图形学需要的数学

向量

设
$$ec{a}=(x_1,y_1,z_1),ec{b}=(x_2,y_2,z_2)$$

$$ec{a} \cdot ec{b} = |ec{a}| |ec{b}| \cos heta = x_1 y_1 + x_2 y_2 + x_3 y_3$$

$$|ec{a} imesec{b}|=|ec{a}||ec{b}|\sin heta$$

$$ec{a} imesec{b} = egin{array}{ccc} i & j & k \ x_1 & y_1 & z_1 \ x_2 & y_2 & z_2 \ \end{array} = (y_1z_2 - y_2z_1, x_2z_1 - x_1z_2, x_1y_z - x_2y_1)$$

设平面 $S_1=A_1x+B_1y+C_1z+D=0, S_2=A_2x+B_2y+C_2z+D=0$, 当 $S_1\cap S_2=L\neq\emptyset$

$$egin{cases} ec{n_1} = (A_1, B_1, C_1) \ ec{n_2} = (A_2, B_2, C_2) \end{cases}$$

则有

$$egin{cases} ec{l} \cdot ec{n_1} = 0 \ ec{l} \cdot ec{n_2} = 0 \end{cases}$$

即

$$ec{l}=ec{n_1} imesec{n_2}$$

多元函数

函数f(x, y, z)

全微分

$$\mathrm{d}f = \frac{\partial f}{\partial x} \mathrm{d}x + \frac{\partial f}{\partial y} \mathrm{d}y + \frac{\partial f}{\partial z} \mathrm{d}z$$

梯度

$$igtriangledown f = rac{\partial f}{\partial x}ec{e_x} + rac{\partial f}{\partial y}ec{e_y} + rac{\partial f}{\partial z}ec{e_z}$$

线元

$$d\vec{r} = \mathrm{d}x \cdot \vec{e_x} + \mathrm{d}y \cdot \vec{e_y} + \mathrm{d}z \cdot \vec{e_z}$$

有

$$\mathrm{d}f = \nabla f \cdot \mathrm{d}\vec{r}$$

类比一维情况df(x)=f'(x)dx,以直观上的理解, ∇f 是变化率,dr是变化量 x=X(u,v,w),y=Y(u,v,w),z=Z(u,v,w),链式法则 $\frac{\partial f}{\partial u}=\frac{\partial f}{\partial x}\frac{\partial x}{\partial u}+\frac{\partial f}{\partial y}\frac{\partial y}{\partial u}+\frac{\partial f}{\partial z}\frac{\partial z}{\partial u}$ $\frac{\partial f}{\partial v}=\frac{\partial f}{\partial x}\frac{\partial x}{\partial v}+\frac{\partial f}{\partial y}\frac{\partial y}{\partial v}+\frac{\partial f}{\partial z}\frac{\partial z}{\partial v}$ $\frac{\partial f}{\partial w}=\frac{\partial f}{\partial x}\frac{\partial x}{\partial w}+\frac{\partial f}{\partial y}\frac{\partial y}{\partial w}+\frac{\partial f}{\partial z}\frac{\partial z}{\partial w}$

散度

散度是针对矢量场,表示各个方向上的曲面上是否有逃逸

$$F(x, y, z) = F_x(x, y, z)\vec{e_x} + F_y(x, y, z)\vec{e_y} + F_z(x, y, z)\vec{e_z}$$

以x方向为例

对于三维矢量函数

$$X_- = -F_x(x,y,z) \mathrm{d}y \mathrm{d}z$$
 $X_+ = F_x(x+\mathrm{d}x,y,z) \mathrm{d}y \mathrm{d}z$

即

$$X_+ + X_- = [F_x(x+\mathrm{d} x,y,z) - F(x,y,z)] \mathrm{d} y \mathrm{d} z = rac{\partial F_x}{\partial x} \mathrm{d} x \mathrm{d} y \mathrm{d} z$$

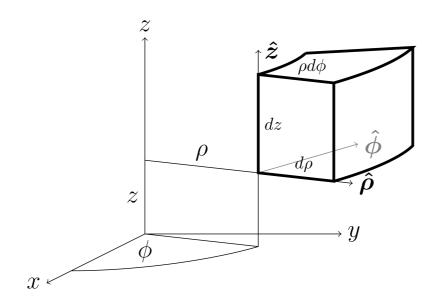
六个方向和为

$$\left(\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}\right) dx dy dz = \left(\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}\right) dV$$

散度是针对某点的逃逸的标量,消去体积元 $\mathrm{d}V$,即

$$\mathbf{div}F = rac{\partial F_x}{\partial x} + rac{\partial F_y}{\partial y} + rac{\partial F_z}{\partial z} = \mathbf{\nabla} \cdot F$$

柱坐标系



$$\begin{cases} x = \rho \cos \varphi \\ x = \rho \sin \varphi \\ z = z \end{cases}$$

基向量

$$\left\{ egin{aligned} ec{e_{
ho}} &= (\cosarphi, \sinarphi, 0) \ ec{e_{arphi}} &= (-\sinarphi, \cosarphi, 0) \ ec{z} &= (0, 0, 1) \end{aligned}
ight.$$

线元

$$\mathrm{d}\vec{r} = \mathrm{d}
ho\cdot\vec{e_{
ho}} +
ho\mathrm{d}arphi\cdot\vec{e_{arphi}} + \mathrm{d}z\cdot\vec{e_{z}}$$

xy面元

$$dS = \rho d\rho d\theta$$

体元

$$dV = \rho d\rho d\theta dz$$

根据

$$\mathrm{d}f = igtriangledown f\cdot \mathrm{d}ec{r}$$

根据基向量互相正交

$$igtriangledown f = rac{\partial f}{\partial
ho} ec{e_
ho} + rac{1}{
ho} rac{\partial f}{\partial arphi} ec{e_arphi} + rac{\partial f}{\partial z} ec{e_z}$$

由于

$$\triangle = \nabla \cdot \nabla$$

即

$$\triangle = \left(\frac{\partial}{\partial \rho} d\rho + \frac{\partial}{\partial \varphi} d\varphi + \frac{\partial}{\partial z} dz\right) \cdot \left(\frac{\partial}{\partial \rho} d\rho + \frac{\partial}{\partial \varphi} d\varphi + \frac{\partial}{\partial z} dz\right)$$

展开化简,根据链式法则,带入下面计算的结果

$$rac{\partial ec{e_i}}{\partial j} \cdots (i \in \{
ho, arphi, z\}, j \in \{
ho, arphi, z\})$$

化简得

$$riangle = rac{1}{
ho}rac{\partial}{\partial
ho} + rac{\partial^2}{\partial
ho^2} + rac{1}{
ho^2}rac{\partial^2}{\partialarphi^2} + rac{\partial^2}{\partial z^2}$$

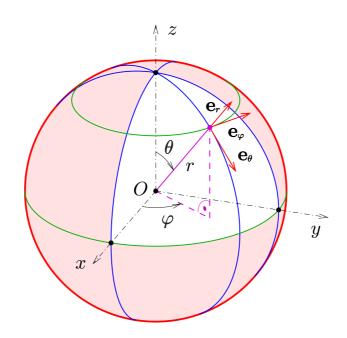
凑微分

$$riangle = rac{1}{
ho}rac{\partial}{\partial
ho}(
horac{\partial}{\partial
ho}) + rac{1}{
ho^2}rac{\partial^2}{\partialarphi^2} + rac{\partial^2}{\partial z^2}$$

即

$$riangle f = rac{1}{
ho}rac{\partial}{\partial
ho}(
horac{\partial f}{\partial
ho}) + rac{1}{
ho^2}rac{\partial^2 f}{\partialarphi^2} + rac{\partial^2 f}{\partial z^2}$$

球坐标系



$$\begin{cases} x = r \sin \theta \cos \varphi \\ y = r \sin \theta \sin \varphi \\ z = r \cos \theta \end{cases}$$

基向量

$$\begin{cases} \vec{e_r} = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta) \\ \vec{e_\theta} = (\cos\theta\cos\varphi, \cos\theta\sin\varphi, -\sin\theta) \\ \vec{e_\varphi} = (-\sin\varphi, \cos\varphi, 0) \end{cases}$$

由于

$$\vec{r} = (r\sin\theta\cos\varphi, r\sin\theta\sin\varphi, r\cos\theta)$$

即

$$\left| \frac{\partial \vec{r}}{\partial r} \right| = \left| (\sin \theta \cos \varphi, \sin \theta \sin \varphi, \cos \theta) \right| = r$$

$$\left| \frac{\partial \vec{r}}{\partial \theta} \right| = \left| (r \cos \theta \cos \varphi, r \cos \theta \sin \varphi, -r \sin \theta) \right| = r$$

$$\left| \frac{\partial \vec{r}}{\partial \varphi} \right| = \left| (-r \sin \theta \sin \varphi, r \sin \theta \cos \varphi, 0) \right| = r \sin \theta$$

线元

$$\mathrm{d}\vec{r} = \mathrm{d}r\cdot\vec{e_r} + r\mathrm{d}\theta\cdot\vec{e_\theta} + r\sin\theta\mathrm{d}arphi\cdot\vec{e_arphi}$$

根据

$$\mathrm{d}f = \bigtriangledown f \cdot \mathrm{d}\vec{r}$$

即

$$igtriangledown f = rac{\partial f}{\partial r}ec{e_r} + rac{1}{r}rac{\partial f}{\partial heta}ec{e_ heta} + rac{1}{r\sin heta}rac{\partial f}{\partial arphi}ec{e_arphi}$$

由于

$$\triangle = \nabla \cdot \nabla$$

即

$$\triangle = \left(\frac{\partial}{\partial r}\vec{e_r} + \frac{1}{r}\frac{\partial}{\partial \theta}\vec{e_\theta} + \frac{1}{r\sin\theta}\frac{\partial}{\partial \varphi}\vec{e_\varphi}\right) \cdot \left(\frac{\partial}{\partial r}\vec{e_r} + \frac{1}{r}\frac{\partial}{\partial \theta}\vec{e_\theta} + \frac{1}{r\sin\theta}\frac{\partial}{\partial \varphi}\vec{e_\varphi}\right)$$

展开化简,根据链式法则,带入下面计算的结果

$$rac{\partial ec{e}_i}{\partial j} \cdots (i \in \{r, heta, arphi\}, j \in \{r, heta, arphi\})$$

化简得

$$riangle = rac{2}{r}rac{\partial}{\partial r} + rac{\partial^2}{\partial r^2} + rac{\cos heta}{r^2\sin heta}rac{\partial}{\partial heta} + rac{1}{r^2}rac{\partial^2}{\partial heta^2} + rac{1}{r^2\sin^2 heta}rac{\partial^2}{\partial arphi^2}$$

凑微分

$$riangle = rac{1}{r^2}rac{\partial}{\partial r}(r^2rac{\partial}{\partial r}) + rac{1}{r^2\sin heta}rac{\partial}{\partial heta}(\sin hetarac{\partial}{\partial heta}) + rac{1}{r^2\sin^2 heta}rac{\partial^2}{\partial arphi^2}$$

即

$$riangle = rac{1}{r^2}rac{\partial}{\partial r}(r^2rac{\partial f}{\partial r}) + rac{1}{r^2\sin heta}rac{\partial}{\partial heta}(\sin hetarac{\partial f}{\partial heta}) + rac{1}{r^2\sin^2 heta}rac{\partial^2 f}{\partial arphi^2}$$

曲线积分

标量曲线积分

类比求曲线物体的质量

$$dS = \sqrt{(\mathrm{d}x)^2 + (\mathrm{d}y)^2} = \sqrt{1 + (y')^2} \cdot \mathrm{d}x = \sqrt{1 + (x')^2} \cdot \mathrm{d}y \cdots (\mathrm{d}x > 0, \mathrm{d}y > 0)$$

$$\int_L f(x,y) \mathrm{d}S$$

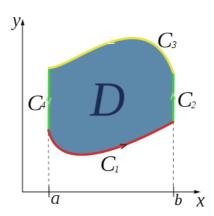
矢量曲线积分

类比求曲线运动的功

$$F(x,y) = f_x(x,y) ec{e_x} + f_y(x,y) ec{e_y}$$
 $dS = \mathrm{d}x \cdot ec{e_x} + \mathrm{d}y \cdot ec{e_y}$ $\int_L F(x,y) \mathrm{d}S = \int_L f_x(x,y) \mathrm{d}x + f_y(x,y) \mathrm{d}y$

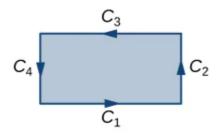
Green's theorem

Green's theorem是环路积分与二重积分转换的工具



将区域D分成无限小的下面的方格,由于中间相邻的曲线积分都被抵消,因此可以看作全体方格的曲线积分的和

对于单个方格



$$C_1 = P \mathrm{d} x$$
 $C_2 = (Q + rac{\partial Q}{\partial x} \mathrm{d} x) \mathrm{d} y$ $C_3 = (P + rac{\partial P}{\partial y} \mathrm{d} y)(-\mathrm{d} x)$ $C_4 = Q(-\mathrm{d} y)$

即

$$C_1+C_2+C_3+C_4=igg(rac{\partial P}{\partial x}-rac{\partial P}{\partial y}igg)\mathrm{d}x\mathrm{d}y$$

得到Green's theorem

$$\oint_L P \mathrm{d}x + Q \mathrm{d}y = \iint_D \left(rac{\partial Q}{\partial x} - rac{\partial P}{\partial y}
ight) \mathrm{d}x \mathrm{d}y$$

对于A到B的路径 L_1

$$\int_L P \mathrm{d}x + Q \mathrm{d}y$$

补充任意一段B到A的路径 L_2 使其形成环

$$\int_{L_1} P \mathrm{d}x + Q \mathrm{d}y + \int_{L_2} P \mathrm{d}x + Q \mathrm{d}y = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \mathrm{d}x \mathrm{d}y$$

当满足

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 0$$

即

$$\int_{L_1} P \mathrm{d}x + Q \mathrm{d}y = -\int_{L_2} P \mathrm{d}x + Q \mathrm{d}y$$

因此路径 L_1 的积分与任意A到B的路径 L_2 的积分都相等,即路径无关

曲面积分

标量曲面积分

类比求曲面物体的质量

设曲面z = Z(x, y), 面微元dS在xy平面上投影为

 $\mathrm{d}x\mathrm{d}y$

该曲面等价于向量函数 $\vec{v}(x,y)=(x,y,Z(x,y))$

$$\frac{\partial \vec{v}}{\partial x} = \left(1, 0, \frac{\partial Z}{\partial x}\right)$$

$$\frac{\partial \vec{v}}{\partial y} = \left(0, 1, \frac{\partial Z}{\partial y}\right)$$

代入有

$$dS = |rac{\partial ec{v}}{\partial x} imes rac{\partial ec{v}}{\partial y}| \mathrm{d}x \mathrm{d}y = \left| \left(-rac{\partial Z}{\partial x}, -rac{\partial Z}{\partial y}, 1
ight)
ight| \mathrm{d}x \mathrm{d}y = \sqrt{\left(rac{\partial Z}{\partial x}
ight)^2 + \left(rac{\partial Z}{\partial y}
ight)^2 + 1} \; \mathrm{d}x \mathrm{d}y$$

即

$$\iint_{S} f(x, y, Z(x, y)) \sqrt{\left(\frac{\partial Z}{\partial x}\right)^{2} + \left(\frac{\partial Z}{\partial y}\right)^{2} + 1} \, \mathrm{d}x \mathrm{d}y$$

矢量曲面积分

类比求曲面物体通过不同截面的流量,推导类比矢量曲线积分,注意方向正负

$$\iint_S f(X(y,z),y,z) \mathrm{d}y \mathrm{d}z + f(x,Y(x,z),z) \mathrm{d}x \mathrm{d}z + f(x,y,Z(x,y)) \mathrm{d}x \mathrm{d}y$$