Topic 13: Trigonometry

1) The Basics:

a) Solving Problems:

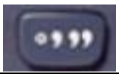
Steps when answering questions

1. Draw a good-sized diagram.
2. Fill in as much information as you can first e.g. the 3rd angle in a triangle where you're given the other 2 angles
3. Label what you're looking for.
4. Is there a right-angled triangle I can use?
   If Yes => Pythagoras Thm, SOHCAHTOA (Section 2 below)
   If No => Go to step 5
5. Do I know an angle and its opposite side?
   If Yes => Sine Rule (Section 3a below)
   If No => Cosine Rule (Section 3b below)

b) Calculator Use:

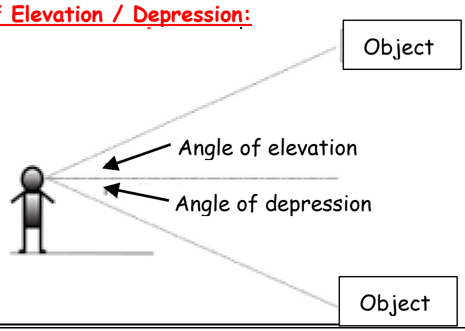
Notes:

- Make sure your calculator is in 'Degree' mode i.e. there is a 'DEG' or a 'D' on the top of your screen.
- If you know the angle, and you want to find Sin, Cos or Tan of it, you can just type it in straight.
  e.g. sin52 = $\boxed{\text{SIN}}$ $\boxed{52}$ $\boxed{=}$ 0.788
- When looking for an angle, then you need to use the SHIFT or 2ndF button in the top left corner of the calculator.
  e.g. Cos A = 0.4534
  => A = $\boxed{\text{SHIFT}}$ $\boxed{\text{COS}}$ $\boxed{0.4534}$ $\boxed{=}$ 63.04°
- To change between degrees and degrees and minutes as well. The button on the Casio calculator for doing that is:



Press this after getting the answer.

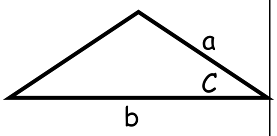
c) Angles of Elevation / Depression:



d) Clinometer

- We can measure angles of elevation / depression using a **clinometer**, as shown below:

e) Area of a Triangle:



- Area can be found when we know **2 sides and the angle in between** the 2 sides:

$$Area = \frac{1}{2}abSinC$$

2) Right Angled Triangles:

a) Pythagoras' Theorem:

Notes:

- We can use **Pythagoras' Theorem** if we know two sides of a right-angled triangle and we want to find the third side i.e.

$$H^2 = O^2 + A^2$$

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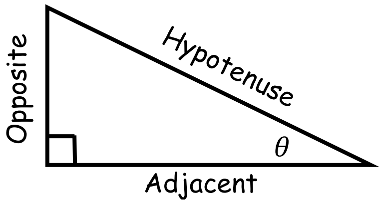
- Make sure and label the hypotenuse correctly when using this theorem.



b) Sine, Cosine, Tan Ratios:

Notes:

- 'θ' is a Greek letter called 'theta'. It is often used to represent angles.
- Another way to remember the sin, cos and tan ratios is Silly Old Harry, Caught A Herring, Trawling Off America (SOHCAHTOA)

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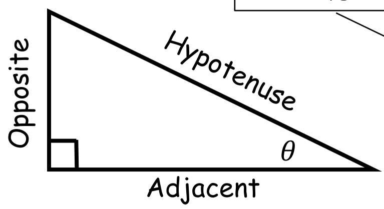


$$sin\theta = \frac{OPP}{HYP}$$

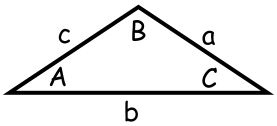
$$cos\theta = \frac{ADJ}{HYP}$$

$$tan\theta = \frac{OPP}{ADJ}$$

3) Non-Right Angled Triangles:

Sine Rule:

- Used if you know a side and its opposite angle
- Side 'a' must be across from angle 'A' and the same for 'b' and 'B'



$$\frac{a}{SinA}=\frac{b}{SinB}=\frac{c}{SinC}$$

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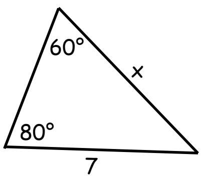
Or

$$\frac{SinA}{a}=\frac{SinB}{b}=\frac{SinC}{c}$$

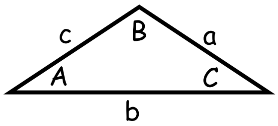
Not in Tables

Example:

Find x in the diagram below. 	$\frac{a}{SinA}=\frac{b}{SinB}$ $\frac{x}{Sin80}=\frac{7}{Sin60}$ $x(Sin60)=7(Sin80)$ (Cross Multiply) => $x=\frac{7(Sin80)}{Sin60}$ (÷ both sides by sin60) => $x=7.96$

Cosine Rule:

- Used if Sine Rule can't be used
- The side you label 'a' **must** be across from the angle you label 'A'. Label the unknown side 'a' or label the unknown angle 'A'.



$$a^2=b^2+c^2-2bccosA$$

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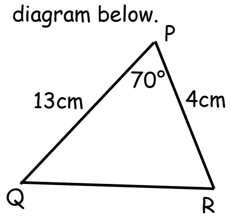
Or

$$cosA=\frac{b^2+c^2-a^2}{2bc}$$

Not in Tables

Example:

Find |QR| in the diagram below. 	Label unknown side 'a' => 70 angle = 'A' $a^2=b^2+c^2-2bccosA$ $a^2=(13)^2+(4)^2-2(13)(4)ccos70$ $a^2=185-35.57$ $a^2=149.43$ $a=\sqrt{149.43}$ $a=12.22$

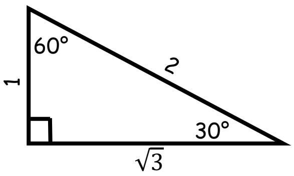
4) Special Angles/Unit Circle:

a) Special Angles:

- Use the table below (pg 13 of Tables) to write down the sin, cos or tan of the angles shown, in the form $\frac{a}{b}$

A (degrees)	0°	90°	180°	270°	30°	45°	60°
A (radians)	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$
cos A	1	0	-1	0	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$
sin A	0	1	0	-1	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$
tan A	0	-	0	-	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$

- Useful to know the right-angled triangles these ratios come from. e.g.



Sin 30 = $\frac{OPP}{HYP}=\frac{1}{2}$

Cos 30 = $\frac{ADJ}{HYP}=\frac{\sqrt{3}}{2}$

Tan 60 = $\frac{OPP}{ADJ}=\frac{\sqrt{3}}{1}$

- Can also to simplify expressions into surd form

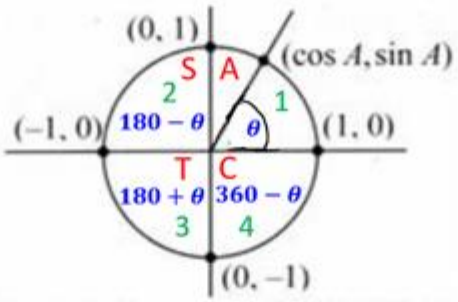
Example: Write $cos30+sin30$ in surd form.

$$cos30+sin60=\frac{\sqrt{3}}{2}+\frac{1}{2}=\frac{\sqrt{3}+1}{2}$$

b) Unit Circle:

Notes:

- Need to be able to write sin, cos and tan of angles that are bigger than 90 in surd form, without a calculator.



Examples: Write i) sin 150 and ii) cos225 iii) sin 300 in surd form

i) 150 in quadrant 2 => will be positive for sin

Ref Angle = $180-\theta=150$ => $\theta=30°$

=> sin150 = + sin30 = $+\frac{1}{2}$

ii) 225 in quadrant 3 => will be negative for cos

Ref Angle = $180+\theta=225$ => $\theta=45°$

=> cos225 = - cos 45 = $-\frac{1}{\sqrt{2}}$

iii) 300 in quadrant 4 => will be negative for sin

Ref Angle = $360-\theta=300$ => $\theta=60°$

=> sin300 = - sin60 = $-\frac{\sqrt{3}}{2}$

5) Radians/Sectors:

a) Radians:

Notes:

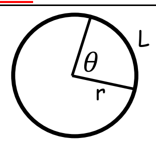
- An alternative way of measuring angles.
- To convert between radians and degrees, we use:

π radians = 180°

Example: i) Convert 145° to radians and ii) Convert $\frac{3\pi}{4}$ to degrees

i)	ii)
180° = π radians 1° = $\frac{\pi}{180}$ $(\div 180)$ 145° = $\frac{\pi}{180}$ x 145 $(\times\ 145)$ = $\frac{29\pi}{36}$ radians	$\frac{3\pi}{4}$ = $\frac{3(180)}{4}$ = 135°

b) Sectors:



θ in degs	θ in rads
$L = 2\pi r(\frac{\theta}{360})$	$L = r\theta$
$A = \pi r^2(\frac{\theta}{360})$	$A = \frac{1}{2}r^2\theta$

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Example: Find i) the area and ii) length of a sector of radius 5cm with angle $\frac{2\pi}{3}$ radians at the centre.

i)

$$A = \frac{1}{2}r^2\theta$$

$$A = \frac{1}{2}(5)^2(\frac{2\pi}{3})$$

= 216.18cm^2

ii)

$$l = r\theta$$

$$l = (5)(\frac{2\pi}{3})$$

= 10.47cm

6) Periodic Functions/Sin, Cos Graphs:

a) Periodic Functions:

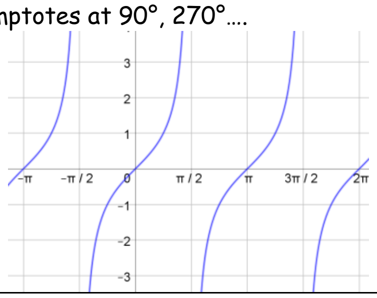
Notes:

- A function that repeats itself every so often.
- The **range** of a function are the lowest and biggest the y-values can be.....written in the form [Ymin, Ymax]
- The **period** is how often the function repeats itself.

b) Graphs of Tangent Function:

Notes:

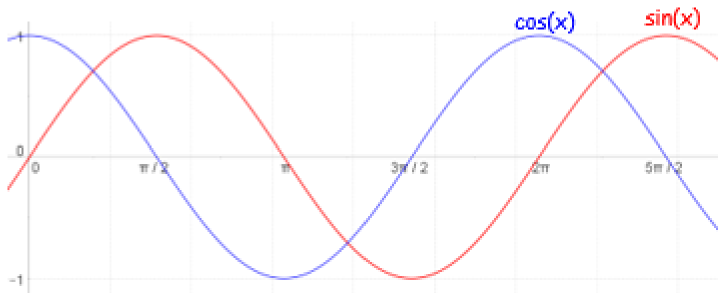
- Asymptotes at 90°, 270°....



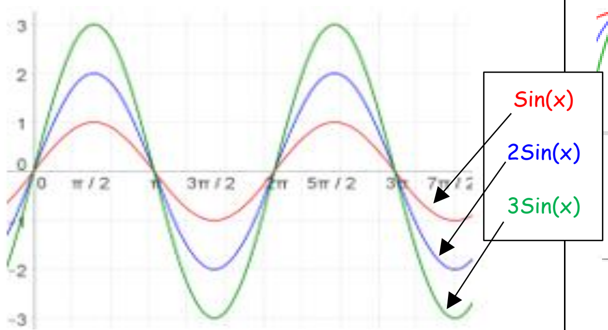
c) Graphs of Sine/Cosine Functions:

Notes:

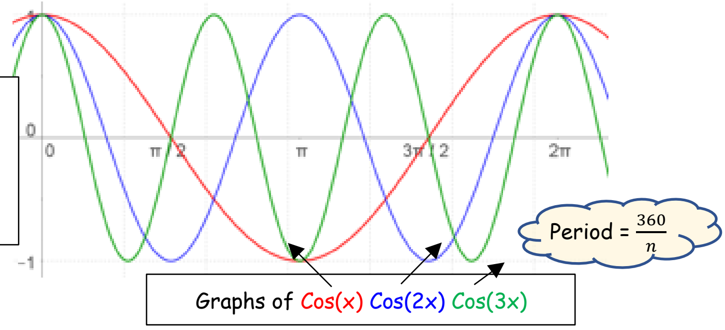
- Range of both is [-1, 1].
- Period of both is 360°.



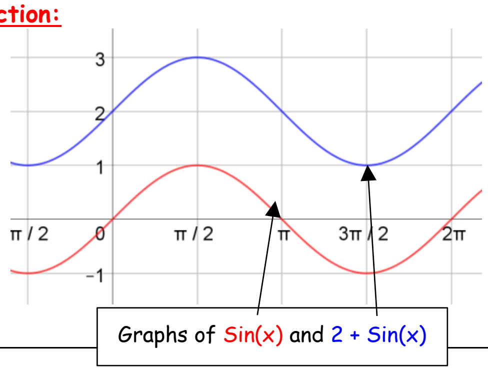
d) Effect of changing 'a' in aSin(nx) or aCos(nx):



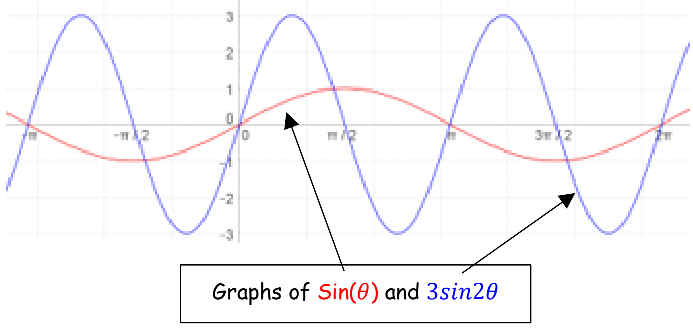
e) Effect of changing 'n' in aSin(nx) or aCos(nx):



f) Effect of adding a constant to Sin/Cos function:



Example: Graph the function $y = 3sin2\theta$, $-\pi \le \theta \le \pi$.



7) Solving Trig Equations:

Steps:

1. Ignore the sign and calculate the reference angle
2. Use the sign to decide what 2 quadrants your answers are in.
3. Use unit circle to find the angle in each of the 2 quadrants.
4. If the angle is a double or triple angle e.g. 3A, then add on multiples of 360 to the answers from step 3.
5. Divide your list of answers by the coefficient of the angle we're looking for, to get a list of solutions.
6. Eliminate any answers outside the range stated in the question.

Example 1: Solve $\cos B = \frac{1}{\sqrt{2}}$, where $0 \leq A \leq 360°$.

Step 1: Ignore the sign and calculate reference angle i.e.

Reference Angle θ = $\cos^{-1}\frac{1}{\sqrt{2}}$ = 45°

Step 2: Use the sign to decide what quadrants your answers are in:

Cosine is positive in Quadrants 1 and 4

Step 3: Use the unit circle to find the angle in each of the 2 quadrants from step 2.

Quadrant 1: θ = 45°

Quadrant 4: 360 - θ = 360 - 45 = 315°

Example 2: Solve $\sin 2A = -\frac{1}{2}$, where $0 \leq A \leq 360°$.

Step 1: Ignore the sign and calculate reference angle i.e.

Reference Angle θ = $\sin^{-1}\frac{1}{2}$ = 30°

Step 2: Use the sign to decide what quadrants your answers are in:

Sine is negative => Quadrants 3 and 4

Step 3: Use the unit circle to find the angle in each of the 2 quadrants from step 2.

Quad 3: 180 + θ = 210° Quad 4: 360 - θ = 330°

Step 4: If the angle is a double or triple angle e.g. 3A, then add on multiples of 360 to the answers from step 3.

210°, 330°, 570°, 690°, 930°.........

Step 5: Divide your list of answers by the coefficient of the angle we're looking for, to get a list of solutions…in this case '2'

105°, 115°, 285°, 345°, 465°........

Step 6: Eliminate any answers outside the required range:

- Only interested here in angles between 0 and 360° so we exclude the 465°, so our final solution is:

A = 105°, 115°, 285°, 345°

8) Trig Identities:

Compound Angle Formulae

1. $\cos(A + B) = cosAcosB - sinAsinB$
2. $\cos(A - B) = cosAcosB + sinAsinB$
3. $\sin(A + B) = sinAcosB + cosAsinB$
4. $\sin(A - B) = sinAcosB - cosAsinB$
5. $\tan(A + B) = \frac{\tan A + \tan B}{1 - tanAtanB}$
6. $\tan(A - B) = \frac{\tan A - \tan B}{1 + tanAtanB}$

Double Angle Formulae

7. $\cos 2A = cos^2A - sin^2A$
8. $\tan 2A = \frac{2\tan A}{1 - tan^2A}$
9. $\sin 2A = 2sinAcosA$
10. $cos^2A = \frac{1}{2}(1 + cos2A)$
11. $\cos 2A = \frac{1 - tan^2A}{1 + tan^2A}$
12. $sin^2A = \frac{1}{2}(1 - cos2A)$
13. $\sin 2A = \frac{2tanA}{1 + tan^2A}$

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Products to Sums and Differences

14. $2cosAcosB = \cos(A + B) + \cos(A - B)$
15. $2sinAcosB = \sin(A + B) + \sin(A - B)$
16. $2sinAsinB = \cos(A - B) - \cos(A + B)$
17. $2cosAsinB = \sin(A + B) - \sin(A - B)$

Sums and Differences to Products

18. $cosA + cosB = 2\cos\frac{A + B}{2}\cos\frac{A - B}{2}$
19. $cosA - cosB = -2\sin\frac{A + B}{2}\sin\frac{A - B}{2}$
20. $sinA + sinB = 2\sin\frac{A + B}{2}\cos\frac{A - B}{2}$
21. $sinA - sinB = 2\cos\frac{A + B}{2}\sin\frac{A - B}{2}$

22. $cos^2A + sin^2A = 1$

Example 1: Express cos75 in surd form, without using a calculator.

- Break up the angle into a combination of the special angles 35° and 40° => cos75 = cos(45 + 30)
- Use identity 1 to rewrite the expression on the right above as:

cos(A + B) = cosAcosB - sinAsinB

cos(45 + 30) = cos45cos30 - sin45sin30

$= \left(\frac{1}{\sqrt{2}}\right)\left(\frac{\sqrt{3}}{2}\right) + \left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{2}\right)$

$= \left(\frac{\sqrt{3}}{2\sqrt{2}}\right) + \left(\frac{1}{2\sqrt{2}}\right)$

$= \frac{\sqrt{3} + 1}{2\sqrt{2}}$

Example 2: If tan(A + B) = 8 and tanA = 2, find the value of tanB.

- Use identity 5 to rewrite an expression for tan(A + B)

tan(A + B) = $\frac{tanA + tanB}{1 - tanAtanB}$ => $\frac{tanA + tanB}{1 - tanAtanB} = 8$

- Cross-multiplying gives:

$tanA + tanB = 8(1 - tanAtanB)$

=> $tanA + tanB = 8 - 8tanAtanB$

- We can now fill in the value of tanA, which we were given in the question and solve for tanB:

=> $2 + tanB = 8 - 8(2)tanB$

=> $2 + tanB = 8 - 16tanB$

=> $tanB = \frac{6}{17}$