Differential calculus

Background mathematics review

David Miller

Differential calculus

First derivative

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First derivative

For some function $y(x)$

The (first) derivative is the

"slope"

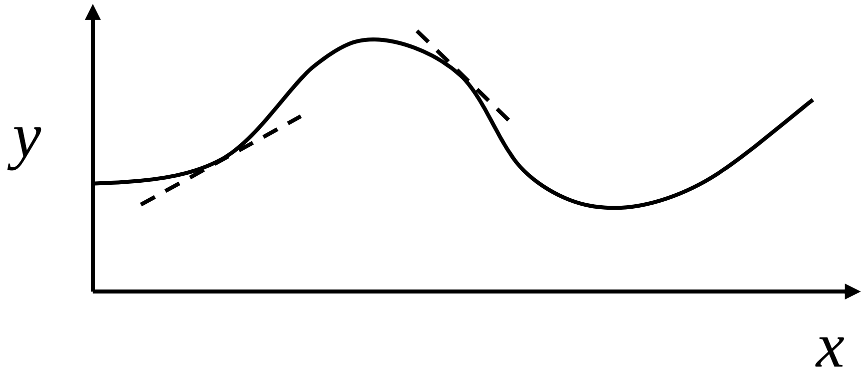
gradient or

rate of change

of y as we change x

If for some small "infinitesimal" change in x, called dx

y changes by some small "infinitesimal" amount dy



the first derivative is written

$$y'(x) = \frac{dy}{dx}$$

The "ratio" notation on the right is "Leibniz notation"

First derivative

The derivative at some specific point x_1 can be written

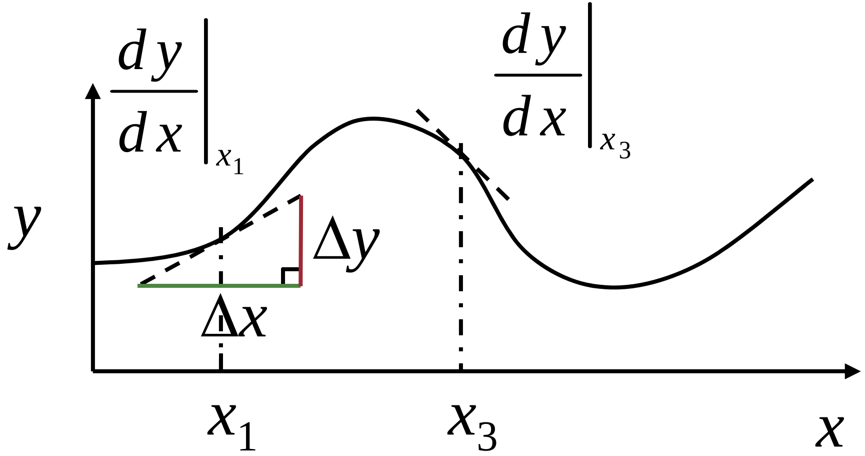
$$y'(x_1) \equiv \left.\frac{dy}{dx}\right|_{x_1}$$

The value of the derivative is the slope of the "tangent" line

the dashed line in the figure

at that point

Equal in value to the tangent $\frac{\Delta y}{\Delta x}$



First derivative

Looking at the slope

$$\frac{y\left(x_1+\frac{\Delta x}{2}\right)-y\left(x_1-\frac{\Delta x}{2}\right)}{\Delta x}$$

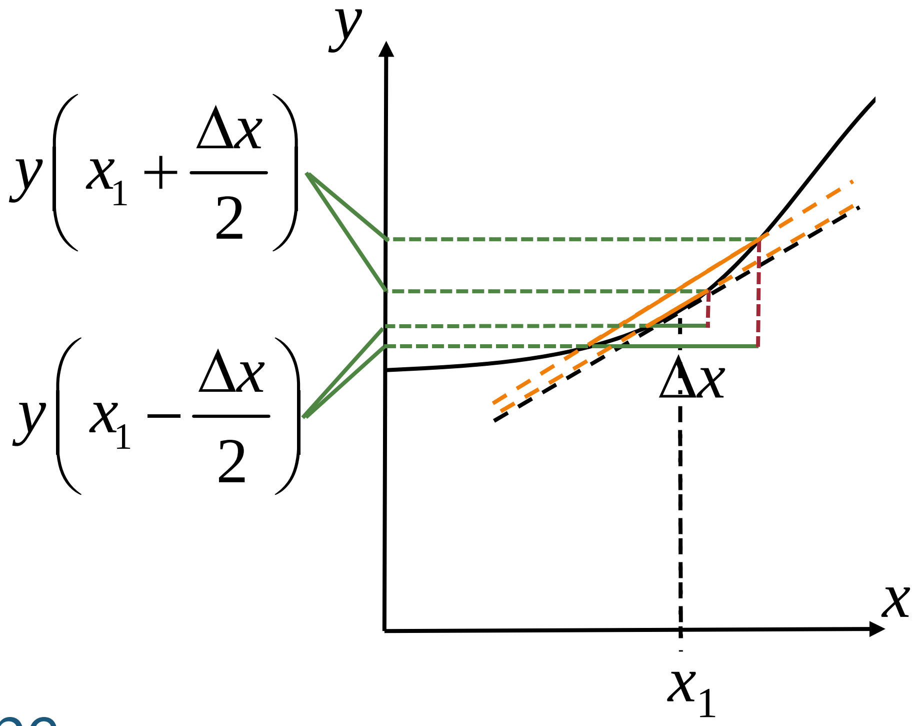
of the "orange" line

as we reduce Δx

the "orange" line slope

becomes closer to the slope

of the "black" tangent line



First derivative

In the limit as Δx becomes very small

i.e., in the limit as Δx "tends to zero"

$$\lim_{\Delta x \to 0}$$

this ratio becomes the (first) derivative

$$\left.\frac{dy}{dx}\right|_{x_1} \equiv \lim_{\Delta x \to 0} \frac{y\left(x_1 + \frac{\Delta x}{2}\right) - y\left(x_1 - \frac{\Delta x}{2}\right)}{\Delta x}$$

Sign of first derivative

If y increases as we increase x

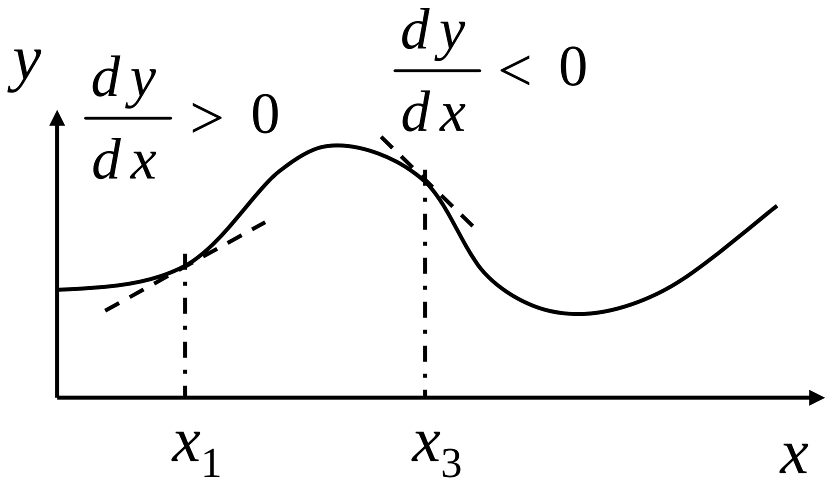
$$\frac{dy}{dx} > 0$$

sloping up to the right

If y decreases as we increase x

$$\frac{dy}{dx} < 0$$

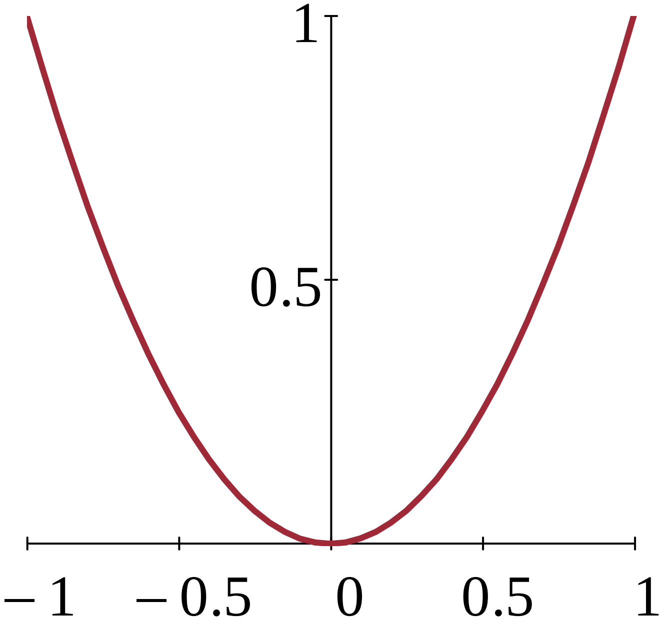
sloping down to the right



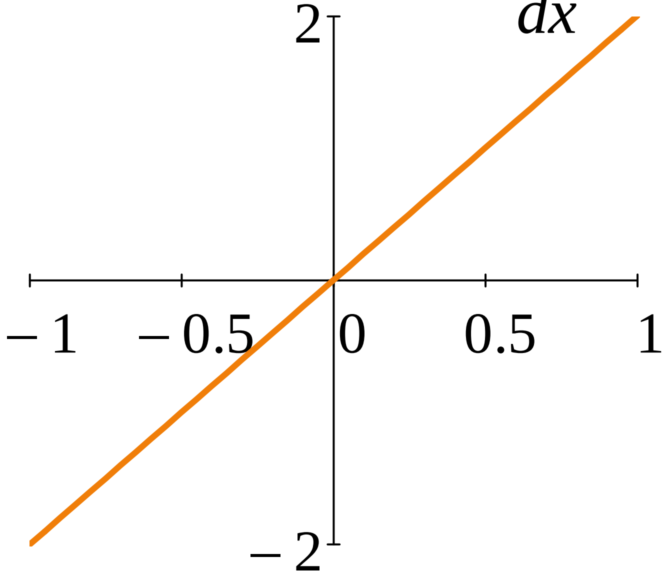
Derivative of a power

$$\boxed{\frac{d}{dx}x^n = nx^{n-1}}$$

$y = x^2$



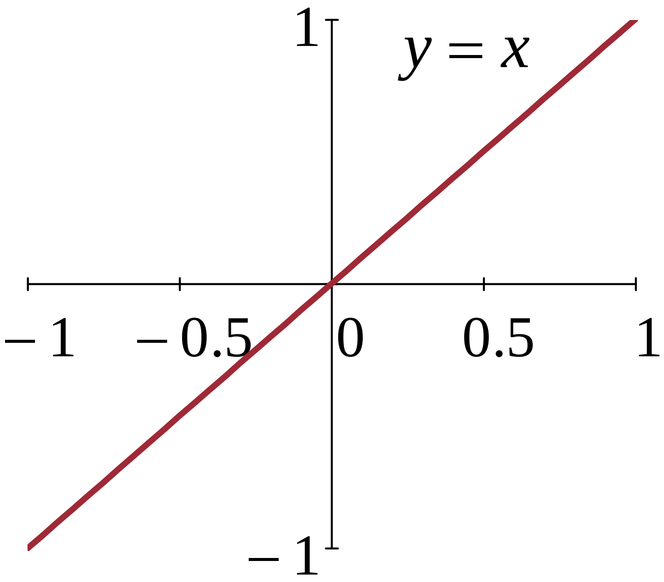
$\frac{dy}{dx} = 2x$

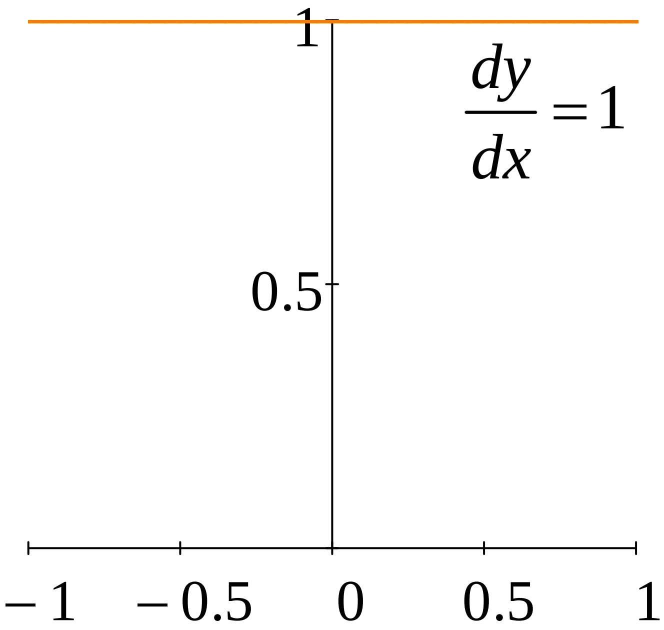


Derivative of a power

The derivative of a straight line is a constant

The straight line has a constant slope

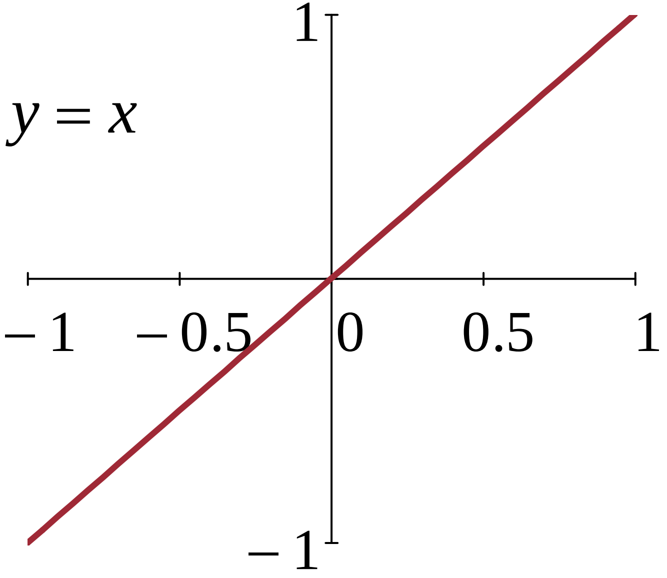


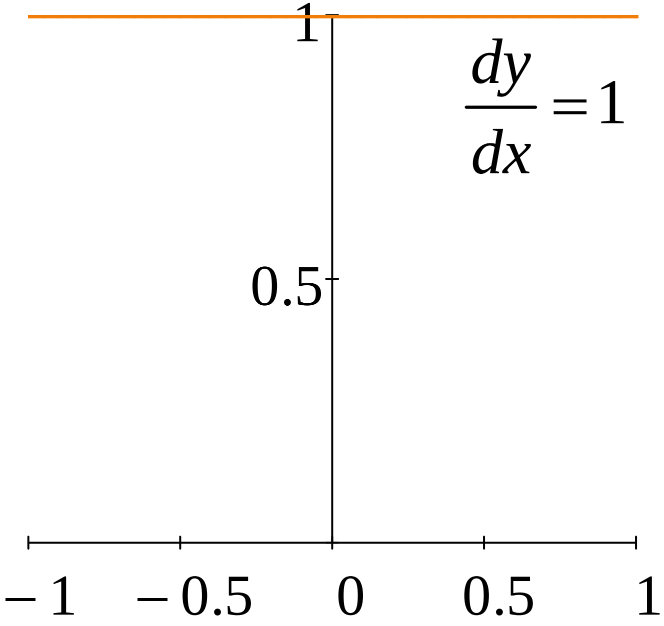


Derivative of a power

The derivative does not depend on the "height"

All these lines have the same slope





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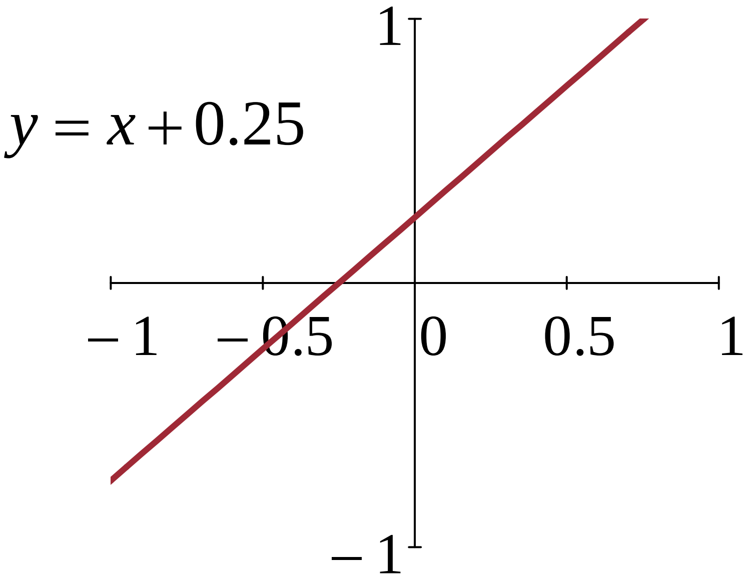

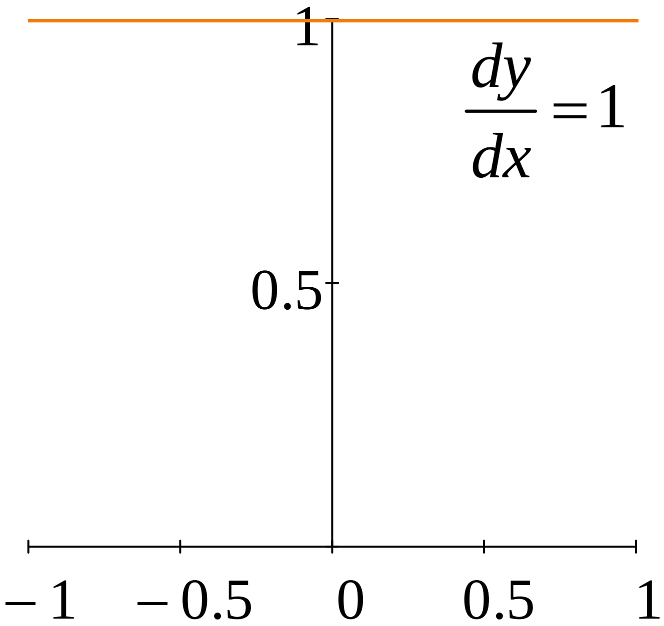

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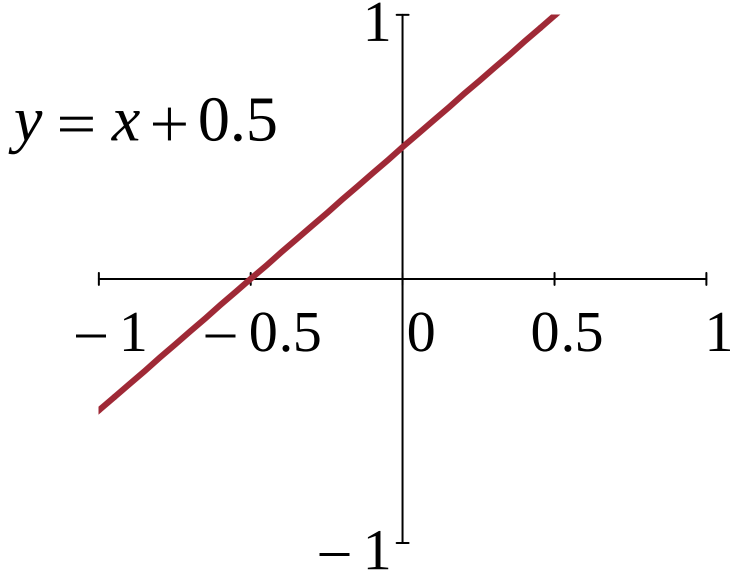
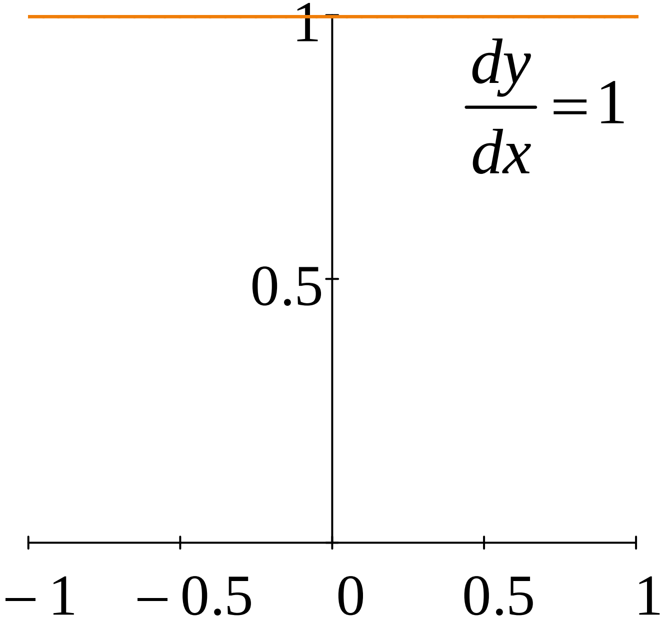
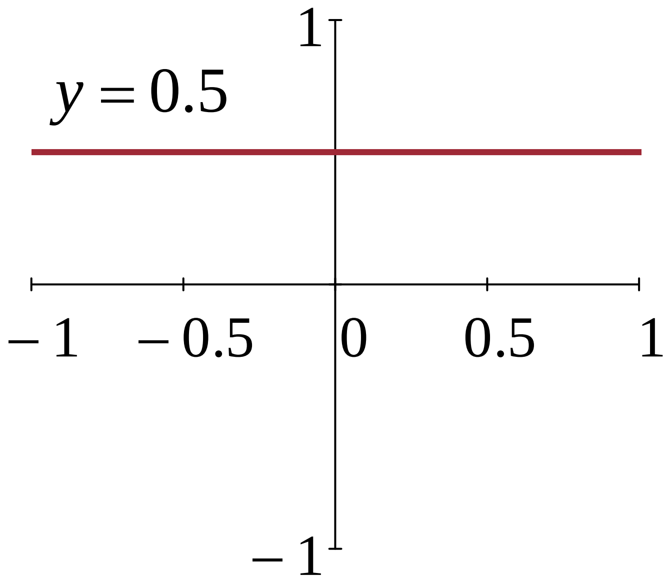


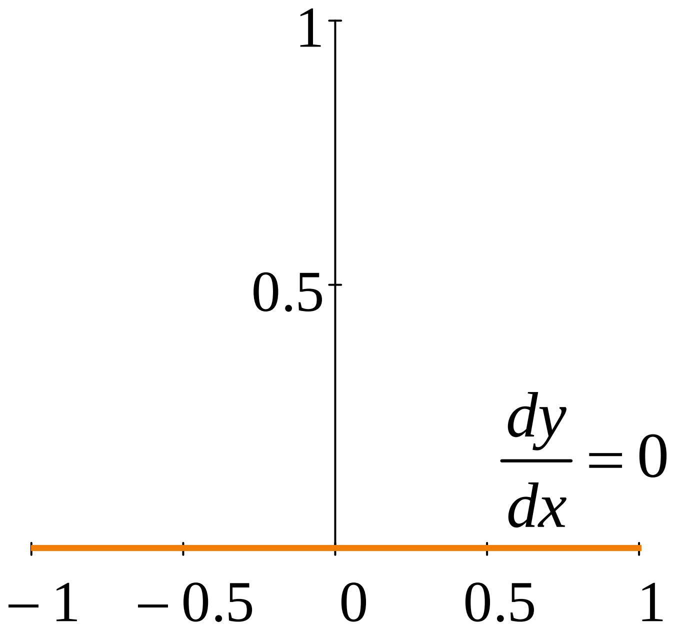


Derivative of a power

The derivative of a constant is zero

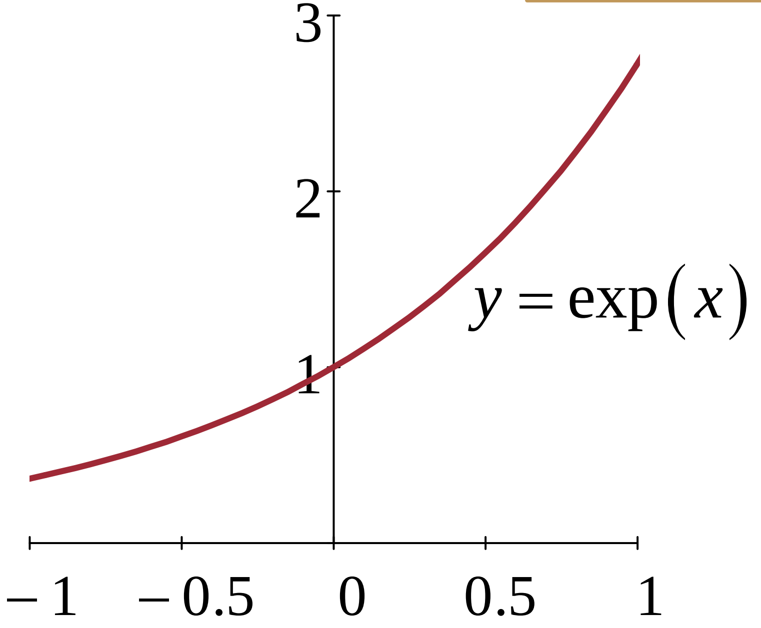
y is not changing with x

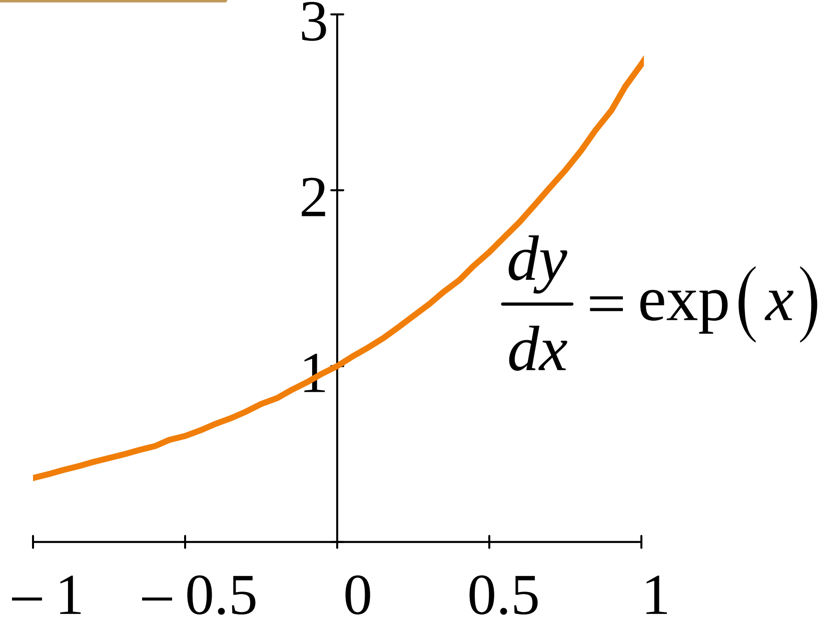




Derivative of an exponential

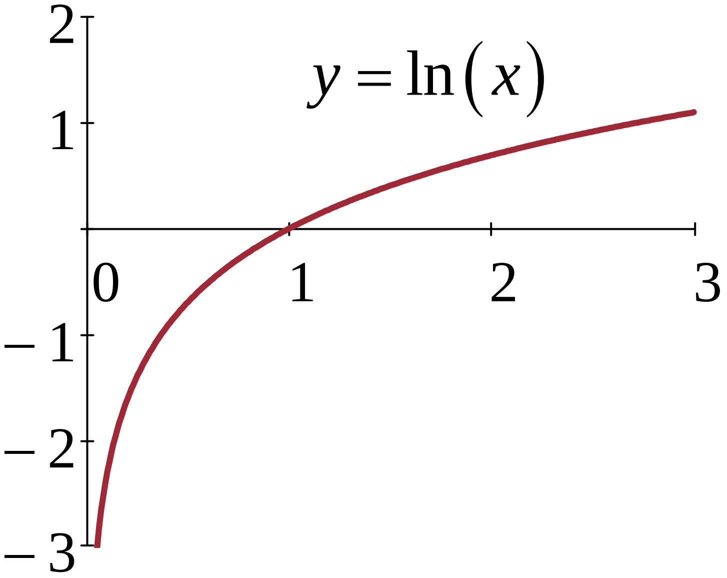
$$\boxed{\frac{d}{dx}\exp(x) = \exp(x)}$$

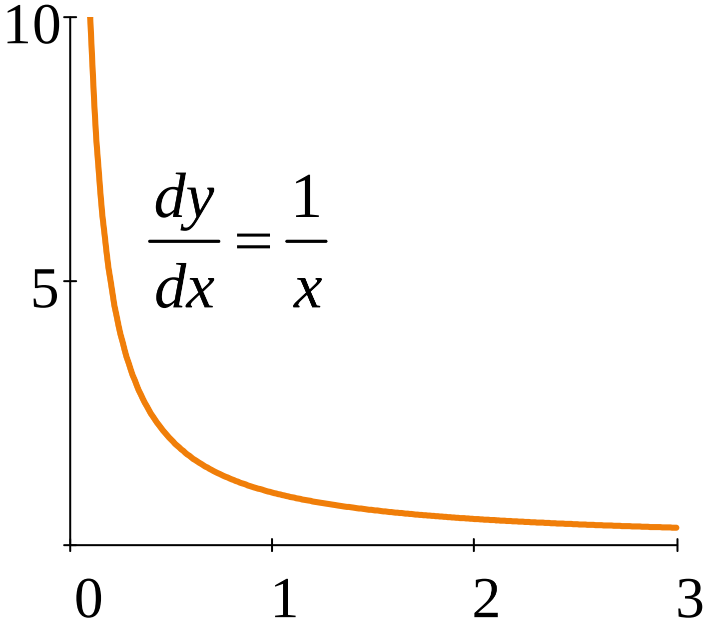




Derivative of a logarithm

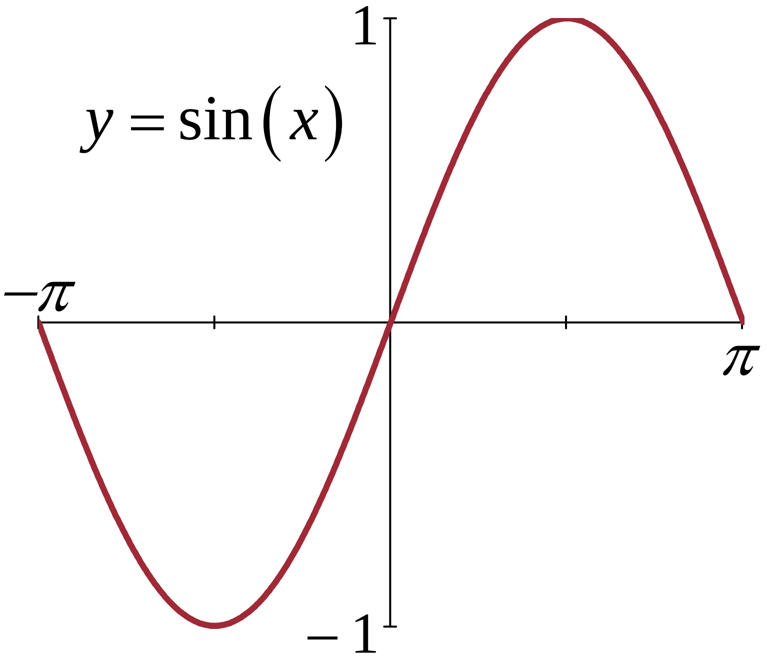
$$\boxed{\frac{d}{dx}\ln(x) = \frac{1}{x}}$$

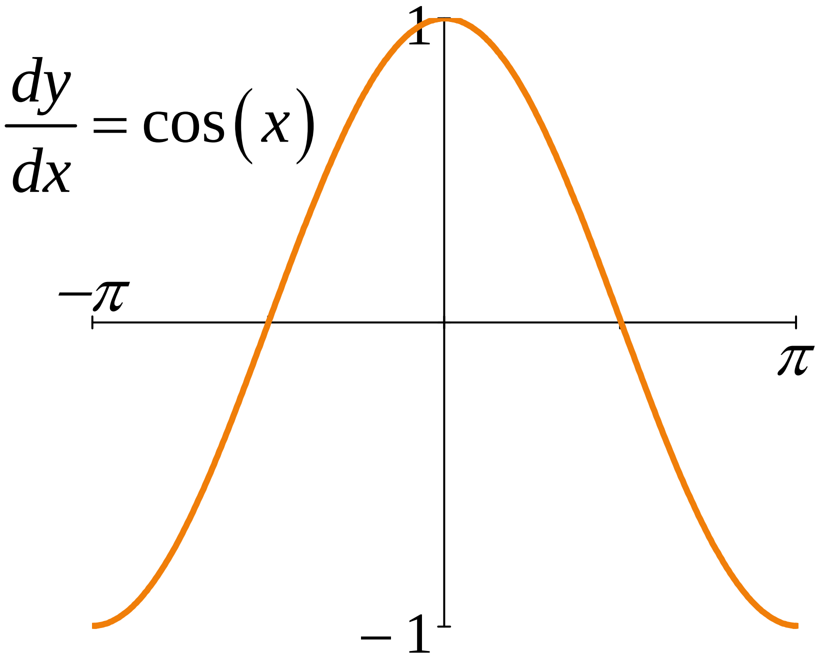




Derivatives of sine and cosine

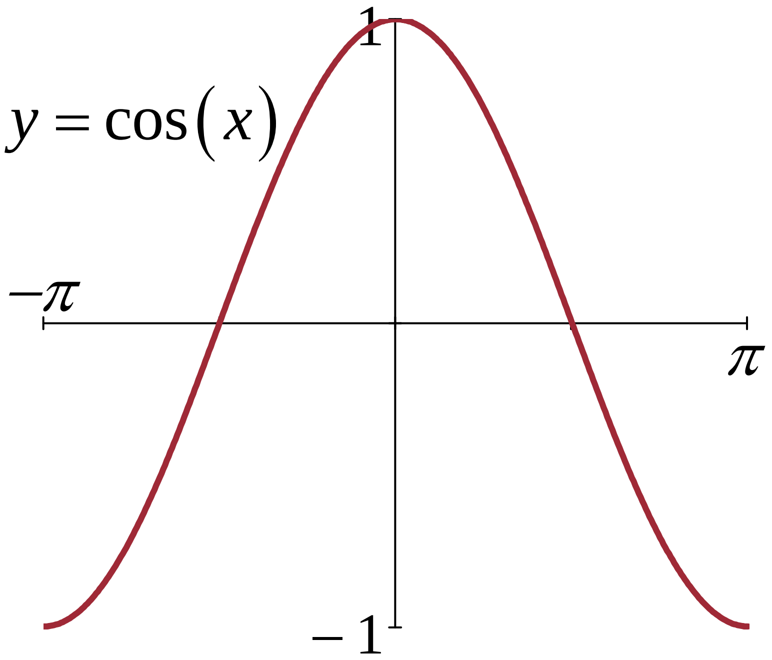
$$\boxed{\frac{d}{dx}\sin(x) = \cos(x)}$$

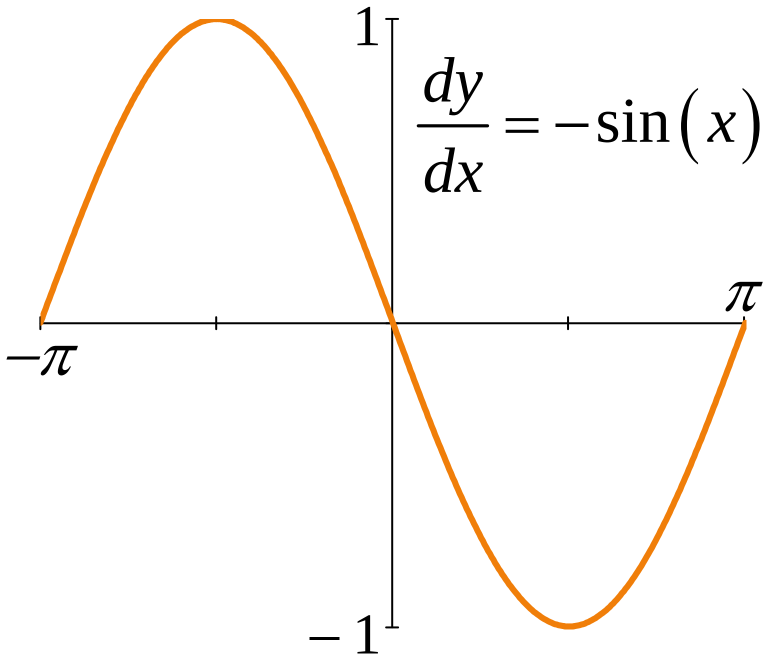


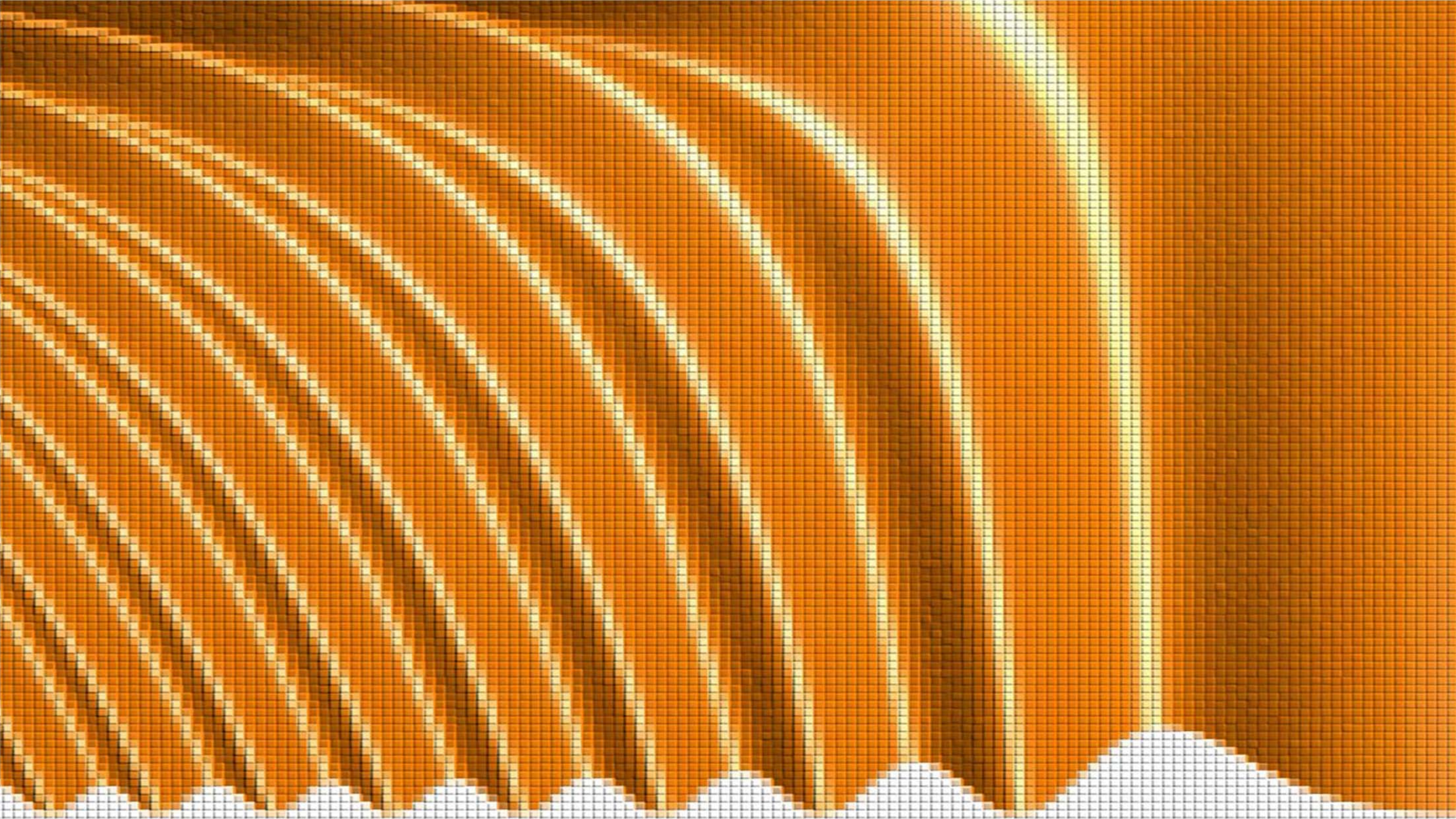


Derivatives of sine and cosine

$$\boxed{\frac{d}{dx}\cos(x) = -\sin(x)}$$







Differential calculus

Second derivative

Background mathematics review

David Miller

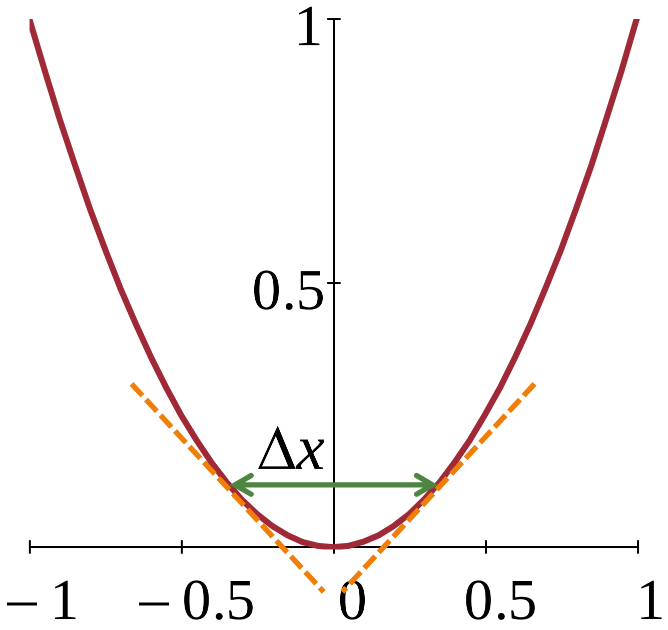
Second derivative

The second derivative is

The derivative of the derivative

$$y''(x) \equiv \frac{d^2 y}{dx^2} \equiv \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

The rate of change of the derivative or "slope"



Second derivative

The slope at $-\Delta x/2$ is, for small Δx

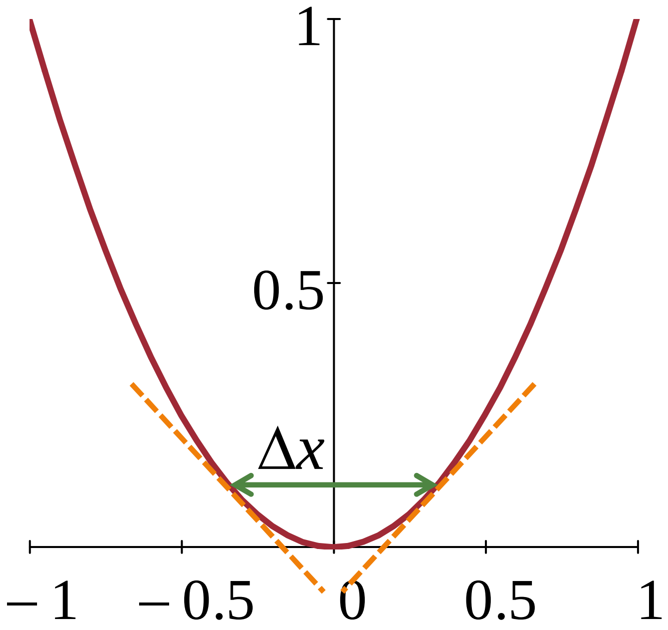
$$\left.\frac{dy}{dx}\right|_{-\Delta x/2} \simeq \frac{y(0)-y(-\Delta x)}{\Delta x}$$

And similarly at $\Delta x/2$

$$\left.\frac{dy}{dx}\right|_{\Delta x/2} \simeq \frac{y(\Delta x)-y(0)}{\Delta x}$$

So

$$\frac{d^2y}{dx^2} \equiv \frac{d}{dx}\left(\frac{dy}{dx}\right) = \lim_{\Delta x\to 0}\frac{\left.\frac{dy}{dx}\right|_{\Delta x/2} - \left.\frac{dy}{dx}\right|_{-\Delta x/2}}{\Delta x}$$



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And similarly at $\Delta x/2$

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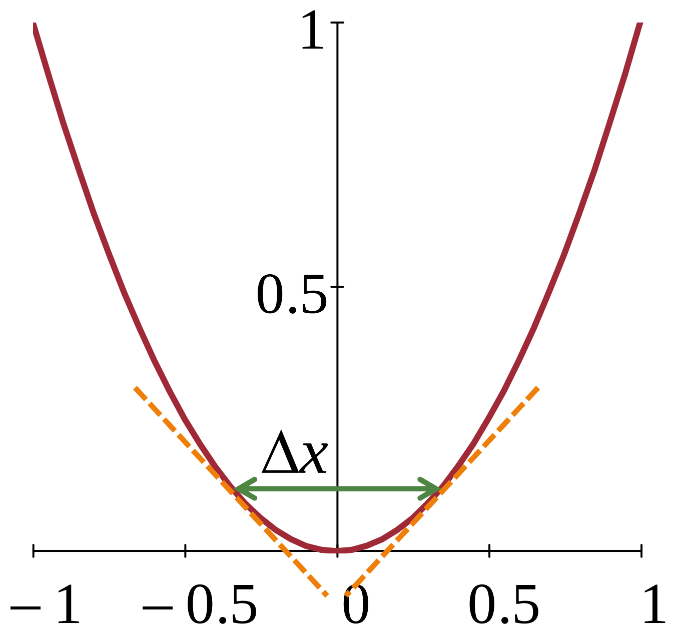
So



$$\frac{d^2y}{dx^2} \equiv \frac{d}{dx}\left(\frac{dy}{dx}\right) = \lim_{\Delta x\to 0}\frac{1}{\Delta x}\left[\frac{y(\Delta x)-y(0)}{\Delta x} - \left\{\frac{y(0)-y(-\Delta x)}{\Delta x}\right\}\right]$$

Second derivative

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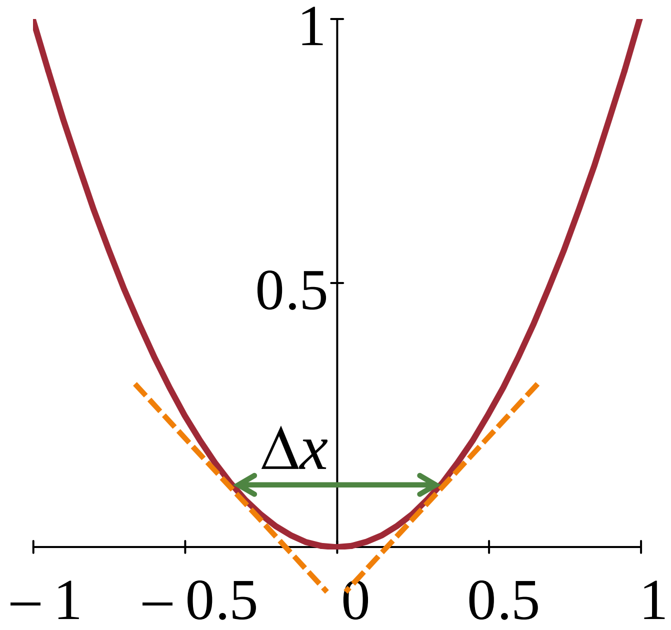
$$\left.\frac{dy}{dx}\right|_{-\Delta x/2} \simeq \frac{y(0)-y(-\Delta x)}{\Delta x}$$

And similarly at $\Delta x/2$

$$\left.\frac{dy}{dx}\right|_{\Delta x/2} \simeq \frac{y(\Delta x)-y(0)}{\Delta x}$$

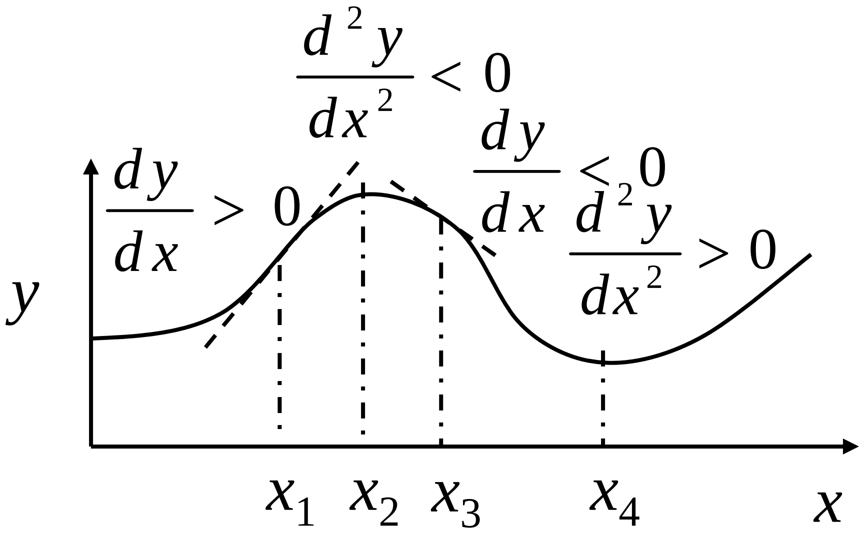
So

$$\boxed{\frac{d^2y}{dx^2} \equiv \frac{d}{dx}\left(\frac{dy}{dx}\right) = \lim_{\Delta x\to 0}\frac{y(\Delta x)-2y(0)+y(-\Delta x)}{(\Delta x)^2}}$$



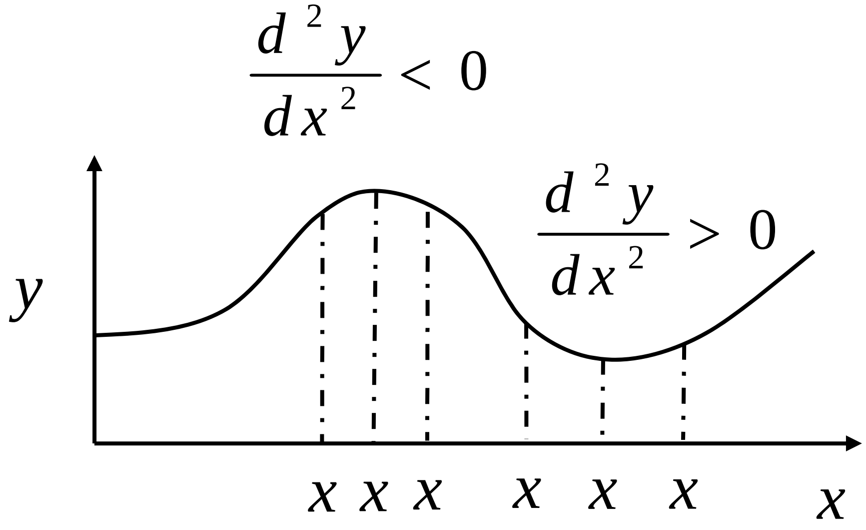
Sign of second derivative

- Going from a positive first derivative
  - To a negative first derivative
    - Gives a negative second derivative
- Going from a negative first derivative
  - To a positive first derivative
    - Gives a positive second derivative



Sign of second derivative

- Any region where the first derivative is decreasing with increasing x
  - Has a negative second derivative
- Any region where the first derivative is increasing with increasing x
  - Has a positive second derivative



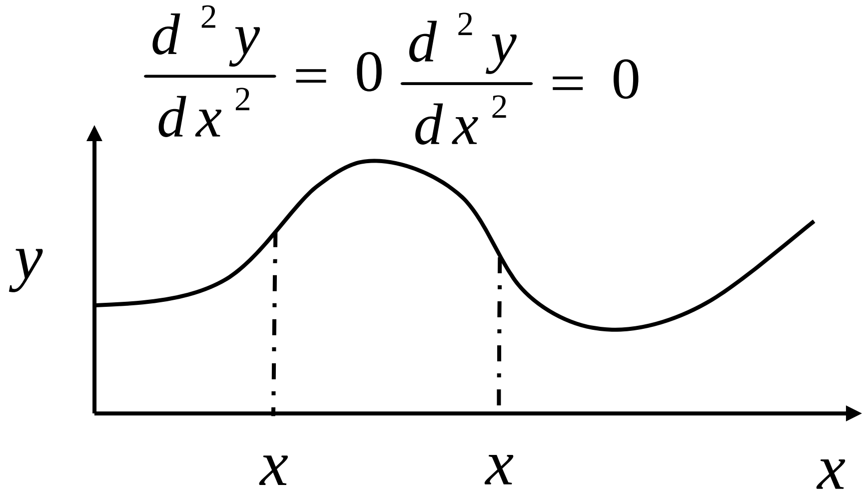
Sign of second derivative

Points where the derivative is neither increasing or decreasing

- i.e., second derivative is changing sign

correspond to zero second derivative

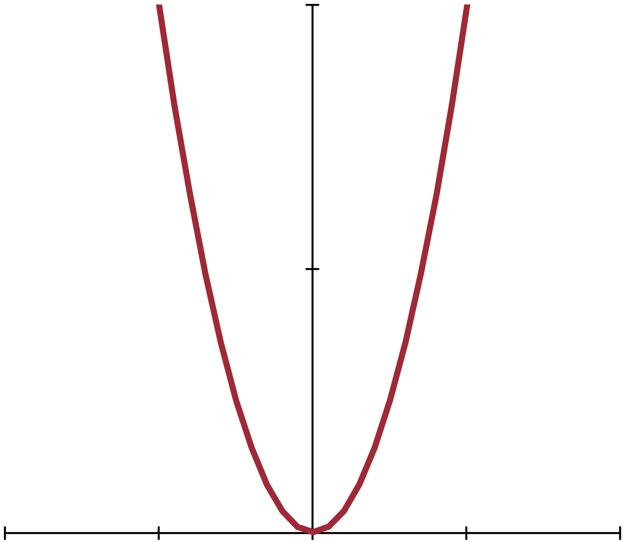
- Known as
  - inflection points



Curvature

The second derivative can be thought of as the "curvature" of a function

Large positive curvature



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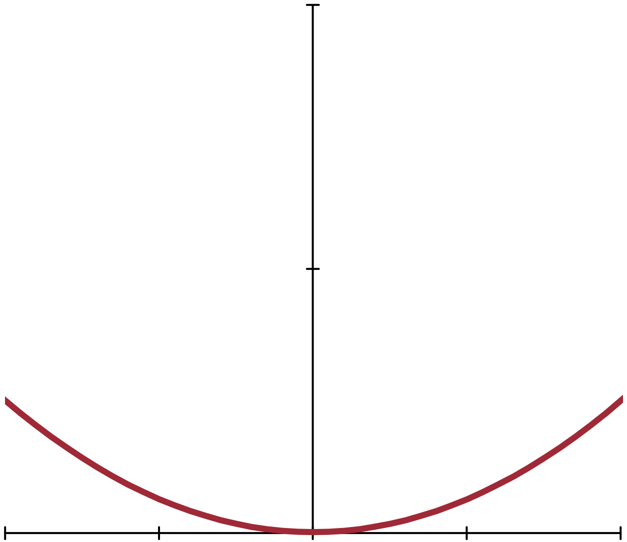

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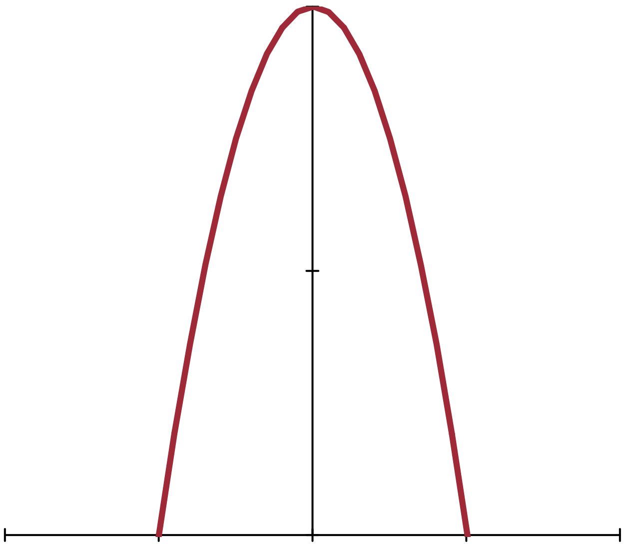

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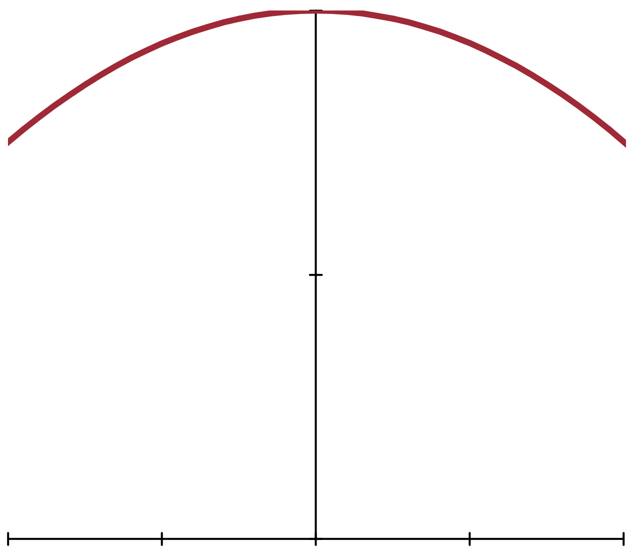
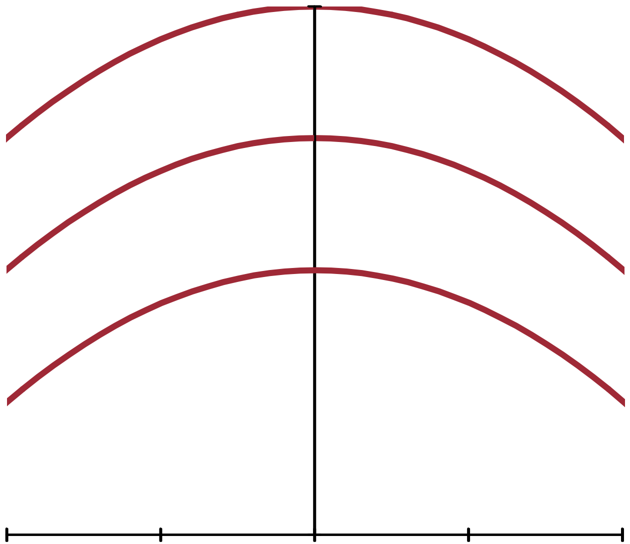
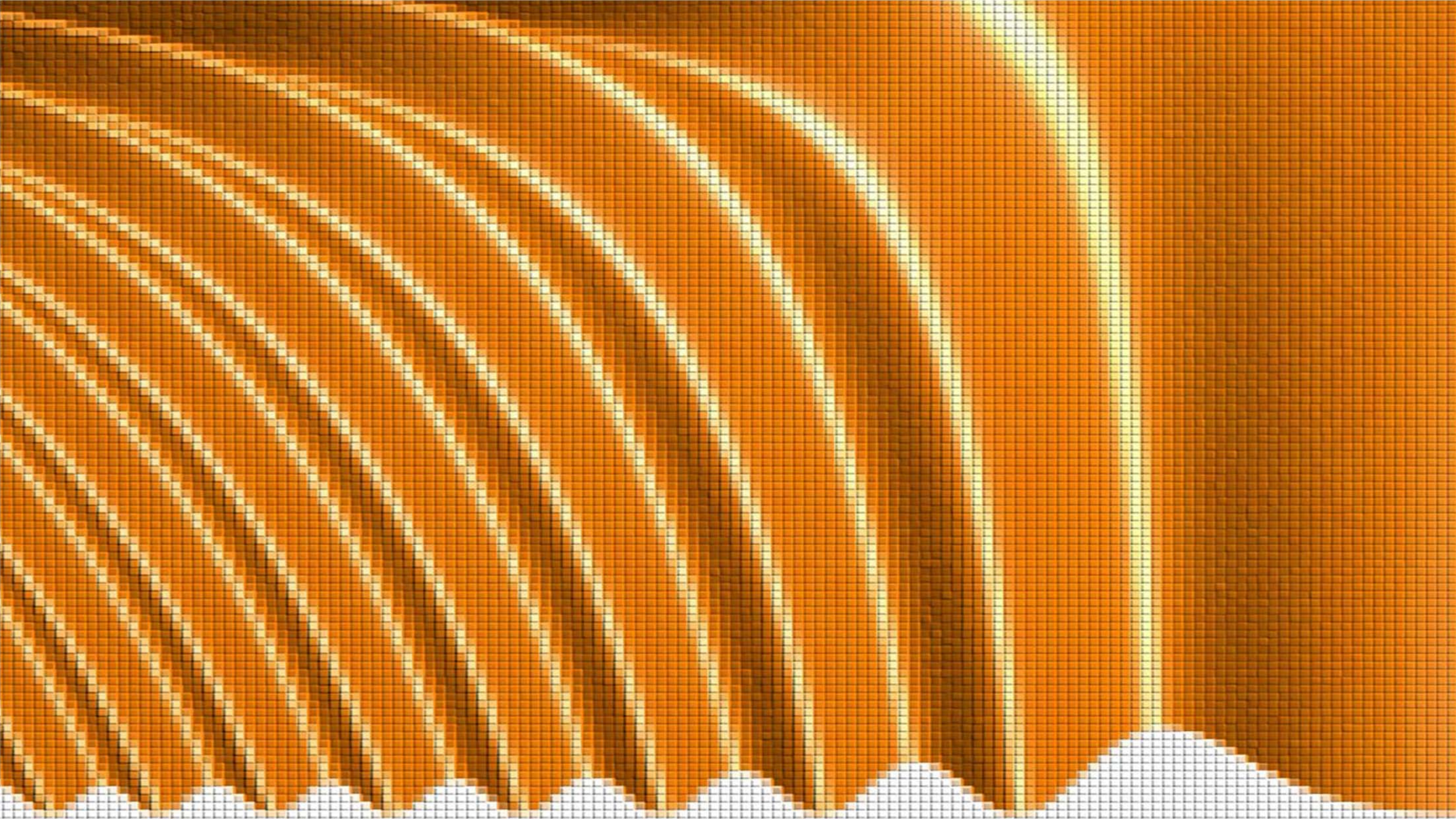


Curvature

- The value of the curvature does not depend on the "height" of the function
  - All these curves have the same curvature





Differential calculus

Linearity and differentiation rules

Background mathematics review

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Linearity – linear superposition

For two functions

$$u(x) \text{ and } v(x)$$

The derivative of the sum is the sum of the derivatives

$$\frac{d}{dx}\left[u(x)+v(x)\right]=\frac{du}{dx}+\frac{dv}{dx}$$

Example

$$f(x)=x+\ln x$$

Split into

$$u(x)=x \qquad v(x)=\ln x$$

So

$$\frac{du}{dx}=1 \qquad \frac{dv}{dx}=\frac{1}{x}$$

So

$$f'(x)=\frac{d(u+v)}{dx}=1+\frac{1}{x}$$

Linearity – multiplying by a constant

For a function $u(x)$

The derivative of

a constant a times a function

is

a times the derivative

$$\frac{d}{dx}[au] = a\frac{du}{dx}$$

Example

$$f(x) = a\sqrt{x} \equiv a\,x^{1/2}$$

Split into

$$a \qquad u(x) = x^{1/2}$$

So

$$\frac{du}{dx} = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}$$

So

$$f'(x) = a\frac{du}{dx} = \frac{a}{2\sqrt{x}}$$

Linearity

An operation or function $f(x)$ is linear if

$$f(y+z) = f(y) + f(z)$$

"linear superposition" or "additivity" condition

and

$$f(ax) = a\,f(x)$$

"multiplication by a constant" (or formally "homogeneity of degree one") condition

Example of nonlinear operation

The function

$$f(x) = x^2$$

does not represent a linear operation

$$f(y+z) = (y+z)^2 = y^2 + z^2 + 2yz$$

But

$$f(y) + f(z) = y^2 + z^2$$

So for this function

$$f(x+y)$$

is not in general equal to

$$f(x) + f(y)$$

Product rule

For two functions

$$u(x) \text{ and } v(x)$$

The derivative of the product is

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

Example

$$f(x) = x^2 \sin x$$

Split into

$$u(x) = x^2 \quad v(x) = \sin x$$

So

$$\frac{du}{dx} = 2x \qquad \frac{dv}{dx} = \cos x$$

So

$$f'(x) = \frac{d(uv)}{dx} = x^2 \cos x + 2x \sin x$$

Quotient rule

For two functions

$$u(x) \text{ and } v(x)$$

The derivative of the ratio or quotient is

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

Example

$$f(x) = \frac{x^3}{1+x^2}$$

Split into

$$u(x) = x^3 \quad v(x) = 1 + x^2$$

So

$$\frac{du}{dx} = 3x^2 \quad \frac{dv}{dx} = 2x$$

So

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{\left(1+x^2\right)\times 3x^2 - x^3 \times 2x}{\left(1+x^2\right)^2}$$

Quotient rule

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$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{3x^2\left(1+x^2\right) - x^3 \times 2x}{\left(1+x^2\right)^2}$$

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So

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{3x^2\left(1+x^2\right) - 2x^4}{\left(1+x^2\right)^2}$$

Quotient rule

For two functions

$$u(x) \text{ and } v(x)$$

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So

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{3x^2}{1+x^2} - \frac{2x^4}{\left(1+x^2\right)^2}$$

Chain rule

For two functions

$$f(y) \text{ and } g(x)$$

The derivative of the "function of a function"

Can be split into a product

$$\frac{d}{dx}f(g(x)) = \left(\frac{df}{dg}\right)\times\left(\frac{dg}{dx}\right)$$

Example

$$h(x) = (1+x^2)^2$$

Split into

$$g(x) = 1+x^2 \qquad f(y) = y^2$$

Chain rule

For two functions

$$f(y) \text{ and } g(x)$$

The derivative of the "function of a function"

Can be split into a product

$$\frac{d}{dx} f\left(g\left(x\right)\right) = \left(\frac{df}{dg}\right) \times \left(\frac{dg}{dx}\right)$$

Example

$$h(x) = \left(1 + x^2\right)^2$$

Split into

$$g(x) = 1 + x^2 \qquad f(g) = g^2$$

So

$$\frac{dg}{dx} = 2x \qquad \frac{df(g)}{dg} = 2g$$

So

$$\frac{dh}{dx} = 2\left(1 + x^2\right) \times 2x = 4x\left(1 + x^2\right)$$

Chain rule

For two functions

$$f(y) \text{ and } g(x)$$

The derivative of the "function of a function"

Can be split into a product

$$\frac{d}{dx} f(g(x)) = \left(\frac{df}{dg}\right) \times \left(\frac{dg}{dx}\right)$$

Example

$$h(x) = \exp(a\,x)$$

Split into

$$g(x) = a\,x \qquad f(g) = \exp(g)$$

So

$$\frac{dg}{dx} = a \qquad \frac{df(g)}{dg} = \exp(g)$$

So

$$\frac{dh}{dx} = \exp(ax) \times a = a\exp(ax)$$

Chain rule

For two functions

$$f(y) \text{ and } g(x)$$

The derivative of the "function of a function"

Can be split into a product

$$\frac{d}{dx} f\left(g(x)\right) = \left(\frac{df}{dg}\right) \times \left(\frac{dg}{dx}\right)$$

Example

$$h(x) = \exp\left(a\,x^2\right)$$

Split into

$$g(x) = a\,x^2 \quad f(g) = \exp(g)$$

So

$$\frac{dg}{dx} = 2ax \quad \frac{df(g)}{dg} = \exp(g)$$

So

$$\frac{dh}{dx} = \exp\left(ax^2\right) \times 2ax = 2ax\exp\left(ax^2\right)$$