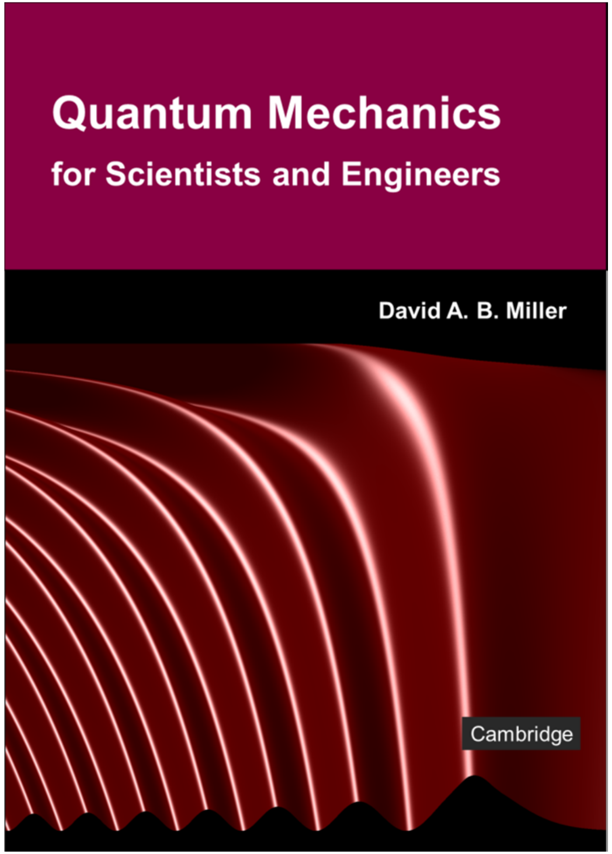
6.3 Operators and quantum mechanics

Slides: Video 6.3.3 General form of the uncertainty principle

Text reference: Quantum Mechanics for Scientists and Engineers

Section 5.2 (up to "Position-momentum uncertainty principle")



Operators and quantum mechanics

General form of the uncertainty principle

Quantum mechanics for scientists and engineers

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General form of the uncertainty principle

First, we need to set up the concepts of the mean and variance of an expectation value

Using $\bar{A}$ to denote the mean value of a quantity A

we have, in the bra-ket notation

for a measurable quantity associated with the Hermitian operator $\hat{A}$

when the state of the system is $|f\rangle$

$$\bar{A} \equiv \langle A \rangle = \langle f | \hat{A} | f \rangle$$

General form of the uncertainty principle

Let us define a new operator $\Delta\hat{A}$

associated with the difference between the measured value of A and its average value

$$\Delta\hat{A} = \hat{A} - \bar{A}$$

Strictly, we should write $\Delta\hat{A} = \hat{A} - \bar{A}\hat{I}$

but we take such an identity operator to be understood

Note that this operator is also Hermitian

General form of the uncertainty principle

Variance in statistics is the "mean square" deviation from the average

To examine the variance of the quantity A

we examine the expectation value of the operator $(\Delta\hat{A})^2$

Expanding the arbitrary function $|f\rangle$

on the basis of the eigenfunctions $|\psi_i\rangle$ of $\hat{A}$

i.e., $|f\rangle = \sum_i c_i |\psi_i\rangle$

we can formally evaluate the expectation value of $(\Delta\hat{A})^2$ when the system is in state $|f\rangle$

General form of the uncertainty principle

We have

$$\left\langle (\Delta\hat{A})^2 \right\rangle = \langle f \,|(\Delta\hat{A})^2|\, f \rangle = \left(\sum_i c_i^* \langle \psi_i | \right) \left(\hat{A} - \bar{A} \right)^2 \left(\sum_j c_j |\psi_j\rangle \right)$$

$$= \left(\sum_i c_i^* \langle \psi_i | \right) \left(\hat{A} - \bar{A} \right) \left(\sum_j c_j \left(A_j - \bar{A} \right) |\psi_j\rangle \right)$$

$$= \left(\sum_i c_i^* \langle \psi_i | \right) \left(\sum_j c_j \left(A_j - \bar{A} \right)^2 |\psi_j\rangle \right) = \sum_i |c_i|^2 \left(A_i - \bar{A} \right)^2$$

General form of the uncertainty principle

Because the $|c_i|^2$ are the probabilities that the system is found, on measurement, to be in the state $|\psi_i\rangle$

and $(A_i - \bar{A})^2$ for that state simply represents

the squared deviation of the value of the quantity A

from its average value

then by definition

$$\overline{(\Delta A)^2} \equiv \left\langle \left(\Delta\hat{A}\right)^2 \right\rangle = \left\langle \left(\hat{A} - \bar{A}\right)^2 \right\rangle = \left\langle f \left| \left(\hat{A} - \bar{A}\right)^2 \right| f \right\rangle$$

is the mean squared deviation for the quantity A

on repeatedly measuring the system prepared in state $|f\rangle$

General form of the uncertainty principle

In statistical language

the quantity $\overline{(\Delta A)^2}$ is called the variance

and the square root of the variance

which we can write as $\Delta A \equiv \sqrt{\overline{(\Delta A)^2}}$

is the standard deviation

In statistics

the standard deviation gives a

well-defined measure of

the width of a distribution

General form of the uncertainty principle

We can also consider some other quantity B

associated with the Hermitian operator $\hat{B}$

$$\overline{B} = \langle B \rangle = \langle f | \hat{B} | f \rangle$$

and, with similar definitions

$$\overline{(\Delta B)^2} \equiv \left\langle \left(\Delta \hat{B}\right)^2 \right\rangle = \left\langle \left(\hat{B} - \overline{B}\right)^2 \right\rangle = \left\langle f \left| \left(\hat{B} - \overline{B}\right)^2 \right| f \right\rangle$$

So we have ways of calculating the uncertainty in the measurements of the quantities A and B

when the system is in a state $| f \rangle$

to use in a general proof of the uncertainty principle

General form of the uncertainty principle

Suppose two Hermitian operators $\hat{A}$ and $\hat{B}$ do not commute

and have a commutation rest $\hat{C}$

as defined above in $\left[\hat{A},\hat{B}\right]=i\hat{C}$

Consider, for some arbitrary real number α, the number

$$G(\alpha)=\left\langle\left(\alpha\Delta\hat{A}-i\Delta\hat{B}\right)f\middle|\left(\alpha\Delta\hat{A}-i\Delta\hat{B}\right)f\right\rangle\geq 0$$

By $\left|\left(\alpha\Delta\hat{A}-i\Delta\hat{B}\right)f\right\rangle$ we mean the vector $\left(\alpha\Delta\hat{A}-i\Delta\hat{B}\right)|f\rangle$

written this way to emphasize it is simply a vector

so it must have an inner product with itself

that is greater than or equal to zero

General form of the uncertainty principle

So
$$G(\alpha)=\left\langle f\left|\left(\alpha\Delta\hat{A}-i\Delta\hat{B}\right)^{\dagger}\left(\alpha\Delta\hat{A}-i\Delta\hat{B}\right)\right|f\right\rangle\ (\geq 0)$$

$$=\left\langle f\left|\left(\alpha\Delta\hat{A}^{\dagger}+i\Delta\hat{B}^{\dagger}\right)\left(\alpha\Delta\hat{A}-i\Delta\hat{B}\right)\right|f\right\rangle$$

By Hermiticity
$$=\left\langle f\left|\left(\alpha\Delta\hat{A}+i\Delta\hat{B}\right)\left(\alpha\Delta\hat{A}-i\Delta\hat{B}\right)\right|f\right\rangle$$

$$=\left\langle f\left|\alpha^{2}\left(\Delta\hat{A}\right)^{2}+\left(\Delta\hat{B}\right)^{2}-i\alpha\left(\Delta\hat{A}\Delta\hat{B}-\Delta\hat{B}\Delta\hat{A}\right)\right|f\right\rangle$$

$$=\left\langle f\left|\alpha^{2}\left(\Delta\hat{A}\right)^{2}+\left(\Delta\hat{B}\right)^{2}-i\alpha\left[\Delta\hat{A},\Delta\hat{B}\right]\right|f\right\rangle$$

$$=\left\langle f\left|\alpha^{2}\left(\Delta\hat{A}\right)^{2}+\left(\Delta\hat{B}\right)^{2}+\alpha\hat{C}\right|f\right\rangle$$

General form of the uncertainty principle

So

$$G(\alpha) = \langle f | \alpha^2 (\Delta\hat{A})^2 + (\Delta\hat{B})^2 + \alpha\hat{C} | f \rangle \quad (\geq 0)$$

$$= \alpha^2 \overline{(\Delta A)^2} + \overline{(\Delta B)^2} + \alpha\bar{C}$$

where $\bar{C} \equiv \langle C \rangle = \langle f | \hat{C} | f \rangle$

By a simple (though not obvious) rearrangement

$$G(\alpha) = \overline{(\Delta A)^2} \left[\alpha + \frac{\bar{C}}{2\overline{(\Delta A)^2}} \right]^2 + \overline{(\Delta B)^2} - \frac{(\bar{C})^2}{4\overline{(\Delta A)^2}} \geq 0$$

General form of the uncertainty principle

But

$$G(\alpha) = \overline{(\Delta A)^2}\left[\alpha + \frac{\bar{C}}{2\overline{(\Delta A)^2}}\right]^2 + \overline{(\Delta B)^2} - \frac{\left(\bar{C}\right)^2}{4\overline{(\Delta A)^2}} \geq 0$$

must be true for arbitrary α

so it is true for $\alpha = -\dfrac{\bar{C}}{2\overline{(\Delta A)^2}}$

which sets the first term equal to zero

so $\overline{(\Delta A)^2}\,\overline{(\Delta B)^2} \geq \dfrac{\left(\bar{C}\right)^2}{4}$ or $\Delta A \Delta B \geq \dfrac{\left|\bar{C}\right|}{2}$

General form of the uncertainty principle

So, for two operators $\hat{A}$ and $\hat{B}$

corresponding to measurable quantities A and B

for which $\left[\hat{A}, \hat{B}\right] = i\hat{C}$

in some state $|f\rangle$ for which $\bar{C} \equiv \langle C \rangle = \langle f | \hat{C} | f \rangle$

we have the uncertainty principle

$$\boxed{\Delta A \Delta B \geq \frac{\left|\bar{C}\right|}{2}}$$

where ΔA an ΔB are the standard deviations of the values of A and B we would measure

General form of the uncertainty principle

Only if the operators $\hat{A}$ and $\hat{B}$ commute

i.e., $\left[\hat{A},\hat{B}\right]=0$ (or, strictly, $\left[\hat{A},\hat{B}\right]=0\hat{I}$)

or if they do not commute, i.e., $\left[\hat{A},\hat{B}\right]=i\hat{C}$

but we are in a state $\left|f\right\rangle$ for which $\left\langle f\left|\hat{C}\right|f\right\rangle=0$

is it possible for both A and B simultaneously to have exact measurable values