Topic 11: Probability

1) The Basics of Counting:

a) Fundamental Principle Of Counting:

If one event has **m** possible outcomes and a second event has **n** possible outcomes, then there are **m** x **n** total possible outcomes for the two events together.

e.g. 2 starters and 5 main courses

=> 10 possible dinner options

b) A Deck Of Cards:

- 52 Cards in a deck
- 4 suits: Spades (black), Clubs (black), Hearts (red) and Diamonds (red)
- Picture Cards: Jack, Queen and King in each suit (12 in total)

c) Different Strategies:

1) We can simply list all possible outcomes.

2) We can make out a **two-way table**, if there are more than two trials.

e.g. tossing a coin two or more times

3) Sometimes it can be useful to make out a **tree diagram**, for showing all possible outcomes of two or more trials.

e.g. chance of picking one yellow and a blue bead from a bag of 6 yellow, 5 blue

2) Permutations/Arrangements:

- The number of ways of rearranging n objects is given by the formula:

$$= n!$$

$$n! = n(n-1)(n-2) \ldots\ldots (3)(2)(1)$$

Example 1: Find the number of ways of rearranging the letters of the word MATHS i) with no restrictions ii) beginning with *A*

i) 5 letters => 5 objects to rearrange

=> 5! = 120

ii) beginning with an 'A' means we have to fix the A in the first position and rearrange the remaining 4 letters

=> 4! = 24

Example 2: Find the number of ways of rearranging the letters of the work MARINES i) with no restrictions ii) if the vowels have to be together

i) 7 letters => 7 objects to rearrange => 7! = 5040

ii) 3 vowels have to be together

=> put 3 vowels together and treat as 1 object

=> we now have 4 consonants and 1 block of vowels to rearrange

=> 5 objects = 5! = 120

=> the three vowels in the vowel block can be rearranged in 3! ways

=> The total no. of Arrangements = 5! X 3! = 720

3) Basics of Probability:

a) Definition of Probability:

- The probability of an event occurring is:

$$\frac{\textit{number of successful outcomes}}{\textit{total number of outcomes}}$$

e.g. bag with 5 red and 4 green beads

P(Green) = $\frac{4}{9}$

Note:

- Probability values must be between 0 and 1 (see scale below)

0	¼	½	¾	1
0%	25%	50%	75%	100%
0.00	0.25	0.5	0.75	1.00
Impossible	Unlikely	Evens Chance	Likely	Certain

b) Terminology:

1. A **trial** is an act of doing an experiment in probability e.g. tossing a coin
2. An **outcome** is one of the possible results of the trial e.g. a 6 when throwing a die
3. A **sample space** is the set of all possible outcomes in a trial.
4. An **event** is the occurrence of one or more specific outcomes.
5. **Probability** is the measure of the chance of an event happening.
6. The **expected frequency** is:

   = (the no. of trials) x (relative frequency or probability)

c) Relative Frequency and Carrying Out Experiments:

- We can carry out an experiment or trials to estimate the probability of an event occurring.

  e.g. throwing a die to see how many 6's we get
- If you throw a die 20 times and a 6 comes up 3 times we could estimate the probability of throwing a 6 to be $\frac{3}{20}$.
- This estimate we get from carrying out trials, is called the **Relative Frequency**.
- The more trials that are done, the closer the relative frequency gets to the actual probability.

d) Expected Frequency:

= No. Of Trials X Relative Freq/Probability

Example 1: A die is tossed 600 times, how many times would you expect to roll a 1?

P(Throwing a 1) = $\frac{1}{6}$

=> Expected Freq of a '1' = 600 x $\frac{1}{6}$ = 100

4) Combinations:

Notes:

- A **selection** of objects in any order i.e. the **order is important**.
  E.g. to select a team of 5 from a panel of 8 players (A, B, C, D, E, F, G, and H), the team ABCDE would be the same team as BCADE

$^{n}c_r$ = the total number of ways of choosing r objects from n

- The button on your calculator for calculating combinations is:

To calculate $^{8}c_5$ press '8' and then 'Shift + the button shown' and then '5' and '=' to get 56.

- Two other properties of combinations are:

$$\binom{n}{r} = \binom{n}{n-r} \qquad \binom{n}{r} = \frac{n!}{r!\,(n-r)!}$$

- There is also a quick way of calculating $^{n}c_r$, which can often be done without using a calculator:

$$\frac{Countdown\ from\ n\ on\ top\ (r\ terms)}{Count\ up\ from\ r\ on\ bottom\ (r\ terms)}$$

E.g. $^{12}c_5 = \frac{12 \times 11 \times 10 \times 9 \times 8}{1 \times 2 \times 3 \times 4 \times 5} = \frac{11 \times 9 \times 8}{1} = 792$

Example 1: A 5th year student has to choose 4 subjects out of a possible 7 for Leaving Cert {French, Accounting, Biology, German, Physics, Chemistry and Applied Maths). How many different choices can they make if:
i) there are no restrictions
ii) French has to be included
iii) French can't be included
iv) Applied Maths and Chemistry are included but Biology cannot?

i) There are no restrictions => choosing 4 from 7: $^{7}c_4 = 35$
ii) If French has to be included then they have to choose 3 other subjects from the 6 remaining: $^{6}c_3 = 20$
iii) If French can't be included then they have to choose 4 from 6: $^{6}c_4 = 15$ or else we could subtract the answer to (ii) from (i) i.e. 35 - 20 = 15
iv) If Applied Maths and Chemistry have to be included but Biology cannot be, then the choice will be 2 subjects out of the remaining 4 i.e. $^{4}c_2 = 6$

Example 2: A table tennis club in a school has 12 members: 7 boys and 5 girls. A team of 4 has to be selected to represent the school. How many different teams can be selected:
i) if there are no restrictions
ii) if there has to be more girls than boys on the team?
i) With no restrictions, the number of selections will be:

$$^{12}c_4 = 495 \text{ teams}$$

ii)
- Has to be more girls than boys on team:
=> (4 girls AND 0 boy) OR (3 girls AND 1 boy)
=> Total number of selections = $(\,^{5}c_4 \times {}^{7}c_0\,) + (\,^{5}c_3 \times {}^{7}c_1\,)$
= (5 x 1) + (10 x 7)
= 5 + 70 = 75 teams

5) Set Theory and Probability:

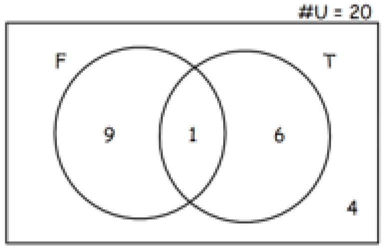
Notes:

- Sets can be used to help solve probability problems.
- Remember that A ∩ B represents A **AND** B whereas A ∪ B represents A **OR** B.

Example 1: 20 people were asked if they preferred Facebook or Twitter. 10 said Facebook, 7 said Twitter and 4 said neither. A person is selected at random from the group, what is the probability that the person selected:
i) chose Facebook and Twitter
ii) chose Facebook or Twitter
iii) chose Facebook only

- Firstly, we need to draw a Venn Diagram to represent the problem.
- 4 people chose neither => 16 people chose Facebook or Twitter
- As 10 chose Facebook and 7 chose Twitter, that means 1 person must have chosen both
- The Venn Diagram for this problem is shown on the right.

#U = 20



i) P(Chose Facebook AND Twitter) = $F \cap T = \frac{1}{20}$

ii) P(Chose Facebook OR Twitter) = $F \cup T = \frac{16}{20} = \frac{4}{5}$

iii) P(Chose Facebook Only) = $\frac{9}{20}$

6) Combined Events/Bernoulli Trials:

a) Combined Events:

Remember:

$$AND = \times$$

$$OR = +$$

Example: The probability of Paul scoring a free throw is 0.8. What is the probability of:

i) scoring three free throws in a row
ii) scoring the first and missing the next two
iii) scoring two of the three free throws

i) P(Score 1st AND Score 2nd AND Score 3rd) = 0.8 x 0.8 x 0.8 = 0.512
ii) P(Score 1st AND Miss 2nd AND Miss 3rd) = 0.8 x 0.2 x 0.2 = 0.032
iii) P(Score 1st AND 2nd AND Miss 3rd) OR (Miss 1st AND Score 2nd AND 3rd) OR (Score 1st AND Miss 2nd AND Score 3rd)
= (0.8 x 0.8 x 0.2) + (0.2 x 0.8 x 0.8) + (0.8 x 0.2 x 0.8) = 0.128 + 0.128 + 0.128 = 0.384

b) Bernoulli Trials:

Notes:

- Experiment that satisfies the 4 conditions:
  - There are a fixed number of repeated trials
  - There are only two outcomes: success and failure
  - The trials are independent.
  - The probability of success in each trial is constant (Let p = probability of success, and q = probability of failure and q = 1 – p)
- Examples of Bernoulli Trials would be:
  - Tossing a coin or shooting free throws (Hit or Miss)

Example: The probability that a person hits a target with a dart is 30%. If they throw 3 successive darts, what is the probability that the dart hits the target twice or more times?

P(Hits Twice) = (Miss 1st AND Hits 2nd AND 3rd) OR (Hit 1st AND 2nd AND Miss 3rd) OR (Hit 1st AND Miss 2nd AND Hit 3rd)
= (0.7 x 0.3 x 0.3) + (0.3 x 0.3 x 0.7) + (0.3 x 0.7 x 0.3) = 0.189

P(Hits Thrice) = Hits 1st AND Hits 2nd AND Hits 3rd
= 0.3 x 0.3 x 0.3 = 0.027
P(Hits Twice OR Thrice) = Hits Twice OR Hits Thrice
= 0.189 + 0.027
= 0.216

c) Binomial Distribution:

Steps:

1. Write down the number of trials, n.
2. Calculate p and q. (q = 1 – p)
3. Let r = number of successes required.
4. Use the formula:

See Tables pg33

$$P(r\ successes) = ({}^{n}c_r)p^r.q^{n-r}$$

Example 1: A die is tossed 10 times. What is the probability of exactly 4 sixes?

Step 1: No. of Trials: n = 10
Step 2: p = $\frac{1}{6}$ and q = 1 - $\frac{1}{6}$ = $\frac{5}{6}$
Step 3: r = 4
Step 4: P(4) = ${}^{10}c_4(\frac{1}{6})^4(\frac{5}{6})^6$ = 0.054

Example 2: A basketball player made 80% free throws in the season. Find the probability that tonight he i) misses for the 1st time on his 5th free throw ii) makes his 1st basket on the 4th free throw iii) makes 1st basket on one of his first 3 free throws

i) Note for this part that we don't need the ${}^{n}c_r$ term
Step 1: No. of Trials: n = 5
Step 2: p = 0.8 and q = 1 – 0.8 = 0.2
Step 3: r = 4
Step 4: P(4) = $(0.8)^4(0.1)^1$ = 0.08192

ii) Again, we don't need the ${}^{n}c_r$ term
Step 1: No. of Trials: n = 4
Step 2: p = 0.2 and q = 1 – 0.2 = 0.8 (in this case success is a miss => p = 0.2)
Step 3: r = 3
Step 4: P(3) = $(0.2)^3(0.8)^1$ = 0.0064

iii) In this part, he could make the free throw on the 1st throw, or the 2nd throw, o the 3rd throw, so we need the ${}^{n}c_r$ term this time:
Step 1: No. of Trials: n = 3
Step 2: p = 0.8 and q = 1 – 0.8 = 0.2
Step 3: r = 1
Step 4: P(1) = ${}^{3}c_1(0.8)^1(0.2)^2$ = 0.09

7) Mutually Exclusive Events/Conditional Probability/Independent Events:

a) Mutually Exclusive Events:

Notes:

- Normally add two probabilities in an **OR** situation.
- If the events do overlap, we adjust the rule:

If there is an overlap:

$$P(A\ OR\ B) = P(A) + P(B) - P(A\ AND\ B)$$

If there is no overlap:

$$P(A\ OR\ B) = P(A) + P(B)$$

- Events with no overlap are **mutually exclusive** so 2nd rule above applies in those situations.
- Mutually exclusive events cannot happen at the same time.

Example: Two standard dice are rolled and summed together. What is the probability of rolling:

i) a 2 or an 8

ii) a 2 or an odd number

iii) a 3 or a prime number

	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

Solution:

i) As you couldn't throw a 2 and an 8 at the same time, the two events here are mutually exclusive so our answer will be:

$$P(A) + P(B) = \frac{1}{36} + \frac{5}{36} = \frac{6}{36} = \frac{1}{6}$$

ii) Again, as 2 is an even number, these two events are mutually exclusive, so our answer will be:

$$P(A) + P(B) = \frac{1}{36} + \frac{18}{36} = \frac{19}{36}$$

iii) The two events in this case are not mutually exclusive, as 3 is also a prime number, so it would be double counted if we simply added the two probabilities together, so our answer will be:

$$P(A) + P(B) - P(A\ AND\ B) = \frac{2}{36} + \frac{15}{36} - \frac{2}{36} = \frac{15}{36} = \frac{5}{12}$$

b) Conditional Probability:

Notes:

- When there are **conditions on a probability** (probability of A **given** B has occurred):

$$P(A\,|\,B) = \frac{\#(A \cap B)}{\#(B)} = \frac{P(A \cap B)}{P(B)}$$

and it follows also that

Note the matching Bs

$$P(A \cap B) = P(A\,|\,B).P(B)$$

- Watch for the word **'given'**.

Example 1: In a college, 25% of students failed Maths, 15% failed Chemistry and 10% failed both Maths and Chemistry. Student selected at random.

i) Find P(M), the probability that the student failed Maths

ii) Find P(C|M), the probability that the student failed Chemistry, **given** that they failed Maths

i) 25% failed Maths => P(Failed Maths) = 0.25.

ii) 10% failed both subjects => P(M ∩ C) = 10% = 0.1

$$\Rightarrow P(C\,|\,M) = \frac{P(C \cap M)}{P(M)} = \frac{0.1}{0.25} = \frac{2}{5}$$

Example 2: 3 cards chosen at random. Probability that all 3 cards are clubs **given** that 2 of them are known to be picture cards?

- Let A = exactly two of the cards are picture cards

=> choose 2 picture cards from the 12 picture cards and 1 other card from the 40 non-picture cards => ${}^{12}C_2 \times {}^{40}C_1 = 66 \times 40 = 2640$

- Let B = All Clubs

- So $A \cap B$ is that exactly two are picture cards and all are clubs

  => choose 2 from the 3 clubs picture cards and 1 from the 10 other clubs => ${}^{3}C_2 \times {}^{10}C_1 = 3 \times 10 = 30$

- We can now use the rule above:

$$P(B\,|\,A) = \frac{\#(B \cap A)}{\#(A)} = \frac{30}{2640} = \frac{1}{88}$$

c) Independent Events:

Notes:

- Two events are independent if one event **does not affect** the outcome of the other.
  E.g. drawing 2 cards from a deck, find probability of getting 2 clubs, i) with replacement and ii) without replacement.
- If there is replacement, then the two events would be independent of each other.
- If there is no replacement, then the 2nd card is drawn from 51 cards, so the 1st outcomes has an effect on the 2nd outcome => not independent.
- If two events are independent:

$$P(A\,|\,B) = P(A)$$

$$P(B\,|\,A) = P(B)$$

$$P(A \cap B) = P(A) \times P(B)$$

Example: Blue and a white die are thrown. E is the event that the number on the blue die is 3 greater than the white die and F is the event that the total of the numbers on the two dice is 7.

i) Find P(E), P(F) and $P(E \cap F)$. ii) Investigate if E and F are independent.

i) Sample space would be {(1, 1), (1, 2), (1, 3)………(6, 6)}, which has 36 outcomes.

- E is event that the number on blue die is 3 greater than the white

  => successful outcomes are: {(4, 1), (5, 2), (6, 3)}.

  $\Rightarrow P(E) = \frac{3}{36} = \frac{1}{12}$

- F is event that the total of the 2 numbers on the dice sum to 7 => successful outcomes are {(1, 6), (2, 5), (3, 4), (4, 3), (5, 2), (6, 1)}

  $\Rightarrow P(F) = \frac{6}{36} = \frac{1}{6}$

- $E \cap F$ is the event that the number on the blue die is 3 greater than the white and that the total of the two numbers is 7

=> the successful outcomes are {(5, 2)} $\Rightarrow P(E \cap F) = \frac{1}{36}$

- Can use any of the three rules above to check for independence, but in this case rule 3 works best: i.e. $P(A \cap B) = P(A) \times P(B)$

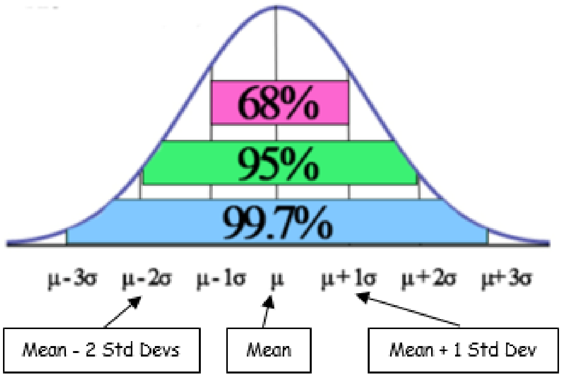
We can see that $\frac{1}{36} \neq \frac{1}{12} \times \frac{1}{6}$ => E and F are not independent

8) Normal Distributions/Z-Scores:

a) Area Under Normal Distribution:

Notes:

- 100% of the data is under the Normal Curve
  => the area under the Normal curve = 1



b) Z-Scores:

Notes:

- To calculate the z-score for a particular data value we use the formula:

$$z = \frac{x-\mu}{\sigma}$$

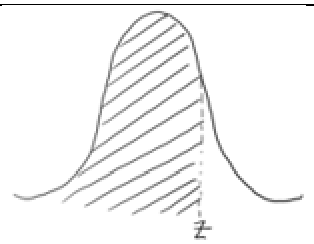
See pg34 Tables book

where μ is the mean and σ is the standard deviation

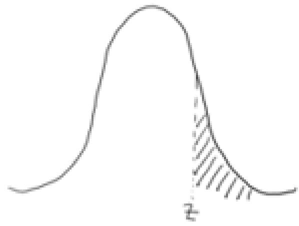
c) Finding Probabilities from the Tables:

Notes:

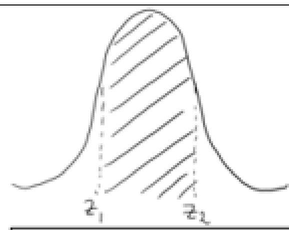
- Area to the **LEFT** of a particular z-score given on pg 36/37 of the Tables => can find probability of event occurring for ANY z-score.



Shaded Area = Area to the Left of z



Shaded Area = 1 – (Area to the Left of z)



Shaded Area = Area to the Left of z_2 – Area to the left of z_1

Examples: Assuming that z is normally distributed with mean 0 and standard deviation 1, find: i) $P(z < 1.46)$ ii) $P(z \geq 2.43)$ iii) $P(z \geq -0.32)$ iv) $P(z < -0.732)$ v) $P(-1.1 < z < 1.64)$

i) Want the area to the left of a positive z-score, so can read that straight from the tables: 0.9279 = 92.79%

ii) Want the area to the right of a positive z-score, so subtract the area to the left of that score from 1:
=> = 1 - $P(z < 2.43)$ = 1 - .9925 = 0.0075 = 0.75%

iii) As curve is symmetrical the area above -0.32 would be the same as the area below +0.32, so: $P(z \leq 0.32)$ = 0.6255 = 62.55%

iv) As curve is symmetrical, the area to the left of z = -0.732, would be the same as the area above z = 0.732
=> = 1 - $P(z < 0.732)$ = 1 – 0.7673 = 0.2327

v) For this part, we will need to work out:

$P(z < 1.64) - P(z < -1.1)$
$= P(z < 1.64) - P(z \geq -1.1)$
$= P(z < 1.64) - [1 - P(z \leq 1.1)]$
$= P(z < 1.64) - 1 + P(z \leq 1.1)$
= 0.9495 – 1 + 0.8643 = 0.8138 = 81.38%

d) Problem Solving:

Steps:

1. Convert the values to z-scores, using the formula.
2. Draw a rough sketch of the curve, and shade the required region.
3. Look up the tables to find the probability.

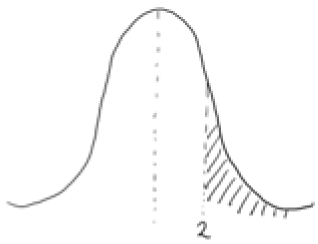
Example: The heights of men in Ireland are normally distributed with a mean of 168cm and a standard deviation of 6cm. Find the probability that an Irishman selected at random will be greater than 180cm in height.

Step 1: Convert the values to z-scores, using the formula.

$$z = \frac{x-\mu}{\sigma}$$

$$\Rightarrow z = \frac{180-168}{6} = 2$$

Step 2: Draw a rough sketch of the curve and shade the required region.



Step 3: Look up the tables to find the probability.

- The area of the shaded region will be:
  = 1 – $P(z < 2)$
  = 1 – 0.9772
  = 0.0228 = 2.28%

9) Expected Value:

Notes:

- A way of determining if a bet is fair, good or bad.

E(X) = All Outcomes X Probability of Each Outcome

- If E(X) = 0 => Bet is **Fair**
- If E(X) > 0 => Bet is **Good**
- If E(X) < 0 => Bet is **Bad**

Example 1: Game costs €3 to play. If a person rolls a 2, they win €12. If they roll a 1, 3 or 5, they get their money back. If they throw a 4 or 6, they lose their money. Good bet or not?

Probability of rolling any number is $\frac{1}{6}$

=> If a 2 is thrown: $\frac{1}{6} \times 12$ = €2

If a 1, 3 or 5 is thrown: $\frac{3}{6} \times 3$ = €1.50

If a 4 or 6 is thrown: = $\frac{2}{6} \times 0$ = €0

Cost of Game = €3 => E(X) = €3.50 - €3 = €0.50

On average, player could expect to win 50cents => a **good bet**.

Example 2: A friend of yours offers you a bet: you have to bet €2. Then you pick a card from a pack. If you choose the Ace of Spades, you win €50 and if you pick a Hearts card, you win €5. Is this a good bet? Justify your answer with reference to the expected value.

- Let X represent the winnings in euro in this example, so the probability distribution will be:

x	P(x)	x. P(x)
€5	$\frac{1}{4}$	$\frac{5}{4}$
€50	$\frac{1}{52}$	$\frac{25}{26}$

- So, the expected value for the winnings would be: = $\sum x.P(x) = \frac{5}{4} + \frac{25}{26} = \frac{115}{52} = €2.20$

It costs €2.00 to play each game, so this is a **good bet**, as you stand to **win €0.20 on average**. (€2.20 - €2.00)