Matrices

Background mathematics review

David Miller

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Matrix notation

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Matrix notation

A matrix is, first of all, a rectangular array of numbers

- An $M \times N$ matrix has
  - M rows (here 2)
    - Rows are horizontal
  - N columns (here 3)
    - Columns are vertical

The array is enclosed in square brackets

$$\hat{A} = \begin{bmatrix} 2 & 1 & -3 \\ 6 & -5 & 4 \end{bmatrix}$$

This is a 2×3 rectangular matrix

Symbol for a matrix

As a symbol for a matrix

- we could just use a capital letter, like A
  - Here, we need to distinguish matrices
    - and other linear operators
  - from numbers and simple variables
    - so we put a "hat" over a symbol $\hat{A}$ representing a matrix
      - which distinguishes a matrix symbol when we write it by hand

$$\hat{A} = \begin{bmatrix} 2 & 1 & -3 \\ 6 & -5 & 4 \end{bmatrix}$$

Rectangular and square matrices

Because all matrices are, by definition, rectangular

- when we say a matrix is rectangular
  - we almost always mean it is not a *square* matrix
    - one with equal numbers of rows and columns

$$\hat{B} = \begin{bmatrix} 1.5 & -0.5i \\ 0.5i & 1.5 \end{bmatrix}$$

This is a 2×2 square matrix

Rectangular and square matrices

The numbers or "elements" in a matrix can be

real, imaginary, or complex

The elements are indexed in "row-column" order

B_{12} is the element (value $-0.5i$) in the first row and second column

We often use the same letter, here B, for the matrix and for its elements

or the lower case version, e.g., b_{12}

$$\hat{B} = \begin{bmatrix} 1.5 & -0.5i \\ 0.5i & 1.5 \end{bmatrix}$$

This is a 2×2 square matrix

Diagonal elements

The "leading diagonal" of a matrix

or just the "diagonal"

is the diagonal from top left to bottom right

Elements on the diagonal

here those with value 1.5

are called diagonal elements

Elements not on the diagonal

here those with value 0.5i and -0.5i

are called off-diagonal elements

$$\hat{B} = \begin{bmatrix} 1.5 & -0.5i \\ 0.5i & 1.5 \end{bmatrix}$$

This is a 2×2 square matrix

Vectors

In the matrix algebra version of vectors

- which are matrices of size 1 in one of their directions
  - we must specify whether a vector is a
    - row vector
      - a matrix with one row
    - or a column vector
      - a matrix with one column

$$c = [4, -2, 5, 7]$$

$$d = \begin{bmatrix} 2+3i \\ -5+2i \\ 4-i \\ -7-6i \end{bmatrix}$$

Transpose

An important manipulation for matrices and vectors is

- the transpose
  - denoted by a superscript "T"
    - a reflection about a diagonal line
      - from top left to bottom right for a matrix

Algebraically

$$\left(\hat{A}^T\right)_{mn} = \hat{A}_{nm}$$

$$\hat{A} = \begin{bmatrix} 2 & 1 & -3 \\ 6 & -5 & 4 \end{bmatrix}$$

$$\hat{A}^T = \begin{bmatrix} 2 & 6 \\ 1 & -5 \\ -3 & 4 \end{bmatrix}$$

Transpose

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$$\hat{B}^T = \begin{bmatrix} 1.5 & 0.5i \\ -0.5i & 1.5 \end{bmatrix}$$

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      - or at 45° for a vector

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$$d^T = [2+3i \quad -5+2i \quad 4-i \quad -7-6i]$$

Hermitian transpose or adjoint

Another common manipulation is the
- "Hermitian adjoint", "Hermitian transpose", or "conjugate transpose"
  - denoted by a superscript "†"
    - pronounced "dagger"
- a reflection about a diagonal line
  - from top left to bottom right for a matrix or at 45° for a vector
    - and taking the complex conjugate of all the elements

$$\hat{D} = \begin{bmatrix} 1.5 & -0.5i \\ 0.3i & 1.5 \end{bmatrix}$$

$$\hat{D}^\dagger = \begin{bmatrix} 1.5 & -0.3i \\ 0.5i & 1.5 \end{bmatrix}$$

$$\left(\hat{D}^\dagger\right)_{mn} = \hat{D}^*_{nm}$$

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$$d = \begin{bmatrix} 2+3i \\ -5+2i \\ 4-i \\ -7-6i \end{bmatrix}$$

$$d^\dagger = \begin{bmatrix} 2-3i & -5-2i & 4+i & -7+6i \end{bmatrix}$$

Hermitian matrix

A matrix is said to be

Hermitian

if it is equal to its own Hermitian adjoint

i.e.,

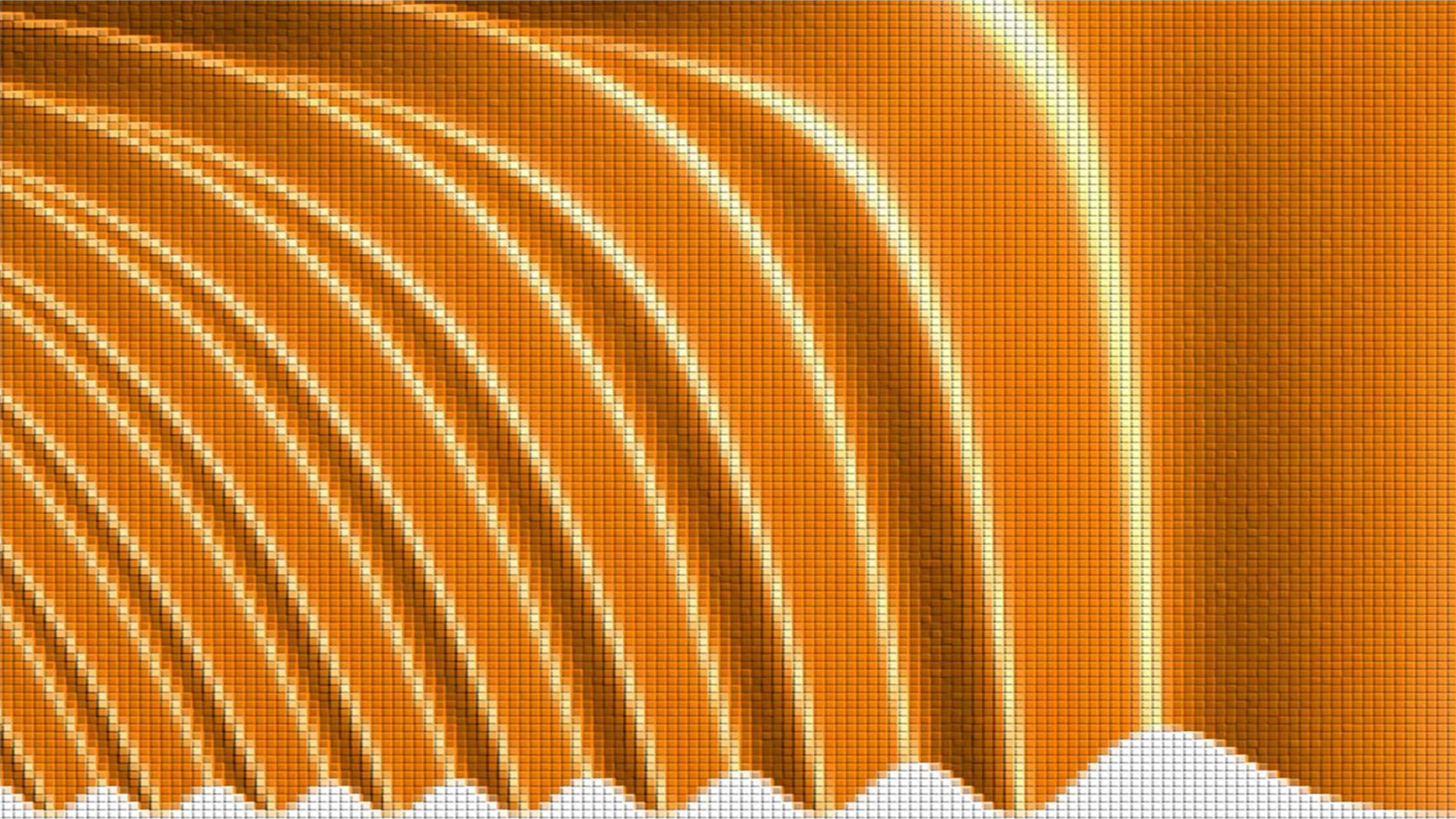
$$\hat{B}^\dagger = \hat{B}$$

or, element by element

$$\left(\hat{B}^\dagger\right)_{nm} = \left(\hat{B}\right)_{nm}$$

$$\hat{B} = \begin{bmatrix} 1.5 & -0.5i \\ 0.5i & 1.5 \end{bmatrix}$$

$$\hat{B}^\dagger = \begin{bmatrix} 1.5 & -0.5i \\ 0.5i & 1.5 \end{bmatrix} = \hat{B}$$



Matrices

Matrix algebra

Background mathematics review

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Adding and subtracting matrices

If two matrices are the same size

- i.e., the same numbers of rows and columns
  - we can add or subtract them by
    - adding or subtracting the individual matrix elements
      - one by one

$$\hat{F} = \begin{bmatrix} 1 & i \\ 2 & 1-3i \end{bmatrix} \qquad \hat{G} = \begin{bmatrix} 5 & 4i \\ -6 & 7+8i \end{bmatrix}$$

$$\hat{K} = \hat{F} + \hat{G}$$

Adding and subtracting matrices

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$$= \begin{bmatrix} 1+5 & i+4i \\ 2-6 & 1+7-3i+8i \end{bmatrix}$$

Adding and subtracting matrices

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$$\hat{F} = \begin{bmatrix} 1 & i \\ 2 & 1-3i \end{bmatrix} \quad \hat{G} = \begin{bmatrix} 5 & 4i \\ -6 & 7+8i \end{bmatrix}$$

$$\hat{K} = \hat{F} + \hat{G}$$

$$= \begin{bmatrix} 1+5 & i+4i \\ 2-6 & 1+7-3i+8i \end{bmatrix}$$

$$= \begin{bmatrix} 6 & 5i \\ -4 & 8+5i \end{bmatrix}$$

Multiplying a vector by a matrix

$$\begin{array}{cc} \text{matrix} & \text{vector} \\ \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} & \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix} \end{array}$$

- Suppose we want to multiply a column vector by a matrix
  - The number of rows in the vector
    - must match the number of columns in the matrix
- This is generally true for matrix-matrix multiplication
  - The number of rows in the matrix on the right
    - must match the number of columns in the matrix on the left

Multiplying a vector by a matrix

First we put the vector "sideways" on top of the matrix

then multiply element by element

and add to get the first element of the resulting vector

Move down

and repeat for the next row

$$\begin{array}{cc} \text{matrix} & \text{vector} \\ \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} & \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix} \end{array}$$

$$\begin{bmatrix} \mathbf{50} \\ \end{bmatrix} \leftarrow \begin{array}{c} \begin{array}{ccc} \mathbf{7} & \mathbf{8} & \mathbf{9} \\ \times & \times & \times \end{array} \\ \begin{bmatrix} \mathbf{1} & \mathbf{2} & \mathbf{3} \\ 4 & 5 & 6 \end{bmatrix} \end{array}$$

$$\begin{aligned} & 1\times 7 \\ & +2\times 8 \\ & +3\times 9 \\ & =50 \end{aligned}$$

$$\begin{bmatrix} 50 \\ \mathbf{122} \end{bmatrix} \leftarrow \begin{bmatrix} 1 & 2 & 3 \\ \mathbf{7} & \mathbf{8} & \mathbf{9} \\ \times & \times & \times \\ \mathbf{4} & \mathbf{5} & \mathbf{6} \end{bmatrix}$$

$$\begin{aligned} & 4\times 7 \\ & +5\times 8 \\ & +6\times 9 \\ & =122 \end{aligned}$$

Multiplying a vector by a matrix

First we put the vector "sideways" on top of the matrix

then multiply element by element

and add to get the first element of the resulting vector

Move down

and repeat for the next row

We can also write this multiplication

with a sum over the repeated index

$$\begin{array}{cc} \text{matrix} & \text{vector} \\ \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} & \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix} \end{array}$$

$$\begin{bmatrix} 50 \\ 122 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}$$

$$d = \hat{A} \quad c$$

$$d_m = \sum_n A_{mn} c_n$$

Multiplying a matrix by a matrix

To multiply a matrix by a matrix

- repeat this operation for each column of the matrix on the right
  - working from left to right

Write down the resulting columns in the resulting matrix

- also working from left to right

Summation notation

- sums over the repeated index

$$\begin{bmatrix} 50 & 14 \\ 122 & 32 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 7 & 1 \\ 8 & 2 \\ 9 & 3 \end{bmatrix}$$

$$\hat{R} = \hat{B} \qquad \hat{A}$$

$$R_{mp} = \sum_n B_{mn} A_{np}$$

Vector – vector products

An "inner" product

- of a row and a column vector
  - collapses two vectors to a number
    - analogous to geometrical vector dot product

An "outer product"

- of a column and a row vector
  - generates a square matrix

$$32 = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$$

$$f = \quad c \quad d$$

$$f = \sum_n c_n d_n$$

$$\begin{bmatrix} 4 & 8 & 12 \\ 5 & 10 & 15 \\ 6 & 12 & 18 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$$

$$\hat{F} = d \quad c$$

$$F_{mp} = d_m c_p$$

Matrix algebra properties

Matrix algebra, like normal algebra

is associative

and has distributive properties

but matrix multiplication is

not in general commutative

as is easily proved by

example

$$\left(\hat{C}\hat{B}\right)\hat{A} = \hat{C}\left(\hat{B}\hat{A}\right)$$

$$\hat{A}\left(\hat{B}+\hat{C}\right) = \hat{A}\hat{B}+\hat{A}\hat{C}$$

$\hat{B}\hat{A} \neq \hat{A}\hat{B}$ in general

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} = \begin{bmatrix} 19 & 22 \\ 43 & 50 \end{bmatrix}$$

$$\begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 23 & 34 \\ 31 & 46 \end{bmatrix}$$

Multiplying a matrix by a number

Multiplying a matrix by a number

means we multiply every element of the matrix by that number

$$2\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix} = \begin{bmatrix}2 & 4\\ 6 & 8\end{bmatrix}$$

Also, we can take out a common factor from every element

multiplying the matrix by that factor

$$\begin{bmatrix}2 & 4\\ 6 & 8\end{bmatrix} = 2\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}$$

Such results are easily proved in summation notation

e.g., for matrix vector multiplication

where $B_{mn} = \alpha A_{mn}$

$$d_m = \alpha \sum_n A_{mn} c_n = \sum_n \alpha A_{mn} c_n$$

$$= \sum_n \left(\alpha A_{mn}\right) c_n = \sum_n B_{mn} c_n$$

Multiplying a matrix by a number

Since number multiplication is commutative

we can move simple factors around arbitrarily in matrix products

e.g., for a number α and a vector c

$$\alpha \hat{A}\hat{B}c = \hat{A}\alpha\hat{B}c = \hat{A}\hat{B}\alpha c$$

This result is also easily proved using summation notation

Hermitian adjoint of a product

The Hermitian adjoint of a product

is the reversed product of the Hermitian adjoints

$$\boxed{\left(\hat{A}\hat{B}\right)^{\dagger} = \hat{B}^{\dagger}\hat{A}^{\dagger}}$$

We can prove this using summation notation

Suppose $\hat{R} = \hat{A}\hat{B}$ so that $R_{mp} = \sum_n A_{mn}B_{np}$ and so

$$\left(\left(\hat{A}\hat{B}\right)^{\dagger}\right)_{pm} = (\hat{R}^{\dagger})_{pm} = R_{mp}^{*} = \sum_n (A_{mn}B_{np})^{*} = \sum_n A_{mn}^{*}B_{np}^{*}$$

$$= \sum_n (\hat{A}^{\dagger})_{nm}(\hat{B}^{\dagger})_{pn} = \sum_n (\hat{B}^{\dagger})_{pn}(\hat{A}^{\dagger})_{nm} = \left(\hat{B}^{\dagger}\hat{A}^{\dagger}\right)_{pm}$$

Inverse of a matrix

For ordinary algebra, the reciprocal or inverse of a number or variable x is $1/x$ or x^{-1}

which has the obvious property $x^{-1}x = \dfrac{x}{x} = 1$

For a matrix, if it has an inverse $\hat{A}^{-1}$

it has the property $\hat{A}^{-1}\hat{A} = \hat{I}$

where $\hat{I}$ is called the identity matrix

which is the "diagonal" matrix with "1" for all diagonal elements

and zeros for all other elements

Identity matrix

For example, the 3x3 identity matrix is

$$\hat{I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The identity matrix in a given multiplication has to be the right size

so we do not typically bother to state the size of the identity matrix

For any matrix $\hat{A}$

we can write $\hat{A}\hat{I} = \hat{I}\hat{A} = \hat{A}$

Like the number 1 in ordinary algebra

the identity matrix is almost trivial

but is very important