图形学需要的数学

向量

设$\vec{a} = (x_1, y_1, z_1), \vec{b} = (x_2, y_2, z_2)$

$$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos\theta = x_1 y_1 + x_2 y_2 + x_3 y_3$$

$$|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}| \sin\theta$$

$$\vec{a} \times \vec{b} = \begin{vmatrix} i & j & k \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix} = (y_1 z_2 - y_2 z_1, x_2 z_1 - x_1 z_2, x_1 y_z - x_2 y_1)$$

设平面$S_1 = A_1 x + B_1 y + C_1 z + D = 0, S_2 = A_2 x + B_2 y + C_2 z + D = 0$，当$S_1 \cap S_2 = L \neq \emptyset$

$$\begin{cases} \vec{n_1} = (A_1, B_1, C_1) \\ \vec{n_2} = (A_2, B_2, C_2) \end{cases}$$

则有

$$\begin{cases} \vec{l} \cdot \vec{n_1} = 0 \\ \vec{l} \cdot \vec{n_2} = 0 \end{cases}$$

即

$$\vec{l} = \vec{n_1} \times \vec{n_2}$$

多元函数

函数$f(x, y, z)$

全微分

$$\mathrm{d}f = \frac{\partial f}{\partial x}\mathrm{d}x + \frac{\partial f}{\partial y}\mathrm{d}y + \frac{\partial f}{\partial z}\mathrm{d}z$$

梯度

$$\bigtriangledown f = \frac{\partial f}{\partial x}\vec{e_x} + \frac{\partial f}{\partial y}\vec{e_y} + \frac{\partial f}{\partial z}\vec{e_z}$$

线元

$$d\vec{r} = \mathrm{d}x \cdot \vec{e_x} + \mathrm{d}y \cdot \vec{e_y} + \mathrm{d}z \cdot \vec{e_z}$$

有

$$\mathrm{d}f = \bigtriangledown f \cdot \mathrm{d}\vec{r}$$

> 类比一维情况$\mathrm{d}f(x) = f'(x)\mathrm{d}x$，以直观上的理解，$\bigtriangledown f$是变化率，$\mathrm{d}\vec{r}$是变化量

$x = X(u, v, w), y = Y(u, v, w), z = Z(u, v, w)$，链式法则

$$\frac{\partial f}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial u}$$

$$\frac{\partial f}{\partial v} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial v}$$

$$\frac{\partial f}{\partial w} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial w} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial w} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial w}$$

散度

散度是针对矢量场，表示各个方向上的曲面上是否有逃逸

对于三维矢量函数

$$F(x, y, z) = F_x(x, y, z)\vec{e_x} + F_y(x, y, z)\vec{e_y} + F_z(x, y, z)\vec{e_z}$$

以x方向为例

$$X_- = -F_x(x, y, z)\mathrm{d}y\mathrm{d}z$$

$$X_+ = F_x(x + \mathrm{d}x, y, z)\mathrm{d}y\mathrm{d}z$$

即

$$X_+ + X_- = [F_x(x + \mathrm{d}x, y, z) - F(x, y, z)]\mathrm{d}y\mathrm{d}z = \frac{\partial F_x}{\partial x}\mathrm{d}x\mathrm{d}y\mathrm{d}z$$

六个方向和为

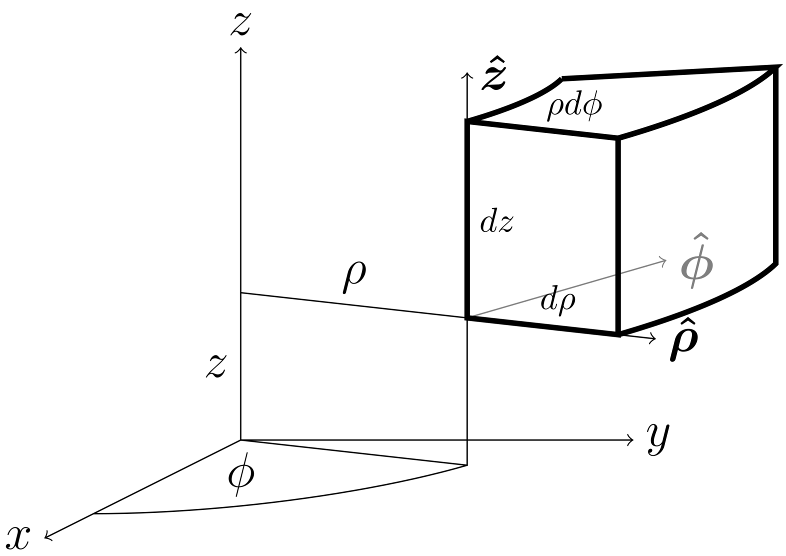
$$\left(\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}\right)\mathrm{d}x\mathrm{d}y\mathrm{d}z = \left(\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}\right)\mathrm{d}V$$

散度是针对某点的逃逸的标量，消去体积元$\mathrm{d}V$，即

$$\mathbf{div}F = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} = \bigtriangledown \cdot F$$

坐标系

柱坐标系



$$\begin{cases} x = \rho\cos\varphi \\ x = \rho\sin\varphi \\ z = z \end{cases}$$

基向量

$$\begin{cases} \vec{e_\rho} = (\cos\varphi, \sin\varphi, 0) \\ \vec{e_\varphi} = (-\sin\varphi, \cos\varphi, 0) \\ \vec{z} = (0, 0, 1) \end{cases}$$

线元

$$\mathrm{d}\vec{r} = \mathrm{d}\rho\cdot\vec{e_\rho} + \rho\mathrm{d}\varphi\cdot\vec{e_\varphi} + \mathrm{d}z\cdot\vec{e_z}$$

xy面元

$$\mathrm{d}S = \rho\mathrm{d}\rho\mathrm{d}\theta$$

体元

$$\mathrm{d}V = \rho\mathrm{d}\rho\mathrm{d}\theta\mathrm{d}z$$

根据

$$\mathrm{d}f = \triangledown f\cdot\mathrm{d}\vec{r}$$

根据基向量互相正交

$$\bigtriangledown f=\frac{\partial f}{\partial\rho}\vec{e_\rho}+\frac{1}{\rho}\frac{\partial f}{\partial\varphi}\vec{e_\varphi}+\frac{\partial f}{\partial z}\vec{e_z}$$

由于

$$\bigtriangleup=\bigtriangledown\cdot\bigtriangledown$$

即

$$\bigtriangleup=(\frac{\partial}{\partial\rho}\mathrm{d}\rho+\frac{\partial}{\partial\varphi}\mathrm{d}\varphi+\frac{\partial}{\partial z}\mathrm{d}z)\cdot(\frac{\partial}{\partial\rho}\mathrm{d}\rho+\frac{\partial}{\partial\varphi}\mathrm{d}\varphi+\frac{\partial}{\partial z}\mathrm{d}z)$$

展开化简，根据链式法则，带入下面计算的结果

$$\frac{\partial\vec{e_i}}{\partial j}\cdots(i\in\{\rho,\varphi,z\},j\in\{\rho,\varphi,z\})$$

化简得

$$\bigtriangleup=\frac{1}{\rho}\frac{\partial}{\partial\rho}+\frac{\partial^2}{\partial\rho^2}+\frac{1}{\rho^2}\frac{\partial^2}{\partial\varphi^2}+\frac{\partial^2}{\partial z^2}$$

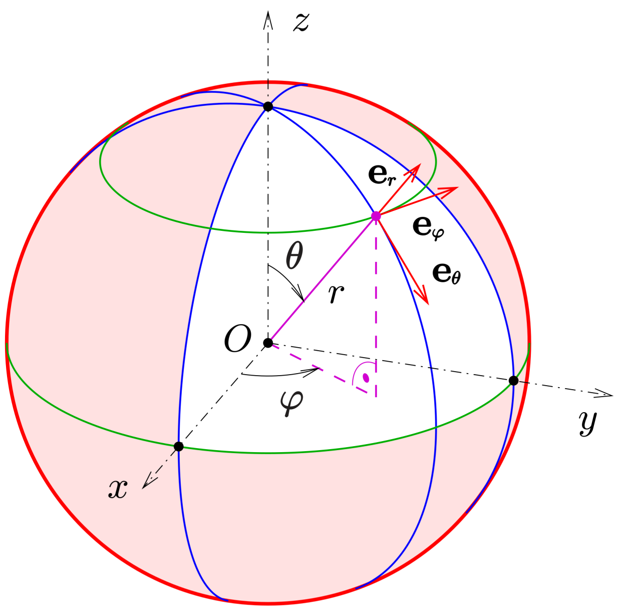
凑微分

$$\bigtriangleup=\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho\frac{\partial}{\partial\rho})+\frac{1}{\rho^2}\frac{\partial^2}{\partial\varphi^2}+\frac{\partial^2}{\partial z^2}$$

即

$$\bigtriangleup f=\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho\frac{\partial f}{\partial\rho})+\frac{1}{\rho^2}\frac{\partial^2 f}{\partial\varphi^2}+\frac{\partial^2 f}{\partial z^2}$$

球坐标系



$$
\begin{cases}
x = r\sin\theta\cos\varphi \\
y = r\sin\theta\sin\varphi \\
z = r\cos\theta
\end{cases}
$$

基向量

$$
\begin{cases}
\vec{e_r} = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta) \\
\vec{e_\theta} = (\cos\theta\cos\varphi, \cos\theta\sin\varphi, -\sin\theta) \\
\vec{e_\varphi} = (-\sin\varphi, \cos\varphi, 0)
\end{cases}
$$

由于

$$
\vec{r} = (r\sin\theta\cos\varphi, r\sin\theta\sin\varphi, r\cos\theta)
$$

即

$$
\left|\frac{\partial\vec{r}}{\partial r}\right| = |(\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)| = r
$$

$$
\left|\frac{\partial\vec{r}}{\partial\theta}\right| = |(r\cos\theta\cos\varphi, r\cos\theta\sin\varphi, -r\sin\theta)| = r
$$

$$
\left|\frac{\partial\vec{r}}{\partial\varphi}\right| = |(-r\sin\theta\sin\varphi, r\sin\theta\cos\varphi, 0)| = r\sin\theta
$$

线元

$$
\mathrm{d}\vec{r} = \mathrm{d}r\cdot\vec{e_r} + r\mathrm{d}\theta\cdot\vec{e_\theta} + r\sin\theta\mathrm{d}\varphi\cdot\vec{e_\varphi}
$$

根据

$$
\mathrm{d}f = \bigtriangledown f\cdot\mathrm{d}\vec{r}
$$

即

$$
\bigtriangledown f = \frac{\partial f}{\partial r}\vec{e_r} + \frac{1}{r}\frac{\partial f}{\partial\theta}\vec{e_\theta} + \frac{1}{r\sin\theta}\frac{\partial f}{\partial\varphi}\vec{e_\varphi}
$$

由于

$$
\triangle = \bigtriangledown\cdot\bigtriangledown
$$

即

$$
\triangle = (\frac{\partial}{\partial r}\vec{e_r} + \frac{1}{r}\frac{\partial}{\partial\theta}\vec{e_\theta} + \frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}\vec{e_\varphi})\cdot(\frac{\partial}{\partial r}\vec{e_r} + \frac{1}{r}\frac{\partial}{\partial\theta}\vec{e_\theta} + \frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}\vec{e_\varphi})
$$

展开化简，根据链式法则，带入下面计算的结果

$$\frac{\partial \vec{e_i}}{\partial j}\cdots(i\in\{r,\theta,\varphi\},j\in\{r,\theta,\varphi\})$$

化简得

$$\triangle=\frac{2}{r}\frac{\partial}{\partial r}+\frac{\partial^2}{\partial r^2}+\frac{\cos\theta}{r^2\sin\theta}\frac{\partial}{\partial\theta}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}+\frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}$$

凑微分

$$\triangle=\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial}{\partial r})+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial}{\partial\theta})+\frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}$$

即

$$\triangle=\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial f}{\partial r})+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial f}{\partial\theta})+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial\varphi^2}$$

曲线积分

标量曲线积分

类比求曲线物体的质量

$$dS=\sqrt{(\mathrm{d}x)^2+(\mathrm{d}y)^2}=\sqrt{1+(y')^2}\cdot\mathrm{d}x=\sqrt{1+(x')^2}\cdot\mathrm{d}y\cdots(\mathrm{d}x>0,\mathrm{d}y>0)$$

$$\int_L f(x,y)\mathrm{d}S$$

矢量曲线积分

类比求曲线运动的功

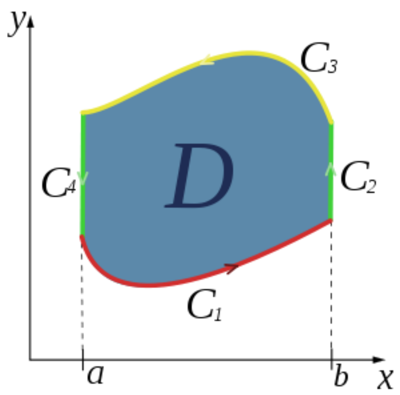
$$F(x,y)=f_x(x,y)\vec{e_x}+f_y(x,y)\vec{e_y}$$

$$dS=\mathrm{d}x\cdot\vec{e_x}+\mathrm{d}y\cdot\vec{e_y}$$

$$\int_L F(x,y)\mathrm{d}S=\int_L f_x(x,y)\mathrm{d}x+f_y(x,y)\mathrm{d}y$$

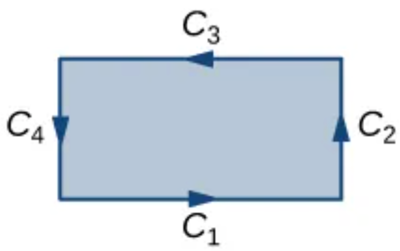
Green’s theorem

Green's theorem是环路积分与二重积分转换的工具



将区域D分成无限小的下面的方格，由于中间相邻的曲线积分都被抵消，因此可以看作全体方格的曲线积分的和

对于单个方格



$$C_1 = P\mathrm{d}x$$

$$C_2 = (Q + \frac{\partial Q}{\partial x}\mathrm{d}x)\mathrm{d}y$$

$$C_3 = (P + \frac{\partial P}{\partial y}\mathrm{d}y)(-\mathrm{d}x)$$

$$C_4 = Q(-\mathrm{d}y)$$

即

$$C_1 + C_2 + C_3 + C_4 = \left(\frac{\partial P}{\partial x} - \frac{\partial P}{\partial y}\right)\mathrm{d}x\mathrm{d}y$$

得到Green's theorem

$$\oint_L P\mathrm{d}x + Q\mathrm{d}y = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathrm{d}x\mathrm{d}y$$

对于A到B的路径L_1

$$\int_L P\mathrm{d}x + Q\mathrm{d}y$$

补充任意一段B到A的路径L_2使其形成环

$$\int_{L_1} P\mathrm{d}x + Q\mathrm{d}y + \int_{L_2} P\mathrm{d}x + Q\mathrm{d}y = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathrm{d}x\mathrm{d}y$$

当满足

$$\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}=0$$

即

$$\int_{L_1}P\mathrm{d}x+Q\mathrm{d}y=-\int_{L_2}P\mathrm{d}x+Q\mathrm{d}y$$

因此路径L_1的积分与任意A到B的路径L_2的积分都相等，即路径无关

曲面积分

标量曲面积分

类比求曲面物体的质量

设曲面$z=Z(x,y)$，面微元dS在xy平面上投影为

$$\mathrm{d}x\mathrm{d}y$$

该曲面等价于向量函数$\vec{v}(x,y)=(x,y,Z(x,y))$

$$\frac{\partial \vec{v}}{\partial x}=\left(1,0,\frac{\partial Z}{\partial x}\right)$$

$$\frac{\partial \vec{v}}{\partial y}=\left(0,1,\frac{\partial Z}{\partial y}\right)$$

代入有

$$dS=|\frac{\partial \vec{v}}{\partial x}\times\frac{\partial \vec{v}}{\partial y}|\mathrm{d}x\mathrm{d}y=\left|\left(-\frac{\partial Z}{\partial x},-\frac{\partial Z}{\partial y},1\right)\right|\mathrm{d}x\mathrm{d}y=\sqrt{\left(\frac{\partial Z}{\partial x}\right)^2+\left(\frac{\partial Z}{\partial y}\right)^2+1}\,\mathrm{d}x\mathrm{d}y$$

即

$$\iint_S f(x,y,Z(x,y))\sqrt{\left(\frac{\partial Z}{\partial x}\right)^2+\left(\frac{\partial Z}{\partial y}\right)^2+1}\,\mathrm{d}x\mathrm{d}y$$

矢量曲面积分

类比求曲面物体通过不同截面的流量，推导类比矢量曲线积分，注意方向正负

$$\iint_S f(X(y,z),y,z)\mathrm{d}y\mathrm{d}z+f(x,Y(x,z),z)\mathrm{d}x\mathrm{d}z+f(x,y,Z(x,y))\mathrm{d}x\mathrm{d}y$$