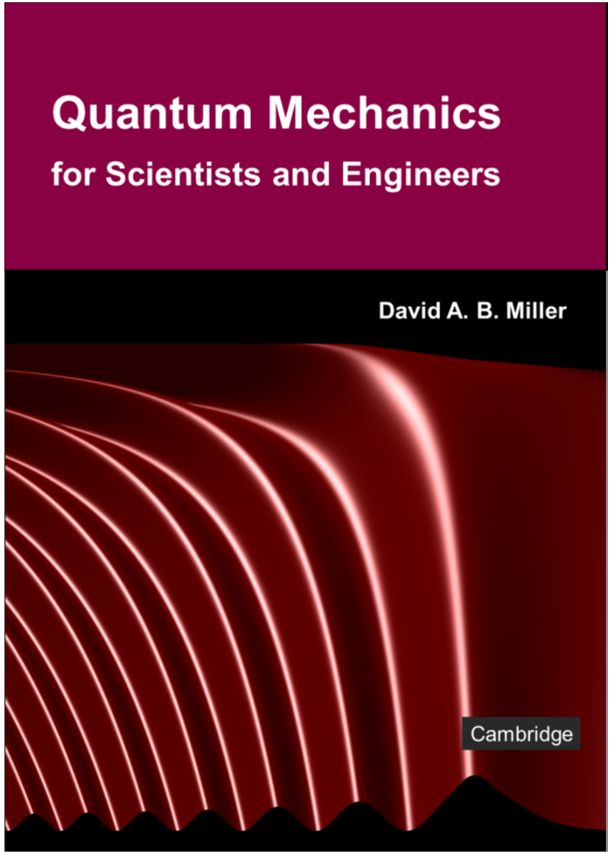
6.1 Types of linear operators

Slides: Video 6.1.5 Inverse and unitary operators

Text reference: Quantum Mechanics for Scientists and Engineers

Sections 4.9 – 4.10 (up to "Changing the representation of vectors")



Types of linear operators

Inverse and unitary operators

Quantum mechanics for scientists and engineers

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Inverse operator

For an operator $\hat{A}$ operating on an arbitrary function $\left| f \right\rangle$

then the inverse operator, if it exists

is that operator $\hat{A}^{-1}$ such that

$$\left| f \right\rangle = \hat{A}^{-1}\hat{A}\left| f \right\rangle$$

Since the function $\left| f \right\rangle$ is arbitrary

we can therefore identify

$$\hat{A}^{-1}\hat{A} = \hat{I}$$

Since the operator can be represented by a matrix

finding the inverse of the operator reduces to

finding the inverse of a matrix

Projection operator

For example, the projection operator

$$\hat{P} = |f\rangle\langle f|$$

in general has no inverse

because it projects all input vectors

onto only one axis in the space

the one corresponding to the specific vector $|f\rangle$

Unitary operators

A unitary operator, $\hat{U}$, is one for which

$$\hat{U}^{-1} = \hat{U}^{\dagger}$$

that is, its inverse is its Hermitian adjoint

The Hermitian adjoint is formed by

reflecting on a -45° line and taking the complex conjugate

$$\begin{bmatrix} u_{11} & u_{12} & u_{13} & \cdots \\ u_{21} & u_{22} & u_{23} & \cdots \\ u_{31} & u_{32} & u_{33} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}^{\dagger} = \begin{bmatrix} u_{11}^{*} & u_{21}^{*} & u_{31}^{*} & \cdots \\ u_{12}^{*} & u_{22}^{*} & u_{32}^{*} & \cdots \\ u_{13}^{*} & u_{23}^{*} & u_{33}^{*} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

Conservation of length for unitary operators

Note first that it can be shown generally that

for two matrices $\hat{A}$ and $\hat{B}$ that can be multiplied

$$\left(\hat{A}\hat{B}\right)^{\dagger} = \hat{B}^{\dagger}\hat{A}^{\dagger}$$

(this is easy to prove using the summation notation for matrix or vector multiplication)

That is, the Hermitian adjoint of the product is

the "flipped round" product of the Hermitian adjoints

Explicitly, for matrix-vector multiplication

$$\left(\hat{A}|h\rangle\right)^{\dagger} = \langle h|\hat{A}^{\dagger}$$

Conservation of length for unitary operators

Consider the unitary operator $\hat{U}$ and vectors $|f_{old}\rangle$ and $|g_{old}\rangle$

We form two new vectors by operating with $\hat{U}$

$$|f_{new}\rangle = \hat{U}|f_{old}\rangle \text{ and } |g_{new}\rangle = \hat{U}|g_{old}\rangle$$

Then $\langle g_{new}| = \langle g_{old}|\hat{U}^\dagger$

So $$\langle g_{new}|f_{new}\rangle = \langle g_{old}|\hat{U}^\dagger\hat{U}|f_{old}\rangle = \langle g_{old}|\hat{U}^{-1}\hat{U}|f_{old}\rangle = \langle g_{old}|\hat{I}|f_{old}\rangle$$
$$= \langle g_{old}|f_{old}\rangle$$

The unitary operation does not change the inner product

So, in particular $\langle f_{new}|f_{new}\rangle = \langle f_{old}|f_{old}\rangle$

the length of a vector is not changed by a unitary operator