Discrete Mathematics

Jonathan Gribbins

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1 Logic & Proofs & Circuits

1.1 Propositional Logic

A **proposition** is a declarative sentence that is **either** true or false, but not both. To represent **propositional variables** to denote propositions—conventionally these are p, q, r, s,... . A proposition has a **truth value** of either T or F, based on whether it is *true* or *false*.

Propositional logic or **propositional calculus** is the area of logic that deals with these propositions. New propositions, called **compound propositions**, are statements built out of already established statements through the use of *logical operators*.

Theorem 1. *Let p be a proposition. The negation of p is $\neg p$, and this is the statement "it is not the case that p." The statement $\neg p$ is read "not p", and the truth value of $\neg p$ is the opposite of p.*

The logical operators that are used to connect two or more propositions and create a new proposition are called **connectives**.

Theorem 2. *Let p and q be propositions. The **conjunction** of p and q is denoted $p \wedge q$, and is said "p and q." This proposition is true if and only if both p and q are true, and is false otherwise..*

Theorem 3. *Let p and q be propositions. The **disjunction** of p and q is denoted $p \vee q$, and is said "p or q." This proposition is false if both p and q are false, and is true otherwise.*

The *or* in the disjunction connective is the English *inclusive or*, which means "either or both." On the contrary, the *exclusive or* is the or that mean "either but not both." The exclusive or is given by an altogether different connective.

Theorem 4. *Let p and q be propositions. The **exclusive** of p and q is denoted $p \oplus q$, and is said "p or q, but not both." This proposition is true only if one of p or q is true. It is false if both are true or if both are false.*

You can also combine propositions with **conditional statements.** These are statements that connect two different propositions and are true based on the truth of the statements.

Theorem 5. *Let p and q be propositions. The **conditional statement** $p \to q$ is the propositions "if p then q," and is only false when p is true and q is false. In this statement, p is the **hypothesis** and q is the **conclusion.** This connective is also called an **implication.***

There are various ways to state this implication. Most of them are simple and are easily recognized as $p \to q$, but there are a couple notably confusing ones. For instance, "q only if p" and "q unless ¬p" both mean $p \to q$.

In addition to the conditional statement that $p \to q$, there are three other commonplace conditional statements. These are:

Contrapositive: The contrapositive of $p \to q$ is the proposition $\neg q \to \neg p$. The truth value of the contrapositive is always the same as the truth value of the original implication.

Converse: The converse of $p \to q$ is the proposition $q \to p$. The converse has the same truth values of the inverse.

Inverse: The inverse of $p \to q$ is the proposition $\neg p \to \neg q$. The inverse the same truth values of the converse.

When two different statements have the same truth values, they are called **equivalent**. So the contrapositive and the implication are equivalent, and the inverse and converse are equivalent.

Biconditional statements are statements whose truth is based upon the conditions of two propositions.

Theorem 6. *The **biconditional statement** $p \iff q$ is the proposition "p if and only if q." This statement is only true if $p \to q$ and $q \to p$ have the same truth value, otherwise it is false. Biconditional statements are also called **bi-implications**, and $p \iff q$ is sometimes also written as "p iff q."*

Truth tables are tables that display different values of propositions and how these values effect a certain propositional statement. For instance, a statement like $p \to q$ would have a truth table like

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

The resultant statement of these truth tables are generally much longer than $p \to q$, and in this case, the statement may be broken down into component parts and "built up" from smaller statements.

These different connectives can be combined to create longer logical expressions with different truth values.

Truth values can also be represented with **bits**. Bits are values of 1 or 0, where 1 represents true and 0 represents false. When using bits, connectives are generally represented as AND for $\wedge$, OR for $\vee$, and XOR for $\oplus$. A **bitstring** is a string of bits (like 000101110), and these connectives can be used on bitstrings as **bitwise connectives**. These bitwise connectives take in two strings of equal length and use the operation on corresponding bits. For instance,

$$101110 \oplus 111000 \to 010110.$$

1.2 Proofs