Math 523 HW8

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Use the method of Examples 1 and 2 to compute these integrals.

1. $\int_{-\infty}^{\infty} \frac{x^4}{1+x^8} dx$

Let $f = z^4/(1+z^8)$. This function has isolated singularities where $z^8 = -1$, i.e. at $z = e^{i(\pi/8+k\pi/4)}$, with integer k. The singularities that lie in the domain U are the points $e^{i\pi/8}$, $e^{3i\pi/8}$, $e^{5i\pi/8}$, $e^{7i\pi/8}$. By the Residue Theorem, this integral is equal to the sum of these residues multiplied by $2\pi i$.

By theorem, the residue at z_0 of a rational function F/G is equal to $F(z_0)/G'(z_0)$, so by letting $F = z^4$ and $G = 1 + z^8$, we may calculate our residues.

- $\text{Res}(f; e^{i\pi/8}) = 1/8e^{3i\pi/8}$
- $\text{Res}(f; e^{3i\pi/8}) = 1/8e^{9i\pi/8}$
- $\text{Res}(f; e^{5i\pi/8}) = 1/8e^{15i\pi/8}$
- $\text{Res}(f; e^{7i\pi/8}) = 1/8e^{21i\pi/8}$

Summing and multiplying these gives us

$$\int_{-\infty}^{\infty} \frac{x^4}{1+x^8} dx = \frac{i\pi}{4}(e^{-3i\pi/8} + e^{-9i\pi/8} + e^{-15i\pi/8} + e^{-21i\pi/8}).$$

2. $\int_{-\infty}^{\infty} \frac{x^2}{x^4-4x^2+5} dx$

Let $f(z) = \frac{z^2}{z^4-4z^2+5}$. This function has isolated singularities at $z = \pm\sqrt{2+i}$ and $z = \pm\sqrt{2-i}$. The singularities that lie in U are $-\sqrt{2-i}$ and $\sqrt{2+i}$.

By theorem, the residue at z_0 of a rational function F/G is equal to $F(z_0)/G'(z_0)$, so by letting $F = z^2$ and $G = z^4 - 4z^2 + 5$, we may calculate our residues. This formula shows that $\text{Res}(f, \sqrt{2+i}) = \frac{\sqrt{2+i}}{4i}$ and $\text{Res}(f, -\sqrt{2-i}) = \frac{\sqrt{2-i}}{4i}$. Summing and multiplying (by 2 πi) these values gives us

$$\int_{-\infty}^{\infty} \frac{x^2}{x^4 - 4x^2 + 5} = 2i\pi \left(\frac{\sqrt{2+i} + \sqrt{2-i}}{4i} \right).$$

3. $\int_{-\infty}^{\infty} \frac{dx}{(x^2+a^2)(x^2+b^2)}, \ a, b > 0$

Let $f(z) = 1/(z^2 + a^2)(z^2 + b^2)$. This function has isolated singularities at $z = \pm bi$ and $z = \pm ai$. As a, b are positive, ai and bi lie in U.

By theorem, the residue at z_0 of a rational function F/G is equal to $F(z_0)/G'(z_0)$, so by letting $F = 1$ and $G = (z^2 + a^2)(z^2 + b^2)$, we may calculate our residues. This formula gives us $\text{Res}(f, ai) = \frac{1}{2ai}\frac{1}{b^2-a^2}$ and $\text{Res}(f, bi) = \frac{1}{2bi}\frac{1}{a^2-b^2}$. Summing and multiplying (by $2\pi i$) these values gives us

$$\int_{-\infty}^{\infty} \frac{dx}{(x^2 + a^2)(x^2 + b^2)} = \frac{\pi}{ab(a + b)}.$$

Use the method of Example 7 to compute these integrals.

9. $\int_0^{2\pi} \frac{d\theta}{(2-\sin\theta)^2}$

With the substitution $z = e^{i\theta}$, it follows that $d\theta = dz/iz$ and $\sin\theta = \frac{1}{2i}(z - (1/z))$. This gives

$$\int_0^{2\pi} \frac{d\theta}{(2 - \sin\theta)^2} = \int_{|z|=1} \frac{dz}{zi(2 - \frac{z}{2i} + \frac{1}{2iz})^2}$$

$$= \frac{1}{i} \int_{|z|=1} \frac{dz}{\frac{9z}{2} - \frac{2z^2}{i} + \frac{2}{i} - \frac{z^3}{4} - \frac{1}{4z}} = 8i\pi \left\{ \frac{1}{2\pi i} \int_{|z|=1} \frac{zdz}{18iz^2 - 8z^3 + 8z - iz^4 - i} \right\}.$$

The integrand (call it f) of this integral has one pole within $|z| < 1$, at $-i(\sqrt{3} - 2)$, so we have

$$\int_0^{2\pi} \frac{d\theta}{(2 - \sin\theta)^2} = 8i\pi \text{Res}(f, -i(\sqrt{3} - 2)) = 8i\pi \frac{-i}{6\sqrt{3}} = \frac{4\pi}{3\sqrt{3}}.$$

10. $\int_0^{2\pi} \frac{d\theta}{(1+\beta\cos\theta)^2}, \ -1 < \beta < 1$

With the substitution $z = e^{i\theta}$, it follows that $d\theta = dz/iz$ and $\cos\theta = \frac{1}{2}(z + (1/z))$. This gives

$$\int_0^{2\pi} \frac{d\theta}{(1 + \beta\cos\theta)^2} = \int_{|z|=1} \frac{dz}{iz(1 + \frac{\beta z}{2} + \frac{\beta}{2z})^2}$$

$$= \frac{1}{i} \int_{|z|=1} \frac{dz}{z + \beta z^2 + \beta + \beta^2 z/2 + \beta^2 z^3/4 + \beta^2/4} = \frac{4}{i} \int_{|z|=1} \frac{zdz}{\beta^2 z^4 + 4\beta z^3 + (2\beta^2 + 4)z^2 + 4\beta z + \beta^2}$$

$$= 8\pi \left\{ \frac{1}{2\pi i} \int_{|z|=1} \frac{zdz}{(\beta z^2 + 2z + \beta)^2} \right\}.$$

The integrand f of this integral has isolated singularities in $|z| < 1$ at $z = \frac{-1}{\beta} \pm \sqrt{1/\beta^2 - 1}$, which are complex conjugates, so we have

$$\int_0^{2\pi} \frac{d\theta}{(1 + \beta\cos\theta)^2} = 8\pi|\text{Res}(f, \frac{-1}{\beta} + \sqrt{1/\beta^2 - 1})| = \frac{8\pi(1/\beta^3)}{(1/\beta^2 - 1)^{3/2}}.$$

12. $\int_0^{2\pi} \sin^{2k}\theta d\theta$

With the substitution $z = e^{i\theta}$, it follows that $d\theta = dz/iz$ and $\sin\theta = \frac{1}{2i}(z - (1/z))$. This gives

$$\int_0^{2\pi} \sin^{2k}\theta d\theta = \int_{|z|=1} \frac{1}{(2i)^{2k}}(z - 1/z)^{2k}$$

$$= \int_{|z|=1} \frac{1}{(2i)^k} \frac{z^{2k}}{z^{2k}}(z - 1/z)^{2k} = \int_{|z|=1} \frac{1}{(2i)^{2k}} \left(\frac{z^2 - 1}{z}\right)^{2k}.$$

This integrand (f) has an isolated singularity at $z = 0$, so we have

$$\int_0^{2\pi} \sin^{2k}\theta d\theta = 2\pi i \text{Res}(f, 0) = \frac{(2k)!\pi}{(k!)^2 2^{2k-1}}.$$