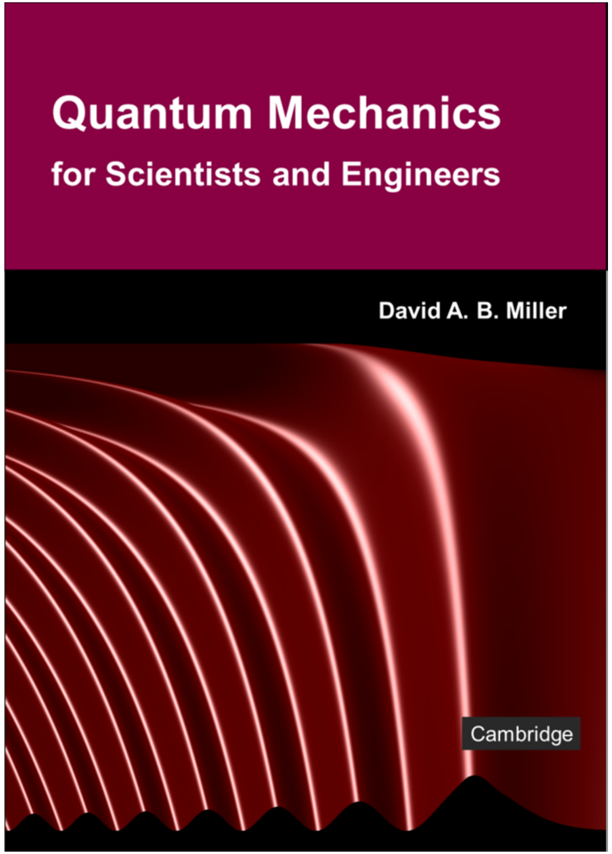
3.2 Finite well and harmonic oscillator

Slides: Video 3.2.2 The finite potential well

Text reference: Quantum Mechanics for Scientists and Engineers

Section 2.9



Particles in potential wells

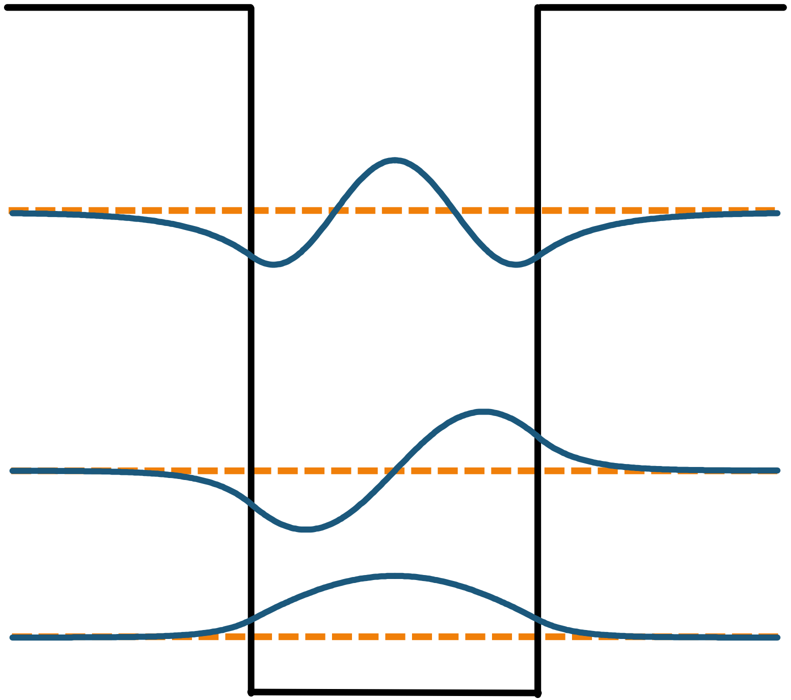
The finite potential well

Quantum mechanics for scientists and engineers

David Miller

Insert video here (split screen)

Finite potential well



Lesson 7

Particles in potential wells

Insert number 2

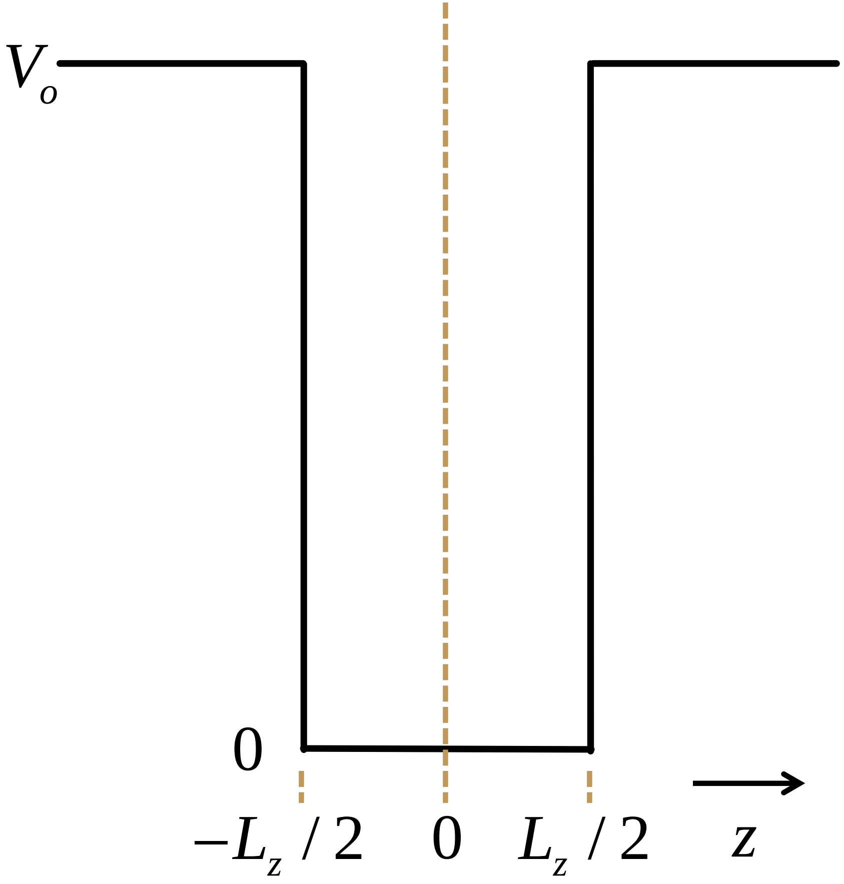
Particle in a finite potential well

We will choose the height of the potential barriers as V_o

with 0 potential energy at the bottom of the well

The thickness of the well is L_z

Now we will choose the position origin in the center of the well



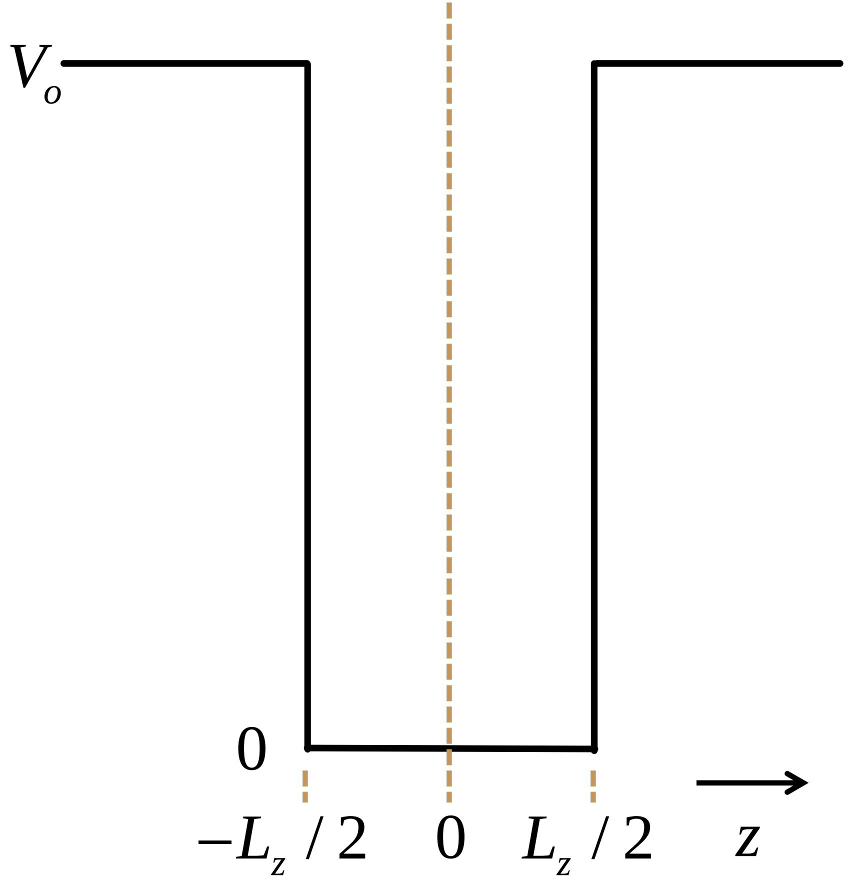
Particle in a finite potential well

If there is an eigenenergy E for which there is a solution

then we already know what form the solution has to take

sinusoidal in the middle

exponentially decaying on either side



Particle in a finite potential well

For some eigenenergy E

with $k = \sqrt{2mE/\hbar^2}$

and $\kappa = \sqrt{2m\left(V_o - E\right)/\hbar^2}$

for $z < -L_z/2$

$$\psi(z) = G\exp(\kappa z)$$

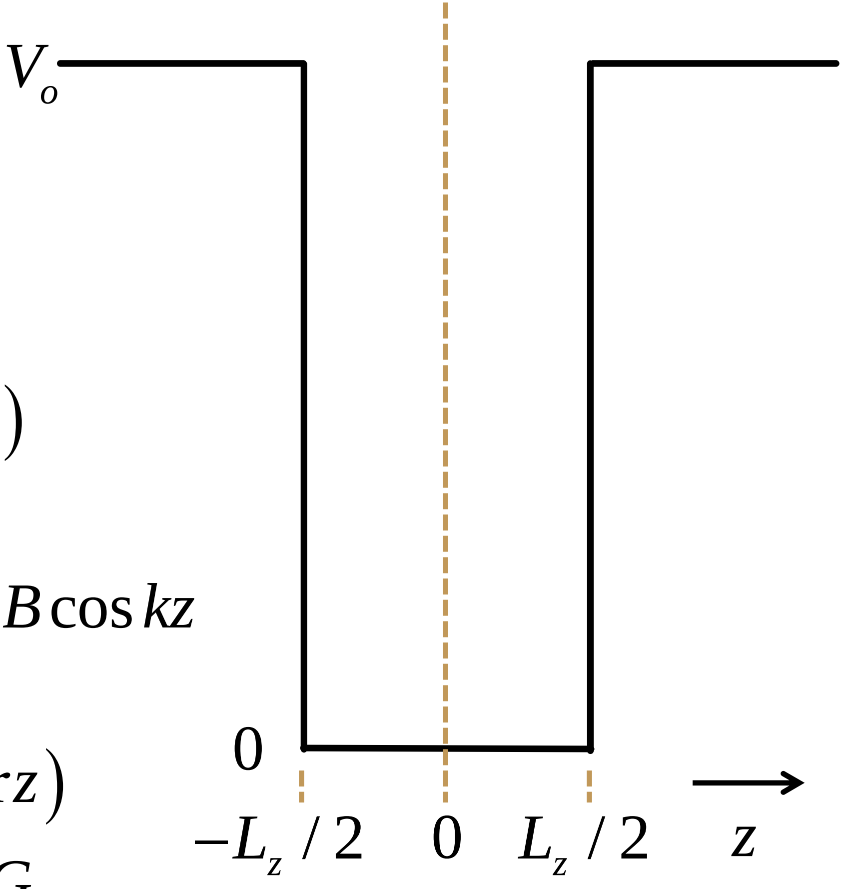
for $-L_z/2 < z < L_z/2$

$$\psi(z) = A\sin kz + B\cos kz$$

for $z > L_z/2$

$$\psi(z) = F\exp(-\kappa z)$$

with constants A, B, F, and G



Particle in a finite potential well

Now we need to apply the boundary conditions to solve for the unknown coefficients

constants A, B, F, and G

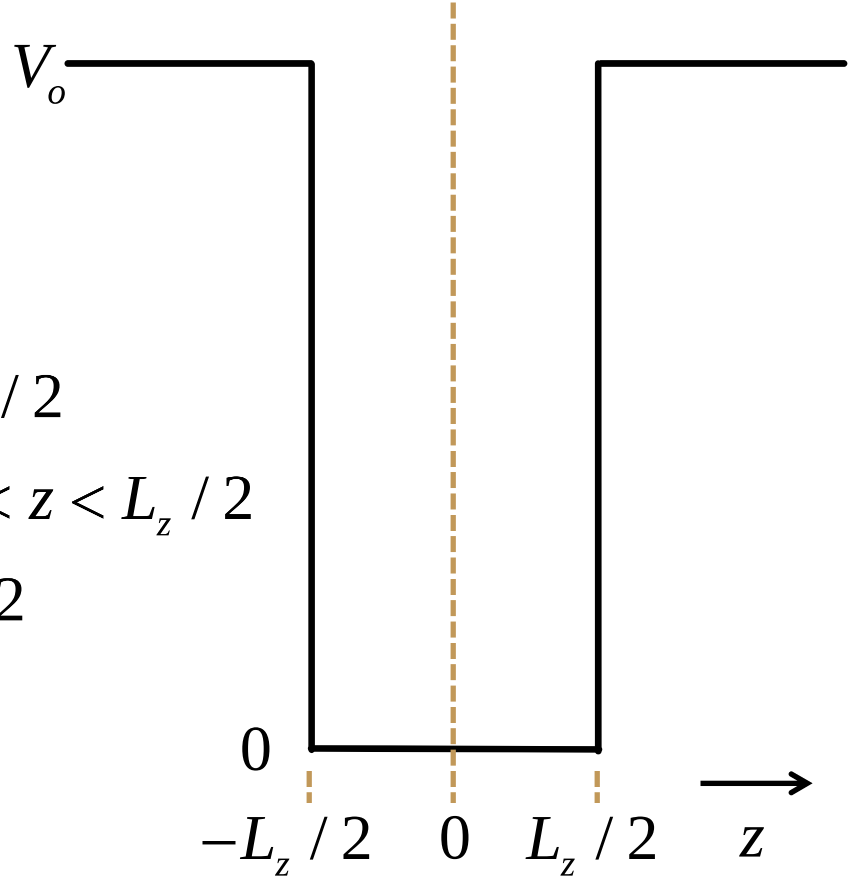
$$\psi(z) = G\exp(\kappa z) \qquad z < -L_z/2$$

$$\psi(z) = A\sin kz + B\cos kz \qquad -L_z/2 < z < L_z/2$$

$$\psi(z) = F\exp(-\kappa z) \qquad z > L_z/2$$

or at least three of them

the fourth could be found by normalization



Particle in a finite potential well

From continuity of the wavefunction at $z = L_z / 2$

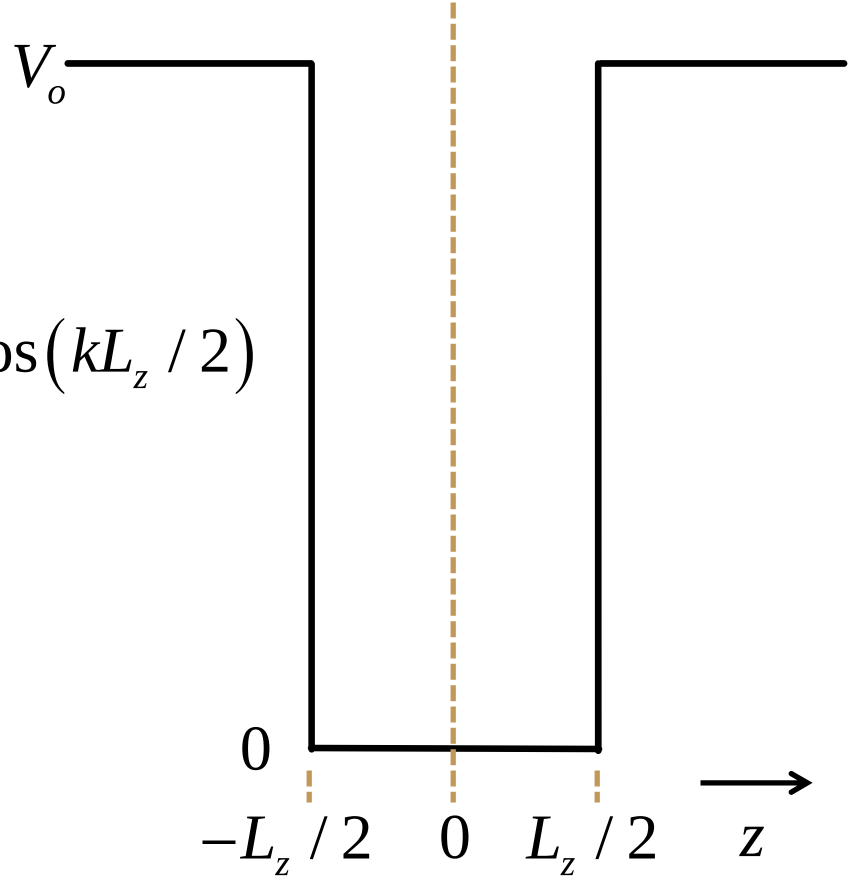
$$\psi\left(L_z / 2\right) = F \exp\left(-\kappa L_z / 2\right)$$
$$= A \sin\left(k L_z / 2\right) + B \cos\left(k L_z / 2\right)$$

Writing $X_L = \exp\left(-\kappa L_z / 2\right)$

$$S_L = \sin\left(k L_z / 2\right)$$
$$C_L = \cos\left(k L_z / 2\right)$$

gives

$$F X_L = A S_L + B C_L$$



Particle in a finite potential well

Similarly at $z = -L_z / 2$

$$GX_L = -AS_L + BC_L$$

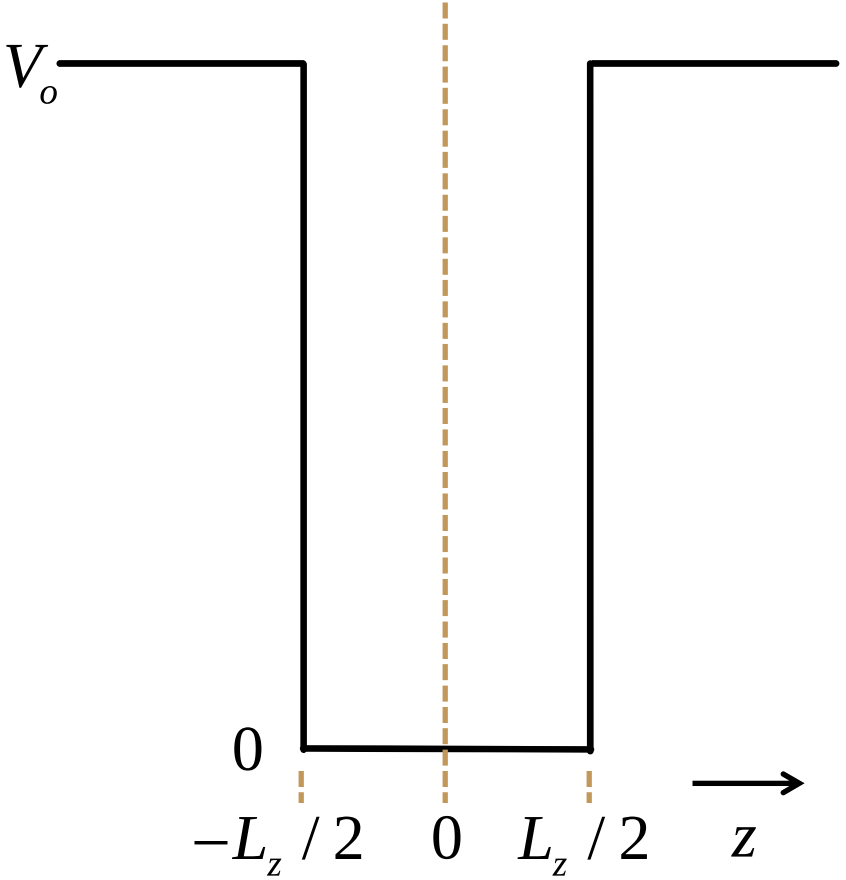
Continuity of the derivative gives

at $z = -L_z / 2$

$$\frac{\kappa}{k} GX_L = AC_L + BS_L$$

at $z = L_z / 2$

$$-\frac{\kappa}{k} FX_L = AC_L - BS_L$$



Particle in a finite potential well

So we have four relations

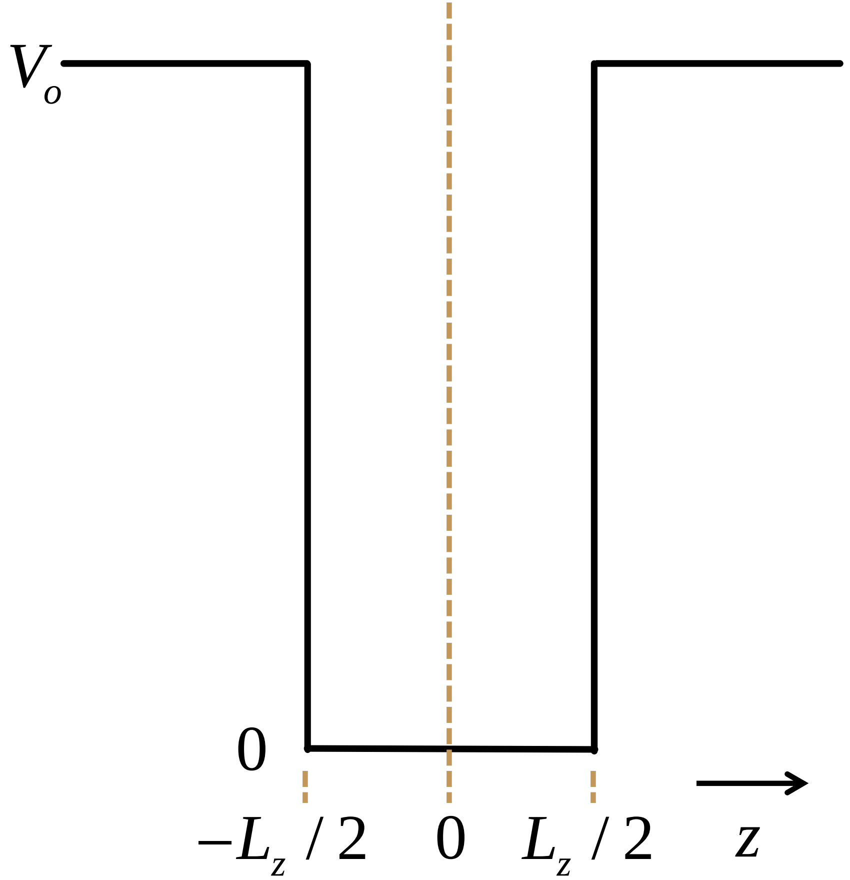
$$GX_L = -AS_L + BC_L$$

$$FX_L = AS_L + BC_L$$

$$\frac{\kappa}{k} GX_L = AC_L + BS_L$$

$$-\frac{\kappa}{k} FX_L = AC_L - BS_L$$

Now we need to find what solutions are compatible with these



Particle in a finite potential well

Adding $GX_L = -AS_L + BC_L$

$$FX_L = AS_L + BC_L$$

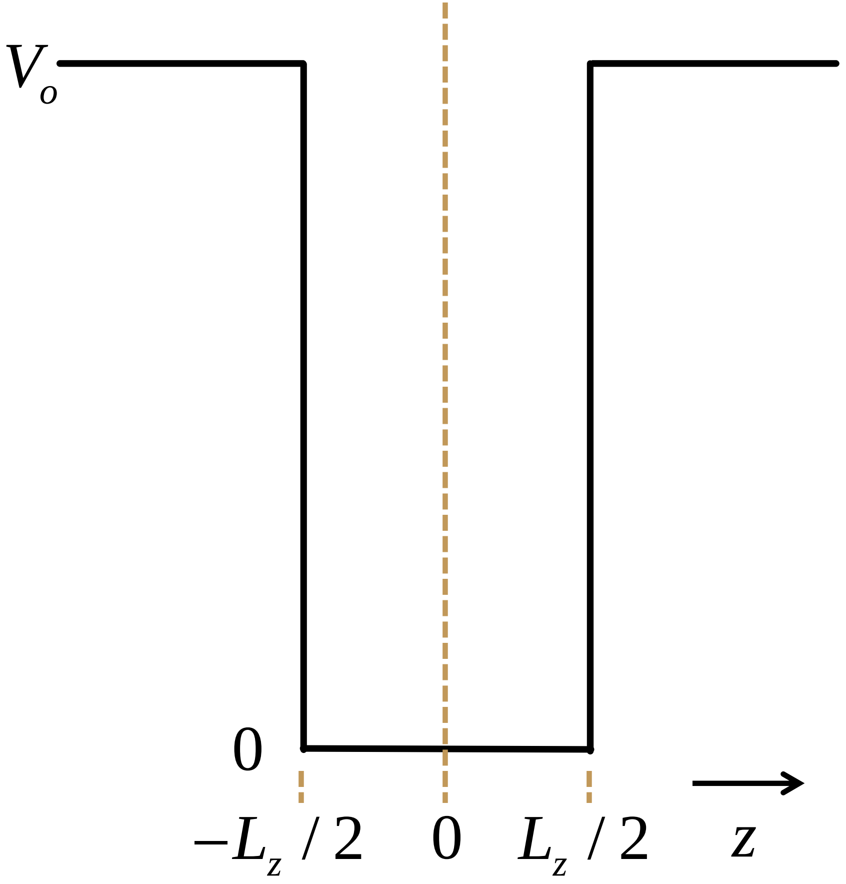
gives $2BC_L = \left(F+G\right)X_L$

Subtracting

$$-\frac{\kappa}{k}FX_L = AC_L - BS_L$$

from $\frac{\kappa}{k}GX_L = AC_L + BS_L$

gives $2BS_L = \frac{\kappa}{k}\left(F+G\right)X_L$



Particle in a finite potential well

As long as $F \neq -G$

we can divide

$$2BS_L = \frac{\kappa}{k}(F+G)X_L$$

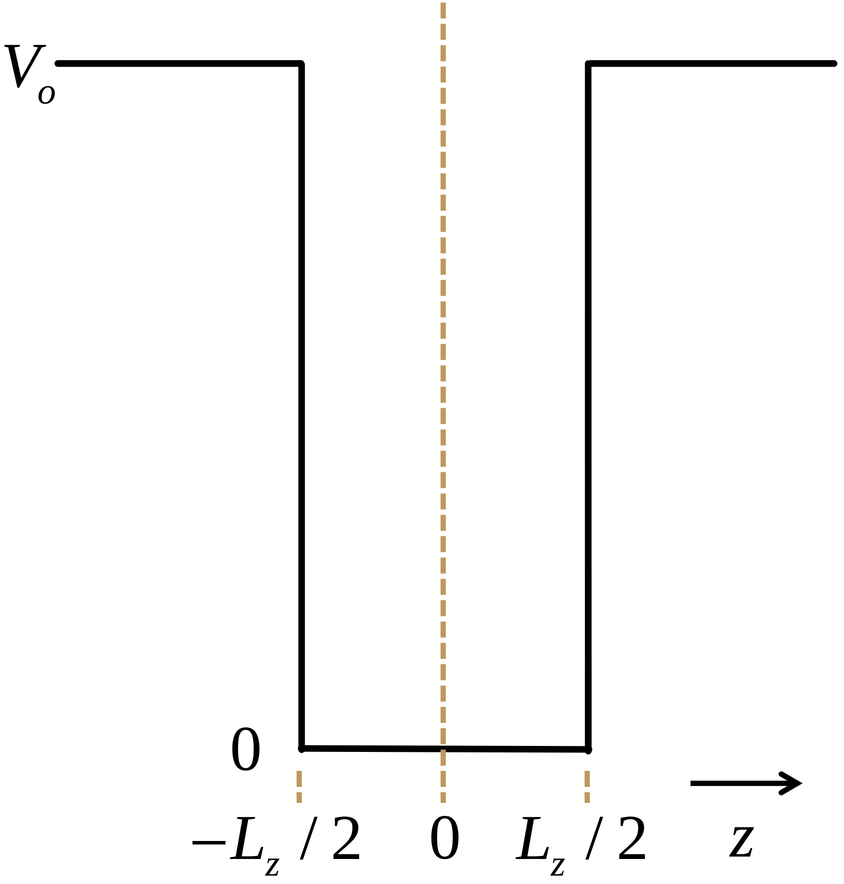
by

$$2BC_L = (F+G)X_L$$

to obtain

$$\tan(kL_z/2) = \kappa/k$$

This relation is effectively a condition for eigenvalues



Particle in a finite potential well

Subtracting $GX_L = -AS_L + BC_L$

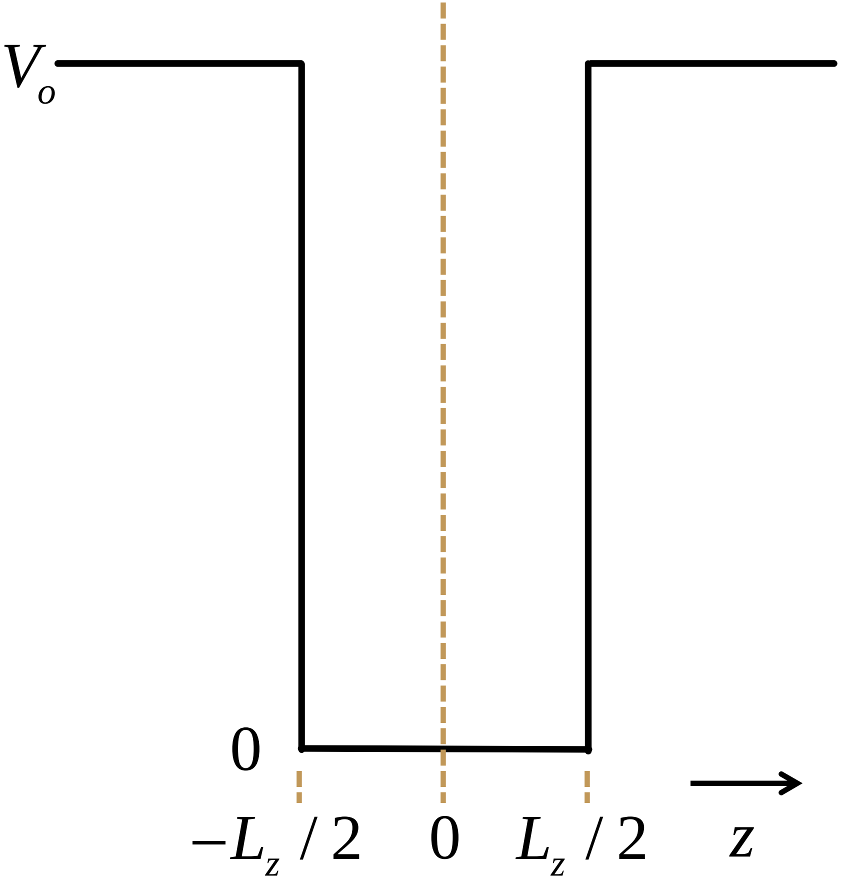
from $FX_L = AS_L + BC_L$

gives $2AS_L = (F - G)X_L$

Adding $-\frac{\kappa}{k}FX_L = AC_L - BS_L$

and $\frac{\kappa}{k}GX_L = AC_L + BS_L$

gives $2AC_L = -\frac{\kappa}{k}(F - G)X_L$



Particle in a finite potential well

Similarly, as long as $F \neq G$

we can divide

$$2AC_L = -\frac{\kappa}{k}\left(F - G\right)X_L$$

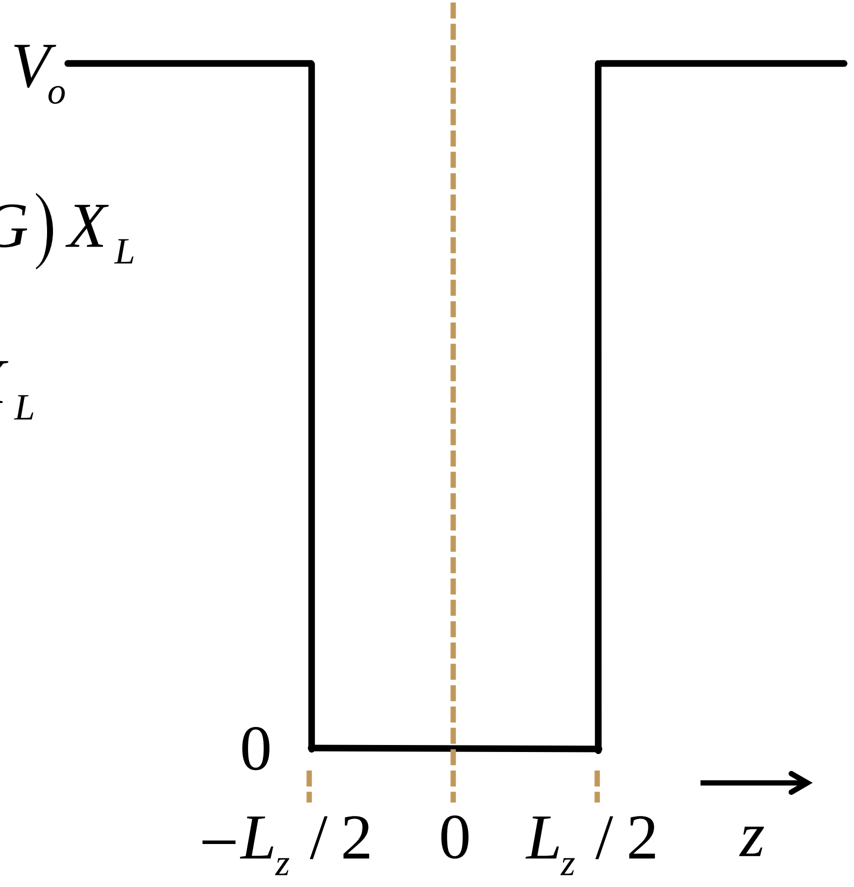
by

$$2AS_L = \left(F - G\right)X_L$$

to obtain

$$-\cot\left(kL_z/2\right) = \kappa/k$$

This relation is also effectively a condition for eigenvalues



Particle in a finite potential well

For any case other than $F = G$

which leaves $\tan\left(kL_z / 2\right) = \kappa / k$

but not $-\cot\left(kL_z / 2\right) = \kappa / k$

or $F = -G$

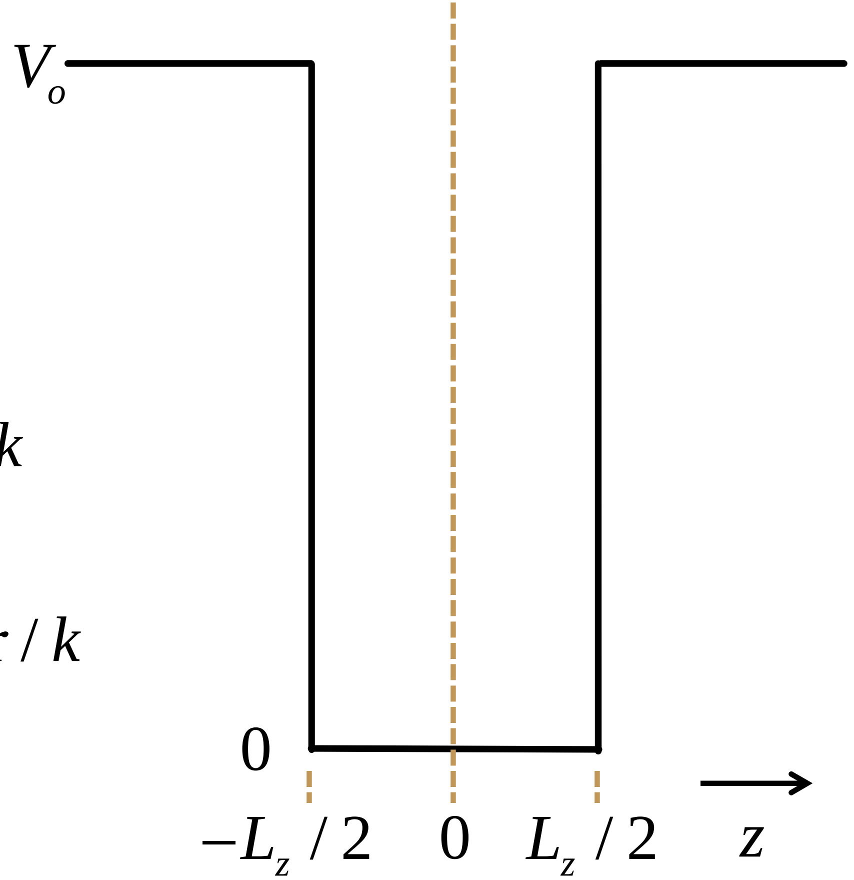
which leaves $-\cot\left(kL_z / 2\right) = \kappa / k$

but not $\tan\left(kL_z / 2\right) = \kappa / k$

then the solutions $\tan\left(kL_z / 2\right) = \kappa / k$

and $-\cot\left(kL_z / 2\right) = \kappa / k$

are contradictory



Particle in a finite potential well

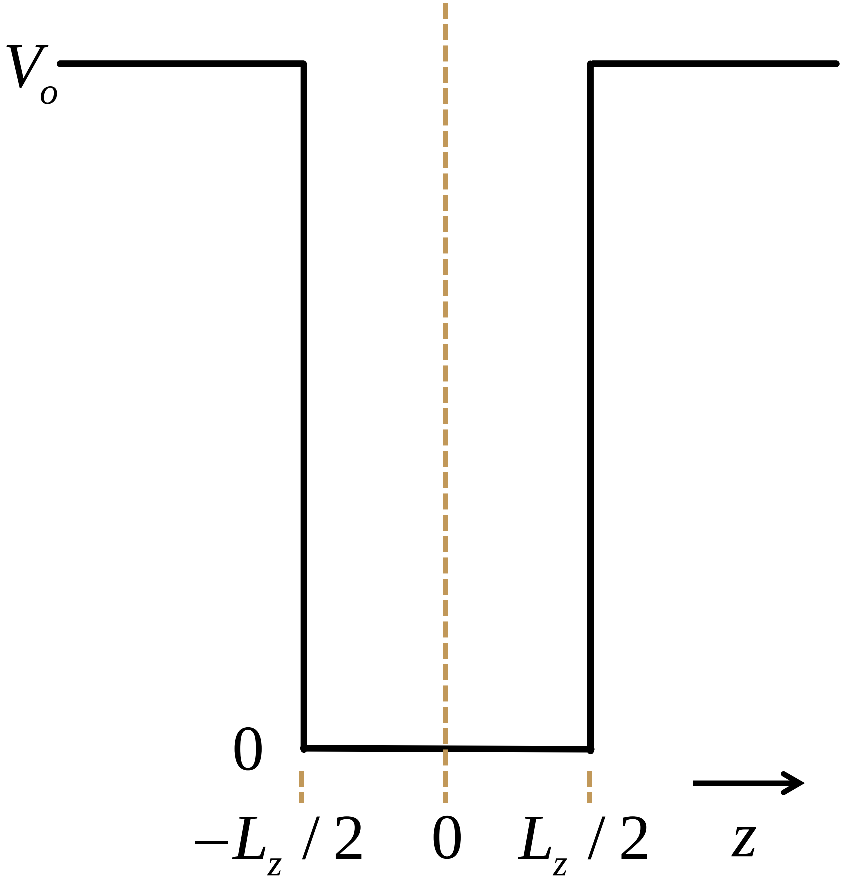
So the only possibilities are

1 - $F = G$

and $\tan\left(kL_z / 2\right) = \kappa / k$

2 – $F = -G$

and $-\cot\left(kL_z / 2\right) = \kappa / k$



Particle in a finite potential well

1 - $F = G$

and $\tan\left(kL_z / 2\right) = \kappa / k$

Note from $2AS_L = \left(F - G\right) X_L$

and $2AC_L = -\frac{\kappa}{k}\left(F - G\right) X_L$

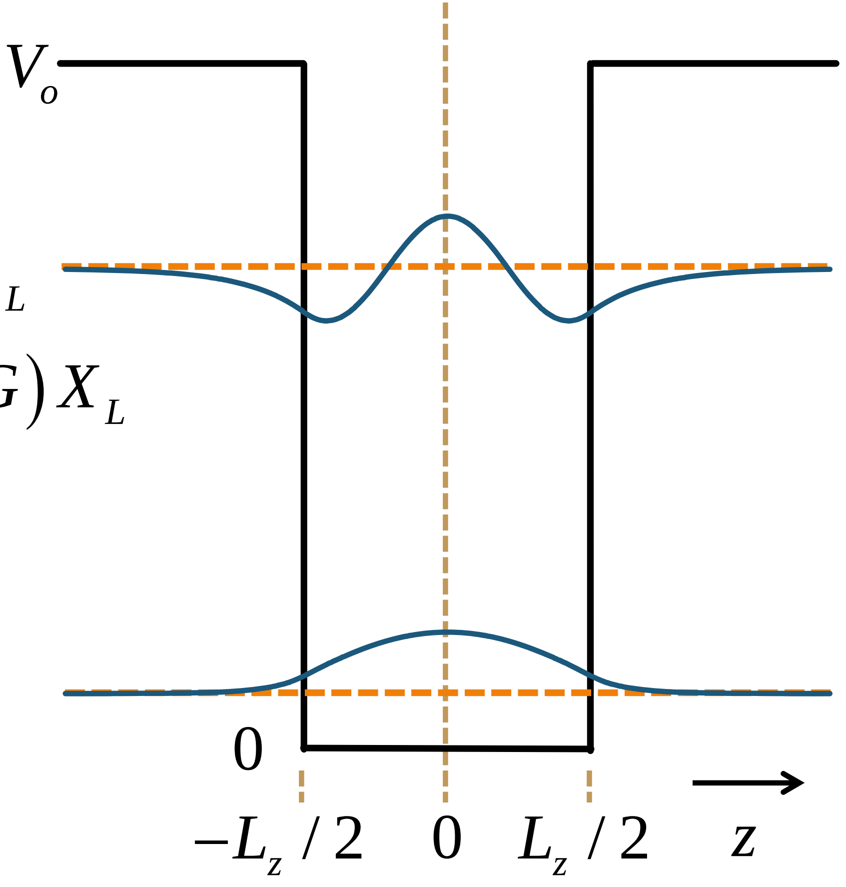
S_L and C_L cannot both be 0

so $A = 0$

Hence in the well we have

$$\psi\left(z\right) \propto \cos kz$$

which is an even function



Particle in a finite potential well

1 - $F = -G$

and $-\cot\left(kL_z / 2\right) = \kappa / k$

Note from $2BC_L = \left(F + G\right) X_L$

and $2BS_L = \dfrac{\kappa}{k}\left(F + G\right) X_L$

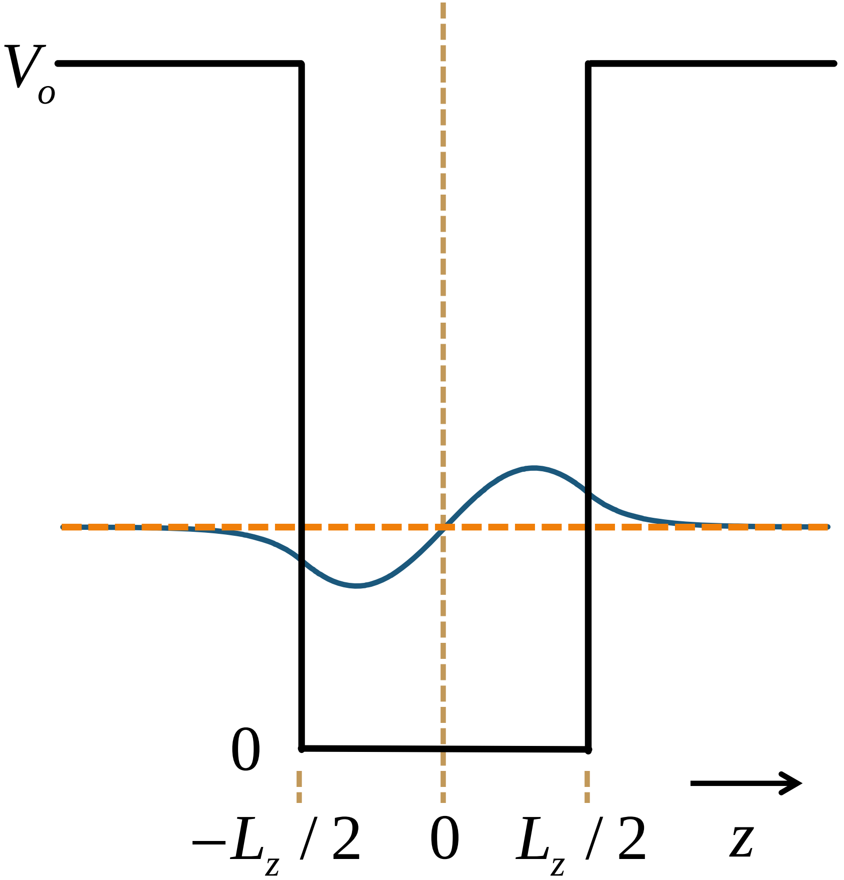
S_L and C_L cannot both be 0

so $B = 0$

Hence in the well we have

$$\psi\left(z\right) \propto \sin kz$$

which is an odd function



Particle in a finite potential well

Though we have found the nature of the solutions

we have not yet formally solved for the eigenenergies E

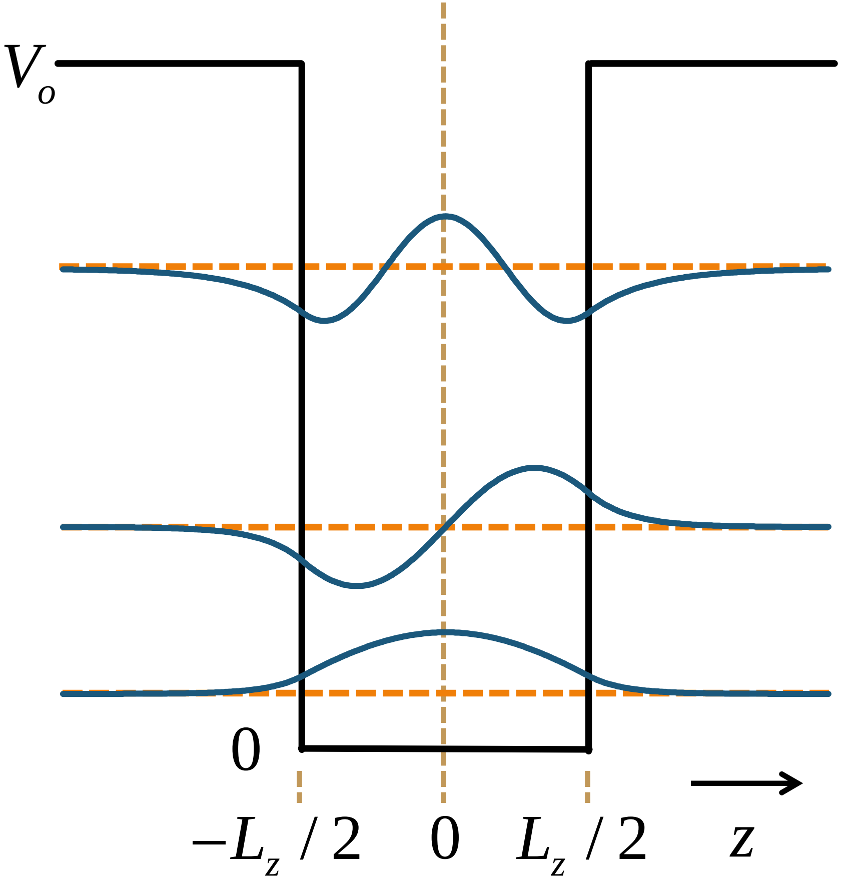
and hence for k and κ

We do this by solving

$$\tan\left(kL_z/2\right) = \kappa/k$$

and

$$-\cot\left(kL_z/2\right) = \kappa/k$$



Solving for the eigenenergies

Change to "dimensionless" units

Use the energy of the first level in the "infinite" potential well width L_z

$$E_1^\infty = \frac{\hbar^2}{2m}\left(\frac{\pi}{L_z}\right)^2$$

leading to a dimensionless eigenenergy

$$\varepsilon \equiv E / E_1^\infty$$

and a dimensionless barrier height

$$v_o \equiv V_o / E_1^\infty$$

Also

$$k = \sqrt{2mE/\hbar^2} = (\pi/L_z)\sqrt{E/E_1^\infty} = (\pi/L_z)\sqrt{\varepsilon}$$

$$\kappa = \sqrt{2m(V_o - E)/\hbar^2} = (\pi/L_z)\sqrt{(V_o - E)/E_1^\infty} = (\pi/L_z)\sqrt{v_o - \varepsilon}$$

Solving for the eigenenergies

Consequently $\dfrac{\kappa}{k} = \sqrt{\dfrac{V_o - E}{E}} = \sqrt{\dfrac{v_o - \varepsilon}{\varepsilon}}$

$$\frac{kL_z}{2} = \frac{\pi}{2}\sqrt{\frac{E}{E_1^\infty}} = \frac{\pi}{2}\sqrt{\varepsilon} \quad \text{and} \quad \frac{\kappa L_z}{2} = \frac{\pi}{2}\sqrt{\frac{V_o - E}{E_1^\infty}} = \frac{\pi}{2}\sqrt{v_o - \varepsilon}$$

So $\tan\left(kL_z/2\right) = \kappa/k$ becomes $\tan\left[\left(\pi/2\right)\sqrt{\varepsilon}\right] = \sqrt{\left(v_o - \varepsilon\right)/\varepsilon}$

or $\boxed{\sqrt{\varepsilon}\tan\left[\left(\pi/2\right)\sqrt{\varepsilon}\right] = \sqrt{\left(v_o - \varepsilon\right)}}$

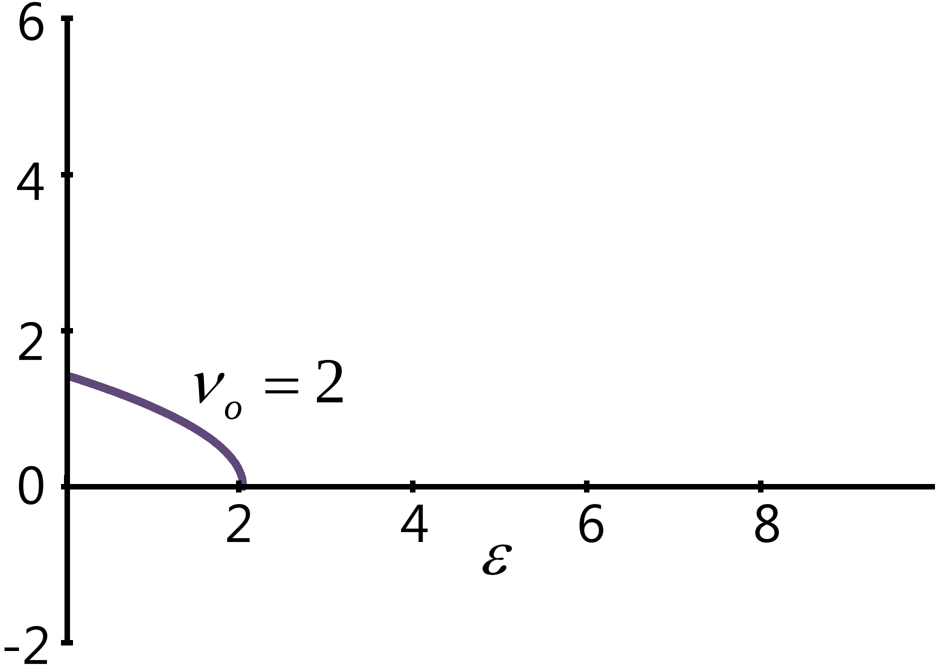
and $-\cot\left(kL_z/2\right) = \kappa/k$ becomes $-\cot\left[\left(\pi/2\right)\sqrt{\varepsilon}\right] = \sqrt{\left(v_o - \varepsilon\right)/\varepsilon}$

or $\boxed{-\sqrt{\varepsilon}\cot\left[\left(\pi/2\right)\sqrt{\varepsilon}\right] = \sqrt{\left(v_o - \varepsilon\right)}}$

Graphical solution

Choose a specific well depth v_o and plot the curve

$$\sqrt{(v_o - \varepsilon)}$$

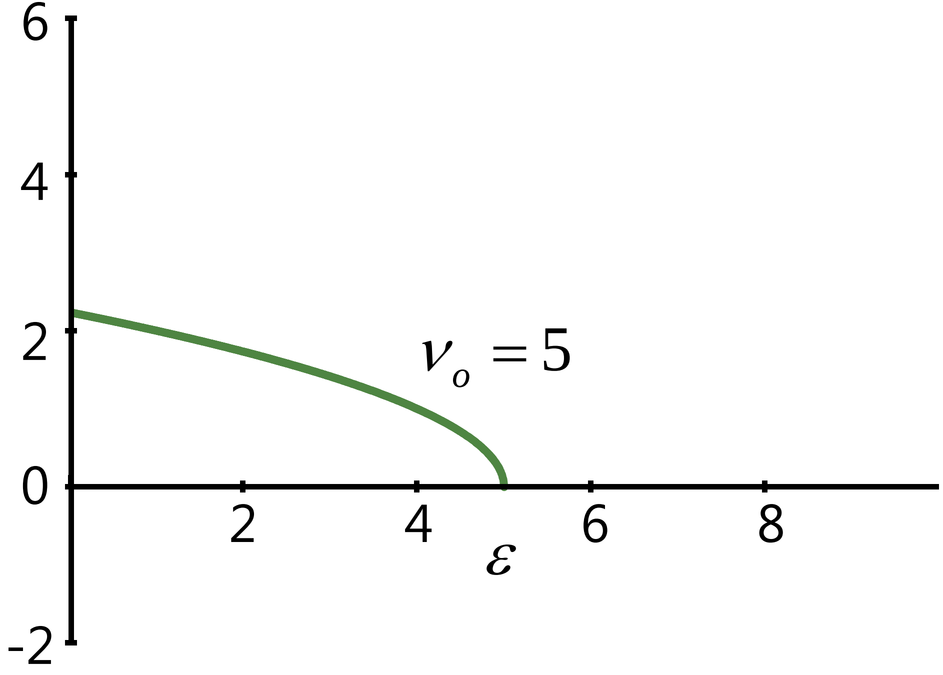


Graphical solution

Choose a specific well depth v_o

and plot the curve

$$\sqrt{\left(v_o - \varepsilon\right)}$$

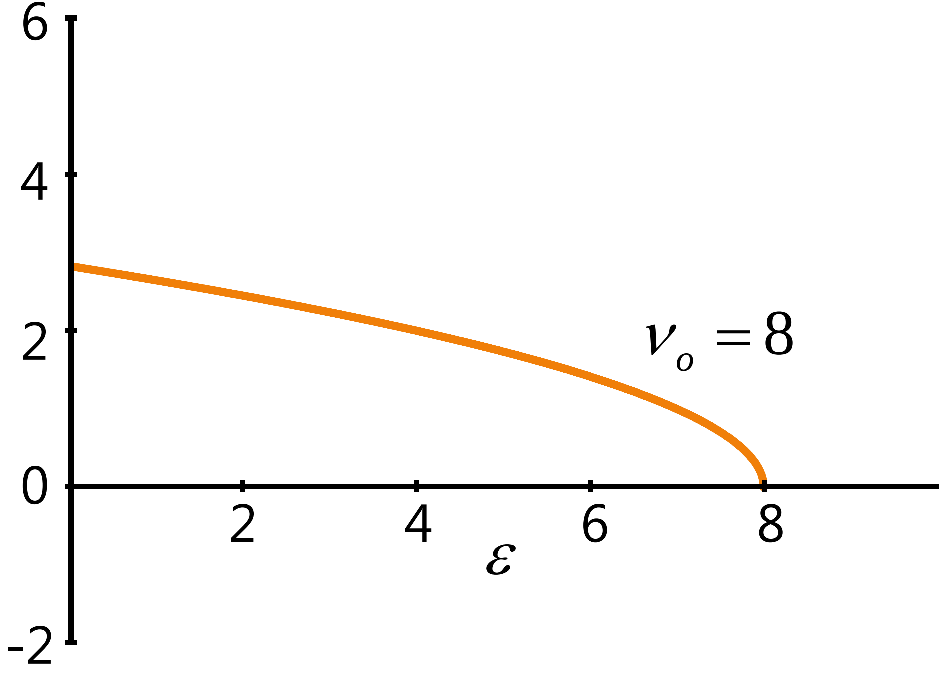


Graphical solution

Choose a specific well depth v_o

and plot the curve

$$\sqrt{(v_o - \varepsilon)}$$



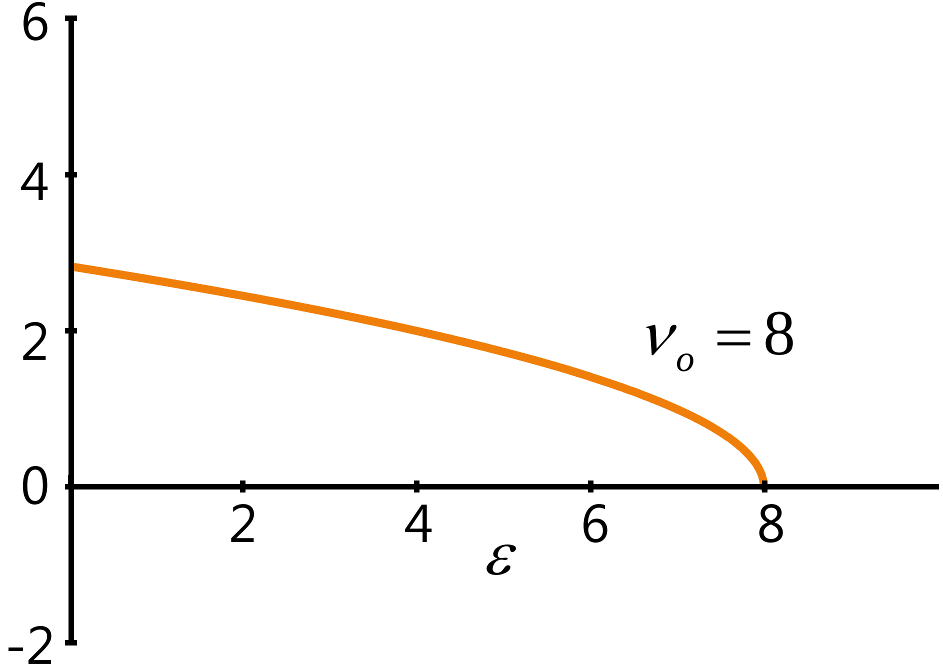
Graphical solution

Choose a specific well depth v_o

and plot the curve

$$\sqrt{(v_o - \varepsilon)}$$

Now add the curves



Graphical solution

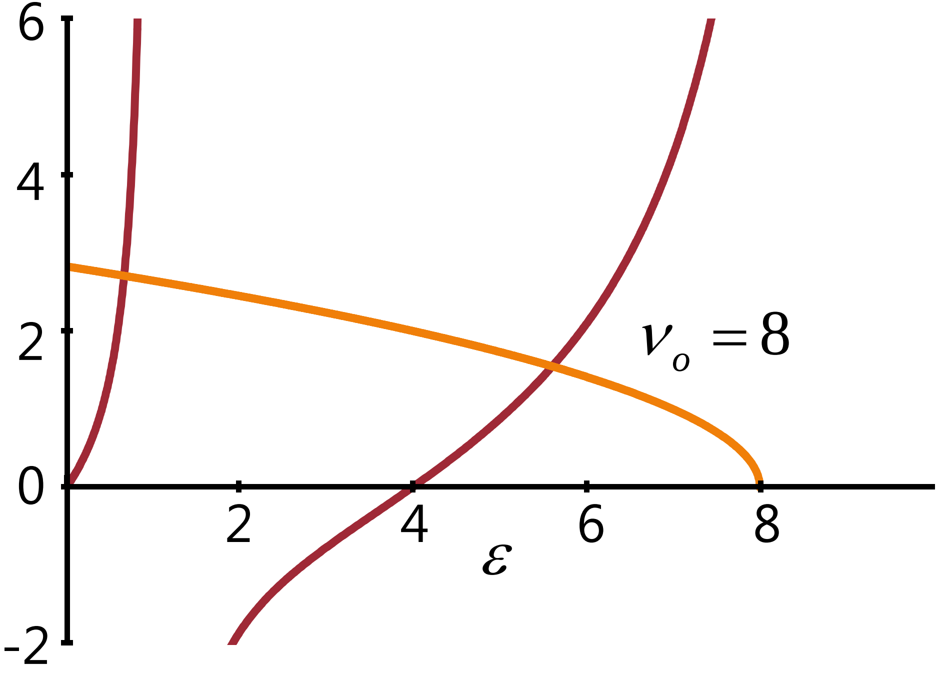
Choose a specific well depth v_o

and plot the curve

$$\sqrt{(v_o - \varepsilon)}$$

Now add the curves

$$\sqrt{\varepsilon}\tan\left(\frac{\pi}{2}\sqrt{\varepsilon}\right)$$



Graphical solution

Choose a specific well depth v_o

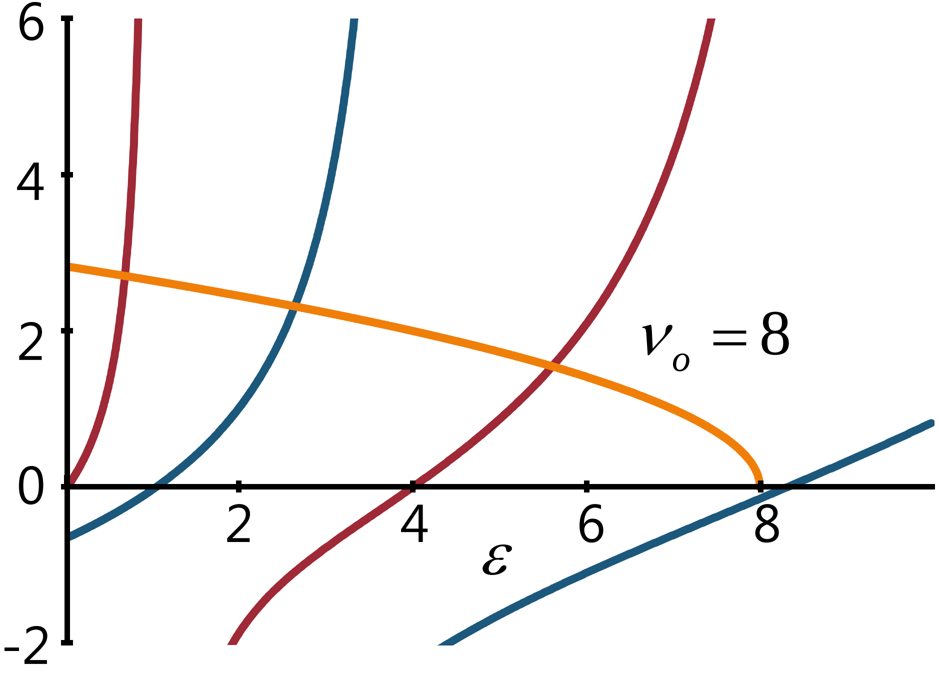
and plot the curve

$$\sqrt{(v_o - \varepsilon)}$$

Now add the curves

$$\sqrt{\varepsilon}\tan\left(\frac{\pi}{2}\sqrt{\varepsilon}\right)$$

$$-\sqrt{\varepsilon}\cot\left(\frac{\pi}{2}\sqrt{\varepsilon}\right)$$



Graphical solution

For a specific v_o

the solutions are the values of ε at the intersections of

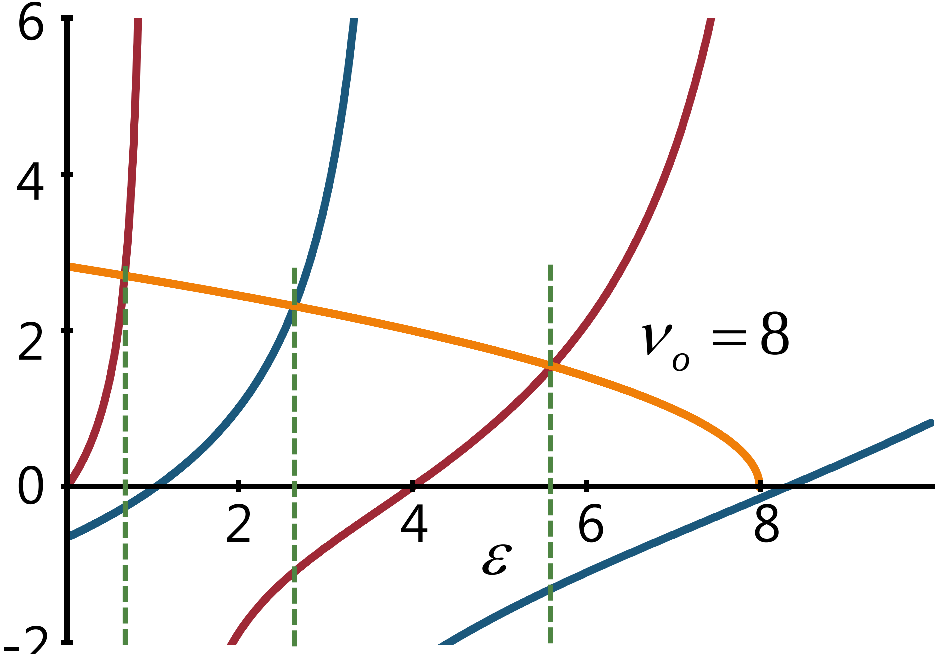
$$\sqrt{(v_o - \varepsilon)}$$

and

$$\sqrt{\varepsilon}\tan\left(\frac{\pi}{2}\sqrt{\varepsilon}\right)$$

or

$$-\sqrt{\varepsilon}\cot\left(\frac{\pi}{2}\sqrt{\varepsilon}\right)$$



$\varepsilon = 0.663 \quad 2.603 \quad 5.609$

Solutions

These are the solutions for a well depth V_o of $8E_1^\infty$

Note that

they are all

lower energies

than the corresponding solutions for the infinitely deep well of the same width