Probability and Statistics: Lesson 4
Continuous Random Variables

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1 Lesson 3 Addendum: Discrete Random Variables

1.1 Definitions

- Binomial Distribution: $X \sim Binomial(n, p) = X_1 + X_2 + \cdots + X_n$ where $X_i \sim Bernoulli(p)$.

1.2 Examples

1. Flip a coin n times
   - X is the number of heads
   - $Ran(X) = \{0, 1, 2, \ldots, n\}$
   - pdf:

$$
\begin{aligned}
&p_X(k) = P(X = k) \\
&p_X(0) = (1 - p^n) \\
&p_X(1) = \binom{n}{1} p^1 (1-p)^{n-1} \\
&p_X(2) = \binom{n}{2} p^2 (1-p)^{n-2} \\
&p_X(k) = \binom{n}{k} p^k (1-p)^{n-k}
\end{aligned}
$$

   - cdf:

$$
\begin{aligned}
&F_X(k) = P(X \leq k) \\
&F_X(0) = P(X = 0) = (1-p)^n \\
&F_X(1) = P(X \leq 1) = (1-p)^n + \binom{n}{1} p^1 (1-p)^{n-1} \\
&F_X(2) = P(X \leq 2) = (1-p)^n + \binom{n}{1} p^1 (1-p)^{n-1} + \binom{n}{2} p^2 (1-p)^{n-2} \\
&F_X(k) = \sum_{i=0}^{k} \binom{n}{i} p^i (1-p)^{n-i}
\end{aligned}
$$

- Expected Value:

$$E[X] = \mu = \sum_x x \cdot p_X(x)$$

- Variance:

$$\begin{aligned} Var(X) &= \Sigma_x (x-\mu)^2 \cdot p_X(x) \\ &= E(x^2) - \mu^2 \end{aligned}$$

2 Continuous Random Variables

2.1 Definitions

- Continuous Random Variable: a random variable that can take on any value in an interval of the real numbers.
- Range of a Continuous Random Variable: the set of all possible values of X. $Ran(X) = \{x \in \mathbb{R} | p_X(x) > 0\}$. A union of continuous intervals.
- Probability Density Function: $f_X(x)$ is a function such that $f_X(x) \geq 0$ and $\int_{-\infty}^{\infty} f_X(x)dx = 1$.
- $\int_a^b f_X(x)dx = P(a \leq X \leq b)$: area under the curve between a and b.

2.2 Uniform Distribution

- Uniform Distribution: $X \sim Uniform(a, b)$ where $a < b$.
- $Ran(X) = (a, b)$
- pdf:

$$\begin{aligned} f_X(x) &= c \\ \int_a^b f_X(x)dx &= 1 \\ \int_a^b cdx &= 1 \\ c(b-a) &= 1 \\ c &= \frac{1}{b-a} \\ f_X(x) &= \frac{1}{b-a} \end{aligned}$$

- cdf:

$$
\begin{aligned}
F_X(x) &= P(X \leq x) \\
&= \int_{-\infty}^{x} f_X(t)dt \\
&= \int_{a}^{x} \frac{1}{b-a}dt \\
&= \frac{t}{b-a}\Big|_a^x \\
&= \frac{x-a}{b-a}
\end{aligned}
$$

- Expected Value:

$$
\begin{aligned}
E[X] &= \int_{-\infty}^{\infty} x \cdot f_X(x)dx \\
&= \int_{a}^{b} x \cdot \frac{1}{b-a}dx \\
&= \frac{x^2}{2(b-a)}\Big|_a^b \\
&= \frac{b^2-a^2}{2(b-a)} \\
&= \frac{b+a}{2}
\end{aligned}
$$

- Variance:

$$
\begin{aligned}
Var(X) &= E(X^2) - \mu^2 \\
&= \int_{-\infty}^{\infty} x^2 \cdot f_X(x)dx - \mu^2 \\
&= \int_{a}^{b} x^2 \cdot \frac{1}{b-a}dx - \mu^2 \\
&= \frac{x^3}{3(b-a)}\Big|_a^b - \mu^2 \\
&= \frac{b^3-a^3}{3(b-a)} - \mu^2 \\
&= \frac{b^2+ab+a^2}{3} - \mu^2 \\
&= \frac{b^2+2ab+a^2-3\mu^2}{3} \\
&= \frac{b^2+2ab+a^2-3\frac{(b+a)^2}{4}}{3} \\
&= \frac{4b^2+8ab+4a^2-3b^2-6ab-3a^2}{12} \\
&= \frac{b^2-2ab+a^2}{12} \\
&= \frac{(b-a)^2}{12}
\end{aligned}
$$

2.2.1 Note

$$\frac{b^n - a^n}{b - a} = \sum_{i=0}^{n-1} a^i b^{n-1-i}$$

2.3 Exponential Distributon

- Exponential Distribution: $X \sim Exponential(\lambda)$ where $\lambda > 0$.
- $Ran(X) = (0, \infty)$
- pdf:

$$f_X(x) = \lambda e^{-\lambda x}$$

- cdf:

$$\begin{aligned} F_X(x) &= P(X \leq x) \\ &= \int_{-\infty}^{x} f_X(t)dt \\ &= \int_{0}^{x} \lambda e^{-\lambda t} dt \\ &= -e^{-\lambda t}\Big|_0^x \\ &= -e^{-\lambda x} + 1 \\ &= 1 - e^{-\lambda x} \end{aligned}$$

- Expected Value:

$$\begin{aligned} E(X) &= \int_{-\infty}^{\infty} x \cdot f_X(x)dx \\ &= \frac{1}{\lambda} \end{aligned}$$