Probability and Statistics:
Lessons 3 and 5 Continued, Homework 2

Morgan McCarty

13 July 2023

1 Lessons 3 and 5 Continued

1.1 Examples

1. There is a dice game which costs \$5 to play.
   - Two fair dice are rolled
   - If the result is a <u>double</u> (both dice have the same value) then the player wins \$10 (net \$5)
   - If the result is <u>odd</u> then the player wins \$$n$ where n is the sum of the two dice
   - X is the player's net winnings (or losses)
   - Is the game fair? ($E(X) = 0$)
   - pdf:

k	-5	5	-2	0	2	4	6
$p_X(k)$	$\frac{12}{36}$	$\frac{6}{36}$	$\frac{2}{36}$	$\frac{4}{36}$	$\frac{6}{36}$	$\frac{4}{36}$	$\frac{2}{36}$

   - Expected Value:

$$\begin{aligned} E(X) &= \sum_k k \cdot p_X(k) \\ &= -5 \cdot \frac{12}{36} + 5 \cdot \frac{6}{36} + (-2) \cdot \frac{2}{36} + 0 \cdot \frac{4}{36} + 2 \cdot \frac{6}{36} + 4 \cdot \frac{4}{36} + 6 \cdot \frac{2}{36} \\ &= \frac{6}{36} \end{aligned}$$

2. Flip a coin n times
   - X is the number of heads, p is the probability of heads
   - $Ran(X) = \{0, 1, 2, \ldots, n\}$
   - $X = X_1 + X_2 + \cdots + X_n$ where $X_i \sim Bernoulli(p)$
   - $E(X_i) = p$
   - $E(X) = n \cdot p$
   - $Var(X_i) = p(1 - p)$
   - $Var(X) = n \cdot p(1 - p)$
   - Binomial distributions are sums of Bernoulli distributions

2 Homework 2 Notes

1. In the World Series, two teams play a series of games until one team gets four wins. Suppose two evenly matched teams are playing (each team has an equal probability of winning). Let X be the number of games player until there is a winner. Find $E(X)$ and $Var(X)$.

   - $X =$ the number of games player until there is a winner.
   - $Ran(X) = \{4, 5, 6, 7\}$
   - pdf:

k	4	5	6	7
$p_X(k)$	$2 \cdot \binom{3}{0} \cdot \frac{1}{4}^4$	$2 \cdot \binom{4}{1} \cdot \frac{1}{2}^5$	$\ldots$	$\ldots$

2. Four people ($\{A, B, C, D\}$) form a random line

   - X is the number of people between A and B, Y is the number of people between C and D

     (a) Find the Joint pdf of X and Y and then the Marginal pdfs.
       - $Ran(X) = \{0, 1, 2\}$
       - $Ran(Y) = \{0, 1, 2\}$
       - $P(X = 0, Y = 0) = \frac{2!2!2!}{4!}$ ($ABCD$, $BACD$, $ABDC$, $BADC$, $CDAB$, $DCAB$, $CDBA$, $DCBA$)
       - Joint pdf:

$\downarrow Y X \rightarrow$	0	1	2	$p_Y(y)$
0	$\frac{8}{24}$	0	$\ldots$	$\ldots$
1	$\ldots$	$\ldots$	$\ldots$	$\ldots$
2	$\ldots$	$\ldots$	$\ldots$	$\ldots$
$p_X(x)$	$\ldots$	$\ldots$	$\ldots$	

     (b) Calculate the covariance of X and Y
       - $Cov(X, Y) = E(XY) - E(X)E(Y)$
       - Covariance should be negative $\rightarrow$ as X increases, Y decreases and vice versa

3. Suppose N people throw their hats into the air, find the expected number of people who caught their own hat.

   - X_i: # of own hats caught by the i-th person
   - $X = X_1 + X_2 + \cdots + X_N$ (total # of people who catch their own hat)
   - $E(X) = E(X_1) + E(X_2) + \cdots + E(X_N)$
   - pdf:

k	0	1
$p_{X_i}(k)$	$1 - \frac{1}{N}$	$\frac{1}{N}$

4. In a series of coin flips with a biased coin where $P(H) = p$.

   - X is the number of flips until the first head
   - $p_X(k) = (1-p)^{k-1}p$ for $k \in 1, 2, 3, \ldots$

(a) Find $E(X)$

$$\begin{aligned} E(X) &= \sum_{k=1}^{\infty} k \cdot (1-p)^{k-1} p \\ &= p \sum_{k=1}^{\infty} k \cdot (1-p)^{k-1} \end{aligned}$$

Given:

$$\sum_{k=1}^{\infty} k \cdot q^{k-1} = \frac{1}{(1-q)^2}$$

(b) Y is the number of flips until the second head. Find $P(Y = k)$.

$$P(Y = k) = \binom{k-1}{1} \dots$$

(c) Find $E(Y)$

$$E(Y) = E(X_1) + E(X_2) \text{ where } X_1 \sim Geometric(p) \text{ and } X_2 \sim Geometric(p)$$