Math 523 HW 5

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Section 2.3

In Exercises (1) to (4), evaluate the given integral using Cauchy's Formula or Theorem.

(1) $\int_{|z|=1} \frac{z}{(z-2)^2} dz$

On the domain of $\mathbb{C} \setminus \{2\}$, the integrand of $\int_{|z|=1} \frac{z}{(z-2)^2} dz$ is analytic, and 2 is not in the interior of $|z| = 1$, so $\int_{|z|=1} \frac{z}{(z-2)^2} dz = 0$.

(2) $\int_{|z|=2} \frac{e^z}{z(z-3)} dz$

The integrand of $\int_{|z|=2} \frac{e^z}{z(z-3)} dz$ is not defined on $z \in \{0, 3\}$, which is part of the interior of $|z| = 2$, so we cannot apply Cauchy's Theorem. Because $z_0 = 0$ is the interior point in this integral, we have $f(0) = \frac{1}{2\pi i} \int_{|z|=2} \frac{e^z/(z-3)}{z-0}$, $-1/3 = \frac{1}{2\pi i} \int_{|z|=2} \frac{e^z}{z(z-3)} dz$, $\int_{|z|=2} \frac{e^z}{z(z-3)} dz = -2\pi i/3$.

(3) $\int_{|z+1|=2} \frac{z^2}{4-z^2} dz$

The integrand of $\int_{|z+1|=2} \frac{z^2}{4-z^2} dz$ is not analytic inside of $|z + 1| = 2$, so we cannot apply Cauchy's Theorem. We can rearrange $\int_{|z+1|=2} \frac{z^2}{4-z^2} dz$ to $\int_{|z+1|=2} \frac{-z^2/(2+z)}{z-2} dz$, so it is clear that $\int_{|z+1|=2} \frac{z^2}{4-z^2} dz = 2\pi i f(2) = 2\pi i \times 2^2/(2 + 2) = 2\pi i$.

(4) $\int_{|z|=1} \frac{\sin z}{z} dz$

By Cauchy's Formula, $\int_{|z|=1} \frac{\sin z}{z} dz = \int_{|z|=1} \frac{\sin z}{z-0} = 2\pi i \sin 0 = 0$.

In Exercises (5) to (8), evaluate the definite trigonometric integral making use of the technique of Examples 6 and 7 in this section.

(5) $\int_0^{2\pi} \frac{d\theta}{2+\cos\theta}$

Letting $z = e^{i\theta}$, we have

$$\cos\theta = \frac{1}{2}\left(z + \frac{1}{z}\right),$$

$$d\theta = \frac{1}{i}\frac{dz}{z}.$$

Through substitution, we have

$$\int_0^{2\pi} \frac{d\theta}{2+\cos\theta} = \int_{|z|=1} \frac{1}{2+\frac{1}{2}\left(z+\frac{1}{z}\right)} \frac{1}{i}\frac{dz}{z} = \int_{|z|=1} \frac{2dz}{4iz+iz^2+i}.$$

This factors to

$$\frac{2}{i}\int_{|z|=1} \frac{dz}{(z-(\sqrt{3}-2))(z+(\sqrt{3}+2))}.$$

Setting $p=\sqrt{3}-2$ and $q=-\sqrt{3}-2$ note that p is within the radius 1 circle, and q is not. The function $(z-q)^{-1}$ is analytic within this circle, so Cauchy's Formula states

$$\frac{1}{2\pi i}\int_{|z|=1} \frac{dz}{(z-q)(z-p)} = \frac{1}{p-q} = \frac{1}{2\sqrt{3}},$$

so our integral gives us

$$\int_0^{2\pi} \frac{d\theta}{2+\cos\theta} = \frac{2\pi}{\sqrt{3}}.$$

(6) $\int_0^{2\pi} \frac{d\theta}{3+\sin\theta+\cos\theta}$

Letting $z=e^{i\theta}$, we have

$$\cos\theta = \frac{1}{2}\left(z+\frac{1}{z}\right),$$

$$\sin\theta = \frac{1}{2i}\left(z-\frac{1}{z}\right)$$

,

$$d\theta = \frac{1}{i}\frac{dz}{z}.$$

Through substitution, we have

$$\int_0^{2\pi} \frac{d\theta}{3+\sin\theta+\cos\theta} = \int_{|z|=1} \frac{dz}{iz\left(3+\frac{1}{2}(z+1/z)+\frac{1}{2i}(z-1/z)\right)} = \int_{|z|=1} \frac{2dz}{z^2+iz^2+6iz-1+1}$$

$$= \frac{2}{1+i}\int_{|z|=1} \frac{dz}{z^2+3z+3iz+i}.$$

The roots of this are

$$\frac{1}{2}\left(-3-3i\pm\sqrt{7}+i\sqrt{7}\right),$$

so by the logic of the previous question, we have

$$\int_0^{2\pi} \frac{d\theta}{3+\sin\theta+\cos\theta} = \frac{2\pi}{\sqrt{7}}.$$

(8) $\int_0^\pi \frac{d\theta}{1+\sin^2\theta}$

Letting $z = e^{i\theta}$, we have

$$d\theta = \frac{1}{i}\frac{dz}{z},$$

$$\sin^2\theta = \left(\frac{1}{2i}\left(z - 1/z\right)\right)^2 = -\frac{1}{4}\left(z^2 - 2 + 1/z^2\right).$$

Therefore,

$$\int_0^\pi \frac{d\theta}{1+\sin^2\theta} = \int_{|z|=1, \mathrm{Re}\ z \geq 0} \frac{4dz}{iz^3 + 2iz - \frac{1}{z}} = \frac{4}{i}\int \frac{zdz}{z^4 + 2z^2 - 1} = \int \frac{zdz}{(}.$$

The roots of this are

In Exercises (9) to (12), evaluate the given integral using the technique of Example 10; indicate which theorem or device you used to obtain your answer.

(9) $\int_\gamma \frac{dz}{z^2}$, where γ is any curve in Re $z > 0$ joining $1 - i$ to $1 + i$.

The integrand $f(z) = 1/z^2$ is the derivative of $F(z) = -1/z$. This is valid everywhere except $z = 0$, so our domain is valid. The integral

$$\int_\gamma f(z)dz = \int_\gamma F'(z)dz$$

$$= F(1+i) - F(1-i)$$

$$= \frac{1}{1+i} - \frac{1}{1-i}$$

$$= -i.$$

(10) $\int_\gamma \left(z + \frac{1}{z}\right) dz$, where γ is any curve in Im $z > 0$ joining $-4 + i$ to $6 + 2i$.

The integrand $f(z) = z + 1/z$ is the derivative of $F(z) = \frac{1}{2}z^2 + \log z$, which is a valid antiderivative outside of $z = 0$, which means our curve is valid. We then have

$$\int_\gamma f(z)dz = \int_\gamma F'(z)dz$$

$$= F(6+2i) - F(-4+1)$$

$$= 16 + 12i + \log 6 + 2i - 15/2 + 4i - \log -4 + i$$

$$= 17/2 + 16i + \log\left(\frac{6+2i}{-4+1}\right).$$

(11) $\int_\gamma e^z dz$, where γ is the semicircle from -1 to 1 passing through i.

The integrand $f(z) = e^z$ is the derivative of the function $F(z) = e^z$, which is valid everywhere, so we have

$$\begin{aligned}\int_\gamma f(z)dz &= \int_\gamma F'(z)dz \\ &= F(1) - F(-1) \\ &= e - 1/e.\end{aligned}$$

(12) $\int_\gamma \sin z dz$, where γ is any curve joining i to π.

The integrand $f(z) = \sin z$ is the derivative of the function $F(z) = -\cos z$, which is analytic everywhere. We then have

$$\begin{aligned}\int_\gamma f(z)dz &= \int_\gamma F'(z)dz \\ &= F(\pi) - F(i) \\ &= -1 - \cos i.\end{aligned}$$