Complex Functions

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February 7, 2020

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1 The Complex Plane

1.1 The Complex Numbers and the Complex Plane

The complex plane is how we visualize the separate real and imaginary components of the complex variable z. A **complex number** is an expression of the form

$$z = x + iy,$$

where x and y are real numbers and i satisfies the rule

$$(i)^2 = (i)(i) = -1.$$

The number x is called the **real part** of z and is written

$$x = \text{Re } z.$$

The number y, despite the fact that it is also a real number, is called the **imaginary part** of z and is written

$$z = \text{Im } z.$$

The **modulus**, or **absolute value** of z is defined by

$$|z| = \sqrt{x^2 + y^2},\ z = x + iy.$$

A complex number $z = x+iy$ corresponds to the point $P(x, y)$ in the xy-plane. The modulus of z, then, is just the distance from the point $P(x, y)$ to the origin, which is 0. These three inequalities hold true:

$$|x| \leq |z|,\ |y| \leq |z|,\ |z| \leq |x| + |y|.$$

The **complex conjugate** of $z = x + iy$ is given by

$$\bar{z} = x - iy.$$

Addition, subtraction, multiplication, and division of complex numbers follow the ordinary rules of arithmetic. For

$$z = x + iy \text{ and } w = s + it$$

we have

$$z + w = (x + s) + i(y + t),$$

$$z - w = (x - s) + i(y - t),$$

$$zw = (xs - yt) + i(xt + ys), \text{ and}$$

$$\frac{z}{w} = \frac{\bar{w}z}{\bar{w}w} = \frac{(xs + yt) + i(ys - xt)}{s^2 + t^2}.$$

We can also represent complex numbers in the complex plane via polar coordinates—

$$x = r\cos\,\theta \text{ and } y = r\,\sin\,\theta.$$

Since $r = |z|$ by definition of the modulus of a complex number, so

$$z = |z|(\cos\,\theta + i\,\sin\,\theta).$$

This is called the **polar representation** of z. For two complex numbers,

$$z = |z|(\cos\,\theta + i\,\sin\,\theta) \text{ and } w = |w|(\cos\,\phi + i\,\sin\,\phi),$$

we have

$$zw = |z||w|(\cos\,(\theta+\phi) + i\,\sin\,(\theta+\phi)) \text{ and}$$
$$\frac{z}{w} = \left(\frac{|z|}{|w|}\right)(\cos\,(\theta-\phi) + i\,\sin\,(\theta-\phi)).$$

From this, we have **De Moivre's Theorem:**

$$(\cos\,\theta + i\,\sin\,\theta)^n = \cos\,n\theta + i\,\sin\,n\theta.$$

We define an **argument** of the nonzero complex number z to be any angle θ for which

$$z = |z|(\cos\,\theta + i\,\sin\,\theta),$$

whether or not it lies in the range $[0, 2\pi)$; we write $\theta = \arg\,z$. We define **Arg** z to be the number $\theta_0 \in [-\pi, \pi)$ such that

$$z = |z|(\cos\,\theta_0 + i\,\sin\,\theta_0).$$

We then have

$$\mathbf{Arg}\,(zw) = \mathbf{Arg}\,z + \mathbf{Arg}\,w \pmod{2\pi}.$$

In summary, the xy-plane is a natural interpretation of a complex variable, and the rules of the xy-plane can be made to fit this depiction. We then call this plane the **complex plane**, the x-axis the **real axis**, and the y-axis the **imaginary axis**.

1.2 Complex Roots and Circles in the Complex Plane

Complex roots follow directly from rules established in the previous section. A complex number z that satisfies the equation $z^n = w$ (with $z = (|z|, \theta)$ and $w = (|w|, \phi)$) is called the ***n*th root of *w***. This nth root of w follows these equations:

$$|z|^n = |w|,\ \cos\,n\theta = \cos\,\phi,\ \sin\,n\theta = \sin\,\phi.$$

A circle around a point p of radius r is given by the equation

$$|z - p| = r,$$

as this is the set of points r distance from p.

1.3 Subsets of the Plane

The set consisting of all points z satisfying $|z - z_0| < R$ is called the **open disc** of radius R centered at z_0. A point w_0 in a set D in the complex plane is called an **interior point** of D if there is some open disc centered at w_0 that lies entirely within D. A set D is called **open** if all of its points are interior points. A point p is a **boundary point** of a set S if every open disc centered at p containing both points of S and not of (S). The set of all boundary points of a set S is called the **boundary** of S. A set C is **closed** if it contains its boundary.

Theorem: A set D is open if and only if it contains no point of its boundary. A set C is closed if and only if its complement $D = \{z : z \notin C\}$ is open.

A **polygonal curve** is the union of a finite number of directed line segments $\mathbf{P}_1, \mathbf{P}_2, ..., \mathbf{P}_n$, where the terminal point of one is the initial point of the next (excluding the last). An open set D is **connected** if each pair p, q of points in D may be joined by a polygonal curve lying entirely in D. An open connected set is called a **domain**. A set S is **convex** if the line segment **pq** joining each p, q in S also lies in S. An **open half-plane** is the set of points strictly to one side of a straight line. A **closed half-plane** is an open half-plane, with the inclusion of the defining straight line.

1.4 Functions ad Limits

A **function** of the complex variable z is a rule that assigns a complex number to each z within some specified set D; D is called the **domain of definition** of the function. The collection of all possible values of the function is called the **range** of the function. We frequently write $w = f(z)$ to distinguish the dependent and independent variables.

Let $\{z_n\}_{n=1}^{\infty}$ be a sequence of complex numbers. We say that $\{z_n\}$ has the complex number A as a **limit**, or that $\{z_n\}$ **converges** to A, and we write

$$\lim_{n\to\infty} z_n = A, \text{ or } z_n \to A$$

if, given any positive number ϵ, there is an integer N such that

$$|z_n - A| < \epsilon, \text{ for all } n \geq N.$$

A sequence that does not converge, for any reason whatsoever, is called **divergent**. Additionally, for $z_n = x_n + iy_n$ and $A = s + it$, then $z_n \to A$ if and only if $x_n \to s$ and $y_n \to t$. Additionally, if $z_n \to A$, then $|z_n| \to |A|$, and for two convergent series $z_n \to A$ and $w_n \to B \neq 0$, $\forall \alpha, \beta \in \mathbb{C},\ \alpha z_n + \beta w_n \to \alpha A + \beta B,\ \alpha z_n w_n \to \alpha AB,\ \alpha z_n / \beta w_n \to \alpha A / \beta B$.

We say that a function f defined on a subset $S \subseteq \mathbb{C}$ has a **limit** L at the point $z_0 \in S$ or in the boundary of S, and we write

$$\lim_{z\to z_0} f(z) = L \text{ or } f(z) \to L \text{ as } z \to z_0$$

if, given $\epsilon > 0$, there is a $\delta > 0$ such that

$$|f(z) - L| < \epsilon \text{ whenever } z \in S \text{ and } |z - z_0| < \delta.$$

We say that a function f has a **limit L at** ∞, and we write

$$\lim_{z \to \infty} f(z) = L$$

if, given $\epsilon > 0$, there is a large number M such that

$$|f(z) - L| < \epsilon \text{ whenever } z \geq M.$$

Note that this only requires that $|z|$ be large; there is no restriction on arg z. The arithmetic rules with constants and limits of sequences hold with limits of functions.

Suppose again that f is a function defined on a subset S on the complex plane. If $z_0 \in S$, then f is **continuous** at z_0 if

$$\lim_{z \to z_0} f(z) = f(z_0).$$

That is, f is continuous at z_0 if the values of $f(z)$ get arbitrarily close to the value $f(z_0)$, so long as z is in S and z is sufficiently close to z_0. If it happens that f is continuous at all points on S, we say f is **continuous on $\boldsymbol{S}$**. The function f is continuous at ∞ if $f(\infty)$ is defined and $\lim_{z \to \infty} f(z) = f(\infty)$.

The sum of an **infinite series of complex numbers** is practically the same as the sum of an infinite series of real numbers. We define the **$\boldsymbol{n}$th partial sum by**

$$s_n = \sum_{j=1}^{n} = z_1 + ... + z_n, \; n = 1, 2, ...$$

If the sequence $\{s_n\}$ has a limit s, then we say that the infinite series $\sum_{j=1}^{\infty} z_j$ **converges** and has sum s; this is written

$$\sum_{j=1}^{\infty} z_j = s.$$

If this does not have a limit, we say that this series **diverges**. As with sequences, the real and imaginary parts of the sum must converge to the real and imaginary parts of the limits of the sum.

1.5 The Exponential, Logarithm, and Trigonometric Function

The exponential function is one of the most important functions in complex analysis. Its definition is this:

$$e^{x+iy} = e^x(\cos\, y + i\, \sin\, y).$$

The form $\exp(z)$ is sometimes used, especially if z is particularly complicated. For any two complex numbers, z and w, we have

$$e^{z+w} = e^z e^w.$$

Additionally,

$$|e^z| = e^{\mathrm{Re}\ z}, \text{ and } \textbf{Arg}\ e^z = \mathrm{Im}\ z.$$

The inverse of the exponential function is the logarithm function. For a nonzero complex number z, we define **log** z to be any complex number w with $e^w = z$.

$$\log z = \ln |z|; ; ; ; +i \arg z.$$