Proc. Roy. Soc. A. **306**, 91–118 (1968)
Printed in Great Britain

Atomic collisions and local thermodynamic equilibrium in stellar atmospheres†

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(*Communicated by M. J. Seaton, F.R.S.—Received* 10 *January* 1968)

Theoretical and observational evidence for the role of collisions with electrons and neutral atoms in establishing the populations of atomic energy levels in stellar atmospheres are reviewed, with special reference to the problem of defining the range of validity of the approximation of local thermodynamic equilibrium. The discussion refers to stellar continuous spectra, absorption lines and emission lines believed to come from extended envelopes surrounding certain hot stars.

1. Introduction

One of the chief aims of stellar spectroscopy is the diagnosis of interesting parameters such as effective temperature, surface gravity, velocity fields and the abundances of the elements. These parameters are related to observable features (continuum intensity at different wavelengths, and profiles or total intensities of absorption lines) by a series of theoretical assumptions on the structure of stellar atmospheres. The most natural assumptions to begin with are (*a*) that the atmosphere is in radiative equilibrium, (*b*) that it is in hydrostatic equilibrium, and (*c*) that it is in a steady state. These assumptions are each self-consistent within quite a wide range (though not the whole range) of stellar atmospheric parameters, and whether or not all of them are valid in any particular case is ultimately decided by observation.

To predict a stellar spectrum for comparison with observation, it is necessary to use assumptions such as (*a*), (*b*) and (*c*) to construct a 'model atmosphere' in which the kinetic temperature, the electron pressure and the distribution of atoms over their various states of excitation and ionization are given as functions of optical depth. The most logical procedure for doing this is to formulate the equations of statistical equilibrium expressing the constancy of the population of each atomic state, write down equations of radiative transfer for each of the important spectral lines and continua, and solve them subject to conditions (*a*) and (*b*).

This rigorous procedure requires knowledge of all the radiative and collisional rates for the processes affecting the statistical equilibrium, and furthermore leads to a complicated system of coupled differential equations which have only been solved in certain special cases and mostly quite recently by using modern electronic

† Based on a review presented at the International Astronomical Union Colloquium on Atomic Collision Processes held at the Joint Institute for Laboratory Astrophysics, Boulder, Colorado, July 1966.

computers. In practice, therefore, both the diagnostic problem and the construction of a model are often enormously simplified by introducing the assumption of local thermodynamic equilibrium (l.t.e.), according to which the atomic level populations are given at every point in the atmosphere by applying the Boltzmann and Saha equations with the local temperature and density. Now in a steady state the distribution of kinetic energy among electrons, atoms and ions is accurately Maxwellian under a wide range of conditions, because elastic collisions are much more frequent than other interactions (Bohm & Aller 1947), and the electron kinetic temperature is virtually equal to that of the atoms and ions (Bhatnagar, Krook, Menzel & Thomas 1955). Consequently l.t.e. gives a very good approximation if the relevant atomic states are populated and depopulated much more frequently by collisions than by radiative processes, since collisions couple the level population to the local kinetic temperature. The radiation field, on the other hand, depends on the populations (and hence temperatures) in the whole portion of the atmosphere that is 'visible' from the particular point being considered. In many laboratory plasmas, electron densities are high ($> 10^{14}\,\mathrm{cm}^{-3}$) and radiation fields weak, so that the levels responsible for, say, the Balmer lines have their l.t.e. populations relative to protons and electrons (Griem 1964). In stellar atmospheres, however, where the electron densities are generally $\leqslant 10^{14}\,\mathrm{cm}^{-3}$ and radiation fields strong, it is frequently the latter which dominate, except at such great optical depths that radiative transitions are effectively cancelled out by the inverse processes. Consequently the assumption of l.t.e. in stellar atmospheres is not really self-consistent. This point has been particularly stressed by Thomas, Jefferies and their collaborators (see Thomas & Athay 1961; Thomas 1965*a*), who have developed a non-l.t.e. theory which is especially useful in discussing problems in which the departures from l.t.e. are large and radiative transfer effects important, e.g. those arising in the solar chromosphere and in the theory of strong spectral lines. However, as an approximation to what happens at photospheric levels—i.e. at those somewhat deeper levels where stellar continua, weak absorption lines and the wings of strong absorption lines arise—the l.t.e. approximation has generally proved to be accurate enough to be a very useful diagnostic tool. The present article reviews the justification for this statement, the reasons why it is true (in which atomic collisions play varying roles in different circumstances) and the effects of collisions on continua and absorption lines in general, particularly in the spectrum of the Sun; some aspects of this problem have previously been discussed from a somewhat different point of view by Unsöld (1963). We shall also make some comments on the role of collisions in extended envelopes surrounding certain hot stars with emission lines of hydrogen and helium.

2. Stellar continuous spectra

An atmosphere having geometrical depth much smaller than the stellar radius can be considered as though it were stratified in plane parallel layers. In this case, a combination of the assumptions of radiative equilibrium and l.t.e. applied to a grey atmosphere (opacity independent of wavelength) leads to the prediction that temperature T increases with optical depth τ according to the law†

$$T^4 = \tfrac{3}{4}T_{\text{eff}}^4[\tau + q(\tau)] \simeq \tfrac{3}{4}T_{\text{eff}}^4[\tau + \tfrac{2}{3}], \tag{1}$$

where T_{eff} is a constant (the effective temperature) and is defined by force-fitting Stefan's law to the total flux at the surface, $\mathscr{F} = \sigma T_{\text{eff}}^4$. In the non-grey case (opacity variable with wavelength), an approximate relation similar to (1) can be derived in terms of a suitably defined mean optical depth $\bar{\tau}$; a more exact relation $T(\tau_\nu)$ in terms of monochromatic optical depth can be obtained numerically by iteration.

In general the continuous opacity consists of two components: absorption, in which a photon causes photo-ionization, for example, and its energy is transferred to the electron gas through collisions; and scattering, in which the photon is re-emitted at nearly the same frequency. One can define an absorption coefficient κ_ν and scattering coefficient σ_ν such that

$$\mathrm{d}\tau_\nu = -\rho(\kappa_\nu + \sigma_\nu)\,\mathrm{d}z$$

where ρ is the density. Examples of scattering are Thomson scattering in very hot stars (O stars and B super-giants) and Rayleigh scattering in cool giants (late K or M); but continuous scattering plays a very minor role in stars of intermediate spectral types A, F, G and moderate or low luminosity such as the Sun (dwarf G2). The emergent flux is given by the integral

$$\mathscr{F}_\nu = 2\pi \int_0^\infty S_\nu(\tau_\nu)\,E_2(\tau_\nu)\,\mathrm{d}\tau_\nu, \tag{2}$$

where
$$E_2(x) \equiv \int_1^\infty y^{-2}\,\mathrm{e}^{-xy}\,\mathrm{d}y \simeq \tfrac{1}{2}\sqrt{3}\,\mathrm{e}^{-x\sqrt{3}},$$

and the 'source function' $S_\nu(\tau_\nu)$ is defined as the ratio of emission per unit volume into one steradian to absorption per unit length and is given in l.t.e. by

$$S_\nu(\tau_\nu) \equiv \frac{j_\nu}{4\pi(\kappa_\nu + \sigma_\nu)\,\rho} = \frac{\kappa_\nu}{\kappa_\nu + \sigma_\nu} B_\nu[T(\tau_\nu)] + \frac{\sigma_\nu}{\kappa_\nu + \sigma_\nu} J_\nu(\tau_\nu), \tag{3}$$

where $B_\nu(T)$ is the Planck function and J_ν the specific intensity averaged over direction. It is through the B_ν term in (3) and relations (1) and (2) that measurements of intensity in stellar continua give information on T_{eff}; for convenience we neglect σ_ν in what follows so that, in l.t.e., $S_\nu = B_\nu(T)$. In the case of the Sun, more detailed information on $S_\nu(\tau_\nu)$ can be derived from limb-darkening observations, which thus provide an empirical check on the assumptions.

Wildt (1956) showed on thermodynamic grounds that S_ν cannot be equal to B_ν in those layers of a grey radiative atmosphere where the net outward flux σT_{eff}^4 is comparable to the total flux, i.e. in the visible layers of a star. The thermodynamic argument does not give a means of readily estimating the departure of S_ν from B_ν,

† Optical depth in a stellar atmosphere may be defined in terms of the attenuation of an upward-directed beam of radiation when it emerges after emission from the relevant layer: $I = I_0\,\mathrm{e}^{-\tau}$.

but this can be done on the basis of a microscopic examination of the processes of absorption and re-emission. Consider a transition between a narrow band of energy states u belonging to an upper level and another narrow band of energy states l belonging to a lower level; each level may be either continuous or discrete, and the two bands are considered to be so narrow that the variation of the emission and absorption coefficients across either band is negligible. Then the source function is immediately given in terms of the Einstein coefficients (cf. Milne 1930):

$$S_\nu \equiv \frac{j_{ul}}{4\pi\kappa_{lu}\rho} = \frac{N_u A_{ul}}{N_l B_{lu} - N_u B_{ul}} = \frac{2h\nu^3}{c^2} 1 \Big/ \left(\frac{N_l/g_l}{N_u/g_u} - 1\right), \tag{4}$$

where g_l, g_u are the statistical weights of the bands. The general expression (4) reduces to $B_\nu(T)$ if N_l/N_u has its thermal-equilibrium value; it follows that, for free–free absorption between two continuum states, $S_\nu = B_\nu(T)$, where T is the electron temperature, to a high degree of accuracy, because of the Maxwell velocity distribution. For bound-free absorption, it is convenient to introduce dimensionless non-equilibrium factors, b_i, following Menzel (1937), such that the population of the bound level i is

$$N_i = b_i N_e N_+ f(T),$$

where N_e, N_+ are the densities of electrons and positive ions (or neutral atoms if state i belongs to a negative ion) and $f(T)$ is given by substituting the local electron temperature T in the Boltzmann and Saha equations ($b_i = 1$ in l.t.e.). Hence the free-bound source function is

$$S_{ki} = \frac{2h\nu^3}{c^3} \frac{1}{b_i e^{h\nu/kT} - 1}, \tag{5}$$

where we use the suffix k to denote a continuum state. S_{ki} will thus approximate to $B_\nu(T)$ either: (i) if the statistical equilibrium of state i relative to the continuum is governed by collisional processes; or (ii) if it is governed by radiation fields which for some reason approximate to $B_\nu(T)$ in the relevant ranges of wavelength. It will be shown below that condition (i) normally holds in the case of H^-, which provides the main source of continuous opacity in stars (other than some supergiants) of spectral types between A5 and K5 (8000 °K $\geqslant T_{\text{eff}} \geqslant$ 3800 °K). For hotter stars, in which continuous absorption is mainly due to photo-ionization of neutral hydrogen, the situation is more complicated, but here it seems that (ii) is satisfied to quite a high degree of accuracy in radiative equilibrium because of the frequency of collisional interactions of highly excited levels among themselves and with the continuum. The two cases are discussed below.

2·1. *The case of* H^-

In solar-type stars, hydrogen is mainly neutral and electrons are supplied either by a small degree of ionization of hydrogen or by a high degree of single ionization of metals, which are present in the Sun with a total atomic abundance of about 10^{-4} relative to hydrogen. Typical densities in the photosphere are $N_H = 10^{16}$ to $10^{17}\,\text{cm}^{-3}$ for neutral hydrogen and $N_e = 10^{12}$ to $10^{13}\,\text{cm}^{-3}$ for electrons. The absorption and emission of radiation in the visible part of the spectrum mainly occur through photo-detachment and recombination of the H^- ion:

$$H^- + h\nu(\geqslant 0{\cdot}747\,\mathrm{eV}) \rightleftharpoons H + e,$$

while in the infrared beyond 1·6 μm, radiative transfer is due to free–free transitions of the H^- system.

The statistical equilibrium of H^- is governed by three types of interaction: radiative, collisions with electrons and collisions involving neutral particles. Calculations by Geltman (1960) show that the rate of detachment by electron collisions is negligible in comparison with that of photo-detachment in the visible layers of the Sun; but the large value of N_H/N_e suggests that collisions with neutral particles may be much more important. In 1959, A. Dalgarno pointed out that the rate coefficient for the associative detachment reaction

$$H^- + H \rightarrow H_2 + e$$

was likely to be of the order of $10^{-9{\cdot}5}\,\mathrm{cm^3\,s^{-1}}$, and this figure implies that associative detachments are more frequent than photo-detachments for optical depths τ_0 in the visible region (λ 5000) exceeding 10^{-3} (Pagel 1959), i.e. at all depths contributing appreciably to the continuum. A slightly higher rate coefficient for associative detachment was estimated afterwards by McDowell (1961).

Associative detachment and its inverse lead to a close coupling between the dissociation equilibria of H^- and H_2, so that a proper discussion of the solar continuum source function requires an investigation of the molecular equilibrium of H_2. This was carried out by Unsöld (1963), who pointed out the importance of three-body collisions

$$3H \rightarrow H + H_2$$

in this context and estimated that the rate of formation of H_2 by this reaction was so much greater than that by associative detachment from H^- that N_{H^-}/N_H is correctly given by Saha's equation over a wide range of conditions.

Advances in atomic physics in recent years have made it worthwhile to re-examine the rates of various reactions entering the H^- and H_2 equilibria (Lambert 1965; Lambert & Pagel 1968). New calculations of electron resonances in the H_2^- ion (Bardsley, Herzenberg & Mandl 1966) have enabled Dalgarno & Browne (1967) to give a revised estimate for the rate of associative detachment

$$k_H = 1{\cdot}3 \times 10^{-9}\,\mathrm{cm^3\,s^{-1}}$$

at solar photospheric temperatures, the dependence on temperature being very slight. The theoretical coefficient for $T = 300\,^\circ$K exceeds that recently measured in the laboratory (Schmeltekopf, Fehsenfeld & Ferguson 1697) by a factor of 1·5, so that an assumed value

$$k_H = 1{\cdot}0 \times 10^{-9}\,\mathrm{cm^3\,s^{-1}} \qquad (6)$$

for the Sun is unlikely to be in error by as much as a factor of 2.

For the three-body reaction, Unsöld's estimate of the rate coefficient now appears to be substantially too high in view of experimental measurements in shock tubes at temperatures between 3000 and 5000 °K (Gardiner & Kistiakowsky 1961; Patch 1962; Rink 1962; Sutton 1962; Jacobs, Giedt & Cohen 1967). These measurements still scatter over a factor 7, but a fair average would suggest a rate coefficient

$$k_{3\mathrm{H}} \simeq 10^{-32}\theta \,\mathrm{cm}^6\,\mathrm{s}^{-1}$$

(where $\theta \equiv 5040/T$) or about 1/100 of the value used by Unsöld. Other relevant reactions that we have considered are charge neutralization (Bates & Lewis 1955)

$$\mathrm{H}^- + \mathrm{H}^+ \rightarrow \mathrm{H}(1s) + \mathrm{H}(ns, p \text{ or } d)$$

and decomposition of molecules such as OH (Kaufman & Del Greco 1963)

$$\mathrm{OH} + \mathrm{H} \rightarrow \mathrm{H}_2 + \mathrm{O}.$$

However, the first of these reactions is negligible in comparison with associative detachment for stars of spectral type later than F 5, while the second is negligible in comparison with three-body collisions for spectral types earlier than K 2. It is readily shown that the rate of formation of H_2 by direct recombination falls short of that due to three-body collisions by many orders of magnitude at photospheric densities because the stabilizing transition to the ground state is highly forbidden (Gould & Salpeter 1963).

Assuming that the equilibria of H^- and H_2 are governed entirely by photoionization, associative detachment and three-body collisions, together with the inverse reactions, we suppose that the densities N_{H}, N_e and the electron temperature T are specified at some optical depth and write

$$N_{\mathrm{H}^-} = b_- N^*_{\mathrm{H}^-}, \quad N_{\mathrm{H}_2} = b_2 N^*_{\mathrm{H}_2},$$

where the 'stars' refer to densities computed assuming l.t.e. Complications introduced by the fact that associated detachment leads to the formation of H_2 in highly excited vibrational states are discussed by Lambert and Pagel (1968) and shown to have only a minor effect on the value of b_-. We now introduce dimensionless numbers α, β, W, defined by the equations

$$N^3_{\mathrm{H}} k_{3\mathrm{H}} = \alpha N^*_{\mathrm{H}^-} 4\pi \int_0^\infty \alpha_\nu B_\nu(T)\,\mathrm{d}\nu/h\nu$$

$$N_{\mathrm{H}} N_{\mathrm{H}^-} k_{\mathrm{H}} = \beta N^*_{\mathrm{H}^-} 4\pi \int_0^\infty \alpha_\nu B_\nu(T)\,\mathrm{d}\nu/h\nu$$

$$\int_0^\infty \alpha_\nu J_\nu \,\mathrm{d}\nu/h\nu = W \int_0^\infty \alpha_\nu B_\nu(T)\,\mathrm{d}\nu/h\nu,$$

where α_ν is the absorption cross-section of H^- (corrected for stimulated emission), and use the steady-state conditions for H^- and H_2 to derive the values of b_- and b_2:

$$b_- = \frac{1+\alpha+\alpha/\beta}{W+\alpha+W\alpha/\beta} \simeq \frac{1+1/\beta}{1+W/\beta}, \quad b_2 = \frac{1+\alpha+W\alpha/\beta}{W+\alpha+W\alpha/\beta} \simeq 1;$$

since $\alpha \geqslant 100$ for $\tau_0 \geqslant 10^{-3}$ in the Sun and $0{\cdot}8 \leqslant W \leqslant 1{\cdot}6$. With k_{H} given by (6), β increases from about 2 at $\tau_0 = 10^{-3}$ to 30 at $\tau_0 = 10^{-1}$, so that b_- in the photosphere ($\tau_0 > 0{\cdot}005$) and low chromosphere does not depart substantially from unity even when W does. It follows that $S_\nu = B_\nu(T)$ in the visible and infra-red continua of solar-type stars, and also that the non-l.t.e. radiative mechanism for heating the low chromosphere put forward by Cayrel (1963) can lead to only a small rise in electron temperature.

Turning to stars other than the Sun, we find that the l.t.e. condition is safely fulfilled in dwarf stars cooler than about spectral type F0($T_{\text{eff}} = 7200\,°\text{K}$). In K giants, associative detachment becomes unimportant in the very outer layers, $\tau_0 \leqslant 0{\cdot}03$, but departures from l.t.e. in these layers will not have any appreciable effect on the observable continuum from a star. For this and other reasons we disagree with the suggestion by Strom (1967) that the infra-red colour indices of K giants are affected by departures from l.t.e. (Lambert & Pagel 1968). It is more difficult to make predictions about the ultraviolet solar spectrum ($\lambda < 3600\,$Å), where the continuous absorption is due to neutral metals and possibly other, unknown, sources, rather than to H^-.

2·2. *Early-type stars with neutral hydrogen opacity*

The case of H^- is especially simple because there is only one bound level and the dissociative equilibrium is governed by collisions. In a hotter star of type A0 ($T_{\text{eff}} \simeq 10^4\,°\text{K}$) or earlier, the absorption is due to neutral hydrogen which has many energy levels and correspondingly complicated equations of statistical equilibrium. Since hydrogen is now ionized to an extent of 10 % or more (Mihalas 1965), electrons are at least comparable in abundance with neutral hydrogen atoms and consequently more effective in promoting collisional interactions; but, at energies corresponding to optical or u.v. wavelengths, these interactions are generally greatly exceeded in frequency by radiative transitions.

The problem of early-type stellar atmospheres in radiative equilibrium and in a steady state has been considered by Lecar (1964, 1965), Kalkofen (1964), Strom & Kalkofen (1966, 1967), Kalkofen & Strom (1966), Mihalas (1967*a*, *b*) and Mihalas & Stone (1968). The approach used by Lecar consisted in computing a model under the assumption of l.t.e. and then calculating steady-state atomic populations by solving the equations of statistical equilibrium for hydrogen, taking into account ten energy levels and the continuum and assuming collisional and radiative rates based on the l.t.e. model. Lecar employed experimental cross-sections σ_{12} and σ_{1k} for excitation of Lyman-α and Lyman continuum respectively measured by Fite and his collaborators (1958, 1959) and estimated cross-sections for higher levels using an approximation developed by Milford (1960) for bound–bound transitions and the classical formula of Thomson (1912) for bound-free transitions:

$$Q_{ik}(E) \equiv \frac{\sigma_{ik}}{\pi a_0^2} = \left(\frac{E_{1k}}{E_{ik}}\right)^2 \left(\frac{1}{\epsilon} - \frac{1}{\epsilon^2}\right), \tag{7}$$

where E_{ik} is the ionization energy from level i and ϵ (> 1) is the energy of the electron in units of E_{ik}. For $E_{ik} > kT$ (or 1 eV for an A0 star), Lecar derives the following formula for the ratio of collisional ionizations† to the corresponding photo-ionizations

† A general semi-empirical formula for collisional ionization rates has recently been given by Lotz (1967).

in a Planckian radiation field:

$$\frac{C_{ik}}{P_{ik}} = \frac{N_e}{4\times 10^{17}}\left(\frac{E_{1k}}{kT}\right)^{\frac{1}{2}} i^7. \tag{8}$$

At the optical depths contributing most to the visible flux, Mihalas's models show that the electron density in main-sequence A0 stars ($\log_{10} g \simeq 4$) is somewhat above $10^{14}\,\mathrm{cm}^{-3}$, so that $C_{ik}/P_{ik} > 1$ when $i > 2$, in the Paschen and higher continua; for giant stars, collisions start to dominate in the Brackett continuum. Since the line transitions almost balance out owing to high local opacity, one may therefore expect l.t.e. populations relative to $N_e N_+$ to prevail among the higher levels with $i > 3$, but departures for the Lyman and Balmer levels. Lecar (1964) in fact found negligible departures from l.t.e. in all level populations for mean optical depth $\bar{\tau} > 10^{-4}$, even when collisions were neglected altogether, but subsequent investigation taking the line transfer problem into account (Lecar 1965) showed that this result was in effect the first step of an iteration process which does not reach convergence. In other words, the l.t.e. solution is self-consistent to a certain degree of approximation, but its accuracy is not demonstrated by this method.

The reasons for the failure of this approach are connected with the large scattering terms in the line source-functions and the enormous difference in opacity or optical free path between the lines and continuum (Lecar 1965; Mihalas 1967*a*). On the other hand, the latter feature suggests an immediate simplification, since radiative transitions in the lines are virtually cancelled out by the inverse transitions in those layers that contribute significantly to the continuum (Kalkofen 1964; cf. also § 3·1). Consequently it is a good approximation to omit the radiative transitions between bound levels in the steady-state equations and by this means the iterative procedure is made convergent. Strom & Kalkofen (1966) used this method to compute a grid of models for main-sequence stars with 9000 °K < T_{eff} < 15 000 °K (spectral types A 0 to B 5) allowing departures from l.t.e. in the first three levels of hydrogen. Collisional transitions between bound levels were first treated in a schematic way by altering the bound-free rates, but afterwards they were included explicity (Kalkofen & Strom 1966) using the formula given by Seaton (1962) on the basis of an empirical adjustment of the Bethe approximation. When this was done, the departures from l.t.e. were found to be smaller than in the more schematic treatment, and quite sensitive to the collision cross-sections, but probably large enough to be measured.

A more detailed treatment of this problem has been carried out by Mihalas (1967 *a*), who uses a similar method, but allows for departures from l.t.e. in ten levels of the hydrogen atom instead of only three and estimates what are reasonably likely to be upper and lower limits to the collision cross-sections using respectively a modification of the Bethe approximation and theoretical results due to Saraph (1964). With either set of cross-sections, the predicted departures from l.t.e. are greatly reduced, apparently because collisions between bound states (especially for $\Delta n = \pm 1$) have an important effect in coupling the lower levels to the higher ones which are themselves closely coupled to the continuum. When the higher levels are assumed to have exactly their l.t.e. populations, collisional transitions between them are cancelled out by detailed balancing. Further papers (Mihalas 1967*b*; Mihalas & Stone 1968) take into account a more detailed theory of transfer in the Lyman continuum and departures from l.t.e. in He I and He II. The results are that, for a series of models with $10\,000\,^{\circ}\mathrm{K} \leqslant T_{\mathrm{eff}} \leqslant 36\,000\,^{\circ}\mathrm{K}$ and $\log_{10} g = 4$ (main sequence) or 3 (between giants and supergiants), departures from l.t.e. populations exceeding a few per cent are confined to a very thin layer, $\tau_0 < 10^{-3}$, and their effects on such measurable features as the jump in the continuous spectrum at the Balmer limit are in most cases too small to be detected observationally. (The main exception, within this range of parameters, is provided by luminous A0 giants, for which the logarithm D_B of the intensity ratio at the Balmer jump is reduced from 0·64 (l.t.e.) to 0·54 (non-l.t.e.)). In the Lyman continuum, however, b_1 can reach values of 4 or 5 at appreciable optical depths (Mihalas 1967*b*); here the exact numerical results are not final because the assumption of complete opacity in the lines may not be valid.

An observational test for the presence of departures from l.t.e. has been pointed out by Strom & Kalkofen (1967). Although the Balmer discontinuity on its own is sensitive to effective temperature and rotation, the ratio of Balmer to Paschen discontinuities is quite insensitive to these factors and chiefly depends on the departures from l.t.e. Predicted values of D_B/D_P, for $T_{\mathrm{eff}} = 10\,000\,^{\circ}\mathrm{K}$ and $\log g = 4$, are as follows: Mihalas (1967*a*)—l.t.e., 6·6; steady state, 6·5. Strom & Kalkofen (1967)—l.t.e., 5·9; steady state 4·5.

Photographic observations by Bloch & Tcheng Mao-Lin (1956) give a mean value for D_B/D_P of 5·0 for stars covering a wide range of spectral type, and this is regarded by Strom & Kalkofen as observational confirmation of the departures from l.t.e. predicted by them. This conclusion is to be treated with reserve, however, because some uncertainties arise in comparing a photographic spectral tracing with the predictions from a model, especially in the presence of overlapping lines of the Balmer series and when the jumps in intensity are small. For the particular case of Vega, both discontinuities have nearly their maximum values and they have been measured photo-electrically as well as photographically, with the following results:

photographic	$D_B = 0{\cdot}52$	(Chalonge & Divan 1952)
	$D_P = 0{\cdot}093$	(Bloch & Tcheng Mao-Lin 1956)
	$D_B/D_P = 5{\cdot}6$	
photoelectric	$D_B = 0{\cdot}54$	(Bahner 1963)
	$D_P = 0{\cdot}092$	(Hall & Williams 1942)
	$D_B/D_P = 5{\cdot}9$	

Mihalas (1966) has noted that the photoelectric value of D_B gives better agreement with a line-blanketed model in l.t.e. than the photographic value, but in neither case do the observations of D_B/D_P differ significantly from l.t.e. predictions, especially if one takes the l.t.e. figure given by Strom & Kalkofen.

The results so far can be summarized by saying that, directly or indirectly, collisions seem to ensure that l.t.e. is a good approximation in discussions of the optical and infra-red continua of main-sequence stars and most normal giants in the range of spectral types B 0 to K 5 or so. For supergiants, where atmospheric densities are low, and in the Lyman continuum (which, while unobservable in the case of most stars of the Galaxy, has an observable effect on the ionization of gaseous nebulae), departures from l.t.e. may become appreciable.

3. Absorption lines

Absorption lines occur in stellar spectra because the selective absorption acts as a blanket on radiation at line frequencies and causes it to have a reduced probability of escaping from the boundary in comparison with frequencies in the neighbouring continuum.

The intensity at some point of the line profile depends on the ratio l_ν/κ of selective to continuous opacity (we now drop the use of suffixes for continuum quantities that have negligible variation over the line profile) and on the manner in which the energy of an absorbed photon is subsequently lost to the atom. Assuming l.t.e. to apply in the continuum, the specific intensity $I_\nu(\tau, \mu)$, where μ is the cosine of the angle between the beam and the outward normal to the surface, varies with depth according to the equation of radiative transfer

$$\mu \frac{\mathrm{d}I_\nu}{\mathrm{d}\tau} = \frac{1+l_\nu}{\kappa} I_\nu - B - \frac{l_\nu}{\kappa} S_\nu, \tag{9}$$

where S_ν is now the source function for line radiation at the particular point of the profile being considered. The equation of transfer can be formally solved to yield either the emergent intensity from some point of the solar disk ($\mu = 1$ at the centre of the disk and $\mu < 1$ towards the limb):

$$I_\nu(0, \mu) = \int_0^\infty \left(B + \frac{l_\nu}{\kappa} S_\nu\right) \mathrm{e}^{-(t_\nu+\tau)/\mu} \frac{\mathrm{d}\tau}{\mu}, \tag{10}$$

where t_ν is the selective optical depth ($\mathrm{d}t_\nu = (l_\nu/\kappa)\,\mathrm{d}\tau$), or the fractional depression below the continuum:

$$R_\nu(0, \mu) = \frac{1}{I_{\text{cont}}(0, \mu)} \int_0^\infty [\Phi(\tau, \mu) - S_\nu(\tau)\, \mathrm{e}^{-\tau/\mu}]\, \mathrm{e}^{-t_\nu/\mu} \frac{\mathrm{d}t_\nu}{\mu} \tag{11}$$

$$= \int_0^\infty g(\tau, \mu)\, \mathrm{e}^{-t_\nu/\mu}\, \mathrm{d}t_\nu, \tag{12}$$

where $$\Phi(\tau, \mu) \equiv \int_\tau^\infty B(\tau')\, \mathrm{e}^{-\tau'/\mu}\, \mathrm{d}\tau'/\mu.$$

$g(\tau, \mu)$ is called the 'weighting function' and $I_{\text{cont}}(0, \mu) = \Phi(0, \mu)$. Integrated flux from a star corresponds roughly to $\mu \simeq 3^{-\frac{1}{2}}$; an exact analogue of (12) in closed form cannot be given for this case unless $t_\nu \to 0$.

The older discussions of line formation refer to two extreme cases which basically represent different assumptions about S_ν: pure absorption, or $S_\nu = B$; and coherent scattering (i.e. scattering without change of frequency), for which

$$S_\nu \simeq J_\nu \equiv \frac{1}{2}\int_{-1}^{1} I_\nu\, \mathrm{d}\mu.$$

In either case, l_ν/κ has most commonly been assumed to be given by the Boltzmann and Saha equations; together with $S_\nu = B$, this constitutes the assumption of l.t.e.

Pure absorption actually corresponds to a case in which the atom, excited through the absorption of a photon, is mostly de-excited by undergoing a superelastic collision before it has time to radiate; this situation does not normally occur in stellar atmospheres, but the assumption $S_\nu = B$ is often made for simplicity when re-radiation can occur to many levels. It will be seen below that this is a tolerably good device for getting an approximately right answer in photospheric layers, as long as it is not taken too seriously. For strong resonance lines, and for certain strong subordinate lines like $H\alpha$, the absorbed radiation is predominantly re-radiated at a nearby frequency in the same line, i.e. it is scattered; but changes in frequency due to thermal Doppler effects and collision damping make the scattering non-coherent (Holstein 1947; Woolley & Stibbs 1953; Thomas 1957, 1965*a*) and it is in fact quite a good approximation to assume that S_ν is independent of frequency across the line profile except in line wings due to natural damping (for discussion see Hummer 1968 and references cited there). We make this assumption in what follows, noting that it permits us to write for two bound levels L (lower) and U (upper).

$$S_\nu = S_{UL} = \frac{2h\nu^3}{c^2}\frac{1}{(b_L/b_U)\,\mathrm{e}^{h\nu/kT}-1} = B(T_{\mathrm{exc},\,UL}), \tag{13}$$

where $T_{\mathrm{exc},\,UL}$ is an 'excitation temperature' derived by force-fitting Boltzmann's equation to the populations of the two levels.

The formation of lines by scattering has been discussed by (among others) Eddington (1926, 1929), Milne (1930) and Strömgren (1935), all of whom assumed coherent scattering, and more recently by Thomas (1957, 1965*a*) whose approach represents a considerable advance on the earlier work and will be closely followed here. Consider first a simplified atomic model with two bound levels and a continuum. For each level there is an equation of statistical equilibrium:

$$N_L[C_{LU}+B_{LU}\bar{J}+\delta_1]-N_U[A_{UL}+B_{UL}\bar{J}+C_{UL}] = N_e N_+ R_{kL}+\delta_2, \tag{14}$$

$$-N_L[C_{LU}+B_{LU}\bar{J}]+N_U[A_{UL}+B_{UL}\bar{J}+C_{UL}+\delta_3] = N_e N_+ R_{kU}+\delta_4, \tag{15}$$

where C_{LU}, C_{UL} represent collisional excitation and de-excitation rates, A_{UL}, B_{UL}, B_{LU} are Einstein coefficients, $\bar{J}$ is the mean intensity averaged over the line profile, $\bar{J} \equiv \int l_\nu J_\nu \,\mathrm{d}\nu / \int l_\nu \,\mathrm{d}\nu$, and R_{kL}, R_{kU} are recombination rates. For the two-level atom, the δ's have the following values:

$$\delta_1 = P_{Lk}+C_{Lk}(1-1/b_L), \tag{16}$$

$$\delta_3 = P_{Uk}+C_{Uk}(1-1/b_U), \tag{17}$$

$$\delta_2 = \delta_4 = 0, \tag{18}$$

where P_{Lk}, P_{Uk} are photo-ionization rates and C_{Lk}, C_{Uk} collisional ionization rates. Solution of (14) and (15), together with (13), then yields the expression for the source function

$$S_{UL} = \frac{\bar{J}+\epsilon B(T)+\eta B^*}{1+\epsilon+\eta}, \tag{19}$$

where
$$\epsilon = \frac{C_{UL}(1-\mathrm{e}^{-h\nu/kT})}{A_{UL}} \simeq \frac{C_{UL}}{A_{UL}}, \tag{20}$$

$$\eta = \frac{\delta_3}{A_{UL}} \frac{R_{kL}+\delta_2}{R_{kL}+\delta_2+R_{kU}+\delta_4}\left[1-\frac{g_L}{g_U}\frac{\delta_1}{\delta_3}\frac{R_{kU}+\delta_4}{R_{kL}+\delta_2}\right] \simeq \frac{P_{Uk}}{A_{UL}}, \tag{21}$$

$$B^* = \frac{2h\nu^3}{c^2}\bigg/\left(\frac{R_{kL}+\delta_2}{R_{kU}+\delta_4}\frac{g_U}{g_L}\frac{\delta_3}{\delta_1}-1\right) \simeq B(T_r), \tag{22}$$

where T_r is a temperature parameter characterizing the distribution of energy in the radiation field between the two photo-ionization limits. The emergent radiation field is calculated by combining (19) with the equation of radiative transfer (9), the two being coupled together through the $\bar{J}$ term, and solving the transfer equation subject to the boundary conditions $I_\nu(0, \mu < 0) = 0$, $I_\nu(\tau, \mu) \to B(\tau)$ as $\tau \to \infty$.

The two terms $\epsilon B(T)$ and ηB^* in the numerator of (19) represent processes populating the upper level of the transition, through collisional excitations (cf. Eddington 1926) or recombinations (cf. Strömgren 1935) respectively, and are therefore called 'source terms' by Thomas, who refers to ϵ and η in the denominator, representing the converse processes, as 'sink terms' and to the scattering term, $\bar{J}$, as a 'reservoir'.

For resonance lines in, say, the solar spectrum, both ϵ and η are small but for strong lines the source and sink terms together control the value of $\bar{J}$ near the surface. Approximate values of ϵ for permitted transitions are given by a general formula due to van Regemorter (1962):

$$\epsilon = 20{\cdot}6\lambda^3 N_e T^{-\frac{1}{2}} P(\Delta E/kT), \tag{23}$$

which is based on the Bethe approximation with effective Gaunt factors deduced from experimental results discussed by Seaton (1962). The function $P(\Delta E/kT)$, representing an integrated Gaunt factor, is tabulated by van Regemorter for neutral atoms and ions: in the Sun, $\Delta E/kT \simeq 5$ at $\lambda 5000$ and $P(\Delta E/kT) \simeq 0{\cdot}04$ and $0{\cdot}20$ in the two cases, so that for neutral atoms $\epsilon \simeq 2\times10^{-3}$ if $N_e = 10^{12}$. (A semi-classical calculation by Woolley & Stibbs (1953) gives values corresponding to $P(\Delta E/kT) \simeq 0{\cdot}1$.) A corresponding formula for η can only be given in the case of hydrogenic atoms (see equation (34) below and Thomas 1965*a*), but many cases of interest can be computed in detail from the work on photo-ionization cross-sections by Burgess & Seaton (1960) and Peach (1967).

Classification of various spectral transitions according to which are the dominant source and sink terms under particular sets of physical conditions has been discussed by Thomas (1965*a*). In the solar spectrum, the Balmer lines are photoelectrically controlled in the chromosphere (for $H\alpha$, $\eta > \epsilon$ if $N_e \leqslant 2\times10^{12}$), but we would predict from equation (34) below that $\epsilon > \eta$ in the deep photospheric layers where their collision and Stark-broadened wings are formed. Other lines that have been studied —Ca^+ H and K and the Na D lines—appear to be collisionally controlled, i.e. $\epsilon \gg \eta$ and $\epsilon B \gg \eta B^*$, while Lyman α has a mixed classification ($\epsilon < \eta$, but $\epsilon B > \eta B^*$). The chief significance of this difference lies in the fact that, if $\epsilon B \gg \eta B^*$, the level population in the outer parts of the atmosphere is influenced by the local electron temperature through collisions and so, for example, a doubly reversed emission core may appear at the centre of a strong line as a consequence of high kinetic temperature in the upper chromosphere (Jefferies & Thomas 1960), whereas in the case of photoelectric control, the central intensity of the line only depends on the radiation field emerging from the photosphere. In deep layers, where the radiation fields are in any case nearly Planckian, the distinction is much less significant.

3·1. *Formation of strong absorption lines*

When the value of the selective optical depth at the line centre, t_0, is sufficiently large, line radiation is effectively imprisoned and radiative transitions are balanced locally by the inverse transitions. (How large t_0 has to be is discussed below.) Under these conditions there is radiative detailed balancing in the line and (if t_0 is sufficiently large in comparison with the scale length of variation in physical conditions; cf. Thomas 1965*a*)

$$S_{UL} = \bar{J} = \frac{\epsilon B + \eta B^*}{\epsilon + \eta}. \tag{24}$$

This situation can also be expressed by introducing the 'net radiative bracket' (Thomas 1960) which represents the departure from radiative balancing. The net downward transition rate per unit volume is

$$\begin{aligned} N_U A_{UL}[NRB]_{UL} &= N_U(A_{UL} + B_{UL}\bar{J}) - N_L B_{LU}\bar{J} \\ &= N_U A_{UL}(1 - \bar{J}/S_{UL}). \end{aligned} \tag{25}$$

For decreasing t_0, $\bar{J}$ and S steadily decrease by virtue of the transfer equation. In effect, a scattered photon has a probability of only $\frac{1}{2}$ of being scattered in the outward direction and so the population of the upper level becomes smaller and smaller towards the surface. This tendency is limited by the source terms. Taking ϵ to represent the whole source term, and neglecting the continuum, solutions of the transfer equation for a uniform isothermal atmosphere with $B = 1$ by Jefferies (quoted by Thomas 1965*a*), Avrett (1965) and Avrett & Hummer (1965) give†

$$I_0(0, \mu) \simeq S(0) \simeq \epsilon^{\frac{1}{2}}, \tag{26}$$

a result that also follows from the formula given by coherent-scattering theory (Eddington 1926, 1929). Thus for the Na D lines, $N_e \simeq 10^{11}$, $\epsilon \simeq 2 \times 10^{-4}$ and the central intensity would be between 1 and 2 % according to (26). Accurate numerical calculations (Chamaraux 1967; Mugglestone 1965) give values between 5·9 and 2·9 % for $D1$ according to whether a chromospheric temperature rise is or is not included; the observed value is 5·0 % (Waddell 1962). In each case the electron collision cross-section was based on measurements by Haft (1933) placed on an absolute scale by fitting to a Born calculation at high energy (Bates, Fundaminsky, Leech & Massey 1950; Johnson 1965). The agreement with observation is encouraging, but does not completely exclude the possibility suggested by Plaskett (1955) that neutral-atom collisions make a contribution to ϵ; such a contribution, however, is unlikely to exceed the electron contribution by an order of magnitude, and it may well be negligible.

† Methods of solving the transfer equation for non-coherent scattering are described systematically by Hummer & Rybicki (1967).

Detailed calculations of the K emission core and Lyman-α are much more difficult to carry out because too little is known about conditions in the upper chromosphere (cf. Hearn 1967). The problem is of especial interest because of the remarkable correlations that exist between the width of the K-emission core and stellar luminosity (Wilson & Bappu 1957) and between its intensity and stellar age and rotation (Wilson 1963, 1966).

Collisional de-excitations and photo-ionizations (sink terms) limit the number of times that a photon is scattered before it is destroyed and added to the pool of thermal energy of the gas (cf. Jefferies 1960). The optical path traversed by an average photon before this happens is called the 'thermalization length', Λ, and represents the optical distance over which conditions in one part of an atmosphere influence another part. Thus the condition for local opacity leading to (24) in a uniform atmosphere is $t_0 \pi^{\frac{1}{2}} > \Lambda$, while in a stratified atmosphere the condition is that the scale-length of variation (in terms of $t_0 \pi^{\frac{1}{2}}$) exceeds Λ.

Values of Λ in a uniform atmosphere (with $\epsilon \gg \eta$) have been computed by Hummer (1964), Avrett (1965) and Avrett & Hummer (1965); see also Hummer & Stewart (1966). If continuous absorption is negligible, then for a purely Doppler-broadened profile

$$\Lambda \simeq 1/\epsilon, \tag{27}$$

while for a Voigt profile with Doppler broadening and collision damping

$$\Lambda \simeq a\epsilon^{-2} \quad \text{if} \quad a > \epsilon, \tag{28}$$

where a is the damping parameter $\gamma/4\pi\Delta\nu_D$; for coherent scattering $\Lambda \simeq \epsilon^{-\frac{1}{2}}$, i.e. it is much smaller. If the continuum is important ($\pi^{\frac{1}{2}} l_0/\kappa \leqslant 1/\epsilon$), then the effective value of ϵ is increased and the source function saturates in any case for $\tau \geqslant 1$.

For a real strong line, formed in a situation where ϵ and B both increase with depth, relations (27) and (28) can give only a very rough idea. Taking $\epsilon = 10^{-3}$, $a \simeq 10^{-2}$ as representative of the inner wings of, say, the solar Na D lines, $S \to B$ for $t_0 \simeq 10^4$. The level corresponding to this has $t_\nu = 1$ (where the greatest contribution to $I_\nu(0, 1)$ arises) at an interval of 8 Doppler widths or 0·3 Å from the centre of the line, and so we expect $S \simeq B$ in computing the intensities of the line wings beyond this point, where the depression below the continuum is about 40 %.

Semi-empirical investigations of the solar source function in the sodium D-lines (Jefferies & Curtis 1965; Mugglestone 1965) and in Ca I λ6162 (Gompertz & Hindmarsh 1963) indicate $S \simeq B$ even considerably closer to the line centre than 8 Doppler widths. Since the source function in a uniform atmosphere is always less than B in the neighbourhood of the boundary, it is not surprising that the departures from l.t.e. are reduced when B is itself allowed to decrease outwards (cf. Athay & Skumanich 1967). This conclusion is confirmed by the detailed theoretical results of Johnson (1964, 1965) and of Chamaraux for the D-lines, which give $S \simeq B$ (closer than 10 %) for $t_0 \geqslant 10^2$; this corresponds to the level of formation of a part of the line profile only 2·2 Doppler widths or 0·09 Å from the line centre, where the depression is 80 %. There is no theoretical justification, however, for the view put forward by Holweger (1967) that l.t.e. holds even much closer to the centres of strong lines than this, and that the 5 % central intensity implies the presence of very low electron temperature (3900°) in the chromosphere; it seems on the contrary to require a temperature rise.

3·2. *Interlocking in the sodium D-lines*

The two-level atom treatment constitutes an oversimplification in cases such as that of the D-lines where there are two or more atomic energy levels (in this case two upper levels) very close together. In such a case, the close-lying levels are interlocked, e.g. by collisions with neutral atoms, and consequently the source functions in the two lines should be equal to each other at considerably smaller depths than that at which they both become equal to $B(T)$ (Jefferies 1960; Waddell 1962; Jefferies 1965). If we call the common lower level 1 and the two upper levels 2 and 3, then the relevant thermalization length is the distance that a photon in one of the lines travels before being converted into a photon in the other line and is given by

$$t_0 \simeq A_{21}/C_{32}, \tag{29}$$

where A, C represent radiative and collisional rates respectively and we assume $A_{21} = A_{31} \gg A_{32}$; $C_{31} \simeq C_{21} \ll C_{32}$.

Waddell (1962) tested the equality of source functions S_1, S_2 in the two D-lines by comparing the residual intensity $I_2(0,\mu)$ of D_2 at small angles from the centre of the solar disk with the corresponding quantity $I_1(0,\frac{1}{2}\mu)$ for D_1 at larger angles. For $l_\nu/\kappa \gg 1$, equation (10) shows that these two intensities are equal if the source functions $S_1(\tau)$, $S_2(\tau)$ are equal at all relevant geometrical depths, since $t_2 = 2t_1$. Waddell found that $I_2(0,\mu) = I_1(0,\mu/2)$ within the observational error of 0·1 % even at the very centres of the lines and concluded that the two source functions are identical for $t_0 \geq 0{\cdot}3$, say. The congruence of the profiles has been confirmed by Jefferies & Curtis (1965) for the D-lines and extended by Waddell (1963*a*) to the components of the Mg *b* triplet.

The significance of this result has been discussed further by Waddell (1963*b*), who concludes from a solution of the equations of radiative transfer using the Eddington approximation that (29) with $t_0 \leq 1$ is a necessary as well as a sufficient condition for equality of the source functions. The most effective contributor to C_{32} seems to be neutral hydrogen, for which cross-sections approaching $100\pi a_0^2$ or $10^{-14}\,\mathrm{cm}^2$ may occur in exciting the hyperfine transition of hydrogen (Purcell & Field 1956; Burgess, Field & Michie 1960). For the reaction

$$\mathrm{Na}(^2P_{\frac{3}{2}}) + \mathrm{H} \rightarrow \mathrm{Na}\,(^2P_{\frac{1}{2}}) + \mathrm{H}$$

which concerns us here, a preliminary calculation by Zare & Bender (1965) suggests a cross-section of about $60\pi a_0^2$. The corresponding rate coefficient is

$$k_\mathrm{H} = \sigma\bar{v} \simeq 5 \times 10^{-9}\,\mathrm{cm}^3\,\mathrm{s}^{-1}.$$

The centres of the D-lines are formed at an optical depth $\tau_0 \simeq 10^{-6}$ in the continuum, with a neutral hydrogen density $N_\mathrm{H} \simeq 10^{14}\,\mathrm{cm}^{-3}$, so that

$$C_{32} \simeq 5 \times 10^5\,\mathrm{s}^{-1}.$$

This falls short of the radiative transition probability by a factor of about 100, so that one was faced with an apparent dilemma which gave rise to some controversy

(Athay 1964; Waddell 1964). The problem appears to have been resolved by Avrett (1966), who points out that the effective conversion length is reduced by the similarity between the two source functions which exists in any case, even in the uncoupled problem, and carries out detailed calculations for a number of different assumptions as to the values of the parameters. In certain cases, the two source functions actually cross over near $t_0 = 1$, and calculations of the line profiles with $\epsilon = 10^{-4}$ and $C_{31}/A_{31} = 10^{-2}$ give perfect congruence, provided that the temperature does not increase too rapidly outwards, basically because the two lines have the same ϵ and values of t_ν at any depth differing by only a factor 2. It thus appears that existing knowledge of collision cross-sections is sufficient to account for the observations. The Ca II and Mg II doublets behave differently from the D-lines (Goldberg 1965 and subsequent discussion), because transitions involving other levels than 1, 2, 3 are of greater importance; in these cases the source functions may be proportional, but not equal (Zirker 1965).

3·3. *Weak and medium-strong lines and abundance determinations*

For most spectral lines, the two-level atomic model needs to be generalized, but the same notation can be used if the symbols η and B^* are re-interpreted by changing the significance of $\delta_1 \ldots \delta_4$ to include net rates of transition between the levels L, U of the spectral line being considered and other levels of the atom (Thomas & Athay 1961; Thomas 1965*a*). η and ηB^* now represent all indirect routes between the upper and lower levels of the line, and the δ's become

$$\delta_1 = P_{Lk} + \sum_{L<j\neq U} C_{Lj}(1-b_j/b_L) + \sum_{L>i} A_{Li}[NRB]_{Li},$$

$$\delta_2 = \sum_{L<j\neq U} N_j A_{jL}[NRB]_{jL} + \sum_{L>i} N_i C_{iL}(1-b_L/b_i),$$

$$\delta_3 = P_{Uk} + \sum_{U<j} C_{Uj}(1-b_j/b_U) + \sum_{U>i\neq L} A_{Ui}[NRB]_{Ui},$$

$$\delta_4 = \sum_{U<j} N_j A_{jU}[NRB]_{jU} + \sum_{U>i\neq L} N_i C_{iU}(1-b_U/b_i),$$

where the sums over collisional terms in δ_1, δ_3 are understood to include the continuum. The factors $(1-b_j/b_U)$, etc., convert the total rates of collisional excitation into net rates and are referred to as 'net collisional brackets'. The additional radiative and collisional terms vanish under conditions of local opacity and l.t.e. respectively.

The expression (19) for the source function is unchanged (using the new δ's), but now we may have $\eta > 1$ because of the extra radiative term in δ_3. Whether for this reason, or simply because a line is faint, S is no longer governed by radiative transfer in the line itself, but depends on what is happening in other lines and in the continuum. Furthermore, it now becomes of greater importance to consider the value of l_ν/κ, which fixes the relation between τ and t_ν and depends on N_L or b_L. (In certain cases the calculation of l_ν/κ presents no special problem because effectively all atoms of the element are in state L, e.g. for resonance lines of singly ionized metals and neutral oxygen in the Sun or of most neutral elements in M-type stars.)

The form of the equations of statistical equilibrium and radiative transfer in this general case is discussed by Kalkofen (1965). The problem is generally a non-linear one owing to the dependence of terms in the denominator of (19) on various radiation fields and level populations, and its solution requires at least approximate estimates of the net radiative and collisional rates between all levels (including the continuum). As a result of these two types of difficulty, solutions have so far only been obtained—taking into account a finite number of energy levels—for a few relatively simple cases relevant to solar and stellar abundance determinations: lithium and beryllium (Johnson 1963); sodium (Johnson 1964, 1965; Chamaraux 1967); and magnesium (A. M. Wilson 1966).

As far as photospheric levels ($\tau_0 > 0{\cdot}005$) are concerned, the results of these calculations (which mainly apply to the Sun and similar stars) show that the departures from l.t.e. are very small, so that abundance determinations from absorption lines are unlikely to be noticeably in error from this cause. This result is extended to more complex atoms by a considerable amount of observational evidence to be described below, so that it seems fair to say that the task facing future investigations into the statistical equilibria of atomic levels is to explain why there is (or appears to be) so little deviation from l.t.e. in these layers. Part of the reason arises from the fact that abundances are usually derived from equivalent widths (i.e. integrated intensities) of faint lines, or of stronger lines whose intensities are related to those of faint lines through a curve of growth which allows for saturation effects (represented by the factor $e^{-t_\nu/\mu}$ in (11)). The faint lines, and the wings of strong lines (which play a much greater part than their central cores in fixing the equivalent width), are formed at reasonably large optical depths ($0{\cdot}05 < \tau_0 < 1$) where departures from l.t.e. are always smaller than in the neighbourhood of the surface, partly because of the high opacity in strong lines and partly because the outward-decreasing temperature gradient ensures $J \simeq B$ in weak lines and in the photo-ionization continua.

A simple case illustrating some of the physical factors involved is that of a very weak resonance line due to a 'trace' element such as lithium λ 6707 (Johnson 1963). Here the two-level atom approximation is adequate and $\epsilon, \eta \ll 1$ so that the dominant term in the source function (19) is $\bar{J} = J$ (continuum); there is no transfer problem. For $0{\cdot}05 \leqslant \tau_0 \leqslant 0{\cdot}1$ in the Sun, J (6707) is exactly equal to $B(T)$ (Pierce & Waddell 1961), although deviations up to a factor of 1·5 can occur at other wavelengths. For $\lambda < 6700$, $J > B$, but $\Phi(\tau, 1) \geqslant 4B(\tau)$ for $\tau_0 = 0{\cdot}05$, so that the error in R_ν from equation (11) is considerably less than the factor 1·5; but a factor of 2 might be reached in the infra-red beyond 1 μm. One might generalize this result by saying that $S = J \simeq B$ is an adequate approximation for all faint lines if one is not too far away from the energy maximum in the continuous spectrum.

To determine l_ν/κ, we solve the equations of statistical equilibrium (14) and (15) for the two levels and derive

$$\begin{aligned}\frac{N_1}{N_e N_+} &= \frac{R_{k1} + C_{k1} + (R_{k2} + C_{k2})/(1+\eta')}{(B_{12}J + C_{12})\,\eta'/(1+\eta') + P_{1k} + C_{1k}} \\ &\simeq \frac{R_{k1} + (R_{k2} + C_{k2})}{P_{1k} + (B_{12}J + C_{12})\,\eta'}\end{aligned} \tag{30}$$

where $$\eta' = (P_{2k} + C_{2k})/(A_{21} + B_{21}\bar{J} + C_{21}) \simeq P_{2k}/A_{21} \simeq 10^{-3}.$$

The relevant rates are given by Johnson (1963) for $\tau_0 = 1$, using collision cross-sections computed by Milford (1960) and photo-ionization cross-sections from Allen (1955) and Burgess & Seaton (1960). The resulting rates are

$$R_{k1} = 2 \times 10^{-13}$$

$$R_{k2} + C_{k2} = 7 \times 10^{-14}, \quad C_{k2} \simeq \tfrac{1}{2} R_{k2}$$

$$P_{1k} = 7 \times 10^{3} \quad \text{if} \quad J = B \text{ at } \lambda\, 2300,$$

$$(B_{12} \bar{J} + C_{12})\, \eta' = 3 \times 10^{3}, \quad C_{12} \simeq 0{\cdot}1 B_{12} \bar{J}.$$

Equation (30) is thus dominated by the recombination and photo-ionization terms and the accuracy of the value of l_ν/κ computed assuming l.t.e. depends on the relation between J and B in the photo-ionization continuum near $\lambda\, 2300$. Whether collisions are included or not makes hardly any difference to the result, especially in view of the fact that conditions at $\tau_0 = 0{\cdot}1$ are more relevant than those at $\tau_0 = 1$, so that the collision rates should be reduced. Since the opacity at $\lambda\, 2300$ exceeds that at $\lambda\, 6700$ by about an order of magnitude, Johnson concludes that

$$J(\lambda\, 2300) \simeq B(\lambda\, 2300)$$

in the relevant layers; the agreement between calculated and observed solar fluxes near $\lambda\, 2300$ (Goldberg 1967) suggests that this is probably true within a factor of 2 or so. Somewhat similar situations exist for other elements that are present in such low abundance that none of their transitions affects any part of the solar radiation field. However, the ultraviolet portion of this radiation field, which largely governs the amount of the atom in the neutral state, itself depends on the ionization equilibria of the abundant elements magnesium, aluminium, calcium and silicon, which provide the continuous opacity in this spectral region. An extreme under-population of neutral states in, say, magnesium would lead to a similar under-population in the ground state of lithium, and at present we have to rely primarily on observations in order to say that this does not occur.

A somewhat more complicated case for which detailed calculations have been carried out is that of sodium, which can be represented by a model atom with five bound levels and continuum (Johnson 1964, 1965; Chamaraux 1967). Chamaraux gives a detailed discussion of the sources of atomic data employed and results for model atmospheres both with and without a chromospheric temperature rise (which affects only the central intensities of the D-lines). The method used is an iterative one, in which the relevant rates are first calculated assuming l.t.e. and then the equations of transfer are solved by the method of Fox (1957) and Feautrier (1964) to yield the next iteration. Source functions in the two D-lines are assumed to be equal. With high chromospheric temperatures, the departure factors b_n are large in the chromosphere above the height of the assumed temperature minimum near $\tau_0 = 0{\cdot}01$ because the level populations are here governed by the radiation field from the photosphere. For the same reason they fall below 1 at the temperature minimum. At $\tau_0 = 0{\cdot}1$, all the b-factors become equal to $0{\cdot}8$, rising to $1{\cdot}0$ for $\tau_0 \geqslant 1$. The main effects of the collisions are: (a) to keep the source functions equal or almost equali n the D-lines, (b) to keep these source functions equal to the Planck function in the region where the D-lines are opaque, and (c) to relate the central intensities of the D-lines to the chromospheric temperature; the other results would be affected to only a minor degree if collisions were ignored (Johnson 1964). As far as equivalent widths are concerned, departures from l.t.e. are quite negligible.

The case of magnesium is of interest because of its major contribution to the ultraviolet opacity for $\lambda < 2500$. Preliminary calculations by A. M. Wilson (1966) give results somewhat analogous to those found for sodium, but with modifications due to the presence of the weakly coupled singlet and triplet systems. A departure from l.t.e. by a factor of the order of 2 could be present in the region contributing most to the ultraviolet.

From an empirical point of view, the adequacy of the l.t.e. approximation for abundance determinations in stars of spectral types F, G and K has been tested and usually confirmed in numerous different ways:

1. *Relative intensities of faint solar lines of neutral metals having different excitation potentials* (Müller & Mutschlecner 1964; Warner 1964; Pagel 1965). Force-fitting of these intensities to a Boltzmann distribution gives a quantity sometimes referred to as the 'excitation temperature', which, however, is distinct from T_{exc} in equation (13). A comparison with the relative intensities predicted by a solar model based on continuum observations† (Goldberg, Müller & Aller 1960) is shown in figure 1, which is based on oscillator strengths of Corliss & Bozman (1962) and Corliss & Warner (1964) corrected for lines of high upper excitation potential following Warner & Cowley (1967). The agreement between the predicted and observed trends with (lower) excitation potential is remarkably good in the λ4500–6500 spectral range where data are best; this implies that the source functions are approximately equal to the Planck function (cf. Pagel 1965) and it excludes the large departures of the source functions from l.t.e. that were inferred semi-empirically by Pecker (1959), Pecker & Vogel (1960) and other authors from central intensities of weak and medium strong lines. Reasons why these source functions are in error have been discussed by Pecker & Roddier (1965) and by Cayrel (1965).

2. *Centre-limb variations in equivalent widths.* Müller & Mutschlecner (1964) found good agreement between abundances determined from equivalent widths measured at the centre of the solar disk and at $\mu = 0{\cdot}5$ and $0{\cdot}3$ near the limb. Since the limb observations represent shallower layers of the photosphere than those carried out at the centre of the disk, one would expect any departures from l.t.e. to be noticeably greater.

3. *Comparison of abundances determined from different states of ionization.* Such comparisons have been carried out for Ti I and Ti II (Teplitskaya 1964) and for Fe I and Fe II (Aller, O'Mara & Little 1964; Warner 1967) and give good agreement. This result has recently been extended to several other elements of the iron group

† This model assumes no temperature rise in the chromosphere. While the actual extent of such a rise is rather uncertain at the present time, it is clear from the foregoing discussion that the rise should be ignored in any attempt to predict line intensities using an l.t.e. theory.

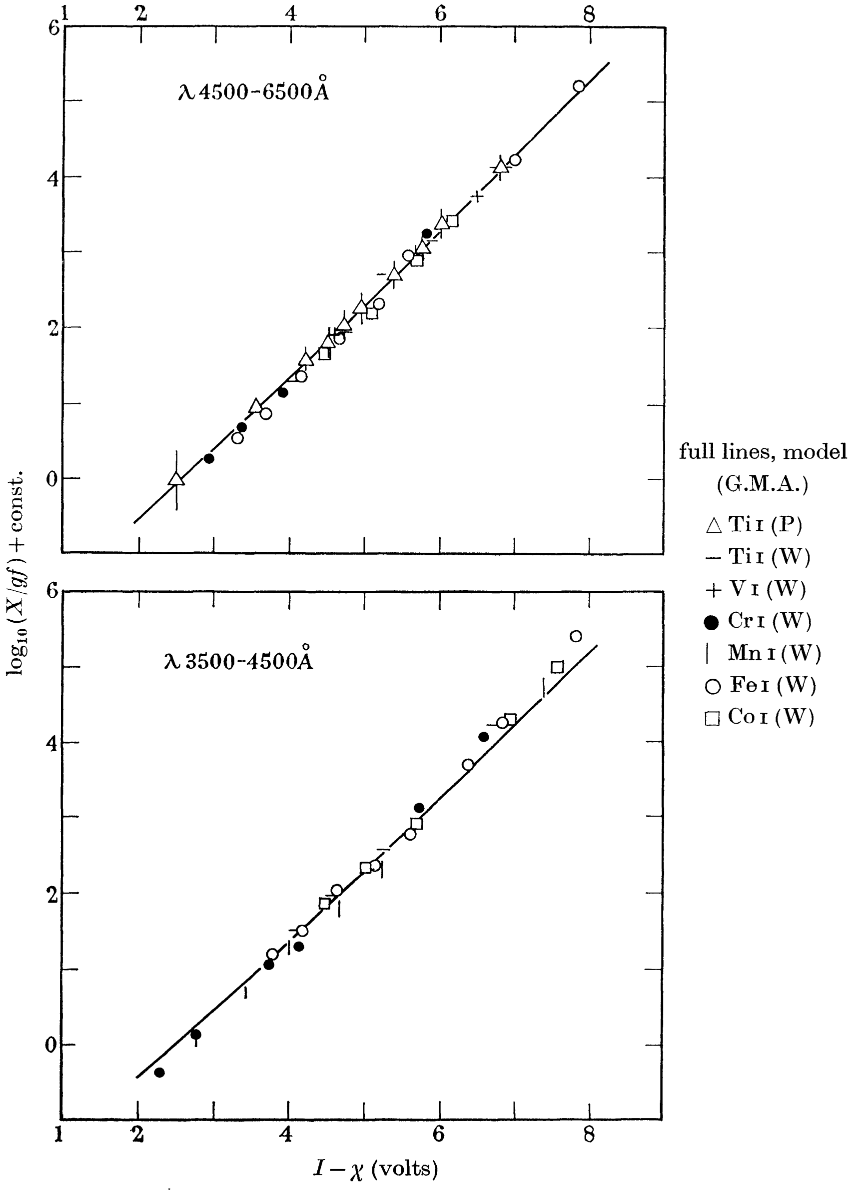

FIGURE 1. Dependence of strengths of solar absorption lines of neutral iron-group elements on excitation potential. Logarithms of X/gf, where X is equal to the equivalent width in units of the wavelength for an unsaturated line, plotted against $I-\chi$, the difference between ionization potential and lower excitation potential. The plotted points are based on results due to Warner (1964: 'W') and Pagel (unpublished: 'P'), the latter being shown with error bars. Oscillator strengths are from Corliss & Bozman (1962) and Corliss & Warner (1964), but Corliss & Bozman's correction for high upper excitation potentials has been removed from the published values in accordance with the conclusions of Warner & Cowley (1967). The theoretical curves, based on the model of Goldberg, Müller & Aller (1960: 'G.M.A.') assuming l.t.e., are taken from data given by Goldberg & Pierce (1959) and are virtually identical for all of these metals, apart from a vertical shift. The curves approximate quite closely to straight lines with slope $\theta_{\text{exc}} = 0{\cdot}96$ (the 'excitation temperature').

by Warner (1968). This confirms the l.t.e. estimates of the degree of ionization (as well as of the source function) within the accuracy of the scales of absolute f-values.

4. *Agreement of solar abundance ratios of* C:N:O derived from both atomic and molecular lines (Lambert 1968) with those determined from solar cosmic rays (Biswas, Fichtel & Guss 1966).

5. *Comparison of abundances* determined on the assumption of l.t.e. among stars in a cluster which are expected to have identical chemical composition. Conti, Wallerstein & Wing (1965) found identical abundances in this manner for main-sequence stars of the Hyades with spectral types ranging from A3 to K0.

6. *Variation of excitation temperature with effective temperature for G and K stars*. Pagel (1966) has compared observed excitation temperatures (measured differentially relative to the Sun) with those predicted by Cayrel & Jugaku (1963) using model atmospheres based on a simple scaling of the solar $\theta(\tau_0)$ relation and finds tolerable over-all agreement except in the case of certain supergiants. Minor deviations can be explained by minor departures either from l.t.e. or from the scaled models.

There are also some outstanding discrepancies in observational results, of which the principal ones are the following:

7. *Excitation temperatures of metallic lines in early A-type stars* Wright (1967) has shown that the excitation temperatures found from relative intensities of Fe I lines in early A-type stars are anomalously low. It is quite possible that this result is due to departures from l.t.e., though there may also be other explanations.

8. *Comparison of photospheric and coronal abundances of iron*. Pottasch (1964*a*) has deduced solar abundances from rocket spectra of the extreme ultraviolet region which show certain discrepancies with those deduced from photospheric absorption lines, notably in the case of iron, for which the coronal abundance comes out to be about 20 times higher. In disagreement with this result, Dupree & Goldberg (1967) have shown that existing ultraviolet observations are consistent with photospheric abundances of oxygen, silicon and iron when dielectronic recombination and certain other factors are taken into account; but a high abundance of iron can also be deduced from the intensities of forbidden lines of Fe ions relative to the electron-scattered continuum in the visible and near infra-red (Pottasch 1964*b*) and in this case the interpretation seems to be more straightforward. A similar discrepancy exists in the case of meteorites (Urey 1967).

9. *Forbidden* [Fe II] *absorption lines*. Swings (1965) has identified [Fe II] lines in absorption on the solar disk and finds, using transition probabilities calculated by Garstang (1962), that their intensities lead to an iron abundance 10 times higher than is deduced from permitted lines of Fe I and Fe II and approximating to the 'coronal' value. A similar discrepancy exists in the case of Arcturus (Gasson & Pagel 1966).

In view of observations 1–6 above, and especially the agreement between permitted lines of Fe I and Fe II, it seems very unlikely that there could be errors of a factor of 10 in the photospheric iron abundance owing to departures from l.t.e.

As far as the corona is concerned, it is not impossible that iron is concentrated in the outer layers from which the forbidden lines are observed by thermal diffusion processes (Seaton 1964), while the [Fe II] lines are extremely faint and their intensities can perhaps have been overestimated in both the Sun and Arcturus. The discrepancies, therefore, do not alter our previous conclusion as to the utility of the l.t.e. approximation in F to K stars. In hotter stars, no detailed checks are available, but Mihalas's results for the continuum, and the large values of ϵ predicted for the visual range at high temperatures by van Regemorter (1962), suggest that departures will not be large for lines that are formed reasonably deep in the atmosphere. However, this condition is not fulfilled either by the lines of neutral metals or in some cases by the strong lines of helium, silicon, etc., that are prominent in the spectra of B-type stars (Underhill 1966).

3·4. *The blanketing effect of absorption lines*

In a star of given effective temperature T_{eff}, the presence of absorption lines forces up the level of the continuum in between (Milne 1923); if the continuum is formed in l.t.e., one can describe this effect by saying that the atmosphere is warmed up by radiation sent back from the selectively absorbing layers. The continuum between the lines then resembles that from a star with effective temperature T'_{eff} where

$$T^4_{\text{eff}} \simeq (1-w)\,T'^4_{\text{eff}}, \tag{31}$$

w being the fraction of the total energy that is removed by the lines. This 'back-warming' effect is a direct result of the conservation of energy, and it occurs whatever the mechanism of re-emission in the lines.

When literally pure absorption occurs, i.e. the absorbing atom is de-excited by superelastic collisions, line blanketing has the additional effect of lowering the temperature close to the boundary below the value that would have occurred in the presence of continuous absorption alone. This is because, in the radiative equilibrium condition that emission = absorption, assuming l.t.e.,

$$\int(\kappa_\nu B_\nu + l_\nu S_\nu)\,\mathrm{d}\nu = \int(\kappa_\nu + l_\nu)\,B_\nu\,\mathrm{d}\nu = \int(\kappa_\nu + l_\nu)\,J_\nu\,\mathrm{d}\nu, \tag{32}$$

J_ν takes on particularly small values where $\kappa_\nu + l_\nu$ is particularly large. It is shown by Cayrel (1966) that the r.h.s. of equation (32)

$$\begin{aligned}\int(\kappa_\nu + l_\nu)\,J_\nu\,\mathrm{d}\nu &\simeq \int\kappa_\nu J_\nu\,\mathrm{d}\nu + \tfrac{1}{2}\int l_\nu B_\nu(T'_0)\,\mathrm{d}\nu\\ &\simeq \int\kappa_\nu B_\nu(T_0)\,\mathrm{d}\nu + \tfrac{1}{2}\int l_\nu B_\nu(T'_0)\,\mathrm{d}\nu\end{aligned}$$

where T'_0, T_0 are the values of the boundary temperature with and without line blanketing respectively and $\int\kappa_\nu J_\nu\,\mathrm{d}\nu$ is altered but little by the presence of lines in view of (31). Hence (32) becomes

$$\int(\kappa_\nu + \tfrac{1}{2}l_\nu)\,B_\nu(T'_0)\,\mathrm{d}\nu \simeq \int\kappa_\nu B_\nu(T_0)\,\mathrm{d}\nu,$$

and so $T'_0 < T_0$, e.g. for the Sun, $T_0 \simeq 4750°$ and $T'_0 \simeq 3400°$ if pure absorption is assumed (Böhm 1954).

If, on the other hand, the lines are formed by scattering, equation (32) is replaced by the condition

$$\int(\kappa_\nu B_\nu + l_\nu S_\nu)\,\mathrm{d}\nu = \int(\kappa_\nu B_\nu + l_\nu J_\nu)\,\mathrm{d}\nu = \int(\kappa_\nu + l_\nu)\,J_\nu\,\mathrm{d}\nu$$

or

$$\int\kappa_\nu B_\nu\,\mathrm{d}\nu = \int\kappa_\nu J_\nu\,\mathrm{d}\nu$$

as in the absence of lines. The distinction between the effects of the two kinds of re-emission on the boundary temperature was first pointed out by Chandrasekhar (1935) and it is expressed physically by saying that it is only through the effect of inelastic and superelastic collisions between the absorbing atoms and the surrounding electron gas that the low radiation intensity in the lines can affect the electron temperature.

Although, as has been shown, the l.t.e. calculation provides a good approximation to the intensities of solar absorption lines, this is certainly not due to the presence of a high degree of collisional coupling between the atoms and the gas: the predominant mechanisms are non-coherent scattering (for resonance lines) and fluorescence (for subordinate lines), the latter being for the present purpose much more like scattering than like pure absorption. The coupling due to photo-ionization from the upper level is probably insignificant in comparison with continuous absorption. Taking into account only the scattering and collisional terms in the expression for the line source function (19), the radiative equilibrium condition becomes

$$\int\left(\kappa_\nu+\frac{l_\nu\,\epsilon}{1+\epsilon}\right)B_\nu(T_0')\,\mathrm{d}\nu=\int\left(\kappa_\nu+l_\nu\,\frac{\epsilon}{1+\epsilon}\right)J_\nu\,\mathrm{d}\nu, \qquad (33)$$

and the lowering of the boundary temperature is greatly reduced in comparison with the pure absorption case (Cayrel 1964, 1966; Thomas 1965*b*). From the use of van Regemorter's formula for ϵ, it appears that only the strong lines of Mg II and Ca II contribute significantly and that $T_0' > 4400°$ (Frisch 1966). Whether the minimum solar temperature actually falls short of 4750° by an amount that can be attributed to line blanketing is not clear at the present time.

In very cool stars, where the main contribution to the blanketing effect is due to vibration-rotation transitions of the H_2O molecule (Auman 1967), it is possible that ϵ is quite large owing to the small spacing between the relevant energy levels and that therefore the lowering of the boundary temperature predicted on the basis of pure absorption really occurs. It would be valuable if estimates of ϵ such as have been made for electron collisions by van Regemorter (1962) could be extended to collisions between molecules and other molecules or neutral hydrogen atoms.

4. Emission lines in the spectra of early-type stars

Following Beals (1934, 1940), it is often assumed that emission lines in the spectra of hot stars of the Of, Wolf-Rayet, nova, P Cygni and Be types are largely due to the Zanstra fluorescence mechanism which occurs in gaseous nebulae: photo-ionization by ultraviolet stellar radiation in an extended envelope, followed by recombination to excited as well as ground levels. Further contributions come from scattering when the envelope is moving (Sobolev 1947; Rottenberg 1952) and in some cases from I. S. Bowen's selective fluorescence process. These mechanisms are plausible in view of the actual appearance of a nebular shell in the later stages of novae and in view of the presence in many cases of selective excitation effects consistent with the presence of a diluted stellar radiation field (for a review see Underhill 1966). On the other hand, certain difficulties in the application of this hypothesis to Wolf-Rayet stars, where the selective effects are usually absent, have led to the suggestion that collisional excitation effects may be of primary importance, as in the upper solar chromosphere (Thomas 1947, 1949; Münch 1950). In either case, the conditions of excitation and ionization are far removed from l.t.e.

Physical conditions in Wolf-Rayet atmospheres are rather uncertain, but several estimates exist with regard to the expanding shells round P Cygni stars and the rotating shells round Be stars in which emission lines arise (Wellmann 1952; Boyarchuk 1958; Kogure 1959, 1961; Pagel 1960). Typically one has an electron density $N_e \simeq 10^{11}$ to $10^{12}\,\mathrm{cm}^{-3}$ and stellar radiation geometrically diluted by a factor $W \simeq 0{\cdot}1$. If there is radiative equilibrium, then the electron temperature of a pure hydrogen nebula is of the same order as or slightly less than the effective temperature of the star (assumed to radiate like a black body with temperature T_{eff}) if collisional effects are ignored (Baker, Menzel & Aller 1938) and is limited to a value of the order of 20 000 °K or less if they are included (Mestel 1954; Hummer 1963), because collisional ionizations followed by emission of radiation act as a kind of thermostat. It thus seems reasonable to take $T = T_{\mathrm{eff}} = 20\,000$ °K as a representative situation in what follows.

The hydrogen and helium spectra in certain emission-line stars like γ Cas and P Cygni seem to be explicable to a considerable extent by the same sort of statistical equilibrium theory as was developed by Baker & Menzel (1938) for hydrogen in gaseous nebulae (Wellmann 1952; Burbidge & Burbidge 1955; Pagel 1960), since their envelopes are largely transparent—even in at least the higher Balmer lines—as a result of the high velocity of expansion. In this case (assuming opacity in the Lyman lines), equations (8) and (23) indicate that the statistical equilibria of the sixth and higher levels are largely governed by collisional transitions to other levels and to the continuum, so that one expects $b_n \simeq 1$ for $n \geqslant 6$. Essentially the same result is obtained for these high levels if only radiative transitions are taken into account (Pagel 1960). For the lower states, on the other hand, radiative transitions dominate the statistical equilibrium, as in the nebular case, despite the fact that the electron density is about 10^7 times higher in the circumstellar envelopes. The main exception to this statement is that the high-rate coefficient for the $2s \rightarrow 2p$ transition due to proton collisions (Seaton 1955) will ensure that degenerate sublevels are populated in accordance with their statistical weights.

For most Be stars, there is self-absorption in the Balmer lines (Burbidge & Burbidge 1955; Kogure 1959, 1961; Pottasch 1961) and so there is a problem of radiative transfer coupled with the statistical-equilibrium equations, similar to that posed by absorption lines (but complicated further by the motion of the envelope). For large optical depth in the Balmer lines, therefore, we investigate whether the source function (19) is controlled by the collisional term (ϵ) or the photo-ionization term (η). We take the approximate form

$$\eta \simeq P_{Uk}/A_{UL},$$

justifying the two-level approximation by assuming that all net radiative and collisional brackets are small; this is likely to be nearly enough true to give the right order of magnitude for η. Taking $\alpha = \alpha_0(\lambda/\lambda_0)^3$, where α_0, λ are the threshold photo-ionization cross-section and wavelength respectively, and assuming stimulated emission factors to have their l.t.e. values, we have

$$\eta \simeq \frac{8\pi W \alpha_0 c}{\lambda_0^3 A_{UL}} E_1(x_0), \tag{34}$$

where $x_0 \equiv hc/\lambda_0 kT_{\text{eff}}$ and $E_1(x)$ is the first exponential integral function. α_0 and A_{UL} are taken from Allen (1955) and ϵ is deduced from van Regemorter's formula, giving the following results:

$$\begin{aligned} H\alpha, \quad & \epsilon = 0{\cdot}004(N_e/10^{12}), \quad \eta = 0{\cdot}025(W/0{\cdot}1) \\ H\beta, \quad & \left.\begin{aligned}\epsilon &= 0{\cdot}001(N_e/10^{12}), \\ \epsilon &= 0{\cdot}2(N_e/10^{12}),\end{aligned}\right\} \eta = 0{\cdot}092(W/0{\cdot}1), \\ \text{Paschen } \alpha, \quad & \end{aligned}$$

etc. Thus $\eta > \epsilon$ for the Balmer lines if $(N_e/W) < 6 \times 10^{13}$ and $\eta B^* > \epsilon B(T)$ if, in addition, $T \leqslant T_{\text{eff}}$; interlocking effects are unimportant if $(N_e/W) < 5 \times 10^{12}$. For this reason, the treatment of the transfer problem for $H\alpha$ and $H\beta$ by Pottasch (1961), in which collisions are taken into account, yields values for the Balmer decrement under typical conditions that do not differ greatly from those computed by Kogure (1959) neglecting collisions. For extreme values of the dilution factor, photo-ionizations become negligible and $H\beta$ quanta are mostly degraded into $H\alpha$ and $P\alpha$; Pottasch shows that this trend is limited by collisions when $W < 0{\cdot}01$ (with $N_e = 10^{11}$), but such an extreme situation does not seem to be reached in the actual cases. Hence the neglect of collisions in the treatment of the Balmer-line transfer problem by Sobolev and Kogure (and in that of the optically thin case by Pagel 1960) does not have a marked effect on the results; the presence of a strong ionizing radiation field in P Cygni and Be envelopes compensates for the increase in density as compared with a gaseous nebula and makes the situation rather different from those prevailing in either the solar chromosphere or supernova remnants. A complete treatment of radiative transfer in a moving stellar envelope remains an interesting and difficult problem.

This article has been influenced by the severe criticisms of an earlier version by Dr R. N. Thomas, who is not, however, to be held responsible for any of the views expressed here. I am grateful to Dr D. L. Lambert for his collaboration on the H^- problem and to Professor M. J. Seaton, F.R.S., for his interest and encouragement

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