

Math 523 HW 4

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Section 2.1

Use the rules for differentiation to find the derivatives of **(2)** and **(4)**.

(2) $z^2 + 10z$

$$\begin{aligned}(z^2+10z)' &= \lim_{h \rightarrow 0} \frac{(z+h)^2 + 10(z+h) - z^2 - 10z}{h} = \lim_{h \rightarrow 0} \frac{z^2 + 2zh + h^2 + 10z + 10h - z^2 - 10z}{h} \\ &= \lim_{h \rightarrow 0} \frac{2zh + h^2 + 10h}{h} = \lim_{h \rightarrow 0} 2z + h + 10 = 2z + 10.\end{aligned}$$

(4) $[\cos(z^2)]^3$

Note that

$$[\cos(z^2)]^3 = \frac{1}{8} \left(e^{iz^2} + e^{-iz^2} \right)^3,$$

so

$$\begin{aligned}\{[\cos(z^2)]^3\}' &= \left\{ \frac{1}{8} \left(e^{iz^2} + e^{-iz^2} \right)^3 \right\}' \\ \implies \{[\cos(z^2)]^3\}' &= \frac{3}{8} \left(e^{iz^2} + e^{-iz^2} \right)^2 \times \left(2zie^{iz^2} - 2zie^{-iz^2} \right) \\ \implies \{[\cos(z^2)]^3\}' &= \frac{3zi}{4} \times 4[\cos(z^2)]^2 \times 2i \sin(z^2) \\ \implies \{[\cos(z^2)]^3\}' &= -6z \cos^2(z^2) \sin(z^2).\end{aligned}$$

For each function f listed in Exercises **(8)** and **(10)**, find an analytic function F with $F' = f$.

(8) $f(z) = z - 2$

$F(z) = \frac{1}{2}z^2 - 2z$ is analytic with $F' = f$.

(10) $f(z) = \sin z \cos z$

$F(z) = \frac{1}{2}(\sin z)^2$ is analytic with $F' = f$.

(14) Let $P(z) = A(z - z_1)\dots(z - z_n)$, where A and z_1, \dots, z_n are complex numbers and $A \neq 0$. Show that

$$\frac{P'(z)}{P(z)} = \sum_{j=1}^n \frac{1}{z - z_j}, \quad z \neq z_1, \dots, z_n.$$

(16) Find the derivative of the **linear fractional transformation** $T(z) = (az+b)/(cz+d)$, $ad \neq bc$. In what way does the condition $ad - bc \neq 0$ enter? Conclude that $T'(z)$ is never zero, $z \neq -d/c$.

$$\begin{aligned} T'(z) &= \left(\frac{az+b}{cz+d} \right)' = \lim_{h \rightarrow 0} \frac{\frac{a(z+h)+b}{c(z+h)+d} - \frac{az+b}{cz+d}}{h} = \lim_{h \rightarrow 0} \frac{\frac{az+ah+b}{cz+ch+d} - \frac{az+b}{cz+d}}{h} \\ &= \lim_{h \rightarrow 0} \frac{\frac{(az+ah+b)(cz+d) - (az+b)(cz+ch+d)}{(cz+ch+d)(cz+d)}}{h} = \lim_{h \rightarrow 0} \frac{\frac{(acz^2+aczh+bcz+adz+adh+bd) - (acz^2+aczh+adz+bcz+bch+bd)}{(cz+ch+d)(cz+d)}}{h} \\ &= \lim_{h \rightarrow 0} \frac{\frac{adh-bch}{(cz+ch+d)(cz+d)}}{h} = \lim_{h \rightarrow 0} \frac{ad-bc}{(cz+ch+d)(cz+d)} = \frac{ad-bc}{(cz+d)^2}. \end{aligned}$$

The condition $ad - bc \neq 0$ enters when taking the limit as $h \rightarrow 0$ —if $ad - bc = 0$, then there would be additional h in the denominator of the limit, and the limit would not converge. Therefore, $T'(z)$ can never be zero, and $z \neq -d/c$, as that would cause $T'(z)$ to not converge to a limit.

(18) Show that $h(z) = \bar{z}$ is not analytic on any domain. (**Hint:** check the Cauchy-Riemann equations.)

If $h(z) = \bar{z}$, where $h(z) = u(x, y) + iv(x, y)$ were analytic, we would have $u_x = v_y$ and $u_y = -v_x$. We also have $h(z) = x - iy$, so $u_x = 1 \neq v_y$, as $v_y = -1$. Therefore, $h(z) = \bar{z}$ cannot be analytic.

(20) Let $f = u + iv$ be analytic. In each of the following, find v given u .

(20a) $u = x^2 - y^2$

$u_x = 2x$, $u_y = -2y$, so $v_y = 2x$, $v_x = 2y$. Therefore, $v = 2xy$.

(20b) $u = \frac{x}{x^2+y^2}$

Section 2.2

In exercises (2) and (4), use Theorem 2 or Example 4 to find the radius of convergence of the following power series.

(2) $\sum_{k=0}^{\infty} \frac{(k!)^2}{(2k)!} (z-2)^k$

$$\begin{aligned} \frac{1}{R} &= \lim_{k \rightarrow \infty} \left| \frac{a_{k+1}}{a_k} \right| = \lim_{k \rightarrow \infty} \left| \frac{((k+1)!)^2 / (2k+2)!}{(k!)^2 / (2k)!} \right| \\ &= \lim_{k \rightarrow \infty} \left| \frac{(k+1)^2}{(2k+2)(2k+1)} \right| = \lim_{k \rightarrow \infty} \left| \frac{k^2 + 2k + 1}{4k^2 + 6k + 2} \right| = \frac{1}{4} \implies R = 4. \end{aligned}$$

(4) $\sum_{k=0}^{\infty} (-1)^k z^{2k}$

Letting $w = z^2$, we have $\sum_{k=0}^{\infty} (-1)^k z^{2k} = \sum_{k=0}^{\infty} (-1)^k w^k$, so we can say

$$\frac{1}{R} = \lim_{n \rightarrow \infty} \left| \frac{(-1)^{k+1}}{(-1)^k} \right| = 1 \implies R = 1.$$

In exercises (8) and (10), find the power series about the origin for the given function.

(8) $z^2 \cos z$

$$z^2 \sin z = z^2 \sum_{n=0}^{\infty} (-1)^n z^{2n+1} / (2n+1)! = \sum_{n=0}^{\infty} (-1)^n z^{2n+3} / (2n+1)!.$$

(10) $\frac{1+z}{1-z}, |z| < 1$

In exercises (14), (16), and (18), find a “closed form” (that is, a simple expression) for each of the given power series.

(14) $\sum_{n=0}^{\infty} \frac{z^{2n}}{n!}$

(16) $\sum_{n=1}^{\infty} n(z-1)^{n-1}$

(18) $\sum_{n=2}^{\infty} n(n-1)z^n$

(22)

(22a) If $f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$ has radius of convergence $R > 0$ and if $f(z) = 0$ for all $z, |z-z_0| < r \leq R$, show that $a_0 = a_1 = \dots = 0$.

(22b) If $F(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$ and $G(z) = \sum_{n=0}^{\infty} b_n (z-z_0)^n$ are equal on some disc $|z-z_0| < r$, show that $a_n = b_n$ for all n .