Math 521 HW 4

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2.2.2.

Verify, using the definition of convergence of a sequence, that the following sequences converge to the proposed limit.

(a) $\lim \frac{2n+1}{5n+4} = \frac{2}{5}$.

Given arbitrary $\epsilon > 0$, let $N_\epsilon \in \mathbb{N}$ satisfy $N_\epsilon > \frac{20\epsilon+3}{25\epsilon}$. This sequence converges if all $n \in \mathbb{N}$ greater than or equal to N_ϵ satisfy the inequality

$$\left|\frac{2n-1}{5n-4} - \frac{2}{5}\right| = \left|\frac{3}{25n-20}\right| < \epsilon.$$

This inequality that holds for $n \geq N_\epsilon$ can be modified to get

$$n > \frac{20\epsilon+3}{25\epsilon} \implies 25n{-}20 > \frac{3}{\epsilon} \implies \frac{3}{25n-20} < \epsilon \implies \left|\frac{3}{25n-20}\right| < \epsilon \implies \left|\frac{2n-1}{5n-4} - \frac{2}{5}\right| < \epsilon,$$

so the requisite inequality holds if $n \geq N_\epsilon$, so the sequence converges to $\frac{2}{5}$.

(b) $\lim \frac{2n^2}{n^3+3} = 0$.

Given arbitrary $\epsilon > 0$, let $N_\epsilon \in \mathbb{N}$ satisfy $N_\epsilon > \frac{2}{\epsilon}$. For this sequence to converge, whenever $n \in \mathbb{N}$ is greater than or equal to N_ϵ the inequality

$$\left|\frac{2n^2}{n^3+3}\right| < \epsilon$$

must hold. Note that

$$\left|\frac{2n^2}{n^3+3}\right| < \left|\frac{2n^2}{n^3}\right|$$

holds for all $n \in \mathbb{N}$, so if $\left|\frac{2n^2}{n^3}\right| = \left|\frac{2}{n}\right| < \epsilon$, then $\left|\frac{2n^2}{n^3+3}\right| < \epsilon$, and the sequence must converge. With a little arithmetic, we have

$$n > \frac{2}{\epsilon} \implies \frac{1}{n} < \frac{\epsilon}{2} \implies \frac{2}{n} < \epsilon \implies \left|\frac{2n^2}{n^3}\right| < \epsilon \implies \left|\frac{2n^2}{n^3+3}\right| < \epsilon,$$

so this sequence converges to 0.

(c) $\lim \frac{\sin(n^2)}{\sqrt[3]{n}} = 0$.

Note that the range of the sine function is the interval $[-1, 1]$, so for all n, the inequality

$$\left|\frac{\sin(n^2)}{\sqrt[3]{n}}\right| \leq \left|\frac{1}{\sqrt[3]{n}}\right|$$

holds. Given arbitrary $\epsilon > 0$, let $N_\epsilon \in \mathbb{N}$ satisfy $N_\epsilon > \frac{1}{\epsilon^3}$. For this sequence to converge, whenever $n \in \mathbb{N}$ greater than or equal to N_ϵ must satisfy the inequality

$$\left|\frac{\sin(n^2)}{\sqrt[3]{n}}\right| < \epsilon.$$

From the inequality set on n and N_ϵ, we have

$$n > \frac{1}{\epsilon^3} \implies \frac{1}{n} < \epsilon^3 \implies \frac{1}{\sqrt[3]{n}} < \epsilon \implies \left|\frac{1}{\sqrt[3]{n}}\right| < \epsilon \implies \left|\frac{\sin(n^2)}{\sqrt[3]{n}}\right| < \epsilon,$$

so this sequence converges to 0.

2.2.4.

Give an example of each or state that the request is impossible. For any that are impossible, give a compelling argument for why that is the case.

(a) A sequence with an infinite number of ones that does not converge to one.

The sequence ($a_n = \{0, 1, 0, 1, 0, 1, ...\}$) i.e. the sequence where a_n is 0 when n is odd and 1 when n is even has an infinite number of ones but does not converge to one.

(b) A sequence with an infinite number of ones that converges to a limit that is not equal to one.

There is no such sequence. To argue against this case, let (a_n) be a sequence with infinite ones and a proposed limit $a \neq 1$. Let the difference between this a and 1 be equal to δ. Therefore, there are an infinite number of elements in the sequence such that $|a_n - a| = \delta$. Selecting $\epsilon = \delta/2$ provides us with the fact that an infinite number of elements in this sequence satisfy $|a_n - a| \geq \epsilon$. As there is no way to insure that the infinite ones stop in this sequence, there can be no N_ϵ such that all $n \geq N_\epsilon$ satisfy $|a_n - a| < \epsilon$ and as such, this cannot converge to 1.

(c) A divergent sequence such that for every $n \in \mathbb{N}$ it is possible to find n consecutive ones somewhere in the sequence.

The sequence $(a_n) = \{1, 0, 1, 1, 0, 1, 1, 1, 0, ...\}$ (the sequence of subsequent natural number amount of 1s separated by a 0) is divergent (as it doesn't converge to 0 or 1) and has an n-length string of consecutive 1s for all n.

2.2.5.

Let $[[x]]$ be the greatest integer less than or equal to x. For each sequence, find $\lim a_n$ and verify it with the definition of convergence.

(a) $a_n = [[5/n]]$.

For this sequence, $\lim a_n$ is equal to 0. To prove this, take an arbitrary $\epsilon > 0$, and take $N_\epsilon > 5$. For all $n \geq N_\epsilon$, we have $|[[5/n]]| < \epsilon$, as $[[5/n]]$ for $n \geq 6$ is equal to 0 and less than ϵ.

(b) $a_n = [[(12 + 4n)/3n]]$.

For this sequence, $\lim a_n$ is equal to 1. To prove this, take arbitrary $\epsilon > 0$ and $N_\epsilon > 6$. For all $n \geq N_\epsilon$, we have $|[[(12 + 4n)/3n]] - 1| = 0 < \epsilon$, as $[[(12 + 4n)/3n]] = 1$ for all $n > 6$.

The statement following Definition 2.2.3 that the "smaller the ϵ-neighborhood, the larger N may have to be" does not hold in all cases, as the N in this case is independent of the ϵ.

2.2.6.

Prove Theorem 2.2.7. To get started, assume $(a_n) \to a$ and also that $(a_n) \to b$. Now argue that $a = b$.

Assume that for some arbitrary sequence (a_n), both $(a_n) \to a$ and $(a_n) \to b$ hold true. This means that for both a and b, the statements

$$\forall \epsilon > 0,\ \exists N_\epsilon \in \mathbb{N} \text{ such that } n \geq N_\epsilon \implies |a_n - a| < \epsilon \text{ and}$$

$$\forall \epsilon > 0,\ \exists N_\epsilon \in \mathbb{N} \text{ such that } n \geq N_\epsilon \implies |a_n - b| < \epsilon.$$

Fixing arbitrary ϵ, we get separate $N_{\epsilon,b}$ and $N_{\epsilon,a}$ that satisfy these statements. The larger of these two numbers guarantees that for all n greater than or equal to it satisfy these inequalities, so $n \geq N_{\epsilon,a}$ and $n \geq N_{\epsilon,b} \implies |a_n - a| < \epsilon$ and $|a_n - b| < \epsilon$. This means that (a_n) converges to equality at some point to both a and b by the definition of equality, and by the transitivity of equality, this implies that $a = b$.

HW4.1:

For each $n \in \mathbb{N}$, let $x_n = 2 + \frac{(-1)^n}{n}$. Prove $x_n \to 2$.

Given arbitrary $\epsilon > 0$, let $N_\epsilon > \frac{1}{n}$. For this sequence to converge, whenever $n \in \mathbb{N}$ is greater or equal to N_ϵ, the following inequality must hold

$$\left|2 - \frac{(-1)^n}{n} - 2\right| < \epsilon \implies \left|\frac{(-1)^n}{n}\right| < \epsilon \implies \frac{1}{n} < \epsilon.$$

Since our n is greater than $\frac{1}{\epsilon}$, this inequality must hold for desired n, so this sequence does converge to 2.

HW4.2:

For each $n \in \mathbb{N}$, let $x_n = (-1)^n$. Prove that (x_n) does not converge to any real number.

For the sake of contradiction, let $x_n \to a$. This means that for all $\epsilon > 0$, $\exists N_\epsilon$ such that $n \geq N_\epsilon \implies |(-1)^n - a| < \epsilon$. Let the value $|1 - a| = \delta$. If $\delta > 0$, then setting $\epsilon = \delta/2$ provides an $\epsilon > 0$ for which there is no satisfactory N_ϵ. If $\delta = 0$, then $|-1 - a| = 2$, and setting $\epsilon = 1$ provides a counterexample. Therefore, this sequence can not converge to any real number.