Calculation Logics of CVA and CVA Greeks

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Abstract

We consider some types of calculation logic of CVA and CVA Greeks. And we analysis the character of each logic by using toy model.

1 Setting of Toy model

We use the notation that $x \vee 0 := \max(x, 0)$. We disregard hazard rate and discount for simplicity.

1.1 Underlying Process

$$dX(t) = X(t)(\mu dt + \sigma dW(t)), \tag{1}$$

$$X(0) = x_0. \tag{2}$$

1.2 Exposure

We consider an European derivative with maturity T and payoff function F. Then the Exposure of it is

$$V(t, x) = E[F(X(T))|X(t) = x]. \tag{3}$$

1.3 CVA

In the setting above, the CVA is defined as

$$\mathrm{CVA} = \int_0^T E[V(t, X(t) \vee 0]dt. \tag{4}$$

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2 Approximation Method

We set grid of time as $0 = t_0 < t_1 < \cdots < t_n = T$ and denote $V(t_i, x)$ by $V_i(x)$. Then approximation of CVA by discretization of integral with respect to time is following.

$$\mathrm{CVA} = \sum_{i=0}^{n-1} E[V_{i+1}(X(t_{i+1})) \vee 0](t_{i+1} - t_i). \tag{5}$$

If we know the analytic expression of exposure function V_i, we can approximate (5) simply as

$$\mathrm{CVA} = \sum_{i=0}^{n-1}(t_{i+1} - t_i)\left(\sum_{\ell=1}^{L}(V_{i+1}(X_\ell(t_{i+1})) \vee 0)\right). \tag{6}$$

Otherwise, we have to approximate the exposure function.

2.1 Approximation of Exposure

For each $i = 1, \ldots, n$, we approximate exposure function $V_i(x)$ by $\tilde{V}_i$. The most popular approximation is LSM whose procedure is as follows.

Definition 1 *Let $\psi_k, k = 1, \ldots, K$, be given basis functions. And we define $\tilde{V}_i^{LSM}$ by*

$$\tilde{V}_i^{LSM}(x) = \sum_{k=1}^{K} \beta_k^{(i)} \psi_k(x), \tag{7}$$

where $\beta_k^{(i)}, k = 1, \ldots, K$ is calculated as follows.

- Ψ : $K \times K$ matrix

$$\Psi^{(i)} = \left(\frac{1}{L}\sum_{\ell=1}^{L} \psi_p(X_\ell(t_i))\psi_q(X_\ell(t_i))\right)_{p,q=1}^{K} \tag{8}$$

- $v^{(i)}$: $K \times 1$ vector

$$v^{(i)} = \left(\frac{1}{L}\sum_{\ell=1}^{L} \psi_p(X_\ell(t_i))f(X_\ell)\right)_{p=1}^{K} \tag{9}$$

- $\beta^{(i)}$: $K \times 1$ vector

$$\beta^{(i)} = \Psi^{(i)} v^{(i)} \tag{10}$$

2.2 Approximation of CVA

When the approximated exposure function $\tilde{V}_i$ is obtained, we consider two types of approximation methods of CVA, 'Explicit Method' and 'Implicit Method'. 'Explicit Method' is the simple method that we substitute V_i by $\tilde{V}_i$ in (6). 'Implicit Method' is proposed in [1] and it is shown that Implicit Method is more efficient approximation than Explicit Method mathematically. The essential difference between them are that 'Explicit Method' uses the approximated exposure function directly, but 'Implicit method' does not. Implicit Method uses exposure function only to judge whether exposure is positive or negative. So it does not use the value of approximated exposure itself. The actual definitions are as follows.

Definition 2 (Explicit Method)

$$C_{Explicit} = \sum_{i=0}^{n-1}(t_{i+1} - t_i)\left(\frac{1}{L}\sum_{\ell=1}^{L}\tilde{V}_i(X_\ell(t_i)) \vee 0\right). \tag{11}$$

Notice that the expected positive exposure is written as follows.

$$\begin{aligned} E[V_i(X(t_i)) \vee 0] &= E[V_i(X(t_i))1_{\{V_i(X(t_i))>0\}}] && (12)\\ &= E[E[F(X(T))|\mathcal{F}_{t_i}]1_{\{V_i(X(t_i))>0\}}] && (13)\\ &= E[F(X(T))1_{\{V_i(X(t_i))>0\}}]. && (14) \end{aligned}$$

Based on this expression, we define the Implicit Method as follows.

Definition 3 (Implicit Method)

$$C_{Implicit} = \sum_{i=0}^{n-1}(t_{i+1} - t_i)\left(\frac{1}{L}\sum_{\ell=1}^{L}F(X_\ell(T))1_{\{\tilde{V}_i(X_\ell(t_i))>0\}}\right). \tag{15}$$

3 Calculation of Greeks

We consider the differentiation of CVA with respect to the parameter θ such that $\theta = x_0$ or σ. When we show explicitly the dependence of θ for arbitrary function $f(x)$, then we denote it by $f(x;\theta)$.

3.1 Greeks Approximation Using *Explicit Method*

The Greeks of CVA about θ, G_θ is as follows.

$$G_\theta = \frac{\partial}{\partial\theta}\sum_{i=0}^{n-1}(t_{i+1} - t_i)E[V_{i+1}(X(t_{i+1};\theta);\theta) \vee 0] \tag{16}$$

Notice that

$$\begin{aligned} &\frac{\partial}{\partial\theta}(V_{i+1}(X(t_{i+1};\theta);\theta) \vee 0) && (17)\\ =&1_{\{V_{i+1}(X(t_{i+1};\theta);\theta)>0\}}\frac{\partial V_{i+1}}{\partial x}(X(t_{i+1};\theta);\theta)\frac{\partial X}{\partial\theta}(t_{i+1};\theta) && (18)\\ +&\frac{\partial V_{i+1}}{\partial\theta}(X(t_{i+1};\theta);\theta) && (19) \end{aligned}$$

Now we consider the following variations of Approximation of Greeks.

Definition 4 (Explicit, Exposure Shock) *To calculate $G^{Shock}_{Explicit}$, we also shock the parameter θ in the V_{i+1}. For LSM, it means that we recalculate the coefficient of LSM with shocked parameter θ.*

$$G^{Shock}_{Explicit} = \sum_{i=0}^{n-1}(t_{i+1}-t_i)\frac{1}{L}\sum_{\ell=1}^{L} \tag{20}$$

$$\{1_{\{V_{i+1}(X(t_{i+1};\theta);\theta)>0\}}\frac{\partial \tilde{V}_{i+1}}{\partial x}(X(t_{i+1};\theta);\theta)\frac{\partial X}{\partial \theta}(t_{i+1};\theta) \tag{21}$$

$$+\frac{\partial \tilde{V}_{i+1}}{\partial \theta}(X(t_{i+1};\theta);\theta)\} \tag{22}$$

Definition 5 (Explicit, Coefficient Fix) *To calculate $G^{Fix}_{Explicit}$, we fix the parameter θ in V_{i+1}. For LSM, it means that we do not recalculate the coefficient of LSM.*

$$G^{Fix}_{Explicit} = \sum_{i=0}^{n-1}(t_{i+1}-t_i)\frac{1}{L}\sum_{\ell=1}^{L} \tag{23}$$

$$\{1_{\{\tilde{V}_{i+1}(X(t_{i+1};\theta);\theta)>0\}}\frac{\partial \tilde{V}_{i+1}}{\partial x}(X(t_{i+1};\theta);\theta)\frac{\partial X}{\partial \theta}(t_{i+1};\theta)\} \tag{24}$$

3.2 Greeks Approximation Using *Implicit Method*

The CVA Greeks formula has an expression by using Implicit Method.

$$\frac{\partial}{\partial \theta}\text{CVA} = \frac{\partial}{\partial \theta}\sum_{i=0}^{n-1} E[F(X(T;\theta)1_{\{V_{i+1}(X(t_{i+1};\theta);\theta)>0\}}](t_{i+1}-t_i) \tag{25}$$

$$= \sum_{i=0}^{n-1}(t_{i+1}-t_i)E[(\frac{\partial}{\partial \theta}F(X(T;\theta))1_{\{V_{i+1}(X(t_{i+1};\theta);\theta)>0\}} \tag{26}$$

$$+ F(X(T;\theta)\delta_0(V_{i+1}(X(t_{i+1};\theta))(\frac{\partial}{\partial \theta}V_{i+1}(X(t_{i+1};\theta);\theta))] \tag{27}$$

Then, notice that

$$E[F(X(T;\theta)\delta_0(V_{i+1}(X(t_{i+1};\theta))(\frac{\partial}{\partial \theta}V_{i+1}(X(t_{i+1};\theta);\theta))] \tag{28}$$

$$=E[V_{i+1}(X(t_{i+1};\theta))\delta_0(V_{i+1}(X(t_{i+1};\theta))(\frac{\partial}{\partial \theta}V_{i+1}(X(t_{i+1};\theta);\theta))] \tag{29}$$

$$=0. \tag{30}$$

So we have

$$\frac{\partial}{\partial \theta}\text{CVA} = \sum_{i=0}^{n-1}(t_{i+1}-t_i)E[(\frac{\partial}{\partial \theta}F(X(T;\theta))1_{\{V_{i+1}(X(t_{i+1};\theta);\theta)>0\}}. \tag{31}$$

Definition 6 (Implicit) *Using (31), we do not need to differentiate exposure function $\tilde{V}_i$.*

$$G_{Implicit} = \sum_{i=0}^{n-1}(t_{i+1}-t_i)\frac{1}{L}\sum_{\ell=1}^{L}(\frac{\partial}{\partial \theta}F(X_\ell(T;\theta))1_{\{\tilde{V}_{i+1}(X_\ell(t_{i+1};\theta);\theta)>0\}}. \tag{32}$$

3.3 Pros and Cons of Each Greeks Methods

- *Explicit, Fix* type is very good for computational cost. But Vega is not appropriate because the term (19) is omitted for this type. On the other hand, delta is calculated appropriately because of the following relation.

$$\begin{cases} \frac{\partial V(x;\theta)}{\partial\theta} = 0, if\theta = x_0, \\ \frac{\partial V(x;\theta)}{\partial\theta} \neq 0, if\theta = \sigma. \end{cases} \tag{33}$$

- *Explicit, Shock* type is not efficient for computational cost, but all Delta and Vega can be calculated appropriately.

- *Implicit* type is good for computational cost, and all Delta and Vega can be calculated appropriately. But CVA value of Implicit Method might be negative.

4 Modification of CVA and CVA Greeks

The Calculation logic of XVA is often different from that of derivative pricing because XVA is needed to be calculated quickly. So XVA calculation is less accurate than derivative pricing. On the other hand, it is said that it is efficient to modify XVA value by the price of derivative and this method is often used in practical. In this section, we assume that we know the true value of derivative price and Greeks. Then, we consider the logic to modify XVA and XVA Greeks and check the efficiency of modification.

4.1 Modification of CVA

Let $\varepsilon_V = V_0(x_0) - \tilde{V}_0(x_0)$. We define modified Exposure $\tilde{V}_i^{Modify}$ as

$$\tilde{V}_i^{Modify}(x) = \tilde{V}_i(x) + \varepsilon_V, \tag{34}$$

Then, we define $C_{Explicit}^{Modify}$ as

$$C_{Explicit}^{Modify} = \sum_{i=0}^{n-1}(t_{i+1} - t_i)\left(\frac{1}{L}\sum_{\ell=1}^{L}\tilde{V}_i^{Modify}(X_\ell(t_i)) \vee 0\right). \tag{35}$$

4.2 Modification of Greeks

We also modifying Greeks of CVA. Let $D_\theta(t_i)$ be future differential of Exposure at t_i.

$$D_\theta(t_i) = \frac{\partial}{\partial\theta}(V_i(X(t_i;\theta);\theta) \vee 0) \tag{36}$$

$$=1_{\{V_i(X(t_i;\theta);\theta)>0\}}\frac{\partial V_i}{\partial x}(X(t_i;\theta);\theta)\frac{\partial X}{\partial\theta}(t_i;\theta) \tag{37}$$

$$+\frac{\partial V_i}{\partial\theta}(X(t_i;\theta);\theta). \tag{38}$$

Let $\tilde{D}_\theta(t_i)$ be an approximation of $D_\theta(t_i)$ using approximated exposure function which is fixed when differentiation.

$$\tilde{D}_\theta(t_i) = \frac{\partial \tilde{V}_i}{\partial x}(X(t_i;\theta);\theta)\frac{\partial X}{\partial \theta}(t_i;\theta). \tag{39}$$

Let $\varepsilon_D = D_\theta(t_0) - \tilde{D}_\theta(t_0)$ which means the difference of true Greeks of Derivative and approximated Greeks of XVA model. Then we define a modified future derivative by

$$\tilde{D}_\theta^{Modify}(t_i) = \tilde{D}_\theta(t_i) + \varepsilon_D. \tag{40}$$

Using modified exposure $\tilde{V}_i$ and modified future differential $\tilde{D}_\theta^{Modify}$, we define modified CVA Greeks by

$$G_{Explicit}^{Modify} = \sum_{i=0}^{n-1}(t_{i+1} - t_i)E[1_{\{\tilde{V}_{i+1}^{Modify}(X(t_{i+1};\theta);\theta)>0\}}\tilde{D}_\theta^{Modify}(t_{i+1})]. \tag{41}$$

References

[1] Morimoto, Y., "Application of Stochastic Mesh Method to Efficient Approximation of CVA", arXiv:1510.04588, 2015.