A Swim in Fluid Dynamics

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Abstract

I studied fluid dynamics to be able to relate the divergence of a vector field under fluid flow and the incompressibility or compressibility of the fluid.

1 Introduction

We will investigate the motion of particles in a fluid via their velocity vector field as well as its derivative with respect to t. The phase portrait of the solution of the differential equation shows that the trajectories of the particles move in a sink. Next, we found a velocity vector field to suit an incompressible fluid, with a stretch factor of 1.

2 Finding Solution of Vector Field

$\vec{U}(\mathbf{y})$ is a velocity vector field that describes a fluid under a flow.The differential equation $\frac{d\mathbf{y}}{dt} = \vec{U}\big(\mathbf{y}(t)\big)$ describes the change of position of a molecule with the initial condition $y(0) = (a, b, c)$

2.1 IVP

The equation $\vec{U}(\tilde{\mathbf{y}}) = -2xi + (-2y + z)j + (y - 2z)k$ can be written as the matrix

$$\begin{pmatrix} -2 & 0 & 0 \\ 0 & -2 & 1 \\ 0 & 1 & -2 \end{pmatrix} = \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix}$$

The differential equation given is $\frac{dy}{dt} = \vec{U}((\mathbf{y}(t))$ with the initial value $\tilde{\mathbf{y}}(0) = (a, b, c)$, therefore

$$\frac{d\mathbf{y}}{dt} = \begin{pmatrix} -2 & 0 & 0 \\ 0 & -2 & 1 \\ 0 & 1 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

The Fundamental Theorem of Ordinary Differential Equations states that there is a unique solution to this IVP To solve for the eigenvalues, the determinate is taken with Λ being subtracted from the diagonal of the matrix

$$det \begin{pmatrix} -2-\Lambda & 0 & 0 \\ 0 & -2-\Lambda & 1 \\ 0 & 1 & -2-\Lambda \end{pmatrix}$$

Two polynomials come out to $(-2-\Lambda)(\lambda^2+4\lambda+3)$ which can be factored to $(-2-\Lambda)(\lambda+1)(\lambda+3)$. So, our values of λ are

$$\lambda_1 = -2 \quad \Lambda_2 = -1 \quad \lambda_3 = -3$$

After finding the eigenvalues, the next step is to find the corresponding eigenvectors. By subtracting the eigenvalue along the diagonal and getting the resulting matrix in reduced row echelon form, the values for x_1, x_2, and x_3 are found.

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

x_1 is free, $x_2 = 0x_1$, and $x_3 = 0x_1$, so the first eigenvector is $\begin{pmatrix}1\\0\\0\end{pmatrix}$. Repeat for all eigenvalues, and the eigenvectors are

$$E_1 = \begin{pmatrix}1\\0\\0\end{pmatrix} E_2 = \begin{pmatrix}0\\1\\1\end{pmatrix} E_3 = \begin{pmatrix}0\\-1\\1\end{pmatrix}$$

The general solution to the IVP is

$$\vec{\mathbf{Y}}(t) = c_1 e^{-2t}\begin{pmatrix}1\\0\\0\end{pmatrix} + c_2 e^{-t}\begin{pmatrix}0\\1\\1\end{pmatrix} + c_3 e{-}3t\begin{pmatrix}0\\-1\\1\end{pmatrix}$$

Then, solve for the constants by using the initial conditions given by $x = (a, b, c)$

$$\left(\left[\begin{array}{ccc|c} 1 & 0 & 0 & a \\ 0 & 1 & -1 & b \\ 0 & 1 & 1 & c \end{array}\right]\right)$$

in reduced row echelon form it becomes

$$\left(\left[\begin{array}{ccc|c} 1 & 0 & 0 & a \\ 0 & 1 & 0 & \frac{c+b}{2} \\ 0 & 0 & 1 & \frac{c-b}{2} \end{array}\right]\right) \quad c_1 = a,\ c_2 = \frac{c+b}{2},\ c_3 = \frac{c-b}{2}$$

The solution of the differential equation is

$$\vec{Y}(t) = ae^{-2t}\begin{pmatrix}1\\0\\0\end{pmatrix} + \frac{c+b}{2}e^{-t}\begin{pmatrix}0\\1\\1\end{pmatrix} + \frac{c-b}{2}e^{-3t}\begin{pmatrix}0\\-1\\1\end{pmatrix}$$

Figure 1 shows the trajectories of particles. When t is coming from $-\infty$ the e^{-3t} term dominates, as t gets closer to ∞ the e^{-2t} term dominates, and as t approachces ∞ e^{-t} dominates

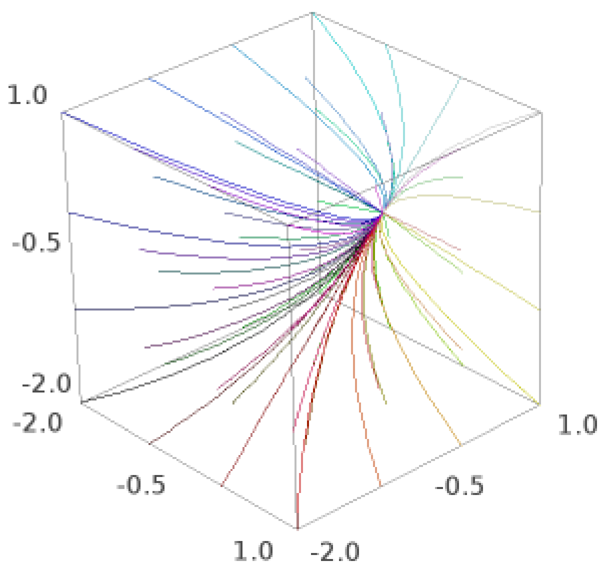

Figure 1: The solution of the differential equation is a sink.

3 Stretch Factor

Continuing with the vector field $\vec{U}$, consider the transformation F_t, describing the mapping of x to $\mathbf{y}(t)$.

$$F_t = \begin{pmatrix} ae^{-2t} \\ \frac{c+b}{2}e^{-t} - \frac{c-b}{2}e^{-3t} \\ \frac{c+b}{2}e^{-t} + \frac{c-b}{2}e^{-3t} \end{pmatrix} = \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} = DF_t = \begin{pmatrix} e^{-2t} & 0 & 0 \\ 0 & \frac{1}{2}e^{-t} + \frac{1}{2}e^{-3t} & \frac{1}{2}e^{-t} - \frac{1}{2}e^{-3t} \\ 0 & \frac{1}{2}e^{-t} - \frac{1}{2}e^{-3t} & \frac{1}{2}e^{-t} + \frac{1}{2}e^{-3t} \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

The stretch factor $S(t) = det(DF_t)$

$$DF_t = \begin{pmatrix} \frac{\partial x}{\partial a} & \frac{\partial x}{\partial b} & \frac{\partial x}{\partial c} \\ \frac{\partial y}{\partial a} & \frac{\partial y}{\partial b} & \frac{\partial y}{\partial c} \\ \frac{\partial z}{\partial a} & \frac{\partial z}{\partial b} & \frac{\partial z}{\partial c} \end{pmatrix} \quad det(DF_t) = e^{-2t} det \begin{pmatrix} \frac{1}{2}e^{-t} + \frac{1}{2}e^{-3t} & \frac{1}{2}e^{-t} - \frac{1}{2}e^{-3t} \\ \frac{1}{2}e^{-t} - \frac{1}{2}e^{-3t} & \frac{1}{2}e^{-t} + \frac{1}{2}e^{-3t} \end{pmatrix}$$

$$= e^{-2t}((\frac{1}{2}e^{-t} + \frac{1}{2}e^{-3t})(\frac{1}{2}e^{-t} + \frac{1}{2}e^{-3t}) - (\frac{1}{2}e^{-t} - \frac{1}{2}e^{-3t})(\frac{1}{2}e^{-t} - \frac{1}{2}e^{-3t}))$$

$$S(t) = e^{-6t}$$

The determinant of the Jacobi matrix is 1 at $t = 0$. This can also be shown in the mapping of $F_t : x \mapsto \mathbf{y}(t)$. The transformation matrix(Jacobi matrix) at $t = 0$ is the identity matrix, meaning that at the instant the fluid starts flowing there is no change in volume.

4 Change in Stretch Factor

We reduce the dimension of the vector field to x and y to simplify our computation of the change in the stretch factor $\frac{dS}{dt}$. Our goal is to find a relationship between $S(t)$ and $\frac{dS}{dt}$.

$$S(t) = det(DF_t) = det \begin{pmatrix} \frac{\partial x}{\partial a}\Big|_{(x(t),y(t))} & \frac{\partial x}{\partial b}\Big|_{(x(t),y(t))} \\ \frac{\partial y}{\partial a}\Big|_{(x(t),y(t))} & \frac{\partial y}{\partial b}\Big|_{(x(t),y(t))} \end{pmatrix}$$

Find the determinant of DF_t

$$det(DF_t) = \Big((\frac{\partial x}{\partial a})(\frac{\partial y}{\partial b}) - (\frac{\partial x}{\partial b})(\frac{\partial y}{\partial a})\Big) = S(t)$$

$$\frac{dS}{dt} = \frac{\partial}{\partial t}\Big((\frac{\partial x}{\partial a})(\frac{\partial y}{\partial b}) - (\frac{\partial x}{\partial b})(\frac{\partial y}{\partial a})\Big)$$

$$\frac{dS}{dt} = (\frac{\partial^2 x}{\partial t \partial a})(\frac{\partial y}{\partial b}) + (\frac{\partial^2 y}{\partial t \partial b})(\frac{\partial x}{\partial a}) - (\frac{\partial^2 x}{\partial t \partial b})(\frac{\partial y}{\partial a}) - (\frac{\partial^2 y}{\partial t \partial a})(\frac{\partial x}{\partial b})$$

$$\frac{\partial x}{\partial t} = P(x(t), y(t)) \quad \frac{\partial y}{\partial t} = Q(x(t), y(t))$$

We can replace any partial derivatives of x or y with respect to t with P or Q, respectively.

$$\frac{dS}{dt} = (\frac{\partial P}{\partial a})(\frac{\partial y}{\partial b}) + (\frac{\partial Q}{\partial b})(\frac{\partial x}{\partial a}) - (\frac{\partial P}{\partial b})(\frac{\partial y}{\partial a}) - (\frac{\partial Q}{\partial a})(\frac{\partial x}{\partial b})$$

The chain rule can be used to find the partial derivatives of the components of $\vec{U}$ with respect to a and b

$$\frac{\partial P}{\partial a} = \frac{\partial P}{\partial x}\frac{\partial x}{\partial a} + \frac{\partial P}{\partial y}\frac{\partial y}{\partial a} \quad \frac{\partial P}{\partial b} = \frac{\partial P}{\partial x}\frac{\partial x}{\partial b} + \frac{\partial P}{\partial y}\frac{\partial y}{\partial b}$$

$$\frac{\partial Q}{\partial a} = \frac{\partial Q}{\partial x}\frac{\partial x}{\partial a} + \frac{\partial Q}{\partial y}\frac{\partial y}{\partial a} \quad \frac{\partial Q}{\partial b} = \frac{\partial Q}{\partial x}\frac{\partial x}{\partial b} + \frac{\partial Q}{\partial y}\frac{\partial y}{\partial b}$$

$$\frac{dS}{dt} = (\frac{\partial P}{\partial x}\frac{\partial x}{\partial a} + \frac{\partial P}{\partial y}\frac{\partial y}{\partial a})\frac{\partial y}{\partial b} - (\frac{\partial P}{\partial x}\frac{\partial x}{\partial b} + \frac{\partial P}{\partial y}\frac{\partial y}{\partial b})\frac{\partial y}{\partial a} + (\frac{\partial Q}{\partial x}\frac{\partial x}{\partial a} + \frac{\partial Q}{\partial y}\frac{\partial y}{\partial a})\frac{\partial x}{\partial a} - (\frac{\partial Q}{\partial x}\frac{\partial x}{\partial b} + \frac{\partial Q}{\partial y}\frac{\partial y}{\partial b})\frac{\partial x}{\partial a}$$

$$= \frac{\partial P}{\partial x}(\frac{\partial x}{\partial a}\frac{\partial y}{\partial b} - \frac{\partial x}{\partial b}\frac{\partial y}{\partial a}) + \frac{\partial P}{\partial y}(\frac{\partial y}{\partial a}\frac{\partial y}{\partial b} - \frac{\partial y}{\partial b}\frac{\partial y}{\partial a}) + \frac{\partial Q}{\partial x}(\frac{\partial x}{\partial b}\frac{\partial x}{\partial a} - \frac{\partial x}{\partial a}\frac{\partial x}{\partial b}) + \frac{\partial Q}{\partial y}(\frac{\partial y}{\partial b}\frac{\partial x}{\partial a} - \frac{\partial y}{\partial a}\frac{\partial x}{\partial b})$$

Two terms become zero and we are left with

$$\frac{\partial P}{\partial x}(\frac{\partial x}{\partial a}\frac{\partial y}{\partial b}-\frac{\partial x}{\partial b}\frac{\partial y}{\partial a})+\frac{\partial Q}{\partial y}(\frac{\partial y}{\partial b}\frac{\partial x}{\partial a}-\frac{\partial y}{\partial a}\frac{\partial x}{\partial b})$$

Factor

$$(\frac{\partial x}{\partial a}\frac{\partial y}{\partial b}-\frac{\partial x}{\partial b}\frac{\partial y}{\partial a})(\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y})$$

If $\vec{U}=\Big(P\big(x(t)\big),Q\big(x(t)\big)\Big)$ then the divergence of $\vec{U}$ can be defined as $\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}$. $S(t)$ is the determinate of the Jacobi matrix. Therefore the change in the stretch factor with respect to time is

$$\frac{dS}{dt}=S(t)\big(div(\vec{U})\big)$$

5 Finding Divergence

Divergence of $\vec{U}$ is the partial derivative of the x component with respect to x added to the partial derivative of the y component with respect to y, so

$$div(\vec{U})=\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}=\frac{\partial(-2x)}{\partial x}+\frac{\partial(-2y+z)}{\partial y}=-4$$

Using the result from Change in Stretch Factor we find $S(t)=e^{-6t}$ and $\frac{dS}{dt}=-4(e^{-6t})$

6 Incompressible Fluids

An incompressible fluid has its volume conserved in fluid flow, so under the transformation the stretch factor $S(t)$ is 1 and the change in the stretch factor $\frac{dS}{dt}$ is 0. Using the general solution from Problem e, we can find $\vec{U}(y)$ with the parameter α. The velocity vector field is given as

$$\vec{U}(y)=\alpha x\vec{i}+(-2y+z)\vec{j}+(y-2z)\vec{k}$$

First, solve for the divergence

$$div\big(\vec{U}(y)\big)=\alpha+-2+-2$$

Then apply the equation found in Change in Stretch Factor to solve for α

$$0=(\alpha+-4)1 \qquad \alpha=4$$

The velocity vector field for an incompressible fluid is

$$\vec{U}(y)=4x\vec{i}+(-2y+z)\vec{j}+(y-2z)\vec{k}$$

With the new vector field $\vec{U}(y)$, we repeat the same steps from problem 1 and 2 in order to find a solution. Solve for the eigenvalues

$$det\begin{pmatrix}4-\Lambda & 0 & 0\\ 0 & -2-\Lambda & 1\\ 0 & 1 & -2-\Lambda\end{pmatrix}$$

$$(4-\Lambda)(\lambda^2+4\lambda+3)(4-\Lambda)(\lambda+1)(\lambda+3)$$

$$\lambda_1=4\Lambda_2=-1\lambda_3=-3$$

The eigenvectors can be found by finding the reduced row echelon form of the matrix $\vec{U}(y)$ with λ subtracted on the diagonal

$$E_1=\begin{pmatrix}1\\0\\0\end{pmatrix}E_2=\begin{pmatrix}0\\1\\1\end{pmatrix}E_3=\begin{pmatrix}0\\-1\\1\end{pmatrix}$$

The eigenvectors end up being the the same from the solution of the compressible fluid we found previously. Therefore, the general solution for the incompressible fluid is

$$\vec{Y}(t)=c_1e^{4t}\begin{pmatrix}1\\0\\0\end{pmatrix}+c_2e^{-t}\begin{pmatrix}0\\1\\1\end{pmatrix}+c_3e^{-3t}\begin{pmatrix}0\\-1\\1\end{pmatrix}$$

6.1 Stretch Factor of Incompressible Fluid

F_t is the mapping of $x \mapsto \mathbf{y}(t)$ and can be defined as

$$F_t = \begin{pmatrix} ae^{4t} \\ \frac{c+b}{2}e^{-t} - \frac{c-b}{2}e^{-3t} \\ \frac{c+b}{2}e^{-t} + \frac{c-b}{2}e^{-3t} \end{pmatrix} = \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} \quad DF_t = \begin{pmatrix} e^{4t} & 0 & 0 \\ 0 & \frac{1}{2}e^{-t} + \frac{1}{2}e^{-3t} & \frac{1}{2}e^{-t} - \frac{1}{2}e^{-3t} \\ 0 & \frac{1}{2}e^{-t} - \frac{1}{2}e^{-3t} & \frac{1}{2}e^{-t} + \frac{1}{2}e^{-3t} \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

The stretch factor $S(t) = det(DF_t)$

$$DF_t = \begin{pmatrix} \frac{\partial x}{\partial a} & \frac{\partial x}{\partial b} & \frac{\partial x}{\partial c} \\ \frac{\partial y}{\partial a} & \frac{\partial y}{\partial b} & \frac{\partial y}{\partial c} \\ \frac{\partial z}{\partial a} & \frac{\partial z}{\partial b} & \frac{\partial z}{\partial c} \end{pmatrix} \quad det(DF_t) = e^{4t} det \begin{pmatrix} \frac{1}{2}e^{-t} + \frac{1}{2}e^{-3t} & \frac{1}{2}e^{-t} - \frac{1}{2}e^{-3t} \\ \frac{1}{2}e^{-t} - \frac{1}{2}e^{-3t} & \frac{1}{2}e^{-t} + \frac{1}{2}e^{-3t} \end{pmatrix}$$

$$= e^{4t}((\frac{1}{2}e^{-t} + \frac{1}{2}e^{-3t})(\frac{1}{2}e^{-t} + \frac{1}{2}e^{-3t}) - (\frac{1}{2}e^{-t} - \frac{1}{2}e^{-3t})(\frac{1}{2}e^{-t} - \frac{1}{2}e^{-3t}))$$

$$S(t) = 1$$

In the phase portrait below, as curves comes from $-\infty$, the e^{-3t} dominates. As the origin is approached, e^{-t} dominates. When t approaches ∞, e^{4t} dominates, as seen in the phase portrait as the straight forest green lines.

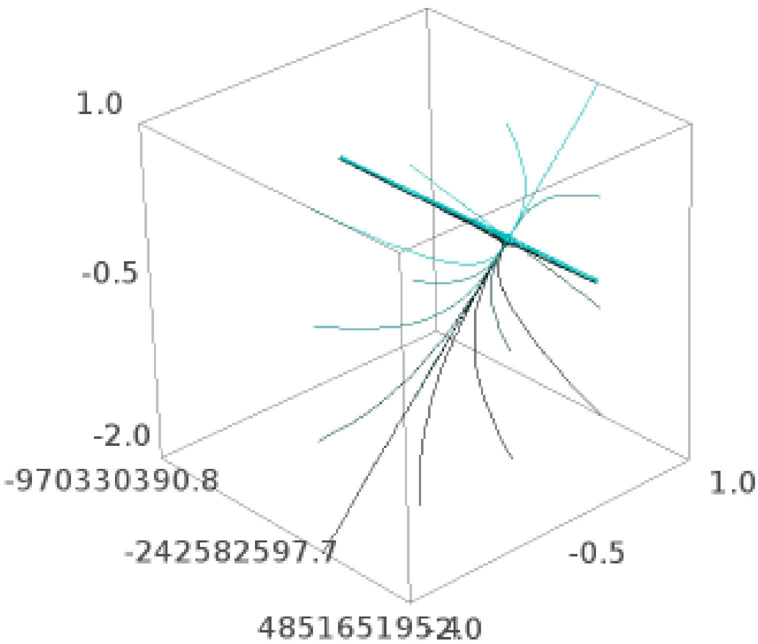

Figure 2: The solution is a saddle

7 Conclusion

We found a relationship between the divergence of the vector field and the stretch factor of the transformation. This information allowed us to find the stretch factor, which varied with t. The solution of the differential equation was found using eigenvalues and eigenvectors, allowing us to create a phase portrait of the sink. Using the relationship of divergence and the stretch factor we were able to find the vector field of an incompressible fluid, whose volume is conserved in motion.