Coordinates and vectors

Background mathematics review

David Miller

Coordinates and vectors

Coordinate axes and vectors

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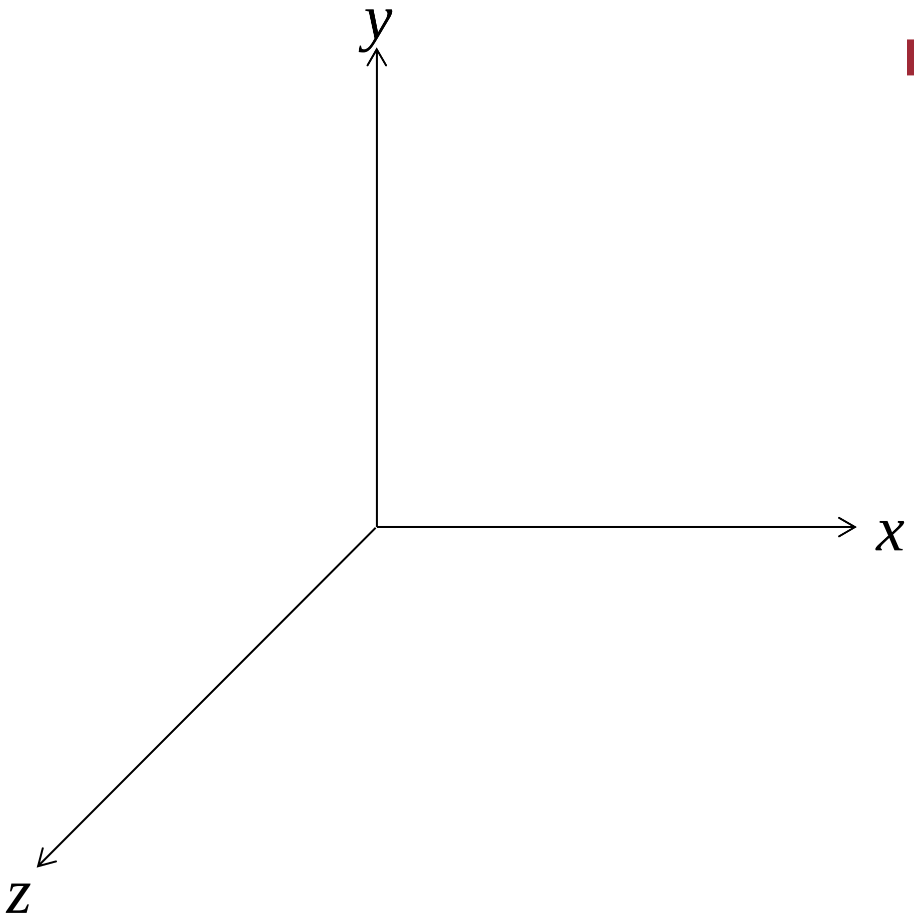
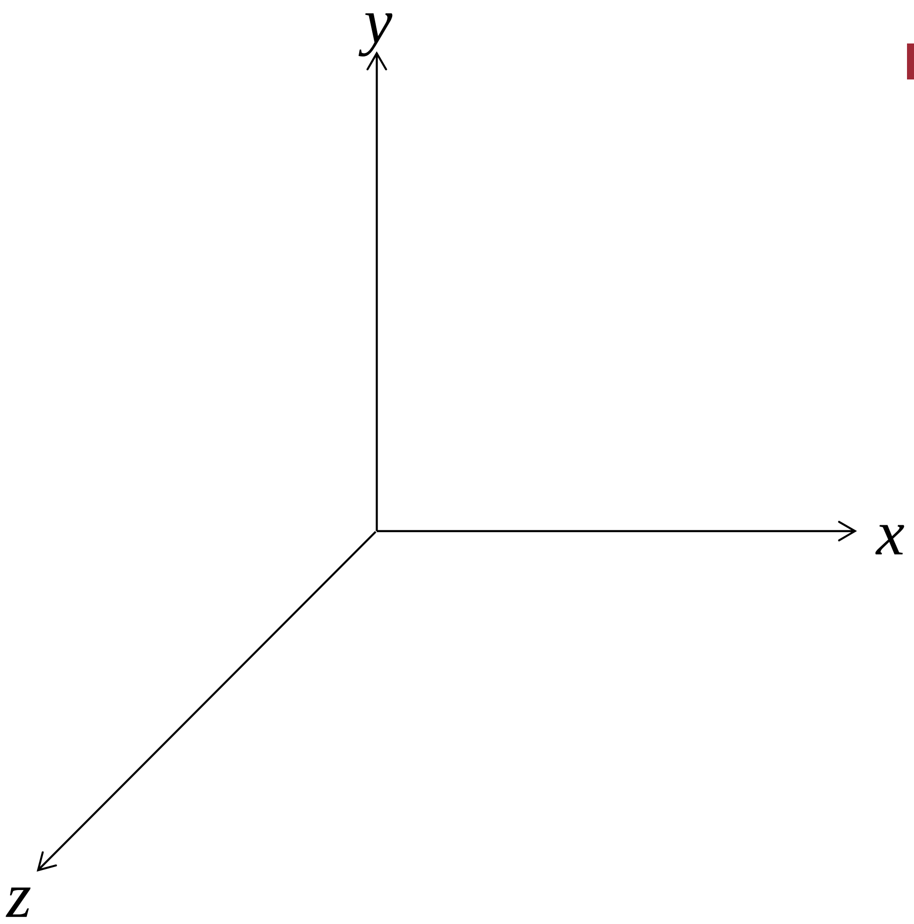


Coordinate axes and vectors

Ordinary geometry

- Three "axes" x, y, and z
  - All at right angles
    - "Cartesian" axes
      - (from René Descartes)

Lines or directions at right angles are also called orthogonal



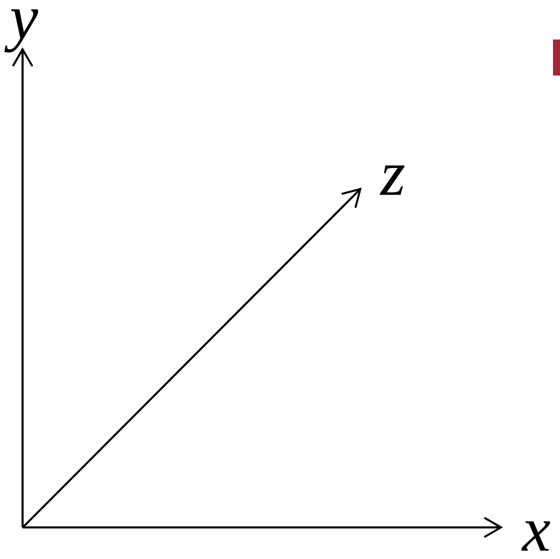
Coordinate axes and vectors

Right-handed axes

- Using your right hand
  - Thumb x
  - Index (“first”) finger y
  - Middle finger z

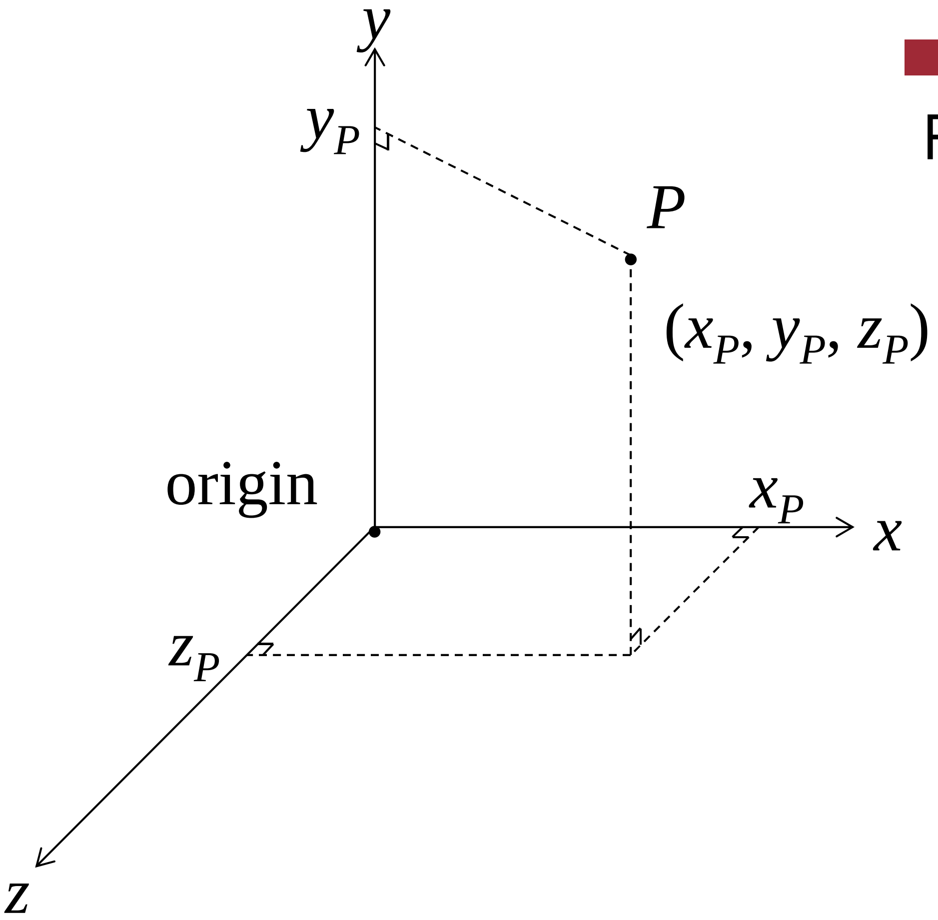
No matter how you now rotate your whole hand

- the axes remain right-handed



Coordinate axes and vectors

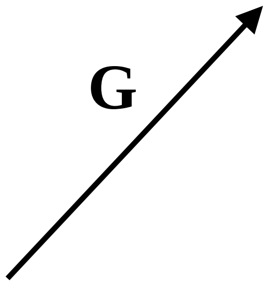
- If you use your left hand
  - Thumb x
  - Index ("first") finger y
  - Middle finger z
    - give left-handed axes
- No rotation of this entire set of left-handed axes will ever make it right-handed
- We use right hand axes unless otherwise stated



Coordinate axes and vectors

For some point P in space

- The corresponding "projections" onto the coordinate axes give
  - Cartesian coordinates
    - x_P, y_P, and z_P,
      - relative to the origin of the axes
        - Sometimes written
          - (x_P, y_P, z_P)



Coordinate axes and vectors

A vector is something with

- a magnitude
  - such as a length
- and a direction

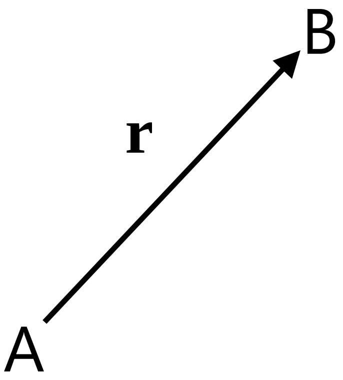
Usually written in "bold" font

- e.g., **G**
- Sometimes $\underline{G}$ or $\vec{G}$

And shown as an "arrow"

- With "length" and direction

Coordinate axes and vectors



A vector could be

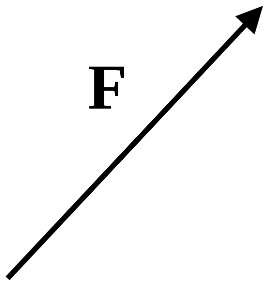
the distance

and

direction

you need to walk to get from A to B

Coordinate axes and vectors



A vector could be

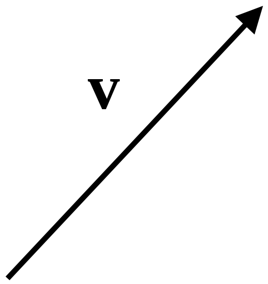
A force

how hard you are pushing

and

what direction you are pushing

Coordinate axes and vectors



A vector could be

- A velocity
  - how fast you are going (speed)
    - e.g., the number on your car speedometer
      - and
  - what direction you are going in
    - e.g., on a compass

Coordinate axes and vectors

$r \quad F \quad v$

An ordinary number

which has no direction

is called a "scalar"

Distance

how hard you push

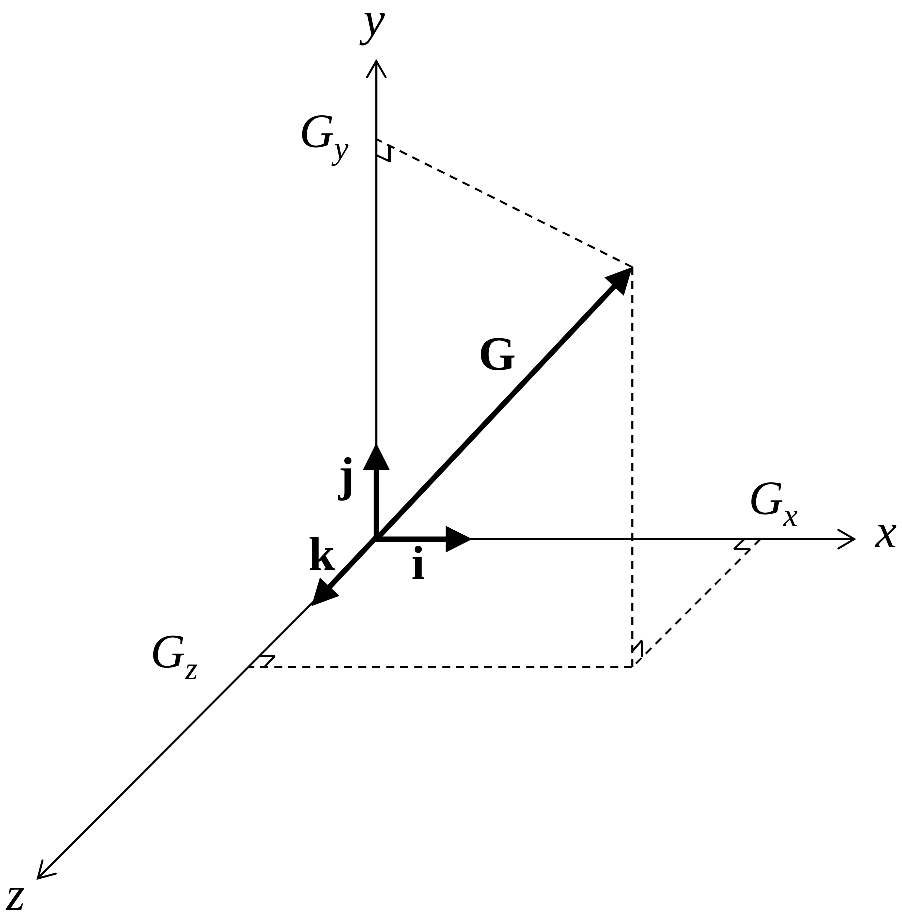
speed

are all scalars

Scalars are in ordinary fonts

Usually italic in printing

Coordinate axes and vectors

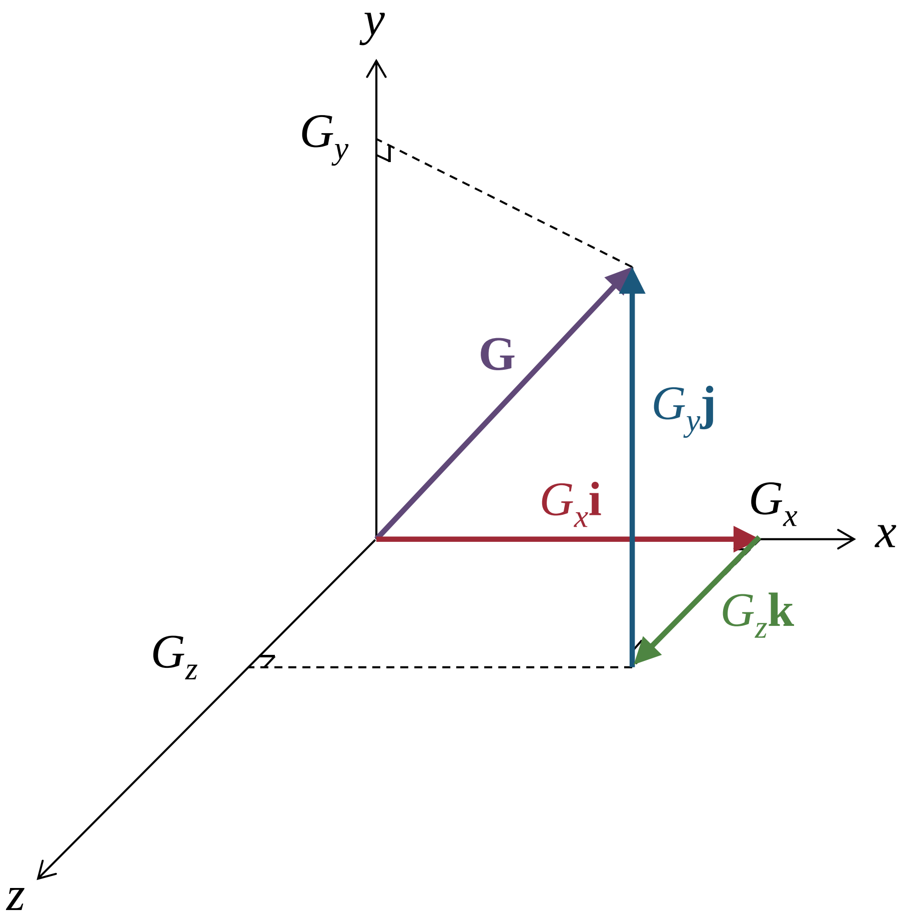


A vector has "components" along three orthogonal axes

- G_x, G_y, and G_z

We can also define vectors of unit length along each axis

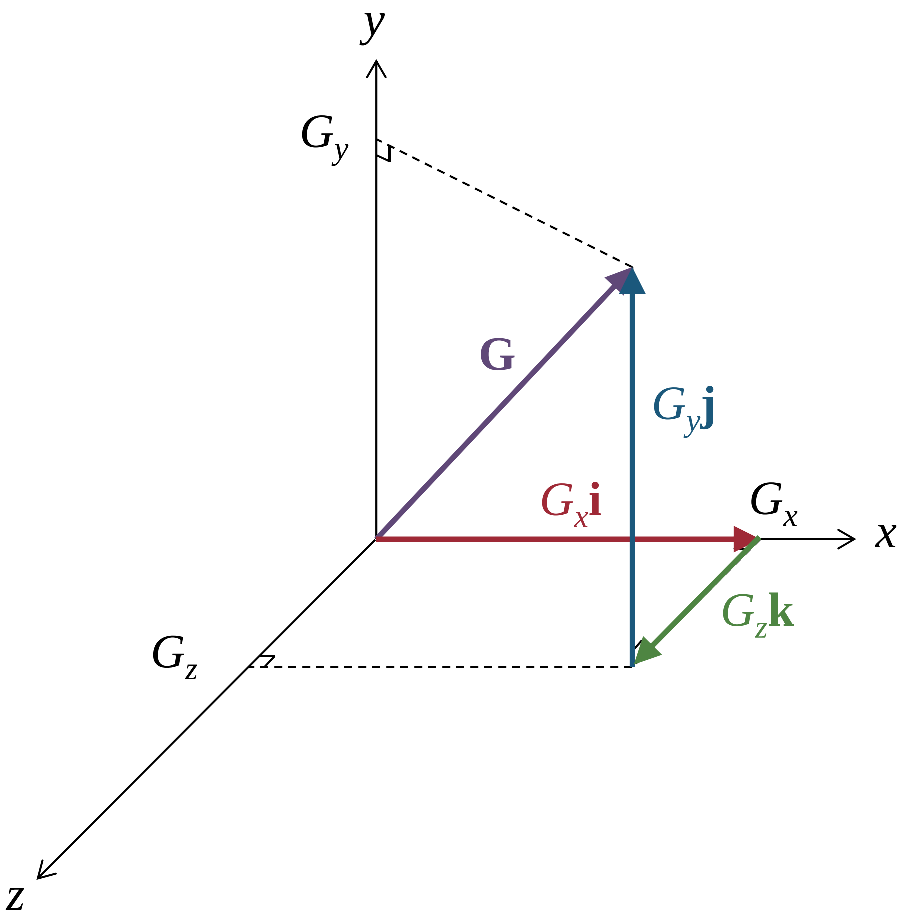
- $\mathbf{i}$ – unit vector along x
- $\mathbf{j}$ – unit vector along y
- $\mathbf{k}$ – unit vector along z



Coordinate axes and vectors

Then we can write

$$G_x\,\mathbf{i}$$

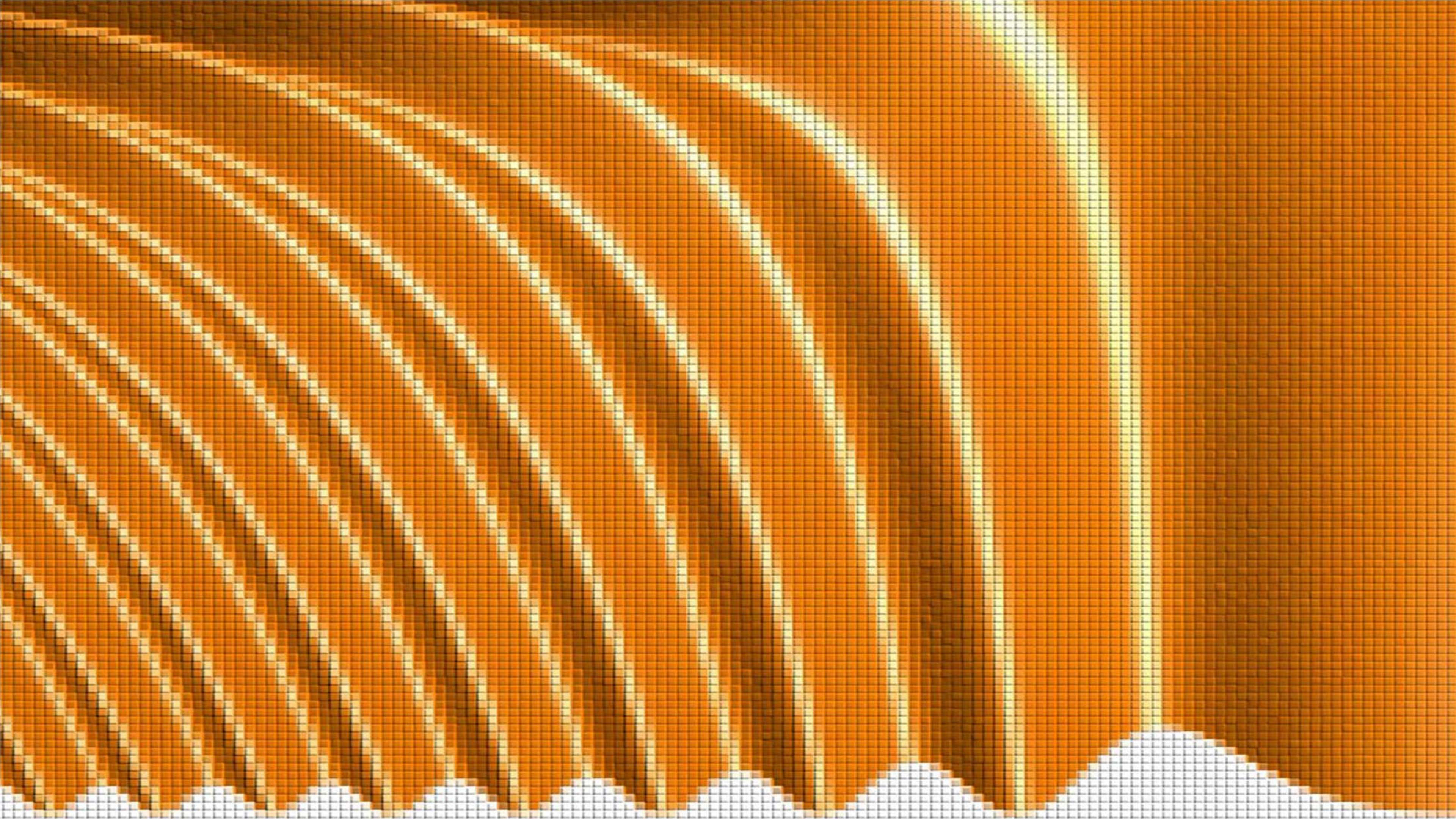


Coordinate axes and vectors

Then we can write

$$\mathbf{G} = G_x\,\mathbf{i} + G_y\,\mathbf{j} + G_z\mathbf{k}$$

making the final vector up by adding its vector components



Coordinates and vectors

Operations with vectors

Background mathematics review

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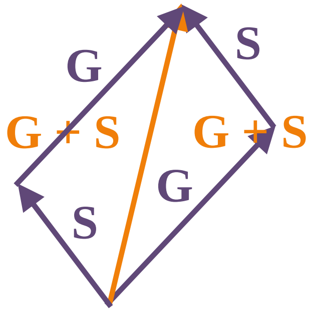
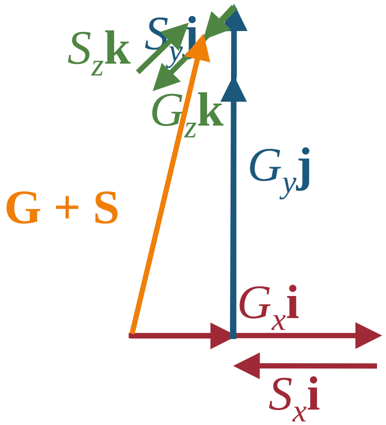
Adding vectors

To add vectors

- graphically
  - connect them head to tail in any order



Adding vectors



To add vectors

algebraically

add them component by component

$$\mathbf{G}+\mathbf{S}=G_x\mathbf{i}+G_y\mathbf{j}+G_z\mathbf{k}$$

$$+S_x\mathbf{i}+S_y\mathbf{j}+S_z\mathbf{k}$$

$$=\left(G_x+S_x\right)\mathbf{i}+\left(G_y+S_y\right)\mathbf{j}+\left(G_z+S_z\right)\mathbf{k}$$

Multiplying vectors

Two kinds of multiplications or “products” for geometrical vectors

- Dot product

  $$\mathbf{a} \cdot \mathbf{b}$$

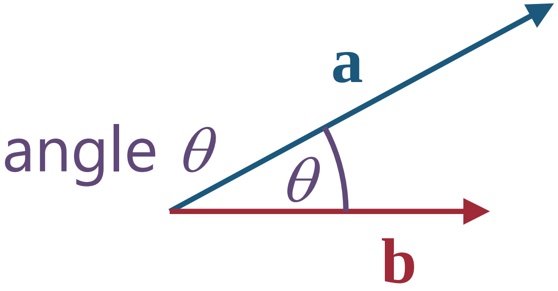
  - Gives a scalar result

- Cross product

  $$\mathbf{a} \times \mathbf{b}$$

  - Gives a vector result

Vector dot product



One formula for the dot product is

$$\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta \equiv ab\cos\theta$$

Here the "modulus" sign "| |" means we take the length of the vector

$$|\mathbf{a}| = a$$

Note that

$$\mathbf{a}\cdot\mathbf{b} = \mathbf{b}\cdot\mathbf{a}$$

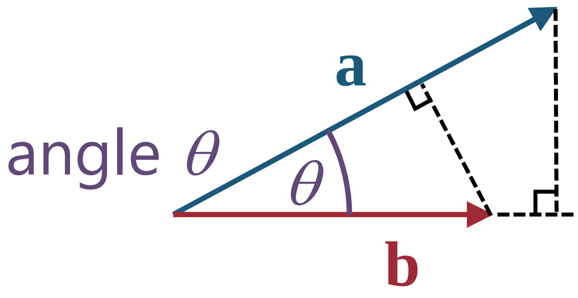
Also

$$\mathbf{a}\cdot\mathbf{a} = a^2$$

So

$$a = \sqrt{\mathbf{a}\cdot\mathbf{a}}$$

Vector dot product



One formula for the dot product is

$$\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta \equiv ab\cos\theta$$

We can think of $|\mathbf{a}||\mathbf{b}|\cos\theta$ as

The projection of vector $\mathbf{b}$ onto the direction of vector $\mathbf{a}$

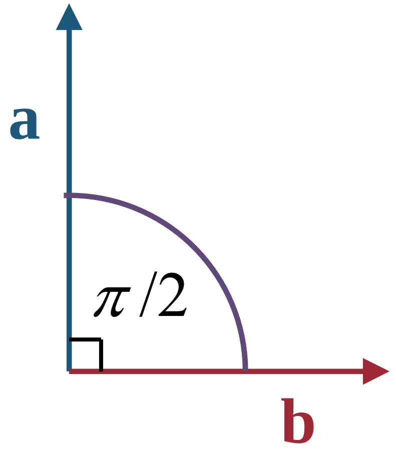
Multiplied by the length of $\mathbf{a}$

or

The projection of vector $\mathbf{a}$ onto the direction of vector $\mathbf{b}$

Multiplied by the length of $\mathbf{b}$

Vector dot product



One formula for the dot product is

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta \equiv ab\cos\theta$$

Note that

for two vectors at right angles

$$\theta = \pi/2 \equiv 90^\circ$$

and

$$\cos(\pi/2) = 0$$

so

the dot product is zero

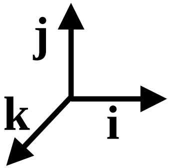
Vector dot product

The unit vectors along the coordinate directions are all orthogonal (at right angles)

So all their dots products with one another are zero

$$\mathbf{i}\cdot\mathbf{j}=0 \quad \mathbf{i}\cdot\mathbf{k}=0 \quad \mathbf{j}\cdot\mathbf{k}=0$$

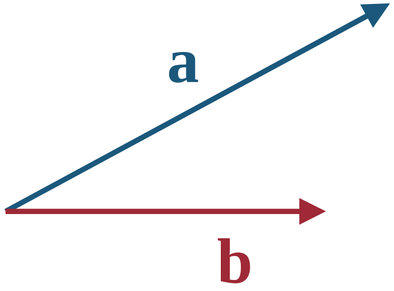
$$\mathbf{j}\cdot\mathbf{i}=0 \quad \mathbf{k}\cdot\mathbf{i}=0 \quad \mathbf{k}\cdot\mathbf{j}=0$$



Also, since these are unit length vectors, by definition

$$\mathbf{i}\cdot\mathbf{i}=1 \quad \mathbf{j}\cdot\mathbf{j}=1 \quad \mathbf{k}\cdot\mathbf{k}=1$$

Vector dot product



Since

$$\mathbf{i}\cdot\mathbf{j}=0 \quad \mathbf{i}\cdot\mathbf{k}=0 \quad \mathbf{j}\cdot\mathbf{k}=0$$

$$\mathbf{j}\cdot\mathbf{i}=0 \quad \mathbf{k}\cdot\mathbf{i}=0 \quad \mathbf{k}\cdot\mathbf{j}=0$$

Forming the dot product algebraically

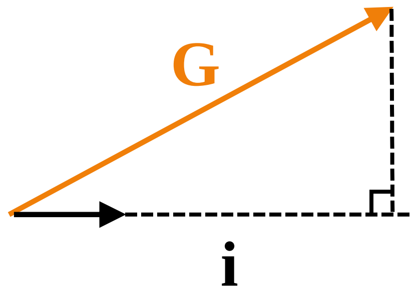
$$\mathbf{a}\cdot\mathbf{b}=\left(a_x\mathbf{i}+a_y\mathbf{j}+a_z\mathbf{k}\right)\cdot\left(b_x\mathbf{i}+b_y\mathbf{j}+b_z\mathbf{k}\right)$$

gives

$$\mathbf{a}\cdot\mathbf{b}=a_xb_x+a_yb_y+a_zb_z$$

which is an equivalent formula for the dot product

Vector dot product

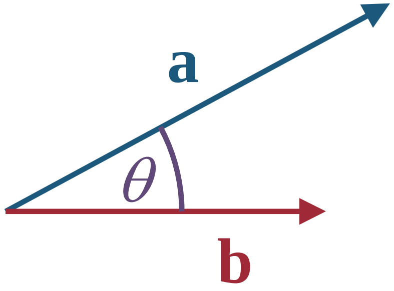


The components of a vector can be found by

- taking the dot product
  - with the unit vectors along the coordinate directions
    - For example

$$\mathbf{G} \cdot \mathbf{i} = \left(G_x \mathbf{i} + G_y \mathbf{j} + G_z \mathbf{k} \right) \cdot \mathbf{i} = G_x$$

Vector cross product



For two vectors

$$\mathbf{a} = a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k}$$

$$\mathbf{b} = b_x\mathbf{i} + b_y\mathbf{j} + b_z\mathbf{k}$$

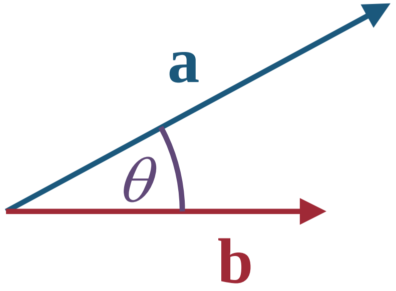
the vector cross product is

$$\mathbf{a} \times \mathbf{b} = \mathbf{n}|\mathbf{a}||\mathbf{b}|\sin\theta \equiv \mathbf{n}ab\sin\theta$$

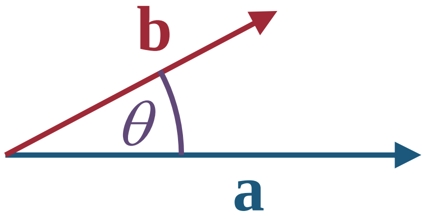
n is a unit vector with a direction given by the

right hand screw rule

Right hand screw rule



$\mathbf{a} \times \mathbf{b}$ gives vector $\mathbf{n}$ away from you



$\mathbf{a} \times \mathbf{b}$ gives vector $\mathbf{n}$ towards you

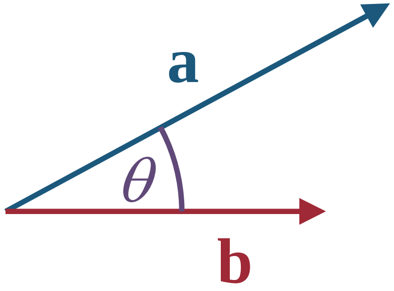
Imagine you have a corkscrew

- With an ordinary right-handed thread
  - with its handle lined up along vector $\mathbf{a}$

Now rotate the handle so it lines up with vector $\mathbf{b}$

- The direction, in or out, that the corkscrew moved is the direction of the vector $\mathbf{n}$

Vector cross product



Note that

$$\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$$

If we have to turn clockwise to go from $\mathbf{a}$ to $\mathbf{b}$

So the corkscrew goes "in"

So $\mathbf{n}$ points "inwards"

Then we have to turn anti-clockwise to go from $\mathbf{b}$ to $\mathbf{a}$

So the corkscrew goes "out"

So $\mathbf{n}$ point "outwards"

Vector cross product

An equivalent algebraic formula for the vector cross product is

$$\mathbf{a}\times\mathbf{b}=\left(a_y b_z-a_z b_y\right)\mathbf{i}+\left(a_z b_x-a_x b_z\right)\mathbf{j}+\left(a_x b_y-a_y b_x\right)\mathbf{k}$$

A short-hand way of writing this is

$$\mathbf{a}\times\mathbf{b}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ a_x & a_y & a_z\\ b_x & b_y & b_z\end{vmatrix}$$

which is the same as the determinant notation used with matrix algebra