Math 521 Homework 2

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1 1.3.1.

1.1 (a) Write a formal definition in the Style of Definition 1.3.2 for the infimum or greatest lower bound of a set.

A real number s is the *greatest lower bound* for a set $A \subseteq \mathbb{R}$ if it meets the following two criteria:

1. s is a lower bound for A
2. if b is any lower bound for a, then $b \leq s$.

1.2 (b) Now, state and prove a version of Lemma 1.3.8 for greatest lower bounds

Lemma 1.3.8 states that for some $s \in \mathbb{R}$, with s as an upper bound for some set $A \subseteq \mathbb{R}$,

$$s = sup\ A \iff \forall \epsilon > 0, \exists a \in A,\ s - \epsilon < a.$$

A version of 1.3.8 for greatest lower bounds would state that for some lower bound for $A \subseteq \mathbb{R}$, $c \in \mathbb{R}$,

$$c = inf\ A \iff \forall \epsilon > 0, \exists a \in A,\ c + \epsilon > a.$$

Direct proof of ($\implies$):

Assume that $c = inf\ A$, for some $A \subseteq \mathbb{R}$. Therefore, c is the greatest lower bound of A, so any other lower bound of A is either lesser or equal to c. Now, for any ϵ, $c + \epsilon$ must not be a lower bound of A, as $c + \epsilon$ is strictly greater than c. From this, there must be some $a \in A,\ c + \epsilon > a$, from the definition of a lower bound.

Proof by contrapositive and contradiction of ($\impliedby$):

To begin this proof, we assume that for some $c \in \mathbb{R}$ and for some fixed $A \subseteq \mathbb{R}$,

$$\forall \epsilon > 0, \exists a \in A,\ c + \epsilon > a.$$

We are given that c is a lower bound of A by hypothesis, so we must show that any other lower bounds of A are less than or equal to c. Any lower bound greater than c, s, can be expressed as $s = c + \epsilon$, for some $\epsilon > 0$, and by assumption, there must be some $a \in A$ that is less than this other, greater, lower bound. Therefore, c must be the greatest lowest bound of A.

2 1.3.2. Give an example of each of the following, or state that the request is impossible.

2.1 (a) A set B with inf B $\geq$ sup B.

B = {0} has sup B = inf B = 0.

2.2 (b) A finite set that contains its infimum but not its supremum.

This request is impossible, as a finite set must contain its supremum.

2.3 (c) A bounded subset of $\mathbb{Q}$ that contains its supremum, but not its infimum.

Let A = $\{x \in \mathbb{Q} : -\sqrt{2} > x \geq 0\}$. This set is bounded above and below, and sup A = 0 $\in$ A, while inf A = $\sqrt{2} \notin$ A.

3 1.3.5. As in Example 1.3.7, let $A \subseteq \mathbb{R}$ be nonempty and bounded above, and let $c \in \mathbb{R}$. This time define the set $cA = \{ca : a \in A\}$.

3.1 (a) If $c \geq 0$, show that sup(cA) = c sup(A).

First, let c = 0. Therefore, cA = 0, and sup(cA) = 0 = 0 sup(A), for all A (bounded and nonempty). Now, we look at the case when c > 0. Let sup(A) = a, which is the lowest number which is greater than or equal to all elements of A. Formally, $\forall b \in A,\ b \leq a$. When this inequality is multiplied by c (as is allowed for positive c), we get $\forall b \in A,\ cb \leq ca$, which implies $\forall d \in cA,\ d \leq ca$, by definition of cA, implicating that sup(cA) = c sup(A).

3.2 (b) Postulate a similar type of statement for sup(cA) for the case c < 0.

Postulate: for c < 0, sup(cA) = c inf(-A). Observe that sup(-A) = -inf(A), so by the proof in (a), sup(cA) = c inf(-A) from some working and substitutions.

4 1.3.8. Compute, without proofs, the suprema and infima (if they exist) of the following sets:

4.1 (a) $\{m/n : m, n \in \mathbb{N}, m < n\}$.

The supremum of this set is 1, and the infimum of this set is 0.

4.2 (b) $\{(-1)^m/n : m, n \in \mathbb{N}\}$.

The supremum of this set is 1, and the infimum of this set is -1.

4.3 $\{n/(3n+1) : n \in \mathbb{N}\}$.

The supremum of this set is 1/3, and the infimum of this set is 1/4.

4.4 $\{m/(m+n) : m, n \in \mathbb{N}\}$.

The supremum of this set is 1, and the infimum of the set is 0.

5 1.3.9.

5.1 (a) If sup A < sup B, show that there exists an element $b \in B$ that is an upper bound for A.

Let x = sup A, and let y = sup B. By definition of supremum, we have x is greater or equal to all elements of A, and y is greater of equal to all elements of B. The assertion that there is an upper bound for A in B (called b) states that said element is greater than or equal to all elements in A. Let us assume that there is no element $b \in B$, which is an upper bound for A. This means that $\forall b \in B, \exists a \in A,\ b < a$. By Lemma 1.3.8, we have $\forall \epsilon > 0, \exists c \in A,\ x - \epsilon < c$ and $\forall \epsilon > 0, \exists d \in B,\ y - \epsilon < d$. Choosing the same ϵ for both of these we have existing elements c in A and d in B that satisfy $x - \epsilon > c$ and $y - \epsilon > d$, which implies $x > c + \epsilon$ and $y > d + \epsilon$, and because $x > y$, we have $c > d$, which shows that the prior assumptions lead to the existence of an upper bound b $\in$ A.

5.2 (b) Give an example to show that this is not almays the case if we only assume sup A $\leq$ sub B.

Let A = B = $\{x \in \mathbb{R} : 0 \leq x < 1\}$ have sup A $\leq$ sup B, but there is no upper bound for A contained in B.