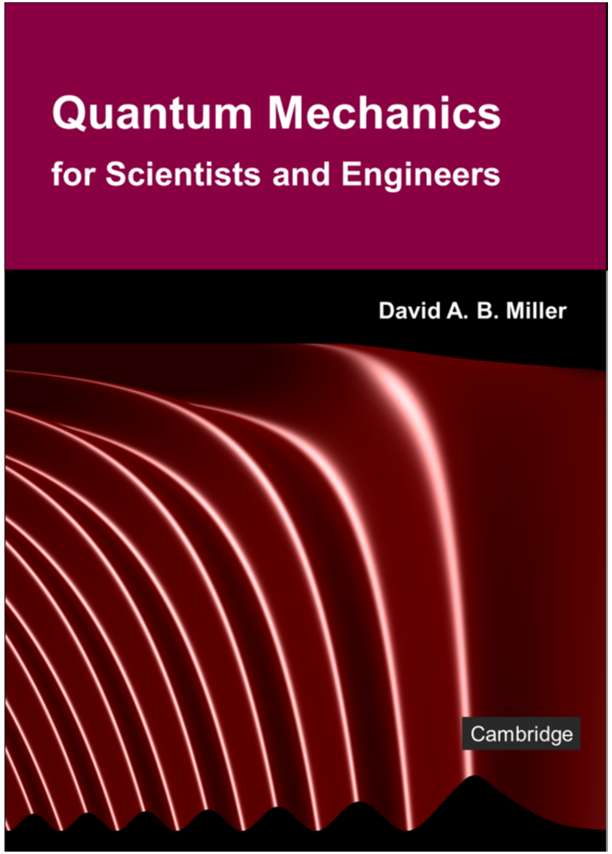
6.1 Types of linear operators

Slides: Video 6.1.1 Bilinear expansion of operators

Text reference: Quantum Mechanics for Scientists and Engineers

Section 4.6



Types of linear operators

Bilinear expansion of operators

Quantum mechanics for scientists and engineers

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Bilinear expansion of linear operators

We know that we can expand functions in a basis set

as in $f(x) = \sum_n c_n \psi_n(x)$ or $\left| f(x) \right\rangle = \sum_n c_n \left| \psi_n(x) \right\rangle$

What is the equivalent expansion for an operator?

We can deduce this from our matrix representation

Consider an arbitrary function *f*, written as the ket $\left| f \right\rangle$

from which we can calculate a function g

written as the ket $\left| g \right\rangle$

by acting with a specific operator $\hat{A}$

$$\left| g \right\rangle = \hat{A} \left| f \right\rangle$$

Bilinear expansion of linear operators

We expand g and f on the basis set ψ_i

$$|g\rangle = \sum_i d_i |\psi_i\rangle \qquad |f\rangle = \sum_j c_j |\psi_j\rangle$$

From our matrix representation of $|g\rangle = \hat{A}|f\rangle$

we know that $d_i = \sum_j A_{ij} c_j$

and, by definition of the expansion coefficient

we know that $c_j = \langle \psi_j | f \rangle$

so $d_i = \sum_j A_{ij} \langle \psi_j | f \rangle$

Bilinear expansion of linear operators

Substituting $d_i = \sum_j A_{ij} \langle \psi_j | f \rangle$ back into $|g\rangle = \sum_i d_i |\psi_i\rangle$ gives

$$|g\rangle = \sum_{i,j} A_{ij} \langle \psi_j | f \rangle |\psi_i\rangle$$

Remember that $\langle \psi_j | f \rangle \equiv c_j$ is simply a number

so we can move it within the multiplicative expression

Hence we have $|g\rangle = \sum_{i,j} A_{ij} |\psi_i\rangle \langle \psi_j | f \rangle = \left[\sum_{i,j} A_{ij} |\psi_i\rangle \langle \psi_j | \right] | f \rangle$

But $|g\rangle = \hat{A} | f \rangle$ and $|g\rangle$ and $| f \rangle$ are arbitrary, so

$$\hat{A} \equiv \sum_{i,j} A_{ij} |\psi_i\rangle \langle \psi_j |$$

Bilinear expansion of linear operators

This form

$$\hat{A} \equiv \sum_{i,j} A_{ij} \left|\psi_i\right\rangle\left\langle\psi_j\right|$$

is referred to as

a "bilinear expansion" of the operator $\hat{A}$

on the basis $\left|\psi_i\right\rangle$

and is analogous to the linear expansion of a vector on a basis

Any linear operator that operates within the space can be written this way

Bilinear expansion of linear operators

Though the Dirac notation is more general and elegant

for functions of a simple variable where

$$g(x) = \int \hat{A} f(x_1)\, dx_1$$

we can analogously write the bilinear expansion in the form

$$\hat{A} \equiv \sum_{i,j} A_{ij} \psi_i(x) \psi_j^*(x_1)$$

Outer product

An expression of the form

$$\hat{A} \equiv \sum_{i,j} A_{ij} \left| \psi_i \right\rangle \left\langle \psi_j \right|$$

contains an *outer* product of two vectors

An inner product expression of the form $\langle g | f \rangle$

results in a single, complex number

An outer product expression of the form $| g \rangle \langle f |$

generates a matrix

Outer product

$$|g\rangle\langle f| = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ \vdots \end{bmatrix} \begin{bmatrix} c_1^* & c_2^* & c_3^* & \cdots \end{bmatrix} = \begin{bmatrix} d_1 c_1^* & d_1 c_2^* & d_1 c_3^* & \cdots \\ d_2 c_1^* & d_2 c_2^* & d_2 c_3^* & \cdots \\ d_3 c_1^* & d_3 c_2^* & d_3 c_3^* & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

The specific summation $\hat{A} \equiv \sum_{i,j} A_{ij} |\psi_i\rangle\langle\psi_j|$

- is actually, then, a sum of matrices
  - In the matrix $|\psi_i\rangle\langle\psi_j|$
    - the element in the ith row and the jth column is 1
      - All other elements are zero