Math 534 HW 7

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(1) An isomorphism from a group to itself, i.e. an isomorphism $\alpha : G \to G$, is called an automorphism. Suppose that G is a finite abelian group which has no elements of order 2. Show that the function $\alpha(g) = g^2$ is an automorphism of G. Show by example that the result doesn't hold in the case that G is infinite.

For α to be an automorphism, it must be an isomorphism from G to itself, and as $g \in G \implies g^2 = \alpha(g) \in G$, this function is $G \to G$. For α to be an isomorphism, it must both be a bijection and satisfy

$$\alpha(gh) = \alpha(g)\alpha(h), \text{ for all g, h in G.}$$

- Proof that α is bijective. Note that, because G is finite, injectivity of α is equivalent to bijectivity (and surjectivity); also note that G is abelian. Let $g, h \in G$ with $\alpha(g) = \alpha(h) \implies g^2 = h^2$. This directly implies that $g = h$, so this function is injective, which implies that it is bijective.

- Proof that α is an homomorphism. Let $g, h \in G$. Then, $\alpha(gh) = ghgh = gghh = g^2h^2 = \alpha(g)\alpha(h)$, so this is a homomorphism.

Therefore, α is an automorphism on G.

An example that the result doesn't hold in the case that G is infinite is provided by the group $(\mathbb{Z}, +)$. There is no element $b \in \mathbb{Z}$ that satisfies $b + b = 3$, so this cannot be an automorphism (however it is abelian without any elements of order 2).

(2) Consider the group $G = (\mathbb{Z}, +)$.

(2a) Show that G is isomorphic to the proper subgroup $H = \{2k : k \in \mathbb{Z}\}$ of even elements.

Let $\phi : G \to H$ be defined by multiplying elements in G by 2.

Proof that ϕ is injective. Let $g, h \in G$ such that $\phi(g) = \phi(h) \implies 2g = 2h \implies g = h$.

Proof that ϕ is surjective. Let $g \in H$. This implies that $g = 2k$, for some $k \in \mathbb{Z}$, which means that $k \in G$, and $g = 2k = \phi(k)$.

Proof that ϕ is a homomorphism. Let $g, h \in G$. Then, $\phi(g+h) = 2(g+h) = 2g + 2h = \phi(g) + \phi(h)$, so ϕ is a homomorphism.

As ϕ is a bijective homomorphism, it is an isomorphism, and these groups are isomorphic.
(2b) Show that there are in fact infinitely many subgroups of G to which it is isomorphic.

Let $a \neq 0 \in \mathbb{Z}$, and let $H_a = \{ak : k \in \mathbb{Z}\} \leq G$. As $\mathbb{Z} \setminus \{0\}$ is infinite, there are an infinite amount of H_a. We will now show that G is isomorphic to any H_a.

Let $\phi : G \to H_a$ be defined by multiplying elements in G by a.

Proof that ϕ is injective. Let $g, h \in G$ such that $\phi(g) = \phi(h) \implies ag = ah \implies g = h$.

Proof that ϕ is surjective. Let $g \in H$. This implies that $g = ak$, for some $k \in \mathbb{Z}$, which means that $k \in G$, and $g = ak = \phi(k)$.

Proof that ϕ is a homomorphism. Let $g, h \in G$. Then, $\phi(g+h) = a(g+h) = ag + ah = \phi(g) + \phi(h)$, so ϕ is a homomorphism.

As ϕ is a bijective homomorphism, it is an isomorphism, and G is isomorphic to an arbitrary H_a. As there are an infinite amount of H_a, G has an infinite amount of subgroups that are isomorphic to G.

(3) Consider the group $\mathbb{R}^\times = (\mathbb{R} \setminus \{0\}, \times)$ of non-zero real numbers under multiplication. Prove that this group is not isomorphic to $(\mathbb{R}, +)$.

Let $\phi : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ be an isomorphism between these two sets. Consider $\phi(1) = 0 = \phi(-1 \times -1) = \phi(-1) + \phi(-1) \implies \phi(-1) = \phi(1)$ and $-1 \neq 1$, which means that ϕ is not bijective, and as such cannot be an isomorphism.

(4) Explain how $\mathfrak{S}_8$ contains subgroups isomorphic to $\mathbb{Z}/15, (\mathbb{Z}/16)^\times$, and D_8. Here, D_8 denotes the group of symmetries of a regular convex octagon.

$\mathfrak{S}_8$ contains subgroups isomorphic to:

- $\mathbb{Z}/15$ because all elements in $\mathbb{Z}/15$ that are even are equal to their quotient by 2 squared, and so permutations on the odd elements of $\mathbb{Z}/15$ permute their corresponding even elements.
- $(\mathbb{Z}/16)^\times$ because this group has order 8, and so this isomorphism exists by Cayley's theorem.

- D_8 has symmetry between the "front" and "back" of the octagon, each with eight different positions, so it is "similar" to an object with eight possible settings.