# Complex Functions

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# 1 The Complex Plane

## 1.1 The Complex Numbers and the Complex Plane

The complex plane is how we visualize the separate real and imaginary components of the complex variable $z$. A **complex number** is an expression of the form

$$z = x + iy,$$

where $x$ and $y$ are real numbers and $i$ satisfies the rule

$$(i)^2 = (i)(i) = -1.$$

The number $x$ is called the **real part** of $z$ and is written

$$x = \text{Re } z.$$

The number $y$, despite the fact that it is also a real number, is called the **imaginary part** of $z$ and is written

$$z = \text{Im } z.$$

The **modulus**, or **absolute value** of $z$ is defined by

$$|z| = \sqrt{x^2 + y^2},\ z = x + iy.$$

A complex number $z = x+iy$ corresponds to the point $P(x, y)$ in the $xy$-plane. The modulus of $z$, then, is just the distance from the point $P(x, y)$ to the origin, which is 0. These three inequalities hold true:

$$|x| \leq |z|,\ |y| \leq |z|,\ |z| \leq |x| + |y|.$$

The **complex conjugate** of $z = x + iy$ is given by

$$\bar{z} = x - iy.$$

Addition, subtraction, multiplication, and division of complex numbers follow the ordinary rules of arithmetic. For

$$z = x + iy \text{ and } w = s + it$$

we have

$$z + w = (x + s) + i(y + t),$$

$$z - w = (x - s) + i(y - t),$$

$$zw = (xs - yt) + i(xt + ys),\ \text{and}$$

$$\frac{z}{w} = \frac{\bar{w}z}{\bar{w}w} = \frac{(xs + yt) + i(ys - xt)}{s^2 + t^2}.$$

We can also represent complex numbers in the complex plane via polar coordinates—

$$x = r\cos\theta \text{ and } y = r\ \sin\theta.$$

Since $r = |z|$ by definition of the modulus of a complex number, so

$$z = |z|(\cos\,\theta + i\,\sin\,\theta).$$

This is called the **polar representation** of $z$. For two complex numbers,

$$z = |z|(\cos\,\theta + i\,\sin\,\theta) \text{ and } w = |w|(\cos\,\phi + i\,\sin\,\phi),$$

we have

$$zw = |z||w|(\cos\,(\theta+\phi) + i\,\sin\,(\theta+\phi)) \text{ and}$$
$$\frac{z}{w} = \left(\frac{|z|}{|w|}\right)(\cos\,(\theta-\phi) + i\,\sin\,(\theta-\phi)).$$

From this, we have **De Moivre's Theorem:**

$$(\cos\,\theta + i\,\sin\,\theta)^n = \cos\,n\theta + i\,\sin\,n\theta.$$

We define an **argument** of the nonzero complex number $z$ to be any angle $\theta$ for which

$$z = |z|(\cos\,\theta + i\,\sin\,\theta),$$

whether or not it lies in the range $[0, 2\pi)$; we write $\theta = \arg\,z$. We define **Arg** $z$ to be the number $\theta_0 \in [-\pi, \pi)$ such that

$$z = |z|(\cos\,\theta_0 + i\,\sin\,\theta_0).$$

We then have

$$\mathbf{Arg}\ (zw) = \mathbf{Arg}\ z + \mathbf{Arg}\ w\ (\mathrm{mod}\ 2\pi).$$

In summary, the $xy$-plane is a natural interpretation of a complex variable, and the rules of the $xy$-plane can be made to fit this depiction. We then call this plane the **complex plane**, the $x$-axis the **real axis**, and the $y$-axis the **imaginary axis**.

## 1.2 Complex Roots and Circles in the Complex Plane

Complex roots follow directly from rules established in the previous section. A complex number $z$ that satisfies the equation $z^n = w$ (with $z = (|z|, \theta)$ and $w = (|w|, \phi)$) is called the ***n*th root of *w***. This $n$th root of $w$ follows these equations:

$$|z|^n = |w|,\ \cos\,n\theta = \cos\,\phi,\ \sin\,n\theta = \sin\,\phi.$$

A circle around a point $p$ of radius $r$ is given by the equation

$$|z - p| = r,$$

as this is the set of points $r$ distance from $p$.

## 1.3 Subsets of the Plane

The set consisting of all points $z$ satisfying $|z - z_0| < R$ is called the **open disc** of radius $R$ centered at $z_0$. A point $w_0$ in a set $D$ in the complex plane is called an **interior point** of $D$ if there is some open disc centered at $w_0$ that lies entirely within $D$. A set $D$ is called **open** if all of its points are interior points. A point $p$ is a **boundary point** of a set $S$ if every open disc centered at $p$ containing both points of $S$ and not of (S). The set of all boundary points of a set $S$ is called the **boundary** of $S$. A set $C$ is **closed** if it contains its boundary.

**Theorem:** A set $D$ is open if and only if it contains no point of its boundary. A set $C$ is closed if and only if its complement $D = \{z : z \notin C\}$ is open.

A **polygonal curve** is the union of a finite number of directed line segments $\mathbf{P}_1, \mathbf{P}_2, ..., \mathbf{P}_n$, where the terminal point of one is the initial point of the next (excluding the last). An open set $D$ is **connected** if each pair $p, q$ of points in $D$ may be joined by a polygonal curve lying entirely in $D$. An open connected set is called a **domain**. A set $S$ is **convex** if the line segment **pq** joining each $p, q$ in $S$ also lies in $S$. An **open half-plane** is the set of points strictly to one side of a straight line. A **closed half-plane** is an open half-plane, with the inclusion of the defining straight line.

## 1.4 Functions ad Limits

A **function** of the complex variable $z$ is a rule that assigns a complex number to each $z$ within some specified set $D$; $D$ is called the **domain of definition** of the function. The collection of all possible values of the function is called the **range** of the function. We frequently write $w = f(z)$ to distinguish the dependent and independent variables.

Let $\{z_n\}_{n=1}^{\infty}$ be a sequence of complex numbers. We say that $\{z_n\}$ has the complex number $A$ as a **limit**, or that $\{z_n\}$ **converges** to $A$, and we write

$$\lim_{n\to\infty} z_n = A, \text{ or } z_n \to A$$

if, given any positive number $\epsilon$, there is an integer $N$ such that

$$|z_n - A| < \epsilon, \text{ for all } n \geq N.$$

A sequence that does not converge, for any reason whatsoever, is called **divergent**. Additionally, for $z_n = x_n + iy_n$ and $A = s + it$, then $z_n \to A$ if and only if $x_n \to s$ and $y_n \to t$. Additionally, if $z_n \to A$, then $|z_n| \to |A|$, and for two convergent series $z_n \to A$ and $w_n \to B \neq 0$, $\forall \alpha, \beta \in \mathbb{C},\ \alpha z_n + \beta w_n \to \alpha A + \beta B,\ \alpha z_n w_n \to \alpha AB,\ \alpha z_n/\beta w_n \to \alpha A/\beta B$.

We say that a function $f$ defined on a subset $S \subseteq \mathbb{C}$ has a **limit** $L$ at the point $z_0 \in S$ or in the boundary of $S$, and we write

$$\lim_{z\to z_0} f(z) = L \text{ or } f(z) \to L \text{ as } z \to z_0$$

if, given $\epsilon > 0$, there is a $\delta > 0$ such that

$$|f(z) - L| < \epsilon \text{ whenever } z \in S \text{ and } |z - z_0| < \delta.$$

We say that a function $f$ has a **limit L at** $\infty$, and we write

$$\lim_{z \to \infty} f(z) = L$$

if, given $\epsilon > 0$, there is a large number $M$ such that

$$|f(z) - L| < \epsilon \text{ whenever } z \geq M.$$

Note that this only requires that $|z|$ be large; there is no restriction on arg $z$. The arithmetic rules with constants and limits of sequences hold with limits of functions.

Suppose again that $f$ is a function defined on a subset $S$ on the complex plane. If $z_0 \in S$, then $f$ is **continuous** at $z_0$ if

$$\lim_{z \to z_0} f(z) = f(z_0).$$

That is, $f$ is continuous at $z_0$ if the values of $f(z)$ get arbitrarily close to the value $f(z_0)$, so long as $z$ is in $S$ and $z$ is sufficiently close to $z_0$. If it happens that $f$ is continuous at all points on $S$, we say $f$ is **continuous on $S$**. The function $f$ is continuous at $\infty$ if $f(\infty)$ is defined and $\lim_{z \to \infty} f(z) = f(\infty)$.

The sum of an **infinite series of complex numbers** is practically the same as the sum of an infinite series of real numbers. We define the **$n$th partial sum by**

$$s_n = \sum_{j=1}^{n} = z_1 + ... + z_n, \ n = 1, 2, ....$$

If the sequence $\{s_n\}$ has a limit $s$, then we say that the infinite series $\sum_{j=1}^{\infty} z_j$ **converges** and has sum s; this is written

$$\sum_{j=1}^{\infty} z_j = s.$$

If this does not have a limit, we say that this series **diverges**. As with sequences, the real and imaginary parts of the sum must converge to the real and imaginary parts of the limits of the sum.

## 1.5 The Exponential, Logarithm, and Trigonometric Function

The exponential function is one of the most important functions in complex analysis. Its definition is this:

$$e^{x+iy} = e^x(\cos\, y + i\, \sin\, y).$$

The form $\exp(z)$ is sometimes used, especially if $z$ is particularly complicated. For any two complex numbers, $z$ and $w$, we have

$$e^{z+w} = e^z e^w.$$

Additionally,

$$|e^z| = e^{\text{Re } z}, \text{ and } \mathbf{arg}\ e^z = \text{Im } z.$$

The inverse of the exponential function is the logarithm function. For a nonzero complex number $z$, we define **log** $z$ to be any complex number $w$ with $e^w = z$. This is given by

$$\log z = \ln |z| + i \arg z.$$

This provides a set of answers, due to the argument of a complex number being a set of numbers varying by $2\pi$. To avoid the ambiguity of mod $2\pi$, we define **Log** $z$ as

$$\text{Log } z = \ln |z| + i\ \mathbf{Arg}\ z.$$

The definition of $\log z$ allows us to complete the discussion of roots for complex powers. For a nonzero complex number $a$, we define $a^z$ by the rule

$$a^z = e^{z \log a}.$$

The trigonometric functions of $z$ are defined in terms of the exponential function. We begin with the cosine and sine of $z$:

$$\cos z = \frac{1}{2}(e^{iz} + e^{-iz})$$
$$\sin z = \frac{1}{2i}(e^{iz} - e^{-iz}).$$

The other four trigonometric functions are defined in terms of $\sin z$ and $\cos z$:

$$\tan z = \frac{\sin z}{\cos z},\ \cot z = \frac{\cos z}{\sin z},\ \sec z = \frac{1}{\cos z},\ \csc z = \frac{1}{\sin z}.$$

The rules of these trigonometric functions over the reals hold for the complex definitions.

The inverse trigonometric functions are defined in this way:

Let $w = \sin z = \frac{1}{2i}(e^{iz} - e^{-iz})$, so $e^{2iz} - 2iwe^{iz} - 1 = 0$, which is a quadratic of the variable $e^{iz}$. By the quadratic equation, we have $e^{iz} = iw + \sqrt{1-w^2}$, or $z = -i\log(iw + \sqrt{1-w^2})$,—uniquely, we can have

$$\text{Arcsin } z = -i\text{Log}(iz + \sqrt{1-z^2})$$

and

$$\text{Arccos } z = -i\text{Log}(z + \sqrt{z^2-1})$$

and

$$\text{Arctan } z = \frac{i}{2}\text{Log}(\frac{1-iz}{1+iz}),$$

with proper logarithms and roots.

## 1.6 Line Integrals and Green's Theorem

A **curve** $\gamma$ is a continuous complex-valued function $\gamma(t)$ defined for $t$ is some interval $[a, b]$ in the real axis. The curve $\gamma$ is **simple** if $\gamma(t_1) \neq \gamma(t_2)$ whenever $a \leq t_1 < t_2 < b$, and it is **closed** if $\gamma(a) = \gamma(b)$. The **Jordan Curve Theorem** asserts that the complement of the range of a curve that is both simple and closed consists of two disjoint open connected sets, one bounded and the other unbounded. The bounded piece is the **inside** of the curve and the outside is the **outside**. Suppose $\gamma$ is a curve; separating the complex number $\gamma(t)$ into its real and imaginary parts and write $\gamma(t) = x(t) + iy(t), t \in [a, b]$ These two functions may or may not be differentiable. If they are differentiable at some $t_0$, we say $\gamma(t)$ differentiable at $t_0$, and we can say $\gamma'(t_0) = x'(t_0) + iy'(t_0)$. A curve $\gamma$ is **smooth** if $\gamma'(t)$ exists and is continuous on $[a, b]$, with the derivatives at $a$ and $b$ being taken from the right and left, respectively. A curve is **piecewise smooth** if it is composed of a finite number of smooth curves, the end of one coinciding with the beginning of the next. Each curve $\gamma$ is **oriented** by increasing $t$. The **reverse orientation** of $\gamma$ begins at the end of $\gamma$ and ends at the beginning of $\gamma$—this is denoted by $-\gamma$ and is parametrized by $-\gamma(t) = \gamma(a+b-t),\ a \leq t \leq b$.

Suppose $g(t) = x(t) + iy(t)$ is a continuous complex function on $[a, b]$. We define the **integral of g over [a,b]** by

$$\int_a^b g(t)dt = \int_a^b x(t)dt + i\int_a^b y(t)dt.$$

Suppose that $\gamma$ is a smooth curve and $u$ is a continuous function on the range of $\gamma$. We define the **line integral of $u$ along** $\gamma$ by

$$\int_\gamma u(z)dz = \int_a^b u(\gamma(t))\gamma'(t)dt.$$

Line integrals have the same properties that definite integrals on the real numbers have. We define the line integral on a piecewise smooth curve $\gamma = \gamma_1 + \gamma_2 + ... + \gamma_n$ of the same function to be

$$\int_\gamma u(z)dz = \sum_{j=i}^{n} \int_{\gamma_j} u(z)dz,$$

with the integral inside the sum being computed in the same way as previously demonstrated. The length of a curve $\gamma$ is given by

$$\text{length } (\gamma) = \int_a^b |\gamma'(t)|dt.$$

From this, we have the inequality

$$\left|\int_\gamma u(z)dz\right| \leq \left(\max_{z\in\gamma} |u(z)|\right) \text{length } (\gamma).$$

Let $\Omega$ be a domain bounded by a boundary $\Gamma = \gamma_1 + ... + \gamma_n$. The line integral over the boundary is given by

$$\int_\Gamma f(z)dz = \sum_{j=1}^{n} \int_{\gamma_j} f(z)dz.$$

Green's Theorem relates this line integral to the integral of a related function over $\Omega$. Ve assume that there is an open set containing both $\Omega$ and $\Gamma$ where $f$ has continuous partial derivatives with respect to $x$ and $y$. That is, if $f = u(x,y) + iv(x,y)$ then

$$\frac{\partial f}{\partial x} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x},\ \frac{\partial f}{\partial y} = \frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y},$$

where all of these are continuous on the open set. Green's Theorem states

$$\int_\Gamma f(z)dz = i \int \int_\Omega \left( \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \right) dxdy.$$

# 2 Basic Properties of Analytic Functions

## 2.1 Analytic and Harmonic Functions; the Cauchy-Riemann Equations

A function $f$ defined for $z$ in a domain $D$ is **differentiable** at a point $z_0$ in $D$ if

$$\lim_{z \to z_0} \frac{f(z) - f(z_0}{z - z_0} = \lim_{h \to 0} \frac{f(z_0 + h) - f(z_0}{h}$$

exists; the limit, if it exists, is denoted by $f'(z_0)$. If $f$ is differentiable at each point of the domain $D$, then $f$ is called **analytic** in $D$. A function analytic on the whole complex plane is called **entire**. The properties of differentiation established in real calculus hold here.

The **Cauchy-Riemann equations** state that if $f = u + iv$ is analytic on a domain $D$, then throughout $D$,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

**Laplace's equation** is

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

If a function satisfies this equation on $D$, then it is called **harmonic** on $D$—the real and imaginary parts of an analytic function are always harmonic. A function $x$, which for a harmonic function $y$ satisfies $x - y = C$ is called a **harmonic conjugate** of $y$. Additionally, if a function is analytic on a disc centered around some point, then said function is differentiable at that point.

## 2.2 Power Series

A **power series** in $z$ is an infinite series of the form

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n,$$

where $a_0, a_1, ...$ are complex numbers, called the **coefficients** of the series; $z_0$ is fixed and is called the **center** of the series. If for some $z_1 \neq z_1$ has

$$\sum_{n=0}^{\infty} a_n (z_1 - z_0)^n$$

converges, then all $z$ that satisfies $|z - z_0| < |z_1 - z_0|$ have

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n$$

absolutely convergent. For some power series $\sum a_n(z-z_0)^n$, there is a unique number $R$ such that

$$|z-z_0| < R \implies \sum_{n=0}^{\infty} a_n(z-z_0)^n \text{ converges}$$

and

$$|z-z_0| > R \implies \sum_{n=0}^{\infty} a_n(z-z_0)^n \text{ diverges.}$$

The number $R$ is called the **radius of convergence** for said power series. The derivative of a power series $f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n$ is given by

$$f'(z) = \sum_{n=1}^{\infty} n a_n (z-z_0)^{n-1},$$

and this is defined for (and analytic) inside of the disc of convergence.