Probability and Statistics: Lesson 2
Conditional Probability and Bayes Theorem

Morgan McCarty

05 July 2023

1 Lesson 1 Review

1.1 Review Definitions

1. Sample Space (S): the set of all possible outcomes
2. Events: a subset of the Sample Space - e.g. $A \subseteq S$

1.2 Review Set Operation Notation

For sets A and $B \subseteq S$:

1. Complement: $A^c = \{x \in S | x \notin A\}$
2. Intersection: $A \cap B = \{x \in S | x \in A \text{ and } x \in B\}$
3. Union: $A \cup B = \{x \in S | x \in A \text{ or } x \in B\}$

1.3 Probability Function

1.3.1 Informally

$P(A)$ is the probability that event A occurs with output in the range $[0, 1]$

1.3.2 Formally

For a function f the domain is the set of all possible inputs (A) and the codomain is the set of all possible outputs (B)
i.e. $f : A \to B$
For a probability function P the domain is the power set (the set of all subsets of a set) of the Sample Space ($\mathcal{P}(S)$) and the codomain is the set of all real numbers in the range $[0, 1]$
i.e. $P : \mathcal{P}(S) \to [0, 1]$

1.4 Axioms and Properties of a Probability Function

Given some probabilities, it is possible to calculate other probabilities

1.4.1 Basic Principle of Probability

Assuming equally likely outcomes, the probability of an event A is:

$$P(A) = \frac{\text{number of outcomes favorable to } A}{\text{total number of possible outcomes}} \tag{1}$$

or

$$P(A) = \frac{|A|}{|S|} \tag{2}$$

1.5 Counting

There are two main rules of counting, the addition and multiplication rules which are mostly self-explanatory

2 Conditional Probability (of event A given event B [has occurred])

[S is our Sample Space; A and B are events in S]

2.1 Basic Notation

$$P(A|B) = \frac{P(A \cap B)}{P(B)} \tag{3}$$

This is the probability of event A given that event B has occurred

2.2 Definitions

1. Contracted Sample Space: B is the new Sample Space as if B has occurred only events in B are possible

2. Product Law: $P(A \cap B) = P(A|B)P(B)$

2.3 $P(A|B)$ in Prior Terms

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{\frac{|A \cap B|}{|S|}}{\frac{|B|}{|S|}} = \frac{|A \cap B|}{|B|} \tag{4}$$

2.4 Examples

1. In a town of people, among all voters during a recent election, how many are first-time voters who voted for "Bob"?

   (a) S is the set of all voters

   (b) A is the set of all first-time voters

   (c) B is the set of all voters who voted for "Bob"

   (d) $A \cap B^c = 260$

   (e) $A \cap B = 465$

(f) $A^c \cap B = 1123$

(g) $A^c \cap B^c = 1507$

(h) $P(B) = \frac{|B|}{|S|} = \frac{465+1123}{260+465+1123+1507} = \frac{1588}{3355}$

(i) $P(B|A) = \frac{|A \cap B|}{|A|} = \frac{465}{260+465} = \frac{465}{725}$

2. In a bag of 9 chips: 3 are **red** and 6 are **white**.

3. Chips are drawn at random:

   (a) If a **red** chip is drawn, it is placed back in the bag.

   (b) If a **white** chip is drawn, it is placed back in the bag along with a new **red** chip.

4. What is the probability that, if two chips are drawn, both are **red**?

   (a) A is the event that the first chip is **red**

   (b) B is the event that the second chip is **red**

   (c) $P(A) = \frac{3}{9} = \frac{1}{3}$

   (d) $P(A^c) = \frac{6}{9} = \frac{2}{3}$

   (e) $P(B|A) = \frac{4}{10}$

   (f) $P(B^c|A) = \frac{6}{10}$

   (g) $P(B|A^c) = \frac{3}{9}$

   (h) $P(B^c|A^c) = \frac{6}{9}$

   (i) Therefore $P(A \cap B)$ (the probability that both chips are **red**) $= P(A)P(B|A) = \frac{3}{9} \times \frac{4}{10} = \frac{2}{15}$

5. What is the probability the second chip is **red**? (i.e. $P(B)$)

$$\begin{aligned}
B &= (A \cap B) \sqcup (A^c \cap B) \\
P(B) &= P(A \cap B) + P(A^c \cap B) \\
&= P(A)P(B|A) + P(A^c)P(B|A^c) \\
&= \frac{3}{9} \times \frac{4}{10} + \frac{6}{9} \times \frac{3}{9} \\
&= \frac{2}{15} + \frac{2}{9} \\
&\approx 0.3556
\end{aligned}$$

2.5 Law of Total Probability

This generalizes to:

$$\begin{aligned}
P(B) &= P(A_1/capB) + P(A_2 \cap B) + \cdots + P(A_n \cap B) \\
&= P(A_1)P(B|A_1) + P(A_2)P(B|A_2) + \cdots + P(A_n)P(B|A_n)
\end{aligned}$$

or

$$P(B) = \sum_{i=1}^{n} P(A_i)P(B|A_i) \tag{5}$$

This is a weighted average of the conditional probabilities of B given A_i for all i.

2.6 Examples

1. There are 3 suppliers of computer chips.
   Supplier A supplies 50% of the chips, supplier B supplies 40%, and supplier C supplies 10%.
   Chips from supplier A fail 1% of the time, chips from supplier B fail 2% of the time, and chips from supplier C fail 5% of the time.
   If a random chip is chosen, what is the probability that it will fail? (i.e. $P(F)$)

   (a) A is the event that the chip is from supplier A
   (b) B is the event that the chip is from supplier B
   (c) C is the event that the chip is from supplier C
   (d) F is the event that the chip fails
   (e) $P(A) = 0.5$ and $P(B) = 0.4$ and $P(C) = 0.1$
   (f) $P(F|A) = 0.01$ and $P(F^c|A) = 0.99$
   (g) $P(F|B) = 0.02$ and $P(F^c|B) = 0.98$
   (h) $P(F|C) = 0.05$ and $P(F^c|C) = 0.95$
   (i) Therefore $P(F) = P(A)P(F|A) + P(B)P(F|B) + P(C)P(F|C) = 0.5 \times 0.01 + 0.4 \times 0.02 + 0.1 \times 0.05 = 0.018$

2. In a standard deck of cards where two cards are drawn sequentially without being revealed:

   (a) What is the probability that the first card is a **heart**?
   $P(A) = \frac{13}{52} = \frac{1}{4}$
   (b) What is the probability that the second card is a **heart** given that the first card is a **heart**?
   $P(B|A) = \frac{12}{51}$
   (c) What is the probability that the second card is a **heart**?
   $P(B) = P(A)P(B|A) + P(A^c)P(B|A^c) = \frac{1}{4} \times \frac{12}{51} + \frac{3}{4} \times \frac{13}{51} = \frac{1}{4}$
   "What we don't know doesn't matter", i.e. if we do not know whether the first card is a **heart** or not, the probability that the second card is a **heart** is still $\frac{1}{4}$

3 Bayes' Theorem

3.1 What is the Inverted Conditional Probability?

i.e. given $P(B|A)$, what is $P(A|B)$?

$$\begin{aligned} P(A|B) &= \frac{P(A \cap B)}{P(B)} \\ &= \frac{P(B|A)P(A)}{P(B)} \\ &= \frac{P(B|A)P(A)}{P(B|A)P(A) + P(B|A^c)P(A^c)} \end{aligned}$$

3.2 Definition

$$P(A|B) = \frac{P(B|A)P(A)}{P(B|A)P(A) + P(B|A^c)P(A^c)} \tag{6}$$

or

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)} \tag{7}$$

3.3 Examples

1. Continuing the chip supplier example: if a chip fails, what is the probability that it came from supplier A?

   (a) We need to find $P(A|F)$

$$
\begin{aligned}
P(A|F) &= \frac{P(F|A)P(A)}{P(F)} \\
&= \frac{0.01 \times 0.5}{.018} \\
&= \frac{0.005}{0.018} \\
&= \frac{5}{18}
\end{aligned}
$$

2. If someone is asked whether they smoke and they say they do not, what is the probability they actually do smoke?
   S is the event that the person smokes
   N is the event that the person says they do not smoke
   $P(S) = 0.25$, $P(N|S) = 0.3$, and $P(N|S^c) = 1$

   (a) We need to find $P(S|N)$

$$
\begin{aligned}
P(S|N) &= \frac{P(S \cap N)}{P(N)} \\
&= \frac{P(N|S)P(S)}{P(N)} \\
&= \frac{0.25 \times 0.3}{0.25 \times 0.3 + 0.75 \times 1} \\
&= \frac{0.075}{0.075 + 0.75} \\
&= \frac{0.075}{0.825} \\
&= \frac{1}{11}
\end{aligned}
$$

   (b) We can also find $P(S^c|N)$

$$
\begin{aligned}
P(S^c|N) &= \frac{P(S^c \cap N)}{P(N)} \\
&= \frac{P(N|S^c)P(S^c)}{P(N)} \\
&= \frac{0.75 \times 1}{0.25 \times 0.3 + 0.75 \times 1} \\
&= \frac{0.75}{0.075 + 0.75} \\
&= \frac{0.75}{0.825} \\
&= \frac{10}{11}
\end{aligned}
$$