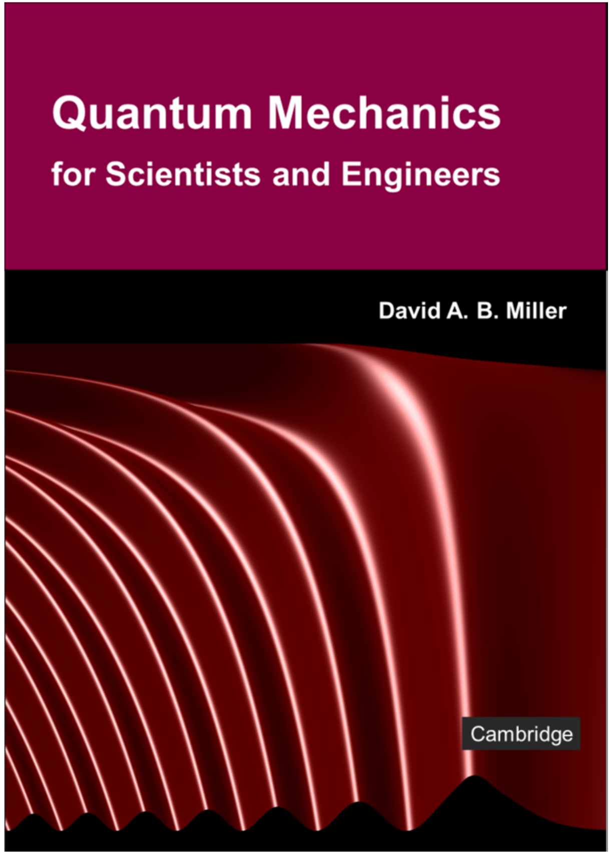
4.1 Time evolution of superpositions

Slides: Video 4.1.2 Superposition for the particle in a box

Text reference: Quantum Mechanics for Scientists and Engineers

Section 3.6 (“Simple linear superposition in an infinite potential well”)



Time evolution of superpositions

Superposition for a particle in a box

Quantum mechanics for scientists and engineers

David Miller

Superposition for a particle in a box

Suppose we have an infinitely deep potential well

a "particle in a box"

with the particle in a linear superposition

for example, with equal parts of the first and second states of the well

$$\Psi(z,t)=\frac{1}{\sqrt{L_z}}\left[\exp\left(-i\frac{E_1}{\hbar}t\right)\sin\left(\frac{\pi z}{L_z}\right)+\exp\left(-i\frac{E_2}{\hbar}t\right)\sin\left(\frac{2\pi z}{L_z}\right)\right]$$

Superposition for a particle in a box

Note for each eigenfunction in the superposition

it is multiplied by the appropriate complex exponential time-varying function

$$\exp\left(-i\frac{E_n}{\hbar}t\right)$$

This superposition is also normalized

$$\Psi(z,t)=\frac{1}{\sqrt{L_z}}\left[\exp\left(-i\frac{E_1}{\hbar}t\right)\sin\left(\frac{\pi z}{L_z}\right)+\exp\left(-i\frac{E_2}{\hbar}t\right)\sin\left(\frac{2\pi z}{L_z}\right)\right]$$

Superposition for a particle in a box

From this superposition

$$\Psi(z,t)=\frac{1}{\sqrt{L_z}}\left[\exp\left(-i\frac{E_1}{\hbar}t\right)\sin\left(\frac{\pi z}{L_z}\right)+\exp\left(-i\frac{E_2}{\hbar}t\right)\sin\left(\frac{2\pi z}{L_z}\right)\right]$$

we can multiply it by its complex conjugate

to get the probability density

$$\left|\Psi(z,t)\right|^2 =$$

$$\frac{1}{L_z}\left[\sin^2\left(\frac{\pi z}{L_z}\right)+\sin^2\left(\frac{2\pi z}{L_z}\right)+2\cos\left(\frac{E_2-E_1}{\hbar}t\right)\sin\left(\frac{\pi z}{L_z}\right)\sin\left(\frac{2\pi z}{L_z}\right)\right]$$

Superposition for a particle in a box

$$\left[\exp\left(-i\frac{E_1}{\hbar}t\right)\sin\left(\frac{\pi z}{L_z}\right)+\exp\left(-i\frac{E_2}{\hbar}t\right)\sin\left(\frac{2\pi z}{L_z}\right)\right]$$

multiplied by its complex conjugate

$$\times\left[\exp\left(+i\frac{E_1}{\hbar}t\right)\sin\left(\frac{\pi z}{L_z}\right)+\exp\left(+i\frac{E_2}{\hbar}t\right)\sin\left(\frac{2\pi z}{L_z}\right)\right]$$

$$=\boxed{\sin^2\left(\frac{\pi z}{L_z}\right)}+\boxed{\sin^2\left(\frac{2\pi z}{L_z}\right)}+\sin\left(\frac{\pi z}{L_z}\right)\sin\left(\frac{2\pi z}{L_z}\right)\boxed{\left[\boxed{\exp\left(i\frac{E_2-E_1}{\hbar}t\right)}+\boxed{\exp\left(-i\frac{E_2-E_1}{\hbar}t\right)}\right]}$$

$$=\sin^2\left(\frac{\pi z}{L_z}\right)+\sin^2\left(\frac{2\pi z}{L_z}\right)+\boxed{2\cos\left(\frac{E_2-E_1}{\hbar}t\right)}\sin\left(\frac{\pi z}{L_z}\right)\sin\left(\frac{2\pi z}{L_z}\right)$$

Superposition for a particle in a box

Note this probability density

$$|\Psi(z,t)|^2 =$$

$$\frac{1}{L_z}\left[\sin^2\left(\frac{\pi z}{L_z}\right)+\sin^2\left(\frac{2\pi z}{L_z}\right)+2\cos\left(\frac{E_2-E_1}{\hbar}t\right)\sin\left(\frac{\pi z}{L_z}\right)\sin\left(\frac{2\pi z}{L_z}\right)\right]$$

has a part that is oscillating in time

at an angular frequency $\omega_{21}=(E_2-E_1)/\hbar=3E_1/\hbar$

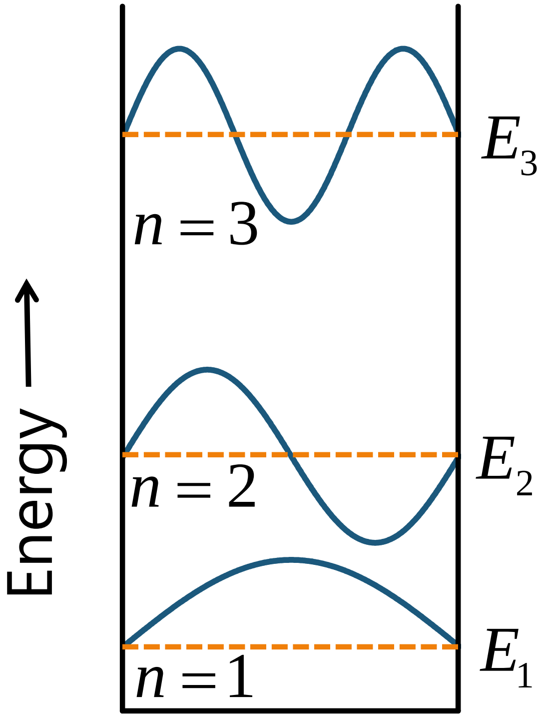
Note also that the absolute energy origin does not matter here for this measurable quantity

only the energy difference E_2-E_1 matters

Particle in a box

As a reminder

- here are the first few particle-in-a-box energy levels
  - and their associated wavefunctions
    - plotted with the orange dashed lines as horizontal axes



Superposition

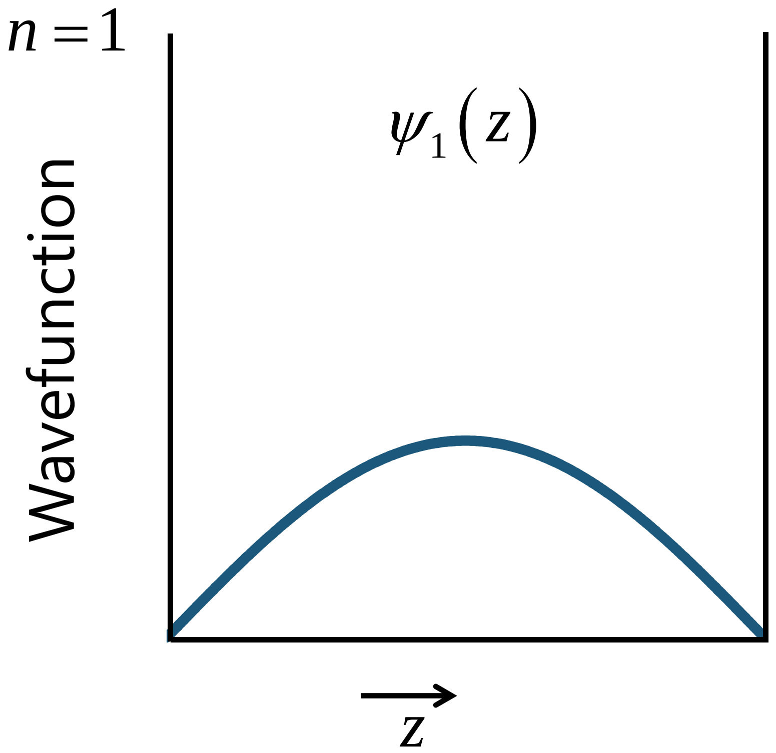
The $n=1$ spatial eigenfunction

$\psi_1(z)$

is plotted here

with the bottom of the box

as its horizontal axis



Superposition

For the probability density

$$\left|\psi_1(z)\right|^2$$

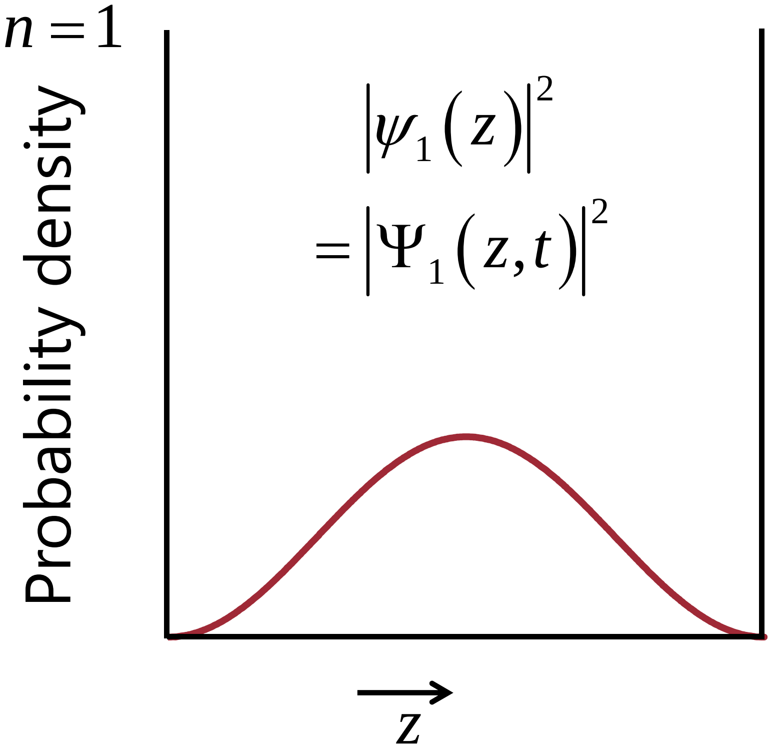
note the different shape

Multiplying by the time dependent factor gives

$$\Psi_1(z,t) = \exp\left(-i\frac{E_1}{\hbar}t\right)\psi_1(z)$$

The probability densities are the same

$$\left|\Psi_1(z,t)\right|^2 = \left|\psi_1(z)\right|^2$$



Superposition

Similarly

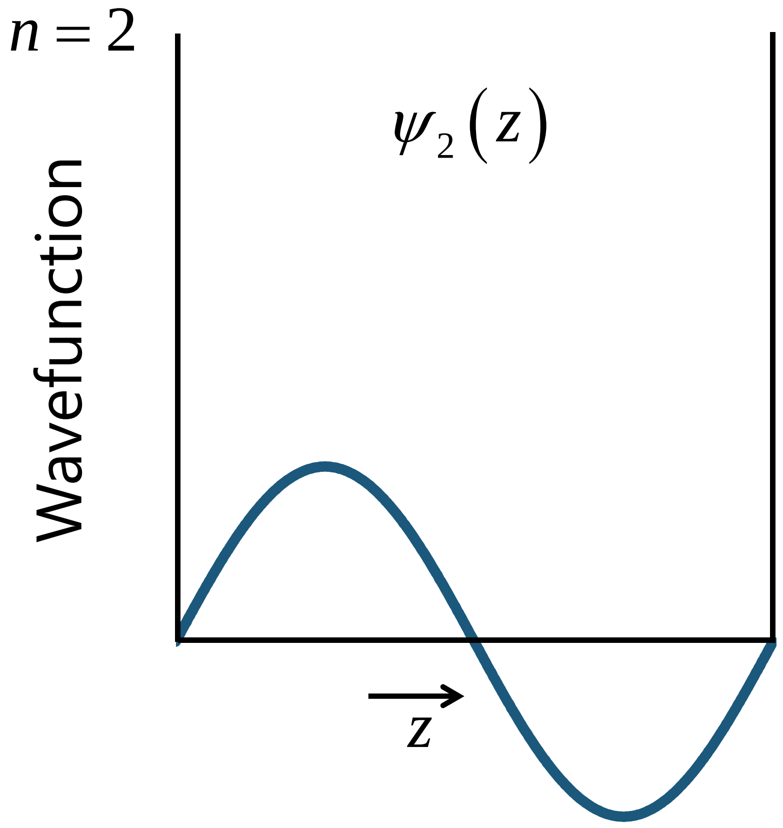
The $n = 2$ spatial eigenfunction

$\psi_2(z)$

is plotted here

with the bottom of the box

as its horizontal axis



Superposition

The probability density

$$\left|\psi_2(z)\right|^2$$

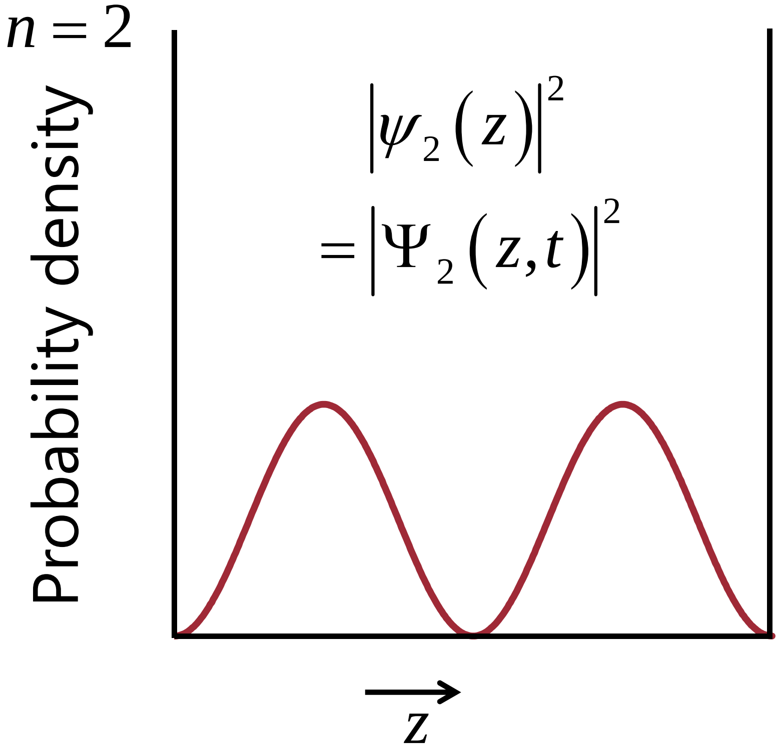
is a positive function

Multiplying by the time dependent factor gives

$$\Psi_2(z,t) = \exp\left(-i\frac{E_2}{\hbar}t\right)\psi_2(z)$$

The probability densities are the same

$$\left|\Psi_2(z,t)\right|^2 = \left|\psi_2(z)\right|^2$$



Superposition

An equal superposition of the two oscillates

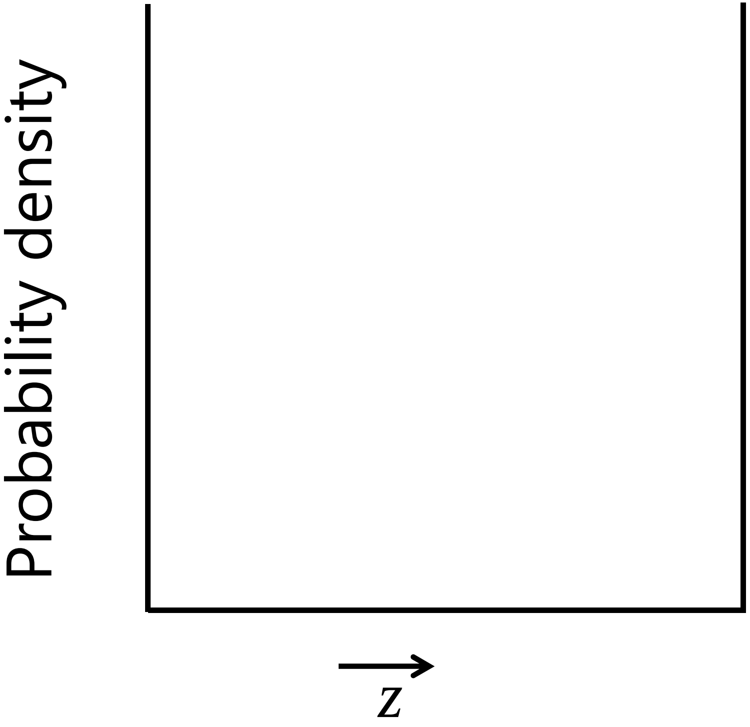
at the angular frequency

$$\omega_{21} = \left(E_2 - E_1\right)/\hbar = 3E_1/\hbar$$

$$\left|\Psi\left(z,t\right)\right|^2 = \left|\Psi_1\left(z,t\right) + \Psi_2\left(z,t\right)\right|^2$$

$$= \left|\psi_1\left(z\right)\right|^2 + \left|\psi_2\left(z\right)\right|^2$$

$$+2\cos\left(\frac{E_2 - E_1}{\hbar}t\right)\psi_1\left(z\right)\psi_2\left(z\right)$$



Superposition

An equal superposition of the two oscillates

at the angular frequency

$$\omega_{21} = \left(E_2 - E_1\right) / \hbar = 3E_1 / \hbar$$

$$\left|\Psi(z,t)\right|^2 = \left|\Psi_1(z,t) + \Psi_2(z,t)\right|^2$$

$$= \left|\psi_1(z)\right|^2 + \left|\psi_2(z)\right|^2$$

$$+2\cos\left(\frac{E_2 - E_1}{\hbar}t\right)\psi_1(z)\psi_2(z)$$