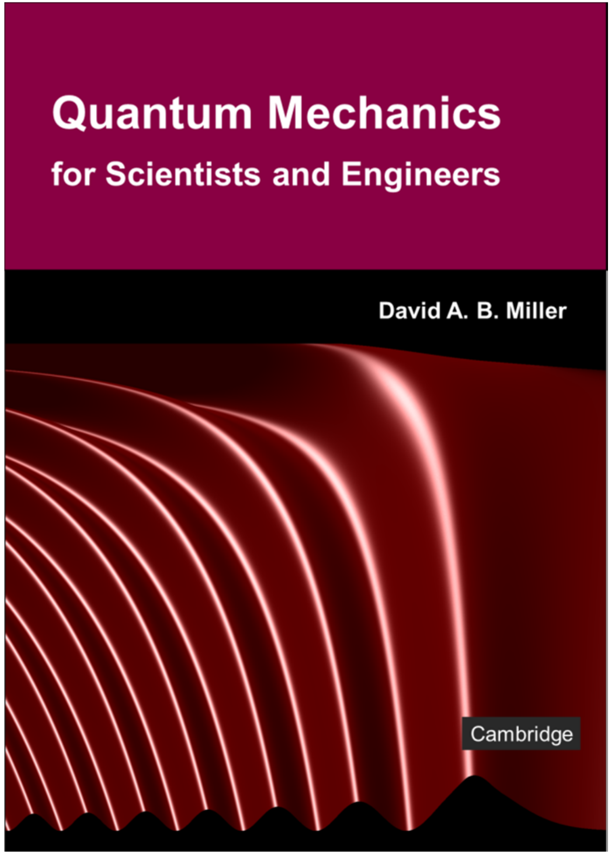
4.2 Wavepackets

Slides: Video 4.2.6 Electron wavepackets

Text reference: Quantum Mechanics for Scientists and Engineers

Section 3.7 (“Examples of motion of wavepackets”)



Wavepackets

Electron wavepackets

Quantum mechanics for scientists and engineers

David Miller

Constructing a wavepacket

We can construct a "wavepacket" by putting together a linear superposition of energy eigensolutions

For a free electron or a similar particle of mass m

the individual eigensolutions are plane waves

For propagation in the z direction, these are of the form

$$\Psi_k(z,t) \propto \exp\left\{-i\left[\frac{E(k)}{\hbar}t - kz\right]\right\} = \exp\{-i[\omega(k)t - kz]\}$$

for some chosen value of k, and hence of

energy $E(k) = \dfrac{\hbar^2 k^2}{2m}$ and frequency $\omega(k) = \dfrac{E(k)}{\hbar}$

Constructing a wavepacket

Since

a linear superposition of such plane wave solutions of the time-dependent Schrödinger equation

is also a solution

we can have a "wavepacket"

$$\Psi_{WP}(z,t) \propto \sum_k a_k \Psi_k(z,t) = \sum_k a_k \exp\{-i[\omega(k)t - kz]\}$$

for some set of values of k in our sum

and some chosen amplitudes a_k for each such plane wave

Gaussian wavepacket

One convenient and useful set of k values and amplitudes a_k to choose is

a set of equally spaced k values

with Gaussian amplitudes or "weights" for a_k

$$\Psi_G(z,t) \propto \sum_k \exp\left[-\left(\frac{k-\bar{k}}{2\Delta k}\right)^2\right] \exp\{-i[\omega(k)t - kz]\}$$

Here $\bar{k}$ is the center of the distribution of k values

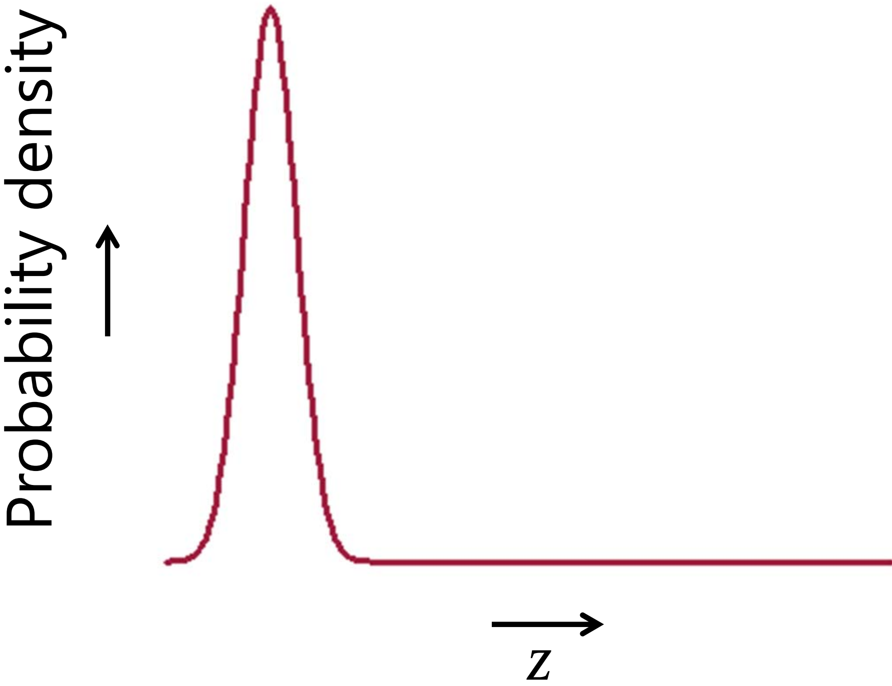
Δk is a width parameter for the Gaussian function

Note this gives a "pulse" that is also Gaussian in space

Gaussian wavepacket

Our Gaussian wavepacket is also Gaussian in space

- As we let time evolve
  - simply adding up the terms in our wavepacket sum at each time
    - we can see the wavepacket propagate
      - moving to the right
        - and getting wider



Gaussian wavepacket

A wavepacket that increases in width as it propagates

is said to be "dispersing"

It gets wider because

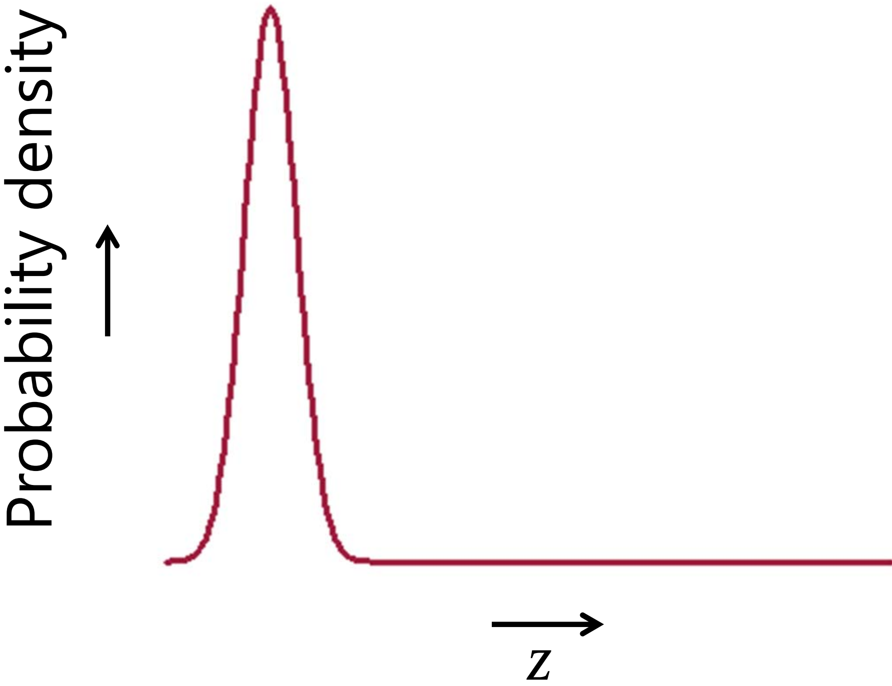
the change in ω with k

is not even linear

(here it is quadratic)

an effect known as

group velocity dispersion



Wavepacket at a barrier

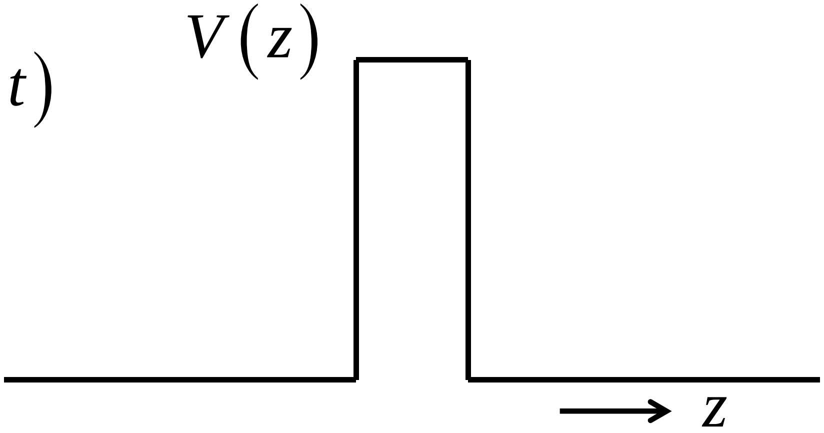
Suppose we want to understand a wavepacket hitting a barrier from the left

We proceed in the same way

but now we use

a superposition of the energy eigenfunctions $\Psi_{Bk}(z,t)$ of the Schrödinger equation

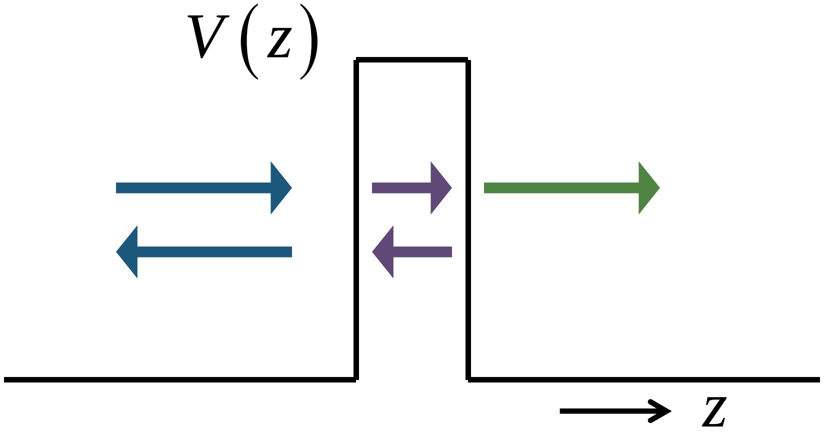
with a barrier potential $V(z)$



Eigensolutions with a barrier

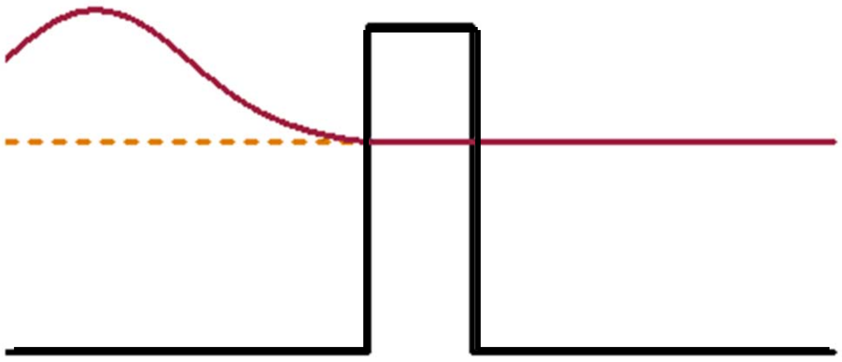
The energy eigensolutions $\Psi_{Bk}(z,t)$

- for a particle incident from the left are
  - forward and backward propagating waves on the left
    - "forward" and "backward" exponentials inside the barrier
      - forward propagating waves on the right



Wavepacket at a barrier

- Constructing a Gaussian-weighted linear superposition of solutions
  - with equally-spaced k-values on the left
    - here centered round the k-value corresponding to the dashed orange line
      - gives an approximately Gaussian "pulse" on the left to start with



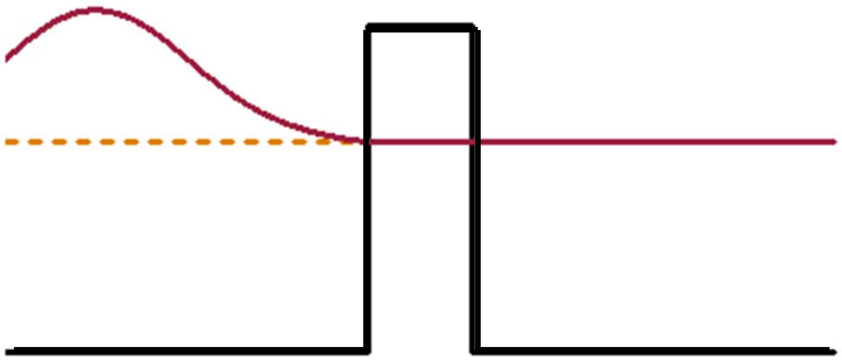
Wavepacket at a barrier

Now as time evolves, the "pulse" moves to the right

- where it partially bounces off
  - and partially transmits

Note that there is a significant probability of

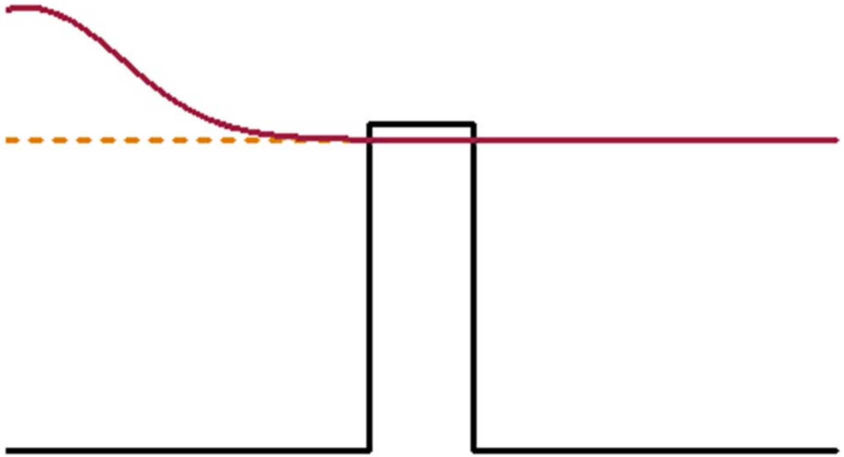
- finding the particle in the barrier
  - while the pulse is "hitting" it



Wavepacket at a barrier

With the particle incident at a higher average energy

- the transmitted pulse is stronger
- and the reflected pulse is weaker



Wavepacket at a barrier

- Even with energies above the barrier
  - there is still significant reflection of the pulse
    - wave are generally reflected off of any changes in the potential