Math 523 HW 4

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Section 2.1

Use the rules for differentiation to find the derivatives of **(2)** and **(4)**.

(2) $z^2 + 10z$

$$(z^2+10z)' = \lim_{h\to 0} \frac{(z+h)^2 + 10(z+h) - z^2 - 10z}{h} = \lim_{h\to 0} \frac{z^2 + 2zh + h^2 + 10z + 10h - z^2 - 10z}{h}$$

$$= \lim_{h\to 0} \frac{2zh + h^2 + 10h}{h} = \lim_{h\to 0} 2z + h + 10 = 2z + 10.$$

(4) $[\cos\,(z^2)]^3$

Note that

$$[\cos\,(z^2)]^3 = \frac{1}{8}\left(e^{iz^2} + e^{-iz^2}\right)^3,$$

so

$$\left\{[\cos\,(z^2)]^3\right\}' = \left\{\frac{1}{8}\left(e^{iz^2} + e^{-iz^2}\right)^3\right\}'$$

$$\implies \left\{[\cos\,(z^2)]^3\right\}' = \frac{3}{8}\left(e^{iz^2} + e^{-iz^2}\right)^2 \times \left(2zie^{iz^2} - 2zie^{-iz^2}\right)$$

$$\implies \left\{[\cos\,(z^2)]^3\right\}' = \frac{3zi}{4} \times 4[\cos\,(z^2)]^2 \times 2i\sin\,(z^2)$$

$$\implies \left\{[\cos\,(z^2)]^3\right\}' = -6z\cos^2\,(z^2)\,\sin\,(z^2).$$

For each function f listed in Exercises **(8)** and **(10)**, find an analytic function F with $F' = f$.

(8) $f(z) = z - 2$

$F(z) = \frac{1}{2}z^2 - 2z$ is analytic with $F' = f$.

(10) $f(z) = \sin\, z\, \cos\, z$

$F(z) = \frac{1}{2}\,(\sin\, z)^2$ is analytic with $F' = f$.

(14) Let $P(z) = A(z - z_1)...(z - z_n)$, where A and $z_1, ...z_n$ are complex numbers and $A \neq 0$. Show that

$$\frac{P'(z)}{P(z)} = \sum_{j=1}^{n} \frac{1}{z - z_j},\ z \neq z_1, ..., z_n.$$

Since $P(z)$ and $P'(z)$ have a common term $A \neq 0$, we can disregard that term completely. We then have, by way of the product rule (and the fact that $(z - z_j)' = 1$), $P'(z)/A = \sum_{j=1}^{n} \prod_{k \neq j}(z - z_k)$, and $P(z)/A = \prod_{k=1}^{n}(z - z_k)$, so

$$(P'(z)/A)/(P(z)/A) = \frac{P'(z)}{P(z)} = \sum_{j=1}^{n} \frac{1}{z - z_j},$$

as the $P'(z)$ terms exclude the jth term.

(16) Find the derivative of the **linear fractional transformation** $T(z) = (az+b)/(cz+d), ad \neq bc$. In what way does the condition $ad - bc \neq 0$ enter? Conclude that $T'(z)$ is never zero, $z \neq -d/c$.

$$T'(z) = \left(\frac{az + b}{cz + d}\right)' = \lim_{h \to 0} \frac{\frac{a(z+h)+b}{c(z+h)+d} - \frac{az+b}{cz+d}}{h} = \lim_{h \to 0} \frac{\frac{az+ah+b}{cz+ch+d} - \frac{az+b}{cz+d}}{h}$$

$$= \lim_{h \to 0} \frac{\frac{(az+ah+b)(cz+d)-(az+b)(cz+ch+d)}{(cz+ch+d)(cz+d)}}{h} = \lim_{h \to 0} \frac{\frac{(acz^2+aczh+bcz+adz+adh+bd)-(acz^2+aczh+adz+bcz+bch+bd)}{(cz+ch+d)(cz+d)}}{h}$$

$$= \lim_{h \to 0} \frac{\frac{adh-bch}{(cz+ch+d)(cz+d)}}{h} = \lim_{h \to 0} \frac{ad - bc}{(cz + ch + d)(cz + d)} = \frac{ad - bc}{(cz + d)^2}.$$

The condition $ad - bc \neq 0$ enters when taking the limit as $h \to 0$—if $ad - bc = 0$, then there would be additional h in the denominator of the limit, and the limit would not converge. Therefore, $T'(z)$ can never be zero, and $z \neq -d/c$, as that would cause $T'(z)$ to not converge to a limit.

(18) Show that $h(z) = \bar{z}$ is not analytic on any domain. (**Hint:** check the Cauchy-Riemann equations.)

If $h(z) = \bar{z}$, where $h(z) = u(x, y) + iv(x, y)$ were analytic, we would have $u_x = v_y$ and $u_y = -v_x$. We also have $h(z) = x - iy$, so $u_x = 1 \neq v_y$, as $v_y = -1$. Therefore, $h(z) = \bar{z}$ cannot be analytic.

(20) Let $f = u + iv$ be analytic. In each of the following, find v given u.

(20a) $u = x^2 - y^2$

$$u_x = 2x,\ u_y = -2y,\ \text{so } v_y = 2x,\ v_x = 2y. \text{ Therefore, } v = 2xy + C.$$

(20b) $u = \frac{x}{x^2+y^2}$

$$u_x = \frac{y^2 - x^2}{(x^2+y^2)^2)},\ u_y = -\frac{2xy}{(x^2+y^2)^2},\ \text{so } v_y = \frac{y^2 - x^2}{(x^2+y^2)^2)},\ v_x = \frac{2xy}{(x^2+y^2)^2}.$$
$$\text{Therefore, } v = -\frac{y}{x^2+y^2} + C.$$

Section 2.2

In exercises **(2)** and **(4)**, use Theorem 2 or Example 4 to find the radius of convergence of the following power series.

(2) $\sum_{k=0}^{\infty} \frac{(k!)^2}{(2k)!}(z-2)^k$

$$\frac{1}{R} = \lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty}\left|\frac{((k+1)!)^2/(2k+2)!}{(k!)^2/((2k)!)}\right|$$
$$= \lim_{k\to\infty}\left|\frac{(k+1)^2}{(2k+2)(2k+1)}\right| = \lim_{k\to\infty}\left|\frac{k^2+2k+1}{4k^2+6k+2}\right| = \frac{1}{4} \implies R = 4.$$

(4) $\sum_{k=0}^{\infty}(-1)^k z^{2k}$

Letting $w = z^2$, we have $\sum_{k=0}^{\infty}(-1)^k z^{2k} = \sum_{k=0}^{\infty}(-1)^k w^k$, so we can say

$$\frac{1}{R} = \lim_{n\to\infty}\left|\frac{(-1)^{k+1}}{(-1)^k}\right| = 1 \implies R = 1.$$

In exercises **(8)** and **(10)**, find the power series about the origin for the given function.
(8) $z^2\cos z$

$$z^2\sin\, z = z^2 \sum_{n=0}^{\infty}(-1)^n z^{2n+1}/(2n+1)! = \sum_{n=0}^{\infty}(-1)^n z^{2n+3}/(2n+1)!.$$

(10) $\frac{1+z}{1-z},\ |z| < 1$

$$\frac{1}{1-z} = 1 + z + z^2 + ... = \sum_{n=0}^{\infty} z^n \text{ so } \frac{1+z}{1-z} = 1 + \sum_{n=1}^{\infty} 2z^n.$$

In exercises **(14)**, **(16)**, and **(18)**, find a "closed form" (that is, a simple expression) for each of the given power series.

(14) $\sum_{n=0}^{\infty} \frac{z^{2n}}{n!}$

Since $e^2 = \sum \frac{z^n}{n!}$, then

$$e^{z^2} = \sum_{n=0}^{\infty} \frac{z^{2n}}{n!}.$$

(16) $\sum_{n=1}^{\infty} n(z-1)^{n-1}$

Note that

$$\sum_{n=1}^{\infty} n(z-1)^{n-1} = \left(\sum_{n=0}^{\infty}(z-1)^n\right)',$$

where the second function is the power series of $\frac{1}{z-2}$, so

$$\sum_{n=1}^{\infty} n(z-1)^{n-1} = \left(\frac{1}{2-z}\right)' = \frac{1}{(2-z)^2}.$$

(18) $\sum_{n=2}^{\infty} n(n-1)z^n$

This sum is the second derivative of the sum $\sum_{n=0}^{\infty} z^n$ multiplied by z^2. This original sum is $\frac{1}{1-z}$, so

$$\sum_{n=2}^{\infty} n(n-1)z^n = z^2(\left(\frac{1}{1-z}\right)'' = \frac{2z^2}{(1-z)^3}.$$

(22)

(22a) If $f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n$ has radius of convergence $R > 0$ and if $f(z) = 0$ for all $z, |z-z_0| < r \leq R$, show that $a_0 = a_1 = ... = 0$.

Given an arbitrary $z \neq z_0$ in the radius of convergence with $|z-z_0| < r \leq R$ (for some r), we have $f(z) = \sum a_n(z-z_0)^n = 0$. We can say that $z - z_0 = \epsilon$, so we have $\sum a_n\epsilon^n = 0$.

If $\epsilon > 0$, then we have $\sum a_n \epsilon^n = a_0 + a_1\epsilon + a_2\epsilon^2 + ...$, which cannot be zero when any of these terms are nonzero, so they must all be 0 for $f(z) = 0$ to hold. If $\epsilon < 0$, let $\delta = -\epsilon$, so we have $\sum a_n(-\delta)^n = a_0 - a_1\delta + a_2\delta^2 = 0$, which implies that the even and odd sums of this are equivalent, i.e. $\sum_n^\infty a_{2n}\epsilon^{2n} = \sum_n^\infty a_{2n+1}\epsilon^{2n+1}$, which cannot be true for non-uniform zero a_n.

(22b) If $F(z) = \sum_{n=0}^\infty a_n(z - z_0)^n$ and $G(z) = \sum_{n=0}^\infty b_n(z - z_0)^n$ are equal on some disc $|z - z_0| < r$, show that $a_n = b_n$ for all n.

For the sake of contradiction, assume that $F(z) = G(z)$ and $a_n \neq b_n$ for all n. Letting A be the nonempty set of $n \in \mathbb{N}$ such that $a_n \neq b_n$. We then have $F(z) - G(z) = 0$, which is a separate sum $0 = \sum_{n\in A}(a_n - b_n)(z - z_0)^n$, with $a_n - b_n \neq 0$, $z - z_0 \neq 0$, which cannot be true due to the same logic in **(22a)**.