Math 521 HW 5

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Exercise 2.3.1. Let $x_n \geq 0$ for all $n \in \mathbb{N}$.

2.3.1.a. If $(x_n) \to 0$, show that $(\sqrt{x_n}) \to 0$.

Given arbitrary $\epsilon > 0$, let $N_\epsilon \in \mathbb{N}$ such that all $n \geq N_\epsilon$ satisfy

$$|x_n| < \epsilon^2.$$

This implies

$$|\sqrt{x_n}||\sqrt{x_n}| < \epsilon^2 \implies |\sqrt{x_n}|^2 < \epsilon^2 \implies |\sqrt{x_n}| < \epsilon.$$

Therefore $(\sqrt{x_n}) \to 0$.

2.3.1.b. If $(x_n) \to x$, show that $(\sqrt{x_n}) \to \sqrt{x}$.

Given arbitrary $\epsilon > 0$, let $N_\epsilon \in \mathbb{N}$ such that all $n \geq N_\epsilon$ satisfy

$$|x_n - x| < \epsilon^2 \implies |\sqrt{x_n} - \sqrt{x}||\sqrt{x_n} + \sqrt{x}| < \epsilon^2,$$

and since

$$|\sqrt{x_n} - \sqrt{x}| \leq |\sqrt{x_n} + \sqrt{x}|, \text{ then } |\sqrt{x_n} - \sqrt{x}|^2 \leq |x_n - x| < \epsilon^2$$
$$\implies |\sqrt{x_n} - \sqrt{x}|^2 < \epsilon^2 \implies |\sqrt{x_n} - \sqrt{x}| < \epsilon,$$

so this sequence converges to $\sqrt{x}$.

Exercise 2.3.2. Using only Definition 2.2.3 (no Algebraic Limit Theorem), prove that if $(x_n) \to 2$, then

2.3.2.a. $(\frac{2x_n - 1}{3}) \to 1$;

Given $\epsilon > 0$, let $N \in \mathbb{N}$ such that all $n \geq N$ implies that

$$|x_n - 2| < 3\epsilon/2.$$

Therefore, we have

$$\left|\frac{2}{3}\right| |x_n - 2| < \epsilon \implies \left|\frac{2}{3}x_n - \frac{4}{3}\right| < \epsilon \implies \left|\frac{2x_n - 4}{3}\right| < \epsilon$$

$$\implies \left|\frac{2x_n - 1}{3} - 1\right| < \epsilon,$$

so $\left(\frac{2x_n-1}{3}\right)$ converges to 1.

2.3.2.b. $(1/x_n) \to 1/2$.

Since (x_n) is a convergent sequence, it is bounded—we can set M equal to the lowest real number such that $x_n \leq M$, $\forall n \in \mathbb{N}$. Given $\epsilon > 0$, let $N \in \mathbb{N}$ such that all $n \geq N$ implies

$$|x_n - 2| < 2|M|\epsilon.$$

We then have

$$|2x_n| \left|\frac{1}{x_n} - \frac{1}{n}\right| < 2|M|\epsilon,$$

and because $x_n \leq M$ we have

$$|2M| \left|\frac{1}{x_n} - \frac{1}{2}\right| < 2|M|\epsilon \implies \left|\frac{1}{x_n} - \frac{1}{2}\right| < \epsilon,$$

so this sequence converges to 1/2.

Exercise 2.3.3 (Squeeze Theorem). Show that if $x_n \leq y_n \leq z_n$ for all $n \in \mathbb{N}$, and if $\lim\, x_n = \lim\, z_n = l$, then $\lim\, y_n = l$ as well.

We will prove this by contradiction in two cases. Take $x_n \leq y_n \leq z_n$ for all $n \in \mathbb{N}$, $\lim x_n = \lim z_n = l$, and let $\lim y_n = k \neq l$.

First, we will prove that k cannot be less than l, by contradiction. Assume that $k > l$. Then, (by the Order Theorem) because both z_n and y_n are convergent sequences with $y_n \leq z_n$ for all n, $k \leq l$, which contradicts our assumptions, so k cannot be less than l.

Similarly, assume $k < l$. This is contradicted by x_n and y_n converging and $x_n \leq y_n$ for all n, which implies $l \leq k$. This is a contradiction, so k cannot be less than l either.

Therefore, $k = l$.

2.3.7. Give an example of each of the following, or state that such a request is impossible by referencing the proper theorem(s):

2.3.7.a. sequences (x_n) and (y_n), which both diverge, but whose sum (x_n+y_n) converges;

The sequences $(x_n) = ((-1)^n)$ and $(y_n) = ((-1)^{n+1})$ both diverge, yet their sum $((-1)^n + (-1)^{n+1}) = (0) \to 0$.

2.3.7.b. sequences (x_n) and (y_n), where (x_n) converges, (y_n) diverges, and $(x_n + y_n)$ converges;

This cannot converge by the Algebraic Limit Theorem. If $(x_n + y_n)$ and (x_n) converge, then (y_n) must converge because $(x_n + y_n - x_n) = (y_n)$ would be the sum of two convergent sequences, yet (y_n) is assumed to be divergent.

2.3.7.c. a convergent sequence (b_n) with $b_n \neq 0$ for all n such that $(1/b_n)$ diverges;

This is impossible by the Algebraic Limit Theorem—the uniform sequence of 1 converges to 1 and the sequence (b_n) converges to some $b \in \mathbb{R}$, so $(1/b_n)$ must converge to $1/b$, and as such cannot diverge.

2.3.7.d. an unbounded sequence (a_n) and a convergent sequence (b_n) with $(a_n - b_n)$ bounded;

This cannot be true because we have the difference between an unbounded and bounded set, which cannot be bounded. An unbounded set is naturally non-convergent, so this cannot converge. We have $\forall M_a \in \mathbb{R}$, there is some a_n such that $|a_n| > M_a$ and some M_b such that all b_n satisfy $|b_n| \leq M_b$. We then have $M_a - M_b < a_n - b_n$ because all real numbers have some a_n larger, and b_n is some finite number (so this difference is unbounded and as such divergent).

2.3.7.e. two sequences (a_n) and (b_n), where $(a_n b_n)$ and (a_n) converge but (b_n) does not.

If (a_n) is uniformly 0 and (b_n) any divergent series, we have $(a_n b_n)$ uniformly zero, which converges.

Exercise 2.3.9.

2.3.9.a. Let (a_n) be a bounded (not necessarily convergent) sequence, and assume $\lim b_n = 0$. Show that $\lim (a_n b_n) = 0$. Why are we not allowed to use the Algebraic Limit Theorem to prove this?

Let M be the bound of (a_n), i.e., for all $n \in \mathbb{N}$, $|a_n| \leq M$. Given $\epsilon > 0$, let $N \in \mathbb{N}$ such that all $n \geq N$ satisfies

$$|b_n| < \epsilon/M.$$

This implies

$$M|b_n| < \epsilon \implies |b_n||a_n| < \epsilon \implies |a_n b_n - 0| < \epsilon,$$

so this converges to 0.

We are not allowed to use the Algebraic Limit Theorem in this case because the Algebraic Limit Theorem requires convergent sequences for its conclusions, and (a_n) is not necessarily convergent.

2.3.9.b. Can we conclude anything about the convergence of $(a_n b_n)$ if we assume that (b_n) converges to some nonzero limit b?

We can not conclude anything about the convergence of $(a_n b_n)$ assuming the convergence of (b_n) to some $b \neq 0$ if we do not know the convergence of (a_n). For instance, take $(a_n) = ((-1)^n)$. For this, we have non-convergent $(a_n b_n)$, but for some convergent (a_n), we have convergent $(a_n b_n)$.

2.3.9.c. Use (a) to prove Theorem 2.3.3, part (iii), for the case when $a = 0$.

Letting $(b_n) \to a = 0$, and (a_n) convergent (to b) and bounded by M (to satisfy the two convergent sequences required by hypothesis). Then, given $\epsilon > 0$, let $N \in \mathbb{N}$ such that all $n \geq N$ satisfies

$$|b_n| < \epsilon/M.$$

This implies

$$M|b_n| < \epsilon \implies |b_n||a_n| < \epsilon \implies |a_n b_n - 0| < \epsilon,$$

so we have $\lim(a_n b_n) = 0 = ab$.