INTEGRATION ON ARTIN TORIC STACKS
AND EULER CHARACTERISTICS

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ABSTRACT. There is a well-developed intersection theory on smooth Artin stacks with quasi-affine diagonal. However, for Artin stacks whose diagonal is not quasi-finite, the notion of the degree of a Chow cycle is not defined. In this paper we propose a definition for the degree of a cycle on Artin toric stacks whose underlying toric varieties are complete. As an application we define the Euler characteristic of an Artin toric stack with complete good moduli space, extending the definition of the orbifold Euler characteristic. An explicit combinatorial formula is given for 3-dimensional Artin toric stacks.

1. INTRODUCTION

Let $\mathcal{X}$ be complete Deligne-Mumford stack. There is a well-developed intersection theory on such stacks [Vis, EG98, Kre] which includes the notion of the degree of the 0-cycle. Over an algebraically closed field of characteristic 0, if x is a point of $\mathcal{X}$ with stabilizer G_x, then the degree of the 0-cycle $[x] \in A_0(\mathcal{X})$ is $1/|G_x|$. As a consequence one can compute the integral of an algebraic cohomology class on a complete Deligne-Mumford stack. This theory of integration plays a crucial role in Gromov-Witten, Donaldson-Thomas and other invariants in modern algebraic geometry.

A natural problem is to extend the notion of degree to stacks with nonfinite diagonal. While such stacks are not separated (and hence cannot be proper over the ground field), there is a class of Artin stacks which in many ways behave like complete Deligne-Mumford stacks. Specifically we consider Artin stacks which are generically Deligne-Mumford and have a complete good moduli space (in the sense of [Al]). If $\mathcal{X}$ is such a stack, then there is a well-defined pushforward of Grothendieck groups $p_*\colon K(\mathcal{X}) \to K(pt) = \mathbb{Z}$, thereby defining Euler characteristics of sheaves on such stacks. Thus it is natural to expect there to be a corresponding pushforward for algebraic cycles as well as a Riemann-Roch theorem relating the K-theoretic and the Chow-theoretic pushforward via some analogue of the Chern character.

The purpose of the present paper is to take a first step in this direction by defining the degree of a 0-cycle on a toric Artin stack in the sense of [BCS]. Our definition is based on our previous paper [EM], where we showed that if $\mathcal{X}(\mathbf{\Sigma})$ is a toric Artin stack with complete good moduli space $X(\Sigma)$, then there is a canonical

birational morphism of toric stacks $\mathcal{X}(\mathbf{\Sigma}') \xrightarrow{f} \mathcal{X}(\mathbf{\Sigma})$ such that $\mathcal{X}(\mathbf{\Sigma}')$ is Deligne-Mumford and such that the induced morphism of the underlying toric varieties is proper and birational. The degree of a 0-cycle on $\mathcal{X}(\mathbf{\Sigma})$ is then defined as the degree of its pullback to the Deligne-Mumford stack $\mathcal{X}(\mathbf{\Sigma}')$.

As an application we define *the stacky Euler characteristic* of an Artin toric stack to be the degree of the Euler class of the tangent bundle stack, and we give a combinatorial formula for this Euler characteristic in dimension 3.

Along the way we prove that the integral Chow ring of an Artin toric stack equals the Stanley–Reisner ring of the associated fan. Iwanari [Iwa] proves this for Deligne-Mumford toric stacks; our proof is different.

2. Toric stacks and Stanley–Reisner rings

In this section, we recall the definition of toric stacks and in the process establish the notation we will use for toric stacks. As in [Iwa], for simplicity we will restrict our attention to stacky fans for which N is torsion free. A *stacky fan* $\mathbf{\Sigma} = (N, \Sigma, \{v_1, \ldots, v_n\})$ consists of:

1. a finitely generated free abelian group $N = \mathbb{Z}^d$,
2. a (not neccesarily simplicial) fan $\Sigma \subset N_{\mathbb{Q}}$, whose rays we will denote by $\rho_1, \ldots, \rho_n$,
3. for every $1 \le i \le n$, an element $v_i \in N$ such that v_i lies on the ray ρ_i.

2.1. Toric stacks via the Cox construction.

We now recall the construction [BCS] of the toric stack $\mathcal{X}(\mathbf{\Sigma})$ associated to a stacky fan $\mathbf{\Sigma} = (N, \Sigma, \{v_1, \ldots, v_n\})$. Let $S = \mathbb{C}[X_\rho \mid \rho \in \Sigma(1)]$, where $\Sigma(1)$ denotes the rays in Σ. Define the ideal (of S) $B(\Sigma) = (X_{\widehat{\sigma}} \mid \sigma \in \Sigma)S$, where $X_{\widehat{\sigma}} = \prod_{\rho \in \Sigma(1) \smallsetminus \sigma} X_\rho$. The Cox space $C(\Sigma)$ is defined to be $\operatorname{Spec} S \smallsetminus V(B(\Sigma))$, where $V(B(\Sigma))$ is the vanishing locus of the ideal $B(\Sigma)$. Let

$$G(\mathbf{\Sigma}) = \{(t_\rho)_{\rho \in \Sigma(1)} \in (\mathbb{C}^*)^n \mid \prod_{\rho \in \Sigma(1)} t_\rho^{\langle m, v_\rho \rangle} = 1 \;\; \forall m \in M\},$$

where $M = \operatorname{Hom}(N, \mathbb{Z})$ and $\langle \cdot, \cdot \rangle : M \times N \to \mathbb{Z}$ is the natural pairing. The group $G(\mathbf{\Sigma})$ acts on $C(\Sigma)$ by restricting the natural action of $(\mathbb{C}^*)^n$ on $C(\Sigma) \subseteq \mathbb{C}^n = \operatorname{Spec} S$. The *toric stack* $\mathcal{X}(\mathbf{\Sigma})$ associated to the stacky fan $\mathbf{\Sigma} = (N, \Sigma, \{v_1, \ldots, v_n\})$ is the quotient stack $[C(\Sigma)/G(\mathbf{\Sigma})]$.

2.2. Chow rings of Artin toric stacks.

Our first result is a description of the Chow ring of an Artin toric stack in terms of the combinatorics of its stacky fan.

Definition 2.1 ([Iwa, Definition 2.1]). Given a stacky fan $\mathbf{\Sigma} = (N, \Sigma, \{v_\rho\}_{\rho \in \Sigma(1)})$, let $S(\Sigma) = \mathbb{Z}[x_\rho \mid \rho \in \Sigma(1)]$. Let $I_{\mathbf{\Sigma}}$ be the ideal of $S(\Sigma)$ generated by $\sum_{\rho \in \Sigma(1)} \langle m, v_\rho \rangle x_\rho$ as m ranges over M. Let J_Σ be the ideal of $S(\Sigma)$ generated by monomials $x_{\rho_1} \cdots x_{\rho_s}$ such that the rays $\rho_1, \ldots, \rho_s$ are not contained in any cone in Σ:

$$(1) \qquad\qquad J_\Sigma = \langle x_{\rho_1} \cdots x_{\rho_s} \mid \{\rho_1, \ldots, \rho_s\} \nsubseteq \tau \in \Sigma \rangle.$$

We will call J_Σ the Stanley–Reisner ideal of Σ. Following [Iwa, Definition 2.1] (but not exactly the possibly more standard terminology in [CLS, Definition 12.4.10]; see also [MS]), define the Stanley–Reisner ring $SR(\mathbf{\Sigma})$ to be

$$(2) \qquad\qquad SR(\mathbf{\Sigma}) = S(\Sigma)/(I_{\mathbf{\Sigma}} + J_\Sigma).$$

Suppose a group G acts on a smooth scheme X, and let $\mathcal{X} = [X/G]$ be the resulting quotient stack. The integral Chow ring $A^*(\mathcal{X})$ of $\mathcal{X}$ is the G-equivariant Chow ring $A_G^*(X)$ of X [EG98]. We prove that the integral Chow ring of an Artin toric stack $\mathcal{X}(\Sigma)$ is the Stanley–Reisner ring of the stacky fan Σ. Iwanari [Iwa] proved the same result for toric Deligne-Mumford stacks. Our proof is different: rather than computing the equivariant Chow groups directly from the definition, we use the excision sequence for equivariant Chow groups.

Proposition 2.2. *Let $\boldsymbol{\Sigma} = (N, \Sigma, \{v_\rho\}_{\rho \in \Sigma(1)})$ be a stacky fan. There is an isomorphism between the integral Chow ring $A^*(\mathcal{X}(\boldsymbol{\Sigma}))$ of the smooth toric Artin stack $\mathcal{X}(\boldsymbol{\Sigma})$ and the Stanley–Reisner ring $SR(\boldsymbol{\Sigma})$, sending $[V(X_\rho)]_{G(\boldsymbol{\Sigma})} \to x_\rho$. Here $[V(X_\rho)]_{G(\boldsymbol{\Sigma})} \in A_{G(\boldsymbol{\Sigma})}^*(C(\boldsymbol{\Sigma})) = A^*(\mathcal{X}(\boldsymbol{\Sigma}))$ denotes the equivariant fundamental class of the divisor $V(X_\rho)$ of $C(\boldsymbol{\Sigma})$.*

Proof. As before, let $S(\Sigma) = \mathbb{Z}[x_\rho \mid \rho \in \Sigma(1)]$, $S = \mathbb{C}[X_\rho \mid \rho \in \Sigma(1)]$, and $G = G(\boldsymbol{\Sigma})$.

The excision sequence in equivariant Chow groups is the exact sequence

$$A_G^*(V(B(\Sigma))) \to A_G^*(\operatorname{Spec} S) \to A_G^*(C(\Sigma)) \to 0.$$

Since $\operatorname{Spec} S$ is an affine bundle over $\mathrm{pt} = \operatorname{Spec} \mathbb{C}$, we have $A_G^*(\operatorname{Spec} S) = A_G^*(\mathrm{pt}) = S(\Sigma)/I_{\boldsymbol{\Sigma}}$. By the following lemma, the image of $A_G^*(V(B(\Sigma)))$ in $A_G^*(\operatorname{Spec} S)$ is the Stanley–Reisner ideal J_Σ (equation (1)). Hence $A_G^*(C(\Sigma)) = S(\Sigma)/(I_{\boldsymbol{\Sigma}} + J_\Sigma) = SR(\boldsymbol{\Sigma})$, the Stanley–Reisner ring of Σ. □

Lemma 2.3. *Let $B(\Sigma) = (X_{\widehat{\sigma}} \mid \sigma \in \Sigma)S$ be as in Section 2.1. Then we have a primary decomposition*

$$(3) \qquad B(\Sigma) = \bigcap_{\{\rho_1, \dots, \rho_s\} \nsubseteq \tau \in \Sigma} (X_{\rho_1}, \dots, X_{\rho_s}),$$

where the intersection ranges over sets of rays $\{\rho_1, \dots, \rho_s\}$ that are not contained in some cone in Σ.

Proof. First we show the "$\subseteq$" inclusion. Fix $\sigma \in \Sigma$. For any $\{\rho_1, \dots, \rho_s\}$ not contained in a cone in Σ, there is some ρ_i (where $1 \leq i \leq s$) such that $\rho_i \notin \sigma$ (for otherwise $\{\rho_1, \dots, \rho_s\} \subseteq \sigma$). Hence $X_{\widehat{\sigma}} \in (X_{\rho_1}, \dots, X_{\rho_s})$, and hence we have proved the "$\subseteq$" inclusion.

To show the opposite inclusion, let RI denote the ideal on the right-hand side of equation (3). Since RI is a monomial ideal, it suffices to show that monomials in RI belong to $B(\Sigma)$; i.e., we need to show if a monomial $X_{\rho_1} \cdots X_{\rho_s}$ lies in RI, then it lies in $B(\Sigma)$. We will show the contrapositive, namely that if $X_{\rho_1} \cdots X_{\rho_s} \notin B(\Sigma)$, then $X_{\rho_1} \cdots X_{\rho_s} \notin RI$. If $X_{\rho_1} \cdots X_{\rho_s} \notin B(\Sigma)$, then the set $\Sigma(1) \smallsetminus \{\rho_1, \dots, \rho_s\}$ is not contained in any cone of Σ (because if $\Sigma(1) \smallsetminus \{\rho_1, \dots, \rho_s\} \subseteq \tau(1)$ for some cone $\tau \in \Sigma$, then taking complements gives $\{\rho_1, \dots, \rho_s\} \supseteq \Sigma(1) \smallsetminus \tau(1)$, and hence $X_{\rho_1} \cdots X_{\rho_s}$ would be a multiple (in S) of $X_{\widehat{\tau}} \in B(\Sigma)$). If we let $\{q_1, \dots, q_t\} = \Sigma(1) \smallsetminus \{\rho_1, \dots, \rho_s\}$, then $X_{\rho_1} \cdots X_{\rho_s} \notin (X_{q_1}, \dots, X_{q_t})$, and hence $X_{\rho_1} \cdots X_{\rho_s} \notin RI$. □

In the remainder of this article, by virtue of the isomorphism $SR(\boldsymbol{\Sigma}) \simeq A^*(\mathcal{X}(\boldsymbol{\Sigma}))$ given in Proposition 2.2, we will write x_ρ to mean $[V(X_\rho)]_{G(\boldsymbol{\Sigma})} \in A^*(\mathcal{X}(\boldsymbol{\Sigma}))$ (where $\rho \in \Sigma(1)$).

3. Integration on complete toric Deligne-Mumford stacks

In this section, we study intersection theory on complete toric Deligne-Mumford stacks. In this case, we have an isomorphism $\phi : A^*(\mathcal{X}(\boldsymbol{\Sigma}))_\mathbb{Q} \to A^*(X(\Sigma))_\mathbb{Q}$ [EG98], and for the rest of this paper, we will work with the rational (as opposed to integral) Chow ring $A^*(\mathcal{X}(\boldsymbol{\Sigma}))_\mathbb{Q}$ of toric stacks.

Definition 3.1 ($D_{\sigma,\boldsymbol{\Sigma}}$, stacky multiplicity of a cone)**.** Let N be a lattice, let σ be the simplicial cone $N_\mathbb{R}$, and let $s = \dim \sigma$. Suppose $v_1, \ldots, v_s$ are elements of N such that $\sigma = \mathbb{R}_{\geq 0}\langle v_1, \ldots, v_s \rangle$ (in particular, each v_i is a lattice point on exactly one ray of σ). Let $N_\sigma = \mathbb{Z}\langle N \cap \sigma \rangle$ be the lattice generated by $N \cap \sigma$. We define the (stacky) multiplicity of the stacky cone $\overline{\sigma} = (N, \sigma, \{v_1, \ldots, v_s\})$ to be the order of $N_\sigma / \mathbb{Z}\langle v_1, \ldots, v_s \rangle$, and denote this positive integer by $D_{\overline{\sigma}}$. Let $\boldsymbol{\Sigma} = (N, \Sigma, \{v_1, v_2, \ldots, v_n\})$ be a stacky fan and σ a simplicial cone in Σ. Label the v_i so that $v_1, \ldots, v_s$ are on the rays of σ. Let $\overline{\sigma} = (N, \sigma, \{v_1, \ldots, v_s\})$. Define the stacky multiplicity of σ with respect to $\boldsymbol{\Sigma}$ to be $D_{\overline{\sigma}}$, and denote this quantity by $D_{\sigma,\boldsymbol{\Sigma}}$.

Proposition 3.2 ([EG03, p. 3749])**.** *Let $\boldsymbol{\Sigma}$ be a complete simplicial stacky fan, and let $\pi : \mathcal{X}(\boldsymbol{\Sigma}) \to X(\Sigma)$ be the coarse moduli space of $\mathcal{X}(\boldsymbol{\Sigma})$. Given distinct rays $\rho_1, \ldots, \rho_s$ of Σ forming a cone $\sigma = \langle \rho_1, \ldots, \rho_s \rangle$ in Σ, let $V(\sigma) \subseteq X(\Sigma)$ be the orbit closure associated to σ. Let $\phi : A^*(\mathcal{X}(\boldsymbol{\Sigma}))_\mathbb{Q} \to A^*(X(\Sigma))_\mathbb{Q}$ be the isomorphism of Chow rings ([EG98, Theorems 3, 4]). Then*

$$(4) \qquad \phi(x_{\rho_1} \cdots x_{\rho_s}) = \frac{1}{D_{\sigma,\boldsymbol{\Sigma}}}[V(\sigma)],$$

where $D_{\sigma,\boldsymbol{\Sigma}}$ is the stacky multiplicity of σ in $\boldsymbol{\Sigma}$ (Definition 3.1).

3.1. **Self-intersections on toric Deligne-Mumford stacks.** If $\rho_1, \ldots, \rho_s$ are not distinct, then to compute $\phi(x_{\rho_1} \cdots x_{\rho_s})$, we can express $x_{\rho_1} \cdots x_{\rho_s}$ as a rational linear combination of monomials with no factor x_ρ appearing more than once, and then we can apply Proposition 3.2 to compute ϕ. The following theorem gives a procedure for finding such a linear combination.

Theorem 3.3 (Recursive formula for self-intersections)**.** *Let $\boldsymbol{\Sigma} = (\mathbb{Z}^d, \Sigma, \{v_\rho\}_{\rho \in \Sigma(1)})$ be a complete simplicial stacky fan. Let $\rho_1, \ldots, \rho_s$ be distinct rays in Σ, and let $a_1, \ldots, a_s$ be positive integers and assume $a_{i_0} \geq 2$ for some $1 \leq i_0 \leq s$. If $\rho_1, \ldots, \rho_s$ are not contained in a cone in Σ, then $\prod_{i=1}^s x_{\rho_i} = 0$ (in $A^*(\mathcal{X}(\boldsymbol{\Sigma}))$) and hence $\prod_{i=1}^s x_{\rho_i}^{a_i} = 0$. Otherwise, fix a (maximal) d-dimensional cone $\sigma = \langle \rho_1, \ldots, \rho_s, \rho_{s+1}, \ldots, \rho_d \rangle \in \Sigma(d)$ containing $\rho_1, \ldots, \rho_s$. Let $C = \{\rho \in \Sigma(1) \smallsetminus \{\rho_1, \ldots, \rho_d\} \mid \langle \rho, \rho_1, \ldots, \rho_s \rangle \in \Sigma\}$ (note $C \neq \emptyset$ since Σ is complete).*

For a ray $\rho \in C$, let $b_{\rho_i, \rho}$ be rational numbers such that

$$(5) \qquad v_\rho + \sum_{i=1}^d b_{\rho_i, \rho} v_{\rho_i} = 0.$$

(Such a relation always exists and is unique since $v_{\rho_1}, \ldots, v_{\rho_d}$ form a basis of $\mathbb{R}^d$.)
Then

$$(6) \qquad \prod_{i=1}^s x_{\rho_i}^{a_i} = \sum_{\rho \in C} b_{\rho_{i_0}, \rho} x_\rho x_{\rho_{i_0}}^{a_{i_0} - 1} \prod_{i=1, i \neq i_0}^s x_{\rho_i}^{a_i}$$

in $A^*(\mathcal{X}(\boldsymbol{\Sigma}))_\mathbb{Q}$.

Remark 3.4. Note that on the right-hand side of equation (6), the exponent of $x_{\rho_{i_0}}$ is one less than on the left-hand side, but for $i \neq i_0$ (and $1 \leq i \leq s$) the exponent of x_{ρ_i} is the same on both sides. Since there was nothing special about x_{ρ_1} (other than $a_1 \geq 2$), we can repeatedly apply Theorem 3.3 to express any monomial $\prod_{i=1}^{s} x_{\rho_i}^{a_i}$ as a rational linear combination of monomials with no factor x_ρ appearing more than once. We can then apply Proposition 3.2 to compute ϕ. Hence Theorem 3.3 gives a method to compute the integral of (or more generally, ϕ applied to) any Chow cohomology class on $\mathcal{X}(\mathbf{\Sigma})$. We give an example after the proof.

Proof. For notational convenience, assume $i_0 = 1$. Let $P = x_{\rho_1}^{a_1-1} \prod_{i=2}^{s} x_{\rho_i}^{a_i}$. For every $m \in M$, we have $\sum_{\rho \in \Sigma(1)} \langle m, v_\rho \rangle x_\rho = 0$ in the Chow ring $A^*(\mathcal{X}(\mathbf{\Sigma}))_\mathbb{Q}$. Multiplying by P we get

$$(7) \qquad \sum_{\rho \in C} \langle m, v_\rho \rangle x_\rho P + \sum_{i=1}^{d} \langle m, v_{\rho_i} \rangle x_{\rho_i} P = 0,$$

where we used $x_\rho P = 0$ for $\rho \notin C \cup \{\rho_1, \ldots, \rho_d\}$ (by Proposition 2.2). By equation (5)

$$v_\rho = -\sum_{i=1}^{d} b_{\rho_i, \rho} v_{\rho_i}.$$

Substituting this expression for v_ρ in the first term of equation (7) gives

$$(8) \qquad \sum_{\rho \in C} \langle m, -\sum_{i=1}^{d} b_{\rho_i, \rho} v_{\rho_i} \rangle x_\rho P + + \sum_{i=1}^{d} \langle m, v_{\rho_i} \rangle x_{\rho_i} P = 0,$$

and collecting the coefficients of $\langle m, v_{\rho_i} \rangle$ gives

$$(9) \qquad \sum_{i=1}^{d} \langle m, v_{\rho_i} \rangle (x_{\rho_i} P - \sum_{\rho \in C} b_{\rho_i, \rho} x_\rho P) = 0.$$

Equation (9) holds for all $m \in M$. Since $v_{\rho_1}, \ldots, v_{\rho_d}$ are linearly independent, we conclude that $x_{\rho_i} P - \sum_{\rho \in C} b_{\rho_i, \rho} x_\rho P = 0 \in A^*(\mathcal{X}(\mathbf{\Sigma}))_\mathbb{Q}$ for $1 \leq i \leq d$.

In particular, taking $i = 1$, we conclude $x_{\rho_1} P = \sum_{\rho \in C} b_{\rho_1, \rho} x_\rho P$, i.e.

$$\prod_{i=1}^{s} x_{\rho_i}^{a_i} = \sum_{\rho \in C} b_{\rho_1, \rho} x_\rho x_{\rho_1}^{a_1-1} \prod_{i=2}^{s} x_{\rho_i}^{a_i}.$$

$\square$

Example 3.5. Let $v_1 = (1, 0, 1)$, $v_2 = (0, 2, 1)$, $v_3 = (-1, 0, 1)$, $v_4 = (0, -1, 1)$, $v_5 = (0, 0, -1)$. Let $\Sigma \subseteq \mathbb{R}^3$ be the fan whose maximal cones are (the $\mathbb{R}_{\geq 0}$-spans of) $\sigma = \langle v_1, v_2, v_3, v_4 \rangle$, and for $(i, j) \in \{(1, 2), (2, 3), (3, 4), (1, 4)\}$, $\sigma_{ij5} = \langle v_i, v_j, v_5 \rangle$. The only non-simplicial cone is σ, and its star subdivision is formed by splitting it into four 3-cones, each having the ray through $v_0 = v_1 + v_2 + v_3 + v_4 = (0, 1, 4)$. The cones in $\Sigma_\sigma(3) \smallsetminus \Sigma$ are $\sigma_{0ij} = \langle v_0, v_i, v_j \rangle$ for $(i, j) \in \{(1, 2), (2, 3), (3, 4), (1, 4)\}$. Let $\mathbf{\Sigma} = (\mathbb{Z}^3, \Sigma, \{v_1, v_2, v_3, v_4, v_5\})$ and $\mathbf{\Sigma}_\sigma = (\mathbb{Z}^3, \Sigma_\sigma, \{v_0, v_1, v_2, v_3, v_4, v_5\})$ be the corresponding toric stack. For brevity, let $Dijk = D_{\sigma_{ijk}, \mathbf{\Sigma}_\sigma}$. Computing the determinants of the appropriate triples of lattice vectors v_i, we get that $D125 = D235 = 2$, $D345 = D145 = 1$, $D012 = D023 = 7$, $D034 = D014 = 5$.

For brevity let $x_i = x_{\langle v_i \rangle} \in A^*(\mathcal{X}(\mathbf{\Sigma_s}))_\mathbb{Q}$. We use Theorem 3.3 to compute $\phi(x_0^2)$ and $\phi(x_0^3)$. We will extend v_0 to the basis v_0, v_1, v_2. Then $C = \{\langle v_3 \rangle, \langle v_4 \rangle\}$.

A calculation shows that equation (5) reads $v_3 + \frac{-4}{7}v_0 + v_1 + \frac{2}{7}v_2 = 0$ and $v_4 + \frac{-3}{7}v_0 + 0 \cdot v_1 + \frac{5}{7}v_2 = 0$. Hence equation (6) reads $x_0^3 = \frac{-4}{7}x_0^2 x_3 + \frac{-3}{7}x_0^2 x_4$. We have to do another iteration to simplify $x_0^2 x_3$ and $x_0^2 x_4$. This time, we extend the basis v_0, v_3 to v_0, v_3, v_4. Then equation (5) becomes $v_1 + \frac{-2}{5}v_0 + v_3 + \frac{-2}{5}v_4 = 0$, and $v_2 + \frac{-3}{5}v_0 + 0 \cdot v_3 + \frac{7}{5}v_4 = 0$. Then equation (6) gives $x_0^2 x_3 = \frac{-3}{5}x_0 x_2 x_3$ and $x_0^2 x_4 = \frac{-2}{5}x_0 x_1 x_4$. Hence $x_0^3 = \frac{12}{35}x_0 x_2 x_3 + \frac{6}{35}x_0 x_2 x_4$, and we have eliminated all the self-intersections. Then by Proposition 3.2, $\int_{\mathcal{X}(\mathbf{\Sigma}_\sigma)} x_0^3 = \phi(x_0^3) = \frac{12}{35}\frac{1}{7} + \frac{6}{35}\frac{1}{5} = \frac{102}{1225}$.

Similarly, $x_0^2 = \frac{-4}{7}x_0 x_3 + \frac{-3}{7}x_0 x_4$. Applying Proposition 3.2 gives the relation $[V(\rho_0)]^2 = \frac{-4}{7}[V(\langle v_0, v_3 \rangle)] + \frac{-3}{7}\frac{1}{5}[V(\langle v_0, v_4 \rangle)]$ in $A^*(X(\Sigma_\sigma))_{\mathbb{Q}}$ in the rational Chow group of the toric variety $X(\Sigma_\sigma)$.

As a special case of Theorem 3.3, we recover a formula for $x_{\rho_1}^2 x_{\rho_2} \cdots x_{\rho_{d-1}}$, equivalent to the one given in [CLS, Prop. 6.4.4]:

Corollary 3.6. *Let $\mathbf{\Sigma} = (\mathbb{Z}^d, \Sigma, \{v_\rho\}_{\rho \in \Sigma(1)})$ be a complete simplicial stacky fan. Let τ^+ and τ^- be distinct (maximal) d-dimensional cones of Σ such that $\sigma = \tau^+ \cap \tau^-$ is a cone of dimension $d-1$. Let ρ^+ and ρ^- be the two rays of Σ such that $\tau^+ = \langle \rho^+, \sigma \rangle$ and $\tau^- = \langle \rho^-, \sigma \rangle$. Let β^+, β^-, b_ρ be rational numbers such that we have a relation*

$$(10) \qquad \beta^+ v_{\rho^+} + \beta^- v_{\rho^-} + \sum_{\rho \in \sigma(1)} b_\rho v_\rho = 0.$$

Then for any $\rho \in \sigma(1)$,

$$x_\rho x_\sigma = \frac{b_\rho}{\beta^+} x_{\rho^+} x_\sigma$$

in $A^*(\mathcal{X}(\mathbf{\Sigma}))$.

Proof. Simply take $s = d-1$, $a_1 = 2$, $a_i = 1$ for $2 \leq i \leq s$ in Theorem 3.3. $\qquad \square$

Remark 3.7. Applying the ring isomorphism $\phi : A^*(\mathcal{X}(\mathbf{\Sigma}))_{\mathbb{Q}} \to A^*(X(\Sigma))_{\mathbb{Q}}$ to the result in Corollary 3.6 gives $\phi(x_\rho)\phi(x_\sigma) = \frac{b_\rho}{\beta^+}\phi(x_{\rho^+}x_\sigma)$. By Proposition 3.2, this equality becomes

$$[V(\rho)] \cdot \left[\frac{1}{D_{\sigma,\mathbf{\Sigma}}}V(\sigma)\right] = \frac{b_\rho}{\beta^+}\left[\frac{1}{D_{\tau^+,\mathbf{\Sigma}}}V(\tau^+)\right],$$

which rearranges to the formula given in [CLS, Prop. 6.4.4]:

$$[V(\rho)] \cdot [V(\sigma)] = \frac{b_\rho}{\beta^+}\frac{D_{\sigma,\mathbf{\Sigma}}}{D_{\tau^+,\mathbf{\Sigma}}}[V(\tau^+)],$$

for any $\rho \in \sigma(1)$.

3.2. **Euler characteristic of toric Deligne-Mumford stacks.**

Definition 3.8. If $\mathcal{X}$ is an n-dimensional smooth complete Deligne-Mumford stack, we define the Euler characteristic of $\mathcal{X}$ as $\chi(\mathcal{X}) := \int_{\mathcal{X}} c_n(\mathbb{T})$, where $\mathbb{T}$ is the tangent bundle of $\mathcal{X}$.

Remark 3.9. The Euler characteristic we define is a rational number and should not be confused with the orbifold Euler characteristic defined by physicists. There is an orbifold version of the Gauss-Bonet theorem which implies that our definition is equivalent to the following topological definition for stacks of the form $[M/G]$,

where G is a finite group and M is a manifold. The manifold M has a CW decomposition where the stabilizer has constant order g_c on the interior of each cell c. Then $\chi([M/G]) := \sum_c \frac{(-1)^{\dim c}}{g_c}$. Since every Deligne-Mumford stack (with finite stabilizer) is locally of the form $[M/G]$, the definition of a Euler characteristic can be extended to arbitrary Deligne-Mumford stacks, but we do not need this here.

The following is well-known, but we include a proof for lack of a suitable reference.

Proposition 3.10. *Let $\mathbf{\Sigma} = (N, \Sigma, \{v_1, v_2, \ldots, v_n\})$ be a complete simplicial stacky fan (i.e. Σ is simplicial and complete). Let $\Sigma_{\max}$ denote the set of maximal cones in Σ. For a maximal cone $\sigma \in \Sigma_{\max}$, let D_σ denote the multiplicity of σ. Then*

$$\chi(\mathcal{X}(\mathbf{\Sigma})) = \sum_{\sigma \in \Sigma_{\max}} \frac{1}{D_{\sigma, \mathbf{\Sigma}}}.$$

Proof. Let $\mathcal{X} = \mathcal{X}(\mathbf{\Sigma}) = [C/G]$, where $C = C(\Sigma) \subseteq \mathbb{A}^n = \operatorname{Spec} \mathbb{C}[X_1, \ldots, X_n]$ is the Cox space of Σ and $G = G(\mathbf{\Sigma})$ is as in Section 2.1. The coarse moduli space of $\mathcal{X}$ is $X = X(\Sigma)$, the toric variety associated to Σ. The equivariant tangent bundle T_C is $\oplus \mathcal{O}(X_i)$ ([EG03, p. 3751]), so

$$\chi(\mathcal{X}(\mathbf{\Sigma})) = \int_C \prod_{i=1}^n (1 + x_i) = \int_X \phi(\prod(1 + x_i)).$$

If $v_{i_1}, \ldots, v_{i_s}$ are not contained in a cone of Σ, then $[V(X_{i_1}, \ldots, X_{i_s})]_G = 0$ in $A_G^*(C)$ (Proposition 2.2). Hence if we expand the product $\prod_{i=1}^n (1 + x_i)$, the only monomials $x_{i_1} \cdots x_{i_s}$ that can contribute to $\int_X \phi(\prod 1 + x_i)$ are ones where $v_{i_1}, \ldots, v_{i_s}$ span a maximal cone (we require maximal cones since the integral only considers the dimension 0 part of the cycle). In this case, the image of $V(X_{i_1}, \ldots, X_{i_s})$ is a point in $X(\Sigma)$, and Proposition 3.2 tells us to assign a factor of $\frac{1}{D_{\sigma, \mathbf{\Sigma}}}$ to the class [pt] of this point. The usual identification of $A_0(X(\Sigma))$ with $\mathbb{Z}$ sending [pt] $\in A^*(X(\Sigma))$ to $1 \in \mathbb{Z}$ gives the result. $\qquad \square$

Remark 3.11. Note that Proposition 3.10 recovers the result (e.g. [CLS, Theorem 12.3.9]) that the Euler characteristic of a smooth toric variety is the number of maximal cones.

The following result expresses how the Euler characteristic increases after a stacky star subdivision [EM, Definition 4.1] of a simplicial stacky fan.

Lemma 3.12. *Let $\mathbf{\Sigma} = (\mathbb{Z}^d, \Sigma, \{v_1, \ldots, v_n\})$ be a complete simplicial stacky fan and let σ be a cone in Σ. Let $\mathbf{\Sigma}_\sigma = (\mathbb{Z}^d, \Sigma_\sigma, \{v_0, v_1, \ldots, v_n\})$ be the stacky fan formed by stacky star subdivision of $\mathbf{\Sigma}$ with respect to σ. Then $\chi(\mathcal{X}(\mathbf{\Sigma}_\sigma)) = \chi(\mathcal{X}(\mathbf{\Sigma})) + \frac{s-1}{D_{\sigma, \mathbf{\Sigma}}}$, where $s = \dim \sigma$.*

Proof. Label the v_i such that $\sigma = \langle v_1, \ldots, v_s \rangle$. Hence $v_0 = v_1 + \cdots + v_s$. For $1 \leq i \leq s$, let $\sigma_i = \langle v_1, \ldots, \widehat{v_i}, \ldots, v_s \rangle$ be the cone of Σ_σ spanned by v_0 and all the rays of σ except for $\langle v_i \rangle$. Then by properties of determinants, we have $D_{\sigma_i, \mathbf{\Sigma}_\sigma} = D_{\sigma, \mathbf{\Sigma}}$. Then apply Proposition 3.10. $\qquad \square$

4. Integration on Artin toric stacks

We now come to the main definitions of the paper. Let $\mathbf{\Sigma}$ be a (not necessarily simplicial) stacky fan and let $\mathcal{X}(\mathbf{\Sigma})$ be the associated Artin toric stack. By [EM, Theorem 5.2] there is a simplicial stacky fan $\mathbf{\Sigma}'$ canonically obtained from $\mathbf{\Sigma}$ by stacky star subdivisions and a commutative diagram of stacks and toric varieties

$$
\begin{array}{ccc}
\mathcal{X}(\mathbf{\Sigma}') & \overset{f}{\to} & \mathcal{X}(\mathbf{\Sigma}) \\
\pi' \downarrow & & \pi \downarrow \\
X(\Sigma') & \overset{g}{\to} & X(\Sigma)
\end{array}
$$

where f is birational, π' is finite and g is proper and birational.

Definition 4.1. If $\alpha \in A^*(\mathcal{X}(\mathbf{\Sigma}))$, define $\pi_*\alpha := g_*\pi'_*f^*\alpha$. In particular, if the associated toric variety $X(\Sigma)$ is complete, we define for any $\alpha \in A^d(\mathcal{X}(\mathbf{\Sigma}))$

$$
\int_{\mathcal{X}(\mathbf{\Sigma})} \alpha := \int_{\mathcal{X}(\mathbf{\Sigma}')} f^*\alpha.
$$

4.1. **The Euler characteristic of an Artin toric stack with complete good moduli space.** We now turn to the definition of the Euler characteristic of an Artin toric stack. If $\mathcal{X} = [X/G]$ is a smooth Artin stack, then the tangent bundle stack to $\mathcal{X}$ equals the vector bundle stack $\mathbb{T}\mathcal{X} = [TX/\operatorname{Lie}(G)]$. The restriction of $\mathbb{T}\mathcal{X}$ to the open set $\mathcal{X}^{DM}$ where $\mathcal{X}$ is Deligne-Mumford is the usual tangent bundle to $\mathcal{X}^{DM}$. By identifying the Chow groups of $\mathcal{X}$ with the G-equivariant Chow groups of X, we may define the Chern series of $\mathbb{T}\mathcal{X}$ as the formal series $c_t(\mathbb{T}\mathcal{X}) = c_t(TX)c_t(\operatorname{Lie}(G))^{-1} \in \prod_{i=0}^{\infty} A_G^i(X) \otimes \mathbb{Z}[[t]]$. Moreover, when G is diagonalizable the Lie algebra of G is a trivial G-module, so $c_t(\operatorname{Lie}(G)) = 1$ and thus the Chern series of $\mathbb{T}\mathcal{X}$ is actually a polynomial.

We can extend the definition of the Euler characteristic to toric Artin stacks.

Definition 4.2. Let $\mathbf{\Sigma}$ be a stacky fan and suppose that the associated toric variety $X(\Sigma)$ has dimension d. Then we set $\chi(\mathcal{X}(\mathbf{\Sigma})) := \int_{\mathcal{X}(\mathbf{\Sigma})} c_d(\mathbb{T}\mathcal{X})$.

4.2. **Formulas for Euler characteristics of 3-dimensional toric Artin stacks.** As an application of Theorem 3.3, we derive a formula for the Euler characteristic of a 3-dimensional toric Artin stack.

Lemma 4.3. *Let $\mathbf{\Sigma} = (\mathbb{Z}^3, \Sigma, \{v_1, \ldots, v_n\})$ be a 3-dimensional complete stacky fan with only one nonsimplicial cone, call it σ, and label the v_i so that $\sigma = \mathbb{R}_{\geq 0}\langle v_1, \ldots, v_s \rangle$. Let $\mathbf{\Sigma}_\sigma = (\mathbb{Z}^3, \Sigma_\sigma, \{v_0, v_1, \ldots, v_n\})$ be the stacky fan formed by the stacky star subdivision of $\mathbf{\Sigma}$ with respect to σ, and let $f : \mathcal{X}(\mathbf{\Sigma}_\sigma) \to \mathcal{X}(\mathbf{\Sigma})$ be the induced morphism. For $0 \leq i \leq n$, let $y_i \in A^*(\mathcal{X}(\mathbf{\Sigma}_\sigma))$ be the equivariant fundamental class of the coordinate hyperplanes restricted to the Cox space $C(\Sigma_\sigma)$. Let $\Sigma_\sigma(3) \smallsetminus \Sigma(3)$ be the set of maximal cones in Σ_σ that are not in Σ. Then the difference between the Euler characteristic of $\mathcal{X}(\mathbf{\Sigma})$ and the Euler characteristic of its simplicialization $\mathcal{X}(\mathbf{\Sigma}_\sigma)$ is given by*

$$
\tag{11} \chi(\mathcal{X}(\mathbf{\Sigma})) - \chi(\mathcal{X}(\mathbf{\Sigma}_\sigma))
$$

$$
\tag{12} = \sum_{\tau \in \Sigma_\sigma(3) \smallsetminus \Sigma(3)} \frac{s-3}{D_{\tau,\Sigma_\sigma}} + \binom{s-1}{2} \int_{\mathcal{X}(\mathbf{\Sigma}_\sigma)} y_0^2 (\sum_{1 \leq i \leq s} y_i) + \binom{s}{3} \int_{\mathcal{X}(\mathbf{\Sigma}_\sigma)} y_0^3.
$$

Proof. With the notation as in Section 2.1, for $1 \leq i \leq s$, we have $f^*(x_i) = y_0 + y_i$. Also, for $i \geq s$, $f^*(x_i) = y_i$. Then $\chi(\mathcal{X}(\mathbf{\Sigma})) = \int_{\mathcal{X}(\mathbf{\Sigma}_\sigma)} f^*(\prod_{i=1}^n (1 + x_i)) = \int_{\mathcal{X}(\mathbf{\Sigma}_\sigma)} (\prod_{i=1}^s (1 + y_0 + y_i)) \prod_{i>s} (1 + y_i)$.

By Proposition 2.2 and equations (2) and (1), we have that $y_{i_1} \cdots y_{i_t} = 0$ in $A_G^*(C)$ if $v_{i_1}, \ldots, v_{i_t}$ are not contained in a cone. In particular, $y_0 y_i = 0$ for any $i > s$. Hence when we expand $(\prod_{i=1}^s (1 + y_0 + y_i)) \prod_{i>s} (1 + y_i)$ and collect the degree 3, we get (note that some monomials in the following equation are zero)

$$\int_{\mathcal{X}(\mathbf{\Sigma}_\sigma)} \left(\sum_{0<i<j<k} y_i y_j y_k + \sum_{0<j<k\leq s} \binom{s-2}{1} y_0 y_j y_k + \sum_{0<k\leq s} \binom{s-1}{2} y_0^2 y_k + \binom{s}{3} y_0^3 \right).$$

We have $(\sum_{0<i<j<k} y_i y_j y_k) + (\sum_{0<j<k\leq s} y_0 y_j y_k) = \chi(\mathcal{X}(\mathbf{\Sigma}_\sigma))$. $\qquad\square$

Remark 4.4. The above proof easily extends to the case when Σ has more than one nonsimplicial cone. Suppose σ_1 and σ_2 are two different nonsimplicial cones of Σ. Then the above proof shows that the equivariant fundamental class of the exceptional divisor of the blowup of $V(\sigma_1)$ (resp. $V(\sigma_2)$) has nonzero product only with divisors coming from rays of σ_1 (resp. σ_2).

By applying Theorem 3.3, we can give a combinatorial formula to the self-intersection integrals on the right-hand side of equation (11).

Fix i satisfying $1 \leq i \leq s$. We will give a formula for $\int_{\mathcal{X}(\mathbf{\Sigma}_\sigma)} y_0^2 y_i$. Let v_i^+ and v_i^- be the two lattice vectors (among $v_1, \ldots, v_s$) such that $\tau_i^+ = \mathbb{R}_{\geq 0}\langle v_i^+, v_0, v_i \rangle$ and $\tau_i^- = \mathbb{R}_{\geq 0}\langle v_i^-, v_0, v_i \rangle$ are the two maximal cones in Σ_σ having $\gamma_i = \mathbb{R}_{\geq 0}\langle v_0, v_i \rangle$ as a common face. Since τ_i^+ and τ_i^- are simplicial, there is a relation

(13) $$\beta_i^+ v_i^+ + b_{0,i} v_0 + b_i v_i + \beta_i^- v_i^- = 0,$$

where $\beta_i^+, b_{0,i}, b_i, \beta_i^-$ are rational numbers. Then by Corollary 3.6,

(14) $$\int_{\mathcal{X}(\mathbf{\Sigma}_\sigma)} y_0^2 y_i = \frac{b_{0,i}}{\beta_i^+ D_{\tau_i^+, \mathbf{\Sigma}_\sigma}} = \frac{b_{0,i}}{\beta_i^- D_{\tau_i^-, \mathbf{\Sigma}_\sigma}}.$$

Finally, to compute the triple self-intersection integral $\int_{\mathcal{X}(\mathbf{\Sigma}_\sigma)} y_0^3$, one could use Theorem 3.3. However, because we have restricted the dimension of Σ to 3, we can instead use a relation of rational equivalence and reduce the computation to the case of double self-intersection integrals just computed. Namely, pick some $m_\sigma \in M$ such that $\langle m_\sigma, v_0 \rangle \neq 0$ (there exists such an m_σ since $v_0 \neq 0$ since v_0 lies in the interior of σ). Then the rational equivalence relation $div(\chi^{m_\sigma}) = 0$ can be written as $\sum_{0 \leq i \leq n} \langle m_\sigma, v_i \rangle y_i = 0$. Hence

$$y_0 = \frac{-1}{\langle m_\sigma, v_0 \rangle} \sum_{1 \leq i \leq n} \langle m_\sigma, v_i \rangle y_i.$$

Multiplying by y_0^2 and using the fact that $y_0 y_i = 0$ for $i > s$ gives

$$\int_{\mathcal{X}(\mathbf{\Sigma}_\sigma)} y_0^3 = \frac{-1}{\langle m_\sigma, v_0 \rangle} \int_{\mathcal{X}(\mathbf{\Sigma}_\sigma)} \sum_{1 \leq i \leq s} \langle m_\sigma, v_i \rangle y_0^2 y_i.$$

Using Remark 4.4, we combine the above formulas to deduce the following result:

Theorem 4.5. *Let* $\boldsymbol{\Sigma} = (\mathbb{Z}^3, \Sigma, \{v_\rho\}_{\rho \in \Sigma(1)})$ *be a 3-dimensional complete stacky fan. Let NS be the set of nonsimplicial cones of Σ. Let*

$$\boldsymbol{\Sigma}_{\mathbf{simp}} = (\mathbb{Z}^3, \Sigma_{\mathbf{simp}}, \{v_\rho\}_{\rho \in \Sigma_{\mathbf{simp}}(1)})$$

be the stacky fan formed by the stacky star subdivision of $\boldsymbol{\Sigma}$ with respect to the cones in NS, and let $f : \mathcal{X}(\boldsymbol{\Sigma}_{\mathbf{simp}}) \to \mathcal{X}(\boldsymbol{\Sigma})$ be the induced morphism.

For $\sigma \in NS$, let $s_\sigma = |\sigma(1)|$ be the number of rays in σ, let $v_\sigma = \sum_{\rho \in \sigma(1)} v_\rho$, let $m_\sigma \in M$ be such that $\langle m_\sigma, v_\sigma \rangle \neq 0$, and regard $\Sigma_\sigma \subseteq \Sigma_{\mathbf{simp}}$. For each $\rho \in \sigma(1)$ let v_ρ^+ and v_ρ^- be the two lattice vectors such that $\tau_\rho^+ = \mathbb{R}_{\geq 0}\langle v_\rho^+, v_\sigma, v_\rho \rangle$ and $\tau_\rho^- = \mathbb{R}_{\geq 0}\langle v_\rho^-, v_\sigma, v_\rho \rangle$ are the two maximal cones in Σ_σ having $\gamma_\rho = \mathbb{R}_{\geq 0}\langle v_\sigma, v_\rho \rangle$ as a common face. Let $\beta_\rho^+, b_{\sigma,\rho}, b_\rho, \beta_\rho^-$ be rational numbers such that

$$(15) \qquad \beta_\rho^+ v_\rho^+ + b_{\sigma,\rho} v_\sigma + b_\rho v_\rho + \beta_\rho^- v_\rho^- = 0.$$

Let $\Sigma_\sigma(3) \smallsetminus \Sigma(3)$ be the set of maximal cones in Σ_σ that are not in Σ. Let $D_{\sigma,\boldsymbol{\Sigma}}$ be as in Definition 3.1. Then the difference between the Euler characteristic of $\mathcal{X}(\boldsymbol{\Sigma})$ and the Euler characteristic of its simplicialization $\mathcal{X}(\boldsymbol{\Sigma}_{\mathbf{simp}})$ is given by

$$\chi(\mathcal{X}(\boldsymbol{\Sigma})) - \chi(\mathcal{X}(\boldsymbol{\Sigma}_{\mathbf{simp}}))$$

$$= \sum_{\sigma \in NS} \left(\sum_{\tau \in \Sigma_\sigma(3) \smallsetminus \Sigma(3)} \frac{s_\sigma - 3}{D_{\tau,\Sigma_\sigma}} + \sum_{\rho \in \sigma(1)} \left(\binom{s_\sigma - 1}{2} - \binom{s_\sigma}{3} \frac{\langle m_\sigma, v_\rho \rangle}{\langle m_\sigma, v_\sigma \rangle} \right) \frac{b_{\sigma,\rho}}{\beta_\rho^+ D_{\tau_\rho^+, \Sigma_\sigma}} \right).$$

As an example, we use this formula to compute the Euler characteristic of the nonsimplicial toric stack considered in Example 3.5.

Example 4.6. We continue with Example 3.5, and let $f : \mathcal{X}(\boldsymbol{\Sigma}_\sigma) \to \mathcal{X}(\boldsymbol{\Sigma})$ be the morphism induced by the stacky star subdivision of Σ with respect to σ. The multiplicities of the maximal cones are $D125 = D235 = 2$, $D345 = D145 = 1$, $D012 = D023 = 7$, $D034 = D014 = 5$. Hence by Proposition 3.10, we have

$$\chi(\mathcal{X}(\boldsymbol{\Sigma}_\sigma)) = \frac{1}{7} + \frac{1}{7} + \frac{1}{5} + \frac{1}{5} + \frac{1}{2} + \frac{1}{2} + \frac{1}{1} + \frac{1}{1} = \frac{129}{35}.$$

Also, $NS = \{\sigma\}$ and $s_\sigma = 4$, so

$$\sum_{\tau \in \Sigma_\sigma(3) \smallsetminus \Sigma(3)} \frac{s_\sigma - 3}{D_{\tau,\Sigma_\sigma}} = \frac{1}{7} + \frac{1}{7} + \frac{1}{5} + \frac{1}{5} = \frac{24}{35}.$$

Since $v_\sigma = v_0 = (0, 1, 4)$, we may take $m_\sigma = (0, 1, 0)$, so $\langle m_\sigma, v_\sigma \rangle = 1$. There are four rays in $\sigma(1)$, namely $\rho_i = \mathbb{R}_{\geq 0}\langle v_i \rangle$ for $i = 1, 2, 3, 4$, and for each of them we compute

$$(16) \qquad \left(\binom{s_\sigma - 1}{2} - \binom{s_\sigma}{3} \frac{\langle m_\sigma, v_\rho \rangle}{\langle m_\sigma, v_\sigma \rangle} \right) \frac{b_{\sigma,\rho}}{\beta_\rho^+ D_{\tau_\rho^+, \boldsymbol{\Sigma}_\sigma}}.$$

For $\rho = \rho_1$, we have $\gamma_{\rho_1} = \mathbb{R}_{\geq 0}\langle v_0, v_1 \rangle$, take $v_\rho^+ = v_2$, $v_\rho^- = v_4$, and set $\beta_\rho^+ = 1$. Then equation (15) becomes $v_2 + b_{\sigma,\rho_1} v_0 + b_{\rho_1} v_1 + \beta_4 v_4 = 0$. Substituting in the coordinates of each v_i, this equation becomes a system of three equations in the three unknowns $b_{\sigma,\rho_1}, b_{\rho_1}, \beta_4$. Solving we get $b_{\sigma,\rho_1} = \frac{-3}{5}$ (and $b_{\rho_1} = 0, \beta_4 = \frac{7}{5}$, but we won't need them). Hence for $\rho = \rho_1$, equation (16) becomes

$$(17) \qquad \left(\binom{4-1}{2} - \binom{4}{3} \frac{0}{1} \right) \frac{\frac{-3}{5}}{1 \cdot 7} = \frac{-9}{35}.$$

Similarly, computations for $\rho = \rho_2, \rho_3, \rho_4$ yield $\frac{20}{49}, \frac{-9}{35}, \frac{-14}{25}$, respectively.

Hence the Euler characteristic $\chi(\mathcal{X}(\boldsymbol{\Sigma})) = \frac{129}{35} + \frac{24}{35} - \frac{9}{35} + \frac{20}{49} - \frac{9}{35} - \frac{14}{25} = \frac{4539}{1225}$.

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