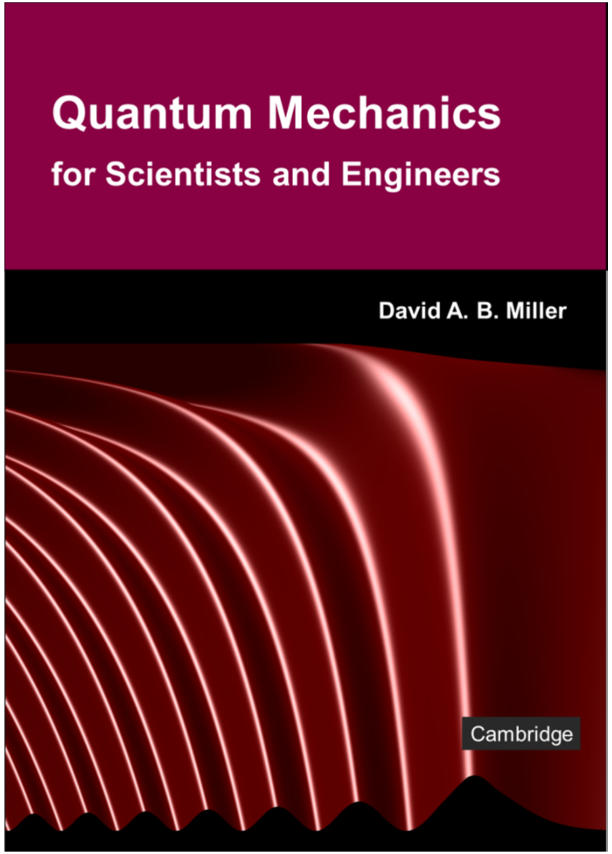
2.1 Wave propagation

Slides: Video 2.1.1 Plane waves and interference

Text reference: Quantum Mechanics for Scientists and Engineers

Section B.4



Wave propagation

Plane waves and interference

Quantum mechanics for scientists and engineers

David Miller

Wave equation in 3 dimensions

Generalizing to 3 dimensions, the wave equation becomes

$$\nabla^2 \phi - \frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2} = 0$$

where

$$\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

or equivalently, with unit vectors $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, and $\hat{\mathbf{z}}$ in the corresponding directions

$$\nabla^2 \equiv \nabla \cdot \nabla = \left(\hat{\mathbf{x}}\frac{\partial}{\partial x} + \hat{\mathbf{y}}\frac{\partial}{\partial y} + \hat{\mathbf{z}}\frac{\partial}{\partial z} \right) \cdot \left(\hat{\mathbf{x}}\frac{\partial}{\partial x} + \hat{\mathbf{y}}\frac{\partial}{\partial y} + \hat{\mathbf{z}}\frac{\partial}{\partial z} \right)$$

Plane wave solutions

We can check that a monochromatic (one frequency) "plane wave" of the form $\exp\left[i\left(\mathbf{k}\cdot\mathbf{r}-\omega t\right)\right]$

where $\mathbf{r}=x\,\hat{\mathbf{x}}+y\,\hat{\mathbf{y}}+z\,\hat{\mathbf{z}}$ and $\mathbf{k}=k_x\,\hat{\mathbf{x}}+k_y\,\hat{\mathbf{y}}+k_z\,\hat{\mathbf{z}}$

is a solution when $k=\omega/c$

First note that

$$\nabla\exp\left[i\left(\mathbf{k}\cdot\mathbf{r}-\omega t\right)\right]=\left(\hat{\mathbf{x}}\frac{\partial}{\partial x}+\hat{\mathbf{y}}\frac{\partial}{\partial y}+\hat{\mathbf{z}}\frac{\partial}{\partial z}\right)\exp\left\{i\left[k_x x+k_y y+k_z z-\omega t\right]\right\}$$

$$=i\left(k_x\hat{\mathbf{x}}+k_y\hat{\mathbf{y}}+k_z\hat{\mathbf{z}}\right)\exp\left\{i\left[k_x x+k_y y+k_z z-\omega t\right]\right\}$$

$$=i\mathbf{k}\exp\left[i\left(\mathbf{k}\cdot\mathbf{r}-\omega t\right)\right]$$

Plane wave solutions

So

$$\begin{aligned}
\nabla^2 \exp\left[i\left(\mathbf{k}\cdot\mathbf{r}-\omega t\right)\right] &= \nabla\cdot\nabla\exp\left[i\left(\mathbf{k}\cdot\mathbf{r}-\omega t\right)\right] \\
&= \nabla\cdot\left(i\mathbf{k}\exp\left[i\left(\mathbf{k}\cdot\mathbf{r}-\omega t\right)\right]\right) \\
&= i\left(k_x\frac{\partial}{\partial x}+k_y\frac{\partial}{\partial y}+k_z\frac{\partial}{\partial z}\right)\exp\left[i\left(\mathbf{k}\cdot\mathbf{r}-\omega t\right)\right] \\
&= -\left(k_x^2+k_y^2+k_z^2\right)\exp\left[i\left(\mathbf{k}\cdot\mathbf{r}-\omega t\right)\right] \\
&= -k^2\exp\left[i\left(\mathbf{k}\cdot\mathbf{r}-\omega t\right)\right]
\end{aligned}$$

Plane wave solutions

Since $\dfrac{\partial^2}{\partial t^2}\exp\left[i\left(\mathbf{k}\cdot\mathbf{r}-\omega t\right)\right]=-\omega^2\exp\left[i\left(\mathbf{k}\cdot\mathbf{r}-\omega t\right)\right]$

then, with $\nabla^2\exp\left[i\left(\mathbf{k}\cdot\mathbf{r}-\omega t\right)\right]=-k^2\exp\left[i\left(\mathbf{k}\cdot\mathbf{r}-\omega t\right)\right]$

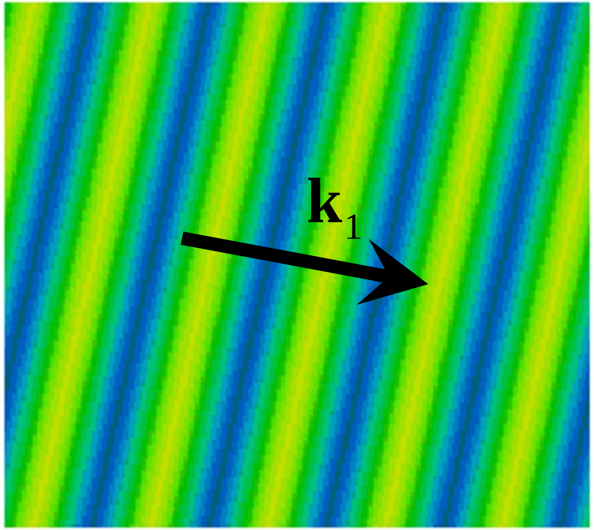
and choosing $k=\omega/c$

$$\nabla^2\exp\left[i\left(\mathbf{k}\cdot\mathbf{r}-\omega t\right)\right]-\frac{1}{c^2}\frac{\partial^2\exp\left[i\left(\mathbf{k}\cdot\mathbf{r}-\omega t\right)\right]}{\partial t^2}$$

$$=\left(-k^2+\omega^2/c^2\right)\exp\left[i\left(\mathbf{k}\cdot\mathbf{r}-\omega t\right)\right]=\left(-k^2+k^2\right)\exp\left[i\left(\mathbf{k}\cdot\mathbf{r}-\omega t\right)\right]=0$$

So $\exp\left[i\left(\mathbf{k}\cdot\mathbf{r}-\omega t\right)\right]$ is indeed a solution for any vector direction $\mathbf{k}$ provided $k=\omega/c$

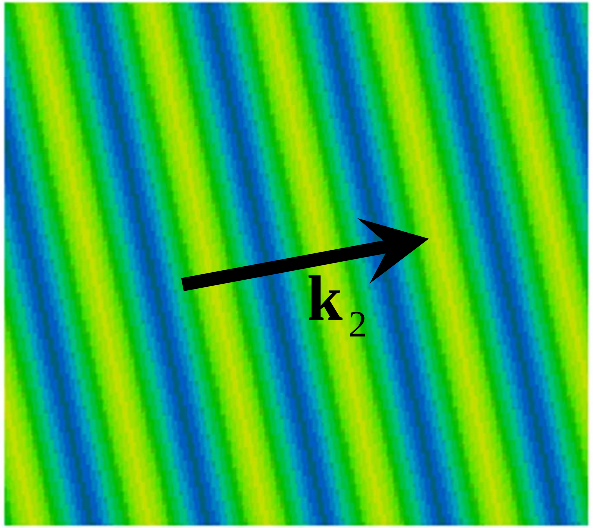
Wave interference



One solution is

the plane wave with wavevector $\mathbf{k}_1$

Wave interference



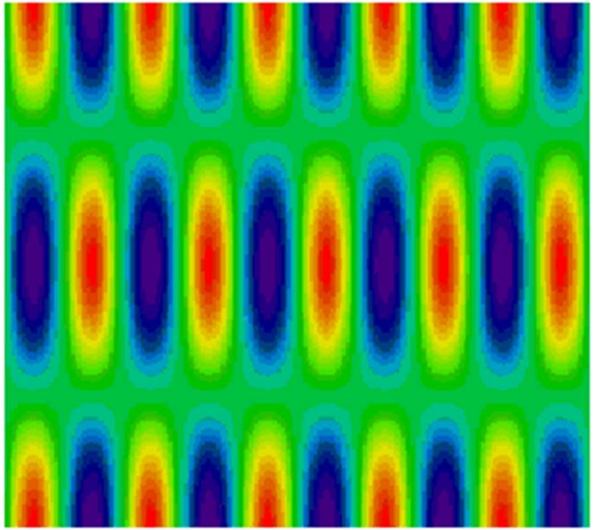
One solution is

the plane wave with wavevector $\mathbf{k}_1$

Another solution is

the plane wave with wavevector $\mathbf{k}_2$

Wave interference



One solution is

the plane wave with wavevector $\mathbf{k}_1$

Another solution is

the plane wave with wavevector $\mathbf{k}_2$

Because the wave equation is linear

the sum is also a solution

showing interference