Math 534 HW 8

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(1) Let $H, K \leq G$ be two subgroups of a given group G. Show that for $a \in G$ the coset $a(H \cap K)$ is equal to $(aH) \cap (aK)$, i.e. is equal to the intersection of the cosets aH and aK.

Let $ax \in a(H \cap K)$. By definition of cosets, $x \in H \cap K$. This implies that $ax \in aH$ and $ax \in aK$, so $ax \in (aH) \cap (aK)$. Now, let $ax \in (aH) \cap (aK)$, which means that $ax \in aH \implies x \in H$ and $ax \in aK \implies x \in K$. Therefore, $x \in H \cap K \implies ax \in a(H \cap K)$. Because $a(H \cap K) \subseteq (aH) \cap (aK)$ and $(aH) \cap (aK) \subseteq a(H \cap K)$, these two sets are equal.

(2) Use the Theorem of Lagrange (and its consequences) to show that $|(\mathbb{Z}/n)^{\times}|$ is always even when $n > 2$.

Let $n > 2, n \in \mathbb{N}$. The group $G = (\mathbb{Z}/n)^{\times}$ then consists of the numbers relatively prime to n. Consider the group generated by the element $n - 1 \in G$ (this element must be in G because $n - 1$ is relatively prime to n for $n > 2$!). This group $H = \langle n - 1 \rangle = \{1, n - 1\}$, as $(n-1)^2 = n^2 - 2n + 1 \equiv 1 \mod n$. As this group has order 2, by Lagrange's Theorem, $|G|$ must be even as the order of a subgroup of G must divide the order of G.

(3) Suppose G is a finite abelian group and $|G|$ is odd. Show that the product of all the elements in G is the identity. Is the same true if $|G|$ is even?

Let $a \in G$ such that $|a| = k + 1$, and let $|G| = n$. As $|G|$ is odd, there are no elements with even orders. Consider the cyclic group generated by a, which is $\{e, a, a^2, ..., a^k\}$. Now, consider its $n/(k + 1)$ cosets, which partitions the entirety of G, so the product of all of their elements is the product of all elements of G.

First, consider the powers of a in this product. The product over $\langle a \rangle$ is equal to $a^{k(k+1)/2}$, and there are $n/(k + 1)$ cosets, so the power of a after multiplying these cosets is $a^{nk/2} = e$, as an element of G to the power of a multiple of $|G|$ is equal to the identity, by the Theorem of Lagrange.

For any element $b \in G$, if $b \langle a \rangle$ is a coset of $\langle a \rangle$, then $b^{-1} \langle a \rangle$ must also be a coset of $\langle a \rangle$, as $b \notin \langle a \rangle \implies b^{-1} \notin \langle a \rangle$. The coset $b \langle a \rangle = \{b, ba, ba^2, ..., ba^k\}$, and the power of b in the product over this coset is b^{k+1}. As $b^{-1} \langle a \rangle$ is also a coset, $b^{-(k+1)}$ is also present in the product, so the total product over this set must be equal to the identity of G.

This does not hold true if $|G|$ is even. Consider the set $\{1, 2\}$ under multiplication modulo 3, which has the product of $2 \neq 1 = e$.

(4) Let $|G| = 8$. Prove that G has an element of order 2.

Let $g \in G$. If $|g| = 8$, there must be an element of order 2 in G, namely g^4. If $|g| = 4$, there must be an element of order 2 in G, namely g^2. If $|g| = 1$, then $g = e$ and there must be some other element in G of order $2, 4$, or 8, which all imply that there is some element with order 2 in G, so there must be an element of order 2 in G.

(5) Let G be a group and let $H, K \leq G$ be subgroups satisfying $|H| = 20$ and $|K| = 28$. Prove that $H \cap K$ is abelian. (Hint: Start by computing the order of $H \cap K$. We've seen that $H \cap K$ is a subgroup of G, but it can also be viewed as a subgroup of...).

The group $H \cap K$ can be viewed as a subgroup of either H or K, so the order of $H \cap K$ must divide the order of both H and K. The only common factors that the orders of H and K share are 2 and 4. If $|H \cap K| = 2$, then this group can be represented by the set $\{e, a\}$ which is clearly abelian as $ea = ae = a$. If $|H \cap K| = 4$, then all elements of said group must be of order 2 or 4. If they are of order 4, then the group is cyclic and necessarily abelian. If the elements are of order 2, then this group can be described by the set $\{e, a, b, ab\}$, where $a^2 = b^2 = (ab)^2 = e$, and it must be abelian. Therefore, $H \cap K$ must be abelian.