

Probability and Statistics: Lesson 1

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1 General Overview

1.1 Definitions

1. Experiment: procedure with undetermined outcomes
2. Sample Space: (S.S. or S) set of all possible outcomes
3. Set: a collection of things
4. Countable: can be put in one-to-one correspondence with the natural numbers (integers are countable)
5. Discrete: finite or countable
6. Continuous: uncountable (in opposition to discrete)
7. Universal Set: set of all possible outcomes equivalent to the sample space in a Probability experiment

1.2 Symbols

- \in : $x \in S$: x is an element of S
- \notin : $x \notin S$ x is not an element of S

1.3 Examples

- Experiment: flip a coin
 - Sample Space: $\{H, T\}$
Sample Space is **finite**
- Experiment: flip a coin until we get a tails
 - Sample Space: $\{T, HT, HHT, HHHT, \dots\}$
Sample Space is **infinite**, but **countable**
- Experiment: pick a number in the interval $[0, 1]$
 - Sample Space: $[0, 1]$ or $\{x \in \mathbb{R} \mid 0 \leq x \leq 1\}$
Sample Space is **infinite**, and **not countable**

2 Events

2.1 Definitions

1. Subset: a set whose elements are all contained in another (super)set, additionally every set is a subset of itself and the empty set is a subset of every set
2. Event: a subset of the sample space

2.2 Symbols

- \subseteq : $A \subseteq B$: A is a subset of B
- \subset : $A \subset B$: A is a proper subset of B (at least one element of B is not in A)
- \emptyset : the empty set

2.3 Examples

- Rolling a six-sided die
 - $S = \{1, 2, 3, 4, 5, 6\}$
Sample Space is **finite**
 - Events:
 - * Event of rolling even numbers: $A = \{2, 4, 6\}$
 - * Event of rolling a “6”: $B = \{6\}$
 - * Event of rolling a prime number: $C = \{2, 3, 5\}$
 - * Event of rolling a number 7 or greater: $D = \emptyset$

3 Set Operations

3.1 Definitions

Given sets A and B :

1. Complement: $A^c = \{x \mid x \in S \text{ and } x \notin A\}$
2. Intersection: $A \cap B = \{x \mid x \in A \text{ and } x \in B\}$
3. Union: $A \cup B = \{x \mid x \in A \text{ or } x \in B\}$
4. Disjoint: if $A \cap B = \emptyset$, then A and B are disjoint
5. DeMorgan's Laws: $(A \cup B)^c = A^c \cap B^c$ and $(A \cap B)^c = A^c \cup B^c$

3.2 Symbols

Assume $A, B \subseteq S$

- A^c : complement of A
- $A \cap B$: intersection of A and B
- $A \cup B$: union of A and B
- $A \sqcup B$: disjoint union of A and B (i.e. $A \cap B = \emptyset$)

4 The Probability Function

4.1 Definitions

1. $\underline{P(A)}$: probability of event A

4.2 Kolmogorov's Axioms

1. Axiom 1: $P(A) \geq 0$
2. Axiom 2: $P(S) = 1$
3. Axiom 3: If A, B are disjoint, then $P(A \cup B) = P(A \sqcup B) = P(A) + P(B)$

4.3 Derived Properties

1. $P(A^c) = 1 - P(A)$

Proof:

$$A \sqcup A^c = S$$

$$P(S) = P(A) + P(A^c) \quad [\text{by Axiom 3}]$$

$$1 = P(A) + P(A^c) \quad [\text{by Axiom 2}]$$

$$P(A^c) = 1 - P(A) \quad \square$$

2. $P(\emptyset) = 0$

Proof:

$$\emptyset \sqcup S = S$$

$$P(S) = P(\emptyset) + P(S) \quad [\text{by Axiom 3}]$$

$$1 = P(\emptyset) + 1 \quad [\text{by Axiom 2}]$$

$$P(\emptyset) = 0 \quad \square$$

3. $P(A) = P(A \cap B) + P(A \cap B^c)$

Proof:

$$A = A \cap S$$

$$= A \cap (B \sqcup B^c) \quad [\text{by definition of complement}]$$

$$= (A \cap B) \sqcup (A \cap B^c) \quad [\text{by distribution}]$$

$$P(A) = P(A \cap B) + P(A \cap B^c) \quad \square [\text{by Axiom 3}]$$

4. If $A \subseteq B$, then $P(A) \leq P(B)$

Proof:

$$B = S \cap B \quad [\text{by definition}]$$

$$= (A \sqcup A^c) \cap B \quad [\text{by definition}]$$

$$= (A \sqcup A^c) \cap (A \cup B) \quad [\text{by definition of subset}]$$

$$= A \sqcup (B \cap A^c) \quad [\text{by distribution}]$$

$$P(B) = P(A) + P(B \cap A^c) \quad [\text{by Axiom 3}]$$

$$\geq P(A) + 0 \quad [\text{by Axiom 1}]$$

$$\geq P(A) \quad \square$$

5. $P(A) \leq 1$

Proof:

$$\begin{aligned}
 A \sqcup A^c &= S && \text{[by definition]} \\
 P(S) &= P(A) + P(A^c) && \text{[by Axiom 3]} \\
 1 &= P(A) + P(A^c) && \text{[by Axiom 2]} \\
 P(A) &\leq 1 && \square
 \end{aligned}$$

6. Union Rule: $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Proof:

$$\begin{aligned}
 A \cup B &= A \sqcup (B \cap A^c) && \text{[by distribution]} \\
 P(A \cup B) &= P(A) + P(B \cap A^c) && \text{[by Axiom 3]} \\
 P(B) &= P(B \cap A) + P(B \cap A^c) && \text{[by derived property 3]} \\
 P(B \cap A^c) &= P(B) - P(B \cap A) && \text{[by algebra]} \\
 P(A \cup B) &= P(A) + P(B) - P(B \cap A) && \square
 \end{aligned}$$

4.4 Examples

- Example 1

- $P(A) = 0.4, P(B) = 0.5, P(A \cap B) = 0.1$
- Determine probability only A occurs:
 $P(A \cap B^c) = P(A) - P(A \cap B) = 0.4 - 0.1 = 0.3$
- Determine probability A or B occurs:
 $P(A \cup B) = P(A) + P(B) - P(A \cap B) = 0.4 + 0.5 - 0.1 = 0.8$
- Determine probability A xor B occurs:
 $P(A \cup B) - P(A \cap B) = 0.8 - 0.1 = 0.7$
or $P(A \cap B^c) + P(B \cap A^c) = 0.3 + 0.4 = 0.7$
- Determine probability neither A nor B occurs:
 $P((A \cup B)^c) = 1 - P(A \cup B) = 1 - 0.8 = 0.2$

5 Basic Principle of Probability

5.1 Definitions

1. Cardinality: the number of elements in a set

5.2 Symbols

- $|A|$: cardinality of A

5.3 The Principle

If every outcome of S.S. is equally likely, then:

$$P(A) = \frac{\text{number of outcomes in } A}{\text{number of outcomes in } S}$$

or

$$P(A) = \frac{|A|}{|S|}$$

5.4 Examples

1. Example 1

- for a fair die roll, what is the probability of rolling an even number?

$$P(A) = \frac{|A|}{|S|} = \frac{3}{6} = \frac{1}{2} = .5$$

2. Example 2

- If you roll two fair dice, what is the probability of rolling a sum greater or equal to 9?

$$A = \{36, 45, 54, 55, 56, 63, 64, 65, 66\}$$

$$P(A) = \frac{|A|}{|S|} = \frac{10}{36} = \frac{5}{18} \approx .278$$

6 Counting and Probability

6.1 Rules

1. Addition Rule
2. Multiple Rule

6.2 Examples

1. Example 1

- In a standard deck of cards, what is the probability of drawing a face card or a black ace?

$$A = \{A_{\spadesuit}, A_{\clubsuit}\} + \{J_{\spadesuit}, J_{\clubsuit}, J_{\heartsuit}, J_{\diamondsuit}, Q_{\spadesuit}, Q_{\clubsuit}, Q_{\heartsuit}, Q_{\diamondsuit}, K_{\spadesuit}, K_{\clubsuit}, K_{\heartsuit}, K_{\diamondsuit}\}$$

$$2 * 1 \text{ (multiple rule) aces} + \text{(addition rule) } 4 * 3 \text{ face cards} = 16 \text{ cards}$$

$$P(A) = \frac{|A|}{|S|} = \frac{14}{52} = \frac{7}{26} \approx .269$$