Math 534 HW 5

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(1) Let $\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 1 & 3 & 5 & 4 & 6 \end{pmatrix}$ and $\beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 1 & 2 & 4 & 3 & 5 \end{pmatrix}$.

(1a) Compute the following: $\alpha^{-1}, \beta^{-1}, \alpha\beta$, and $\beta\alpha$.

$$\alpha^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 1 & 3 & 5 & 4 & 6 \end{pmatrix}.$$

$$\beta^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 5 & 4 & 6 & 1 \end{pmatrix}.$$

$$\alpha\beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 2 & 1 & 5 & 3 & 4 \end{pmatrix}.$$

$$\beta\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 6 & 2 & 3 & 4 & 5 \end{pmatrix}.$$

(1b) Write α and β as products of disjoint cycles.

$$\alpha = (12)(45) \text{ and } \beta = (16532).$$

(1c) Compute $\beta^{-1}\alpha\beta$ and $\alpha^{-1}\beta\alpha$ and write them of products of disjoint cycles. How do their "cycle structures" compare to those of α and β (respectively)?

$$\beta^{-1}\alpha\beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 3 & 2 & 6 & 5 & 4 \end{pmatrix} = (23)(46).$$

$$\alpha^{-1}\beta\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 6 & 1 & 3 & 5 & 4 \end{pmatrix} = (12643).$$

Both $\beta^{-1}\alpha\beta$ and α are comprised of two disjoint 2-cycles and both $\alpha^{-1}\beta\alpha$ and β are comprised of one 5-cycle.

(2) How many elements of order 4 are there in $\mathfrak{S}_6$? How many of order 2? Justify your answers.

All elements of order 4 in $\mathfrak{S}_6$ have the disjoint cyclic forms $(abcd)$ or $(abcd)(ef)$.

The first case, $(abcd)$ has $6 \times 5 \times 4 \times 3 = 360$ possible elements, without disregarding identical cycles with different letter placement. For each actual unique cycle $(abcd)$, can be written as $(abcd)$, $(bcda)$, $(cdab)$, or $(dabc)$, so the amount of one 4-cycle elements of $\mathfrak{S}_6$ are $360/4 = 90$.

The second case, $(abcd)(ef)$ also has 90 cases, as there is no "choice" for the final 2-cycle, as there are only two elements left to choose from.

This means that there are a total of 180 order 4 elements in $\mathfrak{S}_6$.

Elements of order 2 in $\mathfrak{S}_6$ are those able to be expressed in the forms (ab), $(ab)(cd)$, or $(ab)(cd)(ef)$. For the first case, there are $6 \times 5 = 30$ combinations, with half of them being duplicates, so 15 total of the form (ab). For the second case, there are $6 \times 5 \times 4 \times 3 = 360$ combinations, with 3/4 of them being duplicates (and half of them being in different cycle order), so 45 of the form $(ab)(cd)$. For the the third case, there are $6! = 720$ combinations before removing duplicates. 7/8) are identical up to rearrangement, and 5/6 are identical up to cycle placement, so we have 15 total of this form.

This means that there are a total of 75 elements of order 2 in $\mathfrak{S}_6$.

(3) A *perfect shuffle* is performed on a deck by splitting the deck into two halves (the top and bottom half, assuming an even number of cards), and then interweaving the decks so that every other card comes from the same half. There are two ways to do this: an *out shuffle*, which preserves the first and last cards of the deck, and an **in shuffle**, which does not.

(3a) Show that after 8 perfect **out shuffles**, a deck of 52 cards is returned to its original position, but no fewer number of perfect out shuffles will do this.

Let the out shuffle permutation be defined by

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & ... & 49 & 50 & 51 & 52 \\ 1 & 3 & 5 & 7 & ... & 46 & 48 & 50 & 52 \end{pmatrix}$$

, which maps $n \in 1, 2, 3, ..., 50, 51, 52$ to $2n-1$ if $n \leq 26$, and to $2n \mod 52$ for $n > 26$. The disjoint cyclic representation of σ is then

$$\sigma = (1)(2\ 3\ 5\ 9\ 17\ 33\ 14\ 27)(4\ 7\ 13\ 25\ 49\ 46\ 40\ 28)(6\ 11\ 21\ 41\ 30\ 8\ 15\ 29)$$

$$(10\ 19\ 37\ 22\ 43\ 34\ 16\ 31)(12\ 23\ 45\ 38\ 24\ 47\ 42\ 32)(18\ 35)(20\ 39\ 26\ 51\ 50\ 48\ 44\ 36)(52),$$

which has cycles of length 1, 8, 8, 8, 8, 8, 2, 8, and 1. The order of this element is then equal to the least common multiple of all of these, which is 8, which shows that 8 perfect out shuffles returns a deck of cards to its original position.

(3b) How many perfect **in shuffles** does it take to return a deck of 10 cards back to its original position?

The permutation done by an in shuffle of a deck of ten cards can either be defined by

$$\sigma_1 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 1 & 3 & 5 & 7 & 9 & 10 & 8 & 6 & 4 & 2 \end{pmatrix}$$

(4) Let (ab) and (cd) be distinct 2-cycles in $\mathfrak{S}_n$. Show that these elements commute if and only if they are disjoint. Using this, show that $\mathfrak{S}_n$ is not abelian if $n \geq 3$.

(5) Let $\alpha = (123)(145) \in \mathfrak{S}_5$. Compute α^{99}.

(6) Show that we cannot find an element $\sigma \in \mathfrak{S}_7$ so that $\sigma^2 = (1234)$. *Hint: what would be the possible orders of such an element?*