Math 534 HW 4

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(1) Let G be a cyclic group that has exactly 3 subgroups: G itself, the trivial subgroup, and a proper subgroup of order 7. Find $|G|$, making sure to justify your answer.

The cyclic group of order 49 is the only group with such properties. The only divisors of 49 are $1, 7$, and 49, which correspond to the trivial subgroup, the proper subgroup of order 7, and G, respectively.

(2) Let G be an abelian group with $|G| = 35$, and suppose that every element $g \in G$ satisfies the equality $g^{35} = e$. Prove that G is cyclic. (hint: I claim the result follows once you have your hands on an element of order 5 and an element of order 7. Think of the possible of orders of elements in the group, and deduce that you must have elements as above. Finally, at some point you might use the fact that neither 4 nor 6 divide 34.)

Any element in G must have order 1, 5, 7, or 35, as each element raised to the 35th power is the identity. Now, we want some element $a \in G$ such that $|a| = 5$ and $b \in G$ such that $|b| = 7$.

Assume that there is no such a in G. This means that the elements of G all have order 7 or order 35, and if any element has order 35, then it generates G and as such G is cyclic. Therefore, there must be 34 elements of G that are order 7. However, the amount of elements of order 7 in G must be a multiple of $\phi(7) = 6$, but 6 does not divide 34. We can then say that $a \in G$ and $|a| = 5$.

By the same argument, assume there is no such necessary $b \in G$ such that $|b| = 7$. We then have 34 elements of order 5 in G, but $\phi(5) = 4$ does not divide 34. This means that we can have $b \in G$ with $|b| = 5$.

From these two elements, we can show that $|ab| = 35$ and a such generates G. We have $(ab)^{35} = e$, by hypothesis. We now must show that the order of ab cannot be 5, 7, or 1. Because G is abelian, we have

$$(ab) \neq e \implies |ab| \neq 1$$

$$(ab)^5 = a^5 b^5 = eb^5 \neq e \implies |ab| \neq 5$$

$$(ab)^7 = a^7 b^7 = a^7 e \neq e \implies |ab| \neq 7,$$

and so the smallest positive number k such that $(ab)^k = e$ must be 35, and so ab generates G and G is cyclic.

(3) Let G be a group.

(3a) Let $H \leq G$ and $K \leq G$ be subgroups. Show that $H \cap K \subseteq G$ is a subgroup of G.

In order to prove that $H \cap K$ is a subgroup of G, we must prove that

1. $H \cap K$ is closed.
2. $H \cap K$ contains the identity.
3. $a \in H \cap K \implies a^{-1} \in H \cap K$.

We are going to do this $\longrightarrow$

1. To prove that $H \cap K$ is closed, we must take some arbitrary $a, b \in H \cap K$ and show that $ab \in H \cap K$. For $a, b \in H \cap K$, they must both be in both H and K—because these are closed by definition of subgroups, $ab \in H$ and $ab \in K$, so $ab \in H \cap K$. Therefore, $H \cap K$ is closed.
2. To prove $H \cap K$ has the identity, observe that both H and K contain the identity, so their intersection must also contain the identity.
3. To prove $a \in H \cap K \implies a^{-1} \in H \cap K$, take some element $a \in H \cap K$. This means $a \in H \implies a^{-1} \in H$ and $a \in K \implies a^{-1} \in H$, so $a^{-1} \in H \cap K$.

Therefore, this intersection must be a subgroup of G.

(3b) Let $a, b \in G$ such that $|a|$ and $|b|$ are finite and relatively prime. Show that $\langle a \rangle \cap \langle b \rangle = \{e\}$.

We will prove this point by contradiction. Assume that $|a|$ and $|b|$ are relatively prime and that $\langle a \rangle \cap \langle b \rangle \neq \{e\}$. We will show that this second assumption implies that $\langle a \rangle \cap \langle b \rangle$ is not a subgroup, which raises a contradiction (by (3b)). If $\langle a \rangle \cap \langle b \rangle \neq \{e\}$, then there must be some element shared between $\langle a \rangle$ and $\langle b \rangle$—i.e. for some $m, n \in \mathbb{N}$, $a^m = b^n$. Therefore, there must be some $a^k \in \langle a \rangle$ such that $a^k = b$, which implies that $\langle a \rangle = \langle b \rangle = \langle a \rangle \cap \langle b \rangle$. This raises a contradiction however—$|a| \neq |b|$, but we now have $|\langle a \rangle| = |a| = |\langle b \rangle| = |b|$, so our initial assumption must be false—i.e. $\langle a \rangle \cap \langle b \rangle$ must be equal to $\{e\}$.

(4) Let $G = \mathbb{Z}/30$.

(4a) How many distinct subgroups of G are there?

$\mathbb{Z}/30$ has 8 subgroups:

The subgroup of order 1: $\{0\} = \langle 0 \rangle$;

The subgroup of order 2: $\{0, 15\} = \langle 15 \rangle$;

The subgroup of order 3: $\{0, 10, 20\} = \langle 10 \rangle$;

The subgroup of order 5: $\{0, 6, 12, 18, 24\} = \langle 6 \rangle$;

The subgroup of order 6: $\{0, 5, 10, 15, 20, 25\} = \langle 5 \rangle$;

The subgroup of order 10: $\{0, 3, 6, 9, 12, 15, 18, 21, 24, 27\} = \langle 3 \rangle$;

The subgroup of order 15: $\{0, 2, 4, 6, 8, 10, 12, 14, 16, 18, 20, 22, 24, 26, 28\} = \langle 2 \rangle$;

The subgroup of order 30: $\mathbb{Z}/30$.

(4b) Draw the lattice of subgroups of G.