Analysis I

Morgan Gribbins

Contents

1 The Real Numbers

1.1 The Axiom of Completeness

The **axiom of completeness** states that *every nonempty set of real numbers that is bounded above has a least upper bound.*

A set $A \subseteq \mathbb{R}$ is **bounded above** if there exists a number $b \in \mathbb{R}$ such that $\forall a \in A,\ a \leq b$. The number b is called an **upper bound** for A. Similarly, the set A is **bounded below** if there exists a **lower bound** $l \in \mathbb{R}$ satisfying $\forall a \in A,\ l \leq a$.

A real number s is the **least upper bound** for a set $A \subseteq \mathbb{R}$ if it meets the following two criteria:

1. s is an upper bound for A;
2. if b is any upper bound for A, then $s \leq b$.

The least upper bound is also called the **supremum** of the set A, and is called $s = \sup A$. The **greatest lower bound** is defined similarly, and is called the infimum, with $l = \inf A$. Both suprema and infima are unique. A real number a_0 is a **maximum** on a set A if a_0 is an element of A and $a_0 = \sup A$. A **minimum** is an element of A that is also the infimum of A.

Assuming $s \in \mathbb{R}$ is an upper bound for a set $A \subseteq \mathbb{R}$, then $s = \sup A \iff \forall \epsilon > 0,\ \exists a \in A,$ such that $s - \epsilon < a$.

1.2 Consequences of Completeness

The **Archimedean property** states that:

1. Given any number $x \in \mathbb{R}$, there exists an $n \in \mathbb{N}$ such that $n > x$.
2. Given any real number $y > 0$, there exists an $n \in \mathbb{N}$ such that $1/n < y$.

Additionally, for every two real number a and b with $a < b$, there exists some $q \in b\mathbb{Q}$ (or in the set of irrational numbers) such that $a < r < b$.

1.3 Cardinality

A function $f : A \to B$ is **one-to-one** (injective) if $a_1 \neq a_2$ in A implies that $f(a_1) \neq f(a_2)$ in B. The function f is **onto** (surjective) if, given any $b \in B$, there exist an $a \in A$ such that $f(a) = b$. If f is both onto and one-to-one, then it is called **bijective**.

The set A **has the same cardinality** as B if there exists a bijection $f : A \to B$. In this case, we write $A \sim B$. A set A is **countable** if $\mathbb{N} \sim A$. An infinite set that is not countable is called an **uncountable** set. $\mathbb{Q}$ is countable and $\mathbb{R}$ is uncountable. Additionally, subsets of countable sets are countable and countable unions of countable sets are countable.

2 Sequences and Series

2.1 The Limit of a Sequence