_7 = 0$

$\to x_2 - 4x_3 + 2x_4 + x_7 = 0$

$\to x_5 + 2x_6 - 2x_7 = 0$

a) pivot columns are 1, 2, 5

b) basis for row space of A are the pivot rows of A.rref() not A!

$$\boxed{\begin{matrix}(1,0,4,4,0,1,-2)\\(0,1,-4,2,0,0,1)\\(0,0,0,0,1,2,-2)\end{matrix}}$$

dimension of row space is 3

c) Pivot variables are x_1, x_2, x_5

Free variables are the rest: x_3, x_4, x_6, x_7

To find nullspace, use A.rref() to solve for pivot variables in terms of free variables

$$x_1 = -4x_3 - 4x_4 - x_6 + 2x_7$$ (solve row 1 of A.rref() for x_1)

$$x_2 = 4x_3 - 2x_4 - x_7$$

$$x_3 = x_3$$

$$x_4 = x_4$$

$$x_5 = -2x_6 + 2x_7$$

$$x_6 = x_6$$

$$x_7 = x_7$$

$$\vec{x} = \begin{bmatrix}-4\\4\\1\\0\\0\\0\\0\end{bmatrix} x_3 + \begin{bmatrix}-4\\-2\\0\\1\\0\\0\\0\end{bmatrix} x_4 + \begin{bmatrix}-1\\0\\0\\0\\-2\\1\\0\end{bmatrix} x_6 + \begin{bmatrix}2\\-1\\0\\0\\2\\0\\1\end{bmatrix} x_7$$

$$\boxed{\vec{x} = \begin{bmatrix} -4 \\ 4 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} x_3 + \begin{bmatrix} -4 \\ -2 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} x_4 + \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix} x_6 + \begin{bmatrix} 2 \\ -1 \\ 0 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix} x_7}$$

So these 4 vectors form a basis for the nullspace of A

dimension of null space is 4

(Let me copy the matrix from before here:)

```
A=random_matrix(ZZ, 5, 7, algorithm='echelonizable', rank=3, upper_bound=8)
```

```
A
```

```
[ 1  1  0  6  3  7 -7]
[-1  0 -4 -4 -1 -3  4]
[ 0  0  0  0  1  2 -2]
[ 1  0  4  4  0  1 -2]
[ 1  1  0  6  2  5 -5]
```

```
A.rref()
```

```
[ 1  0  4  4  0  1 -2]
[ 0  1 -4  2  0  0  1]
[ 0  0  0  0  1  2 -2]
[ 0  0  0  0  0  0  0]
[ 0  0  0  0  0  0  0]
```

d) Find basis for column space of A

Answer: columns of A corresp to pivot columns:

not A.rref()!

$$\boxed{\begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ 1 \\ 0 \\ 2 \end{bmatrix}}$$

e) Find a basis for row space of A that consists of rows of A

note that answer from part b does not satisfy this condition!

Solution: Find column space of A^T = A.transpose()

```
A.transpose()
```

```
[ 1 -1  0  1  1]
[ 1  0  0  0  1]
[ 0 -4  0  4  0]
[ 6 -4  0  4  6]
[ 3 -1  1  0  2]
[ 7 -3  2  1  5]
[-7  4 -2 -2 -5]
```

```
A.transpose().rref()
```

$$\begin{array}{c}\begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\ \begin{matrix} 1 & \; 2 & \; 3 & \; 4 & \; 5 \end{matrix}\end{array}$$

Pivot columns are 1, 2, 3 so columns 1, 2, 3 of A^T form a basis of column space of A^T, which means rows 1, 2, 3 of A form a basis for row space

$$\boxed{\begin{bmatrix} 1 \\ 1 \\ 0 \\ 6 \\ 3 \\ 7 \\ -7 \end{bmatrix} \quad \begin{bmatrix} -1 \\ 0 \\ -4 \\ -4 \\ -1 \\ -3 \\ 4 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 2 \\ -2 \end{bmatrix}}$$

(These are also rows 1, 2, 3 of A)