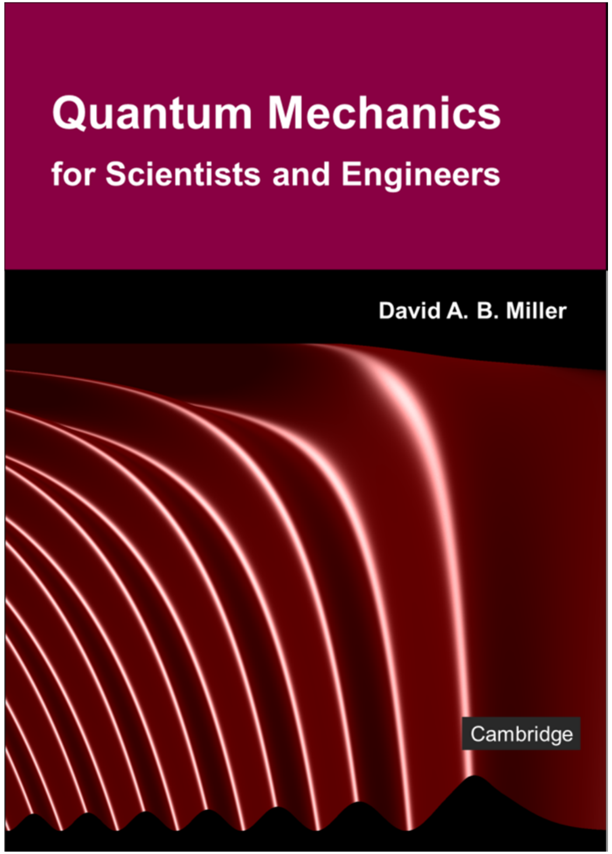
3.3 The time-dependent Schrödinger equation

Slides: Video 3.3.6 Linear superposition

Text reference: Quantum Mechanics for Scientists and Engineers

Section 3.4 – 3.5



The time-dependent Schrödinger equation

Linear superposition

Quantum mechanics for scientists and engineers

David Miller

Linearity of Schrödinger's equation

The time-dependent Schrödinger equation is linear in the wavefunction Ψ

$$-\frac{\hbar^2}{2m}\nabla^2\Psi(\mathbf{r},t)+V(\mathbf{r},t)\Psi(\mathbf{r},t)=i\hbar\frac{\partial\Psi(\mathbf{r},t)}{\partial t}$$

One reason is that no higher powers of Ψ appear anywhere in the equation

A second reason is that Ψ appears in every term

there is no additive constant term anywhere

Linearity of Schrödinger's equation

Linearity requires two conditions

obeyed by Schrödinger's time-dependent equation

1 - If Ψ is a solution, then so also is $a\Psi$, where a is any constant

2 - If Ψ_a and Ψ_b are solutions, then so also is $\Psi_a + \Psi_b$

A consequence of these two conditions is that

$$\Psi_c(\mathbf{r},t) = c_a \Psi_a(\mathbf{r},t) + c_b \Psi_b(\mathbf{r},t)$$

where c_a and c_b are (complex) constants

is also a solution

Linear superposition

The fact that

$$\Psi_c(\mathbf{r},t) = c_a\Psi_a(\mathbf{r},t) + c_b\Psi_b(\mathbf{r},t)$$

is a solution if Ψ_a and Ψ_b are solutions

is the property of

linear superposition

To emphasize

linear superpositions of solutions of the time-dependent Schrödinger equation

are also solutions

Time-dependence and expansion in eigenstates

We know that

if the potential V is constant in time

each of the energy eigenstates $\psi_n(\mathbf{r})$

with eigenenergy E_n

is separately a solution of the time-dependent Schrödinger equation

provided we remember to multiply by the right complex exponential factor

$$\Psi_n(\mathbf{r},t) = \exp(-iE_n t/\hbar)\psi_n(\mathbf{r})$$

Time-dependence and expansion in eigenstates

Now we also know that the set of eigenfunctions of problems we will consider is a complete set

so the wavefunction at $t=0$ can be expanded in them

$$\Psi(\mathbf{r},0)=\sum_n a_n \psi_n(\mathbf{r})$$

where the a_n are the expansion coefficients

But we know that a function that starts out as $\psi_n(\mathbf{r})$

will evolve in time as $\Psi_n(\mathbf{r},t)=\exp(-iE_n t/\hbar)\psi_n(\mathbf{r})$

so, by linear superposition, the solution at time t is

$$\Psi(\mathbf{r},t)=\sum_n a_n \Psi_n(\mathbf{r},t)=\sum_n a_n \exp(-iE_n t/\hbar)\psi_n(\mathbf{r})$$

Time-dependence and expansion in eigenstates

Hence, for the case where the potential V does not vary in time

$$\Psi(\mathbf{r},t)=\sum_n a_n \Psi_n(\mathbf{r},t)=\sum_n a_n \exp(-iE_n t/\hbar)\psi_n(\mathbf{r})$$

is the solution of the time-dependent equation

with the initial condition $\Psi(\mathbf{r},0)=\psi(\mathbf{r})=\sum_n a_n \psi_n(\mathbf{r})$

Hence, if we expand the wavefunction at time $t=0$ in the energy eigenstates

we have solved for the time evolution of the state just by adding up the above sum