Classically Simulating Quantum Circuits with Local Depolarizing Noise

Yasuhiro Takahashi[1,a)] Yuki Takeuchi[1] Seiichiro Tani[1]

Abstract : We study the effect of noise on the classical simulatability of quantum circuits defined by computationally tractable (CT) states and efficiently computable sparse (ECS) operations. Examples of such circuits, which we call CT-ECS circuits, are IQP, Clifford Magic, and conjugated Clifford circuits. This means that there exist various CT-ECS circuits such that their output probability distributions are anti-concentrated and not classically simulatable in the noise-free setting (under plausible assumptions). First, we consider a noise model where a depolarizing channel with an arbitrarily small constant rate is applied to each qubit at the end of computation. We show that, under this noise model, if an approximate value of the noise rate is known, any CT-ECS circuit with an anti-concentrated output probability distribution is classically simulatable. This indicates that the presence of small noise drastically affects the classical simulatability of CT-ECS circuits. Then, we consider an extension of the noise model where the noise rate can vary with each qubit, and provide a similar sufficient condition for classically simulating CT-ECS circuits with anti-concentrated output probability distributions. Details can be found in [13].

Keywords : quantum computing, quantum circuit, classical simulation, local depolarizing noise

1. Background and Main Results

A key step toward realizing a large-scale universal quantum computer is to demonstrate quantum computational supremacy [10], i.e., to perform computational tasks that are classically hard. As such a task, many researchers have focused on simulating quantum circuits, or more concretely, sampling the output probability distributions of quantum circuits. They have shown that, under plausible complexity-theoretic assumptions, this task is classically hard for various quantum circuits that seem easier to implement than universal ones. However, these classical hardness results have been obtained in severely restricted settings, such as a noise-free setting with additive approximation [3], [5], [14], [17] and a noise setting with multiplicative approximation [8]: the former requires us to sample the output probability distribution of a quantum circuit with additive error and the latter to sample it under a noise model with multiplicative error. Thus, there is great interest in considering the above task in a more reasonable setting.

We study the classical simulatability of quantum circuits in a noise setting with additive approximation, which requires us to sample the output probability distribution of a quantum circuit under a noise model with additive error. This setting is more reasonable than the noise-free setting since the presence of noise is unavoidable in realistic situations. Moreover, our setting is more reasonable than the noise setting with multiplicative approximation in the sense that we adopt a more realistic notion of approximation [1], [5], although noise we deal with in this paper is more restrictive than that in [8].

We consider a noise model where a depolarizing channel D_ε with an arbitrarily small constant rate $0 < \varepsilon < 1$ is applied to each qubit at the end of computation. This channel leaves a qubit unaffected with probability $1 - \varepsilon$ and replaces its state with the completely mixed one with probability ε. We call this model noise model **A**. We also consider its extension where the noise rate can vary with each qubit. More concretely, when we have n qubits, D_{ε_j} is applied to the j-th qubit at the end of computation for

1 NTT Communication Science Laboratories, NTT Corporation, 3-1 Morinosato-Wakamiya, Atsugi, Kanagawa 243-0198, Japan
a) yasuhiro.takahashi.rb@hco.ntt.co.jp

any $1 \leq j \leq n$. We call this model noise model **B**. Although these noise models are simple, analyzing them is a meaningful step toward studying more general noise models, such as one where noise exists before and after each gate in a quantum circuit [9]. This is because, for example, this general model is equivalent to noise model **A** when we focus on instantaneous quantum polynomial-time (IQP) circuits, which are described below, with a particular type of intermediate noise [6].

A representative example of a quantum circuit that is not classically simulatable in the noise-free setting is an IQP circuit, which consists of Z-diagonal gates sandwiched by two Hadamard layers. In fact, there exists an IQP circuit such that its output probability distribution is anti-concentrated and not classically samplable in polynomial time with certain constant accuracy in l_1 norm (under plausible assumptions) [5]. On the other hand, Bremner et al. [6] assumed noise model **A** and showed that, if the *exact* value of the noise rate is known, any IQP circuit with an anti-concentrated output probability distribution is classically simulatable in the sense that the resulting probability distribution is classically samplable in polynomial time with arbitrary constant accuracy in l_1 norm. This indicates that, under noise model **A**, if the exact value of the noise rate is known, the presence of small noise drastically affects the classical simulatability of IQP circuits.

In this paper, first, under a weaker assumption on the knowledge of the noise rate, we extend Bremner et al.'s result to a new class of quantum circuits defined by two concepts: computationally tractable (CT) states and efficiently computable sparse (ECS) operations [16]. Examples of such circuits, which we call CT-ECS circuits, are IQP circuits, Clifford Magic circuits [17], and conjugated Clifford circuits [3]. This means that there exist various CT-ECS circuits such that their output probability distributions are anti-concentrated and not classically simulatable in the sense described above for IQP circuits. Constant-depth quantum circuits [2], [4], [15] are also CT-ECS circuits and not classically simulatable, although we do not know whether their output probability distributions are anti-concentrated. We omit the explanation of CT states and ECS operations, which can be found in [13], [16], but, as depicted in Fig. 1(a), a CT-ECS circuit on n qubits is a polynomial-size quantum circuit $C = VU$ such that $U|0^n\rangle$ is CT and $V^\dagger Z_j V$ is ECS for any $1 \leq j \leq n$, where Z_j is a Pauli-Z operation on the j-th qubit. After performing C, we perform Z-basis measurements on all qubits. The CT-ECS circuit C under noise model **B** is depicted in Fig. 1(b).

Our first result assumes noise model **A**, which corresponds to the case where $\varepsilon_j = \varepsilon$ for any $1 \leq j \leq n$ in Fig. 1(b). We show that, if an *approximate* value of the noise rate is known, any CT-ECS circuit with an anti-concentrated output probability distribution is classically simulatable:

Theorem 1 (informal) Let C be an arbitrary CT-ECS circuit on n qubits such that its output probability distribution p is anti-concentrated, i.e.,

$$\sum_{x \in \{0,1\}^n} p(x)^2 \leq \frac{\alpha}{2^n}$$

for some known constant $\alpha \geq 1$. We assume that a depolarizing channel with (possibly unknown) constant rate $0 < \varepsilon < 1$ is applied to each qubit after performing C, which yields the probability distribution $\widetilde{p}_{\mathrm{A}}$. Moreover, we assume that it is possible to choose a constant λ such that

$$1 \leq \frac{\varepsilon}{\lambda} \leq 1 + c,$$

where c is a certain constant depending on α. Then, $\widetilde{p}_{\mathrm{A}}$ is classically samplable in polynomial time with constant accuracy in l_1 norm.

If ε is known, we can choose $\lambda = \varepsilon$. This case with IQP circuits is Bremner et al.'s result [6]. As described above, there exist various CT-ECS circuits such that their output probability distributions are anti-concentrated and not classically simulatable in the noise-free setting (under plausible assumptions). Thus, Theorem 1 indicates that, under noise model **A**, if an *approximate* value of the noise rate is known, the presence of small noise drastically affects the classical simulatability of CT-ECS circuits.

Theorem 1 assumes noise model **A** where noise exists only at the end of computation, but, in some cases, it also holds under an input-noise model. For example, Theorem 1 holds for IQP circuits under a noise model where D_ε is applied to each qubit only at the *start* of computation, although Bu and Koh's main result implies a similar property of IQP circuits [7]. Moreover, Theorem 2, which is described below and assumes noise model **B**, also holds for IQP circuits under an input-noise model where the noise rate can vary with each qubit. Our main result is similar in spirit to Bu and Koh's, which provides classical algorithms for simulating Clifford circuits with nonstabilizer product input states (corresponding to input-noise models). However, in general, it is difficult to relate the output probability distributions of Clifford circuits un-

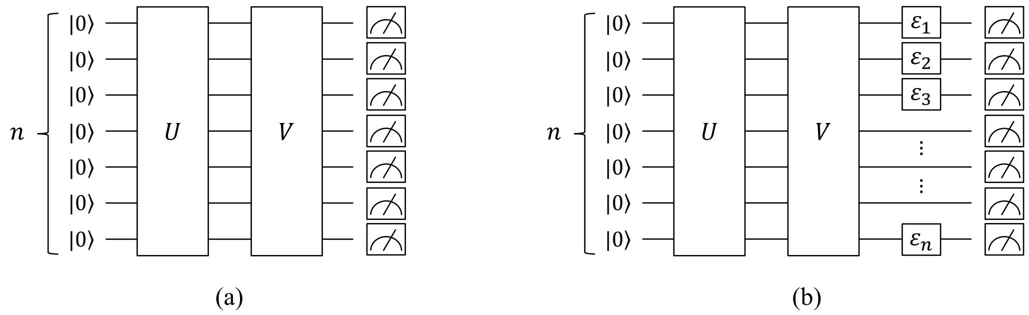

Fig. 1 (a): CT-ECS circuit $C = VU$, where $U|0^n\rangle$ is CT and $V^\dagger Z_j V$ is ECS for any $1 \leq j \leq n$. After performing C, we perform Z-basis measurements on all qubits. (b): CT-ECS circuit $C = VU$ under noise model **B**, where ε_j represents the depolarizing channel D_{ε_j} for any $1 \leq j \leq n$.

der input-noise models to those of CT-ECS circuits under output-noise models.

Our main focus is on a noise setting, but, from a purely theoretical point of view, it is valuable to consider the noise-free setting. The proof method of Theorem 1 is based on computing the Fourier coefficients of an output probability distribution, and is useful in the noise-free setting. In fact, the proof method shows that, when only $O(\log n)$ qubits are measured, any quantum circuit in a class of CT-ECS circuits on n qubits is classically simulatable. More precisely, its output probability distribution is classically samplable in polynomial time with polynomial accuracy in l_1 norm. This class of CT-ECS circuits is defined by a restricted version of ECS operations, and includes IQP, Clifford Magic, conjugated Clifford, and constant-depth quantum circuits. It is known that the above property or a similar one holds for these quantum circuits, but the proofs provided have depended on each circuit class [3], [4], [11], [15]. Our analysis unifies the previous ones and clarifies a class of quantum circuits for which the above property holds.

Our second result assumes noise model **B**, which is depicted in Fig. 1(b). For classically simulating CT-ECS circuits with anti-concentrated output probability distributions, we provide a sufficient condition, which is similar to Theorem 1:

Theorem 2 (informal) Let C be an arbitrary CT-ECS circuit on n qubits such that its output probability distribution p is anti-concentrated, i.e.,

$$\sum_{x\in\{0,1\}^n} p(x)^2 \leq \frac{\alpha}{2^n}$$

for some known constant $\alpha \geq 1$. We assume that a depolarizing channel with (possibly unknown) constant rate $0 < \varepsilon_j < 1$ is applied to the j-th qubit after performing C for any $1 \leq j \leq n$, which yields the probability distribution $\widetilde{p}_{\mathrm{B}}$. Moreover, we assume that it is possible to choose a constant $\lambda_{\min}$ such that

$$1 \leq \frac{\varepsilon_{\min}}{\lambda_{\min}} \leq 1 + c,$$

where $\varepsilon_{\min} = \min\{\varepsilon_j | 1 \leq j \leq n\}$ and c is a certain constant depending on α, and we assume that it is possible to choose a constant λ_j such that

$$0 \leq \varepsilon_j - \lambda_j \leq c\lambda_{\min}$$

for any $1 \leq j \leq n$ with $\varepsilon_j \neq \varepsilon_{\min}$. Then, $\widetilde{p}_{\mathrm{B}}$ is classically samplable in polynomial time with constant accuracy in l_1 norm.

To the best of our knowledge, this is the first analysis of the classical simulatability of quantum circuits under noise model **B**. Theorem 2 indicates that, under this noise model, if approximate values of the minimum noise rate and the other noise rates are known, the presence of small noise drastically affects the classical simulatability of CT-ECS circuits.

2. Overview of Techniques

To prove Theorem 1, we generalize Bremner et al.'s proof for IQP circuits [6]. The first key point is to provide a general method for approximating the Fourier coefficients of the output probability distribution p. The probability distribution $\widetilde{p}_{\mathrm{A}}$, which we want to approximate, can be simply represented by the noise rate ε and the Fourier coefficients $\widehat{p}(s)$ of p for all $s \in \{0,1\}^n$ [12]. We show that, for any CT-ECS circuit on n qubits, there exists a polynomial-time classical algorithm for approximating each of the low-degree Fourier coefficients, i.e., $\widehat{p}(s)$ for any $s = s_1 \cdots s_n \in \{0,1\}^n$ with $\sum_{j=1}^n s_j = O(1)$. Bremner et al. [6] showed that such an algorithm exists for IQP circuits through a direct calculation of the Fourier coefficients for them. In contrast, we first provide a general relation between a quantum circuit and the Fourier coefficients of its output probability distribution. We then approximate each of the low-degree Fourier coefficients by combining this general relation with Nest's classical algorithm for approximating the inner product represented by a CT state and an ECS operation [16]. This general relation seems to be a new tool to investigate the output probability distribution of a quantum circuit and thus may be of independent interest.

The second key point is to approximate $\widetilde{p}_{\mathrm{A}}$ using an approximate value λ of ε. We define a *function* q that seems to be close to $\widetilde{p}_{\mathrm{A}}$ on the basis of its representation with ε and $\widehat{p}(s)$ for all $s \in \{0,1\}^n$. In contrast to Bremner et al.'s setting, we do not know ε. Thus, we define q using λ and an appropriate number of the low-degree Fourier coefficients. This number depends on λ, the constant α associated with the anti-concentration assumption, and the desired approximation accuracy. We then evaluate the approximation accuracy of q. Here, we need to care about the error caused by the difference between λ and ε. We upper-bound this error using the anti-concentration assumption.

To prove Theorem 2, we represent the effect of noise under noise model **B** as the effect of noise under noise model **A** with rate $\varepsilon_{\min}$ and the remaining effects. We do this by transforming the representation of the probability distribution $\widetilde{p}_{\mathrm{B}}$, which we want to approximate, with several basic properties of noise operators on real-valued functions over $\{0,1\}^n$ [12]. The obtained representation means that, to sample $\widetilde{p}_{\mathrm{B}}$, it suffices to sample the probability distribution $\widetilde{p}_{\mathrm{A}}$ (resulting from p) under noise model **A** with rate $\varepsilon_{\min}$ and then to classically simulate noise corresponding to the remaining effects. By Theorem 1, $\widetilde{p}_{\mathrm{A}}$ is classically simulatable. Moreover, we can classically simulate noise corresponding to the remaining effects using the approximate values of $\varepsilon_{\min}$ and the other noise rates.

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