# 作业三

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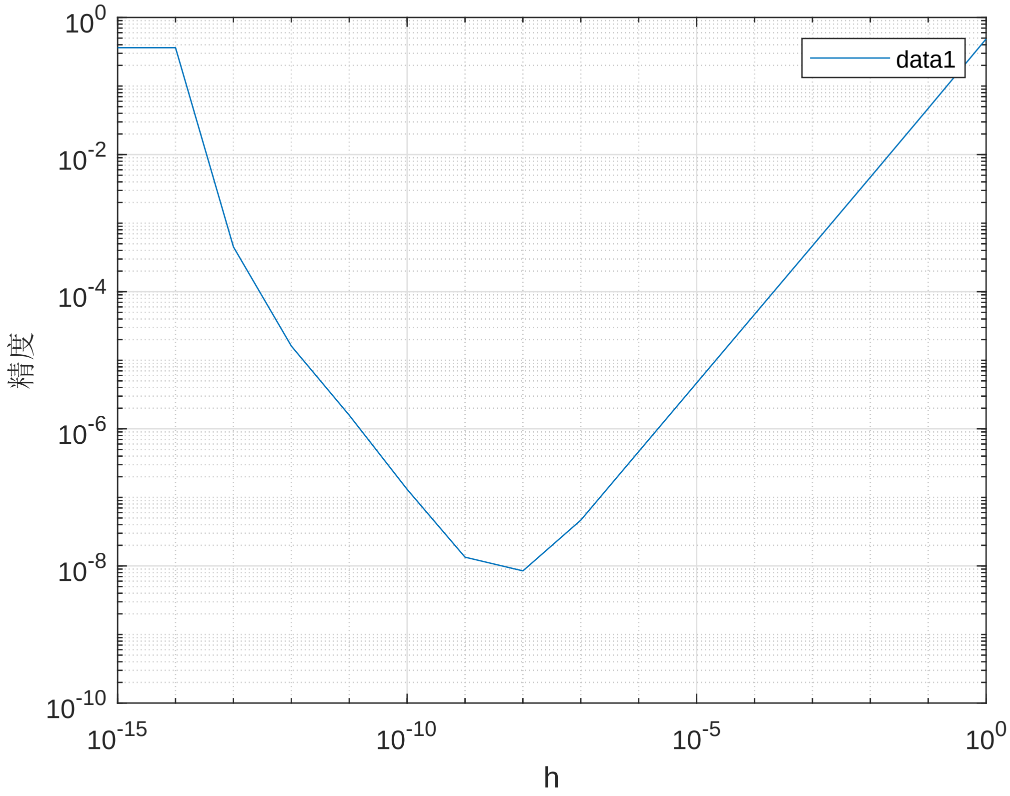
第一题 (a)

MATLAB 程序显示如下：

```
clear,clc
for k = 0:15
    h = 10^(-k);
    fx = fdiff(1.2,h);
    labels(k+1,1) = h;
    labels(k+1,2) = abs(fx-cos(1.2));
end
hchange(labels);

function fx = fdiff(x0,h)
syms x;
f(x) = sin(x);
fx = (f(x0+h)-f(x0))/h;
end
function hchange(labels)
loglog(labels(:,1),labels(:,2));
% 记录横轴纵轴的数据画图
xlabel('h');
ylabel('精度');
```

```
grid on;
legend
end
```




(b)

$$f(x_0) = f(x_0) \tag{1}$$

$$f(x_0 + h) = f(x_0) + hf'(x_0) + \tfrac{h^2}{2!}f''(x_0) + O(h^3) \tag{2}$$

由 ((2) - (1)) / h 得:

$\Rightarrow f'(x_0) = \frac{f(x_0+h)-f(x_0)}{h} - \frac{h}{2!}f''(x_0) + O(h^2)$

记 $N_1(h) = \frac{f(x_0+h)-f(x_0)}{h}$,

$$f'(x_0) = N_1(h) - \tfrac{h}{2!}f''(x_0) + O(h^2) \tag{3}$$

$$f'(x_0) = N_1(\tfrac{h}{2}) - \tfrac{1}{2}(\tfrac{h}{2!})f''(x_0) + O(h^2) \tag{4}$$

2*(4) - (3) 得:

$$f'(x_0) = 2N_1(\tfrac{h}{2}) - N_1(h) + O(h^2) \approx N_2(h)$$

$$N_2(h) = 2N_1(\tfrac{h}{2}) - N_1(h)$$

继续外推：

for j = 2 : n

$$N_j(h) = 2N_{j-1}(\frac{h}{2}) - N_{j-1}(h)$$

$$f'(x_0) = N_j(\frac{h}{2})$$

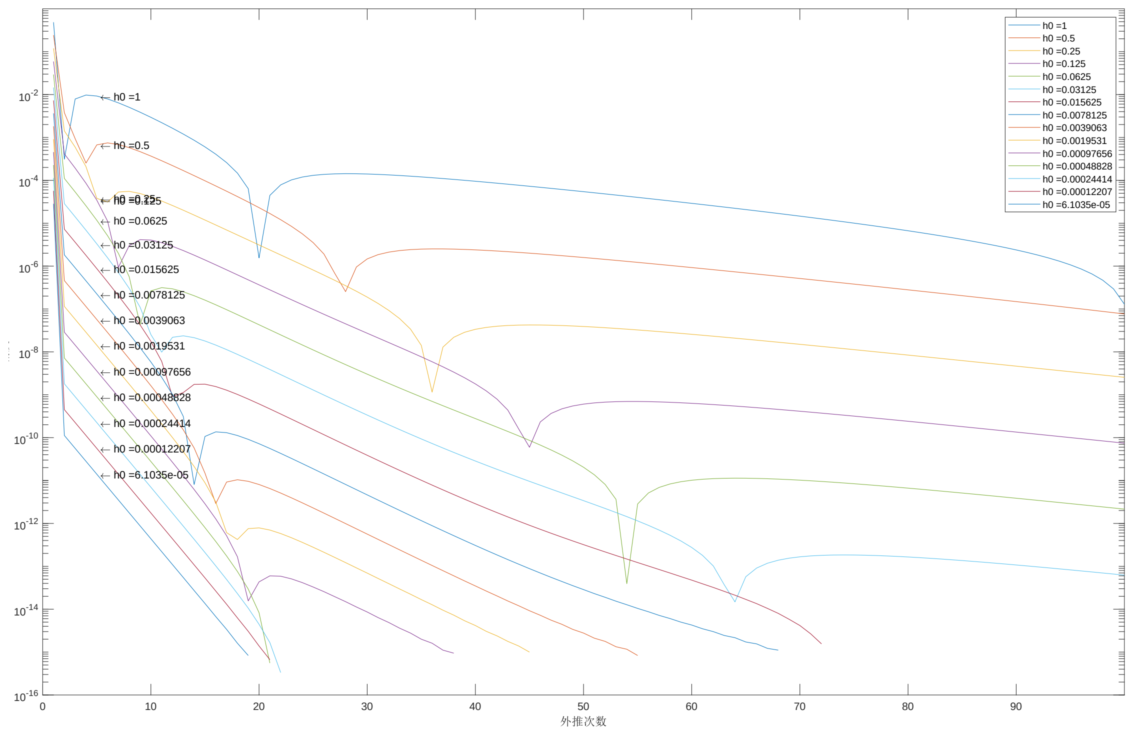
end

(c)

Matlab 程序显示如下：

```
for n = 0:14
    disp(n);
    h0 = 2^(-n);
    nearest = RFD(h0);
    N(n+1,1) = nearest;
    hold on
end
N
function nearest = RFD(h0)
eps = 1e-15;
%设置误差界
syms x;
syms h;
x0 = 1.2;
f(x) = sin(x);
N(h) = (f(x0+h)-f(x0))/h;
k0 = 1;
d = double(abs(N(h0)-N(h0/2)));
fx = double(N(h0));
labels(1,1) = 1;
labels(1,2) = abs(fx - cos(1.2));
for k0 = 2: 100
    if d<=eps || abs(fx - cos(1.2))<=eps
        break;
    else
        N(h) = 2*N(h/2)-N(h);
        d = double(abs(N(h0)-N(h0/2)));
```

```
        fx = double(N(h0));
        labels(k0,1) = k0;
        labels(k0,2) = abs(fx - cos(1.2));
        labels(k0,3) = fx;
    end
end
[Y,I] = min(labels,[],1);
nearest = labels(I(1,2),3);
name1 = ['h0 =',num2str(h0)];
semilogy(labels(:,1),labels(:,2),'DisplayName',name1);
% 记录横轴纵轴的数据画图
name2 = [' \leftarrow ',name1]; %w 的数字指向线
text(5,labels(5,2),name2);
xlabel('外推次数');
ylabel('精度');
legend
end
```




N = 0.362357883861802 0.362357831111555 0.362357755629859 0.362357754416836

0.362357754476634 0.362357754476688 0.362357754476672 0.362357754476673 0.362357754476674 0.362357754476675 0.362357754476673 0.362357754476674 0.362357754476674 0.362357754476673 0.362357754476673

第二题 (a)

$h = \frac{b-a}{h} = \frac{2\pi}{m}$

$I = \sum_{k=0}^{n} A_k f(x_k)$

$T_n = \frac{h}{2}\sum_{k=1}^{n}(f(x_k)+f(x_{k-1})) = \frac{h}{2}(f(a)+f(b)+2\sum_{k=1}^{n-1} f(x_k))$

$E_{T_n} = I - I_n = -\frac{h^3}{12}\sum_{k=0}^{n-1} f''(\eta_k) = -\frac{b-a}{12}h^2 f''(\eta)$

由此，带入已知条件：

(1)

$\Rightarrow |E| = |\frac{2\pi}{12}\frac{(2\pi)^2}{m^2}(cosr\eta)^{(2)}| = |\frac{3\pi^3}{4m^2}r^2 cosr\eta| \leq \frac{3\pi^3 r^2}{4m^2}$

$r = km$ 时，$|E| \leq \frac{3\pi^3 k^2}{4}$

(2)

$\Rightarrow |E| = |\frac{2\pi}{12}\frac{(2\pi)^2}{m^2}(sinr\eta)^{(2)}| = |\frac{3\pi^3}{4m^2}r^2 sinr\eta| \leq \frac{3\pi^3 r^2}{4m^2}$

$r = km$ 时，$|E| \leq \frac{3\pi^3 k^2}{4}$

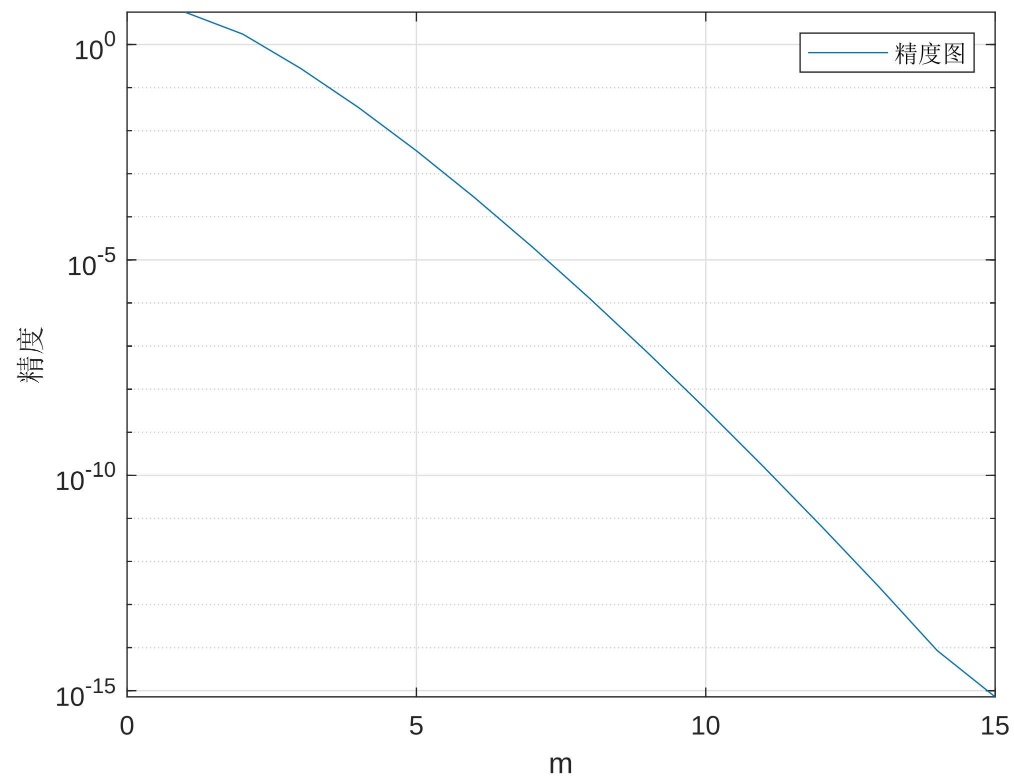
故，误差都有界，因此能提高精度。

MATLAB(b) 程序显示如下：

```
err = 1;
m = 1;
while err>= 1e-15
    disp(m);
    err = intg(m);
    labels(m,1) = m;
    labels(m,2) = err;
    m = m+1;
end
errors(labels);

function err = intg(m)
syms x;
f(x) = exp(cos(x));
h = 2*pi/m;
```

```
T = 0;
for k = 1:m
    T = T + h/2*(f(-pi+k*h)+f(-pi+(k-1)*h));
end
ext = 7.954926521012845274513219665329394328161342771816638573400595955383360608164694666995137357228568774;
      %用Wolfram alpha的在线计算器算得
err = abs(ext - T);
end
function errors(labels)
semilogy(labels(:,1),labels(:,2),'DisplayName','精度图');
% 记录横轴纵轴的数据画图
xlabel('m');
ylabel('精度');
grid on;
legend
```




第三题 (a)

令 $y'(t)=f(t,y(t))$，

$\Rightarrow y(t_n)=y(t_{n-p})+\int_{t_{n-p}}^{t_n} f(t,y(t))dt$

用 Lagrange 插值多项式近似 $f(t,y(t))$，

$y(t_n)=y(t_{n-p})+\int_{t_{n-p}}^{t_n} L_{n-q}^{n}(t)dt$

其中，$L_{n-q}^{n}(t)dt=\sum_{k=n-q}^{n} f_k l_k(t)$

$f_k=f(t_k,y_k)$

$\Rightarrow y_n=y_{n-p}+\sum_{k=n-q}^{n} f_k\int_{t_{n-p}}^{t_n} l_k(t)dt$

当 p = 2, n = n+1 时，

$y_{n+1}=y_{n-1}+f_{n-1}\int_{t_{n-1}}^{t_n+1} l_{n-1}(t)dt+f_n\int_{t_{n-1}}^{t_n+1} l_n(t)dt+f_{n+1}\int_{t_{n-1}}^{t_n+1} l_{n+1}(t)dt$

其中，记步长为 $h$，

$\int_{t_{n-1}}^{t_n+1} l_{n-1}(t)dt=\int_{t_{n-1}}^{t_n+1}\frac{(t-t_n)(t-t_{n+1})}{(t_{n-1}-t_n)(t_{n-1}-t_{n+1})}dt=\frac{1}{3}h$

$\int_{t_{n-1}}^{t_n+1} l_n(t)dt=\int_{t_{n-1}}^{t_n+1}\frac{(t-t_{n-1})(t-t_{n+1})}{(t_n-t_{n-1})(t_n-t_{n+1})}dt=\frac{4}{3}h$

$\int_{t_{n-1}}^{t_n+1} l_{n+1}(t)dt=\int_{t_{n-1}}^{t_n+1}\frac{(t-t_{n-1})(t-t_n)}{(t_{n+1}-t_{n-1})(t_{n+1}-t_n)}dt=\frac{1}{3}h$

$\Rightarrow y_{n+1}=y_{n-1}+\frac{h}{3}f_{n+1}+\frac{4h}{3}f_n+\frac{h}{3}f_{n-1}$

(b)

若 $y_{n-1}=y(x_{n-1}),y_n=y(x_n),y_{n+1}=y(x_{n+1})$，则有：

$y_{n+1}=y(x_{n-1})+\frac{h}{3}f_{n+1}+\frac{4h}{3}f_n+\frac{h}{3}f_{n-1}$

依微分方程有：

$y_{n+1}=y(x_{n-1})+\frac{h}{3}(y'(x_{n+1})+4y'(x_n)+y'(x_{n-1}))$

$T_{n+1}=y(x_{n+1})-y_{n+1}=y(x_{n+1})-y(x_{n-1})-\frac{h}{3}(y'(x_{n+1})+4y'(x_n)+y'(x_{n-1}))$

$y(x_{n+1})=y(x_n)+hy'(x_n)+\frac{h^2}{2!}y''(x_n)+\frac{h^3}{3!}y'''(x_n)+O(h^4)$

$y(x_{n-1})=y(x_n)-hy'(x_n)+\frac{h^2}{2!}y''(x_n)-\frac{h^3}{3!}y'''(x_n)+O(h^4)$

$y'(x_{n+1})=y'(x_n)+hy''(x_n)+\frac{h^2}{2!}y'''(x_n)+O(h^3)$

$y'(x_{n-1})=y'(x_n)-hy''(x_n)+\frac{h^2}{2!}y'''(x_n)-O(h^3)$

带入得，

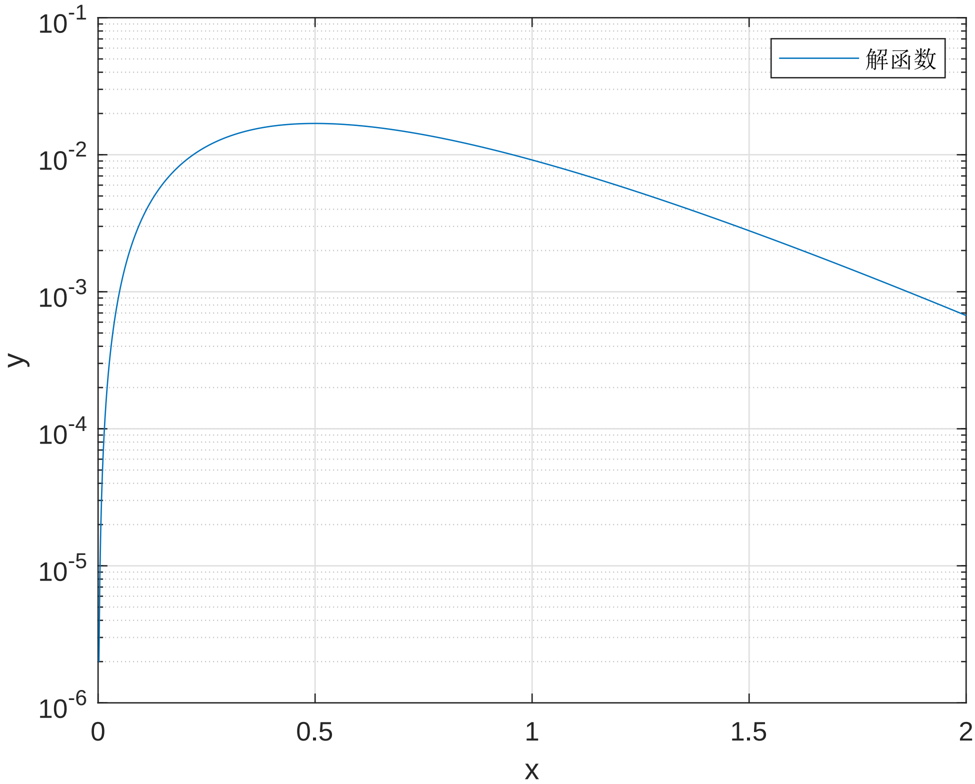
$T_{n+1}=O(h^4)$

$\Rightarrow$ 三阶格式

MATLAB（c）程序显示如下：

```
clear,clc
syms x;
syms y;
f(x,y) = x*exp(-4*x) - 4*y;
h = 2/1000;
%步长2/1000
y0 = 0;
k1 = f(0,0);
y1 = y0 + h*f(h/2,h/2 * k1);
k11 = f(h,h);
y2 = y1 + h*f(h+h/2,y1+h/2 * k11);
labels(1,1) = 0;
labels(1,2) = y0;
labels(2,1) = h;
labels(2,2) = y1;
labels(3,1) = h+h;
labels(3,2) = y2;
for k = 4:1001
    yk = labels(k-2,2) + h/3 *(7*f((k-2)*h,labels(k-1,2))-
         2*f((k-3)*h,labels(k-2,2))+ f((k-4)*h,labels(k-3,
         2)));
    yk = labels(k-2,2) + h/3 *(f((k-1)*h,yk) + 4*f((k-2)*h,
         labels(k-1,2)) + f((k-3)*h,labels(k-2,2)));
    labels(k,1) = (k-1)*h;
    labels(k,2) = yk;
    disp(k);
end
fx(labels);
function fx(labels)
semilogy(labels(:,1),labels(:,2),'DisplayName','解函数');
% 记录横轴纵轴的数据画图
xlabel('x');
ylabel('y');
grid on;
legend
```

```
end
```




(d)

由题：

$\int y'dx = \int(xe^{-4x} - 4y)dx$

$\Rightarrow y = -\frac{1}{16}e^{4x}(4x+1) - 4yx + C$

带入 y(0) = 0 得：

$C = \frac{1}{16}$

$\Rightarrow y = \frac{1}{16}e^{-4x} + \frac{1}{16}(4x+1)^{-1}$

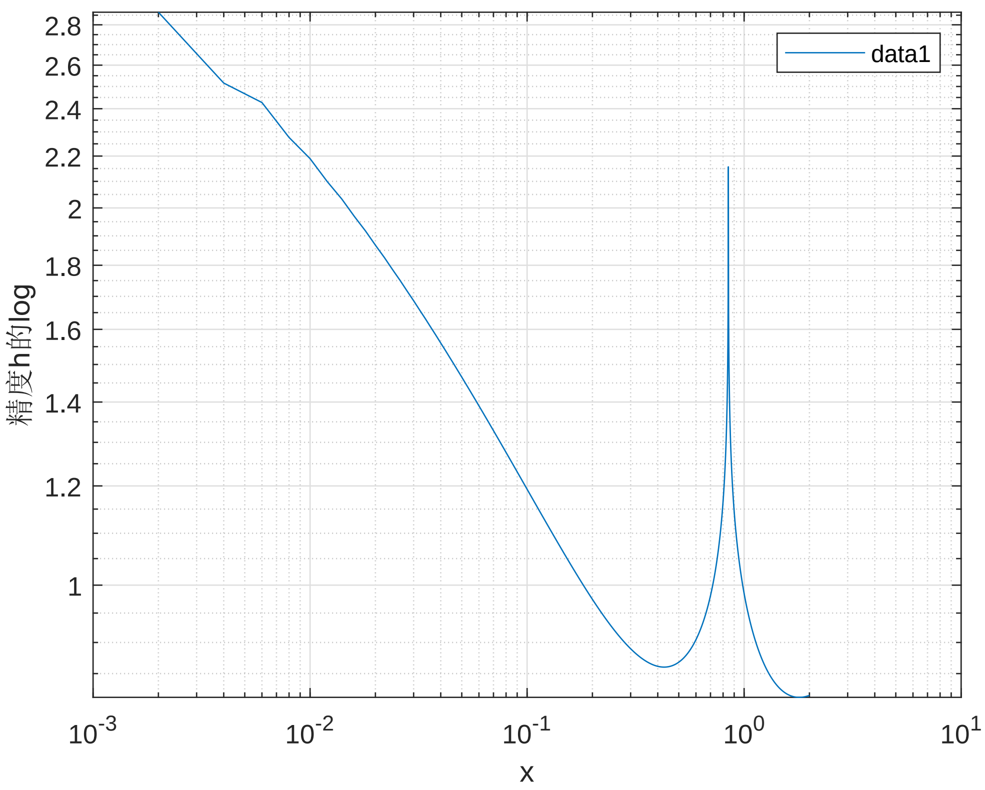
在 x=2 这一点

err = 0.006252051077894

Matlab（d）程序显示如下：

```
f(x) = -1/16 * exp(-4*x) + 1/16 * 1/(4*x + 1);
yk = double(yk);
err = double(abs (yk-f(2)));
```

```
format long
err
for k = 1:1001
    er(k,1) = labels(k,1);
    er(k,2) = log(abs(labels(k,2) - f((k-1)*h)))/log(h);
end
erro(er);
function erro(labels)
loglog(labels(:,1),labels(:,2));
% 记录横轴纵轴的数据画图
xlabel('x');
ylabel('精度h的log');
grid on;
legend
end
```




由于每一步需要用自身的 y 点进行迭代，但自身又是估计得出，因此精度和阶数达不到预期。