Math 521 HW 7

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Exercise 2.4.7. Let (a_n) be a bounded sequence.

2.4.7.a. Prove that the sequence defined by $y_n = \sup\{a_k : k \geq n\}$.

Each y_n has the property that $y_n \geq a_k$ for $k \geq n$. Additionally, for all $k \in \mathbb{N}$, $|a_n| \leq M$ for some M. All $|y_n|$ are then less than or equal to M, so this sequence is also bounded. For this sequence to converge by the Monotonous Convergence Theorem, it must then be monotonous. We would like to show that $y_n \geq y_{n+1}$ i.e. $\sup\{a_k : k \geq n\} \geq \sup\{a_k : k \geq n+1\}$. y_n is greater than all $a_k,\ k \geq n$, and y_{n+1} is greater than all $a_k,\ k \geq n+1$, yet y_n is greater than or equal to all elements that y_{n+1} is greater than or equal to, so $y_n \geq y_{n+1}$. Because this sequence is bounded and monotone, $\lim y_n$ exists.

2.4.7.b. The limit superior of (a_n), or $\limsup a_n$ is defined by

$$\limsup a_n = \lim y_n,$$

where y_n is the sequence from part (a) of this exercise. Provide a reasonable definition for $\liminf a_n$ and briefly explain why it always exists for any bounded sequence.

Define $x_n = \inf\{a_k : k \geq n\}$. The limit inferior of (a_n) is then defined by

$$\liminf a_n = \lim x_n.$$

For a bounded sequence, x_n is then also bounded as every $|x_n| \leq |a_k| \leq M$ (for some M) for all n and $k \geq n$. This sequence is also increasing (and necessarily convergent) as $\forall n \in \mathbb{N}$, $x_n \leq x_{n+1}$, as $x_n = \inf\{a_k : k \geq n\} \leq \inf\{a_k : k \geq n+1\} = x_{n+1}$. Therefore this sequence is convergent.

2.4.7.c. Prove that $\liminf a_n \leq \limsup a_n$, for every bounded sequence, and give an example of a sequence for which the inequality is strict.

For a bounded set, the limit superior and limit inferior both converge. To prove that the limit inferior is less than or equal to the limit superior, we will take an arbitrary n and compare $y_n = \sup\{a_k : k \geq n$ and $x_n = \inf\{a_k : k \geq n\}$. As $y_n \geq a_k \geq x_n$ for all $k \geq n$, then for all n, $y_n \geq x_n$, so by the Order Limit Theorem, $\liminf a_n \leq \limsup a_n$.

This inequality is strict in the case where $a_n = (-1)^n$, as $\liminf a_n = -1 < \limsup a_n = 1$.

2.4.7.d. Show that $\liminf a_n = \limsup a_n$ if and only if $\lim a_n$ exists. In this case, all three share the same value.

Proof of ($\implies$). Assume that $\lim a_n = a$ exists. This implies that a_n is bounded, and as such, $\liminf a_n = b$ and $\limsup a_n = c$ exist. Due to the convergence of $\lim a_n$, we have given $\epsilon > 0$, there exists $n \in \mathbb{N}$ such that $k \geq n$ implies

$$|a_k - a| < \epsilon.$$

As $\inf\{a_n : n \geq k\} \leq a_k$, we have

$$|\inf\{a_n : n \geq k\} - a| < \epsilon,$$

so $\liminf a_n = \lim a_n$. Using the earlier inequality, we have

$$\begin{aligned}|a_k - a| &= |a_k - \sup\{a_n : n \geq k\} + \sup\{a_n : n \geq k\} - a| \\ &\leq |a_k - \sup\{a_n : n \geq k\}| + |\sup\{a_n : n \geq k\} - a|.\end{aligned}$$

As a result of being a supremum, we have $\exists s \in \mathbb{R}$ such that $\sup\{a_n : n \geq k\} - s < a_k \implies \sup\{a_n : n \geq k\} - a_k < s$, so this expression is less that $s + |\sup\{a_n : n \geq k\} - a| < \epsilon$, so this converges for correct choice of ϵ, so $\limsup a_n = \lim a_n = \liminf a_n$.

Proof of ($\impliedby$). Assume that $\liminf a_n = \limsup a_n$. As $x_n \leq a_n \leq y_n$ for all n, and $\lim x_n = \lim y_n$, $\lim a_n$ exists and is equal to the other two limits by the squeeze theorem.

Exercise 2.4.8. For each series, find an explicit formula for the sequence of partial sums and determine if the series converges.

2.4.8.b. $\sum_{n=1}^{\infty} 1/n(n+1)$.

$$s_k = \sum_{n=1}^{k} 1/n(n+1) \implies s_1 = 1/2, s_2 = 2/3, s_3 = 3/4, ..., s_k = k/k+1.$$

The sequence $k/k+1$ is increasing and bounded, so this series converges.

2.4.8.c. $\sum_{n=1}^{\infty} \log\left(\frac{n+1}{n}\right)$.

$$s_k = \sum_{n+1}^{k} \log\left(\frac{n+1}{n}\right) \implies s_1 = \log 2, s_2 = \log 3, s_3 = \log 4, ..., s_k = \log k + 1.$$

This sequence is unbounded, so it is not convergent. The series does not converge.

Exercise 2.5.1. Give an example of each of the following, or argue that such a request is impossible.

2.5.1.a. A sequence that has a subsequence that is bounded but contains no subsequence that converges.

This is not possible, as the Bolzano-Weierstrass Theorem states that all bounded sequences have a convergent subsequence. This bounded subsequence thus has a convergent subsequence. This convergent subsequence must be a subsequence of the original sequence, so the original sequence must have a convergent subsequence.

2.5.1.b. A sequence that does not contain 0 or 1 as a term but contains subsequences converging to each of these values.

The subsequence $a_n = 1/(n+1)$ (if n odd), $1 + 1/n$ (if n even) has subsequences converging to both of these values (the even and odd subsequences) but contains neither.

2.5.1.c. A sequence that contains subsequences converging to every point in the infinite set $\{1, 1/2, 1/3, 1/4, 1/5, ...\}$.

The sequence $a_n = \{1, 1/2, 1, 1/2, 1/3, 1, 1/2, 1/3, 1/4, 1, ...\}$ contains convergent subsequences to each of these elements.

2.5.1.d. A sequence that contains subsequences converging to every point in the infinite set $\{1, 1/2, 1/3, 1/4, 1/5, ...\}$, and no subsequences converging to points outside of this set.

This is not possible, as a set with a sequence converging to $1/n$ for all n must also converge to 0, which is not in said set.

Exercise 2.5.6. Use a similar strategy to the one in Example 2.5.3 to show that $\lim b^{1/n}$ exists for all $b \geq 0$ and find the value of the limit.

We will break this into two cases, $b < 1$ and $b > 1$.

Sidenote: $b = 0$ implies that all $b^{1/n} = 0^{1/n} = 0$, so this converges to 0.

Case 1, $b < 1$. $b \leq b^{1/2} \leq b^{1/3} \leq ... \leq 1$, so this sequence is bounded and increasing, and as such, convergent. We will say that $\lim b^{1/n} = l$. The limit of the subsequence $b^{1/2n} = l$, as subsequences of convergent sequences converge to the same limit. By the Algebraic Limit Theorem, we also have $\lim b^{1/2n} = l = \sqrt{l}$, so $l = 0$ or $l = 1$. However, as it is increasing it cannot converge to 0. Therefore, $\lim b^{1/n} = 1$.

Case 2, $b > 1$. $b \geq b^{1/2} \geq b^{1/3} \geq ... \geq 1$, so this sequence is bounded and decreasing, and as such, convergent. We will say that $\lim b^{1/n} = l$. The limit of the subsequence $b^{1/2n} = l$, as subsequences of convergent sequences converge to the same limit. By the Algebraic Limit

Theorem, we also have $\lim b^{1/2n} = l = \sqrt{l}$, so $l = 0$ or $l = 1$. However, as it is bounded below by 1, it cannot converge to 0. Therefore, $\lim b^{1/n} = 1$.

HW7.1: Let a_n be bounded and let

$$S = \{s \in \mathbb{R} : \exists \text{ a subsequence } (a_{n_k}) \text{ converging to } s\}.$$

This is called the set of subsequential limits. Bolzano-Weierstrass Theorem implies that there is at least one convergent subsequence, so $S \neq \emptyset$. Show S is bounded and $\limsup a_n = \sup(S)$.

Because (a_n) is a bounded sequence, there exists some $M \in \mathbb{R}$ such that $|a_n| \leq M$ for all n. If S is unbounded, then there is some (a_{n_k}) such that it converges to a limit greater than all $m \in \mathbb{R}$, and as such cannot converge to some s. Therefore, S must be bounded. Take some arbitrary subsequence (a_{n_k}). $\limsup a_n$ is then the limit of the sequence $y_n = \sup\{a_k : k \geq n\}$. Each individual y_n, (as $n \leq n_k$, as shown in lecture) is greater than or equal to all a_{n_k}, so by the Order Limit Theorem, $\limsup a_n \geq \lim a_{n_k}$ for arbitrary (a_{n_k}), so $\limsup a_n = \sup(S)$.