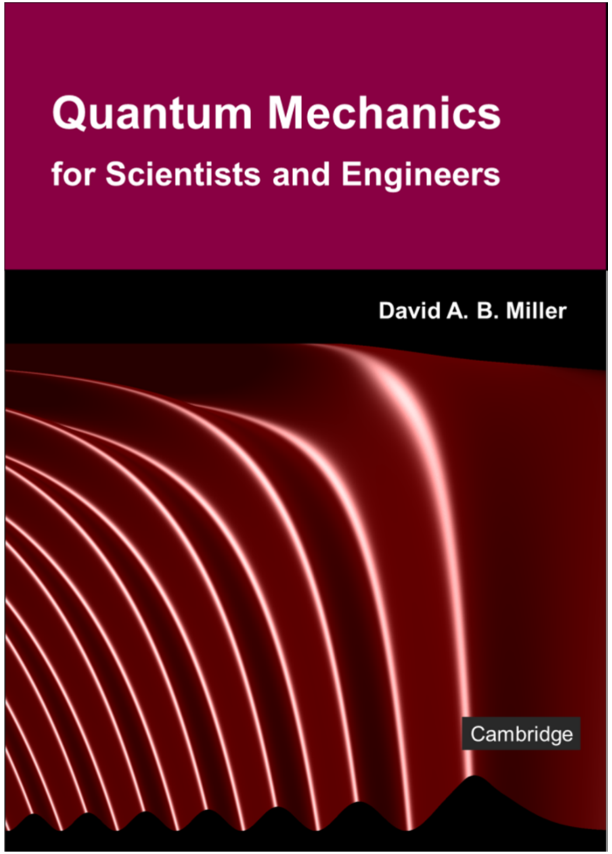
7.1 Angular momentum

Slides: Video 7.1.3 Angular momentum eigenfunctions

Text reference: Quantum Mechanics for Scientists and Engineers

Section 9.1 (remainder)



Angular momentum

Angular momentum eigenfunctions

Quantum mechanics for scientists and engineers

David Miller

Spherical polar coordinates

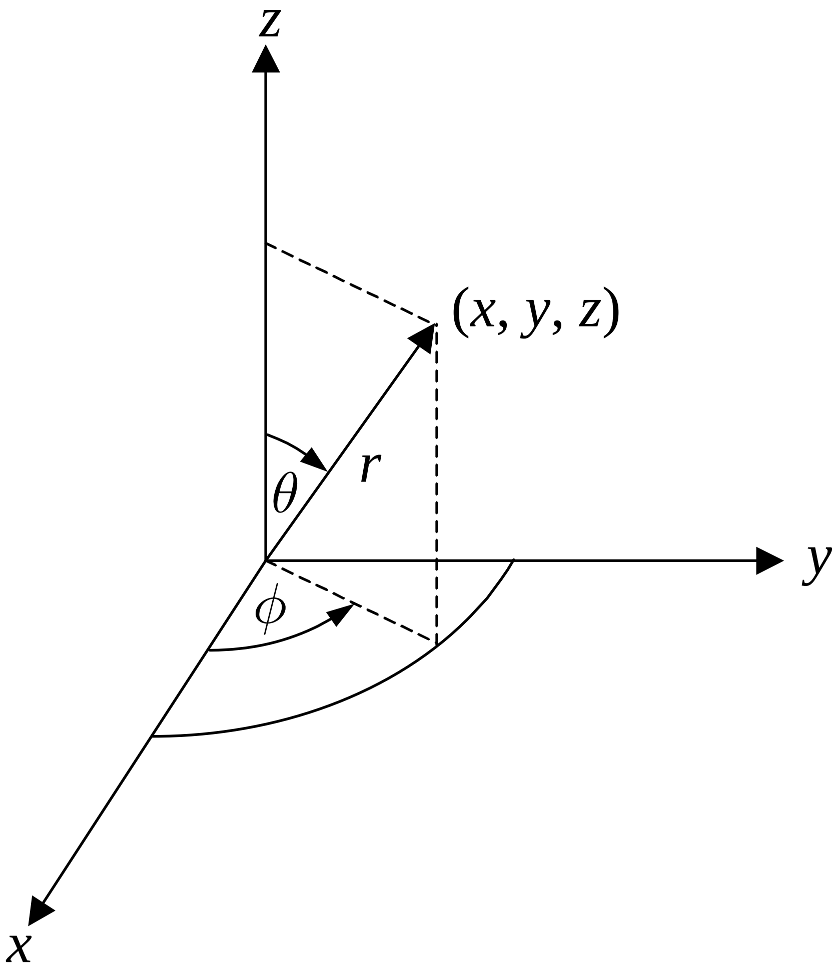
The relation between
spherical polar and
Cartesian coordinates is

$$x = r \sin\theta \cos\phi$$

$$y = r \sin\theta \sin\phi$$

$$z = r \cos\theta$$

θ is the polar angle, and
ϕ is the azimuthal angle



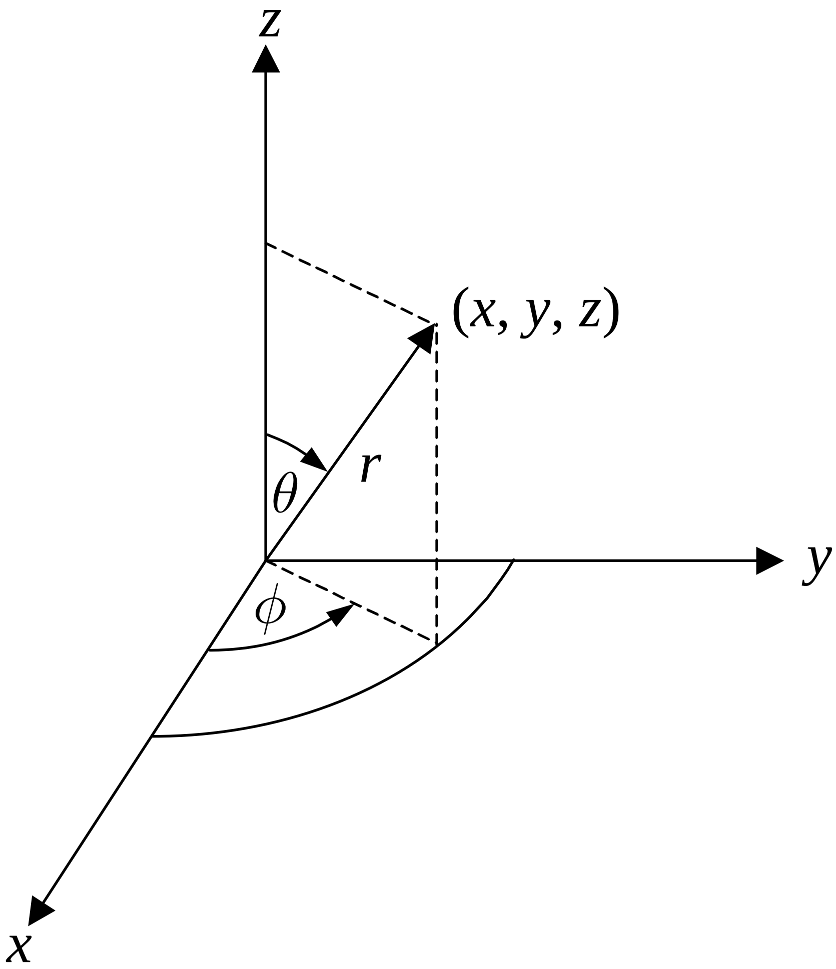
Spherical polar coordinates

In inverse form

$$r = \sqrt{x^2 + y^2 + z^2}$$

$$\theta = \sin^{-1}\left(\frac{\sqrt{x^2 + y^2}}{\sqrt{x^2 + y^2 + z^2}}\right)$$

$$\phi = \tan^{-1}\left(\frac{y}{x}\right)$$



Angular momentum in spherical polar coordinates

With these definitions of spherical polar coordinates

and with standard partial derivative relations of the form

$$\frac{\partial}{\partial x} \equiv \frac{\partial r}{\partial x}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial x}\frac{\partial}{\partial \theta} + \frac{\partial \phi}{\partial x}\frac{\partial}{\partial \phi}$$

for each of the Cartesian coordinate directions

we can rewrite the angular momentum operator components in spherical polar coordinates

Angular momentum in spherical polar coordinates

From $\hat{L}_x\hat{L}_y - \hat{L}_y\hat{L}_x = \left[\hat{L}_x, \hat{L}_y\right] = i\hbar\hat{L}_z$ $\quad \hat{L}_y\hat{L}_z - \hat{L}_z\hat{L}_y = \left[\hat{L}_y, \hat{L}_z\right] = i\hbar\hat{L}_x$

and $\hat{L}_z\hat{L}_x - \hat{L}_x\hat{L}_z = \left[\hat{L}_z, \hat{L}_x\right] = i\hbar\hat{L}_y$

we obtain

$$\hat{L}_x = i\hbar\left(\sin\phi\frac{\partial}{\partial\theta} + \cot\theta\cos\phi\frac{\partial}{\partial\phi}\right)$$

$$\hat{L}_y = i\hbar\left(-\cos\phi\frac{\partial}{\partial\theta} + \cot\theta\sin\phi\frac{\partial}{\partial\phi}\right)$$

$$\hat{L}_z = -i\hbar\frac{\partial}{\partial\phi}$$

L_z eigenfunctions and eigenvalues

Using $\hat{L}_z = -i\hbar \dfrac{\partial}{\partial \phi}$

we solve for the eigenfunctions and eigenvalues of $\hat{L}_z$

The eigen equation is

$$\hat{L}_z \Phi(\phi) = m\hbar \Phi(\phi)$$

where $m\hbar$ is the eigenvalue to be determined

The solution of this equation is

$$\Phi(\phi) = \exp(im\phi)$$

L_z eigenfunctions and eigenvalues

The requirements that the wavefunction and its derivative are

- continuous when we return to where we started
  - i.e., for $\phi = 2\pi$
    - mean that m must be an integer
      - positive or negative or zero

Hence we find that

- the angular momentum around the z axis is quantized
  - with units of angular momentum of $\hbar$