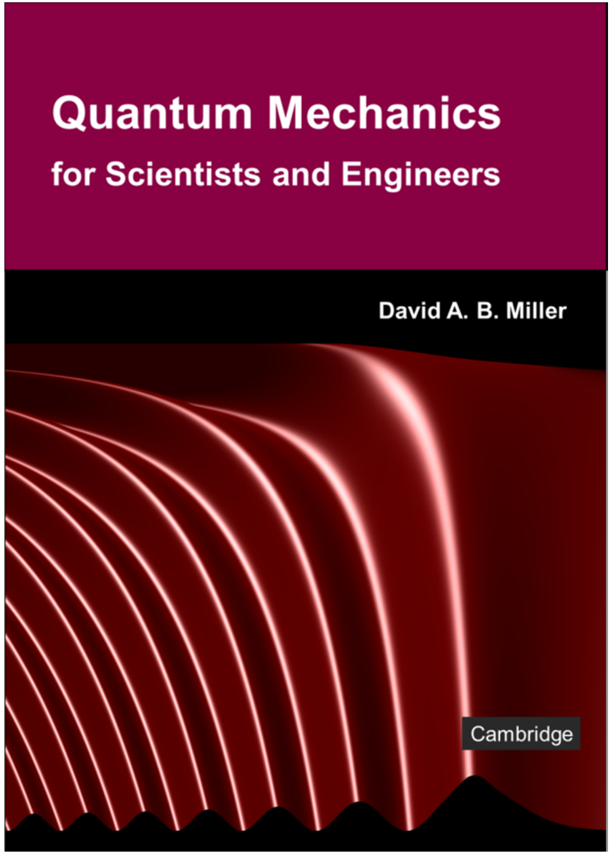
9.2 Time-dependent perturbation theory

Slides: Video 9.2.3 Simple oscillating perturbations

Text reference: Quantum Mechanics for Scientists and Engineers

Section 7.2 (first part)



Time-dependent perturbation theory

Simple oscillating perturbations

Quantum mechanics for scientists and engineers

David Miller

Simple oscillating perturbations

One very useful case is for oscillating perturbations

where a perturbation is varying sinusoidally in time

also called a "harmonic" perturbation

as in the harmonic oscillator

for example, a monochromatic electromagnetic wave

with an electric field in, say, the z direction

$$\mathsf{E}(t) = \mathsf{E}_o\left[\exp(-i\omega t) + \exp(i\omega t)\right] = 2\mathsf{E}_o\cos(\omega t)$$

where ω is a positive (angular) frequency

We consider this here in

first-order time-dependent perturbation theory

Simple oscillating perturbations

With $\mathsf{E}(t) = \mathsf{E}_o\left[\exp(-i\omega t) + \exp(i\omega t)\right] = 2\mathsf{E}_o \cos(\omega t)$

for an electron, the electrostatic energy in this field, relative to position $z = 0$

gives a perturbing Hamiltonian

$$\hat{H}_p(t) = e\mathsf{E}(t)z = \hat{H}_{po}\left[\exp(-i\omega t) + \exp(i\omega t)\right]$$

where, in this case

$$\hat{H}_{po} = e\mathsf{E}_o z$$

which is a time-independent operator

This perturbing Hamiltonian is called

the electric dipole approximation

Simple oscillating perturbations

We will presume that this perturbing Hamiltonian is only "on" for some finite time

For simplicity, we presume that

the perturbation starts at time $t = 0$

and ends at time $t = t_o$

so formally we have

$$\hat{H}_p(t) = 0, t < 0$$

$$= \hat{H}_{po}\left[\exp(-i\omega t) + \exp(i\omega t)\right], 0 < t < t_o$$

$$= 0, t > t_o$$

Simple oscillating perturbations

We are interested in the case where

- for times before $t = 0$
  - the system is in some specific energy eigenstate $|\psi_m\rangle$

Time-dependent perturbation theory will tell us

- with what probability the system
  - will make transitions into other states

With this choice

- all of the initial expansion coefficients $a_n^{(0)}$ are zero
  - except $a_m^{(0)}$
    - which has the value 1

Simple oscillating perturbations

With this simplification of the initial state to $|\psi_m\rangle$

the first order perturbation solution

$$\dot{a}_q^{(1)}(t) = \frac{1}{i\hbar}\sum_n a_n^{(0)} \exp(i\omega_{qn}t)\langle\psi_q|\hat{H}_p(t)|\psi_n\rangle$$

becomes $\dot{a}_q^{(1)}(t) = \frac{1}{i\hbar}\exp(i\omega_{qm}t)\langle\psi_q|\hat{H}_p(t)|\psi_m\rangle$

Now we substitute the perturbing Hamiltonian

$$\begin{aligned}\hat{H}_p(t) &= 0,\ t < 0 \\ &= \hat{H}_{po}\left[\exp(-i\omega t) + \exp(i\omega t)\right],\ 0 < t < t_o \\ &= 0,\ t > t_o\end{aligned}$$

Simple oscillating perturbations

With that substitution

and integrating over time

from time 0 to time t_o

$$a_q^{(1)}(t > t_o) = \frac{1}{i\hbar}\int_0^{t_0} \langle \psi_q | \hat{H}_p(t_1) | \psi_m \rangle \exp(i\omega_{qm} t_1)\, dt_1$$

$$= \frac{1}{i\hbar}\langle \psi_q | \hat{H}_{po} | \psi_m \rangle \int_0^{t_o} \left\{ \exp\left[i(\omega_{qm} - \omega)t_1\right] + \exp\left[i(\omega_{qm} + \omega)t_1\right] \right\} dt_1$$

Simple oscillating perturbations

So $a_q^{(1)}\left(t > t_o\right)$

$$= -\frac{1}{\hbar}\left\langle \psi_q \left| \hat{H}_{po} \right| \psi_m \right\rangle \left\{ \frac{\exp\left(i\left(\omega_{qm} - \omega\right)t_o\right) - 1}{\omega_{qm} - \omega} + \frac{\exp\left(i\left(\omega_{qm} + \omega\right)t_o\right) - 1}{\omega_{qm} + \omega} \right\}$$

$$= \frac{t_o}{i\hbar}\left\langle \psi_q \left| \hat{H}_{po} \right| \psi_m \right\rangle \left\{ \begin{aligned} &\exp\left[i\left(\omega_{qm} - \omega\right)t_o / 2\right] \frac{\sin\left[\left(\omega_{qm} - \omega\right)t_o / 2\right]}{\left(\omega_{qm} - \omega\right)t_o / 2} \\ &+ \exp\left[i\left(\omega_{qm} + \omega\right)t_o / 2\right] \frac{\sin\left[\left(\omega_{qm} + \omega\right)t_o / 2\right]}{\left(\omega_{qm} + \omega\right)t_o / 2} \end{aligned} \right\}$$

Simple oscillating perturbations

The function $\operatorname{sinc}(x) = \sin(x)/x$

peaks at 1 for $x = 0$

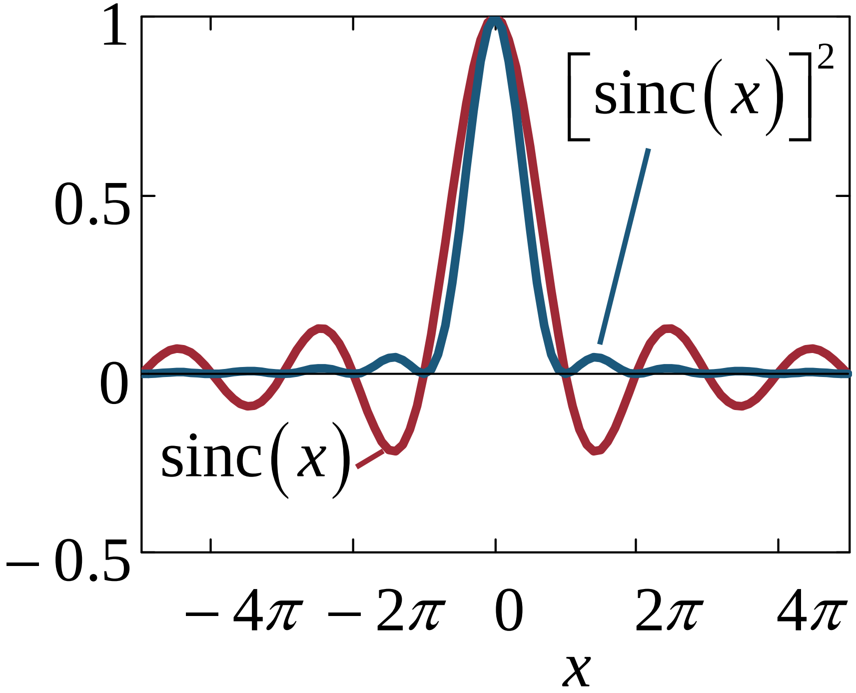
It is only large for $x \simeq 0$

so, e.g., $\dfrac{\sin\left[\left(\omega_{qm} - \omega\right)t_o/2\right]}{\left(\omega_{qm} - \omega\right)t_o/2}$

is strongly resonant

with relatively strong contributions

only for frequency ω close to ω_{qm}



Simple oscillating perturbations

We have now calculated the new state for times $t > t_o$

which is, to first order

$$|\Psi\rangle \simeq \exp\left(-iE_m t/\hbar\right)|\psi_m\rangle + \sum_q a_q^{(1)}(t > t_o)\exp\left(-iE_q t/\hbar\right)|\psi_q\rangle$$

with the $a_q^{(1)}(t > t_o)$ given by our preceding expression

Now that we have established our approximation to the new state

we can start calculating

the time dependence of measurable quantities