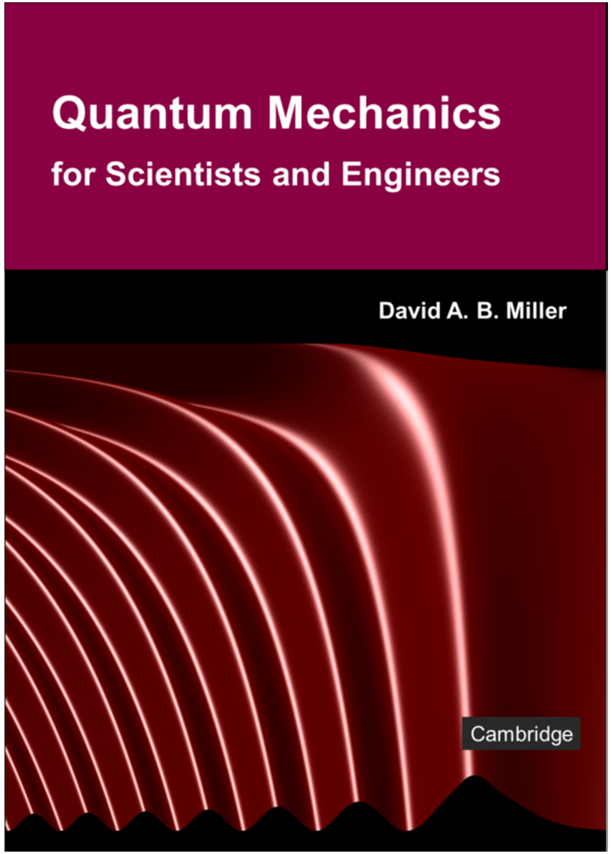
3.3 The time-dependent Schrödinger equation

Slides: Video 3.3.2 Rationalizing the time-dependent Schrödinger equation

Text reference: Quantum Mechanics for Scientists and Engineers

Sections 3.1 – 3.2



The time-dependent Schrödinger equation

Rationalizing the time-dependent Schrödinger equation

Quantum mechanics for scientists and engineers

David Miller

Relation between energy and frequency

The relation between

energy

and

frequency

for

photons

is $\boxed{E = h\nu = \hbar\omega}$

Relation between energy and frequency

The relation between

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quantum mechanics

is

$$\boxed{E = h\nu = \hbar\omega}$$

Rationalizing the time-dependent equation

We want a time-dependent wave equation

- for a particle with mass m
  - with this relation $E = h\nu = \hbar\omega$ between energy and frequency

We might also reasonably want it to have plane wave solutions

- e.g., of the form $\exp\left[i\left(kz - \omega t\right)\right]$
  - when we have some specific energy E
    - and when we are in a uniform potential

Rationalizing the time-dependent equation

Schrödinger postulated the time-dependent equation

$$-\frac{\hbar^2}{2m}\nabla^2\Psi(\mathbf{r},t)+V(\mathbf{r},t)\Psi(\mathbf{r},t)=i\hbar\frac{\partial\Psi(\mathbf{r},t)}{\partial t}$$

Note that for a uniform potential

e.g., $V=0$ for simplicity

with $E=\hbar\omega$ and $k=\sqrt{2mE/\hbar^2}$

waves of the form

$$\exp\left[-i(\omega t\pm kz)\right]\equiv\exp\left[-i\left(\frac{Et}{\hbar}\pm kz\right)\right]\equiv\exp\left(-i\frac{Et}{\hbar}\right)\exp(\mp ikz)$$

are indeed solutions

Rationalizing the time-dependent equation

In his time-dependent equation

$$-\frac{\hbar^2}{2m}\nabla^2\Psi(\mathbf{r},t)+V(\mathbf{r},t)\Psi(\mathbf{r},t)=i\hbar\frac{\partial\Psi(\mathbf{r},t)}{\partial t}$$

Schrödinger chose a sign for the right hand side

which means that a wave with a spatial part

$$\propto \exp(ikz)$$

is definitely going in the positive z direction

That wave, including its time dependence

would be of the form (for $V=0$)

$$\exp\left[i\left(kz-Et/\hbar\right)\right]$$

Compatibility with the time-independent equation

Before examining the time-dependent equation further

- first we should check that it is compatible with the time-independent equation

The time-independent equation could apply if we had states of definite energy E, an eigenenergy

- Suppose we had some corresponding eigenfunction $\psi(\mathbf{r})$ so that

$$-\frac{\hbar^2}{2m}\nabla^2\psi(\mathbf{r})+V(\mathbf{r})\psi(\mathbf{r})=E\psi(\mathbf{r})$$

Compatibility with the time-independent equation

As it stands

this solution $\psi(\mathbf{r})$ is not a solution of the time-dependent equation

$$-\frac{\hbar^2}{2m}\nabla^2\Psi(\mathbf{r},t)+V(\mathbf{r},t)\Psi(\mathbf{r},t)=i\hbar\frac{\partial\Psi(\mathbf{r},t)}{\partial t}$$

Putting $\psi(\mathbf{r})$ in here for $\Psi(\mathbf{r},t)$ does not work

because $\psi(\mathbf{r})$ has no time-dependence

the right hand side is zero

whereas it should be $E\psi(\mathbf{r})$

how do we resolve this?

Compatibility with the time-independent equation

Suppose that, instead of proposing the solution $\psi(\mathbf{r})$

we propose $\Psi(\mathbf{r},t)=\psi(\mathbf{r})\exp(-iEt/\hbar)$

$$-\frac{\hbar^2}{2m}\nabla^2\Psi(\mathbf{r},t)+V(\mathbf{r})\Psi(\mathbf{r},t)$$

$$=-\frac{\hbar^2}{2m}\nabla^2\psi(\mathbf{r})\exp(-iEt/\hbar)+V(\mathbf{r})\psi(\mathbf{r})\exp(-iEt/\hbar)$$

$$=\left[-\frac{\hbar^2}{2m}\nabla^2\psi(\mathbf{r})+V(\mathbf{r})\psi(\mathbf{r})\right]\exp(-iEt/\hbar)=E\psi(\mathbf{r})\exp(-iEt/\hbar)$$

$$=E\Psi(\mathbf{r},t)$$

so $\Psi(\mathbf{r},t)=\psi(\mathbf{r})\exp(-iEt/\hbar)$ solves the time-independent Schrödinger equation

Compatibility with the time-independent equation

Similarly, knowing that $\psi(\mathbf{r})$ solves the time-independent equation with energy E

substituting $\Psi(\mathbf{r},t)=\psi(\mathbf{r})\exp(-iEt/\hbar)$

in the time-dependent equation gives

$$-\frac{\hbar^2}{2m}\nabla^2\Psi(\mathbf{r},t)+V(\mathbf{r})\Psi(\mathbf{r},t)=i\hbar\frac{\partial\Psi(\mathbf{r},t)}{\partial t}=i\hbar\frac{\partial}{\partial t}\left[\psi(\mathbf{r})\exp(-iEt/\hbar)\right]$$

$$=i\hbar\psi(\mathbf{r})\frac{\partial}{\partial t}\left[\exp(-iEt/\hbar)\right]=i\hbar\psi(\mathbf{r})\left[-i\frac{E}{\hbar}\right]\exp(-iEt/\hbar)=E\,\Psi(\mathbf{r},t)$$

so $\Psi(\mathbf{r},t)=\psi(\mathbf{r})\exp(-iEt/\hbar)$ solves the time-dependent Schrödinger equation

Compatibility with the time-independent equation

So every solution $\psi(\mathbf{r})$ of the time-independent Schrödinger equation, with eigenenergy E

is also a solution of the time-dependent equation

as long as we always multiply it by a factor $\exp(-iEt/\hbar)$

If $\psi(\mathbf{r})$ is a solution of the time-independent Schrödinger equation, with eigenenergy E

then $\Psi(\mathbf{r},t) = \psi(\mathbf{r})\exp(-iEt/\hbar)$

is a solution of both the time-independent

and the time-dependent Schrödinger equations

making these two equations compatible

Oscillations and time-independence

If we propose a solution

$$\Psi(\mathbf{r},t)=\psi(\mathbf{r})\exp(-iEt/\hbar)$$

to a time-independent problem

can this represent something that is stable in time?

Yes! - measurable quantities associated with this state

are stable in time!

e.g., probability density

$$\left|\Psi(\mathbf{r},t)\right|^2=\left[\exp(+iEt/\hbar)\psi^*(\mathbf{r})\right]\times\left[\exp(-iEt/\hbar)\psi(\mathbf{r})\right]=\left|\psi(\mathbf{r})\right|^2$$