Math 521 HW 8

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Exercise 2.5.7. Extend the result proved in Example 2.5.3 to the case $|b| < 1$; that is, show $\lim(b^n) = 0$ if and only if $-1 < b < 1$.

Proof of ($\implies$) by contrapositive. Assume that $|b| \geq 1$. This gives us 4 different cases.

Take the case $b = 1$. This means that $b^n = 1$ for all n, so $\lim(b^n) = 1 \neq 0$.

Take the case $b > 1$. This implies that the sequence b^n is unbounded, so this sequence is divergent.

Take the case $b = -1$. This means that $b^n = (-1)^n$, which alternates between -1 and 1 and is divergent.

Take the case $b < -1$. This implies that the sequence b^n is unbounded, so this sequence is divergent.

Therefore, $|b| \geq 1 \implies \lim(b^n) \neq 0$.

Proof of ($\impliedby$). Assume that $-1 < b < 1$. This implies that for all n, $-1 < b^n < 1$, so this sequence is bounded. Additionally, we have $|b^n| \geq |b^{n+1}|$, as $|b| < 1$. Given $\epsilon > 0$, let $N \in \mathbb{N}$ such that $b^N < \epsilon$. We then have, for $n \geq N$,

$$|b^n| < \epsilon,$$

so this sequence converges to 0.

Exercise 2.5.9. Let (a_n) be a bounded sequence, and define the set

$$S = \{x \in \mathbb{R} : x < a_n \text{ for infinitely many terms } a_n\}.$$

Show that there exists as subsequence (a_{n_k}) converging to $s = \sup S$.

As S is nonempty and bounded above (as a_n is bounded), we can say $s = \sup S$. Let $\epsilon_k = 1/k$. By definition of a supremum, there must exists some $b_k \in S$ such that $b_k > s - 1/k$. As there are an infinite amount of a_n greater than b_k and s, there must be some a_{n_k} between $s - 1/k$ and s, so there exists an increasing ($1/k$ decreasing) bounded sequence with all

$a_{n_k} < s$, so there exists a subsequence (a_{n_k}) converging to s.

Exercise 2.6.2. Give an example of each of the following, or argue that such a request is impossible.

(a) A Cauchy sequence that is not monotone.

A Cauchy sequence that is not monotone is $(-1)^n/n$, which varies by $(2n+1)/(n^2+n)$, which goes to zero (so its Cauchy), but it is not monotone.

(b) A Cauchy sequence with an unbounded subsequence.

This is not possible. A Cauchy sequence is necessarily bounded, so all of its subsequences are bounded.

(c) A divergent monotone sequence with a Cauchy subsequence.

This is not possible. A divergent monotone sequence must be unbounded, so a subsequence of this sequence must be unbounded. As this is monotone and unbounded, all subsequences are unbounded and not Cauchy.

(d) An unbounded sequence containing a subsequence that is Cauchy.

The sequence $a_n = (1, 0, -1, 0, 2, 0, -2, ...)$ has Cauchy subsequence uniformly composed of 0s, but is unbounded.

Exercise 2.6.3. If (x_n) and (y_n) are Cauchy sequences, then one easy way to prove that $(x_n + y_n)$ is Cauchy is to use the Cauchy Criterion. By Theorem 2.6.4, (x_n) and (y_n) must be convergent, and the Algebraic Limit Theorem then implies that $(x_n + y_n)$ is convergent and hence Cauchy.

(a) Give a direct argument that $(x_n + y_n)$ is a Cauchy sequence that does not use the Cauchy Criterion or the Algebraic Limit Theorem.

As x_n and y_n Cauchy, given $\epsilon/2 > 0$, there exists $N, M \in \mathbb{N}$ such that $b \geq a \geq N$ and $d \geq c \geq M$ implies

$$|x_a - x_b| < \epsilon/2$$
$$|y_c - y_d| < \epsilon/2.$$

Adding these gives us

$$|x_a - x_b| + |y_c - y_d| < \epsilon/2 + \epsilon/2$$
$$\implies |x_a - x_b + y_c - y_d| = |(x_a + y_c) - (x_b + y_d)| < \epsilon,$$

so for $m \geq n \geq \max(N, M)$, we have

$$|(x_n + y_n) - (x_m - y_m)| < \epsilon,$$

which means that this sequence is Cauchy.

(b) Do the same for the product $(x_n y_n)$.

As x_n and y_n Cauchy, given $\sqrt{\epsilon} > 0$, there exists $N, M \in \mathbb{N}$ such that $b \geq a \geq N$ and $d \geq c \geq M$ implies

$$|x_a - x_b| < \sqrt{\epsilon}$$
$$|y_c - y_d| < \sqrt{\epsilon}.$$

This implies that

$$|x_a - x_b||y_c - y_d| < \epsilon \implies |y_c(x_a - x_b) - y_d(x_a - x_b)| < \epsilon,$$

which implies that for some $m \geq n \geq \max(N, M)$,

$$|(x_n y_n) - (x_m y_m)| < \epsilon,$$

so the sequence is Cauchy.

Exercise 2.6.4. Let (a_n) and (b_n) be Cauchy sequences. Decide whether each of the following sequences is a Cauchy sequence, justifying each conclusion.

(a) $c_n = |a_n - b_n|$

This sequence is Cauchy. The sequence $a_n - b_n$ is Cauchy, which implies $||a_n - b_n| - |a_m - b_m|| \geq |(a_n - b_n) - (a_m - b_m)| < \epsilon$, so c_n is Cauchy.

(b) $c_n = (-1)^n a_n$

This sequence is not necessarily Cauchy. The sequence $a_n = 1$ results in a non-Cauchy c_n, for instance.

(c) $c_n = [[a_n]]$, where $[[x]]$ refers to the greatest integer less than or equal to x.

This sequence is not necessarily Cauchy. The sequence $a_n = 1 + (-1)^n/n$ is Cauchy, but $[[a_n]]$ alternates between 0 and 1, based on the value of n.