Math 523 HW8

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Use the method of Examples 1 and 2 to compute these integrals.

1. $\int_{-\infty}^{\infty} \frac{x^4}{1+x^8} dx$

Let $f = z^4/(1+z^8)$. This function has isolated singularities where $z^8 = -1$, i.e. at $z = e^{i(\pi/8+k\pi/4)}$, with integer k. The singularities that lie in the domain U are the points $e^{i\pi/8}$, $e^{3i\pi/8}$, $e^{5i\pi/8}$, $e^{7i\pi/8}$. By the Residue Theorem, this integral is equal to the sum of these residues multiplied by $2\pi i$.

By theorem, the residue at z_0 of a rational function F/G is equal to $F(z_0)/G'(z_0)$, so by letting $F = z^4$ and $G = 1 + z^8$, we may calculate our residues.

- $\text{Res}(f; e^{i\pi/8}) = 1/8e^{3i\pi/8}$
- $\text{Res}(f; e^{3i\pi/8}) = 1/8e^{9i\pi/8}$
- $\text{Res}(f; e^{5i\pi/8}) = 1/8e^{15i\pi/8}$
- $\text{Res}(f; e^{7i\pi/8}) = 1/8e^{21i\pi/8}$

Summing and multiplying these gives us

$$\int_{-\infty}^{\infty} \frac{x^4}{1+x^8} dx = \frac{i\pi}{4}(e^{-3i\pi/8} + e^{-9i\pi/8} + e^{-15i\pi/8} + e^{-21i\pi/8}).$$

2. $\int_{-\infty}^{\infty} \frac{x^2}{x^4-4x^2+5} dx$

Let $f(z) = \frac{z^2}{z^4-4z^2+5}$. This function has isolated singularities at $z = \pm\sqrt{2+i}$ and $z = \pm\sqrt{2-i}$. The singularities that lie in U are $-\sqrt{2-i}$ and $\sqrt{2+i}$.

By theorem, the residue at z_0 of a rational function F/G is equal to $F(z_0)/G'(z_0)$, so by letting $F = z^2$ and $G = z^4 - 4z^2 + 5$, we may calculate our residues. This formula shows that $\text{Res}(f, \sqrt{2+i}) = \frac{\sqrt{2+i}}{4i}$ and $\text{Res}(f, -\sqrt{2-i}) = \frac{\sqrt{2-i}}{4i}$. Summing and multiplying (by 2 πi) these values gives us

$$\int_{-\infty}^{\infty} \frac{x^2}{x^4 - 4x^2 + 5} = 2i\pi \left(\frac{\sqrt{2+i} + \sqrt{2-i}}{4i} \right).$$

3. $\int_{-\infty}^{\infty} \frac{dx}{(x^2+a^2)(x^2+b^2)}, \ a, b > 0$

Let $f(z) = 1/(z^2 + a^2)(z^2 + b^2)$. This function has isolated singularities at $z = \pm bi$ and $z = \pm ai$. As a, b are positive, ai and bi lie in U.

By theorem, the residue at z_0 of a rational function F/G is equal to $F(z_0)/G'(z_0)$, so by letting $F = 1$ and $G = (z^2 + a^2)(z^2 + b^2)$, we may calculate our residues. This formula gives us $\text{Res}(f, ai) = \frac{1}{2ai}\frac{1}{b^2-a^2}$ and $\text{Res}(f, bi) = \frac{1}{2bi}\frac{1}{a^2-b^2}$. Summing and multiplying (by $2\pi i$) these values gives us

$$\int_{-\infty}^{\infty} \frac{dx}{(x^2 + a^2)(x^2 + b^2)} = \frac{\pi}{ab(a+b)}.$$

Use the method of Example 7 to compute these integrals.

9. $\int_0^{2\pi} \frac{d\theta}{(2-\sin\theta)^2}$

10. $\int_0^{2\pi} \frac{d\theta}{(1+\beta\cos\theta)^2}, \ \ -1 < \beta < 1$

12. $\int_0^{2\pi} \sin^{2k}\theta d\theta$