Math 523 HW 7

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Section 2.5

In Exercises 2 through 6, locate and describe the isolated singularities.

(2) $\frac{z^2}{\sin z}$

This function has an isolated singularity at $z = 0$, and this is a removable singularity, with a value of 0 at this point.

(4) $\pi \cot \pi z$

This function has isolated singularities at $2\pi k, k \in \mathbb{Z}$. These are poles of order 1.

(6) $\frac{e^z - 1}{e^{2z} - 1}$

This function has an isolated singularity at $z = 0$. This is a removable singularity, with a value of $1/2$ at this point.

In Exercises 8 through 12, find the Laurent series for the given function about the indicated point. Also, give the residue of the function at the point.

(8) $\frac{z^2}{z^2-1}$, $z_0 = 1$

$$z^2 = 1 + 2(z-1) + (z-1)^2,\ z^2 - 1 = 2(z-1) + (z-1)^2 \implies$$

$$\frac{z^2}{z^2-1} = (1-2(z-1)+(z-1)^2)\frac{1}{2(z-1)+(z-1)^2} = \left(\frac{1}{2(z-1)} - 1 + \frac{1}{2}(z-1)^2\right)\frac{1}{1-(-\frac{1}{2}(z-1)}$$

$$\implies \frac{z^2}{z^2-1} = \frac{1/2}{(z-1)} + 3/4 + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2^{n+2}}(z-1)^n.$$

This function has residue $1/2$ at $z = 1$.

(10) $\frac{z}{(\sin z)^2}$, $z_0 = 0$ (four terms of the Laurent series)

Let $g(z) = \frac{z}{(\sin z)^2}$. This function has a pole at $z = 0$ of order 1, so its Laurent series is of the form

$$g(z) = a_{-1}/z + a_0 + a_1 z + a_2 z^2 + ..., \implies$$
$$(\sin z)^2 g(z) = z = (z^2 - z^4/3 + 2z^6/45 - z^8/315 + ...)(a_{-1}/z + a_0 + a_1 z + ...)$$
$$z = a_{-1}z + a_0 z^2 + (a_1 - a_{-1}/3)z^3 + (a_2 - a_0/3)z^4 + ...$$
$$\implies g(z) = 1/z + z/3 + z^3/15 + 2z^5/189 + ...,$$

which has residue 1 at $z = 0$.

(12) $\frac{1}{e^z-1}$, $z_0 = 0$ (four terms of the Laurent series)

Let $g(z) = \frac{1}{e^z-1}\}$, then we have $1 = (e^z - 1)g(z)$,

$$1 = (z + z^2/2! + z^3/3! + ...)(a_{-1}/z + a_0 + a_1 z + ...)$$

$$1 = a_{-1} + (a_0 + a_{-1}/2)z + (a_1 + a_0/2 + a_{-1}/6)z^2 + (a_2 + a_1/2 + a_0/6 + a_{-1}/24)z^3 + ...$$
$$\implies g(z) = 1/z - 1/2 + z/12 - z^3/720 + ...,$$

so the residue of this function is 1 at $z = 0$.

(22) Find the Laurent series about $z_0 = 0$ for the following functions, valid in the indicated regions.

(22a) $e^{1/z}$ in $0 < |z| < \infty$

$$e^{1/z} = 1 + \frac{1}{z} + \frac{1}{2z^2} + \frac{1}{6z^3} + ... = \sum_{n=0}^{\infty} \frac{1}{n!z^n}.$$

(22b) $z^4 \sin(1/z)$ in $0 < |z| < \infty$

$$z^4 \sin(1/z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!z^{2n-3}}.$$

(22c) $\frac{1}{z-1} - \frac{1}{z+1}$ in $2 < |z| < \infty$

$$\frac{1}{z-1} - \frac{1}{z+1} = \frac{2}{z^2-1} = -2 - 2z^2 - ... = \sum_{n=0}^{\infty} -2z^{2n}.$$