Sums, products and power series

Background mathematics review

David Miller

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Sum, factorial and product notations

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Summation notation

If we want to add a set of numbers a_1, a_2, a_3, and a_4, we can write

$$S = a_1 + a_2 + a_3 + a_4$$

or we can use summation notation

$$S = \sum_{j=1}^{4} a_j \equiv \sum_{j=1,2,3,4} a_j \equiv \sum_{j=1,\ldots,4} a_j$$

If the range of j is obvious

$$S = \sum_j a_j$$

Here j is an "index"

plural of index – "indexes" or "indices"

Example – arithmetic series

For a set of numbers spaced by a constant amount,

an "arithmetic progression or sequence"

i.e., where the nth number is $a_n = a_1 + (n-1)d$

e.g., the numbers 4, 7, 10, and 13

i.e., $a_1 = 4$, $a_2 = 7$, $a_3 = 10$ and $a_4 = 13$

with, therefore $d = 3$ and $m = 4$ terms in the progression

gives the "series" (sum of the terms)

$$a_1 + a_2 + a_3 + a_4 = 4 + 7 + 10 + 13 = 34$$

Example – arithmetic series

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$$a_1 + a_2 + a_3 + a_4 \equiv \sum_{j=1}^{4} a_j = \sum_{j=1}^{m} \left[a_1 + (j-1)d \right]$$

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$$a_1 + a_2 + a_3 + a_4 = \sum_{j=1}^{m} \left[a_1 + (j-1)d \right] = m\frac{(a_1 + a_4)}{2}$$

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$$a_1 + a_2 + a_3 + a_4 = \boxed{\sum_{j=1}^{m}\left[a_1 + (j-1)d\right] = m\frac{(a_1 + a_m)}{2}} = 4 \times \frac{(4+13)}{2} = 34$$

Example – geometric series

With a constant ratio between successive terms,

a "geometric progression or sequence"

i.e., where the nth term is $a_n = a_1 r^{n-1}$

e.g., the numbers 3, 6, 12, and 24

i.e., $a_1 = 3$, $a_2 = 6$, $a_3 = 12$ and $a_4 = 24$

with, therefore $r = 2$ and $m = 4$ terms in the progression

gives the "series" (sum of the terms)

$$a_1 + a_2 + a_3 + a_4 = 3 + 6 + 12 + 24 = 45$$

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$$a_1 + a_2 + a_3 + a_4 \equiv \sum_{j=1}^{4} a_j = \sum_{j=1}^{m} a_1 r^{(j-1)}$$

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$$a_1 + a_2 + a_3 + a_4 \equiv \sum_{j=1}^{4} a_j = \sum_{j=1}^{m} a_1 r^{(j-1)} = a_1 \frac{1-r^m}{1-r}$$

Example – geometric series

With a constant ratio between successive terms,

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i.e., where the nth term is $a_n = a_1 r^{n-1}$

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with, therefore $r = 2$ and $m = 4$ terms in the progression

gives the "series" (sum of the terms)

$$a_1 + a_2 + a_3 + a_4 \equiv \boxed{\sum_{j=1}^{4} a_j = \sum_{j=1}^{m} a_1 r^{(j-1)} = a_1 \frac{1-r^m}{1-r}} = 3 \times \frac{1-16}{1-2} = 3 \times 15 = 45$$

Summation over multiple indexes

We can extend the summation notation

Suppose we have two lists of numbers

a_1, a_2, a_3, a_4 and b_1, b_2, b_3

and we want to add up all the products

$$\begin{aligned} R = {} & a_1 b_1 + a_1 b_2 + a_1 b_3 \\ & + a_2 b_1 + a_2 b_2 + a_2 b_3 \\ & + a_3 b_1 + a_3 b_2 + a_3 b_3 \\ & + a_4 b_1 + a_4 b_2 + a_4 b_3 \end{aligned}$$

Summation over multiple indexes

Then we can write

$$R = a_1 b_1 + a_1 b_2 + a_1 b_3 = \sum_{j=1}^{4}\left(\sum_{k=1}^{3} a_j b_k\right)$$
$$+a_2 b_1 + a_2 b_2 + a_2 b_3$$
$$+a_3 b_1 + a_3 b_2 + a_3 b_3$$
$$+a_4 b_1 + a_4 b_2 + a_4 b_3$$

and, because order of addition does not matter

in various equivalent notations

$$R = \sum_{j=1}^{4}\sum_{k=1}^{3} a_j b_k = \sum_{k=1}^{3}\sum_{j=1}^{4} a_j b_k \equiv \sum_{j,k} a_j b_k \equiv \sum\nolimits_{j,k} a_j b_k$$

Factorial notation

Quite often, we need a convenient way of writing the product of successive integers

e.g., $1 \times 2 \times 3 \times 4$

We can write this as

$$1 \times 2 \times 3 \times 4 \equiv 4!$$

called "four factorial"

and using the "exclamation point" "!"

The notation is obvious for most other cases

Note, though, that we choose

$$0! = 1$$

Product notation

Generally, when we want to write the product of various successive terms

$$a_1 \times a_2 \times a_3 \times a_4$$

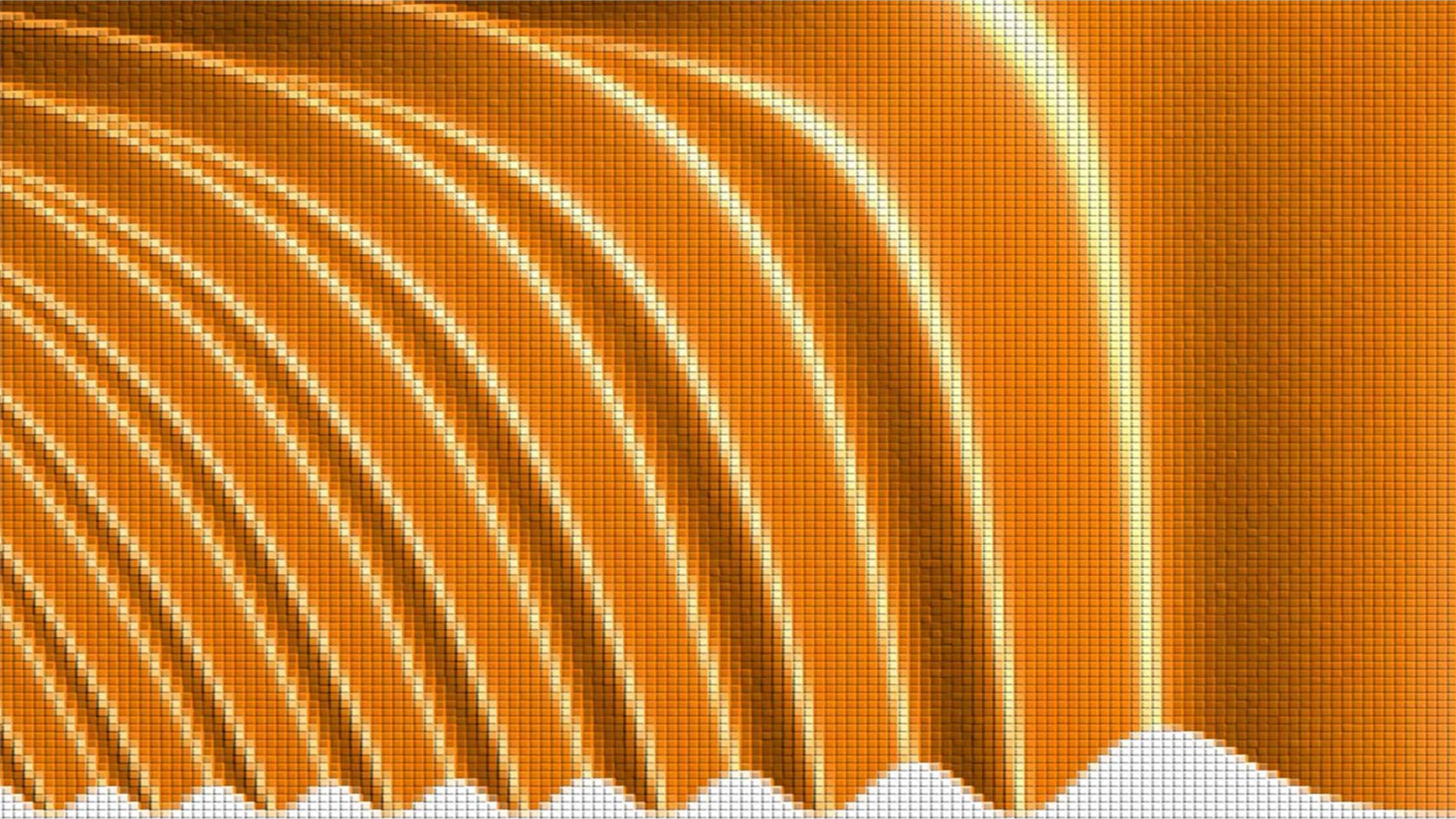
by analogy with the summation notation

we can use the "product" notation

$$a_1 \times a_2 \times a_3 \times a_4 \equiv \prod_{j=1}^{4} a_j$$

For example, for all integers $n \geq 1$

$$n! \equiv \prod_{p=1}^{n} p$$



Sums, products and power series

Power series

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Analytic functions and power series

For a very broad class of the functions in physics

we presume they are "analytic"

(except possibly at some "singularities")

i.e., at least for some range of values of the argument x near some point x_o

the function $f(x)$ can be arbitrarily well approximated by a "power series"

i.e., with $x_o = 0$ for simplicity we have

$$f(x) \equiv a_o + a_1 x + a_2 x^2 + a_3 x^3 \cdots$$

which may be an infinite series

The "ellipsis" "…" means we omit writing some terms explicitly

Analytic functions and power series

For a very broad class of the functions in physics

we presume they are "analytic"

(except possibly at some "singularities")

i.e., at least for some range of values of the argument x near some point x_o

the function $f(x)$ can be arbitrarily well approximated by a "power series"

i.e., generally

$$f(x) \equiv a_o + a_1(x - x_o) + a_2(x - x_o)^2 + a_3(x - x_o)^3 \cdots$$

Maclaurin series

Taking the simplest case of $x_o = 0$ first

$$f(x) \equiv a_o + a_1 x + a_2 x^2 + a_3 x^3 \cdots$$

Obviously at $x = 0$

$$f(0) = a_o \text{ so, trivially, } a_o = f(0)$$

Now,

$$f'(x) \equiv \frac{df}{dx} = a_1 + 2a_2 x + 3a_3 x^2 \cdots$$

so

$$f'(0) \equiv \left.\frac{df}{dx}\right|_0 = a_1$$

Maclaurin series

Continuing

$$f''(x) \equiv \frac{d^2 f}{dx^2} = 2a_2 + 3\times 2\times a_3 x \cdots = 2!\times a_2 + 3!\times a_3 x \cdots$$

so

$$f''(0) \equiv \left.\frac{d^2 f}{dx^2}\right|_0 = 2!a_2 \cdots$$

and

$$f'''(0) \equiv \left.\frac{d^3 f}{dx^3}\right|_0 = 3!a_3 \cdots$$

and so on

Maclaurin and Taylor series

Continuing gives the Maclaurin series

$$f(x) = f(0) + \frac{x}{1!}\frac{df}{dx}\bigg|_0 + \frac{x^2}{2!}\frac{d^2 f}{dx^2}\bigg|_0 + \ldots + \frac{x^n}{n!}\frac{d^n f}{dx^n}\bigg|_0 + \ldots$$

Repeating the same procedure around $x = x_o$ with

$$f(x) \equiv a_o + a_1(x - x_o) + a_2(x - x_o)^2 + a_3(x - x_o)^3 \cdots$$

gives the Taylor series

$$f(x) =$$

$$f(x_o) + \frac{(x - x_o)}{1!}\frac{df}{dx}\bigg|_{x_o} + \frac{(x - x_o)^2}{2!}\frac{d^2 f}{dx^2}\bigg|_{x_o} + \ldots + \frac{(x - x_o)^n}{n!}\frac{d^n f}{dx^n}\bigg|_{x_o} + \ldots$$

Example power series expansions

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots$$

$$\exp(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

Power series for approximations

Maclaurin and Taylor series allow approximations for small ranges of the argument about a point

Examples for small x

Approximations to "first order in x"

$$1/(1+x) \simeq 1 - x \qquad \exp(x) \simeq 1 + x \qquad \sin x \simeq x$$

$$\sqrt{1+x} \simeq 1 + x/2 \qquad \ln(1+x) \simeq x \qquad \tan x \simeq x$$

Note lowest order dependence on x for $\cos x$ is "second order"

$$\cos x \simeq 1 - x^2/2$$