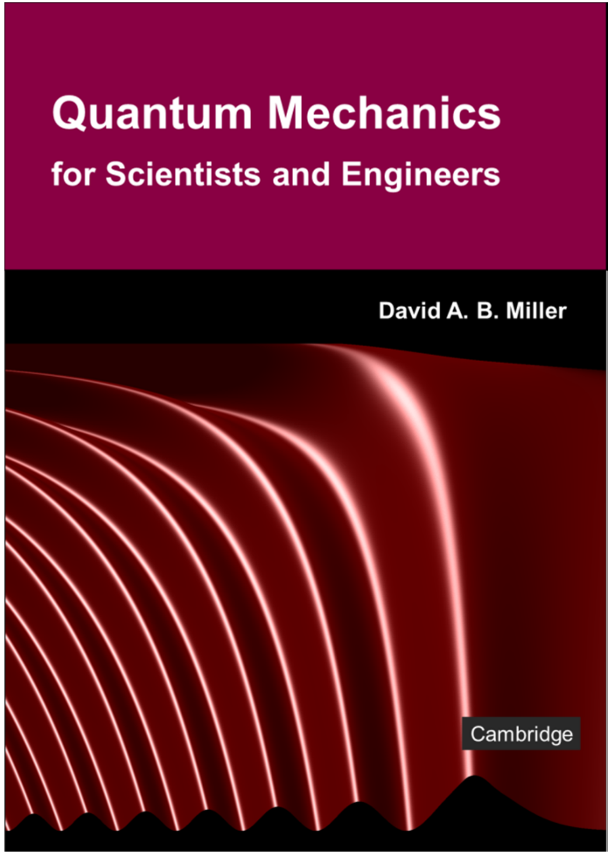
7.1 Angular momentum

Slides: Video 7.1.1 Angular momentum operators

Text reference: Quantum Mechanics for Scientists and Engineers

Chapter 9 introduction and Section 9.1 (first part)



Angular momentum

Angular momentum operators

Quantum mechanics for scientists and engineers

David Miller

Angular momentum operators - preview

We will have operators corresponding to angular momentum about different orthogonal axes

$\hat{L}_x$, $\hat{L}_y$, and $\hat{L}_z$

though they will not commute with one another

in contrast to the linear momentum operators

for the different coordinate directions

$\hat{p}_x$, $\hat{p}_y$, and $\hat{p}_z$

which do commute

Angular momentum operators - preview

We will, however, find another useful angular momentum operator, $\hat{L}^2$

which does commute separately with each of $\hat{L}_x$, $\hat{L}_y$, and $\hat{L}_z$

The eigenfunctions for $\hat{L}_x$, $\hat{L}_y$, and $\hat{L}_z$ are simple

Those for $\hat{L}^2$

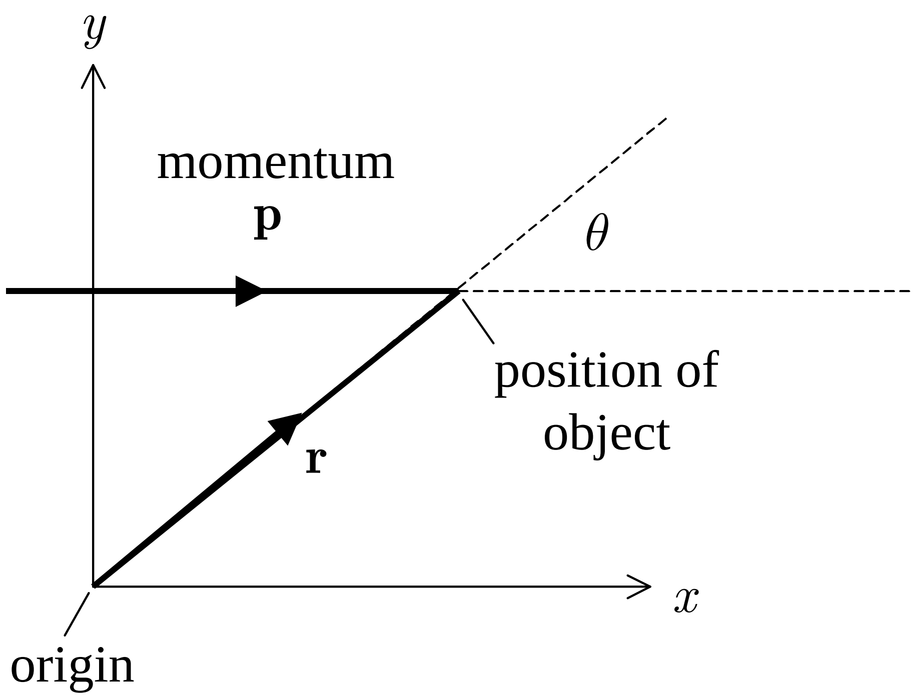
the spherical harmonics, are more complicated

but can be understood relatively simply

and form the angular shapes of the hydrogen atom orbitals

Classical angular momentum

The classical angular momentum
of a small object
of (vector) linear momentum $\mathbf{p}$
centered at a point given by the vector displacement $\mathbf{r}$ relative to some origin
is $\mathbf{L} = \mathbf{r} \times \mathbf{p}$



Vector cross product

As usual

$$\mathbf{C} = \mathbf{A} \times \mathbf{B} \equiv \mathbf{c} AB \sin\theta \equiv \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

$$\equiv \mathbf{i}(A_y B_z - A_z B_y) - \mathbf{j}(A_x B_z - A_z B_x) + \mathbf{k}(A_x B_y - A_y B_x)$$

where $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ are unit vectors in x, y, and z directions

and A_x is the component of $\mathbf{A}$ in the x direction

and similarly for the y and z directions

and the components of $\mathbf{B}$

Vector cross product

In

$$\mathbf{C} = \mathbf{A} \times \mathbf{B} \equiv \mathbf{c} AB \sin\theta \equiv \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

$$\equiv \mathbf{i}(A_y B_z - A_z B_y) - \mathbf{j}(A_x B_z - A_z B_x) + \mathbf{k}(A_x B_y - A_y B_x)$$

$\mathbf{C}$ is perpendicular to the plane of $\mathbf{A}$ and $\mathbf{B}$

just as the z axis is perpendicular to the plane containing the x and y axes in right-handed axes

θ is the angle between the vectors $\mathbf{A}$ and $\mathbf{B}$

$\mathbf{c}$ is a unit vector in the direction of the vector $\mathbf{C}$

Vector cross product

Note that, in

$$\mathbf{C} = \mathbf{A} \times \mathbf{B} \equiv \mathbf{c} AB \sin\theta \equiv \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

$$\equiv \mathbf{i}(A_y B_z - A_z B_y) - \mathbf{j}(A_x B_z - A_z B_x) + \mathbf{k}(A_x B_y - A_y B_x)$$

the ordering of the multiplications in the second line is chosen to work also for operators instead of numbers for one or other vector

the sequence of multiplications in each term is always in the sequence of the rows from top to bottom

Angular momentum operators

With classical angular momentum

$$\mathbf{L} = \mathbf{r} \times \mathbf{p}$$

we can explicitly write out the various components

$$L_x = yp_z - zp_y \quad L_y = zp_x - xp_z \quad L_z = xp_y - yp_x$$

Now we can propose a quantum mechanical angular momentum operator $\hat{\mathbf{L}}$

based on substituting the position and momentum operators

$$\hat{\mathbf{L}} = \hat{\mathbf{r}} \times \hat{\mathbf{p}} = -i\hbar\left(\mathbf{r} \times \nabla\right)$$

and similarly write out component operators

Angular momentum operators

Analogously, we obtain three operators

$$\hat{L}_x = \hat{y}\hat{p}_z - \hat{z}\hat{p}_y = -i\hbar\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y} \right)$$

$$\hat{L}_y = \hat{z}\hat{p}_x - \hat{x}\hat{p}_z = -i\hbar\left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z} \right)$$

$$\hat{L}_z = \hat{x}\hat{p}_y - \hat{y}\hat{p}_x = -i\hbar\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x} \right)$$

which are each Hermitian

and so, correspondingly, is the operator $\hat{\mathbf{L}}$ itself

Commutation relations

The operators corresponding to individual coordinate directions obey commutation relations

$$\hat{L}_x\hat{L}_y - \hat{L}_y\hat{L}_x = \left[\hat{L}_x, \hat{L}_y\right] = i\hbar\hat{L}_z$$

$$\hat{L}_y\hat{L}_z - \hat{L}_z\hat{L}_y = \left[\hat{L}_y, \hat{L}_z\right] = i\hbar\hat{L}_x$$

$$\hat{L}_z\hat{L}_x - \hat{L}_x\hat{L}_z = \left[\hat{L}_z, \hat{L}_x\right] = i\hbar\hat{L}_y$$

These individual commutation relations can be written in a more compact form

$$\hat{\mathbf{L}} \times \hat{\mathbf{L}} = i\hbar\hat{\mathbf{L}}$$

Commutation relations

Unlike operators for position and for linear momentum

- the different components of this angular momentum operator do not commute with one another

Though a particle can have simultaneously a well-defined position in both the x and y directions

- or have simultaneously a well-defined momentum in both the x and y directions

a particle cannot in general simultaneously have a well-defined angular momentum component in more than one direction