Probability and Statistics: Lesson 1

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1 General Overview

1.1 Definitions

1. Experiment: procedure with undetermined outcomes
2. Sample Space: (S.S. or S) set of all possible outcomes
3. Set: a collection of things
4. Countable: can be put in one-to-one correspondence with the natural numbers (integers are countable)
5. Discrete: finite or countable
6. Continuous: uncountable (in opposition to discrete)
7. Universal Set: set of all possible outcomes equivalent to the sample space in a Probability experiment

1.2 Symbols

- $\in$: $x \in S$: x is an element of S
- $\notin$: $x \notin S$ x is not an element of S

1.3 Examples

- Experiment: flip a coin
  - Sample Space: $\{H, T\}$
    Sample Space is **finite**
- Experiment: flip a coin until we get a tails
  - Sample Space: $\{T, HT, HHT, HHHT, \cdots\}$
    Sample Space is **infinite**, but **countable**
- Experiment: pick a number in the interval $[0, 1]$
  - Sample Space: $[0, 1]$ or $\{x \in \mathbb{R} \mid 0 \leq x \leq 1\}$
    Sample Space is **infinite**, and **not countable**

2 Events

2.1 Definitions

1. Subset: a set whose elements are all contained in another (super)set, additionally every set is a subset of itself and the empty set is a subset of every set
2. Event: a subset of the sample space

2.2 Symbols

- $\subseteq$: $A \subseteq B$: A is a subset of B
- $\subset$: $A \subset B$: A is a proper subset of B (at least one element of B is not in A)
- $\emptyset$: the empty set

2.3 Examples

- Rolling a six-sided die
  - $S = \{1, 2, 3, 4, 5, 6\}$
    Sample Space is **finite**
  - Events:
    - Event of rolling even numbers: $A = \{2, 4, 6\}$
    - Event of rolling a "6": $B = \{6\}$
    - Event of rolling a prime number: $C = \{2, 3, 5\}$
    - Event of rolling a number 7 or greater: $D = \emptyset$

3 Set Operations

3.1 Definitions

Given sets A and B:

1. Complement: $A^c = \{x \mid x \in S \text{ and } x \notin A\}$
2. Intersection: $A \cap B = \{x \mid x \in A \text{ and } x \in B\}$
3. Union: $A \cup B = \{x \mid x \in A \text{ or } x \in B\}$
4. Disjoint: if $A \cap B = \emptyset$, then A and B are disjoint
5. DeMorgan's Laws: $(A \cup B)^c = A^c \cap B^c$ and $(A \cap B)^c = A^c \cup B^c$

3.2 Symbols

Assume $A, B \subseteq S$

- A^c: complement of A
- $A \cap B$: intersection of A and B
- $A \cup B$: union of A and B
- $A \sqcup B$: disjoint union of A and B (i.e. $A \cap B = \emptyset$)

4 The Probability Function

4.1 Definitions

1. $\underline{P(A)}$: probability of event A

4.2 Kolmolgorov's Axioms

1. Axiom 1: $P(A) \geq 0$
2. Axiom 2: $P(S) = 1$
3. Axiom 3: If A, B are disjoint, then $P(A \cup B) = P(A \sqcup B) = P(A) + P(B)$

4.3 Derived Properties

1. $P(A^c) = 1 - P(A)$
   Proof:
   $$\begin{aligned} A \sqcup A^c &= S \\ P(S) &= P(A) + P(A^c) && \text{[by Axiom 3]} \\ 1 &= P(A) + P(A^c) && \text{[by Axiom 2]} \\ P(A^c) &= 1 - P(A) && \square \end{aligned}$$
2. $P(\emptyset) = 0$
   Proof:
   $$\begin{aligned} \emptyset \sqcup S &= S \\ P(S) &= P(\emptyset) + P(S) && \text{[by Axiom 3]} \\ 1 &= P(\emptyset) + 1 && \text{[by Axiom 2]} \\ P(\emptyset) &= 0 && \square \end{aligned}$$
3. $P(A) = P(A \cap B) + P(A \cap B^c)$
   Proof:
   $$\begin{aligned} A &= A \cap S \\ &= A \cap (B \sqcup B^c) && \text{[by definition of complement]} \\ &= (A \cap B) \sqcup (A \cap B^c) && \text{[by distribution]} \\ P(A) &= P(A \cap B) + P(A \cap B^c) && \square \text{ [by Axiom 3]} \end{aligned}$$
4. If $A \subseteq B$, then $P(A) \leq P(B)$
   Proof:
   $$\begin{aligned} B &= S \cap B && \text{[by definition]} \\ &= (A \sqcup A^c) \cap B && \text{[by definition]} \\ &= (A \sqcup A^c) \cap (A \cup B) && \text{[by definiton of subset]} \\ &= A \sqcup (B \cap A^c) && \text{[by distribution]} \\ P(B) &= P(A) + P(B \cap A^c) && \text{[by Axiom 3]} \\ &\geq P(A) + 0 && \text{[by Axiom 1]} \\ &\geq P(A) && \square \end{aligned}$$

5. $P(A) \leq 1$
   Proof:

$$\begin{aligned}
A \sqcup A^c &= S && \text{[by definition]} \\
P(S) &= P(A) + P(A^c) && \text{[by Axiom 3]} \\
1 &= P(A) + P(A^c) && \text{[by Axiom 2]} \\
P(A) &\leq 1 && \square
\end{aligned}$$

6. Union Rule: $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
   Proof:

$$\begin{aligned}
A \cup B &= A \sqcup (B \cap A^c) && \text{[by distribution]} \\
P(A \cup B) &= P(A) + P(B \cap A^c) && \text{[by Axiom 3]} \\
P(B) &= P(B \cap A) + P(B \cap A^c) && \text{[by derived property 3]} \\
P(B \cap A^c) &= P(B) - P(B \cap A) && \text{[by algebra]} \\
P(A \cup B) &= P(A) + P(B) - P(B \cap A) && \square
\end{aligned}$$

4.4 Examples

- Example 1
  - $P(A) = 0.4$, $P(B) = 0.5$, $P(A \cap B) = 0.1$
  - Determine probability only A occurs:
    $P(A \cap B^c) = P(A) - P(A \cap B) = 0.4 - 0.1 = 0.3$
  - Determine probability A or B occurs:
    $P(A \cup B) = P(A) + P(B) - P(A \cap B) = 0.4 + 0.5 - 0.1 = 0.8$
  - Determine probability A xor B occurs:
    $P(A \cup B) - P(A \cap B) = 0.8 - 0.1 = 0.7$
    or $P(A \cap B^c) + P(B \cap A^c) = 0.3 + 0.4 = 0.7$
  - Determine probability neither A nor B occurs:
    $P((A \cup B)^c) = 1 - P(A \cup B) = 1 - 0.8 = 0.2$

5 Basic Principle of Probability

5.1 Definitions

1. Cardinality: the number of elements in a set

5.2 Symbols

- $|A|$: cardinality of A

5.3 The Principle

If every outcome of S.S. is equally likely, then:

$$P(A) = \frac{\text{number of outcomes in } A}{\text{number of outcomes in } S}$$

or

$$P(A) = \frac{|A|}{|S|}$$

5.4 Examples

1. Example 1

   - for a fair die roll, what is the probability of rolling an even number?
     $P(A) = \frac{|A|}{|S|} = \frac{3}{6} = \frac{1}{2} = .5$

2. Example 2

   - If you roll two fair dice, what is the probability of rolling a sum greater or equal to 9?
     $A = \{36, 45, 54, 55, 56, 63, 64, 65, 66\}$
     $P(A) = \frac{|A|}{|S|} = \frac{10}{36} = \frac{5}{18} \approx .278$

6 Counting and Probability

6.1 Rules

1. Addition Rule

2. Multiple Rule

6.2 Examples

1. Example 1

   - In a standard deck of cards, what is the probability of drawing a face card or a black ace?
     $A = \{A\spadesuit, A\clubsuit, A\heartsuit, A\diamondsuit\} + \{J\spadesuit, J\clubsuit, J\heartsuit, J\diamondsuit, Q\spadesuit, Q\clubsuit, Q\heartsuit, Q\diamondsuit, K\spadesuit, K\clubsuit, K\heartsuit, K\diamondsuit\}$
     $4 * 1$ (multiple rule) aces + (addition rule) $4 * 3$ face cards = 16 cards
     $P(A) = \frac{|A|}{|S|} = \frac{16}{52} = \frac{4}{13} \approx .308$