Integral calculus

Background mathematics review

David Miller

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Integration in one variable

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Area under a curve

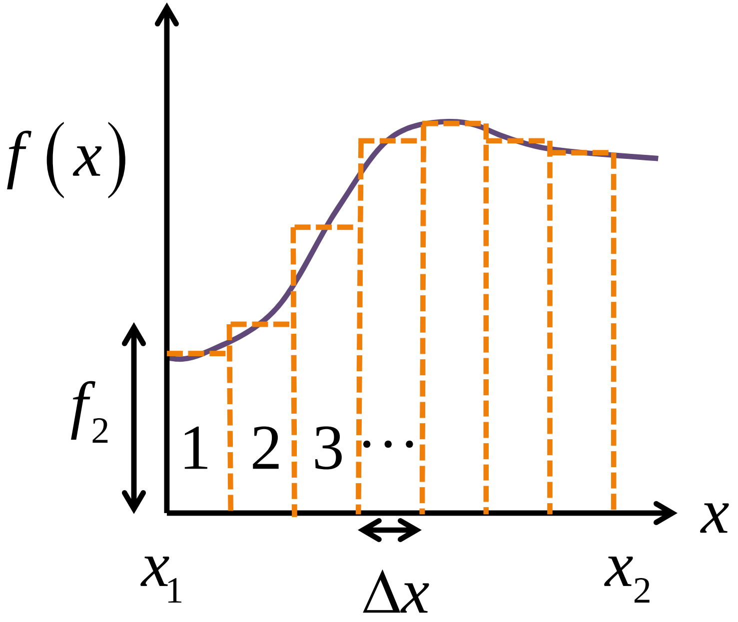
Integration can be thought of as the area under a curve

Approximately

divide area under the curve into rectangles of width Δx

Add up the areas, $f_j \Delta x$, i.e., $\sum_j f_j \Delta x$

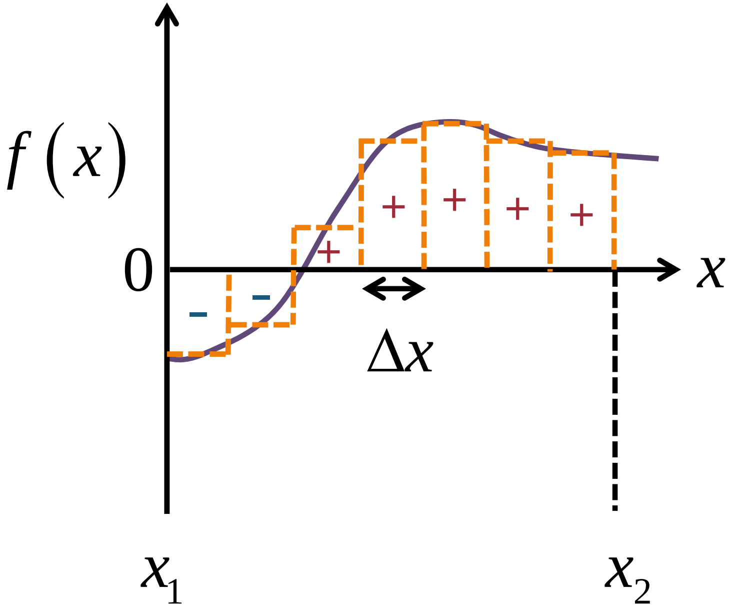
The integral is the limit as we make the rectangles thinner and thinner



$$\int_{x_1}^{x_2} f(x)\,dx \equiv \lim_{\Delta x \to 0} \left(\sum_j f_j \Delta x \right)$$

"Area" under a curve

- We can extend the idea even for negative values of $f(x)$
  - Then we have some negative "areas"
    - but we still add them up to get the integral
- The integral of a non-zero function can be zero
  - for equal "area" magnitudes above and below the axis



$$\int_{x_1}^{x_2} f(x)\,dx \equiv \lim_{\Delta x \to 0}\left(\sum_j f_j \Delta x\right)$$

Notation for integrals

It is common to think of the

integral sign $\int$ and the associated "infinitesimal" dx

as effectively being "brackets"

enclosing everything that has to be integrated (the "integrand")

Alternatively

we can regard everything with the same variable as the infinitesimal (here, x) as being the integrand

$$\int_{x_1}^{x_2} f(x)\,dx \equiv \int_{x_1}^{x_2} dx\, f(x)$$

Limits and integrals

An integral with definite lower and upper bounds for the integration

is called a "definite integral"

e.g., $\int_{x_1}^{x_2} f(x)\,dx$

Sometimes formally an "indefinite integral" without defined limits is useful

as in an analytic result

e.g., $\int x^2 dx = \frac{1}{3}x^3 + C$

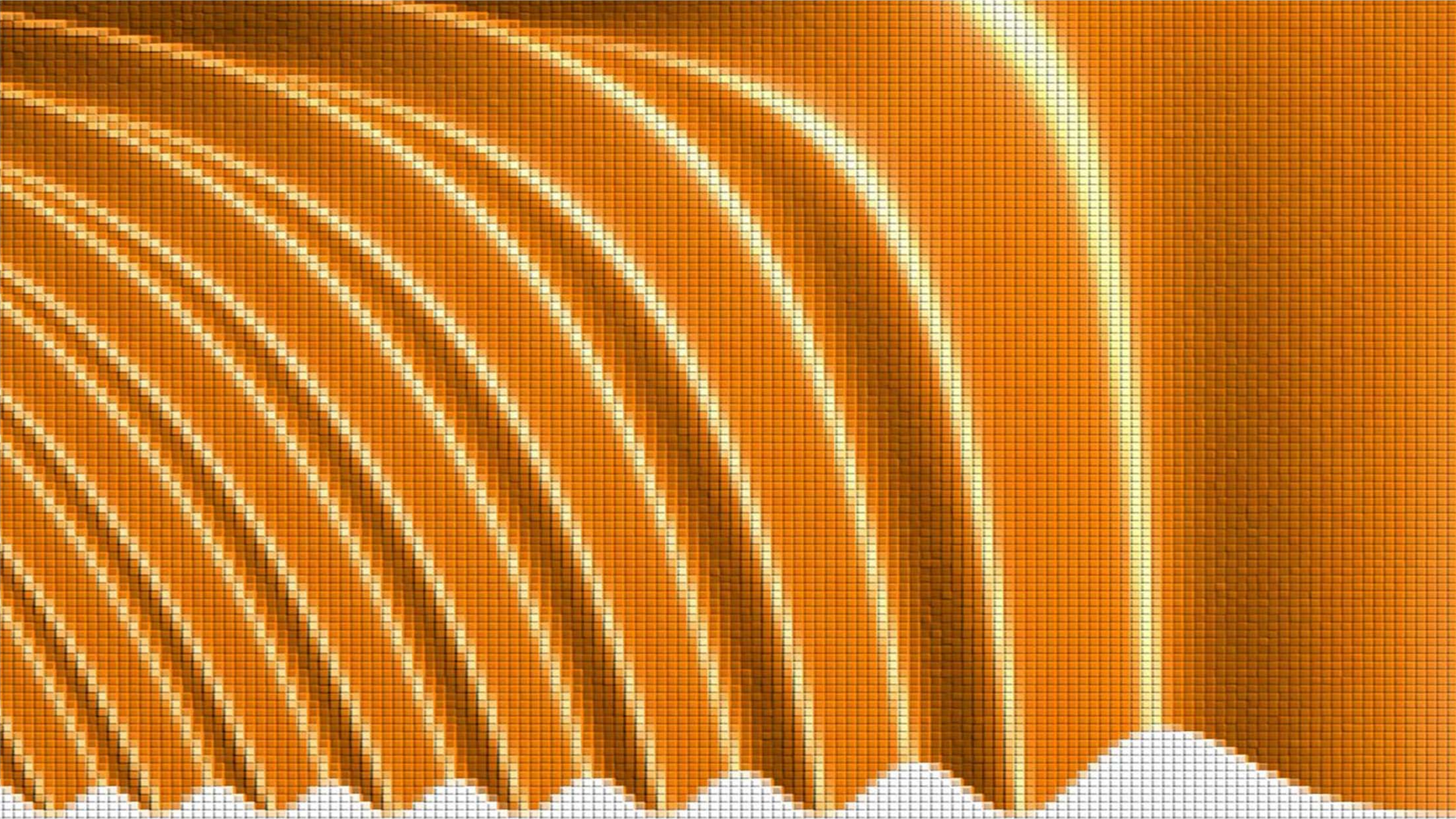
but then the result is arbitrary within an additional "constant of integration" (here C)

Relation between integration and differentiation

- Integration and differentiation are inverse operations of one another
  - Specifically

$$\int_a^b \left(\frac{df}{dx}\right) dx = f(b) - f(a)$$

    - which is known as the
      - "fundamental theorem of calculus"
- Sometimes (but not often)
  - the integral is called the "antiderivative"



Integral calculus

Volume integration

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Volume integration

If the volume of the bricks is ΔV

- then adding up all these small volumes
  - each "labeled" by some index j
    - would give the total volume

In the limit of very small bricks

- we would get the integral that gives us the total volume

$$V = \int_V dV = \lim_{\Delta V \to 0} \left(\sum_j \Delta V \right)$$

Volume integral notation

Various notations are used for the volume of the "infinitesimal" bricks

$$\int_V dV \equiv \int_V d\mathbf{r} \equiv \int_V d^3\mathbf{r}$$

- all of which can be confusing
  - using dV is not very specific about integration variables
    - and V is the total volume also
  - $d\mathbf{r}$ does not have the right dimensions (not meters cubed)
  - $d^3\mathbf{r}$ can be viewed as having the correct dimensions
    - but, like $d\mathbf{r}$, seems to imply a vector
      - and the volume is actually scalar

Volume integral of a quantity

We can integrate some quantity that is a function of position $\mathbf{r}$

- such as
  - the density $\rho(\mathbf{r})$ (kg/m^3)
    - to get the total mass m_{tot} (kg) in the volume

$$m_{tot} = \int_V \rho(\mathbf{r}) d^3\mathbf{r}$$

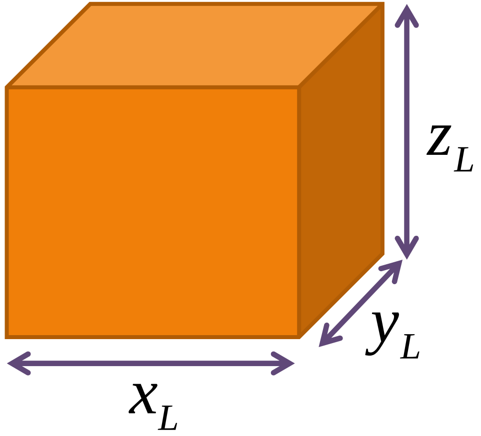
Reducing to one-dimensional integrals

To evaluate volume integrals

- we would like to reduce them to a set of nested one-dimensional integrals

This can be done at least for simple volumes

- such as a cuboid
  - a volume with rectangular faces



$$\int_V dV = \int_{x_c}^{x_c+x_L} \int_{y_c}^{y_c+y_L} \int_{z_c}^{z_c+z_L} dz dy dx = \int_{x_c}^{x_c+x_L} \int_{y_c}^{y_c+y_L} z_L dy dx = \int_{x_c}^{x_c+x_L} z_L y_L dx = z_L y_L x_L = V$$

This is not always possible for other volumes

Surface integrals

We can also perform integrals over surfaces

by dividing a surface S into "patches" of area ΔA

and similarly taking the limit of small patches

$$A = \int_S dA = \lim_{\Delta A \to 0} \left(\sum_j \Delta A \right)$$

We can use similar notations $\int_S dA \equiv \int_S d\mathbf{r} \equiv \int_V d^2\mathbf{r}$

with similar confusions

where $\mathbf{r}$ is position on the surface