Differential equations

Background mathematics review

David Miller

Differential equations

First-order differential equations

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Differential equations in one variable

A differential equation involves derivatives of some function

E.g.,

$$\frac{dy}{dx} = ay$$

where a is some real number

The solution to this equation is

$$y = A\exp(ax)$$

where A is an arbitrary constant

It is easy to verify solutions

but it can be harder to find them

First order differential equations

$\frac{dy}{dx} = ay$ is a "first order" differential equation

no derivatives higher than first order

The solution

$$y = A\exp(ax)$$

has one undetermined constant A

and is called a "general solution"

Undetermined constants become fixed using

boundary conditions

Boundary condition

Suppose a ramp has a slope

$$\frac{dy}{dx} = 0.4y$$

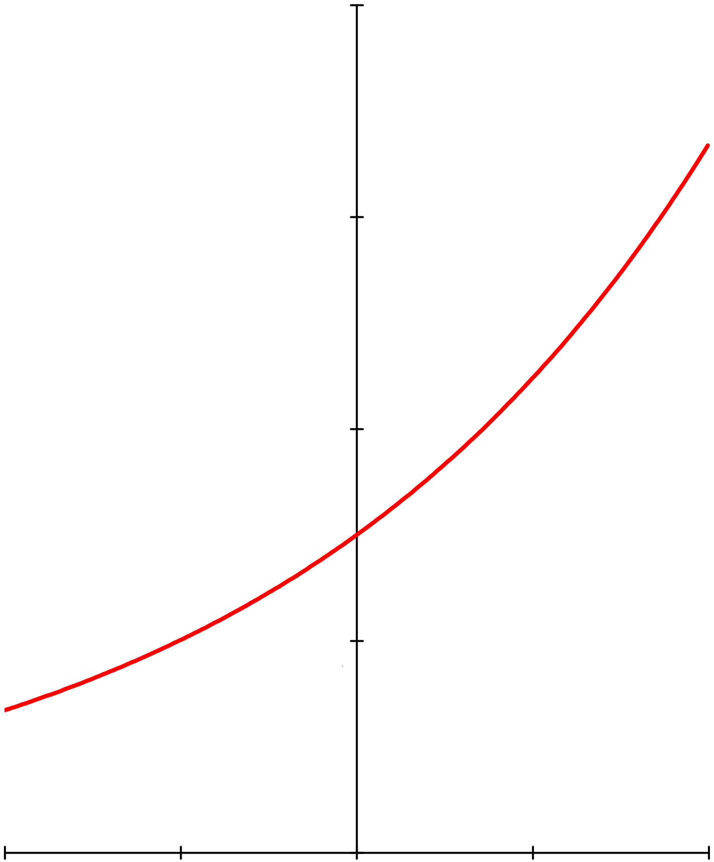
The general solution is

$$y = A\exp(0.4x)$$

If we also know the boundary condition that

at $x = 0,\ y = 1.5$

then, since $\exp(0) = 1$



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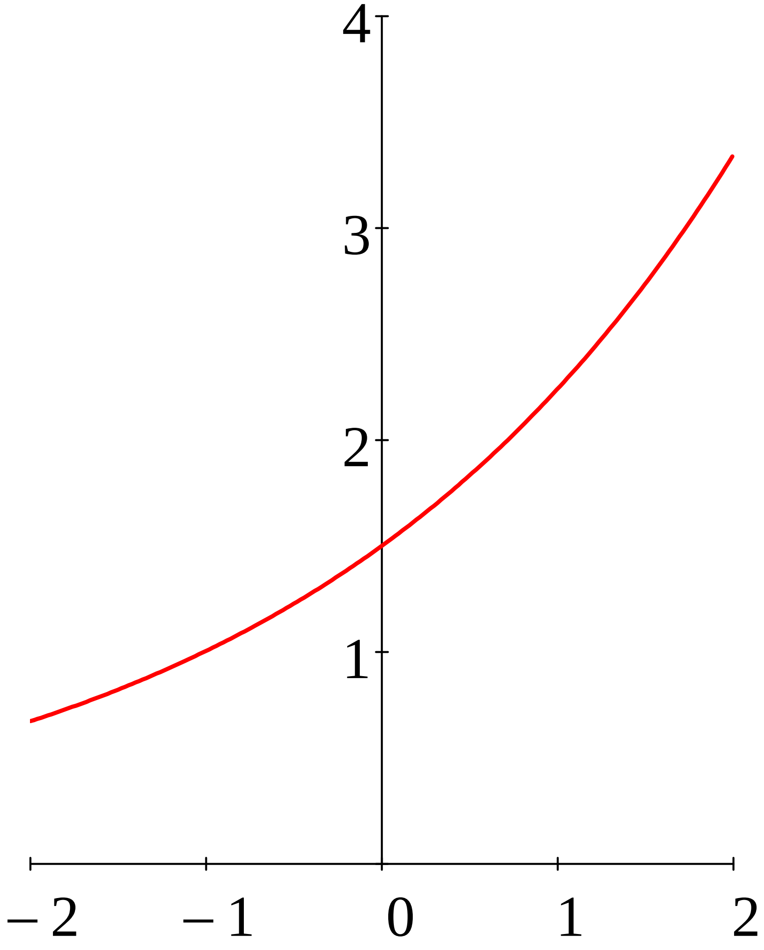
$$y = 1.5\exp(0.4x)$$



Boundary condition

With the same kind of ramp, i.e.,

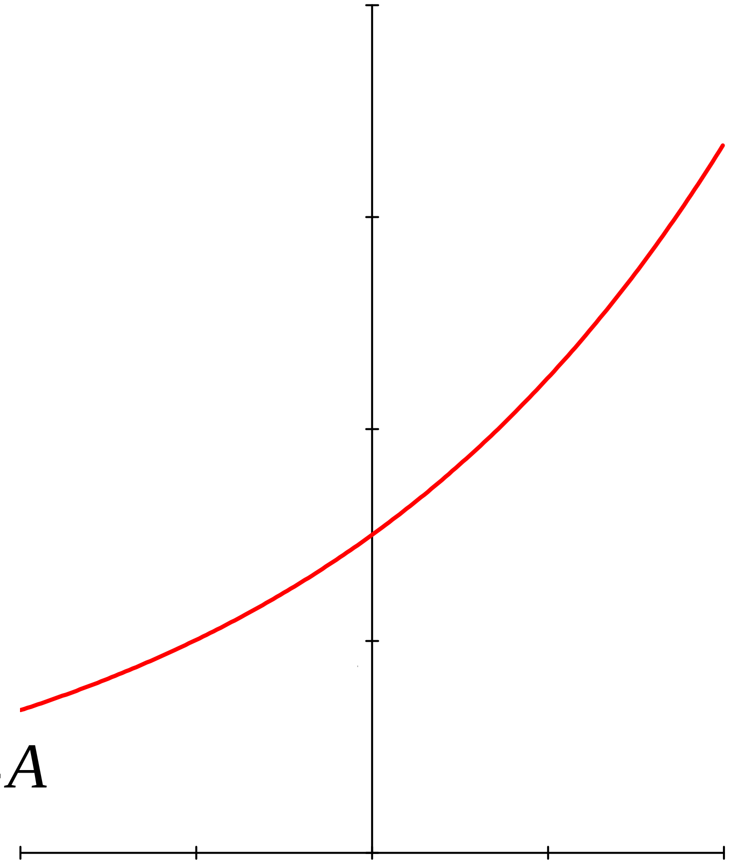
$$\frac{dy}{dx} = 0.4y \quad \text{so} \quad y = A\exp(0.4x)$$

suppose we know instead that

at $x = 0$, $\dfrac{dy}{dx} = 0.6$

Since $\dfrac{dy}{dx} = 0.4A\exp(0.4x)$,

then $\left.\dfrac{dy}{dx}\right|_{x=0} = 0.6 = 0.4A\exp(0) = 0.4A$



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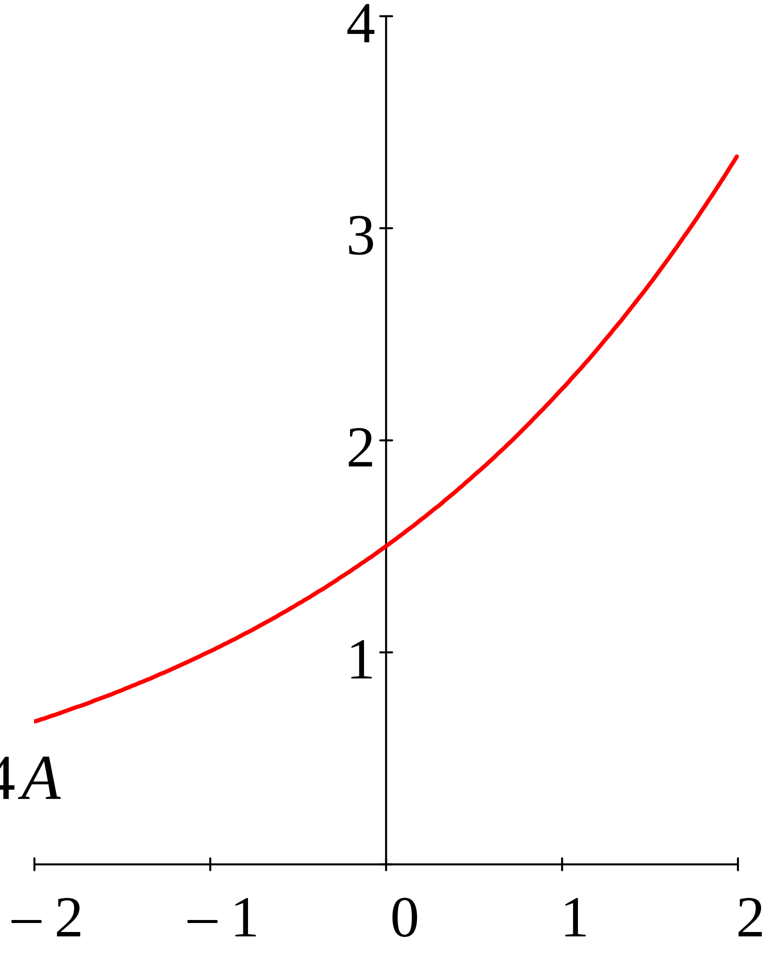
then $\left.\frac{dy}{dx}\right|_{x=0} = 0.6 = 0.4A\exp(0) = 0.4A$

so $A = 0.6/0.4 = 1.5$



Boundary condition types

Boundary conditions that specify the
value of a function
at some position or “boundary”
are called
“Dirichlet” boundary conditions

Boundary conditions that specify the
derivative or slope of a function
at some position or “boundary”
are called
“Neumann” boundary conditions

Imaginary first derivative

Note that the equation

$$\frac{dy}{dx} = iby$$

with b real

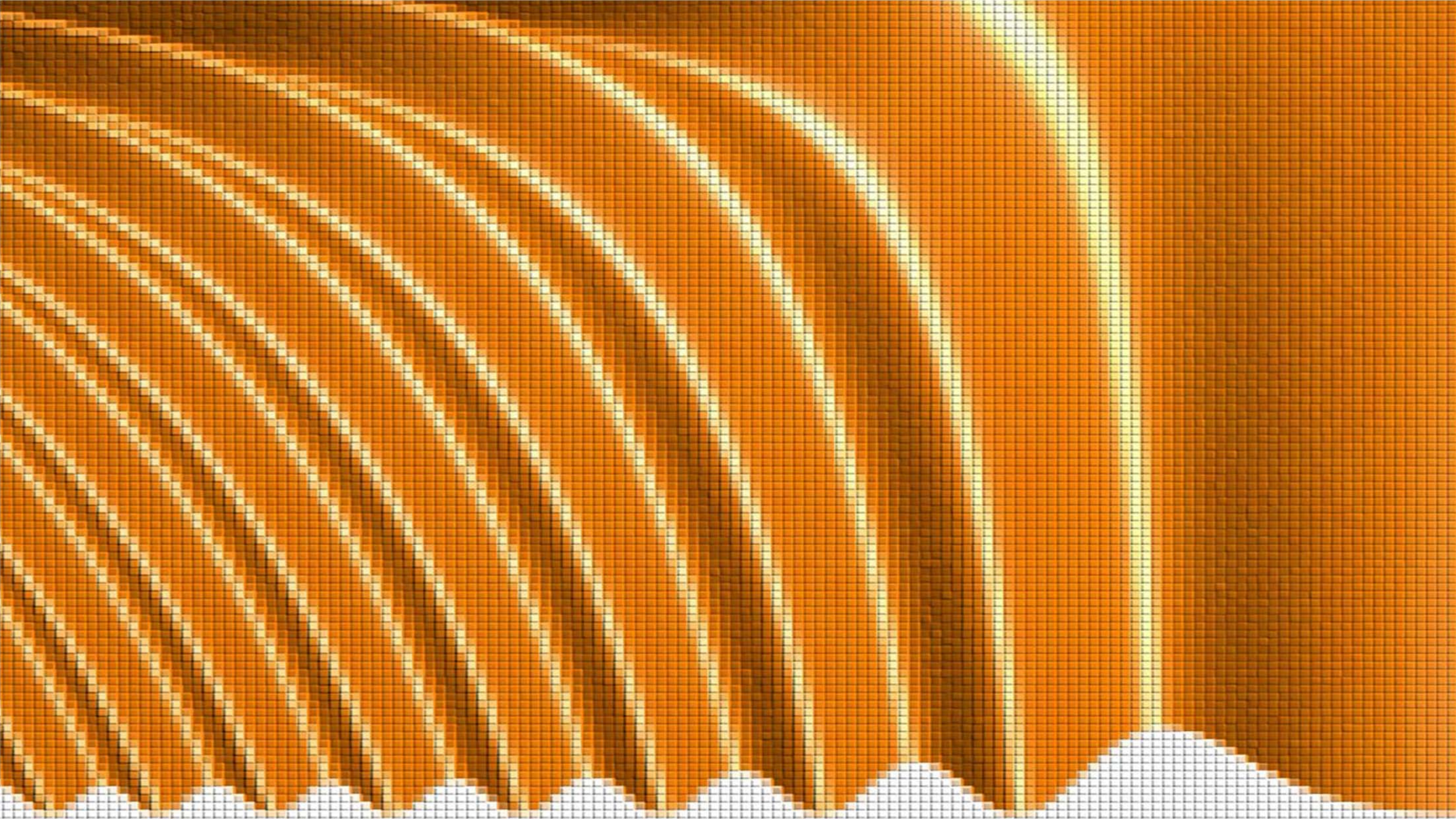
has the general solution

$$y = A\exp(ibx) \equiv A(\cos bx + i\sin bx)$$

Note that

neither $\cos bx$ nor $\sin bx$

is a solution of this equation



Differential equations

Second-order differential equations

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Second-order differential equations

A second-order differential equation

contains no derivatives higher than second order, e.g.,

first derivatives $\dfrac{dy}{dx}$

second derivatives $\dfrac{d^2y}{dx^2}$

but no higher derivatives

such as third order $\dfrac{d^3y}{dx^3}$

Simple harmonic oscillator equation

One simple and important equation is $$\frac{d^2y}{dt^2} = -\omega^2 y$$

Any of the functions $\exp(i\omega t)$ $\exp(-i\omega t)$ $\cos(\omega t)$ $\sin(\omega t)$

is a solution

If we think of t as time

these all represent oscillations

with "angular frequency" ω

or frequency $f = \omega / 2\pi$

hence the name

General solutions

- Since this is a second order equation
  - the general solution needs two arbitrary constants
    - Possible general solutions include
      - where A, B, C, D, F, and Φ are arbitrary constants
- Any of these solutions can be rewritten as any other one
  - they are all equivalent

$$y = A\cos(\omega t) + B\sin(\omega t)$$

$$y = C\exp(i\omega t) + D\exp(-i\omega t)$$

$$y = F\sin(\omega t + \Phi)$$

Notation for time derivatives

Since derivatives with respect to time arise in many situations

there is a shorthand notation

using dots above the variable being differentiated with respect to time

with the number of dots indicating the order of differentiation

e.g., $\dot{a} \equiv \frac{da}{dt}$ and $\ddot{a} = \frac{d^2 a}{dt^2}$

Helmholtz equation

With the same mathematics

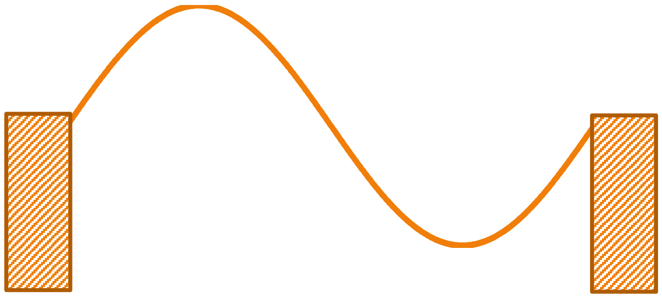
but with the second derivative in position z instead of time t

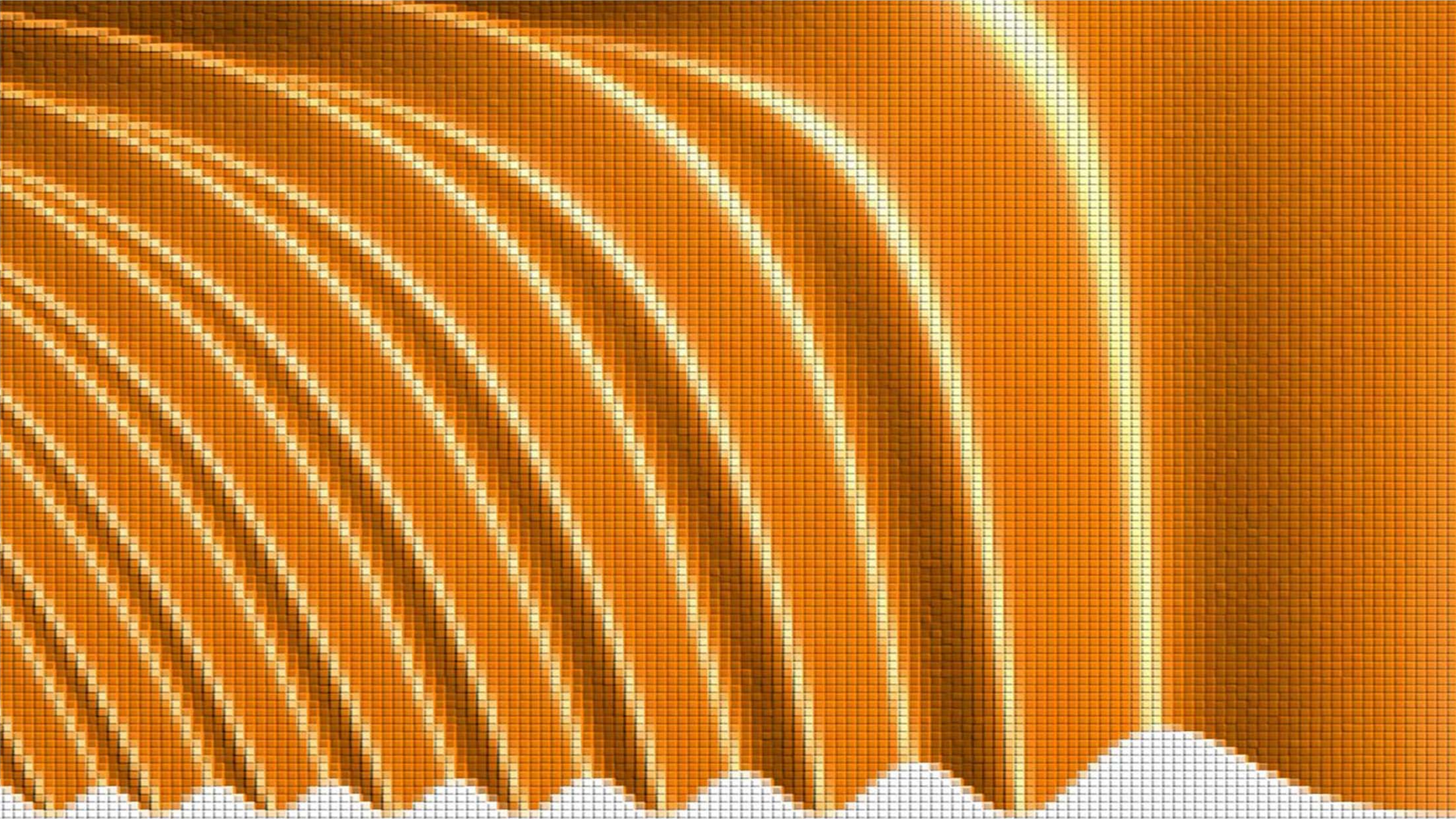
$$\frac{d^2y}{dz^2} = -k^2 z \text{ or equivalently } \frac{d^2y}{dz^2} + k^2 z = 0$$

which describes oscillations in space rather than time

an example of a "wave equation"

Example – amplitude of a "standing wave" on a string





Differential equations

Partial differential equations

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Classical one-dimensional scalar wave equation

The equation

$$\frac{\partial^2 \phi(z,t)}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2 \phi(z,t)}{\partial t^2} = 0$$

relates how strongly curved a function ϕ is (from the second spatial derivative)

to the second time (or temporal) derivative

We could write it more completely as

$$\left.\frac{\partial^2 \phi(z,t)}{\partial z^2}\right|_t - \frac{1}{c^2}\left.\frac{\partial^2 \phi(z,t)}{\partial t^2}\right|_z = 0$$

but usually we will not bother to do so

Classical one-dimensional scalar wave equation

For this equation $\dfrac{\partial^2 \phi(z,t)}{\partial z^2} - \dfrac{1}{c^2}\dfrac{\partial^2 \phi(z,t)}{\partial t^2} = 0$

we can easily check that any functions of the form $f(z-ct)$ or $g(z+ct)$ are solutions

E.g., by the chain rule, at some time t_o with $s = z - ct_o$

$$\frac{\partial f(z-ct_o)}{\partial z} = \left.\frac{df(s)}{ds}\right|_{s=z-ct_o} \times \left.\frac{\partial s}{\partial z}\right|_{t=t_o} = \left.\frac{df(s)}{ds}\right|_{s=z-ct_o}$$

$$\frac{\partial f^2(z-ct)}{\partial z^2} = \frac{\partial}{\partial z}\left[\left.\frac{df(s)}{ds}\right|_{s=z-ct_o}\right] = \left.\frac{d^2 f(s)}{ds^2}\right|_{s=z-ct_o} \times \left.\frac{\partial s}{\partial z}\right|_{t=t_o} = \left.\frac{d^2 f(s)}{ds^2}\right|_{s=z-ct_o}$$

Classical one-dimensional scalar wave equation

Similarly, at some position z_o with $s = z_o - ct$

$$\frac{\partial f(z - ct_o)}{\partial t} = \left.\frac{df(s)}{ds}\right|_{s=z_o-ct} \left.\frac{\partial s}{\partial t}\right|_{z=z_o} = -c\left.\frac{df(s)}{ds}\right|_{s=z_o-ct}$$

$$\frac{\partial f^2(z-ct)}{\partial t^2} = \frac{\partial}{\partial t}\left[-c\left.\frac{df(s)}{ds}\right|_{s=z_o-ct}\right] = \left.\frac{-c\,d^2 f(s)}{ds^2}\right|_{s=z_o-ct} \times \left.\frac{\partial s}{\partial z}\right|_{z=z_o} = \left.\frac{c^2 d^2 f(s)}{ds^2}\right|_{s=z_o-ct}$$

So, at some position z_o and time t_o

$$\frac{\partial^2 f}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2 f}{\partial t^2} = \frac{d^2 f(s)}{ds^2} - \frac{1}{c^2}\left\{\frac{c^2 d^2 f(s)}{ds^2}\right\} = 0$$

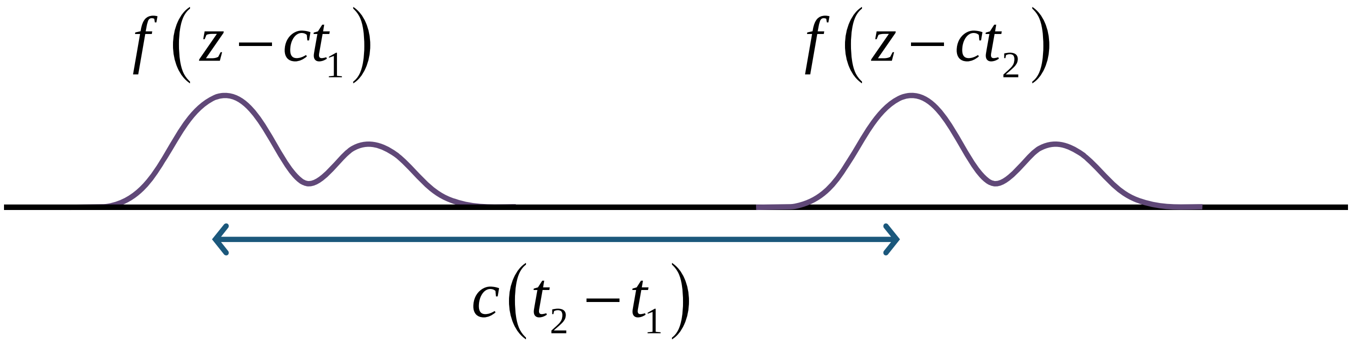
as required for any $f(z-ct)$ to be a solution

Wave equation solutions – forward waves

The difference between

the function $f(z-ct_1)$ and the function $f(z-ct_2)$

is that $f(z-ct_2)$ is shifted to the "right" by $c(t_2-t_1)$



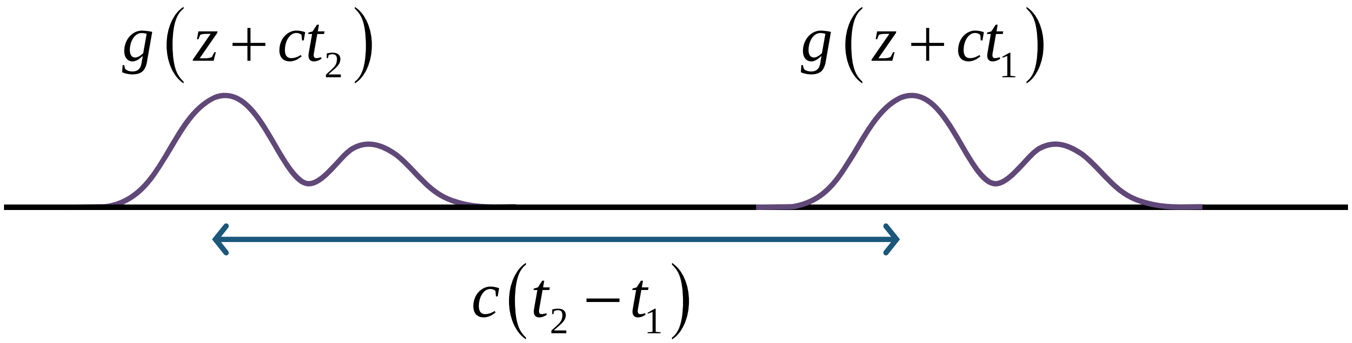
The wave moves to the right with velocity c

Wave equation solutions – backward waves

The difference between

the function $g(z+ct_1)$ and the function $g(z+ct_2)$

is that $g(z+ct_2)$ is shifted to the "left" by $c(t_2-t_1)$



The wave moves to the left with velocity c

Signs and propagation directions

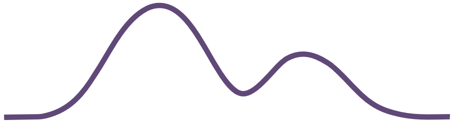
It is not the absolute sign of z or ct in $f(z-ct)$ or in $g(z+ct)$ that matters

only the relative sign of z and ct

$f(ct-z)$ is still a wave going to the "right"

though its shape is the opposite of $f(z-ct)$

$f(z-ct)$



$f(ct-z)$

Monochromatic waves

Often we are interested in waves oscillating at one specific (angular) frequency ω

i.e., temporal behavior of the form

$$T(t) = \exp(i\omega t),\ \exp(-i\omega t),\ \cos(\omega t),\ \sin(\omega t)$$

or any combination of these

Then writing $\phi(z,t) \equiv Z(z)T(t)$, we have $\dfrac{\partial^2 \phi}{\partial t^2} = -\omega^2 \phi$

leaving a wave equation for the spatial part

$$\boxed{\frac{d^2 Z(z)}{dz^2} + k^2 Z(z) = 0}$$ where $k^2 = \dfrac{\omega^2}{c^2}$

the Helmholtz wave equation

Wave equations and linearity

The wave equation $$\frac{\partial^2 \phi(z,t)}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2 \phi(z,t)}{\partial t^2} = 0$$

and therefore also the Helmholtz equation

$$\frac{d^2 Z(z)}{dz^2} + k^2 Z(z) = 0$$

are linear

If $\phi_1(z,t)$ and $\phi_2(z,t)$ are both solutions

so also is any combination

$$a\phi_1(z,t) + b\phi_2(z,t)$$

"linear superposition"

Standing waves

An equal combination of forward and backward waves, e.g.,

$$\phi(z,t) = \sin(kz - \omega t) + \sin(kz + \omega t)$$

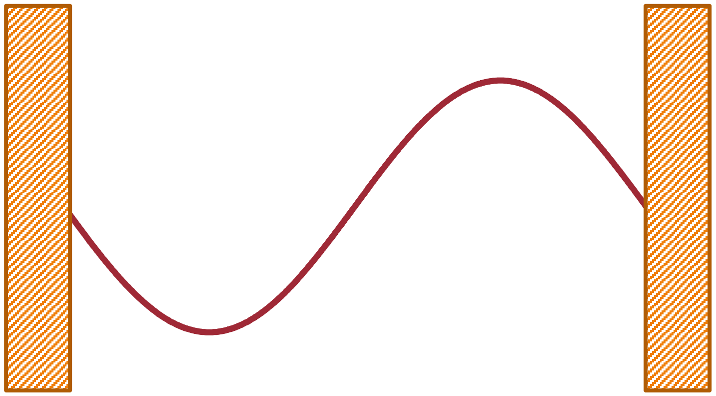
$$\equiv 2\cos(\omega t)\sin(kz)$$

where $k = \omega / c$

gives "standing waves"

E.g., for a rope tied to two walls a distance L apart

with $k = 2\pi / L$ and $\omega = 2\pi c / L$



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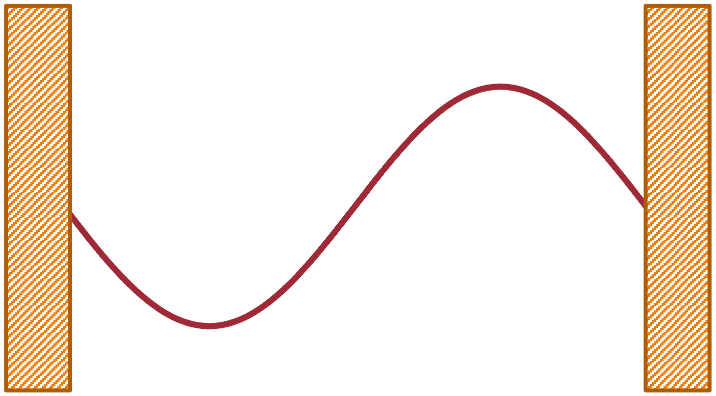

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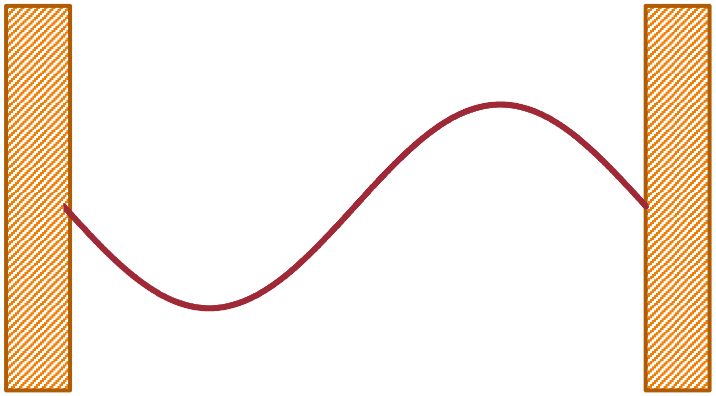

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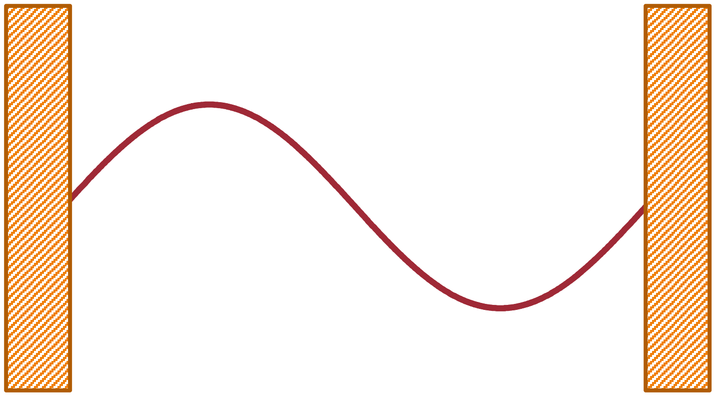

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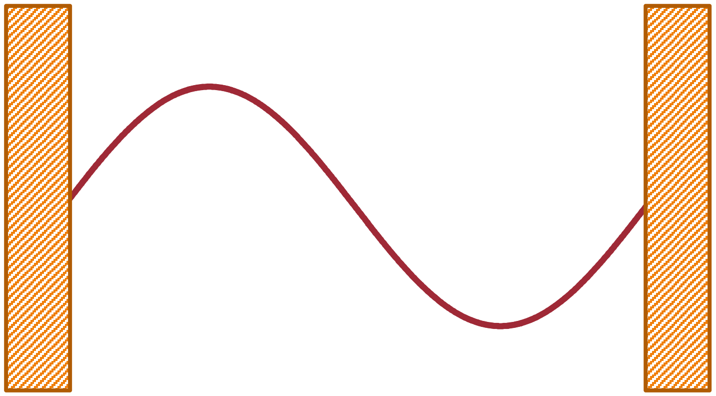

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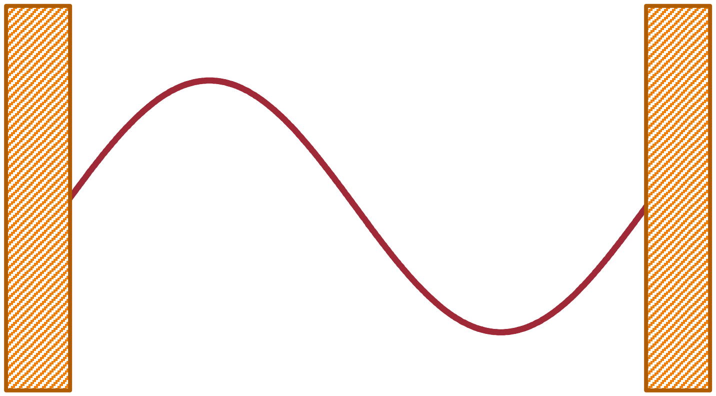

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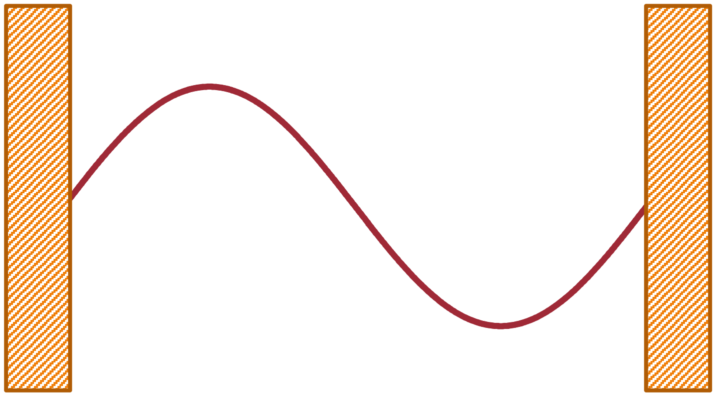

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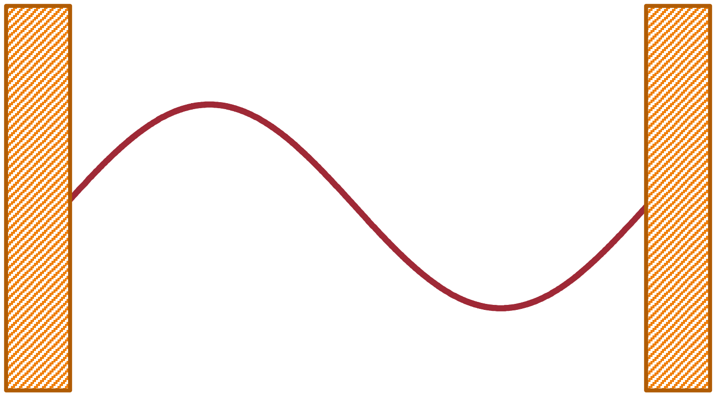

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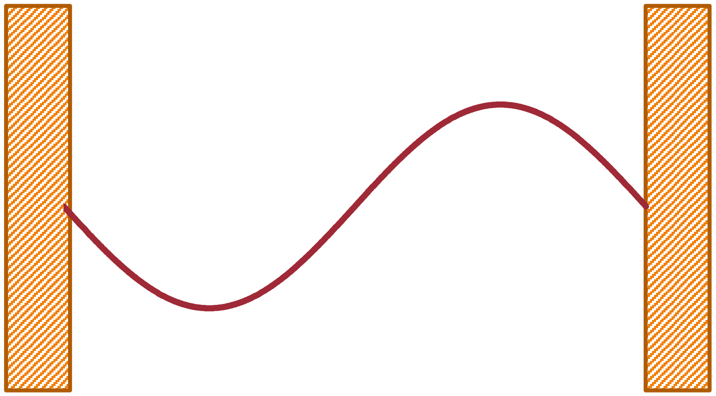

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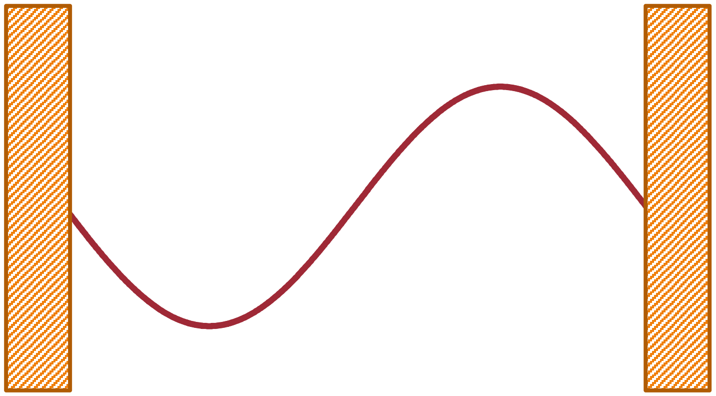

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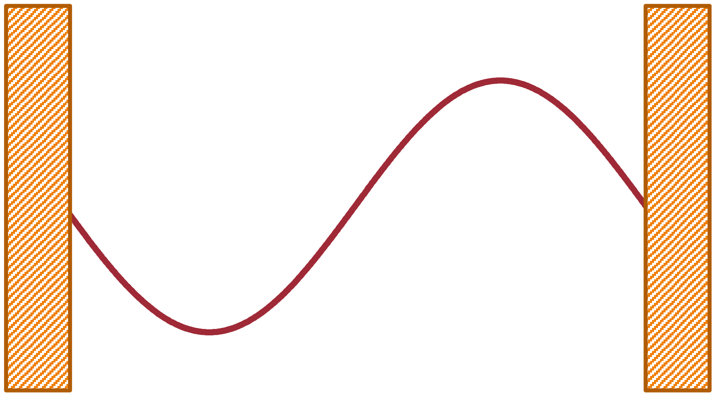

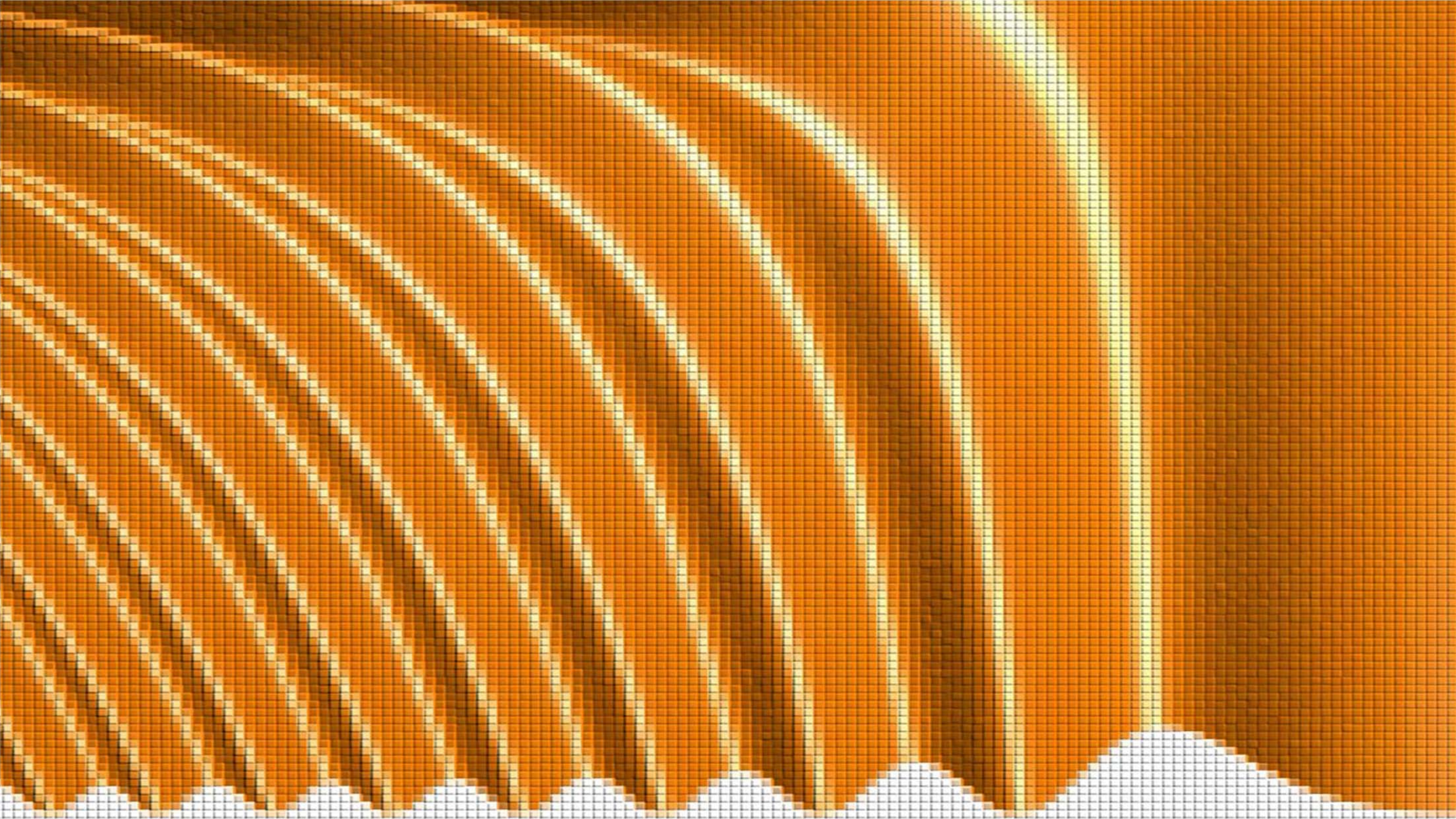

Differential equations

Laplacian and gradient operators

Background mathematics review

David Miller

Laplacian operator

The Laplacian operator can be defined

for ordinary Cartesian coordinates

$$\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

A convenient way to say it is "del squared"

Sometimes it is written Δ instead of ∇^2

though we will not use this notation

Wave equation in three dimensions

We can propose a three-dimensional wave equation

$$\frac{\partial^2\phi(x,y,z,t)}{\partial x^2}+\frac{\partial^2\phi(x,y,z,t)}{\partial y^2}+\frac{\partial^2\phi(x,y,z,t)}{\partial z^2}-\frac{1}{c^2}\frac{\partial\phi(x,y,z,t)}{\partial t^2}=0$$

which we can write more compactly as

$$\nabla^2\phi(x,y,z,t)-\frac{1}{c^2}\frac{\partial\phi(x,y,z,t)}{\partial t^2}=0$$

or just as

$$\nabla^2\phi-\frac{1}{c^2}\frac{\partial\phi}{\partial t^2}=0$$

With some simplifying assumptions

it describes many acoustic and electromagnetic waves

Gradient operator

The gradient of a scalar function $f(x,y,z)$ is

$$\nabla f \equiv \operatorname{grad} f = \mathbf{i}\frac{\partial f}{\partial x} + \mathbf{j}\frac{\partial f}{\partial y} + \mathbf{k}\frac{\partial f}{\partial z}$$

For a function $h(x,y)$ in two dimensions, such as the height of a hill

we could define a two-dimensional gradient

$$\nabla_{xy} h = \mathbf{i}\frac{\partial h}{\partial x} + \mathbf{j}\frac{\partial h}{\partial y}$$

giving the magnitude

and the vector direction

of the largest slope

Gradient notation

Note that

1) though ∇f has no **bold font** or other vector notation

   it is a vector quantity

2) we do allow ourselves to put subscripts on it for clarity on how many and what coordinates we are considering

   as in ∇_{xy} to represent a two-dimensional gradient

The symbol ∇ can be called "del" or "nabla"