Math 521 HW 11

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Exercise 3.3.9. Follow these steps to prove the final implication in Theorem 3.3.8.

Assume K satisfies (i) and (ii), and let $\{O_\lambda : \lambda \in \Lambda\}$ be an open cover for K. For contradiction, let's assume that no finite subcover exists. Let I_0 be a closed interval containing K.

3.3.9.a. Show that there exists a nested sequence of closed intervals ... $\subseteq I_2 \subseteq I_1 \subseteq I_0$ with the property that, for each n, $I_n \cap K$ cannot be finitely covered and $\lim |I_n| = 0$.

As K cannot be finitely covered, $I_n \cap K \subseteq K$ must not be able to be finitely covered, as this would imply that K itself can be finitely covered, so these intervals with such properties must exist.

3.3.9.b. Argue that there exists an $x \in K$ such that $x \in I_n$ for all n.

By the Nested Interval Property, there must exists one $x \in K$ such that $x \in \bigcap_{n\in\mathbb{N}} I_n$, because of the established definitions of each I_n.

3.3.9.c. Because $x \in K$, there must exist an open set O_{λ_0} from the original collection that contains x as an element. Explain how this leads to the desired contradiction.

This leads to the desired contradiction as this implies that K can be finitely covered, because this process may be replicated for each other element of K, so all elements of K may be finitely covered.

Exercise 4.2.2. For each stated limit, find the largest possible δ-neighborhood that is a proper response to the given ϵ challenge.

4.2.2.a. $\lim_{x\to 3}(5x-6) = 9$, where $\epsilon = 1$.

Given $\epsilon = 1$, the desired δ allows the implication

$$|x-3| < \delta \implies |5x-6-9| = |5x-15| < 1.$$

Multiplying $|x-3| < \delta$ by 5 gives

$$|5||x-3| = |5x-15| < 5\delta = 1 \implies \delta = 0.2.$$

4.2.2.b. $\lim_{x\to 4}\sqrt{x} = 2$, where $\epsilon = 1$.

Given $\epsilon = 1$, the desired δ allows the implication

$$|x-4| < \delta \implies |\sqrt{x}-2| < 1.$$

Factoring $|x-4|$ provides the inequality $|(\sqrt{x}-2)(\sqrt{x}+2)| < \delta$. This implies that

$$|\sqrt{x}-2| < \delta/|\sqrt{x}+2| \leq \delta/5 = 1 \implies \delta = 5.$$

4.2.2.c. $\lim_{x\to\pi}[[x]] = 3$, where $\epsilon = 1$.

Given $\epsilon = 1$, the desired δ allows the implication

$$|x-\pi| < \delta \implies |[[x]]-3| < 1.$$

By definition of $f(x) = [[x]]$, $|[[x]]-3| < 1$ occurs only when $3 \leq x < 4$. Therefore, the greatest δ that implies this result is $\pi - 3$.

4.2.2.d. $\lim_{x\to\pi}[[x]] = 3$, where $\epsilon = .01$.

Given $\epsilon = .01$, the desired δ allows the implication

$$|x-\pi| < \delta \implies |[[x]]-3| < .01.$$

By definition of $f(x) = [[x]]$, $|[[x]]-3| < .01$ when $3 \leq x < 4$. Therefore, the requisite $\delta = 3 - \pi$.

Exercise 4.2.5. Use Definition 4.2.1 to supply a proper proof for the following limit statements.

4.2.5.a. $\lim_{x\to 2}(3x+4) = 10$.

Given $\epsilon > 0$, let $\delta = \epsilon/3$. We then have

$$|x-2| < \delta = \epsilon/3 = |3||x-2| = |3x-6| = |3x+4-10| < \epsilon,$$

so $\lim_{x\to 2}(3x+4) = 10$.

4.2.5.b. $\lim_{x\to 0} x^3 = 0$.

Given $\epsilon > 0$, let $\delta = \epsilon^{1/3}$. We then have

$$|x-0| < \delta = \epsilon^{1/3} \implies |x|^3 = |x^3| = |x^3 = 0| < \epsilon,$$

so $\lim_{x\to 0} x^3 = 0$.

4.2.5.c. $\lim_{x \to 2}(x^2 + x - 1) = 5$.

Given $\epsilon > 0$, let $\delta = \epsilon/6$ or 1, whichever is lower. We then have

$$|x-2| < \delta \implies |x-2||x+3| < 6\delta \implies |x^2+x-1-5| < 6\delta \leq \epsilon,$$

so $\lim_{x \to 2}(x^2 + x - 1) = 5$.

4.2.5.d. $\lim_{x \to 3} 1/x = 1/3$.

Given $\epsilon > 0$, let $\delta = 12\epsilon$, and examine the δ-neighborhood $V_1(4)$. We then have

$$|x-3| < 12\epsilon \implies |x-3|/|3x| < \epsilon \implies \left|\frac{1}{x} - \frac{1}{3}\right| < \epsilon,$$

so $\lim_{x \to 3} 1/x = 1/3$.

Exercise 4.3.5. Show using Definition 4.3.1 that if c is an isolated point of $A \subseteq \mathbb{R}$, then $f : A \to \mathbb{R}$ is continuous at c.

Let c be an isolated point of $A \subseteq \mathbb{R}$. Given $\epsilon > 0$, let $\delta > 0$ such that the set $V_\delta(c) = \{c\}$. We will now consider every point in this set. $|c - c| = 0 < \delta$, and $f(c) - f(c)| = 0 < \epsilon$, so this function is continuous at c.

Exercise 4.3.6. Provide an example of each or explain why the request is impossible.

4.3.6.a. Two functions f and g, neither of which is continuous at 0 but such that $f(x)g(x)$ and $f(x) + g(x)$ are continuous at 0.

Let $f(x) = -1$ if $x > 0$ and $f(x) = 1$ if $x \leq 0$, and $g(x) = -f(x)$. Neither of these are continuous at 0, but their product is uniformly -1, and their sum is uniformly 0.

4.3.6.b. A function $f(x)$ continuous at 0 and $g(x)$ not continuous at 0 such that $f(x) + g(x)$ is continuous at 0.

This is not possible by the Algebraic Continuity Theorem. If f and $f + g$ continuous, then $f + g - f = g$ must be continuous.

4.3.6.c. A function $f(x)$ continuous at 0 and $g(x)$ not continuous at 0 such that $f(x)g(x)$ is continuous at 0.

This is not possible by the Algebraic Continuity Theorem. If f and fg continuous, then $fg/f = g$ must be continuous.

4.3.6.d. A function $f(x)$ not continuous at 0 such that $f(x) + 1/f(x)$ is continuous at 0.

Let $f(x) = 2$ for $x > 0$, $f(x) = 1/2$ otherwise. $f(x) + 1/f(x)$ is then uniformly $3/2$ and continuous at 0.

4.3.6.e. A function $f(x)$ not continuous at 0 such that $[f(x)]^3$ is continuous at 0.

Note that the function $g(x) = x^{1/3}$ is continuous everywhere. By Continuity of Compositions of Functions, $g(f(x)^3) = f(x)$ must be continuous, so this is not possible.

Exercise 4.3.9. Assume $h : \mathbb{R} \to \mathbb{R}$ is continuous on $\mathbb{R}$ and let $K = \{x : h(x) = 0\}$. Show that K is a closed set.

Let c be a limit point of K. This implies that $\forall \epsilon > 0$, $V_\epsilon(c)$ intersects K at some points other than c. As h is continuous, for each $\epsilon > 0$, there exists $\delta > 0$ such that $|x - a| < \delta$ implies $|h(x) - h(a)| = |h(a)| < \epsilon$, if $x \in K$. Therefore, if c is a limit point of K, then for all $\epsilon > 0$, there exists $\delta > 0$ such that $|x - c| < \delta \implies |h(x) - h(c)| = |h(c)| < \epsilon$, which is equivalent to $h(c) = 0$ (by definition of equality), so $c \in K$.