Analysis I

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Contents

1 The Real Numbers

1.1 The Axiom of Completeness

The **axiom of completeness** states that *every nonempty set of real numbers that is bounded above has a least upper bound.*

A set $A \subseteq \mathbb{R}$ is **bounded above** if there exists a number $b \in \mathbb{R}$ such that $\forall a \in A,\ a \leq b$. The number b is called an **upper bound** for A. Similarly, the set A is **bounded below** if there exists a **lower bound** $l \in \mathbb{R}$ satisfying $\forall a \in A,\ l \leq a$.

A real number s is the **least upper bound** for a set $A \subseteq \mathbb{R}$ if it meets the following two criteria:

1. s is an upper bound for A;
2. if b is any upper bound for A, then $s \leq b$.

The least upper bound is also called the **supremum** of the set A, and is called $s = \sup A$. The **greatest lower bound** is defined similarly, and is called the infimum, with $l = \inf A$. Both suprema and infima are unique. A real number a_0 is a **maximum** on a set A if a_0 is an element of A and $a_0 = \sup A$. A **minimum** is an element of A that is also the infimum of A.

Assuming $s \in \mathbb{R}$ is an upper bound for a set $A \subseteq \mathbb{R}$, then $s = \sup A \iff \forall \epsilon > 0,\ \exists a \in A,$ such that $s - \epsilon < a$.

1.2 Consequences of Completeness

The **Archimedean property** states that:

1. Given any number $x \in \mathbb{R}$, there exists an $n \in \mathbb{N}$ such that $n > x$.
2. Given any real number $y > 0$, there exists an $n \in \mathbb{N}$ such that $1/n < y$.

Additionally, for every two real number a and b with $a < b$, there exists some $q \in b\mathbb{Q}$ (or in the set of irrational numbers) such that $a < r < b$.

1.3 Cardinality

A function $f : A \to B$ is **one-to-one** (injective) if $a_1 \neq a_2$ in A implies that $f(a_1) \neq f(a_2)$ in B. The function f is **onto** (surjective) if, given any $b \in B$, there exist an $a \in A$ such that $f(a) = b$. If f is both onto and one-to-one, then it is called **bijective**.

The set A **has the same cardinality** as B if there exists a bijection $f : A \to B$. In this case, we write $A \sim B$. A set A is **countable** if $\mathbb{N} \sim A$. An infinite set that is not countable is called an **uncountable** set. $\mathbb{Q}$ is countable and $\mathbb{R}$ is uncountable. Additionally, subsets of countable sets are countable and countable unions of countable sets are countable.

2 Sequences and Series

2.1 The Limit of a Sequence

A **sequence** is a function whose domain is $\mathbb{N}$. A sequence (a_n) **converges** to a real number a if, for every positive number ϵ, there exists an $N \in \mathbb{N}$ such that whenever $n \geq N$ it follows that $|a_n - a| < \epsilon$. This is notated as $\lim a_n = a$.

Given a real number $a \in \mathbb{R}$ and a positive number $\epsilon > 0$, the set

$$V_\epsilon(a) = \{x \in \mathbb{R} : |x - a| < \epsilon\}$$

is called the **ϵ-neighborhood of** a.

The limit of a sequence, when it exists, must be unique.

A sequence that does not converge is said to **diverge**.

2.2 The Algebraic and Order Limit Theorems

A sequence (x_n) is **bounded** if there exists a number $M > 0$ such that $|x_n| \leq M$ for all $n \in \mathbb{N}$. Every convergent sequence is bounded.

The Algebraic Limit Theorem states that for two convergent sequences with $\lim a_n = a$ and $\lim b_n = b$,

1. $\lim ca_n = ca,$ for all $c \in \mathbb{R}$;
2. $\lim a_n + b_n = a + b$;
3. $\lim a_n b_n = ab$;
4. $\lim a_n / b_n = a/b,$ provided $b \neq 0$.

The Order Limit Theorem states that for two convergent sequences with $\lim a_n = a$ and $\lim b_n = b$,

1. If $a_n \geq 0$ for all $n \in \mathbb{N}$, then $a \geq 0$.
2. If $a_n \leq b_n$ for all $n \in \mathbb{N}$, then $a \leq b$.
3. If there exists $c \in \mathbb{R}$ for which $c \leq b_n$ for all $n \in \mathbb{N}$, then $c \leq b$. Similarly, if $a_n \leq c$ for all $n \in \mathbb{N}$, then $a \leq c$.