Math 521 HW 9

Morgan Gribbins

Exercise 2.7.1. Proving the Alternating Series Test amounts to showing that the sequence of partial sums

$$s_n = a_1 - a_2 + ... \pm a_n$$

converges.

2.7.1.a. Prove the Alternating Series Test by showing that (s_n) is a Cauchy sequence.

Given $\epsilon > 0$, let $N \in \mathbb{N}$ large enough such that a_n with $n \geq N$ satisfies $|a_{n+1}| < \epsilon$. Given $m \geq n \geq N$, we then examine the expression

$$|s_m - s_n| = |a_{n+1} - a_{n+2} + ... \pm a_m|.$$

As $a_{n+1} \leq a_n$ for all n, this absolute value is less than $|a_{n+1}|$, so we have

$$|s_m - s_n| \leq |a_{n+1}| < \epsilon,$$

so this series is Cauchy.

2.7.1.b. Supply another proof for this result using the Nested Interval Property.

2.7.1.c. Consider the subsequences (s_{2n}) and (s_{2n+1}), and show how the Monotone Convergence Theorem leads to a third proof for the Alternating Series Test.

The subsequence $(s_{2n}) = a_1 - a_2 + ... - a_{2n}$, and the subsequence $(s_{2n+1}) = a_1 - a_2 + ... - a_{2n} + a_{2n+1}$ and as $a_n \geq 0$ for all n, $s_{2n+1} \geq s_{2n}$. Note that, because a_n is decreasing, all $a_2 \leq s_n \leq a_1$, and so all subsequences are bounded above and below. s_{2n} and s_{2n+1} both converge, as they are both monotone, and their difference converges to 0, so they converge to the same limit, so this sequence is convergent.

Exercise 2.7.4. Give an example to show that it is possible for both $\sum x_n$ and $\sum y_n$ to diverge but for $\sum x_n y_n$ to converge.

The sum $\sum 1/n$ diverges, but $\sum 1/n^2$ converges.

Exercise 3.2.3. Decide whether the following sets are open, closed, or neither. If a set is not open, find a point in the set for which there is no ϵ-neighborhood contained in the set.

If a set is not closed, find a limit point that is not contained in the set.

3.2.3.a. $\mathbb{Q}$

This set is not open, as 0 has no ϵ-neighborhoods entirely contained in $\mathbb{Q}$ centered at it. It is also not closed, as the sequence $1, 1.4, 1.41, ...$ (of rational numbers increasingly approaching without surpassing $\sqrt{2}$) converges to $\sqrt{2}$, which is not contained in $\mathbb{Q}$.

3.2.3.b. $\mathbb{N}$

$\mathbb{N}$ is not open, as 1 has no ϵ-neighborhoods entirely contained in $\mathbb{N}$ centered at it. It is, however, closed, as its complement is open.

3.2.3.c. $\{x \in \mathbb{R} : x > 0\}$

This set is open (as its complement is closed), but not closed. The sequence $1, 1/2, 1/3, ...$ is entirely contained in this set, but converges outside of this set; therefore, 0 is a limit point not contained in the set.

3.2.3.d. $(0, 1] = \{x \in \mathbb{R} : 0 < x \leq 1\}$

This set is neither open nor closed. 1 has no ϵ-neighborhoods entirely contained in this set, and the sequence $1, 1/2, 1/3, ...$ is entirely in this set, yet converges outside of it, so 0 is a limit point not in this set.

3.2.3.e. $\{1 + 1/4 + 1/9 + ... + 1/n^2 : n \in \mathbb{N}\}$

This set is neither open nor closed, as there are no ϵ-neighborhoods centered at 1 entirely contained in the set, and the number $\frac{\pi^2}{6}$ is the limit of the sequence $1, 1+1/4, 1+1/4+1/9, ...$, but is not contained in the set.