Math 534 HW 3

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(1)

Let G be a group, and let $H < G$ and $K < G$ be two proper subgroups. Show that $G \neq H \cup K$ (as sets).

Let $a, b \in G$, with the properties that $a \in H$, $a \notin K$, $b \in K$, and $b \notin H$. If no such two elements exist, then $\forall a \in H, a \in K$ or $\forall b \in K, b \in H$, which means that one of these groups is a subgroup of the other, and their union must not be equal to G as it is equal to either H or K (the union of a set and its subset is equal to the set). Because of the existence of these elements in G, ab must also be in G. However, since only one a or b is in either H or K (and, in groups, $xy \neq xz$ for distinct y, z), ab cannot be in either of these sets. Therefore, the union of these two sets does not contain ab, so $G \neq H \cup K$.

(2)

Let G be a group which contains two distinct elements $a, b \in G$ which commute (i.e. $ab = ba$) and satisfy $|a| = 2 = |b|$. Prove that G contains a subgroup $H \leq G$ of order 4.

The subgroup $H = \{e, a, b, ab\}$ has order four and is a subgroup of G. This subgroup is closed (proof by exhaustion):

- Closure of e:
  - $ee = e \in H$
  - $ea = a \in H$
  - $eb = b \in H$
  - $e(ab) = ab \in H$
- Closure of a:
  - $ae = a \in H$
  - $aa = e \in H$
  - $ab = ab \in H$
  - $a(ab) = (aa)b = b \in H$

- Closure of b:
  - $be = b \in H$
  - $ba = ab \in H$
  - $bb = e \in H$
  - $b(ab) = (bb)a = a \in H$
- Closure of ab:
  - $(ab)e = ab \in H$
  - $(ab)a = b(aa) = b \in H$
  - $(ab)b = a(bb) = a \in H$
  - $(ab)(ab) = (aa)(bb) = e \in H$

It also contains the identity e, and each element has an inverse $((e)^{-1} = e,\ (a)^{-1} = a,\ (b)^{-1} = b, (ab)^{-1} = ab)$, so it is certainly a subgroup of G of order 4.

(3)

Denote the set of 2×2 matrices with entries in $\mathbb{R}$ by $M_2(\mathbb{R})$. Define matrix multiplication as

$$\begin{Bmatrix} a & b \\ c & d \end{Bmatrix} \begin{Bmatrix} w & x \\ y & z \end{Bmatrix} = \begin{Bmatrix} aw + by & ax + bz \\ cw + dy & cx + dz \end{Bmatrix}.$$

(a)

Let $GL_2(\mathbb{R}) = \left\{ \begin{Bmatrix} a & b \\ c & d \end{Bmatrix} \in M_2(\mathbb{R}) : ad - bc \neq 0 \right\}$. Prove that $GL_2(\mathbb{R})$ is a group.

Proof that $GL_2(\mathbb{R})$ is closed:

For two arbitrary elements of $GL_2(\mathbb{R})$, their multiplication is defined by

$$\begin{Bmatrix} a & b \\ c & d \end{Bmatrix} \begin{Bmatrix} w & x \\ y & z \end{Bmatrix} = \begin{Bmatrix} aw + by & ax + bz \\ cw + dy & cx + dz \end{Bmatrix}.$$

For this resultant matrix to be in the group, it must satisfy

$$(aw + by)(cx + dz) - (ax + bz)(cw + dy) \neq 0.$$

Additionally, it is provided by definition of an element of this group that $ad - bc \neq 0$ and $wz - xy \neq 0$. We will assume the negation to prove this:

$$(aw + by)(cx + dz) - (ax + bz)(cw + dy) = 0$$

$$\implies acwx + adwz + bcxy + bdyz - acwx - adxy - bcwz - bdyz = 0$$

$$\implies adwz + bcxy - adxy - bcwz = 0$$
$$\implies ad(wz - xy) - bc(wz - xy) = 0$$
$$\implies ad - bc = 0 \text{ or } wz - xy = 0,$$

both of which contradict our assumptions, so this equation cannot be true and this group must be closed.

Proof that $GL_2(\mathbb{R})$ is associative:
Let $A = \begin{Bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{Bmatrix}$, $B = \begin{Bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{Bmatrix}$, and $C = \begin{Bmatrix} c_1 & c_2 \\ c_3 & c_4 \end{Bmatrix}$. We will now show that $(AB)C = A(BC)$.

$$AB = \begin{Bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{Bmatrix} \begin{Bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{Bmatrix} = \begin{Bmatrix} a_1b_1 + a_2b_3 & a_1b_2 + a_2b_4 \\ a_3b_1 + a_4b_3 & a_3b_2 + a_4b_4 \end{Bmatrix},$$

$$(AB)C = \begin{Bmatrix} a_1b_1 + a_2b_3 & a_1b_2 + a_2b_4 \\ a_3b_1 + a_4b_3 & a_3b_2 + a_4b_4 \end{Bmatrix} \begin{Bmatrix} c_1 & c_2 \\ c_3 & c_4 \end{Bmatrix}$$

$$= \begin{Bmatrix} a_1b_1c_1 + a_2b_3c_1 + a_1b_2c_3 + a_2b_4c_3 & a_1b_1c_2 + a_2b_3c_2 + a_1b_1c_4 + a_2b_4c_4 \\ a_3b_1c_1 + a_4b_3c_1 + a_3b_2c_3 + a_4b_4c_3 & a_3b_1c_2 + a_4b_3c_2 + a_3b_2c_4 + a_4b_4c_4 \end{Bmatrix}.$$

$$BC = \begin{Bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{Bmatrix} \begin{Bmatrix} c_1 & c_2 \\ c_3 & c_4 \end{Bmatrix} = \begin{Bmatrix} b_1c_1 + b_2c_3 & b_1c_2 + b_2c_4 \\ b_3c_1 + b_4c_3 & b_3c_2 + b_4c_4 \end{Bmatrix},$$

$$A(BC) = \begin{Bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{Bmatrix} \begin{Bmatrix} b_1c_1 + b_2c_3 & b_1c_2 + b_2c_4 \\ b_3c_1 + b_4c_3 & b_3c_2 + b_4c_4 \end{Bmatrix}$$

$$= \begin{Bmatrix} a_1b_1c_1 + a_2b_3c_1 + a_1b_2c_3 + a_2b_4c_3 & a_1b_1c_2 + a_2b_3c_2 + a_1b_1c_4 + a_2b_4c_4 \\ a_3b_1c_1 + a_4b_3c_1 + a_3b_2c_3 + a_4b_4c_3 & a_3b_1c_2 + a_4b_3c_2 + a_3b_2c_4 + a_4b_4c_4 \end{Bmatrix},$$

so $(AB)C = A(BC)$.

Proof that there is an identity element in $GL_2(\mathbb{R})$:
Let $A = \begin{Bmatrix} a & b \\ c & d \end{Bmatrix} \in GL_2(\mathbb{R})$ and $I = \begin{Bmatrix} 1 & 0 \\ 0 & 1 \end{Bmatrix} \in GL_2(\mathbb{R})$. Then, we have

$$AI = \begin{Bmatrix} a & b \\ c & d \end{Bmatrix} \begin{Bmatrix} 1 & 0 \\ 0 & 1 \end{Bmatrix} = \begin{Bmatrix} a & b \\ c & d \end{Bmatrix} = A$$

and

$$IA = \begin{Bmatrix} 1 & 0 \\ 0 & 1 \end{Bmatrix} \begin{Bmatrix} a & b \\ c & d \end{Bmatrix} = \begin{Bmatrix} a & b \\ c & d \end{Bmatrix} = A,$$

so this element of $GL_2(\mathbb{R})$ is the identity.

Proof that each $A \in GL_2(\mathbb{R})$ has an inverse:
Let $A = \begin{Bmatrix} a & b \\ c & d \end{Bmatrix}$. The inverse of A is equal to $\frac{1}{ad-bc} \begin{Bmatrix} d & -b \\ -c & a \end{Bmatrix}$, as

$$AA^{-1} = \frac{1}{ad - bc} \begin{Bmatrix} a & b \\ c & d \end{Bmatrix} \begin{Bmatrix} d & -b \\ -c & a \end{Bmatrix} = \frac{1}{ad - bc} \begin{Bmatrix} ad - bc & 0 \\ 0 & ad - bc \end{Bmatrix} = \begin{Bmatrix} 1 & 0 \\ 0 & 1 \end{Bmatrix} = I$$

and

$$A^{-1}A = \frac{1}{ad-bc}\begin{Bmatrix} d & -b \\ -c & a \end{Bmatrix}\begin{Bmatrix} a & b \\ c & d \end{Bmatrix} = \frac{1}{ad-bc}\begin{Bmatrix} ad-bc & 0 \\ 0 & ad-bc \end{Bmatrix} = \begin{Bmatrix} 1 & 0 \\ 0 & 1 \end{Bmatrix} = I,$$

so each matrix in $GL_2(\mathbb{R})$ has an inverse.

Since $GL_2(\mathbb{R})$ satisfies all of the group axioms, it is a group.

(b)

Is $GL_2(\mathbb{R})$ abelian? If so, prove it, and if not, give an example indicating why it isn't.

Let $A = \begin{Bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{Bmatrix}$ and $B = \begin{Bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{Bmatrix}$. Then, we have

$$AB = \begin{Bmatrix} a_1b_1 + a_2b_3 & a_1b_2 + a_2b_4 \\ a_3b_1 + a_4b_3 & a_3b_2 + a_4b_4 \end{Bmatrix}$$

and

$$BA = \begin{Bmatrix} a_1b_1 + a_3b_2 & a_2b_1 + a_4b_2 \\ a_1b_3 + a_3b_4 & a_2b_3 + a_4b_4 \end{Bmatrix}.$$

It is evident that these are not always the same, so this group is not abelian. As an example, let $A = \begin{Bmatrix} 1 & 1 \\ 0 & 1 \end{Bmatrix}$ and $B = \begin{Bmatrix} 1 & 0 \\ 1 & 1 \end{Bmatrix}$.$AB$ is then equal to $\begin{Bmatrix} 2 & 1 \\ 1 & 1 \end{Bmatrix}$ while BA is equal to $\begin{Bmatrix} 1 & 1 \\ 1 & 2 \end{Bmatrix}$.

(c)

Prove that $SL_2 = \left\{ \begin{Bmatrix} a & b \\ c & d \end{Bmatrix} \in M_2(\mathbb{R}) : ad - bc = 1 \right\}$ is a subgroup of $GL_2(\mathbb{R})$.

All elements in SL_2 necessarily must be in $GL_2(\mathbb{R}))$ as $ad - bc = 1 \implies ad - bc \neq 0$.
Proof that SL_2 is closed:

Let $A = \begin{Bmatrix} a & b \\ c & d \end{Bmatrix}$ and $B = \begin{Bmatrix} w & x \\ y & z \end{Bmatrix}$ Their product is then given by

$$\begin{Bmatrix} a & b \\ c & d \end{Bmatrix}\begin{Bmatrix} w & x \\ y & z \end{Bmatrix} = \begin{Bmatrix} aw + by & ax + bz \\ cw + dy & cx + dz \end{Bmatrix}.$$

For this resultant matrix to be in the group, it must satisfy

$$(aw + by)(cx + dz) - (ax + bz)(cw + dy) = 1.$$

Additionally, it is provided by definition of an element of this group that $ad - bc = 1$ and $wz - xy = 1$. We will compute out the requisite equation to check if it is satisfied:

$$(aw + by)(cx + dz) - (ax + bz)(cw + dy) = 1$$

$$\begin{aligned}
\implies acwx + adwz + bcxy + bdyz - acwx - adxy - bcwz - bdyz = 1 \\
\implies adwz + bcxy - adxy - bcwz = 1 \\
\implies ad(wz - xy) - bc(wz - xy) = 0 \\
\implies (ad - bc)(wx - yz) = 1.
\end{aligned}$$

Because both $ad - bc$ and $wx - yz$ are equal to one, their product is also equal to one, and so this equation and the inclusion of $AB \in SL_2$ holds.

Proof that SL_2 has an identity:

$\begin{Bmatrix} 1 & 0 \\ 0 & 1 \end{Bmatrix}$ is the identity and is in SL_2, as $(1)(1) - (0)(0) = 1$.

Proof that SL_2 has inverses:

Let $A = \begin{Bmatrix} a & b \\ c & d \end{Bmatrix} \in SL_2$. $A^{-1} = \begin{Bmatrix} d & -b \\ -c & a \end{Bmatrix}$. This is also in SL_2 as $(da) - (bc) = 1$ holds because $ad - bc = 1$ due to the inclusion of A in SL_2.

Therefore, SL_2 is a subgroup of $GL_2(\mathbb{R})$.

(d)

Compute the orders of $\begin{Bmatrix} 0 & -1 \\ 1 & 1 \end{Bmatrix}$ and $\begin{Bmatrix} 1 & -1 \\ 0 & 1 \end{Bmatrix}$.

Let I be the identity matrix.

$$\begin{Bmatrix} 0 & -1 \\ 1 & 1 \end{Bmatrix} \neq I$$

$$\begin{Bmatrix} 0 & -1 \\ 1 & 1 \end{Bmatrix}\begin{Bmatrix} 0 & -1 \\ 1 & 1 \end{Bmatrix} = \begin{Bmatrix} -1 & -1 \\ 1 & 0 \end{Bmatrix} \neq I$$

$$\begin{Bmatrix} 0 & -1 \\ 1 & 1 \end{Bmatrix}\begin{Bmatrix} 0 & -1 \\ 1 & 1 \end{Bmatrix}\begin{Bmatrix} 0 & -1 \\ 1 & 1 \end{Bmatrix} = \begin{Bmatrix} -1 & 0 \\ 0 & -1 \end{Bmatrix} \neq I$$

$$\begin{Bmatrix} 0 & -1 \\ 1 & 1 \end{Bmatrix}\begin{Bmatrix} 0 & -1 \\ 1 & 1 \end{Bmatrix}\begin{Bmatrix} 0 & -1 \\ 1 & 1 \end{Bmatrix}\begin{Bmatrix} 0 & -1 \\ 1 & 1 \end{Bmatrix} = \begin{Bmatrix} 0 & 1 \\ -1 & -1 \end{Bmatrix} \neq I$$

$$\begin{Bmatrix} 0 & -1 \\ 1 & 1 \end{Bmatrix}\begin{Bmatrix} 0 & -1 \\ 1 & 1 \end{Bmatrix}\begin{Bmatrix} 0 & -1 \\ 1 & 1 \end{Bmatrix}\begin{Bmatrix} 0 & -1 \\ 1 & 1 \end{Bmatrix}\begin{Bmatrix} 0 & -1 \\ 1 & 1 \end{Bmatrix} = \begin{Bmatrix} 1 & 1 \\ -1 & 0 \end{Bmatrix} \neq I$$

$$\begin{Bmatrix} 0 & -1 \\ 1 & 1 \end{Bmatrix}\begin{Bmatrix} 0 & -1 \\ 1 & 1 \end{Bmatrix}\begin{Bmatrix} 0 & -1 \\ 1 & 1 \end{Bmatrix}\begin{Bmatrix} 0 & -1 \\ 1 & 1 \end{Bmatrix}\begin{Bmatrix} 0 & -1 \\ 1 & 1 \end{Bmatrix}\begin{Bmatrix} 0 & -1 \\ 1 & 1 \end{Bmatrix} = \begin{Bmatrix} 1 & 0 \\ 0 & 1 \end{Bmatrix} = I,$$

so

$$\left|\begin{Bmatrix} 0 & -1 \\ 1 & 1 \end{Bmatrix}\right| = 6.$$

Note that for positive integer n,

$$\begin{Bmatrix} 1 & -1 \\ 0 & 1 \end{Bmatrix} \begin{Bmatrix} 1 & -n \\ 0 & 1 \end{Bmatrix} = \begin{Bmatrix} 1 & -n-1 \\ 0 & 1 \end{Bmatrix},$$

so this matrix has infinite order as no positive amount of combinations of itself will lead to the identity matrix.

(4)

Show that a group of order 3 must be cyclic.

Proof by contradiction:

Let G be a non-cyclic group of order 3. We can write the group as $G = \{e, a, b\}$. As G is non-cyclic, we can say that b is not a power of a. Therefore, aa must equal e, as it cannot equal b or e. Since $ae = a$, we must have $ab = b$, as it cannot be any other element in G. However, this is a contradiction, as a is not the identity element but is simultaneously acting as the identity for b, so this group must be cyclic (and $b = a^2$).

(5)

Let G be a group and let $g \in G$ have infinite order, i.e. $|g| = \infty$. For $k, l \in \mathbb{Z}$, show that the cyclic groups generated by g^k and g^l are equal if and only if $k = \pm l$.

($\implies$) Proof by contradiction:

Since g has infinite order, $\forall n, m \in Z$, $g^n = g^m$ holds true if and only if $n = m$, as $g^{n-m} = e$ implies that $n - m = 0 \implies n = m$. Assume that $\langle g^k \rangle = \langle g^l \rangle$. For sake of contradiction, assume that $k \neq \pm l$. This implies that for all elements $g^{nk} \in \langle g^k \rangle$, there is an equivalent element $g^{ml} \in \langle g^l \rangle$ such that $g^{nk} = g^{ml} \implies g^{nk-ml} = g^0 = e \implies nk = ml$. We now have the result that $\forall n \in \mathbb{Z}$, $\exists m \in \mathbb{Z}$ such that $nk = ml$ while $k \neq \pm l$. This is not possible, so $m = pml$.

($\impliedby$) Direct proof:

Let $m = \pm l$. The group generated by g^k is then all g to integer multiples of k or $\pm l$, which certainly holds true. We can also see this when listing the group elements:

$$\langle g^k \rangle = \{..., g^{-2k}, g^{-k}, e, g^k, g^{2k}, ...\} = \{..., g^{\pm 2l}, g^{\pm l}, e, g^{\mp l}, g^{\mp 2l}\} = \langle g^l \rangle.$$