Elementary mathematical expressions

Quadratic equations

$$a^2 - b^2 = (a+b)(a-b) \tag{1}$$

The solutions to the general quadratic equation

$$ax^2 + bx + c = 0 \tag{2}$$

are

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \tag{3}$$

Taylor and Maclaurin series (power-series expansion)

The Taylor series

$$f(x) = f(a) + \frac{(x-a)}{1!}\left.\frac{df}{dx}\right|_a + \frac{(x-a)^2}{2!}\left.\frac{d^2 f}{dx^2}\right|_a + \ldots + \frac{(x-a)^n}{n!}\left.\frac{d^n f}{dx^n}\right|_a + \ldots \tag{4}$$

gives a useful way of approximating a function near to some specific point $x = a$, giving a power-series expansion in $(x-a)^n$ for the function near that point.

The Maclaurin series

$$f(x) = f(0) + \frac{x}{1!}\left.\frac{df}{dx}\right|_0 + \frac{x^2}{2!}\left.\frac{d^2 f}{dx^2}\right|_0 + \ldots + \frac{x^n}{n!}\left.\frac{d^n f}{dx^n}\right|_0 + \ldots \tag{5}$$

is a special case of the Taylor series where we are expanding around the point $x = 0$.

Power-series expansions of common functions

For small *a*, the Maclaurin expansions of various common functions are, to first order

$$\sqrt{1+a} \simeq 1 + a/2 + \ldots \tag{6}$$

$$\frac{1}{1+a} \simeq 1 - a + \ldots \tag{7}$$

$$\sin a \simeq a + \ldots \tag{8}$$

$$\tan a \simeq a + \ldots \tag{9}$$

$$\cos a \simeq 1 - \frac{a^2}{2} + \ldots \tag{10}$$

$$\exp a \simeq 1 + a + \ldots \tag{11}$$

Sine and cosine addition and product formulae

$$\sin^2(\alpha)+\cos^2(\alpha)=1 \tag{12}$$

$$\sin(\alpha\pm\beta)=\sin(\alpha)\cos(\beta)\pm\cos(\alpha)\sin(\beta) \tag{13}$$

$$\sin(2\alpha)=2\sin\alpha\cos\alpha \tag{14}$$

$$\cos(\alpha\pm\beta)=\cos(\alpha)\cos(\beta)\mp\sin(\alpha)\sin(\beta) \tag{15}$$

$$\cos(2\alpha)=\cos^2(\alpha)-\sin^2(\alpha)=2\cos^2(\alpha)-1=1-2\sin^2(\alpha) \tag{16}$$

$$\cos^2(\alpha)=\frac{1}{2}\left[1+\cos(2\alpha)\right] \tag{17}$$

$$\sin^2(\alpha)=\frac{1}{2}\left[1-\cos(2\alpha)\right] \tag{18}$$

$$\cos(\alpha)\cos(\beta)=\frac{1}{2}\left[\cos(\alpha-\beta)+\cos(\alpha+\beta)\right] \tag{19}$$

$$\sin(\alpha)\sin(\beta)=\frac{1}{2}\left[\cos(\alpha-\beta)-\cos(\alpha+\beta)\right] \tag{20}$$

$$\sin(\alpha)\cos(\beta)=\frac{1}{2}\left[\sin(\alpha-\beta)+\sin(\alpha+\beta)\right] \tag{21}$$

$$\cos(\alpha)+\cos(\beta)=2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right) \tag{22}$$

$$\sin(\alpha)+\sin(\beta)=2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right) \tag{23}$$

$$\cos(\alpha)-\cos(\beta)=-2\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right) \tag{24}$$

$$\sin(\alpha)-\sin(\beta)=2\cos\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right) \tag{25}$$

Differential calculus

Product rule

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx} \tag{26}$$

Quotient rule

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2} \tag{27}$$

Chain rule

$$\frac{d}{dx} f\left(g\left(x\right)\right) = \left(\frac{df}{dg}\right) \times \left(\frac{dg}{fx}\right) \tag{28}$$

Derivatives of elementary functions

$$\frac{d}{dx} x^n = nx^{n-1} \tag{29}$$

$$\frac{d}{dx}\exp(ax) = a\exp(ax) \tag{30}$$

$$\frac{d}{dx}\ln(x) = \frac{1}{x} \tag{31}$$

$$\frac{d}{dx}\sin(x) = \cos(x) \tag{32}$$

$$\frac{d}{dx}\cos(x) = -\sin(x) \tag{33}$$

$$\frac{d\sin^{-1}x}{dx} = \frac{1}{\sqrt{1-x^2}} \tag{34}$$

$$\frac{d\tan^{-1}x}{dx} = \frac{1}{1+x^2} \tag{35}$$

Integral calculus

Integration by parts

$$\int_a^b f(x)\left(\frac{dg(x)}{dx}\right)dx = \left[f(x)g(x)\right]_a^b - \int_a^b \left(\frac{df(x)}{dx}\right)g(x)\,dx \tag{36}$$

where we use the common notation

$$\left[h(x)\right]\Big|_a^b = h(b) - h(a) \tag{37}$$

and, specifically, here

$$\left[f(x)g(x)\right]_a^b = f(b)g(b) - f(a)g(a) \tag{38}$$

Some definite integrals

$$\int_0^\pi \sin^2(nx)\,dx = \frac{\pi}{2} \tag{39}$$

$$\begin{aligned}\int_0^\pi (x-\pi/2)\sin(nx)\sin(mx)\,dx &= \frac{-4nm}{(n-m)^2(n+m)^2}\,,\ \text{ for } n+m \text{ odd} \\ &= 0\,,\ \text{ for } n+m \text{ even}\end{aligned} \tag{40}$$

$$\int_0^\pi \sin(\theta)\cos(2\theta)\,d\theta = -2/3 \tag{41}$$

$$\int_0^\pi \sin(2\theta)\cos(\theta)\,d\theta = 4/3 \tag{42}$$

$$\int_0^\pi \sin^3\theta\, d\theta = \frac{4}{3} \tag{43}$$

$$\int_0^\infty t^{1/2}\exp(-t)\,dt = \frac{\sqrt{\pi}}{2} \tag{44}$$

$$\int_{-\infty}^\infty \frac{\sin x}{x}\,dx = \pi \tag{45}$$

$$\int_{-\infty}^\infty \left(\frac{\sin x}{x}\right)^2 dx = \pi \tag{46}$$

$$\int_{-\infty}^\infty \exp(-x^2)\,dx = \sqrt{\pi} \tag{47}$$

$$\int_{-\infty}^\infty \frac{1}{1+x^2}\,dx = \pi \tag{48}$$

Partial differentiation

For a function $h(x,y)$ that is a function of two independent variables x and y, the partial derivative, often stated as "partial d h by d x" or, more explicitly, "partial d h by d x at constant y", and written as

$$\frac{\partial h}{\partial x} \equiv \left.\frac{\partial h}{\partial x}\right|_y \tag{49}$$

is the derivative of h with respect to x with the y variable held at a constant value. That value can also be explicitly stated, for example, as in the notation

$$\left.\frac{\partial h}{\partial x}\right|_{y=y_o} \tag{50}$$

which would be the partial derivative taken at the specific value $y = y_o$.

Higher partial derivatives can be formed similarly, as in the notations

$$\frac{\partial^2 h}{\partial x^2} \equiv \left.\frac{\partial^2 h}{\partial x^2}\right|_y \tag{51}$$

and, for the "cross derivative",

$$\frac{\partial^2 h}{\partial x \partial y} \equiv \left.\frac{\partial}{\partial x}\right|_y \left.\frac{\partial h}{\partial y}\right|_x \tag{52}$$

Provided all the various first derivatives and the two cross-derivatives in the two different orders both exist, we can interchange the order of the partial differentiations in the cross-derivative; that is,

$$\frac{\partial^2 h}{\partial x \partial y} = \frac{\partial^2 h}{\partial y \partial x} \tag{53}$$

For small or infinitesimal changes dx in x and dy in y, the resulting total change in h or *differential* or *exact differential* is written

$$dh = \left.\frac{\partial h}{\partial x}\right|_y dx + \left.\frac{\partial h}{\partial y}\right|_x dy \tag{54}$$

If x and y are both functions of some other variable t, then the *total derivative* dh/dt is given by

$$\frac{dh}{dt} = \left.\frac{\partial h}{\partial x}\right|_y \left(\frac{dx}{dt}\right) + \left.\frac{\partial h}{\partial y}\right|_x \left(\frac{dy}{dt}\right) \tag{55}$$

If x and y are each themselves functions of two variables a and b, then we can write

$$\left.\frac{\partial h}{\partial a}\right|_b = \left.\frac{\partial h}{\partial x}\right|_y \left.\frac{\partial x}{\partial a}\right|_b + \left.\frac{\partial h}{\partial y}\right|_x \left.\frac{\partial y}{\partial a}\right|_b \tag{56}$$

Because this works for any function of *x* and *y* (for which all appropriate derivative exist), we can write

$$\left.\frac{\partial}{\partial a}\right|_b = \left.\frac{\partial x}{\partial a}\right|_b \left.\frac{\partial}{\partial x}\right|_y + \left.\frac{\partial y}{\partial a}\right|_b \left.\frac{\partial}{\partial y}\right|_x \tag{57}$$

which can be used to change partial derivatives from one coordinate system to another.

Vector calculus

Cartesian coordinates

The the ∇ operator, which occurs in various different vector calculus operators, is known as *del* or *nabla*, can be written as

$$\nabla \equiv \mathbf{i}\frac{\partial}{\partial x}+\mathbf{j}\frac{\partial}{\partial y}+\mathbf{k}\frac{\partial}{\partial z} \tag{58}$$

in Cartesian coordinates, with **i, j**, and **k** as unit vectors in the x, y, and z directions respectively.

The *gradient* operator operates on a scalar function $f(x,y,z)$ to give a vector whose magnitude and direction are the slope or gradient of the scalar function at the point of interest. In Cartesian coordinates

$$\text{grad } f = \nabla f = \mathbf{i}\frac{\partial f}{\partial x}+\mathbf{j}\frac{\partial f}{\partial y}+\mathbf{k}\frac{\partial f}{\partial z} \tag{59}$$

The *Laplacian* operator, also known as *del squared*, operates on a scalar function, giving a scalar result. It is written in Cartesian coordinates as

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}+\frac{\partial^2 f}{\partial z^2} \tag{60}$$

The operator $\nabla\cdot\nabla$, sometimes also written as ∇^2, can operate on a vector function, in which case, in Cartesian coordinates, we have

$$\left(\nabla\cdot\nabla\right)\mathbf{F} = \mathbf{i}\frac{\partial^2 F_x}{\partial x^2}+\mathbf{j}\frac{\partial^2 F_y}{\partial y^2}+\mathbf{k}\frac{\partial^2 F_z}{\partial z^2} \tag{61}$$

In Cartesian coordinates, the *divergence* of a vector **F** is defined as

$$\text{div}\,\mathbf{F} = \nabla\cdot\mathbf{F} = \frac{\partial F_x}{\partial x}+\frac{\partial F_y}{\partial y}+\frac{\partial F_z}{\partial z} \tag{62}$$

In Cartesian coordinates, the *curl* of a vector **F** is defined as

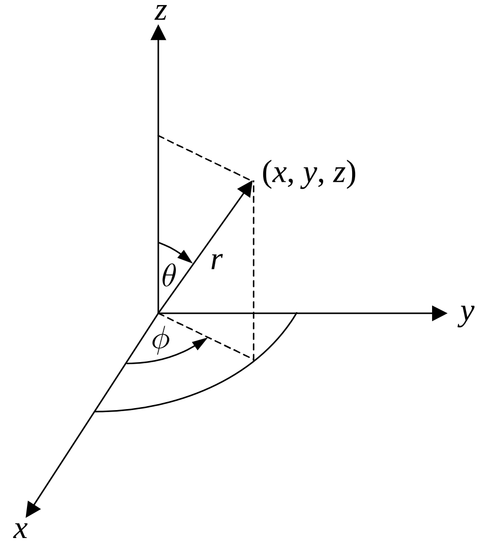
$$\text{curl}\,\mathbf{F} \equiv \nabla\times\mathbf{F} = \left(\frac{\partial F_z}{\partial y}-\frac{\partial F_y}{\partial z}\right)\mathbf{i}+\left(\frac{\partial F_x}{\partial z}-\frac{\partial F_z}{\partial x}\right)\mathbf{j}+\left(\frac{\partial F_y}{\partial x}-\frac{\partial F_x}{\partial y}\right)\mathbf{k} \tag{63}$$

or in the equivalent "determinant" shorthand form,

$$\nabla\times\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix} \tag{64}$$

Spherical polar coordinates

In spherical polar coordinates, which can be defined as in the following diagram



with

$$x = r\sin\theta\cos\phi \tag{65}$$

$$y = r\sin\theta\sin\phi \tag{66}$$

$$z = r\cos\phi$$

the *gradient* can be written

$$\nabla f = \frac{\partial f}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}\hat{\phi} \tag{67}$$

the *Laplacian* can be written

$$\nabla^2 f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial f}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial \phi^2} \tag{68}$$

the *divergence* can be written

$$\nabla\cdot\mathbf{F} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2F_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}\left(F_\theta\sin\theta\right) + \frac{1}{r\sin\theta}\frac{\partial F_\phi}{\partial \phi} \tag{69}$$

and the *curl* can be written

$$\nabla\times\mathbf{F} = \begin{vmatrix} \dfrac{\hat{\mathbf{r}}}{r^2\sin\theta} & \dfrac{\hat{\theta}}{r\sin\theta} & \dfrac{\hat{\phi}}{r} \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial \theta} & \dfrac{\partial}{\partial \phi} \\ F_r & rF_\theta & r\sin\theta F_\phi \end{vmatrix} \tag{70}$$

Vector calculus identities

$$\nabla \cdot (\nabla \times \mathbf{F}) = 0 \tag{71}$$

$$\nabla \times \nabla f = 0 \tag{72}$$

$$\nabla \times (\nabla \times \mathbf{F}) = \nabla (\nabla \cdot \mathbf{F}) - (\nabla \cdot \nabla)\mathbf{F} \tag{73}$$

$$\nabla \cdot (\mathbf{F} \times \mathbf{G}) = -\mathbf{F} \cdot (\nabla \times \mathbf{G}) + \mathbf{G} \cdot (\nabla \times \mathbf{F}) \tag{74}$$

$$\begin{aligned} \Psi \nabla^2 \Psi^* - \Psi^* \nabla^2 \Psi &= \Psi \nabla^2 \Psi^* + \nabla \Psi \nabla \Psi^* - \nabla \Psi \nabla \Psi^* - \Psi^* \nabla^2 \Psi \\ &= \nabla \cdot \left(\Psi \nabla \Psi^* - \Psi^* \nabla \Psi \right) \end{aligned} \tag{75}$$