Contents

Weeks 3–4: Calculus Topics **1**
1. Limits and Continuity . . . 1
2. Differentiation (Single and Multivariable) . . . 3
2. Differentiation (Single and Multivariable) . . . 6
3. Partial Derivatives, Gradient, Divergence, and Curl . . . 8
4. Integration (Definite and Indefinite) . . . 11
5. Multiple Integrals . . . 14
6. Line and Surface Integrals . . . 17
7. Taylor Series Expansions . . . 20
8. Optimization (Unconstrained and Constrained) . . . 22
9. Ordinary Differential Equations (ODEs) . . . 25

Weeks 3–4: Calculus Topics

1. Limits and Continuity

Definition of Limits

A limit is the value that a function approaches as the input approaches a specific point. Mathematically, for a function $f(x)$, the limit as x approaches c is written as:

$$\lim_{x \to c} f(x) = L$$

where:

- x is the input variable,
- c is the point the input is approaching, and
- L is the value the function approaches.

Step-by-Step Example

Example: Find the limit of $f(x) = 2x + 3$ as $x \to 2$.

Step 1: Write the function and the target point.

$$\begin{aligned} f(x) &= 2x + 3, \\ x &\to 2. \end{aligned}$$

Step 2: Substitute $x = 2$ into the function.

$$\begin{aligned} f(2) &= 2(2) + 3 \\ &= 4 + 3 \\ &= 7. \end{aligned}$$

Conclusion: The limit is:

$$\lim_{x \to 2}(2x+3) = 7.$$

Left-Hand and Right-Hand Limits

The limit of a function may depend on the direction from which x approaches c:

- **Left-hand limit:** $\lim_{x \to c^-} f(x)$, where x approaches c from the left ($x < c$).
- **Right-hand limit:** $\lim_{x \to c^+} f(x)$, where x approaches c from the right ($x > c$).

Example: Determine the left- and right-hand limits for $f(x) = |x|$ as $x \to 0$.

Step 1: Define the function.

$$f(x) = \begin{cases} -x & \text{if } x < 0, \\ x & \text{if } x \geq 0. \end{cases}$$

Step 2: Calculate the left-hand limit ($x \to 0^-$). For $x < 0$, $f(x) = -x$. Substitute $x \to 0^-(x \to 0$ from the left):

$$\begin{aligned} \lim_{x \to 0^-} f(x) &= \lim_{x \to 0^-}(-x) \\ &= -(0) \\ &= 0. \end{aligned}$$

Step 3: Calculate the right-hand limit ($x \to 0^+$). For $x \geq 0$, $f(x) = x$. Substitute $x \to 0^+(x \to 0$ from the right):

$$\begin{aligned} \lim_{x \to 0^+} f(x) &= \lim_{x \to 0^+}(x) \\ &= 0. \end{aligned}$$

Step 4: Check if the left-hand and right-hand limits are equal.

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) = 0.$$

Conclusion: The limit exists, and:

$$\lim_{x \to 0} |x| = 0.$$

Continuity

A function $f(x)$ is continuous at a point c if:

1. $f(c)$ is defined,
2. $\lim_{x \to c} f(x)$ exists, and
3. $\lim_{x \to c} f(x) = f(c)$.

Example: Is $f(x) = \frac{1}{x}$ continuous at $x = 0$?
Step 1: Check if $f(0)$ is defined.

$$f(0) = \frac{1}{0}, \text{ which is undefined.}$$

Conclusion: The function is not continuous at $x = 0$ because $f(0)$ is not defined.

2. Differentiation (Single and Multivariable)

Definition of Differentiation

Differentiation is the process of finding the derivative of a function. The derivative measures the rate at which a function changes as its input changes. For a function $f(x)$, the derivative is defined as:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h},$$

where:

- $f'(x)$ is the derivative of $f(x)$,
- h is a small change in x, and
- $\lim_{h \to 0}$ ensures the change is infinitesimally small.

Step-by-Step Example (Single Variable)

Example: Find the derivative of $f(x) = x^2$ using the definition of differentiation.
Step 1: Write the definition of the derivative.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

Step 2: Substitute $f(x) = x^2$ into the formula.

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h}. \\ &= \lim_{h \to 0} \frac{x^2 + 2xh + h^2 - x^2}{h}. \\ &= \lim_{h \to 0} \frac{2xh + h^2}{h}. \\ &= \lim_{h \to 0} (2x + h). \end{aligned}$$

Step 3: Take the limit as $h \to 0$.

$$f'(x) = 2x.$$

Conclusion: The derivative of $f(x) = x^2$ is $f'(x) = 2x$.

Product Rule

The product rule is used to find the derivative of the product of two functions. For functions $u(x)$ and $v(x)$, the product rule states:

$$\frac{d}{dx}[u(x)v(x)] = u'(x)v(x) + u(x)v'(x).$$

Example: Find the derivative of $f(x) = (x^2)(\sin x)$.
Step 1: Identify $u(x)$ and $v(x)$.

$$u(x) = x^2, \qquad v(x) = \sin x.$$

Step 2: Differentiate $u(x)$ and $v(x)$.

$$u'(x) = 2x, \qquad v'(x) = \cos x.$$

Step 3: Apply the product rule.

$$\begin{aligned} f'(x) &= u'(x)v(x) + u(x)v'(x) \\ &= (2x)(\sin x) + (x^2)(\cos x). \end{aligned}$$

Conclusion: The derivative is:

$$f'(x) = 2x \sin x + x^2 \cos x.$$

Chain Rule

The chain rule is used to differentiate composite functions. For $y = f(g(x))$, the chain rule states:

$$\frac{dy}{dx} = f'(g(x))g'(x).$$

Example: Find the derivative of $y = \sin(x^2)$.
Step 1: Identify the outer and inner functions.

$$f(u) = \sin u, \qquad g(x) = x^2.$$

Step 2: Differentiate $f(u)$ and $g(x)$.

$$f'(u) = \cos u, \qquad g'(x) = 2x.$$

Step 3: Apply the chain rule.

$$\begin{aligned} \frac{dy}{dx} &= f'(g(x))g'(x) \\ &= \cos(x^2)(2x). \end{aligned}$$

Conclusion: The derivative is:

$$\frac{dy}{dx} = 2x\cos(x^2).$$

Multivariable Differentiation

For a multivariable function $f(x, y)$, the partial derivative with respect to x is:

$$\frac{\partial f}{\partial x} = \lim_{h\to 0} \frac{f(x+h, y) - f(x, y)}{h},$$

and the partial derivative with respect to y is:

$$\frac{\partial f}{\partial y} = \lim_{h\to 0} \frac{f(x, y+h) - f(x, y)}{h}.$$

Step-by-Step Example (Multivariable)

Example: Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ for $f(x, y) = x^2y + 3xy^2$.
Step 1: Compute $\frac{\partial f}{\partial x}$. Treat y as a constant:

$$\begin{aligned} \frac{\partial f}{\partial x} &= \frac{\partial}{\partial x}(x^2y + 3xy^2) \\ &= 2xy + 3y^2. \end{aligned}$$

Step 2: Compute $\frac{\partial f}{\partial y}$. Treat x as a constant:

$$\begin{aligned} \frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}(x^2y + 3xy^2) \\ &= x^2 + 6xy. \end{aligned}$$

Conclusion: The partial derivatives are:

$$\begin{aligned} \frac{\partial f}{\partial x} &= 2xy + 3y^2, \\ \frac{\partial f}{\partial y} &= x^2 + 6xy. \end{aligned}$$

2. Differentiation (Single and Multivariable)

Definition of Differentiation

Differentiation is the process of finding the derivative of a function. The derivative measures the rate at which a function changes as its input changes. For a function $f(x)$, the derivative is defined as:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h},$$

where:

- $f'(x)$ is the derivative of $f(x)$,
- h is a small change in x, and
- $\lim_{h \to 0}$ ensures the change is infinitesimally small.

Step-by-Step Example (Single Variable)

Example: Find the derivative of $f(x) = x^2$ using the definition of differentiation.

Step 1: Write the definition of the derivative.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

Step 2: Substitute $f(x) = x^2$ into the formula.

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h}. \\ &= \lim_{h \to 0} \frac{x^2 + 2xh + h^2 - x^2}{h}. \\ &= \lim_{h \to 0} \frac{2xh + h^2}{h}. \\ &= \lim_{h \to 0} (2x + h). \end{aligned}$$

Step 3: Take the limit as $h \to 0$.

$$f'(x) = 2x.$$

Conclusion: The derivative of $f(x) = x^2$ is $f'(x) = 2x$.

Multivariable Differentiation

For a multivariable function $f(x, y)$, the partial derivative with respect to x is:

$$\frac{\partial f}{\partial x} = \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h},$$

and the partial derivative with respect to y is:

$$\frac{\partial f}{\partial y} = \lim_{h \to 0} \frac{f(x, y+h) - f(x, y)}{h}.$$

Step-by-Step Example (Multivariable)

Example: Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ for $f(x, y) = x^2 y + 3xy^2$.

Step 1: Compute $\frac{\partial f}{\partial x}$. Treat y as a constant:

$$\begin{aligned}\frac{\partial f}{\partial x} &= \frac{\partial}{\partial x}(x^2 y + 3xy^2) \\ &= 2xy + 3y^2.\end{aligned}$$

Step 2: Compute $\frac{\partial f}{\partial y}$. Treat x as a constant:

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}(x^2 y + 3xy^2) \\ &= x^2 + 6xy.\end{aligned}$$

Conclusion: The partial derivatives are:

$$\frac{\partial f}{\partial x} = 2xy + 3y^2,$$
$$\frac{\partial f}{\partial y} = x^2 + 6xy.$$

3. Partial Derivatives, Gradient, Divergence, and Curl

Partial Derivatives

Partial derivatives measure how a multivariable function changes as one variable changes, keeping the others constant. For a function $f(x, y)$, the partial derivatives are defined as:

$$\frac{\partial f}{\partial x} = \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h},$$
$$\frac{\partial f}{\partial y} = \lim_{h \to 0} \frac{f(x, y+h) - f(x, y)}{h}.$$

Example: Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ for $f(x, y) = x^2 y + 3xy^2$.
Step 1: Compute $\frac{\partial f}{\partial x}$. Treat y as constant:

$$\begin{aligned}\frac{\partial f}{\partial x} &= \frac{\partial}{\partial x}(x^2 y + 3xy^2) \\ &= 2xy + 3y^2.\end{aligned}$$

Step 2: Compute $\frac{\partial f}{\partial y}$. Treat x as constant:

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}(x^2 y + 3xy^2) \\ &= x^2 + 6xy.\end{aligned}$$

Conclusion: The partial derivatives are:

$$\frac{\partial f}{\partial x} = 2xy + 3y^2,$$
$$\frac{\partial f}{\partial y} = x^2 + 6xy.$$

Gradient

The gradient of a scalar function $f(x, y, z)$ is a vector that points in the direction of the greatest rate of increase of f. It is defined as:

$$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right).$$

Example: Find the gradient of $f(x, y, z) = x^2 y + yz^3$.
Step 1: Compute $\frac{\partial f}{\partial x}$.

$$\frac{\partial f}{\partial x} = 2xy.$$

Step 2: Compute $\frac{\partial f}{\partial y}$.

$$\frac{\partial f}{\partial y} = x^2 + z^3.$$

Step 3: Compute $\frac{\partial f}{\partial z}$.

$$\frac{\partial f}{\partial z} = 3yz^2.$$

Conclusion: The gradient is:

$$\nabla f = (2xy, x^2 + z^3, 3yz^2).$$

Divergence

The divergence of a vector field $\mathbf{F} = (F_1, F_2, F_3)$ is a scalar that measures the net rate of flow out of a point. It is defined as:

$$\nabla \cdot \mathbf{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}.$$

Example: Find the divergence of $\mathbf{F} = (x^2, 3xy, z^3)$.
Step 1: Compute $\frac{\partial F_1}{\partial x}$.

$$\frac{\partial F_1}{\partial x} = \frac{\partial}{\partial x}(x^2) = 2x.$$

Step 2: Compute $\frac{\partial F_2}{\partial y}$.

$$\frac{\partial F_2}{\partial y} = \frac{\partial}{\partial y}(3xy) = 3x.$$

Step 3: Compute $\frac{\partial F_3}{\partial z}$.

$$\frac{\partial F_3}{\partial z} = \frac{\partial}{\partial z}(z^3) = 3z^2.$$

Conclusion: The divergence is:

$$\nabla \cdot \mathbf{F} = 2x + 3x + 3z^2 = 5x + 3z^2.$$

Curl

The curl of a vector field $\mathbf{F} = (F_1, F_2, F_3)$ is a vector that measures the rotation of the field around a point. It is defined as:

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}.$$

Example: Find the curl of $\mathbf{F} = (yz, xz, xy)$.

Step 1: Set up the determinant.

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & xz & xy \end{vmatrix}.$$

Step 2: Expand the determinant.

$$\begin{aligned} \nabla \times \mathbf{F} = \mathbf{i} &\left(\frac{\partial(xy)}{\partial y} - \frac{\partial(xz)}{\partial z} \right) \\ - \mathbf{j} &\left(\frac{\partial(xy)}{\partial x} - \frac{\partial(yz)}{\partial z} \right) \\ + \mathbf{k} &\left(\frac{\partial(xz)}{\partial x} - \frac{\partial(yz)}{\partial y} \right). \end{aligned}$$

Step 3: Compute each term.

$$\begin{aligned} \frac{\partial(xy)}{\partial y} &= x, & \frac{\partial(xz)}{\partial z} &= x, \\ \frac{\partial(xy)}{\partial x} &= y, & \frac{\partial(yz)}{\partial z} &= y, \\ \frac{\partial(xz)}{\partial x} &= z, & \frac{\partial(yz)}{\partial y} &= z. \end{aligned}$$

Step 4: Substitute and simplify.

$$\begin{aligned} \nabla \times \mathbf{F} &= \mathbf{i}(x - x) - \mathbf{j}(y - y) + \mathbf{k}(z - z) \\ &= \mathbf{0}. \end{aligned}$$

Conclusion: The curl is:

$$\nabla \times \mathbf{F} = \mathbf{0}.$$

4. Integration (Definite and Indefinite)

Definition of Integration

Integration is the process of finding the integral of a function, which represents the accumulation of quantities. Integrals can be categorized as:

- **Indefinite integral:** Represents a family of functions and includes an arbitrary constant C. It is written as:

$$\int f(x)\,dx = F(x) + C,$$

  where $F'(x) = f(x)$.

- **Definite integral:** Represents the area under the curve of a function $f(x)$ over an interval $[a, b]$. It is written as:

$$\int_a^b f(x)\,dx = F(b) - F(a),$$

  where $F'(x) = f(x)$.

Step-by-Step Example (Indefinite Integral)

Example: Find $\int (2x + 3)\,dx$.

Step 1: Break the integral into separate terms.

$$\int (2x + 3)\,dx = \int 2x\,dx + \int 3\,dx.$$

Step 2: Use the power rule for integration. The power rule states:

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C, \text{ for } n \neq -1.$$

Applying this:

$$\int 2x\,dx = 2 \cdot \frac{x^{1+1}}{1+1} = x^2,$$

$$\int 3\,dx = 3x.$$

Step 3: Combine the results and add the constant of integration.

$$\int (2x + 3)\,dx = x^2 + 3x + C.$$

Conclusion: The indefinite integral is:

$$\int (2x + 3)\,dx = x^2 + 3x + C.$$

Step-by-Step Example (Definite Integral)

Example: Find $\int_1^3 (2x+3)\,dx$.

Step 1: Compute the indefinite integral. From the previous example:

$$\int (2x+3)\,dx = x^2 + 3x + C.$$

Step 2: Apply the limits of integration. Evaluate $F(x) = x^2 + 3x$ at $x = 3$ and $x = 1$:

$$F(3) = 3^2 + 3(3) = 9 + 9 = 18,$$
$$F(1) = 1^2 + 3(1) = 1 + 3 = 4.$$

Step 3: Subtract the results.

$$\int_1^3 (2x+3)\,dx = F(3) - F(1) = 18 - 4 = 14.$$

Conclusion: The definite integral is:

$$\int_1^3 (2x+3)\,dx = 14.$$

Integration by Parts

Integration by parts is a technique used to integrate the product of two functions. It is based on the formula:

$$\int u\,dv = uv - \int v\,du,$$

where u and dv are parts of the integrand.

Example: Find $\int xe^x\,dx$.

Step 1: Choose u and dv. Let:

$$u = x, \qquad dv = e^x\,dx.$$

Step 2: Differentiate u and integrate dv.

$$du = dx, \qquad v = e^x.$$

Step 3: Apply the integration by parts formula.

$$\begin{aligned}\int xe^x\,dx &= uv - \int v\,du \\ &= xe^x - \int e^x\,dx \\ &= xe^x - e^x + C.\end{aligned}$$

Conclusion: The integral is:

$$\int xe^x\,dx = e^x(x-1) + C.$$

Integration by Substitution

Integration by substitution is used to simplify an integral by substituting a part of the integrand with a single variable. It is based on the chain rule.

Example: Find $\int (3x^2+1)^5(6x)\,dx$.

Step 1: Choose a substitution. Let:

$$u = 3x^2 + 1.$$

Step 2: Differentiate u.

$$\frac{du}{dx} = 6x \implies du = 6x\,dx.$$

Step 3: Rewrite the integral in terms of u. Substitute u and du into the integral:

$$\int (3x^2+1)^5(6x)\,dx = \int u^5\,du.$$

Step 4: Solve the integral. Use the power rule for integration:

$$\int u^5\,du = \frac{u^{5+1}}{5+1} + C = \frac{u^6}{6} + C.$$

Step 5: Substitute back $u = 3x^2 + 1$.

$$\int (3x^2+1)^5(6x)\,dx = \frac{(3x^2+1)^6}{6} + C.$$

Conclusion: The integral is:

$$\int (3x^2+1)^5(6x)\,dx = \frac{(3x^2+1)^6}{6} + C.$$

5. Multiple Integrals

Definition of Multiple Integrals

Multiple integrals are used to calculate quantities over regions in two or more dimensions. Common types include:

- **Double integrals:** Integrals over a two-dimensional region, often used to calculate area, volume, or mass. Written as:

  $$\iint_R f(x,y)\,dA,$$

  where $dA = dx\,dy$ or $dy\,dx$ is the infinitesimal area element.

- **Triple integrals:** Integrals over a three-dimensional region, used to calculate volume, mass, or other quantities. Written as:

  $$\iiint_R f(x,y,z)\,dV,$$

  where $dV = dx\,dy\,dz$.

Step-by-Step Example (Double Integral)

Example: Compute the double integral $\iint_R (x+y)\,dA$, where R is the rectangle defined by $0 \le x \le 2$ and $0 \le y \le 3$.

Step 1: Write the integral with limits.

$$\iint_R (x+y)\,dA = \int_0^2 \int_0^3 (x+y)\,dy\,dx.$$

Step 2: Integrate with respect to y. Treat x as a constant while integrating with respect to y:

$$\begin{aligned}\int_0^3 (x+y)\,dy &= \int_0^3 x\,dy + \int_0^3 y\,dy \\ &= x\int_0^3 1\,dy + \left.\frac{y^2}{2}\right|_0^3 \\ &= x[3-0] + \frac{3^2}{2} - \frac{0^2}{2} \\ &= 3x + \frac{9}{2}.\end{aligned}$$

Step 3: Integrate with respect to x. Now integrate $3x + \frac{9}{2}$ with respect to x over $[0, 2]$:

$$\begin{aligned}
\int_0^2 \left(3x + \frac{9}{2}\right) dx &= \int_0^2 3x\, dx + \int_0^2 \frac{9}{2}\, dx \\
&= \frac{3x^2}{2}\bigg|_0^2 + \frac{9}{2}x\bigg|_0^2 \\
&= \frac{3(2)^2}{2} - \frac{3(0)^2}{2} + \frac{9}{2}(2) - \frac{9}{2}(0) \\
&= 6 + 9 = 15.
\end{aligned}$$

Conclusion: The value of the double integral is:

$$\iint_R (x + y)\, dA = 15.$$

Step-by-Step Example (Triple Integral)

Example: Compute the triple integral $\iiint_R z\, dV$, where R is the rectangular box defined by $0 \le x \le 1$, $0 \le y \le 2$, and $0 \le z \le 3$.

Step 1: Write the integral with limits.

$$\iiint_R z\, dV = \int_0^1 \int_0^2 \int_0^3 z\, dz\, dy\, dx.$$

Step 2: Integrate with respect to z.

$$\begin{aligned}
\int_0^3 z\, dz &= \frac{z^2}{2}\bigg|_0^3 \\
&= \frac{3^2}{2} - \frac{0^2}{2} \\
&= \frac{9}{2}.
\end{aligned}$$

Step 3: Integrate with respect to y. Since $\frac{9}{2}$ is constant with respect to y:

$$\begin{aligned}
\int_0^2 \frac{9}{2}\, dy &= \frac{9}{2} \cdot \int_0^2 1\, dy \\
&= \frac{9}{2} \cdot [2 - 0] \\
&= 9.
\end{aligned}$$

Step 4: Integrate with respect to x. Since 9 is constant with respect to x:

$$\begin{aligned}
\int_0^1 9\, dx &= 9 \cdot \int_0^1 1\, dx \\
&= 9 \cdot [1 - 0] \\
&= 9.
\end{aligned}$$

Conclusion: The value of the triple integral is:

$$\iiint_R z\, dV = 9.$$

Applications of Multiple Integrals

- **Double integrals:** Used to calculate areas, volumes, center of mass, and moment of inertia in two-dimensional regions.
- **Triple integrals:** Used to calculate volumes, mass, and other physical properties in three-dimensional regions.

6. Line and Surface Integrals

Definition of Line Integrals

A line integral is used to calculate the integral of a function along a curve. Line integrals are of two types:

- **Scalar line integrals:** Measure the accumulation of a scalar field along a curve C.
$$\int_C f(x, y, z)\, ds,$$
where ds is the arc length differential.

- **Vector line integrals:** Measure the work done by a vector field $\mathbf{F}$ along a curve C.
$$\int_C \mathbf{F} \cdot d\mathbf{r},$$
where $d\mathbf{r}$ is the differential displacement vector along the curve.

Scalar Line Integral Example

Example: Compute the line integral $\int_C x^2\, ds$, where C is the curve defined by $y = x^2$, $0 \leq x \leq 1$.

Step 1: Parameterize the curve. Let $x = t$ and $y = t^2$, where $0 \leq t \leq 1$. The arc length differential is given by:

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$$

Step 2: Compute $\frac{dx}{dt}$ and $\frac{dy}{dt}$.

$$\frac{dx}{dt} = 1, \qquad \frac{dy}{dt} = 2t.$$

Step 3: Compute ds.

$$ds = \sqrt{(1)^2 + (2t)^2}\, dt = \sqrt{1 + 4t^2}\, dt.$$

Step 4: Rewrite the integral. Substitute $x = t$ and ds into the integral:

$$\int_C x^2\, ds = \int_0^1 t^2 \sqrt{1 + 4t^2}\, dt.$$

Step 5: Solve the integral (numerical approximation). This integral does not have a simple closed-form solution, but it can be evaluated numerically:

$$\int_C x^2\, ds \approx 0.290.$$

Vector Line Integral Example

Example: Compute $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F} = (x, y)$ and C is the straight line from $(0, 0)$ to $(1, 1)$.

Step 1: Parameterize the curve. Let $x = t$, $y = t$, where $0 \leq t \leq 1$. Then:

$$d\mathbf{r} = \left(\frac{dx}{dt}, \frac{dy}{dt}\right) dt = (1, 1)dt.$$

Step 2: Rewrite the vector field.

$$\mathbf{F}(x, y) = (t, t).$$

Step 3: Compute $\mathbf{F} \cdot d\mathbf{r}$.

$$\mathbf{F} \cdot d\mathbf{r} = (t, t) \cdot (1, 1) = t + t = 2t.$$

Step 4: Evaluate the integral.

$$\begin{aligned}
\int_C \mathbf{F} \cdot d\mathbf{r} &= \int_0^1 2t \, dt \\
&= 2 \cdot \frac{t^2}{2}\bigg|_0^1 \\
&= \frac{1^2}{1} - \frac{0^2}{1} = 1.
\end{aligned}$$

Conclusion: The line integral is 1.

Definition of Surface Integrals

A surface integral is used to calculate the integral of a function over a surface S in three-dimensional space. Surface integrals can measure scalar quantities (like flux of a scalar field) or vector quantities (like flux of a vector field):

- **Scalar surface integrals:** Measure the accumulation of a scalar field over a surface.

  $$\iint_S f(x, y, z) \, dS,$$

  where dS is the surface area element.

- **Vector surface integrals:** Measure the flux of a vector field $\mathbf{F}$ through a surface.

  $$\iint_S \mathbf{F} \cdot \mathbf{n} \, dS,$$

  where $\mathbf{n}$ is the unit normal vector to the surface.

Scalar Surface Integral Example

Example: Compute $\iint_S z\,dS$, where S is the plane $z = 1 + x + y$, $0 \leq x \leq 1$, $0 \leq y \leq 1$.

Step 1: Compute the surface area element dS. The formula for dS is:

$$dS = \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\,dx\,dy.$$

Step 2: Compute partial derivatives.

$$\frac{\partial z}{\partial x} = 1, \qquad \frac{\partial z}{\partial y} = 1.$$

$$dS = \sqrt{1 + 1^2 + 1^2}\,dx\,dy = \sqrt{3}\,dx\,dy.$$

Step 3: Rewrite the integral.

$$\iint_S z\,dS = \int_0^1 \int_0^1 (1 + x + y)\sqrt{3}\,dx\,dy.$$

Step 4: Evaluate the integral. Factor out $\sqrt{3}$ and integrate:

$$\begin{aligned}
\iint_S z\,dS &= \sqrt{3} \int_0^1 \int_0^1 (1 + x + y)\,dx\,dy \\
&= \sqrt{3} \int_0^1 \left[\int_0^1 1\,dx + \int_0^1 x\,dx + \int_0^1 y\,dx\right] dy \\
&= \sqrt{3} \int_0^1 \left[1 + \frac{1}{2} + y\right] dy \\
&= \sqrt{3} \left[y + \frac{y^2}{2} + \frac{y^2}{2}\Bigg|_0^1\right] \\
&= \sqrt{3} \cdot \frac{3}{2} = \frac{3\sqrt{3}}{2}.
\end{aligned}$$

Conclusion: The surface integral is $\frac{3\sqrt{3}}{2}$.

7. Taylor Series Expansions

Definition of Taylor Series

The Taylor series is a way to approximate a smooth function $f(x)$ around a point $x = a$ as an infinite sum of terms derived from the derivatives of $f(x)$ at a. It is given by:

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f^{(3)}(a)}{3!}(x-a)^3 + \cdots$$

where:

- $f(a)$ is the value of the function at $x = a$,
- $f'(a), f''(a), \ldots$ are the first, second, and higher-order derivatives of $f(x)$ evaluated at $x = a$,
- $(x - a)$ is the displacement from the expansion point a,
- $n!$ is the factorial of n, defined as $n! = n \cdot (n-1) \cdot \cdots \cdot 1$.

Maclaurin Series

A special case of the Taylor series is the Maclaurin series, where $a = 0$:

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \cdots$$

Step-by-Step Example: Taylor Series Expansion

Example: Find the Taylor series expansion of $f(x) = e^x$ around $x = 0$ (Maclaurin series).

Step 1: Compute the derivatives of $f(x)$.

$$f(x) = e^x, \qquad f'(x) = e^x, \qquad f''(x) = e^x, \qquad f^{(n)}(x) = e^x.$$

Step 2: Evaluate the derivatives at $x = 0$.

$$f(0) = e^0 = 1, \qquad f'(0) = e^0 = 1, \qquad f''(0) = e^0 = 1, \qquad \ldots$$

Step 3: Write the series expansion. Using the formula for the Maclaurin series:

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \cdots$$

Substitute the values of $f^{(n)}(0)$:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

Conclusion: The Taylor series for $f(x) = e^x$ around $x = 0$ is:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

Step-by-Step Example: Approximation Using Taylor Series

Example: Approximate $\sin(x)$ around $x = 0$ up to the third order term (Maclaurin series).

Step 1: Compute the derivatives of $f(x) = \sin(x)$.

$$f(x) = \sin(x), \quad f'(x) = \cos(x), \quad f''(x) = -\sin(x), \quad f^{(3)}(x) = -\cos(x).$$

Step 2: Evaluate the derivatives at $x = 0$.

$$f(0) = \sin(0) = 0, \quad f'(0) = \cos(0) = 1, \quad f''(0) = -\sin(0) = 0, \quad f^{(3)}(0) = -\cos(0) = -1.$$

Step 3: Write the series expansion.

$$\sin(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \cdots$$

Substitute the values:

$$\sin(x) \approx 0 + 1 \cdot x + \frac{0}{2!}x^2 - \frac{1}{3!}x^3 + \cdots$$

Simplify:

$$\sin(x) \approx x - \frac{x^3}{6}.$$

Conclusion: Up to the third-order term, $\sin(x)$ is approximated as:

$$\sin(x) \approx x - \frac{x^3}{6}.$$

Applications of Taylor Series

- **Approximating functions:** Taylor series provide polynomial approximations for complex functions.
- **Solving differential equations:** Used to find series solutions.
- **Error estimation:** Higher-order terms indicate the accuracy of approximations.

8. Optimization (Unconstrained and Constrained)

Unconstrained Optimization

Unconstrained optimization is the process of finding the maximum or minimum of a function $f(x)$ without any restrictions on the variables.

First-Order Condition (FOC)

The critical points of $f(x)$ are found by solving:

$$\nabla f(x) = 0,$$

where $\nabla f(x)$ is the gradient of $f(x)$. These points are potential maxima, minima, or saddle points.

Second-Order Condition (SOC)

To classify a critical point, examine the Hessian matrix H (the matrix of second partial derivatives):

- If H is positive definite, the point is a local minimum.
- If H is negative definite, the point is a local maximum.
- If H is indefinite, the point is a saddle point.

Step-by-Step Example (Unconstrained Optimization)

Example: Find the minimum of $f(x, y) = x^2 + y^2 - 4x - 6y + 13$.

Step 1: Compute the gradient.

$$\frac{\partial f}{\partial x} = 2x - 4, \qquad \frac{\partial f}{\partial y} = 2y - 6.$$

$$\nabla f(x, y) = (2x - 4, 2y - 6).$$

Step 2: Solve $\nabla f(x, y) = 0$.

$$2x - 4 = 0 \implies x = 2,$$
$$2y - 6 = 0 \implies y = 3.$$

The critical point is $(2, 3)$.

Step 3: Compute the Hessian matrix.

$$H = \begin{bmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}.$$

Step 4: Classify the critical point. The Hessian matrix is positive definite (all eigenvalues are positive), so $(2, 3)$ is a local minimum.

Step 5: Verify the value of $f(x, y)$ at $(2, 3)$.

$$f(2, 3) = 2^2 + 3^2 - 4(2) - 6(3) + 13 = 4 + 9 - 8 - 18 + 13 = 0.$$

Conclusion: The minimum value of $f(x, y)$ is 0 at $(2, 3)$.

Constrained Optimization

Constrained optimization involves finding the extrema of a function $f(x, y, \dots)$ subject to constraints such as $g(x, y, \dots) = 0$ or $h(x, y, \dots) \leq 0$.

Lagrange Multipliers (Equality Constraints)

To solve:

$$\text{Maximize/minimize } f(x, y) \text{ subject to } g(x, y) = 0,$$
$$\text{form the Lagrangian: } \mathcal{L}(x, y, \lambda) = f(x, y) + \lambda g(x, y),$$

where λ is the Lagrange multiplier. Solve:

$$\nabla \mathcal{L}(x, y, \lambda) = 0.$$

Step-by-Step Example (Constrained Optimization)

Example: Maximize $f(x, y) = xy$ subject to $x^2 + y^2 = 1$.

Step 1: Form the Lagrangian.

$$\mathcal{L}(x, y, \lambda) = xy + \lambda(1 - x^2 - y^2).$$

Step 2: Compute the partial derivatives of $\mathcal{L}$.

$$\frac{\partial \mathcal{L}}{\partial x} = y - 2\lambda x,$$
$$\frac{\partial \mathcal{L}}{\partial y} = x - 2\lambda y,$$
$$\frac{\partial \mathcal{L}}{\partial \lambda} = 1 - x^2 - y^2.$$

Step 3: Solve the system of equations.

$$y - 2\lambda x = 0, \tag{1}$$
$$x - 2\lambda y = 0, \tag{2}$$
$$1 - x^2 - y^2 = 0. \tag{3}$$

From (1): $y = 2\lambda x$. Substitute into (2):

$$x - 2\lambda(2\lambda x) = 0 \implies x(1 - 4\lambda^2) = 0.$$

So $x = 0$ or $\lambda = \pm\frac{1}{2}$. Solving for y using $x^2 + y^2 = 1$:

$$x = 0 \implies y = \pm 1,$$
$$\lambda = \pm\frac{1}{2} \implies x = \pm\frac{\sqrt{2}}{2},\ y = \pm\frac{\sqrt{2}}{2}.$$

Step 4: Evaluate $f(x, y)$ at each point.

$$f(0, \pm 1) = 0,$$
$$f\left(\pm\frac{\sqrt{2}}{2}, \pm\frac{\sqrt{2}}{2}\right) = \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{2}}{2} = \frac{1}{2}.$$

Conclusion: The maximum value of $f(x, y)$ is $\frac{1}{2}$, occurring at $\left(\pm\frac{\sqrt{2}}{2}, \pm\frac{\sqrt{2}}{2}\right)$.

9. Ordinary Differential Equations (ODEs)

Definition of Ordinary Differential Equations

An ordinary differential equation (ODE) is an equation involving a function $y(x)$ and its derivatives. It is written as:

$$F\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \dots\right) = 0.$$

Classification of ODEs:

- **Order:** The order of an ODE is the highest derivative of $y(x)$ that appears in the equation.
- **Linearity:** An ODE is linear if y and its derivatives appear to the power of 1 and are not multiplied together.

First-Order ODEs

A first-order ODE has the form:

$$\frac{dy}{dx} = f(x, y).$$

Example: Solve $\frac{dy}{dx} = 3x^2$.

Step 1: Separate variables (if applicable). Here, y depends only on x, so we can directly integrate:

$$\int \frac{dy}{dx}\, dx = \int 3x^2\, dx.$$

Step 2: Solve the integral. Use the power rule for integration:

$$\int 3x^2\, dx = x^3 + C,$$

where C is the constant of integration.

Step 3: Write the solution.

$$y = x^3 + C.$$

Conclusion: The general solution is $y = x^3 + C$.

Example: Solve the IVP $\frac{dy}{dx} = 2x$, $y(0) = 1$.

Step 1: Integrate the ODE.

$$y = \int 2x\, dx = x^2 + C.$$

Step 2: Use the initial condition. Substitute $x = 0$ and $y = 1$:

$$1 = 0^2 + C \implies C = 1.$$

Conclusion: The solution is $y = x^2 + 1$.

Second-Order ODEs

A second-order ODE involves the second derivative $\frac{d^2y}{dx^2}$ and is written as:

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = g(x).$$

Example: Solve $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = 0$.

Step 1: Write the characteristic equation.

$$r^2 + 3r + 2 = 0.$$

Step 2: Solve the characteristic equation. Factorize:

$$(r+1)(r+2) = 0 \implies r = -1,\ r = -2.$$

Step 3: Write the general solution.

$$y = C_1 e^{-x} + C_2 e^{-2x},$$

where C_1 and C_2 are constants.

Example: Solve $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = e^x$.

Step 1: Solve the homogeneous equation.

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = 0.$$

The characteristic equation is:

$$r^2 - 2r + 1 = 0 \implies (r-1)^2 = 0 \implies r = 1 \text{ (double root)}.$$

The solution to the homogeneous equation is:

$$y_h = (C_1 + C_2 x)e^x.$$

Step 2: Solve the non-homogeneous equation. Assume a particular solution $y_p = Axe^x$. Substituting into the ODE:

$$\frac{d^2}{dx^2}(Axe^x) - 2\frac{d}{dx}(Axe^x) + Axe^x = e^x.$$

Simplify and solve for A:

$$A = \frac{1}{2}.$$

Step 3: Write the general solution.

$$y = y_h + y_p = (C_1 + C_2 x)e^x + \frac{1}{2}xe^x.$$

Applications of ODEs

- **Physics:** Modeling motion, electrical circuits, and wave propagation.
- **Biology:** Population dynamics and spread of diseases.
- **Engineering:** Control systems and fluid dynamics.

Example: Damped Harmonic Oscillator

The equation for a damped harmonic oscillator is:

$$\frac{d^2x}{dt^2} + 2\gamma\frac{dx}{dt} + \omega_0^2 x = 0,$$

where γ is the damping coefficient and ω_0 is the natural frequency.

Step 1: Solve the characteristic equation.

$$r^2 + 2\gamma r + \omega_0^2 = 0.$$

Solutions depend on the discriminant $\Delta = \gamma^2 - \omega_0^2$:

- **Overdamped:** $\Delta > 0$, two real roots.
- **Critically damped:** $\Delta = 0$, double root.
- **Underdamped:** $\Delta < 0$, complex roots.

Step 2: Write the solution for underdamped motion. If $\Delta < 0$, the solution is:

$$x(t) = e^{-\gamma t}(C_1 \cos(\omega t) + C_2 \sin(\omega t)),$$

where $\omega = \sqrt{\omega_0^2 - \gamma^2}$.