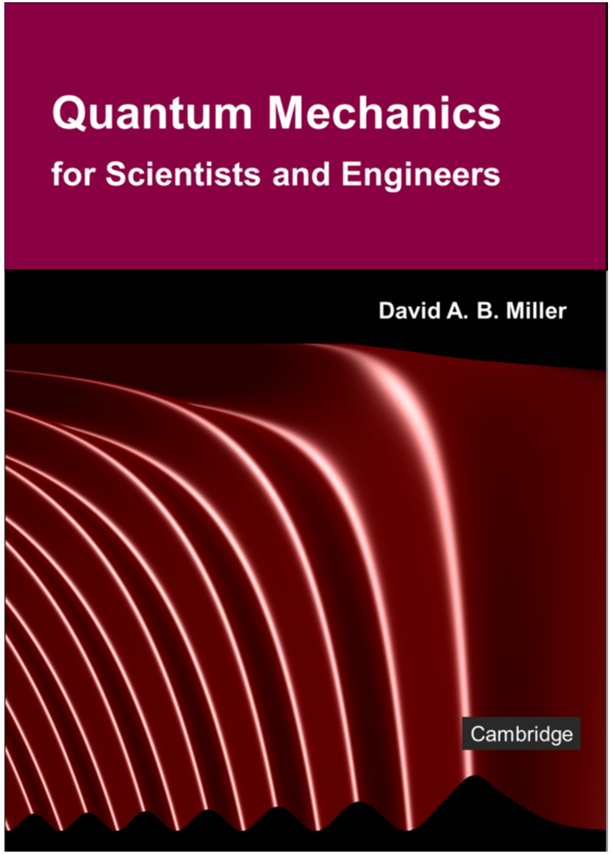
4.2 Wavepackets

Slides: Video 4.2.2 Group velocity

Text reference: Quantum Mechanics for Scientists and Engineers

Section 3.7 ("Group velocity" first part)



Wavepackets

Group velocity

Quantum mechanics for scientists and engineers

David Miller

Group velocity

Consider two waves at different frequencies ω_1 and ω_2

and suppose that the wave velocity v is the same independent of frequency

Then the corresponding wavevector magnitude

$$k = \omega / v$$

is the same for both waves

i.e., $k_1 = \omega_1 / v \qquad k_2 = \omega_2 / v$

Group velocity

If we take two such waves

of equal amplitudes

and add them together

then we get

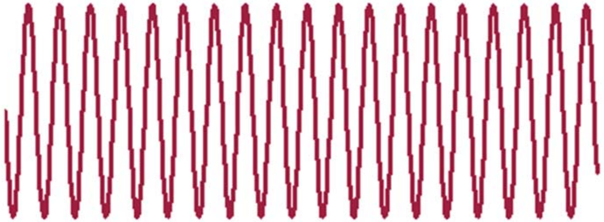
spatial beats

a "spatial envelope"

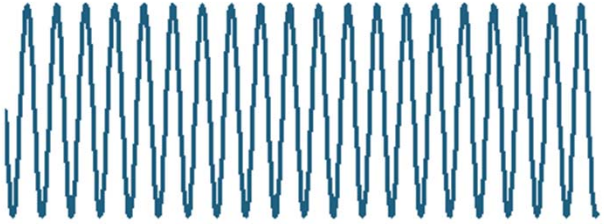
The "envelope" moves at the same speed as the wave

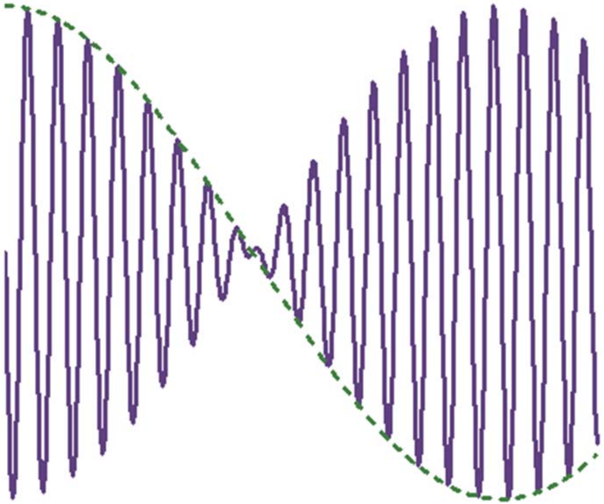
here we chose $v = 1$ for illustration

$$\omega_1 = 0.4$$
$$k_1 = 0.4$$



$$\omega_2 = 0.425$$
$$k_2 = 0.425$$





Group velocity

Algebraically, for two waves at different frequencies

one at frequency $\omega+\delta\omega$ and wavevector $k+\delta k$

one at frequency $\omega-\delta\omega$ and wavevector $k-\delta k$

using complex exponential waves, we get a total wave

$$f(z,t)$$

$$=\exp\left\{-i\left[\left(\omega+\delta\omega\right)t-\left(k+\delta k\right)z\right]\right\}+\exp\left\{-i\left[\left(\omega-\delta\omega\right)t-\left(k-\delta k\right)z\right]\right\}$$

$$=\exp\left[-i\left(\omega t-kz\right)\right]\left\{\exp\left[-i\left(\delta\omega t-\delta kz\right)\right]+\exp\left[+i\left(\delta\omega t-\delta kz\right)\right]\right\}$$

$$=2\cos(\delta\omega t-\delta kz)\exp\left[-i\left(\omega t-kz\right)\right]$$

Group velocity

The algebraic form

$f(z,t) =$

$2\cos(\delta\omega t - \delta k z)\exp\left[-i\left(\omega t - kz\right)\right]$

describes a cosine envelope

multiplying the wave

(here we show the real part)

Note here, because $k = \omega / v$ and $v = \omega / k$

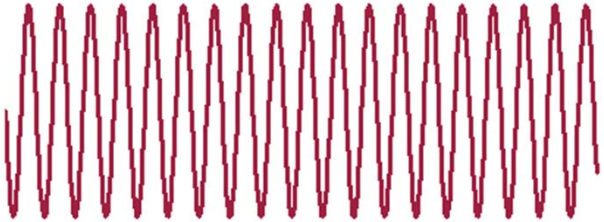
then $\delta k = \delta\omega / v$ and $v = \delta\omega / \delta k$

so the envelope and the wave

move at the same speed

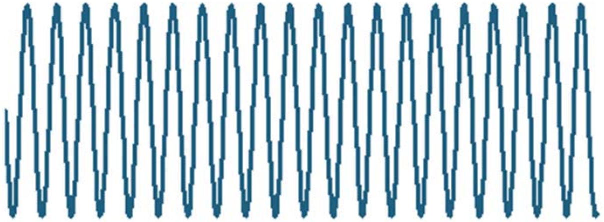
$\omega_1 = 0.4$

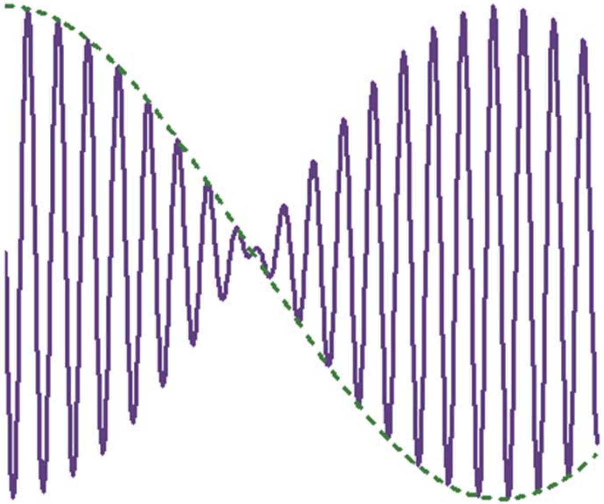
$k_1 = 0.4$



$\omega_2 = 0.425$

$k_2 = 0.425$





Group velocity

But suppose the wave velocity is different for different frequencies

- e.g., suppose the higher frequency wave has a slower velocity
  - so a more than proportionately larger k
    - Then the "envelope velocity"

$$v_g = \delta\omega / \delta k$$

which we will call the **group velocity**

is not the same as the underlying wave velocity

Group velocity

If the higher frequency wave has a lower wave velocity, e.g.,

$$v_2 = \frac{\omega_2}{k_2} = \frac{0.425}{0.4375} = \frac{34}{35} \simeq 0.97$$

then the "envelope" moves at

$$v_g = \delta\omega / \delta k \equiv 0.025 / 0.0375 = 2/3$$

The underlying wave moves at the average speed

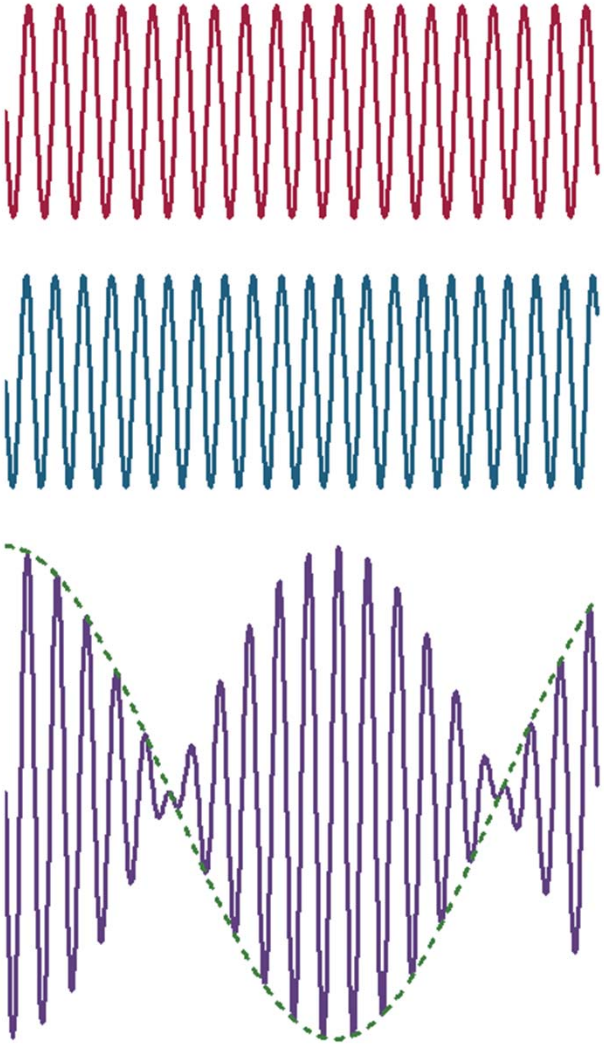
$$\frac{\omega_1 + \omega_2}{2} \bigg/ \frac{k_1 + k_2}{2} = \frac{\omega_1 + \omega_2}{k_1 + k_2} = \frac{66}{67} \simeq 0.985$$

$\omega_1 = 0.4$

$k_1 = 0.4$

$\omega_2 = 0.425$

$k_2 = 0.4375$



Group velocity and phase velocity

We define group velocity as the limit as δk and $\delta\omega$ go to zero

$$v_g = \frac{d\omega}{dk}$$

This concept applies generally for the effective velocity of pulses or "wavepackets" in dispersive media

For clarity, we call

$$v_p = \frac{\omega}{k}$$

for a given frequency ω

the "phase velocity"

Group velocity and phase velocity

$$\boxed{v_g = \frac{d\omega}{dk}}$$

group velocity

$$\boxed{v_p = \frac{\omega}{k}}$$

phase velocity