Math 523 HW 6

Morgan Gribbins

Section 2.4

In Exercises 2 to 8, give the order of each of the zeros of the given function.

(2) $(e^z - 1)^2$

This function has zeros at $z = 2\pi k$, $k \in \mathbb{Z}$, each with order 2, as the first derivative is 0 at these zeros, and the 2nd derivative does not yield zero.

(4) $(z^2 - 4z + 4)^3$

This function has its only zero at $z = 2$, which has order 6.

(6) $\log(1 - z)$, $|z| < 1$

This function has its only zero at $z = 0$, and this has order 1.

(8) $\frac{z}{z^2+1}$

This function has its only zero at $z = 0$, and this has order 1.

In Exercises 10 and 12, find the power-series expansion about the given point for each of the functions; find the largest disc in which the series is valid.

(10) e^z about $z_0 = \pi i$

$$e^z = \sum_{n=0}^{\infty} (z - \pi i)^n / n!,$$

which is entire.

(12) $\frac{(z^2}{1-z}$ about $z_0 = 0$

$$\frac{z^2}{1 - z} = \sum_{n=0}^{\infty} z^{n+2},$$

which is valid within the disc $|z| < 1$.

(25) Find all solutions to the differential equation

$$f''(z) + \beta^2 f(z) = 0, \ \ f \text{ is an entire function.}$$

Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$. The differential equation is then

$$\sum_{n=2}^{\infty} a_n(n^2 - n)z^{n-2} + \beta^2 \sum_{n=0}^{\infty} a_n z^n = 0$$

$$\implies (\beta^2 a_0 + 2a_2) + z(\beta^2 a_1 + 6a_3) + z^2(\beta^2 a_2 + 12a_4) + z^3(\beta^2 a_3 + 20a_5) + ... = 0$$

$$\implies \sum_{n=0}^{\infty}(\beta^2 a_n + (n^2 + 3n + 2)a_{n+2})z^n = 0$$

$$\implies \beta^2 a_n = (-n^2 - 3n - 2)a_{n+2} \implies a_2 = -\beta^2 a_0/2, \ a_3 = -\beta^2 a_1/6,$$

and the rest of the terms are recursively provided by this rule.

(26) Use the technique of Exercise 25 to give the solutions of these differential equations:
(26a) $f''(z) - 3f'(z) + 2f(z) = 0; \ a_0 = 1, \ a_1 = 2.$

Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$. This equation is then

$$\sum_{n=2}^{\infty}(n^2 - n)a_n z^{n-2} - \sum_{n=1}^{\infty} 3na_n z^{n-1} + \sum_{n=0}^{\infty} 2a_n z^n = 0$$

$$\implies (2a_0 - 3a_1 + 2a_2) + z(2a_1 - 6a_2 + 6a_3) + z^2(2a_2 - 9a_3 + 12a_4) + z^3(2a_3 - 12a_4 + 20a_5) + ... = 0.$$

This means that

$$2a_n - 3(n+1)a_{n+1} + (n^2 + 3n + 2)a_{n+2} = 0.$$

With $a_0 = 1$ and $a_1 = 2$, the rest of these terms may be recursively solved.

(26c) $f''(z) + 2f'(z) + f(z) = 0; \ a_0 = 0, \ a_1 = 1.$

Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$. This equation is then

$$\sum_{n=2}^{\infty}(n^2 - n)a_n z^{n-2} + \sum_{n=1}^{\infty} 2na_n z^{n-1} + \sum_{n=0}^{\infty} a_n z^n = 0$$

$$\implies (a_0 + 2a_1 + 2a_2) + z(a_1 + 4a_2 + 6a_3) + z^2(a_2 + 6a_3 + 12a_4) + z^3(a_3 + 8a_4 + 20a_5) + ... = 0.$$

This means that

$$a_n + 2(n+1)a_{n+1} + (n^2 + 3n + 2)a_{n+2} = 0.$$

With $a_0 = 0$ and $a_1 = 2$, the rest of these terms may be recursively solved.