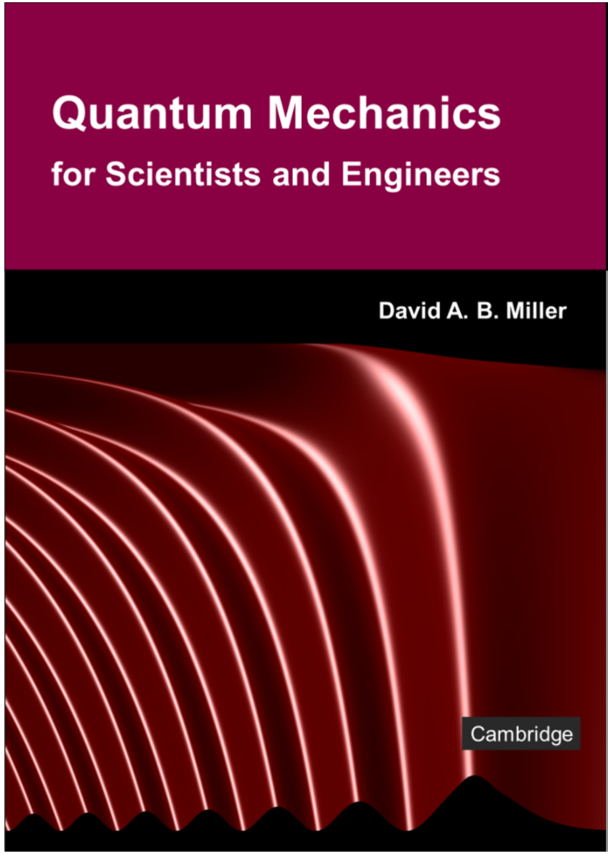
5.1 Uncertainty principle and particle current

Slides: Video 5.1.1 Momentum, position, and the uncertainty principle

Text reference: Quantum Mechanics for Scientists and Engineers

Sections 3.12 – 3.13



Uncertainty principle and particle current

Momentum, position and the uncertainty principle

Quantum mechanics for scientists and engineers

David Miller

Momentum and the momentum operator

For momentum

we write an operator $\hat{p}$

We postulate this can be written as

$$\boxed{\hat{p} \equiv -i\hbar\nabla}$$

with

$$\nabla \equiv \mathbf{x}_o \frac{\partial}{\partial x} + \mathbf{y}_o \frac{\partial}{\partial y} + \mathbf{z}_o \frac{\partial}{\partial z}$$

where $\mathbf{x}_o$, $\mathbf{y}_o$, and $\mathbf{z}_o$ are unit vectors in the x, y, and z directions

Momentum and the momentum operator

With this postulated form $\hat{p} \equiv -i\hbar\nabla$ we find that

$$\frac{\hat{p}^2}{2m} \equiv -\frac{\hbar^2}{2m}\nabla^2$$

and we have a correspondence between the classical notion of the energy E

$$E = \frac{p^2}{2m} + V$$

and the corresponding Hamiltonian operator of the Schrödinger equation

$$\hat{H} = -\frac{\hbar^2}{2m}\nabla^2 + V = \frac{\hat{p}^2}{2m} + V$$

Momentum and the momentum operator

Note that

$$\hat{p}\exp(i\mathbf{k}\cdot\mathbf{r}) = -i\hbar\nabla\exp(i\mathbf{k}\cdot\mathbf{r}) = \hbar\mathbf{k}\exp(i\mathbf{k}\cdot\mathbf{r})$$

This means the plane waves $\exp(i\mathbf{k}\cdot\mathbf{r})$ are the eigenfunctions of the operator $\hat{p}$

with eigenvalues $\hbar\mathbf{k}$

We can therefore say for these eigenstates that

the momentum is $\mathbf{p} = \hbar\mathbf{k}$

Note that $\mathbf{p}$ is a vector, with three components with scalar values

not an operator

Position and the position operator

For the position operator

- the postulated operator is almost trivial when we are working with functions of position
  - It is simply the position vector, $\mathbf{r}$, itself

At least when we are working in a representation that is in terms of position

- we therefore typically do not write $\hat{\mathbf{r}}$
  - though rigorously we should

The operator for the z-component of position would, for example, also simply be z itself

The uncertainty principle

Here we illustrate the position-momentum uncertainty principle by example

We have looked at a Gaussian wavepacket before

- We could write this as a sum over waves of different k-values, with Gaussian weights
  - or we could take the limit of that process by using an integration

$$\Psi_G(z,t) \propto \int_k \exp\left[-\left(\frac{k-\bar{k}}{2\Delta k}\right)^2\right] \exp\left\{-i\left[\omega(k)t - kz\right]\right\} dk$$

The uncertainty principle

We could rewrite

$$\Psi_G(z,t) \propto \int_k \exp\left[-\left(\frac{k-\bar{k}}{2\Delta k}\right)^2\right] \exp\left\{-i\left[\omega(k)t - kz\right]\right\} dk$$

at time t = 0 as

$$\Psi(z,0) = \int_k \Psi_k(k) \exp(ikz)\, dk$$

where

$$\Psi_k(k) \propto \exp\left[-\left(\frac{k-\bar{k}}{2\Delta k}\right)^2\right]$$

The uncertainty principle

In

$$\Psi_k(k) \propto \exp\left[-\left(\frac{k-\bar{k}}{2\Delta k}\right)^2\right]$$

$\Psi_k(k)$ is the representation of the wavefunction in k space

$|\Psi_k(k)|^2$ is the probability P_k

strictly, the probability density

that if we measured the momentum of the particle

actually the z component of momentum

it would be found to have value $\hbar k$

The uncertainty principle

With

$$\Psi_k(k) \propto \exp\left[-\left(\frac{k-\bar{k}}{2\Delta k}\right)^2\right]$$

then this probability (density) of finding a value $\hbar k$ for the momentum would be

$$P_k = \left|\Psi_k(k)\right|^2 \propto \exp\left[-\frac{\left(k-\bar{k}\right)^2}{2\left(\Delta k\right)^2}\right]$$

This Gaussian corresponds to the statistical Gaussian probability distribution

with standard deviation Δk

The uncertainty principle

Note also that $\Psi(z,0) = \int_k \Psi_k(k) \exp(ikz)\, dk$

is the Fourier transform of $\Psi_k(k)$

and, as is well known

the Fourier transform of a Gaussian is a Gaussian

specifically here

$$\Psi(z,0) \propto \exp\left[-(\Delta k)^2 z^2\right]$$

The uncertainty principle

If we want to rewrite

$$|\Psi(z,0)|^2 \propto \exp\left[-2(\Delta k)^2 z^2\right]$$

in the standard form

$$|\Psi(z,0)|^2 \propto \exp\left[-\frac{z^2}{2(\Delta z)^2}\right]$$

where the parameter Δz

would now be the standard deviation

in the probability distribution for z

then $\Delta k \Delta z = 1/2$

The uncertainty principle

From $\Delta k \Delta z = 1/2$

if we now multiply by $\hbar$ to get the standard deviation we would measure in momentum

we have

$$\Delta p \Delta z = \frac{\hbar}{2}$$

which is the relation between the standard deviations we would see in

measurements of position and measurements of momentum

The uncertainty principle

This relation

$$\Delta p \Delta z = \frac{\hbar}{2}$$

is as good as we can get for a Gaussian

- For example
  - a Gaussian pulse will broaden in space as it propagates
    - even though the range of k values remains the same

The uncertainty principle

It also turns out that the Gaussian shape

is the one with the minimum possible

product of Δp and Δz

So quite generally

$$\boxed{\Delta p \Delta z \geq \frac{\hbar}{2}}$$

which is the uncertainty principle

for position and momentum in

one direction

The uncertainty principle in Fourier analysis

Uncertainty principles are well known in Fourier analysis

- One cannot simultaneously have both
  - a well defined frequency and
  - a well defined time

If a signal is a short pulse

- it is necessarily made up out of a range of frequencies

$$\Delta\omega\Delta t \geq \frac{1}{2}$$

- The shorter the pulse is
  - the larger the range of frequencies