Math 523 HW 9

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Use the technique of Example 2 to determine the number of zeroes of f in the first quadrant.

1. $f(z) = z^2 - z + 1$

We examine $f(z)$ on the quarter circle of radius $R >> 0$ in the first quadrant bounded by the real and imaginary axes. On the segment $0 \le x \le R$, $f(x) = x^2 - x + 1$, which is real and positive. On the quarter circle $z = Re^{it}, 0 \le t \le \pi/2$,

$$f(Re^{it}) = R^2 e^{2it}\left(1 - \frac{1}{Re^{it}} + \frac{1}{R^2 e^{2it}}\right) = R^2 e^{2it}(1 + \gamma),$$

where $|\gamma| \le 2/R < \epsilon$ for R large. Thus, $\arg f(Re^{it})$ is approximately $\arg(e^{2it}) = 2t$ for large R, so $\arg f(Re^{it})$ increases from 0 to about π as t increases from 0 to $\pi/2$. On the segment $z = iy, R \ge y \ge 0$,

$$f(iy) = -y^2 - iy + 1.$$

Thus, as y decreases from R to 0, $f(iy)$ moves from the third quadrant to $z = 1$, and $\arg f(z)$ increases by π, so as z traverses the contour, $\arg f(z)$ increases by exactly 2π, and so $f(z)$ has exactly one zero in the first quadrant.

2. $f(z) = z^4 - 3z^2 + 3$

We examine $f(z)$ on the quarter circle of radius $R >> 0$ in the first quadrant bounded by the real and imaginary axes. On the segment $0 \le x \le R$, $f(x) = x^4 - 3x^2 + 3$, which is real and positive. On the quarter circle $z = Re^{it}, 0 \le t \le \pi/2$,

$$f(Re^{it}) = R^4 e^{4it}\left(1 - \frac{3}{R^2 e^{2it}} + \frac{3}{R^4 e^{4it}}\right) = R^4 e^{4it}(1 + \gamma),$$

where $|\gamma| \le 6/R < \epsilon$ for R large. Thus, $\arg f(Re^{it})$ is approximately $\arg(e^{4it}) = 4t$ for large R, so $\arg f(Re^{it})$ increases from 0 to about 2π as t increases from 0 to $\pi/2$. On the segment $z = iy, R \ge y \ge 0$,

$$f(iy) = y^4 + 3iy^2 + 3.$$

Thus, as y decreases from R to 0, $f(iy)$ moves from the first quadrant to $z = 3$, and $\arg f(z)$ increases by 0, so as z traverses the contour, $\arg f(z)$ increases by exactly 2π, and so $f(z)$

has exactly one zero in the first quadrant.

3. $f(z) = z^3 - 3z + 6$

We examine $f(z)$ on the quarter circle of radius $R >> 0$ in the first quadrant bounded by the real and imaginary axes. On the segment $0 \le x \le R$, $f(x) = x^3 - 3x + 6$, which is real and positive. On the quarter circle $z = Re^{it}, 0 \le t \le \pi/2$,

$$f(Re^{it}) = R^3 e^{3it}\left(1 - \frac{3}{Re^{it}} + \frac{6}{R^3 e^{3it}}\right) = R^3 e^{3it}(1+\gamma),$$

where $|\gamma| \le 7/R < \epsilon$ for R large. Thus, $\arg f(Re^{it})$ is approximately $\arg(e^{3it}) = 3t$ for large R, so $\arg f(Re^{it})$ increases from 0 to about $3\pi/2$ as t increases from 0 to $\pi/2$. On the segment $z = iy, R \ge y \ge 0$,

$$f(iy) = -iy^3 - 3iy + 6.$$

Thus, as y decreases from R to 0, $f(iy)$ moves from the fourth quadrant to $z = 6$, and $\arg f(z)$ increases by $\pi/2$, so as z traverses the contour, $\arg f(z)$ increases by exactly 2π, and so $f(z)$ has exactly one zero in the first quadrant.

4. $f(z) = z^2 + iz + 2 + i$

We examine $f(z)$ on the quarter circle of radius $R >> 0$ in the first quadrant bounded by the real and imaginary axes. On the segment $0 \le x \le R$, $f(x) = x^2 + ix + 2 + i$, which traverses from $2+i$ to $R^2+2+i(R+1)$, so $\arg f(x)$ changes by $-\arg(2+i)$. On the quarter circle $z = Re^{it}, 0 \le t \le \pi/2$,

$$f(Re^{it}) = R^2 e^{2it}\left(1 + \frac{i}{Re^{it}} + \frac{2+i}{R^3 e^{3it}}\right) = R^2 e^{2it}(1+\gamma),$$

where $|\gamma| \le 4/R < \epsilon$ for R large. Thus, $\arg f(Re^{it})$ is approximately $\arg(e^{2it}) = 2t$ for large R, so $\arg f(Re^{it})$ increases from 0 to about π as t increases from 0 to $\pi/2$. On the segment $z = iy, R \ge y \ge 0$,

$$f(iy) = -y^2 - y + 2 + i.$$

Thus, as y decreases from R to 0, $f(iy)$ moves from the third quadrant to $z = 2+i$, and $\arg f(z)$ increases by $-\pi + \arg(2+i)$, so as z traverses the contour, $\arg f(z)$ increases by exactly 2π, and so $f(z)$ has no zeros in the first quadrant.

5. $f(z) = z^9 + 5z^2 + 3$

We examine $f(z)$ on the quarter circle of radius $R >> 0$ in the first quadrant bounded by the real and imaginary axes. On the segment $0 \le x \le R$, $f(x) = x^9 + 5x^2 + 3$, which is real and positive. On the quarter circle $z = Re^{it}, 0 \le t \le \pi/2$,

$$f(Re^{it}) = R^9 e^{9it}\left(1 + \frac{5}{Re^{7it}} + \frac{3}{R^9 e^{9it}}\right) = R^9 e^{9it}(1+\gamma),$$

where $|\gamma| \leq 8/R < \epsilon$ for R large. Thus, $\arg f(Re^{it})$ is approximately $\arg(e^{9it}) = 9t$ for large R, so $\arg f(Re^{it})$ increases from 0 to about $9\pi/2$ as t increases from 0 to $\pi/2$. On the segment $z = iy, R \geq y \geq 0$,

$$f(iy) = iy^9 - 5y^2 + 3.$$

Thus, as y decreases from R to 0, $f(iy)$ moves from the first quadrant to $z = 3$, and $\arg f(z)$ increases by $-\pi/2$, so as z traverses the contour, $\arg f(z)$ increases by exactly 4π, and so $f(z)$ has exactly two zero in the first quadrant.

18. Extend Formula 4 to prove the following. Let g be analytic on a domain containing γ and its inside. Then

$$\frac{1}{2\pi i}\int_\gamma \frac{h'(z)}{h(z)} g(z)dz = \sum_{i=1}^{N} g(z_i) - \sum_{j=1}^{M} g(w_j),$$

where $z_1, ..., z_N$ are the zeroes of h and $w_1, ..., w_M$ are the poles of h inside γ, each listed according to its multiplicity.

By the Residue Theorem,

$$\frac{1}{2\pi i}\int_\gamma \frac{h'(z)}{h(z)} g(z)dz = \sum_{i=1}^{N} \operatorname{Res}\left(\frac{h'}{h} g; z_i\right),$$

where z_i are the poles of $\frac{h'}{h} g$. Note that this set of poles is the union of the set of zeros of $h(z)$ and poles of $h(z)$ inside γ. By theorem, a residue of $\frac{h'(z)g(z)}{h(z)}$ at z_i is equal to $\frac{h'(z_i)g(z_i)}{h'(z_i} = g(z_i)$. Therefore,

$$\frac{1}{2\pi i}\int_\gamma \frac{h'(z)}{h(z)} g(z)dz = \sum_{i=1}^{N} g(z_i) - \sum_{j=1}^{M} g(w_j),$$

with the prior definition of z_i and w_j.