Probability and Statistics: Lesson 5
Joint Densities: Discrete

Morgan McCarty

12 July 2023

1 Comparison of Discrete and Continuous Random Variables

1.1 Discrete

- $Ran(X)$: discrete set (finite/countable)
- pdf: $P(X = x), x \in Ran(X)$
- e.g.
  - $P(X = 1), P(X = 2), P(X = 3), \ldots, P(X = 6) = \frac{1}{6}$ for a fair die

1.2 Continuous

- $Ran(X)$: continuous interval (or union of such intervals) e.g. $(0, 1)$
- pdf: $f_X(x)$ such that $P(a \leq X \leq b) = \int_a^b f_X(x)dx$, piecewise continuous
- e.g.
  - $X \sim Uniform(0, 1)$
  - $P(\frac{1}{4} < X < \frac{1}{2}) = \int_{\frac{1}{4}}^{\frac{1}{2}} 1 \; dx = \frac{1}{4}$
- or
  - $X \sim Exponential(\lambda)$ (λ is the rate of decay)
  - $f_X(x) = \lambda e^{-\lambda x}$ for $x, \lambda \geq 0$
  - $P(a < X < b) = \int_a^b \lambda e^{-\lambda x} \; dx = e^{-\lambda a} - e^{-\lambda b}$

2 Lesson 4 Continued

2.1 Derivation of $E(X) = \frac{1}{\lambda}$

$$\begin{aligned}
X &\sim Exponential(\lambda) \\
E(X) &= \int_0^\infty x\lambda e^{-\lambda x}\ dx \\
&= \lambda \int_0^\infty x e^{-\lambda x}\ dx \\
&\text{Use integration by parts} \\
\int x^n e^{ax}\ dx &= \frac{n!}{a^{n+1}} \\
E(X) &= \lambda \frac{1!}{\lambda} \\
&= \frac{1}{\lambda}
\end{aligned}$$

- $Var(X)$:

$$\begin{aligned}
Var(X) &= E(X^2) - \mu^2 \\
&= \int_0^\infty x^2 \lambda e^{-\lambda x}\ dx - \frac{1}{\lambda^2} \\
&= \frac{2}{\lambda^2} - \frac{1}{\lambda^2} \\
&= \frac{1}{\lambda^2}
\end{aligned}$$

3 Lesson 5: Joint Densities: Discrete

- A Joint Density occurs when we are interested in multiple variables at the same time.
- e.g. X and Y.

3.1 Example

- Assume we have a fair 3-way spinner (i.e. a spinner with 3 equal sectors with equal probability of landing on each sector).
- We spin the spinner twice.
- Let X be the sum of the two spins. Let Y be the absolute difference of the two spins.
- $S =$

$$\begin{bmatrix} (2,1), & (2,2), & (2,3), \\ (1,1), & (1,2), & (1,3), \\ (3,1), & (3,2), & (3,3) \end{bmatrix}$$

- $Ran(X) = \{2, 3, 4, 5, 6\}$
- $Ran(Y) = \{0, 1, 2\}$

- Joint pdf $f_{X,Y}(x,y) = P(X=x, Y=y) = p_{X,Y}(x,y)$

$\downarrow Y\ X \rightarrow$	2	3	4	5	6	$p_Y(y)$
0	$\frac{1}{9}$	0	$\frac{1}{9}$	0	$\frac{1}{9}$	$\frac{3}{9}$
1	0	$\frac{2}{9}$	0	$\frac{2}{9}$	0	$\frac{4}{9}$
2	0	0	$\frac{2}{9}$	0	0	$\frac{2}{9}$
$p_X(x)$	$\frac{1}{9}$	$\frac{2}{9}$	$\frac{3}{9}$	$\frac{2}{9}$	$\frac{1}{9}$	

- Marginal pdfs: $p_X(x) = \sum_y p_{X,Y}(x,y)$ and $p_Y(y) = \sum_x p_{X,Y}(x,y)$

3.2 Independence

X and Y are independent if and only if $p_{X,Y}(x,y) = p_X(x) \cdot p_Y(y)$ for all x, y.

- From the example:
  - X, Y are not independent because $p_{X,Y}(2,1) = \frac{1}{9} \neq \frac{1}{9} \cdot \frac{4}{9} = p_X(2) \cdot p_Y(1)$

3.3 Expected Value

$E[g(X,Y)] = \sum_x \sum_y g(x,y) \cdot p_{X,Y}(x,y)$

3.4 Covariance

$Cov(X,Y) = E[(X-\mu_X)(Y-\mu_Y)] = E(XY) - \mu_X \mu_Y$

- $Cov(X,Y) = 0$ if X, Y are independent.
- $Cov(X,Y) > 0$ if X, Y are positively correlated.
- $Cov(X,Y) < 0$ if X, Y are negatively correlated.

$$E(XY) = \sum_x \sum_y x \cdot y \cdot p_{X,Y}(x,y) \tag{1}$$

3.5 Theorems

1. $E(aX + bY + c) = aE(X) + bE(Y) + c$
2. $Var(aX + bY + c) = a^2 \cdot Var(X) + b^2 \cdot Var(Y) + 2ab \cdot Cov(X,Y)$

3.6 IID

$X_1, X_2, \ldots, X_n$ are independent and identically distributed (IID) if and only if $pdf(X_1) = pdf(X_2) = \cdots = pdf(X_n)$ and $X_1, X_2, \ldots, X_n$ are independent.

3.6.1 Consequences

- Sample Sum: $S_n = X_1 + X_2 + \cdots + X_n$
- $E(S_n) = n \cdot \mu$
- $Var(S_n) = n \cdot \sigma^2$