Planning and Scheduling: CSP – Constraint Satisfaction Problems

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Acknowledgements

- These slides refer to Chapter 5 of the textbook:
  S. Russell and P. Norvig:
  Artificial Intelligence: A Modern Approach
  Prentice Hall, 2003, 2nd Edition (or more recent edition)
- These slides are an adaptation of slides by Min-Yen Kan
- The contributions of these authors are gratefully acknowledged.

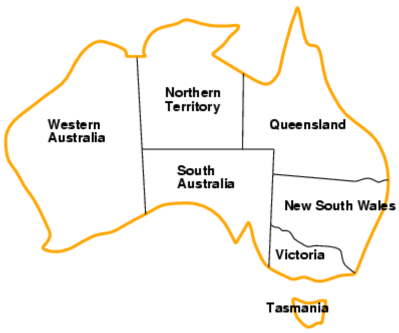
Outline

- Constraint Satisfaction Problems (CSP)
- Backtracking search for CSPs
- Local search for CSPs

Constraint Satisfaction Problems (CSPs)

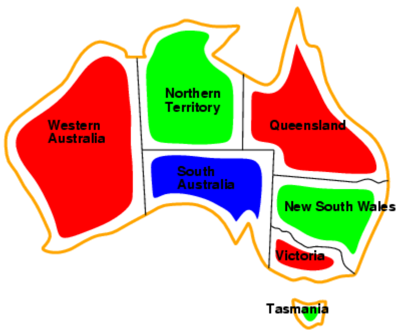
- Standard search problem:
  - state is a "black box“ – any data structure that supports
    - successor function,
    - heuristic function, and
    - goal test
- CSP:
  - state is defined by **variables** X_i with **values** from **domain** D_i
  - goal test is a set of **constraints** specifying allowable combinations of values for subsets of variables
- Simple example of a **formal representation language**
- Allows useful **general-purpose algorithms** with more power than standard search algorithms

Example: Map-Coloring



- Variables WA, NT, Q, NSW, V, SA, T
- Domains Di = {red,green,blue}
- Constraints: adjacent regions must have different colors
- e.g., WA ≠ NT, or (WA,NT) in {(red,green),(red,blue),(green,red),(green,blue),(blue,red),(blue,green)}

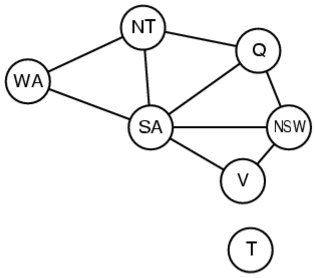
Example: Map-Coloring



- Solutions are complete and consistent assignments, e.g., WA = red, NT = green, Q = red, NSW = green, V = red, SA = blue, T = green

Constraint Graph

- Binary CSP: each constraint relates two variables
- Constraint graph: nodes are variables, arcs are constraints



Varieties of CSPs

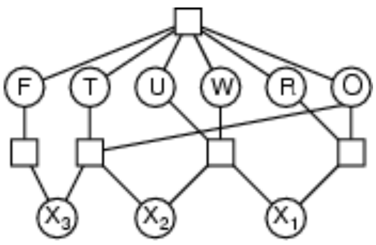
- Discrete variables
  - finite domains:
    - n variables, domain size d → $O(d^n)$ complete assignments
    - e.g., Boolean CSPs, incl. Boolean satisfiability (NP-complete)
  - infinite domains:
    - integers, strings, etc.
    - e.g., job scheduling, variables are start/end days for each job
    - need a constraint language, e.g., StartJob1 + 5 ≤ StartJob3
- Continuous variables
  - e.g., start/end times for Hubble Space Telescope observations
  - linear constraints solvable in polynomial time by linear programming

Varieties of Constraints

- **Unary** constraints involve a single variable,
  - e.g., SA ≠ green
- **Binary** constraints involve pairs of variables,
  - e.g., SA ≠ WA
- **Higher-order** constraints involve 3 or more variables,
  - e.g., crypto-arithmetic column constraints

Example: Cryptarithmetic

$$\begin{array}{cccc} & T & W & O \\ + & T & W & O \\ \hline F & O & U & R \end{array}$$



- Variables: F T U W R O X1 X2 X3
- Domains: {0,1,2,3,4,5,6,7,8,9}
- Constraints:
  - Alldiff (F,T,U,W,R,O)
  - O + O = R + 10 · X1
  - X1 + W + W = U + 10 · X2
  - X2 + T + T = O + 10 · X
  - X3 = F, T ≠ 0, F ≠ 0

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Real-World CSPs

- Assignment problems
  - e.g., who teaches what class
- Timetabling problems
  - e.g., which class is offered when and where?
- Transportation scheduling
- Factory scheduling

- Notice that many real-world problems involve real-valued variables

Standard Search Formulation (Incremental)

- Let's start with the straightforward approach, then fix it
- States are defined by the values assigned so far
  - **Initial state**: the empty assignment { }
  - **Successor function**: assign a value to an unassigned variable that does not conflict with current assignment
    - → fail if no legal assignments
  - **Goal test**: the current assignment is complete
- This is the same for all CSPs
- Every solution appears at depth n with n variables
  → use depth-first search
- Path is irrelevant, so can also use complete-state formulation
- b = (n - l)d at depth l, hence n! · d^n leaves

Backtracking Search

- Variable assignments are **commutative**, i.e.,
  - [WA = red then NT = green] same as [NT = green then WA = red]
- Only need to consider assignments to a single variable at each node
  - → b = d and there are d^n leaves
- Depth-first search for CSPs with single-variable assignments is called backtracking search
- Backtracking search is the basic uninformed algorithm for CSPs
- Can solve n-queens for n ≈ 25

Backtracking Search

```
function BACKTRACKING-SEARCH( csp) returns a solution, or failure
   return RECURSIVE-BACKTRACKING({}, csp)
function RECURSIVE-BACKTRACKING( assignment, csp) returns a solution, or failure
   if assignment is complete then return assignment
   var ← SELECT-UNASSIGNED-VARIABLE( Variables[csp], assignment, csp)
   for each value in ORDER-DOMAIN-VALUES(var, assignment, csp) do
      if value is consistent with assignment according to Constraints[csp] then
         add { var = value } to assignment
         result ← RECURSIVE-BACKTRACKING(assignment, csp)
         if result ≠ failue then return result
         remove { var = value } from assignment
   return failure
```

Backtracking Example

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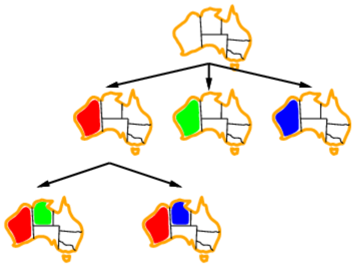

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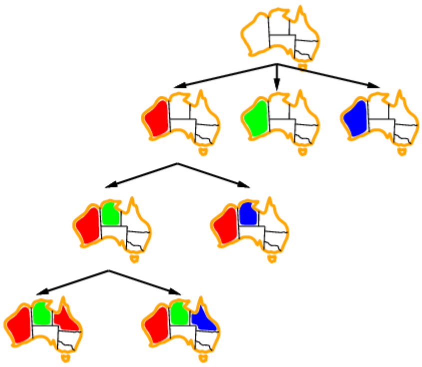
Backtracking Example



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Improving Backtracking Efficiency

- General-purpose methods can give huge gains in speed:
  - Which variable should be assigned next?
  - In what order should its values be tried?
  - Can we detect inevitable failure early?

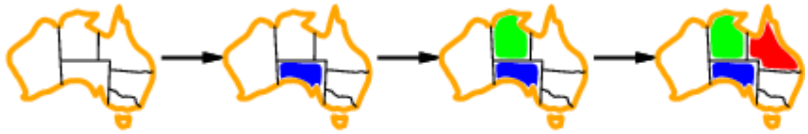
Most Constrained Variable

- Most constrained variable:
  - choose the variable with the fewest legal values

- a.k.a. minimum remaining values (MRV) heuristic

Most Constraining Variable

- Tie-breaker among most constrained variables
- Most constraining variable:
  - choose the variable with the most constraints on remaining variables



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Least Constraining Value

- Given a variable, choose the least constraining value:
  - the one that rules out the fewest values in the remaining variables

- Combining these heuristics makes 1000 queens feasible

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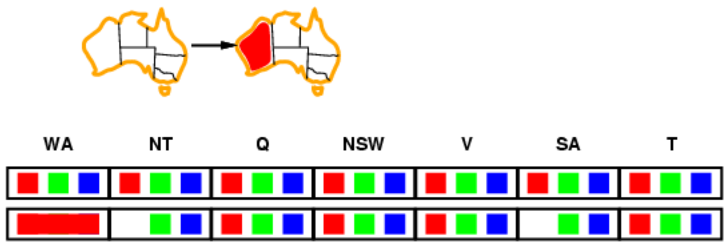
Forward Checking

- Idea:
  - Keep track of remaining legal values for unassigned variables
  - Terminate search when any variable has no legal values



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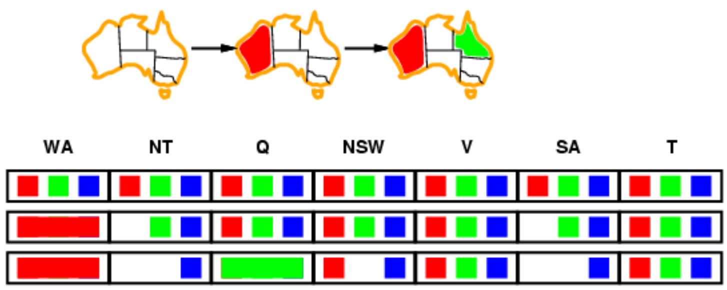

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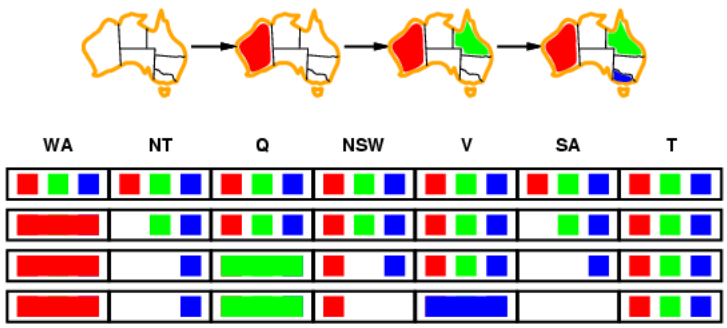
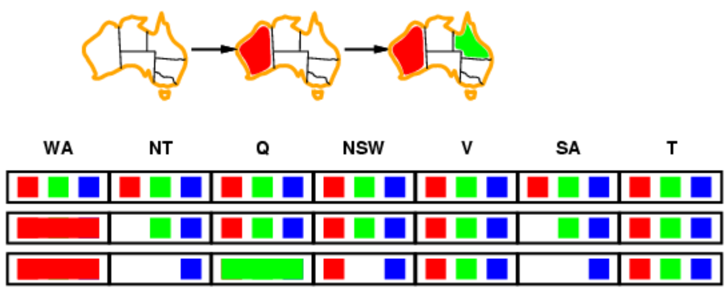


Constraint Propagation

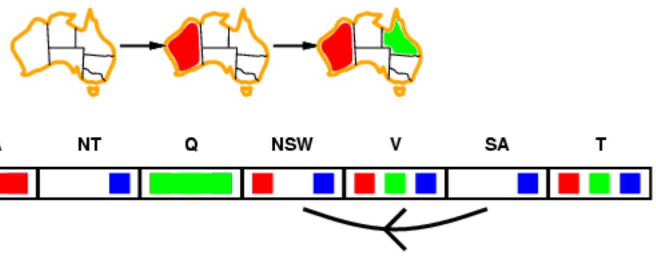
- Forward checking propagates information from assigned to unassigned variables, but doesn't provide early detection for all failures:



- NT and SA cannot both be blue!
- Constraint propagation repeatedly enforces constraints locally

Arc Consistency

- Simplest form of propagation makes each arc **consistent**
- X → Y is consistent iff
  for **every** value x of X there is **some** allowed y



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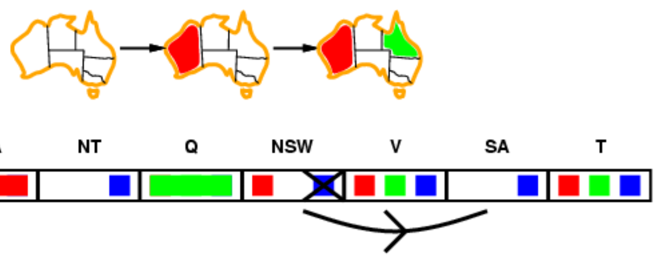

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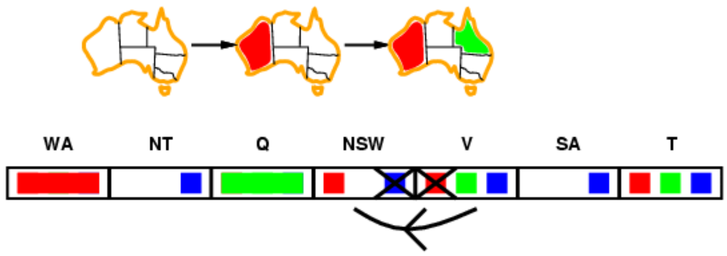

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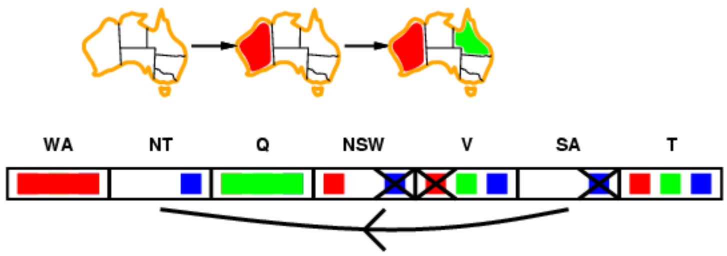


- If X loses a value, neighbors of X need to be rechecked
- Arc consistency detects failure earlier than forward checking
- Can be run as a preprocessor or after each assignment

Arc Consistency Algorithm AC-3

function AC-3(*csp*) **returns** the CSP, possibly with reduced domains
  inputs: *csp*, a binary CSP with variables $\{X_1, X_2, \ldots, X_n\}$
  local variables: *queue*, a queue of arcs, initially all the arcs in *csp*

  while *queue* is not empty **do**
    $(X_i, X_j) \leftarrow$ REMOVE-FIRST(*queue*)
    if RM-INCONSISTENT-VALUES(X_i, X_j) **then**
      for each X_k **in** NEIGHBORS[X_i] **do**
        add (X_k, X_i) to *queue*

---

function RM-INCONSISTENT-VALUES(X_i, X_j) **returns** true iff remove a value
  removed $\leftarrow$ *false*
  for each x **in** DOMAIN[X_i] **do**
    if no value y in DOMAIN[X_j] allows (x,y) to satisfy constraint(X_i, X_j)
      then delete x from DOMAIN[X_i]; *removed* $\leftarrow$ *true*
  return *removed*

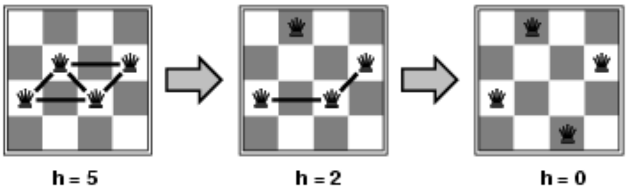
- Time complexity: $O(n^2d^3)$

Local Search for CSPs

- Hill-climbing, simulated annealing typically work with "complete" states,
  i.e., all variables assigned
- To apply to CSPs:
  - allow states with unsatisfied constraints
  - operators **reassign** variable values
- Variable selection: randomly select any conflicted variable
- Value selection by **min-conflicts** heuristic:
  - choose value that violates the fewest constraints
  - i.e., hill-climb with h(n) = total number of violated constraints

Example: 4-Queens

- States: 4 queens in 4 columns (4^4 = 256 states)
- Actions: move queen in column
- Goal test: no attacks
- Evaluation: h(n) = number of attacks



- Given random initial state, can solve n-queens in almost constant time for arbitrary n with high probability (e.g., n = 10,000,000)

Summary

- CSPs are a special kind of problem:
  - states defined by values of a fixed set of variables
  - goal test defined by constraints on variable values
- Backtracking = depth-first search with one variable assigned per node
- Variable ordering and value selection heuristics help significantly
- Forward checking prevents assignments that guarantee later failure
- Constraint propagation (e.g., arc consistency) does additional work to constrain values and detect inconsistencies
- Iterative min-conflicts is usually effective in practice