View

V矩阵用于将模型变换到Camera空间

通常依赖下列参数

- Camera位置(x_p, y_p, z_p)
- Up方向(x_u, y_u, z_u)
- LookAt方向(x_l, y_l, z_l)

对于观察者而言，模型做任何变换等价与Camera做对应的逆变换

初始态V_0没有任何变换，即在原点往$-z$轴方向看，Up方向y轴

- Camera位置$(0, 0, 0)$
- Up方向$(0, 1, 0)$
- LookAt方向$(0, 0, -1)$

$V \rightarrow V_0$等价于

- Translate
    - $(x_p, y_p, z_p) \rightarrow (0, 0, 0)$
- Rotate
    - X
      $(x_l, y_l, z_l) \times (x_u, y_u, z_u) = (x_r, y_r, z_r) \rightarrow (0, 0, -1) \times (0, 1, 0) = (1, 0, 0)$
    - Y $(x_u, y_u, z_u) \rightarrow (0, 1, 0)$
    - Z $(x_l, y_l, z_l) \rightarrow (0, 0, -1)$

Translate为

$$V_t = \begin{bmatrix} 1 & 0 & 0 & -x_p \\ 0 & 1 & 0 & -y_p \\ 0 & 0 & 1 & -z_p \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Rotate不好直接写出，考虑逆变换的旋转矩阵

$$\begin{cases} V_r^{-1} \times (1, 0, 0, 0)^T = (x_r, y_r, z_r, 0) \\ V_r^{-1} \times (0, 1, 0, 0)^T = (x_u, y_u, z_u, 0) \\ V_r^{-1} \times (0, 0, -1, 0)^T = (x_l, y_l, z_l, 0) \end{cases}$$

得到

$$V_r^{-1} = \begin{bmatrix} x_r & x_u & -x_l & 0 \\ y_r & y_u & -y_l & 0 \\ z_r & z_u & -z_l & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

通常取旋转矩阵V_r为正交矩阵

$$V_r^{-1} = V_r^T$$

即

$$V_r = \begin{bmatrix} x_r & y_r & z_r & 0 \\ x_u & y_u & z_u & 0 \\ -x_l & -y_l & -z_l & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

因此

$$V = V_r V_t = \begin{bmatrix} x_r & y_r & z_r & -(x_p x_r + y_p y_r + z_p z_r) \\ x_u & y_u & z_u & -(x_p x_u + y_p y_u + z_p z_u) \\ -x_l & -y_l & -z_l & x_p x_l + y_p y_l + z_p z_l \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

实际上构建V矩阵是通过下列参数控制的

$$\begin{cases} (x_p, y_p, z_p) = position \\ (w_x, w_y, w_z) = world\ up \\ (x_l, y_l, z_l) = look\ at \end{cases}$$

带入即可求得

$$\begin{cases} (x_r, y_r, z_r) = look\ at \times\ world\ up \\ (x_u, y_u, z_u) = (x_r, y_r, z_r) \times\ look\ at \end{cases}$$

Projection

Orthographic

本质上是下面的映射

$$R^3 \in \{[l, r] \times [t, b] \times [n, f]\} \to R^3 \in \{[-1, 1]^3\}$$

- Translate $(0, 0, 0)$
- Scale $(2, 2, 2)$

$$P_o = \begin{bmatrix} \frac{2}{r-l} & 0 & 0 & 0 \\ 0 & \frac{2}{t-b} & 0 & 0 \\ 0 & 0 & \frac{2}{f-n} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 0 & -\frac{r+l}{2} \\ 0 & 1 & 0 & -\frac{t+b}{2} \\ 0 & 0 & 1 & -\frac{f+n}{2} \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{2}{r-l} & 0 & 0 & -\frac{r+l}{r-l} \\ 0 & \frac{2}{t-b} & 0 & -\frac{t+b}{t-b} \\ 0 & 0 & \frac{2}{f-n} & -\frac{f+n}{f-n} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

P_o有点复杂，对于渲染屏幕宽度为(W, H)，通常我们取

$$\begin{cases} l = -\frac{W}{2} \\ r = \frac{W}{2} \\ b = -\frac{H}{2} \\ t = \frac{H}{2} \end{cases}$$

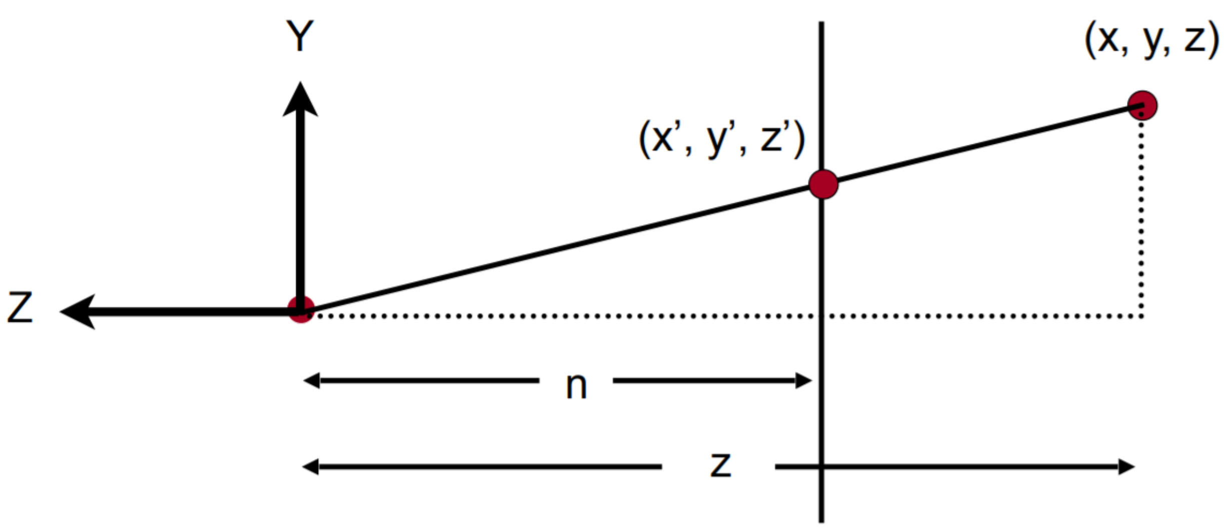
可以化简为

$$P_o = \begin{bmatrix} \frac{2}{W} & 0 & 0 & 0 \\ 0 & \frac{2}{H} & 0 & 0 \\ 0 & 0 & \frac{2}{f-n} & -\frac{f+n}{f-n} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Perspective

Perspective投影可以由Orthographic投影经过变换$P_{o \to p}$得到

对于Homogeneous coordination下面经过P_o变换得到的3D Point$(x, y, z, 1)$，经过$P_{o \to p}$得到3D Point$(x', y', z', 1)$



根据相似三角形

$$\begin{cases} x' = \frac{nx}{z} = P_{o\to p}[0] \cdot (x,y,z,1) \\ y' = \frac{ny}{z} = P_{o\to p}[1] \cdot (x,y,z,1) \end{cases}$$

不希望得到的$P_{o\to p}$中含有坐标相关的变量z

因此根据Homogeneous vector的定义，把变换后的坐标改写一下（这里左右两边的z'是不一样的，这里简写了）

$$\left(\frac{nx}{z}, \frac{ny}{z}, z', 1\right) \to (nx, ny, z', z)$$

即

$$\begin{cases} nx = P_{o\to p}[0] \cdot (x,y,z,1) \\ ny = P_{o\to p}[1] \cdot (x,y,z,1) \\ z = P_{o\to p}[3] \cdot (x,y,z,1) \end{cases}$$

得到

$$\begin{cases} P_{o\to p}[0] = (n,0,0,0) \\ P_{o\to p}[1] = (0,n,0,0) \\ P_{o\to p}[3] = (0,0,1,0) \end{cases}$$

near面上的点$P_{o\to p}$变换前后不变，far面上的中心点$P_{o\to p}$变换前后位置不变

$$\begin{cases} P_{o\to p} \times (x,y,n,1)^T \to (xn, yn, n^2, n) \\ P_{o\to p} \times (0,0,f,1)^T \to (0,0,f^2,f) \end{cases}$$

即

$$\begin{cases} P_{o\to p}[2] \cdot (x,y,n,1) = n^2 \\ P_{o\to p}[2] \cdot (0,0,f,1) = f^2 \end{cases}$$

待定系数，求解得到

$$P_{o\to p}[2] = (0,0,f+n,-fn)$$

即

$$P_{o\to p} = \begin{bmatrix} n & 0 & 0 & 0 \\ 0 & n & 0 & 0 \\ 0 & 0 & f+n & -fn \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

因此有

$$P_p = P_o \times P_{o\to p} = \begin{bmatrix} \frac{2n}{W} & 0 & 0 & 0 \\ 0 & \frac{2n}{H} & 0 & 0 \\ 0 & 0 & \frac{f+n}{f-n} & -\frac{2fn}{f-n} \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

实际上构建P_p矩阵是通过下列参数控制的

$$\begin{cases} aspect = \frac{W}{H} \\ \frac{H}{2n} = \tan(\frac{fov}{2}) \\ n = n \\ f = f \end{cases}$$

即

$$\frac{2n}{W} = \frac{2}{H \cdot aspect} \cdot \frac{H}{2\tan(\frac{fov}{2})} = \frac{1}{aspect \cdot \tan(\frac{fov}{2})}$$

$$\frac{2n}{H} = \frac{1}{\tan(\frac{fov}{2})}$$

因此

$$P_p = \begin{bmatrix} \frac{1}{aspect \cdot \tan(\frac{fov}{2})} & 0 & 0 & 0 \\ 0 & \frac{1}{\tan(\frac{fov}{2})} & 0 & 0 \\ 0 & 0 & \frac{f+n}{f-n} & -\frac{2fn}{f-n} \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

这是glm里面的glm::perspectiveLH_NO函数返回的projection matrix

01Depth

上面的深度$z \in [-1, 1]$，实际上Vulkan之类的API的$z \in [0, 1]$，因此

$$P_o = \begin{bmatrix} \frac{2}{W} & 0 & 0 & 0 \\ 0 & \frac{2}{H} & 0 & 0 \\ 0 & 0 & \frac{1}{f-n} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -n \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{2}{W} & 0 & 0 & 0 \\ 0 & \frac{2}{H} & 0 & 0 \\ 0 & 0 & \frac{1}{f-n} & -\frac{n}{f-n} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

此时

$$P_p = \begin{bmatrix} \frac{2}{W} & 0 & 0 & 0 \\ 0 & \frac{2}{H} & 0 & 0 \\ 0 & 0 & \frac{f}{f-n} & -\frac{fn}{f-n} \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

使用控制参数表示

$$P_p = \begin{bmatrix} \frac{1}{aspect\cdot\tan(\frac{fov}{2})} & 0 & 0 & 0 \\ 0 & \frac{1}{\tan(\frac{fov}{2})} & 0 & 0 \\ 0 & 0 & \frac{f}{f-n} & -\frac{fn}{f-n} \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

这是glm里面的glm::perspectiveLH_ZO函数返回的projection matrix

RightHand

右手系于左手系相比，就是z轴反向了，而xy轴保持不变

因此直接给原矩阵右乘一个变换矩阵即可

$$P_{right} = P_p \times \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

这是glm里面的glm::perspectiveRH_NO和glm::perspectiveRH_ZO函数返回的projection matrix