Powers, logs, exponentials and complex numbers

Background mathematics review

David Miller

Powers, logs, exponentials and complex numbers

Powers

Background mathematics review

David Miller

Powers

Remember from elementary algebra

$$2 \times 2 \times 2 \equiv 2^3$$

which is "2 to the power 3"

- The "3" here can also be called the "exponent"
  - the power to which the number "2" is raised

Powers

Multiplying by the same number raises the power

$$2^3 \times 2 = 2^4$$

Dividing by the same number lowers the power

$$2^3 / 2 = 2^2$$

Following this logic $2^2 / 2 = 2^1 = 2$

and $2^1 / 2 = 2^0 = 1$

Generalizing

any number to the power zero is 1

$$x^0 = 1$$

Powers

Continuing

$$1/2 = 2^0/2 \equiv 2^{-1}$$

and so on for further negative powers

$$\frac{1}{2} \times \frac{1}{2} = \left(\frac{1}{2}\right)^2 \equiv \frac{1}{2^2} = 2^{-1}/2 = 2^{-2} = \frac{1}{4}$$

Generalizing,

for any number x and any power a

$$x^{-a} = \frac{1}{x^a}$$

Reciprocal

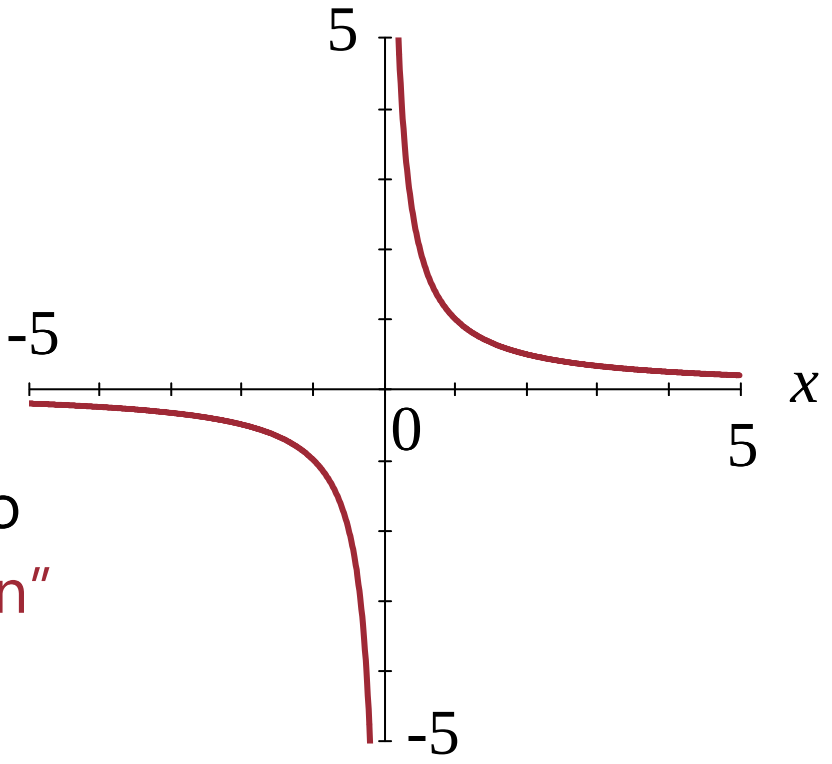
$$\frac{1}{x} \equiv x^{-1}$$

is called the "reciprocal of x"

It becomes arbitrarily large in magnitude as x goes towards zero

- Loosely, it "explodes at the origin"
- Rigorously, it is "singular" at the origin

Negative powers generally have this property



Squares and square roots

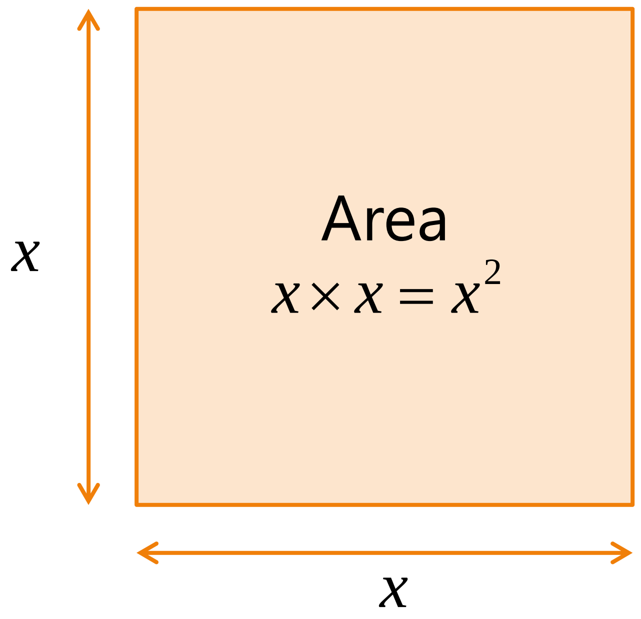
Multiplying a number x by itself

$$x \times x = x^2$$

is called

"taking the square"

Because it gives the area of a square of "side" x



Squares and square roots

For some number x

the number $\sqrt{x}$ that, when multiplied by itself gives x

is called the

"square root" of x

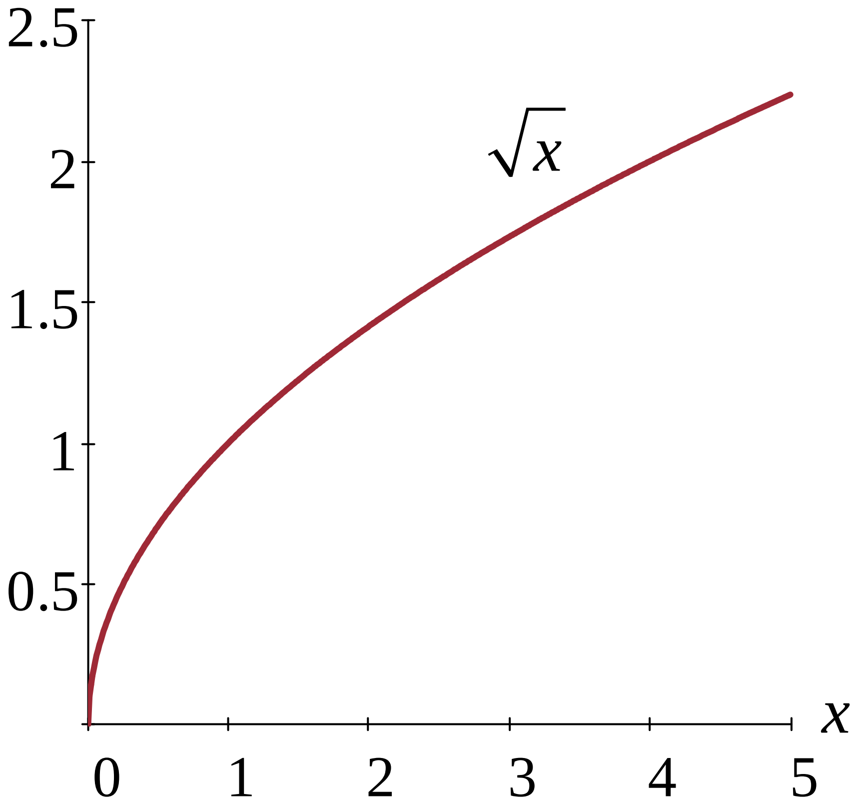
$$\sqrt{x} \times \sqrt{x} = x$$

e.g.,

$$\sqrt{4} = 2$$

$\sqrt{}$ is the "radical" sign

Also $\sqrt{x} \equiv x^{1/2}$



Square root

Note that

If $\sqrt{x} \times \sqrt{x} = x$

So also $\left(-\sqrt{x}\right) \times \left(-\sqrt{x}\right) = x$

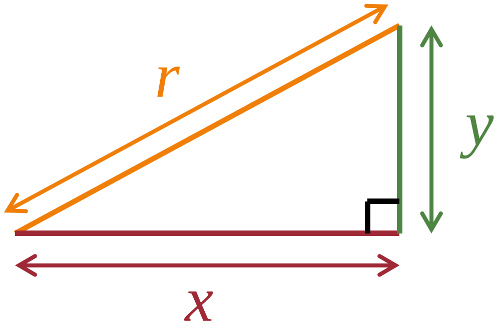
2 is the square root of 4

So also -2 is the square root of 4

Conventionally, we presume we mean the positive square root unless otherwise stated

But we always have both positive and negative versions of the square root

Distance and Pythagoras's theorem



Pythagoras's theorem gives

$$r^2 = x^2 + y^2$$

or equivalently

$$r = \sqrt{x^2 + y^2}$$

where we always take the positive square root

so r is a distance

and is always positive

Quadratics and roots

For a quadratic equation of form

$$ax^2 + bx + c = 0$$

the solutions or "roots" are

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

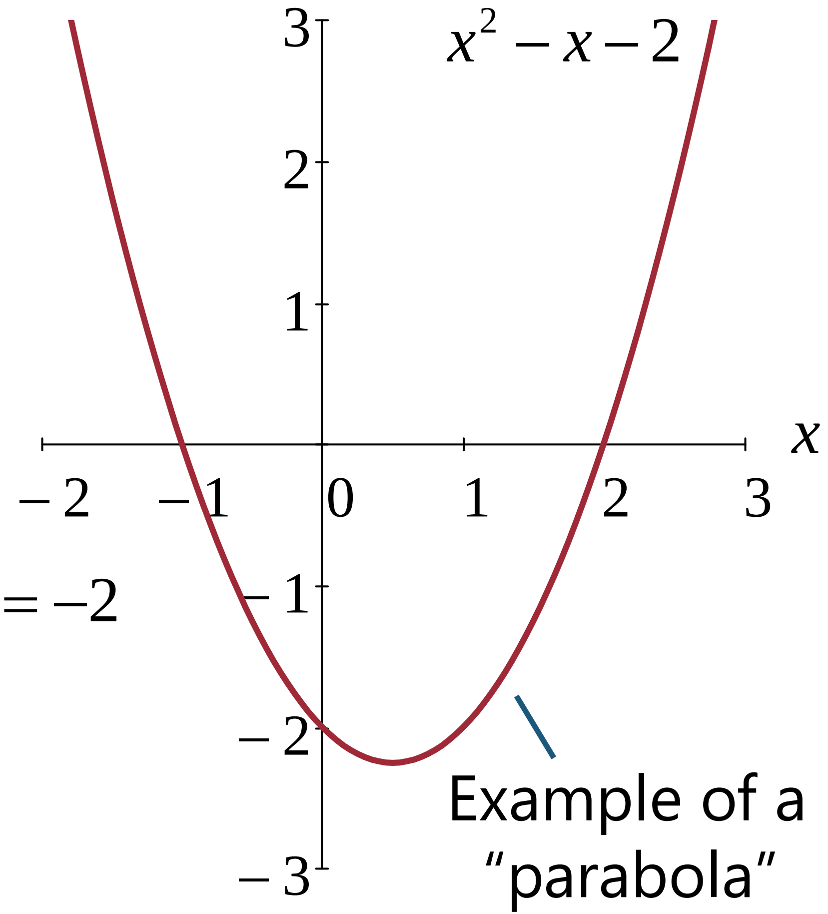
For $x^2 - x - 2 = 0$, $a = 1$, $b = -1$ and $c = -2$

So

$$x = \frac{1 \pm \sqrt{1+8}}{2} = \frac{1 \pm 3}{2} = 2 \text{ or } -1$$

Note

$$x^2 - x - 2 = (x-2)(x+1)$$



Powers of powers

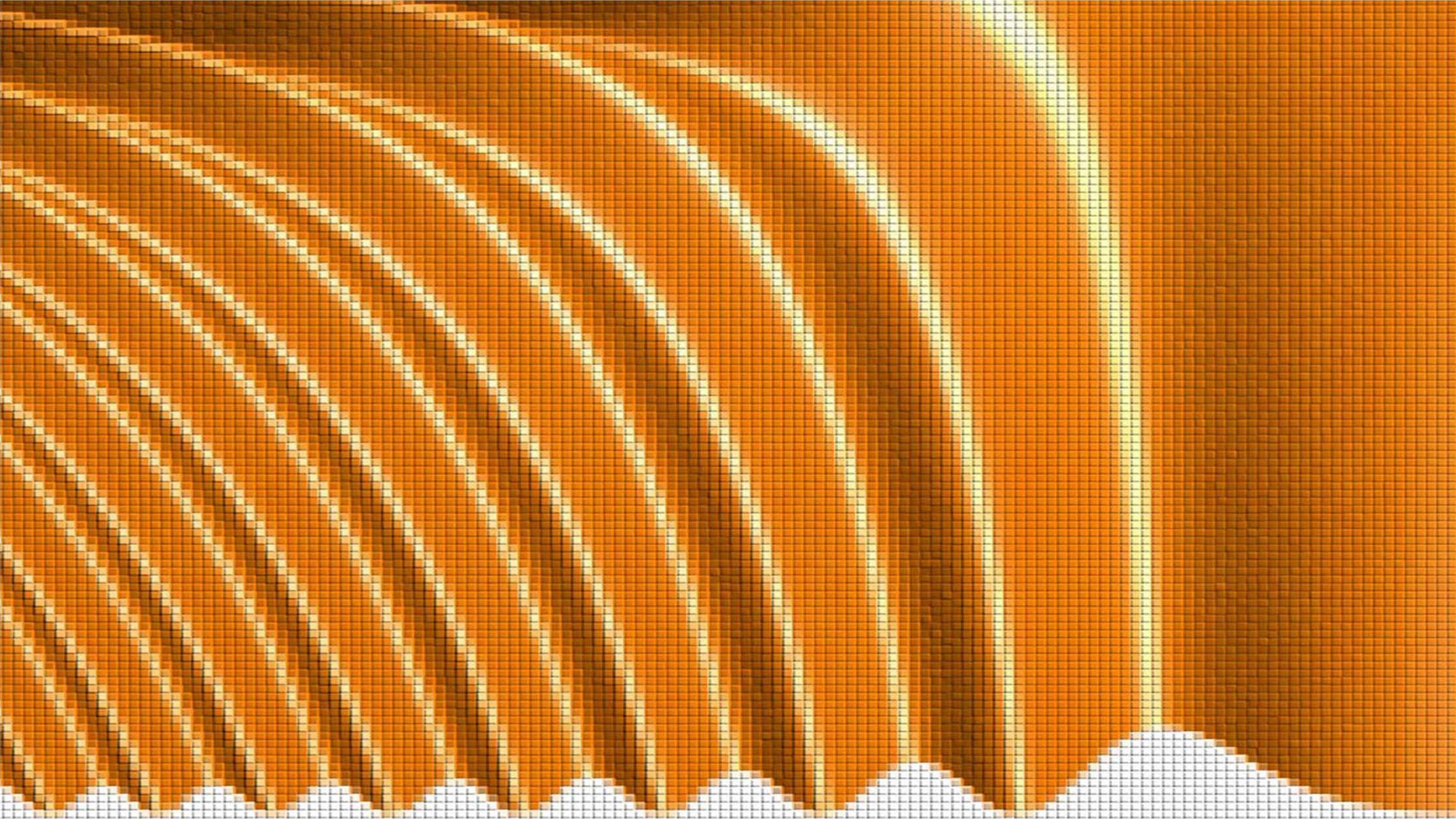
To raise a power to a power

Multiply the powers, e.g.,

$$\left(2^3\right)^2 = (2\times 2\times 2)\times(2\times 2\times 2) = 2^6$$

Generalizing

$$\left(a^b\right)^c = a^{bc}$$



Powers, logs, exponentials and complex numbers

Logarithms and exponentials

Background mathematics review

David Miller

Powers and logarithms

The inverse operation of raising to a power is

taking the "logarithm" (log for short)

With logarithms, we need to specify the "base" of the logarithm

Our example was

"2 to the power 3" is 8

$$2 \times 2 \times 2 \equiv 2^3 = 8$$

The "logarithm to the base 2 of 8" is 3

$$\log_2 8 = 3$$

Powers and logarithms

Generalizing

a chain of n numbers g multiplied together

is

$$g \times g \times \cdots \times g \equiv g^n$$

and

n is the "log to the base g" of g^n

$$n = \log_g \left(g^n \right)$$

Powers and logarithms

Though we created these ideas using integer numbers of multiplications n

- we can generalize to non-integers
  - For some arbitrary (real, positive, and non-zero) number b,
    - we can write

$$a = \log_g b$$

which means

$$g^a = b$$

Powers and logarithms

We note, for example, that

$$(2\times 2)\times(2\times 2\times 2)=2^2\times 2^3=32=2^5=2^{2+3}$$

Note that in multiplying we have added the exponents

Generalizing $g^a\times g^b=g^{a+b}$

Equivalently, if we write

$A=g^a$ so $\log_g A=a$ and $B=g^b$ so $\log_g B=b$

Then

$$\log_g(A\times B)=\log_g(g^a\times g^b)=\log_g\left(g^{a+b}\right)=a+b=\log_g A+\log_g B$$

multiplying numbers is equivalent to adding their logarithms

Bases for logarithms

- When logarithms are used for calculations
  - Typically base 10 is used
- Base 10 logarithms are often used by engineers in expressing power ratios
  - in practice using “decibels” (abbreviated dB)
    - which are 10 times the logarithm of the ratio

Bases for logarithms

E.g., for an amplifier with

an output power P_{out} that is 100 times larger than the input power P_{in}

Gain (in dB)=10 $\log_{10}(P_{out}/P_{in})$

i.e.,

Gain (in dB)=10 $\log_{10}(100)$=10x2=20dB

E.g., for an amplifier with a gain of 2

Noting that $\log_{10}2 \cong 0.301$

Gain of times 2 (in dB)=10 $\log_{10}2 \cong$ 3dB

Changing bases of logarithms

Suppose

$$\log_{10} b = a \quad \text{i.e.,} \quad b = 10^a$$

Now, by definition

$$10 = 2^{\log_2 10}$$

So

$$b = \left(2^{\log_2 10}\right)^a = 2^{(\log_2 10)\times a}$$

So

$$\log_2 b = (\log_2 10)\times a = (\log_2 10)\times(\log_{10} b)$$

Generalizing, and dropping parentheses and "×"

$$\boxed{\log_c b = \log_c d \ \log_d b}$$

Bases for logarithms

Sometimes "$\log$" means "$\log_{10}$"

- e.g., on a calculator keyboard

Another common base is "base 2" (i.e., "$\log_2$")

- e.g., in computer science because of binary numbers

Fundamental physical science and mathematics almost always uses logs to the base "e"

$$e \simeq 2.71828\ 18284\ 59045\ 23536$$

- e is the "base of the natural logarithms"

Notations with "e"

Logs to base "e" are called "natural logarithms"

- "$\log_e$" (sometimes just "log") or "ln"
  - letter "l" for "logarithm" and letter "n" for "natural"

To avoid confusion with other uses of "e"

- e.g., for the charge on an electron

And to avoid superscript characters

- we use the "exponential" notation

$$\exp(x) \equiv e^x$$

- Also means this can be referred to as the "exponential" function

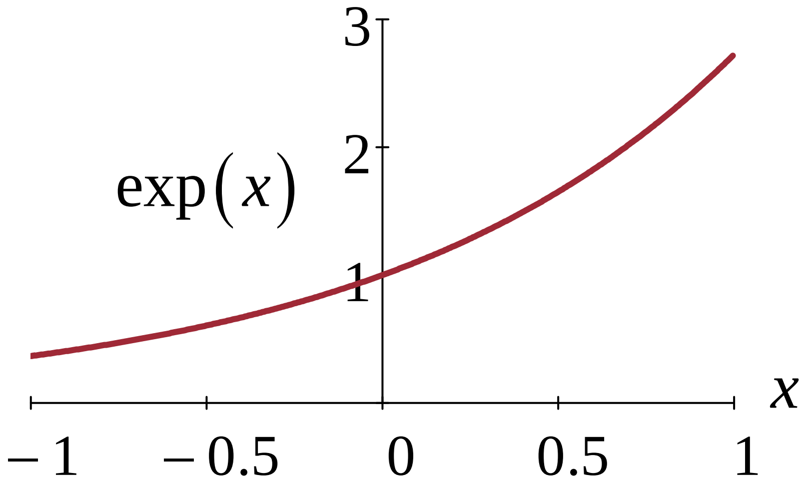
Exponential and logarithm

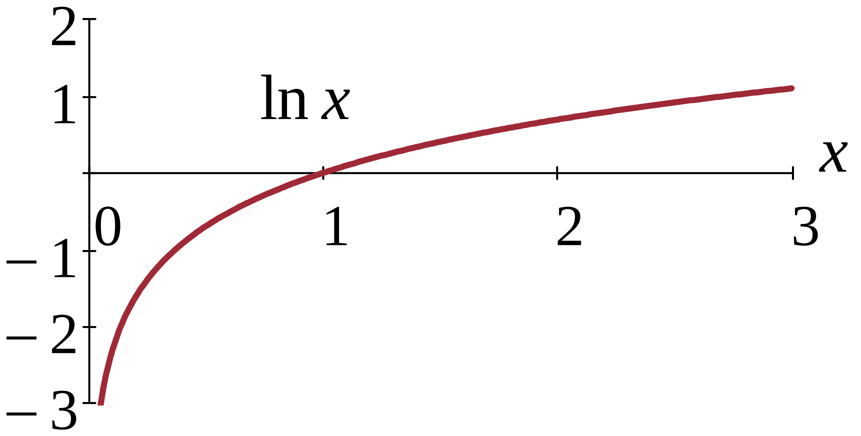
Exponential function

- For larger negative arguments
  - Gets closer and closer ("asymptotes") to the *x* axis
- For larger positive arguments
  - Grows faster and faster

Logarithm

- For smaller positive arguments
  - arbitrarily large and negative





Exponentials and logarithms

Note all the following formulas

Which follow from the discussions above

$$\exp(a)\exp(b) = \exp(a+b)$$

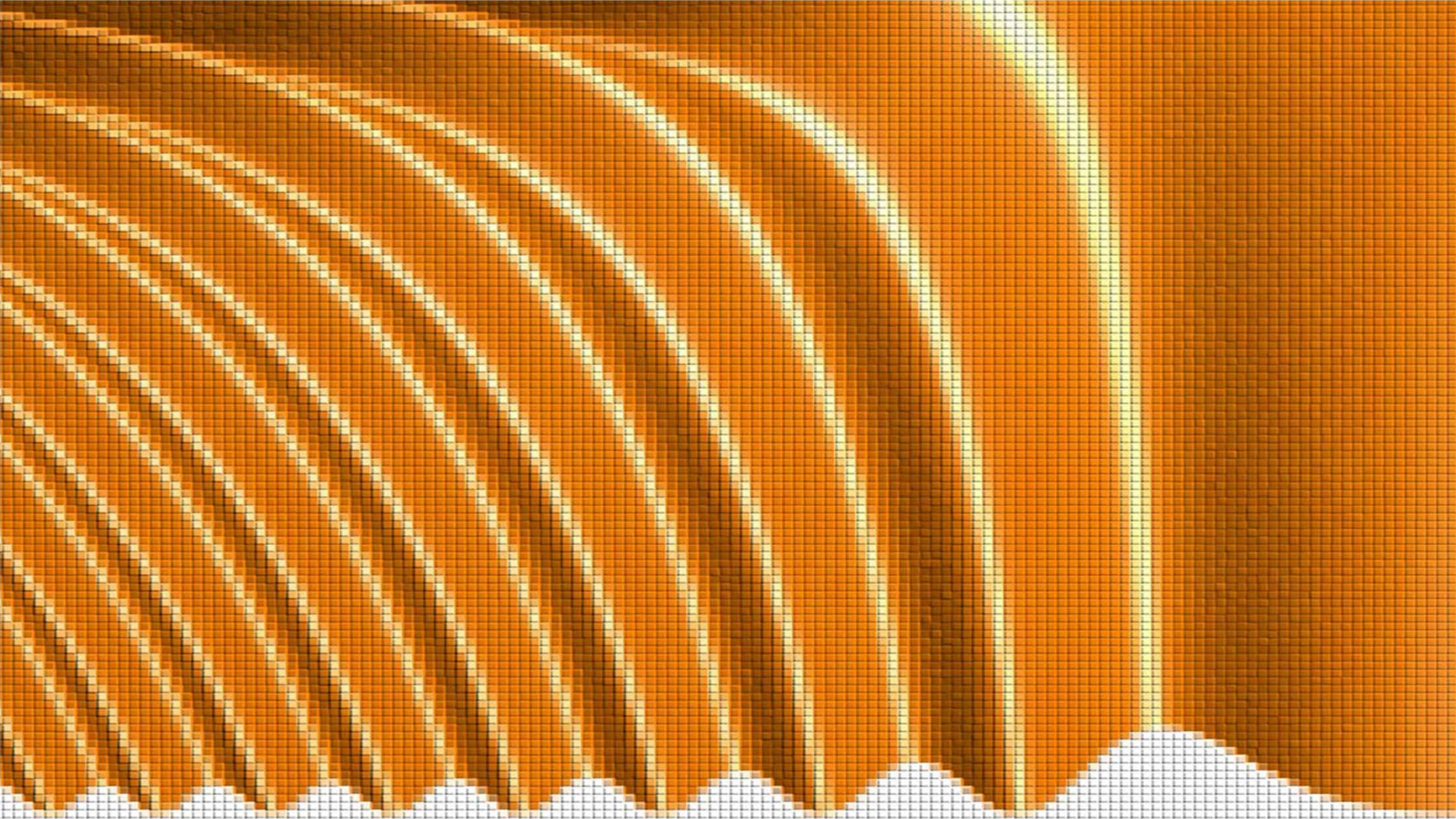
$$\ln(ab) = \ln(a) + \ln(b)$$

$$\frac{1}{\exp(a)} = \exp(-a)$$

$$\ln(1/a) = -\ln(a)$$

$$\exp\left[\ln(a)\right] = a$$

$$\ln\left[\exp(a)\right] = a$$



Powers, logs, exponentials and complex numbers

Imaginary and complex numbers

Background mathematics review

David Miller

Square root of minus one

In ordinary real numbers

- no number multiplied by itself
  - gives a negative result
- Equivalently
  - There is no (real) square root of a negative number

$$2 \times 2 = 4$$

$$(-2) \times (-2) = 4$$

If, however, we choose to define an entity that we call the square root of minus one

- We can write square roots of negative numbers
  - We obtain a very useful algebra

Square root of minus one

Define $i = \sqrt{-1}$ so $i^2 = -1$ and $(-i)^2 = -1$

Also, common engineering notation is $j = \sqrt{-1}$

Any number proportional to i is called

an imaginary number

e.g., $4i$, $3.74i$, $i\pi$

Common to put the "i" after numbers,
but before variables or constants

Can write the square root of any negative number using i

$$\sqrt{-4} = \sqrt{(-1) \times 4} = \sqrt{-1} \times \sqrt{4} = i \times 2 \equiv 2i$$

Complex numbers

A number that can be written

$$g = a + ib$$

where a and b are both real numbers

is called a "complex number"

a is called the "real part" of g

$$a = \mathrm{Re}(g)$$

b is called the "imaginary part" of g

$$b = \mathrm{Im}(g)$$

Complex conjugate and modulus

The "complex conjugate" has the sign of the imaginary part reversed

- And is indicated by a superscript "*"

$$g^* \equiv a - ib$$

Multiplying g by g^* gives a positive number

- Called the "modulus squared" of g

$$|g|^2 = g^* g = g g^*$$

- The (positive) square root of this is called the "modulus" of g

$$|g| = +\sqrt{|g|^2}$$

Important complex number identities

Note for the modulus squared

$$|g|^2 = gg^* = (a+ib)(a-ib) = a^2 - iab + iba - i^2b^2 = a^2 + b^2$$

i.e., $\boxed{|g|^2 = a^2 + b^2}$

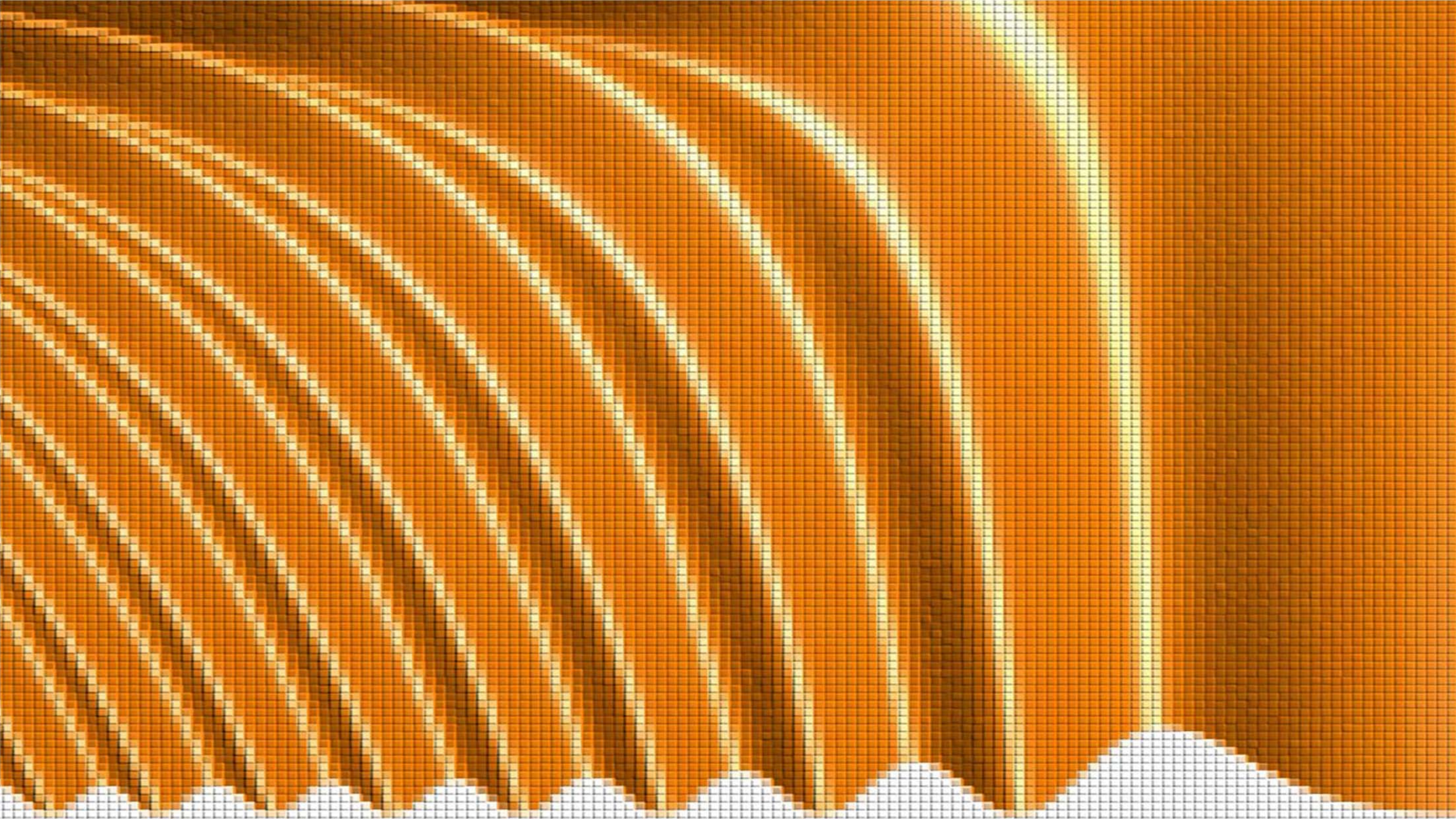
and for the reciprocal

$$g = \frac{1}{c+id} = \frac{1}{(c+id)}\frac{(c-id)}{(c-id)} = \frac{c-id}{c^2+d^2} = \frac{c}{c^2+d^2} - i\frac{d}{c^2+d^2}$$

i.e.,

$$g = \boxed{\frac{1}{c+id} = \frac{c}{c^2+d^2} - i\frac{d}{c^2+d^2}}$$

Still a sum of real and imaginary parts



Powers, logs, exponentials and complex numbers

Euler's formula and the complex plane

Background mathematics review

David Miller

Euler's formula

Euler's formula is the remarkable result

$$\boxed{\exp(i\theta) = \cos\theta + i\sin\theta}$$

A major practical algebraic reason for use of complex numbers in engineering

Exponentials much easier to manipulate than sines and cosines

Some results from Euler's formula

Using Euler's formula

$$\boxed{\exp(i\theta) = \cos\theta + i\sin\theta}$$

Note that

$$\exp(-i\theta) \equiv \exp(i[-\theta]) = \cos(-\theta) + i\sin(-\theta) = \cos(\theta) - i\sin(\theta)$$

so $$\boxed{[\exp(i\theta)]^* = \exp(-i\theta)}$$

Also

$$\exp(i\theta)\exp(-i\theta) = \exp(i\theta - i\theta) = \exp(0) = 1$$

so

$$[\cos\theta + i\sin\theta][\cos\theta - i\sin\theta]$$

$$= \cos^2\theta + i\sin\theta\cos\theta - i\cos\theta\sin\theta - i^2\sin^2\theta = \boxed{\cos^2\theta + \sin^2\theta = 1}$$

Complex exponential or polar form

For any complex number $g = a + ib$

we can write

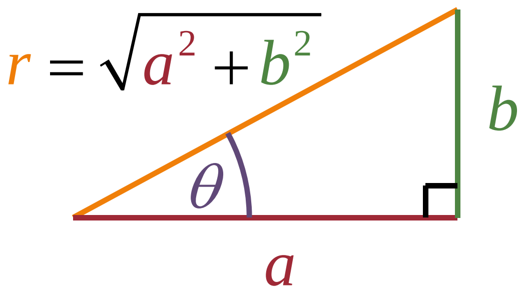
$$g = |g|\frac{a+ib}{|g|} = |g|\left(\frac{a}{\sqrt{a^2+b^2}} + i\frac{b}{\sqrt{a^2+b^2}}\right)$$

which we can write in the form

$$g = |g|(\cos\theta + i\sin\theta)$$

so any complex number can be written in the form

$$\boxed{g = |g|\exp(i\theta)}$$



Complex plane

Propose a "complex plane"

- horizontal "real axis"
- vertical "imaginary axis"

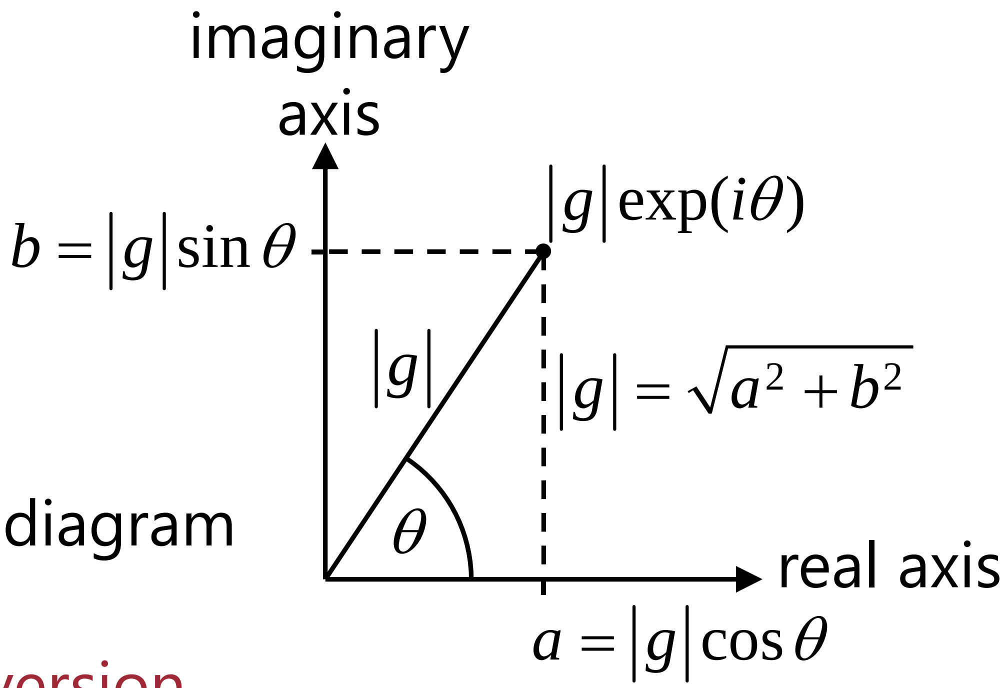
Then any complex number

$$g = a + ib = |g|\exp(i\theta)$$

- is a point on this plane

Sometimes called an Argand diagram

- Sometimes θ is called the "argument" of this polar version of a complex number, $\theta = \arg(g)$



Multiplication in polar representation

In the polar representation $g = |g|\exp(i\theta)$

- To multiply two numbers
  - Multiply the moduli and add the angles
    - I.e., with

$$h = |h|\exp(i\phi)$$

$$g \times h = |g|\exp(i\theta) \times |h|\exp(i\phi)$$

$$= |g||h|\exp\left(i[\theta + \phi]\right)$$

*n*th roots of unity

Note that the number $\exp(2\pi i / n)$ when raised to the *n*th power is 1 (unity)

$$\left[\exp\left(\frac{2\pi i}{n}\right)\right]^n = \exp\left(\frac{2\pi i}{n} \times n\right) = \exp(2\pi i) = 1$$

Many different complex numbers when raised to the *n*th power can give 1

But this specific one is conventionally called the *n*th root of unity

$$\sqrt[n]{1} = \exp(2\pi i / n)$$

Algebraic results for complex numbers

All the following useful algebraic identities are easily proved from the complex exponential form

$$(gh)^* = g^* h^*$$

$$\left(\frac{1}{gh}\right)^* = \frac{1}{g^* h^*}$$

$$\left(\frac{g}{h}\right)^* = \frac{g^*}{h^*}$$

Sine and cosine addition formulas

Sine and cosine sum and difference formulas are easily deduced from the complex exponential form

e.g., $\exp(2i\theta) = \cos 2\theta + i\sin 2\theta$

$$= \exp(i\theta)\exp(i\theta) = [\cos\theta + i\sin\theta]^2$$

$$= \cos^2\theta - \sin^2\theta + 2i\sin\theta\cos\theta$$

Equating real parts gives

$$\cos 2\theta = \cos^2\theta - \sin^2\theta$$

Equating imaginary parts gives

$$\sin 2\theta = 2\sin\theta\cos\theta$$