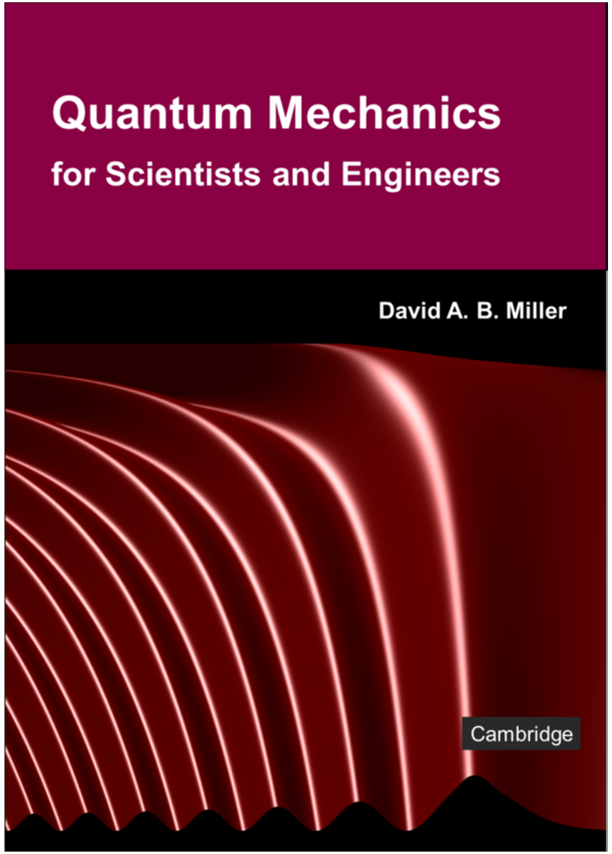
4.1 Time evolution of superpositions

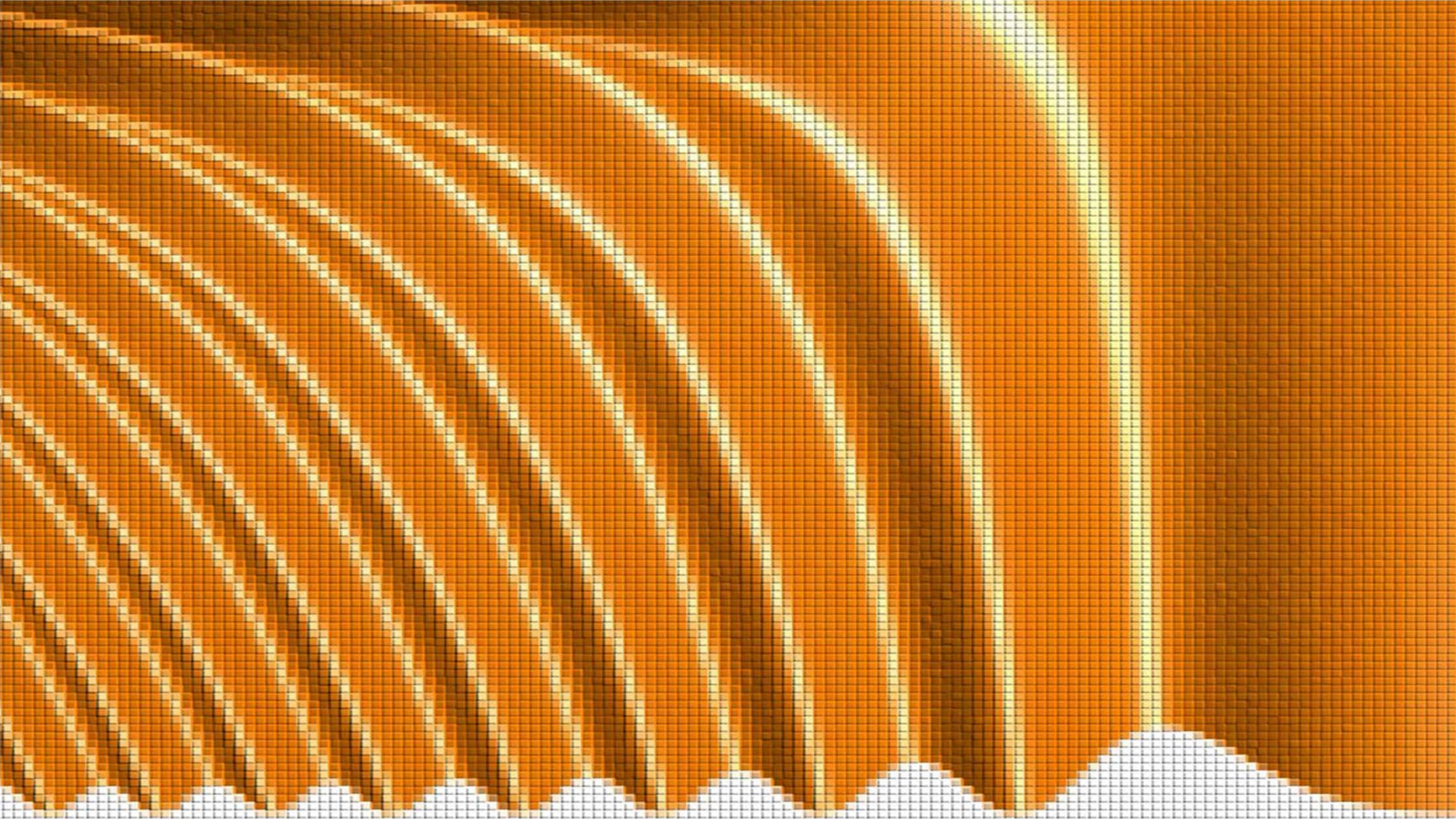
Slides: Video 4.1.1 Introduction to time evolution of superpositions



Time evolution of superpositions

Quantum mechanics for scientists and engineers

David Miller

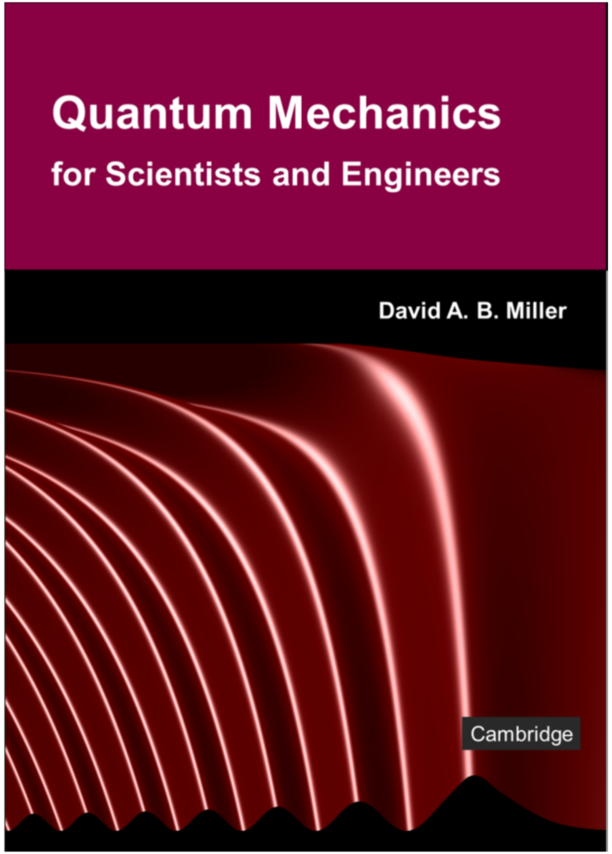


4.1 Time evolution of superpositions

Slides: Video 4.1.2 Superposition for the particle in a box

Text reference: Quantum Mechanics for Scientists and Engineers

Section 3.6 (“Simple linear superposition in an infinite potential well”)



Time evolution of superpositions

Superposition for a particle in a box

Quantum mechanics for scientists and engineers

David Miller

Superposition for a particle in a box

Suppose we have an infinitely deep potential well

a "particle in a box"

with the particle in a linear superposition

for example, with equal parts of the first and second states of the well

$$\Psi(z,t)=\frac{1}{\sqrt{L_z}}\left[\exp\left(-i\frac{E_1}{\hbar}t\right)\sin\left(\frac{\pi z}{L_z}\right)+\exp\left(-i\frac{E_2}{\hbar}t\right)\sin\left(\frac{2\pi z}{L_z}\right)\right]$$

Superposition for a particle in a box

Note for each eigenfunction in the superposition

it is multiplied by the appropriate complex exponential time-varying function

$$\exp\left(-i\frac{E_n}{\hbar}t\right)$$

This superposition is also normalized

$$\Psi(z,t)=\frac{1}{\sqrt{L_z}}\left[\exp\left(-i\frac{E_1}{\hbar}t\right)\sin\left(\frac{\pi z}{L_z}\right)+\exp\left(-i\frac{E_2}{\hbar}t\right)\sin\left(\frac{2\pi z}{L_z}\right)\right]$$

Superposition for a particle in a box

From this superposition

$$\Psi(z,t) = \frac{1}{\sqrt{L_z}}\left[\exp\left(-i\frac{E_1}{\hbar}t\right)\sin\left(\frac{\pi z}{L_z}\right) + \exp\left(-i\frac{E_2}{\hbar}t\right)\sin\left(\frac{2\pi z}{L_z}\right)\right]$$

we can multiply it by its complex conjugate

to get the probability density

$$\left|\Psi(z,t)\right|^2 =$$

$$\frac{1}{L_z}\left[\sin^2\left(\frac{\pi z}{L_z}\right) + \sin^2\left(\frac{2\pi z}{L_z}\right) + 2\cos\left(\frac{E_2 - E_1}{\hbar}t\right)\sin\left(\frac{\pi z}{L_z}\right)\sin\left(\frac{2\pi z}{L_z}\right)\right]$$

Superposition for a particle in a box

$$\left[\exp\left(-i\frac{E_1}{\hbar}t\right)\sin\left(\frac{\pi z}{L_z}\right)+\exp\left(-i\frac{E_2}{\hbar}t\right)\sin\left(\frac{2\pi z}{L_z}\right)\right]$$

multiplied by its complex conjugate

$$\times\left[\exp\left(+i\frac{E_1}{\hbar}t\right)\sin\left(\frac{\pi z}{L_z}\right)+\exp\left(+i\frac{E_2}{\hbar}t\right)\sin\left(\frac{2\pi z}{L_z}\right)\right]$$

$$=\boxed{\sin^2\left(\frac{\pi z}{L_z}\right)}+\boxed{\sin^2\left(\frac{2\pi z}{L_z}\right)}+\sin\left(\frac{\pi z}{L_z}\right)\sin\left(\frac{2\pi z}{L_z}\right)\boxed{\left[\boxed{\exp\left(i\frac{E_2-E_1}{\hbar}t\right)}+\boxed{\exp\left(-i\frac{E_2-E_1}{\hbar}t\right)}\right]}$$

$$=\sin^2\left(\frac{\pi z}{L_z}\right)+\sin^2\left(\frac{2\pi z}{L_z}\right)+\boxed{2\cos\left(\frac{E_2-E_1}{\hbar}t\right)}\sin\left(\frac{\pi z}{L_z}\right)\sin\left(\frac{2\pi z}{L_z}\right)$$

Superposition for a particle in a box

Note this probability density

$$|\Psi(z,t)|^2 =$$

$$\frac{1}{L_z}\left[\sin^2\left(\frac{\pi z}{L_z}\right)+\sin^2\left(\frac{2\pi z}{L_z}\right)+2\cos\left(\frac{E_2-E_1}{\hbar}t\right)\sin\left(\frac{\pi z}{L_z}\right)\sin\left(\frac{2\pi z}{L_z}\right)\right]$$

has a part that is oscillating in time

at an angular frequency $\omega_{21}=\left(E_2-E_1\right)/\hbar=3E_1/\hbar$

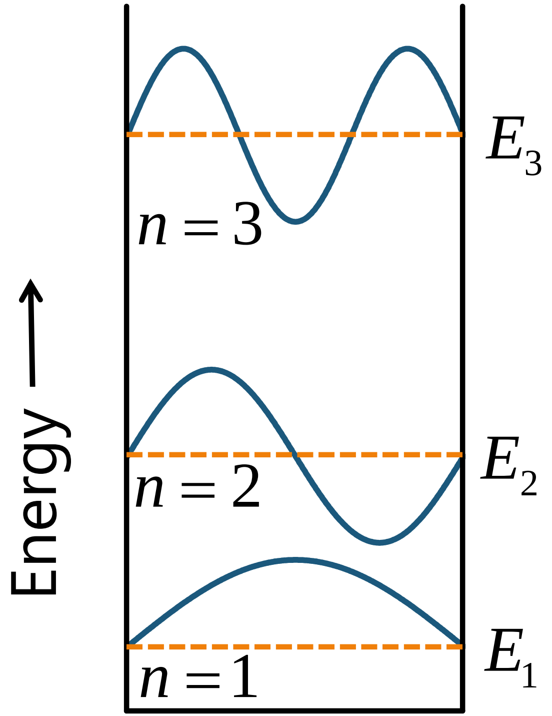
Note also that the absolute energy origin does not matter here for this measurable quantity

only the energy difference E_2-E_1 matters

Particle in a box

As a reminder

- here are the first few particle-in-a-box energy levels
  - and their associated wavefunctions
    - plotted with the orange dashed lines as horizontal axes



Superposition

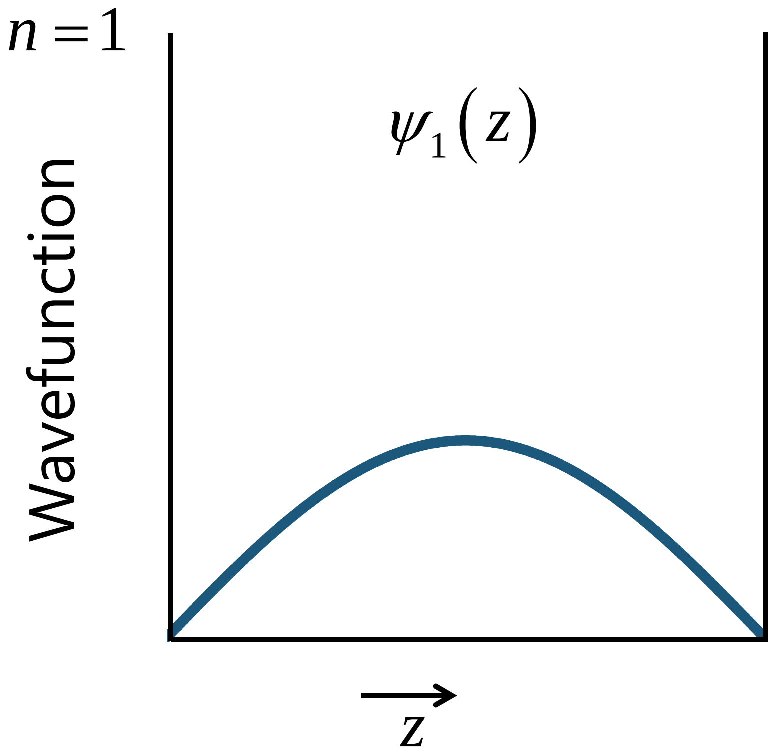
The $n=1$ spatial eigenfunction

$\psi_1(z)$

is plotted here

with the bottom of the box

as its horizontal axis



Superposition

For the probability density

$$\left|\psi_1(z)\right|^2$$

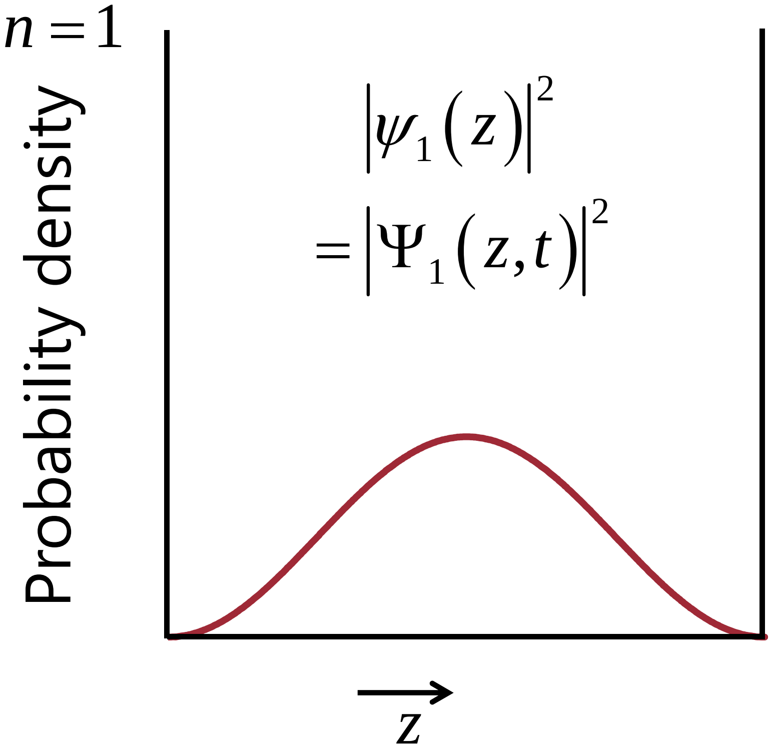
note the different shape

Multiplying by the time dependent factor gives

$$\Psi_1(z,t) = \exp\left(-i\frac{E_1}{\hbar}t\right)\psi_1(z)$$

The probability densities are the same

$$\left|\Psi_1(z,t)\right|^2 = \left|\psi_1(z)\right|^2$$



Superposition

Similarly

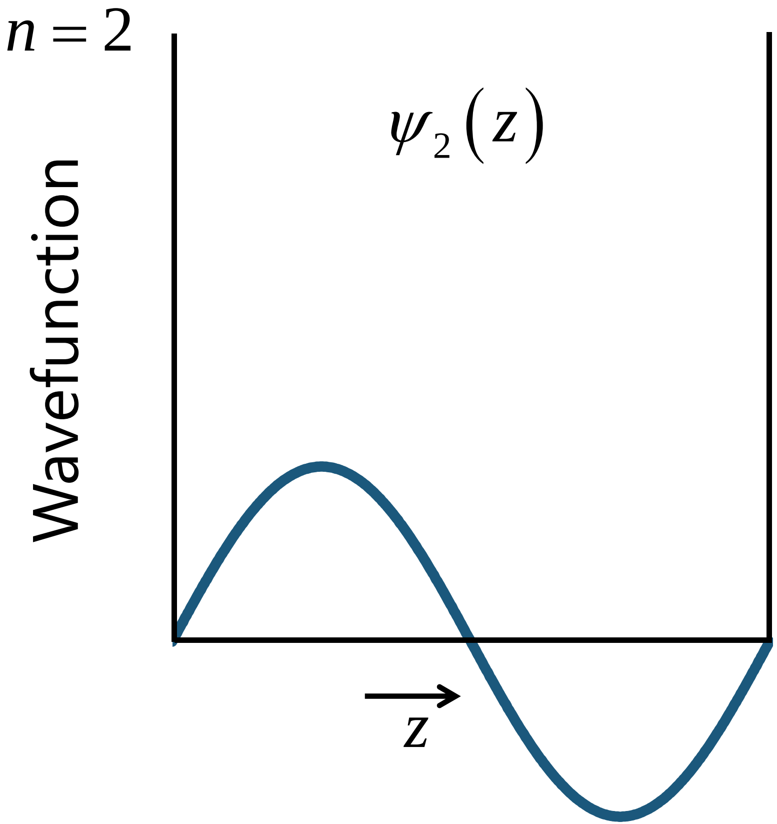
The $n = 2$ spatial eigenfunction

$\psi_2(z)$

is plotted here

with the bottom of the box

as its horizontal axis



Superposition

The probability density

$$\left|\psi_2(z)\right|^2$$

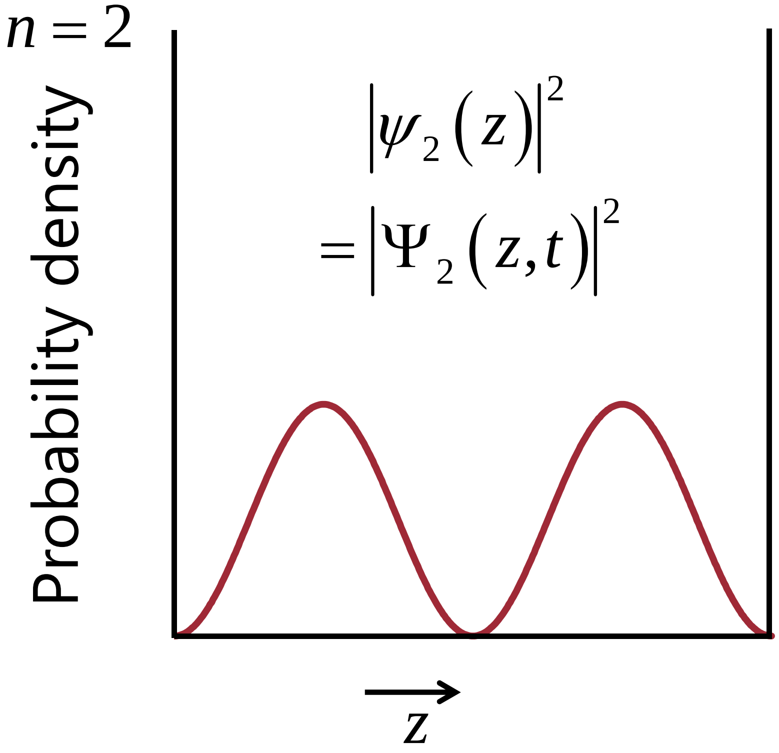
is a positive function

Multiplying by the time dependent factor gives

$$\Psi_2(z,t) = \exp\left(-i\frac{E_2}{\hbar}t\right)\psi_2(z)$$

The probability densities are the same

$$\left|\Psi_2(z,t)\right|^2 = \left|\psi_2(z)\right|^2$$



Superposition

An equal superposition of the two oscillates

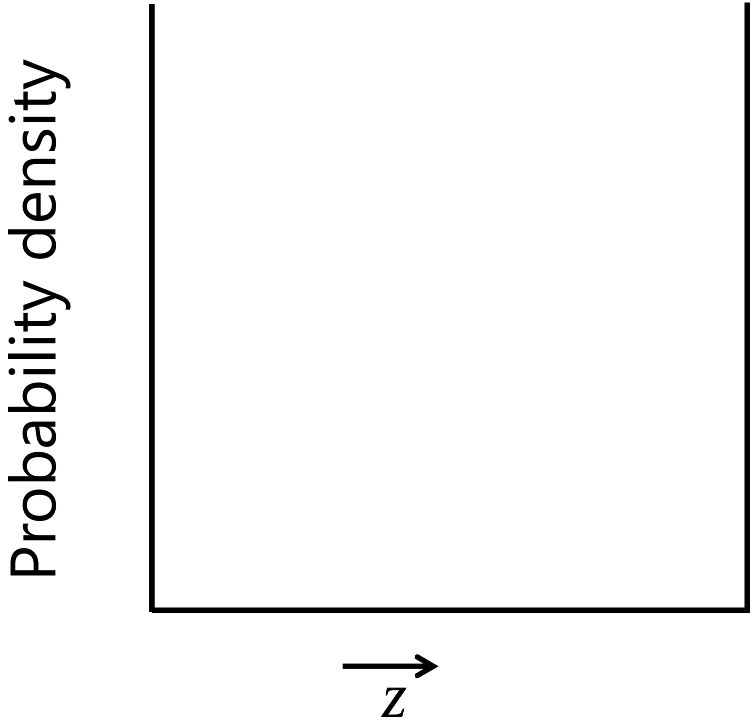
at the angular frequency

$$\omega_{21} = \left(E_2 - E_1\right)/\hbar = 3E_1/\hbar$$

$$\left|\Psi\left(z,t\right)\right|^2 = \left|\Psi_1\left(z,t\right) + \Psi_2\left(z,t\right)\right|^2$$

$$= \left|\psi_1\left(z\right)\right|^2 + \left|\psi_2\left(z\right)\right|^2$$

$$+2\cos\left(\frac{E_2 - E_1}{\hbar}t\right)\psi_1\left(z\right)\psi_2\left(z\right)$$



Superposition

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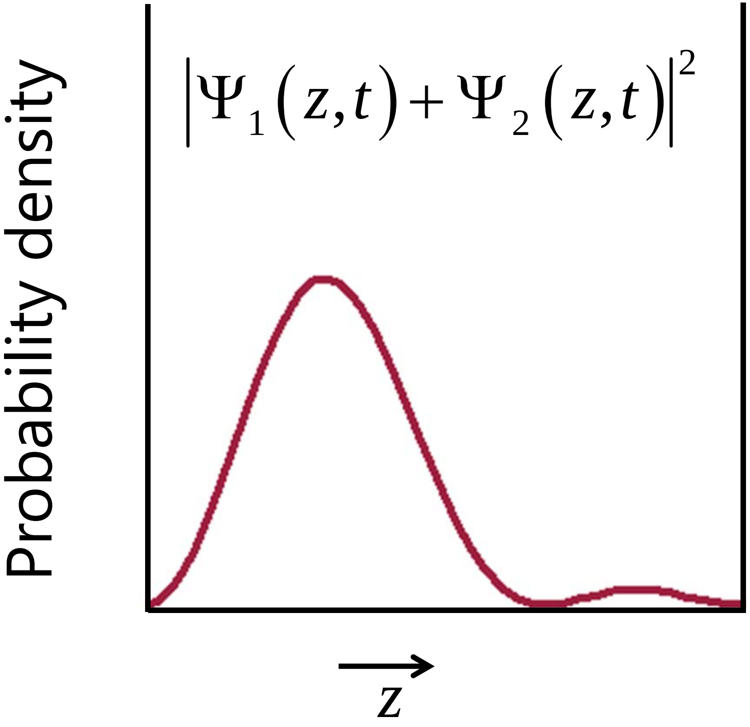
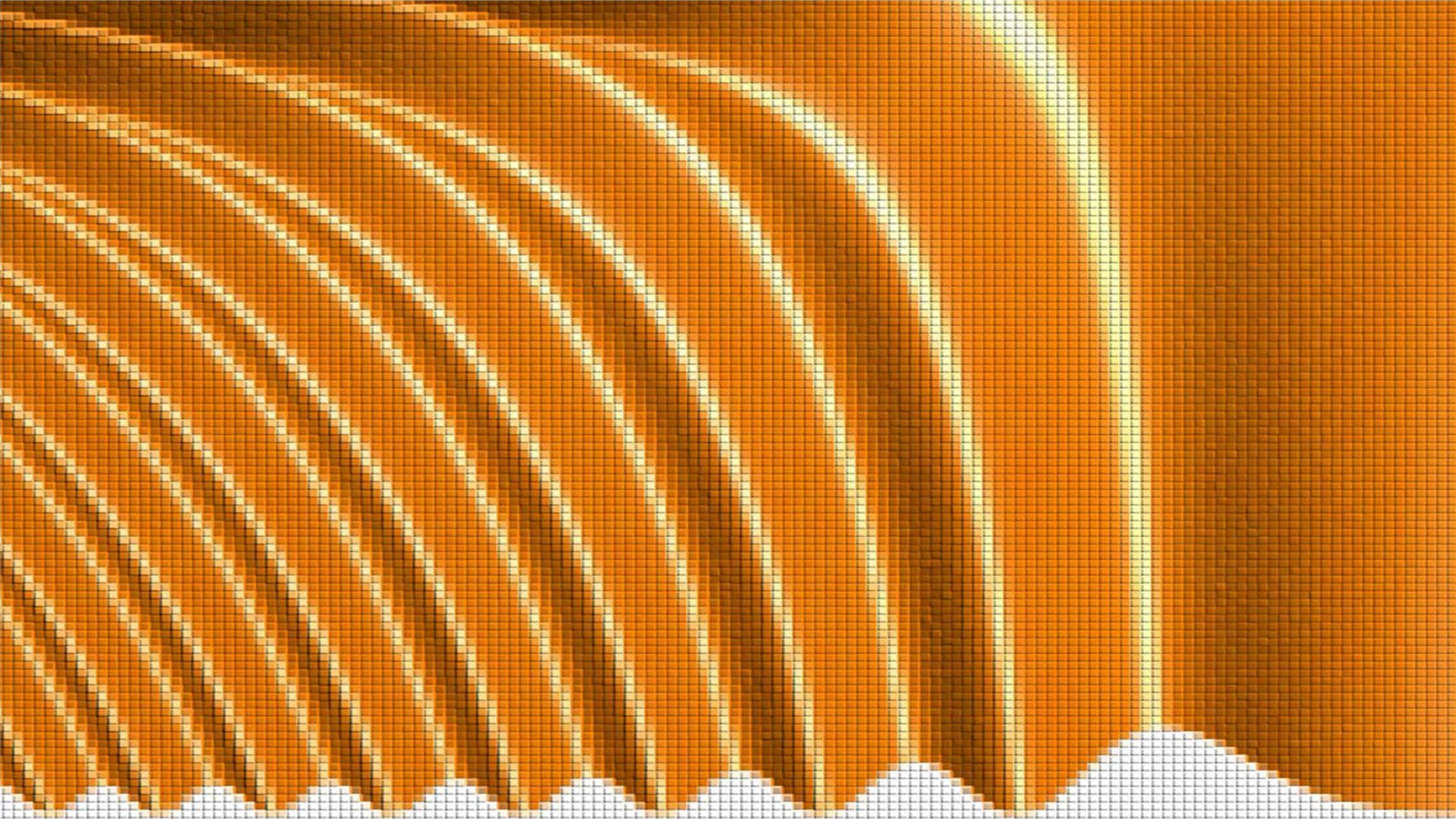
$$\omega_{21} = \left(E_2 - E_1\right)/\hbar = 3E_1/\hbar$$

$$\left|\Psi\left(z,t\right)\right|^2 = \left|\Psi_1\left(z,t\right) + \Psi_2\left(z,t\right)\right|^2$$

$$= \left|\psi_1\left(z\right)\right|^2 + \left|\psi_2\left(z\right)\right|^2$$

$$+2\cos\left(\frac{E_2 - E_1}{\hbar}t\right)\psi_1\left(z\right)\psi_2\left(z\right)$$



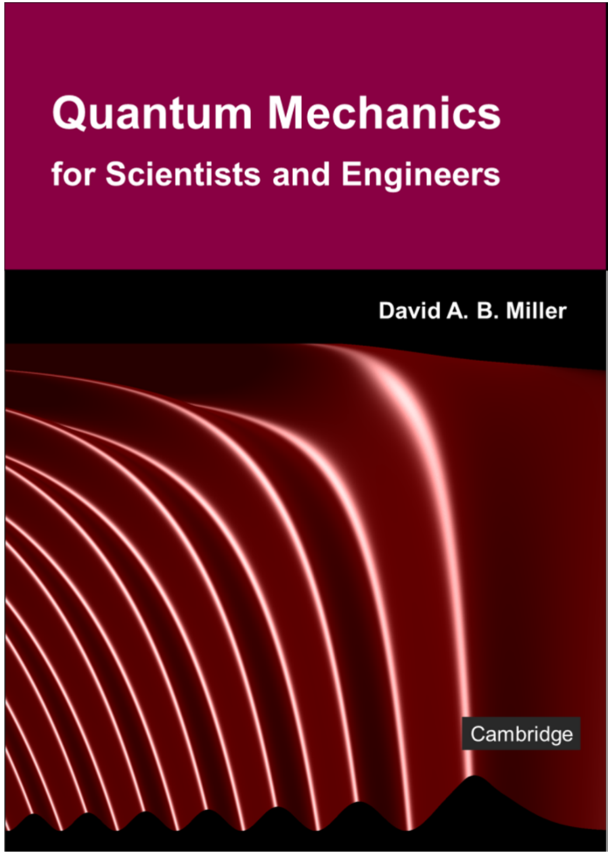


4.1 Time evolution of superpositions

Slides: Video 4.1.4 Superposition for the harmonic oscillator

Text reference: Quantum Mechanics for Scientists and Engineers

Section 3.6 ("Harmonic oscillator example")



Time evolution of superpositions

Superposition for the harmonic oscillator

Quantum mechanics for scientists and engineers

David Miller

Superpositions and oscillation

Quite generally

- if we make a linear combination of two energy eigenstates
  - with energies E_a and E_b
    - the resulting probability distribution
      - will oscillate at the (angular) frequency

$$\omega_{ab} = \left| E_a - E_b \right| / \hbar$$

Superpositions and oscillation

So, if we have a superposition wavefunction

$$\Psi_{ab}(\mathbf{r},t) = c_a \exp\left(-i\frac{E_a}{\hbar}t\right)\psi_a(\mathbf{r}) + c_b \exp\left(-i\frac{E_b}{\hbar}t\right)\psi_b(\mathbf{r})$$

then the probability distribution will be

$$\left|\Psi_{ab}(\mathbf{r},t)\right|^2 = \left|c_a\right|^2\left|\psi_a(\mathbf{r})\right|^2 + \left|c_b\right|^2\left|\psi_b(\mathbf{r})\right|^2$$

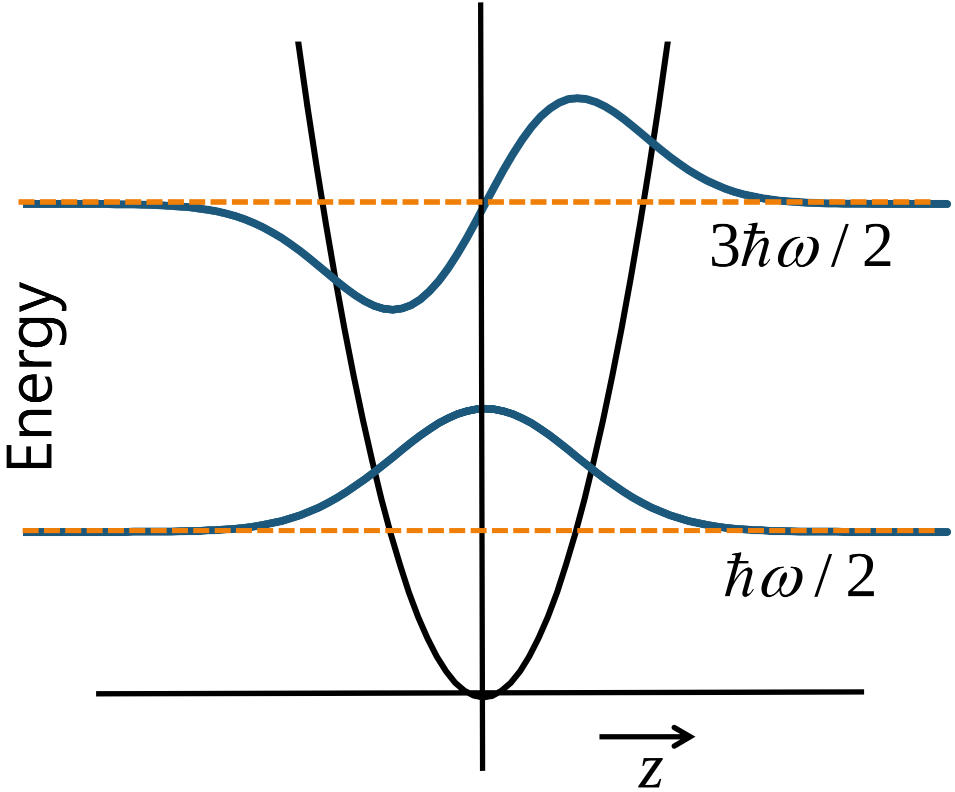
$$+2\left|c_a^*\psi_a^*(\mathbf{r})c_b\psi_b(\mathbf{r})\right|\cos\left[\frac{(E_a - E_b)t}{\hbar} - \theta_{ab}\right]$$

where $\theta_{ab} = \arg\left(c_a\psi_a(\mathbf{r})c_b^*\psi_b^*(\mathbf{r})\right)$

Harmonic oscillator

As a reminder

- here are the first two harmonic energy levels
  - and their associated wavefunctions
    - plotted with the orange dashed lines as horizontal axes



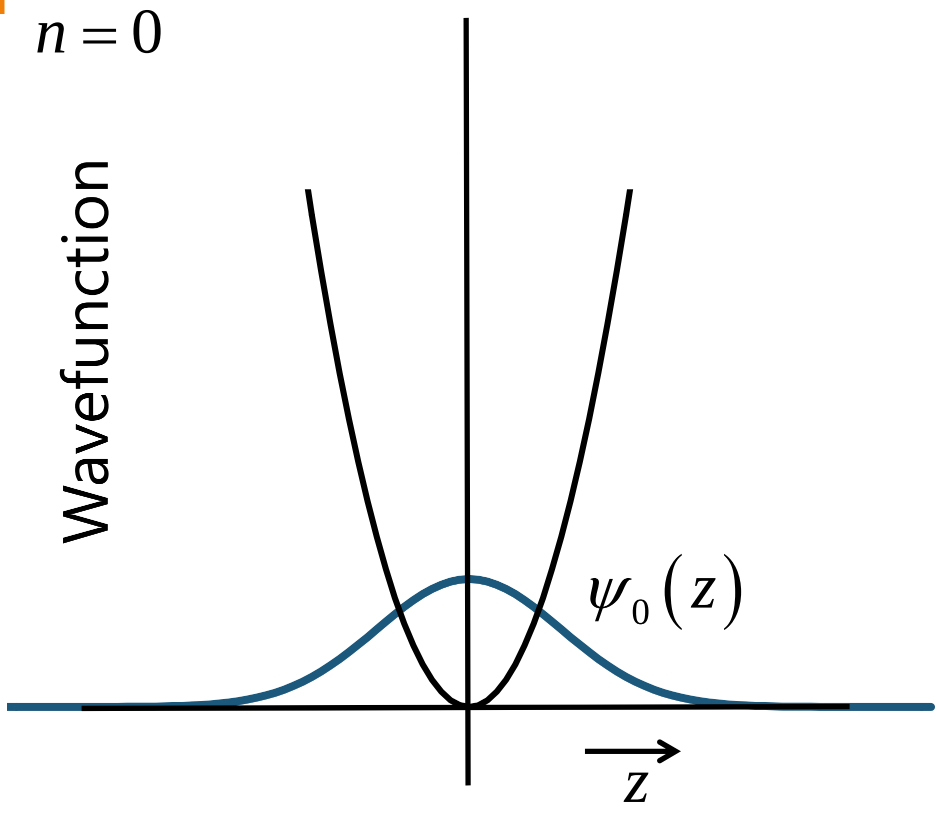
Superposition

The $n = 0$ spatial eigenfunction

$\psi_0(z)$

is plotted here

with the bottom of the parabolic well as its horizontal axis



Superposition

For the probability density

$$\left|\psi_0(z)\right|^2$$

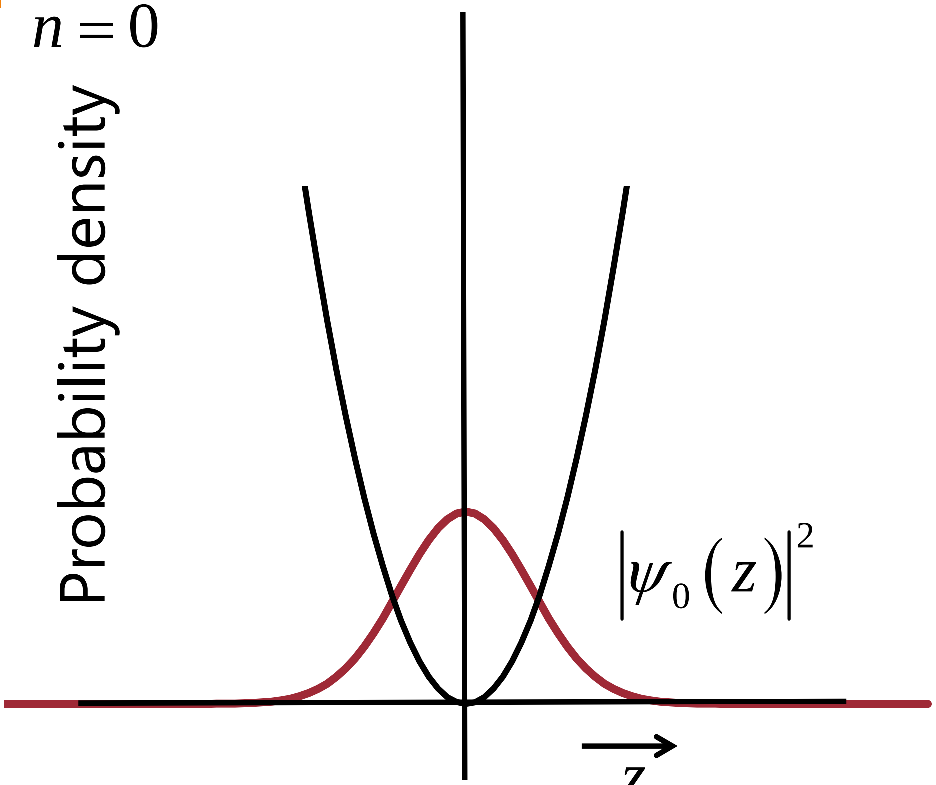
note the narrower shape

Multiplying $\psi_0(z)$ by the time dependent factor gives

$$\Psi_0(z,t) = \exp\left(-i\frac{E_0}{\hbar}t\right)\psi_0(z)$$

The probability densities are the same

$$\left|\Psi_0(z,t)\right|^2 = \left|\psi_0(z)\right|^2$$



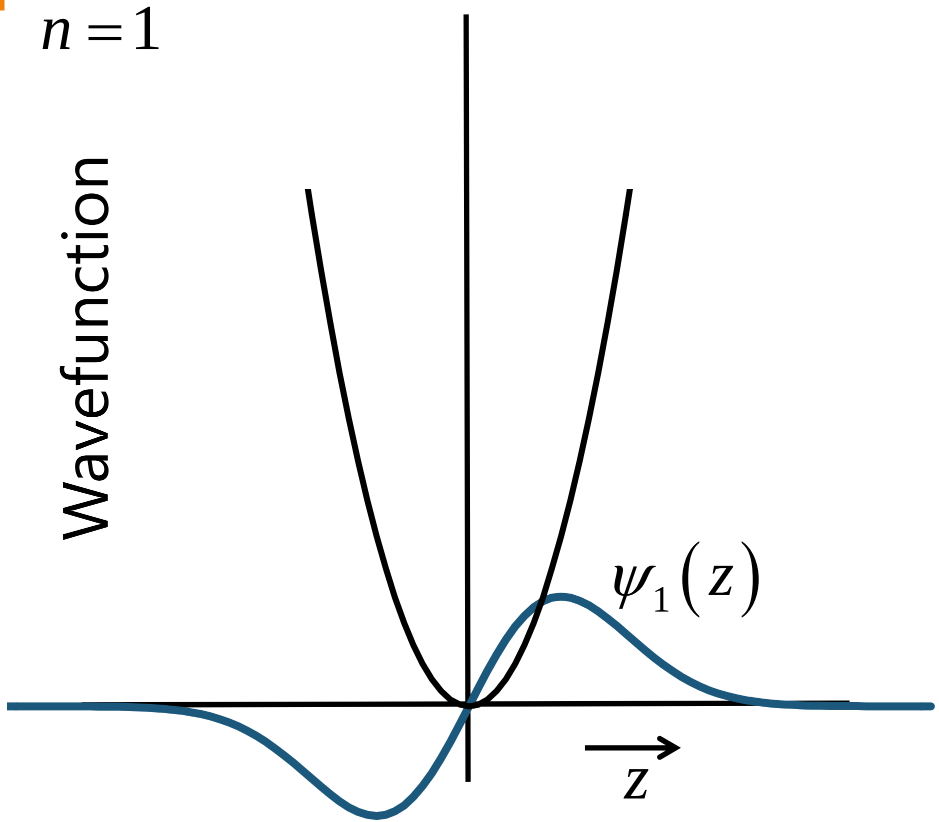
Superposition

The $n = 1$ spatial eigenfunction

$\psi_1(z)$

is plotted here

with the bottom of the parabolic well as its horizontal axis



Superposition

For the probability density

$$\left|\psi_1(z)\right|^2$$

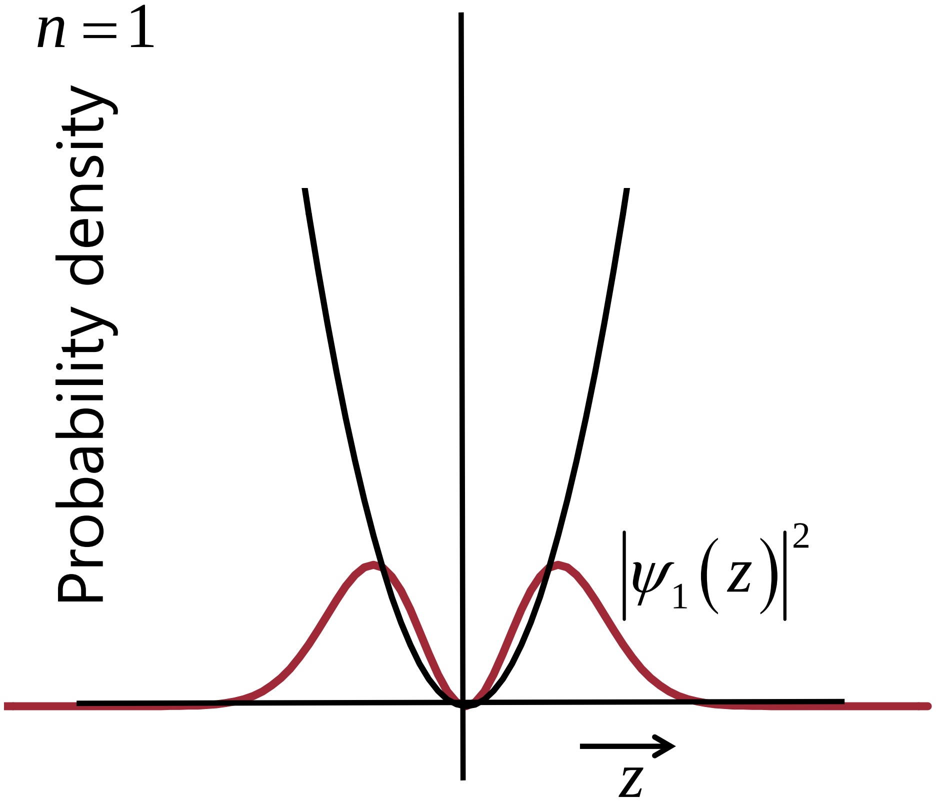
note it is positive

Multiplying by the time dependent factor gives

$$\Psi_1(z,t) = \exp\left(-i\frac{E_1}{\hbar}t\right)\psi_1(z)$$

The probability densities are the same

$$\left|\Psi_1(z,t)\right|^2 = \left|\psi_1(z)\right|^2$$



Superposition

An equal superposition of the two oscillates

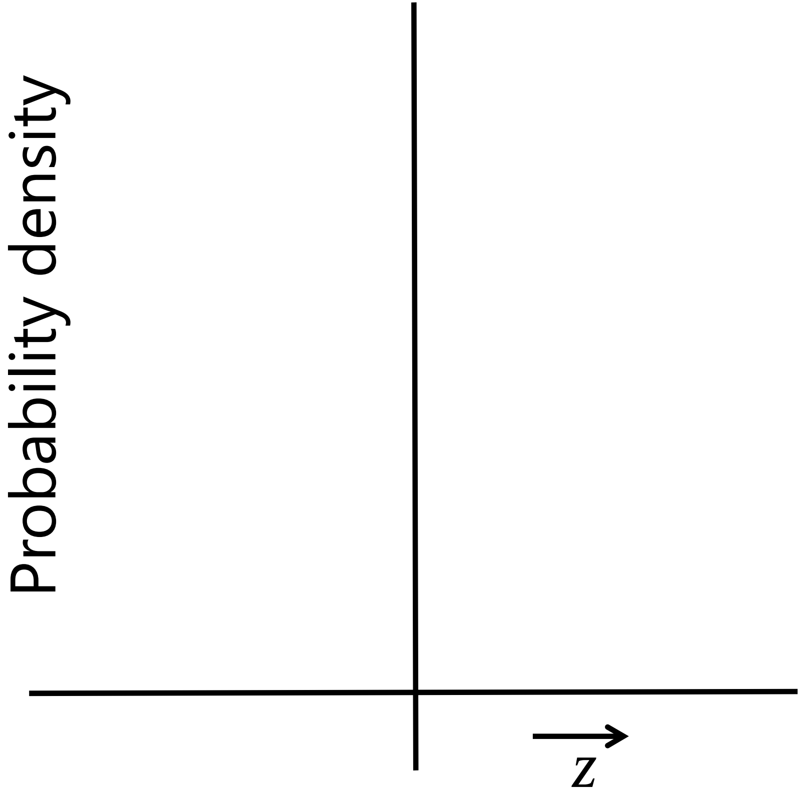
at the angular frequency

$$\omega = \left(E_1 - E_0\right)/\hbar$$

$$\left|\Psi\left(z,t\right)\right|^2 = \left|\Psi_0\left(z,t\right)+\Psi_1\left(z,t\right)\right|^2$$

$$= \left|\psi_0\left(z\right)\right|^2 + \left|\psi_1\left(z\right)\right|^2$$

$$+2\cos\left(\omega t\right)\psi_0\left(z\right)\psi_1\left(z\right)$$



Superposition

An equal superposition of the two oscillates

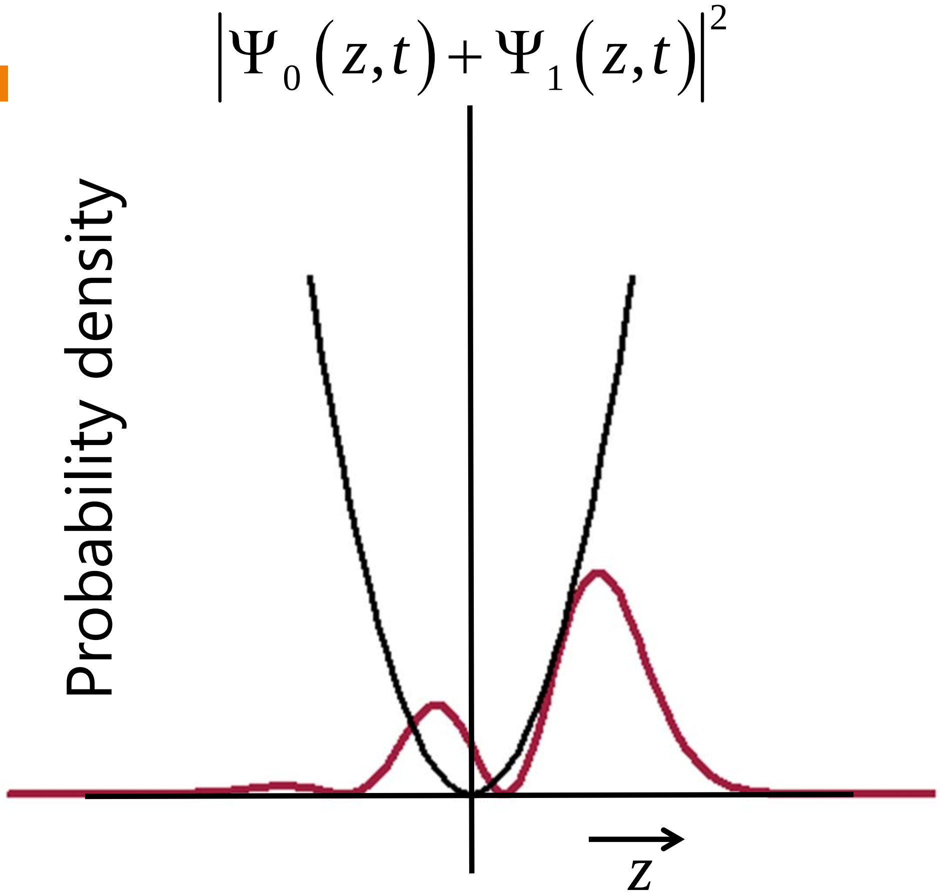
at the angular frequency

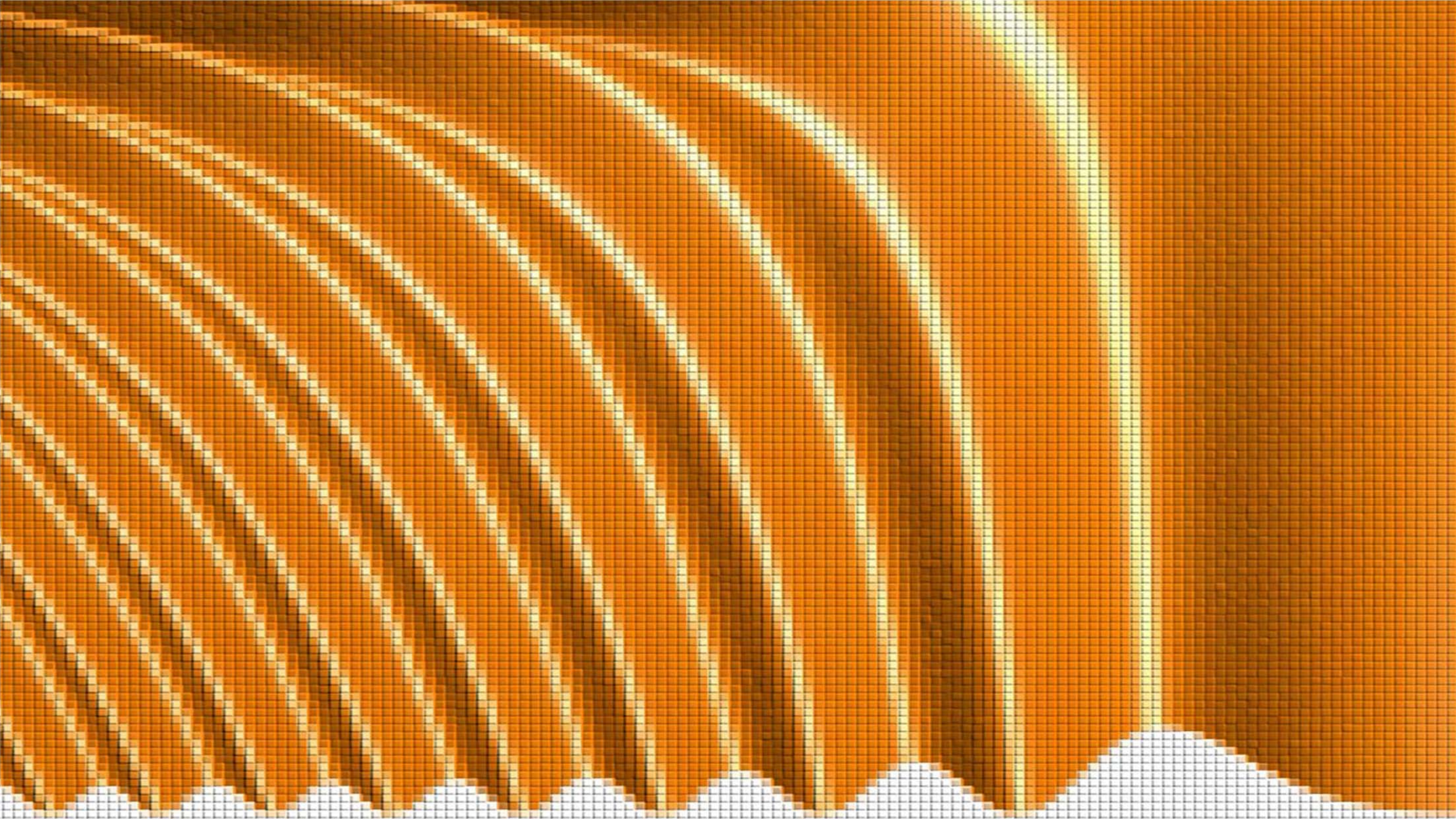
$$\omega = \left(E_1 - E_0\right) / \hbar$$

$$\left|\Psi\left(z,t\right)\right|^2 = \left|\Psi_0\left(z,t\right) + \Psi_1\left(z,t\right)\right|^2$$

$$= \left|\psi_0\left(z\right)\right|^2 + \left|\psi_1\left(z\right)\right|^2$$

$$+2\cos\left(\omega t\right)\psi_0\left(z\right)\psi_1\left(z\right)$$



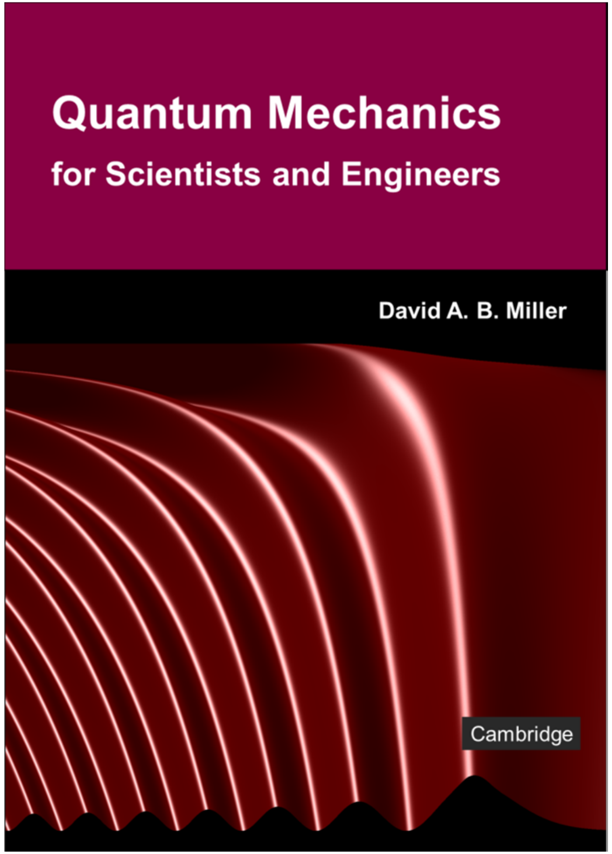


4.1 Time evolution of superpositions

Slides: Video 4.1.6 The coherent state

Text reference: Quantum Mechanics for Scientists and Engineers

Section 3.6 (“Coherent state”)



Time evolution of superpositions

The coherent state

Quantum mechanics for scientists and engineers

David Miller

The coherent state

The coherent state for a harmonic oscillator of frequency ω is

$$\Psi_N(\xi,t)=\sum_{n=0}^{\infty}c_{Nn}\exp\left[-i\left(n+\frac{1}{2}\right)\omega t\right]\psi_n(\xi)$$

where

$$c_{Nn}=\sqrt{\frac{N^n\exp(-N)}{n!}}$$

and the $\psi_n(\xi)$ are the harmonic oscillator eigenstates

The coherent state

Incidentally, note that for the expansion coefficients c_{Nn}

$$\left|c_{Nn}\right|^2 = \frac{N^n \exp(-N)}{n!}$$

This is the Poisson distribution from statistics

with mean N and standard deviation $\sqrt{N}$

We will make no direct use of this here

but in the end it explains, e.g., the Poissonian distribution of photons in a laser beam

Coherent state

Coherent state oscillations

with

$$\Psi_N(\xi,t) =$$

$$\sum_{n=0}^{\infty} c_{Nn} \exp\left[-i\left(n+\frac{1}{2}\right)\omega t\right]\psi_n(\xi)$$

$$c_{Nn} = \sqrt{\frac{N^n \exp(-N)}{n!}}$$

Coherent state

Coherent state oscillations

with

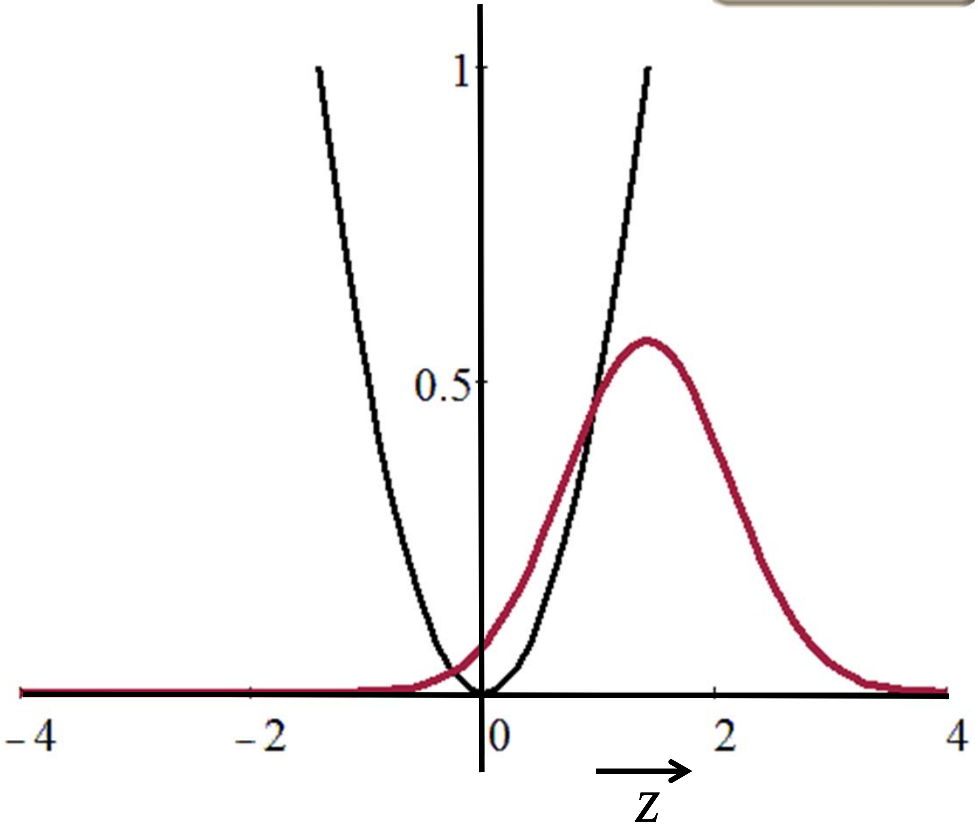
$$\Psi_N(\xi,t) =$$

$$\sum_{n=0}^{\infty} c_{Nn} \exp\left[-i\left(n+\frac{1}{2}\right)\omega t\right]\psi_n(\xi)$$

$$c_{Nn} = \sqrt{\frac{N^n \exp(-N)}{n!}}$$

$\left|\Psi_N(\xi,t)\right|^2$

$N = 1$



Coherent state

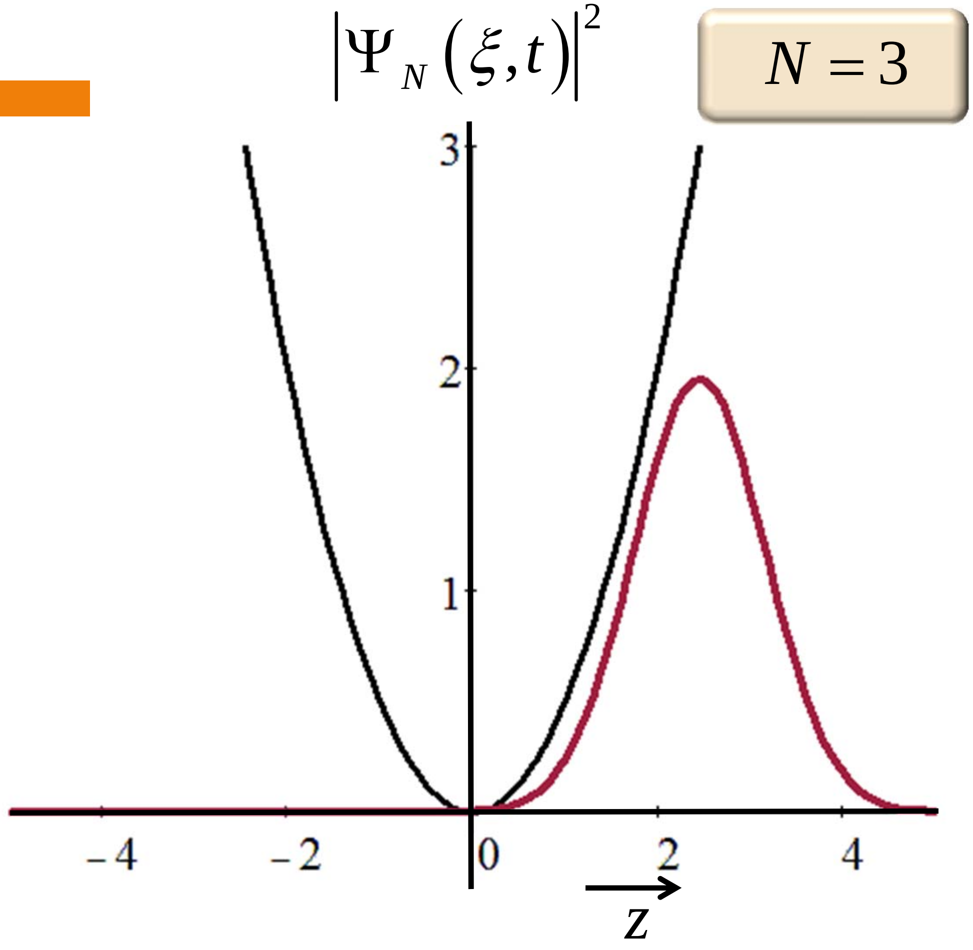
Coherent state oscillations

with

$$\Psi_N(\xi,t) =$$

$$\sum_{n=0}^{\infty} c_{Nn} \exp\left[-i\left(n+\frac{1}{2}\right)\omega t\right]\psi_n(\xi)$$

$$c_{Nn} = \sqrt{\frac{N^n \exp(-N)}{n!}}$$



Coherent state

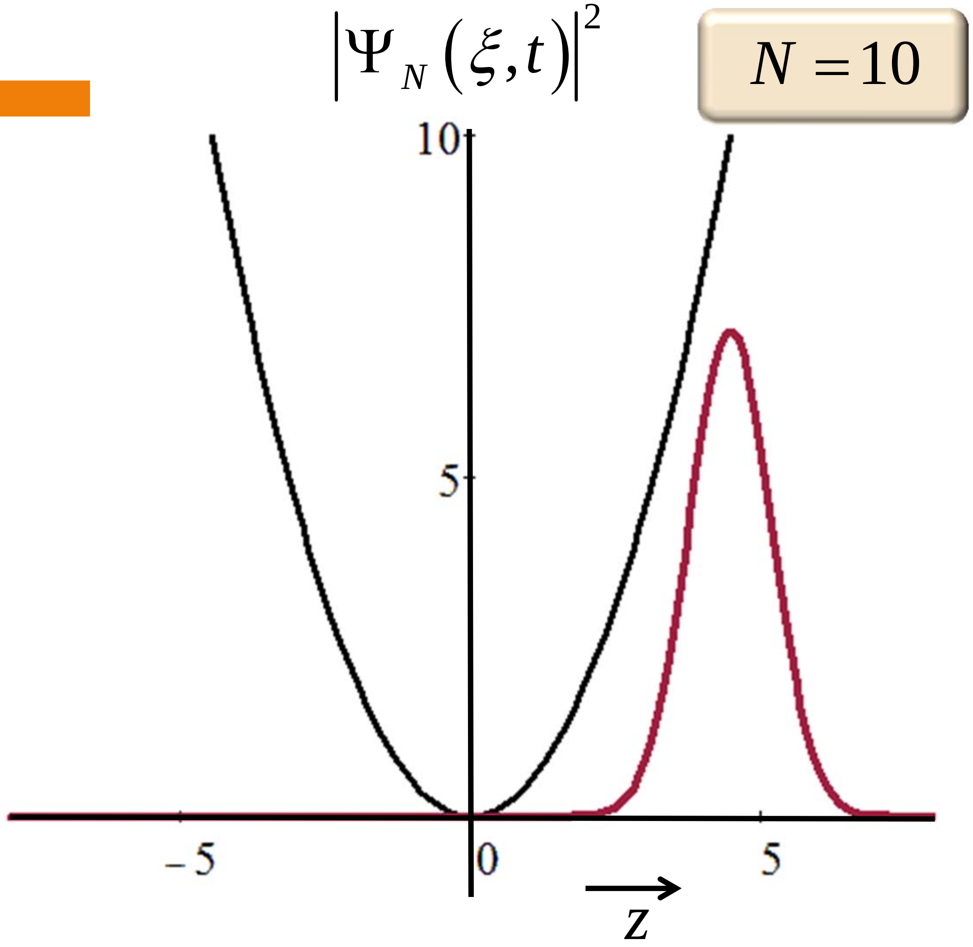
Coherent state oscillations

with

$$\Psi_N(\xi,t) =$$

$$\sum_{n=0}^{\infty} c_{Nn} \exp\left[-i\left(n+\frac{1}{2}\right)\omega t\right]\psi_n(\xi)$$

$$c_{Nn} = \sqrt{\frac{N^n \exp(-N)}{n!}}$$



Coherent state

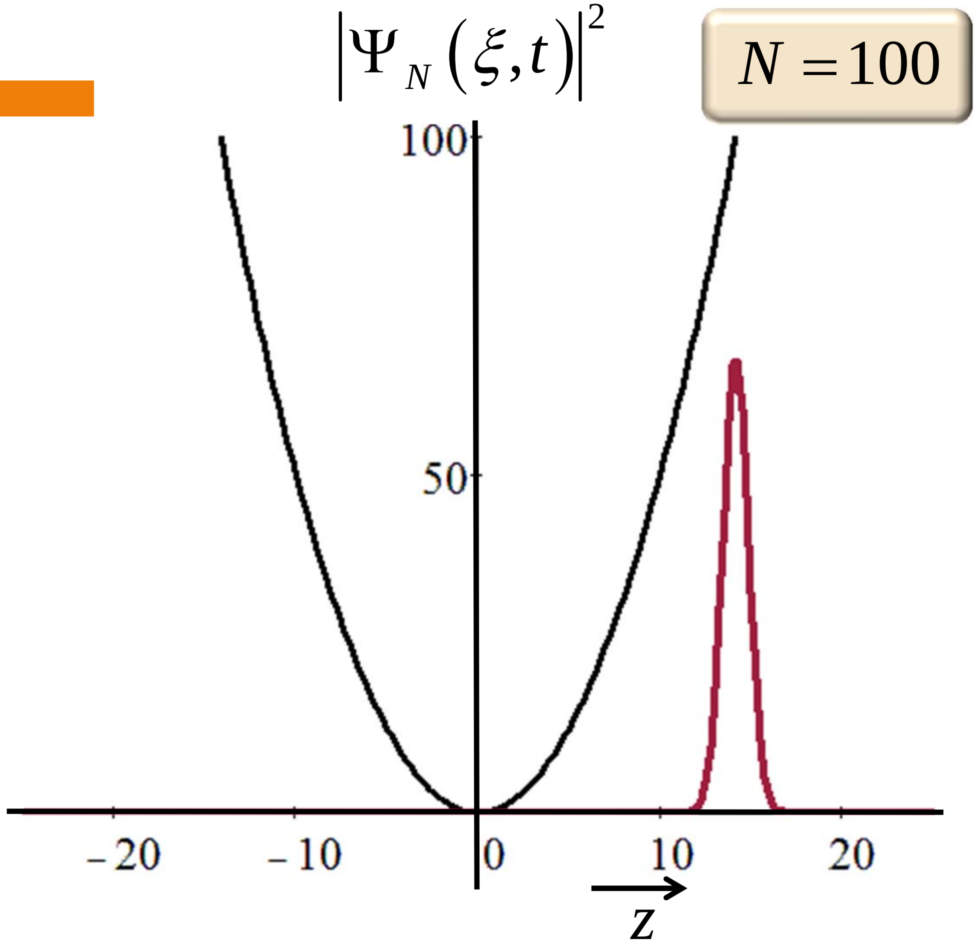
Coherent state oscillations

with

$$\Psi_N(\xi,t) =$$

$$\sum_{n=0}^{\infty} c_{Nn} \exp\left[-i\left(n+\frac{1}{2}\right)\omega t\right]\psi_n(\xi)$$

$$c_{Nn} = \sqrt{\frac{N^n \exp(-N)}{n!}}$$



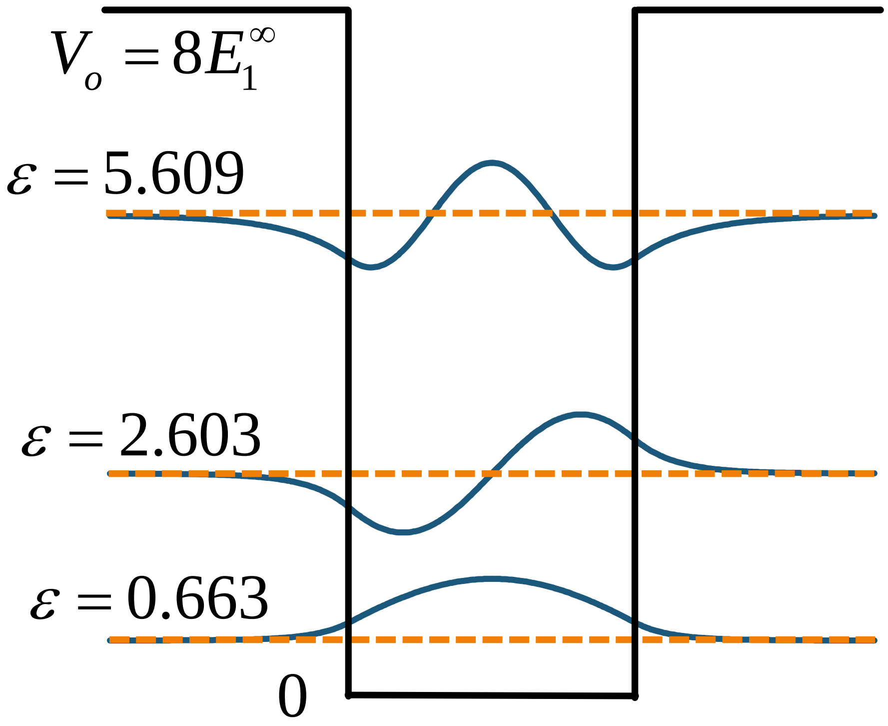
Finite well superposition

Make an equal superposition of the first three states of a finite potential well

as in our previous example

Because the energies are not rationally related

the superposition never repeats



Finite well superposition

Make an equal superposition of the first three states of a finite potential well

as in our previous example

Because the energies are not rationally related

the superposition never repeats

e.g., in the probability density in time