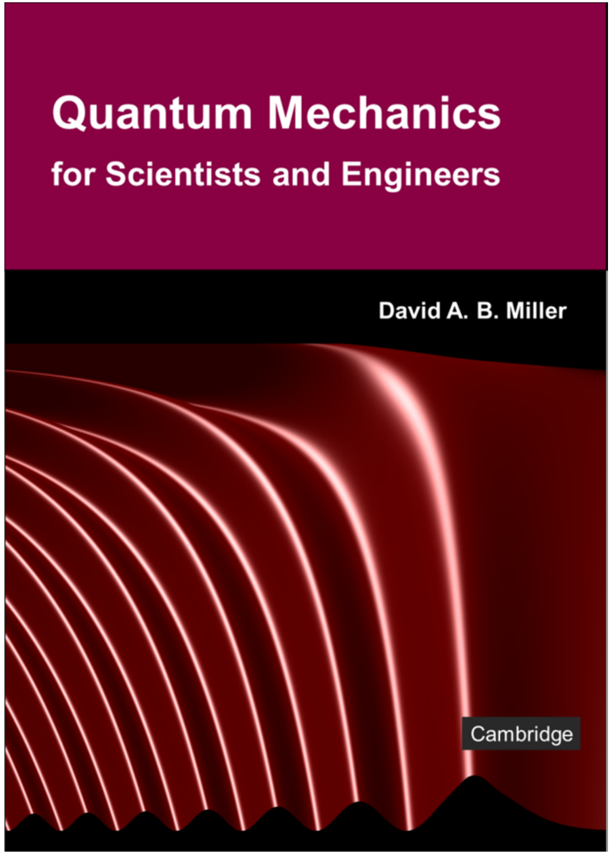
6.3 Operators and quantum mechanics

Slides: Video 6.3.1 Hermitian operators in quantum mechanics

Text reference: Quantum Mechanics for Scientists and Engineers

Section 5.1



Operators and quantum mechanics

Hermitian operators in quantum mechanics

Quantum mechanics for scientists and engineers

David Miller

Commutation of Hermitian operators

For Hermitian operators $\hat{A}$ and $\hat{B}$ representing physical variables

- it is very important to know if they commute
  - i.e., is $\hat{A}\hat{B} = \hat{B}\hat{A}$?

Remember that

- because these linear operators obey the same algebra as matrices
  - in general operators do not commute

Commutator

For quantum mechanics, we formally define an entity

$$\boxed{\left[\hat{A},\hat{B}\right]=\hat{A}\hat{B}-\hat{B}\hat{A}}$$

This entity is called the commutator

An equivalent statement to saying $\hat{A}\hat{B}=\hat{B}\hat{A}$

is then $\left[\hat{A},\hat{B}\right]=0$

Strictly, this should be written $\left[\hat{A},\hat{B}\right]=0\hat{I}$

where $\hat{I}$ is the identity operator

but this is usually omitted

Commutation of operators

If the operators do not commute

then $\left[\hat{A},\hat{B}\right]=0$ does not hold

and in general we can choose to write

$$\left[\hat{A},\hat{B}\right]=i\hat{C}$$

where $\hat{C}$ is sometimes referred to as

the remainder of commutation or

the commutation rest

Commuting operators and sets of eigenfunctions

Operators that commute share the same set of eigenfunctions

and

operators that share the same set of eigenfunctions commute

We will now prove both of these statements

Commuting operators and sets of eigenfunctions

Suppose that operators $\hat{A}$ and $\hat{B}$ commute

and suppose the $|\psi_n\rangle$ are the eigenfunctions of $\hat{A}$

with eigenvalues A_i

Then $\hat{A}\hat{B}|\psi_i\rangle = \hat{B}\hat{A}|\psi_i\rangle = \hat{B}A_i|\psi_i\rangle = A_i\hat{B}|\psi_i\rangle$

i.e., $\hat{A}\left[\hat{B}|\psi_i\rangle\right] = A_i\left[\hat{B}|\psi_i\rangle\right]$

But this means that the vector $\hat{B}|\psi_i\rangle$

is also the eigenvector $|\psi_i\rangle$ or is proportional to it

i.e., for some number B_i

$$\hat{B}|\psi_i\rangle = B_i|\psi_i\rangle$$

Commuting operators and sets of eigenfunctions

This kind of relation $\hat{B}|\psi_i\rangle = B_i|\psi_i\rangle$

holds for all the eigenfunctions $|\psi_i\rangle$

so these eigenfunctions

are also the eigenfunctions of the operator $\hat{B}$

with associated eigenvalues B_i

Hence we have proved the first statement that

operators that commute share the same set of eigenfunctions

Note that the eigenvalues A_i and B_i are not in general equal to one another

Commuting operators and sets of eigenfunctions

Now we consider the statement

operators that share the same set of eigenfunctions commute

Suppose that the Hermitian operators $\hat{A}$ and $\hat{B}$

share the same complete set $|\psi_n\rangle$ of eigenfunctions

with associated sets of eigenvalues A_n and B_n respectively

Then

$$\hat{A}\hat{B}|\psi_i\rangle = \hat{A}B_i|\psi_i\rangle = A_iB_i|\psi_i\rangle$$

and similarly

$$\hat{B}\hat{A}|\psi_i\rangle = \hat{B}A_i|\psi_i\rangle = B_iA_i|\psi_i\rangle$$

Commuting operators and sets of eigenfunctions

Hence, for any function $|f\rangle$

which can always be expanded in this complete set of functions $|\psi_n\rangle$ i.e., $|f\rangle = \sum_i c_i |\psi_i\rangle$

we have

$$\hat{A}\hat{B}|f\rangle = \sum_i c_i A_i B_i |\psi_i\rangle = \sum_i c_i B_i A_i |\psi_i\rangle = \hat{B}\hat{A}|f\rangle$$

Since we have proved this for an arbitrary function

we have proved that the operators commute

hence proving the statement

operators that share the same set of eigenfunctions commute