Partial differentiation

Background mathematics review

David Miller

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Partial derivatives

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Partial derivative

Suppose we have a function $f(x,y)$ of two variables

Then we can have

"the gradient of f as a function of x with y held constant"

which we write $\left.\frac{\partial f}{\partial x}\right|_y$ or sometimes just $\frac{\partial f}{\partial x}$

"the partial derivative of f with respect to x with y constant"

"partial d f by d x"

∂ - "partial d"

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Partial derivative

We have a surface

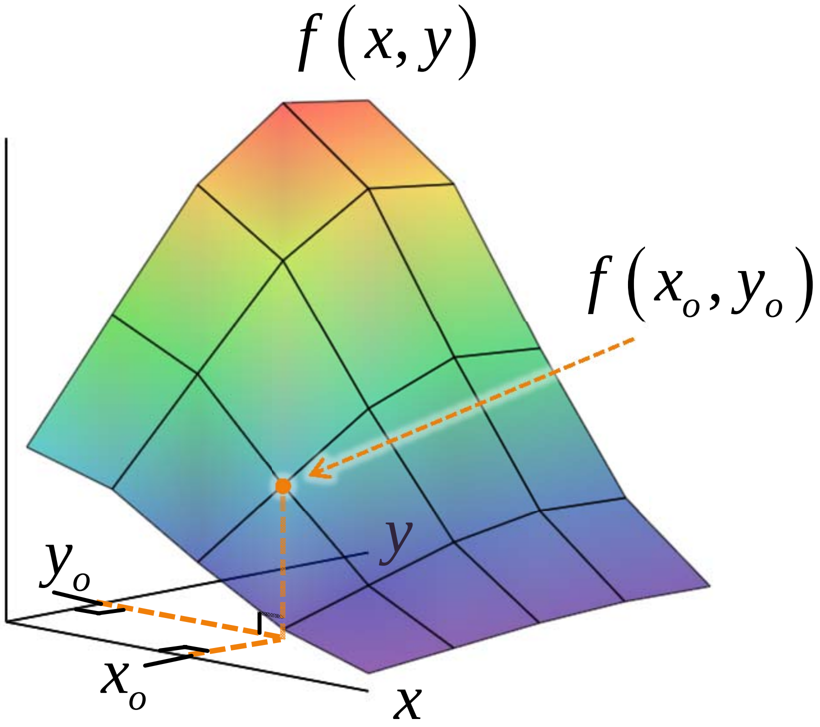
with height

$$f(x, y)$$

varying with the coordinates x and y

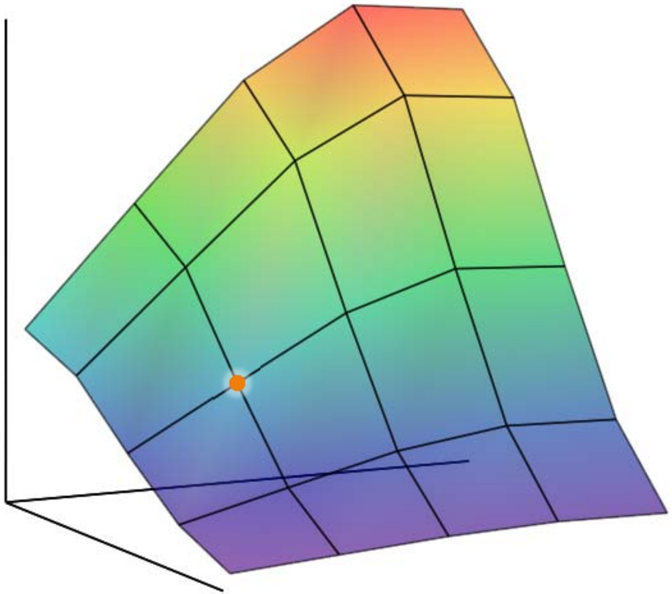
At a specific position (x_o, y_o)

the height is $f(x_o, y_o)$



Partial derivative

$$f(x, y)$$



Partial derivative

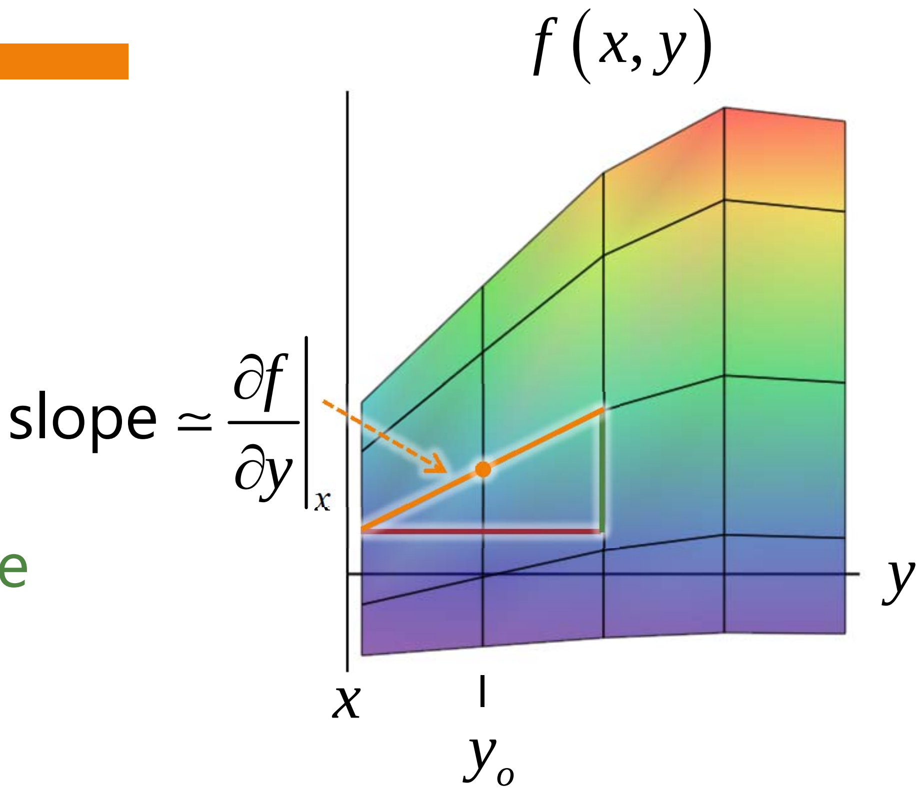
Along the y direction

at point (x_o, y_o)

the rate of change of height with y

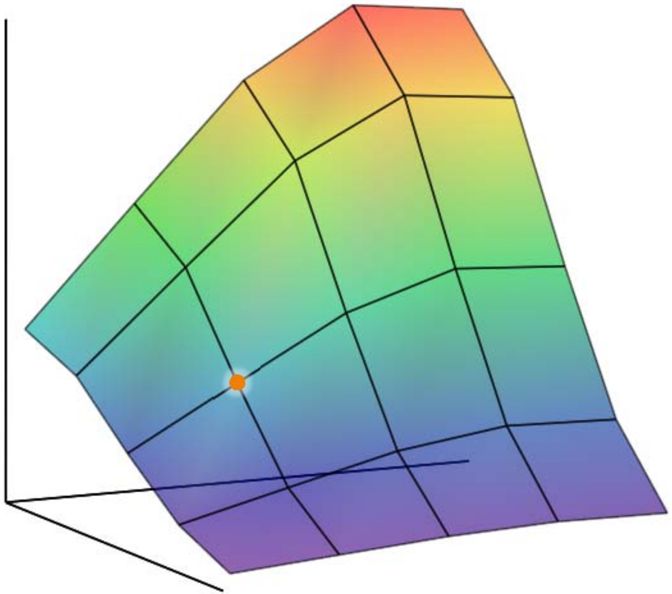
at constant x

is the slope of the orange line



Partial derivative

$f(x, y)$



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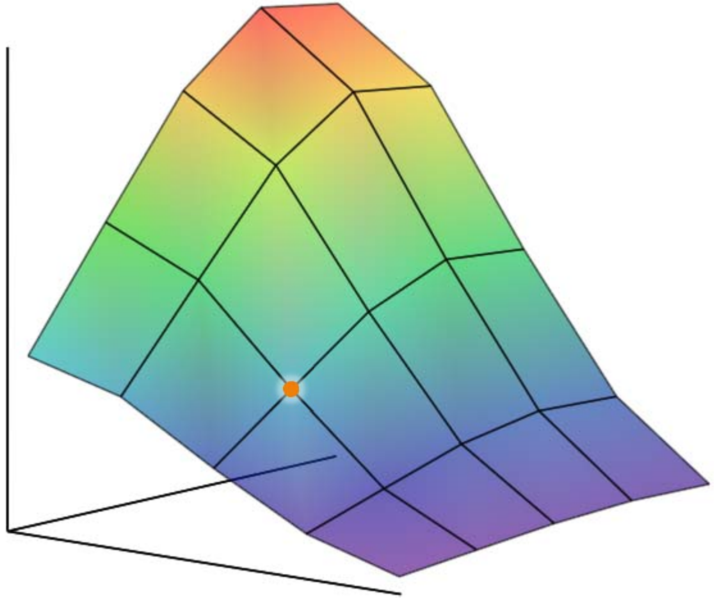


Partial derivative

Along the x direction

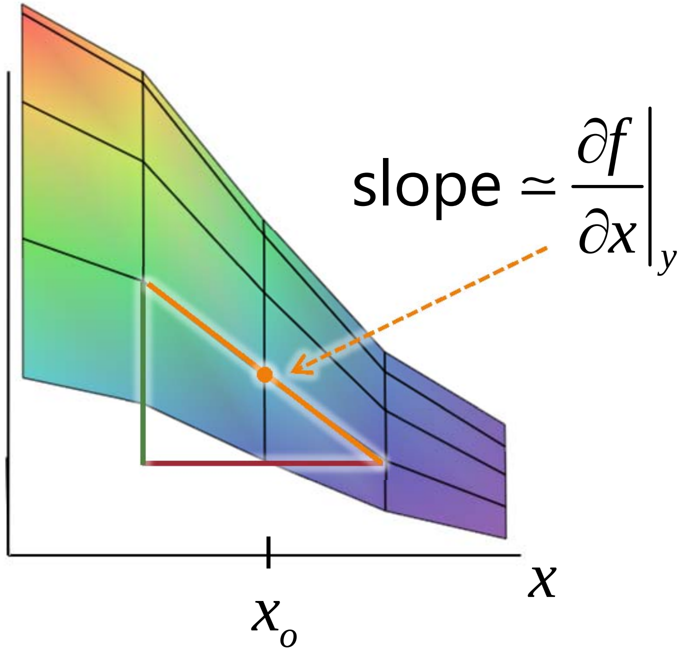
at point (x_o, y_o)

the rate of change of height with x

at constant y

is the slope of the orange line

$f(x,y)$



Second partial derivatives

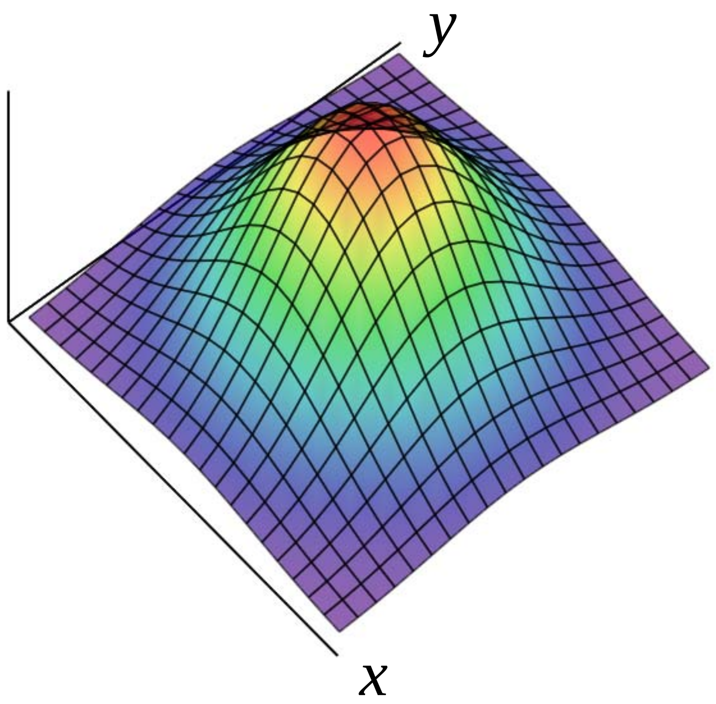
We can also form second (and higher order) partial derivatives

$$\frac{\partial^2 f}{\partial x^2} \equiv \left.\frac{\partial^2 f}{\partial x^2}\right|_y$$

is a second derivative

and is a measure of the curvature of the function in the x direction

$f(x, y)$



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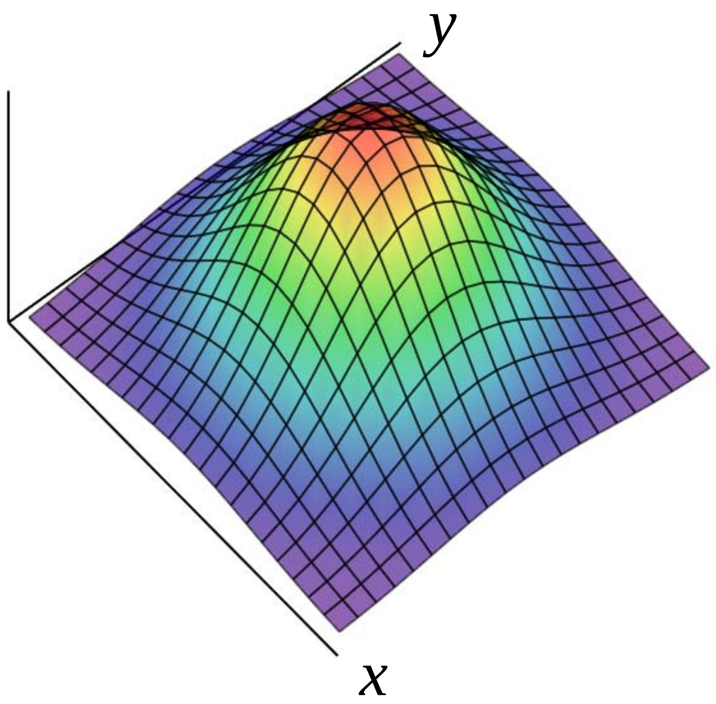

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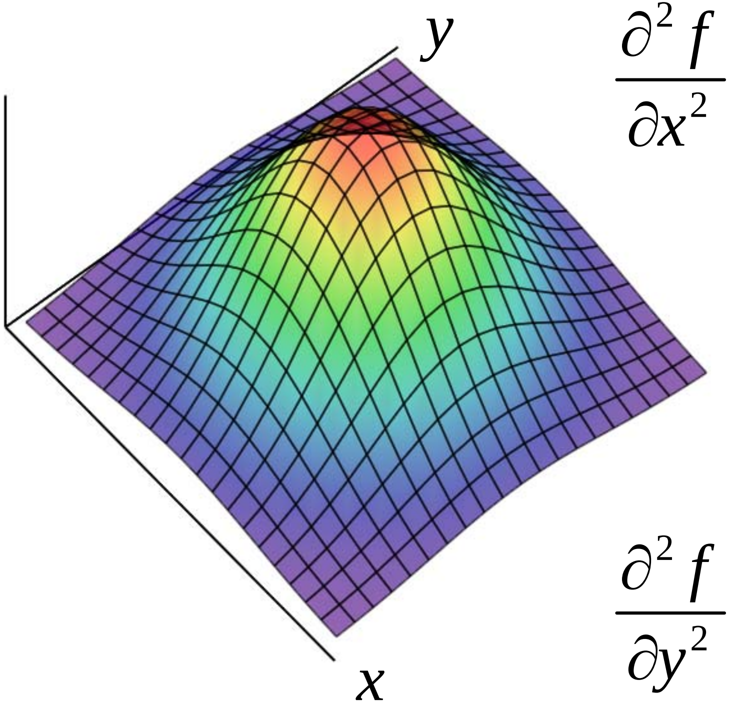

$$\frac{\partial^2 f}{\partial x^2} < 0$$

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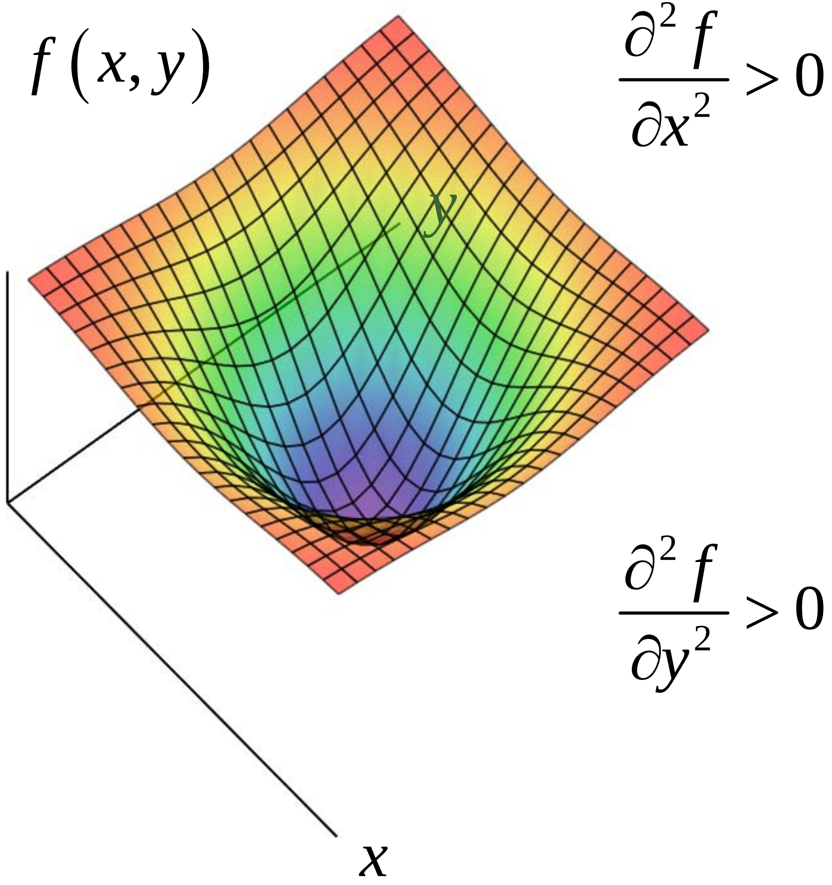

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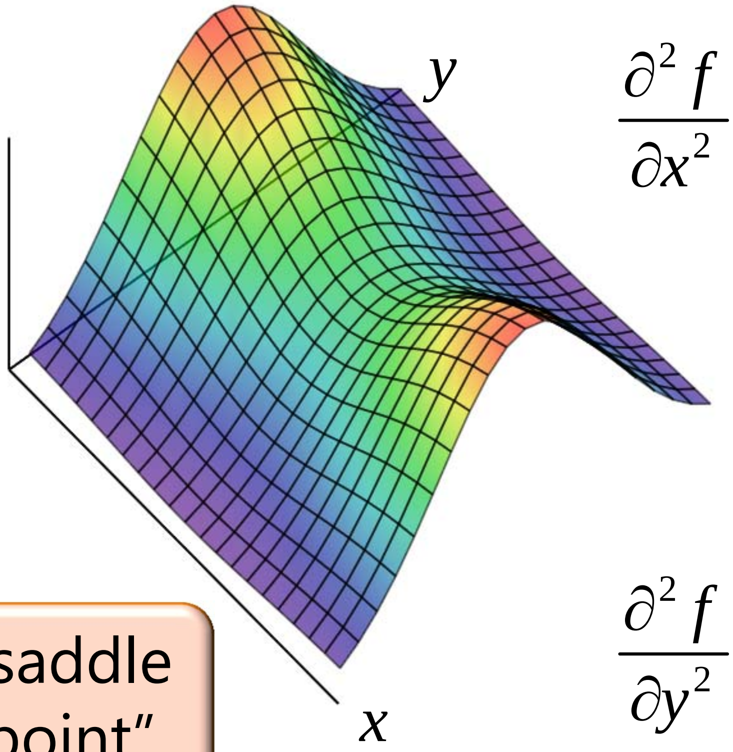


$$\frac{\partial^2 f}{\partial x^2} > 0$$

$$\frac{\partial^2 f}{\partial y^2} < 0$$

"saddle point"

Cross derivative

The “cross derivative”

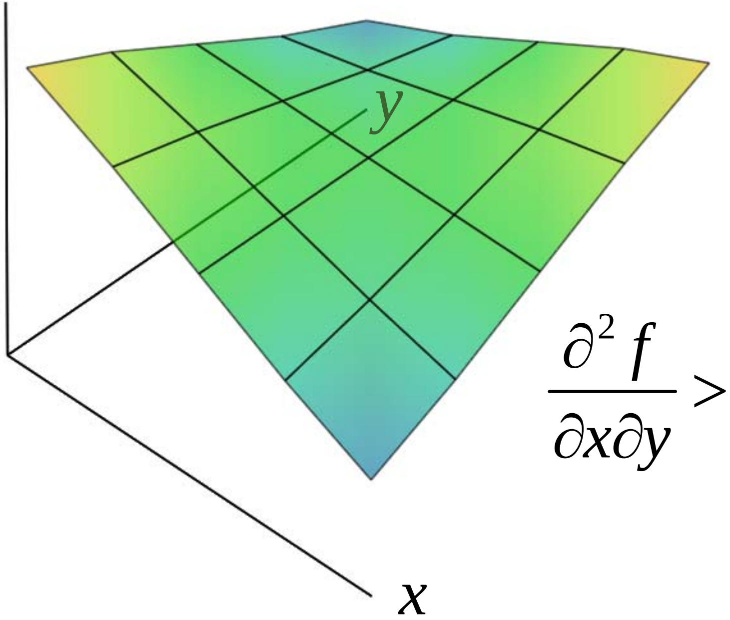
$$\frac{\partial^2 f}{\partial x \partial y} \equiv \left.\frac{\partial}{\partial x}\right|_y \left.\frac{\partial f}{\partial y}\right|_x$$

is a measure of how “warped” a surface is

Note that, for ordinary smooth functions

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$$

$f(x,y)$



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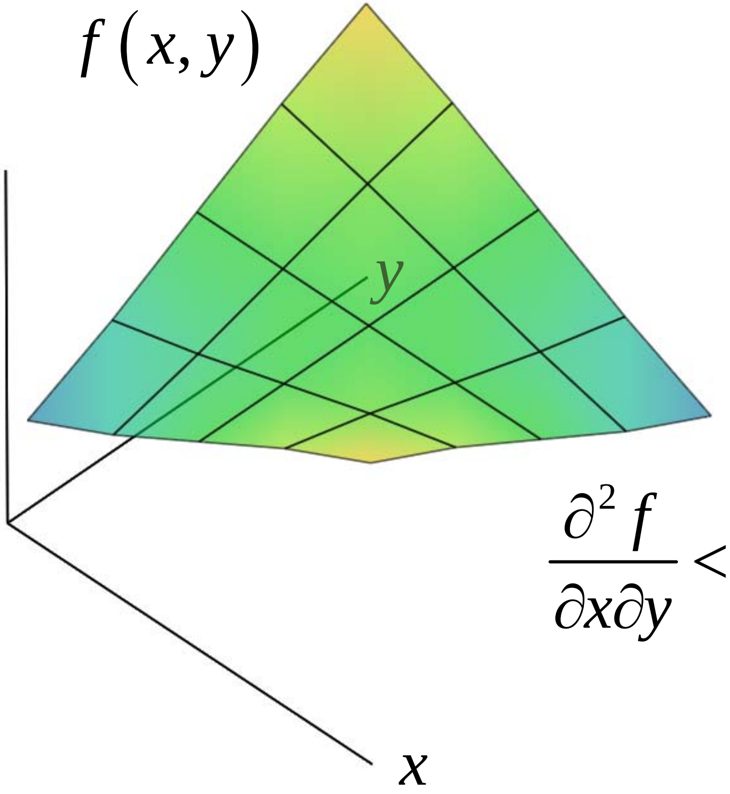
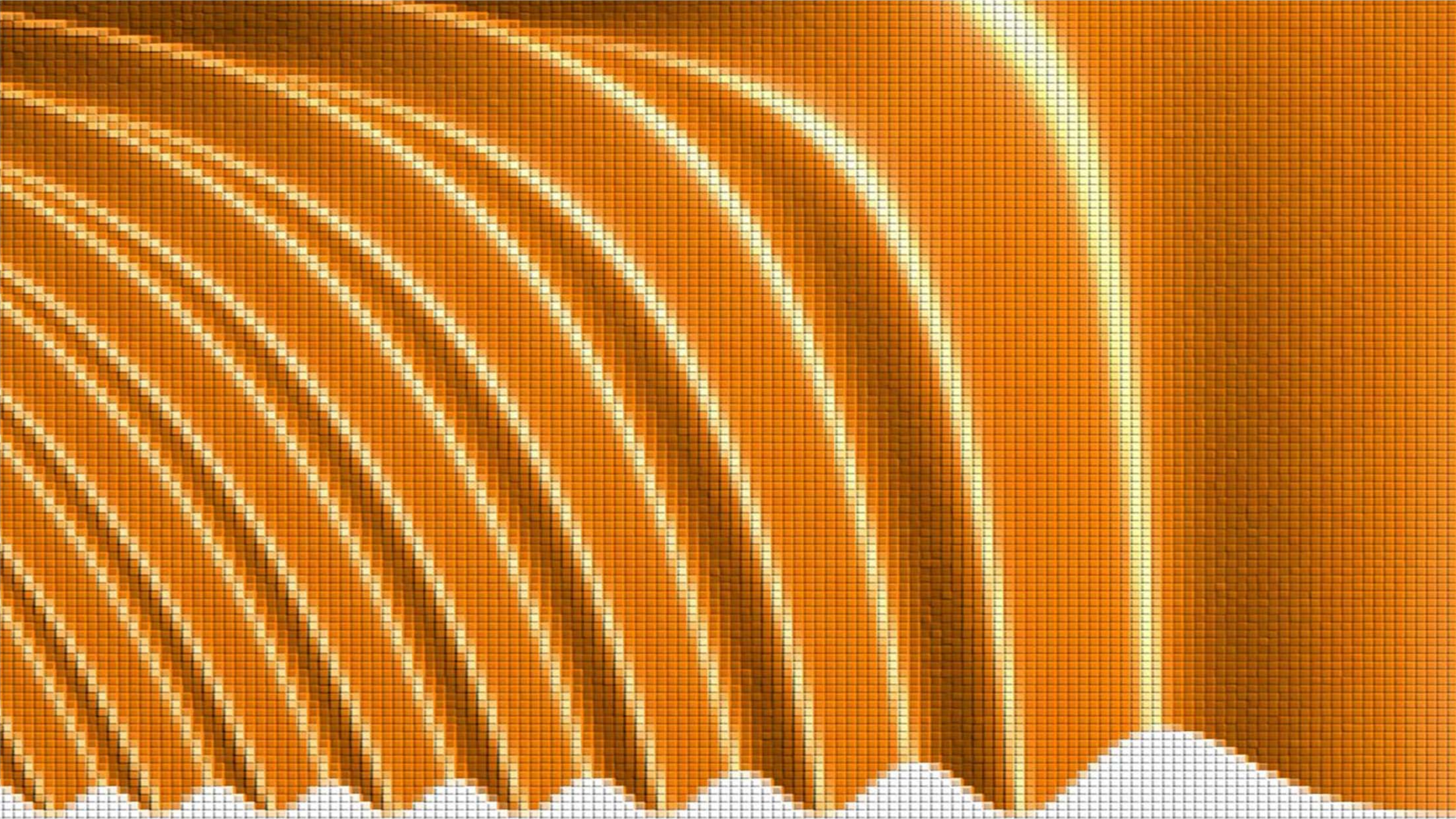


$$\frac{\partial^2 f}{\partial x \partial y} < 0$$



Partial differentiation

Differentials

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Differential

Look at one "patch" of a function $f(x, y)$

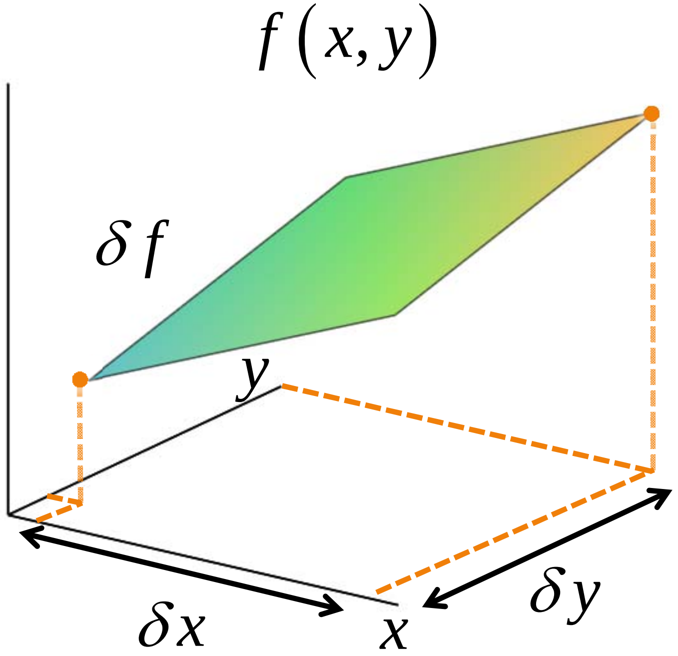
- δx long in the x direction
- δy long in the y direction

How much does the "height" change, i.e., δf,

as we move from

one corner

to the other?



Differential

As we move by δx along x

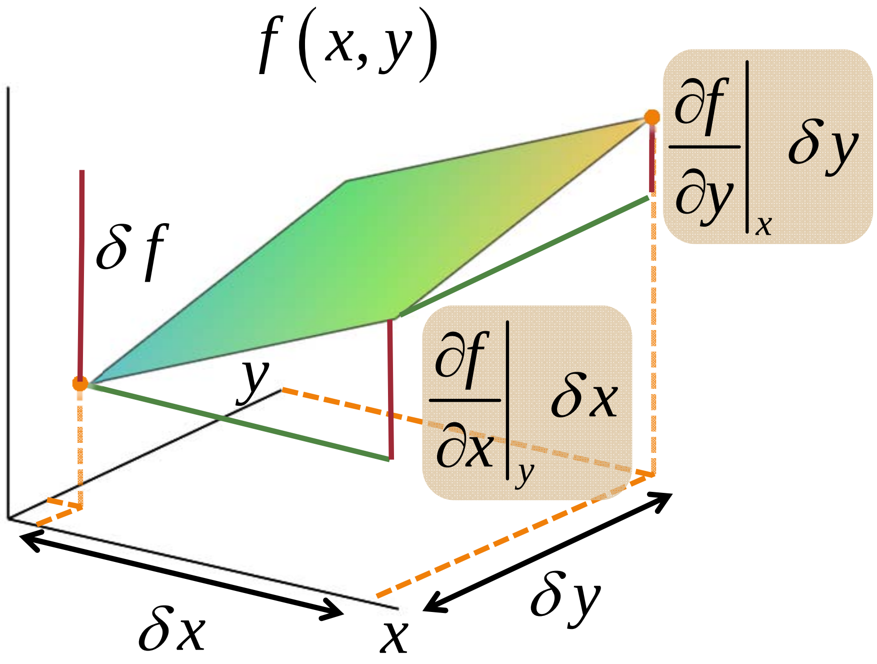
the "height" changes $\left.\frac{\partial f}{\partial x}\right|_y \delta x$

As we move by δy along y

the "height" changes $\left.\frac{\partial f}{\partial y}\right|_x \delta y$

so the total change in "height"

is $\delta f \simeq \left.\frac{\partial f}{\partial x}\right|_y \delta x + \left.\frac{\partial f}{\partial y}\right|_x \delta y$



Differential

In the limit as we make δx and δy very small

i.e., infinitesimal

$$\delta f \simeq \left.\frac{\partial f}{\partial x}\right|_y \delta x + \left.\frac{\partial f}{\partial y}\right|_x \delta y$$

becomes

$$df = \left.\frac{\partial f}{\partial x}\right|_y dx + \left.\frac{\partial f}{\partial y}\right|_x dy$$

which is called

a differential

or sometimes an "exact differential"

Total derivative

We suppose we know the slopes of the hill in the two coordinate directions

$$\left.\frac{\partial f}{\partial x}\right|_y \quad \text{and} \quad \left.\frac{\partial f}{\partial y}\right|_x$$

We presume we also know how fast we are moving in the x and y directions

$$v_x = \frac{dx}{dt} \quad \text{and} \quad v_y = \frac{dy}{dt}$$

So, in some small time δt

we will have moved amounts δx and δy in the x and y directions

$$\delta x \simeq \frac{dx}{dt}\delta t \quad \text{and} \quad \delta y \simeq \frac{dy}{dt}\delta t$$

Total derivative

So, using the differential idea

$$\delta f \simeq \left.\frac{\partial f}{\partial x}\right|_y \delta x + \left.\frac{\partial f}{\partial y}\right|_x \delta y$$

we have

$$\delta f \simeq \left.\frac{\partial f}{\partial x}\right|_y \left(\frac{dx}{dt}\right)\delta t + \left.\frac{\partial f}{\partial y}\right|_x \left(\frac{dy}{dt}\right)\delta t$$

or

$$\frac{\delta f}{\delta t} \simeq \left.\frac{\partial f}{\partial x}\right|_y \left(\frac{dx}{dt}\right) + \left.\frac{\partial f}{\partial y}\right|_x \left(\frac{dy}{dt}\right)$$

or, in the limit of small δt

we have the "total derivative"

$$\frac{df}{dt} = \left.\frac{\partial f}{\partial x}\right|_y \left(\frac{dx}{dt}\right) + \left.\frac{\partial f}{\partial y}\right|_x \left(\frac{dy}{dt}\right)$$

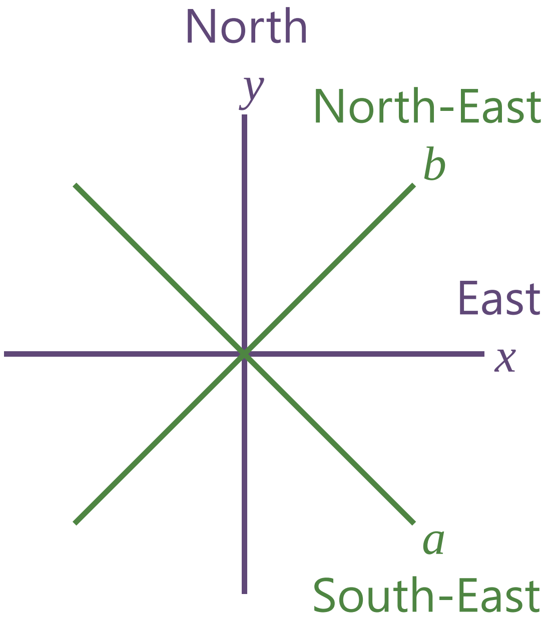
Changing coordinates for partial derivatives

Suppose we want the slopes of a hill $f(x,y)$ along the South-East (a) and North-East (b) directions

instead of along the East (x) and North (y) directions

but we only know the slopes along the East (x) and North (y) directions

$$\left.\frac{\partial f}{\partial x}\right|_y \quad \text{and} \quad \left.\frac{\partial f}{\partial y}\right|_x$$



Changing coordinates for partial derivatives

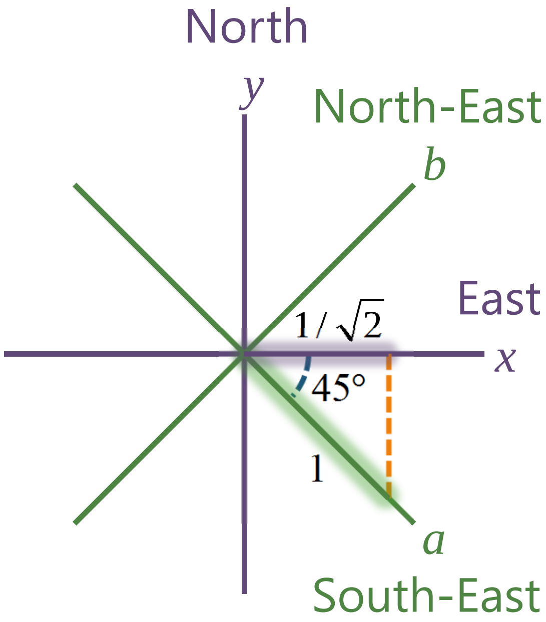
We do know that

if we move South-East by one unit

we move East by $1/\sqrt{2}$ units

since $\cos 45° = 1/\sqrt{2}$

i.e., $\left.\frac{\partial x}{\partial a}\right|_b = \frac{1}{\sqrt{2}}$



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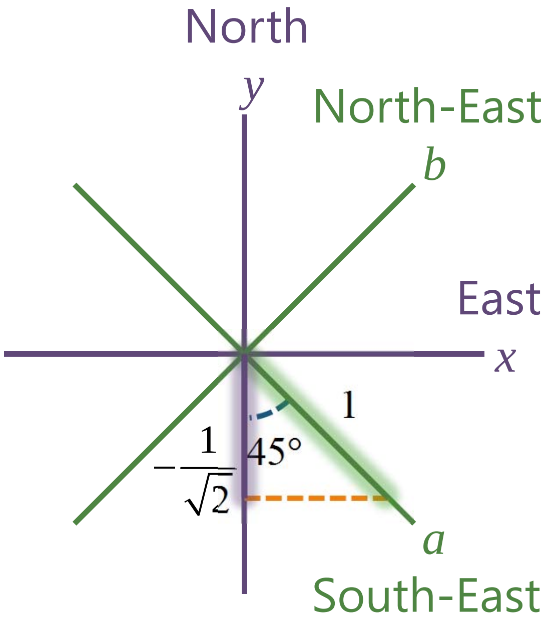

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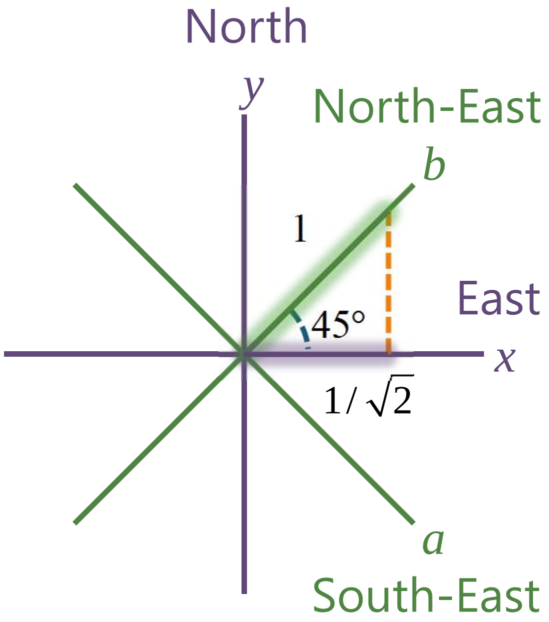
$$\left.\frac{\partial y}{\partial a}\right|_b = -\frac{1}{\sqrt{2}} \qquad \left.\frac{\partial x}{\partial b}\right|_a = \frac{1}{\sqrt{2}}$$



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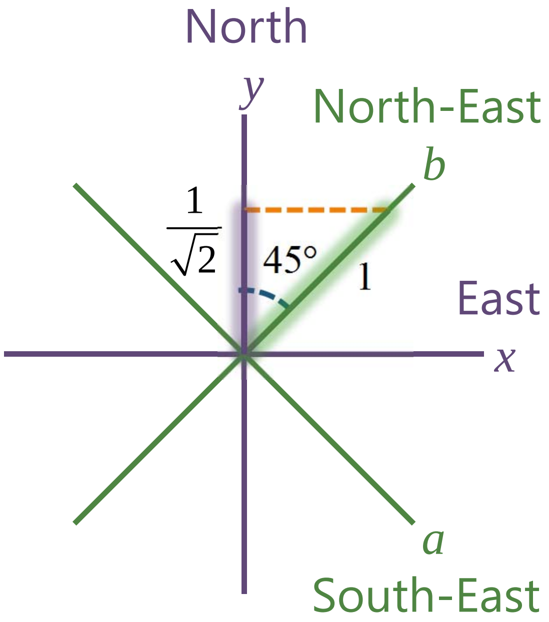

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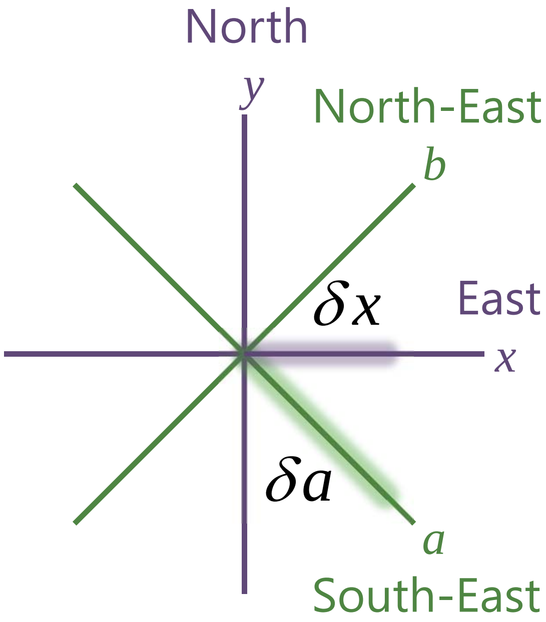
Changing coordinates for partial derivatives

Suppose we make a small movement δa along the a (South-East) direction and no movement along the b (North-East) direction

Then $\delta x \simeq \left.\dfrac{\partial x}{\partial a}\right|_b \delta a \quad \left(= \dfrac{1}{\sqrt{2}} \delta a \right)$

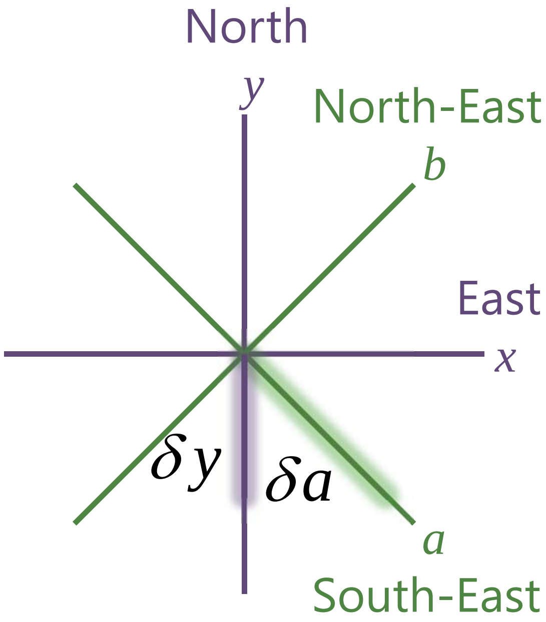


Changing coordinates for partial derivatives

Suppose we make a small movement δa along the a (South-East) direction and no movement along the b (North-East) direction

Then $\delta x \simeq \left.\dfrac{\partial x}{\partial a}\right|_b \delta a \quad \left(= \dfrac{1}{\sqrt{2}} \delta a \right)$

and $\delta y \simeq \left.\dfrac{\partial y}{\partial a}\right|_b \delta a \quad \left(= -\dfrac{1}{\sqrt{2}} \delta a \right)$



Changing coordinates for partial derivatives

With these results

$$\delta x \simeq \left.\frac{\partial x}{\partial a}\right|_b \delta a \;\left(=\frac{1}{\sqrt{2}}\delta a\right) \qquad \delta y \simeq \left.\frac{\partial y}{\partial a}\right|_b \delta a \;\left(=-\frac{1}{\sqrt{2}}\delta a\right)$$

the resulting change δf in the value of the function $f(x,y)$

from this movement along the a (South-East) direction

is $$\delta f \simeq \left.\frac{\partial f}{\partial x}\right|_y \left.\frac{\partial x}{\partial a}\right|_b \delta a + \left.\frac{\partial f}{\partial y}\right|_x \left.\frac{\partial y}{\partial a}\right|_b \delta a$$

Changing coordinates for partial derivatives

Starting from

$$\delta f \simeq \left.\frac{\partial f}{\partial x}\right|_y \left.\frac{\partial x}{\partial a}\right|_b \delta a + \left.\frac{\partial f}{\partial y}\right|_x \left.\frac{\partial y}{\partial a}\right|_b \delta a$$

dividing by δa gives

$$\frac{\delta f}{\delta a} \simeq \left.\frac{\partial f}{\partial x}\right|_y \left.\frac{\partial x}{\partial a}\right|_b + \left.\frac{\partial f}{\partial y}\right|_x \left.\frac{\partial y}{\partial a}\right|_b$$

taking the limit of small δa and noting that this is all done at constant b

$$\left.\frac{\partial f}{\partial a}\right|_b = \left.\frac{\partial f}{\partial x}\right|_y \left.\frac{\partial x}{\partial a}\right|_b + \left.\frac{\partial f}{\partial y}\right|_x \left.\frac{\partial y}{\partial a}\right|_b$$

Since $\left.\frac{\partial x}{\partial a}\right|_b$ and $\left.\frac{\partial y}{\partial a}\right|_b$ are just numbers

we can move them to get

$$\left.\frac{\partial f}{\partial a}\right|_b = \left.\frac{\partial x}{\partial a}\right|_b \left.\frac{\partial f}{\partial x}\right|_y + \left.\frac{\partial y}{\partial a}\right|_b \left.\frac{\partial f}{\partial y}\right|_x$$

Changing coordinates for partial derivatives

Since

$$\left.\frac{\partial f}{\partial a}\right|_b = \left.\frac{\partial x}{\partial a}\right|_b \left.\frac{\partial f}{\partial x}\right|_y + \left.\frac{\partial y}{\partial a}\right|_b \left.\frac{\partial f}{\partial y}\right|_x$$

holds for any function $f(x,y)$

provided it is suitably differentiable

we can write more generally

$$\left.\frac{\partial}{\partial a}\right|_b = \left.\frac{\partial x}{\partial a}\right|_b \left.\frac{\partial}{\partial x}\right|_y + \left.\frac{\partial y}{\partial a}\right|_b \left.\frac{\partial}{\partial y}\right|_x$$

which is a general way of changing the coordinates for a partial derivative