Matrix eigenequations

Background mathematics review

David Miller

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Linear equations and matrices

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Linear equations and matrices

Suppose we have equations for two straight lines

$$A_{11}x + A_{12}y = c_1$$

$$A_{12}x + A_{22}y = c_2$$

Note, if you are used to the form

$$y = mx + c$$

we can rewrite this as

$$y = (-A_{11} / A_{12})x + (c_1 / A_{12})$$

so these are equivalent

We can rewrite these equations as the one matrix equation

$$\hat{A}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

or, with b_1 instead of x and b_2 instead of y

in summation form

$$\sum_{n=1}^{2} A_{mn} b_n = c_m$$

Linear equation solutions

With the linear equations in matrix form

$$\hat{A}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

- we can formally solve them if we know the inverse $\hat{A}^{-1}$
  - multiplying by $\hat{A}^{-1}$

$$\hat{A}^{-1}\hat{A}\begin{bmatrix} x \\ y \end{bmatrix} = \hat{A}^{-1}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

Since $\hat{A}^{-1}\hat{A} = \hat{I}$

- we have the solution
  - the intersection point (x, y) of the lines

$$\begin{bmatrix} x \\ y \end{bmatrix} = \hat{A}^{-1}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

Solving linear equations and inverting a matrix are the same operation

Determinant

If the determinant of a matrix is not zero

then the matrix has an inverse

and if a matrix has an inverse,

the determinant of the matrix is not zero

A nonzero determinant is a

necessary and sufficient condition for a matrix to be invertible

Determinant of a matrix

The determinant of a matrix $\hat{A}$ is written in one of two notations

$$\det\left(\hat{A}\right) = \begin{vmatrix} A_{11} & A_{12} & \cdots & A_{1N} \\ A_{21} & A_{22} & \cdots & A_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ A_{N1} & A_{N2} & \cdots & A_{NN} \end{vmatrix}$$

There are two complete formulas for calculating it

- Leibniz's formula
- Laplace's formula

and many numerical techniques to calculate it

- we will not give these general formulas or methods here

Determinant of a 2x2 matrix

For a 2x2 matrix

$$\det\left(\hat{A}\right) = \begin{vmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{vmatrix} = A_{11}A_{22} - A_{12}A_{21}$$

we add the product on the leading diagonal

and subtract the product on the other diagonal

Determinant of a 3x3 matrix

For a 3x3 matrix, we have

$$\det\left(\hat{A}\right) = \begin{vmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{vmatrix} = \left\{ \begin{array}{l} A_{11}\left(A_{22}A_{33} - A_{23}A_{32}\right) \\ -A_{12}\left(A_{21}A_{33} - A_{23}A_{31}\right) \\ +A_{13}\left(A_{21}A_{32} - A_{22}A_{31}\right) \end{array} \right\}$$

$$\begin{array}{c} + \\ \begin{vmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{vmatrix} \end{array} \qquad \begin{array}{c} - \\ \begin{vmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{vmatrix} \end{array} \qquad \begin{array}{c} + \\ \begin{vmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{vmatrix} \end{array}$$

General form of determinant

If we multiply out the 3x3 determinant expression

$$\det\left(\hat{A}\right) = A_{11}\left(A_{22}A_{33} - A_{23}A_{32}\right) - A_{12}\left(A_{21}A_{33} - A_{23}A_{31}\right) + A_{13}\left(A_{21}A_{32} - A_{22}A_{31}\right)$$

$$= A_{11}A_{22}A_{33} - A_{11}A_{23}A_{32} - A_{12}A_{21}A_{33} + A_{12}A_{23}A_{31} + A_{13}A_{21}A_{32} - A_{13}A_{22}A_{31}$$

we see that each term, e.g., $A_{12}A_{23}A_{31}$ contains a different element from each row

and the elements in each term are never from the same column

but we always have one element from each row and each column in each term

General form of the determinant

We see that this form

$$\det\left(\hat{A}\right) = A_{11}A_{22}A_{33} - A_{11}A_{23}A_{32} - A_{12}A_{21}A_{33} + A_{12}A_{23}A_{31} + A_{13}A_{21}A_{32} - A_{13}A_{22}A_{31}$$

contains every possible term with one element from each row

all from different columns

this is a general property of determinants

General form of the determinant

- To understand how to construct a determinant
  - it only remains to find the sign of the terms
- To do so
  - count the number of adjacent row (or column) “swaps” required
    - to get all the elements in the term onto the leading diagonal
      - if that number is even, the sign is “+”
      - if that number is odd, the sign is “-”

Sign of determinant terms

For the term $A_{11}A_{23}A_{32}$

$$\begin{vmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{vmatrix} \Rightarrow \begin{vmatrix} A_{11} & A_{12} & A_{13} \\ A_{31} & A_{32} & A_{33} \\ A_{21} & A_{22} & A_{23} \end{vmatrix}$$

we have to perform 1 row swap

1 is an odd number

so the sign of this term in the determinant is negative

Sign of determinant terms

For the term $A_{12}A_{23}A_{31}$

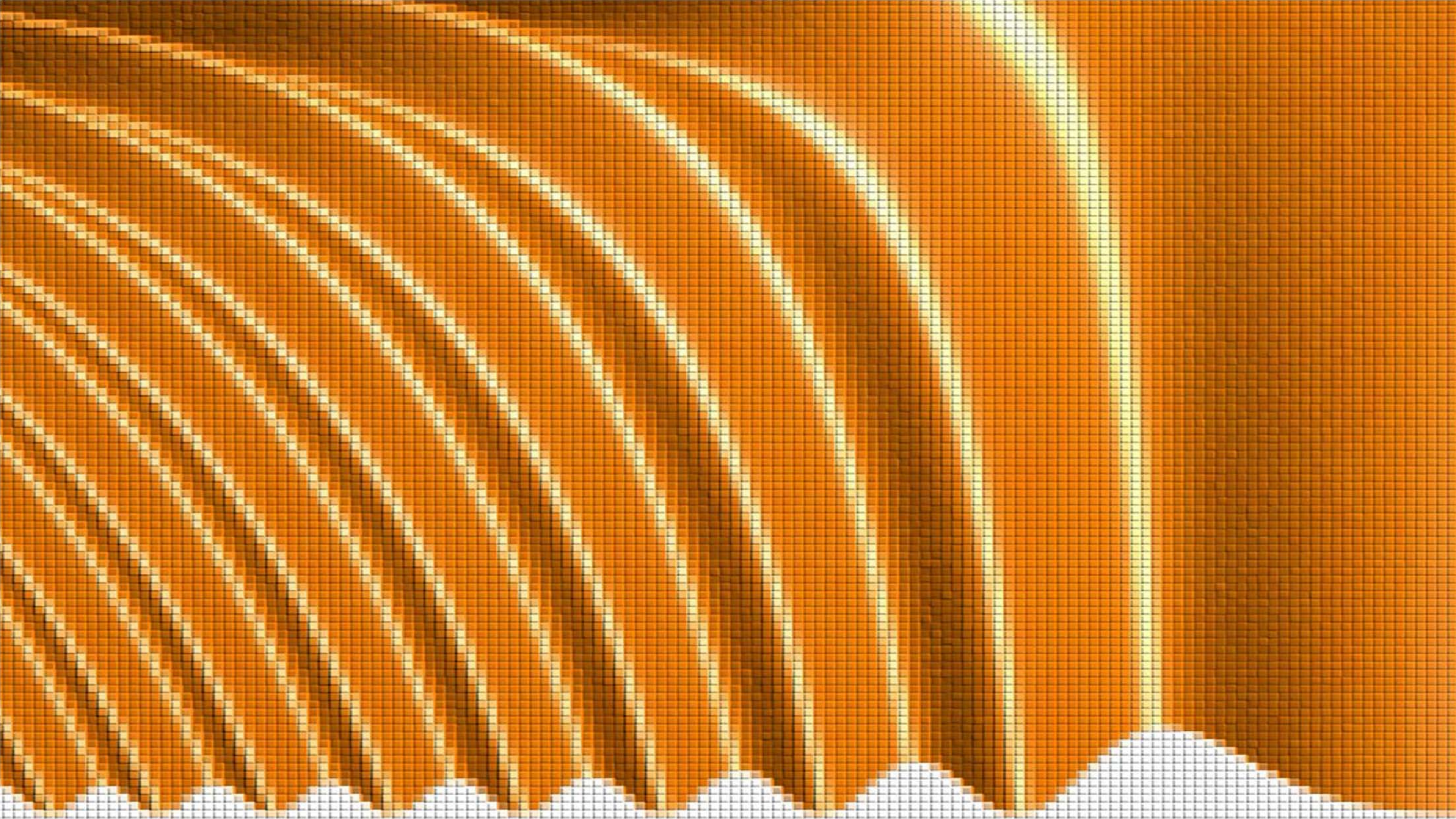
$$\begin{vmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{vmatrix} \Rightarrow \begin{vmatrix} A_{11} & A_{12} & A_{13} \\ A_{31} & A_{32} & A_{33} \\ A_{21} & A_{22} & A_{23} \end{vmatrix} \Rightarrow \begin{vmatrix} A_{31} & A_{32} & A_{33} \\ A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \end{vmatrix}$$

we have to perform 2 row swaps

2 is an even number

so the sign of this term in the determinant is positive

This is actually Leibniz’s determinant formula



Matrix eigenequations

Eigenvalues and eigenvectors

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Matrix eigenequation

An equation of the form

$$\hat{A}d = \lambda d$$

where d is a vector, λ is a number, and $\hat{A}$ is a square matrix

is called an

eigenequation

with eigenvalue λ

and eigenvector d

If there are solutions

they may only exist for specific values of λ

Solving an eigenequation

We can rewrite $\hat{A}d = \lambda d$ as

$$\hat{A}d = \lambda \hat{I}d$$

where we have introduced the identity matrix $\hat{I}$

which we can always do because $\hat{I}d = d$

So

$$\left(\hat{A} - \lambda \hat{I}\right)d = 0$$

(strictly, the 0 here is a vector with elements 0)

so, writing $\hat{B} = \hat{A} - \lambda \hat{I}$ we have

$$\hat{B}d = 0$$

Solving an eigenequation

Now, for $\hat{B}d = 0$ to have any solutions for any non-zero vector d

the matrix $\hat{B}$ cannot have an inverse

if it did have an inverse $\hat{B}^{-1}$

$$\hat{B}^{-1}\hat{B}d = \hat{I}d = d = \hat{B}^{-1}0$$

but any (finite) matrix multiplying a zero vector must give a zero vector

so there is no non-zero solution d

Hence, by *reductio ad absurdum*,

$\hat{B}$ has no inverse

Solving an eigenequation

The fact that $\hat{B} = \hat{A} - \lambda\hat{I}$ has no inverse means

from the properties of the determinant

$$\det\left(\hat{A} - \lambda\hat{I}\right) = 0$$

This equation will allow us to construct

a "secular equation"

whose solutions will give the eigenvalues λ

From those we will deduce the corresponding eigenvectors d

Solving an eigenequation

Suppose we want to find the eigenvalues and eigenvectors

if they exist

of the matrix $\hat{A} = \begin{bmatrix} 1.5 & -0.5i \\ 0.5i & 1.5 \end{bmatrix}$

So we write the determinant condition for finding eigenvalues

$$\det\left(\hat{A} - \lambda \hat{I}\right) = \det\left(\begin{bmatrix} 1.5 & -0.5i \\ 0.5i & 1.5 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Solving an eigenequation

Now

$$\det\left(\begin{bmatrix}1.5 & -0.5i \\ 0.5i & 1.5\end{bmatrix} - \lambda\begin{bmatrix}1 & 0 \\ 0 & 1\end{bmatrix}\right) = \det\left(\begin{bmatrix}1.5 & -0.5i \\ 0.5i & 1.5\end{bmatrix} - \begin{bmatrix}\lambda & 0 \\ 0 & \lambda\end{bmatrix}\right)$$

$$= \det\left(\begin{bmatrix}1.5-\lambda & -0.5i \\ 0.5i & 1.5-\lambda\end{bmatrix}\right)$$

So our secular equation becomes, from $\det\left(\hat{A} - \lambda\hat{I}\right) = 0$

$$\left(1.5-\lambda\right)^2 - (0.5i)\left(-0.5i\right) = \left(1.5-\lambda\right)^2 - 0.25 = 0$$

i.e., $\lambda^2 - 3\lambda + 2 = 0$

Solving an eigenequation

Solving this quadratic equation $\lambda^2 - 3\lambda + 2 = 0$

gives roots

$\lambda_1 = 1$ and $\lambda_2 = 2$

Now that we know the eigenvalues

we substitute them back into the eigenequation

and deduce the corresponding eigenvectors

Solving an eigenequation

Our eigenequation $\hat{A}d = \lambda d$ is, explicitly

$$\begin{bmatrix} 1.5 & -0.5i \\ 0.5i & 1.5 \end{bmatrix}\begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = \lambda \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$$

where now, for a given eigenvalue λ

we are trying to find d_1 and d_2

so we know the corresponding eigenvector

Rewriting gives

$$\begin{bmatrix} 1.5-\lambda & -0.5i \\ 0.5i & 1.5-\lambda \end{bmatrix}\begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Solving an eigenequation

Now evaluating $\begin{bmatrix} 1.5-\lambda & -0.5i \\ 0.5i & 1.5-\lambda \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

for a specific eigenvalue

say, the first one, $\lambda_1 = 1$

gives $\begin{bmatrix} 0.5 & -0.5i \\ 0.5i & 0.5 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

or, as linear equations

$$0.5d_1 - 0.5id_2 = 0$$

$$0.5id_1 + 0.5d_2 = 0$$

Solving an eigenequation

From either one of these equations

$$0.5d_1 - 0.5id_2 = 0 \quad 0.5id_1 + 0.5d_2 = 0$$

we can now deduce the eigenvector

Either equation gives us $d_2 = id_1$

We are free to choose one of the elements

say, choose $d_1 = 1$

which gives the eigenvector v_1

$$v_1 = \begin{bmatrix} 1 \\ i \end{bmatrix}$$

Using $\lambda_2 = 2$ and similar mathematics

gives the other eigenvector v_2

$$v_2 = \begin{bmatrix} 1 \\ -i \end{bmatrix}$$

Solving eigenequations

For larger matrices with eigensolutions

- e.g., $N \times N$
  - we have correspondingly higher order polynomial secular equations
    - which can have N eigenvalues and eigenvectors
- e.g., a 3×3 matrix can have 3 eigenvalues and eigenvectors

Note eigenvectors can be multiplied by any constant and still be eigenvectors