

Probability and Statistics: Lesson 5

Joint Densities: Discrete

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0.1 Comparison of Discrete and Continuous Random Variables

0.2 Discrete

- $Ran(X)$: discrete set (finite/countable)
- pdf: $P(X = x), x \in Ran(X)$
- e.g.
 - $P(X = 1), P(X = 2), P(X = 3), \dots, P(X = 6) = \frac{1}{6}$ for a fair die

0.3 Continuous

- $Ran(X)$: continuous interval (or union of such intervals) e.g. $(0, 1)$
- pdf: $f_X(x)$ such that $P(a \leq X \leq b) = \int_a^b f_X(x)dx$, piecewise continuous
- e.g.
 - $X \sim Uniform(0, 1)$
 - $P(\frac{1}{4} < X < \frac{1}{2}) = \int_{\frac{1}{4}}^{\frac{1}{2}} 1 \, dx = \frac{1}{4}$
- or
 - $X \sim Exponential(\lambda)$ (λ is the rate of decay)
 - $f_X(x) = \lambda e^{-\lambda x}$ for $x, \lambda \geq 0$
 - $P(a < X < b) = \int_a^b \lambda e^{-\lambda x} \, dx = e^{-\lambda a} - e^{-\lambda b}$

1 Lesson 4 Continued

1.1 Derivation of $E(X) = \frac{1}{\lambda}$

$$X \sim \text{Exponential}(\lambda)$$

$$E(X) = \int_0^{\infty} x \lambda e^{-\lambda x} dx$$

$$= \lambda \int_0^{\infty} x e^{-\lambda x} dx$$

Use integration by parts

$$\int x^n e^{ax} dx = \frac{n!}{a^{n+1}}$$

$$E(X) = \lambda \frac{1!}{\lambda}$$

$$= \frac{1}{\lambda}$$

- $Var(X)$:

$$\begin{aligned} Var(X) &= E(X^2) - \mu^2 \\ &= \int_0^{\infty} x^2 \lambda e^{-\lambda x} dx - \frac{1}{\lambda^2} \\ &= \frac{2}{\lambda^2} - \frac{1}{\lambda^2} \\ &= \frac{1}{\lambda^2} \end{aligned}$$

2 Lesson 5: Joint Densities: Discrete

- A Joint Density occurs when we are interested in multiple variables at the same time.
- e.g. X and Y .

2.1 Example

- Assume we have a fair 3-way spinner (i.e. a spinner with 3 equal sectors with equal probability of landing on each sector).
- We spin the spinner twice.
- Let X be the sum of the two spins. Let Y be the absolute difference of the two spins.
- $S =$

$$\begin{bmatrix} (2, 1), & (2, 2), & (2, 3), \\ (1, 1), & (1, 2), & (1, 3), \\ (3, 1), & (3, 2), & (3, 3) \end{bmatrix}$$

- $Ran(X) = \{2, 3, 4, 5, 6\}$
- $Ran(Y) = \{0, 1, 2\}$

- Joint pdf $f_{X,Y}(x, y) = P(X = x, Y = y) = p_{X,Y}(x, y)$

| $\downarrow Y \ X \rightarrow$ | 2 | 3 | 4 | 5 | 6 | $p_Y(y)$ |
|--------------------------------|---------------|---------------|---------------|---------------|---------------|---------------|
| 0 | $\frac{1}{9}$ | 0 | $\frac{1}{9}$ | 0 | $\frac{1}{9}$ | $\frac{3}{9}$ |
| 1 | 0 | $\frac{2}{9}$ | 0 | $\frac{2}{9}$ | 0 | $\frac{4}{9}$ |
| 2 | 0 | 0 | $\frac{2}{9}$ | 0 | 0 | $\frac{2}{9}$ |
| $p_X(x)$ | $\frac{1}{9}$ | $\frac{2}{9}$ | $\frac{3}{9}$ | $\frac{2}{9}$ | $\frac{1}{9}$ | |

- Marginal pdfs: $p_X(x) = \sum_y p_{X,Y}(x, y)$ and $p_Y(y) = \sum_x p_{X,Y}(x, y)$

2.2 Independence

X and Y are independent if and only if $p_{X,Y}(x, y) = p_X(x) \cdot p_Y(y)$ for all x, y .

- From the example:

$$- X, Y \text{ are not independent because } p_{X,Y}(2, 1) = \frac{1}{9} \neq \frac{1}{9} \cdot \frac{4}{9} = p_X(2) \cdot p_Y(1)$$

2.3 Expected Value

$$E[g(X, Y)] = \sum_x \sum_y g(x, y) \cdot p_{X,Y}(x, y)$$

2.4 Covariance

$$Cov(X, Y) = E[(X - \mu_X)(Y - \mu_Y)] = E(XY) - \mu_X \mu_Y$$

- $Cov(X, Y) = 0$ if X, Y are independent.
- $Cov(X, Y) > 0$ if X, Y are positively correlated.
- $Cov(X, Y) < 0$ if X, Y are negatively correlated.

$$E(XY) = \sum_x \sum_y x \cdot y \cdot p_{X,Y}(x, y) \tag{1}$$

2.5 Theorems

1. $E(aX + bY + c) = aE(X) + bE(Y) + c$
2. $Var(aX + bY + c) = a^2 \cdot Var(X) + b^2 \cdot Var(Y) + 2ab \cdot Cov(X, Y)$

2.6 IID

X_1, X_2, \dots, X_n are independent and identically distributed (IID) if and only if $pdf(X_1) = pdf(X_2) = \dots = pdf(X_n)$ and X_1, X_2, \dots, X_n are independent.

2.6.1 Consequences

- Sample Sum: $S_n = X_1 + X_2 + \dots + X_n$
- $E(S_n) = n \cdot \mu$
- $Var(S_n) = n \cdot \sigma^2$